Some new cases of the Breuil-Schneider conjecture

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1 Introduction

The main motivation for this thesis, is to search for p-adic Langlands correspondence. The conjecture in question was first proposed by Breuil and Schneider in [BS07]. This conjecture may be seen as a first evidence to a p-Langlands correspondence. For a brief survey of this conjecture one may consult [Sor15]. This introduction is strongly influenced by this survey.

The aim of this work is to deduce some new cases of Breuil-Schneider conjecture using the patching construction of [CEG$^+$16].

1.1 Notation

Let p a prime number such that $p \nmid 2n$. Let F be a finite extension of $\mathbb{Q}_p$ with a finite residue field k_F. Let $\mathcal{O}_F$ be its complete discrete valuation ring, let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_F$ with uniformizer ϖ, and let $q = |\mathcal{O}_F/\varpi\mathcal{O}_F|$. Let $G = GL_n(F)$.

Let E be a finite extension of $\mathbb{Q}_p$ (the field of coefficients), $\mathcal{O}$ the ring of integers of E and $\mathbb{F}$ the residue field. Fix a residual Galois representation $\overline{r} : G_F \longrightarrow GL_n(\mathbb{F})$ of the local Galois group $G_F := \mathrm{Gal}(\bar{F}/F)$. We assume that E is large enough to contain all the embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$.

Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$. In the literature, smooth representations of $GL_n(F)$ are studied on complex vector spaces, hence the coefficients are in $\overline{\mathbb{Q}}_p$. The section 3.13 [CEG$^+$16] provides a framework which shows that those results, which are valid in the classical theory over complex numbers, have an analogue over E by faithfully flat descent $\overline{\mathbb{Q}}_p/E$. For instance if π is an irreducible representation of $GL_n(F)$ with coefficients in E, after extending scalars to larger extension E'/E we may assume that π is absolutely irreducible and hence do the base change to $\overline{\mathbb{Q}}_p$. In this way all the results stated with E-coefficients can be proven over $\overline{\mathbb{Q}}_p$ without loss of generality.

In this thesis I follows the notation and conventions of [CEG$^+$16](cf. 1.8), unless otherwise is stated.

1.2 The Breuil-Schneider conjecture

Now suppose $r : G_F \to GL_n(E)$ is a potentially semi-stable lift of $\overline{r}$, with Hodge-Tate weights $\mathrm{HT}_\kappa = \{i_{\kappa,1} < \ldots < i_{\kappa,n}\}$, for each embedding $\kappa : F \hookrightarrow E$. By Fontaine's recipe one associates an n-dimensional Weil-Deligne $WD(r)$

representation to r with coefficients in $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$. Let rec denote the classical local Langlands correspondence with coefficients in $\mathbb{C}$ normalized in a way such that the central character of an irreducible smooth representation of $GL_n(F)$ corresponds to the determinant of the associated Weil-Deligne representation via the local class field theory. This is compatible with convention in the book [HT01], see Lemma VII.2.6. Let rec_p denote the local Langlands correspondence over $\overline{\mathbb{Q}}_p$, defined by $\iota \circ \mathrm{rec}_p = \mathrm{rec} \circ \iota$ and define $r_p(\pi) = \mathrm{rec}_p(\pi \otimes |\det|)$. Let $\pi_{sm}(r)$ an irreducible smooth representation of $GL_n(F)$ with coefficients in E defined by $\pi_{sm}(r) = r_p^{-1}(WD(r)^{F-ss})$, where $F-ss$ denotes the Frobenius semi-simplification of $WD(r)$. Assume that $\pi_{sm}(r)$ is generic, i.e. admits a Whittaker model. Then there is a model of $\pi_{sm}(r)$ with coefficients in E, denoted again $\pi_{sm}(r)$ which a smooth irreducible E-representation of $GL_n(F)$. We say that r is generic when $\pi_{sm}(r)$ is generic. In the case when $\pi_{sm}(r)$ is not generic, we need to do some modifications, see [BS07] for more details. Indeed, by Bernstein-Zelevinsky classification, $\pi_{sm}(r)$ is a Langlands quotient and there is a unique parabolic induction, denoted $\pi_{gen}(r)$, such that $\pi_{gen}(r) \twoheadrightarrow \pi_{sm}(r)$. This representation has a model over E, which we will denote again by $\pi_{gen}(r)$.

To the multi-set $\{\mathrm{HT}_\kappa\}_{\kappa: F \hookrightarrow E}$ one can attach an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}_p}(GL_n/F)$, which we evaluate at E to get an algebraic representation $\pi_{alg}(r)$ of $GL_n(F)$. More precisely, for each κ, let $\pi_{alg,\kappa}(r)$ be the irreducible algebraic representation of $GL_n(F)$ of highest weight $\{-i_{\kappa,1}, \ldots, -i_{\kappa,n}+n-1\}$ relative to the upper-triangular Borel. Then define $\pi_{alg}(r) = \bigotimes_\kappa \pi_{alg,\kappa}(r)$, with $GL_n(F)$ acting diagonally. This is the representation $L_\xi \otimes_{\mathcal{O}} E$ with notation of section 1.8[CEG+16], with $\xi_{\kappa,j} = -i_{\kappa,j} + j - 1$.

Define: $BS(r) := \pi_{gen}(r) \otimes_E \pi_{alg}(r)$.

The conjecture, which we state in the generic case, predicts that irreducible locally algebraic representations of G admit integral structures if and only if they are related to Galois representations. More precisely:

Conjecture 1.1. *Let π be an absolutely irreducible generic representation of $GL_n(F)$ and σ an irreducible algebraic representation of algebraic group $\mathrm{Res}_{F/\mathbb{Q}_p} GL_n/F$, both having coefficients in E. Then the following statements are equivalent:*

(1) $\pi \otimes_E \sigma$ admits a G-invariant norm.

(2) *There is a potentially semi-stable Galois representation* $r : G_F \to GL_n(E)$ *such that* $\pi = \pi_{sm}(r)$ *and* $\sigma = \pi_{alg}(r)$.

The implication (1) $\Rightarrow$ (2), was proven by Hu in full generality in his paper [Hu09]. The converse is still largely open. In [CEG$^+$16], the authors prove many cases of this conjecture by constructing an admissible unitary E-Banach space representation $V(r)$ of $GL_n(F)$, such that the locally algebraic vectors in $V(r)$ are isomorphic to $BS(r)$ as $GL_n(F)$. In [CEG$^+$16] the authors assume that r is potentially crystalline. This corresponds to the case when the monodromy operator N of the Weil-Deligne representation $WD(r)$ is zero.

In this thesis we extend the methods of [CEG$^+$16] to handle the case, when r is potentially semi-stable. This corresponds to the case when the monodromy operator N of the Weil-Deligne representation $WD(r)$ is allowed to be arbitrary. We will be mostly concerned with the case when the Galois representation r is generic, in this case $BS(r) = \pi_{sm}(r) \otimes_E \pi_{alg}(r)$ is irreducible. We prove, that the locally algebraic vectors of $V(r)$ are isomorphic to $BS(r)$. This will allow us to deduce new cases for the implication (2) $\Rightarrow$ (1).

Note that asking for a norm amounts to asking for a lattice. Indeed given a norm $\|.\|$, look at the unit ball of a lattice Λ. Conversely, given a lattice Λ, look at its "gauge", i.e. $\|x\| = q_E^{-v_\Lambda(x)}$ ($q_E = |\mathbb{F}|$), where $v_\Lambda(x)$ is the largest ν such that $x \in \varpi_E^\nu \Lambda$ (ϖ_E is a uniformizer of E). Thus we are looking for integral structures in $BS(r)$.

1.3 Typical representations

Let $\mathcal{R}(G)$ be the category of all smooth E-representations of G. We denote by $i_P^G : \mathcal{R}(M) \longrightarrow \mathcal{R}(G)$ the normalized parabolic induction functor, where $P = MN$ is a parabolic subgroup of G with Levi subgroup M. Let $\overline{P}$ be the opposite parabolic.

We are given an inertial class $\Omega = [M, \rho]_G$, where ρ is a supercuspidal representation of M and $D = [M, \rho]_M$. To any inertial class Ω we may associate a full subcategory $\mathcal{R}^\Omega(G)$ of $\mathcal{R}(G)$, such that $(\pi, V) \in \mathrm{Ob}(\mathcal{R}^\Omega(G))$ if every irreducible G-subquotient π_0 of π appear as a composition factor of $i_P^G(\rho \otimes \omega)$ for ω some unramified character of M and P some parabolic subgroup of G with Levi factor M. The category $\mathcal{R}^\Omega(G)$ is called a Bernstein component of $\mathcal{R}(G)$. We will say that a representation π is in Ω if π is an object of $\mathcal{R}^\Omega(G)$.

Let J be a compact open subgroup of G and let λ be an irreducible representation of J. We say that (J, λ) is an Ω-type if for every irreducible representation $(\pi, V) \in \mathrm{Ob}(\mathcal{R}^{\Omega}(G))$, V is generated by the λ-isotypical component of V as G-representation.

Let $\mathcal{R}_{\lambda}(G)$ be a full subcategory of $\mathcal{R}(G)$ such that $(\pi, V) \in \mathrm{Ob}(\mathcal{R}_{\lambda}(G))$ if and only if V is generated by V^{λ} (the λ-isotypical component of V) as G-representation.

Let K be a maximal compact open subgroup of G containing J. We say that an irreducible representation σ of K is *typical for* Ω if for any irreducible representation π of G, $\mathrm{Hom}_K(\sigma, \pi) \neq 0$ implies that π is an object in $\mathcal{R}^{\Omega}(G)$.

Define $\mathcal{H}(G, \lambda) := \mathrm{End}_G(\text{c–Ind}_J^G \lambda)$. Then for any Ω-type (J, λ), by Theorem 4.2 (ii)[BK98], the functor:

$$\begin{array}{rcccl} \mathfrak{M}_{\lambda} & : & \mathcal{R}_{\lambda}(G) & \rightarrow & \mathcal{H}(G, \lambda) - Mod \\ & & \pi & \mapsto & \mathrm{Hom}_J(\lambda, \pi) = \mathrm{Hom}_G(\text{c–Ind}_J^G \lambda, \pi) \end{array}$$

induces an equivalence of categories. Since (J, λ) is an Ω-type, we have $\mathcal{R}^{\Omega}(G) = \mathcal{R}_{\lambda}(G)$.

Write $\mathfrak{Z}_{\Omega}$ for the centre of category $\mathcal{R}^{\Omega}(G)$ and $\mathfrak{Z}_D$ for the centre of category $\mathcal{R}^{D}(M)$, which is defined the same way as $\mathcal{R}^{\Omega}(G)$. Recall that the centre of a category is the ring of endomorphisms of the identity functor. For example the centre of the category $\mathcal{H}(G, \lambda) - Mod$ is $Z(\mathcal{H}(G, \lambda))$, where $Z(\mathcal{H}(G, \lambda))$ is the centre of the ring $\mathcal{H}(G, \lambda)$. We will call $\mathfrak{Z}_{\Omega}$ a Bernstein centre.

When $G = GL_n(F)$, the types can be constructed in an explicit manner (cf. [BK93], [BK98] and [BK99]) for every Bernstein component. Moreover, Bushnell and Kutzko have shown that $\mathcal{H}(G, \lambda)$ is naturally isomorphic to a tensor product of affine Hecke algebras of type A.

The simplest example of a type is $(I, 1)$, where I is Iwahori subgroup of G and 1 is the trivial representation of I. In this case $\Omega = [T, 1]_G$, where T is the subgroup of diagonal matrices. We will refer to example as the Iwahori case.

In [SZ99] section 6 (just above proposition 2) the authors define irreducible K-representations $\sigma_{\mathcal{P}}(\lambda)$, where $\mathcal{P}$ is partition valued functions with compact support (cf. section 2 [SZ99]). One has the decomposition :

$$\mathrm{Ind}_J^K \lambda = \bigoplus_{\mathcal{P}} \sigma_{\mathcal{P}}(\lambda)^{\oplus m_{\mathcal{P},\lambda}} \tag{1}$$

where the summation runs over partition valued functions with compact support. The integers $m_{\mathcal{P},\lambda}$ are finite and we call them multiplicity of $\sigma_{\mathcal{P}}(\lambda)$.

There is a natural partial ordering, as defined in [SZ99], on the partition valued functions. Let $\mathcal{P}_{max}$ be the maximal partition valued function and let $\mathcal{P}_{min}$ the minimal one. Define $\sigma_{max}(\lambda) := \sigma_{\mathcal{P}_{max}}(\lambda)$ and $\sigma_{min}(\lambda) := \sigma_{\mathcal{P}_{min}}(\lambda)$. Both $\sigma_{max}(\lambda)$ and $\sigma_{min}(\lambda)$ occur in $\mathrm{Ind}_J^K \lambda$ with multiplicity 1.

In the Iwahori case, $\sigma_{min}(\lambda)$ is the inflation of Steinberg representation of $GL_n(k_F)$ to K and $\sigma_{max}(\lambda)$ is the trivial representation.

The representation theory of affine Hecke algebras has been studied by Rogawski in [Rog85] and we use these results to prove Proposition 1.2 and 1.3 below.

Proposition 1.2. *Let $\mathcal{P}$ be a partition valued function and let $\sigma_{\mathcal{P}}(\lambda)$ as defined in Section 6 [SZ99]. Let π be an irreducible generic representation, with type (J, λ). The following statement are equivalent:*

1. $\mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), \pi) \neq 0$ *and* $\mathrm{Hom}_K(\sigma_{\mathcal{P}'}(\lambda), \pi) = 0$*, for all partitions valued functions $\mathcal{P}'$ such that $\mathcal{P} < \mathcal{P}'$.*

2. $\pi = i_P^G(L(\Delta_1) \otimes \ldots \otimes L(\Delta_k))$*, where P is the standard parabolic associated to the partition valued function $\mathcal{P}$ and all the segments Δ_i are not pairwise linked.*

Moreover if $\sigma_{\mathcal{P}}(\lambda)$ satisfies the equivalent properties above, it occurs with multiplicity one in π.

As a consequence of the proposition 1.2, given any smooth irreducible generic representation π of G we determine the shape of monodromy operator N of the Weil-Deligne representation corresponding to π via classical local Langlands, by knowing for which $\mathcal{P}$, we have $\mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), \pi) \neq 0$. If $\mathrm{Hom}_K(\sigma_{max}(\lambda), \pi) \neq 0$ then the associated Weil-Deligne representation to π has a zero monodromy operator.

The next proposition generalizes the Theorem 3.7 [CEG$^+$16],

Proposition 1.3. *Let π be an irreducible representation in the Berstein component Ω (having type (J, λ)). We have then,* $\mathrm{Hom}_K(\sigma_{min}(\lambda), \pi) \neq 0$ *if and only if π is generic.*

Finally we prove:

Theorem 1.4. *If σ occurs with multiplicity one in $\mathrm{Ind}_J^K\lambda$, then*

$$\mathfrak{Z}_\Omega \simeq \mathrm{End}_G(\text{c–}\mathrm{Ind}_K^G\sigma).$$

In particular the previous theorem applies to $\sigma = \sigma_{max}(\lambda)$ (Theorem 3.7 [CEG+16]) and also to $\sigma = \sigma_{min}(\lambda)$.

1.4 Locally algebraic vectors

Let $\mathbf{v} = \{\mathrm{HT}_\kappa\}_{\kappa:F\hookrightarrow E}$ be a multiset of all Hodge-Tate weights and let $\tau : I_F \to GL_n(E)$ be an inertial type, i.e. τ is a representation of I_F with open kernel which extends to a representation of the Weil group W_F of F, where I_F is the inertia subgroup of G_F. We let $R^\square_{\tilde{\mathfrak{p}}}$ denote the universal $\mathcal{O}$-lifting ring of $\bar{r}$. Then there is a ring $R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min}) := R^\square_{\bar{r}}(\tau, \mathbf{v})$, which is the unique reduced and p-torsion free quotient of $R^\square_{\tilde{\mathfrak{p}}}$ corresponding to potentially semi-stable lifts of weight σ_{alg} (i.e. of weight $\mathbf{v}$) and inertial type τ. This ring was constructed in [Kis08]. Moreover there is a "universal admissible filtered (φ, N)-module" $D^\square_{\bar{r}}(\tau, \mathbf{v})$ which is a locally free $R^\square_{\bar{r}}(\tau, \mathbf{v})[1/p] \otimes_{\mathbb{Q}_p} F_0$-module of rank n, where F_0 is a maximal subfield of F such that F/F_0 is totally ramified. The module $D^\square_{\bar{r}}(\tau, \mathbf{v})$ comes equipped with a universal Frobenius, denoted by φ.

Let σ_{alg} the restriction to K of $\pi_{alg}(r)$. Define $\sigma_{min} := \sigma_{min}(\lambda) \otimes \sigma_{alg}$ and $\mathcal{H}(\sigma_{min}) := \mathrm{End}_G(\text{c–}\mathrm{Ind}_K^G\sigma_{min})$.

We have fixed a type τ, so there is a finite extension L of F such that the restriction of every Galois representation r_x to G_L is semi-stable. Let L_0 its maximal unramified subfield of L. We assume $[L_0 : \mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L_0, E)|$ and we let p^f be the cardinality of the residue field of L_0. By universal we mean that the specialization D_x of $D^\square_{\bar{r}}(\tau, \mathbf{v})$ at the closed point x of $R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ with residue field E_x is an admissible filtered $(\varphi, N, \mathrm{Gal}(L/F))$-module attached to the Galois representation r_x given by the point x.

On the other hand one may show using the classical local Langlands correspondence that τ determines a Bernstein component $\mathcal{R}^\Omega(G)$. We prove that there is a map that interpolates the local Langlands correspondence, more precisely:

Theorem 1.5. *There is an E-algebra homomorphism*

$$\eta : \mathcal{H}(\sigma_{min}) \longrightarrow R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$$

such that for any closed point x of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ with residue field E_x, the action of $\mathfrak{Z}_\Omega$ on the smooth G-representation $\pi_{sm}(r_x)$ factors as η composed with the evaluation map $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p] \longrightarrow E_x$.

This result generalizes Theorem 4.1 [CEG+16] (i.e. if we restrict to the crystalline locus the two maps coincide), however the proof does not follow methods of this paper. Instead we give an explicit construction of this map.

We will sketch the construction of η in the Iwahori case, in this case $L = F$ because the lifts we consider are semi-stable. By Satake isomorphism and Theorem 1.4, we have $\mathcal{H}(\sigma_{\min}) \simeq E[\theta_1, \ldots, \theta_{n-1}, (\theta_n)^{\pm 1}]$, where θ_r is a double coset operator $\left[K \begin{pmatrix} \varpi I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} K\right]$. All the details of this construction will be discussed in the section 5.2.2.

If for an embedding κ the Hodge-Tate weights are $i_{\kappa,1} < \ldots < i_{\kappa,n}$ define $\xi_{\kappa,j} = -i_{\kappa,j} + (j-1)$. Then the map $\eta : \mathcal{H}(\sigma_{min}) \longrightarrow R^{\square}_{\bar{r}}(\tau, \mathbf{v})[1/p]$ is given by the assignment

$$\theta_r \mapsto \varpi^{-\sum_\kappa \sum\limits_{i=r}^{n} \xi_{\kappa,j}} q^{\frac{r(1-r)}{2}} \mathrm{Tr}(\bigwedge^r \varphi^f)$$

For $\mathcal{P}$ any partition valued function, define $\sigma_{\mathcal{P}} := \sigma_{\mathcal{P}}(\lambda) \otimes \sigma_{alg}$, where $\sigma_{\mathcal{P}}(\lambda)$ was defined above and σ_{alg} is the restriction to K of an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}_p} GL_n$ given by the Hodge-Tate weights. Fix a K-stable $\mathcal{O}$-lattice $\sigma^{\circ}_{\mathcal{P}}$ in $\sigma_{\mathcal{P}}$. Set

$$M_\infty(\sigma^\circ_{\mathcal{P}}) := \left(\mathrm{Hom}^{cont}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ_{\mathcal{P}})^d)\right)^d$$

where M_∞ is R_∞-module constructed in section 2 [CEG+16] by patching process and $(.)^d = \mathrm{Hom}^{cont}_{\mathcal{O}}(., \mathcal{O})$ denotes the Shikhof dual. Since $\sigma^\circ_{\mathcal{P}}$ is a free $\mathcal{O}$-module of finite rank, it follows from the proof of Theorem 1.2 of [ST02] that Schikhof duality induces an isomorphism

$$\mathrm{Hom}^{cont}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ_{\mathcal{P}})^d) \simeq \mathrm{Hom}_K(\sigma^\circ_{\mathcal{P}}, (M_\infty)^d)$$

and Frobenius reciprocity gives

$$\mathrm{Hom}_K(\sigma^\circ_{\mathcal{P}}, (M_\infty)^d) \simeq \mathrm{Hom}_G(\text{c–Ind}^G_K \sigma^\circ_{\mathcal{P}}, (M_\infty)^d).$$

The action of $\mathfrak{Z}_\Omega$ on $\text{c–Ind}^G_K \sigma_{\mathcal{P}}$ induces an action on $M_\infty(\sigma^\circ_{\mathcal{P}})[1/p]$.

To any closed point of $x \in \text{m-Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ we can attach a partition valued function $\mathcal{P}_x$, which encodes information about the shape of monodromy operator of the admissible filtered (φ, N)-module D_x. We prove that there is a reduced p-torsion free quotient $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$ of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$, such that $x \in \text{m-Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})[1/p]$ if and only if $\mathcal{P}_x \geq \mathcal{P}$. When $\sigma_{\mathcal{P}} = \sigma_{min}$, the ring corresponds to all the potentially semi-stable lift and this is compatible with the notation introduced at the beginning. The other extreme case is $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})$, this ring parametrizes all the potentially crystalline lifts.

As a part of patching construction we know that R_∞ is an $R^{\square}_{\tilde{\mathfrak{p}}}$-algebra. We define $R_\infty(\sigma_{\mathcal{P}})' := R_\infty \otimes_{R^{\square}_{\tilde{\mathfrak{p}}}} R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$. Let $R_\infty(\sigma_{\mathcal{P}})$ be the quotient of R_∞ which acts faithfully on $M_\infty(\sigma^\circ_{\mathcal{P}})$. The usual commutative algebra arguments underlying the Taylor-Wiles-Kisin method, as in [CEG+16], show that $M_\infty(\sigma^\circ_{\mathcal{P}})$ is a maximal Cohen-Macaulay module over $R_\infty(\sigma_{\mathcal{P}})$. Moreover we prove an important result about the support of $M_\infty(\sigma^\circ_{min})$:

Proposition 1.6. *1. The module $M_\infty(\sigma^\circ_{min})[1/p]$ is locally free of rank one over the regular locus of* $\operatorname{Spec} R_\infty(\sigma_{min})[1/p]$.

2. $\operatorname{Spec} R_\infty(\sigma_{min})[1/p]$ *is a union of irreducible components of* $\operatorname{Spec} R_\infty(\sigma_{min})'[1/p]$.

The components appearing in the second statement of the Proposition 1.6 are termed *automorphic components*. The proof of the proposition above is similar to Lemma 4.18 [CEG+16]. The action of $\mathfrak{Z}_\Omega$ on $M_\infty(\sigma^\circ_{min})[1/p]$ induces an E-algebra homomorphism:

$$\alpha : \mathfrak{Z}_\Omega \longrightarrow \operatorname{End}_{R_\infty[1/p]}(M_\infty(\sigma^\circ_{min})[1/p])$$

From the Proposition 1.6, we deduce that:

Theorem 1.7. *We have the following commutative diagram:*

$$\begin{tikzcd}
(\operatorname{Spec} R_\infty(\sigma_{min})[1/p])^{reg} \arrow[r, "\alpha^\sharp"] \arrow[d, hook] & \operatorname{Spec} \mathcal{H}(\sigma_{min}) \\
\operatorname{Spec} R_\infty(\sigma_{min})[1/p] \arrow[r, "can"] & \operatorname{Spec} R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p], \arrow[u]
\end{tikzcd}$$

where $(\operatorname{Spec} R_\infty(\sigma_{min})[1/p])^{reg}$ *is the regular locus of* $\operatorname{Spec} R_\infty(\sigma_{min})[1/p]$ *and $\alpha^\sharp$ the map induced by α.*

Just as in §4.28 [CEG$^+$16], the main technique is to convert information on locally algebraic vectors in the completed cohomology into commutative algebra statements about the module $M_\infty(\sigma^\circ_{min})$ using results on K-typical representations that we have explained in the previous section.

Let x be a closed E-valued point of Spec $R_\infty(\sigma_{min})[1/p]$. The corresponding Galois representation r_x is given by the homomorphism $x : R^\square_{\tilde{\mathfrak{p}}} \to \mathcal{O}$, which we extend arbitrarily to homomorphism $x : R_\infty \to \mathcal{O}$. Then

$$V(r_x) := \mathrm{Hom}^{cont}_{\mathcal{O}}(M_\infty \otimes_{R_\infty, x} \mathcal{O}, E) \tag{2}$$

is an admissible unitary E-Banach space representation of G. The main result of this thesis is the following theorem:

Theorem 1.8. *Let x be a closed E-valued point of* Spec $R_\infty(\sigma_{min})[1/p]$*, such that $\pi_{sm}(r_x)$ is generic and irreducible. Then*

$$V(r_x)^{l.alg} \simeq \pi_{sm}(r_x) \otimes \pi_{alg}(r_x),$$

where $(.)^{l.alg}$ denotes the subspace of locally algebraic vectors.

Since the action of G on $V(r_x)$ is unitary, we obtain:

Corollary 1.9. *Suppose $p \nmid 2n$, and that $r : G_F \longrightarrow GL_n(E)$ is a generic potentially semi-stable Galois representation of regular weight. If r correspond to a closed point $x \in$ Spec $R_\infty(\sigma_{min})[1/p]$, then $\pi_{sm}(r) \otimes \pi_{alg}(r)$ admits a non-zero unitary admissible Banach completion.*

It is conjectured in [CEG$^+$16] that $V(r_x)$ depends only on the Galois representation r_x and that $r_x \mapsto V(r_x)$ realizes the hypothetical p-adic local Langlands correspondence. Our Theorem 1.8 provides further evidence of this conjecture.

1.5 The organisation of the thesis

The sections 2 and 3 deal with questions concerning the representations of $GL_n(F)$. In the section 2 we will study the specialization of the projective generator of the category of E-representations of $GL_n(F)$ with type (J, λ) at maximal ideals of the centre of this category. This will allow us to identify the centre with some commutative Hecke algebra (Corollary 2.18).

After this we will focus on generic representations of $GL_n(F)$. In the section 3 we will introduce some combinatorial tools which will allow us to describe a particular class of simple modules over Iwahori Hecke algebras, which correspond to generic representations via the usual equivalence of categories. Then we will translate the results obtained for modules in the language of representations. Then using the theory developed in [BK99], we will treat the general case building up from the Iwahori case. The results from the two previous sections will be used in section 5 to compute locally algebraic vectors of $V(r)$, where r is a generic Galois representation.

Then in section 4 we will prove that if D is a weakly admissible (φ, N)-module and if we set $N = 0$ then there is a filtration on D, the underlying φ-module of D, such that the φ-module then D is again weakly admissible, by writing down explicitly the admissibility condition. In this section we will introduce some notation that will be used in the next section devoted to the locally algebraic vectors. We will also recall some facts about Weil-Deligne representations and weakly admissible (φ, N)-modules.

After these preparations we will be able to compute locally algebraic vectors of $V(r)$, in section 5. This is the heart of this thesis. We will begin by recalling a few useful facts about locally algebraic vectors in section 5.1. Then, in section 5.2, we will prove that there is a ring homomorphism $\mathcal{H}(\sigma_{min}) \longrightarrow R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$, which interpolates the usual Langlands correspondence. Moreover this ring homomorphism can be described in a very explicit manner. First we will prove the existence of this ring homomorphism via a commutative diagram in section 5.2.1. Then, in section 5.2.2, we will show that it can actually be given by a very explicit formula, in the Iwahori case. The derivation of this formula provides in fact an other proof that such a map exist and that it is unique, satisfying the required properties. In the section 5.3 we will introduce a stratification of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$ with respect to the partition valued functions, which will help us to study the support of $M_{\infty}(\sigma^{\circ}_{min})$. The goal of the section 5.4 will be to prove that the action of $\mathcal{H}(\sigma_{min})$ on $M_{\infty}(\sigma^{\circ}_{min})$ is compatible with the interpolation map constructed in section 5.2. This will be stated in more precise manner as Theorem 5.14 and the results about the support of $M_{\infty}(\sigma^{\circ}_{min})$ will be given in section 5.5. With all this in hand we will be able to prove the key result in section 5.6, the Theorem 5.22.

In the last part, section 6, we will deduce few theorems. The theorems stated here are a direct consequence of the results derived in the section 5. The theorem 6.1 is the generalization of Theorem 5.3[CEG$^+$16] to potentially semi-stable case, i.e. in this case r_x is a potentially semi-stable Galois representation which lies on an automorphic component, such that $\pi_{sm}(r_x)$

is generic. Then Theorem 6.9 allows to deduce the existence of G-invariant norm on $BS(r_x)$ for some potentially crystalline points such that r_x is not generic and lies on the automorphic component. In particular the generic case (i.e. the Galois representation is generic) also follows from Theorem 6.9. We will finish this section with an example where we can deduce the existence of G-invariant norm on $BS(r_x)$, without assuming that x lies on an automorphic component. However we are forced to make an assumption that certain set is Zariski closed. We hope to remove this assumption in future work. Moreover, in this example this G-invariant norm does not come from a restriction of a G-invariant norm on a parabolic induction of a unitary character.

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2 Hecke algebras

Recall that $G := GL_n(F)$. Throughout this section we fix a Bernstein component $\Omega = [\rho, M]_G$ and an Ω-type (J, λ).

Denote by W the vector space on which the representation λ is realized. Next, let $(\check{\lambda}, W^\vee)$ denote the contragradient of (λ, W).

Then by (2.6) [BK99], the Hecke algebra $\mathcal{H}(G, \lambda) := \mathrm{End}_G(\text{c–}\mathrm{Ind}_J^G \lambda)$ can be identified with the space of compactly supported functions $f : G \longrightarrow \mathrm{End}_E(W^\vee)$ such that $f(j_1.g.j_2) = \check{\lambda}(j_1) \circ f(g) \circ \check{\lambda}(j_2)$, with $j_1,\ j_2 \in \lambda$ and $g \in G$ and the multiplication of two elements f_1 and f_2 is given by the convolution:

$$f_1 * f_2(g) = \int_G f_1(x) \circ f_2(x^{-1}g)dx$$

For $u \in \mathrm{End}_E(W^\vee)$, we write $\check{u} \in \mathrm{End}_E(W)$ for the transpose of u with respect of the canonical pairing between W and $W^\vee$. This gives $(\check{\lambda}(j))^\vee = \lambda(j)$, for $j \in J$. For $f \in \mathcal{H}(G, \lambda)$, define $\check{f} \in \mathcal{H}(G, \check{\lambda})$, by $\check{f}(g) = f(g^{-1})^\vee$, for all $g \in G$.

Recall that K is a maximal compact open subgroup of G containing J. Let $\widehat{K}$ denote the set of all isomorphism classes of irreducible representations of K. In order to simplify the notation, the decomposition (1), from the section 1.3:

$$\mathrm{Ind}_J^K \lambda = \bigoplus_{\mathcal{P}} \sigma_{\mathcal{P}}(\lambda)^{\oplus m_{\mathcal{P},\lambda}},$$

will be written as,

$$\mathrm{Ind}_J^K \lambda = \bigoplus_{\sigma \in \widehat{K}} \sigma^{\oplus m_\sigma} \tag{3}$$

The integers m_σ are the multiplicties of σ's in $\mathrm{Ind}_J^K \lambda$. It follows that :

$$\text{c–}\mathrm{Ind}_J^G \lambda = \text{c–}\mathrm{Ind}_K^G(\mathrm{Ind}_J^K \lambda) = \bigoplus_{\sigma \in \widehat{K}} (\text{c–}\mathrm{Ind}_K^G \sigma)^{\oplus m_\sigma} \tag{4}$$

The main result is the following, if $m_\sigma = 1$ then:

$$\mathfrak{Z}_\Omega \simeq \mathrm{End}_G(\text{c–}\mathrm{Ind}_K^G \sigma)$$

this is the statement of Corollary 2.18.

This section is organised in the following manner. In the section 2.1 we will recall some facts about representations of G and prove few easy lemmas.

Next, in section 2.2, we will prove some results about Bernstein centre. Those results will allow us to study the specialization of a projective generator at maximal ideals that belong to some dense set. This will be achieved in section 2.3. Then in sections 2.4 and 2.5 we collect some technical results that will be needed in the next section. Then in section 2.6 we will prove the main result of this chapter. In the last section we will describe the specialization of a projective generator for any maximal ideal, in the Iwahori case when $n = 2$.

2.1 Classical results and commutative algebra

We will start stating a few very useful results and we will introduce more notation. Combining together theorem in section VI.4.4.(p.232) and the first lemma in section VI.10.3. (p.311) both in [Ren10], we get following theorem:

Theorem 2.1. *$\mathcal{H}(G, \lambda)$ is a free finitely generated $\mathfrak{Z}_D$-module.*

The following result is the proved in section VI.10.3. [Ren10] (p.314), just before the statement of a theorem:

Lemma 2.2.

$$\mathfrak{Z}_\Omega = \mathfrak{Z}_D^{W(D)}$$

where

$$W(D) = \left\{g \in G | g^{-1}Mg = M \text{ and } [M, \rho^g]_M = D\right\} /M$$

The following lemma is a direct consequence from [Bou85a] Chapitre 5 §1.9:

Lemma 2.3. *$\mathfrak{Z}_D$ is a $\mathfrak{Z}_\Omega$-module(algebra) of finite type.*

Write χ for algebra homomorphism $\chi : \mathfrak{Z}_\Omega \to E$. Let $\mathfrak{m} = \mathrm{Ker}(\mathfrak{Z}_\Omega \xrightarrow{\chi} E)$ a maximal ideal of $\mathfrak{Z}_\Omega$ and $\kappa(\mathfrak{m})$ the residue field which is isomorphic to E. From now on we will always identify an algebra homomorphism $\chi : \mathfrak{Z}_\Omega \to E$ and a maximal ideal $\mathfrak{m} = \mathrm{Ker}(\mathfrak{Z}_\Omega \xrightarrow{\chi} E)$ of $\mathfrak{Z}_\Omega$.

Lemma 2.4. *Let A and B be two E-algebras. Let G a finite group acting on A and H another finite group acting on B, so that $G \times H$ acts on $A \otimes_E B$. Then the invariants under action of $G \times H$ are $(A \otimes_E B)^{G \times H} = (A^G) \otimes_E (B^H)$*

Proof. It is easy to see that $(A \otimes_E B)^{G\times H} \supseteq (A^G) \otimes_E (B^H)$. Moreover $(A \otimes_E B)^{G\times H} = ((A \otimes_E B)^{G\times\{1\}})^{\{1\}\times H}$. It would be enough then to prove $(A \otimes_E B)^{G\times\{1\}} \subseteq A^G \otimes_E B$. Let $\sum_i a_i \otimes b_i$ be any element of $A \otimes_E B$ invariant by action of $G \times \{1\}$. Since B is also a E-vector space, we may assume without any loss of generality that all the b_i are linearly independent, then $\forall g \in G$:

$$(g, 1).\sum_i a_i \otimes b_i = \sum_i a_i \otimes b_i$$

this equation can be put in the form:

$$0 = \sum_i (g.a_i - a_i) \otimes b_i$$

therefore for all i, $g.a_i = a_i$, because all the b_i are linearly independent. This proves the lemma. $\square$

Lemma 2.5. *Let A be a commutative E-algebra, which is also a Jacobson ring. Let $f \in A$, $f \neq 0$ which is a not a zero divisor in A. Then the set* m-Spec$(A[\frac{1}{f}])$ *is Zariski dense in* Spec(A)*.*

Proof. The sets $D(g) := \{\mathfrak{p} \in \mathrm{Spec}(A) | g \notin \mathfrak{p}\}$, $g \in A$, from a basis of open neighbourhoods for the topology on Spec(A). The set of all maximal ideal m-Spec(A) is Zariski dense in Spec(A), because A is Jacobson. Hence m-Spec(A) intersects any open set of Spec(A) non trivially. Then m-Spec$(A[\frac{1}{f}]) = \text{m-Spec}(A) \cap D(f) \neq 0$, and $\forall g \in A$ such that $g \neq 0$:

$$\begin{aligned}\text{m-Spec}(A[\frac{1}{f}]) \cap D(g) &= \text{m-Spec}(A) \cap D(f) \cap D(g)\\ &= \text{m-Spec}(A) \cap D(fg) \neq 0\end{aligned}$$

because $fg \neq 0$ since f is a non-zero divisor in A. This proves that the set m-Spec$(A[\frac{1}{f}])$ intersects any non-trivial open subset of Spec(A). $\square$

Lemma 2.6. *Let $Z := E[X_1, \ldots, X_e]$ and $S := Z^{\mathfrak{S}_e}$, where the symmetric group $\mathfrak{S}_e$ acts by permutation of variables, i.e. $\sigma \in \mathfrak{S}_e$ acts by $\sigma.X_i = X_{\sigma(i)}$. Let $s_i := \sum_{1\leq j_1<\ldots<j_i\leq e} X_{j_1} \ldots X_{j_i}$ be the elementary symmetric polynomial, then $S \simeq E[s_1, \ldots, s_e]$. We know that Z is a free S-module of rank $e!$ with basis given by monomials $X^\nu := X_1^{\nu(1)} \ldots X_e^{\nu(e)}$, such that $0 \leq \nu(i) < i$ for $1 \leq i \leq e$. Let $\Delta = \det(\mathrm{tr}_{Z/S}(X^\mu.X^\nu))_{\mu,\nu}$ and let $d = \prod_{i<j}(X_i - X_j)^2$. Then Δ is some power of d.*

Proof. According to [Bou03] IV.§6.1 Theorem 1 c) Z is a free S-module of rank $e!$. Let's first prove that d is irreducible. Assume that $d = d_1 d_2 = \prod_{i \neq j}(X_i - X_j)$ with d_1 and d_2 both in S and have positive degree. Let $T = \{(i,j) | i \neq j\}$. Since Z is an UFD, then by uniqueness of factorization we have:

$$d_k = c_k \prod_{(i,j) \in T_k} (X_i - X_j)$$

where $k \in \{1, 2\}$ and $c_k \in E$. The subsets T_k of T are such that $T_k \neq \emptyset$, $T_1 \cup T_2 = T$ and $T_1 \cap T_2 = \emptyset$. Since $d_k \in S$, then $\forall \sigma \in \mathfrak{S}_e$ we have that $\sigma.d_k = d_k$, then

$$\prod_{(i,j) \in T_k)} (X_{\sigma(i)} - X_{\sigma(j)}) = \prod_{(i,j) \in T_k} (X_i - X_j)$$

again by uniqueness of factorization in Z, we may identify factors on both sides. In particular we have that if $(i,j) \in T_k$ then for any permutation σ we have that $(\sigma(i), \sigma(j)) \in T_k$. This implies that $T \subseteq T_k$, a contradiction.

The map $f : \operatorname{Spec} Z \to \operatorname{Spec} S$ induced by an embedding $S \hookrightarrow Z$ is étale at a point x if and only if it is unramified at x. However the zero locus of Δ, $V(\Delta)$, is equal to the set of points where the map f is ramified (i.e. is not étale), by definition of the discriminant. The map f is not étale when $X_i = X_j$ for $i \neq j$, this is the zero locus of d. Since d is irreducible in S, it follows that Δ is some power of d. $\square$

2.2 Properties of Bernstein centre

In this section we will work with $\overline{\mathbb{Q}}_p$-coefficients, i.e. $E = \overline{\mathbb{Q}}_p$. Our goal is to determine $\mathfrak{Z}_D \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$ when $\mathfrak{m}$ varies through a dense set of maximal ideals in $\operatorname{Spec} \mathfrak{Z}_\Omega$.

Let's first describe the action of $W(D)$ on $\mathfrak{Z}_D$. Let $\mathcal{X}(M)$ be the group of unramified characters on M and $\mathcal{X}(M)(\rho) = \{\chi \in \mathcal{X}(M) | \rho \simeq \rho \otimes \chi\}$. Let M° be the intersection of kernels of the characters $\chi \in \mathcal{X}(M)$ and let T be the intersection of the kernels of the $\chi \in \mathcal{X}(M)(\rho)$. The restriction to T induces a bijection $\mathcal{X}(M)/\mathcal{X}(M)(\rho) \simeq \mathcal{X}(T)$. Let $Irr(D)$ be the set of irreducible representations in D. Every such a representation is of the form $\rho \otimes \chi$ for $\chi \in \mathcal{X}(M)$. Thus we have a bijection $\mathcal{X}(M)/\mathcal{X}(M)(\rho) \simeq Irr(D)$, $\chi \mapsto \rho \otimes \chi$. Composing it with previous bijection we get a bijection $Irr(D) \simeq \mathcal{X}(T)$. Now $\mathcal{X}(T)$ is naturally isomorphic to the set of E-algebra homomorphisms from $E[T/M^\circ]$ to E. It is explained in [Ren10] section V.4.4 that we have an identification $\mathfrak{Z}_D \simeq E[T/M^\circ]$, so that the bijection $Irr(D) \simeq \mathcal{X}(T)$ induces a natural bijection between $Irr(D)$ and m-Spec$(\mathfrak{Z}_D)$. The group $W(D)$ acts on $Irr(D)$ by conjugation. For each $w \in W(D)$ let $\xi \in \mathcal{X}(M)$ be any character such that $\rho^w \simeq \rho \otimes \xi$, and let ξ_w the restriction of ξ to T. If $\chi \in \mathcal{X}(M)$ then $(\rho \otimes \chi)^w \simeq \rho \otimes \chi^w.\xi$. Thus the action of $W(D)$ on $\mathcal{X}(T)$ is given by $w.\chi = \chi^w.\xi_w$. It is immediate that the induced action on $E[T/M^\circ] \simeq \mathfrak{Z}_D$ is given by $w.(tM^\circ) = \xi_w(t)^{-1}t^w M^\circ$.

Lemma 2.7. *An E-algebras homomorphism $\mathrm{X} : \mathfrak{Z}_D \to E$ can be lifted to an unramified character $\overline{\chi}$ of M, i.e. we have a surjective map:*

$$\mathcal{X}(M) \twoheadrightarrow \mathrm{Hom}_{E-alg}(\mathfrak{Z}_D, E)$$

This map has the following description, given an unramified character $\overline{\chi}$ of M, we can associate to it a E-algebras homomorphism $\mathrm{X} : \mathfrak{Z}_D \to E$, defined as:

$$\begin{array}{rcccc} \mathrm{X} & : & \mathfrak{Z}_D & \to & E \\ & & z & \mapsto & z(\overline{\chi}) \end{array}$$

where $z(\overline{\chi})$ is a scalar by which z acts on one dimensional representation $\overline{\chi}$ of M.

Proof. By the description of the action of $W(D)$ on $\mathfrak{Z}_D$, above this lemma, we have the following isomorphisms:

$$\mathcal{X}(M)/\mathcal{X}(M)(\rho) \simeq \mathcal{X}(T) \simeq \mathrm{Hom}_{E-alg}(\mathfrak{Z}_D, E)$$

hence a surjective map $\mathcal{X}(M) \longrightarrow \mathrm{Hom}_{E-alg}(\mathfrak{Z}_D, E)$. $\square$

Let $\mathfrak{m} = \mathrm{Ker}(Z \xrightarrow{\overline{\chi}} E)$ a maximal ideal of $\mathfrak{Z}_\Omega$ and $\kappa(\mathfrak{m})$ the residue field which is isomorphic to E.

Lemma 2.8. *There is a dense set in* m-Spec $\mathfrak{Z}_\Omega$ *of maximal ideals* $\mathfrak{m} \in$ m-Spec $\mathfrak{Z}_\Omega$*, such that:*

$$\mathfrak{Z}_D \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq \prod_{k=1}^{|W(D)|} \kappa(\mathfrak{M}_i)$$

where $\mathfrak{M}_i$ are maximal ideals of $\mathfrak{Z}_D$ above $\mathfrak{m}$, and $\kappa(\mathfrak{M}_i)$ the residue fields. Moreover this dense set is the set of maximal ideals such that the local discriminant is not zero and $\mathfrak{Z}_D$ is free over $\mathfrak{Z}_\Omega$ of rank $|W(D)|$.

