Locally analytic representations in the moduli spaces of Lubin-Tate

Dissertation

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Abstract

Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathfrak{o}$, and let H_0 be a formal $\mathfrak{o}$-module of finite height over a separable closure of the residue class field of K. The Lubin-Tate moduli space X_m classifies deformations of H_0 equipped with level-m-structure. In this thesis, we study a particular type of p-adic representations originating from the action of $\mathrm{Aut}(H_0)$ on certain equivariant vector bundles on the generic fibre of X_m. We show that, for arbitrary level m, the Fréchet space of the global sections of these vector bundles is dual to a locally K-analytic representation of $\mathrm{Aut}(H_0)$ generalizing previous results of J. Kohlhaase in the case $K = \mathbb{Q}_p$ and $m = 0$. To get a better understanding of these representations, we compute their locally finite vectors. Essentially, all locally finite vectors arise from the global sections over the projective space via pullback along the Gross-Hopkins period map.

Zusammenfassung der Dissertation

Sei K eine endliche Erweiterung von $\mathbb{Q}_p$ mit dem Ring der ganzen Zahlen $\mathfrak{o}$, und sei H_0 ein formaler $\mathfrak{o}$-Modul von endlicher Höhe über einem separablen Abschluß des Restklassenkörpers von K. Der Lubin-Tate-Modulraum X_m klassifiziert die Deformationen von H_0 zusammen mit Level-m-Struktur. In dieser Doktorarbeit studieren wir einen besonderen Typ p-adischer Darstellungen, die sich aus der Aktion von $\mathrm{Aut}(H_0)$ auf bestimmten äquivarianten Vektorbündeln auf der generischen Faser von X_m ergeben. Für alle Level m zeigen wir, daß der Fréchet-Raum der globalen Schnitte dieser Vektorbündel dual zu einer lokal K-analytischen Darstellung von $\mathrm{Aut}(H_0)$ ist und die vorherigen Ergebnisse von J. Kohlhaase im den Fall $K = \mathbb{Q}_p$ und $m = 0$ verallgemeinern. Als ein erster Schritt, um diese Darstellungen besser zu verstehen, berechnen wir ihre lokal endlichen Vektoren. Im Wesentlichen entstehen alle lokal endlichen Vektoren durch Zurückziehen von globalen Schnitten des projektiven Raums über den Gross-Hopkins-Periodenmorphismus.

Contents

Introduction

Let p be a prime number and K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathfrak{o}$, uniformizer ϖ and residue class field k. Let $\breve{K}$ denote the completion of the maximal unramified extension of K and $\breve{\mathfrak{o}}$ denote its ring of integers. We fix a one dimensional formal $\mathfrak{o}$-module H_0 of height h over a separable closure k^{sep} of k and consider a problem of its deformations to local $\breve{\mathfrak{o}}$-algebras with residue class field k^{sep}. Extending the work of Lubin-Tate, Drinfeld showed that the functor of deformations together with level-m-structure is representable by a smooth affine formal $\breve{\mathfrak{o}}$-scheme $X_m := \text{Spf}(R_m)$. Further, the formal scheme X_0 commonly known as *the Lubin-Tate moduli space* is non-canonically isomorphic to the formal spectrum $\text{Spf}(\breve{\mathfrak{o}}[[u_1, \dots, u_{h-1}]])$. The Lubin-Tate moduli space X_0 plays a pivotal role in understanding the arithmetic of the local field K. For $h = 1$, it realizes the main theorem of local class field theory by explicitly constructing the totally ramified part of the maximal abelian extension of K.

The generic fibre X_m^{rig} of X_m has a structure of a rigid analytic variety over $\breve{K}$, and is a finite étale covering of the $(h-1)$-dimensional open unit polydisc X_0^{rig} over $\breve{K}$. The *Lubin-Tate tower* $\varprojlim_m X_m^{\text{rig}}$ carries actions of important groups which is what makes it an interesting object to study. For every $m \geq 0$, there are natural commuting actions of the groups $\Gamma := \text{Aut}(H_0)$ and $G_0 := GL_h(\mathfrak{o})$ on the universal deformation ring R_m, where Γ is isomorphic to the group $\mathfrak{o}_{B_h}^{\times}$ of units in the maximal order of the central K-division algebra B_h of invariant $1/h$. By functoriality, these group actions pass on to the rigid spaces X_m^{rig} and their global sections $R_m^{\text{rig}} := \mathcal{O}_{X_m^{\text{rig}}}(X_m^{\text{rig}})$. The G_0-action on X_m^{rig} factors through a quotient by the m-th principal congruence subgroup $G_m := 1 + \varpi^m M_h(\mathfrak{o})$ making $X_m^{\text{rig}} \longrightarrow X_0^{\text{rig}}$ a Galois covering with Galois group G_0/G_m. Allowing deformations of H_0 with quasi-isogenies of arbitrary heights in the moduli problem, the generalized Lubin-Tate tower admits an action of the triple product group $GL_h(K) \times B_h^{\times} \times W_K$, where W_K is the Weil group of K, and its l-adic étale cohomology realizes both the Jacquet-Langlands correspondence and the local Langlands correspondence for $GL_h(K)$ as shown by Harris and Taylor (cf. [**HT01**]). On the other hand, the action of Γ on $R_0 \simeq \breve{\mathfrak{o}}[[u_1, \dots, u_{h-1}]]$ is also shown to be related to important problems in stable homotopy theory by Devinatz and Hopkins (cf. [**DH95**]). Despite its general interest, the Γ-action on X_m^{rig} is poorly understood. In this thesis, we study the Γ-action in terms of interesting p-adic representations it gives rise to on the global sections of certain equivariant vector bundles on X_m^{rig}.

The Lie algebra $\text{Lie}(\mathbb{H}^{(m)})$ of the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(m)}$ over R_m is a free R_m-module of rank 1 and is equipped with a semilinear Γ-action. For any integer s, the s-fold tensor power $\text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ of $\text{Lie}(\mathbb{H}^{(m)})$ gives rise to a Γ-equivariant line bundle $(\mathcal{M}_m^s)^{\text{rig}}$ on X_m^{rig} with global sections M_m^s isomorphic to $R_m^{\text{rig}} \otimes_{R_m} \text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$. The functoriality induces a Γ-action on M_m^s such that the isomorphism $M_m^s \simeq R_m^{\text{rig}} \otimes_{R_m} \text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ is Γ-equivariant for the diagonal Γ-action on the right. The algebra R_m^{rig} is a nuclear $\breve{K}$-Fréchet space and endows M_m^s with the same type of topology. Note that $\Gamma \simeq \mathfrak{o}_{B_h}^{\times}$ is a compact locally K-analytic group, i.e. a compact Lie group over a p-adic field K. The $\breve{K}$-linear Γ-representation M_m^s is then an example of a representation of a p-adic analytic group on a p-adic locally convex vector space. Such representations have been systematically studied by Schneider and Teitelbaum in a series of articles including [**ST02**] and [**ST03**]. Motivated by the questions in the p-adic Langlands program, they introduce a category of *locally analytic representations*. This is a subcategory of continuous representations of Γ on locally convex $\breve{K}$-vector spaces large enough to contain smooth and finite dimensional algebraic representations, and which can be algebraized.

A locally analytic representation V is defined by the property that, for each $v \in V$, the orbit map $\Gamma \longrightarrow V$, $\gamma \mapsto \gamma(v)$ is locally on Γ given by a convergent power series with coefficients in V. These representations can be analysed by viewing them as modules over the *distribution algebra* $D(\Gamma, \breve{K})$ of $\breve{K}$-valued locally analytic distributions on Γ. A fundamental theorem [**ST02**], Corollary 3.4 of locally analytic representation theory says that the category of locally analytic Γ-representations on $\breve{K}$-vector spaces of compact type is anti-equivalent to the category of continuous $D(\Gamma, \breve{K})$-modules on nuclear $\breve{K}$-Fréchet spaces via the duality functor. We now state our main result regarding the Γ-representations M_m^s (cf. Theorem 3.3.6 and Theorem 3.4.7):

Theorem A. *For all $m \geq 0$ and $s \in \mathbb{Z}$, the action of Γ on the nuclear $\breve{K}$-Fréchet space M_m^s extends to a continuous action of the distribution algebra $D(\Gamma, \breve{K})$. Therefore, its strong topological $\breve{K}$-linear dual $(M_m^s)'_b$ is a locally K-analytic representation of Γ.*

We point out that the continuity and the differentiability of the Γ-action on $R_0^{\mathrm{rig}} = M_0^0$ was already known by the work of Gross and Hopkins (cf. [**GH94**], Proposition 19.2 and Proposition 24.2). In [**Koh14**], Theorem 3.5, Kohlhaase first proves the local analyticity of the Γ-action on M_0^s when $K = \mathbb{Q}_p$ making previous results of Gross and Hopkins more precise. Our result generalizes Kohlhaase's theorem to any finite base extension K of $\mathbb{Q}_p$ and also extends it to the higher levels m. The proof of the above theorem essentially follows the same approach as the one of Kohlhaase. Using continuity of the Γ-action on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ obtained in Theorem 2.2.9, one shows that it remains continuous after passing to rigidification (cf. Proposition 3.3.1 and Proposition 3.4.2). Together with the structure theory of the distribution algebra $D(\Gamma, \breve{K})$, this yields local $\mathbb{Q}_p$-analyticity of the Γ-action on M_m^s. To show that it is indeed locally K-analytic, for level $m = 0$, we make use of the explicit locally K-analytic action of Γ on the sections M_D^s of our line bundle over the *Gross-Hopkins fundamental domain* D. The local K-analyticity at level 0 then implies the local K-analyticity at higher levels because the covering morphisms are étale.

The Gross-Hopkins' *p-adic period map* $\Phi : X_0^{\mathrm{rig}} \longrightarrow \mathbb{P}_{\breve{K}}^{h-1}$ constructed in [**GH94**] turns out to be a crucial tool to understand the action of Γ on the Lubin-Tate moduli space. The period map Φ is an étale surjective morphism of rigid analytic spaces which is also Γ-equivariant for an explicitly known linear action of Γ on the projective space. It is constructed in such a way that the line bundle $(\mathcal{M}_m^s)^{\mathrm{rig}}$ is a pullback of $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ along the composition $\Phi_m : X_m^{\mathrm{rig}} \xrightarrow[\text{map}]{\text{covering}} X_0^{\mathrm{rig}} \xrightarrow{\Phi} \mathbb{P}_{\breve{K}}^{h-1}$ (cf. Remark 3.1.16). The fundamental domain D is an affinoid subdomain of X_0^{rig} on which Φ is injective. This allows us to obtain explicit formulae for the action of Γ on M_D^s and also for the action of the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\Gamma)$ on M_D^s (cf. Proposition 3.1.13, Remark 3.1.15 and Lemma 4.1.3).

Using the Lie algebra action, we prove very first results concerning the structure of the global sections M_m^s of equivariant vector bundles as $\breve{K}$-linear representations of Γ. A *locally finite vector* in M_m^s is a vector contained in a finite dimensional sub-H-representation of M_m^s for some open subgroup $H \subseteq \Gamma$. The set $(M_m^s)_{\mathrm{lf}}$ of all locally finite vectors is again a Γ-representation and is equal to the union of all finite dimensional subrepresentations of M_m^s due to compactness of Γ.

Consider the h-dimensional $\breve{K}$-linear representation $B_h \otimes_{K_h} \breve{K}$ of Γ on which $\Gamma \simeq \mathfrak{o}_{B_h}^\times$ acts by the left multiplication, and let $\breve{K}_m$ denote the m-th Lubin-Tate extension of $\breve{K}$ equipped with the Γ-action via $\mathfrak{o}_{B_h}^\times \xrightarrow{\text{Nrd}} \mathfrak{o}^\times \twoheadrightarrow (\mathfrak{o}/\varpi^m\mathfrak{o})^\times \simeq \mathrm{Gal}(\breve{K}_m/\breve{K})$. Here Nrd denotes the reduced norm. Then our next main result concerning the locally finite vectors in M_m^s shows that they all come from the projective space via pullback (cf. Remark 4.1.14).

Theorem B. *For all $m \geq 0$ and $s \in \mathbb{Z}$, we have*

$$(M_m^s)_{\mathrm{lf}} \simeq \breve{K}_m \otimes_{\breve{K}} \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K}) \simeq \breve{K}_m \otimes_{\breve{K}} \mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}})$$

as $\breve{K}[\Gamma]$-modules with the diagonal Γ-action on the tensor product. Thus, $(M_m^s)_{\mathrm{lf}}$ is zero if $s < 0$ and is a finite dimensional semi-simple locally algebraic Γ-representation if $s \geq 0$.

We prove the above theorem in several parts (cf. Corollary 4.1.13, Theorem 4.2.5, Theorem 4.2.13, Remark 4.2.15). The use of the actions of the groups G_0 and $G^0 := \{g \in GL_h(K) | \det(g) \in \mathfrak{o}^\times\}$ on the Lubin-Tate tower commuting with the Γ-action is a novel feature of the proof. Implicitly, we also make use of Strauch's computation of the geometrically connected components of X_m^{rig} (cf. [**Str08i**]). Namely, it implies that $(M_m^0)_{\mathrm{lf}} = (R_m^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}_m$ (cf. Theorem 4.2.2). The case $m > 0$, $s > 0$ is the most challenging one and involves a careful analysis of locally finite vectors in the sections over a particular admissible covering of X_m^{rig}.

It is noteworthy to mention that the first, well-studied example of p-adic representations stemming from equivariant vector bundles on p-adic analytic spaces did not concern the Lubin-Tate spaces. Rather, the geometric object here was *Drinfeld's upper half space* $\mathcal{X}$ obtained by deleting all K-rational hyperplanes from the projective space $\mathbb{P}_K^{h-1}$. The natural action of $GL_h(K)$ on the projective space stabilizes $\mathcal{X}$. Observe the difference with the Lubin-Tate case where the space is simple looking (open unit polydisc) but the action of the group Γ on it is much more complicated. Restricting any $GL_h(K)$-equivariant vector bundle $\mathcal{F}$ on $\mathbb{P}_K^{h-1}$ to $\mathcal{X}$ gives rise to a locally analytic $GL_h(K)$-representation on the strong dual of the nuclear Fréchet space $\mathcal{F}(\mathcal{X})$ of its global sections (cf. [**Orl08**]). By the work of Orlik-Strauch, the Jordan-Hölder series of these locally analytic representations is explicitly known. The Jordan-Hölder constituents are subrepresentations of parabolic inductions of certain locally algebraic representations (cf. [**OS15**]). Though our results are not as precise, we hope that they lay the groundwork for further study of locally analytic representations arising from the Lubin-Tate moduli spaces. We also mention a closely related work of Chi Yu Lo showing anlyticity of a certain rigid analytic group on a particular closed disc of X_0 in the case of height $h = 2$ (cf. [**Lo15**]).

We conclude this introduction by discussing a few important questions that we were unable to answer.

- A major open question concerning the locally analytic Γ-representations $(M_m^s)_b'$ is whether they are *admissible* or not in the sense of [**ST03**], Section 6. In the language of Section 3.3 and Section 3.4, this would require proving that the $\Lambda(\Gamma_*)_{\breve{K},l}$-modules $M_{m,l}^s$ are finitely generated and the natural maps $\Lambda(\Gamma_*)_{\breve{K},l} \otimes_{\Lambda(\Gamma_*)_{\breve{K},l+1}} M_{m,l+1}^s \longrightarrow M_{m,l}^s$ are isomorphisms for all l. By [**ST03**], Lemma 3.8, this would imply the *coadmissibility* of $D(\Gamma, \breve{K})$-modules M_m^s.
- Similar to the case of Drinfeld's upper half space, one would like to know the Jordan-Hölder series of the representations $(M_m^s)_b'$. In Theorem 4.1.7, for $s \geq 0$, we construct a filtration $M_D^s \supset V_s \supset \{0\}$ on the sections of $(\mathcal{M}_0^s)^{\mathrm{rig}}$ over D by closed Γ-stable subspaces with topologically irreducible quotients, where $V_s \simeq \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K})$. This gives rise

by duality to a filtration

$$\{0\} \subset (M_0^s/V_s)'_b \subset (M_0^s)'_b$$

by locally analytic Γ-representations on the strong dual of the global sections. The quotient $\frac{(M_0^s)'_b}{(M_0^s/V_s)'_b} \simeq (V_s)'_b$ is irreducible as a Γ-representation. However, the irreducibility of $(M_0^s/V_s)'_b$ is not yet clear.

Organization of the thesis:

The first chapter provides a brief overview of several important notions and results in non-archimedean functional analysis. These include locally convex vector spaces of compact type, locally analytic distribution algebra and Schneider-Teitelbaum's fundamental theorem in locally analytic representation theory. On the way, we prove a useful lemma on restriction of scalars (cf. Lemma 1.2.3). In order to explain the Fréchet structure of the distribution algebra, a short introduction to uniform pro-p-groups is given in Section 1.3.

In the second chapter, we present the Lubin-Tate deformation problem with Drinfeld's level-m-structures and define the actions of the groups Γ and G_0 on the universal deformation rings R_m. Although we are mainly interested in the Γ-action, the action of the covering group G_0 is needed later in the fourth chapter to compute locally finite vectors. The main results of this chapter concerning the continuity of the Γ-action on R_m and on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ are Theorem 2.2.8 and Theorem 2.2.9 which generalize [**Koh14**] Theorem 2.4 and [**Koh14**] Theorem 2.6 to higher levels $m > 0$ respectively. This continuity is used crucially later to prove local analyticity. In the last section of this chapter, we pass on to the rigidifications of the formal schemes $X_m = \mathrm{Spf}(R_m)$ and explain the $\breve{K}$-Fréchet algebra structure of their global sections R_m^{rig}. We also introduce our main example M_m^s of $\breve{K}$-linear Γ-representations which arise as global sections of rigid equivariant vector bundles on X_m^{rig}.

The third chapter is devoted to proving that the strong dual of M_m^s is a locally analytic Γ-representation for all s and m. The proof is divided into 3 steps which form Section 3.2, Section 3.3 and Section 3.4 respectively. The key ingredients of the proof are Proposition 3.3.1 and Proposition 3.4.2 exhibiting continuity of the Γ-action on R_m^{rig}. The first section 3.1 of this chapter summarizes Gross-Hopkins' construction of the period morphism Φ and of the fundamental domain D. The interpretation of the Γ-equivariant line bundles $(\mathcal{M}_m^s)^{\mathrm{rig}}$ as the pullback of invertible sheaves $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)$ on the projective space along the covering morphism Φ_m is explained in Remark 3.1.16 at the end.

The final chapter deals with the computation of Γ-locally finite vectors in M_m^s. The case $m = 0$ is solved using the formulae of the Lie algebra action obtained in Lemma 4.1.3, whereas the case $s \leq 0$ is solved using the property of *generic flatness* of the line bundle induced by the Lie algebra of the universal additive extension $\mathbb{E}^{(m)}$ (cf. Remark 3.1.6 and Theorem 4.2.5). For $m > 0$, $s > 0$, we make use of the action of a bigger group $G^0 \times B_h^\times$ on the Lubin-Tate tower. Let $D_m \subset X_m^{\mathrm{rig}}$ denote the inverse image of D under the covering map and Π be a uniformizer of B_h, then it is known that the "g-translates" of D_m with $g \in G^0$ cover $\Phi_m^{-1}(\Phi(D))$, and the "Π-translates" of $\Phi_m^{-1}(\Phi(D))$ cover X_m^{rig}. Using complete reducibility of finite dimensional $\mathfrak{sl}_h(K_h)$-representations and highest weight theory, the problem then reduces to computing invariants of the sections over the aforementioned covering under the upper nilpotent Lie algebra action. This is done in Proposition 4.2.10 and Proposition 4.2.12 before proving the final Theorem 4.2.13.

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Notation and conventions

Throughout the thesis, we fix the following notation. Other notation used in this thesis will be defined as needed along the way.

- $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of non-negative integers respectively. If $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}_0^r$ is an r-tuple of non-negative integers and $T = (T_1, \ldots, T_r)$ is a family of indeterminates for some $r \in \mathbb{N}$, then we set $|\alpha| := \alpha_1 + \ldots + \alpha_r$, and $T^\alpha := T_1^{\alpha_1} \cdots T_r^{\alpha_r}$.
- Unless stated otherwise, all rings are considered to be commutative with identity. A ring extension $A \subseteq B$ will be denoted by $B|A$, and its degree by $[B : A]$ if it is finite and free.
- Let p be a fixed prime number and let K be a finite field extension of $\mathbb{Q}_p$ with the valuation ring $\mathfrak{o}$. We fix a uniformizer ϖ of K and let $k := \mathfrak{o}/\varpi\mathfrak{o}$ denote its residue class field of characteristic p and cardinality q. The absolute value $|\cdot|$ of K is assumed to be normalized through $|p| = p^{-1}$.
- We denote by $\breve{K}$ the completion of the maximal unramified extension of K, and by $\breve{\mathfrak{o}}$ its valuation ring. We denote by σ the Frobenius automorphism of a separable closure k^{sep} of k, as well as its unique lift to a ring automorphism of $\breve{\mathfrak{o}}$ and the induced field automorphism of $\breve{K}$. We also fix an algebraic closure $\overline{\breve{K}}$ of $\breve{K}$ and denote its valuation ring by $\overline{\breve{\mathfrak{o}}}$. The absolute value $|\cdot|$ on K extends uniquely to $\breve{K}$, and to $\overline{\breve{K}}$.
- For a positive integer h, let K_h be the unramified extension of K of degree h, $\mathfrak{o}_h$ be its valuation ring, and B_h be the central K-division algebra of invariant $1/h$. We fix an embedding $K_h \hookrightarrow B_h$ and a uniformizer Π of B_h, satisfying $\Pi^h = \varpi$. Let $\mathrm{Nrd} : B_h \longrightarrow K$ denote the reduced norm of B_h over K.
- Let $\mathcal{C}$ be the category of commutative unital complete Noetherian local $\breve{\mathfrak{o}}$-algebras $R = (R, \mathfrak{m}_R)$ with residue class field k^{sep}.
- For a locally K-analytic group Γ and its Lie algebra $\mathfrak{g}$, we denote their scalar restrictions to $\mathbb{Q}_p$ by $\Gamma_{\mathbb{Q}_p}$ and $\mathfrak{g}_{\mathbb{Q}_p}$ respectively.
- $\mathbb{P}^{h-1}_{\breve{K}}$ always denotes the $(h-1)$-dimensional rigid analytic projective space over $\breve{K}$.

CHAPTER 1

Preliminaries on locally analytic representation theory

Let $L|K$ be a field extension such that L is spherically complete with respect to a non-archimedean absolute value $|\cdot|$ extending the one on K. We let $\mathfrak{o}_L$ denote the ring of integers L. For us, the field K is the "base field", while L is the "coefficient field", i.e. our analytic groups are defined over K, and their representations have coefficients in L. From chapter 2 onwards, the role of L will be played by $\breve{K}$.

1.1. Locally convex vector spaces

Let V be an L-vector space. A *lattice* Λ in V is an $\mathfrak{o}_L$-submodule satisfying the condition that for any vector $v \in V$ there is a non-zero scalar $a \in L^\times$ such that $av \in \Lambda$. The natural map $L \otimes_{\mathfrak{o}_L} \Lambda \longrightarrow V$, $(a \otimes v \longmapsto av)$ is bijective (cf. [**Sch02**], top of page 10). Let $(\Lambda_j)_{j \in J}$ be a non-empty family of lattices in V satisfying the following two conditions:

- for any $j \in J$ and $a \in L^\times$ there exists a $k \in J$ such that $\Lambda_k \subseteq a\Lambda_j$
- for any two $i, j \in J$ there exists a $k \in J$ such that $\Lambda_k \subseteq \Lambda_i \cap \Lambda_j$.

Then the subsets $v + \Lambda_j$ of V with $v \in V$ and $j \in J$ form the basis of a topology on V which is called the *locally convex topology on* V *defined by the family* $(\Lambda_j)_{j \in J}$.

Definition 1.1.1. A *locally convex* L*-vector space* is an L-vector space equipped with a locally convex topology.

The notion of a locally convex topology is equivalent to the notion of a topology defined by a family of seminorms (cf. [**Sch02**], Proposition 4.3 and Proposition 4.4). Thus any normed L-vector space (in particular L-Banach space) is a locally convex L-vector space.

Definition 1.1.2. A locally convex L-vector space is called an L*-Fréchet space* if it is metrizable and complete.

Any countable projective limit of Banach spaces with the projective limit topology is a Fréchet space (cf. [**Sch02**], Proposition 8.1).

A subset B of a locally convex L-vector space V is said to be *compactoid* if for any open lattice $\Lambda \subseteq V$, there are finitely many vectors $v_1, \dots, v_m$ such that $B \subseteq \Lambda + \mathfrak{o}_L v_1 + \dots + \mathfrak{o}_L v_m$. A compactoid and complete $\mathfrak{o}_L$-submodule of V is called *c-compact.* A continuous linear map $f : V \longrightarrow W$ between two locally convex L-vector spaces is called *compact* if there is an open lattice $\Lambda \subseteq V$ such that $f(\Lambda)$ is c-compact in W. Now consider a sequence $V_1 \longrightarrow \dots \longrightarrow V_n \xrightarrow{i_n} V_{n+1} \longrightarrow \dots$ of locally convex L-vector spaces V_n with continuous linear transition maps i_n. The vector space inductive limit $V := \varinjlim V_n$ equipped with the finest locally convex topology such that all the natural maps $j_n : V_n \longrightarrow V$ are continuous is called the *locally convex inductive limit* of this sequence.

Definition 1.1.3. A locally convex L-vector space is called *vector space of compact type* if it is a locally convex inductive limit of a sequence of L-Banach spaces with injective and compact transition maps.

We say a locally convex L-vector space V is *barrelled* if every closed lattice in V is open. Fréchet spaces and vector spaces of compact type are examples of barrelled locally convex vector spaces (cf. [**Sch02**], Examples on page 40).

For locally convex L-vector spaces V and W, the set $\mathcal{L}(V,W)$ of all continuous linear maps from V to W can be equipped with several locally convex topologies (cf. [**Sch02**], Lemma 6.4). We write V'_s and V'_b to denote the dual vector space of V equipped with the *weak* and *strong* topology respectively (cf. [**Sch02**], Examples on page 35). A Hausdorff locally convex L-vector space V is called *reflexive* if the duality map $\delta : V \longrightarrow (V'_b)'_b$ $((\delta(v))(l) := l(v))$ is a topological isomorphism.

Theorem 1.1.4. *The duality functor $V \longmapsto V'_b$ is an anti-equivalence between the category of vector spaces of compact type and the category of nuclear Fréchet spaces.*

Proof. See [**ST02**] Corollary 1.4. □

For the notion of a nuclear locally convex L-vector space, we refer the reader to [**Sch02**], Section 19.

1.2. Locally analytic functions and distributions

Let $(V, \|\cdot\|)$ be an L-Banach space. For any $\varepsilon > 0$ the power series $f(X) = \sum_{\alpha\in\mathbb{N}_0^r} v_\alpha X^\alpha$ in r variables $X = (X_1,\dots,X_r)$ with $v_\alpha \in V$ is called *ε-convergent* if $\lim_{|\alpha|\to\infty} \|v_\alpha\|\varepsilon^{|\alpha|} = 0$. We denote by $\mathcal{F}_\varepsilon(K^r, V)$ the L-vector space of all ε-convergent power series in r variables with coefficients from V. This is a Banach space with respect to the norm $\|f\|_\varepsilon := \max_\alpha \|v_\alpha\|\varepsilon^{|\alpha|}$.

Definition 1.2.1. Let $U \subseteq K^r$ be an open subset. A function $f : U \longrightarrow V$ is called *locally K-analytic*, if for any point $x \in U$ there exists a closed ball $B_\varepsilon(x) \subseteq U$ of radius $\varepsilon > 0$ around x, and a power series $F \in \mathcal{F}_\varepsilon(K^r, V)$ such that $f(y) = F(x-y)$ for all $y \in B_\varepsilon(x)$.

Let M be a Hausdorff topological space. A chart (U,φ) of dimension n for M is an open subset $U \subseteq M$ together with a map $\varphi : U \longrightarrow K^n$ such that $\varphi(U)$ is open in K^n and $\varphi : U \simeq \varphi(U)$ is a homeomorphism. Two charts (U_1,φ_1) and (U_2,φ_2) are said to be compatible if the maps $\varphi_2\circ\varphi_1^{-1} : \varphi_1(U_1\cap U_2) \longrightarrow \varphi_2(U_1\cap U_2)$ and $\varphi_1\circ\varphi_2^{-1} : \varphi_2(U_1\cap U_2) \longrightarrow \varphi_1(U_1\cap U_2)$ are locally K-analytic. Two compatible charts with non-empty intersection have the same dimension. An atlas for M is a collection of compatible charts that cover M. Given an atlas, one can enlarge it by adding all charts compatible with every chart in the given atlas, giving a maximal atlas. The topological space M with such a maximal atlas is called a *locally K-analytic manifold*. In this case, M is said to have dimension n if all charts in its atlas have dimension n.

Definition 1.2.2. Let M be a locally K-analytic manifold and V be an L-Banach space. A function $f : M \to V$ is called *locally K-analytic* if $f\circ\varphi^{-1} : \varphi(U) \to V$ is locally K-analytic for any chart (U,φ) for M.

A map $f : M \longrightarrow N$ between two locally K-analytic manifolds is said to be locally K-analytic if for any point $x \in M$ there exists a chart (U,φ) for M around x and a chart (V,ψ) for N around $f(x)$ such that $f(U) \subseteq V$ and the map $\psi\circ f\circ\varphi^{-1} : \varphi(U) \longrightarrow K^n$ is locally K-analytic.

At several places later, we shall use the following lemma on restriction of scalars.

Lemma 1.2.3. *Let $K'|K$ be a field extension of finite degree d. Let M and N be locally K'-analytic manifolds of dimensions m and n respectively, and $g : M \longrightarrow N$ be a locally K'-analytic map. Then M and N are locally K-analytic manifolds of dimensions md and nd respectively, and g is a locally K-analytic map too.*

Proof. Let $\mathcal{A}' = \{(U_i, \phi_i, K'^m)\}_{i\in I}$ be an atlas for M over K'. We claim that $\mathcal{A} = \{(U_i, \phi_i, K^{md})\}_{i\in I}$ is an atlas for M over K. Since $K'^m \simeq K^{md}$ is a finite dimensional K-vector space, any two vector space norms on it are equivalent. So the topology induced by the supremum norm $\|\cdot\|_{K'^m}$ of K'^m, and that induced by the supremum norm $\|\cdot\|_{K^{md}}$ of K^{md} are the same. Hence, (U_i, ϕ_i, K^{md}) is a chart for every $i \in I$.

