Conformal blocks attached to twisted groups

Dissertation
zur Erlangung des akademischen Grades
Doktors der Naturwissenschaften (Dr.rer.nat.)

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vorgelegt der Fakultät für Mathematik
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Essen 2017

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Datum der mündlichen Prüfung
25.09.2017

CONTENTS

Zusammenfassung der Dissertation

Geometrische Darstellungtheorie erlaubt es uns zu einer einfachen und einfach zusammenhängenden algebraischen Gruppe über einem Körper und einer positiven ganzen Zahl ℓ ein Vektorbündel, genannt Garbe der konformen Blöcke, über $\mathcal{M}_g$, dem Stack, welcher Kurven von Geschlecht g parametrisiert, zu assoziieren. In dieser Doktorarbeit verallgemeinern wir diese Konstruktion indem wir die Gruppe G durch eine verdrehte Gruppe, die von Überlagerungsdaten abhängt, ersetzten. Den Ideen von Balaji und Seshadri folgend erzeugen wir, aus einer (verzweigte) galoissche Überlagerung von Kurven $q\colon \widetilde{X} \to X$ mit galoisscher Gruppe $\Gamma := \mathbb{Z}/p\mathbb{Z}$ und einem Gruppenhomomorphismus $\Gamma \to \mathrm{Aut}(G)$, eine Gruppe $\mathcal{H}$ über X als die Γ-invarianten Untergruppe von $q_*(G \times \widetilde{X})$. Das induziert eine Gruppe $\mathcal{H}_{univ}$ über der universellen Kurve X_{univ} über dem Hurwitz stack $\mathcal{H}\mathrm{ur}(\Gamma, \xi)_g$, der Γ-Überlagerung Kurven parametrisiert. Wir zeigen, dass es möglich ist, in Analogie zum klassichen Fall, zu $\mathcal{H}_{univ}$ und ℓ ein Vektorbündel $\mathcal{H}_\ell(0)_{X_{univ}}$ über $\mathcal{H}\mathrm{ur}(\Gamma, \xi)_g$ zu assoziieren. Den Methoden von Looijenga benutzend beweisen wir darüber hinaus, dass die Haupteigenschaften des klassischen Bündels der konforme Blöcke auch in dieser allgemeineren Situation gelten. Insbesondere beschreiben wir den WZW Zusammenhang, die sogenannte Fortsetzung von Vacua und die Fusionsregeln.

INTRODUCTION

In conformal field theory [**TUY89**], there is a way to associate to a simple and simply connected group G over an algebraically closed field k of characteristic zero, a vector bundle $\mathbb{H}_\ell(0)_{X_{univ}}$, called the *sheaf of conformal blocks*, on $\mathcal{M}_g$, the stack parametrizing smooth curves of genus g. The goal of this thesis is to generalize this construction to the case in which the group G is replaced by a certain type of parahoric Bruhat-Tits group $\mathcal{H}$ arising from cyclic coverings.

Classical conformal blocks. Before going into the details of the content of this thesis, we briefly recall the properties of the sheaf of conformal blocks. Denote by $\mathfrak{g}$ the Lie algebra of G and by P_ℓ the set of integral dominant weights of $\mathfrak{g}$ of level at most ℓ. Let X be a (nodal) curve over Spec(k) of genus g, which is stably marked by the points $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Then, to the $2n$-tuple $(\mathfrak{p}_i, \lambda_i)_{i=1}^n$, with $\lambda_i \in P_\ell$, it is possible to associate a vector space $\mathbb{H}_\ell(\lambda_i)$. This construction extends to families of n-pointed stable curves of genus g, giving rise to the vector bundle $\mathbb{H}_\ell(\lambda_i)_{X_{univ}}$ on $\overline{\mathcal{M}}_{g,n}$. This is what is called the *sheaf of conformal blocks* attached to the weights λ_i's.

In the case in which all the λ_i's are zero, the so called *propagation of vacua* ensures that the associated sheaf of conformal blocks is actually independent of the marked points, hence it descends to $\mathcal{M}_g$. We denote this vector bundle, which is called the *sheaf of covacua*, by $\mathbb{H}_\ell(0)_{X_{univ}}$.

The rank of $\mathbb{H}_\ell(\lambda_i)_{X_{univ}}$ has been computed with the *Verlinde formula* [**TUY89**] [**Fal94**] [**Sor96**]. The main ingredient for this computation consists in the *fusion rules* which control the behaviour of the rank under degeneration of curves. Thanks to this property the computation of the rank is reduced to the case of the projective line $\mathbb{P}^1$ with three marked points.

These sheaves have played an important role in algebraic geometry not only as a tool to study $\overline{\mathcal{M}}_{g,n}$, but also in the study of $\mathrm{Bun}_G(X)$, the stack parametrizing G-bundles on a smooth curve X. In fact, for every $\ell \in \mathbb{N}$ there is a canonical isomorphism

$$H^0(\mathrm{Bun}_G(X), \mathscr{L}^{\otimes \ell})^* \cong \mathbb{H}_\ell(0)_X$$

where $\mathscr{L}$ is the determinant line bundle on $\mathrm{Bun}_G(X)$ [**BL94**] [**KNR94**]. The key point to prove this isomorphism is the *uniformization theorem* which describes $\mathrm{Bun}_G(X)$ as a quotient of the the affine Grassmannian $\mathrm{Gr}(G)$, whose line bundles and the space of their global sections have been described in terms of representations of $\mathfrak{g}$ by Kumar

[**Kum87**] and Mathieu [**Mat88**]. This theorem, which was proved initially by Beauville and Laszlo in [**BL94**] for $G = \mathrm{SL}_n$, has been generalized for parabolic groups by Pauly in [**Pau96**] and by Laszlo and Sorger in [**LS97**]. Finally Heinloth proved the uniformization theorem for $\mathrm{Bun}_{\mathcal{H}}(X)$, for connected parahoric Bruhat-Tits groups $\mathcal{H}$ in [**Hei10**], where he also gave a description of the Picard group of $\mathrm{Bun}_{\mathcal{H}}(X)$. Having in hand the notion of the sheaf of conformal blocks for parahoric Bruhat-Tits groups $\mathcal{H}$ satisfying factorization rules and propagation of vacua, is then the first step to describe the space of global sections $H^0(\mathrm{Bun}_{\mathcal{H}}(X), \mathscr{L})$ of certain line bundles $\mathscr{L}$ on $\mathrm{Bun}_{\mathcal{H}}(X)$ and achieve, in a second time, a Verlinde type formula for $H^0(\mathrm{Bun}_{\mathcal{H}}(X), \mathscr{L})$, as asked by Pappas and Rapoport in [**PR10**].

Parahoric Bruhat-Tits groups arising from coverings. As already mentioned, in our generalization we replace the group G with a parahoric Bruhat-Tits group $\mathcal{H}$ defined over a curve X. Since the group $\mathcal{H}$ depends on the geometry of the curve, our version of the sheaf of conformal blocks will be in general not defined over $\overline{\mathcal{M}}_{g,n}$ but on a moduli space which encodes also the information on $\mathcal{H}$. Inspired by [**BS15**], we restrict ourselves to consider only those groups *arising from coverings* in the following sense. We fix the cyclic group $\Gamma := \mathbb{Z}/p\mathbb{Z}$ of prime order p and a group homomorphism $\rho\colon \Gamma \to \mathrm{Aut}(G)$. Let $q\colon \widetilde{X} \to X$ be a (ramified) Galois covering of nodal curves with Galois group Γ and denote its moduli stack by $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_g$. We remark that in contrast to [**BR11**], we assume that the nodes of X are disjoint from the branch locus $\mathcal{R}$ of q. Then we say that a group $\mathcal{H}$ on X arises from q and ρ if it is isomorphic to the group of Γ-invariants of the Weil restriction of $\widetilde{X} \times_k G$ along q, i.e. $\mathcal{H} = q_*(\widetilde{X} \times_k G)^\Gamma$.

We observe that the groups $\mathcal{H}$ that we consider are parahoric Bruhat-Tits groups which in general are not generically split, while in [**BS15**] the authors only work in the split situation. This reflects the condition that in their paper they only allow Γ to act on G by inner automorphisms, i.e. ρ arises from a group homomorphism $\Gamma \to G$. The following statement is a particular instance of Theorem A.0.7 which generalizes [**BS15**, Theorem 4.1.6].

THEOREM. *Let $q\colon \widetilde{X} \to X$ be a Γ-covering of curves and $\rho\colon \Gamma \to Aut(G)$ be a homomorphism of groups. Set $\mathcal{H} = q_*(X \times G)^\Gamma$. Then the functor $q_*(-)^\Gamma$ induces an equivalence between $Bun_{\mathcal{H}}(X)$ and the stack $Bun^G_{(G,\Gamma)}(\widetilde{X})$ parametrizing G-bundles on $\widetilde{X}$ equipped with an action of* Γ compatible *with the one on G.*

The notion of compatibility stressed in the above Theorem will be clarified in Appendix A in terms of *local type* of (Γ, G)-bundles. It is important to underline that in [**BS15**], it has been shown that all the split parahoric Bruhat-Tits groups are defined by means of Γ-coverings, for Γ a finite, non necessarily cyclic, group acting via inner automorphisms on a constant group G.