Proof. Let's first deal with two particular cases before dealing with general case.
1. Supercuspidal case. In this case we have $M = G$, then $\mathfrak{Z}_D \simeq \mathfrak{Z}_\Omega$. Everything is clear, there is nothing to prove.

2. Simple type case. Assume now that (J, λ) is a simple type, without loss of generality we may assume then $M = GL_k(F)^e$ and $\rho = \pi \otimes \ldots \otimes \pi$ (e times), where π is a supercuspidal representation of $GL_k(F)$.

By Theorem (6.6.2) [BK93], there is a maximal type (J_0, λ_0) of $GL_k(F)$, a field extension Γ of F and a uniquely determined representation Λ of $\Gamma^\times J_0$ such that $\Lambda|J_0 = \lambda_0$ and $\pi = \text{c–Ind}_{\Gamma^\times J_0}^{GL_k(F)} \Lambda$. From Frobenius reciprocity follows a Hecke algebras isomorphism $\mathcal{H}(M, \lambda_M) \simeq \mathcal{H}(\widetilde{J_M}, \widetilde{\lambda_M})$ because any $g \in M$ that intertwines λ_M lies in $\widetilde{J_M}$. Since $\widetilde{J_M}/J_M$ is free abelian group, $\mathcal{H}(\widetilde{J_M}, \widetilde{\lambda_M})$ is commutative, and we have an isomorphism $\mathcal{H}(\widetilde{J_M}, \widetilde{\lambda_M}) \simeq E[\widetilde{J_M}/J_M]$. Therefore we have:

$$\mathfrak{Z}_D \simeq \mathcal{H}(M, \lambda_M) \simeq \mathcal{H}(\widetilde{J_M}, \widetilde{\lambda_M}) \simeq E[\widetilde{J_M}/J_M] \simeq E[(\Gamma^\times J_0)^e/J_0^e]$$

$$\simeq E[(\Gamma^\times J_0/J_0)^e] \simeq E[(\Gamma^\times/\mathcal{O}_\Gamma^\times)^e] \simeq E[((\varpi_\Gamma)^{\mathbb{Z}})^e] \simeq E[X_1^{\pm 1}, \ldots, X_e^{\pm 1}]$$

where $\mathcal{O}_\Gamma$ is the ring of integers of Γ and ϖ_Γ an uniformizer. Since $\rho = \pi \otimes \ldots \otimes \pi$, it follows from the previous description action of $W(D)$ on $\mathfrak{Z}_D$, that we have $\xi = 1$. Then the group $W(D) \simeq \mathfrak{S}_e$ acts by permutation of variables X_i on $\mathfrak{Z}_D \simeq E[X_1^{\pm 1}, \ldots, X_e^{\pm 1}]$.

Let $Z := E[X_1, \ldots, X_e]$ and $S := Z^{\mathfrak{S}_e}$. Let $s_i := \sum\limits_{1 \leq j_1 < \ldots < j_i \leq e} X_{j_1} \ldots X_{j_i}$ be the elementary symmetric polynomial,then $S \simeq E[s_1, \ldots, s_e]$. According to [Bou03] IV.§6.1 Theorem 1 c) Z is a free S-module of rank $e!$ with basis given by monomials $X^\nu := X_1^{\nu(1)} \ldots X_e^{\nu(e)}$, such that $0 \leq \nu(i) < i$ for $1 \leq i \leq e$.

The group $W(D)$ acts by permutation on $\mathfrak{Z}_D$. It follows that $\mathfrak{Z}_\Omega = \mathfrak{Z}_D^{W(D)} \simeq E[X_1^{\pm 1}, \ldots, X_e^{\pm 1}]^{\mathfrak{S}_e} \simeq E[s_1, \ldots, s_{e-1}, s_e^{\pm 1}]$. After a localization with respect of $\{s_e^n\}_{n \geq 0}$ we see that $\mathfrak{Z}_D = Z_{s_e}$ is a free $\mathfrak{Z}_\Omega = S_{s_e}$-module of rank $|W(D)| = e!$ with basis given by monomials $X^\nu := X_1^{\nu(1)} \ldots X_e^{\nu(e)}$, such that $0 \leq \nu(i) < i$ for $1 \leq i \leq e$. Let $d = \prod_{i<j}(X_i - X_j)^2 \in \mathfrak{Z}_\Omega$. By Lemma 2.6 the discriminant is some power of d.

When the specialization $d(\mathfrak{m}) := d \otimes \kappa(\mathfrak{m})$ of d at a maximal ideal $\mathfrak{m}$ is non zero, then $\mathfrak{m}$ is of form $(s_1 - a_1, \ldots, s_e - a_e)$, where the a_1,...,a_e are such that the polynomial $f \in \kappa(\mathfrak{m})[X]$ defined by $f = X^e + \sum\limits_{k=1}^{e} (-1)^k a_k X^{e-k}$ has

e distinct roots, say α_1,..., α_e. Let $w \in W(D) \simeq \mathfrak{S}_e$, set $\mathfrak{M}_w$ the kernel of homomorphism $\mathfrak{Z}_D \longrightarrow E$ sending $X_k \mapsto \alpha_{w(k)}$. Moreover $\mathfrak{M}_w$ is a maximal ideal of $\mathfrak{Z}_D$ above $\mathfrak{m}$. We have a natural surjection :

$$\mathfrak{Z}_D \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \twoheadrightarrow \prod_{w \in W(D)} \kappa(\mathfrak{M}_w)$$

Since $\dim_{\kappa(\mathfrak{m})}(\mathfrak{Z}_D \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) = |W(D)|$, this surjection is an isomorphism of E vector spaces by comparing the dimensions. Then $\mathfrak{Z}_D \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$ is a product of $|W(D)|$ copies of E, since E is assumed to be algebraically closed.

Moreover, the set $S := \{\mathfrak{m} \in \text{m-Spec}(\mathfrak{Z}_\Omega) | d(\mathfrak{m}) \neq 0\} = \text{m-Spec}(\mathfrak{Z}_\Omega[\frac{1}{d}])$ is not empty and Zariski dense, because of the Lemma 2.5.

3. General case. Now let's treat the general case, where the type (J, λ) is semi-simple. We may always assume that $M = \prod_{i=1}^s M_i$ and $\rho = \bigotimes_{i=1}^s \rho_i$, where $M_i = GL_{n_i}(F)^{e_i}$ and ρ_i is a supercuspidal representation of M_i. Define $G_i = GL_{n_i e_i}(F)$, $\Omega_i = [M_i, \rho_i]_{G_i}$, $D_i = [M_i, \rho_i]_{M_i}$. Let $\overline{M}$ be a unique Levi subgroup of G which contains the $N_G(M)$-stabilizer of the inertia class D and is minimal for this property. The section 1.5 in [BK99] applied to $\mathcal{H}(\overline{M}, \lambda_M) \simeq \mathfrak{Z}_D$ gives:

$$\mathfrak{Z}_D \simeq \bigotimes_{i=1}^s \mathfrak{Z}_{D_i}$$

and

$$W(D) \simeq \prod_{i=1}^s W(D_i)$$

The action of $W(D)$ on $\mathfrak{Z}_D$ is such that every $W(D_i)$ acts only on $\mathfrak{Z}_{D_i}$.
An inductive application of Lemma 2.4, to the previous decomposition of $\mathfrak{Z}_D$ gives:

$$\mathfrak{Z}_\Omega \simeq \bigotimes_{i=1}^s \mathfrak{Z}_{\Omega_i}$$

By previous case we have the following non canonical isomorphisms :

$$\mathfrak{Z}_{D_i} \simeq E[X_{1,i}^{\pm 1}, \ldots, X_{e_i,i}^{\pm 1}]$$

$$\mathfrak{Z}_{\Omega_i} \simeq E[X_{1,i}^{\pm 1}, \ldots, X_{e_i,i}^{\pm 1}]^{\mathfrak{S}_{e_i}}$$

We have $W(D_i) \simeq \mathfrak{S}_{e_i}$ and we may assume that $\mathfrak{S}_{e_i}$ acts on $E[X_{1,i}^{\pm 1}, \ldots, X_{e_i,i}^{\pm 1}]$ by permutation, since it is always the case after an appropriate linear change of variables. Moreover $\mathfrak{Z}_{D_i}$ is free $\mathfrak{Z}_{\Omega_i}$-module of rank $e_i!$, and let $d_{e_i} = \prod_{k<l}(X_{k,i} - X_{l,i})^2$.

Then $\mathfrak{Z}_D$ is free $\mathfrak{Z}_\Omega$ module of rank $|W(D)|$ and define $d = \prod_{i=1}^{s} d_{e_i}$, we call d the discriminant. The proof of general case ends exactly in the same way as in the simple type case and the set $S' :=$ m-Spec$(\mathfrak{Z}_\Omega[\frac{1}{d}])$ is not empty and Zariski dense, because of the Lemma 2.5. $\square$

Lemma 2.9. $\mathcal{H}(G,\lambda)$ *is free and finitely generated over* $\mathfrak{Z}_\Omega$.

Proof. It follows from the proof of Lemma 2.8, that $\mathfrak{Z}_D$ is free $\mathfrak{Z}_\Omega$ module of rank $|W(D)|$. Finally the Theorem 2.1 gives the desired result. $\square$

Remark. The lemma above is essentially the same as Lemma 2.1 in [Dat99a].

2.3 Specialization of a projective generator at maximal ideal of Bernstein centre

In this section we compute c–Ind$_J^G\lambda \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$ in terms of parabolic induction for $\mathfrak{m} \in$ Spec $\mathfrak{Z}_\Omega$ a maximal ideal which belongs to some dense set of points in Spec $\mathfrak{Z}_\Omega$. This result is an improvement of Lemma 1.2 in [Dat99a]. The representation c–Ind$_J^G\lambda$ is a projective generator of $\mathcal{R}^\Omega(G)$. In this section we will work with $\overline{\mathbb{Q}}_p$-coefficients, i.e. $E = \overline{\mathbb{Q}}_p$.

Let $\overline{\chi}$ be any lift of X as in Lemma 2.7. Let now $\chi = \mathrm{X}|\mathfrak{Z}_\Omega$. We say that a character χ on $\mathfrak{Z}_\Omega$ is induced from unramified character $\overline{\chi}$ of M.

Once and for all we fix the following notation. Let (J,λ) a type for Ω. There exists a D-type (J_M, λ_M), such that :

1. $J_M = J \cap M$ and $\lambda_M = \lambda|J_M$.

2. J has an Iwahori decomposition $J \simeq (J\cap\overline{N})(J\cap M)(J\cap N)$ such that $\lambda|(J\cap\overline{N})$ and $\lambda|(J\cap N)$ are trivial. Here $\overline{N}$ is unipotent radical of opposite parabolic subgroup $\overline{P}$.

3. For any parabolic subgroup P with Levi component M, there is an element z_P in centre of M contracting strictly N by conjugation such that there is an invertible element in $\mathcal{H}(G,\lambda)$ supported in Jz_PJ.

4. There is a subgroup $\widetilde{J_M}$ of M compact modulo centre of M such that $J_M = \widetilde{J_M} \cap K \cap M$.

5. There is an extension $\widetilde{\lambda_M}$ of λ_M to $\widetilde{J_M}$ such that $\lambda_M = \widetilde{\lambda_M}|J_M$. Moreover $\rho =$ c–Ind$_{\widetilde{J_M}}^M\widetilde{\lambda_M}$ (is irreducible supercupidal) and any $g \in M$ which intertwines λ_M lies in $\widetilde{J_M}$.

The theorem on existence of G-covers((8.3) [BK98]) ensures the conditions 1, 2, 3. The conditions 4 and 5 follow from (5.5)[BK98]. Now we state and prove the main result of this section:

Proposition 2.10. *Let $\chi : \mathfrak{Z}_\Omega \to E$ be an algebra homomorphism corresponding to maximal ideal $\mathfrak{m} = \mathrm{Ker}(\mathfrak{Z}_\Omega \xrightarrow{\chi} E)$ of $\mathfrak{Z}_\Omega$. Then there is a Zariski dense set S in $\mathrm{Spec}(\mathfrak{Z}_\Omega)$ such that:*

$$\mathrm{c\text{--}Ind}_J^G \lambda \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq P(\chi)^{\oplus |W(D)|}$$

$\forall \mathfrak{m} \in S$, where $P(\chi) := i_{\overline{P}}^G(\rho \otimes \overline{\chi})$ is an irreducible parabolic induction of a supercuspidal representation of a Levi subgroup of G and $\overline{\chi}$ some character corresponding to algebra homomorphism $\mathrm{X} : \mathfrak{Z}_D \to E$ such that $\mathfrak{M} = \mathrm{Ker}(\mathrm{X})$ is a maximal ideal of $\mathfrak{Z}_D$ above $\mathfrak{m}$.

Proof. The following argument, that gives an isomorphism between $i_{\overline{P}}^G(\mathrm{c\text{--}Ind}_{J_M}^M \lambda_M)$ and $\mathrm{c\text{--}Ind}_J^G \lambda$, was taken from 1.5 [Dat99b]. We have following commutative diagram:

$$\begin{tikzcd}
\mathcal{R}^\Omega(G) \arrow[r, "\mathfrak{M}_\lambda"] & \mathcal{H}(G,\lambda) - Mod \\
\mathcal{R}^D(M) \arrow[u, "i_{\overline{P}}^G"] \arrow[r, "\mathfrak{M}_{\lambda_M}"] & \mathcal{H}(M,\lambda_M) - Mod, \arrow[u, "-\otimes_{\mathcal{H}(M,\lambda_M)}\mathcal{H}(G,\lambda)"']
\end{tikzcd}$$

where the horizontal arrows are isomorphisms. It follows from this diagram that:

$$\mathfrak{M}_\lambda(i_{\overline{P}}^G(\mathrm{c\text{--}Ind}_{J_M}^M \lambda_M)) = \mathfrak{M}_{\lambda_M}(\mathrm{c\text{--}Ind}_{J_M}^M \lambda_M) \otimes_{\mathcal{H}(M,\lambda_M)} \mathcal{H}(G,\lambda)$$

$$\simeq \mathcal{H}(M,\lambda_M) \otimes_{\mathcal{H}(M,\lambda_M)} \mathcal{H}(G,\lambda) \simeq \mathcal{H}(G,\lambda) \simeq \mathfrak{M}_\lambda(\mathrm{c\text{--}Ind}_J^G \lambda)$$

Hence

$$i_{\overline{P}}^G(\mathrm{c\text{--}Ind}_{J_M}^M \lambda_M) \simeq \mathrm{c\text{--}Ind}_J^G \lambda \tag{5}$$

because the functor $\mathfrak{M}_\lambda$ is an equivalence of categories.
The functor $i_{\overline{P}}^G$ is exact, hence:

$$\mathrm{c\text{--}Ind}_J^G \lambda \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq i_{\overline{P}}^G(\mathrm{c\text{--}Ind}_{J_M}^M \lambda_M \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$

by previous isomorphism of representations. Let's find a decomposition of $\mathrm{c\text{--}Ind}_{J_M}^M \lambda_M \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$. Indeed

$$\mathrm{c\text{--}Ind}_{J_M}^M \lambda_M = \mathrm{c\text{--}Ind}_{\widetilde{J_M}}^M \mathrm{c\text{--}Ind}_{J_M}^{\widetilde{J_M}} \lambda_M = \mathrm{c\text{--}Ind}_{\widetilde{J_M}}^M (\widetilde{\lambda_M} \otimes \mathrm{c\text{--}Ind}_{J_M}^{\widetilde{J_M}} 1)$$

Since $\text{c–Ind}_{J_M}^{M}\lambda_M \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq \text{c–Ind}_{J_M}^{M}\lambda_M \otimes_{\mathfrak{Z}_D} (\mathfrak{Z}_D \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$, it is enough to find the decomposition of $\text{c–Ind}_{J_M}^{M}\lambda_M \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_j)$ $(1 \leq j \leq |W(D)|)$, because of Lemma 2.8.

The functor $\text{c–Ind}_{\widetilde{J_M}}^{M}$ is exact, therefore using projection formula:

$$\text{c–Ind}_{J_M}^{M}\lambda_M \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_i) = \text{c–Ind}_{\widetilde{J_M}}^{M}(\widetilde{\lambda_M} \otimes (\text{c–Ind}_{J_M}^{\widetilde{J_M}}1) \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_j))$$

Let's express $\text{c–Ind}_{J_M}^{\widetilde{J_M}}1 \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_j)$ in terms of more suitable data. Let's drop the index j temporarily and write $\mathfrak{M} := \mathfrak{M}_j$. From Frobenius reciprocity follows a Hecke algebras isomorphism $\mathcal{H}(M, \lambda_M) \simeq \mathcal{H}(\widetilde{J_M}, \widetilde{\lambda_M})$ because any $g \in M$ that intertwines λ_M lies in $\widetilde{J_M}$. Since $\widetilde{J_M}/J_M$ is free abelian group, $\mathcal{H}(\widetilde{J_M}, \widetilde{\lambda_M})$ is commutative, and we have an isomorphism $\mathcal{H}(\widetilde{J_M}, \widetilde{\lambda_M}) \simeq E[\widetilde{J_M}/J_M]$. Therefore we have:

$$\mathfrak{Z}_D \simeq \mathcal{H}(M, \lambda_M) \simeq \mathcal{H}(\widetilde{J_M}, \widetilde{\lambda_M}) \simeq E[\widetilde{J_M}/J_M] \simeq \text{End}_{\widetilde{J_M}}(\text{c–Ind}_{J_M}^{\widetilde{J_M}}1)$$

The representation $\text{c–Ind}_{J_M}^{\widetilde{J_M}}1$ is naturally isomorphic to the space of functions on $\widetilde{J_M}$ which are left invariant by J_M. We have the following canonical isomorphism $\text{c–Ind}_{J_M}^{\widetilde{J_M}}1 \simeq E[\widetilde{J_M}/J_M]$. This shows that $\text{c–Ind}_{J_M}^{\widetilde{J_M}}1$ is free $\mathfrak{Z}_D$-module of rank 1.

The $\mathfrak{M}$ an maximal ideal of $\mathfrak{Z}_D$ is above some maximal ideal $\mathfrak{m} \in \operatorname{Spec} \mathfrak{Z}_\Omega$. We may always assume that $M = \prod_{i=1}^{s} M_i$ and $\rho = \bigotimes_{i=1}^{s} \rho_i$, where $M_i = GL_{n_i}(F)^{e_i}$ and $\rho_i \simeq \pi_i \otimes \ldots \otimes \pi_i$ (e_i times) is a supercuspidal representation of M_i and π_i is a supercuspidal representation of $GL_{n_i}(F)$. Define $G_i = GL_{n_i e_i}(F)$, $\Omega_i = [M_i, \rho_i]_{G_i}$, $D_i = [M_i, \rho_i]_{M_i}$. Then:

$$\mathfrak{Z}_D \simeq E[X_{1,1}^{\pm 1}, \ldots, X_{e_1,1}^{\pm 1}, \ldots, X_{1,s}^{\pm 1}, \ldots, X_{e_s,s}^{\pm 1}]$$

Let

$$\mathfrak{m} = (s_{1,i} - a_{1,i}, \ldots, s_{e_i,i} - a_{e_i,i})_{1 \leq i \leq s}$$

where $s_{k,i}$ are elementary symmetric functions in variables $X_{1,i}, \ldots, X_{e_i,i}$ and $a_{k,i} \in \overline{\mathbb{Q}}_p$. Then

$$\mathfrak{M} = (X_{1,i} - \alpha_{1,i}, \ldots, X_{e_i,i} - \alpha_{e_i,i})_{1 \leq i \leq s}$$

where for each i, $\alpha_{1,i}, \ldots, \alpha_{e_i,i}$ are the e_i distinct roots of polynomial $X^{e_i} + \sum_{k=1}^{e_i} (-1)^k a_k X^{e_i - k}$. We assumed that the extension E is big enough, so without loss of generality we may assume that all those roots lie in E. Let $\overline{\chi} := \overline{\chi}_j$, the unramified character which corresponds to $\mathfrak{M}$. Then $\overline{\chi} = \bigotimes_{i=1}^{s} \overline{\psi}_i$, where $\overline{\psi}_i$ are unramified characters of $M_i = GL_{n_i}(F)^{e_i}$, such that $\overline{\psi}_i = \bigotimes_{k=1}^{e_i} \overline{\psi}_{k,i}$ and if ϖ denotes the uniformizer of F and I the identity matrix of $GL_{n_i}(F)$, $\overline{\psi}_{k,i}(\varpi.I) = \alpha_{k,i}$.

Since c–Ind$_{J_M}^{\widetilde{J_M}} 1$ is a free $\mathfrak{Z}_D$-module of rank one, we have an isomorphism of $\mathfrak{Z}_D$-modules:

$$\text{c–Ind}_{J_M}^{\widetilde{J_M}} 1 \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_j) \simeq \mathfrak{Z}_D/\text{Ker}(\text{X}_j) \simeq \text{Im}(\text{X}_j) \tag{6}$$

where $\mathfrak{M}_i = \text{Ker}(\text{X}_j)$ and the algebra homomorphism $\text{X}_j : \mathfrak{Z}_D \to E$ is such that the unramified character $\overline{\chi}_j$ of M maps to X_j as in Lemma 2.7. It follows from previous description of the maximal ideal $\mathfrak{M} := \mathfrak{M}_j$ and the character $\overline{\chi}_j$ that:

$$\text{Im}(\text{X}_j) = \text{Im}(\overline{\chi}_j)$$

Then from (6) follows that the representation c–Ind$_{J_M}^{\widetilde{J_M}} 1 \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_j)$ is one dimensional and also we have an isomorphism of $\widetilde{J_M}$-representations:

$$\text{c–Ind}_{J_M}^{\widetilde{J_M}} 1 \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_j) \simeq \overline{\chi}_j|\widetilde{J_M}$$

Now using projection formula and previous isomorphism we may write:

$$\text{c–Ind}_{\widetilde{J_M}}^{M}(\widetilde{\lambda_M} \otimes (\text{c–Ind}_{J_M}^{\widetilde{J_M}} 1) \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_i)) \simeq \text{c–Ind}_{\widetilde{J_M}}^{M}(\widetilde{\lambda_M} \otimes \overline{\chi}_i|\widetilde{J_M}) \simeq \rho \otimes \overline{\chi}_j$$

because $\rho = \text{c–Ind}_{\widetilde{J_M}}^{M} \widetilde{\lambda_M}$. So that we have

$$\text{c–Ind}_{J_M}^{M} \lambda_M \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_i) \simeq \rho \otimes \overline{\chi}_i \tag{7}$$

Using (5) and (7) we get:

$$\text{c–Ind}_{J}^{G} \lambda \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq i_P^G(\text{c–Ind}_{J_M}^{M} \lambda_M \otimes_{\mathfrak{Z}_D} (\mathfrak{Z}_D \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})))$$

$$\simeq i_P^G\Big(\bigoplus_{j=1}^{|W(D)|} \text{c–Ind}_{J_M}^{M} \lambda_M \otimes_{\mathfrak{Z}_D} \kappa(\mathfrak{M}_j)\Big) \simeq \bigoplus_{j=1}^{|W(D)|} i_P^G(\rho \otimes \overline{\chi}_j)$$

where the maximal ideal $\mathfrak{m}$ belongs to open dense set S', defined by $S' :=$ m-Spec$(\mathfrak{Z}_\Omega[\frac{1}{d}])$ as in Lemma 2.8.

Let's now prove that all the $i_P^G(\rho\otimes\overline{\chi}_j)$ are irreducible on the subset $S :=$ m-Spec$(\mathfrak{Z}_\Omega[\frac{1}{d\Delta}])$ of S', with $\Delta := \prod_{(k',i')\neq(k,i)}(X_{k',i'} - qX_{k,i})(X_{k,i} - qX_{k',i'})$, for all $1 \le k, k' \le e_i$ and $1 \le i, i' \le s$, and q is the cardinality of the residue field of F. Again by the Lemma 2.5 the set S is dense. Let $\mathfrak{M}$ a maximal ideal of $\mathfrak{Z}_D$ above $\mathfrak{m} \in S$ corresponding to $\overline{\chi}_j$. With the same notation as above, we have then:

$$\rho\otimes\overline{\chi}_j = \bigotimes_{i=1}^{s}\bigotimes_{k=1}^{e_i}(\pi_i\otimes\overline{\psi}_{k,i})$$

By definition of representations π_i, there is no integer m such that $\pi_i \simeq \pi_j \otimes |\det|^m$(for any $i \neq j$) since all the $\alpha_{k,i}$ are distinct (for a fixed i) and $\alpha_{k,i}\alpha_{k',i'}^{-1} \neq q^{\pm1}$(for any couples $(k', i') \neq (k, i)$). Then the segments $\Delta_{k,i} = \pi_i \otimes \overline{\psi}_{k,i}$ are not linked pairwise for any k and i. Then it follows by Bernstein-Zelevisky classification [Zel80], that $i_P^G(\rho\otimes\overline{\chi})$ is irreducible.

We have just proved that if $\overline{\chi}$ is the unramified character of which corresponds to a maximal ideal $\mathfrak{M}$ of $\mathfrak{Z}_D$ above $\mathfrak{m} \in S$, then $i_P^G(\rho\otimes\overline{\chi})$ is irreducible. By construction all the maximal ideals $\mathfrak{M}_i$ (which all lie above $\mathfrak{m} \in S$) are pairwise conjugated by some element $w \in W(D)$, so are the characters $\overline{\chi}_i$. Then for $\mathfrak{m} \in S$ all $i_P^G(\rho\otimes\overline{\chi}_i)$ are irreducible.

Let $\mathfrak{m} \in S$, it follows from Frobenius reciprocity that $\mathrm{Hom}_G(i_P^G(\rho \otimes \overline{\chi}_i), i_P^G(\rho\otimes\overline{\chi}_j)) \neq 0$, for all $1 \le i \le |W(D)|$ and $1 \le j \le |W(D)|$, because there is a $w_{i,j} \in W(D)$ such that $\overline{\chi}_i = \overline{\chi}_j^{w_{i,j}}$. Then for all $1 \le i \le |W(D)|$, $1 \le j \le |W(D)|$, $i_P^G(\rho\otimes\overline{\chi}_i) \simeq i_P^G(\rho\otimes\overline{\chi}_j)$, because all these representations are irreducible on S. Write $P(\chi) := i_P^G(\rho\otimes\overline{\chi}_i)$, for some integer i.

Then on open dense set S we get :

$$\text{c–Ind}_J^G\lambda\otimes_{\mathfrak{Z}_\Omega}\kappa(\mathfrak{m}) \simeq P(\chi)^{\oplus|W(D)|}$$

□

2.4 Intertwining of representations

In this section we collect some useful lemmas, and we continue to assume that $E = \overline{\mathbb{Q}}_p$.

Lemma 2.11. *With the notations of* ***Proposition 2.10****, we have:*

$$\mathrm{Hom}_G(\text{c–Ind}_K^G\sigma, P(\chi)) = \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma\otimes_{\mathfrak{Z}_\Omega}\kappa(\mathfrak{m}), P(\chi))$$

Proof. Let $\psi \in \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma, P(\chi))$, $\mathfrak{m} = \mathrm{Ker}(\mathfrak{Z}_\Omega \xrightarrow{\chi} E)$ a maximal ideal of $\mathfrak{Z}_\Omega$ that kills $P(\chi)$ (by definition) and $\phi \in \mathfrak{m}$. Since $\mathfrak{Z}_\Omega$ acts on both $\text{c–Ind}_K^G\sigma$ and $P(\chi)$, the multiplication by ϕ induces endomorphisms on both $\text{c–Ind}_K^G\sigma$ and $P(\chi)$, denoted ϕ. We have the following commutative diagram:

$$\begin{tikzcd}
\text{c–Ind}_K^G\sigma \arrow[r, "\phi"] \arrow[d, "\psi"] & \text{c–Ind}_K^G\sigma \arrow[d, "\psi"] \\
P(\chi) \arrow[r, "\phi=0"] & P(\chi)
\end{tikzcd}$$

the bottom arrow is 0 because $\mathfrak{m}$ kills $P(\chi)$. Hence one has $\psi \circ \phi = 0$, then $\phi(\text{c–Ind}_K^G\sigma) \subset \mathrm{Ker}\psi$. This inclusion holds for all $\phi \in \mathfrak{m}$ therefore $\mathfrak{m}(\text{c–Ind}_K^G\sigma) \subset \mathrm{Ker}\psi$ and ψ factors through $\mathfrak{m}(\text{c–Ind}_K^G\sigma)$. The factorization is valid for all $\psi \in \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma, P(\chi))$ hence:

$$\mathrm{Hom}_G(\text{c–Ind}_K^G\sigma, P(\chi)) = \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma/\mathfrak{m}(\text{c–Ind}_K^G\sigma), P(\chi))$$

$$= \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), P(\chi))$$

□

Notation : $I_\sigma := \text{c–Ind}_K^G\sigma$, $A(\sigma_1, \sigma_2) := \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma_1, \text{c–Ind}_K^G\sigma_2)$ and $A_\sigma := A(\sigma, \sigma)$.

Lemma 2.12. *We have the following isomorphisms:*

$$\mathrm{Hom}_G(\text{c–Ind}_K^G\sigma_1, \text{c–Ind}_K^G\sigma_2) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$$

$$\simeq \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma_1 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), \text{c–Ind}_K^G\sigma_2 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$
$$\simeq \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma_1, \text{c–Ind}_K^G\sigma_2 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$

Proof. Let $\mathfrak{m} = \mathrm{Ker}(\mathfrak{Z}_\Omega \xrightarrow{\chi} E)$ a maximal ideal of $\mathfrak{Z}_\Omega$. We have an exact sequence:

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{Z}_\Omega \longrightarrow \kappa(\mathfrak{m}) \longrightarrow 0$$

Via the natural action of $\mathfrak{Z}_\Omega$ on I_{σ_2}, we may view $\mathfrak{M}_\lambda(I_{\sigma_2})$ as $\mathfrak{Z}_\Omega$-module. We know that I_{σ_2} is a direct summand of $\text{c–Ind}_J^G\lambda$, then $\mathfrak{M}_\lambda(I_{\sigma_2})$ is a direct summand of $\mathcal{H}(G, \lambda)$. By Theorem 2.1, $\mathcal{H}(G, \lambda)$ is free over $\mathfrak{Z}_D$ and $\mathfrak{Z}_D$ is free over $\mathfrak{Z}_\Omega$ by Lemma 2.8, so $\mathcal{H}(G, \lambda)$ is free over $\mathfrak{Z}_\Omega$. Now $\mathfrak{M}_\lambda(I_{\sigma_2})$ is a direct summand of a free $\mathfrak{Z}_\Omega$-module, hence is also flat over $\mathfrak{Z}_\Omega$. Then tensoring previous exact sequence with $\mathfrak{M}_\lambda(I_{\sigma_2})$, we get an exact sequence:

$$0 \longrightarrow \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \mathfrak{m} \longrightarrow \mathfrak{M}_\lambda(I_{\sigma_2}) \longrightarrow \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \longrightarrow 0$$

applying $\mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}), .)$ (here Hom denotes the homomorphisms in the category of $\mathcal{H}(G,\lambda)$-modules) to previous exact sequence, we get an exact sequence:

$$0 \to \mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}), \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \mathfrak{m}) \to A(\sigma_1,\sigma_2) \to \mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}), \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) \to 0$$

because $\mathfrak{M}_\lambda(I_{\sigma_1})$ is projective, since it is a direct summand of free $\mathfrak{Z}_\Omega$-module by the same argument as above. Since $\mathfrak{M}_\lambda$ is an equivalence of categories we have that $\mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}), \mathfrak{M}_\lambda(I_{\sigma_2})) = A(\sigma_1,\sigma_2)$. Since $\mathfrak{M}_\lambda(I_{\sigma_2})$ is flat, $\mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \mathfrak{m} = \mathfrak{m}.\mathfrak{M}_\lambda(I_{\sigma_2})$. Then

$$\mathrm{Hom}(\mathfrak{M}_\lambda(I_\sigma), \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \mathfrak{m}) = \mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}), \mathfrak{m}.\mathfrak{M}_\lambda(I_{\sigma_2}))$$

$$= \mathfrak{m}.\mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}), \mathfrak{M}_\lambda(I_{\sigma_2})) = \mathfrak{m}.A(\sigma_1,\sigma_2)$$

The same argument as in Lemma 2.11 shows that:

$$\mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}), \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) = \mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$

Since the functor $\mathfrak{M}_\lambda$ is exact it commutes with the tensor product, then it follows that:

$$\mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) =$$

$$= \mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})), \mathfrak{M}_\lambda(I_{\sigma_2} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})))$$
$$= \mathrm{Hom}_G(I_{\sigma_1} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), I_{\sigma_2} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$

Then

$$A(\sigma_1,\sigma_2) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq A(\sigma_1,\sigma_2)/\mathfrak{m}(A(\sigma_1,\sigma_2))$$
$$\simeq A(\sigma_1,\sigma_2)/\mathrm{Hom}(\mathfrak{M}_\lambda(I_\sigma), \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \mathfrak{m})$$
$$\simeq \mathrm{Hom}(\mathfrak{M}_\lambda(I_{\sigma_1}), \mathfrak{M}_\lambda(I_{\sigma_2}) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$
$$\simeq \mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_1 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), \text{c–Ind}_K^G \sigma_2 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$

□

2.5 Computation of multiplicities

As in section 2.3, we assume that E is algebraically closed. Recall that, $m_\sigma := \dim_E \mathrm{Hom}_K(\text{c–Ind}_J^K \lambda, \sigma)$. Now we can deduce the following result from Proposition 2.10:

Corollary 2.13. *Let $\sigma \in \widehat{K}$ and $\chi : \mathfrak{Z}_\Omega \to E$ a algebra homomorphism corresponding to maximal ideal $\mathfrak{m} = \mathrm{Ker}(\mathfrak{Z}_\Omega \xrightarrow{\chi} E)$ of $\mathfrak{Z}_\Omega$. Then there is an integer n_σ and a Zariski dense set S in $\mathrm{Spec}(\mathfrak{Z}_\Omega)$ such that:*

$$\text{c–Ind}_K^G \sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq P(\chi)^{\oplus n_\sigma}$$

$\forall \mathfrak{m} \in S$, where $P(\chi) := i_{\overline{P}}^G(\rho \otimes \overline{\chi})$ an irreducible parabolic induction of a supercuspidal representation of a Levi subgroup of G and $\overline{\chi}$ some character corresponding to algebra homomorphism $\mathrm{X} : \mathfrak{Z}_D \to E$ such that $\mathfrak{M} = \mathrm{Ker}(\mathrm{X})$ is a maximal ideal of $\mathfrak{Z}_D$ above $\mathfrak{m}$.

Moreover we have the following relations of multiplicities :

$$\sum_{\sigma \in \widehat{K}} m_\sigma n_\sigma = |W(D)|$$

$$\sum_{\sigma \in \widehat{K}} m_\sigma^2 = |W(D)|$$

Proof. It follows from decomposition (3) and from Proposition 2.10 that:

$$\bigoplus_{\sigma \in \widehat{K}} (\text{c–Ind}_K^G \sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))^{\oplus m_\sigma} \simeq P(\chi)^{\oplus |W(D)|}$$

Then we also have

$$\bigoplus_{\sigma \in \widehat{K}} (\mathfrak{M}_\lambda(\text{c–Ind}_K^G \sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})))^{\oplus m_\sigma} \simeq \mathfrak{M}_\lambda(P(\chi))^{\oplus |W(D)|}$$

Observe that by Proposition 2.10 the representation $P(\chi)$ is irreducible, in particular is indecomposable. The same observation holds in the category of $\mathcal{H}(G,\lambda)$-modules for $\mathfrak{M}_\lambda(P(\chi))$. Moreover the $\mathcal{H}(G,\lambda)$-module $\mathfrak{M}_\lambda(\text{c–Ind}_K^G \sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$ is of finite length hence by §2, n°5, Theorem 2. a) [Bou12] it can be written as a direct sum of indecomposable modules $I_k(\sigma)$:

$$\mathfrak{M}_\lambda(\text{c–Ind}_K^G \sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) = \bigoplus_k I_k(\sigma)$$

Then, again, by theorem of by Krull-Remak-Schmidt Theorem(§2, n°5, Theorem 2. b) [Bou12]), the decomposition:

$$\bigoplus_{\sigma\in\widehat{K}}\bigoplus_{k} I_k(\sigma)^{\oplus m_\sigma} \simeq \mathfrak{M}_\lambda(P(\chi))^{\oplus|W(D)|},$$

into indecomposable sub-modules is unique up to permutation of factors. This theorem is applicable because all the modules in the direct sum are of finite length. It follows that by the uniqueness of such a decomposition there exists an integer $n_{k,\sigma}$, such that:

$$I_k(\sigma) \simeq \mathfrak{M}_\lambda(P(\chi))^{\oplus n_{k,\sigma}}$$

Then there exists an integer $n_\sigma := \sum_k n_{k,\sigma}$ (that may depend on χ as well) such that:

$$\mathfrak{M}_\lambda(\text{c–Ind}_K^G\sigma \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m})) \simeq \mathfrak{M}_\lambda(P(\chi))^{\oplus n_\sigma}$$

So the same holds for representations:

$$\text{c–Ind}_K^G\sigma \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m}) \simeq P(\chi)^{\oplus n_\sigma}$$

Then by definition of n_σ we have:

$$\sum_{\sigma\in\widehat{K}} m_\sigma n_\sigma = |W(D)|$$

Let's compute $\dim_{\overline{\mathbb{Q}}_p}\mathrm{Hom}_K(\mathrm{Ind}_J^K\lambda, \mathrm{Ind}_J^K\lambda)$ in two different ways. By restriction induction formula we have

$$\mathrm{Res}_J^K\mathrm{Ind}_J^K\lambda = \bigoplus_{\overline{g}\in J\backslash K/J}\mathrm{Ind}_{K\cap J^g}^K\mathrm{Res}_{K\cap J^g}^{J^g}\lambda^g$$

Then combining it with Frobenius reciprocity we get:

$$\mathrm{Hom}_K(\mathrm{Ind}_J^K\lambda, \mathrm{Ind}_J^K\lambda) = \bigoplus_{\overline{g}\in J\backslash K/J}\mathrm{Hom}_{J\cap J^g}(\lambda, \lambda^g)$$

By definition the space $\mathrm{Hom}_{J\cap J^g}(\lambda, \lambda^g)$ is the intertwining space. Assume first that (J, λ) is a simple type. In the course of this proof we will use the same notation from the book [BK93]. Let $\Gamma = F[\beta]/F$ an extension of F, which is denoted E in the chapter 5 of that book. Then according to [BK93] (5.5.11) $g \in G$ intertwines λ if and only if $g \in J\tilde{W}(\mathfrak{B})J$. So $g \in K$ intertwines λ if and only if $g \in J\tilde{W}(\mathfrak{B}) \cap KJ = JW_0(\mathfrak{B})J$ and $|W_0(\mathfrak{B})| = e(\mathfrak{B}|\mathfrak{o}_\Gamma)!$ by construction. In simple type case we have then $\dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, \mathrm{Ind}_J^K \lambda) = e(\mathfrak{B}|\mathfrak{o}_\Gamma)! = |W(D)|$.

In general case we have to deal with semi-simple types. The reference is [BK99]. Let $\overline{M}$ be a unique Levi subgroup of G which contains the $N_G(M)$-stabilizer of the inertia class D and is minimal for this property. The Levi subgroup $\overline{M}$ is the G-stabilizer of a decomposition $V = \bigoplus_{i=1}^s W_i$ of $V \simeq F^n$ as a sum of non-zero subspaces W_i. Set $G_i = \mathrm{Aut}_F(W_i)$. We then have $M = \prod_{i=1}^s M_i$ and $K \cap M = \prod_{i=1}^s K_i$, where $M_i = M \cap G_i = GL_{n_i}(F)^{e_i}$ and $K_i = K \cap G_i$. The type (J_M, λ_M) decomposes as a tensor product of types (J_{M_i}, λ_{M_i}), each of which admits a G_i-cover $(J_{\overline{M}_i}, \lambda_{\overline{M}_i})$ as in [BK99] section 1.4. We put $J_{\overline{M}} = \prod_{i=1}^s J_{\overline{M}_i}$ and $\lambda_{\overline{M}} = \bigotimes_{i=1}^s \lambda_{\overline{M}_i}$. The main theorem asserts that (J, λ) is a G-cover of $(J_{\overline{M}}, \lambda_{\overline{M}})$.

It follows from corollary 1.6 in [BK99], that $g \in G$ intertwines λ if and only if it is of the form $g = j_1 m j_2$, where j_1 and j_2 are in J and $m \in \overline{M}$, which intertwines $\lambda_{\overline{M}}$. The element m can be written as $m = m_1 \otimes \ldots \otimes m_s$, where $m_i \in M_i$ intertwine $\lambda_{\overline{M}_i}$. Then according to [BK93] (5.5.11) $m_i \in M_i$ intertwine $\lambda_{\overline{M}_i}$ if and only if $m \in J_{\overline{M}_i}\tilde{W}(\mathfrak{B}_i)J_{\overline{M}_i}$ (with analogous notations to 5.5[BK93]). This shows that $m \in \overline{M}$ intertwine $\lambda_{\overline{M}}$ if and only if $m \in J_{\overline{M}}(\prod_{i=1}^s \tilde{W}(\mathfrak{B}_i))J_{\overline{M}}$.

The decomposition

$$\mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, \mathrm{Ind}_J^K \lambda) = \bigoplus_{\overline{g} \in J\backslash K/J} \mathrm{Hom}_{J \cap J^g}(\lambda, \lambda^g)$$

shows that

$$\dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, \mathrm{Ind}_J^K \lambda) = |J \setminus \{g \in K | g \text{ intertwines } \lambda\} / J|$$

$$= |J \backslash \{g \in K | g = j_1 m j_2, j_1 \text{ and } j_2 \text{ are in } J \text{ and } m \in \overline{M} \text{ intertwines } \lambda_{\overline{M}}\} / J|$$

$$= |J \setminus \left\{ g \in K | g = j_1 m j_2; j_1, j_2 \in J \text{ and } m \in J_{\overline{M}}(\prod_{i=1}^s \tilde{W}(\mathfrak{B}_i))J_{\overline{M}} \right\} / J|$$

$$= |J \setminus K \cap (J J_{\overline{M}}(\prod_{i=1}^s \tilde{W}(\mathfrak{B}_i))J_{\overline{M}} J)/J|$$

$$= |J \setminus (J J_{\overline{M}}(\prod_{i=1}^s K_i \cap \tilde{W}(\mathfrak{B}_i))J_{\overline{M}} J)/J|$$

$$= |\prod_{i=1}^{s} W_0(\mathfrak{B}_i)| = \prod_{i=1}^{s} |W(D_i)| = |W(D)|$$

Hence in every case

$$\dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, \mathrm{Ind}_J^K \lambda) = |W(D)|$$

We have the following decomposition $\mathrm{Ind}_J^K \lambda = \bigoplus_{\sigma \in \widehat{K}} \sigma^{\oplus m_\sigma}$, by definition of multiplicities. Then

$$\dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, \mathrm{Ind}_J^K \lambda) = \sum_{\sigma \in \widehat{K}} m_\sigma^2$$

□

The rest of this section will be focused on proving that $m_\sigma = n_\sigma$. Let $\mathcal{R}_\lambda(K)$ denote the category of smooth K-representations generated by λ-isotypic subspace.

Lemma 2.14. *The category $\mathcal{R}_\lambda(K)$ is abelian.*

Proof. The category $\mathcal{R}_\lambda(K)$, is a subcategory of a semi-simple category of all K-representations. To prove that $\mathcal{R}_\lambda(K)$ is actually abelian, it would be enough to show that all irreducible subquotiens will be generated by the λ-isotypical subspace.