Now, for any two $i, j \in I$, the map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow K'^m$ is locally K'-analytic, i.e. for every $x \in \phi_i(U_i \cap U_j)$, there exists $\varepsilon_x > 0$ and $s_x(X) = \sum_{\beta'\in\mathbb{N}_0^m}(s_x)_{\beta'}X^{\beta'} \in \mathcal{F}_{\varepsilon_x}(K'^m, K'^m)$ such that for all $y \in B_{\varepsilon_x}^{K'^m}(x)$, $\phi_j \circ \phi_i^{-1}(y) = s_x(y - x)$. Let $\{a_1, a_2, \ldots, a_d\}$ be a basis of K'/K, and let $m_0 := \max\{|a_1|, \ldots, |a_d|\}$. Then $B_{\varepsilon_x/m_0}^{K^{md}}(x) \subseteq B_{\varepsilon_x}^{K'^m}(x)$. For $y \in B_{\varepsilon_x/m_0}^{K^{md}}(x) \subseteq B_{\varepsilon_x}^{K'^m}(x)$, we have

$$s_x(y - x) = \sum_{\beta'\in\mathbb{N}_0^m}(s_x)_{\beta'}(y_1 - x_1)^{\beta'_1} \ldots (y_m - x_m)^{\beta'_m}.$$

Writing z for $(y - x)$, we have

$$\begin{aligned}
s_x(z) &= \sum_{\beta'\in\mathbb{N}_0^m}(s_x)_{\beta'} z_1^{\beta'_1} \ldots z_m^{\beta'_m} \\
&= \sum_{\beta'\in\mathbb{N}_0^m}(s_x)_{\beta'}(z_{11}a_1 + \cdots + z_{1d}a_d)^{\beta'_1} \ldots (z_{m1}a_1 + \cdots + z_{md}a_d)^{\beta'_m} \\
&= \sum_{\beta\in\mathbb{N}_0^{md}}(t_x)_\beta z_{11}^{\beta_1} \ldots z_{1d}^{\beta_d} z_{21}^{\beta_{d+1}} \ldots z_{m(d-1)}^{\beta_{md-1}} z_{md}^{\beta_{md}}.
\end{aligned}$$

Given $\beta \in \mathbb{N}_0^{md}$, the monomial $z_{11}^{\beta_1} \ldots z_{md}^{\beta_{md}}$ in the previous expression appears only in $(z_{11}a_1 + \cdots + z_{1d}a_d)^{(\beta_1+\cdots+\beta_d)} \ldots (z_{m1}a_1 + \cdots + z_{md}a_d)^{(\beta_{(m-1)d+1}+\cdots+\beta_{md})}$ in the expression above it. By comparing coefficients, we get

$$(t_x)_\beta = n_\beta a_1^{(\beta_1+\beta_{d+1}+\cdots+\beta_{(m-1)d+1})} \ldots a_d^{(\beta_d+\beta_{2d}+\cdots+\beta_{md})}(s_x)_{\beta'}$$

for some integer n_β and $\beta' = ((\beta_1 + \cdots + \beta_d), \ldots, (\beta_{(m-1)d+1} + \cdots + \beta_{md}))$. Thus,

$$\|(t_x)_\beta\|_{K^{md}} \leq C\|(t_x)_\beta\|_{K'^m} \leq C\|(s_x)_{\beta'}\|_{K'^m} m_0^{|\beta|}$$

for some constant C. Therefore,

$$\|(t_x)_\beta\|_{K^{md}}\left(\frac{\varepsilon_x}{m_0}\right)^{|\beta|} \leq C\|(s_x)_{\beta'}\|_{K'^m}\varepsilon_x^{|\beta|} = C\|(s_x)_{\beta'}\|_{K'^m}\varepsilon_x^{|\beta'|}.$$

As $|\beta| \to \infty$, $|\beta'| \to \infty$ and the right hand side of the above inequality tends to 0. So the left hand side also tends to 0 as $|\beta| \to \infty$. Hence, $\sum_{\beta\in\mathbb{N}_0^{md}}(t_x)_\beta Y^\beta \in \mathcal{F}_{\varepsilon_x/m_0}(K^{md}, K^{md})$, and the map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow K^{md}$ is locally K-analytic for any $x \in \phi_i(U_i \cap U_j)$. This implies that $\mathcal{A}$ is an atlas for M over K, and M is a locally K-analytic manifold of dimension md.

A similar argument as above can be used to show the local analyticity of the map g over K. □

Let M be a strictly paracompact m-dimensional locally K-analytic manifold. This means that any open covering of M can be refined into a covering by pairwise disjoint open subsets. Let V be a Hausdorff locally convex L-vector space. In this situation the locally convex L-vector space $C^{an}(M, V)$ of all V-valued locally K-analytic functions on M can be defined as follows. A V-index $\mathcal{I}$ on M is a family of triples $\{(D_i, \phi_i, V_i)\}_{i\in I}$ where the D_i are pairwise disjoint open subsets of M which cover M, each $\phi_i : D_i \longrightarrow K^m$ is a chart for M such that $\varphi_i(D_i) = B_{\varepsilon_i}(x_i)$ and $V_i \hookrightarrow V$ is a continuous linear injection from an L-Banach space V_i into V. Let $\mathcal{F}_{\phi_i}(V_i)$ denote the L-Banach space of all functions $f : D_i \longrightarrow V_i$ such that $f \circ \phi_i^{-1}(x) = F_i(x - x_i)$ for some $F_i \in \mathcal{F}_{\varepsilon_i}(K^m, V_i)$. Then we form the locally convex direct product $\mathcal{F}_{\mathcal{I}}(V) := \prod_{i\in I} \mathcal{F}_{\phi_i}(V_i)$ and we define $C^{an}(M,V) := \varinjlim_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(V)$ (cf. [**Sch11**], Section 11).

Definition 1.2.4. The elements of the strong dual $D(M,V) := C^{an}(M,V)'_b$ of $C^{an}(M,V)$ are called *V-valued locally K-analytic distributions on M.*

Denote by $\delta_x \in D(M,L)$ the Dirac distributions defined by $\delta_x(f) := f(x)$. Then there is a unique natural L-linear map $\int : C^{an}(M,V) \longrightarrow \mathcal{L}(D(M,L),V)$ such that for all $x \in M$, $\int(f)(\delta_x) = f(x)$ which can be considered as the integration map (cf. [**ST02**], Theorem 2.2).

Definition 1.2.5. A *locally K-analytic group G* is a locally K-analytic manifold which also carries a group structure such that the multiplication map $G \times G \longrightarrow G$ $((g,h) \longmapsto gh)$ is locally K-analytic.

We note that by [**Sch11**], Proposition 13.6 the inversion $G \longrightarrow G$, $(g \mapsto g^{-1})$, is automatically locally K-analytic.

If G is a locally K-analytic group, then $D(G,L)$ is a unital, associative L-algebra with separately continuous multiplication such that the inclusion $L[G] \hookrightarrow D(G,L)$, $(\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g \delta_g)$, is a homomorphism of L-algebras (cf. [**ST02**], Section 2). One method to explicitly construct elements in $D(G,L)$ is through the Lie algebra $\mathfrak{g}$ of G. From the discussion after [**ST02**], Proposition 2.3, one has the inclusion of algebras $U(\mathfrak{g}) \otimes_K L \hookrightarrow D(G,L)$ where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ over K.

1.3. Structure of the distribution algebra

If G is a compact locally K-analytic group then $D(G,L)$ is an L-Fréchet algebra (cf. [**ST02**], Proposition 2.3). Further, if G is a *uniform pro p-group* then this Fréchet structure can be made very explicit. This is what we will explain next.

Let p be a prime number. A *pro p-group* is a profinite group in which every open normal subgroup has index equal to some power of p. A topological group G is a pro p-group if and only if G is topologically isomorphic to an inverse limit of finite p-groups (cf. [**DDMS03**], Proposition 1.12).

Let G be a pro p-group. Set $P_1(G) := G$ and $P_{i+1}(G) := \overline{P_i(G)^p[P_i(G),G]}$ for $i \geq 1$. Here $P_i(G)^p[P_i(G),G]$ denotes the subgroup of G generated by the pth powers of elements of $P_i(G)$ and by all commutators $[a,b]$ with $a \in P_i(G)$ and $b \in G$; $\overline{X}$ denotes the topological closure of a subset X of G. If G is topologically finitely generated then $P_i(G)$ is open in G for each i and the set $\{P_i(G) | i \geq 1\}$ is a base for the neighbourhoods of identity of G (cf. [**DDMS03**], Proposition 1.16). A pro p-group is called *powerful* if p is odd and $G/\overline{G^p}$ is abelian or if $p = 2$ and $G/\overline{G^4}$ is abelian.

Definition 1.3.1. A pro p-group G is called *uniform* if it is topologically finitely generated, powerful and if $|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$ for all $i \geq 1$.

We call the minimal cardinality of a topological generating set of a uniform pro p-group G the *rank* of G. The following easy lemma will be handy later.

Lemma 1.3.2. *If G is a uniform pro p-group of rank r then $P_i(G)$ is also a uniform pro p-group of rank r for all $i \geq 1$.*

Proof. Since $P_i(G)$ is open and closed in G, it follows from [**DDMS03**], Proposition 1.7 and Proposition 1.11 (i) that $P_i(G)$ is topologically finitely generated pro p-group for all $i \geq 1$. Now [**DDMS03**], Theorem 3.6 (i) tells us that $P_i(G)$ is powerful and $P_i(P_j(G)) = P_{i+j-1}(G)$ implies that it is uniform for all $i \geq 1$. The rank statement follows from [**DDMS03**], Proposition 4.4. $\square$

Locally $\mathbb{Q}_p$-analytic groups have a following group-theoretic characterization due to Lazard:

Theorem 1.3.3. *A topological group G is locally $\mathbb{Q}_p$-analytic if and only if it contains an open subgroup which is a uniform pro-p group.*

Proof. See [**DDMS03**], Theorem 8.32. □

Let G be a uniform pro p-group of rank r and $\{a_1, \ldots, a_r\}$ be an (ordered) minimal topological generating set for G. Then the mapping $(\lambda_1, \ldots, \lambda_r) \mapsto a_1^{\lambda_1} \ldots a_r^{\lambda_r}$ from $\mathbb{Z}_p^r$ to G is a well-defined homeomorphism (cf. [**DDMS03**], Theorem 4.9). Of course, unless G is commutative, this is not an isomorphism of groups. Using the inverse as a global chart for G, we may identify the locally convex L-vector spaces of locally $\mathbb{Q}_p$-analytic functions

$$C^{an}(\mathbb{Z}_p^r, L) \simeq C^{an}(G, L),$$

and also the locally convex L-vector spaces of locally analytic $\mathbb{Q}_p$-distributions

$$D(\mathbb{Z}_p^r, L) \simeq D(G, L)$$

after dualizing. By Amice's theorem, $f \in C^{an}(\mathbb{Z}_p^r, L)$ if and only if the coefficients $c_\alpha \in L$ of its Mahler expansion satisfy $\lim_{|\alpha|\to\infty} |c_\alpha| r^{|\alpha|} = 0$ for some real number $r > 1$ (cf. [**ST03**], page 16). Set $b_i := a_i - 1 \in \mathfrak{o}_L[G] \subset D(G, L)$ for all $1 \leq i \leq r$, and $b^\alpha := b_1^{\alpha_1} \cdots b_r^{\alpha_r}$ for $\alpha \in \mathbb{N}_0^r$. Then it follows that any distribution $\mu \in D(G, L)$ has a unique convergent expansion of the form

$$\mu = \sum_{\alpha \in \mathbb{N}_0^r} d_\alpha b^\alpha$$

with $d_\alpha \in L$ such that $\lim_{|\alpha|\to\infty} |d_\alpha| r^{|\alpha|} = 0$ for all $0 < r < 1$. The family of norms defined by

$$\Big\| \sum_{\alpha \in \mathbb{N}_0^r} d_\alpha b^\alpha \Big\|_r := \sup_{\alpha \in \mathbb{N}_0^r} |d_\alpha| r^{|\alpha|}$$

with $1/p \leq r < 1$ endow $D(G, L)$ a structure of an L-Fréchet algebra (cf. [**ST03**], Proposition 4.2). In fact, the norms $\| \cdot \|_r$ for $1/p \leq r < 1$ are submultiplicative in the sense that $\|\mu\mu'\|_r \leq \|\mu\|_r \|\mu'\|_r$ for all $\mu, \mu' \in D(G, L)$.

1.4. Locally analytic representations

Let G be a finite dimensional locally K-analytic group.

Definition 1.4.1. A *locally K-analytic representation* of G (over L) is a barrelled Hausdorff locally convex L-vector space equipped with a G-action by continuous linear endomorphisms such that for each $v \in V$, the orbit map $\rho_v = (g \mapsto g(v))$ is a V-valued locally K-analytic function on G.

Let V a be locally K-analytic representation of G. Then the Lie algebra $\mathfrak{g}$ of G acts on V by continuous linear endomorphisms as follows : if $\mathfrak{x} \in \mathfrak{g}$ and $v \in V$ then

$$\mathfrak{x}(v) := \frac{d}{dt} \exp(t\mathfrak{x})(v)\Big|_{t=0} = \lim_{t\to 0} \frac{\exp(t\mathfrak{x})(v) - v}{t} \tag{1.4.2}$$

where $\exp : \mathfrak{g} \longrightarrow G$ is the exponential map defined locally around zero on $\mathfrak{g}$. This $\mathfrak{g}$-action extends to an action of the universal enveloping algebra $U(\mathfrak{g})$ on V by continuous linear endomorphisms. Using the integration map, we obtain a $D(G, L)$-module structure on V via $\mu(v) := \int (\rho_v)(\mu)$, which is separately continuous and extends the action of $U(\mathfrak{g})$ on V (cf. [**ST02**], Proposition 3.2).

Theorem 1.4.3 (Schneider-Teitelbaum)**.** *If G is compact then the duality functor $V \longmapsto V'_b$ gives an anti-equivalence between the following categories*

locally K-analytic representations of G on L-vector spaces of compact type with continuous linear G-maps	$\longrightarrow$	continuous $D(G, L)$-modules on nuclear L-Fréchet spaces with continuous $D(G, L)$-module maps

PROOF. See [**ST02**], Corollary 3.4. □

CHAPTER 2

The Lubin-Tate moduli spaces and the group actions

The Lubin-Tate moduli spaces are deformation spaces that parametrize the deformations of formal $\mathfrak{o}$-modules together with level structures. Before explaining this deformation problem precisely, we first recall some basic definitions and results on formal $\mathfrak{o}$-modules.

2.1. Deformations of formal $\mathfrak{o}$-modules with level structures

Definition 2.1.1. Let A be a commutative unital $\mathfrak{o}$-algebra with the structure map $i : \mathfrak{o} \longrightarrow A$. A (one-dimensional) *formal $\mathfrak{o}$-module F over A* is a one-dimensional commutative formal group law $F(X, Y) \in A[[X, Y]]$ together with a ring homomorphism $[\,\cdot\,]_F : \mathfrak{o} \longrightarrow \mathrm{End}(F)$ such that $[a]_F(X) = i(a)X \mod \deg 2$ for all $a \in \mathfrak{o}$. Here $\mathrm{End}(F)$ denotes the ring of endomorphisms of the formal group law F over A.

Higher dimensional formal $\mathfrak{o}$-modules are defined similarly, i.e. a formal $\mathfrak{o}$-module F over A of dimension d for some positive integer d is a d-dimensional commutative formal group law $F(X, Y) = (F_j(X_1, \ldots, X_d, Y_1, \ldots, Y_d))_{1\le j\le d}$ together with a ring homomorphism $\mathfrak{o} \longrightarrow \mathrm{End}(F)$, $a \mapsto [a]_F(X) = (([a]_F)_j(X_1, \ldots, X_d))_{1\le j\le d}$ satisfying $([a]_F)_j(X_1, \ldots, X_d) = i(a)X_j \mod \deg 2$ for all $a \in \mathfrak{o}$ and for all $1 \le j \le d$. However, as we are mostly concerned with the one dimensional formal $\mathfrak{o}$-modules in this thesis, we restrict ourselves to the one-dimensional case in this exposition.

Example 2.1.2.
(1) The additive formal group law $\mathbb{G}_a(X, Y) = X + Y$ is a formal $\mathfrak{o}$-module over any $\mathfrak{o}$-algebra A with the $\mathfrak{o}$-multiplication given by $[a]_{\mathbb{G}_a}(X) = i(a)X$.
(2) The multiplicative formal group law $\mathbb{G}_m(X, Y) = (1 + X)(1 + Y) - 1$ becomes a formal $\mathbb{Z}_p$-module over $\mathbb{Z}_p$ for the $\mathbb{Z}_p$-multiplication given by $[a]_{\mathbb{G}_m}(X) = \sum_{n=1}^{\infty}\binom{a}{n}X^n$. Here $\binom{a}{n} := \frac{a(a-1)\cdots(a-n+1)}{n!} \in \mathbb{Z}_p$ for all $a \in \mathbb{Z}_p$.

Given a homomorphism of $\mathfrak{o}$-algebras $f : A \longrightarrow B$ and a formal $\mathfrak{o}$-module F over A, its base change $F \otimes_A B$ (or pushforward f_*F) is a formal $\mathfrak{o}$-module over B obtained by applying f to the coefficients of $F(X, Y)$. The $\mathfrak{o}$-action $[\,\cdot\,]_{F\otimes_A B}$ on $F \otimes_A B$ is given by the composition of the induced map $\mathrm{End}(F) \xrightarrow[\text{to coeff}]{\text{apply } f} \mathrm{End}(F \otimes_A B)$ with $[\,\cdot\,]_F$.

A homomorphism between formal $\mathfrak{o}$-modules is a homomorphism of formal group laws which commutes with $\mathfrak{o}$-multiplications. By abuse of notation, we denote the endomorphism ring of a formal $\mathfrak{o}$-module F over A again by $\mathrm{End}(F)$ or by $\mathrm{End}_A(F)$ to emphasize the ring A over which the endomorphisms are defined.

Definition 2.1.3. Let $A[[X]]dX$ be the free $A[[X]]$-module of rank 1 generated by dX. For a formal $\mathfrak{o}$-module F over A, the A-module $\omega(F)$ of *invariant differentials on F* is an A-submodule of $A[[X]]dX$ consisting of differentials $f(X)dX$ that satisfy $f(F(X, Y))d(F(X, Y)) = f(X)dX + f(Y)dY$ and $f([a]_F(X))d([a]_F(X)) = i(a)f(X)dX$ for all $a \in \mathfrak{o}$.

It follows from [**GH94**], Proposition 2.2 that $\omega(F)$ is free of rank 1 over A, and generated by $F_X(0,X)^{-1}dX$, where $F_X(X,Y)$ denotes the formal partial derivative of $F(X,Y)$ with respect to X. The dual notion to $\omega(F)$ is the module of *F-invariant A-linear derivations of* $A[[X]]$, which is a free A-module of rank 1 spanned by the derivation $F_X(0,X)\frac{d}{dX}$ (cf. [**GH94**], Section 2).

Definition 2.1.4. The *Lie algebra* $\mathrm{Lie}(F)$ of a formal $\mathfrak{o}$-module F over A is the tangent space $\mathrm{Hom}_A((X)/(X)^2, A)$ of its coordinate ring $A[[X]]$ (equipped with the trivial Lie bracket). By definition of formal modules, the endomorphism $[a]_F(X)$ with $a \in \mathfrak{o}$ acts on $\mathrm{Lie}(F)$ by the multiplication by $i(a)$.

Clearly, $\mathrm{Lie}(F)$ is a free A-module of rank 1 with a basis given by the formal derivative $\frac{d}{dX}$. The map $a\frac{d}{dX} \longmapsto aF_X(0,X)\frac{d}{dX}$ gives a canonical isomorphism from $\mathrm{Lie}(F)$ to the module of F-invariant A-linear derivations of $A[[X]]$. Thus, the Lie algebra $\mathrm{Lie}(F)$ of F can be interpreted as the A-linear dual of the free A-module $\omega(F)$ of invariant differentials on F.

A homomorphism of $\varphi : F \longrightarrow F'$ of formal $\mathfrak{o}$-modules over A induces a natural A-linear map $\mathrm{Lie}(\varphi) : \mathrm{Lie}(F) \longrightarrow \mathrm{Lie}(F')$ between their Lie algebras mapping the basis element $\frac{d}{dX}$ to $\varphi'(0)\frac{d}{dX'}$, where $\varphi'(X)$ is the formal derivative of the formal power series $\varphi(X) \in A[[X]]$. It then follows readily that $\mathrm{Lie}(\varphi \circ \psi) = \mathrm{Lie}(\varphi) \circ \mathrm{Lie}(\psi)$ for any two homomorphisms $\psi : F \longrightarrow F'$ and $\varphi : F' \longrightarrow F''$ of formal $\mathfrak{o}$-modules over A.

A homomorphism $\varphi : F \longrightarrow \mathbb{G}_a$ of formal $\mathfrak{o}$-modules over A gives rise to an invariant differential $d\varphi = \varphi'(X)dX \in \omega(F)$. If A is a flat $\mathfrak{o}$-algebra and $\omega \in \omega(F)$ is a basis, then there exists a unique isomorphism $\varphi : F \xrightarrow{\sim} \mathbb{G}_a$ over $A \otimes_{\mathfrak{o}} K$ such that $d\varphi = \omega$ (cf. [**GH94**], Proposition 3.2). The power series $\varphi(X) \in (A \otimes_{\mathfrak{o}} K)[[X]]$ is called a *logarithm of* F, and can be constructed by formally integrating ω.

Definition 2.1.5. For a formal $\mathfrak{o}$-module F over k^{sep}, either $[\varpi]_F(X) = 0$ or there exists a unique positive integer h such that $[\varpi]_F(X) = f(X^{q^h})$ with $f'(0) \neq 0$ (cf. [**GH94**], Lemma 4.1). In the latter case, we say F has *height* h. For example, the multiplicative formal $\mathbb{Z}_p$-module $\mathbb{G}_m \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$ (cf. Example 2.1.2 (2)) over $\overline{\mathbb{F}}_p$ has height 1.

We now fix a one-dimensional formal $\mathfrak{o}$-module H_0 of finite height h over k^{sep} which is defined over k. The formal module H_0 is unique up to isomorphism, and we have

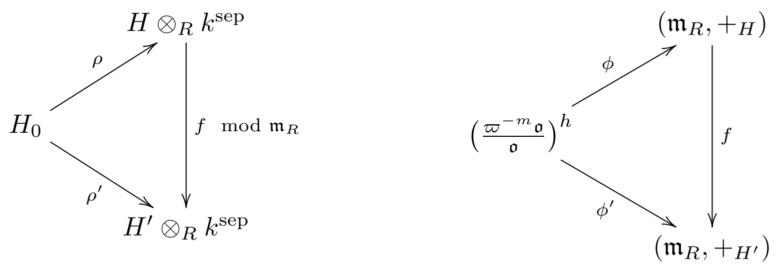
$$\mathrm{End}(H_0) \simeq \mathfrak{o}_{B_h} \tag{2.1.6}$$

where $\mathfrak{o}_{B_h}$ is the valuation ring of the central K-division algebra B_h of invariant $1/h$ (cf. [**Dri74**] Proposition 1.6 and 1.7). We wish to consider the liftings of H_0 to the objects of $\mathcal{C}$ together with certain additional data defined below. Recall that $\mathcal{C}$ denotes the category of complete Noetherian local $\breve{\mathfrak{o}}$-algebras with residue class field k^{sep}.

Definition 2.1.7. Let $R \in \mathrm{Ob}(\mathcal{C})$ and H be a formal $\mathfrak{o}$-module over R.

- A pair (H, ρ), where $\rho : H_0 \xrightarrow{\sim} H \otimes_R k^{\mathrm{sep}}$ is an isomorphism of formal $\mathfrak{o}$-modules over k^{sep}, is called *a deformation of H_0 to R.*
- Denote by $(\mathfrak{m}_R, +_H)$ the abstract $\mathfrak{o}$-module $\mathfrak{m}_R$ in which addition and $\mathfrak{o}$-multiplication are defined as $x +_H y := H(x,y)$ and $ax := [a]_H(x)$ respectively for all $x, y \in \mathfrak{m}_R$, $a \in \mathfrak{o}$. For a non-negative integer m, a *Drinfeld level-m-structure on H* is a homomorphism $\phi : \left(\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}}\right)^h \longrightarrow (\mathfrak{m}_R, +_H)$ of abstract $\mathfrak{o}$-modules such that $\prod_{\alpha \in (\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}})^h}(X - \phi(\alpha))$ divides $[\varpi^m]_H(X)$ in $R[[X]]$.
- We call the triple (H, ρ, ϕ) *a deformation of H_0 to R with level-m-structure* if (H, ρ) is a deformation of H_0 to R and ϕ is a Drinfeld level-m-structure on H.

Two deformations (H, ρ, ϕ) and (H', ρ', ϕ') of H_0 to R with level-m-structures are isomorphic if there is an isomorphism $f : H \xrightarrow{\sim} H'$ of formal $\mathfrak{o}$-modules over R making the following diagrams commutative ($f \mod \mathfrak{m}_R$ is the isomorphism obtained by reducing the coefficients of f modulo $\mathfrak{m}_R$):



For any integer $m \geq 0$, consider the set valued functor $\mathrm{Def}_{H_0}^m : \mathcal{C} \longrightarrow \mathrm{Set}$, which associates to an object R of $\mathcal{C}$ the set of isomorphism classes of deformations of H_0 to R with level-m-structures. For a morphism $\varphi : R \longrightarrow R'$ in $\mathcal{C}$, $\mathrm{Def}_{H_0}^m(\varphi)$ is defined by sending a class $[(H, \rho, \phi)]$ to the class $[(H \otimes_R R', \rho, \varphi \circ \phi)]$. Notice that $\rho : H_0 \xrightarrow{\sim} H \otimes_R k^{\mathrm{sep}} \simeq (H \otimes_R R') \otimes_{R'} k^{\mathrm{sep}}$. We denote the triple $(H \otimes_R R', \rho, \varphi \circ \phi)$ by $\varphi_*(H, \rho, \phi)$ for simplicity.

Theorem 2.1.8 (Lubin-Tate, Drinfeld)**.**

(1) *The functor* $\mathrm{Def}_{H_0}^m$ *is representable by a regular local ring* R_m *of dimension* h *for all* $m \geq 0$.
(2) *For any two integers* $0 \leq m \leq m'$, *the natural transformation* $\mathrm{Def}_{H_0}^{m'} \longrightarrow \mathrm{Def}_{H_0}^m$ *of functors defined by sending a class* $[(H, \rho, \phi)]$ *to* $[(H, \rho, \phi|_{(\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}})^h})]$ *induces a homomorphism of local rings* $R_m \longrightarrow R_{m'}$ *which is finite and flat.*
(3) *The ring* R_0 *is non-canonically isomorphic to the ring* $\breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$ *of formal power series in* $h-1$ *indeterminates over* $\breve{\mathfrak{o}}$.

Proof. See [**Dri74**], Proposition 4.2 and 4.3. □

Definition 2.1.9. The smooth affine formal $\breve{\mathfrak{o}}$-scheme $X_m := \mathrm{Spf}(R_m)$ is called *the Lubin-Tate moduli space of finite level* m. In the literature, often X_0 is referred as *the Lubin-Tate moduli space* because the above theorem was initially proven by Lubin and Tate for the case $m = 0$ and $\mathfrak{o} = \mathbb{Z}_p$.

Let us denote the universal deformation of H_0 to R_m with level-m-structure by the triple $(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})$. Here $\mathbb{H}^{(m)} = \mathbb{H}^{(0)} \otimes_{R_0} R_m$ i.e. the universal deformation $\mathbb{H}^{(m)}$ with level-m-structure is given by the base change of the universal deformation $\mathbb{H}^{(0)}$ without level structure under the map $R_0 \longrightarrow R_m$ induced by Part 2, Theorem 2.1.8. We note that, since $R_0 \simeq \breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$ is an integral domain, the flatness of the map $R_0 \longrightarrow R_m$ implies $R_0 \hookrightarrow R_m$ for all $m \geq 0$.

By the universal property, given an object R of $\mathcal{C}$ and a deformation (H, ρ, ϕ) of H_0 to R with level-m-structure, there is a unique $\breve{\mathfrak{o}}$-linear local ring homomorphism $\varphi : R_m \longrightarrow R$ such that $\mathrm{Def}_{H_0}^m(\varphi)([(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})]) = [\varphi_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})] = [(H, \rho, \phi)]$. The unique isomorphism between the deformations $\varphi_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})$ and (H, ρ, ϕ) over R will be denoted by $[\varphi] : \varphi_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)}) \xrightarrow{\sim} (H, \rho, \phi)$.

2.2. The group actions

For all $m \geq 0$, the functor $\mathrm{Def}^m_{H_0}$ admits natural commuting left actions of the groups $\Gamma := \mathrm{Aut}(H_0)$ and $G_0 := GL_h(\mathfrak{o})$ for which the morphisms $\mathrm{Def}^{m'}_{H_0} \longrightarrow \mathrm{Def}^m_{H_0}$ of functors mentioned in Part 2, Theorem 2.1.8 are equivariant. On R-valued points, they are given by

(2.2.1) $[(H,\rho,\phi)] \longmapsto [(H,\rho\circ\gamma^{-1},\phi)]$ and $[(H,\rho,\phi)] \longmapsto [(H,\rho,\phi\circ g^{-1})]$ for $\gamma\in\Gamma, g\in G_0$.

Because of the representability, these actions give rise to commuting left actions of Γ and G_0 on the universal deformation rings R_m. Indeed, given $\gamma \in \Gamma$, $g \in G_0$, as explained earlier, there are unique $\breve{\mathfrak{o}}$-linear local ring endomorphisms of R_m denoted by the same letters γ and g, and unique isomorphisms

$$[\gamma] : \gamma_*(\mathbb{H}^{(m)},\rho^{(m)},\phi^{(m)}) \xrightarrow{\sim} (\mathbb{H}^{(m)},\rho^{(m)}\circ\gamma^{-1},\phi^{(m)})$$
$$[g] : g_*(\mathbb{H}^{(m)},\rho^{(m)},\phi^{(m)}) \xrightarrow{\sim} (\mathbb{H}^{(m)},\rho^{(m)},\phi^{(m)}\circ g^{-1})$$

of deformations over R_m. Here $g^{-1} \in GL_h(\mathfrak{o})$ acts on the free $(\frac{\mathfrak{o}}{\varpi^m\mathfrak{o}})$-module $(\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}})^h$ by considering it as an $\mathfrak{o}$-module via the natural reduction map $\mathfrak{o} \twoheadrightarrow \frac{\mathfrak{o}}{\varpi^m\mathfrak{o}}$. It follows from the uniqueness that the resulting endomorphisms of R_m are in fact automorphisms.

Remark 2.2.2. For $m \geq 1$, denote by $G_m := 1 + \varpi^m M_h(\mathfrak{o})$ the m-th principal congruence subgroup of G_0. If $g \in G_m$, then by writing $g^{-1} = 1 + \varpi^m g'$, we see that for all $\alpha \in \left(\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}}\right)^h$, $\phi^{(m)}\circ g^{-1}(\alpha) = \phi^{(m)}(1+\varpi^m g'(\alpha)) = \phi^{(m)}(\alpha + g'(\varpi^m\alpha)) = \phi^{(m)}(\alpha)$. As a result, the G_0-action on R_m factors through the quotient group G_0/G_m. For $m' \geq m \geq 0$, the induced action of $G_m/G_{m'}$ makes $R_{m'}[\frac{1}{\varpi}]$ étale and Galois over $R_m[\frac{1}{\varpi}]$ with Galois group $G_m/G_{m'}$ (cf. Theorem 2.1.2 (ii), [**Str08ii**]).

The actions of Γ and G_0 on R_m induce semilinear actions of Γ and G_0 on the Lie algebra $\mathrm{Lie}(\mathbb{H}^{(m)})$ of the universal deformation $\mathbb{H}^{(m)}$. We now describe the Γ-action on $\mathrm{Lie}(\mathbb{H}^{(m)})$; the G_0-action is defined in a same way. Given $\gamma \in \Gamma$, extend the ring automorphism γ of R_m to $R_m[[X]]$ by sending X to itself. This induces a homomorphism

$$\gamma_* : \mathrm{Lie}(\mathbb{H}^{(m)}) \longrightarrow \mathrm{Lie}(\gamma_*\mathbb{H}^{(m)})$$

of additive groups. The isomorphism $[\gamma] : \gamma_*\mathbb{H}^{(m)} \xrightarrow{\sim} \mathbb{H}^{(m)}$ also induces a natural R_m-linear map

$$\mathrm{Lie}([\gamma]) : \mathrm{Lie}(\gamma_*\mathbb{H}^{(m)}) \longrightarrow \mathrm{Lie}(\mathbb{H}^{(m)}).$$

We define $\gamma : \mathrm{Lie}(\mathbb{H}^{(m)}) \longrightarrow \mathrm{Lie}(\mathbb{H}^{(m)})$ as the composite of these two maps i.e. $\gamma := \mathrm{Lie}([\gamma])\circ\gamma_*$.

Given another element $\gamma' \in \Gamma$, let $\gamma'_*[\gamma] : \gamma'_*(\gamma_*\mathbb{H}^{(m)}) \xrightarrow{\sim} \gamma'_*\mathbb{H}^{(m)}$ be the isomorphism obtained by applying γ' to the coefficients of $[\gamma]$. Then $[\gamma']\circ\gamma'_*[\gamma]$ is an isomorphism between the formal $\mathfrak{o}$-modules $(\gamma'\gamma)_*\mathbb{H}^{(m)}$ and $\mathbb{H}^{(m)}$ over R_m. Therefore by uniqueness, we have $[\gamma'\gamma] = [\gamma']\circ\gamma'_*[\gamma]$. One also checks easily that the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{Lie}(\gamma_*\mathbb{H}^{(m)}) & \xrightarrow{\mathrm{Lie}([\gamma])} & \mathrm{Lie}(\mathbb{H}^{(m)}) \\
\downarrow{\scriptstyle \gamma'_*} & & \downarrow{\scriptstyle \gamma'_*} \\
\mathrm{Lie}(\gamma'_*(\gamma_*\mathbb{H}^{(m)})) & \xrightarrow[\mathrm{Lie}(\gamma'_*[\gamma])]{} & \mathrm{Lie}(\gamma'_*\mathbb{H}^{(m)})
\end{array}$$

Then it follows that

$$\begin{aligned}\text{Lie}([\gamma'\gamma])\circ(\gamma'\gamma)_* &= \text{Lie}([\gamma'])\circ\text{Lie}(\gamma'_*[\gamma])\circ\gamma'_*\circ\gamma_* \\ &= \text{Lie}([\gamma'])\circ(\gamma'_*\circ\text{Lie}([\gamma])\circ(\gamma'_*)^{-1})\circ\gamma'_*\circ\gamma_* \\ &= \text{Lie}([\gamma'])\circ\gamma'_*\circ\text{Lie}([\gamma])\circ\gamma_*.\end{aligned} \tag{2.2.3}$$

Thus we obtain an action of Γ (and of G_0) on the additive group $\text{Lie}(\mathbb{H}^{(m)})$ which is semilinear for the action of Γ (and of G_0 respectively) on R_m because γ_* is semilinear. Given a positive integer s, we denote by $\text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ the s-fold tensor product of $\text{Lie}(\mathbb{H}^{(m)})$ over R_m with itself. This is a free R_m-module of rank 1 with a semi-linear action of Γ defined by $\gamma(\delta_1\otimes\cdots\otimes\delta_s) := \gamma(\delta_1)\otimes\cdots\otimes\gamma(\delta_s)$. Set $\text{Lie}(\mathbb{H}^{(m)})^{\otimes 0} := R_m$ and $\text{Lie}(\mathbb{H}^{(m)})^{\otimes s} := \text{Hom}_{R_m}(\text{Lie}(\mathbb{H}^{(m)})^{\otimes(-s)}, R_m)$ if s is a negative integer. In the latter case, a semi-linear action of Γ is defined by $\gamma(\varphi)(\delta_1\otimes\cdots\otimes\delta_{-s}) := \gamma(\varphi(\gamma^{-1}(\delta_1)\otimes\cdots\otimes\gamma^{-1}(\delta_{-s})))$. The semi-linear actions of G_0 on the s-fold tensor products are defined similarly. As before, for all $s\in\mathbb{Z}$, the G_0-action on $\text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ factors through G_0/G_m.