Main results. In order to define the generalized sheaf of conformal blocks, we first of all need to introduce the pointed version of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_g$ and in second place replace P_ℓ with an appropriate set of representations of $\mathcal{H}$. We denote by $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}$ the stack parametrizing Γ-coverings of nodal curves $\widetilde{X} \to X$, where X is marked by a point $\mathfrak{p}$ which is disjoint from the branch locus $\mathcal{R}$. In similar fashion we define $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$ for $n \geq 1$. Let $\mathcal{H}$ be the group on X_{univ} arising from the universal covering $(\widetilde{X}_{univ} \to X_{univ}, \mathfrak{p})$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}$ and the homomorphism $\rho\colon \Gamma \to \mathrm{Aut}(G)$. Set $\mathfrak{h} := \mathrm{Lie}(\mathcal{H})$ and denote by $\mathrm{IrRep}_\ell(\mathfrak{h}|_\mathfrak{p})$ the set of irreducible representations $\mathcal{V}$ of $\mathfrak{h}|_\mathfrak{p}$ of level at most ℓ (Definition 2.2.8).

In Chapter 2 we explain how to associate to each representation $\mathcal{V} \in \mathrm{IrRep}_\ell(\mathfrak{h}|_\mathfrak{p})$, a vector bundle $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$ (Proposition 2.2.9 and Definition 2.2.10). This is called the *sheaf of conformal blocks* attached to $\mathcal{V}$. Similarly, working on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,n}$, we can construct a vector bundle $\mathcal{H}_\ell(\mathcal{V}_1,\dots,\mathcal{V}_n)_{X_{univ}}$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,n}$ attached to the representations $\mathcal{V}_i \in \mathrm{IrRep}_\ell(\mathfrak{h}|_{\mathfrak{p}_i})$ (Section 3.6).

As in the classical case, the Propagation of Vacua holds (Proposition 4.1.1), leading to the following statement.

PROPOSITION. *Let $\mathcal{V}(0)$ be the trivial representation of $\mathfrak{h}|_\mathfrak{p}$. Then the vector bundle $\mathcal{H}_\ell(\mathcal{V}(0))_{X_{univ}}$ is independent of the choice of the marked point, hence it descends to a vector bundle $\mathcal{H}_\ell(0)_{X_{univ}}$ on $\mathcal{H}ur(\Gamma,\xi)_g$.*

Moreover, in Proposition 4.2.2 we formulate fusion rules controlling the rank of the vector bundle under degeneration of the covering:

PROPOSITION. *Let $(q\colon \widetilde{X} \to X, \mathfrak{p}) \in \overline{\mathcal{H}ur}(\Gamma,\xi)_{g,1}(k)$ and let x be a nodal point of X. Let X_N be the partial normalization of X at x so that $q_N\colon \widetilde{X}_N \to X_N$ is a Γ-covering with X_N marked by three marked points. Then for any $\mathcal{W} \in IrRep_\ell(\mathfrak{h}|_\mathfrak{p})$ we have a canonical isomorphism*

$$\mathcal{H}_\ell(\mathcal{W})_X \cong \bigoplus_{\mathcal{V}\in IrRep_\ell(\mathfrak{h}|_x)} \mathcal{H}_\ell(\mathcal{W},\mathcal{V},\mathcal{V}^*)_{X_N}.$$

Insights into the construction and properties of $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$. We now give an overview of how the twisted conformal blocks are defined, generalizing the methods used in [**Kac90**], [**TUY89**] and [**Loo13**]. For simplicity we consider a covering of smooth curves $\widetilde{X} \to X$ which is marked by a point $\mathfrak{p} \in X(k)$. We denote by $\mathfrak{h}_\mathcal{L}$ the restriction of the sheaf of Lie algebras $\mathfrak{h} = \mathrm{Lie}(\mathcal{H})$ to the punctured disc $\mathcal{L} = \mathrm{Spec}(k((t)))$ around the point $\mathfrak{p}$. Observe that since $\mathfrak{p}$ is not a branch point, $\mathfrak{h}|_\mathfrak{p}$ is isomorphic, although non canonically, to $\mathfrak{g}$ and $\mathfrak{h}_\mathcal{L}$ is isomorphic to the affine Lie algebra $\mathfrak{g}\mathcal{L} := \mathfrak{g} \otimes_k k((t))$. It follows that once we choose such an isomorphism, we can use the classical construction [**Kac90**, Chapter 7] to associate to each representation $\mathcal{V} \in \mathrm{IrRep}_\ell(\mathfrak{h}|_\mathfrak{p}) \cong P_\ell$ the integrable highest weight representation $\mathcal{H}_\ell(\mathcal{V})$ of $\widehat{\mathfrak{h}}_\mathcal{L}$, a central extension of $\mathfrak{h}_\mathcal{L} = \mathfrak{h} \otimes_k k((t))$ defined in terms of Killing form and residue pairing. The key point is to see that this construction is actually independent of the isomorphism chosen between $\mathfrak{h}|_\mathfrak{p}$ and $\mathfrak{g}$. Serre duality ensures that the Lie algebra $\mathfrak{h}_\mathcal{A} := \mathfrak{h}|_{X\setminus\mathfrak{p}}$ is a sub Lie algebra of $\widehat{\mathfrak{h}}_\mathcal{L}$, so that we set $\mathcal{H}_\ell(\mathcal{V})_X$ to be space of $\mathfrak{h}_\mathcal{A}$-coinvariants of $\mathcal{H}_\ell(\mathcal{V})$, i.e. the quotient $\mathfrak{h}_\mathcal{A} \backslash \mathcal{H}_\ell(\mathcal{V})$. The construction of the sheaf of conformal blocks runs similarly for any family $(\widetilde{X} \to X, \sigma) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}(S)$, being careful that the isomorphism between $\mathfrak{h}|_{\sigma(S)}$ and $\mathfrak{g} \otimes_k S$ exists only étale locally on S.

Although it is easy to show that $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is coherent (Proposition 2.2.12), it is not immediate from its construction that it is also locally free. Following Looijenga in [**Loo13**], the first step to achieve this result is to generalizes the *WZW connection* defined in terms of conformal field theory:

PROPOSITION (Corollary 3.5.2). *The sheaf $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ on $\overline{\mathcal{H}ur}(\Gamma,\xi)_{g,1}$ is equipped with a projective connection with logarithmic singularities along the boundary Δ_{univ}.*

This shows in particular that $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is a locally free module over $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1}$. The idea is to realise $\mathcal{H}_\ell(\mathcal{V})$ as a Fock-type representation of a Lie algebra of derivations which is a central extension of the sheaf of logarithmic vector fields of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$ along Δ_{univ}. Combining this with the fusion rules, we are able to prove the local freeness on the whole stack $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$ (Corollary 5.2.8). It is then clear that the fusion rules play a double role in the theory of conformal blocks. On one side they contribute to show that $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is locally free on the whole $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$, and on the other side they are a useful tool to reduce the computation to lower genera curves.

ACKNOWLEDGEMENTS

I first of all want to thank my advisor Jochen Heinloth for his supervision and constant encouragement. Thanks for the patience, not only in explaining his points of views and ideas, but also in listening and understanding mines. His guidance in the whole time of the research and writing of this thesis has been essential to me.

I thank Christian Pauly for the discussions we had in Nice and for the many questions he asked. This helped me having a wider understanding on the problems underlying this PhD project. I also wish to thank him for agreeing to act as Gutachter. Thanks to the entire committee of the thesis defence.

Thanks to the many mathematicians whom I have met throughout this time, especially Angela Gibney, Katrin Wendland, Hacen Zelaci for showing their interest in my work and for giving me suggestions and new points of view on the subject.

I thank all the people who are part or have been part of the Essen Seminar for Algebraic Geometry and Arithmetic in these years. Besides sharing their mathematical knowledge and curiosity, they contributed to create a nice atmosphere in the department. In particular I thank Andrea, Barinder, Carlos, Fabian, Giuseppe, Ishai, Matteo, Matti, Michael, Niels, Rodolfo, Stefan and Timo. A special thank goes to Federico, Lorenzo and Tim for the many discussions about math and life.

Many thanks to the new friends who made my staying in Essen more than pleasant, and to the old ones, for teaching me that friendship lasts no matter the distance.

Finally, I would like thank my parents, my brother and my sister for their unconditional support in these years away from home.

NOTATION AND CONVENTIONS

Unless otherwise stated, we fix the following objects.

- An algebraically closed field k of characteristic zero.
- A prime p and for simplicity of notation we denote the group $\mathbb{Z}/p\mathbb{Z}$ by Γ.
- A simple and simply connected algebraic group G over $\mathrm{Spec}(k)$.
- A group homomorphism $\rho\colon \Gamma \to \mathrm{Aut}(G)$.