Indeed any irreducible summand σ of $\mathrm{Ind}_J^K \lambda$ will be generated by its λ-isotypic subspace, because

$$\mathrm{Hom}_K(\sigma, \mathrm{Ind}_J^K \lambda) = \mathrm{Hom}_J(\sigma, \lambda) = \mathrm{Hom}_J(\lambda, \sigma) \neq 0$$

since everything is semi-simple. If M is any module of $\mathcal{H} := \mathcal{H}(K, J, \lambda)$, then by writing M as a quotient of free module, we deduce that $M \otimes_{\mathcal{H}} \mathrm{Ind}_J^K \lambda$ is a quotient of a direct sum of copies of $\mathrm{Ind}_J^K \lambda$. Thus irreducible subquotients of $M \otimes_{\mathcal{H}} \mathrm{Ind}_J^K \lambda$ will be subquotients of $\mathrm{Ind}_J^K \lambda$. But this means that all irreducible subquotiens will be generated by the λ-isotypical subspace. □

Lemma 2.15. *The categories $\mathcal{R}_\lambda(K)$ and $\mathcal{H}(K, J, \lambda) - Mod$ are equivalent.*

Proof. Let $\mathcal{R}_\lambda(K)$ denote the category of smooth K-representations generated by λ-isotypic subspace. First let's prove that the following functors:

$$\begin{array}{cccc} \mathfrak{M}_\lambda & : & \mathcal{R}_\lambda(K) & \rightarrow & \mathcal{H}(K, J, \lambda) - Mod \\ & & \tau & \mapsto & \mathrm{Hom}_J(\lambda, \tau) = \mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, \tau) \end{array}$$

and

$$\begin{array}{rcccc} \mathfrak{T}_\lambda & : & \mathcal{H}(K,J,\lambda)-Mod & \to & \mathcal{R}_\lambda(K) \\ & & M & \mapsto & M \otimes_{\mathcal{H}(K,J,\lambda)} \operatorname{Ind}_J^K \lambda \end{array}$$

define an equivalence of categories. First notice that $\mathfrak{M}_\lambda(\mathfrak{T}_\lambda(\mathcal{H}(K,J,\lambda))) \simeq \mathcal{H}(K,J,\lambda)$ and $\mathfrak{T}_\lambda(\mathfrak{M}_\lambda(\operatorname{Ind}_J^K \lambda)) \simeq \operatorname{Ind}_J^K \lambda$.

We check easily that the following map:

$$\begin{array}{rcccc} \theta_M & : & M & \to & \operatorname{Hom}_J(\lambda, M \otimes_{\mathcal{H}(K,J,\lambda)} \operatorname{Ind}_J^K \lambda) \\ & & m & \mapsto & (\theta_M(m) : v \mapsto m \otimes v) \end{array}$$

is a morphism from identity functor on $\mathcal{H}(K,J,\lambda)-Mod$ to $\mathfrak{M}_\lambda \circ \mathfrak{T}_\lambda$. Let's prove that θ_M is actually an isomorphism. Given any $\mathcal{H}(K,J,\lambda)$-module M, we have an exact sequence:

$$\bigoplus_{i\in I} \mathcal{H}(K,J,\lambda) \longrightarrow \bigoplus_{l\in L} \mathcal{H}(K,J,\lambda) \longrightarrow M \longrightarrow 0$$

The functor $\mathfrak{M}_\lambda \circ \mathfrak{T}_\lambda$is right exact and commutes with direct sums since both functors $\mathfrak{M}_\lambda$ and $\mathfrak{T}_\lambda$ are. It follows that we have the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{i\in I} \mathfrak{M}_\lambda \circ \mathfrak{T}_\lambda(\mathcal{H}(K,J,\lambda)) \arrow[r] & \bigoplus_{l\in L} \mathfrak{M}_\lambda \circ \mathfrak{T}_\lambda(\mathcal{H}(K,J,\lambda)) \arrow[r] & \mathfrak{M}_\lambda \circ \mathfrak{T}_\lambda(M) \arrow[r] & 0 \\ \bigoplus_{i\in I} \mathcal{H}(K,J,\lambda) \arrow[r] \arrow[u] & \bigoplus_{l\in L} \mathcal{H}(K,J,\lambda) \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "\theta_M"] & 0 \end{tikzcd}$$

Since $\mathfrak{M}_\lambda(\mathfrak{T}_\lambda(\mathcal{H}(K,J,\lambda))) \simeq \mathcal{H}(K,J,\lambda)$, the two vertical arrows on the left hand side are isomorphisms. It follows that θ_M is also an isomorphism.

We check easily that the following map:

$$\begin{array}{rcccc} ev_\tau & : & \operatorname{Hom}_J(\lambda,\tau) \otimes_{\mathcal{H}(K,J,\lambda)} \operatorname{Ind}_J^K \lambda & \to & \tau \\ & & \alpha \otimes w & \mapsto & \alpha(w) \end{array}$$

is a morphism from $\mathfrak{T}_\lambda \circ \mathfrak{M}_\lambda$ to identity functor on $\mathcal{R}_\lambda(K)$. Let's prove that ev_τ is actually an isomorphism.

Since λ is a type $\operatorname{Hom}_K(\operatorname{Ind}_J^K \lambda, \tau) = \operatorname{Hom}_J(\lambda,\tau) \neq 0, \forall \tau \in \mathcal{R}_\lambda(K)$. Then $\forall x \in \tau$ there is a morphism $f_x : \operatorname{Ind}_J^K \lambda \longrightarrow \tau$, such that the image contains x. Taking the direct sum over all $x \in \tau$, we get a surjection:

$$\bigoplus_{x\in\tau} \operatorname{Ind}_J^K \lambda \longrightarrow \tau \longrightarrow 0$$

Let κ be the kernel of this morphism, since by previous lemma $\mathcal{R}_\lambda(K)$ is stable by kernels. The same argument gives a surjection $\bigoplus_{y\in\kappa}\mathrm{Ind}_J^K\lambda \longrightarrow \kappa \longrightarrow 0$. Then:

$$\bigoplus_{y\in\kappa}\mathrm{Ind}_J^K\lambda \longrightarrow \kappa \longrightarrow \bigoplus_{x\in\tau}\mathrm{Ind}_J^K\lambda \longrightarrow \tau \longrightarrow 0$$

finally, we get:

$$\bigoplus_{y\in\kappa}\mathrm{Ind}_J^K\lambda \longrightarrow \bigoplus_{x\in\tau}\mathrm{Ind}_J^K\lambda \longrightarrow \tau \longrightarrow 0$$

Using this presentation we conclude that ev_τ is an isomorphism, the same way we did it for θ_M. This etablishes the equivalence of categories $\mathcal{R}_\lambda(K)$ and $\mathcal{H}(K,J,\lambda)-Mod$. □

Lemma 2.16. *Let $\sigma \in \widehat{K}$, where $\widehat{K}$ is a set of all isomorphism classes of irreducible K-representations. Write $m_\sigma := \dim_E \mathrm{Hom}_K(\sigma, \mathrm{Ind}_J^K\lambda)$ for its multiplicity. Then:*

$$m_\sigma = \dim_E \mathrm{Hom}_G(\text{c–}\mathrm{Ind}_K^G\sigma, P(\chi)) = n_\sigma$$

Proof. Without loss of generality we may work with $\overline{\mathbb{Q}}_p$-coefficients, and we will do so. We claim that $\mathrm{Hom}_G(\text{c–}\mathrm{Ind}_J^G\lambda, P(\chi))$ is a free rank one module over algebra $\mathcal{H}(K,J,\lambda) = \{f \in \mathcal{H}(G,J,\lambda)|\mathrm{supp}(f) \subset K\}$. Let's first deal with a particular case before dealing with general case.

1. Simple type case. Assume that λ is a simple type. It follows from (5.6) of [BK93] that there is an extension Γ of F, that allows to define an isomorphism of Hecke algebras $\mathcal{H}(G_\Gamma, I_\Gamma, 1) \simeq \mathcal{H}(G,J,\lambda)$. Let $M = \mathrm{Hom}_G(\text{c–}\mathrm{Ind}_J^G\lambda, P(\chi)) = \mathrm{Hom}_J(\lambda, P(\chi)|J) = \mathrm{Hom}_K(\mathrm{Ind}_J^K\lambda, P(\chi)|K)$, this is an $\mathcal{H}(G,J,\lambda)$-module.

Notice that when $P(\chi) = i_{\overline{P}}^G(\rho\otimes\overline{\chi})$ is irreducible, the $\mathcal{H}(G,J,\lambda)$-module M is simple. The module M is also naturally an $\mathcal{H}(G_\Gamma, I_\Gamma, 1)$-module, and corresponds to an irreducible representation $M\otimes_{\mathcal{H}(G_\Gamma,I_\Gamma,1)} \text{c–}\mathrm{Ind}_{I_\Gamma}^{G_\Gamma}1 \simeq i_{B_\Gamma}^{G_\Gamma}\overline{\chi}_\Gamma$, where $\overline{\chi}_\Gamma$ is an unramified character of Borel subgroup B_Γ of G_Γ, making $i_{B_\Gamma}^{G_\Gamma}\overline{\chi}_\Gamma$ irreducible. Notice that $M = \mathrm{Hom}_J(\lambda, P(\chi)|J)$ does not depend on the character $\overline{\chi}$, so that discussion above is always valid.

$$\mathrm{Hom}_{G_\Gamma}(\text{c–}\mathrm{Ind}_{I_\Gamma}^{G_\Gamma}1, i_{B_\Gamma}^{G_\Gamma}\overline{\chi}_\Gamma) = \mathrm{Hom}_{G_\Gamma}(\text{c–}\mathrm{Ind}_{I_\Gamma}^{G_\Gamma}1, M\otimes_{\mathcal{H}(G_\Gamma,I_\Gamma,1)} \text{c–}\mathrm{Ind}_{I_\Gamma}^{G_\Gamma}1) = M$$

According to description of Hecke algebras in section (5.6) of [BK93] the isomorphism of Hecke algebras $t : \mathcal{H}(G_\Gamma, I_\Gamma, 1) \simeq \mathcal{H}(G, J, \lambda)$ is support preserving, in the sense that $\mathrm{supp}(tf) = J.\mathrm{supp}(f).J$, we have also a natural isomorphism between $\mathcal{H}(K_\Gamma, I_\Gamma, 1) = \{f \in \mathcal{H}(G_\Gamma, I_\Gamma, 1) | \mathrm{supp}(f) \subset K_\Gamma\}$ and $\mathcal{H}(K, J, \lambda) = \{f \in \mathcal{H}(G, J, \lambda) | \mathrm{supp}(f) \subset K\}$. Then we have:

$$\mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, P(\chi)|K) =$$
$$\mathrm{Hom}_G(\text{c–}\mathrm{Ind}_J^G \lambda, P(\chi)) = \mathrm{Hom}_{G_\Gamma}(\text{c–}\mathrm{Ind}_{I_\Gamma}^{G_\Gamma} 1, i_{B_\Gamma}^{G_\Gamma} \overline{\chi}_\Gamma)$$
$$= \mathrm{Hom}_{K_\Gamma}(\text{c–}\mathrm{Ind}_{I_\Gamma}^{K_\Gamma} 1, i_{B_\Gamma \cap K_\Gamma}^{K_\Gamma} 1) = \mathrm{Hom}_{K_\Gamma}(\mathrm{Ind}_{I_\Gamma}^{K_\Gamma} 1, \mathrm{Ind}_{B_\Gamma \cap K_\Gamma}^{K_\Gamma} 1) =$$
$$= \mathrm{Hom}_{K_\Gamma}(\mathrm{Ind}_{I_\Gamma}^{K_\Gamma} 1, (\mathrm{Ind}_{B_\Gamma \cap K_\Gamma}^{K_\Gamma} 1)^{K_1}) = \mathrm{Hom}_{K_\Gamma}(\mathrm{Ind}_{I_\Gamma}^{K_\Gamma} 1, \mathrm{Ind}_{I_\Gamma}^{K_\Gamma} 1)$$
$$= \mathcal{H}(K_\Gamma, I_\Gamma, 1) = \mathcal{H}(K, J, \lambda),$$

where $K_1 = \{g \in G_\Gamma | g \equiv 1 \mod \mathfrak{p}_\Gamma\}$.

2. General case. Let now λ be some general semi-simple type. The second part of Main Theorem of section 8 in [BK99] gives a support preserving Hecke algebra isomorphism $j : \mathcal{H}(\overline{M}, \lambda_M) \to \mathcal{H}(G, \lambda)$(here $\overline{M}$ is a unique Levi subgroup of G which contains the $N_G(M)$-stabilizer of the inertia class D and is minimal for this property), and the section 1.5 gives a tensor product decomposition $\mathcal{H}(\overline{M}, \lambda_M) = \mathcal{H}_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} \mathcal{H}_s$, where $\mathcal{H}_i = \mathcal{H}(G_i, J_i, \lambda_i)$ is an affine Hecke algebras of type A and (J_i, λ_i) is some simple type with G_i some general linear group over a p-adic field.

Let now $M = \mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, P(\chi)|K)$. The same argument as for simple type case shows that M is a simple $\mathcal{H}(G, \lambda)$-module. Then by [Bou12] §12 Proposition 2, M is a quotient of $M_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} M_s$, where M_i is a simple $\mathcal{H}_i$-module. Since $\overline{\mathbb{Q}}_p$ is algebraically closed we have by [Bou12] §12 Theorem 1 part a), that $M_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} M_s$ is a simple $\mathcal{H}_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} \mathcal{H}_s$-module. Then it follows that $M \simeq M_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} M_s$.

Let K_i denote the maximal compact subgroup of G_i and $\mathcal{A}_i := \mathcal{H}(K_i, J_i, \lambda_i)$. Then by simple type case $M_i \simeq \mathcal{A}_i$.

The isomorphism j above is being support preserving, we have then a similar decomposition for $\mathcal{H}(K, J, \lambda)$, namely $\mathcal{H}(K, J, \lambda) \simeq \mathcal{A}_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} \mathcal{A}_s$. Then $M \simeq M_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} M_s \simeq \mathcal{A}_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} \mathcal{A}_s \simeq \mathcal{H}(K, J, \lambda)$

By the discussion above we always have $\mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, P(\chi)|K) = \mathcal{H}(K, J, \lambda)$. Let now τ be subrepresentation of $P(\chi)|K$, generated by λ-isotypic subspace of $P(\chi)|K$. Then by the isomorphisms above we have

$$\mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, \tau) = \mathrm{Hom}_K(\mathrm{Ind}_J^K \lambda, P(\chi)|K) = \mathcal{H}(K, J, \lambda)$$

as isomorphisms of $\mathcal{H}(K,J,\lambda)$-modules. Then by equivalence of categories (as in Lemma 2.16):

$$\mathrm{Ind}_J^K\lambda \simeq \tau \hookrightarrow P(\chi)|K$$

where the first arrow(from left) comes from the discussion above and the second one is a natural inclusion. Applying $\mathrm{Hom}_K(\sigma,.)$ to previous injection and then taking the dimensions of both sides yields an inequality:

$$m_\sigma \leq \dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma, P(\chi))$$

Moreover by Lemma 2.11 we have:

$$\dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_G(I_\sigma, P(\chi)) = \dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_G(I_\sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), P(\chi))$$

Since by Corollary 2.13 we have $I_\sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq P(\chi)^{\oplus n_\sigma}$, then

$$\dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_G(I_\sigma, P(\chi)) = \dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_G(I_\sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), P(\chi))$$

$$= \dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_G(P(\chi)^{\oplus n_\sigma}, P(\chi)) = n_\sigma$$

The inequality $m_\sigma \leq \dim_{\overline{\mathbb{Q}}_p} \mathrm{Hom}_G(\text{c–Ind}_K^G\sigma, P(\chi)) = n_\sigma$ is actually an equality because of the relations

$$\sum_{\sigma \in \widehat{K}} m_\sigma n_\sigma = |W(D)|$$

$$\sum_{\sigma \in \widehat{K}} m_\sigma^2 = |W(D)|$$

which were proven in Corollary 2.13. □

2.6 Consequences

In this section we will deduce few results from previous sections and finally prove that if $m_\sigma = 1$ then :

$$\mathfrak{Z}_\Omega \simeq \mathrm{End}_G(\text{c–Ind}_K^G\sigma)$$

We will prove the isomorphism above over $\overline{\mathbb{Q}}_p$, then by Proposition 3.23 [CEG+16] the isomorphism above holds with E a finite extension of $\mathbb{Q}_p$. We begin with the following proposition:

Proposition 2.17. *Let $\sigma \in \widehat{K}$, where $\widehat{K}$ is a set of all isomorphism classes of irreducible K-representations. Write $m_\sigma := \dim_E \mathrm{Hom}_K(\text{c–Ind}_J^K \lambda, \sigma)$ its multiplicity. Then:*

$$\mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_1, \text{c–Ind}_K^G \sigma_2)_{\mathfrak{m}} \simeq (\mathfrak{Z}_\Omega)_{\mathfrak{m}}^{m_{\sigma_1} m_{\sigma_2}}$$

for all maximal ideals $\mathfrak{m}$ in $\mathfrak{Z}_\Omega$ and $\forall \sigma_1, \sigma_2 \in \widehat{K}$.

Proof. Write $A(\sigma_1, \sigma_2)$ for $\mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_1, \text{c–Ind}_K^G \sigma_2)$. Recall the decomposition (4) :

$$\text{c–Ind}_J^G \lambda = \bigoplus_{\sigma \in \widehat{K}} (\text{c–Ind}_K^G \sigma)^{\oplus m_\sigma}$$

so that:

$$\mathcal{H}(G, \lambda) \simeq \mathrm{End}_G(\text{c–Ind}_J^G \lambda) = \prod_{\sigma_1, \sigma_2 \in \widehat{K}} \mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_1, \text{c–Ind}_K^G \sigma_2)^{m_{\sigma_1} m_{\sigma_2}}$$

Moreover the action of $\mathfrak{Z}_\Omega$ on $A(\sigma_1, \sigma_2)$ by multiplication, makes $A(\sigma_1, \sigma_2)$ into a sub-$\mathfrak{Z}_\Omega$-module of $\mathcal{H}(G, \lambda)$, because of previous decomposition. Let's prove that $A(\sigma_1, \sigma_2)$ is also a locally free finitely generated $\mathfrak{Z}_\Omega$-module.

By the decomposition of $\mathcal{H}(G, \lambda)$, $A(\sigma_1, \sigma_2)$ is also a direct summand of $\mathcal{H}(G, \lambda)$. For every maximal ideal $\mathfrak{m} \in \mathfrak{Z}_\Omega$, $(A(\sigma_1, \sigma_2))_{\mathfrak{m}}$ is a direct summand of $(\mathcal{H}(G, \lambda))_{\mathfrak{m}}$. By Lemma 2.9, $(\mathcal{H}(G, \lambda))_{\mathfrak{m}}$ is free, then $(A(\sigma_1, \sigma_2))_{\mathfrak{m}}$ is a projective $(\mathfrak{Z}_\Omega)_{\mathfrak{m}}$-module, and therefore is free over $(\mathfrak{Z}_\Omega)_{\mathfrak{m}}$ because $(\mathfrak{Z}_\Omega)_{\mathfrak{m}}$ is a local ring. Let $d_{\mathfrak{m}}$ be the rank of $(A(\sigma_1, \sigma_2))_{\mathfrak{m}}$. Then

$$d_{\mathfrak{m}} = \dim_E(A(\sigma_1, \sigma_2)_{\mathfrak{m}} \otimes_{(\mathfrak{Z}_\Omega)_{\mathfrak{m}}} \kappa(\mathfrak{m})) = \dim_E(A(\sigma_1, \sigma_2) \otimes_{\mathfrak{Z}_\Omega} (\mathfrak{Z}_\Omega)_{\mathfrak{m}} \otimes_{(\mathfrak{Z}_\Omega)_{\mathfrak{m}}} \kappa(\mathfrak{m}))$$

$$= \dim_E A(\sigma_1, \sigma_2) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$$

(recall that $\kappa(\mathfrak{m}) \simeq E$).

Let $i \in \{1, 2\}$. Choose now $\mathfrak{m} = \mathrm{Ker}(\mathfrak{Z}_\Omega \xrightarrow{\chi} E) \in S$. By Corollary 2.13 there is an integer n_{σ_i} such that:

$$I_{\sigma_i} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq P(\chi)^{\oplus n_{\sigma_i}}$$

Then

$$\dim_E \mathrm{Hom}_G(I_{\sigma_i}, P(\chi)) = \dim_E \mathrm{Hom}_G(I_{\sigma_i} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), P(\chi))$$

$$= \dim_E \mathrm{Hom}_G(P(\chi)^{\oplus n_{\sigma_i}}, P(\chi)) = n_{\sigma_i}$$

By Lemma 2.16 we have:

$$m_{\sigma_i} = \dim_E \mathrm{Hom}_G(I_{\sigma_i}, P(\chi)) = n_{\sigma_i}$$

Then

$$\mathrm{Hom}_G(I_{\sigma_1} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), I_{\sigma_2} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) \simeq \mathrm{Hom}_G(P(\chi)^{\oplus m_{\sigma_1}}, P(\chi)^{\oplus m_{\sigma_2}})$$
$$\simeq \mathrm{End}_G(P(\chi))^{m_{\sigma_1} m_{\sigma_2}} \simeq (E)^{m_{\sigma_1} m_{\sigma_2}}$$

since by Schur's lemma $\mathrm{End}_G(P(\chi)) = E$. Finally by Lemma 2.12

$$d_{\mathfrak{m}} = \dim_E A(\sigma_1, \sigma_2) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) = \dim_E \mathrm{Hom}_G(I_{\sigma_1} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}), I_{\sigma_2} \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$
$$= m_{\sigma_1} m_{\sigma_2}$$

This proves that $d_{\mathfrak{m}} = m_{\sigma_1} m_{\sigma_2}$, $\forall \mathfrak{m} \in S$. Moreover this equality is true for all $\mathfrak{m}$ since $\mathfrak{m} \mapsto d_{\mathfrak{m}}$ is locally constant function([Bou85b] Chapitre 2 §5.2 Théorème 1 c)) and S is a dense set. □

Now we deduce the result announced in the beginning of this section:

Corollary 2.18. *Let $\sigma \in \widehat{K}$, such that $m_\sigma = 1$. Then*

$$\mathfrak{Z}_\Omega \simeq \mathrm{End}_G(\text{c–Ind}_K^G \sigma)$$

Proof. Take $\sigma := \sigma_1 = \sigma_2$ in the previous proposition. Recall that $A_\sigma := A(\sigma, \sigma)$. There is a canonical map:

$$\phi : \mathfrak{Z}_\Omega \longrightarrow A_\sigma,$$

and $C = \mathrm{Coker}(\phi)$ its cokernel and $K = \mathrm{Ker}(\phi)$ its kernel. Then we have an exact sequence:

$$0 \to K \to \mathfrak{Z}_\Omega \xrightarrow{\phi} A_\sigma \to C \to 0$$

Localizing at any maximal ideal $\mathfrak{m}$, we get an exact sequence:

$$0 \to K_{\mathfrak{m}} \to (\mathfrak{Z}_\Omega)_{\mathfrak{m}} \xrightarrow{\phi_{\mathfrak{m}}} (A_\sigma)_{\mathfrak{m}} \to C_{\mathfrak{m}} \to 0$$

The assumption $m_\sigma = 1$ implies that $A_\sigma / \mathfrak{m} A_\sigma \simeq E \simeq \kappa(\mathfrak{m})$. By construction of ϕ, the map $\phi \otimes \kappa(\mathfrak{m}) : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{m})$ induced by reduction modulo $\mathfrak{m}$ is an isomorphism. Therefore for any maximal ideal $\mathfrak{m}$, $\phi_{\mathfrak{m}} \otimes \kappa(\mathfrak{m})$ is an isomorphism.

Tensoring the previous exact sequence with $\kappa(\mathfrak{m})$ we get :

$$K_{\mathfrak{m}}\otimes_{\mathfrak{Z}_\Omega}\kappa(\mathfrak{m}) \to (\mathfrak{Z}_\Omega)_{\mathfrak{m}}\otimes_{\mathfrak{Z}_\Omega}\kappa(\mathfrak{m}) \xrightarrow{\phi_{\mathfrak{m}}\otimes\kappa(\mathfrak{m})} (A_\sigma)_{\mathfrak{m}}\otimes_{\mathfrak{Z}_\Omega}\kappa(\mathfrak{m}) \to C_{\mathfrak{m}}\otimes_{\mathfrak{Z}_\Omega}\kappa(\mathfrak{m}) \to 0$$

Since $\phi_{\mathfrak{m}}\otimes\kappa(\mathfrak{m})$ is an isomorphism, $C_{\mathfrak{m}}\otimes\kappa(\mathfrak{m}) = 0$. Then we have $C_{\mathfrak{m}} = \mathfrak{m}C_{\mathfrak{m}}$. By Nakayama's lemma $C_{\mathfrak{m}} = 0$. Then $C_{\mathfrak{m}} = 0$, for any maximal ideal $\mathfrak{m}$, implies $C = 0$.

By previous proposition $(A_\sigma)_{\mathfrak{m}} \simeq (\mathfrak{Z}_\Omega)_{\mathfrak{m}}$. Therefore $\phi_{\mathfrak{m}}$ is a surjective morphism between free modules both of rank 1, hence an isomorphism. This implies $K_{\mathfrak{m}} = 0$, since it is true for any maximal ideal $\mathfrak{m}$, we get $K = 0$. $\square$

2.7 Example : GL_2-case

In this section we will work with $\overline{\mathbb{Q}}_p$-coefficients. Let F be a local p-adic field, with maximal ideal $\mathfrak{p}$, uniformizer ϖ and ring of integers $\mathcal{O}_F$. Write G for $GL_2(F)$ and $I = \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \mathfrak{p} & \mathcal{O}_F^\times \end{pmatrix}$ for Iwahori subgroup of G.

In this section Ω has a type $(I, 1)$. Let $\mathcal{H} = \mathrm{End}_G(\text{c–Ind}_I^G 1)$ be the Hecke algebra and Z the centre of this algebra. A representation π is in Ω if and only if π is generated by $\pi^I \neq 0$ as a G-representation.

Since $(I, 1)$ is a type in G, we have $\text{c–Ind}_I^G 1 = i_{\bar{B}}^G(\text{c–Ind}_{I\cap T}^T 1)$ (see 1.5 [Dat99b]), where B is a Borel subgroup $\bar{B}$ oposite Borel subgroup and $T \simeq (F^\times)^2$ the torus. Let ${}^\circ T = T \cap I \simeq (\mathcal{O}_F^\times)^2$.

Let $\mathfrak{m} = \mathrm{Ker}(Z \xrightarrow{\overline{\chi}} \overline{\mathbb{Q}}_p)$ a maximal ideal of $\mathfrak{Z}_\Omega$ and $\kappa(\mathfrak{m})$ the residue field which is isomorphic to $\overline{\mathbb{Q}}_p$. By equivalence of categories between $R^\Omega(G)$ and $\mathcal{H} - Mod$, we have naturally $\mathfrak{Z}_\Omega \simeq Z$.

The equality $i_{\bar{B}}^G(\text{c–Ind}_{{}^\circ T}^T 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) = i_{\bar{B}}^G(\text{c–Ind}_{{}^\circ T}^T 1 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$ holds, for all maximal, ideals, since the functor $i_{\bar{B}}^G$ is exact. Let's write down the decomposition of $\text{c–Ind}_{{}^\circ T}^T 1 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$.

The representation $\text{c–Ind}_{{}^\circ T}^T 1$ is naturally isomorphic to the space of functions on T which are left invariant by ${}^\circ T$. We have the following canonical isomorphisms

$$\text{c–Ind}_{{}^\circ T}^T 1 \simeq \overline{\mathbb{Q}}_p[{}^\circ T\backslash T] \simeq \overline{\mathbb{Q}}_p[(F^\times/\mathcal{O}_F^\times)^2] \simeq \overline{\mathbb{Q}}_p[(\varpi^{\mathbb{Z}})^2] \simeq \overline{\mathbb{Q}}_p[X_1^{\pm 1}, X_2^{\pm 1}] \quad (8)$$

The last isomorphism is given by $(\varpi, 1) \mapsto X_1$ and $(1, \varpi) \mapsto X_2$.

Let $\mathcal{H}_T := \mathrm{End}_T(\text{c–Ind}_{{}^\circ T}^T 1)$. We have $\mathcal{H}_T \simeq \overline{\mathbb{Q}}_p[{}^\circ T\backslash T/{}^\circ T]$. Moreover the right multiplication by $t \in T$, induces an isomorphism:

$$\begin{array}{ccc} \overline{\mathbb{Q}}_p[{}^\circ T\backslash T/{}^\circ T] & \to & \overline{\mathbb{Q}}_p[{}^\circ T\backslash T] \\ t & \mapsto & (x \mapsto tx) \end{array}$$

This shows that $\text{c–Ind}_{{}^\circ T}^T 1$ is a free $\mathcal{H}_T$-module of rank 1 and $\mathcal{H}_T \simeq \overline{\mathbb{Q}}_p[X_1^{\pm 1}, X_2^{\pm 1}]$

Lemma 2.19. *The unramified character χ of T corresponds bijectively to the homomorphism:*

$$\begin{array}{ccccc} X & : & \mathrm{End}_T(\text{c–Ind}_{{}^\circ T}^T 1) & \to & \overline{\mathbb{Q}}_p \\ & & z & \mapsto & z(\chi) \end{array}$$

where $z(\chi)$ is a scalar by which z acts on one dimensional representation χ of T, i.e. the map $\chi \mapsto X$, with χ and X as above is a bijection from the set of unramified characters of T to the set of homomorphisms of $\overline{\mathbb{Q}}_p$-algebras $\mathrm{Hom}_{\overline{\mathbb{Q}}_p-alg}(\mathrm{End}_T(\text{c–Ind}_{{}^\circ T}^T 1), \overline{\mathbb{Q}}_p)$.

Proof. This is a consequence of the Lemma 2.7, since the kernel of this map is trivial. ☐

Let $A := \overline{\mathbb{Q}}_p[X_1, X_2]$ and $S := \overline{\mathbb{Q}}_p[X_1+X_2, X_1X_2]$. Let s_1 denote X_1+X_2 and $s_2 = X_1X_2$. According to [Bou03] IV.§6.1 Theorem 1 c) A is a free S-module of rank 2 with basis $\{1, X_2\}$. Let's now compute the discriminant d in this basis. Consider the matrix $M = \begin{pmatrix} \mathrm{tr}_{A/S}(1) & \mathrm{tr}_{A/S}(X_2) \\ \mathrm{tr}_{A/S}(X_2) & \mathrm{tr}_{A/S}(X_2^2) \end{pmatrix}$, where $\mathrm{tr}_{A/S}(f)$ is the trace of the endomorphism, multiplication by f, $P \mapsto fP$. By definition $d = \det M$, then it follows that $d = 2\mathrm{tr}_{A/S}(X_2^2) - \mathrm{tr}_{A/S}(X_2)^2$. We have $X_2^2 = -s_2.1 + s_1.X_2$, thus the matrix of multiplication by X_2 is given by $\begin{pmatrix} 0 & -s_2 \\ 1 & s_1 \end{pmatrix}$ and $\mathrm{tr}_{A/S}(X_2) = s_1$. It follows that $d = 2s_1.\mathrm{tr}_{A/S}(X_2) - 2s_2.\mathrm{tr}_{A/S}(1) - \mathrm{tr}_{A/S}(X_2)^2 = 2s_1^2 - 4s_2 - s_1^2$. The value of discriminant is $d = s_1^2 - 4s_2 = (X_1 - X_2)^2$.

Thus we are reduced to study the decomposition of $\mathcal{H}_T \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq \mathcal{H}_T/\mathfrak{m}\mathcal{H}_T$. Let $W = \langle 1, w \rangle$ the Weyl group of order 2, where w is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. According to result proved in section VI.10.3. [Ren10], we have $\mathfrak{Z}_\Omega \simeq Z = \mathcal{H}_T^W$.

Notice that $\mathcal{H}_T$ is integral over Z. We would like to study the decomposition of a maximal ideal $\mathfrak{m}$ of Z in $\mathcal{H}_T$. Since $Z = \mathcal{H}_T^W$, we are in the case of application of the Theorem 2, chapter V, § 2 n°2 [Bou85a]. It follows that, a priori, there are 3 possible cases how a maximal ideal $\mathfrak{m}$ of Z behaves in $\mathcal{H}_T$. Even though we are not dealing with Dedekind domains, the situation is quite similar:

1. $\mathfrak{m}$ "splits". More precisely there are exactly two distinct maximal ideals $\mathfrak{M}_1$, $\mathfrak{M}_2$ of $\mathcal{H}_T$. Moreover $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are conjugated by w.
2. $\mathfrak{m}$ stays "inert", meaning that $\mathfrak{m}\mathcal{H}_T$ is a maximal ideal of $\mathcal{H}_T$.
3. $\mathfrak{m}$ is "totally ramified" : there is a unique maximal ideal $\mathfrak{M}$ above $\mathfrak{m}$.

The action of w permutes X_1 and X_2. It follows that $Z \simeq \overline{\mathbb{Q}}_p[X_1 + X_2, (X_1X_2)^{\pm 1}]$. Notice that $\mathcal{H}_T$ is a localization of $\overline{\mathbb{Q}}_p$. Since the localization is flat, $\mathcal{H}_T$ is a free Z-module of rank 2 with basis $\{1, X_2\}$. The discriminant in this basis is again $d = (X_1 - X_2)^2$(same as computed above).

Since $\mathcal{H}_T$ is a free Z-module of rank 2, then $\mathcal{H}_T \otimes_Z \kappa(\mathfrak{m})$ is a two dimensional $\kappa(\mathfrak{m})$-vector space. It follows that the case 2. is impossible because in this case $\mathcal{H}_T \otimes_Z \kappa(\mathfrak{m})$ is one dimensional $\kappa(\mathfrak{m})$-vector space.

We are left to treat the case 1 and 3. In any case, there are two complex numbers a and b such that $\mathfrak{m} = (s_1 - a, s_2 - b)$, then $\mathfrak{M} = (X_1 - \lambda_1, X_2 - \lambda_2)$ where $\{\lambda_1, \lambda_2\}$ is a pair of roots of the equation $X^2 - aX + b = 0$.

Assume that the specialization $d \otimes \kappa(\mathfrak{m})$ of d at maximal ideal $\mathfrak{m}$ is non zero. Since by construction $d \otimes \kappa(\mathfrak{m})$ is the discriminant of $\kappa(\mathfrak{m})$-algebra $\mathcal{H}_T \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m})$ it follows that $\mathcal{H}_T \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m})$ splits as a product of two copies of $\overline{\mathbb{Q}}_p$ (case 1.). In this case $\lambda_1 \neq \lambda_2$ with $\mathfrak{M}_1 = (X_1 - \lambda_1, X_2 - \lambda_2)$ and $\mathfrak{M}_2 = \mathfrak{M}_1^w = (X_1 - \lambda_2, X_2 - \lambda_1)$. Putting all this together we get:

$$\mathcal{H}_T \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m}) \simeq \overline{\mathbb{Q}}_p[X_1^{\pm 1}, X_2^{\pm 1}]/(X_1 - \lambda_1, X_2 - \lambda_2).(X_1 - \lambda_2, X_2 - \lambda_1)$$

$$\simeq \kappa(\mathfrak{M}_1) \times \kappa(\mathfrak{M}_2)$$

where $\kappa(\mathfrak{M}_1)) = \mathcal{H}_T/\mathfrak{M}_1$ and $\kappa(\mathfrak{M}_2)) = \mathcal{H}_T/\mathfrak{M}_2$. Then we get the following decomposition of a representation c–$\mathrm{Ind}_{\circ T}^T 1 \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m})$:

$$\text{c–}\mathrm{Ind}_{\circ T}^T 1 \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m}) \simeq \text{c–}\mathrm{Ind}_{\circ T}^T 1 \otimes_{\mathcal{H}_T} \mathcal{H}_T \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m})$$

$$\simeq (\text{c–Ind}_{\circ T}^{T} 1 \otimes_{\mathcal{H}_T} \kappa(\mathfrak{M}_1)) \oplus (\text{c–Ind}_{\circ T}^{T} 1 \otimes_{\mathcal{H}_T} \kappa(\mathfrak{M}_2))$$

Write $\phi_i := \text{c–Ind}_{\circ T}^{T} 1 \otimes_{\mathcal{H}_T} \kappa(\mathfrak{M}_i)$. Recall that $\text{c–Ind}_{\circ T}^{T} 1$ is a free $\mathcal{H}_T$-module of rank 1. Then we have an isomorphism of $\mathcal{H}_T$-modules:

$$\phi_i \simeq \kappa(\mathfrak{M}_i) = \overline{\mathbb{Q}}_p[X_1^{\pm 1}, X_2^{\pm 1}]/\mathfrak{M}_i \tag{9}$$

Then the representations ϕ_i are one dimensional. In order to determine completely the representation ϕ_i, it would be enough to specify $\phi_i(\varpi, 1)$ and $\phi_i(\varpi, 1)$. The isomorphism (8) with identification $(\varpi, 1) \mapsto X_1$ and $(1, \varpi) \mapsto X_2$ allows to compute $\phi_i(\varpi, 1)$ and $\phi_i(\varpi, 1)$ from (9). Let's do this computation for ϕ_1. Indeed from the isomorphism of $\mathcal{H}_T$-modules:

$$\phi_1 \simeq \overline{\mathbb{Q}}_p[X_1^{\pm 1}, X_2^{\pm 1}]/(X_1 - \lambda_1, X_2 - \lambda_2)$$

we deduce that evaluating the character ϕ_1 is the same as evaluating a $\overline{\mathbb{Q}}_p$-algebra homomorphism $\mathcal{X}_1 : \overline{\mathbb{Q}}_p[X_1^{\pm 1}, X_2^{\pm 1}] \longrightarrow \overline{\mathbb{Q}}_p$ such that $\text{Ker}(\mathcal{X}_1) = (X_1 - \lambda_1, X_2 - \lambda_2)$. Indeed $\phi_1(\varpi, 1) = \mathcal{X}_1(X_1) = \lambda_1$ and $\phi_1(1, \varpi) = \mathcal{X}_1(X_2) = \lambda_2$. Let ψ_i an unramified character of $F^\times$ such that $\psi_i(\varpi) = \lambda_i$. Then $\psi_1 \simeq \psi_1 \otimes \psi_2$ and similarly we get $\psi_2 \simeq \psi_2 \otimes \psi_1$. From now on let $\chi_1 := \psi_1 \otimes \psi_2$ and $\chi_2 := \psi_2 \otimes \psi_1$. Observe that $\chi_1 = \chi_2^w$ and $\chi_2 = \chi_1^w$.

The identification above allow us to write:

$$\text{c–Ind}_{\circ T}^{T} 1 \otimes_{\mathcal{H}_T} \kappa(\mathfrak{M}_i) \simeq \chi_i$$

where χ_i is viewed as an unramified character of T via the identification above, $\mathfrak{M}_i = \text{Ker}(\mathrm{X}_i)$, and the function $\mathrm{X}_i : \mathcal{H}_T \to \overline{\mathbb{Q}}_p$ corresponds bijectively to unramified character χ_i of T, as in Lemma 2.19.

Finally in the first case we have the following decomposition:

$$\text{c–Ind}_{\circ T}^{T} 1 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq \chi_1 \oplus \chi_2$$

where χ_1 and χ_2 are two unramified characters of T corresponding to maximal ideal $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. This gives the desired decomposition:

$$(\text{c–Ind}_{I}^{G} 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) = (i_B^G(\text{c–Ind}_{\circ T}^{T} 1)) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) = i_B^G(\text{c–Ind}_{\circ T}^{T} 1 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$$

$$= i_B^G(\chi_1) \oplus i_B^G(\chi_2)$$

Notice that by Proposition 9.10 [BH06], we have that $\dim \mathrm{End}_G(i_{\bar{B}}^G(\chi)) = 1$ for any unramified character χ, in particular $i_{\bar{B}}^G(\chi)$ is indecomposable. But also we have

$$(\text{c–Ind}_I^G 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) = (\text{c–Ind}_K^G st) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \oplus (\text{c–Ind}_K^G 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}),$$

where st the inflation of Steinberg representation of GL_2 over a finite field. The representation $\text{c–Ind}_K^G \sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$, for $\sigma = 1$ and $\sigma = st$, is indecomposable because $\dim_{\overline{\mathbb{Q}}_p} \mathrm{End}_G(\text{c–Ind}_K^G \sigma \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) = \dim_{\overline{\mathbb{Q}}_p} \mathrm{End}_G(\text{c–Ind}_K^G \sigma) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) = 1$, by Proposition 2.17.

Recall that we have an equivalence of categories between the category $\mathcal{R}^I(G)$ of smooth $\overline{\mathbb{Q}}_p$-representations with non zero Iwahori-fixed vectors and the category of $\mathcal{H}$-modules. Let $F : \mathcal{R}^I(G) \longrightarrow \mathcal{H} - \mathrm{Mod}$ a functor that realizes this equivalence.

Then $F((\text{c–Ind}_I^G 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$ has two different decompositions into indecomposable $\mathcal{H}$-modules. Since $F((\text{c–Ind}_I^G 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$ is a $\mathcal{H}$-module of finite length, then by Krull-Remak-Schmidt Theorem(§2, n°5, Theorem 2. b [Bou06]), there is a permutation $\tau : \{1, 2\} \mapsto \{1, 2\}$ such that:

$$F((\text{c–Ind}_K^G 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) \simeq F(i_{\bar{B}}^G(\chi_{\tau(1)}))$$

and

$$F((\text{c–Ind}_K^G st) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})) \simeq F(i_{\bar{B}}^G(\chi_{\tau(2)}))$$

Then it follows that

$$(\text{c–Ind}_K^G 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq i_{\bar{B}}^G(\chi_{\tau(1)})$$

and

$$(\text{c–Ind}_K^G st) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq i_{\bar{B}}^G(\chi_{\tau(2)})$$

Observe that again by Proposition 9.10 [BH06], we have an equality $\dim \mathrm{Hom}_G(i_{\bar{B}}^G(\chi_{\tau(1)}), i_{\bar{B}}^G(\chi_{\tau(2)})) = 1$. In particular if $i_{\bar{B}}^G(\chi_i)$, were both irreducible we would have $i_{\bar{B}}^G(\chi_1) \simeq i_{\bar{B}}^G(\chi_2)$.

Let's treat the case 3. In this case $d \otimes \kappa(\mathfrak{m}) = 0$.Then with the notations above we have $\lambda_1 = \lambda_2 = \lambda$ and $a = 2\lambda$, $b = \lambda^2$. The maximal ideal $\mathfrak{M} = (X_1 - \lambda, X_2 - \lambda)$ is the only maximal ideal above $\mathfrak{m}$. Notice that in the $\overline{\mathbb{Q}}_p$-algebra $\mathcal{H}_T \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$, we have $X_1 = X_2$. Let $X := X_1 = X_2$, it follows that:

$$\mathcal{H}_T \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq \overline{\mathbb{Q}}_p[X^{\pm 1}]/(X - \lambda)^2$$

Let $\rho := \text{c–Ind}_{\circ T}^{T} 1 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$, ψ an unramified character of $F^\times$ such that $\psi(\varpi) = \lambda$, $\chi = \psi \otimes \psi$ an unramified character of T and let ϕ an additive character of F such that $\phi(\varpi) = \lambda^{-1}$. In this case we have that $\rho(\varpi, 1) = \rho(1, \varpi) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is determined by the condition $(\rho(\varpi, 1) - \lambda.Id)^2 = 0$, hence there is a basis in which:

$$\rho(\varpi, 1) = \rho(1, \varpi) \simeq \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$$

It follows that

$$\text{c–Ind}_{\circ T}^{T} 1 \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \simeq \chi. \begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}$$

and finally:

$$(\text{c–Ind}_{I}^{G} 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) = i_{\bar{B}}^{G}(\chi. \begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix})$$

This induces the following exact sequence for representation of T:

$$0 \longrightarrow \chi \longrightarrow \rho \longrightarrow \chi \longrightarrow 0$$

Then, since the functor $i_{\bar{B}}^{G}$ is exact, it follows that:

$$0 \longrightarrow i_{\bar{B}}^{G}(\chi) \longrightarrow i_{\bar{B}}^{G}(\rho) \longrightarrow i_{\bar{B}}^{G}(\chi) \longrightarrow 0$$

But also we have

$$(\text{c–Ind}_{I}^{G} 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) = (\text{c–Ind}_{K}^{G} st) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}) \oplus (\text{c–Ind}_{K}^{G} 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m})$$

Let $M := F(i_{\bar{B}}^{G}(\chi))$, $I_1 := F((\text{c–Ind}_{K}^{G} 1) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$ and
$I_2 := F((\text{c–Ind}_{K}^{G} st) \otimes_{\mathfrak{Z}_\Omega} \kappa(\mathfrak{m}))$. We have an exact sequence of $\mathcal{H}$-modules:

$$0 \longrightarrow M \longrightarrow I_1 \oplus I_2 \longrightarrow M \longrightarrow 0$$

From the injective map $M \longrightarrow I_1 \oplus I_2$, we get a non-zero map $M \oplus M \longrightarrow I_1 \oplus I_2$ by universal property of $\oplus$. It follows that we have the following commutative diagram, with exact rows, in the category of $\mathcal{H}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "id"] & M \oplus M \arrow[r] \arrow[d, "g"] & M \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & M \arrow[r] & I_1 \oplus I_2 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

If h was a zero map then g would also be a zero map, so by snake lemma h is non-zero. Since $\mathrm{End}(M) \simeq \overline{\mathbb{Q}}_p$, because the corresponding representation to M is indecomposable, it follows h must be a scalar multiple of the identity of M, i.e. an isomorphism. This implies, again by the snake lemma, that g is an isomorphism, i.e. $M \oplus M \simeq I_1 \oplus I_2$. Then by Krull-Remak-Schmidt Theorem, we have that $I_1 \simeq M \simeq I_2$. On the representation side we get:

$$(\text{c–Ind}_K^G st) \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m}) \simeq i_{\bar{B}}^G(\chi)$$

and

$$(\text{c–Ind}_K^G 1) \otimes_{\mathfrak{z}_\Omega} \kappa(\mathfrak{m}) \simeq i_{\bar{B}}^G(\chi)$$

3 Monodromy of irreducible generic representations

Here we give a characterization of the monodromy of the Weil-Deligne representation associated to irreducible representation π of $GL_n(F)$ via a special class of irreducible K-representations that are parametrized by partition valued functions.