Remark 2.2.4. Using that the group actions of Γ and G_0 on R_m commute, one can show that they commute on $\text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ as follows: It suffices to show the commutativity for $s = 1$. Since the G_0-action is defined likewise, we may use (2.2.3) for $\gamma\in\Gamma$ and $g\in G_0$. As a result, we get

$$\text{Lie}([g])\circ g_*\circ\text{Lie}([\gamma])\circ\gamma_* = \text{Lie}([g\gamma])\circ(g\gamma)_* = \text{Lie}([\gamma g])\circ(\gamma g)_* = \text{Lie}([\gamma])\circ\gamma_*\circ\text{Lie}([g])\circ g_*.$$

We are primarily interested in the Γ-action on $\text{Lie}(\mathbb{H}^{(m)})^{\otimes s}$. It follows from (2.1.6) that $\Gamma\simeq\mathfrak{o}_{B_h}^\times$. Recall that the division algebra B_h is a K_h-vector space of dimension h with basis $\{\Pi^i\}_{0\le i\le h-1}$ whose multiplication is determined by the relations $\Pi^h=\varpi$ and $\Pi\lambda=\lambda^\sigma\Pi$ for all $\lambda\in K_h$ (λ^σ denotes the image of λ under the Frobenius automorphism σ). Thus, any $\gamma\in\Gamma=\mathfrak{o}_{B_h}^\times$ can be uniquely written as

$$\gamma=\sum_{i=0}^{h-1}\lambda_i\Pi^i$$

with $\lambda_0\in\mathfrak{o}_h^\times$ and $\lambda_1,\ldots,\lambda_{h-1}\in\mathfrak{o}_h$. The map

$$\begin{aligned}\psi:\Gamma &\longrightarrow K_h^h \\ \sum_{i=0}^{h-1}\lambda_i\Pi^i &\longmapsto(\lambda_0,\lambda_1\ldots\lambda_{h-1})\end{aligned} \tag{2.2.5}$$

identifies Γ with a compact open subset $\mathfrak{o}_h^\times\times\mathfrak{o}_h^{h-1}$ of K_h^h making it into a compact open locally K_h-analytic submanifold of K_h^h. The composition map

$$\psi(\Gamma)\times\psi(\Gamma)\xrightarrow{\psi^{-1}\times\psi^{-1}}\Gamma\times\Gamma\xrightarrow{\text{multiplication}}\Gamma\xrightarrow{\psi}\psi(\Gamma)$$

from an open subset in K_h^{2h} to K_h^h can be easily seen to be locally K-analytic since each component of this map is a composition of a polynomial and a K-linear Frobenius automorphism σ, both being locally K-analytic. Hence, Γ is a locally K-analytic group. Notice that Γ is not locally K_h-analytic group because $\sigma:\mathfrak{o}_h^\times\longrightarrow\mathfrak{o}_h^\times$ is not locally K_h-analytic unless $h=1$.

Alternatively, if $\mathbb{B}_h^\times$ denotes the algebraic group over K defined by $\mathbb{B}_h^\times(A) := (B_h\otimes_K A)^\times$ for any K-algebra A, then the local K-analyticity of the group Γ also follows from the fact that the group of K-valued points $\mathbb{B}_h^\times(K)=B_h^\times$ is a locally K-analytic group and $\mathfrak{o}_{B_h}^\times$ is open in $B_h^\times$.

Now, being a compact and a totally disconnected Hausdorff topological group, Γ is a profinite topological group. A basis of neighbourhoods of the identity is given by the normal subgroups $\Gamma_i := 1 + \varpi^i \mathfrak{o}_{B_h} = 1 + \varpi^i \mathrm{End}(H_0)$, $i \geq 1$ of finite index. We put $\Gamma_0 := \Gamma$.

Our next aim is to show that the Γ-action on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ is continuous, i.e. the action map $\Gamma \times \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ is continuous for the $\mathfrak{m}_{R_m}$-adic topology on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$, and for the product of profinite and $\mathfrak{m}_{R_m}$-adic topology on the left hand side. But, first we need a couple of lemmas. For any two non-negative integers n and m, set $\mathbb{H}_n^{(m)} := \mathbb{H}^{(m)} \otimes_{R_m} (R_m/\mathfrak{m}_{R_m}^{n+1})$. We have $H_0 \xrightarrow{\sim} \mathbb{H}_0^{(m)}$ via $\rho^{(m)}$ for all $m \geq 0$.

Lemma 2.2.6. *If n and m are non-negative integers then the homomorphism of $\mathfrak{o}$-algebras $\mathrm{End}(\mathbb{H}_{n+1}^{(m)}) \longrightarrow \mathrm{End}(\mathbb{H}_n^{(m)})$, induced by reduction modulo $\mathfrak{m}_{R_m}^{n+1}$, is injective.*

Proof. Let $m \geq 0$ be arbitrary. We show by induction on n that the ring homomorphism $i_n : \mathrm{End}(\mathbb{H}_n^{(m)}) \longrightarrow \mathrm{End}(\mathbb{H}_0^{(m)})$, induced by reduction modulo the maximal ideal, is injective for every $n \in \mathbb{N}_0$. The case $n = 0$ is trivial. Let $n \geq 1$ and assume that i_{n-1} is injective. Since H_0 is of height h, we have $[\varpi]_{\mathbb{H}_0^{(m)}}(X) \equiv \overline{u}X^{q^h} \mod \deg q^h + 1$ for some $u \in R_m^{\times}$. Then $[\varpi]_{\mathbb{H}_0^{(m)}} = i_n([\varpi]_{\mathbb{H}_n^{(m)}})$ implies that

$$[\varpi]_{\mathbb{H}_n^{(m)}}(X) \equiv \overline{\varpi}X + \overline{b_2}X^2 + \cdots + \overline{b_{q^h-1}}X^{q^h-1} + \overline{u}X^{q^h} \quad \mod \deg q^h + 1$$

for some $b_2, \cdots, b_{q^h-1} \in \mathfrak{m}_{R_m}$.

Now let $f(X) = \sum_{i=1}^{\infty} \overline{a_i}X^i \in \mathrm{End}(\mathbb{H}_n^{(m)})$ such that $i_n(f) = 0$ i.e. $a_i \in \mathfrak{m}_{R_m}$ for all $i \geq 1$. We need to show that $a_i \in \mathfrak{m}_{R_m}^{n+1}$ for all $i \geq 1$. However the induction hypothesis implies that $a_i \in \mathfrak{m}_{R_m}^n$. Thus $[\varpi]_{\mathbb{H}_n^{(m)}} \circ f = 0$. Since $[\varpi]_{\mathbb{H}_n^{(m)}} \circ f = f \circ [\varpi]_{\mathbb{H}_n^{(m)}}$, we get $a_i u^i \in \mathfrak{m}_{R_m}^{n+1}$ by induction i and hence $a_i \in \mathfrak{m}_{R_m}^{n+1}$ for all $i \geq 1$. □

The above lemma allows us to consider all the $\mathfrak{o}$-algebras $\mathrm{End}(\mathbb{H}_n^{(m)})$ as subalgebras of $\mathrm{End}(\mathbb{H}_0^{(m)})$.

Proposition 2.2.7. *For all $n \geq 0$, $m \geq 0$, the subalgebra $\mathrm{End}(\mathbb{H}_n^{(m)})$ of $\mathrm{End}(\mathbb{H}_0^{(m)})$ contains $\varpi^n \mathrm{End}(\mathbb{H}_0^{(m)})$.*

Proof. Let $m \geq 0$ be arbitrary. We proceed by induction on n, the case $n = 0$ being trivial. Let $n \geq 1$ and assume the assertion to be true for $n - 1$. Let $\varphi \in \varpi^n \mathrm{End}(\mathbb{H}_0^{(m)})$. By induction hypothesis, we have $\varphi \in \varpi \mathrm{End}(\mathbb{H}_{n-1}^{(m)})$. Now for any $\psi \in \mathrm{End}(\mathbb{H}_{n-1}^{(m)})$, choose a power series $\tilde{\psi} \in (R_m/\mathfrak{m}_{R_m}^{n+1})[[X]]$ with trivial constant term such that $\tilde{\psi} \mod \mathfrak{m}_{R_m}^n = \psi$. The power series $\varpi\tilde{\psi} = [\varpi]_{\mathbb{H}_n^{(m)}} \circ \tilde{\psi}$ is a lift of $\varpi\psi = [\varpi]_{\mathbb{H}_{n-1}^{(m)}} \circ \psi$. We claim that $\varpi\tilde{\psi} \in \mathrm{End}(\mathbb{H}_n^{(m)})$ and $(\varpi\psi \longmapsto \varpi\tilde{\psi}) : \varpi\mathrm{End}(\mathbb{H}_{n-1}^{(m)}) \longrightarrow \mathrm{End}(\mathbb{H}_n^{(m)})$ is a well-defined injective map. The proposition then follows from the claim.

First, let us see why $\varpi\tilde{\psi}$ defines an endomorphism of $\mathbb{H}_n^{(m)}$. Since $\psi \in \mathrm{End}(\mathbb{H}_{n-1}^{(m)})$, we have

$$0 = \psi(X +_{\mathbb{H}_{n-1}^{(m)}} Y) -_{\mathbb{H}_{n-1}^{(m)}} \psi(X) -_{\mathbb{H}_{n-1}^{(m)}} \psi(Y) = (\tilde{\psi}(X +_{\mathbb{H}_n^{(m)}} Y) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(X) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(Y)) \mod \mathfrak{m}_{R_m}^n.$$

Thus all the coefficients of the power series $(\tilde{\psi}(X +_{\mathbb{H}_n^{(m)}} Y) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(X) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(Y))$ lie in $\mathfrak{m}_{R_m}^n/\mathfrak{m}_{R_m}^{n+1}$. Since $\varpi \in \mathfrak{m}_{R_m}$ and $(\mathfrak{m}_{R_m}^n)^k \subseteq \mathfrak{m}_{R_m}^{n+1}$ for all integers $k > 1$, we get $[\varpi]_{\mathbb{H}_n^{(m)}} \circ (\tilde{\psi}(X +_{\mathbb{H}_n^{(m)}} Y) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(X) -_{\mathbb{H}_n^{(m)}} \tilde{\psi}(Y)) = 0$. Consequently, $\varpi\tilde{\psi}(X +_{\mathbb{H}_n^{(m)}} Y) = \varpi\tilde{\psi}(X) +_{\mathbb{H}_n^{(m)}}$

$\varpi\tilde{\psi}(Y)$. Similarly one shows that

$$0 = [\varpi]_{\mathbb{H}_n^{(m)}} \circ ([a]_{\mathbb{H}_n^{(m)}} \circ \tilde{\psi} -_{\mathbb{H}_n^{(m)}} \tilde{\psi} \circ [a]_{\mathbb{H}_n^{(m)}}) = [a]_{\mathbb{H}_n^{(m)}} \circ \varpi\tilde{\psi} -_{\mathbb{H}_n^{(m)}} \varpi\tilde{\psi} \circ [a]_{\mathbb{H}_n^{(m)}}$$

for all $a \in \mathfrak{o}$. Therefore $\varpi\tilde{\psi} \in \mathrm{End}(\mathbb{H}_n^{(m)})$.

To see that the above map is well-defined, take another lift $\tilde{\psi}'$ of ψ with trivial constant terms. Then $(\tilde{\psi}' -_{\mathbb{H}_n^{(m)}} \tilde{\psi}) \mod \mathfrak{m}_{R_m}^n = \psi -_{\mathbb{H}_{n-1}^{(m)}} \psi = 0$. Thus $[\varpi]_{\mathbb{H}_n^{(m)}} \circ (\tilde{\varphi}' -_{\mathbb{H}_n^{(m)}} \tilde{\varphi}) = 0$ as above. Hence $\varpi\tilde{\psi}' = \varpi\tilde{\psi}$.

Injectivity is clear because $\varpi\tilde{\psi}_1 = \varpi\tilde{\psi}_2$ implies $\varpi\psi_1 = \varpi\psi_2$ after reduction modulo $\mathfrak{m}_{R_m}^n$. □

Theorem 2.2.8. *For all $n \geq 0$, $m \geq 0$, the induced action of Γ_{n+m} on $R_m/\mathfrak{m}_{R_m}^{n+1}$ is trivial. Thus the map $((\gamma, f) \mapsto \gamma(f)) : \Gamma \times R_m \longrightarrow R_m$ is continuous where the left hand side carries the product topology.*

Proof. Let n and m be arbitrary non-negative integers. Let $\gamma \in \Gamma_{n+m}$ and $\mathrm{pr}_n^{(m)} : R_m \longrightarrow R_m/\mathfrak{m}_{R_m}^{n+1}$ denote the natural projection. Consider the level-m-structure $\phi_n^{(m)} := \mathrm{pr}_n^{(m)} \circ \phi^{(m)}$ on $\mathbb{H}_n^{(m)}$ and consider the deformation $(\mathbb{H}_n^{(m)}, \rho^{(m)} \circ \gamma^{-1}, \phi_n^{(m)})$ of H_0 to $R_m/\mathfrak{m}_{R_m}^{n+1}$ with this level-m-structure. Let $\gamma_n^{(m)} : R_m \longrightarrow R_m/\mathfrak{m}_{R_m}^{n+1}$ denote the unique ring homomorphism for which there exists an isomorphism $[\gamma_n^{(m)}] : (\gamma_n^{(m)})_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)}) \xrightarrow{\sim} (\mathbb{H}_n^{(m)}, \rho^{(m)} \circ \gamma^{-1}, \phi_n^{(m)})$. Note that also the ring homomorphism $\mathrm{pr}_n^{(m)} \circ \gamma : R_m \longrightarrow R_m/\mathfrak{m}_m^{n+1}$ admits an isomorphism of deformations

$$(\mathrm{pr}_n^{(m)} \circ \gamma)_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)}) = (\mathrm{pr}_n^{(m)})_*(\gamma_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})) \xrightarrow{\sim} (\mathbb{H}_n^{(m)}, \rho^{(m)} \circ \gamma^{-1}, \phi_n^{(m)}).$$

Therefore by uniqueness, we have $\mathrm{pr}_n^{(m)} \circ \gamma = \gamma_n^{(m)}$ and $[\gamma_n^{(m)}] = [\gamma] \mod \mathfrak{m}_{R_m}^{n+1}$.

Since the map $(\sigma \mapsto \rho^{(m)} \circ \sigma \circ (\rho^{(m)})^{-1})$ is an isomorphism $\mathrm{End}(H_0) \xrightarrow{\sim} \mathrm{End}(\mathbb{H}_0^{(m)})$ of $\mathfrak{o}$-algebras, Proposition 2.2.7 shows that $\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1} \in 1 + \varpi^m \mathrm{End}(\mathbb{H}_n^{(m)}) \subseteq \mathrm{Aut}(\mathbb{H}_n^{(m)})$. We claim that $(\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}) \circ \phi_n^{(m)} = \phi_n^{(m)}$: Write $\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1} = 1 + \varepsilon\varpi^m$ for some $\varepsilon \in \mathrm{End}(\mathbb{H}_n^{(m)})$ and let $\alpha \in (\frac{\varpi^{-m}\mathfrak{o}}{\mathfrak{o}})^h$ be arbitrary. Then

$$\begin{aligned}(\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1})(\phi_n^{(m)}(\alpha)) = (1 + \varepsilon\varpi^m)(\phi_n^{(m)}(\alpha)) &= \phi_n^{(m)}(\alpha) +_{\mathbb{H}_n^{(m)}} \varepsilon(\varpi^m(\phi_n^{(m)}(\alpha))) \\ &= \phi_n^{(m)}(\alpha) +_{\mathbb{H}_n^{(m)}} \varepsilon(\phi_n^{(m)}(\varpi^m\alpha)) \\ &= \phi_n^{(m)}(\alpha) +_{\mathbb{H}_n^{(m)}} \varepsilon(\phi_n^{(m)}(0)) \\ &= \phi_n^{(m)}(\alpha)\end{aligned}$$

Therefore, the automorphism $\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}$ of $\mathbb{H}_n^{(m)}$ defines an isomorphism of deformations

$$\begin{aligned}(\mathbb{H}_n^{(m)}, \rho^{(m)}, \phi_n^{(m)}) &\simeq (\mathbb{H}_n^{(m)}, (\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}) \circ \rho^{(m)}, (\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}) \circ \phi_n^{(m)}) \\ &= (\mathbb{H}_n^{(m)}, \rho^{(m)} \circ \gamma^{-1}, \phi_n^{(m)}).\end{aligned}$$

However; $(\mathbb{H}_n^{(m)}, \rho^{(m)}, \phi_n^{(m)}) = (\mathrm{pr}_n^{(m)})_*(\mathbb{H}^{(m)}, \rho^{(m)}, \phi^{(m)})$. By uniqueness again, we have $\mathrm{pr}_n^{(m)} = \mathrm{pr}_n^{(m)} \circ \gamma = \gamma_n^{(m)}$. This implies that Γ_{n+m} acts trivially on $R_m/\mathfrak{m}_{R_m}^{n+1}$ and $[\gamma] \mod \mathfrak{m}_{R_m}^{n+1} = \rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1}$.

□

The R_m-module $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ is complete and Hausdorff for the $\mathfrak{m}_{R_m}$-adic topology because it is free of finite rank. By the semi-linearity of the Γ-action, the R_m-submodules $\mathfrak{m}_{R_m}^n \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ are Γ-stable for any non-negative integer n.

Theorem 2.2.9. *Let s, n, m be integers with $n \geq 0$ and $m \geq 0$. The induced action of* Γ_{2n+m+1} *on* $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ *is trivial. Thus the map* $((\gamma,\delta) \mapsto \gamma(\delta)) : \Gamma \times \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ *is continuous where the left hand side carries the product topology.*

Proof. If we assume the assertion to be true for $s = 1$, then by the definition of the action, it is easy to see that it holds for all positive s. On the other hand, let $\overline{\varphi} \in \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes -1}/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes -1} = \mathrm{Hom}_{R_m}(\mathrm{Lie}(\mathbb{H}^{(m)}), R_m)/\mathfrak{m}_{R_m}^{n+1}\mathrm{Hom}_{R_m}(\mathrm{Lie}(\mathbb{H}^{(m)}), R_m)$, and let $\delta \in \mathrm{Lie}(\mathbb{H}^{(m)})$ and $\gamma \in \Gamma_{2n+m+1}$. Then by assumption, $\gamma(\delta) - \delta \in \mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})$. Write $\gamma^{-1}(\delta) = \delta + \sum_{i=1}^{r} \alpha_i\eta_i$ with $\alpha_i \in \mathfrak{m}_{R_m}^{n+1}$ and $\eta_i \in \mathrm{Lie}(\mathbb{H}^{(m)})$. Then

$$\begin{aligned}(\varphi - \gamma(\varphi))(\delta) = \varphi(\delta) - \gamma(\varphi)(\delta) = \varphi(\delta) - \gamma(\varphi(\gamma^{-1}(\delta))) &= \varphi(\delta) - \gamma(\varphi(\delta + \sum_{i=1}^{r}\alpha_i\eta_i)) \\ &= \varphi(\delta) - \gamma(\varphi(\delta)) - \sum_{i=1}^{r}\gamma(\alpha_i)\gamma(\varphi(\eta_i)).\end{aligned}$$

Since $2n+m+1 \geq n+m$, by Theorem 2.2.8, we have $\varphi(\delta) - \gamma(\varphi(\delta)) \in \mathfrak{m}_{R_m}^{n+1}$. Also $\gamma(\alpha_i) \in \mathfrak{m}_{R_m}^{n+1}$. Therefore $(\varphi - \gamma(\varphi))(\delta) \in \mathfrak{m}_{R_m}^{n+1}$. If δ_0 is a basis of $\mathrm{Lie}(\mathbb{H}^{(m)})$ over R_m, and $\psi \in \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes -1}$ is defined by $\psi(\delta_0) = 1$, then $\varphi - \gamma(\varphi) = (\varphi - \gamma(\varphi))(\delta_0)\psi \in \mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes -1}$. Thus $\overline{\varphi} = \gamma(\overline{\varphi})$. A similar argument like this can be used to show that the assertion is true for all higher negative s. Hence it is sufficient to prove the theorem for $s = 1$.

Let $\gamma \in \Gamma_{2n+m+1}$. By identifying $\mathrm{Lie}(\mathbb{H}^{(m)})/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)}) = \mathrm{Lie}(\mathbb{H}_n^{(m)})$, Theorem 2.2.8 and its proof show that the map $\gamma \mod \mathfrak{m}_{R_m}^{n+1} : \mathrm{Lie}(\mathbb{H}_n^{(m)}) \longrightarrow \mathrm{Lie}(\mathbb{H}_n^{(m)})$ is given by $\mathrm{Lie}(\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1})$ where $\rho^{(m)} \circ \gamma^{-1} \circ (\rho^{(m)})^{-1} \in 1 + \varpi^{2n+m+1}\mathrm{End}(\mathbb{H}_0^{(m)}) \subseteq 1 + \varpi^{n+m+1}\mathrm{End}(\mathbb{H}_n^{(m)})$. Therefore it suffices to show that the natural action of $1 + \varpi^{n+m+1}\mathrm{End}(\mathbb{H}_n^{(m)}) \subset \mathrm{End}(\mathbb{H}_n^{(m)})$ on $\mathrm{Lie}(\mathbb{H}_n^{(m)})$ is trivial. However if $\varphi \in \mathrm{End}(\mathbb{H}_n^{(m)})$ and $\delta \in \mathrm{Lie}(\mathbb{H}_n^{(m)})$, then

$$\begin{aligned}(\mathrm{Lie}(1 + \varpi^{n+m+1}\varphi)(\delta))(\overline{X}) = \delta(\overline{(1 + \varpi^{n+m+1}\varphi)(X)}) &= \delta(\overline{X +_{\mathbb{H}_n^{(m)}} \varpi^{n+m+1}\varphi(X)}) \\ &= \delta(\overline{X + \varpi^{n+m+1}\varphi(X)}) = \delta(\overline{X})\end{aligned}$$

because $\varpi^{n+m+1} \in \mathfrak{m}_{R_m}^{n+1}$. □

Remark 2.2.10. The Γ-action on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ gives rise to an action of the group ring $\breve{\mathfrak{o}}[\Gamma]$ on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$. By Theorem 2.2.9, the induced action of $\breve{\mathfrak{o}}[\Gamma]$ on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ factors through $(\breve{\mathfrak{o}}/\varpi^{n+1}\breve{\mathfrak{o}})[\Gamma/\Gamma_{2n+m+1}]$ such that the following diagram with the horizontal action maps and the vertical reduction maps commutes for all n.

$$\begin{array}{ccc}(\breve{\mathfrak{o}}/\varpi^{n+1}\breve{\mathfrak{o}})[\Gamma/\Gamma_{2n+m+1}] \times \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} & \longrightarrow & \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}/\mathfrak{m}_{R_m}^{n+1}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \\ \downarrow & & \downarrow \\ (\breve{\mathfrak{o}}/\varpi^{n}\breve{\mathfrak{o}})[\Gamma/\Gamma_{2(n-1)+m+1}] \times \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}/\mathfrak{m}_{R_m}^{n}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} & \longrightarrow & \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}/\mathfrak{m}_{R_m}^{n}\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}\end{array}$$

Taking projective limits over n, we obtain an action of the Iwasawa algebra $\breve{\mathfrak{o}}[[\Gamma]]$ on $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ that extends the action of Γ.

2.3. Rigidification and the equivariant vector bundles

Berthelot's rigidification functor associates to every locally Noetherian adic formal scheme over $\mathrm{Spf}(\breve{\mathfrak{o}})$ whose reduction is a scheme locally of finite type over $\mathrm{Spec}(k^{\mathrm{sep}})$, a rigid analytic space over $\breve{K}$ (cf. [**Jong95**], Section 7). For an affine formal $\breve{\mathfrak{o}}$-scheme $\mathrm{Spf}(A)$, there is a bijection between the closed points of its generic fibre $\mathrm{Spec}(A \otimes_{\breve{\mathfrak{o}}} \breve{K})$ and the points of the associated rigid analytic space (cf. [**Jong95**], Lemma 7.1.9). Let us denote by X_m^{rig} the rigidification of the affine formal $\breve{\mathfrak{o}}$-scheme $X_m = \mathrm{Spf}(R_m)$ under Berthelot's functor, and by $R_m^{\mathrm{rig}} := \mathcal{O}_{X_m^{\mathrm{rig}}}(X_m^{\mathrm{rig}})$ the global rigid analytic functions on X_m^{rig}.

It follows from the isomorphism $R_0 \simeq \breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$ that X_0^{rig} is isomorphic to the $(h-1)$-dimensional rigid analytic open unit polydisc over $\breve{K}$, and the isomorphism $R_0 \simeq \breve{\mathfrak{o}}[[u_1, \ldots, u_{h-1}]]$ extends to an isomorphism

$$R_0^{\mathrm{rig}} \simeq \Big\{ \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha u^\alpha \mid c_\alpha \in \breve{K} \text{ and } \lim_{|\alpha| \to \infty} |c_\alpha| r^{|\alpha|} = 0 \text{ for all } 0 < r < 1 \Big\} \tag{2.3.1}$$

of $\breve{K}$-algebras. This allows us to view R_0^{rig} as a topological $\breve{K}$-Fréchet algebra whose topology is defined by the family of norms $\| \cdot \|_l$, given by

$$\Big\| \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha u^\alpha \Big\|_l := \sup_{\alpha \in \mathbb{N}_0^{h-1}} \{ |c_\alpha| |\varpi|^{|\alpha|/l} \}$$

for any positive integer l. Let $R_{0,l}^{\mathrm{rig}}$ be the completion of R_0^{rig} with respect to the norm $\| \cdot \|_l$. Then

$$R_{0,l}^{\mathrm{rig}} \simeq \Big\{ \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha u^\alpha \Big| c_\alpha \in \breve{K}, \lim_{|\alpha| \to \infty} |c_\alpha| |\varpi|^{|\alpha|/l} = 0 \Big\}$$

is the $\breve{K}$-Banach algebra of rigid analytic functions on the affinoid subdomain

$$\mathbb{B}_l := \{ x \in X_0^{\mathrm{rig}} \mid |u_i(x)| \leq |\varpi|^{1/l} \text{ for all } 1 \leq i \leq h-1 \}$$

of X_0^{rig}. Further, $R_0^{\mathrm{rig}} \simeq \varprojlim_l R_{0,l}^{\mathrm{rig}}$ is the topological projective limit of the $\breve{K}$-Banach algebras $R_{0,l}^{\mathrm{rig}}$.

By functoriality, X_m^{rig} and R_m^{rig} carry commuting (left) actions of Γ and G_0, and the G_0-action factors through G_0/G_m (cf. Remark 2.2.2). For $m' \geq m \geq 0$, let

$$\pi_{m',m} : X_{m'}^{\mathrm{rig}} \longrightarrow X_m^{\mathrm{rig}}$$

denote the morphism of rigid analytic spaces induced by Part 2, Theorem 2.1.8 and by functoriality. It follows from the properties of the rigidification functor that the morphism $\pi_{m',m}$ is a finite étale Galois covering with Galois group $G_m/G_{m'}$ (cf. [**Jong95**], Section 7). Consequently, the ring extension $R_{m'}^{\mathrm{rig}} | R_m^{\mathrm{rig}}$ is finite Galois with Galois group $G_m/G_{m'}$. Since the group actions commute, we have an action of the product group $\Gamma \times G_0$ on X_m^{rig} (and on R_m^{rig}) by setting $(\gamma, g)(x) := \gamma(g(x)) = g(\gamma(x))$ for $(\gamma, g) \in \Gamma \times G_0$ and $x \in X_m^{\mathrm{rig}}$. We note that all covering morphisms are $(\Gamma \times G_0)$-equivariant.

Since R_0 is a local ring, the finite flat R_0-module R_m is free by [**Mat87**], Theorem 7.10, and has rank $r := |G_0/G_m|$ (cf. Remark 2.2.2). Let $R_{m,l}^{\mathrm{rig}}$ denote the affinoid $\breve{K}$-algebra of the rigid analytic functions on the affinoid subdomain $\mathbb{B}_{m,l} := \pi_{m,0}^{-1}(\mathbb{B}_l)$ of X_m^{rig}. Then by [**Jong95**], Lemma 7.2.2, we have

$$R_{m,l}^{\mathrm{rig}} \simeq R_m \otimes_{R_0} R_{0,l}^{\mathrm{rig}} \tag{2.3.2}$$

as $R_m | R_0$ is finite. Let us fix a basis $\{e_1, \cdots, e_r\}$ of R_m over R_0 and view it as an $R_{0,l}^{\mathrm{rig}}$-basis of $R_{m,l}^{\mathrm{rig}} = R_m \otimes_{R_0} R_{0,l}^{\mathrm{rig}}$. The next lemma shows that $R_{m,l}^{\mathrm{rig}}$ is a $\breve{K}$-Banach algebra with respect to the norm $\|(f_1e_1+\cdots+f_re_r)\|_l := \max_{1\le i\le r}\{\|f_i\|_l\}$, where $f_i \in R^{\mathrm{rig}}_{0,l}$ for all i, by showing that it is indeed an algebra norm.

Lemma 2.3.3. *Let* $f = f_1e_1+\cdots+f_re_r$, $g = g_1e_1+\cdots+g_re_r \in R^{\mathrm{rig}}_{m,l}$. *Then* $\|fg\|_l \le \|f\|_l\|g\|_l$.

PROOF. Let $e_ie_j = \sum_{k=1}^r a_{ijk}e_k$ for all $1\le i,j\le r$. Note that $a_{ijk}\in R_0 = \breve{\mathfrak{o}}[[u_1,\ldots,u_{h-1}]]$ and thus $\|a_{ijk}\|_l \le 1$ for all $1\le i,j,k\le r$. Also $\|\cdot\|_l$ is multiplicative on $R^{\mathrm{rig}}_{0,l}$. Therefore

$$\|fg\|_l = \max_{1\le k\le r}\left\{\Big\|\sum_{1\le i,j\le r} f_ig_ja_{ijk}\Big\|_l\right\} \le \max_{1\le k\le r}\left\{\max_{1\le i,j\le r}\|f_ig_ja_{ijk}\|_l\right\}$$
$$\le \max_{1\le i,j\le r}\{\|f_i\|_l\|g_j\|_l\} \le \|f\|_l\|g\|_l.$$

□

It then follows from [**BGR84**], (6.1.3), Proposition 2 that the affinoid topology on $R^{\mathrm{rig}}_{m,l}$ coincides with Banach topology given by the aforementioned norm $\|\cdot\|_l$. The natural maps $R^{\mathrm{rig}}_{m,l+1}\longrightarrow R^{\mathrm{rig}}_{m,l}$ induced from $R^{\mathrm{rig}}_{0,l+1}\longrightarrow R^{\mathrm{rig}}_{0,l}$ endow the projective limit

$$R^{\mathrm{rig}}_m \simeq \varprojlim_l R^{\mathrm{rig}}_{m,l}$$

with the structure of a $\breve{K}$-Fréchet algebra. Indeed, this projective limit is isomorphic to the $\breve{K}$-algebra of global rigid analytic functions on X^{rig}_m by [**Jong95**], Lemma 7.2.2. Thus we have

$$R^{\mathrm{rig}}_m \simeq R_m \otimes_{R_0} R^{\mathrm{rig}}_0 \tag{2.3.4}$$

as R_m is a finite free R_0-module, and $R^{\mathrm{rig}}_{m,l}$ can be viewed as the Banach completion of R^{rig}_m with respect to the norm $\|\cdot\|_l$ defined as before by $\|(f_1e_1+\cdots+f_re_r)\|_l := \max_{1\le i\le r}\{\|f_i\|_l\}$, with $f_i\in R^{\mathrm{rig}}_0$.