Throughout this thesis we will use the following notation and convention.

- $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ the set of non negative integers.
- k**-Alg** denotes the category of unitary and commutative k-algebras.
- $\mathbf{Sch}_S$ denotes the category of schemes over a fixed scheme S. When $S = \mathrm{Spec}(R)$ we write $\mathbf{Sch}_R := \mathbf{Sch}_{\mathrm{Spec}(R)}$.
- Let H be a finite group acting on a set M. We denote by M^H the set of elements of M which are invariant under the action of Γ. The same notation is used when M is a sheaf.
- Let L be a Lie algebra over a ring R. We denote by UL its universal enveloping algebra, i.e. the associative algebra

  $$UL = \oplus_{n \in \mathbb{N}_0} L^{\otimes_R n} / I,$$

  where I is the ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$ for all $X, Y \in L$. We write $X \circ Y$ for the class of $X \otimes Y$ in UL. The same notation is used for sheaves of Lie algebras.

1 | Preliminaries on groups arising from coverings and Hurwitz stacks

In this chapter we introduce the group schemes associated to coverings as indicated in the introduction. Since we need to work with these groups in families, we will formulate the definition for families of coverings of curves. We obtain in this way the family $\mathcal{H}_{univ}$ over the universal curve X_{univ} over the Hurwitz stack parametrizing coverings of curves.

Definition 1.0.1. Let $\pi\colon X \to S$ be a possibly nodal curve over $S \in \mathbf{Sch}_k$. A *Galois covering* of X with group Γ, called also Γ-*covering*, is the data of

a) a finite, faithfully flat and generically étale map $q\colon \widetilde{X} \to X$ between curves;
b) an isomorphism $\phi\colon \Gamma \cong \mathrm{Aut}_X(\widetilde{X})$;

satisfying the following conditions:

(1) each fibre of $\widetilde{X}$ is a generically étale Γ-torsor over X;
(2) the singular locus of πq, i.e. the set of nodes of $\widetilde{X}$, is contained in the étale locus of q.

We want to attach to any Γ-covering $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S)$ and to the homomorphism $\rho\colon \Gamma \to \mathrm{Aut}(G)$ a group scheme $\mathcal{H}$ over X in the same fashion as in [**BS15**, Section 4]. We remark that Balaji and Seshadri consider ρ to map to the inner automorphisms of G only, i.e. arising from a morphism $\Gamma \to G$. Without imposing that restriction we allow also groups $\mathcal{H}$ which are non-split over the generic point of X.

First of all we consider the scheme $\widetilde{G} := \widetilde{X} \times_k G$ and let $q_*(\widetilde{G})$ be its Weil restriction along q which is defined as

$$q_*\widetilde{G}(T) := \mathrm{Hom}_{\widetilde{X}}(T \times_X \widetilde{X}, \widetilde{G})$$

for every $T \in \mathbf{Sch}_X$. It follows from [**BLR90**, Theorem 4 and Proposition 5, Section 7.6] that $q_*\widetilde{G}$ is representable by a smooth group scheme over X. The actions of Γ on G and on $\widetilde{X}$ induce the action of Γ on $q_*\widetilde{G}$ given by

$$(\gamma \cdot f)(t, \widetilde{x}) := \rho(\gamma)^{-1} f(\gamma(t, \widetilde{x})) = \rho(\gamma)^{-1} f(t, \gamma^*(\widetilde{x}))$$

for all $t \in T$ and $\widetilde{x} \in \widetilde{X}$.

We define $\mathcal{H}$ to be the subgroup of Γ-invariants of $q_*(\widetilde{G})$, i.e.

$$\mathcal{H} := (q_*\widetilde{G})^\Gamma.$$

We denote by $\mathfrak{h}$ the sheaf of Lie algebras of $\mathcal{H}$. Since $\mathcal{H}$ is smooth, as shown in [**Edi92**, Proposition 3.4], $\mathfrak{h}$ is a vector bundle on X which is moreover equipped with a structure of Lie algebra.

REMARK 1.0.2. The action of Γ on G via ρ induces an action on $\mathfrak{g} := \mathrm{Lie}(G)$. We equivalently could have defined $\mathfrak{h}$ as the Lie algebra of Γ-invariants of $q_*(\mathfrak{g} \otimes_k \mathscr{O}_{\widetilde{X}})$.

EXAMPLE 1.0.3. Let $\rho\colon \Gamma = \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathrm{SL}_n)$ be given by $\rho(\gamma)M = (M^t)^{-1}$ and $q\colon \widetilde{X} \to X$ a Γ-covering of smooth curves. The group $\mathcal{H} = (q_*(\mathrm{SL}_n \times \widetilde{X}))^\Gamma$ is the quasi split special unitary group associated to the extension $k(X) \subseteq k(\widetilde{X})$. Observe that only in the case $n = 2$ this action comes from inner automorphisms.

The stack $\mathrm{Bun}_{\mathcal{H}}$ which parametrizes $\mathcal{H}$-bundles on X can be described in terms of G-bundles over $\widetilde{X}$ which admit an action of Γ compatible with ρ. This is a corollary of Theorem A.0.7 which holds in a more general setup and for which we refer to Appendix A.

1.1. Properties of Γ-coverings

We recall in this section the properties of Γ-coverings of curves. Although the main reference is [**BR11**], we make the stronger assumption that the ramification locus of the covering map q consists only of smooth points.

1.1.1. Ramification and branch divisors. Consider a Γ-covering $(f\colon \widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S)$. We define the *ramification divisor* $\widetilde{\mathcal{R}}$ to be the effective Cartier divisor $(p-1)\widetilde{X}^\Gamma$, where $\widetilde{X}^\Gamma$ is the subscheme of $\widetilde{X}$ fixed by Γ. Equivalently, since Γ does not have proper subgroups, $\widetilde{X}^\Gamma$ is the complement of the étale locus of q, which is either empty or an effective Cartier divisor of $\widetilde{X}$. The *reduced branch divisor* $\mathcal{R}$ is the effective divisor given by the image of $\widetilde{X}^\Gamma$ in X. It is the reduced divisor of the proper pushforward $q_*\widetilde{\mathcal{R}}$.

REMARK 1.1.1. If the map q is not étale both divisors $\widetilde{\mathcal{R}}$ and $\mathcal{R}$ are finite and étale over S. This is a proved in [**BR11**, Proposition 3.1.1] for the smooth case only and in [**BR11**, Proposition 4.1.8] for the general situation.

The ramification divisors are naturally related to tangent bundles of X and $\widetilde{X}$. Let $\mathcal{T}_{\widetilde{X}/S}$ be the tangent bundle of $\widetilde{X}$ relative to S, so that its sections are $f^{-1}\mathscr{O}_S$-linear derivations of $\mathscr{O}_{\widetilde{X}}$. Consider its pushforward to X along q and notice that the action of Γ on $q_*\mathscr{O}_{\widetilde{X}}$ induces an action on $q_*\mathcal{T}_{\widetilde{X}/S}$ by sending a derivation D to $\gamma D \gamma^{-1}$. The following lemma, which describes the Γ-invariants of $q_*\mathcal{T}_{\widetilde{X}/S}$, follows from [**BR11**, Proposition 4.1.11] and we report the proof for completeness.

LEMMA 1.1.2. *The sheaf $(q_*\mathcal{T}_{\widetilde{X}/S})^\Gamma$ over X is isomorphic to $\mathcal{T}_{X/S}(-\mathcal{R})$.*

PROOF. Let first observe that the natural map $d(q)\colon \mathcal{T}_{\widetilde{X}/S} \to q^*\mathcal{T}_{X/S}$ identifies $\mathcal{T}_{\widetilde{X}/S}$ with $q^*\mathcal{T}_{X/S}(-\widetilde{\mathcal{R}})$. This is clear outside $\widetilde{\mathcal{R}}$. On the formal neighbourhood $R[[t]]$ of a point $\tilde{x} \in \widetilde{\mathcal{R}}$ the map q is given by sending t to $t\zeta$ for a primitive p-th root of unity ζ. It follows that $d(q)\colon R[[t]]d/dt \to R[[t]]d/d(t^p)$ sends the generator d/dt to $pt^{p-1}d/d(t^p)$, concluding the argument.