We begin by introducing some notation. Let I denote an Iwahori subgroup of G and let $\mathcal{H} = \mathcal{H}(G)$ be the set of compactly supported functions f on G which are left and right inariant by the action of I, that is $f(agb) = f(g)$, $\forall a, b \in I$. The multiplication of such functions is given by convolution with respect to a Haar mesure μ on G such that $\mu(I) = 1$. This gives an algebra structure on $\mathcal{H}$.

Let $\mathcal{R}^I(G)$ be the category of all smooth complex representations of G that are generated by I-invariants. Let $K = GL_n(\mathcal{O}_F)$ be a hyperspecial maximal compact subgroup of G and B Borel subgroup of G such that $K \cap B = K \cap I$. If $P = MN$ standard parabolic subgroup ($P \supseteq B$), where $M = GL_{n_1}(F) \times \ldots \times GL_{n_k}(F)$ is the Levi subgroup and N the unipotent radical, define similarly $\mathcal{R}^{I\cap M}(M)$ as the category of all smooth complex representations of M that are generated by $I \cap M$-invariants. Let $\mathcal{H}(M) = \mathcal{H}(GL_{n_1}(F)) \otimes_E \ldots \otimes_E \mathcal{H}(GL_{n_k}(F))$. We can identify $\mathcal{H}(M)$ as subalgebra of $\mathcal{H}$ since it consists of $I \cap M$-bi-invariant functions via the natural homomorphism $\mathcal{H}(M) \longrightarrow \mathcal{H}(G)$ sending each simple tensor to the product of functions. Recall that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{R}^I(G) \arrow[r, "\mathfrak{M}"] & \mathcal{H} - Mod \\ \mathcal{R}^{I\cap M}(M) \arrow[u, "i_P^G"] \arrow[r, "\mathfrak{M}"] & \mathcal{H}(M) - Mod \arrow[u, "-\otimes_{\mathcal{H}(M)}\mathcal{H}"'] \end{tikzcd} \qquad (*)$$

where the horizontal arrows are equivalence of categories that associates to a representation its I-invariants: $\pi \mapsto \pi^I$ and i_P^G denotes the normalized parabolic induction. See [Bor76] for more details on the equivalence of categories.

In [SZ99] section 6 (just above proposition 2) the authors define irreducible K-representations $\sigma_{\mathcal{P}}(\lambda)$, where $\mathcal{P}$ is partition valued functions with compact support (cf. section 2 [SZ99]). Let π be an irreducible smooth generic representation. The main result of this section is that knowledge of which of the $\sigma_{\mathcal{P}}(\lambda)$'s are contained in π allow us to describe completely the monodromy of the associated Weil-Deligne representation. This statement will be made more precise in the Proposition 3.19 and in section 3.10.

It has been observed by Jack Shotton [Sho16] that Propositions 3.19 can be proved by modifying the proof of Proposition 2 Section 6 [SZ99] in the tempered case. In this thesis we present a different proof of this result by deducing it directly from the work of Rogawski [Rog85] in the Iwahori case and then transferring it to the arbitrary type using Bushnell-Kutzko theory of types.

Now a few words about structure of this section. In the sections from 3.1 to 3.4 we will recall some elementary combinatorics which will allow us to treat the case when π has a trivial type. Then in the sections from 3.5 to 3.7 we consider the Iwahori case, i.e. when the representation π has a trivial type. This will be done by studying simple modules over Iwahori-Hecke algebra. Then in sections 3.8 and 3.9 we will explain how the case of the general type can be deduced from the trivial type case via Bushnell-Kutzko theory. The section 3.10 links the statement of the Proposition 3.19 to the modoromy of the Weil-Deligne representation associated to π via classical local Langlands correspondence. The last section gives a characterization of irreducible generic representations of $GL_n(F)$.

3.1 Weyl group of type A_{n-1}

My reference for general theory of Coxeter groups is [BB05]. Let W denote the symmetric group S_n. Then W is a Coxeter group with Coxeter system (W, S) where $S = \{s_i, 1 \leq i \leq n-1\}$ is the set of simple reflections, and s_i is the transposition $(i, i+1)$. Moreover, W has the following presentation: $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ if $|i-j| > 1$. Each element $w \in W$ can be written as a product of generators (simple reflections). If k is minimal number of generators among all such expressions for w, then k is called the length of w, written $l(w) = k$. Such an expression is called a reduced word (or reduced decomposition or reduced expression) for w.

For each subset T of S, the subgroup W_T generated by T is called a parabolic subgroup of W. It has a Coxeter system (W_T, T) and its length function is induced from l. It has a unique longest element w_T. The longest element of W, usually written w_0, is an involution such that $i \mapsto n+1-i$ or explicitly $w_0 = s_1.(s_2 s_1)\ldots(s_{n-1}\ldots s_1)$.

Let (W, S) be a Coxeter system and $R = \{wsw^{-1} : w \in W, s \in S\}$ its set of reflections(transpositions). Let $u, v \in W$. Then write $u \xrightarrow{t} v$ when $u^{-1}v = t \in R$ and $l(u) < l(v)$. Similarly $u \to v$ means that $u \xrightarrow{t} v$ for some $t \in R$. Now we define the Bruhat order $\leq$ on W by declaring that $u \leq v$ when there exist $u_i \in W$ such that:

$$u = u_0 \to u_1 \to \ldots \to u_{k-1} \to u_k = v$$

3.2 Young Tableaux

A partition of n is a sequence of integers $(\lambda_1, \ldots, \lambda_m)$ such that $\lambda_1 + \ldots + \lambda_m = n$ and $\lambda_1 \geq \ldots \geq \lambda_m > 0$. Given a partition $(\lambda_1, \ldots, \lambda_m)$ of n there is an associated diagram consisting of m rows of boxes in which row i has λ_i boxes. A tableau is a filling of such a diagram with positive integers without repetition such that the entries increase from left to right along rows and from top to bottom down columns. The tableau is called Young tableau or standard tableau if the entries are precisely $\{1, \ldots, n\}$. We write **Shape**(T) for the underlying partition of the tableau T.

We say that a partition λ^c is conjugate of $\lambda = (\lambda_1, \ldots, \lambda_m)$ if it is represented by the reflected diagram of the one associated to λ with respect to the line $y = -x$ with the coordinate of the upper left corner is taken to be $(0, 0)$.

There is a bijection $w \mapsto (P(w), Q(w))$ between S_n and pairs of standard tableaux of the same shape (a partition of n), it is called the Robinson-Schensted correspondence. For description of this correspondence I refer reader to [Knu98] chapter 5 section 5.1.4, or to [Ari00].

Lets record the following easy application of this correspondence in the following lemma:

Lemma 3.1. *Let w_0 the longest element in S_n. Then we have* $P(w_0) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$

Proof. We apply the insertion algorithm described in 5.1.4 [Knu98]. Let $T \leftarrow x$ be a tableau obtained by inserting x into T("bumping algorithm"). We insert successively n into an empty tableau, then insert $n-1$,…,and finally insert 1. We represent this procedure by : $(((\emptyset \leftarrow n) \leftarrow n-1) \leftarrow \ldots) \leftarrow 1$. $\square$

Lemma 3.2. *Let $\mathcal{P} = (n_1, \dots, n_k)$ a partition of n.*
Let $X = \{1, \dots, n\} \setminus \{n_1, n_1 + n_2, \dots, n_1 + \dots + n_k\}$ be the set of integers and $T = \{(i, i+1), i \in X\}$ the corresponding set of simple reflections. Then $\mathbf{Shape}(P(w_T)) = \mathcal{P}^c$ *, where $\mathcal{P}^c$ is the partition conjugate to $\mathcal{P}$.*

Proof. We have that $W_T \simeq S_{n_1} \times \dots \times S_{n_k}$ and $w_T = w_{0,1} \dots w_{0,k}$, where $w_{0,i} = s_{\tilde{n}_{i-1}+1}.(s_{\tilde{n}_{i-1}+2} s_{\tilde{n}_{i-1}+1}) \dots (s_{\tilde{n}_i - 1} \dots s_1)$ is the longest element of $W_i := S_{n_i}$, with $\tilde{n}_i = \sum_{j=1}^{i} n_j$ and $\tilde{n}_0 = 0$. This is a consequence of Lemma 3.1 since w_T is the product of longest elements with disjoint support. Indeed, the i-th column $\begin{array}{|c|}\hline \tilde{n}_{i-1}+1 \\ \hline \vdots \\ \hline \tilde{n}_i \\ \hline \end{array}$, in the tableau $P(w_T)$ is given by $w_{0,i}$ as in Lemma 3.1. Therefore the i-th column has exactly n_i boxes, the result follows. $\square$

We define the a partial ordering on partitions of n. We write $\lambda = (\lambda_1, \dots, \lambda_l) \geqq \mu = (\mu_1, \dots, \mu_m)$ if and only if $\sum\limits_{i=1}^{j} \lambda_i \leq \sum\limits_{i=1}^{j} \mu_i$ for all integers $j = 1, \dots, \max(m, l)$,if $m < l$ we set $\mu_k = 0$ for $m + 1 \leq k \leq l$, similarly if $m > l$ we set $\lambda_k = 0$ for $l + 1 \leq k \leq m$. The smallest partition for this partial order is (n) and the biggest is $(1, \dots, 1)$ (n times 1). This the opposite of the usual order on partitions ([Knu98] chapter 5 section 5.1.4).

3.3 Hecke algebra $\mathcal{H}$ and Kazhdan-Lusztig basis

A detailed account of the structure of $\mathcal{H}$ may be found in [How94]. Here is a brief summary. Let now T be a maximal torus such that $T \subset B$, and $N_G(T)$ the normalizer of T in G. The affine Weyl group $\tilde{W} = N_G(T)/(T \cap I)$ is isomorphic to semidirect product $\mathbb{Z}^n \rtimes W$, and $W = N_G(T)/T$ is just the symmetric group S_n. The Bruhat decomposition for I says that $G = I\tilde{W}I$. For $w \in \tilde{W}$, let T_w denote the characteristic function of double coset IwI. Then $\left\{T_w | w \in \tilde{W}\right\}$ is a basis of $\mathcal{H}$. The finite dimensional algebra $\mathcal{H}_W$ is identified with the subalgebra of $\mathcal{H}$ generated by T_w such that w has a representative in K.

The algebra $\mathcal{H}_W$ is called sometimes a Hecke algebra of a symmetric group because it has the following presentation. Let (W, S) be a Coxeter group, where W is a symmetric group as above. The Hecke algebra of (W, S), denoted $\mathcal{H}_W$, is spanned by elements T_w, $w \in W$, subject to relations:

$$T_x T_y = T_{xy} \text{ if } l(xy) = l(x) + l(y)$$

$$T_s^2 = (q-1)T_s + q \text{ for all } s \in S$$

Let $\mathcal{A} = E[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the algebra of Laurent polynomials in n variables. The Weyl group W acts on $\mathcal{A}$ by permutation of variables : $w(x_i) = x_{w(i)}$. Then it can be shown that $\mathcal{H}$ is generated by $\mathcal{H}_W$ and $\mathcal{A}$ subject to relations:

$$x_i T_{s_j} = T_{s_j} x_i \text{ if } |i-j| > 1$$

$$x_i T_{s_i} = T_{s_i} x_{i+1} - (q-1)x_{i+1} \text{ for all } i = 1, \ldots, n-1$$

$$x_{i+1} T_{s_i} = T_{s_i} x_i + (q-1)x_{i+1} \text{ for all } i = 1, \ldots, n-1$$

Every element $T \in \mathcal{H}$ has a unique expression $T = \sum_{w \in W} a_w T_w$, where $a_w \in \mathcal{A}$ and the centre of $\mathcal{H}$ is $\mathcal{A}^W$.

For $y, w \in W$ such that $y \leq w$ (here the order on W is the Bruhat order as defined above), let $P_{y,w}(q)$ be the Kazhdan-Lusztig polynomial. By Theorem 1.1 in [KL79] $P_{y,w}(q)$ is a polynomial in q of degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w} = 1$. The coefficient of $q^{\frac{1}{2}(l(w)-l(y)-1)}$ in $P_{y,w}(q)$ is denoted $\mu(y, w)$. For $w \in W$, define:

$$C_w = q^{l(w)/2} \sum_{y \leq w} (-1)^{l(w)-l(y)} q^{-l(y)} P_{y,w}(q^{-1}) T_y$$

and

$$C'_w = q^{-l(w)/2} \sum_{y \leq w} P_{y,w}(q) T_y$$

Both $\{C_w | w \in W\}$ and $\{C'_w | w \in W\}$ are a basis for $\mathcal{H}_W$.

3.4 Cell decomposition with respect to Kazhdan-Lusztig order

We use same notation as in [Rog85]. First recall a more convenient definition of Kazhdan-Lusztig order. We say that $x \in W$ is (left-)linked to $y \in W$ if there is a non-zero element $h \in \mathcal{H}$ such that C_x appears with non zero coefficient in the product hC_y, and we write $x \xrightarrow{L} y$. The same way we define $x \xrightarrow{R} y$ (R stands for "right") if there is a non-zero element $h \in \mathcal{H}$ such that C_x appears with non zero coefficient in the product $C_y h$. Now we define $x \leq_L y$ on W by declaring that there exist $u_i \in W$ such that:

$$x = u_0 \xrightarrow{L} u_1 \xrightarrow{L} \ldots \xrightarrow{L} u_{k-1} \xrightarrow{L} u_k = y$$

We define $x \leq_R y$ analogously. We also say that $x \leq_{LR} y$ if there exist $u_i \in W$ such that:

$$x = u_0 \xrightarrow{X_1} u_1 \xrightarrow{X_2} \dots \xrightarrow{X_{k-1}} u_{k-1} \xrightarrow{X_k} u_k = y$$

where $X_i \in \{L, R\}$. We define an equivalence relation by $x \sim_X y$ if $x \leq_X y$ and $y \leq_X x$ with $X \in \{L, R, LR\}$. An equivalence class for $\sim_L$ (resp. $\sim_R$) is called left (rep. right) cell and an equivalence class for $\sim_{LR}$ is called a two sided cell. For an accessible and detailed account of the properties of these partial orderings I refer the reader to [Wil].

As in §5 of [Rog85], for $w \in W$ let J_w be the subspace of $\mathcal{H}_W$ spanned by $\{C_y | y \leq_L x\}$. Let now T be a subset of the set of simple reflections. If $w = w_T$, we abbreviate $J_T := J_{w_T}$. Then the representation of $\mathcal{H}_W$ on the left cell containing w is defined by:

$$J(w) = \frac{J_w}{\sum_{y <_L w} J_y}$$

As a consequence of Theorem 1.4 of [KL79], all the $J(w)$ are irreducible and $J(w)$ is isomorphic to $J(w')$ if and only if $w \sim_{LR} w'$. Moreover $J(w)$ occurs with multiplicity one in J_w.

3.5 Principal series induced from parabolic subgroup

The Steinberg representation of $GL_n(F)$ is

$$St_n = \frac{\mathrm{Ind}_B^G 1}{\sum_{B \subsetneq P} \mathrm{Ind}_P^G 1}$$

Let $K_1 = \{x \in K | x \equiv 1 \mod \mathfrak{p}\}$. First note $\mathcal{H}_W \simeq (\mathrm{Ind}_I^K 1)^I \simeq (\mathrm{Ind}_B^G 1)^I$ then

$$St_n^I \simeq \frac{(\mathrm{Ind}_I^K 1)^I}{\sum_{B \subsetneq P} (\mathrm{Ind}_{(P \cap K)K_1}^K 1)^I}$$

It is known that St_n^I is one dimensional with basis given by $C := \sum\limits_{w \in W} (-q)^{-l(w)} T_w$ by remark after the proof Proposition (4.12) in [Ree92].

Lemma 3.3. *We have $T_s C = -C$, $\forall s \in S$.*

Proof. We have the following relations between generators

$$T_sT_w = \begin{cases} (q-1)T_w + qT_{sw} & \text{if } sw < w \\ T_{sw} & \text{if } sw > w \end{cases}$$

Then

$$T_sC = \sum_{\substack{w\in W\\ sw<w}} (-q)^{-l(w)}((q-1)T_w + qT_{sw}) + \sum_{\substack{w\in W\\ sw>w}} (-q)^{-l(w)}T_{sw}$$

After a change of variable $y = sw$, we get:

$$T_sC = (q-1)\sum_{\substack{w\in W\\ sw<w}} (-q)^{-l(w)}T_w + q\sum_{\substack{w\in W\\ y<sy}} (-q)^{-(l(y)+1)}T_y + \sum_{\substack{w\in W\\ y>sy}} (-q)^{-(l(y)-1)}T_y$$

$$= -\sum_{\substack{w\in W\\ sw<w}} (-q)^{-l(w)}T_w - \sum_{\substack{w\in W\\ y<sy}} (-q)^{-l(y)}T_y = -C$$

□

Following the proof of the Proposition 4.5 in [Rog85] word by word, we get the following statement:

Proposition 3.4. *Let T be a subset of the set of simple reflections. For every $\mathcal{H}_W$-module M, the map $\varphi \mapsto \varphi(C_{w_T})$ induces an isomorphism:*

$$\mathrm{Hom}_{\mathcal{H}_W}(J_T, M) \simeq \{m \in M | C'_s.m = 0, \, \forall s \in T\}$$

We have the following statement as a particular case of the previous proposition:

Proposition 3.5. *We have the identification $J(w_0) = J_{w_0} = St_n^I$. And the $\mathcal{H}_W$-module satisfies the following universal property:*

$$\mathrm{Hom}_{\mathcal{H}_W}((St_n)^I, M) = \{m \in M | C'_s.m = 0, \, \forall s \in S\}$$

Proof. The basis of $J(w)$ consists of elements $\{C_x : x \sim_L w\}$. By Lemma 4.2 (1) [Rog85] if $y \leq_L w_0$ then $y = w_0$, where w_0 is the longest element in the Weyl group. It follows that there is only one element in the equivalence class of w_0. So $J(w_0) = J_{w_0}$, and moreover it is one dimensional generated by $C_{w_0} = q^{l(w_0)/2} \sum_{y \le w_0} (-1)^{l(w_0)-l(y)} T_y = q^{l(w_0)/2} (-1)^{l(w_0)} C$. We have then the following identification $J(w_0) = J_{w_0} = St_n^I$, since by previous lemma St_n^I satisfies also an analogue of the universal property of Proposition 3.4, with $T = S$. $\square$

We have the similar result for an arbitrary parabolic cone J_T. We will see in the next proposition that J_T can be identified with restriction to K of some normalized parabolic induction.

Proposition 3.6. *The set T determines in a unique way a partition of n, denoted $(n_1, \ldots, n_k)$ and this partition correspond to a standard parabolic subgroup P of G. Let I_i the Iwahori subgroup of $GL_{n_i}(F)$. We have the identification*

$$J_T = \mathcal{H}_W \otimes_{\mathcal{H}_{W_T}} (St_{n_1}^{I_1} \otimes \ldots \otimes St_{n_k}^{I_k}) = (\mathrm{Ind}_{(P\cap K)K_1}^K St_{n_1} \otimes \ldots \otimes St_{n_k})^I =$$

$$= (i_P^G((St_{n_1} \otimes \chi_1) \otimes \ldots \otimes (St_{n_k} \otimes \chi_k))|K)^I$$

of $\mathcal{H}_W$-modules. This holds for any unramified characters χ_i, since their restriction to K are trivial.

Proof. Let $(n_1, \ldots, n_k)$ be a partition of n, and let $P = MN$ standard parabolic corresponding to this partition, where M is the Levi subgroup and N the unipotent radical. Let $X = \{1, \ldots, n\} \backslash \{n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_k\}$ be the set of integers and $T = \{(i, i+1), i \in X\}$ the corresponding set of simple reflections. Then $W_T \simeq S_{n_1} \times \ldots \times S_{n_k}$ and $w_T = w_{0,1} \ldots w_{0,k}$, where $w_{0,i} = s_{\tilde{n}_{i-1}+1}.(s_{\tilde{n}_{i-1}+2} s_{\tilde{n}_{i-1}+1}) \ldots (s_{\tilde{n}_i - 1} \ldots s_1)$ is the longest element of $W_i := S_{n_i}$, with $\tilde{n}_i = \sum_{j=1}^{i} n_j$ and $\tilde{n}_0 = 0$.

The representation $i_P^G(St_{n_1} \otimes \chi_1) \otimes \ldots \otimes (St_{n_k} \otimes \chi_k)|K = \mathrm{Ind}_{P\cap K}^K St_{n_1} \otimes \ldots \otimes St_{n_k}$ corresponds to $\mathcal{H}_W$-module $\mathcal{H}_W \otimes_{\mathcal{H}_{W_T}} ((St_{n_1})^{I_1} \otimes \ldots \otimes (St_{n_k})^{I_k})$, where I_i is the Iwahori subgroup of $GL_{n_i}(F)$. Indeed, restricting to K the diagram (*) from the section devoted to notations, we have:

$$(i_P^G(St_{n_1} \otimes \chi_1) \otimes \ldots \otimes (St_{n_k} \otimes \chi_k)|K)^I = (\mathrm{Ind}_{P\cap K}^K St_{n_1} \otimes \ldots \otimes St_{n_k})^I$$

$$= \mathcal{H}_W \otimes_{\mathcal{H}_{W_T}} ((St_{n_1})^{I_1} \otimes \ldots \otimes (St_{n_k})^{I_k})$$

Notice that we also have $(\mathrm{Ind}_{P\cap K}^K St_{n_1} \otimes \ldots \otimes St_{n_k})^I = (\mathrm{Ind}_{(P\cap K)K_1}^K St_{n_1} \otimes \ldots \otimes St_{n_k})^I$. Then by universal property of tensor product we get:

$$\mathrm{Hom}_{\mathcal{H}_W}(\mathcal{H}_W \otimes_{\mathcal{H}_{W_T}} (St_{n_1}^{I_1} \otimes \ldots \otimes St_{n_k}^{I_k}), M) = \mathrm{Hom}_{\mathcal{H}_{W_T}}((St_{n_1}^{I_1} \otimes \ldots \otimes St_{n_k}^{I_k}), M)$$

Since we have $\mathcal{H}_{W_T} = \mathcal{H}_{W_1} \otimes \ldots \otimes \mathcal{H}_{W_k}$, where $\mathcal{H}_{W_i}$ is the Hecke algebra of Weyl group $W_i := S_{n_i}$, by inductive application of Proposition 3.5, we get:

$$\mathrm{Hom}_{\mathcal{H}_{W_T}}((St_{n_1}^{I_1} \otimes \ldots \otimes St_{n_k}^{I_k}), M) = \{m \in M | C'_s.m = 0, \, \forall s \in T\}$$

Therefore by universal property as in Proposition 3.4, we may identify J_T with $\mathcal{H}_W \otimes_{\mathcal{H}_{W_T}} (St_{n_1}^{I_1} \otimes \ldots \otimes St_{n_k}^{I_k}) = (\mathrm{Ind}_{P\cap K}^{K} St_{n_1} \otimes \ldots \otimes St_{n_k})^I$. □

3.6 Relation to the work of Schneider and Zink

Let $X = \{1, \ldots, n\} \setminus \{n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_k\}$ a set of integers associated to a partition $\mathcal{P} = (n_1, \ldots, n_k)$ of n. Define $T = \{(i, i+1), i \in X\}$ to be the corresponding set of simple reflections. Then $W_T \simeq S_{n_1} \times \ldots \times S_{n_k}$. Let P_* standard parahoric corresponding to this partition $\mathcal{P}$, then $P_* = IW_TI$.

We identify the partition valued functions from [SZ99](defined in section 1) and partitions. Indeed, in our setting the supersuspidal support is reduced just to the trivial representation, therefore every relevant partition valued function $\tilde{\mathcal{P}}$(following the notation of [SZ99]) satisfies $\tilde{\mathcal{P}}(1) = \mathcal{P}$, where $\mathcal{P}$ is some partition and $\tilde{\mathcal{P}}(\sigma) = 0$ if $\sigma \neq 1$.Then, we will always identify $\tilde{\mathcal{P}}$ with $\mathcal{P}$.

Define the K-representation

$$\sigma_{\mathcal{P}} = \frac{\pi_{P_*}}{\sum_{P_* \subsetneq P'_*} \pi_{P'_*}}$$

where $\pi_{P_*} = \mathrm{Ind}_{P_*}^{K}(St_{n_1}^{I_1} \otimes \ldots \otimes St_{n_k}^{I_k})$, where I_i are the Iwahori subgroups of $GL_{n_i}(F)$. Similar definition holds for $\pi_{P'_*}$ and the sum runs over all the standard parahoric subgroups that contain strictly P_* and the partition $\mathcal{P}$ is identified with partition valued function as above. This representation is the inflation of a representation over the finite field as defined in [SZ99] section 5 just above Proposition 9. The representation $\sigma_{\mathcal{P}}$, is irreducible and occurs with multiplicity one in π_{P_*}(Proposition in Section 4 of [SZ99]).

In the same way as above these P'_* correspond uniquely to a partition and this partition gives rise to a set T' of simple reflections as well as parabolic subgroup $W_{T'}$ with the longest element $w_{T'}$.

Lemma 3.7. *With the same notations as above, the following statements are equivalent:*

1. $w_{T'} \leq_L w_T$

2. $T \subseteq T'$

3. $W_T \subseteq W_{T'}$

4. $P_* \subseteq P'_*$

Proof. The equivalence $1 \Leftrightarrow 2$ follows from equivalence $(a) \Leftrightarrow (c)$ of Proposition 4.2 in [Rog85]. The rest is a standard exercise in combinatorics of Coxeter groups. $\square$

By equivalence of categories of K-representations and $\mathcal{H}_W$-modules, as in Proposition 3.6, $\pi_{\mathcal{P}_*}$ corresponds to $\mathcal{H}_W$-module J_T. It follows that $\sigma_{\mathcal{P}}$ corresponds to an $\mathcal{H}_W$-module:

$$\frac{J_T}{\sum_{T \subsetneq T'} J_{T'}}$$

By Lemma 3.7 it follows that this module is also

$$\frac{J_T}{\sum_{w_{T'} <_L w_T} J_{T'}} = \frac{J_T}{\sum_{w <_L w_T} J_w} = J(w_T)$$

For any $w \in W$ the irreducible $\mathcal{H}_W$-module $J(w)$ corresponds to some $\sigma_{\tilde{\mathcal{P}}}$. Now the partition $\tilde{\mathcal{P}}$ gives rise to the set $\tilde{T}$ of simple reflections. Then we must have $J(w_{\tilde{T}}) \simeq J(w)$. This is equivalent to $w_{\tilde{T}} \sim_{LR} w$, which in turn is equivalent to $\mathbf{Shape}(P((w)) = \mathbf{Shape}(P((w_{\tilde{T}}))$ by corollary 5.4.2 in [Wil]. By Lemma 3.2 we have then $\tilde{\mathcal{P}} = \mathbf{Shape}(P((w_{\tilde{T}}))^c = \mathbf{Shape}(P((w))^c$. We can summarize all this in the following proposition:

Proposition 3.8. *Let $\mathcal{P} = (n_1, \ldots, n_k)$ be a partition of n. There is an isomorphism of $\mathcal{H}_W$-modules $\sigma^I_{\mathcal{P}} \simeq J(w_T)$, where w_T in the longest element in the parabolic subgroup $W_T \simeq S_{n_1} \times \ldots \times S_{n_k}$.*

Conversely, given $w \in W$ there is an isomorphism of $\mathcal{H}_W$-modules: $J(w) \simeq \sigma^I_{\mathcal{P}}$, where $\mathcal{P} = \mathbf{Shape}(P((w))^c$.

Note that by Lemma 3.7 we have $w_{T'} \leq_L w_T$ if and only if $\mathbf{Shape}(P((w_{T'}))^c \leq \mathbf{Shape}(P((w_T))^c$.

3.7 Trivial type case

Lemma 3.9. $\mathrm{Hom}_{\mathcal{H}_W}(J_T, J(w)) \neq 0$ *if and only if* $sx < x$, $\forall s \in T$ *and* $\forall x \sim_L w$.

Proof. The basis of $J(w)$ consists of elements $\{C_x : x \sim_L w\}$. We also know that by proposition 4.5 [Rog85] we have:

$$\mathrm{Hom}_{\mathcal{H}_W}(J_T, J(w)) = \{m \in J(w) | C'_s.m = 0,\ \forall s \in T\}$$

Then $\mathrm{Hom}_{\mathcal{H}_W}(J_T, J(w)) \neq 0$ if and only if $T_s.C_x = -C_x$, $\forall s \in T$ and $\forall x \sim_L w$.

Since we have the following multiplication formula according to Theorem 3.6.1 in [Wil]:

$$T_s C_x = \begin{cases} -C_x & \text{if } sx < x \\ qC_x + q^{\frac{1}{2}} C_{sx} + q^{\frac{1}{2}} \sum\limits_{\substack{y<x \\ sy<y \\ \mu(x,y)\neq 0}} \mu(x,y) C_y & \text{if } sx > x \end{cases}$$

The conclusion follows.

$\square$

Observe first that if $y <_L w$ then we have an inclusion $J_y \hookrightarrow J_w$. This allows to construct a filtration by proper submodules. It follows that J_w contains $J(v)$ if $v \leq_L w$.

Lemma 3.10. *Assume that* $\mathrm{Hom}_{\mathcal{H}_W}(J_T, J(w)) \neq 0$ *and* $\mathrm{Hom}_{\mathcal{H}_W}(J_T, J(w')) = 0$, $\forall w' \in W$ *such that* $w <_L w'$. *Then* $w \sim_{LR} w_T$.

Proof. The decomposition of $\mathcal{H}_W$-module J_T implies that if J_T contains $J(w)$ ($\mathrm{Hom}_{\mathcal{H}_W}(J_T, J(w)) \neq 0$) then $w \leq_L w_T$. By previous Lemma 3.9 we have $w' < sw'$, $\forall s \in T$ and $w <_L w'$. Replacing $J(w')$ by some isomorphic module $J(v)$, we may then assume, without loss of generality, that w' is the longest element in some parabolic subgroup of W generated by the set T' of simple reflections since the two sided cells are in bijection with partitions and a partition of a given shape determines a longest element of an associated parabolic subgroup. Similarly we may also assume that w is the longest element in some other parabolic subgroup of W generated by the set $\tilde{T}$ of simple reflections. By Lemma 3.7 we have that $T \subseteq \tilde{T}$ as well $T' \subset \tilde{T}$ and T' is a proper subset. Moreover the condition $w' < sw'$ means that the simple reflection s does not occur in the reduced expression of w', since it is true for all $s \in T$ we must have $T \cap T' = \emptyset$. Assume that T is also a proper subset, i.e. $T \neq \tilde{T}$. We have just seen that for all proper subsets we $T' \subset \tilde{T}$ we have $T \cap T' = \emptyset$. Then by assumption we must have $T \cap T = \emptyset$, we arrive at the contradiction. It follows then that $T = \tilde{T}$ and $w = w_T$. The conclusion follows. $\square$

Let π be an irreducible generic representation. This representation can be written as $\pi = i_P^G(St_{n_1} \otimes \chi_1) \otimes \ldots \otimes (St_{n_k} \otimes \chi_k)$, where χ_i are some unramified characters. The parabolic subgroup P of G defines a partition $\mathcal{P} = (n_1, \ldots, n_k)$ of n and segments $\Delta_i = \chi_i \otimes \ldots \otimes \chi_i|.|^{n_i - 1}$ such that the character $\chi = \Delta_1 \otimes \ldots \otimes \Delta_k$ is adapted to this partition. Since π is generic no two segments Δ_i are linked ([Kud94]). Then $\pi^I = J_T(\chi)$, where T is the set of simple reflections obtained from the shape of partition $\mathcal{P}$, following the notation of [Rog85].

Proposition 3.11. *Let π be an irreducible generic representation, with non zero Iwahori fixed vectors. The following statement are equivalent:*

1. $\mathrm{Hom}_{\mathcal{H}_W}(J(w_T), \pi^I) \neq 0$ *and* $\mathrm{Hom}_{\mathcal{H}_W}(J(w), \pi^I) = 0$, $\forall w \in W$ *such that* $w_T <_L w$.

2. *There is a character χ such that $\pi^I = J_T(\chi)$ as $\mathcal{H}_W$-modules.*

Moreover $J(w_T)$ occurs in J_T with multiplicity one.

Proof. For any $v \in W$, we have $\mathrm{Hom}_{\mathcal{H}_W}(J(v), \pi^I) \simeq \mathrm{Hom}_{\mathcal{H}_W}(\pi^I, J(v))$.

$1 \implies 2$. We know that, by Bernstein-Zelevinsky classification, $\pi^I = J_{\tilde{T}}(\chi)$, for some $\tilde{T}$ and some χ. By previous Lemma 3.10 we have that $w_{\tilde{T}} \sim_{LR} w_T$, therefore $T = \tilde{T}$.

$2 \implies 1$. Observe first that if $y <_L w_T$ then we have an inclusion $J_y \hookrightarrow J_T$. This allows to construct a filtration by proper submodules. It follows that J_T contains $J(y)$ if $y \leq_L w_T$.

The multiplicity one statement is consequence of remarks above Proposition 5.1 [Rog85]. $\square$

Since we have $\mathrm{Hom}_K(\sigma_{\mathcal{P}}, \pi) \simeq \mathrm{Hom}_{\mathcal{H}_W}(\sigma_{\mathcal{P}}^I, \pi^I) \simeq \mathrm{Hom}_{\mathcal{H}_W}(J(w_T), \pi^I)$, we can express previous result in the language of representations:

Proposition 3.12. *Let π be an irreducible generic representation, with non zero Iwahori fixed vectors. The following statement are equivalent:*

1. $\mathrm{Hom}_K(\sigma_{\mathcal{P}}, \pi) \neq 0$ *and* $\mathrm{Hom}_K(\sigma_{\mathcal{P}'}, \pi) = 0$*, for all partitions* $\mathcal{P}'$ *such that* $\mathcal{P} < \mathcal{P}'$.

2. $\pi = i_P^G(L(\Delta_1) \otimes \ldots \otimes L(\Delta_k))$*, where* P *is the standard parabolic associated to the partition* $\mathcal{P} = (n_1, \ldots, n_k)$ *and all the segments* Δ_i *are not pairwise linked. Here* $L(\Delta)$ *denotes, as usual, the Langlands quotient of segment* Δ.

   Moreover if $\sigma_{\mathcal{P}}$ *satisfies the equivalent properties above, it occurs with multiplicity one in* π.

Let $\sigma_{min} := \sigma_{\mathcal{P}}$, where $\mathcal{P}$ is minimal for partial ordering on partitions. We can also deduce the following lemma

Lemma 3.13. *Let* π *be an irreducible generic representation such that* $\pi^I \neq 0$*, then* $\mathrm{Hom}_K(\sigma_{min}, \pi) \neq 0$

Proof. By equivalence of categories as above we have $\mathrm{Hom}_K(\sigma_{min}, \pi) = \mathrm{Hom}_{\mathcal{H}_W}(J_{w_0}, \pi^I)$, and $\mathrm{Hom}_{\mathcal{H}_W}(J_{w_0}, \pi^I) \neq 0$ if and only if $\mathrm{Hom}_{\mathcal{H}_W}(\pi^I, J_{w_0}) = \mathrm{Hom}_{\mathcal{H}_W}(\pi^I, J(w_0)) \neq 0$. We saw in the proof of the proposition 3.11 that π^I is identified with J_T for some subset T of S as a $\mathcal{H}_W$-module, hence we have $\mathrm{Hom}_{\mathcal{H}_W}(\pi^I, J(w_0)) = \mathrm{Hom}_{\mathcal{H}_W}(J_T, J(w_0))$. Then by Lemma 3.9, $\mathrm{Hom}_{\mathcal{H}_W}(J_T, J(w_0)) \neq 0$ if and only if $sx < x$, $\forall s \in T$ and $\forall x \sim_L w_0$. Since w_0 is the longest element then the condition $\forall x \sim_L w_0$ says that $x = w_0$ and for all $s \in S$ we have $sw_0 < w_0$. The conclusion follows. $\square$

3.8 Transfer to the simple type case

Let $\Gamma = F[\beta]/F$ a finite field extension, $\mathcal{O}_\Gamma$ its ring of integers and k_Γ the residue field. Define $R = n/[\Gamma : F]$ and we have $R = ef$. Let (J, λ) a simple type in G, where J is a compact open subgroup in G and $\lambda = \kappa \otimes \sigma$ with κ a β-extension and σ the inflation of $\tau \otimes \ldots \otimes \tau$ (e-times), where τ a cuspidal representation of $GL_f(k_\Gamma)$.

Let $W = S_e$, and $\mathcal{H}_W$ the Hecke algebra of Coxeter group (W, S) as before. In this section $B := B(k_\Gamma)$ is the Borel subgroup in $\overline{G}_e = GL_e(k_\Gamma)$ and let $\overline{G} = GL_R(k_\Gamma)$. We will always identify $w \in W$ with a matrix in $\overline{G}_e$ or with a matrix in $\overline{G}$, depending on the context.

Let $\overline{P}$ be a subgroup of $\overline{G}$ consisting of is upper triangular matrices by blocs with bloc sizes $f \times f$. Let $\phi_w \in \mathcal{H}(\overline{G}, \sigma)$ is null outside $\overline{P} w \overline{P}$ such that

$\phi_w(p_1wp_2) = \sigma(p_1)\circ\phi_w(w)\circ\sigma(p_2)$ and $\phi_w(w)(y_1\otimes\ldots\otimes y_e) = y_{w(1)}\otimes\ldots\otimes y_{w(e)}$. The homomorphism of Hecke algebras, as in (5.6.1) [BK93]:

$$\begin{array}{rcccc} \Phi & : & \mathcal{H}_W & \rightarrow & \mathcal{H}(\overline{G},\sigma) \\ & & T_w & \mapsto & \phi_w \end{array}$$

is actually an isomorphism according to Theorem 5.1 in Chapter 1 [How85]. In fact one can carry out a calculation to prove that ϕ_w are generators of $\mathcal{H}(\overline{G},\sigma)$ and they satisfy the same relations as T_w in $\mathcal{H}_W$.

We have the following isomorphisms of Hecke algebras:

$$\mathcal{H}_W \simeq \mathrm{End}_{\overline{G}_e}(\mathrm{Ind}_B^{\overline{G}_e}1)$$

and

$$\mathcal{H}(\overline{G},\sigma) \simeq \mathrm{End}_{\overline{G}}(\mathrm{Ind}_{\overline{P}}^{\overline{G}}\sigma)$$

Let $\mathscr{M}(\overline{G}_e)$ be the category of $\overline{G}_e$-representations and $\mathscr{M}_\omega(\overline{G}_e)$ the full subcategory of $\mathscr{M}(\overline{G}_e)$ of all $\overline{G}_e$-representations whose irreducible constituents all have cuspidal support ω. Define:

$$\begin{array}{ccc} \mathscr{M}_1(\overline{G}_e) & \rightarrow & \mathcal{H}_W \\ \pi & \mapsto & \mathrm{Hom}_{\overline{G}_e}(\mathrm{Ind}_B^{\overline{G}_e}1,\pi) \end{array}$$

$$\begin{array}{rcccc} \Phi_* & : & \mathcal{H}_W - Mod & \rightarrow & \mathcal{H}(\overline{G},\sigma) - Mod \\ & & M & \mapsto & M\otimes_{\mathcal{H}_W}\mathcal{H}(\overline{G},\sigma) \end{array}$$

$$\begin{array}{ccc} \mathcal{H}(\overline{G},\sigma) - Mod & \rightarrow & \mathscr{M}_\omega(\overline{G}) \\ M & \mapsto & M\otimes_{\mathcal{H}(\overline{G},\sigma)}\mathrm{Ind}_{\overline{P}}^{\overline{G}}\sigma \end{array}$$

Let $F_e : \mathscr{M}_1(\overline{G}_e) \rightarrow \mathscr{M}_\omega(\overline{G})$ the composition of these 3 functors.

First notice that $F_e(\mathrm{Ind}_B^{\overline{G}_e}1) = \mathrm{Ind}_{\overline{P}}^{\overline{G}}\sigma$. Let Q be any standard parabolic of $\overline{G}_e$. We obtain a standard parabolic $\tilde{Q}$ of $\overline{G}$ from Q by enlarging each entry of Q to a bloc of size $f\times f$.

Lemma 3.14. *Let Q be a standard parabolic of $\overline{G}_e$ and $\tilde{Q}$ as above, a standard parabolic of $\overline{G}$. Then:*

$$F_e(\mathrm{Ind}_Q^{\overline{G}_e}1) = \mathrm{Ind}_{\tilde{Q}}^{\overline{G}}\sigma$$

Proof. Let W_Q be the parabolic subgroup of W associated to Q. We have

$$F_e(\mathrm{Ind}_Q^{\overline{G}_e}1) = \mathrm{Hom}_{\overline{G}_e}(\mathrm{Ind}_B 1, \mathrm{Ind}_Q^{\overline{G}_e}1) \otimes_{\mathcal{H}(\overline{G},\sigma)} \mathrm{Ind}_{\overline{P}}^{\overline{G}}\sigma$$

$$= \bigoplus_{w \in W/W_Q} \mathrm{Hom}_{B^w \cap Q}(1, 1^w) \otimes_{\mathcal{H}(\overline{G},\sigma)} \mathrm{Ind}_{\overline{P}}^{\overline{G}}\sigma$$

We identify, as usual, $\mathrm{Hom}_{B^w \cap Q}(1, 1^w)$ with the set of functions in $\mathcal{H}_W$ supported on BwQ. Via the isomorphism Φ of Hecke algebras, the set functions in $\mathcal{H}_W$ supported on BwQ is in bijection with the set of functions in $\mathcal{H}(\overline{G}, \sigma)$ supported on $\overline{P}w\tilde{Q}$ and this set is indetified with the intertwining set $\mathrm{Hom}_{\overline{P}^w \cap \tilde{Q}}(\sigma, \sigma^w)$. It follows, that:

$$F_e(\mathrm{Ind}_Q^{\overline{G}_e}1) \simeq \bigoplus_{w \in W/W_Q} \mathrm{Hom}_{\overline{P}^w \cap \tilde{Q}}(\sigma, \sigma^w) \otimes_{\mathcal{H}(\overline{G},\sigma)} \mathrm{Ind}_{\overline{P}}^{\overline{G}}\sigma$$

$$= \mathrm{Hom}_{\overline{G}}(\mathrm{Ind}_{\overline{P}}^{\overline{G}}\sigma, \mathrm{Ind}_{\tilde{Q}}^{\overline{G}}\sigma) \otimes_{\mathcal{H}(\overline{G},\sigma)} \mathrm{Ind}_{\overline{P}}^{\overline{G}}\sigma$$

The result follows. □

Let $st(\tau, e)$ be a representation of $\overline{G}_{fe}$, defined as the unique nondegenerate irreducible representation with cuspidal support $\tau \otimes \ldots \otimes \tau$ (e-times). Since F_e is exact, $F_e(st(1,e)) = \frac{F_e(\mathrm{Ind}_B^{\overline{G}_e}1)}{\sum_{Q \supsetneq B} F_e(\mathrm{Ind}_Q^{\overline{G}_e}1)}$ and by previous lemma, $F_e(st(1,e)) = st(\tau, e)$.