The Γ-action on X^{rig}_0 stabilizes the affinoid subdomains $\mathbb{B}_l$ for all positive integers l: Since $\gamma(u_i)$ belongs to the maximal ideal $(\varpi,u_1,\ldots,u_{h-1})$ of R_0, $\|\gamma(u_i)\|_l \le |\varpi|^{1/l}$ for all $1\le i\le h-1$. This implies that $\|\gamma(f)\|_l \le \|f\|_l$ for all $f\in R^{\mathrm{rig}}_0$. Thus the Γ-action on R^{rig}_0 extends to its completion $R^{\mathrm{rig}}_{0,l}$ for all positive integers l. As a result the affinoid subdomains $\mathbb{B}_{m,l}$ of X^{rig}_m are stable under the $(\Gamma\times G_0)$-action, and the isomorphism (2.3.2) is $(\Gamma\times G_0)$-equivariant for the diagonal $(\Gamma\times G_0)$-action on the right. Therefore the isomorphism (2.3.4) is also $(\Gamma\times G_0)$-equivariant for the diagonal $(\Gamma\times G_0)$-action on the right.

By [**BGR84**], (6.1.3), Theorem 1, the $\breve{K}$-algebra automorphism of an affinoid $\breve{K}$-algebra $R^{\mathrm{rig}}_{m,l}$ corresponding to $(\gamma,g)\in\Gamma\times G_0$ is automatically continuous for its $\breve{K}$-Banach topology defined by the norm $\|\cdot\|_l$. Since the $\breve{K}$-Fréchet topology of R^{rig}_m is given by the family of norms $\|\cdot\|_l$, $l\in\mathbb{N}$, the group Γ acts on R^{rig}_m by continuous $\breve{K}$-algebra automorphisms for all $m\ge 0$.

Remark 2.3.5. A *Lubin-Tate formal* $\mathfrak{o}$*-module* is a formal $\mathfrak{o}$-module F over $\mathfrak{o}$ such that $[\varpi]_F(X)\equiv X^q \mod \varpi$, i.e. $F\otimes_{\mathfrak{o}} k^{\mathrm{sep}}$ has height 1. By Theorem 2.1.8, over $\breve{\mathfrak{o}}$, there exists a unique such formal $\mathfrak{o}$-module up to isomorphism. Let $\breve{K}_m$ denote the finite Galois extension of $\breve{K}$ obtained by adjoining to it the ϖ^m-torsion points of any Lubin-Tate formal $\mathfrak{o}$-module F over $\mathfrak{o}$, i.e.

$$\breve{K}_m = \breve{K}(\{\alpha\in\mathfrak{m}_{\overline{\breve{\mathfrak{o}}}} \mid [\varpi^m]_F(\alpha)=0\}).$$

This is a totally ramified extension of $\breve{K}$ with $\mathrm{Gal}(\breve{K}_m/\breve{K})\simeq(\mathfrak{o}/\pi^m\mathfrak{o})^\times$ and plays a crucial role in local class field theory. It is a non-trivial result of M. Strauch (cf. [**Str08i**], Corollary 3.4 (ii)) that $\breve{K}_m\subset R^{\mathrm{rig}}_m$. In fact, $\breve{K}_m$ is stable under the actions of G_0/G_m and Γ on R^{rig}_m. For $g\in G_0/G_m$,

$\gamma \in \Gamma$ and $\alpha \in \breve{K}_m$, these actions are given by $g(\alpha) = \det(g)^{-1}(\alpha)$ and $\gamma(\alpha) = \mathrm{Nrd}(\gamma)(\alpha)$ viewing $\breve{K}$ as a left $\mathfrak{o}^\times$-module via the map $\mathfrak{o}^\times \twoheadrightarrow (\mathfrak{o}/\varpi^m\mathfrak{o})^\times \simeq \mathrm{Gal}(\breve{K}_m/\breve{K})$ (cf. [**Str08i**], Theorem 4.4). If the height h of H_0 equals 1 then we have the equality $\breve{K}_m = R_m^{\mathrm{rig}}$.

Following [**GH94**], we now recall the following definition:

Definition 2.3.6. A Γ-*equivariant vector bundle* $\mathcal{M}$ on the formal scheme X_m is a locally free $\mathcal{O}_{X_m}$-module $\mathcal{M}$ of finite rank equipped with a (left) Γ-action that is compatible with the Γ-action on X_m. In other words, $\gamma(fs) = \gamma(f)\gamma(s)$ for $\gamma \in \Gamma$ and for the sections f and s of $\mathcal{O}_{X_m}$ and $\mathcal{M}$ respectively over some Γ-stable open subset of X_m. The notion of a G_0-equivariant vector bundle can be defined similarly. A homomorphism $f : \mathcal{M} \longrightarrow \mathcal{N}$ of Γ-equivariant vector bundles on X_m is a homomorphism of $\mathcal{O}_{X_m}$-modules which is Γ-equivariant.

Since X_m is formally affine, a Γ-equivariant vector bundle $\mathcal{M}$ on X_m is completely determined by its global sections $\mathcal{M}(X_m)$. Hence, for all $s \in \mathbb{Z}$ and $m \geq 0$, the free R_m-module $\mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s}$ of rank 1 equipped with a semilinear Γ-action gives rise to a Γ-equivariant line bundle

$$\mathcal{M}_m^s := \mathcal{L}\mathrm{ie}(\mathbb{H}^{(m)})^{\otimes s}$$

on X_m. Its rigidification $(\mathcal{M}_m^s)^{\mathrm{rig}}$ is a locally free $\mathcal{O}_{X_m^{\mathrm{rig}}}$-module of rank 1 by [**Jong95**], 7.1.11. Let

$$M_m^s := (\mathcal{M}_m^s)^{\mathrm{rig}}(X_m^{\mathrm{rig}})$$

denote its global sections. Because of the fact that X_m is affine, the natural map

$$R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow M_m^s \tag{2.3.7}$$

is an isomorphism. By functoriality, Γ acts on $(\mathcal{M}_m^s)^{\mathrm{rig}}$ in such a way that the map (2.3.7) is Γ-equivariant for the diagonal Γ-action on the left and for the Γ-action induced by functoriality on the right. In particular, the Γ-action on M_m^s is semilinear for its action on R_m^{rig}, and thus $(\mathcal{M}_m^s)^{\mathrm{rig}}$ is a *rigid* Γ-*equivariant line bundle* on X_m^{rig}. In a similar fashion, it can be seen that $(\mathcal{M}_m^s)^{\mathrm{rig}}$ is a rigid G_0-equivariant line bundle on X_m^{rig}, and the actions of Γ and G_0 commute (cf. Remark 2.2.4). By functoriality again, the G_0-action on $(\mathcal{M}_m^s)^{\mathrm{rig}}$ factors through G_0/G_m (cf. Remark 2.2.2).

For all s, m and l, set $M_{m,l}^s := (\mathcal{M}_m^s)^{\mathrm{rig}}(\mathbb{B}_{m,l})$. Then $M_{m,l}^s$ is a free $R_{m,l}^{\mathrm{rig}}$-module of rank 1 for which the natural $R_{m,l}^{\mathrm{rig}}$-linear map

$$R_{m,l}^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow M_{m,l}^s \tag{2.3.8}$$

is an isomorphism (cf. [**Jong95**], 7.1.11), and is $(\Gamma \times G_0)$-equivariant for the diagonal $(\Gamma \times G_0)$-action on the left. Endowing M_m^s and $M_{m,l}^s$ with the natural topologies of finitely generated modules over R_m^{rig} and $R_{m,l}^{\mathrm{rig}}$ respectively, makes them a $\breve{K}$-Fréchet space and a $\breve{K}$-Banach space respectively. One then has a topological isomorphism

$$M_m^s \simeq \varprojlim_l M_{m,l}^s$$

for the projective limit topology on the right, and the group $\Gamma \times G_0$ acts on M_m^s by continuous $\breve{K}$-vector space automorphisms for all s and m.

The $\breve{K}$-linear Γ-representations M_m^s so obtained will be the central topic of study in the subsequent chapters.

CHAPTER 3

Locally analytic representations arising from the Lubin-Tate spaces

As seen in the previous chapter, the group $\Gamma = \mathrm{Aut}(H_0)$ acts on the global sections M_m^s of the rigid Γ-equivariant line bundle $(\mathcal{M}_m^s)^{\mathrm{rig}}$ by continuous vector space automorphisms. The goal of this chapter is to show that the strong topological $\breve{K}$-linear dual $(M_m^s)'_b$ with the induced Γ-action is a locally K-analytic representation of Γ in the sense of Definition 1.4.1 for all $s \in \mathbb{Z}$ and levels $m \geq 0$. A strategy to do this is as follows:

Step 1. The Γ-action on the sections M_D^s of $(\mathcal{M}_0^s)^{\mathrm{rig}}$ over a certain Γ-stable affinoid subdomain $D \subset X_0^{\mathrm{rig}}$ is explicitly known. Show that this action is locally K-analytic by direct computations.

Step 2. Following the same approach as in [**Koh14**], show that the Γ-action on $(M_0^s)'_b$ is locally $\mathbb{Q}_p$-analytic. Use Step 1 and the $\mathfrak{g}_{\mathbb{Q}_p}$-equivariant $\breve{K}$-linear embedding $M_0^s \hookrightarrow M_D^s$ to deduce that the Γ-action on $(M_0^s)'_b$ is not only locally $\mathbb{Q}_p$-analytic but also locally K-analytic.

Step 3. For level $m > 0$, first prove that the Γ-action on $(M_m^s)'_b$ is locally $\mathbb{Q}_p$-analytic as done in Step 2. Then using the local K-analyticity of the Γ-action at level 0 and the étaleness of the extension $R_m^{\mathrm{rig}} | R_0^{\mathrm{rig}}$, deduce that the Γ-action on $(M_m^s)'_b$ is locally K-analytic.

3.1. The period morphism and the Gross-Hopkins fundamental domain

In Section 23, [**GH94**], Gross and Hopkins construct an étale surjective morphism $\Phi : X_0^{\mathrm{rig}} \longrightarrow \mathbb{P}_{\breve{K}}^{h-1}$ of rigid analytic varieties which is Γ-equivariant for the action of $\Gamma \hookrightarrow GL_h(K_h)$ by fractional linear transformations on the rigid analytic projective space $\mathbb{P}_{\breve{K}}^{h-1}$. They also define a Γ-stable affinoid subdomain $D \subset X_0^{\mathrm{rig}}$ on which Φ is injective (and hence bijective onto $\Phi(D)$). The morphism Φ is known as *the period morphism*, and D is known as *the Gross-Hopkins fundamental domain.* We now briefly review the construction of the period morphism Φ, and recall the definition of D.

Definition 3.1.1. A sequence $0 \longrightarrow F' \xrightarrow{\alpha} E \xrightarrow{\beta} F \longrightarrow 0$ of formal $\mathfrak{o}$-modules over an $\mathfrak{o}$-algebra A is said to be *exact* if the associated sequence $0 \longrightarrow \mathrm{Lie}(F') \xrightarrow{\mathrm{Lie}(\alpha)} \mathrm{Lie}(E) \xrightarrow{\mathrm{Lie}(\beta)} \mathrm{Lie}(F) \longrightarrow 0$ of free A-modules is exact. A formal $\mathfrak{o}$-module E is then said to be *an extension of F by F'*. With an obvious notion of an isomorphism of extensions, we denote by $\mathrm{Ext}(F, F')$ the set of isomorphism classes of extensions of F by F'.

It follows from [**GH94**], Proposition 9.8 that $\mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a)$ is a free R_m-module of rank $h-1$ for all $m \geq 0$. Let $\mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a)^* := \mathrm{Hom}_{R_m}(\mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a), R_m)$ denote the R_m-linear dual of $\mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a)$, and $(\mathbb{H}^{(m)})' := \mathbb{G}_a \otimes_{R_m} \mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a)^*$ be the associated additive formal $\mathfrak{o}$-module of dimension $h-1$. Then

$$\mathrm{Ext}(\mathbb{H}^{(m)}, (\mathbb{H}^{(m)})') = \mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a) \otimes_{R_m} \mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a)^* = \mathrm{End}_{R_m}(\mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a)).$$

Let $0 \longrightarrow (\mathbb{H}^{(m)})' \xrightarrow{\alpha^{(m)}} \mathbb{E}^{(m)} \xrightarrow{\beta^{(m)}} \mathbb{H}^{(m)} \longrightarrow 0$ be an extension in the class corresponding to the identity homomorphism in $\mathrm{End}_{R_m}(\mathrm{Ext}(\mathbb{H}^{(m)}, \mathbb{G}_a))$. Then the extension $\mathbb{E}^{(m)}$ is unique up to a unique isomorphism, and is a *universal additive extension of* $\mathbb{H}^{(m)}$ in the sense that given any other additive extension E' of $\mathbb{H}^{(m)}$ by an additive formal $\mathfrak{o}$-module F', there exists unique homomorphisms $(\mathbb{H}^{(m)})' \longrightarrow F'$ and $\mathbb{E}^{(m)} \longrightarrow E'$ of formal $\mathfrak{o}$-modules over R_m making all the relevant diagrams commute (cf. [**GH94**], Proposition 11.3).

Remark 3.1.2. Let us recall from [**GH94**], Section 16 how $\mathcal{L}\mathrm{ie}(\mathbb{E}^{(m)})$ and its tensor powers can be viewed as Γ-equivariant vector bundles on X_m. Given $\gamma \in \Gamma$, note that $\gamma_*\mathbb{E}^{(m)}$ is an extension of $\gamma_*\mathbb{H}^{(m)}$ by $\gamma_*(\mathbb{H}^{(m)})'$ obtained by applying $\gamma \in \mathrm{Aut}(R_m)$ to the coefficients of $\mathbb{E}^{(m)}$. The isomorphism $[\gamma] : \gamma_*\mathbb{H}^{(m)} \xrightarrow{\sim} \mathbb{H}^{(m)}$ and the universal property of $\mathbb{E}^{(m)}$ gives us a map $\mathrm{Lie}([\gamma]_{\mathbb{E}^{(m)}}) : \mathrm{Lie}(\gamma_*\mathbb{E}^{(m)}) \longrightarrow \mathrm{Lie}(\mathbb{E}^{(m)})$ which is an isomorphism of free R_m-modules of rank h. As before (cf. Section 2.2), we have a map $\gamma_* : \mathrm{Lie}(\mathbb{E}^{(m)}) \longrightarrow \mathrm{Lie}(\gamma_*\mathbb{E}^{(m)})$ of additive groups induced by the base change $\gamma : R_m[[X_1, \ldots, X_h]] \longrightarrow R_m[[X_1, \ldots, X_h]]$, ($a \mapsto \gamma(a)$, $a \in R_m$ and $X_i \mapsto X_i$ for all $1 \leq i \leq h$). As a result, we obtain a semilinear Γ-action on $\mathrm{Lie}(\mathbb{E}^{(m)})$ given by $\gamma = \mathrm{Lie}([\gamma]_{\mathbb{E}^{(m)}}) \circ \gamma_*$. This Γ-action is extended to $\mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s}$ as before for all $s \in \mathbb{Z}$.

A semilinear G_0-action on $\mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s}$ is defined similarly with the subgroup $G_m \subseteq G_0$ acting trivially. The actions of Γ and G_0 on $\mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s}$ commute. It follows directly from the definitions of the actions that the maps $\mathrm{Lie}(\alpha^{(m)})$ and $\mathrm{Lie}(\beta^{(m)})$ are $(\Gamma \times G_0)$-equivariant.

By rigidification, the exact sequence

$$0 \longrightarrow \mathrm{Lie}((\mathbb{H}^{(m)})')^{\otimes s} \xrightarrow{\mathrm{Lie}(\alpha^{(m)})^{\otimes s}} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s} \xrightarrow{\mathrm{Lie}(\beta^{(m)})^{\otimes s}} \mathrm{Lie}(\mathbb{H}^{(m)})^{\otimes s} \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow (\mathcal{L}\mathrm{ie}((\mathbb{H}^{(m)})')^{\otimes s})^{\mathrm{rig}} \longrightarrow (\mathcal{L}\mathrm{ie}(\mathbb{E}^{(m)})^{\otimes s})^{\mathrm{rig}} \longrightarrow (\mathcal{M}_m^s)^{\mathrm{rig}} \longrightarrow 0$$

of rigid $(\Gamma \times G_0)$-equivariant line bundles on X_m^{rig} for all non-negative s, and for negative s in the opposite direction. Since X_m is an affine formal scheme, by taking global sections, we get an exact sequence

$$0 \longrightarrow R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}((\mathbb{H}^{(m)})')^{\otimes s} \longrightarrow R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s} \longrightarrow M_m^s \longrightarrow 0 \tag{3.1.3}$$

of $\breve{K}$-linear $(\Gamma \times G_0)$-representations for $s \geq 0$, and in the opposite direction for $s < 0$.

The following proposition constitutes a key ingredient in the construction of the period morphism.

Proposition 3.1.4. *The Γ-equivariant line bundle $\mathcal{L}\mathrm{ie}(\mathbb{E}^{(0)})$ is generically flat, i.e. there exists a basis $\{c_0, c_1, \ldots, c_{h-1}\}$ of $R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)})$ over R_0^{rig} such that the $\breve{K}$-subspace of $R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)})$ spanned by c_i's is Γ-stable. Let $B_h \otimes_{K_h} \breve{K}$ be the h-dimensional $\breve{K}$-linear Γ-representation where the action of $\Gamma \simeq \mathfrak{o}_{B_h}^{\times}$ is given by left multiplication. Then we have an isomorphism*

$$R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)}) \simeq R_0^{\mathrm{rig}} \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K}) \tag{3.1.5}$$

of $R_0^{\mathrm{rig}}[\Gamma]$-modules with Γ acting diagonally on both sides.

Proof. The construction of the basis $\{c_i\}_{0 \leq i \leq h-1}$ is given in [**GH94**], Section 21. The isomorphism $R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)}) \simeq R_0^{\mathrm{rig}} \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K})$ is proved in [**GH94**], Proposition 22.4. □

Remark 3.1.6. Since $\mathbb{H}^{(m)} = \mathbb{H}^{(0)} \otimes_{R_0} R_m$, it follows from the universality of $\mathbb{E}^{(m)}$ that $\mathbb{E}^{(m)} = \mathbb{E}^{(0)} \otimes_{R_0} R_m$. The isomorphism $\mathrm{Lie}(\mathbb{E}^{(m)}) \simeq R_m \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)})$ of $R_m[\Gamma \times G_0]$-modules gives rise to an isomorphism $R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)}) \simeq R_m^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)})$ of $R_m^{\mathrm{rig}}[\Gamma \times G_0]$-modules. Then using (2.3.4) and (3.1.5), we have an isomorphism

$$R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)}) \simeq R_m^{\mathrm{rig}} \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K}) \tag{3.1.7}$$

of $R_m^{\mathrm{rig}}[\Gamma \times G_0]$-modules, where Γ and G_0 act diagonally on both sides. The action of G_0 on $B_h \otimes_{K_h} \breve{K}$ by convention is trivial.

For $s > 0$, the s-fold tensor product $(B_h \otimes_{K_h} \breve{K})^{\otimes s}$ of $B_h \otimes_{K_h} \breve{K}$ over $\breve{K}$ with itself is a Γ-representation with the diagonal Γ-action. Set $(B_h \otimes_{K_h} \breve{K})^{\otimes 0} := \breve{K}$ with the trivial Γ-action, and $(B_h \otimes_{K_h} \breve{K})^{\otimes s} := \mathrm{Hom}_{\breve{K}}((B_h \otimes_{K_h} \breve{K})^{\otimes -s}, \breve{K})$ for $s < 0$ equipped with the contragredient Γ-action, i.e. if $\gamma \in \Gamma$, $\varphi \in \mathrm{Hom}_{\breve{K}}((B_h \otimes_{K_h} \breve{K})^{\otimes -s}, \breve{K})$, and $v \in (B_h \otimes_{K_h} \breve{K})^{\otimes -s}$, then $(\gamma(\varphi))(v) = \gamma(\varphi(\gamma^{-1}(v))) = \varphi(\gamma^{-1}(v))$. Since

$$\begin{aligned} R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Hom}_{R_m}(\mathrm{Lie}(\mathbb{E}^{(m)}), R_m) &\simeq \mathrm{Hom}_{R_m^{\mathrm{rig}}}(R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)}), R_m^{\mathrm{rig}}) \\ &\simeq \mathrm{Hom}_{R_m^{\mathrm{rig}}}(R_m^{\mathrm{rig}} \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K}), R_m^{\mathrm{rig}}) \\ &\simeq R_m^{\mathrm{rig}} \otimes_{\breve{K}} \mathrm{Hom}_{\breve{K}}(B_h \otimes_{K_h} \breve{K}, \breve{K}), \end{aligned}$$

the isomorphism (3.1.7) extends to the isomorphism

$$R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s} \simeq R_m^{\mathrm{rig}} \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K})^{\otimes s} \tag{3.1.8}$$

of $R_m^{\mathrm{rig}}[\Gamma \times G_0]$-modules for all $s \in \mathbb{Z}$. To put it differently, the $(\Gamma \times G_0)$-equivariant line bundle $\mathcal{L}\mathrm{ie}(\mathbb{E}^{(m)})^{\otimes s}$ is *generically flat* for all $m \geq 0$ and $s \in \mathbb{Z}$.

Let v_i denote the images of the basis elements c_i under the second map in the short exact sequence (3.1.3) for $s = 1$, $m = 0$. According to [**GH94**], Proposition 23.2, the global sections $\{v_i\}_{0 \leq i \leq h-1}$ of the line bundle $(\mathcal{M}_0^1)^{\mathrm{rig}}$ have no common zeros on X_0^{rig}, and are linearly independent over $\breve{K}$. If $\mathbb{V}$ denotes the $\breve{K}$-subspace of M_0^1 spanned by them, then $\mathbb{V}$ is Γ-stable, and is isomorphic to $B_h \otimes_{K_h} \breve{K}$ as a Γ-representation. Let $\mathbb{P}(\mathbb{V})$ be the projective space of all hyperplanes in $\mathbb{V}$, then the map

$$\begin{aligned} \Phi : X_0^{\mathrm{rig}} &\longrightarrow \mathbb{P}(\mathbb{V}) \\ x &\longmapsto \{v \in \mathbb{V} | v(x) = 0\} \end{aligned}$$

is an étale surjective morphism of rigid analytic spaces, if $\mathbb{P}(\mathbb{V})$ is identified with the $(h-1)$-dimensional rigid analytic projective space $\mathbb{P}_{\breve{K}}^{h-1}$ (cf. [**GH94**], Proposition 23.5). The morphism $\Phi : X_0^{\mathrm{rig}} \longrightarrow \mathbb{P}_{\breve{K}}^{h-1}$ is called *the period morphism.* In homogeneous projective coordinates, it is given by $\Phi(x) = [\varphi_0(x) : \ldots : \varphi_{h-1}(x)]$ where $\varphi_0, \ldots, \varphi_{h-1} \in R_0^{\mathrm{rig}}$ are certain global rigid analytic functions without any common zero. These functions can be constructed from the logarithm $g_0(X) = \sum_{n \geq 0} a_n X^{q^n}$ of the universal formal $\mathfrak{o}$-module $\mathbb{H}^{(0)}$ over R_0 as the limits

$$\begin{aligned} \varphi_0 &:= \lim_{n \to \infty} \varpi^n a_{nh} \\ \varphi_i &:= \lim_{n \to \infty} \varpi^{n+1} a_{nh+i}, \text{ if } 1 \leq i \leq h-1 \end{aligned} \tag{3.1.9}$$

in the Fréchet topology of R_0^{rig} (cf. [**GH94**], (21.6) and (21.13)).

An important property of the period morphism Φ is that it is Γ-equivariant for the Γ-action on $\mathbb{P}^{h-1}_{\breve{K}}$ by fractional linear transformations via following inclusion of groups (cf. [**Koh13**], Remark 1.4):

$$j : \Gamma \hookrightarrow GL_h(K_h)$$

$$\sum_{i=0}^{h-1} \lambda_i \Pi^i \longmapsto \begin{pmatrix} \lambda_0 & \varpi\lambda_1 & \varpi\lambda_2 & \cdots & \cdots & \varpi\lambda_{h-1} \\ \lambda_{h-1}^{\sigma} & \lambda_0^{\sigma} & \lambda_1^{\sigma} & \cdots & \cdots & \lambda_{h-2}^{\sigma} \\ \lambda_{h-2}^{\sigma^2} & \varpi\lambda_{h-1}^{\sigma^2} & \lambda_0^{\sigma^2} & \cdots & \cdots & \lambda_{h-3}^{\sigma^2} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \lambda_1^{\sigma^{h-1}} & \varpi\lambda_2^{\sigma^{h-1}} & \cdots & \cdots & \varpi\lambda_{h-1}^{\sigma^{h-1}} & \lambda_0^{\sigma^{h-1}} \end{pmatrix} \tag{3.1.10}$$

The map j is the group homomorphism coming from the Γ-action via left multiplication on the right K_h-vector space B_h with basis $\{1, \Pi^{h-1}, \Pi^{h-2}, \dots, \Pi\}$. In a sense, the period morphism "linearises" the complicated Γ-action on X_0^{rig}.

The Gross-Hopkins fundamental domain D is the affinoid subdomain of X_0^{rig} defined as follows:

$$D := \left\{ x \in X_0^{\mathrm{rig}} \;\middle|\; |u_i(x)| \leq |\varpi|^{(1-\frac{i}{h})} \text{ for all } 1 \leq i \leq h-1 \right\} \tag{3.1.11}$$

According to [**GH94**], Lemma 23.14, the function φ_0 does not have any zeroes on D, hence is a unit in $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. Setting $w_i := \frac{\varphi_i}{\varphi_0}$ for $1 \leq i \leq h-1$, [**GH94**], Lemma 23.14 implies that the affinoid $\breve{K}$-algebra $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is isomorphic to the generalized Tate algebra.

$$\begin{aligned} \mathcal{O}_{X_0^{\mathrm{rig}}}(D) &\simeq \breve{K}\langle \varpi^{-(1-\frac{1}{h})} w_1, \dots, \varpi^{-(1-\frac{h-1}{h})} w_{h-1} \rangle \\ &:= \left\{ \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \in \breve{K}[[w_1, \dots, w_{h-1}]] \;\middle|\; \lim_{|\alpha| \to \infty} |c_\alpha| |\varpi|^{\sum_{i=1}^{h-1} \alpha_i (1-\frac{i}{h})} = 0 \right\} \end{aligned} \tag{3.1.12}$$

Note that over a field extension L of $\breve{K}$ containing an h-th root of ϖ, this is isomorphic to the Tate algebra $L\langle T_1, \dots, T_{h-1} \rangle$.

It follows from [**FGL08**], Remarque I.3.2 that D is stable under the Γ-action on X_0^{rig}. Also, the Γ-equivariant period morphism Φ restricts to an isomorphism $\Phi : D \xrightarrow{\sim} \Phi(D)$ over D (cf. [**GH94**], Corollary 23.15). As a result, we have an explicit formula for the Γ-action on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ similar to the one of Devinatz-Hopkins (cf. [**Koh13**], Proposition 1.3.):

Proposition 3.1.13. *Fix i with $1 \leq i \leq h-1$, and let $\gamma = \sum_{j=0}^{h-1} \lambda_j \Pi^j \in \Gamma$. Then*

$$\gamma(w_i) = \frac{\sum_{j=1}^{i} \lambda_{i-j}^{\sigma^j} w_j + \sum_{j=i+1}^{h} \varpi \lambda_{h+i-j}^{\sigma^j} w_j}{\lambda_0 + \sum_{j=1}^{h-1} \lambda_{h-j}^{\sigma^j} w_j}. \tag{3.1.14}$$

The group Γ acts on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ by continuous $\breve{K}$-algebra endomorphisms extending the action on R_0^{rig}.

Proof. This is straightforward since γ acts on the projective homogeneous coordinates $[\varphi_0 : \dots : \varphi_{h-1}]$ through right multiplication with the matrix $j(\gamma)$ in (3.1.10). By [**BGR84**], (6.1.3), Theorem 1, the induced $\breve{K}$-algebra endomorphism γ of the affinoid $\breve{K}$-algerba $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is automatically continuous. $\square$

Remark 3.1.15. *A rigidified extension* (E,s) of $\mathbb{H}^{(0)}$ by $\mathbb{G}_a$ is an extension E of $\mathbb{H}^{(0)}$ by $\mathbb{G}_a$ together with a section $s : \mathrm{Lie}(\mathbb{H}^{(0)}) \longrightarrow \mathrm{Lie}(E)$. The set $\mathrm{RigExt}(\mathbb{H}^{(0)}, \mathbb{G}_a)$ of isomorphism classes of rigidified extensions of $\mathbb{H}^{(0)}$ by $\mathbb{G}_a$ is a free R_0-module of rank h, and has a basis $\{g_0, g_1, \ldots, g_{h-1}\}$ where $g_0 \in R_0[[X]]$ is the logarithm of $\mathbb{H}^{(0)}$, and $g_i := \frac{\partial g_0}{\partial u_i}$ for $1 \le i \le h-1$ (cf. [**GH94**], Proposition 9.8). Moreover, the R_0-module $\omega(\mathbb{E}^{(0)})$ of invariant differentials on the universal additive extension is isomorphic to $\mathrm{RigExt}(\mathbb{H}^{(0)}, \mathbb{G}_a)$ (cf. [**GH94**], (11.4)). Thus, $R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{E}^{(0)}) \simeq \mathrm{Hom}_{R_0}(\mathrm{RigExt}(\mathbb{H}^{(0)}, \mathbb{G}_a), R_0^{\mathrm{rig}})$. The functions φ_i in (3.1.9) are precisely $c_i(g_0)$, and the basis dg_0 of $\omega(\mathbb{H}^{(0)})$ is mapped to g_0 under the natural map $\omega(\mathbb{H}^{(0)}) \longrightarrow \omega(\mathbb{E}^{(0)})$. As a result, the global sections v_i and v_j of the line bundle $(\mathcal{M}_0^1)^{\mathrm{rig}}$ (see paragraph after Remark 3.1.6) are related by the relation $\varphi_j v_i = \varphi_i v_j$ for all $0 \le i,j \le h-1$. Consequently, we have $\varphi_j^s v_i^s = \varphi_i^s v_j^s$ in M_0^s. Let $U_i \subset X_0^{\mathrm{rig}}$ be the non-vanishing locus of φ_i, then on $U_i \cap U_j$, we get $v_i^s = \frac{\varphi_i^s}{\varphi_j^s} v_j^s$ and $v_j^s = \frac{\varphi_j^s}{\varphi_i^s} v_i^s$. The U_i's cover X_0^{rig} as the functions φ_i's do not vanish simultaneously at any point on X_0^{rig}. The data $\{U_i, \varphi_i^s \in \mathcal{O}_{U_i}^\times, \frac{\varphi_i^s}{\varphi_j^s} \in \mathcal{O}_{U_i \cap U_j}^\times\}_{0 \le i,j \le h-1}$ represent an element in $H^1(X_0^{\mathrm{rig}}, \mathcal{O}_{X_0^{\mathrm{rig}}}^\times)$ corresponding to the line bundle $(\mathcal{M}_0^s)^{\mathrm{rig}}$. This means that $(\mathcal{M}_0^s)^{\mathrm{rig}}\big|_{U_i} \simeq \mathcal{O}_{U_i}\varphi_i = \mathcal{O}_{X_0^{\mathrm{rig}}}\big|_{U_i}\varphi_i$ for all $0 \le i \le h-1$. In particular, for $i = 0$, we have an isomorphism $M_D^s := (\mathcal{M}_0^s)^{\mathrm{rig}}(D) \simeq \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s$ of $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$-modules which is also Γ-equivariant. The Γ-action on M_D^s is semilinear for its action on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, i.e. $\gamma(f\varphi_0^s) = \gamma(f)\gamma(\varphi_0^s) = \gamma(f)\gamma(\varphi_0)^s$ for $\gamma \in \Gamma$ and $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$.

Remark 3.1.16. The discussion in Remark 3.1.15 shows that the generating global sections v_i's of the line bundle $(\mathcal{M}_0^1)^{\mathrm{rig}}$ are the pullbacks $\Phi^*(\varphi_i)$ of φ_i's along the period morphism Φ for all $0 \le i \le h-1$. As a consequence, it follows that

$$(\mathcal{M}_0^1)^{\mathrm{rig}} \simeq \Phi^* \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(1).$$

By the general properties of the inverse image functor, we then have

$$(\mathcal{M}_0^s)^{\mathrm{rig}} \simeq \Phi^* \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$$

for all $s \in \mathbb{Z}$.