We now pushforward the isomorphism $d(q)\colon \mathcal{T}_{\widetilde{X}/S} \to q^*\mathcal{T}_{X/S}(-\widetilde{\mathcal{R}})$ along q and take Γ-invariants obtaining the isomorphism

$$(q_*d(q))^\Gamma\colon (q_*\mathcal{T}_{\widetilde{X}/S})^\Gamma \to (q_*(q^*\mathcal{T}_{X/S}(-\widetilde{\mathcal{R}})))^\Gamma.$$

Since étale morphisms induce isomorphism on the tangent bundles, this map is an isomorphism outside the branch divisor $\mathcal{R}$. Since by assumption the branch points are smooth, we are left to check that the target of the map equals $\mathcal{T}_{X/S}(-\mathcal{R})$ under the condition that $X \to S$ is smooth. From the smoothness we deduce that $\mathcal{T}_{X/S}$ is locally free, so using the projection formula we obtain that $(q_*(q^*\mathcal{T}_{X/S}(-\widetilde{\mathcal{R}})))^\Gamma \cong \mathcal{T}_{X/S} \otimes q_*(\mathscr{O}(-\widetilde{\mathcal{R}}))^\Gamma$. We are left to prove that $q_*(\mathscr{O}(-\widetilde{\mathcal{R}}))^\Gamma$ is isomorphic to $\mathscr{O}(-\mathcal{R})$. Observe, for this purpose, that the sheaf $q_*\mathscr{O}_{\widetilde{\mathcal{R}}}$ is supported only at $\mathcal{R}$, so we only need to compute that its submodule of Γ-invariants is one dimensional. Let $x \in \mathcal{R}$ and note that the formal neighbourhood of $q_*\mathscr{O}_{\widetilde{\mathcal{R}}}$ at x is isomorphic to

$$R[[t]]/t^{p-1}R[[t]] \cong R \oplus tR \oplus \cdots \oplus t^{p-2}R$$

on which any element of Γ acts multiplying t by a p-th root of unity. It follows that the only invariant submodule is R, hence $(q_*\mathscr{O}_{\widetilde{\mathcal{R}}})^\Gamma \cong \mathscr{O}_{\mathcal{R}}$. □

Hurwitz data. The Hurwitz data provide a description of the action of Γ at the ramification points. Before considering families of curves we take $\widetilde{X} \to X$, a Γ-covering of curves over k. Let $\widetilde{x} \in \widetilde{X}(k)^\Gamma$ be a ramification point and up to the choice of a local parameter t the formal disc around $\widetilde{x}$ is isomorphic to $\mathrm{Spec}(k[[t]])$. Since Γ fixes $\widetilde{x}$, one of its generators acts on $k[[t]]$ by sending t to ζt for a primitive p-th root unity ζ. It follows that the action of Γ on $\mathrm{Spec}(k[[t]])$ is uniquely determined by non trivial characters $\chi_{\widetilde{x}}\colon \Gamma \to k^*$. Let $\mathrm{Char}(\Gamma)^*$ be the set of all non trivial characters of Γ and set $R_+(\Gamma) := \oplus_{\chi \in \mathrm{Char}(\Gamma)^*}\mathbb{Z}\chi$. The *ramification data* or *Hurwitz data* of a Γ-covering $\widetilde{X} \to X$ is the element

$$\xi := \sum_{\widetilde{x} \in \widetilde{X}^\Gamma} \chi_{\widetilde{x}} \in R_+(\Gamma).$$

The *degree* of $\xi = \sum b_i\chi_i$ is $\deg(\xi) := \sum b_i$. Note that $\deg(\xi) = \deg(\widetilde{X}^\Gamma) = \deg(\mathcal{R})$.

REMARK 1.1.3. In the case $\Gamma = \mathbb{Z}/2\mathbb{Z}$, the Hurwitz data encode only the number of points which are fixed.

DEFINITION 1.1.4. Let $\widetilde{X} \to X \to S$ be a Γ-covering with S connected. We say that it has Hurwitz data $\xi \in R_+(\Gamma)$ if ξ is the Hurwitz data of one, hence all ([**BR11**, Lemme 3.1.3]), of its fibres.

We fix for the next two lemmas, a generator γ of Γ and $\zeta \in k$ a primitive p-th root of 1. This identifies the set of characters of Γ with $\{0, \ldots, p-1\}$.

LEMMA 1.1.5. *Denote by $\mathscr{E}_i$ the $\mathscr{O}_X$-submodule of $q_*\mathscr{O}_{\widetilde{X}}$ where γ acts by multiplication by ζ^i. Then*

$$q_*\mathscr{O}_{\widetilde{X}} = \bigoplus_{i=0}^{p-1}\mathscr{E}_i \quad \textit{and} \quad \mathscr{E}_i \otimes \mathscr{E}_{p-i} \cong \mathscr{O}(-\mathcal{R}).$$

PROOF. The action of Γ on $q_*\mathscr{O}_{\widetilde{X}}$ provides the decomposition with $\mathscr{E}_0 \cong \mathscr{O}_X$. For the second statement, the tensor product $\mathscr{E}_i \otimes \mathscr{E}_{p-i}$ is a submodule of $\mathscr{E}_0 \cong \mathscr{O}_X$ as γ acts there as the identity. Outside the branch divisor $\mathcal{R}$ this is an isomorphism so we only need to check what is the image along $\mathcal{R}$. Let $x \in \mathcal{R}$ and call $\widetilde{x} \in \widetilde{\mathcal{R}}$ the point above x so that $\widehat{\mathscr{O}}_{\widetilde{X},\widetilde{x}} \cong R[[t]]$, with $\gamma(t) = \zeta^n t$ with $n \in \{1, \ldots, p-1\}$. If follows that $(\widehat{\mathscr{E}_i})_x \cong t^{i/n} R[[t^p]]$ and $(\widehat{\mathscr{E}}_{p-i})_x \cong t^{(p-i)/n} R[[t^p]]$, where $i/n \in \{1, \ldots, p-1\}$. It follows that $(\widehat{\mathscr{E}_i})_x \otimes (\widehat{\mathscr{E}}_{p-i})_x \cong t^p R[[t^p]]$ which is isomorphic to the completion of $\mathscr{O}(-\mathcal{R})$ at the point x. □

LEMMA 1.1.6. *Denote by* $\mathfrak{g}^{\zeta^i}$ *the submodule of* $\mathfrak{g}$ *where* γ *acts by multiplication by* ζ^i. *The sheaf* $\mathfrak{h}$ *decomposes as*

$$\mathfrak{h} = \bigoplus_{i=0}^{p-1} \mathfrak{g}^{\zeta^{-i}} \otimes_k \mathscr{E}_i.$$

PROOF. As the action of γ on $\mathfrak{g}$ is diagonalizable with eigenvalues belonging to $\{1, \zeta, \cdots, \zeta^{p-1}\}$, we can decompose $\mathfrak{g}$ as $\oplus \mathfrak{g}^{\zeta^{-i}}$. As $\mathfrak{h}$ is the Lie algebra of Γ-invariants of $q_*(\mathscr{O}_{\widetilde{X}} \otimes_k \mathfrak{g}) = q_*\mathscr{O}_{\widetilde{X}} \otimes_k \mathfrak{g}$, we can combine this with the description of $q_*\mathscr{O}_{\widetilde{X}}$ provided by Lemma 1.1.5 to obtain the wanted decomposition of $\mathfrak{h}$. □

1.2. Hurwitz stacks

We define in this section the stack parametrizing Γ-coverings with fixed Hurwitz data $\xi \in R_+(\Gamma)$. Let g be a non negative integer.

Let $f : \widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S$ be a Γ-covering of curves and let $\sigma : S \to X$ be a section of π with $\sigma(S)$ disjoint from the nodes of X and from the branch locus $\mathcal{R}$ of q. The we say that the covering is *stably marked* by σ if $(X, \sigma \cup \mathcal{R})$ is a stably marked curve [**BR11**, Définition 4.3.4. and Proposition 5.1.3]. The same notion holds if we fix more sections. Let $n \in \mathbb{N}$ and fix n pairwise disjoint sections $\{\sigma_i\}_i = 1^n$ of π which are disjoint from the branch locus $\mathcal{R}$ of q. We say that the covering is *stably marked* by $\{\sigma_i\}$ if $(X, \sigma_1 \cup \cdots \cup \sigma_n \cup \mathcal{R})$ is a stably marked curve.

DEFINITION 1.2.1. We define the *Hurwitz stack* $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$ as

$$\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}(S) = \left\langle f : \widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S,\ \{\sigma_j : S \to X\}_{j=1}^n \text{ such that i and ii hold} \right\rangle$$

i. the map $q : \widetilde{X} \to X$ is a Γ-covering of curves with ramification data ξ;
ii. $(X, \{\sigma_j\})$ is an n-marked curve of genus g with $\sigma_j(S)$ disjoint from the branch divisor $\mathcal{R}$ for all j and such that the covering is stably marked by $\{\sigma_j\}$.

When $n = 0$ we omit the subscript and use the notation $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_g$. We denote by $\mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n}$ the open substack of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$ parametrizing Γ-coverings of smooth curves.

WARNING. Although the notation seems to suggest that $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$ is a compactification of $\mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n}$, this is not true because we do not allow ramification and singular points to collide.

REMARK 1.2.2. We want to remark that the role of the ramification data, besides fixing the genus of the curve $\widetilde{X}$ thanks to the Riemann-Hurwitz formula, is to guarantee the connectedness of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$ and $\mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n}$ [**BR11**, Proposition 2.3.9].