Lemma 3.15. *Let L be the Levi subgroup of Q. The following diagram commutes:*

$$\begin{tikzcd}
\mathscr{M}_1(\overline{G}_e) \arrow[r, "F_e"] & \mathscr{M}_\omega(\overline{G}) \\
\mathscr{M}_1(L) \arrow[r, "F_L"] \arrow[u, "\mathrm{Ind}_Q^{\overline{G}_e}"] & \mathscr{M}_{\omega_L}(\tilde{L}) \arrow[u, "\mathrm{Ind}_{\tilde{Q}}^{\overline{G}}"']
\end{tikzcd}$$

where the horizontal arrows are an equivalence of categories.

Proof. It is enough to check that this diagram commutes for every irreducible representation of $\mathscr{M}_1(L)$. By definition, the irreducible representations of $\mathscr{M}_1(L)$ are just unramified characters of L. Moreover, by Lemma 3.14 we have that $F_e(\mathrm{Ind}_Q^{\overline{G}_e}1) = \mathrm{Ind}_{\tilde{Q}}^{\overline{G}}F_L(1)$. The same identity holds if 1 is replaced any unramified character of L, by the same argument as in Lemma 3.14. □

As in trivial type case we identify the partitions and partition valued functions. Let $\tilde{\mathcal{P}}$ be the partition $(e_1 f, \dots, e_k f)$ of R associated to parabolic subgroup $\tilde{Q}$ and $\mathcal{P}$ the partition $(e_1, \dots, e_k)$ of e associated to parabolic subgroup Q. Define $\pi(\tau, \tilde{\mathcal{P}}) = \mathrm{Ind}_{\tilde{Q}}^{\overline{G}} st(\tau, e_1) \otimes \dots \otimes st(\tau, e_k)$ and $\sigma(\tau, \tilde{\mathcal{P}})$ the representation of $\overline{G}$ that occurs in $\pi(\tau, \tilde{\mathcal{P}})$ with multiplicity 1 and not in $\pi(\tau, \mathcal{Q})$ if $\mathcal{Q} > \tilde{\mathcal{P}}$.

Lemma 3.16. *Let Q be a standard parabolic of $\overline{G}_e$ and $\tilde{Q}$ as above, a standard parabolic of $\overline{G}$. Then:*

$$F_e(\sigma(1, \mathcal{P})) = \sigma(\tau, \tilde{\mathcal{P}})$$

Proof. By previous lemma we have that:

$$F_e(\pi(1, \mathcal{P})) = \mathrm{Ind}_{\tilde{Q}}^{\overline{G}} F_L(st(1, e_1) \otimes \dots \otimes st(1, e_k))$$

$$= \mathrm{Ind}_{\tilde{Q}}^{\overline{G}} F_{e_1}(st(1, e_1)) \otimes \dots \otimes F_{e_k}(st(1, e_k)) = \pi(\tau, \tilde{\mathcal{P}})$$

Since F_e is exact:

$$F_e(\sigma(1, \mathcal{P})) = \frac{F_e(\pi(1, \mathcal{P})}{\sum_{\mathcal{Q} < \mathcal{P}} F_e(\pi(1, \mathcal{Q})} = \frac{\pi(\tau, \tilde{\mathcal{P}})}{\sum_{\mathcal{Q} < \mathcal{P}} \pi(\tau, \tilde{\mathcal{Q}})} = \sigma(\tau, \tilde{\mathcal{P}})$$

□

The functor $\mathrm{Ind}_{J_{max}}^{K}(\kappa_{max} \otimes \cdot) : \mathscr{M}_\omega(\overline{G}) \to \mathscr{M}_\lambda(K)$ is an equivalence of categories according the discussion above Proposition 11 in Section 5 [SZ99]. Let $\sigma_{\tilde{\mathcal{P}}}(\lambda) = \mathrm{Ind}_{J_{max}}^{K}(\kappa_{max} \otimes \sigma(\tau, \tilde{\mathcal{P}}))$.

Note that the K-representation $\sigma_{\tilde{\mathcal{P}}}(\lambda)$ to $\mathcal{H}_W$-module $J(w_T)$ where T is the set of simple reflections corresponding to the partition $\mathcal{P}$. Hence we may mirror the result obtained in the category of representations generated by Iwahori fixed vectors to the category of representation admitting a simple type (J, λ):

Proposition 3.17. *Let π be an irreducible generic representation, with simple type (J, λ). The following statement are equivalent:*

1. $\mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), \pi) \neq 0$ *and* $\mathrm{Hom}_K(\sigma_{\mathcal{P}'}(\lambda), \pi) = 0$, *for all partitions valued functions $\mathcal{P}'$ such that $\mathcal{P} < \mathcal{P}'$.*

2. $\pi = i_P^G(L(\Delta_1) \otimes \ldots \otimes L(\Delta_k))$*, where* P *is the standard parabolic associated to the partition valued function* $\mathcal{P}$ *and all the segments* Δ_i *are not pairwise linked.*

Moreover if $\sigma_{\mathcal{P}}$ *satisfies the equivalent properties above, it occurs with multiplicity one in* π*.*

Let $\sigma_{min}(\lambda) := \sigma_{\mathcal{P}}(\lambda)$, where $\mathcal{P}$ is minimal for partial ordering on partition valued functions. Using the isomorphism between Iwahori-Hecke algebra and Hecke algebra of a simple type we can generalize the Lemma 3.13, as follows:

Lemma 3.18. *Let* π *be an irreducible generic representation that has a simple type* (J, λ)*, then* $\mathrm{Hom}_K(\sigma_{min}(\lambda), \pi) \neq 0$

3.9 Semi-simple type case. General case

Let now λ be some general semi-simple type. The second part of Main Theorem of section 8 in [BK98] gives a support preserving Hecke algebra isomorphism $j : \mathcal{H}(\overline{M}, \lambda_M) \to \mathcal{H}(G, \lambda)$(here $\overline{M}$ is a unique Levi subgroup of G which contains the $N_G(M)$-stabilizer of the inertia class D and is minimal for this property), and the section 1.5 gives a tensor product decomposition $\mathcal{H}(\overline{M}, \lambda_M) = \mathcal{H}_1 \otimes_E \ldots \otimes_E \mathcal{H}_s$, where $\mathcal{H}_i = \mathcal{H}(G_i, J_i, \lambda_i)$ is an affine Hecke algebras of type A and (J_i, λ_i) is some simple type with G_i some general linear group over a p-adic field.

Let M be a Levi subgroup of $P = MN$, then $K \cap M = \prod_{i=1}^{s} K_i$, where K_i a maximal compact subgroup of i-th factor in M. By definition, see the end of section 6 in [SZ99], the restriction of K-representation $\sigma_{\mathcal{P}}(\lambda)$ to $K \cap N$ is trivial, and $\sigma_{\mathcal{P}}(\lambda)|K \cap M \simeq \sigma_1 \otimes \ldots \otimes \sigma_s$ where $\sigma_i := \sigma_{\mathcal{P}_i}(\lambda_i)$ with obvious notations.

According to Theorem (8.5.1) in [BK93] the irreducible representation π is of form

$$\pi \simeq \pi_1 \times \ldots \times \pi_s$$

such that π_i is irreducible representation of G_i and contains the simple type (J_i, λ_i). Moreover the supersupidal support of π_i is disjoint from supersupidal support of π_j for $i \neq j$. Then

$$\mathrm{Hom}_K(\sigma, \pi) = \mathrm{Hom}_K(\sigma, \mathrm{Ind}_{K \cap P}^{K}(\pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s))$$

$$= \mathrm{Hom}_{K \cap P}(\sigma_{K \cap M}, \pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s)$$

$$= \mathrm{Hom}_{K\cap M}(\sigma_1 \otimes \ldots \otimes \sigma_s, \pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s)$$

is non zero if and only $\mathrm{Hom}_{K_i}(\sigma_i, \pi_i|K_i)$ is non zero for all i. Then from the simple type case we get the following proposition:

Proposition 3.19. *Let π be an irreducible generic representation, with semi-simple type (J, λ). The following statement are equivalent:*

1. $\mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), \pi) \neq 0$ *and* $\mathrm{Hom}_K(\sigma_{\mathcal{P}'}(\lambda), \pi) = 0$*, for all partitions valued functions $\mathcal{P}'$ such that $\mathcal{P} < \mathcal{P}'$.*

2. $\pi = i_P^G(L(\Delta_1) \otimes \ldots \otimes L(\Delta_k))$*, where P is the standard parabolic associated to the partition valued function $\mathcal{P}$ and all the segments Δ_i are not pairwise linked.*

   Moreover if $\sigma_{\mathcal{P}}$ satisfies the equivalent properties above, it occurs with multiplicity one in π.

Let $\sigma_{min}(\lambda) := \sigma_{\mathcal{P}}(\lambda)$, where $\mathcal{P}$ is minimal for partial ordering on partition valued functions. A general Hecke algebra is a tensor product of Iwahori-Hecke algebras, hence we have the following generalization of the Lemma 3.18 :

Lemma 3.20. *Let π be an irreducible generic representation that has a semi-simple type (J, λ), then* $\mathrm{Hom}_K(\sigma_{min}(\lambda), \pi) \neq 0$

3.10 Relation to monodromy of the associated Weil-Deligne representation

Let π any irreducible generic representation. Let r be the associated Weil-Deligne representation to π by classical local Langlands correspondence.

Then by Bernstein-Zelevinsky classification there are supercuspidal representations π_i $(1 \leq i \leq s)$ and segments $\Delta_{i,j} = (\pi_i \otimes \chi_{i,j}) \otimes \ldots \otimes (\pi_i \otimes \chi_{i,j} \otimes |\det|^{k_{i,j}-1})$ $(1 \leq j \leq r_i)$, where $\chi_{i,j}$ are unramified characters and $k_{i,j}$ are positive integers, such that:

$$\pi \simeq L(\Delta_{1,1}) \times \ldots \times L(\Delta_{1,r_1}) \times \ldots \times L(\Delta_{s,1}) \times \ldots \times L(\Delta_{s,r_s})$$

Notice that, since π is generic, all the segments $\Delta_{i,j}$ and $\Delta_{i',j'}$ are not linked for $i \neq i'$, this means that any permutation of blocs $L(\Delta_{i,1}) \times \ldots \times$

$L(\Delta_{i,r_i})$ gives an isomorphic representation. Then the Weil-Deligne representation is of the form :

$$r = \mathrm{Sp}_{k_{1,1}}(\rho_{1,1}) \oplus \ldots \oplus \mathrm{Sp}_{k_{1,r_1}}(\rho_{1,r_1}) \oplus \ldots \oplus \mathrm{Sp}_{k_{s,1}}(\rho_{s,1}) \oplus \ldots \oplus \mathrm{Sp}_{k_{s,r_s}}(\rho_{s,r_s})$$

where $\rho_{i,j}$ is the irreducible representation corresponding to the supercuspidal representation $\pi_i \otimes \chi_{i,j}$ and the monodromy operator N on r is given by

$$N = N_{1,1} \oplus \ldots \oplus N_{1,r_1} \oplus \ldots \oplus N_{s,1} \oplus \ldots \oplus N_{s,r_s}$$

so N is diagonal by blocs with each bloc $N_{i,j}$ is the full monodromy on the space $\mathrm{Sp}_{k_{i,j}}(\rho_{i,j})$. We see that the shape of the monodromy operator is determined by the size of segments in π as well by their number. In other words the shape of N is determined by a partition valued function $\mathcal{P}$ according to Proposition 3.19, because π always has a type by the main Theorem 1.1 of [BK99].

3.11 Generic representations

We are given an inertial class $\Omega = [M, \rho]_G$, where ρ is a supercuspidal representation of M and an Ω-type (J, λ) with $J \subset K$ a compact open subgroup of G. Write $\mathfrak{Z}_\Omega$ for the centre of Bernstein component of Ω.

Choose a partition valued function $\mathcal{P}^{min}$ which is minimal for partial ordering as in [SZ99]. From now on let $\sigma_{min}(\lambda) := \sigma_{\mathcal{P}^{min}}(\lambda)$ with the notations of section 6 in [SZ99], unless otherwise is specified. The K-representation $\sigma(\tau)$ was defined by choosing a maximal partition valued function, we will denote it $\sigma_{max} := \sigma(\tau)$.

Proposition 3.21. *Let π be an irreducible representation in the Berstein component Ω, then $\mathrm{Hom}_K(\sigma_{min}(\lambda), \pi) \neq 0$ if and only if π is generic.*

Proof. In this proof $\sigma := \sigma_{min}(\lambda)$. If π is a generic representation, then $\mathrm{Hom}_K(\sigma, \pi) \neq 0$ follows from Lemma 3.20.

Without loss of generality we may work with $\overline{\mathbb{Q}}_p$-coefficients, and we will do so. Assume now that $\mathrm{Hom}_K(\sigma, \pi) \neq 0$. Let's first deal with a particular case before the general case.

1. Simple type case. If π is supercuspidal, it is generic and there is nothing to prove. So assume that π contains a simple type (J, λ) which is not maximal. In this case $\Omega = [GL_r(F)^e, \omega \otimes \ldots \otimes \omega]_G$ where the tensor product

$\rho := \omega \otimes \ldots \otimes \omega$ is taken e times and ω is a supersupidal represntation of $GL_r(F)$. According to description of Hecke algebras in section (5.6) of [BK93] there is a support preserving isomorphism of Hecke algebras $\mathcal{H}(G_L, I_L, 1) \simeq \mathcal{H}(G, J, \lambda)$, where L is a well defined extension of F(denoted K in [BK93]), $G_L = GL_e(L)$ with I_L the Iwahori subgroup of G_L and K_L be a maximal compact subgroup of G_L.

The representation π is a Langlands quotient of the form $Q(\Delta_1, \ldots, \Delta_s)$ (previously denoted $L(\Delta_1, \ldots, \Delta_s)$) such that for $i < j$ the segment Δ_i does not precede Δ_j. After twisting π by some unramified character we may assume that all the segments are of the form $\Delta_i = [\omega(\alpha_i), \omega(\alpha_i + e_i - 1)]$, where α_i is some real number and e_i an integer such that $\sum_{i=1}^{s} e_i = e$. Here the notation $\omega(\alpha_i)$ means that $\omega(\alpha_i) := \omega \otimes |\det|^{\alpha_i}$. If $s = 1$ then π is generic. Assume that $s > 1$.

According to Theorem 7.6.20 in [BK93], the diagram

$$\begin{tikzcd} \mathcal{H}(G, J, \lambda) \arrow[r, "\Phi"] & \mathcal{H}(G_L, I_L, 1) \\ \mathcal{H}(M, J_M, \lambda_M) \arrow[u] \arrow[r, "\Phi_1^{\otimes e}"] & \mathcal{H}(T_L, T_L^{\circ}, 1) \arrow[u] \end{tikzcd}$$

is commutative, where the horizontal arrows are support preserving isomorphisms. This diagram in turn produces the following commutative diagram:

$$\begin{tikzcd} \mathcal{R}_\lambda(G) \arrow[r, "\mathrm{Hom}_J(\lambda, \bullet)"] & \mathcal{H}(G, J, \lambda) - Mod \arrow[r] & \mathcal{H}(G_L, I_L, 1) - Mod \arrow[r, "T_\lambda"] & \mathcal{R}_1(G_L) \\ \mathcal{R}_{\lambda_M}(M) \arrow[u, "i_P^G"] \arrow[r, "\mathrm{Hom}_{J_M}(\lambda_M, \bullet)"'] & \mathcal{H}(M, J_M, \lambda_M) - Mod \arrow[u] \arrow[r] & \mathcal{H}(T_L, T_L^{\circ}, 1) - Mod \arrow[u] \arrow[r, "T_1"] & \mathcal{R}_1(T_L) \arrow[u, "i_{B_L}^{G_L}"'] \end{tikzcd}$$

where the horizontal arrows are equivalences of categories, $T_\lambda = \bullet \otimes_{\mathcal{H}(G_L, I_L, 1)}$ c–$\mathrm{Ind}_{I_L}^{G_L} 1$ and $T_1 = \bullet \otimes_{\mathcal{H}(T_L, T_L^{\circ}, 1)}$ c–$\mathrm{Ind}_{T_L^{\circ}}^{T_L} 1$. It follow from this commutative diagram that

$$\Phi(\mathrm{Hom}_J(\lambda, i_P^G(\rho))) \otimes_{\mathcal{H}(G_L, I_L, 1)} \text{c–}\mathrm{Ind}_{I_L}^{G_L} 1$$
$$= i_{B_L}^{G_L}(\Phi_1^{\otimes e}(\mathrm{Hom}_{J_M}(\lambda_M, \rho)) \otimes_{\mathcal{H}(T_L, T_L^{\circ}, 1)} \text{c–}\mathrm{Ind}_{T_L^{\circ}}^{T_L} 1)$$

Observe that the representation c–$\mathrm{Ind}_{T_L^{\circ}}^{T_L} 1$ is canonically a rank 1 free $\mathcal{H}(T_L, T_L^{\circ}, 1)$-module. This observation allows to simplify the right hand side.

Since (J, λ) is a simple type, $\lambda_M = \lambda_0 \otimes \ldots \otimes \lambda_0$ (e times), $J_M = J_0^e$ and (J_0, λ_0) is a maximal simple type for the supercuspidal representation ω

$$\mathrm{Hom}_{J_M}(\lambda_M, \rho) = \mathrm{Hom}_{J_M}(\lambda_0 \otimes \ldots \otimes \lambda_0, \omega|J_0 \otimes \ldots \otimes \omega|J_0)$$
$$= \mathrm{Hom}_{J_0^e}(\lambda_0 \otimes \ldots \otimes \lambda_0, \lambda_0 \otimes \ldots \otimes \lambda_0)$$

The representation $\Phi_1^{\otimes e}(\mathrm{Hom}_{J_M}(\lambda_M, \rho)) \otimes_{\mathcal{H}(T_L, T_L^\circ, 1)} \text{c–Ind}_{T_L^\circ}^{T_L} 1$ is a trivial character of T_L. Then an object $i_P^G \rho$ in $\mathcal{R}_\lambda(G)$ corresponds to an object $i_{B_L}^{G_L} 1$ in $\mathcal{R}_1(G_L)$.

Let F be the composition of all the top horizontal arrows. Hence the functor $F : \mathcal{R}_\lambda(G) \longrightarrow \mathcal{R}_1(G_L)$ from above, is an equivalence of categories. Then

$$\mathrm{Hom}_G(\text{c–Ind}_K^G \sigma, \pi) = \mathrm{Hom}_{G_L}(F(\text{c–Ind}_K^G \sigma), F(\pi))$$

We know that π is an irreducible sub quotient of $i_P^G((\omega \otimes \chi_1 \circ \det) \otimes \ldots (\omega \otimes \chi_e \circ \det))$, where $\chi_1, \ldots, \chi_e$ are some unramified characters. Then by the equivalence of categories described above, $F(\pi)$ is an irreducible sub quotient of $F(i_P^G((\omega \otimes \chi_1 \circ \det) \otimes \ldots (\omega \otimes \chi_e \circ \det))) = i_{B_L}^{G_L}(\chi_1 \otimes \ldots \otimes \chi_e)$. Let now $\Delta = [\omega(\alpha), \omega(\alpha + e - 1)]$, a segment in G, where α is a real number. Then the commutative diagram above shows that the G-representation $i_P^G(\Delta)$ corresponds to G_L-representation $F(i_P^G(\Delta)) = i_{B_L}^{G_L}(\Delta_L)$, where $\Delta_L = [1(\alpha), 1(\alpha + e - 1)]$ is a segment in G_L and 1 is the trivial character of $L^\times$. We know that $i_P^G(\Delta)$ admits a unique irreducible quotient $Q(\Delta)$, so the G-representation $Q(\Delta)$ corresponds to the G_L representation $F(Q(\Delta)) = Q(\Delta_L)$.

The similar argument works with multiple segments. Therefore $F(\pi) = Q(\Delta'_1, \ldots, \Delta'_s)$, where $\Delta'_i = [1(\alpha_i), 1(\alpha_i + e_i - 1)]$ for all i.

Since the isomorphisms of Hecke algebras are support preserving, we also have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{R}_\lambda(G) \arrow[r, "\mathrm{Hom}_J(\lambda,\bullet)"] & \mathcal{H}(G, J, \lambda) - Mod \arrow[r] & \mathcal{H}(G_L, I_L, 1) - Mod \arrow[r, "T_\lambda"] & \mathcal{R}_1(G_L) \\
\mathcal{R}_\lambda(K) \arrow[r, "\mathrm{Hom}_J(\lambda,\bullet)"'] \arrow[u, "\text{c–Ind}_K^G"] & \mathcal{H}(K, J, \lambda) - Mod \arrow[r] \arrow[u] & \mathcal{H}(K_L, I_L, 1) - Mod \arrow[r, "T_{K_L}"'] \arrow[u] & \mathcal{R}_1(K_L) \arrow[u, "\text{c–Ind}_{K_L}^{G_L}"']
\end{tikzcd}$$

where $T_{K_L} = \text{c–Ind}_{I_L}^{G_L} 1$.

If we denote the composition of all the top horizontal arrow by F and the composition of all the bottom horizontal arrow by F_K, then $F(\text{c–Ind}_K^G \sigma) = \text{c–Ind}_{K_L}^{G_L} F_K(\sigma)$. The same argument that computes $F(\pi)$ shows that we also have $F_K(\sigma) = \sigma_{\mathcal{P}^{min}}(\text{trivial}) = st$, where st denotes the inflation of Steinberg representation of GL_n over a finite field. All together we have:

$$\mathrm{Hom}_G(\text{c–Ind}_K^G\sigma, \pi) = \mathrm{Hom}_{G_L}(F(\text{c–Ind}_K^G\sigma), F(\pi))$$

$$= \mathrm{Hom}_{G_L}(\text{c–Ind}_{K_L}^{G_L} F_K(\sigma), Q(\Delta'_1, \dots, \Delta'_s))$$

$$= \mathrm{Hom}_{K_L}(F_K(\sigma), Q(\Delta'_1, \dots, \Delta'_s)|K_L)$$

Observe that $Q(\Delta'_1, \dots, \Delta'_s)$ is generic if and only if π is generic. So we are reduced to consider the case when $(J, \lambda) = (I, 1)$. That's what we will assume from now on. So π is smooth irreducible representation of G such that $\pi^I \neq 0$ and $\sigma = st$. Assume that $\mathrm{Hom}_K(st, \pi) \neq 0$. We identify $\mathfrak{Z}_\Omega \simeq \mathrm{End}_G(\text{c–Ind}_K^G 1)$ via Corollary 2.18. The action of $\mathfrak{Z}_\Omega$ on π gives a ring homomorphism : $\chi : \mathfrak{Z}_\Omega \longrightarrow \mathrm{End}_G(\pi) \simeq \overline{\mathbb{Q}}_p$. Define $\pi' := \text{c–Ind}_K^G 1 \otimes_{\mathfrak{Z}_\Omega, \chi} \overline{\mathbb{Q}}_p$ and let $\mathrm{soc}_G \pi'$ the G-socle of π'.

By corollary 3.11 [CEG$^+$16], $\mathrm{soc}_G \pi'$ is irreducible and generic. Then we have $\mathrm{Hom}_K(st, \mathrm{soc}_G \pi') \neq 0$ by Lemma 3.20. Again by corollary 3.11[CEG$^+$16] we know that π is an irreducible subquotient of π' and $\mathrm{soc}_G \pi'$ is the unique irreducible subquotient of π', i.e. all the other irreducible subquotients of π' are non-generic. Having all this, it would be enough to show that $\mathrm{Hom}_K(st, \pi')$ is one-dimensional.

Indeed, assume that $\mathrm{Hom}_K(st, \pi')$ is one-dimensional. Let π'' be any irreducible subquotient of π', then since the functor $\mathrm{Hom}_K(st, .)$ is exact, we would have that $\mathrm{Hom}_K(st, \pi'')$ is an irreducible subquotient of $\mathrm{Hom}_K(st, \pi')$ and that $\dim \mathrm{Hom}_K(st, \pi'') \leq \dim \mathrm{Hom}_K(st, \pi') = 1$. In particular we have that $\mathrm{Hom}_K(st, \pi') \simeq \mathrm{Hom}_K(st, \mathrm{soc}_G \pi')$. Let π'' be an irreducible subquotient of π', which is not generic, then $\mathrm{Hom}_G(\pi'', \mathrm{soc}_G \pi') = 0$. Since the functor $\mathrm{Hom}_K(st, .)$ is exact, we also have $\mathrm{Hom}(\mathrm{Hom}_K(st, \pi''), \mathrm{Hom}_K(st, \mathrm{soc}_G \pi')) = 0$. Therefore $\mathrm{Hom}_K(st, \pi'')$ and $\mathrm{Hom}_K(st, \mathrm{soc}_G \pi')$ are not isomorphic. We must have $\mathrm{Hom}_K(st, \pi'') = 0$.

If π was not generic and we know that π is a subquotient of π' it would follow by the discussion above that $\mathrm{Hom}_K(st, \pi) = 0$, a contradiction. So π must be generic.

Let's prove now that $\dim \mathrm{Hom}_K(st, \pi') = 1$. It follows from the proof of corollary 3.11 [CEG$^+$16], that π' in an unramified principal series representation. Then,

$$\pi'|K \simeq \mathrm{Ind}_{B \cap K}^K 1$$

Frobenius reciprocity gives:

$$\mathrm{Hom}_K(st, \pi'|K) = \mathrm{Hom}_{B\cap K}(st, 1)$$

Since $K_1 = \{x \in K | x \equiv 1 \mod \mathfrak{p}\}$ acts trivially on st and $I = (K \cap B)K_1$, then

$$\mathrm{Hom}_{B\cap K}(st, 1) = \mathrm{Hom}_I(st, 1)$$

Finally $\dim \mathrm{Hom}_I(st, 1) = \dim st^I = 1$.

2. Semi-simple type case (general case). Let now λ be some general semi-simple type. The second part of Main Theorem of section 8 in [BK98] gives a support preserving Hecke algebra isomorphism $j : \mathcal{H}(\overline{M}, \lambda_M) \to \mathcal{H}(G, \lambda)$(here $\overline{M}$ is a unique Levi subgroup of G which contains the $N_G(M)$-stabilizer of the inertia class D and is minimal for this property), and the section 1.5 gives a tensor product decomposition $\mathcal{H}(\overline{M}, \lambda_M) = \mathcal{H}_1 \otimes_{\overline{\mathbb{Q}}_p} \ldots \otimes_{\overline{\mathbb{Q}}_p} \mathcal{H}_s$, where $\mathcal{H}_i = \mathcal{H}(G_i, J_i, \lambda_i)$ is an affine Hecke algebras of type A and (J_i, λ_i) is some simple type with G_i some general linear group over a p-adic field.

Let M be a Levi subgroup of $P = MN$, then $K \cap M = \prod_{i=1}^s K_i$, where K_i a maximal compact subgroup of i-th factor in M. By definition, see the end of section 6 in [SZ99], the restriction of K-representation σ to $K \cap N$ is trivial, and $\sigma|K \cap M \simeq \sigma_1 \otimes \ldots \otimes \sigma_s$ where $\sigma_i := \sigma_{\mathcal{P}_i^{min}}(\lambda_i)$ with obvious notations.

According to Theorem (8.5.1) in [BK93] the irreducible representation π is of form

$$\pi \simeq \pi_1 \times \ldots \times \pi_s,$$

such that π_i is irreducible representation of G_i and contains the simple type (J_i, λ_i). Moreover the supersupidal support of π_i is disjoint from supersupidal support of π_j for $i \neq j$. Then

$$\mathrm{Hom}_K(\sigma, \pi) = \mathrm{Hom}_K(\sigma, \mathrm{Ind}_{K\cap P}^K(\pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s))$$

$$= \mathrm{Hom}_{K\cap P}(\sigma_{K\cap M}, \pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s)$$

$$= \mathrm{Hom}_{K\cap M}(\sigma_1 \otimes \ldots \otimes \sigma_s, \pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s)$$

is non zero if and only $\mathrm{Hom}_{K_i}(\sigma_i, \pi|K_i)$ is non zero for all i. By simple type case π_i is a generalized steinberg representation. It follows that π is generic. $\square$

Lemma 3.22. *We have* $\dim \mathrm{Hom}_K(\sigma_{min}(\lambda), \pi) = 1$*, for* π *an irreducible generic representation of* G *in* Ω*.*

Proof. Let $x \in$ m-Spec $\mathfrak{Z}_\Omega$ a maximal ideal defined by π. Since π is generic we have that $\mathrm{Hom}_K(\sigma_{min}(\lambda), \pi) \neq 0$ by Proposition 3.21. It follows that we have c–$\mathrm{Ind}_K^G \sigma_{min}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(x) \twoheadrightarrow \pi$. Since the functor $\mathrm{Hom}_K(\sigma_{min}(\lambda), .)$ is exact, we have $\mathrm{Hom}_K(\sigma_{min}(\lambda), \text{c–}\mathrm{Ind}_K^G \sigma_{min}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(x)) \twoheadrightarrow \mathrm{Hom}_K(\sigma_{min}(\lambda), \pi)$. Moreover by Frobenius reciprocity we have that

$$\mathrm{Hom}_K(\sigma_{min}(\lambda), \text{c–}\mathrm{Ind}_K^G \sigma_{min}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(x))$$

$$= \mathrm{Hom}_G(\text{c–}\mathrm{Ind}_K^G \sigma_{min}(\lambda), \text{c–}\mathrm{Ind}_K^G \sigma_{min}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(x))$$

and then by Lemma 2.12:

$$\mathrm{Hom}_K(\sigma_{min}(\lambda), \text{c–}\mathrm{Ind}_K^G \sigma_{min}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(x))$$

$$\simeq \mathrm{Hom}_K(\sigma_{min}(\lambda), \text{c–}\mathrm{Ind}_K^G \sigma_{min}(\lambda)) \otimes_{\mathfrak{Z}_\Omega} \kappa(x)$$

Moreover by Corollary 2.18, $\mathrm{Hom}_K(\sigma_{min}(\lambda), \text{c–}\mathrm{Ind}_K^G \sigma_{min}(\lambda)) \simeq \mathfrak{Z}_\Omega$. Hence we have a surjective map of $\kappa(x)$-vector spaces:

$$\kappa(x) \twoheadrightarrow \mathrm{Hom}_K(\sigma_{min}(\lambda), \pi)$$

Then $1 \geq \dim \mathrm{Hom}_K(\sigma_{min}(\lambda), \pi)$ and this space is non-zero, hence it must be one-dimensional. $\square$

4 Potentially crystalline representations

Let D be a weakly admissible (φ, N)-module. Here we will prove that if we set $N = 0$ then there is a filtration on D, the underlying φ-module of D, such that the φ-module then D is again weakly admissible. First we recall few definitions and then we will study in detail the two dimensional case. Then we will see how some elementary inequalities allow us to deduce this result.

4.1 Notation

Recall that p is a prime number. In this section fix two finite extensions F (the base field) and E (the coefficient field) of $\mathbb{Q}_p$ such that $[F : \mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(F, E)|$ where $\mathrm{Hom}_{\mathbb{Q}_p}(F, E)$ denotes the set of all $\mathbb{Q}_p$-linear embeddings of the field F into the field E. We assume F is contained in an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We denote by $q = p^{f_0}$ the cardinality of the residue field of F and by $F_0 = \mathrm{Frac}(W(\mathbb{F}_q))$ its maximal unramified subfield. If $e := [L : \mathbb{Q}_p]/f_0$, we set $val_F(x) := e.val_{\mathbb{Q}_p}(x)$ (where $val_{\mathbb{Q}_p}(p) := 1$) and $|x|_F := q^{-val_F(x)}$ for any x in a finite extension of $\mathbb{Q}_p$. We denote by $W_F = W(\overline{\mathbb{Q}}_p/F)$ (resp. $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$) the Weil (resp. Galois) group of F and by $\mathrm{rec}_p : W(\overline{\mathbb{Q}}_p/F)^{ab} \to F^\times$ the reciprocity map sending the geometric Frobenius to the uniformizer.

Let L be a finite Galois extension of L and L_0 its maximal unramified subfield. We assume $[L_0 : \mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L_0, E)|$ and we let p^f be the cardinality of the residue field of L_0 and φ_0 be the Frobenius on F (raising to the p each component of the Witt vectors). Consider the following two categories:

1. the category $\mathrm{WD}_{L/F}$ of representations (r, N, V) of the Weil-Deligne group of F on a E-vector space V of finite dimension such that r is unramified when restricted to $W(\overline{\mathbb{Q}}_p/L)$.

2. the category $\mathrm{MOD}_{L/F}$ of quadruples $(\varphi, N, \mathrm{Gal}(L/F), D)$ where D is a free $L_0 \otimes_{\mathbb{Q}_p} E$-module of finite rank endowed with a Frobenius $\varphi : D \to D$, which is ϕ_0-semi-linear bijective map, an $L_0 \otimes_{\mathbb{Q}_p} E$-linear endomorphism $N : D \to D$ such that $N\varphi = p\varphi N$ and an action of $\mathrm{Gal}(L/F)$ commuting with φ and N.

There is a functor (due to Fontaine):

$$\mathrm{WD} : \mathrm{MOD}_{L/F} \longrightarrow \mathrm{WD}_{L/F}$$

The following proposition was proven in [BS07](Proposition 4.1):

Proposition 4.1. *The functor* $\mathrm{WD} : \mathrm{MOD}_{L/F} \longrightarrow \mathrm{WD}_{L/F}$ *is an equivalence of categories.*

Denote MOD a quasi inverse of the functor WD.

If D is an object of $\mathrm{MOD}_{L/F}$, we define:

$$t_N(D) = \frac{1}{[F : L_0]f} val_F(\det_{L_0}(\varphi^f|D))$$

For $\sigma : L \hookrightarrow K$, let $D_L = D \otimes_{L_0} L$ and :

$$D_{L,\sigma} = D_L \otimes_{L \otimes_{\mathbb{Q}_p} E} (L \otimes_{F,\sigma} E)$$

Then one has $D_L \simeq \prod_{\sigma : F \to E} D_{L,\sigma}$. To give an $L \otimes_{\mathbb{Q}_p} E$-submodule $\mathrm{Fil}^i D_L$ of D_L preserved by $\mathrm{Gal}(L/F)$ is the same thing as to give a collection $(\mathrm{Fil}^i D_{L,\sigma})_\sigma$ where $\mathrm{Fil}^i D_{L,\sigma}$ is a free $L \otimes_{F,\sigma} E$-submodule of $D_{L,\sigma}$ (hence a direct factor as $L \otimes_{F,\sigma} E$-module) preserved by the action of $\mathrm{Gal}(L/F)$. If $(\mathrm{Fil}^i D_{L,\sigma})_{\sigma,i}$ is a decreasing exhaustive separated filtration on D_L by $L \otimes_{\mathbb{Q}_p} E$-submodules indexed by $i \in \mathbb{Z}$ and preserved by $\mathrm{Gal}(L/F)$, we define:

$$t_H(D_L) = \sum_\sigma \sum_{i \in \mathbb{Z}} i \dim_L(\mathrm{Fil}^i D_{L,\sigma} / \mathrm{Fil}^{i+1} D_{L,\sigma})$$

Recall that such a filtration is called admissible if $t_H(D_L) = t_N(D)$ and if, for any L_0-vector subspace $D' \subseteq D$ preserved by φ and N with the induced filtration on D'_L, one has $t_H(D'_L) \leq t_N(D')$.

4.2 Weakly admissible modules

4.2.1 Examples

Let's consider the case of semi-stable representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We know that the category of semi-stable representations is equivalent to the category of filtered weakly admissible (φ, N)-modules. In what follows we will deal exclusively with that type of (φ, N)-modules.

Let's examine what happens in two dimensional case. In order to avoid specifying the extension E over which the module D is semi stable we will work with coefficients in $\overline{\mathbb{Q}}_p$. Assume that Hodge-Tate weights are $(0, k-1)$, with $k \geq 2$. The weakly admissible modules are of the form $D = \overline{\mathbb{Q}}_p e_1 \oplus \overline{\mathbb{Q}}_p e_2$ with $\mathrm{Fil}^i D = D$ for $i \leq 0$, $\mathrm{Fil}^1 D = \ldots = \mathrm{Fil}^{k-1} D \neq 0$ and $\mathrm{Fil}^i D = 0$ for $i \geq k$. Let v_p denote the p-adic valuation, such that $v_p(p) = 1$.

Consider the module given by:

$$\begin{cases} \varphi(e_1) & = & p\lambda e_1 \\ \varphi(e_1) & = & \lambda e_2 \\ \mathrm{Fil}^{k-1} D & = & \overline{\mathbb{Q}}_p(e_1 + \mathcal{L} e_2) \\ N(e_1) & = & e_2 \\ N(e_2) & = & 0 \\ \mathcal{L} & \in & \overline{\mathbb{Q}}_p \end{cases}$$

The non-trivial only (φ, N)-stable sub vector spaces is $\overline{\mathbb{Q}}_p e_2$ so the admissibility condition is equivalent to $k - 1 = 2v_p(\lambda) + 1$ and $0 = t_H(\overline{\mathbb{Q}}_p e_2) \leq t_N(\overline{\mathbb{Q}}_p e_2) = v_p(\lambda)$.

If we set $N = 0$ and we want to keep the same φ and the same filtration, then D the form:

$$\begin{cases} \varphi(e_1) & = & p\lambda e_1 \\ \varphi(e_1) & = & \lambda e_2 \\ \mathrm{Fil}^{k-1} D & = & \overline{\mathbb{Q}}_p(e_1 + e_2) \end{cases}$$

Notice this is the same filtration as above but we made a base change $e_2 \mapsto \mathcal{L} e_2$. From the admissibility of (φ, N)-module we get the admissibility of φ-module. Indeed the non-trivial φ-stable sub vector spaces are $\overline{\mathbb{Q}}_p e_1$, $\overline{\mathbb{Q}}_p e_2$ and we have $0 = t_H(\overline{\mathbb{Q}}_p e_1) < t_N(\overline{\mathbb{Q}}_p e_1) = v_p(\lambda) + 1$. This was the only inequality that was left to check.

For 3-dimensional examples the reader may look at [Par16] and check in the same way that for every isomorphism class with non zero monodromy operator, we get a crystalline representation when we kill the monodromy operator.

4.2.2 Inequalities of integers

Lemma 4.2. *Let $i_1 \leq i_2 \leq \ldots \leq i_d$, a sequence of integers in $\mathbb{Z}$ and let $c \in \mathbb{Q}$. Assume that we have*

$$\sum_{j=1}^{d} i_j \leq dc$$

then for all $1 \leq n \leq d$*, we have*

$$\sum_{j=1}^{n} i_j \leq nc$$

Proof. We have that $i_{n+1} \geq i_j$ for all $1 \leq j \leq n$. Then it follows that $ni_{n+1} \geq \sum_{j=1}^{n} i_j$. By a simple calculation this inequality is equivalent to

$$\frac{1}{n+1}\sum_{j=1}^{n+1} i_j - \frac{1}{n}\sum_{j=1}^{n} i_j = \frac{i_{n+1}}{n+1} + (\frac{1}{n+1} - \frac{1}{n})\sum_{j=1}^{n} i_j$$

$$= \frac{1}{n(n+1)}(ni_{n+1} - \sum_{j=1}^{n} i_j) \geq 0$$

Then it follows by induction that

$$\frac{1}{n}\sum_{j=1}^{n} i_j \leq \frac{1}{d}\sum_{j=1}^{d} i_j \leq c$$

The result follows. $\square$

Lemma 4.3. *Let* $i_1 \leq i_2 \leq \ldots \leq i_{d+k}$*, a sequence of integers in* $\mathbb{Z}$ *and let* $c_1, c_2 \in \mathbb{Q}$ *. Assume that we have*

$$\sum_{j=1}^{d} i_j \leq dc_1$$

and

$$\sum_{j=1}^{d} i_j + \sum_{j=d+1}^{d+k} i_j \leq dc_1 + k(c_2 + 1)$$

then for all $1 \leq n \leq k$*, we have*

$$\sum_{j=1}^{d} i_j + \sum_{j=d+1}^{d+n} i_j \leq dc_1 + n(c_2 + 1)$$

Proof. We prove first the desired inequality for $n = 1$. If $i_{d+1} \leq c_2 + 1$ then

$$\sum_{j=1}^{d} i_j + i_{d+1} \leq dc_1 + i_{d+1} \leq dc_1 + (c_2 + 1)$$

If $i_{d+1} \geq c_2 + 1$, then

$$\sum_{j=1}^{d} i_j + k i_{d+1} \leq \sum_{j=1}^{d} i_j + \sum_{j=d+1}^{d+k} i_j \leq dc_1 + k(c_2 + 1)$$

Subtracting $(k-1)i_{d+1} + c_2 + 1 + dc_1$ on both sides of the previous inequality, we get

$$\sum_{j=1}^{d} i_j + i_{d+1} - (dc_1 + c_2 + 1) \leq (k-1)(-i_{d+1} + c_2 + 1) \leq 0$$

Hence in any case we have that

$$\sum_{j=1}^{d} i_j + i_{d+1} \leq dc_1 + (c_2 + 1) = d(c_1 + (c_2 + 1)/d)$$

This proves the lemma for $n = 1$. Now replacing d by $d + 1$, c_1 by $c_1 + (c_2 + 1)/d$ and k by $k - 1$, so that we can repeat the procedure above for $n = 1$, i.e. we start now with inequalities:

$$\sum_{j=1}^{d+1} i_j \leq d(c_1 + (c_2 + 1)/d)$$

and

$$\sum_{j=1}^{d+1} i_j + \sum_{j=d+2}^{d+k} i_j \leq d(c_1 + (c_2 + 1)/d) + (k-1)(c_2 + 1)$$

then proceed as before to get:

$$(\sum_{j=1}^{d} i_j + i_{d+1}) + i_{d+2} \leq d(c_1 + (c_2 + 1)/d) + (c_2 + 1) = dc_1 + 2(c_2 + 1),$$

i.e. an inequality for $n = 2$. We proceed by induction in a similar fashion to prove this lemma. $\square$

By induction from the two previous lemmas we get the following result:

Lemma 4.4. *Let $i_1 \leq i_2 \leq \ldots \leq i_{d_1+\ldots+d_s}$, a sequence of integers in $\mathbb{Z}$ and let $c_i \in \mathbb{Q}$ for all $1 \leq i \leq s$. Assume that we have*

$$\sum_{j=1}^{d_1} i_j \leq d_1 c_1$$

$$\sum_{j=1}^{d_1} i_j + \sum_{j=d_1+1}^{d_1+d_2} i_j \leq d_1 c_1 + d_2 c_2$$

$$\vdots$$

$$\sum_{j=1}^{d_1} i_j + \ldots + \sum_{j=d_1+\ldots d_{s-1}+1}^{d_1+\ldots+d_s} i_j \leq d_1 c_1 + \ldots + d_s c_s$$

then for all $1 \leq k \leq s$ and for all $1 \leq n \leq d_k$, we have

$$\sum_{j=1}^{d_1} i_j + \ldots + \sum_{j=d_1+\ldots d_{k-1}+1}^{d_1+\ldots d_{k-1}+n} i_j \leq d_1 c_1 + \ldots + d_{k-1} c_{k-1} + n c_k$$

4.2.3 General case

Proposition 4.5. *Let $(\varphi, N, \mathrm{Gal}(L/F), D)$ be an object in the category $\mathrm{MOD}_{L/F}$ which has an admissible filtration. Then the object $(\varphi, 0, \mathrm{Gal}(L/F), D)$ has also an admissible filtration. The Hodge-Tate weights of $(\varphi, 0, \mathrm{Gal}(L/F), D)$ and $(\varphi, N, \mathrm{Gal}(L/F), D)$ are the same and those objects have the same action of φ on D.*

Proof. Let $(r, N, V) = \mathrm{WD}(D)$. Assume that the extension E of $\mathbb{Q}_p$ is big enough so that

$$(r, N, V) = \bigoplus_{i=1}^{s} (r_i, N_i, V_i)$$

where (r_i, N_i, V_i) is absolutely indecomposable of dimension d_i.

Let $(\varphi_i, N_i, D_i) = \mathrm{MOD}((r_i, N_i, V_i))$, then it is an absolutely indecomposable object in $\mathrm{MOD}_{L/F}$. Let $D_{i,0} = \mathrm{Ker}(N_i : D_i \to D_i)$ and $\varphi_{i,0} = \varphi_i | D_{i,0}$.