3.2. Local analyticity of the Γ-action on M_D^s

In this section, we show that the orbit map $(\gamma \mapsto \gamma(f\varphi_0^s)) : \Gamma \longrightarrow M_D^s$ explicitly given by Proposition 3.1.13 and Remark 3.1.15 is locally K-analytic for all $f\varphi_0^s \in M_D^s$.

Let $M_h(K_h)$ denote the ring of $h \times h$ matrices with entries from K_h. It carries an induced topology from the identification with $K_h^{h^2}$, which endows it with a structure of a locally analytic K_h-manifold. The subset $GL_h(K_h)$ of invertible matrices is open and forms a locally K_h-analytic group. Consider the subgroup P of $GL_h(K_h)$ defined as follows. It is conjugate to a standard Iwahori subgroup of $GL_h(K_h)$.

$$P := \{a = (a_{ij})_{0 \le i,j \le h-1} \in GL_h(\mathfrak{o}_h) \mid a_{ij}, a_{0k} \in \varpi\mathfrak{o}_h \text{ for all } 1 \le i,j,k \le h-1 \text{ with } i > j\}$$

The conditions on the entries of a matrix in P force all of its diagonal entries to lie in $\mathfrak{o}_h^\times$. Since $B_{|\varpi^2|}(a) \subseteq P$ for any $a \in P$, P is open in $GL_h(K_h)$. Thus, P is a locally K_h-analytic subgroup of $GL_h(K_h)$. The inclusion map $j : \Gamma \hookrightarrow GL_h(K_h)$ mentioned in (3.1.10) has image in P.

Lemma 3.2.1. *The inclusion map* $j : \Gamma \hookrightarrow P$ *in (3.1.10) is locally* K*-analytic.*

PROOF. The global chart for P induced from that for $M_h(K_h)$ sends a in P to

$$(a_{00}, a_{01}, \dots a_{0(h-1)}, a_{10}, a_{11}, \dots, a_{(h-1)(h-2)}, a_{(h-1)(h-1)})$$

in $K_h^{h^2}$. Recall the global chart ψ for Γ from (2.2.5). Using the global charts for both groups, it is easy to see that the corresponding map from the open subset $\psi(\Gamma)$ in K_h^h to $K_h^{h^2}$ is locally K-analytic since each component of this map is either a linear polynomial or a K-linear Frobenius automorphism σ or a composition of both, all being locally K-analytic. As before, we remark that j is generally not locally K_h-analytic because $\sigma : \mathfrak{o}_h^\times \longrightarrow \mathfrak{o}_h^\times$ is not locally K_h-analytic unless $h = 1$. □

The algebra $\mathcal{O}_{X_0^{\text{rig}}}(D)$ is a $\breve{K}$-Banach algebra with respect to the multiplicative norm $\|\cdot\|_D$defined as follows: for $f = \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \in \mathcal{O}_{X_0^{\text{rig}}}(D)$, $\|f\|_D := \sup_{\alpha \in \mathbb{N}_0^{h-1}} |c_\alpha| |\varpi|^{\sum_{i=1}^{h-1} \alpha_i (1 - \frac{i}{h})}$ (cf. [**BGR84**], Section 6.1.5, Proposition 1 and 2). Let P act on $\mathcal{O}_{X_0^{\text{rig}}}(D)$ by $\breve{K}$-linear ring automorphisms by defining

$$a(w_i) := \frac{a_{0i} + \sum_{j=1}^{h-1} a_{ji} w_j}{a_{00} + \sum_{j=1}^{h-1} a_{j0} w_j} \tag{3.2.2}$$

for $a \in P$ and for $1 \le i \le h-1$. This gives an action of P on $\mathcal{O}_{X_0^{\text{rig}}}(D)$ by continuous $\breve{K}$-linear ring automorphisms which, when restricted to Γ via j, coincides with the Γ-action on $\mathcal{O}_{X_0^{\text{rig}}}(D)$ (cf. Proposition 3.1.13). Indeed, note that $a_{00} + \sum_{j=1}^{h-1} a_{j0} w_j = a_{00}(1 + \sum_{j=1}^{h-1} a_{00}^{-1} a_{j0} w_j) \in (\mathcal{O}_{X_0^{\text{rig}}}(D))^\times$ is a unit of norm 1, and $\left\| a_{0i} + \sum_{j=1}^{h-1} a_{ji} w_j \right\|_D = \|w_i\|_D$ by the strict triangle inequality. Altogether, $\|a(w_i)\|_D = \|w_i\|_D$ which ensures that P acts on $\mathcal{O}_{X_0^{\text{rig}}}(D)$ via $\sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \longmapsto \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha a(w_1)^{\alpha_1} \dots a(w_{h-1})^{\alpha_{h-1}}$ in a well-defined way.

We now show that the above action is locally K-analytic.

Lemma 3.2.3. *The map $\iota : P \longrightarrow P$, $(a_{ij})_{0 \le i,j \le h-1} \longmapsto (\iota(a)_{ij})_{0 \le i,j \le h-1}$ defined by*

$$\iota(a)_{ij} = \begin{cases} a_{ij}^{-1}, & \text{if } i = j = 0; \\ a_{ij}, & \text{otherwise} \end{cases}$$

is locally K_h-analytic, and thus locally K-analytic.

PROOF. This follows from [**Sch11**], Proposition 13.6, and the fact that $K_h^\times$ is a locally K_h-analytic group. □

Proposition 3.2.4. *The action of P on $\mathcal{O}_{X_0^{\text{rig}}}(D)$ is locally K_h-analytic, and thus locally K-analytic.*

PROOF. By Lemma 3.2.3, it is enough to show that, for each $f \in \mathcal{O}_{X_0^{\text{rig}}}(D)$, the map $\iota(a) \longmapsto a(f)$ from P to $\mathcal{O}_{X_0^{\text{rig}}}(D)$ is locally K_h-analytic. Consider the open neighbourhood U of 0 in $K_h^{h^2}$ defined as follows:

$$U := \{x = (x_1, x_2, \dots, x_{h^2}) \in \mathfrak{o}_h^{h^2} \mid x_i, x_{qh+r} \in \varpi \mathfrak{o}_h \ \forall\ 2 \le i \le h \text{ and } \forall\ q \ge r \text{ with } q, r > 1 \}$$

Let $T = (T_1, T_2, \dots, T_{h^2})$, and let $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D))$ denote the set of power series in T with coefficients from $\mathcal{O}_{X_0^{\text{rig}}}(D)$ which converge on U, i.e. $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D)) :=$

$$\left\{ \sum_{\alpha \in \mathbb{N}_0^{h^2}} f_\alpha T^\alpha \in \mathcal{O}_{X_0^{\text{rig}}}(D)[[T]] \;\middle|\; \lim_{|\alpha| \to \infty} \|f_\alpha\|_D |\varpi|^{(\alpha_2 + \alpha_3 + \dots + \alpha_h + \alpha_{2h+2} + \alpha_{3h+2} + \alpha_{3h+3} + \dots + \alpha_{(h-1)h+h-1})} = 0 \right\}$$

Like $\mathcal{O}_{X_0^{\text{rig}}}(D)$, $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D))$ is also a $\breve{K}$-Banach algebra with respect to the following multiplicative norm (cf. [**Sch11**], Proposition 5.3):

$$\Big\| \sum_{\alpha \in \mathbb{N}_0^{h^2}} f_\alpha T^\alpha \Big\|_U := \sup_{\alpha \in \mathbb{N}_0^{h^2}} \|f_\alpha\|_D |\varpi|^{(\alpha_2+\alpha_3+\cdots+\alpha_h+\alpha_{2h+2}+\alpha_{3h+2}+\alpha_{3h+3}+\cdots+\alpha_{(h-1)h+h-1})}$$

Under the global chart of P in the proof of 3.2.1, we now show that, for a monomial $w^\alpha \in \mathcal{O}_{X_0^{\text{rig}}}(D)$, the map $\iota(a) \longmapsto a(w^\alpha)$ belongs to $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D))$ for every $\alpha \in \mathbb{N}_0^{h-1}$.

By (3.2.2), we have

$$\begin{aligned} a(w^\alpha) &= a(w_1)^{\alpha_1} \dots a(w_{h-1})^{\alpha_{h-1}} \\ &= \left(\frac{a_{01} + \sum_{j=1}^{h-1} a_{j1} w_j}{a_{00} + \sum_{j=1}^{h-1} a_{j0} w_j} \right)^{\alpha_1} \cdots \left(\frac{a_{0(h-1)} + \sum_{j=1}^{h-1} a_{j(h-1)} w_j}{a_{00} + \sum_{j=1}^{h-1} a_{j0} w_j} \right)^{\alpha_{h-1}} \\ &= \left(\prod_{i=1}^{h-1} \Big(a_{0i} + \sum_{j=1}^{h-1} a_{ji} w_j \Big)^{\alpha_i} \right) (a_{00}^{-1})^{|\alpha|} \Big(1 + a_{00}^{-1} \sum_{j=1}^{h-1} a_{j0} w_j \Big)^{-|\alpha|} \end{aligned}$$

$$= \left(\prod_{i=1}^{h-1} \Big(a_{0i} + \sum_{j=1}^{h-1} a_{ji} w_j \Big)^{\alpha_i} \right) (a_{00}^{-1})^{|\alpha|} \left(\sum_{l=0}^{\infty} \Big(- a_{00}^{-1} \sum_{j=1}^{h-1} a_{j0} w_j \Big)^l \right)^{|\alpha|} \tag{3.2.5}$$

Thus the expression of $a(w^\alpha)$ is a product of $(a_{00}^{-1})^{|\alpha|}$ and two big brackets. The first big bracket in (3.2.5) is a product of polynomials in a_{ij}'s with coefficients from $\mathcal{O}_{X_0^{\text{rig}}}(D)$, and hence is the evaluation at $\iota(a)$ of an element in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D))$. Similarly, $(a_{00}^{-1})^{|\alpha|}$ is the evaluation at $\iota(a)$ of the monomial $T_1^{|\alpha|}$ which belongs to $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D))$. The second big bracket is the $|\alpha|$-th power of a certain geometric series. The l-th term in that series is the evaluation of the polynomial $\big(- T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \big)^l$ at $\iota(a)$, and

$$\Big\| \Big(- T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \Big)^l \Big\|_U = \Big(\Big\| - T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \Big\|_U \Big)^l = |\varpi|^{\frac{l}{h}}$$

Hence, the series $\sum_{l=0}^{\infty} (-T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j)^l$ converges in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D))$, and the map $\iota(a) \longmapsto a(w^\alpha) \in \mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D))$ for every $\alpha \in \mathbb{N}_0^{h-1}$.

Let us calculate the norms $\| \cdot \|_U$ of the above power series corresponding to the terms in the expression (3.2.5) or find an upper bound for them. First, $\|T_1^{|\alpha|}\|_U = 1$. Since

$$\Big\| \sum_{l=0}^{\infty} \Big(- T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \Big)^l \Big\|_U \le \sup_{l \ge 0} \Big\| \Big(- T_1 \sum_{j=1}^{h-1} T_{jh+1} w_j \Big)^l \Big\|_U = \sup_{l \ge 0} |\varpi|^{\frac{l}{h}} = 1,$$

the power series corresponding to the second big bracket in (3.2.5) has the norm ≤ 1. The first big bracket is obtained by evaluating $\prod_{i=1}^{h-1} \Big(T_{i+1} + \sum_{j=1}^{h-1} T_{jh+i+1} w_j \Big)^{\alpha_i}$ at $\iota(a)$, and

$$\Big\| \prod_{i=1}^{h-1} \Big(T_{i+1} + \sum_{j=1}^{h-1} T_{jh+i+1} w_j \Big)^{\alpha_i} \Big\|_U = \prod_{i=1}^{h-1} \Big\| \Big(T_{i+1} + \sum_{j=1}^{h-1} T_{jh+i+1} w_j \Big) \Big\|_U^{\alpha_i} = \prod_{i=1}^{h-1} |\varpi|^{\alpha_i (1 - \frac{i}{h})}$$

Therefore, the power series corresponding to the first big bracket has the norm $|\varpi|^{\sum_{i=1}^{h-1} \alpha_i (1 - \frac{i}{h})}$. So, for every $\alpha \in \mathbb{N}_0^{h-1}$, the map $\iota(a) \longmapsto a(w^\alpha)$ is given by an element in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\text{rig}}}(D))$

whose norm is bounded above by $|\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})}$.

Now for every $f = \sum_{\alpha\in\mathbb{N}_0^{h-1}} c_\alpha w^\alpha \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, we have $a(f) = \sum_{\alpha\in\mathbb{N}_0^{h-1}} c_\alpha(a(w^\alpha))$, and for every $\alpha \in \mathbb{N}_0^{h-1}$, $c_\alpha(a(w^\alpha))$ is represented by a power series in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$ having norm $\leq |c_\alpha||\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})}$. Since, $\lim_{|\alpha|\to\infty} |c_\alpha||\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})} = 0$, we see that the map $\iota(a) \longmapsto a(f)$ from P to $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is given by a convergent power series in $\mathcal{F}_U(K_h^{h^2}, \mathcal{O}_{X_0^{\mathrm{rig}}}(D))$. As a is arbitrary, this implies that the action of P on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is locally K_h-analytic, and thus locally K-analytic by Lemma 1.2.3. □

Proposition 3.2.6. *The $\breve{K}$-vector space $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is a locally K-analytic representation of Γ.*

Proof. This follows from the Lemma 3.2.1 and Proposition 3.2.4. □

The above proposition can be generalized as follows completing the Step 1 of the strategy mentioned in the beginning:

Theorem 3.2.7. *Let s be any integer, then the $\breve{K}$-vector space M_D^s is a locally K-analytic representation of Γ.*

Proof. According to Remark 3.1.15, we have a Γ-equivariant, $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$-linear isomorphism $M_D^s \simeq \mathcal{O}_{X_0^{\mathrm{rig}}}(D).\varphi_0^s$ of free $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$-modules of rank 1. Then M_D^s obtains a structure of a $\breve{K}$-Banach space with respect to the norm defined as $\|f\varphi_0^s\|_{M_D^s} := \|f\|_D$. Since the Γ-action on M_D^s is semilinear for its action on $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, we have $\gamma(f\varphi_0^s) = \gamma(f)\gamma(\varphi_0^s)$ for all $\gamma \in \Gamma$ and $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. Now, as mentioned in the proof of Proposition 3.1.13, $\gamma(\varphi_0^s) = \gamma(\varphi_0)^s = (\lambda_0\varphi_0 + \lambda_{h-1}^\sigma\varphi_1 + \cdots + \lambda_1^{\sigma^{h-1}}\varphi_{h-1})^s = (\lambda_0 + \lambda_{h-1}^\sigma w_1 + \cdots + \lambda_1^{\sigma^{h-1}} w_{h-1})^s\varphi_0^s$. So the orbit map from Γ to M_D^s is given by sending γ to $\gamma(f\varphi_0^s) = \gamma(f)(\lambda_0 + \lambda_{h-1}^\sigma w_1 + \cdots + \lambda_1^{\sigma^{h-1}} w_{h-1})^s\varphi_0^s$. The map $\gamma \mapsto \gamma(f)$ is locally K-analytic by Proposition 3.2.6, and the map $\gamma \mapsto (\lambda_0 + \lambda_{h-1}^\sigma w_1 + \cdots + \lambda_1^{\sigma^{h-1}} w_{h-1})^s\varphi_0^s$ is also locally K-analytic since it is given by a linear polynomial in the coordinates of γ. Thus, the orbit map, being a product of these two maps, is locally K-analytic. Therefore, M_D^s is a locally analytic Γ-representation for all integers s. □

3.3. Local analyticity of the Γ-action on M_0^s

Recall that the free R_0^{rig}-module M_0^s of rank 1 is a $\breve{K}$-Fréchet space, and the group Γ acts on it by continuous $\breve{K}$-linear automorphisms. This induces an action of Γ on the strong topological $\breve{K}$-linear dual $(M_0^s)'_b$ of M_0^s given by

$$\gamma : (M_0^s)'_b \longrightarrow (M_0^s)'_b$$
$$l \longmapsto (\delta \mapsto l(\gamma^{-1}(\delta))).$$

The goal of this section is to show that the above action of Γ on $(M_0^s)'_b$ is locally K-analytic by showing that its dual M_0^s is a continuous $D(\Gamma, \breve{K})$-module.

The continuity of the Γ-action on the universal deformation ring R_0 (cf. Theorem 2.2.8) leads to a continuous Γ-action on R_0^{rig}, i.e. the action map $\Gamma \times R_0^{\mathrm{rig}} \longrightarrow R_0^{\mathrm{rig}}$ is continuous for the Fréchet topology on R_0^{rig} and for the product of profinite and Fréchet topology on the left. This is implied by the next proposition which also forms the main ingredient of the Step 2 of our strategy.

Proposition 3.3.1. *Let n and l be integers with $n \geq 0$ and $l \geq 1$. If $\gamma \in \Gamma_n$, and if $f \in R_0^{\mathrm{rig}}$, then $\|\gamma(f) - f\|_l \leq |\varpi|^{n/l}\|f\|_l$.*

PROOF. Note that $R_{0,l}^{\mathrm{rig}}$ is a generalized Tate algebra over $\breve{K}$ in the variables $(\varpi^{-1/l}u_i)_{1\le i\le h-1}$. Then by [**BGR84**], (6.1.5), Proposition 5, we have

$$\|g\|_l = \sup\left\{|g(x)| \;\middle|\; x\in \mathbb{B}_l\big(\overline{\breve{K}}\big)\right\} \text{ for any } g\in R_{0,l}^{\mathrm{rig}},$$

where

$$\mathbb{B}_l\big(\overline{\breve{K}}\big) = \left\{x\in \big(\overline{\breve{K}}\big)^{h-1} \;\middle|\; |x_i|\le |\varpi|^{1/l} \text{ for all } 1\le i\le h-1\right\}.$$

Let us first prove the assertion for $f = u_i$ for some $1\le i\le h-1$. If $x\in\mathbb{B}_l\big(\overline{\breve{K}}\big)$ and $y=(y_j) := (\gamma(u_j)(x))$, then we need to show that $|x_i - y_i| \le |\varpi|^{(n+1)/l}$ because $\|u_i\|_l = |\varpi|^{1/l}$. Consider the commutating diagram

$$\begin{array}{ccc} R_0 & \xrightarrow{\;\gamma\;} & R_0 \\ & {\scriptstyle f\mapsto f(y)}\searrow \quad \swarrow{\scriptstyle f\mapsto f(x)} & \\ & \overline{\breve{\mathfrak{o}}} & \end{array}$$

of homomorphisms of $\breve{\mathfrak{o}}$-algebras. Choosing $z\in\overline{\breve{\mathfrak{o}}}$ with $|z| = |\varpi|^{1/l}$, we have $x_j\in z\overline{\breve{\mathfrak{o}}}$ for all j. Further, $\varpi\in z\overline{\breve{\mathfrak{o}}}$ because $l\ge 1$. As a consequence, the right oblique arrow of the above diagram maps $\mathfrak{m}_{R_0} = (\varpi, u_1,\ldots,u_{h-1})$ to $z\overline{\breve{\mathfrak{o}}}$. Note that $\gamma(u_j)\in\mathfrak{m}_{R_0}$ so we obtain $y_j\in z\overline{\breve{\mathfrak{o}}}$ as well. Therefore, also the left oblique arrow maps $\mathfrak{m}_{R_0}$ to $z\overline{\breve{\mathfrak{o}}}$. Now consider the induced diagram

$$\begin{array}{ccc} R_0/\mathfrak{m}_{R_0}^{n+1} & \xrightarrow{\;\gamma\;} & R_0/\mathfrak{m}_{R_0}^{n+1} \\ & \searrow \quad \swarrow & \\ & \overline{\breve{\mathfrak{o}}}/(z^{n+1}) & \end{array}$$

According to Theorem 2.2.8, the upper horizontal arrow is the identity. It follows that $x_i - y_i\in z^{n+1}\overline{\breve{\mathfrak{o}}}$, i.e. $|x_i-y_i|\le|\varpi|^{(n+1)/l}$.

We now prove the assertion for $f = u^\alpha$ by induction on $|\alpha|$. The case $|\alpha| = 0$ is trivial. Let $|\alpha|>0$. Choose an index i with $\alpha_i>0$. Define $\beta_j := \alpha_j$ if $j\ne i$, and $\beta_i := \alpha_i - 1$. Then for $x\in\mathbb{B}_l\big(\overline{\breve{K}}\big)$,

$$\begin{aligned} |\gamma(u^\alpha)(x) - u^\alpha(x)| &= |y^\alpha - x^\alpha| = |y_iy^\beta - x_ix^\beta| \\ &\le \max\{|y_i||y^\beta - x^\beta|, |y_i - x_i||x^\beta|\}. \end{aligned}$$

Now $|y_i||y^\beta - x^\beta| \le |\varpi|^{1/l}\|\gamma(u^\beta) - u^\beta\|_l \le |\varpi|^{(n+1)/l}\|u^\beta\|_l = |\varpi|^{n/l}\|u^\alpha\|_l$ by the induction hypothesis and $|y_i-x_i||x^\beta|\le|\varpi|^{(n+1)/l}|\varpi|^{|\beta|/l} = |\varpi|^{n/l}\|u^\alpha\|_l$ as seen above. Thus we obtain $|\gamma(u^\alpha)(x) - u^\alpha(x)|\le|\varpi|^{n/l}\|u^\alpha\|_l$ for all $x\in\mathbb{B}_l\big(\overline{\breve{K}}\big)$ as required.

Therefore if $f = \sum_{\alpha\in\mathbb{N}_0^{h-1}} c_\alpha u^\alpha\in R_0^{\mathrm{rig}}$, then by continuity of γ, we get

$$\begin{aligned} \|\gamma(f) - f\|_l = \Big\|\sum_{\alpha\in\mathbb{N}_0^{h-1}} c_\alpha(\gamma(u^\alpha) - u^\alpha)\Big\|_l &\le \sup_{\alpha\in\mathbb{N}_0^{h-1}} |c_\alpha|\|\gamma(u^\alpha) - u^\alpha\|_l \\ &\le \sup_{\alpha\in\mathbb{N}_0^{h-1}} |c_\alpha||\varpi|^{n/l}\|u^\alpha\|_l = |\varpi|^{n/l}\|f\|_l. \end{aligned}$$

□

As stated in the notation, we write $\Gamma_{\mathbb{Q}_p}$ for Γ when viewed as a locally $\mathbb{Q}_p$-analytic group, and $\mathfrak{g}_{\mathbb{Q}_p}$ for its Lie algebra $\mathfrak{g}$ when considered as a $\mathbb{Q}_p$-vector space. Let $d := [K:\mathbb{Q}_p]$. Since $\Gamma_{\mathbb{Q}_p}$

is a compact locally $\mathbb{Q}_p$-analytic group of dimension $t := dh^2$, it contains an open subgroup Γ_o which is a uniform pro-p group of rank t (cf. Theorem 1.3.3). The subgroups in its lower p-series $P_i(\Gamma_o)$ $(i \geq 1)$ form a basis of open neighbourhoods of the identity in Γ_o and are also uniform pro-p groups of rank t by Lemma 1.3.2. Let n be a positive integer such that $\Gamma_n \subseteq \Gamma_o$. As Γ_n is open in Γ_o, it contains $\Gamma_* := P_i(\Gamma_o)$ for some $i \geq 1$. In what follows, we view Γ_* as a locally $\mathbb{Q}_p$-analytic group.

Let us denote by $\Lambda(\Gamma_*) := \breve{\mathfrak{o}}[[\Gamma_*]]$ the Iwasawa algebra of Γ_* over $\breve{\mathfrak{o}}$. Set $b_i := \gamma_i - 1 \in \Lambda(\Gamma_*)$ and $b^\alpha := b_1^{\alpha_1} \cdots b_t^{\alpha_t}$ for any $\alpha \in \mathbb{N}_0^t$ where $\{\gamma_1, \dots, \gamma_t\}$ is a minimal topological generating set of Γ_*. By [**DDMS03**], Theorem 7.20, any element $\mu \in \Lambda(\Gamma_*)$ admits a unique expansion of the form

$$\mu = \sum_{\alpha \in \mathbb{N}_0^t} d_\alpha b^\alpha \text{ with } d_\alpha \in \breve{\mathfrak{o}} \;\forall\; \alpha \in \mathbb{N}_0^t$$

For any $l \geq 1$, this allows us to define the $\breve{K}$-norm $\|\cdot\|_l$ on the algebra $\Lambda(\Gamma_*)_{\breve{K}} := \Lambda(\Gamma_*) \otimes_{\breve{\mathfrak{o}}} \breve{K}$ through

$$\Big\| \sum_{\alpha \in \mathbb{N}_0^t} d_\alpha b^\alpha \Big\|_l := \sup_{\alpha \in \mathbb{N}_0^t} \{|d_\alpha| |\varpi|^{|\alpha|/l}\} \tag{3.3.2}$$

By [**ST03**], Proposition 4.2, the norm $\|\cdot\|_l$ on $\Lambda(\Gamma_*)_{\breve{K}}$ is submultiplicative. As a consequence, the completion

$$\Lambda(\Gamma_*)_{\breve{K},l} = \Big\{ \sum_{\alpha \in \mathbb{N}_0^t} d_\alpha b^\alpha \mid d_\alpha \in \breve{K},\ \lim_{|\alpha| \to \infty} |d_\alpha| |\varpi|^{|\alpha|/l} = 0 \Big\}$$

of $\Lambda(\Gamma_*)_{\breve{K}}$ with respect to $\|\cdot\|_l$ is a $\breve{K}$-Banach algebra. The natural inclusions $\Lambda(\Gamma_*)_{\breve{K},l+1} \hookrightarrow \Lambda(\Gamma_*)_{\breve{K},l}$ endow the projective limit

$$D(\Gamma_*, \breve{K}) = \varprojlim_l \Lambda(\Gamma_*)_{\breve{K},l}$$

with the structure of a $\breve{K}$-Fréchet algebra. As explained in Section 1.3, the above projective limit is indeed equal to the algebra of $\breve{K}$-valued locally $\mathbb{Q}_p$-analytic distributions on Γ_*. By fixing coset representatives $\{\gamma'_1 = 1, \gamma'_2, \dots, \gamma'_s\}$ of Γ_* in $\Gamma_{\mathbb{Q}_p}$, the natural topological isomorphism $C^{an}(\Gamma_{\mathbb{Q}_p}, \breve{K}) \simeq \prod_{i=1}^s C^{an}(\gamma'_i \Gamma_*, \breve{K})$ of locally convex $\breve{K}$-vector spaces induces a topological isomorphism

$$D(\Gamma_{\mathbb{Q}_p}, \breve{K}) \simeq \bigoplus_{i=1}^s \delta_{\gamma'_i} D(\Gamma_*, \breve{K}) \qquad (\delta_{\gamma'_i}\text{'s are Dirac distributions}) \tag{3.3.3}$$

by dualizing (cf. [**Fé99**], Korollar 2.2.4). This defines a $\breve{K}$-Fréchet algebra structure on $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ given by the family of norms $\|\delta_{\gamma'_1}\mu_1 + \cdots + \delta_{\gamma'_s}\mu_s\|_l := \max_{i=1}^s \{\|\mu_i\|_l\}$ with $l \geq 1$ (cf. [**ST03**], Theorem 5.1).

Note that $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ is not the same as the distribution algebra $D(\Gamma, \breve{K})$ of $\breve{K}$-valued locally K-analytic distributions on Γ. In fact, the natural embedding $C^{an}(\Gamma, \breve{K}) \hookrightarrow C^{an}(\Gamma_{\mathbb{Q}_p}, \breve{K})$ induces a map $D(\Gamma_{\mathbb{Q}_p}, \breve{K}) \longrightarrow D(\Gamma, \breve{K})$ which is a strict surjection and a homomorphism of $\breve{K}$-algebras by [**Koh07**], Lemma 1.3.1. The kernel I of the sujection $D(\Gamma_{\mathbb{Q}_p}, \breve{K}) \twoheadrightarrow D(\Gamma, \breve{K})$ has the following explicit description due to [**Koh07**], Lemma 1.3.2 and Lemma 1.3.3:

(3.3.4) If $i : \mathfrak{g}_{\mathbb{Q}_p} \hookrightarrow D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ denotes the natural inclusion, then I is the closure of the ideal generated by all elements of the form $i(t\mathfrak{x}) - ti(\mathfrak{x})$ with $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$ and $t \in K$.

The quotient topology on $D(\Gamma, \breve{K})$ is the $\breve{K}$-Fréchet topology mentioned in the beginning of Section 1.3.

Theorem 3.3.5. *The action of $\Gamma_{\mathbb{Q}_p}$ on R_0^{rig} extends to a continuous action of the $\breve{K}$-Fréchet algebra $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$, which then factors through a continuous action of $D(\Gamma, \breve{K})$ on R_0^{rig}. Hence the action of Γ on the strong continuous $\breve{K}$-linear dual $(R_0^{\mathrm{rig}})'_b$ of R_0^{rig} is locally K-analytic.*

Proof. First, we show that $R_{0,l}^{\mathrm{rig}}$ is a topological Banach module over the $\breve{K}$-Banach algebra $\Lambda(\Gamma_*)_{\breve{K},l}$ for all $l \geq 1$. To show this, let us prove by induction on $|\alpha|$ that $\|b^\alpha(f)\|_l \leq \|b^\alpha\|_l \|f\|_l$ for any $f \in R_0^{\mathrm{rig}}$. This is clear if $|\alpha| = 0$. Let $|\alpha| > 0$ and let i be the minimal index such that $\alpha_i > 0$. Define $\beta_j := \alpha_j$ if $j \neq i$, and $\beta_i := \alpha_i - 1$. Since $\Gamma_* \subseteq \Gamma_n$, Proposition 3.3.1 and the induction hypothesis imply

$$\begin{aligned} \|b^\alpha(f)\|_l &= \|((\gamma_i - 1)b^\beta)(f)\|_l = \|(\gamma_i - 1)(b^\beta(f))\|_l \leq |\varpi|^{n/l} \|b^\beta(f)\|_l \\ &\leq |\varpi|^{1/l} \|b^\beta\|_l \|f\|_l \leq |\varpi|^{1/l} |\varpi|^{|\beta|/l} \|f\|_l = |\varpi|^{(|\beta|+1)/l} \|f\|_l = \|b^\alpha\|_l \|f\|_l \end{aligned}$$

as required. By Remark 2.2.10, this immediately gives $\|\mu(f)\|_l \leq \|\mu\|_l \|f\|_l$ for all $\mu \in \Lambda(\Gamma_*)_{\breve{K}}$ and $f \in R_0[\frac{1}{\varpi}] = R_0 \otimes_{\breve{\mathfrak{o}}} \breve{K}$. Hence the map $\Lambda(\Gamma_*)_{\breve{K}} \times R_0[\frac{1}{\varpi}] \longrightarrow R_0[\frac{1}{\varpi}]$ $((\mu, f) \mapsto \mu(f))$ is continuous if $\Lambda(\Gamma_*)_{\breve{K}}$ and $R_0[\frac{1}{\varpi}]$ are endowed with the respective $\|\cdot\|_l$-topologies, and if the left hand side carries the product topology. Since $R_0[\frac{1}{\varpi}]$ is dense in $R_{0,l}^{\mathrm{rig}}$, we obtain a map $\Lambda(\Gamma_*)_{\breve{K},l} \times R_{0,l}^{\mathrm{rig}} \longrightarrow R_{0,l}^{\mathrm{rig}}$ by passing to completions. By continuity, it gives $R_{0,l}^{\mathrm{rig}}$ the structure of a topological Banach module over the $\breve{K}$-Banach algebra $\Lambda(\Gamma_*)_{\breve{K},l}$.

Taking projective limits over l, we obtain a continuous map $D(\Gamma_*, \breve{K}) \times R_0^{\mathrm{rig}} \longrightarrow R_0^{\mathrm{rig}}$ giving R_0^{rig} the structure of a continuous module over $D(\Gamma_*, \breve{K})$. Because of the topological isomorphism (3.3.3), R_0^{rig} becomes a continuous module over $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$. To see that the $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$-action on R_0^{rig} factors through a continuous action of $D(\Gamma, \breve{K})$, it suffices from (3.3.4) to check that $i(t\mathfrak{x})(f) = ti(\mathfrak{x})(f)$ for all $t \in K$, $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p} \subseteq D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ and $f \in R_0^{\mathrm{rig}}$. However by Theorem 3.2.7, this holds for all $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ because $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$, being a locally K-analytic Γ-representation, carries an action of the Lie algebra $\mathfrak{g}$. As the K-linear inclusion $R_0^{\mathrm{rig}} \hookrightarrow \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is continuous, it is $\mathfrak{g}_{\mathbb{Q}_p}$-equivariant. Hence the equality $i(t\mathfrak{x})(f) = ti(\mathfrak{x})(f)$ is true for all $f \in R_0^{\mathrm{rig}}$.