In the previous section we explained how to associate to each Γ-covering $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_g(S)$, a group $\mathcal{H}$ (resp. a sheaf of Lie algebras $\mathfrak{h}$) over X. This defines a group $\mathcal{H}_{univ}$ (resp. a sheaf of Lie algebras $\mathfrak{h}_{univ}$) on X_{univ}, where we denote by $\widetilde{X}_{univ} \to X_{univ}$ the universal covering on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_g$. The same construction works on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$, defining $\mathcal{H}_{univ}$ and $\mathfrak{h}_{univ}$ on the universal curve X_{univ} of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$.

REMARK 1.2.3. The complement $\Delta_{univ} := \overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n} \setminus \mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n}$ is a normal crossing divisor. First of all observe that $\Delta_{\overline{\mathcal{M}}_{g,d}} := \overline{\mathcal{M}}_{g,d} \setminus \mathcal{M}_{g,d}$ is a normal crossing divisor: in fact given a nodal curve $X \to \mathrm{Spec}(k)$ with a reduced divisor D of degree d, there exists a versal deformation $\mathcal{X} \to S$ where the locus $\Delta \subset S$ consisting of singular curves is a normal crossing divisor of S [**ACG11**]. We now want to compare the deformation theory of a Γ-covering $(\widetilde{X} \to X, \{\sigma_i\})$ to the one of $(X, \{\sigma_i\})$. Following [**BR11**, Théorème 5.1.5] we see that the natural map $\delta\colon \mathrm{Def}(\widetilde{X} \to X, \{\sigma_i\}) \to \mathrm{Def}(X, \{\sigma_i\} \cup \mathcal{R})$ fails to be an isomorphism only when the intersection between $\mathcal{R}$ and X^{sing} is not empty, but since by assumption we impose that $\mathcal{R} \cap X^{sing} = \emptyset$, in our context this map is always an isomorphism. This then allow to obtain, from the versal deformation $\mathcal{X} \to S$ of $(X, \{\sigma_i\} \cup \mathcal{R})$, the versal deformation $(\widetilde{\mathcal{X}} \to \mathcal{X}, \{\varsigma_i\})$ of $\widetilde{X} \to X$, and hence deduce from the theory of $\overline{\mathcal{M}}_{g,n+\deg(\mathcal{R})}$ that Δ_{univ} is a normal crossing divisor.

The following statement, which is given by [**BR11**, Proposition 2.3.9. and Théorème 6.3.1], describes the properties of the above stacks.

PROPOSITION 1.2.4. *The stacks $\overline{\mathcal{H}ur}(\Gamma, \xi)_{g,n}$ and $\mathcal{H}ur(\Gamma, \xi)_{g,n}$ are smooth Deligne-Mumford stacks which are connected and of finite type over Spec(k).*

Instead of marking the curve X, we can mark the curve $\widetilde{X}$, so that we define.

DEFINITION 1.2.5. For each $S \in \mathbf{Sch}_k$ we set

$$\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)^n_g(S) = \left\langle f\colon \widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S,\ \{\tau_j\colon S \to \widetilde{X}\}_{j=1}^n \text{ such that i and ii hold} \right\rangle$$

i. the map $q\colon \widetilde{X} \to X$ is a Γ-covering of curves with ramification data ξ;
ii. $(\widetilde{X}, \{\tau_j\})$ is an n-marked curve with $q\tau_j(S)$ pairwise disjoint, $\tau_j(S)$ disjoint from $\widetilde{X}^\Gamma$ for all j and such that the covering q is stably marked by $\{q\tau_j\}$.

It follows, from the fact that the image of τ lies in the étale locus of q, that the map

$$\mathbf{Forg}^n_n\colon \mathcal{H}\mathrm{ur}(\Gamma, \xi)^n_g \to \mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n}, \qquad (\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S,\ \{\tau_j\}) \mapsto (\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S,\ \{q\tau_j\})$$

is an étale and surjective morphism of stacks. For any $n \in \mathbb{N}_0$ we also have the forgetful map $\mathbf{Forg}_n\colon \mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n} \to \mathcal{H}\mathrm{ur}(\Gamma, \xi)_g$ and in more generality, for all $n, m \in \mathbb{N}_0$ we have the morphism $\mathbf{Forg}_{n+m,n}\colon \mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n+m} \to \mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n}$ which forgets the last m sections.

Let $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \tau) \in \mathcal{H}\mathrm{ur}(\Gamma, \xi)^1_g(S)$ and write $\sigma := q\tau$. Fixing τ allows us to canonically identify $\mathcal{H}|_{\sigma(S)}$ with $G \times_k S$ as explained in the proof of the following statement.

PROPOSITION 1.2.6. *The section τ induces an isomorphism between $\sigma^*\mathcal{H}$ and $G \times_k S$.*

PROOF. Construct the cartesian diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{\sigma}} & \widetilde{X} \\ \downarrow{\scriptstyle q_S} & & \downarrow{\scriptstyle q} \\ S & \xrightarrow{\sigma} & X \end{array}$$

and since by assumption the image of σ lies in the étale locus of q the left vertical arrow q_S is étale and it has a section given by τ. This implies that $\widetilde{S}$ is isomorphic to $\coprod_{\gamma_i \in \Gamma} S$. Observe that $q_{S*}\widetilde{\sigma}^*(\widetilde{G}) \cong \sigma^* q_*(\widetilde{G})$ and that taking Γ-invariants commutes with restriction along σ. It follows that

$$\sigma^*\mathcal{H} = \left(\sigma^* q_*(\widetilde{G})\right)^\Gamma = \left(q_{S*}\widetilde{\sigma}^*(\widetilde{G})\right)^\Gamma = \left(q_{S*}\left(\coprod_{\gamma_i \in \Gamma} S \times G\right)\right)^\Gamma = \left(\prod_{\gamma_i \in \Gamma} S \times G\right)^\Gamma$$

where $\gamma_j \in \Gamma$ acts on $\prod_{\gamma_i \in \Gamma} S \times G$ by sending $(s_i, g_i)_{\gamma_i}$ to $(s_i, \gamma_j(g_i))_{\gamma_j \gamma_i}$. It follows that the invariant elements are of the form $(s, \gamma_i(g))_{\gamma_i}$ for any $s \in S$ and $g \in G$, so that the projection on any component of $S \times G$ realized an isomorphism between $\sigma^*\mathcal{H}$ and $G \times S$. The map τ selects a preferred component, giving in this way a canonical isomorphism. $\square$

2 | The sheaf of conformal blocks

In this chapter we define the sheaf of conformal blocks $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$ attached to a representation $\mathcal{V}$ of $\sigma^*\mathfrak{h}_{univ}$. To do this, we will define it for any family $(f\colon \widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \sigma)$ over an affine base $S = \mathrm{Spec}(R)$. We will assume moreover that $X \setminus \sigma(S) \to S \to S$ is affine. We will see in Remark 4.1.5 how to drop this assumption.

For the classical definition of the sheaf of conformal blocks attached to a represendetation of $\mathfrak{g}$ one can refer to [**TUY89**] or to [**Loo13**]. We will use the latter as main reference.

Warning. The word *conformal block* has been used in literature to denote either a certain vector bundle or its dual. We use here the word sheaf of conformal blocks to denote what in [**TUY89**] is called the dual of the sheaf of conformal blocks. In [**Loo13**], the author calls this sheaf the *sheaf of covacua*.

Let $X^* := X \setminus \sigma(S)$ and denote by $\mathcal{A}$ the pushforward to S of $\mathscr{O}_{X^*}$, i.e.

$$\mathcal{A} := \pi_* j_{A*} \mathscr{O}_{X^*}$$

where j_A denotes the open immersion $X^* \to X$. Since the map π restricted to X^* is affine we have that $X^* = \mathbf{Spec}(\mathcal{A})$ and that $\mathcal{A} = \pi_* \varinjlim_n \mathcal{I}_\sigma^{-n} = \varinjlim_n \pi_* \mathcal{I}_\sigma^{-n}$ where $\mathcal{I}_\sigma = \mathscr{O}_X(-\sigma(S))$ is the ideal defining $\sigma(S)$.

We denote by $\widehat{\mathcal{O}}$ the formal completion of $\mathscr{O}_X$ along $\sigma(S)$: by definition σ gives a short exact sequence

$$0 \to \mathcal{I}_\sigma \to \mathscr{O}_X \to \mathscr{O}_{\sigma(S)} \to 0$$

of $\mathscr{O}_X$-modules. We define

$$\widehat{\mathcal{O}} := \pi_* \varprojlim_n \mathscr{O}_X \,/ (\mathcal{I}_\sigma)^n = \varprojlim_n \pi_* \mathscr{O}_X \,/ (\mathcal{I}_\sigma)^n$$

which is naturally a sheaf of $\mathscr{O}_S$-modules. We denote by $\mathcal{L}$ the $\mathscr{O}_S$-module

$$\mathcal{L} := \varinjlim_{N \in \mathbb{N}_0} \pi_* \varprojlim_{n \in \mathbb{N}} \mathcal{I}_\sigma^{-N} / \mathcal{I}_\sigma^n$$

which is equipped with a natural filtration $F^N\mathcal{L} = \pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{I}_\sigma^N / \mathcal{I}_\sigma^{N+n}$ for $N \geq 0$ and $F^N\mathcal{L} = \pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{I}_\sigma^N / \mathcal{I}_\sigma^n$ for $N \leq -1$ taking into account the order of the poles or zeros along $\sigma(S)$.