An indecomposable Weil-Deligne representation can be always written in a specific form (cf. 3.1.3 (ii) [Del75]). Then D_i can be written as

$$D_i = D_{i,0} \oplus D_{i,0}(1) \oplus \ldots \oplus D_{i,0}(b_i - 1)$$

where $D_{i,0}(n) \simeq D_{i,0}$ with $\varphi_{i,0}|D_{i,0}(n) = p^n \varphi_{i,0}$, $N_i|D_{i,0} = 0$ and N_i sends $D_{i,0}(n)$ to $D_{i,0}(n-1)$ via identity if $n > 0$. Note that $D_{i,0}$ is absolutely irreducible. Since $\varphi_{i,0}^f$ is $L_0 \otimes_{\mathbb{Q}_p} E$-linear and commutes with $\mathrm{Gal}(L/F)$ $\varphi_{i,0}$, then $\varphi_{i,0}^f$ is a scalar matrix with values in $F^\times$. Let $n_i = \dim D_{i,0}$ and $\varphi_{i,0}^f = \lambda_i.\mathrm{Id}$.

Choose an order on summands such that $\mathrm{val}_F(\lambda_1) \leq \mathrm{val}_F(\lambda_2) \leq \ldots \leq \mathrm{val}_F(\lambda_s)$.

For an embedding σ the Hodge-Tate weights are $i_{\sigma,1} < \ldots < i_{\sigma,n}$. Write $c_i = \frac{1}{[E:L_0]f}\mathrm{val}_F(\lambda_i)$ and $i_j = \sum_\sigma i_{\sigma,j}$.

Notice that the only sub-objects of D_i are $D_{i,0} \oplus \ldots \oplus D_{i,0}(k)$ for $0 \leq k \leq b_i - 1$. Then the admissibility condition of D, for these subobjects, gives us the following inequalities:

1. admissibility for $D_{1,0} \oplus \ldots \oplus D_{1,0}(k)$

$$\sum_{j=1}^{n_1} i_j + \ldots + \sum_{j=kn_1+1}^{(k+1)n_1} i_j \leq n_1 c_1 + \ldots + n_1(c_1 + k)$$

   for $0 \leq k \leq b_1 - 1$,

2. admissibility for $D_1 \oplus D_{2,0} \oplus \ldots \oplus D_{2,0}(k)$

$$\sum_{j=1}^{d_1} i_j + \sum_{j=d_1+1}^{d_1+n_2} i_j + \ldots + \sum_{j=d_1+kn_2+1}^{d_1+(k+1)n_2} i_j \leq$$

$$\leq n_1 c_1 + \ldots + n_1(c_1 + b_1 - 1) + n_2 c_2 + \ldots + n_2(c_2 + k)$$

   for $0 \leq k \leq b_2 - 1$,

$$\vdots$$

s. admissibility for $D_1 \oplus \ldots \oplus D_{s-1} \oplus D_{s,0} \oplus \ldots \oplus D_{s,0}(k)$

$$\sum_{j=1}^{d_1+\ldots+d_{s-1}} i_j + \sum_{j=d_1+\ldots+d_{s-1}+1}^{d_1+\ldots+d_{s-1}+n_s} i_j + \ldots + \sum_{j=d_1+\ldots+d_{s-1}+kn_s+1}^{d_1+\ldots+d_{s-1}+(k+1)n_s} i_j \leq$$

$$\leq \sum_{i=1}^{s-1}\sum_{l=0}^{b_i-1} n_i(c_i+l) + n_s c_s + \ldots + n_s(c_s+k)$$

for $0 \leq k \leq b_s - 1$, with an equality for $k = b_s$.

Then applying Lemma 4.4 to each set of inequalities above, from 1 to s, we get the following intermediate inequalities:

1. For $0 \leq k \leq b_1 - 1$,

$$\sum_{j=1}^{a} i_j \leq ac_1$$

for $1 \leq a \leq n_1$,

$$\vdots$$

$$\sum_{j=1}^{n_1} i_j + \ldots + \sum_{j=kn_1+1}^{a} i_j \leq n_1 c_1 + \ldots + a(c_1+k)$$

for $kn_1 + 1 \leq a \leq (k+1)n_1$,

2. For $0 \leq k \leq b_2 - 1$,

$$\sum_{j=1}^{d_1} i_j + \sum_{j=d_1+1}^{d_1+a} i_j \leq n_1 c_1 + \ldots + n_1(c_1 + b_1 - 1) + ac_2$$

for $d_1 + 1 \leq a \leq n_2$,

$$\vdots$$

$$\sum_{j=1}^{d_1} i_j + \sum_{j=d_1+1}^{d_1+n_2} i_j + \ldots + \sum_{j=d_1+kn_2+1}^{a} i_j \leq$$

$$\leq n_1 c_1 + \ldots + n_1(c_1 + b_1 - 1) + n_2 c_2 + \ldots + a(c_2+k)$$

for $d_1 + kn_2 + 1 \leq a \leq d_1 + (k+1)n_2$,

$$\vdots$$

s. For $0 \leq k \leq b_s - 1$,

$$\sum_{j=1}^{d_1+\ldots+d_{s-1}} i_j + \sum_{j=d_1+\ldots+d_{s-1}+1}^{a} i_j \leq \sum_{i=1}^{s-1} \sum_{l=0}^{b_i-1} n_i(c_i + l) + ac_s$$

for $d_1 + \ldots + d_{s-1} + 1 \leq a \leq d_1 + \ldots + d_{s-1} + n_s$,

$$\vdots$$

$$\sum_{j=1}^{d_1+\ldots+d_{s-1}} i_j + \sum_{j=d_1+\ldots+d_{s-1}+1}^{d_1+\ldots+d_{s-1}+n_s} i_j + \ldots + \sum_{j=d_1+\ldots+d_{s-1}+kn_s+1}^{d_1+\ldots+d_{s-1}+a} i_j \leq$$

$$\leq \sum_{i=1}^{s-1} \sum_{l=0}^{b_i-1} n_i(c_i + l) + n_s c_s + \ldots + a(c_s + k)$$

for $d_1 + \ldots + d_{s-1} + kn_s + 1 \leq a \leq d_1 + \ldots + d_{s-1} + (k+1)n_s$.

By equivalence $(i) \Leftrightarrow (ii)$ of Proposition 3.2 [BS07], all these inequalities tell that there is an admissible filtration on the φ-modules $(\varphi, D^{N=0})$. By construction the φ-modules $(\varphi, D^{N=0})$ has the same Hodge-Tate weights as (φ, N)-module D and both modules inherit the same action of φ. $\square$

5 Locally algebraic vectors

We will begin this section by recalling some notation from [CEG$^+$16]. The globalization of $\overline{r}$ constructed in section 2.1 [CEG$^+$16], provides us with a global imaginary CM field $\tilde{F}$ with maximal totally real subfield $\tilde{F}^+$. We refer the reader to the section 2.1 [CEG$^+$16], for more details and precise definitions. Let S_p denote the set of primes of $\tilde{F}^+$ dividing p. Fix $\mathfrak{p}|p$. For each $v \in S_p$, let $\tilde{v}$ be a choice of a place in $\tilde{F}$ lying above v, as defined in section 2.4 [CEG$^+$16].

Recall that there is a ring $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$, which is the unique reduced and p-torsion free quotient of $R^{\square}_{\tilde{\mathfrak{p}}}$(the universal $\mathcal{O}$-lifting ring of $\overline{r}$) corresponding to potentially semi-stable lifts of weight σ_{alg} and inertial type τ, which was constructed in [Kis08].

For $\mathcal{P}$ any partition valued function, define $\sigma_{\mathcal{P}} := \sigma_{\mathcal{P}}(\lambda) \otimes \sigma_{alg}$, where $\sigma_{\mathcal{P}}(\lambda)$ was defined above and σ_{alg} is the restriction to K of an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}_p} GL_n$ given by the Hodge-Tate weights. Fix a K-stable $\mathcal{O}$-lattice $\sigma^{\circ}_{\mathcal{P}}$ in $\sigma_{\mathcal{P}}$. Set

$$M_\infty(\sigma^{\circ}_{\mathcal{P}}) := \left(\mathrm{Hom}^{cont}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^{\circ}_{\mathcal{P}})^d)\right)^d$$

where M_∞ is $R_\infty[G]$-module constructed in section 2 [CEG$^+$16] by patching process, $(.)^d = \mathrm{Hom}^{cont}_{\mathcal{O}}(., \mathcal{O})$ denotes the Shikhof dual and R_∞ is a complete noetherian local $R^{\square}_{\tilde{\mathfrak{p}}}$-algebra with residue field $\mathbb{F}$. Moreover the module M_∞ is finitely generated over the completed group algebra $R_\infty[[K]]$.

Let x be a closed E-valued point of Spec $R_\infty(\sigma_{min})[1/p]$. The corresponding Galois representation r_x is given by the homomorphism $x : R^{\square}_{\tilde{\mathfrak{p}}} \to \mathcal{O}$, which we extend arbitrarily to homomorphism $x : R_\infty \to \mathcal{O}$. Then

$$V(r_x) := \mathrm{Hom}^{cont}_{\mathcal{O}}(M_\infty \otimes_{R_\infty, x} \mathcal{O}, E) \tag{10}$$

is an admissible unitary E-Banach space representation of G.

The main result of this section is to compute the locally algebraic vectors for $V(r_x)$, a candidate for the p-adic local Langlands correspondence, at the smooth points which lie on some automorphic component.

The arrangement of this part of a thesis is as follows: After recalling few definitions in section 5.1, we will construct the map $\mathcal{H}(\sigma_{min}) \longrightarrow R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$, which interpolates the local Langlands correspondence in section 5.2. Then in section 5.3 we will introduce a stratification of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$ with respect to the partition valued function, which will help us to study the support of $M_\infty(\sigma^\circ_{min})$. The goal of the section 5.4 will be to prove that the action of $\mathcal{H}(\sigma_{min})$ on $M_\infty(\sigma^\circ_{min})$ is compatible with the interpolation map constructed in section 5.2. In order to deal with the monodromy of potentially semi-stable Galois representations we will use results from section 3. We proved in section 3 that the partition valued functions encode information about the monodromy. This will be stated in more precise manner as Theorem 5.14 and the results about the support of $M_\infty(\sigma^\circ_{min})$ will be given in section 5.5. In the section 5.6, the we will compute locally algebraic vectors using a global point where we know the result already. The main result of that section is the Theorem 5.22, which is also the main result of this thesis.

5.1 Locally algebraic vectors. Definition. First properties

In this section I reproduced some parts of the appendix in [ST01]. Let $E/\mathbb{Q}_p$ be a finite extension and V a vector space over E. We begin with a definition, a vector $v \in V$ is termed **locally algebraic** if:

The orbit map of the vector v is locally algebraic, i.e. for $v \in V$, there is a compact open subgroup K_v in G, and a finite dimensional subspace U of V containing the vector v such that K_v leaves U invariant and operates on U via the restriction to K_v of a finite dimensional algebraic representation of the algebraic group scheme $\mathrm{Res}_{F/\mathbb{Q}_p} GL_n$.

Similarly a representation π of $G = GL_n(F)$ on V is called **locally algebraic** if:

1. The restriction of π to any open compact subgroup K of G is a sum of finite dimensional irreducible representations of K.

2. Any vector $v \in V$ is locally algebraic.

We have the following classification of locally algebraic representations of G. The following theorem was taken from the appendix of [ST01]:

Theorem 5.1. *1. Every irreducible locally algebraic representation π of G is the tensor product $\pi = \pi_1 \otimes \pi_2$ of an irreducible algebraic representation π_1 of G and of a smooth irreducible representation π_2 of G.*

2. *Conversely, the tensor product* $\pi = \pi_1 \otimes \pi_2$ *of an irreducible algebraic representation* π_1 *of* G *and of a smooth irreducible representation* π_2 *of* G *is an irreducible locally algebraic representation of* G.

Proof. (Sketch). By the definition of locally algebraic representation, there exists an algebraic representation π_1 of G, a compact open subgroup K_1 of G and a finite dimensional subspace U of π invariant under K_1 such that the action of K_1 on U is the restriction to K_1 of the representation π_1 of G. Clearly, we can assume that π_1 is an irreducible representation of G. Define,

$$\pi_2 = \varinjlim_{K} \mathrm{Hom}_K(\pi_1, \pi)$$

where the direct limit is taken over all the compact open subgroups K of G which have their common intersection as only $\{e\}$. Then one has to prove that the canonical map $\pi_1 \otimes \pi_2 \to \pi$ is G-equivariant and injective. See Theorem 1 in the appendix of [ST01] for more details. □

For any Banach vector space representation V we have the following functor $V \mapsto V^{l.alg}$, where $V^{l.alg}$ is the subspace of locally algebraic vectors in V.

Notation. Let π be an irreducible representation of G, then we will write $\pi^{l.alg} = \pi_{sm} \otimes \pi_{alg}$, where π_{sm} is π_1 of the previous theorem and π_{alg} is π_2.

5.2 Interpolation map

5.2.1 Construction in general case

First we extend few results from section 3 of [CEG$^+$16]. Let π be any irreducible representation, then the action of $\mathfrak{Z}_\Omega$ on π defines a E-algebra morphism $\chi_\pi : \mathfrak{Z}_\Omega \longrightarrow \mathrm{End}_G(\pi) \simeq E$.

Lemma 5.2. *Let* π *be an irreducible generic representation of* $G = GL_n(F)$. *Then by Bernstein-Zelevinsky classification there are pairwise non-isomorphic supercuspidal representations* π_i $(1 \leq i \leq s)$ *and segments* $\Delta_{i,j} = (\pi_i \otimes \chi_{i,j}) \otimes \ldots \otimes (\pi_i \otimes \chi_{i,j} \otimes |\det|^{k_{i,j}-1})$ $(1 \leq j \leq r_i)$, *where* $\chi_{i,j}$ *are unramified characters and* $k_{i,j}$ *are positive integers, such that:*

$$\pi \simeq L(\Delta_{1,1}) \times \ldots \times L(\Delta_{1,r_1}) \times \ldots \times L(\Delta_{s,1}) \times \ldots \times L(\Delta_{s,r_s})$$

Notice that all the segments $\Delta_{i,j}$ and $\Delta_{i',j'}$ are not linked for $i \neq i'$, this means that any permutation of blocs $L(\Delta_{i,1}) \times \ldots \times L(\Delta_{i,r_i})$ gives an isomorphic representation.

Define $\tilde{\Delta}_{i,j} := (\pi_i \otimes \chi_{i,j} \otimes |\det|^{1-k_{i,j}}) \otimes \ldots \otimes (\pi_i \otimes \chi_{i,j})$ and consider it as a representation of a corresponding Levi subgroup. Write,

$$\eta := \tilde{\Delta}_{1,1} \times \ldots \times \tilde{\Delta}_{1,r_1} \times \ldots \times \tilde{\Delta}_{s,1} \times \ldots \times \tilde{\Delta}_{s,r_s}$$

Notice that any permutation of blocs $\tilde{\Delta}_{i,1} \times \ldots \times \tilde{\Delta}_{i,r_i}$ gives a representation isomorphic to η. Then we have:

$$\text{c--Ind}_K^G \sigma_{max} \otimes_{\mathfrak{Z}_\Omega,\chi_\pi} E \simeq \eta$$

Moreover the action of $\mathfrak{Z}_\Omega$ on η is given by the maximal ideal χ_π.

Proof. The result follows by the argument similar to the one given in the proof of corollary 3.11 [CEG+16]. Let π' be G-cosocle of η. Then by Proposition 3.10 [CEG+16], we have $\text{c--Ind}_K^G \sigma_{max} \otimes_{\mathfrak{Z}_\Omega,\chi_{\pi'}} E \simeq \eta$. Since the G-socle of η is irreducible and occurs as a subquotient with multiplicity one, the action of $\mathfrak{Z}_\Omega$ on η factors through a maximal ideal, which is equal to χ_π, as π occurs as subquotient. Since π' is the G-cosocle of η, then π' satisfies the conditions of Proposition 3.10 [CEG+16] and we have $\chi_{\pi'} = \chi_\pi$. $\square$

We are given π an irreducible generic representation as in lemma above. We would like to describe χ_π in more concrete terms. In facts we would like to have a concrete description of the action of $\mathfrak{Z}_\Omega$ on $\pi \otimes |\det|^{\frac{n-1}{2}}$ in terms of eigenvalues of associated Weil-Deligne representation by local Langlands correspondence.

We will construct explicitly an analogous of the map η from Theorem 4.1 [CEG+16] in the potentially semi-stable case.

Let $W := W(k_F)$ be the ring of Witt vectors of the residue field of F, recall that ϖ is a uniformizer of F. Let $F_0 = W(k_F)[1/p]$, then F/F_0 is totally ramified. We will denote by $E(u) \in F_0[u]$ the Eisenstein polynomial of ϖ.

Let $R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min}) := R^\square_{\overline{r}}(\tau, \mathbf{v})$ the unique reduced and p-torsion free quotient of $R^\square_{\tilde{\mathfrak{p}}}$ corresponding to potentially semi-stable lifts of weight σ_{alg} (i.e. of weight $\mathbf{v}$) and inertial type τ, which was constructed in [Kis08] and $\rho^\square_{pst} : G_K \longrightarrow GL_n(R^\square_{\overline{r}}(\tau, \mathbf{v})[1/p])$ the universal lift corresponding to the identity homomorphism $id : R^\square_{\overline{r}}(\tau, \mathbf{v})[1/p] \longrightarrow R^\square_{\overline{r}}(\tau, \mathbf{v})[1/p]$.

It follows from the Theorem (2.5.5) (2) [Kis08] that $R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p]$ is endowed with a universal (φ, N)-module $D_{\bar{r}}^{\square}(\tau, \mathbf{v})$, which is a locally free $R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p] \otimes_{\mathbb{Q}_p} F_0$-module of rank n.

We will continue to use here the notation from the section 4. Let L be an extension of F such that every Galois representation r_x is semi-stable and L_0 its maximal unramified subfield. We assume $[L_0 : \mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L_0, E)|$ and we let p^f be the cardinality of the residue field of L_0.

Our goal here is to construct a canonical map $\mathcal{H}(\sigma_{min}) \longrightarrow R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p]$. We proceed in the following steps:

1. Take a smooth closed point $x \in \mathrm{Spec}\, R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p]$.

2. The point x corresponds is given by an E-algebra homomorphism $x : R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p] \longrightarrow E$ and it corresponds to n-dimensional Galois representation of G_F, denoted V_x. Let $D_{st,L}(V_x) := (B_{st} \otimes_{\mathbb{Q}_p} V_x)^{G_L}$, by construction this is also D_x, the specialization of $D_{\bar{r}}^{\square}(\tau, \mathbf{v})$ at closed point x. The admissible filtered $(\varphi, N, \mathrm{Gal}(L/F))$-module $D_x = D_{st,L}(V_x)$ is equipped with Frobenius endomorphism ϕ_x, which is the specilaization of the universal Frobenius φ on $D_{\bar{r}}^{\square}(\tau, \mathbf{v})$ at x, i.e. $\varphi \otimes \kappa(x) = \phi_x$.

3. Let $\tilde{\pi}$ be an irreducible representation of $GL_n(F)$ such that

$$\mathrm{rec}_p(\tilde{\pi} \otimes |\det|^{\frac{1-n}{2}}) = WD(D_x),$$

here $WD(D_x)$ is the Weil-Deligne representation associated to D_x via Fontaine's recipe. Let $\pi := \mathrm{rec}_p^{-1}(WD(D_x))$.

4. Theorem 1.2.7 [All16] implies that the representation π is generic, because x is a smooth point.

5. Let $\eta := \text{c–Ind}_K^G \sigma_{max} \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} E$, as in Lemma 5.2. By the Lemma 5.2 the action of $\mathfrak{Z}_\Omega$ on $\tilde{\pi} = \pi \otimes |\det|^{\frac{n-1}{2}}$ is identified with the action of $\mathfrak{Z}_\Omega$ on $\eta \otimes |\det|^{\frac{n-1}{2}}$. We will try to understand the action of $\mathfrak{Z}_\Omega$ on $\eta \otimes |\det|^{\frac{n-1}{2}}$.

6. We can interpret the action of $\mathfrak{Z}_\Omega$ on $\eta \otimes |\det|^{\frac{n-1}{2}}$ in terms of eigenvalues of the linearized canonical map obtained from the specialization of the absolute Frobenius φ at point x. For this we use the decomposition of spherical Hecke algebra of semi-simple type as a tensor product of

Iwahori Hecke algebras and this decomposition restricts to $\mathfrak{Z}_\Omega$. Then we use Satake isomorphism on each factor of $\mathfrak{Z}_\Omega$. In the Iwahori case there is just one factor in that tensor product decomposition.

7. From previous step we can "guess" a ring homomorphism $\beta : \mathfrak{Z}_\Omega \longrightarrow R^{\square}_{\bar{r}}(\tau, \mathbf{v})[1/p]$. This map is canonical in the sense that if there was another map β' it would coincide with β on all of the smooth points, and since the smooth points are dense by Theorem 3.3.4 [Kis08], the two maps have to be equal.

8. Finally $\mathcal{H}(\sigma_{min}(\lambda)) \to \mathcal{H}(\sigma_{min})$ given by $f \mapsto f.\sigma_{alg}$ is an isomorphism according to Lemma 1.4 [ST06] and by corollary 2.18 we have a canonical isomorphism $\mathfrak{Z}_\Omega \simeq \mathcal{H}(\sigma_{min}(\lambda))$. Composing β with those isomorphisms gives us the desired map.

Notice that it follows from the step 5 that the map β does not depend on monodromy.

We will prove the following Theorem:

Theorem 5.3. *There is an E-algebra homomorphism*

$$\beta : \mathcal{H}(\sigma_{min}) \longrightarrow R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$$

such that for any closed point x of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ with residue field E_x, the action of $\mathfrak{Z}_\Omega$ on a smooth G-representation $\pi_{sm}(r_x)$ factors as β composed with the evaluation map $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p] \longrightarrow E_x$.

The aim of the next sections is to carry out steps 1.-8. outlined above via very explicit computations. However before we embark on this task, we can give a "less computational" proof of that Theorem. Morally both proofs are based on the Lemma 4.3 [CEG+16]. Let's prove now the Theorem above.

Proof. Consider the following map, obtained by specialisation:

$$\gamma_G : \mathfrak{Z}_\Omega \longrightarrow \prod_{x \in \text{m-Spec}\, R^{\square}_{\bar{r}}(\tau, \mathbf{v})[1/p]} E'_x$$

where γ_G is defined on the factor corresponding to x by evaluating $\mathfrak{Z}_\Omega$ at the closed point in the Bernstein component Ω determined via local Langlands by x, and E'_x/E_x is a sufficiently large finite extension.

Consider as well the following map, also obtained by specialisation:

$$\gamma_{WD} : R^{\square}_{\overline{r}}(\tau, \mathbf{v})[1/p] \longrightarrow \prod_{x \in \text{m-Spec}\, R^{\square}_{\overline{r}}(\tau,\mathbf{v})[1/p]} E'_x$$

The map γ_{WD} is injective, because the ring $R^{\square}_{\overline{r}}(\tau, \mathbf{v})[1/p]$ is reduced and Jacobson.

We have the following diagram:

$$\begin{tikzcd}
\mathfrak{Z}_\Omega \arrow[r, "\gamma_G"] \arrow[dr, dotted, "I"] & \prod_{x \in \text{m-Spec}\, R^{\square}_{\overline{r}}(\tau,\mathbf{v})[1/p]} E'_x \\
W_F \arrow[u, "T"] \arrow[r, dotted, "?"] & R^{\square}_{\overline{r}}(\tau, \mathbf{v})[1/p] \arrow[u, "\gamma_{WD}"']
\end{tikzcd}$$

where $T : W_F \longrightarrow \mathfrak{Z}_\Omega$ be the pseudo-representation constructed in Proposition 3.11 of [Che09] and I is the map that we want to construct. Observe that the Lemma 3.24 [CEG$^+$16] tells us that the Cheneviers $E[\mathfrak{B}]$ is our $\mathfrak{Z}_\Omega$, so that the definition of the map T makes sense.

First we will construct a map ? such that the diagram above commutes. We can apply the Fontaine's recipe to the absolute Frobenius φ on $D^{\square}_{\overline{r}}(\tau, \mathbf{v})$, which is a free $R^{\square}_{\overline{r}}(\tau, \mathbf{v})[1/p] \otimes_{\mathbb{Q}_p} F_0$-module of rank n. Let's recall first this construction in the usual setting.

Let $x \in \text{m-Spec}\, R^{\square}_{\overline{r}}(\tau, \mathbf{v})[1/p]$ be a closed E-valued point. Let D_x and φ_x be the specializations of $D^{\square}_{\overline{r}}(\tau, \mathbf{v})$ and φ at x, respectively. Let L be a finite extension of F where all the Galois representation of the given inertial type τ are semi-stable and L_0 a subfield of L such that L/L_0 is totally ramified.

Then we deduce from the isomorphism

$$L_0 \otimes_{\mathbb{Q}_p} E \simeq \prod_{\sigma_0 : L_0 \hookrightarrow E} E,$$

the isomorphism

$$D_x = \prod_{\sigma_0 : L_0 \hookrightarrow E} D_{\sigma_0},$$

where $D_{\sigma_0} = (0, \ldots, 0, 1_{\sigma_0}, 0, \ldots, 0)D_x$, is the "$\sigma_0$-th coordinate of D_x". Fix now a σ_0. Set $W_x = D_{\sigma_0}$.

Let $w \in W_F$, define $\overline{w}$ to be the image of w in $\mathrm{Gal}(L/F)$ and let $\alpha(w) \in f_0\mathbb{Z}$ such that the action of w on $\overline{\mathbb{F}}_p$ is the $\alpha(w)$-power of the map $(x \mapsto x^p)$.

We can define an endomorphism of D_x by $r_x(w) := \overline{w} \circ \varphi_x^{-\alpha(w)}$, it can be shown that the restriction of $r(w)$ to W_x does not depend on σ_0.

We are interested in the trace of $r_x(w)$, we have trivially

$$\mathrm{Tr}(r_x(w)|D_x) = |\mathrm{Hom}_{\mathbb{Q}_p}(L_0, E)|\mathrm{Tr}(r_x(w)|W_x).$$

However since E is assumed to be large enough we have $|\mathrm{Hom}_{\mathbb{Q}_p}(L_0, E)| = [L_0 : \mathbb{Q}_p]$.

Observe that it makes sense to define for each $w \in W_F$ an endomorphism $r(w) := \overline{w} \circ \varphi^{-\alpha(w)}$ of $D^{\square}_{\overline{r}}(\tau, \mathbf{v})$ and we can also take its trace.

Define now the following map:

$$\begin{array}{rccc} \mathrm{Tr} & : & W_F & \longrightarrow & R^{\square}_{\overline{r}}(\tau, \mathbf{v})[1/p] \\ & & w & \longmapsto & \frac{1}{[L_0:\mathbb{Q}_p]}\mathrm{Tr}(r(w)|D^{\square}_{\overline{r}}(\tau, \mathbf{v})) \end{array}$$

Then by the construction of T, we have $\gamma_G \circ T = \gamma_{WD} \circ \mathrm{Tr}$, i.e. the diagram of sets

$$\begin{tikzcd}
\mathfrak{z}_\Omega \arrow[r, "\gamma_G"] \arrow[dr, dotted, "I"] & \prod\limits_{x \in \text{m-Spec } R^{\square}_{\overline{r}}(\tau,\mathbf{v})[1/p]} E'_x \\
W_F \arrow[u, "T"] \arrow[r, "\mathrm{Tr}"] & R^{\square}_{\overline{r}}(\tau, \mathbf{v})[1/p] \arrow[u, "\gamma_{WD}"']
\end{tikzcd}$$

commutes. Now we can define the map I, in order to so, it suffices to show that the image of $\mathfrak{z}_\Omega$ under γ_G is contained in the image of γ_{WD}. However by Lemma 4.5 [CEG+16] the image of T generates $\mathfrak{z}_\Omega$. Then, any element $a \in \mathfrak{z}_\Omega$ can be written as $a = \sum_i \mu_i T(g_i)$. It follows from the commutative diagram above, that the map I is given by $\sum_i \mu_i T(g_i) \mapsto \sum_i \mu_i \mathrm{Tr}(g_i)$. Let's prove that the map I is a well defined ring homomorphism.

The map I is well defined. Indeed, if we choose two different presentations of an element $a \in \mathfrak{z}_\Omega$, $a = \sum_i \mu_i T(g_i) = \sum_k \lambda_k T(h_k)$ then the elements $\sum_i \mu_i \mathrm{Tr}(g_i)$ and $\sum_k \lambda_k \mathrm{Tr}(h_k)$ should coincide. It is enough to prove that if $\sum_i \mu_i T(g_i) = 0$, then $\sum_i \mu_i \mathrm{Tr}(g_i) = 0$. Indeed, we have $0 = \gamma_G(0) = \gamma_G(\sum_i \mu_i T(g_i)) = \sum_i \mu_i \gamma_G(T(g_i)) = \sum_i \mu_i \gamma_{WD}(\mathrm{Tr}(g_i)) = \gamma_{WD}(\sum_i \mu_i \mathrm{Tr}(g_i))$, then $\sum_i \mu_i \mathrm{Tr}(g_i) = 0$ since γ_{WD} is injective.

Now, we will prove that I is a ring homomorphism. First notice that γ_G and γ_{WD} are already ring homomorphisms. Let any $a, b \in \mathfrak{Z}_\Omega$, then

$$\gamma_{WD}(I(a.b) - I(a).I(b)) = \gamma_{WD}(I(a.b)) - \gamma_{WD}(I(a)).\gamma_{WD}(I(b))$$

$$= \gamma_G(a.b) - \gamma_G(a).\gamma_G(b) = 0$$

Since γ_{WD} is injective it follows that $I(a.b) = I(a).I(b)$. Similarly we get $I(a+b) = I(a) + I(b)$ and $I(1) = 1$.

Let M be the Levi subgroup in the supercuspidal support of any irreducible representation in Ω, and $X(M)$ is the group of unramified characters of M. The group automorphism $X(M) \longrightarrow X(M)$ given by $\chi_M \to \chi_M|\det|^{\frac{(1-n)}{2}}$ gives rise to an E-isomorphism $\operatorname{Spec}\mathfrak{Z}_D \to \operatorname{Spec}\mathfrak{Z}_D$. The latter map is invariant under the $W(D)$-action (the point is that $|\det|$ is invariant under G-conjugation) so it descends to an E-isomorphism $\operatorname{Spec}\mathfrak{Z}_\Omega \to \operatorname{Spec}\mathfrak{Z}_\Omega$. Let $t_W : \mathfrak{Z}_\Omega \to \mathfrak{Z}_\Omega$ denote the induced isomorphism. Now we construct β' as the following composite map:

$$\mathfrak{Z}_\Omega \xrightarrow{t_W} \mathfrak{Z}_\Omega \xrightarrow{I} R_{\overline{r}}^{\square}(\tau, \mathbf{v})[1/p]$$

In order to get the map β as in the statement of the theorem, compose β' with the isomorphisms $\mathcal{H}(\sigma_{min}(\lambda)) \to \mathcal{H}(\sigma_{min})$ and $\mathfrak{Z}_\Omega \simeq \mathcal{H}(\sigma_{min}(\lambda))$. The the desired interpolation property of β', follows from the commutative diagram above. This can be easily be checked on points. $\square$

We will see next, that all the steps in the proof above can be made very explicit.

5.2.2 Construction in the Iwahori case

Assume now, that π has a trivial type $(I, 1)$, i.e. $\pi^I \neq 0$ and $\Omega = [T, 1]_G$. So the inertial type τ is also trivial. Let $\mathcal{H}(\sigma_{min}) := \operatorname{End}_G(\text{c–Ind}_K^G \sigma_{min})$, and by corollary 2.18 we have a canonical isomorphism $\mathfrak{Z}_\Omega \simeq \mathcal{H}(\sigma_{min}(\lambda))$ and also $\mathfrak{Z}_\Omega \simeq \mathcal{H}(\sigma_{max}(\lambda)) = \mathcal{H}(G, K)$. Moreover the map $\mathcal{H}(G, K) \to \mathcal{H}(\sigma_{max})$ given by $f \mapsto f.\sigma_{alg}$. is an isomorphism according to Lemma 1.4 [ST06]. By Satake isomorphism we have $\mathcal{H}(G, K) \simeq E[\theta_1, \ldots, \theta_{n-1}, (\theta_n)^{\pm 1}]$, where θ_r is a double coset operator $\left[K \begin{pmatrix} \varpi I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} K\right]$. Putting all these isomorphisms together we have $\mathcal{H}(\sigma_{min}(\lambda)) \simeq \mathfrak{Z}_\Omega \simeq E[\theta_1, \ldots, \theta_{n-1}, (\theta_n)^{\pm 1}]$. So in order to describe completely the action of $\mathfrak{Z}_\Omega$ on $\eta \otimes |\det|^{\frac{n-1}{2}}$, it would be enough to describe the action of each θ_r.

Let q be the cardinality of residue field $\mathcal{O}_F/\mathfrak{p}_F$ where $\mathcal{O}_F$ is the ring of integers of F and $\mathfrak{p}_F$ the maximal ideal. Let ϖ be a uniformizer of F.

We describe first the action of $\mathfrak{Z}_\Omega$.

Lemma 5.4. *Let $\psi := \psi_1 \otimes \ldots \otimes \psi_n$, an unramified character of torus T, and $\eta = i_B^G(\psi)$. Then θ_r acts on $\eta \otimes |\det|^{\frac{n-1}{2}}$ by a scalar:*

$$q^{\frac{r(1-r)}{2}} \sum_{\lambda_1 < \ldots < \lambda_r} \psi_{\lambda_1}(\varpi) \ldots \psi_{\lambda_r}(\varpi)$$

where the sum is taken through all the integers $1 \leq \lambda_i \leq n$ such that those inequalities are satisfied.

Proof. We follow closely Bump's lecture notes [Bum] on Hecke algebras, and adapts the argument therein for our needs. One may consult section 9, Proposition 40 in [Bum] for more details. It follows from Iwasawa decomposition that the space of K-invariants of $(\eta \otimes |\det|^{\frac{n-1}{2}})^K$ is one dimensional and that space generated by the function $f^\circ : bk \mapsto \delta_B^{1/2}(b)\psi'(b)$, with $b \in B$ and $k \in K$ and $\psi'(b) = \psi_1(b_{11})|b_{11}|^{\frac{n-1}{2}} \ldots \psi_n(b_{nn})|b_{nn}|^{\frac{n-1}{2}}$. Hence $\theta_r.f^\circ = c.f^\circ$, then $c = \theta_r.f^\circ(1)$. Using the a double coset decomposition:

$$K \begin{pmatrix} \varpi I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} K = \bigcup_{\beta \in \Lambda} \beta K,$$

where Λ is a complete set of representatives, we will compute $\theta_r.f^\circ(1)$. We have a freedom of choice for β's, so we can put them in a specific form. More precisely we have

$$K \begin{pmatrix} \varpi I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} K = \bigcup_{S=\{\lambda_1,\ldots,\lambda_r\}} \bigcup_{\beta \in \Lambda_S} \beta K,$$

where $\lambda_1 < \ldots < \lambda_r$ and $\beta \in \Lambda_S$ if and only if the following four conditions are satisfied:

1. β is upper triangular;
2. $\beta_{ii} = \varpi$ if $i \in S$ and $\beta_{ii} = 1$ if $i \notin S$;

3. β_{ij} is any element of $\mathcal{O}_F/\mathfrak{p}_F$ if $i<j$, $i \in S$ and $j \notin S$

4. all other entries are zero

The number of non-zero entries outside diagonal in a matrix $\beta \in \Lambda_S$ is $\sum_{i=1}^{r}(n-r-\lambda_i+i) = r(n-r)+r(r+1)/2-\sum_{i=1}^{r}\lambda_i$, therefore

$$|\Lambda_S| = q^{r(n-r)+r(r+1)/2-\sum_{i=1}^{r}\lambda_i}$$

it follows then,

$$\theta_r.f^\circ(1) = \sum_{\lambda_1<\ldots<\lambda_r}\sum_{\beta\in\Lambda_S} f^\circ(\beta) = \sum_{\lambda_1<\ldots<\lambda_r}|\Lambda_S|f^\circ(\beta)$$

Now let's compute $f^\circ(\beta) = \delta_B^{1/2}(\beta)\psi'(\beta)$. By definition we have

$$\delta_B^{1/2}(\beta) = \prod_{i=1}^{n}|\beta_{ii}|^{\frac{n-2i+1}{2}} = q^{-\sum_{i=1}^{r}\frac{n-2\lambda_i+1}{2}} = q^{-\frac{r(n+1)}{2}+\sum_{i=1}^{r}\lambda_i},$$

and

$$\psi'(\beta) = q^{-\frac{r(n-1)}{2}}\psi_{\lambda_1}(\varpi)\ldots\psi_{\lambda_r}(\varpi)$$

The total power of q is:

$$r(n-r)+r(r+1)/2-\sum_{i=1}^{r}\lambda_i-\frac{r(n+1)}{2}+\sum_{i=1}^{r}\lambda_i-\frac{r(n-1)}{2} = -\frac{r(r-1)}{2}$$

Finally

$$\theta_r.f^\circ(1) = q^{\frac{r(1-r)}{2}}\sum_{\lambda_1<\ldots<\lambda_r}\psi_{\lambda_1}(\varpi)\ldots\psi_{\lambda_r}(\varpi)$$

$\square$

Let $x : R_{\bar{r}}^{\square}(\tau,\mathbf{v})[1/p] \longrightarrow E$ be an E-algebra homomorphism with V_x the corresponding n-dimensional Galois representation of G_F. Here V_x is already semi-stable, so $L=F$ and $f=f_0$ (cf. notation from section 4.1). Let $D_{st}(V_x) := (B_{st}\otimes_{\mathbb{Q}_p}V_x)^{G_F}$, by construction this is also D_x, the specialization of $D_{\bar{r}}^{\square}(\tau,\mathbf{v})$ at closed point x. Then by Proposition 4.1, $WD(D_{st}(V_x))$ is a

Weil-Deligne representation that corresponds to π by Local Langlands correspondence, with normalization as in [HT01]. Assume that π is a generic representation. Let $\eta := \text{c–Ind}_K^G \sigma_{max} \otimes_{\mathfrak{Z}_{\Omega,\chi_\pi}} E$ as in Lemma 5.2. The admissible filtered $(\varphi, N, \mathrm{Gal}(L/F))$-module $D_x = D_{st}(V_x)$ is equipped with Frobenius endomorphism ϕ_x, which is the specilaization of the universal Frobenius φ on $D_{\bar{r}}^{\square}(\tau, \mathbf{v})$ at x, i.e. $\varphi \otimes \kappa(x) = \phi_x$.

Proposition 5.5. *Let $x : R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p] \longrightarrow E$, an E-algebra homomorphism as above. The double coset operator θ_r acts on $\eta \otimes |\det|^{\frac{n-1}{2}}$ (equivalently on $\pi \otimes |\det|^{\frac{n-1}{2}}$) as scalar multiplication by $q^{\frac{r(1-r)}{2}} \mathrm{Tr}(\bigwedge^r (\phi_x)^f)$.*

Proof. With the notations of Lemma 5.2 we have $s = 1$ and $\pi_1 = 1$. Then there is a partition of n, $\sum_{i=1}^t n_i = n$, such that $\pi := L(\Delta_1) \times \ldots \times L(\Delta_t)$, with $\Delta_i = \chi_i \otimes \ldots \otimes \chi_i |\cdot|^{n_i - 1}$ and $\chi_i \chi_j^{-1} \neq |\cdot|^{\pm 1}$ for all $i \neq j$. Then $\eta = \tilde{\Delta}_1 \times \ldots \times \tilde{\Delta}_t$, where $\tilde{\Delta}_i = \chi_i |\cdot|^{1-n_i} \otimes \ldots \otimes \chi_i$. Define $\psi := \psi_1 \otimes \ldots \otimes \psi_n = \tilde{\Delta}_1 \otimes \ldots \otimes \tilde{\Delta}_t$ an unramified character of torus T, so that $\eta \simeq i_B^G(\psi)$.

By previous lemma, θ_r acts on one dimensional space $(\eta \otimes |\det|^{\frac{n-1}{2}})^K$ as scalar multiplication by

$$q^{\frac{r(1-r)}{2}} s_r(\chi_1(\varpi) q^{n_1 - 1}, \ldots, \chi_1(\varpi), \ldots, \chi_t(\varpi) q^{n_t - 1}, \ldots, \chi_t(\varpi))$$

where s_r is the r^{th} symmetric polynomial in n variables.

The eigenvalues of ϕ_x^f are $\chi_1(\varpi)q^{n_1-1}, \ldots, \chi_1(\varpi), \ldots, \chi_t(\varpi)q^{n_t-1}, \ldots, \chi_t(\varpi)$. Then it follows that

$$s_r(\chi_1(\varpi) q^{n_1 - 1}, \ldots, \chi_1(\varpi), \ldots, \chi_t(\varpi) q^{n_t - 1}, \ldots, \chi_t(\varpi))$$

$$= \mathrm{Tr}(\bigwedge^r (\phi_x^f)) = \mathrm{Tr}(\bigwedge^r WD(D_{st}(V_x))(Frob_p))$$

where $Frob_p$ is the geometric Frobenius. Notice that the computations above do not depend on the choice of $Frob_p$. $\square$

If for an embedding σ the Hodge-Tate weights are $i_{\kappa,1} < \ldots < i_{\kappa,n}$, define $\xi_{j,\kappa} = -i_{\kappa,j} + (j-1)$. The highest weight of the algebraic representation σ_{alg} with respect to the upper triangular matrices is given by $\mathrm{diag}(x_1, \ldots, x_n) \mapsto \prod_{j=1}^n \prod_\kappa \kappa(x_j^{\xi_{\kappa,j}})$. Then we have to rescale θ_r by the factor

$$\varpi^{-\sum_\kappa \sum_{j=r}^{n} \xi_{\kappa,j}}$$

in order to be compatible with isomorphism, $\mathcal{H}(\sigma_{min}(\lambda)) \to \mathcal{H}(\sigma_{min})$ given by $f \mapsto f.\sigma_{alg}$.

Define $\tilde{\theta}_r = q^{\frac{r(r-1)}{2}}.\varpi^{-\sum_\kappa \sum\limits_{i=r}^{n} \xi_{\kappa,j}}.\theta_r$. Then we have a canonical isomorphism $\mathcal{H}(\sigma_{min}) \simeq E[\tilde{\theta}_1, \ldots, \tilde{\theta}_{n-1}, (\tilde{\theta}_n)^{\pm 1}]$. We can summarize the results of this section with the following theorem:

Theorem 5.6. *If τ is trivial, then define $\beta : \mathcal{H}(\sigma_{min}) \longrightarrow R^{\square}_{\bar{r}}(\tau, \mathbf{v})[1/p]$ by the assignment*

$$\tilde{\theta}_r \mapsto \varpi^{-\sum_\kappa \sum\limits_{i=r}^{n} \xi_{\kappa,j}} \mathrm{Tr}(\bigwedge^{r} \varphi^f),$$

where φ is the universal Frobenius on $D^{\square}_{\bar{r}}(\tau, \mathbf{v})$. Then the map β is an E-algebra homomorphism and β interpolates local Langlands correspondence, i.e. such that for any closed point x of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ with residue field E_x, the action of $\mathfrak{Z}_\Omega$ on a smooth G-representation $\pi_{sm}(r_x)$ factors as β composed with the evaluation map $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p] \longrightarrow E_x$.

Proof. Since $E[\tilde{\theta}_1, \ldots, \tilde{\theta}_{n-1}, (\tilde{\theta}_n)^{\pm 1}]$ is a polynomial E-algebra, the previous assignment β a ring homomorphism. Moreover the weak admissibility of D_x implies that $val_F(\varpi^{-\sum_\sigma \sum\limits_{i=r}^{n} \xi_{j,\sigma}} \mathrm{Tr}(\bigwedge^r \phi_x^f)) \geq 0$, and then $x(\beta(\theta_r))$ belongs to the ring of integers, for all r. It follows that the image of the map β is contained in the normalization of $R^{\square}_{\bar{r}}(\tau, \mathbf{v})[1/p]$, by Proposition 7.3.6 [dJ95].