Now it follows from [**Sch02**], Proposition 19.9 and the arguments proving the claim on page 98, that the $\breve{K}$-Fréchet space R_0^{rig} is nuclear. Therefore, Theorem 1.4.3 implies that the locally convex $\breve{K}$-vector space $(R_0^{\mathrm{rig}})'_b$ is of compact type and that the action of Γ obtained by dualizing is locally K-analytic. □

The preceding theorem can be generalized as follows. Let Γ_* be a uniform pro-p group contained in Γ_{2n+1} for some positive integer n.

Theorem 3.3.6. *The action of $\Gamma_{\mathbb{Q}_p}$ on M_0^s extends to a continuous action of the $\breve{K}$-Fréchet algebra $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$, which then factors through a continuous action of $D(\Gamma, \breve{K})$ on M_0^s. Hence the action of Γ on the strong continuous $\breve{K}$-linear dual $(M_0^s)'_b$ of M_0^s is locally K-analytic for any $s \in \mathbb{Z}$.*

Proof. Choose a generator δ of the R_0-module $\mathrm{Lie}(\mathbb{H}^{(0)})^{\otimes s}$. Then by (2.3.7) and (2.3.8), $M_0^s = R_0^{\mathrm{rig}}\delta$ and $M_{0,l}^s = R_{0,l}^{\mathrm{rig}}\delta$. The topology on $M_{0,l}^s$ is defined by the norm $\|f\delta\|_l := \|f\|_l$. Let $\gamma(\delta) = f_0\delta$ then by Γ-equivariance, we have $\gamma(f\delta) = \gamma(f)\gamma(\delta) = \gamma(f)f_0\delta$ for all $f\delta \in M_0^s$. Hence

$$\gamma(f\delta) - f\delta = (\gamma(f)f_0 - f)\delta = (\gamma(f)f_0 - ff_0 + ff_0 - f)\delta = ((\gamma(f) - f)f_0 + f(f_0 - 1))\delta \tag{3.3.7}$$

Now if $\gamma \in \Gamma_* \subseteq \Gamma_{2n+1}$ and if $f\delta \in M_0^s$, then $\|\gamma(f) - f\|_l \le |\varpi|^{\frac{2n+1}{l}} \|f\|_l$ by Proposition 3.3.1 and $\gamma(\delta) - \delta = (f_0 - 1)\delta \in \mathfrak{m}_{R_0}^{n+1}\mathrm{Lie}(\mathbb{H}^{(0)})^{\otimes s}$ by Theorem 2.2.9 i.e. $f_0 - 1 \in \mathfrak{m}_{R_0}^{n+1}$. Since $\|y\|_l \le |\varpi|^{1/l}$ for any $y \in \mathfrak{m}_{R_0} = (\varpi, u_1, \cdots, u_{h-1})$, $\|f_0 - 1\|_l \le |\varpi|^{\frac{n+1}{l}}$ and $\|f_0\|_l \le \max\{\|f_0 - 1\|_l, 1\} = 1$. Thus by the multiplicativity of the norm $\|\cdot\|_l$ on R_0^{rig} and by (3.3.7), we have

$$\begin{aligned}\|\gamma(f\delta) - f\delta\|_l = \|(\gamma(f) - f)f_0 + f(f_0 - 1)\|_l &\le \max\{\|(\gamma(f) - f)\|_l\|f_0\|_l, \|f\|_l\|(f_0 - 1)\|_l\} \\ &\le \max\{|\varpi|^{\frac{2n+1}{l}}\|f\|_l, |\varpi|^{\frac{n+1}{l}}\|f\|_l\} \\ &= |\varpi|^{\frac{n+1}{l}}\|f\|_l = |\varpi|^{\frac{n+1}{l}}\|f\delta\|_l\end{aligned}$$

The rest of the proof now proceeds along the same lines as for Theorem 3.3.5. □

3.4. Local analyticity of the Γ-action on M_m^s with $m > 0$

As the title indicates, this section extends the theorems of the previous section to higher levels $m > 0$. The following observation together with the continuity of the Γ-action on R_m and on R_0^{rig} (cf. Theorem 2.2.8 and Proposition 3.3.1 respectively) allows us to show the continuity of the Γ-action on R_m^{rig} for $m > 0$.

Lemma 3.4.1. *For every $m \ge 0$, there exists a positive integer k_m such that $\mathfrak{m}_{R_m}^n \subseteq \mathfrak{m}_{R_0}R_m$ for all $n \ge k_m$.*

Proof. Since R_m is a finite free module over R_0, $R_m/\mathfrak{m}_{R_0}R_m$ is a finite dimensional vector space over $R_0/\mathfrak{m}_{R_0} = k^{\mathrm{sep}}$. Moreover, $R_m/\mathfrak{m}_{R_0}R_m$ is still a Noetherian local ring with the maximal ideal $\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0}R_m$. The powers $(\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0}R_m)^n$, $n \in \mathbb{N}$, of the ideal $\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0}R_m$ form a descending sequence of finite dimensional subspaces which eventually must become stationary. Let k_m be a positive integer such that $(\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0}R_m)^{n+1} = (\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0}R_m)^n$ for all $n \ge k_m$. Then by Nakayama's lemma, for all $n \ge k_m$, $(\mathfrak{m}_{R_m}/\mathfrak{m}_{R_0}R_m)^n = 0$, in other words, $\mathfrak{m}_{R_m}^n \subseteq \mathfrak{m}_{R_0}R_m$. □

Proposition 3.4.2. *Let m, n and l be integers with $m \ge 1$, $l \ge 1$ and $n \ge k_m - 1$ where k_m is as stated in Lemma 3.4.1. If $\gamma \in \Gamma_{n+m}$ and if $f \in R_m^{\mathrm{rig}}$, then $\|\gamma(f) - f\|_l \le |\varpi|^{1/l}\|f\|_l$.*

Proof. Write $f = f_1e_1 + \cdots + f_re_r$ where $\{e_1, \cdots, e_r\}$ is a basis of R_m over R_0 and $f_i \in R_0^{\mathrm{rig}}$ for all $1 \le i \le r$. Let $x_i := \gamma(e_i) - e_i$. Then $x_i \in \mathfrak{m}_{R_m}^{n+1}$ for all $1 \le i \le r$ by Theorem 2.2.8 and thus by Lemma 3.4.1, $x_i \in \mathfrak{m}_{R_0}R_m$ for all $1 \le i \le r$. Since $\|y\|_l \le |\varpi|^{1/l}$ for any $y \in \mathfrak{m}_{R_0} = (\varpi, u_1, \cdots, u_{h-1})$, $\|x_i\|_l \le |\varpi|^{1/l}$ for all $1 \le i \le r$. Now note that

$$\begin{aligned}\|\gamma(f) - f\|_l \le \max_{1\le i\le r}\{\|\gamma(f_ie_i) - f_ie_i\|_l\} &= \max_{1\le i\le r}\{\|\gamma(f_i)\gamma(e_i) - f_ie_i\|_l\} \\ &= \max_{1\le i\le r}\{\|\gamma(f_i)\gamma(e_i) - f_i\gamma(e_i) + f_i\gamma(e_i) - f_ie_i\|_l\} \\ &= \max_{1\le i\le r}\{\|(\gamma(f_i) - f_i)\gamma(e_i) + (\gamma(e_i) - e_i)f_i\|_l\} \\ &= \max_{1\le i\le r}\{\|(\gamma(f_i) - f_i)(e_i + x_i) + x_if_i\|_l\}\end{aligned}$$

Then Lemma 2.3.3 and Proposition 3.3.1 imply that for every $1 \le i \le r$,

$$\begin{aligned}\|(\gamma(f_i) - f_i)(e_i + x_i) + x_if_i\|_l &\le \max\{\|(\gamma(f_i) - f_i)(e_i + x_i)\|_l, \|x_if_i\|_l\} \\ &\le \max\{\|(\gamma(f_i) - f_i)\|_l, \|x_i\|_l\|f_i\|_l\} \\ &\le \max\{|\varpi|^{(n+m)/l}\|f_i\|_l, |\varpi|^{1/l}\|f_i\|_l\} = |\varpi|^{1/l}\|f_i\|_l\end{aligned}$$

where we use that $e_i + x_i = \gamma(e_i) \in R_m$ has $\|\cdot\|_l$-norm less than or equal to 1. Therefore, $\|\gamma(f) - f\|_l \le \max_{1\le i\le r}\{|\varpi|^{1/l}\|f_i\|_l\} = |\varpi|^{1/l}\|f\|_l$. □

We now fix a level $m \geq 1$. As before, we have a uniform pro-p group Γ_o of rank t as an open subgroup of $\Gamma_{\mathbb{Q}_p}$. We also fix a positive integer $n \geq k_m - 1$ such that $\Gamma_{n+m} \subseteq \Gamma_o$. Then Γ_{n+m} contains $\Gamma_* := P_i(\Gamma_o)$ for some $i \geq 1$ which is also a uniform pro-p group of rank t.

Let $\{\gamma_1, \dots, \gamma_t\}$ be an ordered basis of Γ_* and let $b_i := \gamma_i - 1 \in \Lambda(\Gamma_*)$. Then as before, we equip the $\breve{K}$-algebra $\Lambda(\Gamma_*)_{\breve{K}}$ with the sub-multiplicative norm $\|\cdot\|_l$ defined in (3.3.2) for every positive integer l. The natural inclusions $\Lambda(\Gamma_*)_{\breve{K},l+1} \hookrightarrow \Lambda(\Gamma_*)_{\breve{K},l}$ of $\breve{K}$-Banach completions endow the projective limit $D(\Gamma_*, \breve{K}) = \varprojlim_l \Lambda(\Gamma_*)_{\breve{K},l}$ with the structure of a $\breve{K}$-Fréchet algebra which is equal to the algebra of $\breve{K}$-valued locally K-analytic distributions on Γ_*.

Proposition 3.4.3. *For any integer $l \geq 1$, the action of Γ_* on R_m^{rig} extends to $R_{m,l}^{\mathrm{rig}}$ and makes $R_{m,l}^{\mathrm{rig}}$ a topological Banach module over the $\breve{K}$-Banach algebra $\Lambda(\Gamma_*)_{\breve{K},l}$. The action of $\Gamma_{\mathbb{Q}_p}$ on R_m^{rig} extends to a continuous action of the $\breve{K}$-Fréchet algebra $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$.*

Proof. The proof is similar to that of Theorem 3.3.5. First, we prove by induction on $|\alpha|$ that $\|b^\alpha(f)\|_l \leq \|b^\alpha\|_l \|f\|_l$ for any $f \in R_m^{\mathrm{rig}}$. This is clear if $|\alpha| = 0$. Let $|\alpha| > 0$ and let i be the minimal index such that $\alpha_i > 0$. Define $\beta_j := \alpha_j$ if $j \neq i$ and $\beta_i := \alpha_i - 1$. Then Proposition 3.4.2 and the induction hypothesis imply

$$\begin{aligned} \|b^\alpha(f)\|_l &= \|((\gamma_i - 1)b^\beta)(f)\|_l = \|(\gamma_i - 1)(b^\beta(f))\|_l \leq |\varpi|^{1/l}\|b^\beta(f)\|_l \\ &\leq |\varpi|^{1/l}\|b^\beta\|_l\|f\|_l \leq |\varpi|^{1/l}|\varpi|^{|\beta|/l}\|f\|_l = |\varpi|^{(|\beta|+1)/l}\|f\|_l = \|b^\alpha\|_l\|f\|_l \end{aligned}$$

as required. By Remark 2.2.10, this immediately gives $\|\mu(f)\|_l \leq \|\mu\|_l\|f\|_l$ for all $\mu \in \Lambda(\Gamma_*)_{\breve{K}}$ and $f \in R_m[\frac{1}{\varpi}] = R_m \otimes_{\breve{\mathfrak{o}}} \breve{K}$. Hence the map $\Lambda(\Gamma_*)_{\breve{K}} \times R_m[\frac{1}{\varpi}] \longrightarrow R_m[\frac{1}{\varpi}]$ $((\mu, f) \mapsto \mu(f))$ is continuous if $\Lambda(\Gamma_*)_{\breve{K}}$ and $R_m[\frac{1}{\varpi}]$ are endowed with the respective $\|\cdot\|_l$-topologies and if the left hand side carries the product topology. Since $R_m[\frac{1}{\varpi}]$ is dense in $R_{m,l}^{\mathrm{rig}}$, we obtain a map $\Lambda(\Gamma_*)_{\breve{K},l} \times R_{m,l}^{\mathrm{rig}} \longrightarrow R_{m,l}^{\mathrm{rig}}$ by passing to completions. By continuity, it gives $R_{m,l}^{\mathrm{rig}}$ the structure of a topological Banach module over the $\breve{K}$-Banach algebra $\Lambda(\Gamma_*)_{\breve{K},l}$.

Taking projective limits over l, we obtain a continuous map $D(\Gamma_*, \breve{K}) \times R_m^{\mathrm{rig}} \longrightarrow R_m^{\mathrm{rig}}$ giving R_m^{rig} the structure of a continuous module over $D(\Gamma_*, \breve{K})$. Since $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ is topologically isomorphic to the locally convex direct sum $\bigoplus_{\gamma\Gamma_* \in \Gamma_{\mathbb{Q}_p}/\Gamma_*} \delta_\gamma D(\Gamma_*, \breve{K})$, R_m^{rig} is a continuous module over $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$. □

We want to show that the $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$-action on R_m^{rig} factors through a continuous $D(\Gamma, \breve{K})$-action. As mentioned in the Step 3, the idea is to use the local K-analyticity of the Γ-action on R_0^{rig} obtained in Theorem 3.3.5 and the property of étaleness of the extension $R_m^{\mathrm{rig}}|R_0^{\mathrm{rig}}$.

For a ring homomorphism $A \to B$ of commutative unital rings, let $\mathrm{Der}_A(B, B)$ denote the B-module of A-linear derivations from B to B, and let $\Omega_{B/A}$ denote the B-module of differentials of B over A. Recall that B is said to be *formally smooth* over A if for any A-algebra C, for any ideal $I \subset C$ satisfying $I^2 = 0$ and for any A-algebra homomorphism $u : B \to C/I$, there exists a lifting $v : B \to C$ of u making the following diagram commutative.

$$\begin{array}{ccc} B & \xrightarrow{u} & C/I \\ \uparrow & \overset{v}{\searrow} & \uparrow \\ A & \longrightarrow & C \end{array}$$

It is said to be *formally unramified* if there exists atmost one such lifting, and *formally étale* if it is both formally smooth and formally unramified, i.e. if there exists a unique such lifting (cf. [**Sta**, Tag 00UQ]).

Lemma 3.4.4. *Let $A \to B \to C$ be ring homomorphisms such that $B \to C$ is formally étale. Then $\Omega_{B/A} \otimes_B C \simeq \Omega_{C/A}$.*

Proof. Since $B \to C$ is formally smooth, we have $\Omega_{C/A} \simeq \Omega_{C/B} \oplus (\Omega_{B/A} \otimes_B C)$ by [**Sta**, Tag 031K]. However, $\Omega_{C/B} = 0$ by [**Sta**, Tag 00UO] because $B \to C$ is formally unramified. □

Since $R_m[\frac{1}{\varpi}]$ is étale over $R_0[\frac{1}{\varpi}]$ (cf. Remark 2.2.2), $R_m^{\text{rig}} \simeq R_m[\frac{1}{\varpi}] \otimes_{R_0[\frac{1}{\varpi}]} R_0^{\text{rig}}$ is étale over R_0^{rig} by [**Sta**, Tag 00U0], and so is formally étale by [**Sta**, Tag 00UR]. Then by applying Lemma 3.4.4 to the ring homomorphisms $\breve{K} \hookrightarrow R_0^{\text{rig}} \hookrightarrow R_m^{\text{rig}}$, we get $\Omega_{R_m^{\text{rig}}/\breve{K}} \simeq \Omega_{R_0^{\text{rig}}/\breve{K}} \otimes_{R_0^{\text{rig}}} R_m^{\text{rig}}$. Therefore

$$\begin{aligned}\operatorname{Der}_{\breve{K}}(R_0^{\text{rig}}, R_0^{\text{rig}}) &\simeq \operatorname{Hom}_{R_0^{\text{rig}}}(\Omega_{R_0^{\text{rig}}/\breve{K}}, R_0^{\text{rig}}) \\ &\hookrightarrow \operatorname{Hom}_{R_0^{\text{rig}}}(\Omega_{R_0^{\text{rig}}/\breve{K}}, R_m^{\text{rig}}) \\ &\simeq \operatorname{Hom}_{R_m^{\text{rig}}}(\Omega_{R_0^{\text{rig}}/\breve{K}} \otimes_{R_0^{\text{rig}}} R_m^{\text{rig}}, R_m^{\text{rig}}) \\ &\simeq \operatorname{Hom}_{R_m^{\text{rig}}}(\Omega_{R_m^{\text{rig}}/\breve{K}}, R_m^{\text{rig}}) \simeq \operatorname{Der}_{\breve{K}}(R_m^{\text{rig}}, R_m^{\text{rig}}).\end{aligned}$$

In other words,

Lemma 3.4.5. *Any $\breve{K}$-linear derivation from R_0^{rig} to R_0^{rig} extends uniquely to a $\breve{K}$-linear derivation from R_m^{rig} to R_m^{rig}.* □

Theorem 3.4.6. *The action of $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ on R_m^{rig} factors through a continuous action of $D(\Gamma, \breve{K})$ on R_m^{rig}. Hence the action of Γ on the strong continuous $\breve{K}$-linear dual $(R_m^{\text{rig}})'_b$ of R_m^{rig} is locally K-analytic.*

Proof. Recall the inclusion map $i : \mathfrak{g}_{\mathbb{Q}_p} \hookrightarrow D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ from (3.3.4). Because of Theorem 3.4.3, for every $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$, $i(\mathfrak{x}) \in D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ acts on R_m^{rig} by $\breve{K}$-linear vector space endomorphism and for any $f, g \in R_m^{\text{rig}}$, we have

$$\begin{aligned} i(\mathfrak{x})(fg) = \mathfrak{x}(fg) &= \frac{d}{dt}\exp(t\mathfrak{x})(fg)\big|_{t=0} \\ &= \lim_{t\to 0}\frac{\exp(t\mathfrak{x})(fg) - fg}{t} \\ &= \lim_{t\to 0}\frac{\exp(t\mathfrak{x})(f)\exp(t\mathfrak{x})(g) - fg}{t} \\ &= \lim_{t\to 0}\frac{\exp(t\mathfrak{x})(f)\exp(t\mathfrak{x})(g) - g\exp(t\mathfrak{x})(f) + g\exp(t\mathfrak{x})(f) - fg}{t} \\ &= \lim_{t\to 0}\exp(t\mathfrak{x})(f)\Big(\frac{\exp(t\mathfrak{x})(g) - g}{t}\Big) + \lim_{t\to 0} g\Big(\frac{\exp(t\mathfrak{x})(f) - f}{t}\Big) \\ &= f\mathfrak{x}(g) + g\mathfrak{x}(f) = fi(\mathfrak{x})(g) + gi(\mathfrak{x})(f)\end{aligned}$$

Therefore we have a natural map $\partial : i(\mathfrak{g}_{\mathbb{Q}_p}) \longrightarrow \operatorname{Der}_{\breve{K}}(R_m^{\text{rig}}, R_m^{\text{rig}})$. For $t \in K$ and $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$, consider the derivation $\partial(i(t\mathfrak{x}) - ti(\mathfrak{x})) \in \operatorname{Der}_{\breve{K}}(R_m^{\text{rig}}, R_m^{\text{rig}})$. It is zero on R_0^{rig} by Theorem 3.3.5 and thus by Lemma 3.4.5, it is zero on R_m^{rig} i.e. we have

$$0 = \partial(i(t\mathfrak{x}) - ti(\mathfrak{x}))(f) = (i(t\mathfrak{x}) - ti(\mathfrak{x}))(f)$$

for all $t \in K$, $\mathfrak{x} \in \mathfrak{g}_{\mathbb{Q}_p}$ and $f \in R_m^{\text{rig}}$. This means that the action of $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ on R_m^{rig} factors through a continuous action of $D(\Gamma, \breve{K})$ on R_m^{rig} by (3.3.4).

As the $\breve{K}$-Fréchet space R_m^{rig} is topologically isomorphic to $\bigoplus_{i=1}^r R_0^{\mathrm{rig}}$, it follows from [**Sch02**], Proposition 19.7, that R_m^{rig} is nuclear. Therefore, Theorem 1.4.3 implies that the locally convex $\breve{K}$-vector space $(R_m^{\mathrm{rig}})'_b$ is of compact type and that the action of Γ obtained by dualizing is locally K-analytic. □

Similar to the before, Theorem 3.4.6 can be generalized as follows. Fix $m \geq 1$, $n \geq k_m - 1$ and a uniform pro-p group $\Gamma_* \subseteq \Gamma_{2n+m+1}$ with k_m as in Lemma 3.4.1.

Theorem 3.4.7. *The action of $\Gamma_{\mathbb{Q}_p}$ on M_m^s extends to a continuous action of the $\breve{K}$-Fréchet algebra $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$, which then factors through a continuous action of $D(\Gamma, \breve{K})$ on M_m^s. Hence the action of Γ on the strong continuous $\breve{K}$-linear dual $(M_m^s)'_b$ of M_m^s is locally K-analytic for any $s \in \mathbb{Z}$.*

Proof. Using Theorem 2.2.9, Lemma 3.4.1 and Proposition 3.4.2, the proof of the first part of the assertion is similar to that of Theorem 3.3.6.

Observe that the isomorphism $\mathrm{Lie}(\mathbb{H}^{(m)}) \simeq \mathrm{Lie}(\mathbb{H}^{(0)}) \otimes_{R_0} R_m$ is Γ -equivariant for the diagonal Γ-action on the right. Therefore, we have the following Γ-equivariant isomorphisms by (2.3.7)

$$\begin{aligned} M_m^s &\simeq R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{H}^{(m)}) \\ &\simeq R_m^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{H}^{(0)}) \\ &\simeq R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} R_0^{\mathrm{rig}} \otimes_{R_0} \mathrm{Lie}(\mathbb{H}^{(0)}) \\ &\simeq R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} M_0^s \end{aligned}$$

with Γ-acting diagonally on all the tensor products. As a consequence, the $\mathfrak{g}_{\mathbb{Q}_p}$-action on $f \otimes \delta \in M_m^s$ is given by $\mathfrak{x}(f \otimes \delta) = \mathfrak{x}(f) \otimes \delta + f \otimes \mathfrak{x}(\delta)$. However, by Theorem 3.3.6 and Theorem 3.4.6, M_0^s and R_m^{rig} are not only $\mathfrak{g}_{\mathbb{Q}_p}$-modules but also $\mathfrak{g}$-modules. Thus, it follows from (3.3.4) that the $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$-action on M_m^s factors through the continuous $D(\Gamma, \breve{K})$-action as required. □

Remark 3.4.8. By Theorem 3.3.6 and Theorem 3.4.7, we have a continuous action of $D(\Gamma_{\mathbb{Q}_p}, \breve{K})$ on $M_{m,l}^s$ via the natural map $D(\Gamma_*, \breve{K}) \longrightarrow \Lambda(\Gamma_*)_{\breve{K},l}$ for all $s \in \mathbb{Z}$, $m \geq 0$, $l \geq 1$. This action factors through a continuous action of $D(\Gamma, \breve{K})$ by the similar arguments as in the proofs of Theorem 3.3.6 and Theorem 3.4.7. Therefore, the $\breve{K}$-Banach space $M_{m,l}^s$ is a locally K-analytic Γ-representation for all $s \in \mathbb{Z}$, $m \geq 0$, $l \geq 1$ by [**ST02**], discussion before Corollary 3.3.

CHAPTER 4

Locally finite vectors in the global sections of equivariant vector bundles

In this chapter, we study representation-theoretic aspects of our Γ-representations M_m^s. Our results include a complete description of the Γ-locally finite vectors in M_m^s for all $s \in \mathbb{Z}$ and $m \geq 0$.

4.1. Locally finite vectors in the Γ-representations M_0^s

Definition 4.1.1. Let G be a topological group and V be a vector space over a field F equipped with an F-linear G-action. We say that a vector $v \in V$ is *locally finite* (or *G-locally finite*) if there is an open subgroup H of G and a finite dimensional H-stable subspace W of V containing v. It follows easily that the set V_{lf} of all locally finite vectors of V forms a G-stable subspace. V is called a locally finite representation of G if $V_{\mathrm{lf}} = V$.

In [**Eme04**], Proposition-Definition 4.1.8, the notion of a locally finite vector is defined for the vector spaces over a complete non-archimedean field F, and requires locally finite vector v to be contained in a *continuous* finite dimensional H-representation W for its natural Hausdorff topology as a finite dimensional F-vector space. Since all the Γ-representations we are concerned with in this chapter are continuous representations on $\breve{K}$-Fréchet spaces, the continuity condition is automatically satisfied. Thus, the above definition without the continuity condition coincides with the one in [**Eme04**] for our representations.

Remark 4.1.2. If V and W are F-linear G-representations, and if $f : V \longrightarrow W$ is an F-linear G-equivariant map, then clearly $f(V_{\mathrm{lf}}) \subseteq W_{\mathrm{lf}}$.

To calculate locally finite vectors, we make extensive use of the Lie algebra action. Let $\mathfrak{g}$ be the Lie algebra of the Lie group Γ over K and $U(\mathfrak{g})$ be its universal enveloping algebra over K. Note that $\mathfrak{g}$ is isomorphic to the Lie algebra associated with the associative K-algebra B_h. Thus, $\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes_K K_h \simeq \mathfrak{gl}_h(K_h)$. We denote by $\mathfrak{x}_{ij} \in \mathfrak{gl}_h(K_h)$ the matrix with entry 1 at the place (i,j) and zero everywhere else. By Theorem 3.2.7, the Γ-representation $M_D^s \simeq \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s$ carries a continuous linear action of $U(\mathfrak{g}) \otimes_K \breve{K} \simeq U(\mathfrak{gl}_h(K_h)) \otimes_{K_h} \breve{K} \hookrightarrow D(\Gamma, \breve{K})$. Since $GL_h(K_h)$ acts on the projective coordinates $\varphi_0, \ldots, \varphi_{h-1}$ by fractional linear transformations, one can explicitly determine the Lie algebra action directly using the definition (1.4.2). Here we use the description of $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ in terms of power series in w as in (3.1.12).

Lemma 4.1.3. *Let i, j and s be integers with $0 \leq i, j \leq h-1$. Put $w_0 := 1$. If $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ then*

$$\mathfrak{x}_{ij}(f\varphi_0^s) = \begin{cases} w_i \frac{\partial f}{\partial w_j} \varphi_0^s, & \text{if } j \neq 0; \\ (sf - \sum_{l=1}^{h-1} w_l \frac{\partial f}{\partial w_l})\varphi_0^s, & \text{if } i = j = 0; \\ w_i(sf - \sum_{l=1}^{h-1} w_l \frac{\partial f}{\partial w_l})\varphi_0^s, & \text{if } i > j = 0. \end{cases} \tag{4.1.4}$$

Proof. This is exactly same as [**Koh14**], Lemma 4.1, which treats the case $K = \mathbb{Q}_p$. ☐

Given a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{gl}_h(K_h)$, and a $\breve{K}$-linear $\mathfrak{gl}_h(K_h)$-representation W, the $\breve{K}$-subspace of $\mathfrak{h}$-*invariants of* W is the subspace $\{w \in W | \mathfrak{x}(w) = 0 \text{ for all } \mathfrak{x} \in \mathfrak{h}\}$ of W. Let us denote by $\mathfrak{n}$ the Lie subalgebra of $\mathfrak{gl}_h(K_h)$ consisting of strictly upper triangular matrices. For later use, we calculate the $\mathfrak{g}$-invariants and the $\mathfrak{n}$-invariants of $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ in the next lemma using the formulae (4.1.4).

Lemma 4.1.5. $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{g}=0} = \mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{n}=0} = \breve{K}$.

PROOF. Since $\breve{K} \subseteq \mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{g}=0} \subseteq \mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{n}=0}$, it suffices to show that the latter is $\breve{K}$. Now if $f \in \mathcal{O}_{X_0^{\mathrm{rig}}}(D)^{\mathfrak{n}=0}$ then applying the formulae (4.1.4), we get $\mathfrak{x}_{0j}(f) = \frac{\partial f}{\partial w_j} = 0$ for all $1 \leq j \leq h-1$. Therefore, f must be a constant power series. □

We now compute the space $(M_D^s)_{\mathrm{lf}}$ of locally finite vectors in the Γ-representation M_D^s. The key step is the following lemma based on Lemma 4.1.3:

Lemma 4.1.6. *The subspace* $\breve{K}[w_1, \ldots, w_{h-1}]\varphi_0^s$ *of* M_D^s *is contained in* $(U(\mathfrak{g}) \otimes_K \breve{K})(f\varphi_0^s)$ *for any non-zero homogeneous polynomial* $f \in \breve{K}[w_1, \ldots, w_{h-1}]$ *of total degree* $d > s$.

PROOF. Using (4.1.4), we have for all $0 < i, j \leq h-1$

$$\mathfrak{x}_{0j}(f\varphi_0^s) = \frac{\partial f}{\partial w_j}\varphi_0^s$$
$$\mathfrak{x}_{i0}(f\varphi_0^s) = (s-d)w_i f\varphi_0^s.$$

To obtain $g\varphi_0^s$ with a monomial g of total degree $\leq s$, first reduce $f\varphi_0^s$ to φ_0^s by applying suitable $\mathfrak{x}_{0j}$ $(j \neq 0)$ to it iteratively and then apply appropriate $\mathfrak{x}_{i0}$ $(i > 0)$ to φ_0^s to get the desired element $g\varphi_0^s$. To obtain $g\varphi_0^s$ with a monomial g of total degree $> s$, reverse the procedure i.e. first apply appropriate $\mathfrak{x}_{i0}$ $(i > 0)$ to $f\varphi_0^s$ and then reduce the result to $g\varphi_0^s$ by applying suitable $\mathfrak{x}_{0j}$ $(j \neq 0)$ to it. □

Recall that, if V is a topological vector space over a field F with an F-linear action of a group G, then V is called *topologically irreducible* if it is non-zero and if it does not have a closed, non-trivial ($\neq \{0\}, V$) G-stable subspace. For any $s \in \mathbb{Z}$, we define the $\breve{K}$-subspace of M_D^s by

$$V_s := \sum_{|\alpha| \leq s} \breve{K} w^\alpha \varphi_0^s.$$

Note that we have $V_s = 0$ if $s < 0$.

Theorem 4.1.7. *The* Γ*-representation* M_D^s *is topologically irreducible if* $s < 0$*, and if* $s \geq 0$ *then* V_s *is a topologically irreducible sub-representation of* M_D^s *with topologically irreducible quotient* M_D^s/V_s.

PROOF. Case $s < 0$: Let V be a non-zero closed Γ-stable subspace of M_D^s. Let $f_0\varphi_0^s \in V$ where $f_0 = \sum_{\alpha \in \mathbb{N}_0^{h-1}} c_\alpha w^\alpha \neq 0$ and d (≥ 0) be the smallest natural number such that $c_\alpha \neq 0$ for some $\alpha \in \mathbb{N}_0^{h-1}$ with $|\alpha| = d$. Thus,

$$f_0\varphi_0^s = \left(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty} \sum_{|\alpha|=d+i} c_\alpha w^\alpha \right) \varphi_0^s \in V$$

where $\sum_{|\alpha|=d} c_\alpha w^\alpha \neq 0$. For $n > 0$, define a sequence of elements of M_D^s inductively as follows:

$$f_n\varphi_0^s := \frac{1}{n}\Big((d+n-s)f_{n-1}\varphi_0^s + \mathfrak{x}_{00}(f_{n-1}\varphi_0^s)\Big).$$

Since V is closed and Γ-stable, V is stable under the action of the Lie algebra (1.4.2) and thus $f_n \in V$ for all $n \in \mathbb{N}_0$.