REMARK 2.0.1. Recall that when $R = k$, the choice of a local parameter t, i.e. of a generator of $\mathcal{I}_\sigma$, gives an isomorphism $\widehat{\mathcal{O}} \cong k[[t]]$ and hence $\mathcal{L} \cong k((t))$ and so $F^n\mathcal{L} \cong t^n k[[t]]$. In the general case, since $\mathcal{I}_\sigma$ is locally principal, for every $s \in \sigma(S)$ we can find an open covering U of X containing s and such that $\mathcal{I}_\sigma|_U$ is principal. Let denote by S' the open of S given by $\sigma^{-1}(U)$ and by U' the open $U \cap \pi^{-1}S'$. Then $\mathcal{I}_\sigma|_{U'}$ is principal and $\varprojlim_n \mathscr{O}_{U'}/(\mathcal{I}_\sigma|_{U'})^n$ is isomorphic to $\mathscr{O}_{S'}[[t]]$, where t is a generator of $\mathcal{I}_\sigma|_{U'}$. This moreover implies that the completion of $\widehat{\mathcal{O}}$ at a point $s \in S$ is isomorphic to $\widehat{\mathscr{O}}_{S,s}[[t]]$, where $\widehat{\mathscr{O}}_{S,s}$ denotes the completion of $\mathscr{O}_S$ at s.

Denote by $\mathfrak{h}_\mathcal{A}$ the restriction of $\mathfrak{h}$ to the open curve X^* and by $\mathfrak{h}_\mathcal{L}$ the "restriction of $\mathfrak{h}$ to the punctured formal neighbourhood around $\sigma(S)$", and consider both sheaves as $\mathscr{O}_S$-modules naturally equipped with a Lie bracket. In other words we set

$$\mathfrak{h}_\mathcal{A} := \pi_* j_{A*} {j_A}^*(\mathfrak{h}) = \pi_* \left(\varinjlim_{N\in\mathbb{N}_0} \mathcal{I}_\sigma^{-N} \otimes_{\mathscr{O}_X} \mathfrak{h} \right) = \varinjlim_{N\in\mathbb{N}_0} \pi_*(\mathcal{I}_\sigma^{-N} \otimes_{\mathscr{O}_X} \mathfrak{h})$$

$$\mathfrak{h}_\mathcal{L} := \varinjlim_{N\in\mathbb{N}_0} \pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{I}_\sigma^{-N}/\mathcal{I}_\sigma^n \otimes_{\mathscr{O}_X} \mathfrak{h}.$$

The following observations follow from the definitions.

(1) The injective morphism $\mathcal{I}_\sigma^{-N} \to \varprojlim_n \mathcal{I}_\sigma^{-N}/\mathcal{I}_\sigma^n$ induces the inclusion $\mathfrak{h}_\mathcal{A} \to \mathfrak{h}_\mathcal{L}$.

(2) The filtration on $\mathcal{L}$ defines the filtration $F^*\mathfrak{h}_\mathcal{L}$ as

$$F^N(\mathfrak{h}_\mathcal{L}) = \pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{I}_\sigma^N/\mathcal{I}_\sigma^{N+n} \otimes_{\mathscr{O}_X} \mathfrak{h} \quad \text{and} \quad F^{-N}(\mathfrak{h}_\mathcal{L}) = \pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{I}_\sigma^{-N}/\mathcal{I}_\sigma^n \otimes_{\mathscr{O}_X}$$

for all $N \in \mathbb{N}_0$ and we denote $F^0(\mathfrak{h}_\mathcal{L})$ by $\mathfrak{h}_{\widehat{\mathcal{O}}}$.

(3) We could have equivalently defined $\mathfrak{h}_\mathcal{A}$ as the Lie subalgebra of Γ-invariants of $f_*(\mathfrak{g} \otimes_k j_{\widetilde{A}*}\mathscr{O}_{\widetilde{X}*})$ where $j_{\widetilde{A}}$ denotes the open immersion of $\widetilde{X}^* := \widetilde{X} \times_X X^* \to \widetilde{X}$. This follows from the equalities

$${j_A}^*\mathfrak{h} = {j_A}^*(q_*(\mathfrak{g} \otimes_k \mathscr{O}_{\widetilde{X}}))^\Gamma = ({j_A}^* q_*(\mathfrak{g} \otimes_k \mathscr{O}_{\widetilde{X}}))^\Gamma = q_*({j_{\widetilde{A}}}^*(\mathfrak{g} \otimes_k \mathscr{O}_{\widetilde{X}}^*))^\Gamma.$$

Similarly $\mathfrak{h}_\mathcal{L}$ is the Lie subalgebra of Γ-invariants of $\mathfrak{g} \otimes_k \widehat{\mathcal{L}}$, where

$$\widehat{\mathcal{L}} := \varinjlim_N f_* \varprojlim_n (\mathfrak{g} \otimes_k q^*(\mathcal{I}_\sigma^{-N})/q^*(\mathcal{I}_\sigma^n)).$$

REMARK 2.0.2. Since $\sigma(S)$ has trivial intersection with $\mathcal{R}$, we can find an étale cover of S such that $q^{-1}(\sigma(S)) = \coprod_\Gamma S$ or in other terms such that the pull back of $\mathcal{I}_\sigma$ to the cover totally splits, i.e. $q^*\mathcal{I}_\sigma = \prod_{\gamma_i\in\Gamma} \mathcal{I}_{\sigma,i}$. This implies that

$$\mathfrak{h}_\mathcal{L} \cong \left(\mathfrak{g} \otimes_k \bigoplus_{\gamma_i\in\Gamma} (\varinjlim_N f_* \varprojlim_n \mathcal{I}_{\sigma,i}^{-N}/\mathcal{I}_{\sigma,i}^n) \right)^\Gamma$$

which leads to $\mathfrak{h}_\mathcal{L} \cong (\mathfrak{g} \otimes_k (\oplus_{\gamma_i\in\Gamma}\mathcal{L}))^\Gamma$ where the action is given by

$$\gamma_j * ((X_i f_i)_{\gamma_i}) = (\gamma_j(X_i) f_i)_{\gamma_j\gamma_i} \quad \text{for all } X_i \in \mathfrak{g} \text{ and } f_i \in \mathcal{L}.$$

It follows that the invariant elements are combination of elements of the type $(\gamma_i(X)f)_{\gamma_i}$ for $X \in \mathfrak{g}$ and $f \in \mathcal{L}$. For every $i \in \{0, \ldots, p-1\}$, the projection on the i-th component

$$\mathrm{pr}_i \colon \mathfrak{h}_\mathcal{L} \to \mathfrak{g}\mathcal{L} := \mathfrak{g} \otimes_k \mathcal{L}, \quad (\gamma_j(X)f)_{\gamma_j} \mapsto \gamma_i(X)f$$

defines a non canonical isomorphism of sheaves of Lie algebras of $\mathfrak{h}_\mathcal{L}$ with $\mathfrak{g}\mathcal{L}$. The inverse is the map that sends the element Xf of $\mathfrak{g}\mathcal{L}$ to the p-tuple $(\gamma_j(\gamma_i^{-1}(X))f)_{\gamma_j}$.

2.1. The central extension of $\mathfrak{h}_{\mathcal{L}}$

Once we have defined $\mathfrak{h}_{\mathcal{L}}$ and $\mathfrak{h}_{\mathcal{A}}$, in order to define $\mathcal{H}_{\ell}(\mathcal{V})_{X_{univ}}$, we need to extend $\mathfrak{h}_{\mathcal{L}}$ centrally. Following [**Kac90**, Chapter 7], [**TUY89**] and [**Loo13**] we construct this central extension using a normalized Killing form and the residue pairing.

Normalized Killing form. We fix once and for all a maximal torus T of G and a Borel subgroup B of G containing T, or equivalently we fix the root system $R(G,T) = R(\mathfrak{g},\mathfrak{t}) \subseteq \mathfrak{t}^{\vee} := \mathrm{Hom}(\mathfrak{t},k)$ of G and a basis Δ of positive simple roots, where $\mathfrak{t} = \mathrm{Lie}(T)$. Given a root α we denote by $H_{\alpha} \in \mathfrak{t}$ the associated coroot.