As it was observed in the point 7. in the section 5.2.1 such a map interpolates local Langlands correspondence on all the closed points. □

5.3 Local deformation rings

We begin this section with some elementary linear algebra. Those preparatory results will help us to deal with monodromy of potentially semi-stable Galois representations. Indeed we will introduce locally algebraic representations $\sigma_{\mathcal{P}}$, where $\mathcal{P}$ is a partition valued function. The smooth part $\sigma_{\mathcal{P}}(\lambda)$ of $\sigma_{\mathcal{P}}$ was studied in section 3 and it was proven in that section how the monodromy of an irreducible generic representation can be read of the $\sigma_{\mathcal{P}}(\lambda)$'s that it contains. In a similar way, we may study the support of $M_\infty(\sigma^\circ_{min})$ by introducing a stratification that depends on the $\sigma_{\mathcal{P}}$'s. This will be dealt with in the next section. Here we will introduce a stratification of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$ with respect to any partition valued function $\mathcal{P}$, more precisely we will construct the rings $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$, which are reduced, p-torsion free quotient of $R^{\square}_{\overline{r}}(\tau, \mathbf{v})$, satisfying the following property: $x \in \operatorname{Spec} R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})[1/p]$ if and only if $\mathcal{P}_x \geq \mathcal{P}$.

Recall a few facts about partitions. Let $(\lambda_1, \dots, \lambda_l)$ be a partition of n, i.e. we have $n = \lambda_1 + \ldots + \lambda_l$ with $\lambda_1 \geq \ldots \geq \lambda_l > 0$. We say that a partition λ^c is conjugate of $\lambda = (\lambda_1, \dots, \lambda_l)$ if it is represented by the reflected diagram of the one associated to λ with respect to the line $y = -x$ with the coordinate of the upper left corner is taken to be $(0,0)$. We have that $\lambda^c_i = |\{k : \lambda_k \geq i\}|$.

Let M be any field, V a n-dimensional M-vector space and $N : V \to V$ a nilpotent endomorphism. Then the Jordan normal form of N is uniquely determined up conjugacy by a partition $(n_1, \dots, n_t)$, i.e the blocks are ordered by decreasing size $n_1 \geq \ldots \geq n_t$.

Lemma 5.7. *Let M be any field, V a n-dimensional M-vector space, with two nilpotent endomorphisms $N : V \to V$ and $N' : V \to V$. To the endomorphism N (resp. N') corresponds a partition $(n_1, \dots, n_t)$ (resp. $(n'_1, \dots, n'_s)$). Then the following statements are equivalent:*

1. $\forall i$, $\dim \operatorname{Ker}(N^i) \leq \dim \operatorname{Ker}(N'^i)$.
2. $\forall i$, $\sum_{k=1}^{i} n_k \geq \sum_{k=1}^{i} n'_k$.

Proof. The Jordan normal form gives an isomorphism $N \simeq \bigoplus_{k=1}^{t} N_k$, where N_k is a nilpotent operator of maximal rank on a n_k-dimensional vector space. Then:

$$\dim \operatorname{Ker}(N^i_k) = \begin{cases} i, & \text{for } i \leq n_k \\ n_k, & \text{for } i > n_k \end{cases}$$

and $\dim \operatorname{Ker}(N^i) = \sum_{k=1}^{t} \dim \operatorname{Ker}(N^i_k) = \sum_{k=1}^{t} \min(i, n_k)$.
Let $\kappa_j = \dim \operatorname{Ker}(N^j) - \dim \operatorname{Ker}(N^{j-1})$ for $j \geq 1$ and $\dim \operatorname{Ker}(N^0) = 0$. We get $(\kappa_1, \kappa_2, \dots)$ a partition of n and we will call this partition a kernel partition of the nilpotent operator N. By the description of $\dim \operatorname{Ker}(N^i)$ in terms of the partition $(n_1, \dots, n_t)$, we see that $\kappa_i = |\{k : n_k \geq i\}|$. So the partition $(\kappa_1, \kappa_2, \dots)$ is the dual of the partition $(n_1, \dots, n_t)$. Let now $(\kappa'_1, \kappa'_2, \dots)$ the kernel partition of N'. Then the inequalities:

$$\dim \operatorname{Ker}(N^i) = \sum_{j=1}^{i} \kappa_j \leq \dim \operatorname{Ker}(N'^i) = \sum_{j=1}^{i} \kappa'_j,$$

$\forall i$, are equivalent to the inequalities from 2. This concludes the proof. $\square$

Lemma 5.8. *Let A be a commutative ring, V projective finitely generated A-module and $N : V \longrightarrow V$ a nilpotent A-linear operator. Then the set*

$$\left\{\mathfrak{p} \in \operatorname{Spec} A \middle| \dim_{\kappa(\mathfrak{p})}(\operatorname{Coker} N) \otimes_A \kappa(\mathfrak{p}) \geq m\right\}$$

is closed for any integer m.

For a point $x \in \operatorname{Spec} A$, the shape (Jordan normal form) of nilpotent operator $N \otimes \kappa(x)$ is given by a partition $\mathcal{P}_x$ and this partition determines uniquely, up to conjugacy, a Jordan normal form of a nilpotent operator. Define a partial order $\leq$ on partitions which is the reverse of so-called natural or dominance partial order ([Knu98] chapter 5 section 5.1.4). Then for all integers i,

$$\dim_{\kappa(x)}(\operatorname{Coker} N^i) \otimes_A \kappa(x) \leq \dim_{\kappa(y)}(\operatorname{Coker} N^i) \otimes_A \kappa(y)$$

if and only if $\mathcal{P}_x \leq \mathcal{P}_y$.

Proof. Let's prove the first assertion. Let $m_1, \dots m_n$, any set of generators of $C := \operatorname{Coker} N$ over A. It would be enough to prove that the set $U := \left\{\mathfrak{p} \in \operatorname{Spec} A \middle| \dim_{\kappa(\mathfrak{p})} C \otimes_A \kappa(\mathfrak{p}) < n\right\}$ is open. Let $\mathfrak{p} \in \operatorname{Spec} A$ and $\bar{x}_1, \dots, \bar{x}_k$ be a basis of $\kappa(\mathfrak{p})$-vector space $C_\mathfrak{p}/\mathfrak{p}C_\mathfrak{p}$. It follows from Nakayama's lemma the lifts $x_1, \dots, x_k$ to $C_\mathfrak{p}$, form a minimal generating set of $C_\mathfrak{p}$ over $A_\mathfrak{p}$. Write $m_i/1 = \sum_{j=1}^{k}(a_{ij}/b_{ij})x_j$ and let $b = \prod b_{ij}$. For any $\mathfrak{q} \in D(b)$, $x_1, \dots, x_k$ is still a generating set of $C_\mathfrak{q}$ over $A_\mathfrak{q}$. Again, by Nakayama's lemma it follows that $\dim_{\kappa(\mathfrak{p})} C \otimes_A \kappa(\mathfrak{p}) \leq k < n$, so that $D(b) \subseteq U$. Therefore U is open.

The second assertion follows from the previous lemma, because $\dim \operatorname{Ker}(N^i \otimes \kappa(x)) = \dim \operatorname{Coker}(N^i \otimes \kappa(x))$ and we have an isomorphism $\operatorname{Coker}(N^i \otimes \kappa(x)) \simeq (\operatorname{Coker} N^i) \otimes_A \kappa(x)$ since the tensor product is right-exact. $\square$

Recall from previous section that we have an endomorphism φ on $D_{\bar{r}}^{\square}(\tau, \mathbf{v})$. Again by Theorem (2.5.5)[Kis08], there is a universal monodromy operator $N : D_{\bar{r}}^{\square}(\tau, \mathbf{v}) \longrightarrow D_{\bar{r}}^{\square}(\tau, \mathbf{v})$ which is $F_0 \otimes_{\mathbb{Q}_p} R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p]$-linear. Observe, that the monodromy of $WD(r_x)$ is the specialization of N at closed point $x \in \operatorname{Spec} R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p]$.

Let $\mathcal{P}$ be a partition valued function as in [SZ99]. Apply previous lemma with $A = F_0 \otimes_{\mathbb{Q}_p} R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p]$ and $V = D_{\bar{r}}^{\square}(\tau, \mathbf{v})$ to get that the set

$$\left\{x \in \operatorname{Spec} R_{\bar{r}}^{\square}(\tau, \mathbf{v})[1/p] \middle| \mathcal{P}_x \geq \mathcal{P}\right\}$$

$$= \bigcap_{i\geq 1} \left\{ x \in \operatorname{Spec} R^{\square}_{\overline{r}}(\tau, \mathbf{v})[1/p] \,|\, \dim_{\kappa(x)}(\mathrm{Coker} N^i) \otimes_A \kappa(x) \geq m_i \right\}$$

is closed, with $m_i = \sum_\sigma \sum_k \min(i, \mathcal{P}(\sigma)(k))$. Hence this set, is of the form $V(I_{\mathcal{P}})$, where $I_{\mathcal{P}}$ is an ideal in $R^{\square}_{\overline{r}}(\tau, \mathbf{v})$, such that the quotient is reduced. Define the ring $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}}) := R^{\square}_{\overline{r}}(\tau, \mathbf{v})/I_{\mathcal{P}}$, this ring has the following property: $x \in \operatorname{Spec} R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})[1/p]$ if and only if $\mathcal{P}_x \geq \mathcal{P}$. Observe that $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$ is a reduced, p-torsion free quotient of $R^{\square}_{\overline{r}}(\tau, \mathbf{v})$.

For $\mathcal{P}$ maximal partition, which we denote by $\sigma_{\mathcal{P}} = \sigma_{max}$, we get potentially crystalline deformation ring and for $\mathcal{P}$ minimal, which we denote by $\sigma_{\mathcal{P}} = \sigma_{min}$, we get the potentially semi-stable deformation ring $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min}) := R^{\square}_{\overline{r}}(\tau, \mathbf{v})$. It follows from Theorem 3.2 [HH] that the ring $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ is Cohen-Macaulay.

Notice that Theorem (3.3.4) [Kis08] gives the dimension of $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$ and by Theorem (3.3.8) [Kis08] we know that $\dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max}) = \dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$. Since we have, $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min}) \twoheadrightarrow R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}}) \twoheadrightarrow R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})$.

It follows that $\dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max}) \leq \dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}}) \leq \dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$, so that $\dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min}) = \dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$.

5.4 Local-Global compatibility

In this section we will study the support of $M_\infty(\sigma^\circ_{min})$ by introducing a stratification that depends on the $\sigma_{\mathcal{P}}$'s. This will allow us to have finer control on the monodromy operator.

The main result of this section is Theorem 5.14. This result tells us that the action of $\mathcal{H}(\sigma_{min})$ on $M_\infty(\sigma^\circ_{min})$ is compatible with the interpolation map $\mathcal{H}(\sigma_{min}) \longrightarrow R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$, constructed previously. Most of the proofs in this section are very similar to the ones given in the section 4 [CEG+16].

Let $\mathcal{P}$ be a partition valued function. Define $\sigma_{\mathcal{P}} := \sigma_{\mathcal{P}}(\lambda) \otimes \sigma_{alg}$, so that $(\sigma_{\mathcal{P}})_{sm} = \sigma_{\mathcal{P}}(\lambda)$ and $(\sigma_{\mathcal{P}})_{alg} = \sigma_{alg}$, where $\sigma_{\mathcal{P}}(\lambda)$ is a smooth type for K as in Proposition 3.19 and σ_{alg} is the restriction to K of an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}_p} GL_n$. Fix a K-stable $\mathcal{O}$-lattice $\sigma^\circ_{\mathcal{P}}$ in $\sigma_{\mathcal{P}}$. Set

$$M_\infty(\sigma^\circ_{\mathcal{P}}) := \left(\mathrm{Hom}^{cont}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ_{\mathcal{P}})^d)\right)^d$$

where we are considering homomorphisms that are continuous for the profinite topology on M_∞ and the p-adic topology on $(\sigma^\circ_{\mathcal{P}})^d$, and where we equip

$\mathrm{Hom}^{cont}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ_{\mathcal{P}})^d)$ with the p-adic topology. Note that $M_\infty(\sigma^\circ_{\mathcal{P}})$ is an $\mathcal{O}$-torsion free, profinite, linear-topological $\mathcal{O}$-module.

Let $R_\infty(\sigma_{\mathcal{P}})$ be the quotient of R_∞ which acts faithfully on $M_\infty(\sigma^\circ_{\mathcal{P}})$, i.e. $R_\infty(\sigma_{\mathcal{P}}) = M_\infty(\sigma^\circ_{\mathcal{P}})/\mathrm{ann}(M_\infty(\sigma^\circ_{\mathcal{P}}))$. Set $R_\infty(\sigma_{\mathcal{P}})' = R_\infty \otimes_{R^\square_{\tilde{\mathfrak{p}}}} R^\square_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$.

Lemma 5.9. *Let $\mathcal{P}$ be a partition valued function, then $R_\infty(\sigma_{\mathcal{P}})$ is a reduced $\mathcal{O}$-torsion free quotient of $R_\infty(\sigma_{\mathcal{P}})'$. Moreover the module $M_\infty(\sigma^\circ_{\mathcal{P}})$ is Cohen-Macaulay.*

Proof. That $R_\infty(\sigma_{\mathcal{P}})$ is $\mathcal{O}$-torsion free follows immediately from the fact that by definition it acts faithfully on the $\mathcal{O}$-torsion free module $M_\infty(\sigma^\circ_{\mathcal{P}})$.

The fact that it is actually a quotient of $R_\infty(\sigma_{\mathcal{P}})'$ is a consequence of classical local-global compatibility at $\tilde{\mathfrak{p}}$. The proof of this is identical to the proof of Lemma 4.17(1) in [CEG+16]. Even though that proof is written for σ (i.e. σ_{max} with the notation of this thesis), all the details remain unchanged if we replace σ by $\sigma_{\mathcal{P}}$, if we observe that by the local-global compatibility (Theorem 1.1 of [Car14]) the restriction to the local factor at $\tilde{\mathfrak{p}}$ of global Galois representation, coming from a closed point of a Hecke algebra, is potentially semi-stable such that the partition valued function associated to monodromy (as in Lemma 5.8) of this local Galois representation bigger then $\mathcal{P}$.

To prove the remaining assertions first notice that the module $M_\infty(\lambda^\circ)$ is a Cohen-Macaulay module, by Lemma 4.30 [CEG+16]. Then $M_\infty(\sigma^\circ_{\mathcal{P}})$ is also a Cohen-Macaulay module because it is a direct summand of $M_\infty(\lambda^\circ)$. Finally, to see that $R_\infty(\sigma_{\mathcal{P}})$ is reduced, notice that since $R_\infty(\sigma_{\mathcal{P}})'$ is reduced, any non reduced quotient of the same dimension will have an associated prime, which is not minimal. So $M_\infty(\sigma^\circ_{\mathcal{P}})$ is a faithful Cohen-Macaulay module over $R_\infty(\sigma_{\mathcal{P}})$, thus this cannot happen, and so $R_\infty(\sigma_{\mathcal{P}})$ is reduced. $\square$

Let $\mathcal{H}(\sigma^\circ_{min}) := \mathrm{End}_G(\text{c–Ind}^G_K \sigma^\circ_{min})$. Note that since $\sigma^\circ_{\mathcal{P}}$ is a free $\mathcal{O}$-module of finite rank, it follows from the proof of Theorem 1.2 of [ST02] that Schikhof duality induces an isomorphism

$$\mathrm{Hom}^{cont}_{\mathcal{O}[[K]]}(M_\infty, (\sigma^\circ_{\mathcal{P}})^d) \simeq \mathrm{Hom}_K(\sigma^\circ_{\mathcal{P}}, (M_\infty)^d)$$

and Frobenius reciprocity gives

$$\mathrm{Hom}_K(\sigma^\circ_{\mathcal{P}}, (M_\infty)^d) = \mathrm{Hom}_G(\text{c–Ind}^G_K \sigma^\circ_{\mathcal{P}}, (M_\infty)^d).$$

Thus $M_\infty(\sigma^\circ_{\mathcal{P}})$ is equipped with an action of $\mathfrak{Z}_\Omega$ which commutes with the action of R_∞.

When $\sigma_{\mathcal{P}} = \sigma_{min}$, the module $M_\infty(\sigma^\circ_{min})$ is equipped with an action of $\mathcal{H}(\sigma^\circ_{min})$. Such an action of $\mathcal{H}(\sigma^\circ_{min})$ commutes with the action of R_∞. The isomorphism $\mathcal{H}(\sigma_{\min}) \simeq \mathfrak{Z}_\Omega$(Corollary 2.18) and the isomorphism of Lemma 1.4 [ST06], $\mathcal{H}(\sigma_{min}(\lambda)) \to \mathcal{H}(\sigma_{min})$, allow us to define the action of $\mathcal{H}(\sigma^\circ_{min})$ on $M_\infty(\sigma^\circ_{\mathcal{P}})$.

Lemma 5.10. *If $z \in \mathcal{H}(\sigma^\circ_{min})$ is such that $\beta(z) \in R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$, then the action of z on $M_\infty(\sigma^\circ_{\mathcal{P}})$ agrees with the action of $\beta(z)$ via the natural map $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}}) \longrightarrow R_\infty(\sigma_{\mathcal{P}})'$*

Proof. As before, this is a consequence of classical local-global compatibility at $\tilde{\mathfrak{p}}$. The proof of this is identical to the proof of Lemma 4.17(2) in [CEG$^+$16], where we replace σ by $\sigma_{\mathcal{P}}$ and instead of using Lemma 4.17(1) in [CEG$^+$16] we apply Lemma 5.9. $\square$

We will now define the space of algebraic automorphic forms. First recall some notation from [CEG$^+$16]. The globalization constructed in section 2.1 [CEG$^+$16], provides us with a global imaginary CM field $\tilde{F}$ with maximal totally real subfield $\tilde{F}^+$. We refer the reader to this section for the details of these definitions to section 2.1 [CEG$^+$16].

Recall some notation from section 2.3 [CEG$^+$16]. Let $\tilde{G}/\tilde{F}^+$ a certain definite unitary group as defined in the paper [CEG$^+$16]. Let $U = \prod_v U_v$ any compact open subgroup of $\tilde{G}(\mathbb{A}^\infty_{\tilde{F}^+})$. Let S_p denote the set of primes of $\tilde{F}^+$ dividing p. Fix $\mathfrak{p}|p$. Let ξ the weight as in section 1.2 of this thesis and τ the inertial type as in section 1.4. Let $W_{\xi,\tau}$ be the finite free $\mathcal{O}$-module with an action of $\prod_{v\in S_p\setminus\{\mathfrak{p}\}} U_v$.

For any compact open U and any $\mathcal{O}$-module V, let $S_{\xi,\tau}(U,V)$ denote the set of continuous functions

$$f : \tilde{G}(\tilde{F}^+) \setminus \tilde{G}(\mathbb{A}^\infty_{\tilde{F}^+}) \longrightarrow W_{\xi,\tau} \otimes V$$

such that for $g \in \tilde{G}(\mathbb{A}^\infty_{\tilde{F}^+})$ we have $f(gu) = u^{-1}f(g)$ for $u \in U$, where U acts on $W_{\xi,\tau} \otimes V$ via the projection to $\prod_{v\in S_p} U_v$. The space $S_{\xi,\tau}(U,V)$ is called the **space of algebraic modular forms**.

Now we can define ϖ-adically completed cohomology space. For each positive integer m, the compact open subgroups U_m as defined in the beginning of the section 2.3 [CEG+16] have the same level away from $\mathfrak{p}$. Let $U^{\mathfrak{p}}$ denote that common level. Define the ϖ-adically **completed cohomology space**:

$$\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}} := \varprojlim_{s}(\varinjlim_{m} S_{\xi,\tau}(U_m, \mathcal{O}/\varpi^s)_{\mathfrak{m}})$$

The space is equipped with a natural G-action, induced from the action of G on algebraic automorphic forms.

The module M_∞ comes with an action of S_∞(cf. page 27, Section 2.8 [CEG+16]). Recall that by Corollary 2.11 [CEG+16], we have a G-equivariant isomorphism $M_\infty/\mathfrak{a}M_\infty \simeq \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$, where $\mathfrak{a}$ is an ideal in S_∞ generated by some formal variables (cf. page 27, Section 2.8 [CEG+16]). Moreover that isomorphism commutes with $R^{\square}_{\tilde{\mathfrak{p}}}$-action on both sides.

Lemma 5.11. *Let pr : $\mathrm{Spec}\, R_\infty(\sigma_{min})'[1/p] \longrightarrow \mathrm{Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ be the usual projection map induced by $j : R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min}) \longrightarrow R_\infty \otimes_{R^{\square}_{\tilde{\mathfrak{p}}}} R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})$, $x \mapsto 1 \otimes x$. Let $x \in \mathrm{Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ a closed smooth point. Then any $y \in pr^{-1}(x) \subset R_\infty(\sigma_{min})'[1/p]$ is a smooth point of* $\mathrm{Spec}\, R_\infty(\sigma_{min})$.

Proof. This is essentially the first part of the proof of Theorem 4.35 [CEG+16]. □

Proposition 5.12. *If $y \in \mathrm{Spec}\, R_\infty(\sigma_{min})'[1/p] \cap V(\mathfrak{a})$ a closed point, then y is a smooth point of* $\mathrm{Spec}\, R_\infty(\sigma_{min})$ *and $V(r_y)^{l.alg} \simeq \pi_{sm}(r_y) \otimes \pi_{alg}(r_y)$.*

Proof. We follow here quite closely the proof of Theorem 4.35 [CEG+16]. By definition

$$V(r_y) := \mathrm{Hom}^{cont}_{\mathcal{O}}(M_\infty \otimes_{R_\infty,y} \mathcal{O}, E)$$

Since $\mathfrak{a} \subseteq \mathrm{Ker}(y) = \mathfrak{m}_y$, we have that:

$$\mathrm{Hom}^{cont}_{\mathcal{O}}(M_\infty \otimes_{R_\infty,y} \mathcal{O}, E) = \mathrm{Hom}^{cont}_{\mathcal{O}}(M_\infty/\mathfrak{a}M_\infty \otimes_{R_\infty,y} \mathcal{O}, E)$$

Then by Corollary 2.11[CEG+16], we have

$$\mathrm{Hom}^{cont}_{\mathcal{O}}(M_\infty/\mathfrak{a}M_\infty \otimes_{R_\infty,y} \mathcal{O}, E) \simeq \mathrm{Hom}^{cont}_{\mathcal{O}}(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}} \otimes_{R_\infty,y} \mathcal{O}, E)$$

The ideal $\mathfrak{m}_y$ is finitely generated, choose a presentation $\mathfrak{m}_y = (a_1, \dots, a_k)$, then we get an exact sequence of R_∞-modules:

$$R^{\oplus k}_\infty \longrightarrow R_\infty \longrightarrow \mathcal{O} \longrightarrow 0$$

Let $\Pi(\bullet) = \mathrm{Hom}^{cont}_{\mathcal{O}}(\bullet \otimes_{R_\infty} \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}, E)$. Then the functor Π is left exact and contravariant, by Lemma 2.20 of [Paš15]. Apply this functor to the exact sequence above to get the following exact sequence:

$$0 \longrightarrow V(r_y) \longrightarrow \mathrm{Hom}^{cont}_{\mathcal{O}}(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}, E) \xrightarrow{f} \mathrm{Hom}^{cont}_{\mathcal{O}}(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}, E)^{\oplus k}$$

were $f(l) = (l.a_1, \ldots, l.a_k)$. By the exactness we identify

$$V(r_y) \simeq \mathrm{Hom}^{cont}_{\mathcal{O}}(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}, E)[\mathfrak{m}_y],$$

but

$$\begin{aligned}\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} E &\simeq \mathrm{Hom}^{cont}_{\mathcal{O}}(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}, \mathcal{O}) \otimes_{\mathcal{O}} E \\ &\simeq \mathrm{Hom}^{cont}_{\mathcal{O}}(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}, E)\end{aligned}$$

Thus

$$V(r_y)^{l.alg} \simeq (\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O}) \otimes_{\mathcal{O}} E)^{l.alg}[\mathfrak{m}_y]$$

Proposition 3.2.4 of [Eme06] shows that locally algebraic vectors of any given weight are precisely the algebraic automorphic forms of that weight. Hence:

$$(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O}) \otimes_{\mathcal{O}} E)^{l.alg}[\mathfrak{m}_y] \simeq \pi_{sm}(r_y) \otimes \pi_{alg}(r_y)$$

This isomorphism follows from the classical local-global compatibility (Theorem 1.1 of [Car14]). A priori, $\pi_{sm}(r_y) \otimes \pi_{alg}(r_y)$ may appear with some multiplicity. However this multiplicity is seen to be one. Indeed the group $\tilde{G}$ is compact at infinity, so the condition $(*)$ from Theorems 5.4 of [Lab11] is automatically satisfied. We may then apply Theorems 5.4 and 5.9 of [Lab11], where σ, in those Theorems, is our $(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O}) \otimes_{\mathcal{O}} E)^{l.alg}$ and π is an automorphic cuspidal representation of GL_n, which is a base change of σ. Then by the choice of $U^{\mathfrak{p}}$(section 2.3 [CEG+16] for definition of the U_m), the fact that we have fixed the action mod p of the Hecke operators at $\tilde{v}_1$(section 2.3 [CEG+16]) and the irreducibility of the globalization of $\bar{r}$, we see that the multiplicity of $\pi_{sm}(r_y) \otimes \pi_{alg}(r_y)$ is one.

Local factors of π, as in the paragraph above, are generic according to Corollary of Theorem 5.5 [Sha74]. Then by Theorem 5.9 [Lab11], the local factors of $(\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O}) \otimes_{\mathcal{O}} E)^{l.alg}$ are also generic, since $\tilde{G}$, by construction, is quasi-split at all the finite places. It follows that $\pi_{sm}(r_y)$ is generic. Moreover, by Theorem 1.2.7[All16], the closed point $pr(y)$ (pr : $\mathrm{Spec}\, R_\infty(\sigma_{min})'[1/p] \longrightarrow \mathrm{Spec}\, R^{\square}_{\mathfrak{p}}(\sigma_{min})[1/p]$ the usual projection, with notation of Lemma 5.11) is smooth if and only if $\pi_{sm}(r_y)$ is generic. Then by Lemma 5.11 the point y is smooth. $\square$

In order to study the support of $M_\infty(\sigma^\circ_{min})$, we will use the commutative algebra arguments underlying the Taylor-Wiles-Kisin method.

Proposition 5.13. *1. The module $M_\infty(\sigma^\circ_{min})[1/p]$ is locally free of rank one over the regular locus of $R_\infty(\sigma_{min})[1/p]$.*

2. Spec $R_\infty(\sigma_{min})[1/p]$ *is a union of irreducible components of* Spec $R_\infty(\sigma_{min})'[1/p]$.

Proof. **(1)**. Let $\mathfrak{m}$ be a smooth point in the support of $M_\infty(\sigma^\circ_{min})[1/p]$. Since $M_\infty(\sigma^\circ_{min})[1/p]$ is a Cohen-Macaulay module we have $\operatorname{depth} M_\infty(\sigma^\circ_{min})[1/p]_\mathfrak{m} = \dim M_\infty(\sigma^\circ_{min})[1/p]_\mathfrak{m}$. Moreover $\dim M_\infty(\sigma^\circ_{min})[1/p]_\mathfrak{m} = \dim R_\infty(\sigma_{min})[1/p]_\mathfrak{m}$ since $R_\infty(\sigma_{min})$ acts faithfully on $M_\infty(\sigma^\circ_{min})$.

By assumption the ring $R_\infty(\sigma_{min})[1/p]_\mathfrak{m}$ is regular, it follows that the module $M_\infty(\sigma^\circ_{min})[1/p]_\mathfrak{m}$ has a finite projective dimension over this ring. We also have that $\operatorname{depth} R_\infty(\sigma_{min})[1/p]_\mathfrak{m} = \dim R_\infty(\sigma_{min})[1/p]_\mathfrak{m}$. Then by Auslander-Buchsbaum formula (Theorem 19.1 [Mat89]), $M_\infty(\sigma^\circ_{min})[1/p]_\mathfrak{m}$ is free over $R_\infty(\sigma_{min})[1/p]_\mathfrak{m}$. It follows that $M_\infty(\sigma^\circ_{min})[1/p]$ is locally free(i.e. projective) over regular locus of $R_\infty(\sigma_{min})[1/p]$.

Let's check that it is locally free of rank one. Let $x \in \operatorname{Supp} M_\infty(\sigma^\circ_{min})$ and $y \in \operatorname{Supp}(M_\infty(\sigma^\circ_{min})) \cap V(\mathfrak{a})$ a smooth closed point that lies on the same irreducible component V as x. Such a point y always exist because $V(\mathfrak{p}) \cap V(\mathfrak{a}) \neq 0$. Since $M_\infty(\sigma^\circ_{min})[1/p]$ is projective the local rank is constant on irreducible components of the support. It would be enough to compute the local rank at y, which is given by

$$\dim_E M_\infty(\sigma^\circ_{min}) \otimes_{R_\infty} \kappa(y) = \dim_E \operatorname{Hom}_K(\sigma_{min}, V(r_y)^{l.alg}),$$

according to Proposition 2.22 [Paš15]. By Proposition 5.12 we have that $V(r_y)^{l.alg} \simeq \pi_{sm}(r_y) \otimes \pi_{alg}(r_y)$. Moreover, since σ_{alg} is an irreducible representation of a Lie algebra of G, we have

$$\dim_E \operatorname{Hom}_K(\sigma_{min}, V(r_y)^{l.alg}) = \dim_E \operatorname{Hom}_K(\sigma_{min}(\lambda), \pi_{sm}(r_y))$$

Then $\dim_E \operatorname{Hom}_K(\sigma_{min}(\lambda), \pi_{sm}(r_y)) = 1$ by Lemma 3.22, because $\pi_{sm}(r_y)$ is generic.

(2). The proof is the same as in Lemma 4.18 [CEG+16]. $\square$

The action of $\mathfrak{Z}_\Omega$ on $M_\infty(\sigma^\circ_{min})[1/p]$ induces an E-algebra homomorphism:

$$\alpha : \mathfrak{Z}_\Omega \longrightarrow \operatorname{End}_{R_\infty[1/p]}(M_\infty(\sigma^\circ_{min})[1/p])$$

From the Proposition 5.13, we deduce that:

Theorem 5.14. *We have the following commutative diagram:*

$$\begin{tikzcd}
(\mathrm{Spec}\, R_\infty(\sigma_{min})[1/p])^{reg} \arrow[r, "\alpha^\sharp"] \arrow[d, hook] & \mathrm{Spec}\, \mathcal{H}(\sigma_{min}) \\
\mathrm{Spec}\, R_\infty(\sigma_{min})[1/p] \arrow[r, "pr"] & \mathrm{Spec}\, R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p], \arrow[u]
\end{tikzcd}$$

where $(\mathrm{Spec}\, R_\infty(\sigma_{min})[1/p])^{reg}$ *is the regular locus of* $\mathrm{Spec}\, R_\infty(\sigma_{min})[1/p]$, $\alpha^\sharp$ *the map induced by* α *and* $pr : \mathrm{Spec}\, R_\infty(\sigma_{min})'[1/p] \longrightarrow \mathrm{Spec}\, R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ *the usual projection map induced by* $j : R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min}) \longrightarrow R_\infty \otimes_{R^\square_{\tilde{\mathfrak{p}}}} R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min})$, $x \mapsto 1 \otimes x$.

Proof. We proceed here as in the proof of Theorem 4.19 [CEG$^+$16]. It is enough to check all it on points since all the rings are Jacobson and reduced. Let $x : R_\infty(\sigma_{min})[1/p] \to E$ a closed point smooth point. Note firstly that if $z \in \mathcal{H}(\sigma^\circ_{min})$ is such that $\beta(z) \in R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min})$, then $x(\alpha(z)) = x(j(\beta(z)))$ by Lemma 5.10. Since $R_\infty(\sigma_{min})$ is p-torsion free by Lemma 5.9, it is therefore enough to show that $\mathcal{H}(\sigma^\circ_{min})$ is spanned over E by such elements. But, $\mathcal{H}(\sigma^\circ_{min})$ certainly spans $\mathcal{H}(\sigma_{min})$ over E, so it is enough to show that for any element $z \in \mathcal{H}(\sigma^\circ_{min})$, we have $\beta(p^C z) \in R^\square_{\tilde{\mathfrak{p}}}(\sigma_{min})$ for some $C \geq 0$. The latter condition is obviously true, this concludes the proof. $\square$

5.5 Support of patched modules

Let (J, λ) be the type, a locally algebraic representation $\lambda \otimes (\sigma_{alg}|J)$ of J will be again denoted by λ. We have also a patched module $M_\infty(\lambda^\circ) := \left(\mathrm{Hom}^{cont}_{\mathcal{O}[[J]]}(M_\infty, (\lambda^\circ)^d)\right)^d$, where λ° is a J-stable lattice in λ. Define also $R_\infty(\lambda) := R_\infty/\mathrm{ann}(M_\infty(\lambda^\circ))$. We would like to have some statements about a support of patched modules. More precisely, we will prove that $M_\infty(\sigma^\circ_{min})$ and $M_\infty(\lambda^\circ)$ have the same support.

Proposition 5.15.

$$\mathrm{Supp}(M_\infty(\sigma^\circ_{min})) = \mathrm{Supp}(M_\infty(\lambda^\circ))$$

Proof. It follows from decomposition:

$$\mathrm{Ind}_J^K \lambda = \bigoplus_{\mathcal{P}} \sigma_{\mathcal{P}}^{\oplus m_{\mathcal{P}}}$$

that

$$M_\infty(\lambda^\circ) = \bigoplus_{\mathcal{P}} M_\infty(\sigma^\circ_{\mathcal{P}})^{\oplus m_{\mathcal{P}}}.$$

Then $\mathrm{Supp}(M_\infty(\sigma^\circ_{min})) \subseteq \mathrm{Supp}(M_\infty(\lambda^\circ))$. By definition, $\mathrm{Supp}(M_\infty(\sigma^\circ_{min})) = \mathrm{Spec}\, R_\infty(\sigma_{min})$ and also $\mathrm{Supp}(M_\infty(\lambda^\circ)) = \mathrm{Spec}\, R_\infty(\lambda)$. Let $V(\mathfrak{p})$ an irreducible component of the spectrum $\mathrm{Spec}\, R_\infty(\lambda)$. It is enough to find a point $x \in V(\mathfrak{p})$ such that $x \notin V(\mathfrak{q})$ for any minimal prime $\mathfrak{q}$ of $R_\infty(\lambda)$ such that $\mathfrak{q} \neq \mathfrak{p}$ and $x \in \mathrm{Spec}\, R_\infty(\sigma_{min})$. The ideal $\mathfrak{a}$ is generated by a regular sequence $(y_1, \ldots, y_h)$ and $y_1, \ldots, y_h, \varpi$ is a system of parameters for $\mathrm{Spec}\, R_\infty(\lambda)/\mathfrak{p}$. Then by Lemma 3.9 [Paš16], $V(\mathfrak{p})$ contains a closed point $x \in \mathrm{Spec}\, R_\infty(\lambda)/(y_1, \ldots y_h)[1/p]$. The point x is smooth by the Lemma 5.12, hence it does not lies on the intersection of irreducible components.

We have that $x \in \mathrm{Spec}\, R_\infty(\lambda)[1/p] \cap V(\mathfrak{a})$, so it is a closed point of $\mathrm{Supp}(M_\infty(\lambda^\circ))$, then

$$M_\infty(\lambda^\circ) \otimes_{R_\infty} \kappa(x) \neq 0$$

and by Proposition 2.22 [Paš15], we have that:

$$M_\infty(\lambda^\circ) \otimes_{R_\infty} \kappa(x) = \mathrm{Hom}^{cont}_E \big(\mathrm{Hom}_J(\lambda, V(r_x)^{l.alg}), E\big) \neq 0$$

Then by Proposition 5.12 we have that $V(r_x)^{l.alg} \simeq \pi_{sm}(r_x) \otimes \pi_{alg}(r_x)$. Moreover the representation $\pi_{sm}(r_x)$ is generic, it follows then from Proposition 3.21 that we also have $\mathrm{Hom}_K(\sigma_{min}, V(r_x)^{l.alg}) \neq 0$. This means that $x \in \mathrm{Spec}\, R_\infty(\sigma_{min})[1/p] \cap V(\mathfrak{a})$. $\square$

5.6 Computation of locally algebraic vectors

By Proposition 5.15, we have $\mathrm{Supp}(M_\infty(\sigma^\circ_{min})) = \mathrm{Supp}(M_\infty(\lambda^\circ))$. In what follows we always identify these two sets, so we have $\mathrm{Spec}\, R_\infty(\sigma_{min}) = \mathrm{Supp}(M_\infty(\sigma^\circ_{min})) = \mathrm{Supp}(M_\infty(\lambda^\circ)) = \mathrm{Spec}\, R_\infty(\lambda)$. Let $x \in \text{m-Spec}\, R_\infty[1/p]$, such that $V(r_x) \neq 0$. Assume moreover that $x \in \mathrm{Supp}(M_\infty(\lambda^\circ))$ and that the representation $\pi_{sm}(r_x) := r_p^{-1}(WD(r_x))$ is generic and irreducible. By definition $\pi_{sm}(r_x)$ lies in Ω.

As always for any partition valued function $\mathcal{P}$, we will write $\sigma_{\mathcal{P}} := \sigma_{\mathcal{P}}(\lambda) \otimes \sigma_{alg}$, so that $(\sigma_{\mathcal{P}})_{sm} = \sigma_{\mathcal{P}}(\lambda)$ and $(\sigma_{\mathcal{P}})_{alg} = \sigma_{alg}$.

By Proposition 4.33 [CEG+16], we have $V(r_x)^{l.alg} = \pi_x \otimes \pi_{alg}(r_x)$, where π_x is an admissible smooth representation which lies in Ω.

Since $x \in \mathrm{Supp}(M_\infty(\sigma^\circ_{min}))$, then by Proposition 2.22 [Paš15] we have that $0 \neq \mathrm{Hom}_K(\sigma_{min}, V(r_x)^{l.alg}) = \mathrm{Hom}_K(\sigma_{min}(\lambda), \pi_x)$

The action of $\mathfrak{Z}_\Omega$ on $\pi_{sm}(r_x)$ defines a E-algebra morphism $\chi_{sm} : \mathfrak{Z}_\Omega \longrightarrow \mathrm{End}_G(\pi_{sm}(r_x)) \simeq E$, the kernel of such a morphism is a maximal ideal in $\mathfrak{Z}_\Omega$.

The space $M_\infty(\sigma^\circ_{min}) \otimes_{R_\infty} \kappa(x)$ is one dimensional by Proposition 5.13 and the Hecke algebra $\mathcal{H}(\sigma_{min}(\lambda))$ acts on this space by a character $\chi' : \mathcal{H}(\sigma_{min}(\lambda)) \longrightarrow E$. Composing χ' with an isomorphism $\mathfrak{Z}_\Omega \simeq \mathcal{H}(\sigma_{min})$, obtained from Lemma 1.4 [ST06] and Corollary 2.18, we get a a E-algebra morphism $\chi : \mathfrak{Z}_\Omega \longrightarrow E$.

Lemma 5.16. *The E-algebra morphisms χ and χ_{sm} defined above coincide.*

Proof. It follows from Theorem 5.14 and Theorem 5.3 and the isomorphism $\mathfrak{Z}_\Omega \simeq \mathcal{H}(\sigma_{min})$ as above. □

Lemma 5.17. *The representation $\pi_{sm}(r_x)$ is a G-subquotient of π_x.*

Proof. Define $\gamma_x := \text{c–Ind}_K^G(\sigma_{min}(\lambda)) \otimes_{\mathfrak{Z}_\Omega,\chi} E$. Since $x \in \mathrm{Supp}(M_\infty(\sigma^\circ_{min}))$, we have by definition $0 \neq \mathrm{Hom}_K(\sigma_{min}, V(r_x)^{l.alg}) = \mathrm{Hom}_K(\sigma_{min}(\lambda), \pi_x)$. So, there exists a non zero map $\psi : \gamma_x \longrightarrow \pi_x$.

Let π' any irreducible quotient of γ_x, then $\mathrm{Hom}_K(\sigma_{min}(\lambda), \pi') \neq 0$, by Proposition 3.21 π' is generic. It follows that by Corollary 3.11 [CEG+16], the representation π' is the socle of $\text{c–Ind}_K^G\sigma_{max}(\lambda) \otimes_{\mathfrak{Z}_\Omega,\chi} E$. We write it $\pi' \simeq \mathrm{soc}_G(\text{c–Ind}_K^G(\sigma_{max}(\lambda)) \otimes_{\mathfrak{Z}_\Omega,\chi} E)$. Similarly by corollary 3.11 [CEG+16], $\pi_{sm}(r_x) \simeq \mathrm{soc}_G(\text{c–Ind}_K^G(\sigma_{max}(\lambda)) \otimes_{\mathfrak{Z}_\Omega,\chi_{sm}} E)$. By Lemma 5.16 $\chi_{sm} = \chi$, then $\pi' \simeq \pi_{sm}(r_x)$. So at this stage we proved that the cosocle of γ_x, is generic, irreducible and isomorphic to $\pi_{sm}(r_x)$.

Let $\kappa = \mathrm{Ker}(\psi)$, then $\gamma_x/\kappa \hookrightarrow \pi_x$. Let now π' any irreducible quotient of γ_x/κ, in particular π' is a sub-quotient of π_x. Moreover $\gamma_x \twoheadrightarrow \gamma_x/\kappa \twoheadrightarrow \pi'$, so π' is an irreducible quotient of γ_x. By what we have proven above $\pi' \simeq \pi_{sm}(r_x)$. It follow that $\pi_{sm}(r_x)$ is a sub-quotient of π_x. □

Proposition 5.18. *Let x, y be two closed, E-valued points of* $\mathrm{Spec}\, R_\infty(\sigma_{min})[1/p]$, *lying on the same irreducible component. Let $\mathcal{P}$ be a partition valued function. If x is smooth, then*

$$\dim_E \mathrm{Hom}_K(\sigma_{\mathcal{P}}, V(r_x)^{l.alg}) \leq \dim_E \mathrm{Hom}_K(\sigma_{\mathcal{P}}, V(r_y)^{l.alg})$$

Proof. The proof follows the proof of the Proposition 4.34 [CEG+16] by replacing λ with $\sigma_{\mathcal{P}}$ everywhere. □

Lemma 5.19. *Let x, $y \in$ m-Spec $R_\infty(\sigma_{min})[1/p]$ smooth points such that the monodromy operators of $WD(r_x)$ and $WD(r_y)$ are the same and $WD(r_x)|I_F \simeq WD(r_y)|I_F$. Then $\pi_{sm}(r_x)|K \simeq \pi_{sm}(r_y)|K$.*

Proof. Since x and y are both smooth points, the representations $\pi_{sm}(r_x)$ and $\pi_{sm}(r_y)$ are both irreducible and generic. Moreover, it follows from hypotheses that $\pi_{sm}(r_x)$ and $\pi_{sm}(r_y)$ have the same inertial support and as well as the same number and the same size of segments for Bernstein-Zelevisky classification. So if $\pi_{sm}(r_x) = L(\Delta_1) \times \ldots \times L(\Delta_r)$ then there are unramified characters χ_i such that $\pi_{sm}(r_y) = L(\Delta_1 \otimes \chi_1) \times \ldots \times L(\Delta_r \otimes \chi_r)$. Restricting to K, we get $\pi_{sm}(r_x)|K \simeq \pi_{sm}(r_y)|K$. $\square$

Lemma 5.20. *Let $x \in \text{m-Spec}\, R_\infty(\sigma_{min})[1/p]$, such that $\pi_{sm}(r_x)$ is generic, then $x \in \operatorname{Supp} M_\infty(\sigma_{\mathcal{P}_x})$*

Proof. By Lemma 5.17, $\pi_{sm}(r_x)$ is a subquotient of π_{sm}, then for any partition valued function $\mathcal{P}$, we have:

$$\dim_E \operatorname{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \leq \dim_E \operatorname{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm})$$

In particular we have

$$\dim_E \operatorname{Hom}_K((\sigma_{\mathcal{P}_x})_{sm}, \pi_{sm}(r_x)) \leq \dim_E M_\infty(\sigma_{\mathcal{P}_x}) \otimes_{R_\infty} \kappa(x)$$

Since $\dim_E \operatorname{Hom}_K((\sigma_{\mathcal{P}_x})_{sm}, \pi_{sm}(r_x)) \neq 0$ then $\dim_E M_\infty(\sigma_{\mathcal{P}_x}) \otimes_{R_\infty} \kappa(x) \neq 0$. This means that $x \in \operatorname{Supp} M_\infty(\sigma_{\mathcal{P}_x})$. $\square$

Proposition 5.21. *Let x be any point of* $\text{m-Spec}\, R_\infty(\sigma_{min})[1/p]$*. Then $x \in \operatorname{Supp} M_\infty(\sigma^\circ_{\mathcal{P}})$ implies that $\mathcal{P}_x \geq \mathcal{P}$.*

Proof. By Lemma 5.9, the action of R_∞ on $M_\infty(\sigma^\circ_{\mathcal{P}})$ is a reduced torsion free quotient of $R_\infty(\sigma_{\mathcal{P}})'$. So if $x \in \operatorname{Supp} M_\infty(\sigma^\circ_{\mathcal{P}})$ then $x \in \operatorname{Spec} R_\infty(\sigma_{\mathcal{P}})' = \operatorname{Spec} R_\infty \otimes_{R^\square_{\tilde{\mathfrak{p}}}} R^\square_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$ then by definition of $R^\square_{\tilde{\mathfrak{p}}}(\sigma_{\mathcal{P}})$ we have that $\mathcal{P}_x \geq \mathcal{P}$. $\square$

Theorem 5.22. *Let x a closed E-valued point of* $\operatorname{Spec} R_\infty(\sigma_{min})[1/p]$*, such that $\pi_{sm}(r_x)$ is generic and irreducible. Then*

$$V(r_x)^{l.alg} \simeq \pi_{sm}(r_x) \otimes \pi_{alg}(r_x)$$

Proof. By Lemma 5.17 $\pi_{sm}(r_x)$ is a G-subquotient of π_x, and for every partition valued function $\mathcal{P}$

$$\dim_E \operatorname{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \leq \dim_E \operatorname{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_x)$$

Let $y \in \mathrm{Supp}(M_\infty(\lambda)) \cap V(\mathfrak{a})$ a smooth closed point that lies on the same irreducible component $V(\mathfrak{p})$ as x. Such a point y always exist because $V(\mathfrak{p}) \cap V(\mathfrak{a}) \neq 0$. Moreover we have that,

$$V(r_y)^{l.alg} \simeq \pi_{sm}(r_y) \otimes \pi_{alg}(r_y)$$

by Proposition 5.12. Then it follows from Proposition 5.18, that for every partition valued function $\mathcal{P}$, we have

$$\dim_E \mathrm{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_x) = \dim_E \mathrm{Hom}_K((\sigma_{\mathcal{P}}), V(r_x)^{l.alg})$$

$$= \dim_E \mathrm{Hom}_K((\sigma_{\mathcal{P}}), V(r_y)^{l.alg}) = \dim_E \mathrm{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_y))$$

because x and y are both smooth points, lying on the same component. In particular we have that

$$\dim_E M_\infty(\sigma^\circ_{\mathcal{P}}) \otimes_{R_\infty} \kappa(x) = \dim_E M_\infty(\sigma^\circ_{\mathcal{P}}) \otimes_{R_\infty} \kappa(y)$$

Then $y \in \mathrm{Supp}\, M_\infty(\sigma^\circ_{\mathcal{P}})$ if and only if $x \in \mathrm{Supp}\, M_\infty(\sigma_{\mathcal{P}})$.