We prove by induction on n that

$$f_n\varphi_0^s = \left(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i} (-1)^n \binom{i-1}{n} c_\alpha w^\alpha\right)\varphi_0^s \tag{4.1.8}$$

Here the generalized binomial coefficients are defined by $\binom{x}{n} := \frac{x(x-1)\ldots(x-n+1)}{n!}$ for any $x \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The case $n = 0$ is true by definition. Assuming that (4.1.8) holds for $n-1$, we compute using Lemma 4.1.3 that

$$\begin{aligned}
f_n\varphi_0^s &= \frac{1}{n}\Big((d+n-s)f_{n-1}\varphi_0^s + \mathfrak{x}_{00}(f_{n-1}\varphi_0^s)\Big)\\
&= \frac{1}{n}\Bigg((d+n-s)\Big(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i} (-1)^{n-1}\binom{i-1}{n-1} c_\alpha w^\alpha\Big)\varphi_0^s\\
&\qquad + \mathfrak{x}_{00}\Bigg(\Big(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i} (-1)^{n-1}\binom{i-1}{n-1} c_\alpha w^\alpha\Big)\varphi_0^s\Bigg)\Bigg)\\
&= \frac{1}{n}\Bigg((d+n-s)\Big(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i} (-1)^{n-1}\binom{i-1}{n-1} c_\alpha w^\alpha\Big)\varphi_0^s\\
&\qquad + \left(\sum_{|\alpha|=d} (s-d) c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i} (s-(d+i))(-1)^{n-1}\binom{i-1}{n-1} c_\alpha w^\alpha\right)\varphi_0^s\Bigg)\\
&= \left(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i} (n-i)(-1)^{n-1}\frac{1}{n}\binom{i-1}{n-1} c_\alpha w^\alpha\right)\varphi_0^s\\
&= \left(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i} (-1)^{n}\binom{i-1}{n} c_\alpha w^\alpha\right)\varphi_0^s
\end{aligned}$$

We now claim that, as $n \to \infty$, the sequence $f_n\varphi_0^s$ converges to $\left(\sum_{|\alpha|=d} c_\alpha w^\alpha\right)\varphi_0^s$ with respect to the norm $\|\cdot\|_{M_D^s}$ defined in the proof of Theorem 3.2.7: As $f_0 \in \mathcal{O}(D)$, we know that given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}_0^{h-1}$ with $|\alpha| > N_\varepsilon$, we have $|c_\alpha||\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})} < \varepsilon$.

Therefore, $\sup_{|\alpha|>N_\varepsilon} |c_\alpha||\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})} < \varepsilon$. Now for all $n > N_\varepsilon - d$, using (4.1.8), we have

$$\begin{aligned}
\Big\| f_n\varphi_0^s - \Big(\sum_{|\alpha|=d} c_\alpha w^\alpha\Big)\varphi_0^s\Big\|_{M_D^s} &= \Big\|\Big(\sum_{i=1}^{\infty}\sum_{|\alpha|=d+i}(-1)^n\binom{i-1}{n}c_\alpha w^\alpha\Big)\varphi_0^s\Big\|_{M_D^s} \\
&= \Big\|\sum_{i=1}^{\infty}\sum_{|\alpha|=d+i}(-1)^n\binom{i-1}{n}c_\alpha w^\alpha\Big\|_D \\
&= \Big\|\sum_{i=n+1}^{\infty}\sum_{|\alpha|=d+i}(-1)^n\binom{i-1}{n}c_\alpha w^\alpha\Big\|_D \\
&\leq \Big\|\sum_{|\alpha|>d+n} c_\alpha w^\alpha\Big\|_D \\
&= \sup_{|\alpha|>d+n}|c_\alpha||\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})} \\
&\leq \sup_{|\alpha|>N_\varepsilon}|c_\alpha||\varpi|^{\sum_{i=1}^{h-1}\alpha_i(1-\frac{i}{h})} < \varepsilon
\end{aligned}$$

Hence, $f_n\varphi_0^s$ converges to $\Big(\sum_{|\alpha|=d} c_\alpha w^\alpha\Big)\varphi_0^s$ as $n\to\infty$ and $\Big(\sum_{|\alpha|=d} c_\alpha w^\alpha\Big)\varphi_0^s \in V$ because V is closed. Since $\sum_{|\alpha|=d} c_\alpha w^\alpha$ is a non-zero homogeneous polynomial of total degree $d \geq 0 > s$, it follows from Lemma 4.1.6 that $\breve{K}[w_1,\dots,w_{h-1}]\varphi_0^s \subseteq V$. Since $\breve{K}[w_1,\dots,w_{h-1}]\varphi_0^s$ is dense in $M_D^s = \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s$ and V is closed, it follows that $V = M_D^s$. Hence, M_D^s is topologically irreducible for all $s < 0$.

Case $s \geq 0$: First we show that the finite dimensional subspace V_s of M_D^s is stable under the action of Γ: It is sufficient to prove that $\gamma(w^\alpha\varphi_0^s) \in V_s$ for any $w^\alpha\varphi_0^s$ with $|\alpha| \leq s$ and for any $\gamma = \sum_{i=0}^{h-1}\lambda_i\Pi^i \in \Gamma$. In this case, using the action of the matrix (3.1.10) on the projective coordinates $[\varphi_0 : \ldots : \varphi_{h-1}]$, we find

$$\begin{aligned}
\gamma(w^\alpha\varphi_0^s) &= \gamma(w_1^{\alpha_1}\dots w_{h-1}^{\alpha_{h-1}}\varphi_0^s) = \gamma(\varphi_1^{\alpha_1}\dots\varphi_{h-1}^{\alpha_{h-1}}\varphi_0^{s-|\alpha|}) = \gamma(\varphi_1)^{\alpha_1}\dots\gamma(\varphi_{h-1})^{\alpha_{h-1}}\gamma(\varphi_0)^{s-|\alpha|} \\
&= (\varpi\lambda_1\varphi_0+\dots+\varpi\lambda_2^{\sigma^{h-1}}\varphi_{h-1})^{\alpha_1}\dots(\varpi\lambda_{h-1}\varphi_0+\dots+\lambda_0^{\sigma^{h-1}}\varphi_{h-1})^{\alpha_{h-1}}(\lambda_0\varphi_0+\dots+\lambda_1^{\sigma^{h-1}}\varphi_{h-1})^{s-|\alpha|} \\
&= (\varpi\lambda_1+\dots+\varpi\lambda_2^{\sigma^{h-1}}w_{h-1})^{\alpha_1}\dots(\varpi\lambda_{h-1}+\dots+\lambda_0^{\sigma^{h-1}}w_{h-1})^{\alpha_{h-1}}(\lambda_0+\dots+\lambda_1^{\sigma^{h-1}}w_{h-1})^{s-|\alpha|}\varphi_0^s \\
&\in V_s.
\end{aligned}$$

Let V be a non-zero closed Γ-stable subspace of V_s. Then V is stable under the action of the Lie algebra and thus it becomes a module over $U(\mathfrak{g})\otimes_K\breve{K}$. As mentioned in the proof of Lemma 4.1.6, any non-zero element $f\varphi_0^s$ of V can be reduced to φ_0^s by applying suitable $\mathfrak{x}_{0j}$ $(j\neq 0)$ to it iteratively and then φ_0^s can be converted into any monomial of total degree $\leq s$ multiplied with φ_0^s by applying appropriate $\mathfrak{x}_{i0}$ $(i>0)$ to it. Therefore $V = V_s$ and V_s is topologically irreducible.

Now let $\phi : M_D^s \longrightarrow M_D^s/V_s$ be the canonical surjective map and let $W \subset M_D^s/V_s$ be a non-zero, closed Γ-stable subspace. Then $\phi^{-1}(W)$ is a non-zero, closed Γ-stable subspace of M_D^s not equal to V_s.

Let $\Big(\sum_{\alpha\in\mathbb{N}_0^{h-1}} c_\alpha w^\alpha\Big)\varphi_0^s + V_s$ be a non-zero element of W. Then $f_0\varphi_0^s := \Big(\sum_{|\alpha|>s} c_\alpha w^\alpha\Big)\varphi_0^s \neq 0 \in \phi^{-1}(W)$. Let $d > s$ be the smallest natural number such that $c_\alpha \neq 0$ for some $\alpha\in\mathbb{N}_0^{h-1}$ with

$|\alpha| = d$. Thus,

$$f_0\varphi_0^s = \left(\sum_{|\alpha|=d} c_\alpha w^\alpha + \sum_{i=1}^{\infty}\sum_{|\alpha|=d+i} c_\alpha w^\alpha\right)\varphi_0^s \in \phi^{-1}(W)$$

where $\sum_{|\alpha|=d} c_\alpha w^\alpha \neq 0$.

As in the case of $s < 0$, we define a sequence of elements in $\phi^{-1}(W)$ inductively for $n > 0$ as follows:

$$f_n\varphi_0^s := \frac{1}{n}\Big((d+n-s)f_{n-1}\varphi_0^s + \mathfrak{x}_{00}(f_{n-1}\varphi_0^s)\Big).$$

Using exactly the same proof in the previous case of $s < 0$, it can be shown that $f_n\varphi_0^s$ converges to $\Big(\sum_{|\alpha|=d} c_\alpha w^\alpha\Big)\varphi_0^s$ as $n \to \infty$ and $\Big(\sum_{|\alpha|=d} c_\alpha w^\alpha\Big)\varphi_0^s \in \phi^{-1}(W)$ because $\phi^{-1}(W)$ is closed. Since $\sum_{|\alpha|=d} c_\alpha w^\alpha$ is a non-zero homogeneous polynomial of total degree $d > s$, it follows from Lemma 4.1.6 that $\breve{K}[w_1,\ldots,w_{h-1}].\varphi_0^s \subseteq \phi^{-1}(W)$. Since $\breve{K}[w_1,\ldots,w_{h-1}]\varphi_0^s$ is dense in $M_D^s = \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s$ and $\phi^{-1}(W)$ is closed, it follows that $\phi^{-1}(W) = M_D^s$. Hence $W = \phi(M_D^s) = M_D^s/V_s$. Thus M_D^s/V_s is topologically irreducible. □

Corollary 4.1.9. *For all $s \in \mathbb{Z}$, we have*

$$(M_D^s)_{\mathrm{lf}} = V_s.$$

Thus, $(M_D^s)_{\mathrm{lf}}$ is zero if $s < 0$ and is a finite dimensional irreducible Γ-representation if $s \geq 0$.

Proof. Let $s < 0$ and v be a non-zero locally finite vector in the $\breve{K}$-vector space M_D^s. Thus v is contained in a finite dimensional H-subrepresentation W of M_D^s for some open subgroup H of Γ. Being finite dimensional, W is complete and hence closed H-stable subspace. Therefore W is stable under the action of $\mathfrak{g} = \mathrm{Lie}(H)$. Then it follows from the proof of Theorem 4.1.7 (case $s < 0$) that $W = M_D^s$ which is a contradiction since M_D^s is not finite dimensional.

Let $s \geq 0$. It is clear that $V_s \subseteq (M_D^s)_{\mathrm{lf}}$. Suppose there exists a locally finite vector v not contained in V_s. Then v is contained in a finite dimensional H-subrepresentation W of M_D^s for some open subgroup H of Γ. Now $W/(W \cap V_s)$ is a non-zero finite dimensional H-stable subspace of M_D^s/V_s and thus it is closed in M_D^s/V_s. Then it follows from the proof of Theorem 4.1.7 (case $s \geq 0$) that $W/(W \cap V_s) = M_D^s/V_s$ which is again a contradiction due to dimensionality argument. □

A few remarks are in order.

Remark 4.1.10. In the context of Theorem 4.1.7, one might say that the Γ-representation M_D^s is *topologically* of length 1 or 2. However; in this very general situation, the notion *topological length* is problematic since the image of a closed subspace of M_D^s in M_D^s/V_s under the quotient map is generally not closed for the quotient topology on M_D^s/V_s. For this, one would have to work in the category of admissible Banach space representations.

Remark 4.1.11. For $s \geq 0$, the finite-dimensional Γ-representation V_s is also a $\mathfrak{gl}(K_h)$-module. Let $\mathfrak{t} \subset \mathfrak{sl}_h(K_h)$ be the Cartan subalgebra of $\mathfrak{sl}_h(K_h)$ consisting of diagonal matrices, and let $\{\varepsilon_1,\ldots,\varepsilon_{h-1}\}$ be the basis of the root system $(\mathfrak{sl}_h(K_h),\mathfrak{t})$ given by $\varepsilon_i(\mathrm{diag}(t_0,\ldots,t_{h-1})) := t_{i-1}-t_i$. Define the fundamental dominant weight $\chi_0 := \frac{1}{h}\sum_{i=1}^{h-1}(h-i)\varepsilon_i \in \mathfrak{t}^*$. Then, by the same proof as in [**Koh14**], Proposition 4.3, it follows that V_s is an irreducible $\mathfrak{sl}_h(K_h)$-representation of highest weight $s\chi_0$. Although this is stronger than saying that V_s is an irreducible Γ-representation, our result (Theorem 4.1.7) also gives information about the Γ-representation M_D^s when $s < 0$.

The Corollary 4.1.9 leads us to calculate locally finite vectors in the global sections M_0^s over X_0^{rig}. Recall from [**BGR84**], (9.3.4), Example 3, that the rigid analytic projective space $\mathbb{P}_{\breve{K}}^{h-1}$ has a finite admissible covering by the $(h-1)$-dimensional closed unit polydiscs $V_i := \mathrm{Sp}(\breve{K}\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}\rangle)$, $0 \leq i \leq h-1$. If $V_{ij} := \mathrm{Sp}(\breve{K}\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}, (\frac{\varphi_j}{\varphi_i})^{-1}\rangle)$ for $0 \leq i, j \leq h-1$, then gluing the V_i's along the identification $V_{ij} \simeq V_{ji}$ of affinoid subdomains via

$$\breve{K}\Big\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}, \Big(\frac{\varphi_j}{\varphi_i}\Big)^{-1}\Big\rangle = \breve{K}\Big\langle \frac{\varphi_0}{\varphi_j}, \ldots, \frac{\varphi_{h-1}}{\varphi_j}, \Big(\frac{\varphi_i}{\varphi_j}\Big)^{-1}\Big\rangle$$

gives the rigid analytic projective space $\mathbb{P}_{\breve{K}}^{h-1}$. The affinoid covering $\{V_i\}_{0 \leq i \leq h-1}$ allows us to describe the construction of the line bundles $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ on the rigid analytic projective space in a way analogous to the classical construction. For $s \geq 0$, define its sections over the affinoid space V_i

$$\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(V_i) := \breve{K}\Big\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}\Big\rangle \varphi_i^s$$

to be a free module of rank 1 generated by φ_i^s over $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(V_i) = \breve{K}\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}\rangle$, and the transition functions $\psi_{ij} : V_{ij} \xrightarrow{\sim} V_{ji}$ induced by the homomorphisms of affinoid $\breve{K}$-algebras

$$\psi_{ij}^* : \breve{K}\Big\langle \frac{\varphi_0}{\varphi_j}, \ldots, \frac{\varphi_{h-1}}{\varphi_j}, \Big(\frac{\varphi_i}{\varphi_j}\Big)^{-1}\Big\rangle \varphi_j^s \xrightarrow{\text{multiply by } \frac{\varphi_i^s}{\varphi_j^s}} \breve{K}\Big\langle \frac{\varphi_0}{\varphi_i}, \ldots, \frac{\varphi_{h-1}}{\varphi_i}, \Big(\frac{\varphi_j}{\varphi_i}\Big)^{-1}\Big\rangle \varphi_i^s$$

for all $0 \leq i, j \leq h-1$. The above datum gives rise to a locally free $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}$-module $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ of rank 1. For $s < 0$, $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ turns out to be the $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}$-linear dual of $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(-s)$. It then follows easily from the above description that the global sections of $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ are the $\breve{K}$-vector space of homogeneous polynomials of degree s in the variables φ_i's if $s \geq 0$, and are 0 otherwise. The line bundles $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ carry a canonical action of Γ induced by its action on the projective space $\mathbb{P}_{\breve{K}}^{h-1}$.

Now for any $\mathcal{O}_{X_0^{\mathrm{rig}}}$-module $\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}$−module $\mathcal{G}$, there is a bijection between the sets $\mathrm{Hom}_{\mathcal{O}_{X_0^{\mathrm{rig}}}-\mathrm{mod}}(\Phi^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}-\mathrm{mod}}(\mathcal{G}, \Phi_*\mathcal{F})$, where $\Phi : X_0^{\mathrm{rig}} \longrightarrow \mathbb{P}_{\breve{K}}^{h-1}$ is the Gross-Hopkins' period morphism. The morphism $\mathrm{id}_{\Phi^*\mathcal{G}}$ corresponds to the adjunction morphism $\mathrm{ad} : \mathcal{G} \longrightarrow \Phi_*\Phi^*\mathcal{G}$. Let $\mathcal{G} = \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)$ with $s \in \mathbb{Z}$. The period morphism Φ is constructed in such a way that $\Phi^*\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s) \simeq (\mathcal{M}_0^s)^{\mathrm{rig}}$ (cf. Remark 3.1.16). This gives us a map $\mathrm{ad} : \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s) \longrightarrow \Phi_*(\mathcal{M}_0^s)^{\mathrm{rig}}$ of $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}$-modules. Taking global sections, we get a homomorphism of Γ-representations

$$\mathrm{ad}_{\mathbb{P}_{\breve{K}}^{h-1}} : \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\mathbb{P}_{\breve{K}}^{h-1}) \longrightarrow \Phi_*(\mathcal{M}_0^s)^{\mathrm{rig}}(\mathbb{P}_{\breve{K}}^{h-1}) = (\mathcal{M}_0^s)^{\mathrm{rig}}(\Phi^{-1}(\mathbb{P}_{\breve{K}}^{h-1})) = (\mathcal{M}_0^s)^{\mathrm{rig}}(X_0^{\mathrm{rig}}) = M_0^s.$$

Lemma 4.1.12. *The map* $\mathrm{ad}_{\mathbb{P}_{\breve{K}}^{h-1}}$ *is injective.*

Proof. The period morphism Φ, when restricted to the affinoid subdomain D, is an isomorphism. Thus $(\mathcal{M}_0^s)^{\mathrm{rig}}(D) \simeq \Phi^*\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(D) \simeq \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\Phi(D))$. Also we have $(\mathcal{M}_0^s)^{\mathrm{rig}}(D) \simeq \mathcal{O}_{X_0^{\mathrm{rig}}}(D)\varphi_0^s \simeq \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D))\varphi_0^s$. As a result, it follows from the preceding discussion on the line bundles that $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\mathbb{P}_{\breve{K}}^{h-1})$ maps bijectively onto $V_s \subset \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(s)(\Phi(D))$ under the restriction map. The lemma now follows from the following commutative diagram with vertical restriction maps.

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}}) & \xrightarrow{\mathrm{ad}_{\mathbb{P}^{h-1}_{\breve{K}}}} & \Phi_*(\mathcal{M}_0^s)^{\mathrm{rig}}(\mathbb{P}^{h-1}_{\breve{K}}) = M_0^s \\
\big\downarrow & & \big\downarrow \\
\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\Phi(D)) & \underset{\mathrm{ad}_{\Phi(D)}}{\xrightarrow{\simeq}} & \Phi_*(\mathcal{M}_0^s)^{\mathrm{rig}}(\Phi(D)) = (\mathcal{M}_0^s)^{\mathrm{rig}}(D) = M_D^s
\end{array}$$

□

In fact, also the right vertical arrow in the commutative diagram above is injective. Namely, the inclusion $R_0^{\mathrm{rig}} \hookrightarrow \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ gives rise to a Γ-equivariant inclusion $M_0^s \hookrightarrow M_D^s \simeq \mathcal{O}_{X_0^{\mathrm{rig}}}(D) \otimes_{R_0} \mathrm{Lie}(\mathbb{H}^{(0)})^{\otimes s}$ using (2.3.7) and the freeness of $\mathrm{Lie}(\mathbb{H}^{(0)})^{\otimes s}$ as an R_0-module. As $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}})$ is a finite dimensional $\breve{K}$-vector space, we have for $s \geq 0$, $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}}) \subseteq (M_0^s)_{\mathrm{lf}} \subseteq (M_D^s)_{\mathrm{lf}} = V_s$, where the first and the last $\breve{K}$-vector spaces are isomorphic as mentioned in the proof of the previous lemma. Hence,

Corollary 4.1.13. *For all $s \in \mathbb{Z}$, we have an isomorphism of Γ-representations*

$$(M_0^s)_{\mathrm{lf}} = \mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}}) \simeq V_s.$$

Thus, $(M_0^s)_{\mathrm{lf}}$ is zero if $s < 0$ and is a finite dimensional irreducible Γ-representation if $s \geq 0$. □

Remark 4.1.14. From now on, we identify the subrepresentation $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}})$ of M_0^s with V_s. For $s = 1$, the Γ-locally finite subrepresentation V_1 of M_0^1 is the representation $\mathbb{V}$ mentioned in the construction of the period morphism Φ (cf. the paragraph after Remark 3.1.6), and thus is isomorphic to the h-dimensional Γ-representation $B_h \otimes_{K_h} \breve{K}$. Since the Γ-representations $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}})$ and V_s are isomorphic, and $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}})$ is same as the s-th symmetric power $\mathrm{Sym}^s(\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(1)(\mathbb{P}^{h-1}_{\breve{K}}))$ of $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(1)(\mathbb{P}^{h-1}_{\breve{K}})$, we obtain the isomorphism,

$$(M_0^s)_{\mathrm{lf}} = V_s \simeq \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K})$$

of Γ-representations for all $s \geq 0$. This makes the shape of these representations completely explicit.

4.2. Locally finite vectors in the Γ-representations M_m^s with $m > 0$

We compute the locally finite vectors in two parts: $s \leq 0$ and $s > 0$. The idea here is to use the commuting actions of Γ and the finite group G_0/G_m on M_m^s.

Part I : $s \leq 0$

Lemma 4.2.1. *Let G be a finite group acting on an integral domain R by ring automorphisms such that the subring of G-invariants R^G is a perfect field F. Then R is a field and the extension R/F is finite.*

Proof. If $\alpha \in R$ then $\Pi_{\sigma \in G}(t - \sigma(\alpha))$ is a monic polynomial of degree $|G|$ with coefficients in $R^G = F$, and has α as a root. This implies that every nonzero α has a unique inverse, since R

is an integral domain. The second assertion now follows from [**Lang02**], Chapter VI, Lemma 1.7. □

The following result relies on the Strauch's computation of the geometrically connected components of the Lubin-Tate tower (cf. [**Str08i**] and Remark 2.3.5).

Theorem 4.2.2. *For all $m \geq 0$, $(M_m^0)_{\mathrm{lf}} = (R_m^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}_m$.*

Proof. The kernel of the composition map $\Gamma \xrightarrow{\mathrm{Nrd}} \mathfrak{o}^\times \longrightarrow (\mathfrak{o}/\varpi^m\mathfrak{o})^\times$ is an open subgroup of Γ which acts trivially on $\breve{K}_m$ (cf. Remark 2.3.5). Thus $\breve{K}_m \subseteq (R_m^{\mathrm{rig}})_{\mathrm{lf}}$. Notice that $(R_m^{\mathrm{rig}})_{\mathrm{lf}}$ is a subring of R_m^{rig}. Indeed, if $f_1, f_2 \in (R_m^{\mathrm{rig}})_{\mathrm{lf}}$ then there exist open subgroups H_1 and H_2 of Γ and finite dimensional sub-representations V_1 and V_2 of H_1 and H_2 in R_m^{rig} respectively such that $f_1 \in V_1$ and $f_2 \in V_2$. Let $V_1 + V_2 = \{v_1 + v_2 | v_1 \in V_1, v_2 \in V_2\}$ and V_1V_2 be the $\breve{K}$-vector space generated by $\{v_1v_2\}_{v_1\in V_1, v_2\in V_2}$. Then V_1+V_2 and V_1V_2 are finite dimensional sub-representations of the open subgroup $H_1 \cap H_2$ containing $f_1 + f_2$ and f_1f_2 respectively.

The $\breve{K}$-algebra R_m^{rig} carries commuting actions of the groups Γ and G_0/G_m with $(R_m^{\mathrm{rig}})^{G_0/G_m} = R_0^{\mathrm{rig}}$ (cf. [**Koh11**], Theorem 1.4 (i)). Now let $f \in (R_m^{\mathrm{rig}})_{\mathrm{lf}}$ and V be a finite dimensional H-subrepresentation of R_m^{rig} containing f for some open subgroup H of Γ. Let $g \in G_0/G_m$. Then the $\breve{K}$-vector space gV is H-stable since the actions of H and G_0/G_m on V commute. Thus gV is a finite dimensional H-subrepresentation of R_m^{rig} containing gf implying that gf is locally finite. Hence $(R_m^{\mathrm{rig}})_{\mathrm{lf}}$ is stable under the action of G_0/G_m with the ring of invariants $(R_m^{\mathrm{rig}})_{\mathrm{lf}}^{G_0/G_m} = ((R_m^{\mathrm{rig}})^{G_0/G_m})_{\mathrm{lf}} = (R_0^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}$ by Corollary 4.1.13. As G_0/G_m is finite, $(R_m^{\mathrm{rig}})_{\mathrm{lf}}$ is a finite field extension of $\breve{K}$ by Lemma 4.2.1. So it is also finite over $\breve{K}_m$. However it follows from the proof of [**Koh11**], Theorem 1.4 that $\breve{K}_m$ is separably closed in R_m^{rig}. Therefore $(R_m^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}_m$. □

Remark 4.2.3. By Theorem 3.4.6, we have a $\mathfrak{g}$-action on R_m^{rig}. The subspace of $\mathfrak{g}$-invariants $(R_m^{\mathrm{rig}})^{\mathfrak{g}=0}$ forms a subring of R_m^{rig}, and is stable under the action of G_0/G_m because the G_0/G_m-action on R_m^{rig} is continuous and commutes with that of Γ. As said in the proof of Theorem 4.2.2, the kernel of the composition map $\Gamma \xrightarrow{\mathrm{Nrd}} \mathfrak{o}^\times \longrightarrow (\mathfrak{o}/\varpi^m\mathfrak{o})^\times$ is an open subgroup of Γ which acts trivially on $\breve{K}_m$. Thus, $\breve{K}_m \subseteq (R_m^{\mathrm{rig}})^{\mathfrak{g}=0}$. Proceeding similarly as above, we have $((R_m^{\mathrm{rig}})^{\mathfrak{g}=0})^{G_0/G_m} = ((R_m^{\mathrm{rig}})^{G_0/G_m})^{\mathfrak{g}=0} = (R_0^{\mathrm{rig}})^{\mathfrak{g}=0} = \breve{K}$ (cf. Lemma 4.1.5). Then by the same arguments as above, we get $(R_m^{\mathrm{rig}})^{\mathfrak{g}=0} = \breve{K}_m$.

For all integers s, the Γ-equivariant isomorphism $M_m^s \simeq R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} M_0^s$ (cf. proof of Theorem 3.4.7) and the freeness of the R_0^{rig}-module M_0^s give rise to a Γ-equivariant inclusion $M_0^s \subset M_m^s$ of $\breve{K}$-vector spaces. Consequently, we have $(M_0^s)_{\mathrm{lf}} \subseteq (M_m^s)_{\mathrm{lf}}$. Using the above theorem, we see that $(M_m^s)_{\mathrm{lf}}$ is a module over $(R_m^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}_m$, and thus we obtain a natural map

$$\breve{K}_m \otimes_{\breve{K}} (M_0^s)_{\mathrm{lf}} \longrightarrow (M_m^s)_{\mathrm{lf}}$$

of $\breve{K}$-vector spaces. Our objective is to show that this map is an isomorphism of $\breve{K}[\Gamma]$-modules for all integers s.

Lemma 4.2.4. *Suppose V and W are two representations of a topological group G over a field F such that one of them, say W, is finite dimensional. Consider the representation $V \otimes_F W$ with diagonal G-action. Then $(V \otimes_F W)_{\mathrm{lf}} = V_{\mathrm{lf}} \otimes_F W$.*

Proof. We omit the subscript F in $\otimes_F$, as all the tensor products are over F. The inclusion $V_{\mathrm{lf}} \otimes W \subseteq (V \otimes W)_{\mathrm{lf}}$ is clear. Let W^* be the F-linear dual of W equipped with the contragredient G-action i.e. $(gf)(w) = f(g^{-1}w)$ for all $g \in G, w \in W$ and $f \in W^*$. Choose an F-basis

$\{w_1, \ldots, w_d\}$ of W, and let $\{f_1, \ldots, f_d\}$ be the dual basis of W^* (i.e. $f_i(w_j) = \delta_{ij}$). Then the natural evaluation map $W \otimes W^* \longrightarrow F$ ($w \otimes f \mapsto f(w)$) is G-equivariant for the diagonal G-action on the left and for the trivial G-action on the right. Tensoring both sides with V, we get a G-equivariant map $\phi : V \otimes W \otimes W^* \longrightarrow V$ for the diagonal G-action on the left, sending $v \otimes w \otimes f$ to $f(w)v$. Because of its G-equivariance, ϕ maps locally finite vectors to locally finite vectors. Now let $x \in (V \otimes W)_{\mathrm{lf}}$. Then x can be uniquely written as $x = \sum_{i=1}^d x_i \otimes w_i$ for some $x_1, \ldots, x_d \in V$. Since W^* is finite dimensional, $x \otimes f_i \in (V \otimes W)_{\mathrm{lf}} \otimes (W^*)_{\mathrm{lf}} \subseteq (V \otimes W \otimes W^*)_{\mathrm{lf}}$ for all $1 \leq i \leq d$. Hence, $\phi(x \otimes f_i) = x_i \in V_{\mathrm{lf}}$ for all $1 \leq i \leq d$. Therefore, $x \in V_{\mathrm{lf}} \otimes W$. □

The following theorem is based on the property of generic flatness of the line bundles $\mathcal{L}\mathrm{ie}(\mathbb{E}^{(m)})^{\otimes s}$ obtained in Remark 3.1.6.

Theorem 4.2.5. *For all $s < 0$ and for all $m \geq 0$, $(M_m^s)_{\mathrm{lf}} \simeq \breve{K}_m \otimes_{\breve{K}} (M_0^s)_{\mathrm{lf}} = 0$.*

Proof. Recall from (3.1.8) that we have an isomorphism

$$R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s} \simeq R_m^{\mathrm{rig}} \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K})^{\otimes s}$$

of Γ-representations. As a result, using Lemma 4.2.4 together with Theorem 4.2.2, we obtain locally finite vectors in the global sections of $\mathcal{L}\mathrm{ie}(\mathbb{E}^{(m)})^{\otimes s}$, i.e.

$$(R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s})_{\mathrm{lf}} \simeq \breve{K}_m \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K})^{\otimes s}.$$

Then, since $s < 0$, the $(\Gamma \times (G_0/G_m))$-equivariant inclusion

$$M_m^s \subset R_m^{\mathrm{rig}} \otimes_{R_m} \mathrm{Lie}(\mathbb{E}^{(m)})^{\otimes s}$$

from (3.1.3) gives rise to a $(\Gamma \times (G_0/G_m))$-equivariant inclusion

$$(M_m^s)_{\mathrm{lf}} \subseteq \breve{K}_m \otimes_{\breve{K}} (B_h \otimes_{K_h} \breve{K})^{\otimes s}$$

of $\breve{K}$-vector spaces. As the action of $SL_h(\mathfrak{o}/\varpi^m\mathfrak{o}) \subset G_0/G_m$ on the right hand side above is trivial (cf. Remark 2.3.5), we get $(M_m^s)_{\mathrm{lf}} = \left(M_m^s\right)_{\mathrm{lf}}^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})} = \left(\left(M_m^s\right)^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})}\right)_{\mathrm{lf}}$, where the latter equality is due to the fact that the both group actions on M_m^s commute. Therefore,

$$\begin{aligned}(M_m^s)_{\mathrm{lf}} = \left(\left(M_m^s\right)^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})}\right)_{\mathrm{lf}} &\simeq \left(\left(R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} M_0^s\right)^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})}\right)_{\mathrm{lf}} \\ &\simeq \left(\left(R_m^{\mathrm{rig}}\right)^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})} \otimes_{R_0^{\mathrm{rig}}} M_0^s\right)_{\mathrm{lf}} \\ &\simeq \left((\breve{K}_m \otimes_{\breve{K}} R_0^{\mathrm{rig}}) \otimes_{R_0^{\mathrm{rig}}} M_0^s\right)_{\mathrm{lf}} \\ &\simeq \left(\breve{K}_m \otimes_{\breve{K}} M_0^s\right)_{\mathrm{lf}} = \breve{K}_m \otimes_{\breve{K}} (M_0^s)_{\mathrm{lf}} = 0\end{aligned}$$

where the second isomorphism holds because M_0^s is free over R_0^{rig} with trivial G_0/G_m-action, and the third isomorphism holds because $\left(R_m^{\mathrm{rig}}\right)^{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})}$ is Galois over R_0^{rig} with the Galois group isomorphic to $\frac{G_0/G_m}{SL_h(\mathfrak{o}/\varpi^m\mathfrak{o})} \simeq (\mathfrak{o}/\varpi^m\mathfrak{o})^\times \simeq \mathrm{Gal}(\breve{K}_m/\breve{K})$ and $\breve{K}_m \subseteq R_m^{\mathrm{rig}}$. For the second last equality in the above, we use Lemma 4.2.4 again. The final result then follows from Corollary 4.1.13. □

Part II : $s > 0$

To compute the locally finite vectors in M_m^s for $s > 0$, we make use of the action of the group $G^0 := \{g \in GL_h(K) | \det(g) \in \mathfrak{o}^\times\}$ on *the Lubin-Tate tower* $\left(X_m^{\mathrm{rig}}\right)_{m \in \mathbb{N}_0}$ described in [**Str08ii**], Section 2.2.2.