Denote by $(\,|\,)\colon \mathfrak{g} \otimes \mathfrak{g} \to k$ the unique multiple of the Killing form such that $(H_{\theta}|H_{\theta}) = 2$ where θ is the highest root of $\mathfrak{g}$. As $\mathfrak{g}$ is simple, this form gives an isomorphism $(\,|\,)$ between $\mathfrak{g}$ and $\mathfrak{g}^{\vee} := \mathrm{Hom}(\mathfrak{g},k)$. Pulling back this form to $\widetilde{X}$ we obtain $(\,\tilde{|}\,)\colon \widetilde{\mathfrak{g}} \otimes \widetilde{\mathfrak{g}} \to \mathscr{O}_{\widetilde{X}}$, where $\widetilde{\mathfrak{g}} := \mathscr{O}_{\widetilde{X}} \otimes_k \mathfrak{g}$. We push forward $(\,\tilde{|}\,)$ along q obtaining

$$q_*(\,\tilde{|}\,)\colon q_*(\widetilde{\mathfrak{g}}) \otimes q_*(\widetilde{\mathfrak{g}}) \to q_*(\mathscr{O}_{\widetilde{X}})$$

which is Γ-equivariant as the Killing form is invariant under automorphisms of $\mathfrak{g}$. Taking Γ-invariants we obtain the pairing

$$(\,|\,)_{\mathfrak{h}}\colon \mathfrak{h} \otimes_{\mathscr{O}_X} \mathfrak{h} \to \mathscr{O}_X$$

which however is not perfect because of ramification. Combining this with the multiplication morphism $\mathcal{I}_{\sigma}^{-N}/\mathcal{I}_{\sigma}^{N+n} \times \mathcal{I}_{\sigma}^{-N}/\mathcal{I}_{\sigma}^{N+n} \to \mathcal{I}_{\sigma}^{-2N}/\mathcal{I}_{\sigma}^{n}$ and taking the limit on n and N we obtain the perfect pairing $(\,|\,)_{\mathfrak{h}_{\mathcal{L}}}\colon \mathfrak{h}_{\mathcal{L}} \otimes_{\mathcal{L}} \mathfrak{h}_{\mathcal{L}} \to \mathcal{L}$.

Residue pairing. We introduce the sheaf $\theta_{\mathcal{L}/S}$ of continuous derivations of $\mathcal{L}$ which are $\mathscr{O}_S$ linear. Denote its $\mathcal{L}$-dual by $\omega_{\mathcal{L}/S}$: this is the sheaf of continuous differentials of $\mathcal{L}$ relative to $\mathscr{O}_S$.

REMARK 2.1.1. When $\widehat{\mathcal{O}} \cong R[[t]]$ we have that $\theta_{\mathcal{L}/S}$ is isomorphic to $R((t))d/dt$ and $\omega_{\mathcal{L}/S}$ to $R((t))dt$.

The residue map $\mathrm{Res}\colon \omega_{\mathcal{L}/S} \to \mathscr{O}_S$ is computed locally as $\mathrm{Res}(\sum_{i \geq N} \alpha_i t^i dt) = \alpha_{-1}$. Composing this with the canonical morphism ${\mathfrak{h}_{\mathcal{L}}}^{\vee} \times \mathfrak{h}_{\mathcal{L}} \to \mathcal{L}$ we obtain the perfect pairing

$$\mathrm{Res}_{\mathfrak{h}}\colon \omega_{\mathcal{L}/S} \otimes_{\mathcal{L}} {\mathfrak{h}_{\mathcal{L}}}^{\vee} \times \mathfrak{h}_{\mathcal{L}} \to \mathscr{O}_S.$$

The differential of a section. Let $d\colon \mathscr{O}_{\widetilde{X}} \to \Omega_{\widetilde{X}/S}$ be the universal derivation, which induces the morphism $d\colon \mathfrak{g} \otimes_k \mathscr{O}_{\widetilde{X}} \to \mathfrak{g} \otimes_k \Omega_{\widetilde{X}/S}$ by tensoring it with $\mathfrak{g}$. Let $U = X \setminus \{\mathcal{R} \cup X^{\mathrm{sing}}\}$ be the open subscheme of X which is smooth over S and which does not intersect the branch divisor $\mathcal{R}$ of q and call $\widetilde{U} = U \times_X \widetilde{X}$. Once we restrict d to $\widetilde{U}$ and we push it forward along q we obtain the map

$$d\colon q_*(\mathfrak{g} \otimes_k \mathscr{O}_{\widetilde{U}}) \to q_*(\mathfrak{g} \otimes_k \mathscr{O}_{\widetilde{U}}) \otimes_{\mathscr{O}_U} \Omega_{U/S}$$

by using the projection formula. Taking Γ-invariants one obtains $d\colon \mathfrak{h}|_U \to \mathfrak{h}|_U \otimes_U \Omega_{U/S}$ and since $\sigma(S) \subset U$, this induces the map $d\colon \mathfrak{h}_{\mathcal{L}} \to \omega_{\mathcal{L}/S} \otimes_{\mathcal{L}} \mathfrak{h}_{\mathcal{L}}$. We can furthermore compose this map with the morphism $\mathfrak{h}_{\mathcal{L}} \to {\mathfrak{h}_{\mathcal{L}}}^{\vee}$ given by the normalized Killing form $(\,|\,)_{\mathfrak{h}_{\mathcal{L}}}$, obtaining

$$d_{\mathfrak{h}_{\mathcal{L}}}\colon \mathfrak{h}_{\mathcal{L}} \to \omega_{\mathcal{L}/S} \otimes_{\mathcal{L}} {\mathfrak{h}_{\mathcal{L}}}^{\vee}.$$

REMARK 2.1.2. We could have equivalently defined $d_{\mathfrak{h}_{\mathcal{L}}}$ by using the local isomorphism between $\mathfrak{h}_{\mathcal{L}}$ and $\mathfrak{g}\mathcal{L}$. Using this approach, we can describe $d_{\mathfrak{h}_{\mathcal{L}}}$ as the map which associates to the element $Xf \in \mathfrak{g}\mathcal{L}$, the element $df \otimes (X|-)$ belonging to $\omega_{\mathcal{L}/S} \otimes_{\mathcal{L}} (\mathfrak{g}\mathcal{L})^{\vee}$.

REMARK 2.1.3. Given $X, Y \in \mathfrak{h}_{\mathcal{L}}$, we simply write $(dX|Y)$ for $d_{\mathfrak{h}_{\mathcal{L}}}(X)(Y) \in \omega_{L/S}$. Note that the following equality holds $d_{\mathcal{L}}(X|Y)_{\mathfrak{h}_{\mathcal{L}}} = (dX|Y) + (X|dY)$, where $d_{\mathcal{L}} \colon \mathcal{L} \to \omega_{\mathcal{L}/S}$ is the universal derivation.

The central extension of $\mathfrak{h}_{\mathcal{L}}$. We have introduced all the ingredients we needed to be able to define the central extension $0 \to c\mathscr{O}_S \to \widehat{\mathfrak{h}}_{\mathcal{L}} \to \mathfrak{h}_{\mathcal{L}} \to 0$ of $\mathfrak{h}_{\mathcal{L}}$ where c is a formal variable.

DEFINITION 2.1.4. We define the sheaf of Lie algebras $\widehat{\mathfrak{h}}_{\mathcal{L}}$ to be $\mathfrak{h}_{\mathcal{L}} \oplus c\mathscr{O}_S$ as $\mathscr{O}_S$-module, with $c\mathscr{O}_S$ being in the centre of $\widehat{\mathfrak{h}}_{\mathcal{L}}$ and with Lie bracket defined as

$$[X,Y] := [X,Y]_{\mathfrak{h}_{\mathcal{L}}} + c\mathrm{Res}_{\mathfrak{h}}\left(d_{\mathfrak{h}_{\mathcal{L}}}(X) \otimes Y\right) = [X,Y]_{\mathfrak{h}_{\mathcal{L}}} + c\mathrm{Res}\,(dX|Y)$$

for all $X, Y \in \mathfrak{h}_{\mathcal{L}}$.

The Lie algebra $\widehat{\mathfrak{h}}_{\mathcal{L}}$ comes equipped with the filtration $F^N\widehat{\mathfrak{h}}_{\mathcal{L}} = F^N\mathfrak{h}_{\mathcal{L}}$ for all $N \in \mathbb{N}$ and $F^N\widehat{\mathfrak{h}}_{\mathcal{L}} = FNi\mathfrak{h}_{\mathcal{L}} \oplus c\mathscr{O}_S$ for $N \in \mathbb{Z}_{\leq 0}$. As $\mathfrak{h}_{\mathcal{A}} \subset \mathfrak{h}_{\mathcal{L}}$, one might wonder which is the Lie algebra structure induced on $\widehat{\mathfrak{h}}_{\mathcal{A}}$. The following two lemmas tell us that $\widehat{\mathfrak{h}}_{\mathcal{A}}$ is a split extension of $\mathfrak{h}_{\mathcal{A}}$, hence $\mathfrak{h}_{\mathcal{A}}$ is a Lie subalgebra of $\widehat{\mathfrak{h}}_{\mathcal{L}}$.