Taking $\mathcal{P} = \mathcal{P}_x$, by Lemma 5.20 we have that $x \in \mathrm{Supp}\, M_\infty(\sigma^\circ_{\mathcal{P}_x})$ hence $y \in \mathrm{Supp}\, M_\infty(\sigma^\circ_{\mathcal{P}_x})$. Then by Proposition 5.21 $\mathcal{P}_y \geq \mathcal{P}_x$. Exchanging the roles of x and y, we get $\mathcal{P}_y = \mathcal{P}_x$. This means that the monodromy operators of $WD(r_x)$ and $WD(r_y)$ are the same. All together we have:

$$\dim_E \mathrm{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \leq \dim_E \mathrm{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}) =$$

$$= \dim_E \mathrm{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_y))$$

Similarly using the Proposition 4.34 [CEG$^+$16],we get

$$\dim_E \mathrm{Hom}_J(\lambda_{sm}, \pi_{sm}(r_x)) \leq \dim_E \mathrm{Hom}_K(\lambda_{sm}, \pi_x) =$$

$$= \dim_E \mathrm{Hom}_K(\lambda_{sm}, \pi_{sm}(r_y))$$

We have shown that x and y have the same monodromy, then by Lemma 5.19, we get that $\pi_{sm}(r_x)|K \simeq \pi_{sm}(r_y)|K$, so $\pi_{sm}(r_x)|J \simeq \pi_{sm}(r_y)|J$. It follows that the inequality above, is an equality.

We know that the functor $\mathrm{Hom}_J(\lambda_{sm}, .)$ is an exact functor. It follows that $\mathrm{Hom}_J(\lambda_{sm}, \pi_{sm}(r_x))$ is a subquotient of $\mathrm{Hom}_J(\lambda_{sm}, \pi_x)$ in the category of $\mathcal{H}(G, \lambda_{sm})$-modules, because by Lemma 5.17 $\pi_{sm}(r_x)$ is a subquotient of π_x. Since those two $\mathcal{H}(G, \lambda_{sm})$-modules have the same dimension they must be equal. Using the fact that the functor $\mathrm{Hom}_J(\lambda_{sm}, .)$ is an equivalence of categories, we get that $\pi_{sm}(r_x) \simeq \pi_x$. $\square$

Corollary 5.23. *Let* $x \in \text{m-Spec}\, R_\infty(\sigma_{min})[1/p]$ *such that* $\pi_{sm}(r_x)$ *is generic, then* $\mathcal{P}_x \geq \mathcal{P}$ *implies that* $x \in \operatorname{Supp} M_\infty(\sigma^\circ_{\mathcal{P}})$.

Proof. From Proposition 3.19, follows that $\mathcal{P}_x$ is the maximal partition $\mathcal{P}$ such that $\operatorname{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \neq 0$. Then by maximality, $\mathcal{P}_x \geq \mathcal{P}$ implies that $\operatorname{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \neq 0$. However by previous proposition combined with Proposition 2.22 of [Paš15] we have that $\operatorname{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \neq 0$ if and only if $M_\infty(\sigma^\circ_{\mathcal{P}}) \otimes_{R_\infty} \kappa(x) \neq 0$. So $x \in \operatorname{Supp} M_\infty(\sigma^\circ_{\mathcal{P}})$. $\square$

6 Applications

In this section we deduce some theorems from the results of previous sections. Recall from section 5.5, that $\operatorname{Spec} R_\infty(\sigma_{min}) = \operatorname{Supp}(M_\infty(\sigma^\circ_{min})) = \operatorname{Supp}(M_\infty(\lambda^\circ)) = \operatorname{Spec} R_\infty(\lambda)$. In what follows we will not differentiate between these four sets.

6.1 Points on automorphic components

In this section we will prove that if a Galois representation r is generic and corresponds to a closed point lying on an automorphic component then $BS(r)$ admits a G-invariant norm. Let's say a few words about automorphic components. It follows from Proposition 5.13, that $\operatorname{Spec} R_\infty(\sigma_{min})$ is a union of irreducible components of $\operatorname{Spec} R_\infty(\sigma_{min})'$. An irreducible component of $\operatorname{Spec} R_\infty(\sigma_{min})'$ which is also an irreducible component of $\operatorname{Spec} R_\infty(\sigma_{min})$ is called an **automorphic component**.

By Corollary 2.11 [CEG+16], we have $M_\infty/\mathfrak{a}M_\infty \simeq \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$, where the ideal $\mathfrak{a} = (x_1, ..., x_h)$ is generated by a regular sequence $(x_1, ..., x_h)$ on $M_\infty(\sigma^\circ_{min})$. We know that $(\varpi, x_1, \ldots, x_h)$ is a system of parameters for $M_\infty(\sigma^\circ_{min})$. Then by Lemma 3.9 [Paš16], an irreducible component of $\operatorname{Spec} R_\infty(\sigma_{min})$ contains a closed point $x \in \text{m-Spec}(R_\infty(\sigma_{min})/\mathfrak{a})[1/p]$. The set $\text{m-Spec}(R_\infty(\sigma_{min})/\mathfrak{a})[1/p]$ is finite, since the ring $(R_\infty(\sigma_{min})/\mathfrak{a})[1/p]$ is zero-dimensional. This point $x \in \operatorname{Supp}(M_\infty(\sigma^\circ_{min})) \cap V(\mathfrak{a})$ corresponds to a Galois representation attached to an algebraic automorphic forms (cf. section 2.6 [CEG+16]).

Let's outline, briefly, how x gives rise to a Galois representation. By Proposition 5.3.2 [EG14] there is a unique lift of a globalization to the Hecke algebra, then by universal property of a global deformation ring we get a surjective map from this global deformation ring to the Hecke algebra. The point x corresponds to a maximal ideal of this Hecke algebra. Thus x corresponds to a maximal ideal of this global deformation ring via the map from global deformation ring to the Hecke algebra. The maximal ideals of global deformation ring correspond to Galois representations of a number field, restricting it to the decomposition group we get a local Galois representation we have been looking for.

The components of $\operatorname{Spec} R_\infty(\sigma_{min})'$ which contain such a point are precisely the automorphic components. This observation justifies why those components are called automorphic.

Theorem 6.1. *Suppose $p \nmid 2n$, and that $r : G_F \longrightarrow GL_n(E)$ is a generic potentially semi-stable Galois representation of regular weight. If r correspond to a closed point $x \in \operatorname{Spec} R_\infty(\sigma_{min})[1/p]$, then $\pi_{sm}(r) \otimes \pi_{alg}(r)$ admits a non-zero unitary admissible Banach completion.*

Proof. By Theorem 5.22, we have that $\pi_{sm}(r) \otimes \pi_{alg}(r) \simeq V(r)^{l.alg}$, and by Proposition 2.13[CEG+16], $V(r)$ is an admissible unitary Banach space representation, hence a G-invariant norm on $V(r)$ restricts to a G-invariant norm on $\pi_{sm}(r) \otimes \pi_{alg}(r)$. □

Remark. It is expected that $\operatorname{Spec} R_\infty(\sigma_{min})[1/p] = \operatorname{Spec} R_\infty(\sigma_{min})'[1/p]$, i.e. that all the components are automorphic.

6.2 Some potentially crystalline non-generic points

In this section we will investigate the existence of a G-invariant norm on $BS(r)$ in some cases when r is potentially crystalline Galois representation, which is not necessarily generic. A more precise statement will be given in Theorem 6.9. Similarly as for the Theorem 6.1 from the previous section we will embed $BS(r)$ into a unitary E-Banach space representation of G. In section 6.2.1 we will build-up a framework for the proof of this theorem by examining the support of patched modules $M_\infty(\sigma^\circ_{min})$ and $M_\infty(\sigma^\circ_{max})$. Then in section 6.2.2 we will formulate the main result. In the last section we will give an example to illustrate Theorem 6.9.

6.2.1 More on support of patched modules

Let $x \in \text{m-Spec}\, R_\infty(\sigma_{min})[1/p]$ and y be the image of x in $\operatorname{Spec} \mathfrak{Z}_\Omega$ by the map $\alpha^\sharp$ from Theorem 5.14. Define

$$\gamma_x := \text{c–Ind}_K^G \sigma_{max}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(y)$$

and

$$\delta_x := \text{c–Ind}_K^G \sigma_{min}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(y).$$

Let now $x \in \text{m-Spec}\, R_\infty(\sigma_{max})[1/p]$. To the point x corresponds a Galois representation r_x. Let Σ be a subset of $\text{m-Spec}\, R_\infty(\sigma_{max})[1/p]$, consisting of those x such that the representation $\pi_{sm}(r_x)$ is generic. We will prove that the set Σ is a dense subset of $\text{m-Spec}\, R_\infty(\sigma_{max})[1/p]$.

Let $H := \mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_{min}(\lambda), \text{c–Ind}_K^G \sigma_{max}(\lambda))$. Consider the natural evaluation map:

$$ev : H \otimes_{\mathfrak{Z}_\Omega} \text{c–Ind}_K^G \sigma_{min}(\lambda) \to \text{c–Ind}_K^G \sigma_{max}(\lambda),$$

given by $f \otimes v \mapsto f(v)$. It follows from Proposition 2.17, that H is locally free $\mathfrak{Z}_\Omega$-module of rank one.

Recall that $\sigma_{max} := \sigma_{max}(\lambda) \otimes \sigma_{alg}$ and $\sigma_{min} := \sigma_{min}(\lambda) \otimes \sigma_{alg}$. We have an isomorphism:

$$H \simeq \mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_{min}, \text{c–Ind}_K^G \sigma_{max}),$$

thus $\mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_{min}, \text{c–Ind}_K^G \sigma_{max})$ is a locally free $\mathfrak{Z}_\Omega$-module of rank one. Let ϕ be the image of ev by the functor $\mathrm{Hom}_E^{cont}\left(\mathrm{Hom}_G(., (M_\infty)^d[1/p]), E\right)$. Then:

$$\phi : \mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_{min}, \text{c–Ind}_K^G \sigma_{max}) \otimes_{\mathfrak{Z}_\Omega} M_\infty(\sigma_{min}^\circ)[1/p] \longrightarrow M_\infty(\sigma_{max}^\circ)[1/p]$$

is a homomorphism of $R_\infty(\sigma_{min})$-modules. Let r_x be the Galois representations corresponding to the point x. Then by Proposition 4.33 [CEG+16], we have

$$V(r_x)^{l.alg} = \pi_x \otimes \pi_{alg}(r_x),$$

where π_x is some smooth admissible representation in the Bernstein component Ω.

Let X be the set of points x such that $\phi \otimes \kappa(x) \neq 0$.

Lemma 6.2. *By assumption we have that $x \in \mathrm{Supp}(M_\infty(\sigma_{max}^\circ))$. It follows $\mathrm{Hom}_K(\sigma_{max}, V(r_x)^{l.alg}) \neq 0$ so that we have a non-zero map $\gamma_x \to \pi_x$. Then $x \in X$ if and only the composition $\delta_x \xrightarrow{\Delta} \gamma_x \to \pi_x$ is non-zero for some (equivalently any) non-zero $\Delta \in \mathrm{Hom}_G(\delta_x, \gamma_x)$.*

Proof. If $x \in X$, then the specialization

$$\phi \otimes \kappa(x) : M_\infty(\sigma_{min}^\circ) \otimes_{R_\infty} \kappa(x) \to M_\infty(\sigma_{max}^\circ) \otimes_{R_\infty} \kappa(x)$$

is non zero, where $\kappa(x)$ is the residue field at x. However by Proposition 2.22 [Paš15] and Frobenius reciprocity we have:

$M_\infty(\sigma_{min}^\circ) \otimes_{R_\infty} \kappa(x) \simeq \mathrm{Hom}_E^{cont}\left(\mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_{min} \otimes_{\mathfrak{Z}_\Omega} \kappa(y), V(r_x)^{l.alg}), E\right) \simeq$

$$\mathrm{Hom}_E^{cont}\left(\mathrm{Hom}_G(\delta_x \otimes \pi_{alg}(r_x), \pi_x \otimes \pi_{alg}(r_x)), E\right) \simeq \left(\mathrm{Hom}_G(\delta_x, \pi_x)\right)^*$$

where $(.)^* = \mathrm{Hom}_E(., E)$ is the dual of finite dimensional vector spaces and similarly,

$$M_\infty(\sigma^\circ_{max}) \otimes_{R_\infty} \kappa(x) \simeq \left(\mathrm{Hom}_G(\gamma_x, \pi_x)\right)^*.$$

It follows that the map $\phi \otimes \kappa(x)$ is induced by the following map:

$$\begin{array}{ccc} \mathrm{Hom}_G(\delta_x, \gamma_x) \otimes \mathrm{Hom}_G(\gamma_x, \pi_x) & \to & \mathrm{Hom}_G(\delta_x, \pi_x) \\ \Delta \otimes f & \mapsto & f \circ \Delta \end{array}$$

The assertion of this lemma follows. Since $\mathrm{Hom}_G(\delta_x, \gamma_x)$ is one dimensional, any non-zero $\Delta \in \mathrm{Hom}_G(\delta_x, \gamma_x)$ will do. $\square$

Lemma 6.3. *Let $x \in \mathrm{Supp}(M_\infty(\sigma^\circ_{max}))$ be a closed point. The following assertions are equivalent:*

1. *$x \in X$*

2. *The G-equivariant map $\gamma_x \to \pi_x$ is injective.*

Proof. <u>*1* implies *2*</u>. First notice that $\phi \otimes \kappa(x) \neq 0 \implies ev \otimes \kappa(y) \neq 0$. Let $\mathrm{soc}_G(\gamma_x)$ be the G-socle of γ_x and let ι the image of the map $ev \otimes \kappa(y)$. The image ι has a finite length because the representation γ_x is of finite length. Let ν be an irreducible quotient of ι. Then $\mathrm{Hom}_G(\delta_x, \nu) = \mathrm{Hom}_G(\text{c–Ind}_K^G \sigma_{min}(\lambda), \nu) = \mathrm{Hom}_K(\sigma_{min}(\lambda), \nu) \neq 0$ because $ev \otimes \kappa(y) \neq 0$. By Proposition 3.21 the representation ν is generic, but Corollary 3.11 in [CEG$^+$16] says that the only irreducible generic subquotient of γ_x is $\mathrm{soc}_G(\gamma_x)$, so $\nu = \mathrm{soc}_G(\gamma_x)$.

Then the map $ev \otimes \kappa(y)$ factors through $\mathrm{soc}_G(\gamma_x)$ so that the diagram below commutes:

$$\begin{tikzcd} \delta_x \arrow[r] \arrow[d] \arrow[dr] & \gamma_x \\ \iota \arrow[r, two heads] & \mathrm{soc}_G(\gamma_x) \arrow[u, hook] \end{tikzcd}$$

In particular the composition:

$$\delta_x \twoheadrightarrow \mathrm{soc}_G(\gamma_x) \hookrightarrow \gamma_x \longrightarrow \pi_x$$

is non zero, by Lemma 6.2.

If the map $\gamma_x \longrightarrow \pi_x$ is not injective, let κ be it's kernel. Since κ is non zero by assumption it is equal or contains an irreducible representation η. The representation η is also a sub-representation of γ_x. Since $\mathrm{soc}_G(\gamma_x)$ is irreducible it is a unique irreducible sub-representation of γ_x, hence $\eta = \mathrm{soc}_G(\gamma_x)$. Therefore $\mathrm{soc}_G(\gamma_x) \subseteq \kappa$, so the image of $\mathrm{soc}_G(\gamma_x)$ by the map $\gamma_x \longrightarrow \pi_x$ is 0. Since $\mathrm{soc}_G(\gamma_x)$ is irreducible the composite map $\mathrm{soc}_G(\gamma_x) \hookrightarrow \gamma_x \longrightarrow \pi_x$ is injective and 0.

This would imply that the composition:

$$\delta_x \twoheadrightarrow \mathrm{soc}_G(\gamma_x) \hookrightarrow \gamma_x \longrightarrow \pi_x$$

is 0, which is a contradiction. Therefore assertion *2* follows.

2 implies *1*. Since $\mathrm{soc}_G(\gamma_x)$ is generic, by Proposition 3.21 we have that $\mathrm{Hom}_G(\delta_x, \mathrm{soc}_G(\gamma_x)) \neq 0$. Then the composition $\delta_x \twoheadrightarrow \mathrm{soc}_G(\gamma_x) \hookrightarrow \gamma_x \hookrightarrow \pi_x$ is non zero and by Lemma 6.2, $x \in X$.

□

Recall that Σ is a subset of m-Spec $R_\infty(\sigma_{max})[1/p]$, consisting of those x such that the representation $\pi_{sm}(r_x)$ is generic.

Lemma 6.4. *We have the following inclusion* $\Sigma \subseteq X$.

Proof. Let $x \in \Sigma$. Since $x \in \mathrm{Supp}(M_\infty(\sigma^\circ_{max}))$, we have $\mathrm{Hom}_G(\gamma_x, \pi_x) \neq 0$. However, by Corollary 3.12 [CEG+16], we have $\pi_{sm}(r_x) \simeq \gamma_x$. It follows that γ_x is irreducible. Thus any non-zero G-equivariant map $\gamma_x \to \pi_x$ is injective. The assertion follows from the Lemma 6.3. □

Lemma 6.5. *Let S be an equidimensional Noetherian ring, and $\mathfrak{a}$ an ideal in S. Assume that S is Jacobson. Then we have the following assertions*

1. $\dim(S/\mathfrak{a}) = \dim S$ *if and only if* $V(\mathfrak{a}) \cap$ m-Spec S *contains an irreducible component of* m-Spec S.

2. $\dim S/\mathfrak{a} < \dim S$ *if and only if* m-Spec $S \setminus V(\mathfrak{a}) \cap$ m-Spec S *is Zariski dense in* m-Spec S.

Proof. The assertions 1 and 2 are trivially equivalent. Let's prove the first one.

Assume that $\dim(S/\mathfrak{a}) = \dim S$ and write $V(\mathfrak{a}) \cap \text{m-Spec}\, S = \bigcup_i V(\mathfrak{q}_i)$ as a union of irreducible components. Then there is an index i such that

$\dim V(\mathfrak{q}_i) = \dim S$, it follows that $\mathfrak{q}_i$ is actually a minimal prime in S. Then $V(\mathfrak{q}_i)$ is an irreducible component of m-Spec S.

Assume now that $V(\mathfrak{a}) \cap$m-Spec S contains an irreducible component $V(\mathfrak{p})$ of m-Spec S. From inclusions $V(\mathfrak{p}) \subseteq V(\mathfrak{a}) \cap \text{m-Spec}\, S \subseteq \text{m-Spec}\, S$, follows that $\dim V(\mathfrak{p}) \leq \dim V(\mathfrak{a}) \cap \text{m-Spec}\, S \leq \dim S$. Since S is equidimensional $\dim V(\mathfrak{p}) = \dim S$, it follows that $\dim(S/\mathfrak{a}) = \dim S$. $\square$

Lemma 6.6. *The set* $\{x \in \text{m-Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})[1/p] \mid \pi_{sm}(r_x) \textit{ is generic}\}$ *is dense in* m-Spec $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})[1/p]$.

Proof. By Theorem 1.2.7 [All16] $x \in \text{m-Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ is smooth if and only if $WD(r_x)$ is generic and by Lemma 1.1.3 [All16], $WD(r_x)$ is generic if and only if $\pi_{sm}(r_x)$ is generic. Let $\mathcal{S}$ the singular locus of Spec $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$. Then we have that:

$$\{x \in \text{m-Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})[1/p] \mid \pi_{sm}(r_x) \text{ is generic}\}$$
$$= \text{m-Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})[1/p] \setminus (\text{m-Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})[1/p] \cap \mathcal{S})$$

We know that the ring $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$ is equidimensional and the complement of the singular locus is Zariski dense, by Theorem (3.3.4) [Kis08]. Then by previous lemma we have that $\dim \mathcal{S} < \dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$. Moreover we have

$$\dim(\text{m-Spec}\, R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})[1/p] \cap \mathcal{S}) \leq \dim \mathcal{S} < \dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p]$$

and $\dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{min})[1/p] = \dim R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})[1/p]$. We conclude by Lemma 6.5, because the ring $R^{\square}_{\tilde{\mathfrak{p}}}(\sigma_{max})[1/p]$ is also equidimensional by Theorem (3.3.8) [Kis08]. $\square$

Proposition 6.7. Σ *is Zariski dense in* Spec $R_\infty(\sigma_{max})[1/p]$.

Proof. To prove that Σ is Zariski dense in $\text{Spec}(R_\infty(\sigma_{\max})[1/p])$, it would suffice to prove that those points are dense in every irreducible component of $R_\infty(\sigma_{max})[1/p]$. Since the spectrum of this ring is a union of irreducible components of the spectrum of $R_\infty(\sigma_{\max})'[1/p]$ by Proposition 5.13, it is enough to prove it for this ring. The result follows from previous lemma and the fact that closed points are dense in a Jacobson ring. $\square$

Proposition 6.8. *The set* X *is closed if and only if* ϕ *is surjective. In this case* $X = \text{Spec}\, R_\infty(\sigma_{max})[1/p]$.

Proof. If ϕ is surjective then $X = \text{Spec}\, R_\infty(\sigma_{max})[1/p]$.

If X is closed then by Lemma 6.4, it contains the closure of Σ. However by Proposition 6.7, $X = \text{Spec}\, R_\infty(\sigma_{max})[1/p]$, so ϕ is surjective. $\square$

6.2.2 Existence of a G-invariant norm

Recall that throughout this thesis we have fixed a residual Galois representation $\overline{r} : G_F \longrightarrow GL_n(\mathbb{F})$. The patching construction carried out in [CEG+16] associates to $\overline{r}$ a modules $M_\infty(\overline{r}) := M_\infty$ and also the ring $R_\infty(\sigma_{min})(\overline{r}) := R_\infty(\sigma_{min})$.

Let $\rho : G_F \longrightarrow GL_n(E)$ be a potentially semi-stable Galois representation of weight σ_{alg} and inertial type τ. By the theory of Fontaine, the Galois representation ρ corresponds to a filtered admissible $(\varphi, N, \mathrm{Gal}(L/F))$-module $\tilde{D}$. Proposition 4.5 says that there is an admissible filtered φ-module D such that the underlying Weil representation and the Hodge-Tate weights of D are the same as that of $\tilde{D}$, but the monodromy operator is equal to zero. Let r be the Galois representation corresponding to D. This Galois representation is potentially crystalline. Observe that the Galois representations r and ρ may have different reduction mod p since the filtrations on the corresponding (φ, N)-modules may differ.

Let now $\overline{\rho} : G_F \longrightarrow GL_n(\mathbb{F})$ be the reduction mod p of ρ. We can consider a new deformation problem associated to $\overline{\rho}$. Let $\tilde{M}_\infty := M_\infty(\overline{\rho})$, $\tilde{R}_\infty(\sigma_{min}) := R_\infty(\sigma_{min})(\overline{\rho})$ and let $\tilde{R}_\infty(\sigma_{min})' = R_\infty \otimes_{R^\square_{\overline{\mathfrak{p}}}} R^\square_{\overline{\rho}}(\tau, \mathbf{v})$, where the ring $R^\square_{\overline{\rho}}(\tau, \mathbf{v})$ parametrizes all the potentially semi-stable lifts of $\overline{\rho}$ of weight σ_{alg} and inertial type τ.

Theorem 6.9. *Let r and ρ be two Galois representations, as above. Assume that ρ is generic (we do not assume that ρ corresponds to a point lying on an automorphic component) and that $r_x := r$ corresponds to a closed point $x \in \operatorname{Spec} R_\infty(\sigma_{max})[1/p]$ (deformation problem for $\overline{r}$). Furthermore, assume that the equivalent conditions of Proposition 6.8 hold. Then $BS(r_x)$ and $BS(\rho)$ both admit a G-invariant norm. The completions of $BS(r)$ and $BS(\rho)$ with respect to these norms are admissible.*

Proof. Recall that we have $V(r_x)^{l.alg} = \pi_x \otimes \pi_{alg}(r_x)$ by Proposition 4.33 [CEG+16], where π_x is some smooth admissible representation in the Bernstein component Ω. By Corollary 3.11 [CEG+16], the irreducible representation $\pi_{sm}(\rho)$ is the socle of $\text{c–Ind}_K^G \sigma_{max}(\lambda) \otimes_{\mathfrak{Z}_\Omega, \chi_{\pi_{sm}(\rho)}} E$.

Let y be the image of x by the map $\operatorname{Spec} R_\infty(\sigma_{max})[1/p] \longrightarrow \operatorname{Spec} \mathfrak{Z}_\Omega$ (map $\alpha^\sharp$ of Theorem 5.14 or equivalently the map from Theorem 4.19 [CEG+16]). According to Theorem 5.3 and Theorem 5.14, the closed point y depends only on the eigenvalues of the linearised Frobenius φ^f (which acts on both D and $\tilde{D}$). The Galois representation ρ corresponds to a closed point $\tilde{x}$ of m-Spec $R^{\square}_{\bar{\rho}}(\tau, \mathbf{v})[1/p]$. Let $\tilde{y}$ be the image of $\tilde{x}$ by the map Spec $R^{\square}_{\bar{\rho}}(\tau, \mathbf{v})[1/p] \longrightarrow$ Spec $\mathfrak{Z}_\Omega$(Theorem 5.3). The same conclusions hold for $\tilde{y}$.

By construction, D and $\tilde{D}$ have the same action of φ. Then together with the observation above, it follows that $y = \tilde{y}$.

Let $\gamma_x := \text{c–Ind}_K^G \sigma_{max}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(y)$, this is a parabolic induction of a supercuspidal representation of a Levi subgroup of G. Let $\delta_x := \text{c–Ind}_K^G \sigma_{min}(\lambda) \otimes_{\mathfrak{Z}_\Omega} \kappa(y)$. By definition $BS(r_x) = \gamma_x \otimes \pi_{alg}(r_x)$.

Since $y = \tilde{y}$, it follows that $\text{c–Ind}_K^G \sigma_{max}(\lambda) \otimes_{\mathfrak{Z}_\Omega, \chi_{\pi_{sm}(\rho)}} E = \gamma_x$. We conclude that $\pi_{sm}(\rho) = \text{soc}(\gamma_x)$ is the socle of γ_x, and this irreducible representation is generic. Hence by Proposition 3.21, $\text{Hom}_K(\sigma_{min}(\lambda), \text{soc}(\gamma_x)) \neq 0$. It follows that there is a non zero G-equivariant map $\delta_x \longrightarrow \text{soc}(\gamma_x)$. Hence we have a non-zero map $\delta_x \longrightarrow \text{soc}(\gamma_x) \hookrightarrow \gamma_x$.

Since $x \in \text{Supp}(M_\infty(\sigma^\circ_{max}))$, because r_x is potentially crystalline, we have $\text{Hom}_K(\sigma_{max}(\lambda), \pi_x) \neq 0$, therefore there is a non-zero map $\gamma_x \longrightarrow \pi_x$. By Lemma 6.2 and Lemma 6.3 the composition $\delta_x \longrightarrow \text{soc}(\gamma_x) \longrightarrow \gamma_x \longrightarrow \pi_x$ is non-zero if and only if the map $\gamma_x \longrightarrow \pi_x$ is injective. We will prove that $\gamma_x \longrightarrow \pi_x$ is injective.

With the notation introduced in the previous section, we have that $X = \text{Supp}(M_\infty(\sigma^\circ_{max}))$ by Proposition 6.8. Therefore $\gamma_x \longrightarrow \pi_x$ is injective for any closed point $x \in \text{Supp}(M_\infty(\sigma^\circ_{max}))$.

So we have that $BS(r_x) \hookrightarrow V(r_x)^{l.alg}$, by a similar argument. The restriction of the norm on Banach space representation $V(r_x)$ induces a G-invariant norm on $BS(r_x)$. Since $\pi_{sm}(\rho)$ is the socle of γ_x, we have a G-equivariant injection $BS(\rho) \hookrightarrow BS(r_x)$. So we also obtain a G-invariant norm on $BS(\rho)$ by restricting a G-invariant norm on $BS(r_x)$. $\square$

Remark. It is expected that $BS(r_x) \simeq V(r_x)^{l.alg}$.

6.2.3 Example

In this section, we will give an example to illustrate the Theorem 6.9 above. Let ρ and r as in Theorem 6.9. Under assumptions of Corollary 5.5 (2) [CEG+16], it is known that all the components are automorphic. Then in the case when all components are automorphic, Theorem 6.9 can be applied to r whenever the equivalent conditions of Proposition 6.8 hold. I hope to remove this assumption in the future work. This gives us a G-invariant norm on $BS(r)$ and also on $BS(\rho)$. Let's now specify r and ρ.

Let $F = \mathbb{Q}_p$, $n = 3$ and let r, s be two integers such that $0 < r < s$, $r \le p-1$ and $s-r \le p-1$. Let v_p be a valuation $\overline{\mathbb{Q}}_p$ with $v_p(p) = 1$. Assume $p \ne 3$. Let $\tilde{D}$ be an admissible filtered (φ, N)-module with Hodge-Tate weights $0 < r < s$, from Example 3.40 [Par16]:

- $\mathrm{Fil}^r \tilde{D} = E(e_1, e_2)$ and $\mathrm{Fil}^s \tilde{D} = E(e_1)$.
- $N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\varphi = \begin{pmatrix} p\mu & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu \end{pmatrix}$, with $\mu \ne \mu_2 \ne p^{\pm 1}\mu$.
- $r - 1 > v_p(\mu) > (r-1)/2 > 0$ and $1 + v_p(\mu) \ge v_p(\mu_2) \ge v_p(\mu)$.

According to Proposition 3.41 [Par16], the (φ, N)-module above is irreducible, because $r - 1 > v_p(\mu) > (r-1)/2$ and $r > 1$.

Let ρ be a semi-stable Galois representation with Hodge-Tate weights $0 < r < s$ corresponding to $\tilde{D}$. Let P be a standard parabolic subgroup of G corresponding to a partition $(2, 1)$, let χ and χ_2 unramified characters of $\mathbb{Q}_p$ such that $\chi(p) = \mu$ and $\chi_2(p) = \mu_2$. Then

$$\pi_{sm}(\rho) = i_P^G((St \otimes |.|^{-1/2}\chi) \otimes \chi_2) \otimes |\det|,$$

where St is the Steinberg representation of $GL_2(\mathbb{Q}_p)$. Since after killing the monodromy there is not a unique choice of a filtration that makes the underlying φ-module admissible, we may choose r, so that the Galois representation r corresponds to a φ-module D, from Example 2.61 [Par16]:

- $\mathrm{Fil}^r D = E(e_1 + e_2 + e_3, e_2 + 2e_3)$ and $\mathrm{Fil}^s D = E(e_1 + e_2 + e_3)$.
- $N = 0$ and $\varphi = \begin{pmatrix} p\mu & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu \end{pmatrix}$, with $\mu \ne \mu_2 \ne p^{\pm 1}\mu$.
- $s > r > 1 + v_p(\mu) \ge v_p(\mu_2) \ge v_p(\mu) > (r-1)/2 > 0$ and $1 + v_p(\mu) + v_p(\mu_2) + v_p(\mu) = r + s$.

Since $s > 1 + v_p(\mu) \ge v_p(\mu_2) \ge v_p(\mu) > 0$, it follows from Proposition 2.62 [Par16] that this φ-module is irreducible. Then by classical Langlands correspondence:

$$\pi_{sm}(r) = L(|.|^{-1}\chi, \chi_2, \chi) \otimes |\det| = L(\chi, |.|\chi_2, |.|\chi)$$

is a Langlands quotient where the segments χ and $|.|\chi$ are linked.

Proposition 6.10. *Let ρ be a generic semi-stable Galois representation and r be a crystalline Galois representation as above, both having same Hodge-Tate weights $0 < r < s$ and the same action of the Frobenius φ. Let χ and χ_2 be two unramified characters such that $\chi(p) = \mu$ and $\chi_2(p) = \mu_2$, with μ and μ_2 satisfying relations:*

(A) $\mu \neq \mu_2 \neq p^{\pm 1}\mu$ *and* $r > 1 + v_p(\mu) \geq v_p(\mu_2) \geq v_p(\mu) > (r-1)/2 > 0$.

(B) $1 + v_p(\mu) + v_p(\mu_2) + v_p(\mu) = r + s$.

Assume that the equivalent conditions of Proposition 6.8 hold. Then $BS(r) := (i_B^G(\chi \otimes |.|\chi_2 \otimes |.|\chi)) \otimes \pi_{alg}(r)$ admits a G-invariant norm, where B is a Borel subgroup of G and $\pi_{alg}(r)$ is an irreducible algebraic representation of highest weight $\psi(\mathrm{diag}(t_1, t_2, t_3)) = t_1^0.t_2^{-r+1}.t_3^{-s+2}$ with respect to upper triangular Borel B. Moreover, $BS(\rho) := \pi_{sm}(\rho) \otimes \pi_{alg}(r)$ admits also a G-invariant norm and the completions of $BS(r)$ and $BS(\rho)$ with respect to these norms are admissible.

Proof. First observe that $\pi_{sm}(r)$ is an irreducible quotient of $i_B^G(\chi \otimes |.|\chi_2 \otimes |.|\chi)$, hence by definition $BS(r) := (i_B^G(\chi \otimes |.|\chi_2 \otimes |.|\chi)) \otimes \pi_{alg}(r)$. By construction, the Hodge-Tate weights of the Galois representation r lie in the extended Fontaine-Laffaille range. Then Galois representation r corresponds to a point which lies on an automorphic component since by the proof of Corollary 5.5 (2) [CEG$^+$16], all the components are automorphic. We may then apply Theorem 6.9, to prove that $BS(r)$ admits a G-invariant norm. In this setting we obtained a G-invariant norm on $BS(r)$ unconditionally on the assumption that the Galois representation corresponds to a point lying on an automorphic component. Since $BS(\rho) \hookrightarrow BS(r)$, we also obtain a G-invariant norm on $BS(\rho)$ also unconditionally. $\square$

We will prove that the G-invariant norm on $BS(r)$ does not come from a restriction of a norm on a parabolic induction of a unitary character. First, we will prove the following lemma:

Lemma 6.11. *Let π be a smooth admissible representation of G and σ an algebraic irreducible representation of G with highest weight ψ with respect to upper triangular Borel subgroup B of G. Then $(\pi \otimes \sigma)_N \simeq \pi_N \otimes \sigma_N$.*

Proof. Let V denote vector space equipped with a G-action. We will denote by $V(N)$ the space spanned by $n.v - v$, $n \in N$ and $v \in V$, and by $V_N =$

$V/V(N)$. We will identify injective maps with the inclusions. Since $(\pi \otimes \sigma)(N) \subseteq \pi(N) \otimes \sigma(N)$ then we have $(\pi \otimes \sigma)_N \twoheadrightarrow \pi_N \otimes \sigma_N$.

The representation σ is finite dimensional. Let w be the highest weight vector of σ. Observe that σ_N is one dimensional generated by w.

Let $v \in \pi$. Since π is smooth, the vector v is fixed by some compact open $N_0 \subseteq N$. We have also that $\sigma_N = \sigma_{N_0} = E.w$, because this representation is algebraic. Since σ is finite dimensional, we may choose $w_1, \dots w_d$ a basis of σ such that $w_m \in \sigma(N_0)$ for $m \neq d$ and $w_d = w$. Then $w_m \in \sigma(N_0)$ can be written as $w_m = \sum\limits_{k=1}^{d} a_k(n_k - 1)w_k$, where a_k are some scalars and $n_k \in N_0$. It follows that for $m \neq d$:

$$v \otimes w_m = v \otimes \sum_{k=1}^{d} a_k(n_k - 1)w_k = \sum_{k=1}^{d} a_k(v \otimes (n_k - 1)w_k) =$$

$$= \sum_{k=1}^{d} a_k((v \otimes n_k w_k) - v \otimes w_k) = \sum_{k=1}^{d} a_k((n_k v \otimes n_k w_k) - v \otimes w_k) =$$

$$= \sum_{k=1}^{d} a_k(n_k(v \otimes w_k) - v \otimes w_k) = \sum_{k=1}^{d} a_k(n_k - 1)(v \otimes w_k) \in (\pi \otimes \sigma)(N)$$

This shows that $\pi \otimes (\sigma(N)) \subseteq (\pi \otimes \sigma)(N)$. Therefore we get a surjection $\pi \otimes \sigma_N \twoheadrightarrow (\pi \otimes \sigma)_N$. Since N acts trivially on σ_N, this map factors through $\pi_N \otimes \sigma_N$, as

$$\begin{tikzcd}
\pi \otimes \sigma_N \arrow[r, two heads] \arrow[d, two heads] & (\pi \otimes \sigma)_N \\
\pi_N \otimes \sigma_N \arrow[ur, two heads] &
\end{tikzcd}$$

The composition map $\pi_N \otimes \sigma_N \twoheadrightarrow (\pi \otimes \sigma)_N \twoheadrightarrow \pi_N \otimes \sigma_N$ is the identity. This allows us to conclude. $\square$

Proposition 6.12. *Let T be a group of diagonal matrices and N group of unipotent matrices such that both are subgroups of a Borel $B = TN$ as in Proposition 6.11. Let $\theta : T \to \mathcal{O}^\times \to E^\times$ be a unitary character. Then there is no such unitary character θ, such that the we have an embedding $BS(r) \hookrightarrow \mathrm{Ind}_B^G(\theta)_{cont}$ (index cont means that we consider continuous functions in this space). We also have that for any unitary character θ as above, there is no injection $BS(\rho) \hookrightarrow \mathrm{Ind}_B^G(\theta)_{cont}$.*

Proof. Since $BS(r)$ is locally algebraic, we can restrict ourself to the space $\mathrm{Ind}_B^G(\theta)^{l.an}$(the functions in this space are locally analytic, see [Eme07] for definitions), and work in the category of locally analytic representations. According to Theorem 4.2.6 [FdL99], we have a Frobenius reciprocity in the category of locally analytic representations:

$$\mathrm{Hom}_G^{cont}(BS(r), \mathrm{Ind}_B^G(\theta)^{l.an}) \simeq \mathrm{Hom}_B^{cont}(BS(r)|B, \theta)$$

Let $\delta^{1/2} = |.|^{-1/2} \otimes 1 \otimes |.|^{1/2}$ be a square root of the modulus character. Since θ is trivial on N, $BS(r)$ factors through N-coinvariants, then:

$$\mathrm{Hom}_G^{cont}(BS(r), \mathrm{Ind}_B^G(\theta)^{l.an}) \simeq \mathrm{Hom}_T^{cont}((BS(r))_N, \theta)$$

Then Lemma 6.11 allows us to compute:

$$(BS(r))_N = \delta^{1/2}.r_B^G(i_B^G(\chi \otimes |.|\chi_2 \otimes |.|\chi)) \otimes \pi_{alg}(r)_N,$$

where r_B^G is the left adjoint functor of i_B^G in the category of smooth representations. Let $\tilde{\chi} := \chi \otimes |.|\chi_2 \otimes |.|\chi$. Theorem 1.2 [Zel80] tells us that there is a filtration:

$$0 = \tau_0 \subset \tau_1 \subset ... \subset \tau_6 = r_B^G(i_B^G(\tilde{\chi})),$$

such that $\tau_i/\tau_{i-1} \simeq \tilde{\chi}^{w_i}$, where w_i is an element of the symmetric group in 3 letters $\mathfrak{S}_3 = \{w_1, ..., w_6\}$. Let $\psi(\mathrm{diag}(t_1, t_2, t_3)) = t_1^0.t_2^{-r+1}.t_3^{-s+2}$ be the weight of the highest weight representation $\pi_{alg}(r)$ with respect to upper triangular matrices. If we have

$$\mathrm{Hom}_T^{cont}(\delta^{1/2}.\tilde{\chi}^w.\psi, \theta) = 0,$$

for all $w \in \mathfrak{S}_3$, then $\mathrm{Hom}_G^{cont}(BS(r), \mathrm{Ind}_B^G(\theta)^{l.an}) = 0$. It is enough to prove that for any $w \in \mathfrak{S}_3$ the character $\delta^{1/2}.\tilde{\chi}^w.\psi$ is not unitary. Indeed we have that:

$$v := v_p(\delta^{1/2}.\tilde{\chi}^w.\psi(\mathrm{diag}(1, p, 1))) \in$$
$$\{1 - r + v_p(\mu), -r + v_p(\mu), -r + v_p(\mu_2)\},$$

and in all these cases we have $v < 0$ by relations (A) of Proposition 6.10. It follows that the character $\delta^{1/2}.\tilde{\chi}^w.\psi$ does not take its values in $\mathcal{O}^\times$. For $BS(\rho)$ we repeat the same proof. □

The locally algebraic representations $BS(\rho)$ and $BS(r)$ have the same central character. We can easily verify that, by relation (B) of Proposition 6.10, we have $v_p(\tilde{\chi}.\psi(\mathrm{diag}(p,p,p))) = v_p(\mu)+v_p(\mu_2)-1+v_p(\mu)+3-r-s = 2v_p(\mu)+v_p(\mu_2)+1-r-s = 0$, therefore the central character of $BS(r)$ (or of $BS(\rho)$) is unitary.

We have shown that, in this example, $BS(r)$ can not be embedded into $\mathrm{Ind}_B^G(\theta)_{cont}$, with θ unitary. Thus the G-invariant norm on $BS(r)$, obtained by the Theorem 6.9, does not come from a restriction of a G-invariant norm on a parabolic induction of a unitary character. Same conclusions hold for $BS(\rho)$.

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