Given $g \in G^0$ and $m \geq 0$, for every $m' \geq m$ sufficiently large (depending on g), there is a morphism $g_{m',m} : X_{m'}^{\mathrm{rig}} \longrightarrow X_m^{\mathrm{rig}}$ of rigid analytic spaces satisfying the following properties:

(i) For all $g \in G^0$ and for all $n \geq m'' \geq m' \geq m$, we have $g_{n,m} = \pi_{m',m} \circ g_{m'',m'} \circ \pi_{n,m''}$, where recall that $\pi_{m',m} : X_{m'}^{\mathrm{rig}} \longrightarrow X_m^{\mathrm{rig}}$ denotes the covering morphism. In particular, if $g = e$, and if $m = m' = m''$, then we get $e_{n,m} = \pi_{n,m}$ for all $n \geq m$ because $e_{m,m} = \mathrm{id}_{X_m^{\mathrm{rig}}}$ by definition (cf. [**Str08ii**], Section 2.2.2).

(ii) $(gh)_{m'',m} = g_{m',m} \circ h_{m'',m}$ for all $g, h \in G^0$ and for all $m'' \geq m' \geq m$.

(iii) Set $\Phi_m := \Phi \circ \pi_{m,0} : X_m^{\mathrm{rig}} \longrightarrow \mathbb{P}_{\breve{K}}^{h-1}$. Then $\Phi_{m'} = \Phi_m \circ g_{m',m}$ for all $g \in G^0$, $m' \geq m$. In other words, the Gross-Hopkins period morphism is G^0-equivariant for the trivial G^0-action on $\mathbb{P}_{\breve{K}}^{h-1}$.

(iv) All $g_{m',m}$ are Γ-equivariant morphisms.

(v) For $g \in GL_h(\mathfrak{o})$ and $m \geq 0$, $g_{m,m}$ is defined. The gives an action of $GL_h(\mathfrak{o})$ on X_m^{rig} which factors through $GL_h(\mathfrak{o}/\varpi^m\mathfrak{o}) = G_0/G_m$. The induced G_0/G_m-action coincides with the G_0/G_m-action introduced in Section 2.2.

In the above and hereafter, $m' \geq m$ means that m' is sufficiently larger than or equal to m so that $g_{m',m}$ is defined.

Let $D_m := \pi_{m,0}^{-1}(D)$ where D is the Gross-Hopkins fundamental domain D in X_0^{rig}. The admissible open D_m is a Γ-stable affinoid subdomain because $\pi_{m,0}$ is a finite, Γ-equivariant morphism, and D is Γ-stable. For every $g \in G^0$ and $m \geq 0$, we define a g-translate of D_m as $gD_m := g_{m',m}(D_{m'})$ by choosing $m' \geq m$ large enough. Note that this definition is independent of the choice of m', since by property (i), for $m'' \geq m' \geq m$,

$$\begin{aligned} g_{m'',m}(D_{m''}) = g_{m',m}(\pi_{m'',m'}(D_{m''})) &= g_{m',m}(\pi_{m'',m'}(\pi_{m'',0}^{-1}(D))) \\ &= g_{m',m}(\pi_{m'',m'}(\pi_{m'',m'}^{-1}(\pi_{m',0}^{-1}(D)))) = g_{m',m}(D_{m'}), \end{aligned}$$

using that $\pi_{m'',m'}$ is surjective.

Proposition 4.2.6. *The set $\{gD_m\}_{g \in G^0}$ forms an admissible affinoid covering of $\Phi_m^{-1}(\Phi(D))$ consisting of Γ-stable affinoid subdomains.*

Proof. This is a part of the cellular decomposition of the Lubin-Tate tower in [**FGL08**], Proposition I.7.1 relying on [**GH94**], Corollary 23.26. The Γ-stability of gD_m follows from (iv) and that of $D_{m'}$. ☐

Lemma 4.2.7. *For all $g \in G^0$ and $m' \geq m$, the maps $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D)) \longrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ of affinoid $\breve{K}$-algebras induced by the morphisms $D_{m'} \xrightarrow{g_{m',m}} gD_m \xrightarrow{\Phi_m} \Phi(D)$ are injective.*

Proof. By property (iii), the composition $\Phi_m \circ g_{m',m} = \Phi_{m'} = \Phi \circ \pi_{m',0}$ is flat because Φ and $\pi_{m',0}$ are flat. Hence the composition map $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D)) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ of affinoid $\breve{K}$-algebras is flat. However, $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D)) \simeq \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is an integral domain (cf. [**BGR84**], (6.1.5), Proposition 2). So the map $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D)) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ is injective: the multiplication by a non-zero element f on $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D))$ is injective, which remains injective after flat base change. In particular, the image of f in $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ cannot be zero.

To show that the other map $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ is injective, choose $m'' \geq m'$ large enough so that $g_{m'',m'}^{-1} : X_{m''}^{\mathrm{rig}} \longrightarrow X_{m'}^{\mathrm{rig}}$ is defined. Then using properties (i) and (ii), we see that $g^{-1}_{m'',m'}(gD_{m''}) = g^{-1}_{m'',m'}(g_{n,m''}(D_n)) = e_{n,m'}(D_n) = \pi_{n,m'}(D_n) = D_{m'}$ and thus $gD_m = g_{m',m}(D_{m'}) = g_{m',m}(g^{-1}_{m'',m'}(gD_{m''})) = e_{m'',m}(gD_{m''}) = \pi_{m'',m}(gD_{m''})$. In other words,

$$\left(g_{m',m} \circ g^{-1}_{m'',m'}\right)\big|_{gD_{m''}} = \pi_{m'',m}.$$

Hence the induced composition $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'}) \longrightarrow \mathcal{O}_{X_{m''}^{\mathrm{rig}}}(gD_{m''})$ of the maps of affinoid $\breve{K}$-algebras is flat. Now it is not clear if the algebra $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ is an integral domain. However we can decompose gD_m into its finitely many disjoint connected components $gD_m = \bigsqcup_{i=1}^r U_i$ so that each $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ is an integral domain (cf. discussion after [**BGR84**], (9.1.4), Proposition 8 as well as [**Con99**], Lemma 2.1.4). This decomposition also gives a decomposition $gD_{m''} = \bigsqcup_{i=1}^r (\pi_{m'',m}|_{gD_{m''}})^{-1}(U_i)$ of $gD_{m''}$ into disjoint admissible open subsets. By the same argument as in the first paragraph, each map $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i) \longrightarrow \mathcal{O}_{X_{m''}^{\mathrm{rig}}}((\pi_{m'',m}|_{gD_{m''}})^{-1}(U_i))$ is injective. As a consequence, the composition $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_{m''}^{\mathrm{rig}}}(gD_{m''})$ is injective since it is the finite direct product of all these maps. □

Remark 4.2.8. The affinoid subdomain D_m, by definition, is the same as the fibre product $X_m^{\mathrm{rig}} \times_{X_0^{\mathrm{rig}}} D$ for the maps $\pi_{m,0} : X_m^{\mathrm{rig}} \longrightarrow X_0^{\mathrm{rig}}$ and $D \hookrightarrow X_0^{\mathrm{rig}}$. Thus, we have an isomorphism $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m) \simeq R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ because $R_m^{\mathrm{rig}}\big|R_0^{\mathrm{rig}}$ is finite. The Galois group $G_0/G_m = \mathrm{Gal}(R_m^{\mathrm{rig}}\big|R_0^{\mathrm{rig}})$ acts on $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m)$ via $\sum_{i=1}^r f_i \otimes f_i' \mapsto \sum_{i=1}^r g(f_i) \otimes f_i'$ for $g \in G_0/G_m$, which gives an action on $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m)$ by $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$-linear automorphisms. Hence the extension $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m)|\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ is finite Galois with the Galois group G_0/G_m. Consequently, for all $m \geq 0$, the extension of coordinate rings $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m)|\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(\Phi(D))$ induced by the map Φ_m is finite Galois with the same Galois group.

Remark 4.2.9. As both R_m^{rig} and $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$ are $\mathfrak{g}$-modules (cf. Proposition 3.2.6, Theorem 3.4.6), we have a $\mathfrak{g}$-action on $\mathcal{O}_{X_m^{\mathrm{rig}}}(D_m) \simeq R_m^{\mathrm{rig}} \otimes_{R_0^{\mathrm{rig}}} \mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. Namely, if $\mathfrak{x} \in \mathfrak{g}$ then on simple tensors, $\mathfrak{x}(f \otimes f') = \mathfrak{x}(f) \otimes f' + f \otimes \mathfrak{x}(f')$. The $\mathfrak{g}$-action on $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ restricts to the subalgebra $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$, because by Remark 4.2.8, $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ is a Γ-stable submodule of the finitely generated $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(\Phi(D))$-module $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ and hence is closed in $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ by [**BGR84**], (3.7.3), Proposition 1. Denoting by Ad_γ the adjoint automorphism of $\mathfrak{g}$ corresponding to $\gamma \in \Gamma$, we remark that the actions of Γ and $\mathfrak{g}$ on $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ are compatible in the sense that $\gamma(\mathfrak{x}(f)) = \mathrm{Ad}_\gamma(\mathfrak{x})(\gamma(f))$, since the Lie algebra action comes from the action of the distribution algebra $D(\Gamma, \breve{K})$ on R_m^{rig} and $\mathcal{O}_{X_0^{\mathrm{rig}}}(D)$. Using the isomorphism $(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m) \simeq \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \otimes_{R_m^{\mathrm{rig}}} M_m^s$ and Theorem 3.4.7, one obtains that $(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m)$ carries compatible actions of Γ and $\mathfrak{g}$ for all $s \in \mathbb{Z}$.

Proposition 4.2.10. *For all $g \in G^0$ and for all $m \geq 0$, $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{g}=0} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{n}=0}$. All these $\breve{K}$-vector spaces are finite dimensional.*

PROOF. Let $g \in G^0, m \geq 0$ be arbitrary, and $m' \geq m$ so that $g_{m',m}$ is defined. As seen in the proof of Lemma 4.2.7, the composition $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(\Phi(D)) \hookrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \hookrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ is induced by $\Phi_{m'}$. The Γ-equivariance of $g_{m',m}$ and of Φ_m yields the inclusions $\mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(\Phi(D))_{\mathrm{lf}} \hookrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} \hookrightarrow \mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})_{\mathrm{lf}}$ of $\breve{K}$-algebras. The Galois action on $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})$ commutes with the Γ-action. As a result, $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})_{\mathrm{lf}}$ is stable under the Galois action, and $\left(\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})_{\mathrm{lf}}\right)^{G_0/G_{m'}} = \left(\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})^{G_0/G_{m'}}\right)_{\mathrm{lf}} = \mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(\Phi(D))_{\mathrm{lf}} = \mathcal{O}_{X_0^{\mathrm{rig}}}(D)_{\mathrm{lf}} = \breve{K}$ (cf. Corollary 4.1.9). Since $G_0/G_{m'}$

is finite, $\mathcal{O}_{X_{m'}^{\mathrm{rig}}}(D_{m'})_{\mathrm{lf}}$ is integral over $\breve{K}$, and thus $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}}$ is integral over $\breve{K}$. As before, we write $gD_m = \bigsqcup_{i=1}^r U_i$ where U_is are the connected components of gD_m. Let Γ_i be the stabilizer of U_i in Γ; then each Γ_i has a finite index in Γ. Let $\Gamma_o \subseteq \Gamma$ be an open subgroup which is a uniform pro-p group. Then for every i, the intersection $\Gamma_i \cap \Gamma_o$ has a finite index in Γ_o, and thus is open in Γ_o by [**DDMS03**], Theorem 1.17. As a result, $\Gamma_i \cap \Gamma_o$ is open in Γ, and

$$\Gamma_i = \bigcup_{\bar{\gamma} \in \Gamma_i / \Gamma_i \cap \Gamma_o} \gamma(\Gamma_i \cap \Gamma_o)$$

implies that Γ_i is open in Γ for all i. Therefore their intersection $\Gamma' := \bigcap_{i=1}^r \Gamma_i$ is again an open subgroup of Γ.

Now the decomposition $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \simeq \Pi_{i=1}^r \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ of $\breve{K}$-algebras is Γ'-equivariant for the componentwise Γ'-action on the right. Thus the compactness of Γ gives the decomposition $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\Gamma'-\mathrm{lf}} \simeq \Pi_{i=1}^r \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}}$ of locally finite vectors. Denote by K_i the integral closure of $\breve{K}$ in the integral domain $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ for each i. It then follows that K_i is a field extension of $\breve{K}$. Since every projection $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} \longrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}}$ is a surjective $\breve{K}$-algebra homomorphism, the integrality of $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}}$ over $\breve{K}$ implies that $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}}$ is integral over $\breve{K}$ for all i. Therefore, $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}} \subseteq K_i$ for all i. On the other hand, for each i, K_i is Γ'-stable as Γ' acts $\breve{K}$-linearly on $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$. Now for any classical point $x \in U_i$, the composition map $K_i \hookrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i) \twoheadrightarrow \kappa(x)$ is injective, and because $\kappa(x)|\breve{K}$ is finite, $K_i|\breve{K}$ is a finite extension. This gives the other inclusion $K_i \subseteq \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)_{\Gamma'-\mathrm{lf}}$ for all i. Thus we have $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)_{\mathrm{lf}} = \Pi_{i=1}^r K_i$ with each K_i a finite field extension of $\breve{K}$.

We now claim that $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{g}=0} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{n}=0} = \Pi_{i=1}^r K_i$. Note that $\mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ is $\mathfrak{g}$-stable for all i because the projection map $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \twoheadrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)$ of affinoid $\breve{K}$-algebras is surjective, continuous and Γ'-equivariant. Then all arguments in the last two paragraphs carry over to these cases since $\mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D))^{\mathfrak{g}=0} = \mathcal{O}_{\mathbb{P}_{\breve{K}}^{h-1}}(\Phi(D))^{\mathfrak{n}=0} = \breve{K}$ (cf. Lemma 4.1.5). The only thing that remains to be shown is $K_i \subseteq \mathcal{O}_{X_m^{\mathrm{rig}}}(U_i)^{\mathfrak{g}=0}$ for all i : Write $K_i = \breve{K}[\alpha_i]$. Then the set $\{\gamma(\alpha_i)\}_{\gamma \in \Gamma'}$ is finite as Γ' takes α_i to its conjugates. Therefore, the stabilizer Γ'_i of α_i in Γ' has a finite index in Γ', and thus we obtain the open subgroup Γ'_i of Γ which acts trivially on K_i. □

The embedding $\Gamma \hookrightarrow GL_h(K_h)$ in (3.1.10) extends to an embedding $B_h^\times \hookrightarrow GL_h(K_h)$ of locally K-analytic groups via the same map. This yields an action of $B_h^\times$ on $\mathbb{P}_{\breve{K}}^{h-1}$. The Γ-action on X_m^{rig} is extended to the full group $B_h^\times$ by letting $b \in B_h^\times$ act by the action of $(1, b, \tilde{\sigma}^{-\mathrm{val}(\mathrm{Nrd}(b))}) \in GL_h(K) \times B_h^\times \times W_K$ given on page 20 of [**Car90**]. Here $\tilde{\sigma}$ denotes a lift of the Frobenius in the Weil group W_K and val is the normalized valuation of K. The maps Φ_m are equivariant for the extended $B_h^\times$-action for all m.

Lemma 4.2.11. *The set $\{\Pi^i \Phi(D)\}_{0 \le i \le h-1}$ forms an admissible covering of $\mathbb{P}_{\breve{K}}^{h-1}$. Thus, X_m^{rig} has an admissible covering $\{\Pi^i \Phi_m^{-1}(\Phi(D))\}_{0 \le i \le h-1}$ for all $m \ge 0$.*

Proof. This is proved as a part of [**GH94**], Corollary 23.21. □

For $0 \le i \le h-1$, $s \ge 0$, $m \ge 0$, define $N_m^s(i) := (\mathcal{M}_m^s)^{\mathrm{rig}}(\Pi^i \Phi_m^{-1}(\Phi(D)))$ and $A_m(i) := N_m^0(i) = \mathcal{O}_{X_m^{\mathrm{rig}}}(\Pi^i \Phi_m^{-1}(\Phi(D)))$. Note that each $\Pi^i \Phi_m^{-1}(\Phi(D))$ is Γ-stable because the conjugation by Π^{-i} ($\gamma \mapsto \Pi^{-i}\gamma\Pi^i$) is an automorphism of Γ. Therefore, all $A_m(i)$ and $N_m^s(i)$ are Γ-representations.

Because of Proposition 4.2.6, we have the exact diagram

$$A_m(0) \xrightarrow{r} \prod_{g\in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \overset{r_1}{\underset{r_2}{\rightrightarrows}} \prod_{g,g'\in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m \cap g'D_m)$$

with the maps given by $r(f) = \left(f\big|_{gD_m}\right)_{g\in G^0}$, $r_1((f_g)_{g\in G^0}) = \left(f_g\big|_{gD_m\cap g'D_m}\right)_{g,g'\in G^0}$, and $r_2((f_g)_{g\in G^0}) = \left(f_{g'}\big|_{gD_m\cap g'D_m}\right)_{g,g'\in G^0}$. The continuity of the restriction maps $\mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) \longrightarrow \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m \cap g'D_m)$ between affinoid $\breve{K}$-algebras implies that the maps r_1 and r_2 are continuous for the product topology on their source and target. The Remark 4.2.9 allows us to view $\prod_{g\in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ as a $\mathfrak{g}$-module with the componentwise $\mathfrak{g}$-action. Now, $A_m(0)$ can be identified with the kernel of the continuous map

$$\begin{aligned} r_1 - r_2 : \prod_{g\in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m) &\longrightarrow \prod_{g,g'\in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m \cap g'D_m) \\ (f_g)_{g\in G^0} &\longmapsto r_1((f_g)_{g\in G^0}) - r_2((f_g)_{g\in G^0}). \end{aligned}$$

Hence, $A_m(0)$ is a closed Γ-stable subspace of $\prod_{g\in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ as r is Γ-equivariant. Consequently, $A_m(0)$ is stable under the induced $\mathfrak{g}$-action.

Observe that the isomorphism $(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m \cap g'D_m) \simeq \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m \cap g'D_m) \otimes_{R_m^{\mathrm{rig}}} M_m^s$ yields a $\mathfrak{g}$-action on $(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m \cap g'D_m)$ (cf. Theorem 3.4.7 and Remark 4.2.9). The restriction maps $(\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m) \longrightarrow (\mathcal{M}_m^s)^{\mathrm{rig}}(gD_m \cap g'D_m)$ are continuous for the topology of finitely generated Banach modules. Then by the similar argument as in the last paragraph, $N_m^s(0)$ carries a $\mathfrak{g}$-module structure. The $\mathfrak{g}$-action and the Γ-action on $A_m(0)$ and on $N_m^s(0)$ are compatible with each other (cf. Remark 4.2.9).

Now since M_m^s is generated over R_m^{rig} by V_s (cf. (3.1.3), Proposition 3.1.4), $N_m^s(i)$ is generated by V_s as an $A_m(i)$-module for all $0 \le i \le h-1$. Let $A_m(i)^{\mathfrak{g}=0}V_s$ and $A_m(i)^{\mathfrak{g}=0}\varphi_0^s$ denote the $A_m(i)^{\mathfrak{g}=0}$-submodules of $N_m^s(i)$ generated by V_s and φ_0^s respectively.

Proposition 4.2.12. *For all $0 \le i \le h-1$, $s \ge 0$, $m \ge 0$, we have $N_m^s(i)_{\mathrm{lf}} \subseteq A_m(i)^{\mathfrak{g}=0}V_s$ and $(N_m^s(i)_{\mathrm{lf}})^{\mathfrak{n}=0} \subseteq A_m(i)^{\mathfrak{g}=0}\varphi_0^s$.*

Proof. We first show that $N_m^s(0)_{\mathrm{lf}} \subseteq A_m(0)^{\mathfrak{g}=0}V_s$. Note that $\varphi_0 \in \mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(\Phi(D))^\times \hookrightarrow A_m(0)^\times$. This implies that φ_0^s alone generates $N_m^s(0)$ as a free $A_m(0)$-module of rank one. Now let $W \subseteq N_m^s(0)$ be a finite dimensional Γ-stable subspace. As an $\mathfrak{sl}_h(K_h)$-representation, W decomposes as a direct sum of simple $\mathfrak{sl}_h(K_h)$-modules by Weyl's complete reducibility theorem. From highest weight theory, we know that each simple module in the decomposition is generated by an element annihilated by the subalgebra $\mathfrak{n}$ of strictly upper triangular matrices. Now $W^{\mathfrak{n}=0} \subseteq N_m^s(0)^{\mathfrak{n}=0} = (A_m(0)\varphi_0^s)^{\mathfrak{n}=0} = A_m(0)^{\mathfrak{n}=0}\varphi_0^s$ because $\mathfrak{n}\varphi_0^s = 0$ (cf. Lemma 4.1.3). Let $f \in A_m(0)^{\mathfrak{n}=0}$, then $f\big|_{gD_m} \in \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{n}=0} = \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{g}=0}$ for all $g \in G^0$ by Proposition 4.2.10. The $\mathfrak{g}$-linear injection $A_m(0) \hookrightarrow \prod_{g\in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)$ of $\breve{K}$-algebras induces an equality

$$A_m(0)^{\mathfrak{g}=0} = A_m(0) \cap \prod_{g\in G^0} \mathcal{O}_{X_m^{\mathrm{rig}}}(gD_m)^{\mathfrak{g}=0}.$$

Therefore, $f \in A_m(0)^{\mathfrak{g}=0}$, and hence $A_m(0)^{\mathfrak{g}=0} = A_m(0)^{\mathfrak{n}=0}$.

Since the $\mathfrak{sl}_h(K_h)$-simple modules in the decomposition of W are also irreducible as Γ -representations, W is generated over Γ by its $\mathfrak{n}$-invariants and thus we have

$$W = \Gamma.W^{\mathfrak{n}=0} \subseteq \Gamma.(A_m(0)^{\mathfrak{g}=0}\varphi_0^s) = A_m(0)^{\mathfrak{g}=0}V_s.$$

The last equality follows from the observation that $A_m(0)^{\mathfrak{g}=0}$ is Γ-stable and $\Gamma.\varphi_0^s = V_s$. This proves the desired inclusion $N_m^s(0)_{\mathrm{lf}} \subseteq A_m(0)^{\mathfrak{g}=0}V_s$. On the way, we have also seen that $(N_m^s(0)_{\mathrm{lf}})^{\mathfrak{n}=0} \subseteq A_m(0)^{\mathfrak{g}=0}\varphi_0^s$.

If the Γ-action on $\Phi_m^{-1}(\Phi(D))$ is changed via the automorphism $\gamma \mapsto \Pi^{-i}\gamma\Pi^i$, then the map $\Pi^i : \Phi_m^{-1}(\Phi(D)) \xrightarrow{\sim} \Pi^i\Phi_m^{-1}(\Phi(D))$ is a Γ-equivariant isomorphism. We note that the new Γ-action does not change the locally finite vectors in $N_m^s(0)$. Writing $\varphi_h := \varphi_0$ formally, we have an induced isomorphism $\Pi^i : (N_m^s(i))_{\mathrm{lf}} \xrightarrow{\sim} (N_m^s(0))_{\mathrm{lf}}$ mapping φ_0^s to φ_{h-i}^s, and the $\mathfrak{n}$-invariants onto the $\mathfrak{n}_i := \mathrm{Ad}_{\Pi^i}(\mathfrak{n})$-invariants for all $0 \le i \le h-1$. Therefore,

$$\begin{aligned}(N_m^s(i)_{\mathrm{lf}})^{\mathfrak{n}=0} = (\Pi^i)^{-1}((N_m^s(0)_{\mathrm{lf}})^{\mathfrak{n}_i=0}) &\subseteq (\Pi^i)^{-1}((A_m(0)^{\mathfrak{g}=0}V_s)^{\mathfrak{n}_i=0}) \\ &= (\Pi^i)^{-1}(A_m(0)^{\mathfrak{g}=0}V_s^{\mathfrak{n}_i=0}) \\ &= (\Pi^i)^{-1}(A_m(0)^{\mathfrak{g}=0}\varphi_{h-i}^s) \\ &= A_m(i)^{\mathfrak{g}=0}\varphi_0^s.\end{aligned}$$

As before, this also implies $N_m^s(i)_{\mathrm{lf}} \subseteq A_m(i)^{\mathfrak{g}=0}V_s$ for all $0 \le i \le h-1$. □

Theorem 4.2.13. *For all $s \ge 0$, $m \ge 0$, we have an isomorphism*

$$(M_m^s)_{\mathrm{lf}} \simeq \breve{K}_m \otimes_{\breve{K}} V_s \simeq \breve{K}_m \otimes_{\breve{K}} \mathcal{O}_{\mathbb{P}^{h-1}_{\breve{K}}}(s)(\mathbb{P}^{h-1}_{\breve{K}}) \simeq \breve{K}_m \otimes_{\breve{K}} \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K})$$

of Γ-representations for the diagonal Γ-action on the tensor products. The representation $(M_m^s)_{\mathrm{lf}}$ is a finite dimensional semi-simple representation of Γ.

Proof. As before, $(M_m^s)_{\mathrm{lf}}$ is generated over Γ by its $\mathfrak{n}$-invariants. Let $x \in ((M_m^s)_{\mathrm{lf}})^{\mathfrak{n}=0}$. Then, using the preceding proposition, $x\big|_{\Pi^i\Phi_m^{-1}(\Phi(D))} \in (N_m^s(i)_{\mathrm{lf}})^{\mathfrak{n}=0} \subseteq A_m(i)^{\mathfrak{g}=0}\varphi_0^s$ for all $0 \le i \le h-1$. Let $Y_i := \Pi^i\Phi_m^{-1}(\Phi(D))$, and write $x\big|_{Y_i} = f_i\varphi_0^s$ with $f_i \in A_m(i)^{\mathfrak{g}=0}$.

For all $0 \le i,j \le h-1$, we have $\big(f_i\big|_{Y_i\cap Y_j} - f_j\big|_{Y_i\cap Y_j}\big)\varphi_0^s = x\big|_{Y_i\cap Y_j} - x\big|_{Y_i\cap Y_j} = 0$. Now M_m^s is free over the integral domain R_m^{rig}, and contains $\varphi_0^s \neq 0$. Hence the map $(r \mapsto r\varphi_0^s)$ from R_m^{rig} to M_m^s is injective and remains injective after any flat base change. In particular, the map $(r \mapsto r\varphi_0^s) : \mathcal{O}_{X_m^{\mathrm{rig}}}(Y_i \cap Y_j) \longrightarrow (\mathcal{M}_m^s)^{\mathrm{rig}}(Y_i \cap Y_j)$ is injective, and thus $f_i\big|_{Y_i\cap Y_j} = f_j\big|_{Y_i\cap Y_j}$ for all $0 \le i,j \le h-1$. Therefore, by the sheaf axioms, the functions $(f_i)_i$ glue together to a global section $f \in R_m^{\mathrm{rig}}$ and $x = f\varphi_0^s$. Since $f\big|_{Y_i} = f_i \in A_m(i)^{\mathfrak{g}=0}$ for all i, and the map $R_m^{\mathrm{rig}} \hookrightarrow \Pi_{i=0}^{h-1}A_m(i)$ is $\mathfrak{g}$-equivariant, $f \in (R_m^{\mathrm{rig}})^{\mathfrak{g}=0} = \breve{K}_m$ (cf. Remark 4.2.3). Hence $x \in \breve{K}_m\varphi_0^s$. As a result, $(M_m^s)_{\mathrm{lf}} \subseteq \Gamma.(\breve{K}_m\varphi_0^s) = \breve{K}_mV_s$. The other inclusion $\breve{K}_mV_s \subseteq (M_m^s)_{\mathrm{lf}}$ is easy to see as $(M_m^s)_{\mathrm{lf}}$ is a module over $(R_m^{\mathrm{rig}})_{\mathrm{lf}} = \breve{K}_m$, and $V_s = (M_0^s)_{\mathrm{lf}} \subseteq (M_m^s)_{\mathrm{lf}}$.

Now to justify the isomorphism $\breve{K}_m \otimes_{\breve{K}} V_s \simeq \breve{K}_mV_s$, it is enough to show that the natural map

$$\begin{aligned}\breve{K}_m \otimes_{\breve{K}} V_s &\longrightarrow \breve{K}_mV_s \\ \sum_{0\le|\alpha|\le s} c_\alpha(1 \otimes w^\alpha\varphi_0^s) &\longmapsto \sum_{0\le|\alpha|\le s} c_\alpha w^\alpha\varphi_0^s\end{aligned}$$

is injective. Here the set $\{1 \otimes w^\alpha\varphi_0^s\}_{0\le|\alpha|\le s}$ forms a $\breve{K}_m$-basis of $\breve{K}_m \otimes_{\breve{K}} V_s$. By Lemma 4.1.3, we have $\mathfrak{x}_{00}(w^\alpha\varphi_0^s) = (s-|\alpha|)w^\alpha\varphi_0^s$ and $\mathfrak{x}_{ii}(w^\alpha\varphi_0^s) = \alpha_iw^\alpha\varphi_0^s$ for all $1 \le i \le h-1$. Since $\mathfrak{g}$ annihilates $\breve{K}_m$, if $\sum_{0\le|\alpha|\le s} c_\alpha w^\alpha\varphi_0^s = 0$, one can use the above actions of the diagonal matrices iteratively to deduce that each summand $c_\alpha w^\alpha\varphi_0^s$ is zero, and therefore $c_\alpha = 0$ for all $0 \le |\alpha| \le s$.

Unlike $(M_0^s)_{\mathrm{lf}} = V_s$, the space of locally finite vectors $(M_m^s)_{\mathrm{lf}} \simeq \breve{K}_m \otimes_{\breve{K}} V_s$ at level $m > 0$ is not an irreducible Γ-representation as it properly contains the representation V_s. However; it is semi-simple and this can be seen as follows: the action of Γ on $\breve{K}_m$ factors through a finite group (cf. Remark 2.3.5). As a result, $\breve{K}_m$ decomposes into a direct sum $\breve{K}_m \simeq \bigoplus_{i=1}^n W_i$ of irreducible representations. This gives us a decomposition

$$\breve{K}_m \otimes_{\breve{K}} V_s \simeq \bigoplus_{i=1}^{n} (W_i \otimes_{\breve{K}} V_s). \tag{4.2.14}$$

Now we note that $V_s \simeq \mathrm{Sym}^s(B_h \otimes_{K_h} \breve{K})$ is an irreducible algebraic representation of $\Gamma \simeq \mathfrak{o}_{B_h}^\times$ (cf. Theorem 4.1.7, [**Koh14**], Remark 4.4), and $\breve{K}_m$ is a smooth representation of Γ by Remark 4.2.3. Thus every direct summand in (4.2.14) is a tensor product of a smooth irreducible representation and an irreducible algebraic representation of Γ. Such a representation is irreducible by [**ST01**], Appendix by Dipendra Prasad, Theorem 1. As a consequence, $(M_m^s)_{\mathrm{lf}}$ is a semi-simple representation of Γ. □

Remark 4.2.15. We recall from [**Eme04**] Section 4.2 that a vector $v \in M_m^s$ is said to be *locally algebraic* if there exists a finite dimensional algebraic representation W of $\mathbb{B}_h^\times$ and an H-equivariant homomorphism $\phi : W^n \longrightarrow M_m^s$ with $v \in \phi(W^n)$ for some open subgroup $H \subseteq \Gamma$ and $n \in \mathbb{N}$. The set of all locally algebraic vectors $(M_m^s)_{\mathrm{lalg}}$ forms a Γ-stable subspace of M_m^s (cf. [**Eme04**], Proposition-Definition 4.2.6), and it is clear that $(M_m^s)_{\mathrm{lalg}} \subseteq (M_m^s)_{\mathrm{lf}}$. However; Theorem 1 in [**ST01**], Appendix by Dipendra Prasad asserts that the direct summands in the decomposition (4.2.14) of $\breve{K}_m \otimes_{\breve{K}} V_s$ are in fact irreducible locally algebraic representations of Γ. Therefore, our explicit results show that the space of locally finite vectors $(M_m^s)_{\mathrm{lf}} = (M_m^s)_{\mathrm{lalg}} \simeq \breve{K}_m \otimes_{\breve{K}} V_s$ is actually a locally algebraic representation of Γ.

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