LEMMA 2.1.5. *The image of $\mathfrak{h}_{\mathcal{A}}$ via $d_{\mathfrak{h}_{\mathcal{L}}}$ is $\omega_{\mathcal{A}} \otimes {\mathfrak{h}_{\mathcal{A}}}^{\vee}$.*

PROOF. We can restrict to the case of family of smooth curves, as on the singular points the result follows from [**Loo13**, Lemma 5.1] by identifying $\mathfrak{h}$ with $\mathfrak{g}$. Recall from Lemma 1.1.6 that $\mathfrak{h} = \bigoplus_{i=0}^{p-1} \mathfrak{g}^{\zeta^{-i}} \otimes_k \mathscr{E}_i$, and note that the image of $\mathscr{E}_i$ under d is $\mathscr{E}_i(\mathcal{R}) \otimes \Omega_X$. Moreover observe that $(\,|\,)$ gives an isomorphism between $\mathfrak{g}^{\zeta^i}$ and the dual of $\mathfrak{g}^{\zeta^{-i}}$. Since $\mathscr{E}_i \otimes \mathscr{E}_{p-i} \cong \mathscr{O}(-\mathcal{R})$ for $i \neq 0$, the normalized killing form $(\,|\,)_{\mathfrak{h}}$ gives an isomorphism between $\mathfrak{g}^{\zeta^{-i}} \otimes_k \mathscr{E}_i$ and $(\mathfrak{g}^{\zeta^i} \otimes_k \mathscr{E}_{p-i}(\mathcal{R}))^{\vee}$. It follows that

$$d_{\mathfrak{h}_{\mathcal{L}}}(\mathfrak{g}^{\zeta^{-i}} \otimes_k \mathscr{E}_i) = \mathfrak{g}^{\zeta^i} \otimes_k (\mathscr{E}_i(\mathcal{R}))^{\vee} \otimes_{\mathscr{O}_X} \Omega_X = \mathfrak{g}^{\zeta^i} \otimes_k \mathscr{E}_{-i} \otimes_{\mathscr{O}_X} \Omega_X$$

which yields $d_{\mathfrak{h}_{\mathcal{L}}}(\mathfrak{h}_{\mathcal{A}}) = \omega_{\mathcal{A}} \otimes {\mathfrak{h}_{\mathcal{A}}}^{\vee}$. □

LEMMA 2.1.6. *The annihilator of $\mathfrak{h}_{\mathcal{A}}$ with respect to the pairing $Res_{\mathfrak{h}}$, which is denoted $Ann_{Res_{\mathfrak{h}}}(\mathfrak{h}_{\mathcal{A}})$, is $\omega_{\mathcal{A}} \otimes {\mathfrak{h}_{\mathcal{A}}}^{\vee}$.*

PROOF. Before starting with the proof, we remark that this lemma holds if we replace $\mathfrak{h}_{\mathcal{A}}$ with any vector bundle $\mathscr{E}$ on X as it is essentially a consequence of Serre duality. We start by giving a description of the quotient $\mathfrak{h}_{\mathcal{A}} \backslash \mathfrak{h}_{\mathcal{L}}$, as the annihilator of $\mathfrak{h}_{\mathcal{A}}$ will be the dual of that quotient with respect to the residue pairing. The double quotient $\mathfrak{h}_{\mathcal{A}} \backslash \mathfrak{h}_{\mathcal{L}} / F^n\mathfrak{h}_{\mathcal{L}}$ computes $R^1\pi_*(\mathfrak{h} \otimes_{\mathscr{O}_X} \mathcal{I}_\sigma^n)$. It follows that the projective limit $\varprojlim_{n\geq 1} R^1\pi_*(\mathfrak{h} \otimes_{\mathscr{O}_X} \mathcal{I}_\sigma^n))$ equals $\varprojlim_{n\geq 1} \mathfrak{h}_{\mathcal{A}} \backslash \mathfrak{h}_{\mathcal{L}} / F^n\mathfrak{h}_{\mathcal{L}}$ which is $\mathfrak{h}_{\mathcal{A}} \backslash \mathfrak{h}_{\mathcal{L}}$. As the residue pairing gives rise to Serre duality, we know that $R^1\pi_*(\mathfrak{h} \otimes_{\mathscr{O}_X} \mathcal{I}_\sigma^n)$ is isomorphic to the dual of $\pi_*(\Omega_{X/S} \otimes (\mathfrak{h} \otimes \mathcal{I}_\sigma^n)^{\vee})$. It follows that

$$\mathrm{Ann}_{\mathrm{Res}_{\mathfrak{h}}}(\mathfrak{h}_{\mathcal{A}}) = \varinjlim_{n\geq 1} \pi_*\left(\Omega_{X/S} \otimes_{\mathscr{O}_X} (\mathfrak{h} \otimes_{\mathscr{O}_X} \mathcal{I}_\sigma^n)^{\vee}\right).$$

which equals $\omega_{\mathcal{A}/S} \otimes_{\mathcal{A}} {\mathfrak{h}_{\mathcal{A}}}^{\vee}$. □

2.2. Conformal blocks attached to integrable representations

This section is devoted to the definition of the sheaf of conformal blocks. Let $U\widehat{\mathfrak{h}}_{\mathcal{L}}$ denote the universal enveloping algebra of $\widehat{\mathfrak{h}}_{\mathcal{L}}$ and recall that $F^0\mathfrak{h}_{\mathcal{L}} = \pi_* \varprojlim_n \mathfrak{h} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathcal{I}_\sigma^n$, i.e. it is the subalgebra of $\mathfrak{h}_{\mathcal{L}}$ which has no poles along $\sigma(S)$. Observe that this implies that it is also a Lie sub algebra of $\widehat{\mathfrak{h}}_{\mathcal{L}}$.

DEFINITION 2.2.1. For any $\ell \in \mathbb{N}$ we define the *Verma module of level ℓ* to be the left $U\widehat{\mathfrak{h}}_{\mathcal{L}}$-module given by

$$\widetilde{\mathcal{H}_\ell}(0) := U\widehat{\mathfrak{h}}_{\mathcal{L}} \,/\, \Big(U\widehat{\mathfrak{h}}_{\mathcal{L}} \circ F^0\mathfrak{h}_{\mathcal{L}},\, c = \ell\Big).$$

For what follows we will need a generalization of this module attached to certain representations of $\sigma^*\mathfrak{h}$.

DEFINITION 2.2.2. An irreducible finite dimensional representation $\mathcal{V}$ of $\sigma^*\mathfrak{h}$ is a locally free $\mathscr{O}_S$-module which is equipped with an action of $\sigma^*\mathfrak{h}$ which locally étale on S, and up to an isomorphism of $\sigma^*\mathfrak{h}$ with $\mathfrak{g} \times \mathscr{O}_S$, is isomorphic to $V \otimes_k \mathscr{O}_S$ for an irreducible finite dimensional representation V of $\mathfrak{g}$.

Let $\mathcal{V}$ be an irreducible finite dimensional representation of the Lie algebra $\sigma^*\mathfrak{h}$: we will see how this induces a representation of $\widehat{\mathfrak{h}}_{\mathcal{L}}$ with the central element acting as multiplication by $\ell \in \mathbb{N}$. As first step, note that the exact sequence

$$0 \to \mathcal{I}_\sigma \to \mathscr{O}_X \to \mathscr{O}_S \to 0$$

defining $\sigma(S)$ gives rise to the map of Lie algebras $[\star]_{\mathcal{I}_\sigma} \colon F^0\mathfrak{h}_{\mathcal{L}} \to \sigma^*\mathfrak{h}$ induced by the truncation map $\varprojlim_n \mathscr{O}_X/\mathcal{I}_\sigma^n \to \mathscr{O}_X/\mathcal{I}_\sigma$. The action of $\sigma^*\mathfrak{h}$ on $\mathcal{V}$ is then extended to the action of $F^0\widehat{\mathfrak{h}}_{\mathcal{L}} = F^0\mathfrak{h}_{\mathcal{L}} \oplus c\mathscr{O}_S$ by imposing, for every $v \in \mathcal{V}$ and for every $X \in F^0\mathfrak{h}_{\mathcal{L}}$, the relations

$$c * v := \ell v \quad \text{and} \quad X * v := [X]_{\mathcal{I}_\sigma} v.$$

In view of this, once we fix $\ell \in \mathbb{N}$ we always view a representation $\mathcal{V}$ of $\sigma^*\mathfrak{h}$ as a $UF^0\widehat{\mathfrak{h}}_{\mathcal{L}}$-module with the central part acting by multiplication by ℓ.

DEFINITION 2.2.3. For every $\ell \in \mathbb{N}$ we define the *Verma module of level ℓ attached to $\mathcal{V}$* to be left $U\widehat{\mathfrak{h}}_{\mathcal{L}}$-module of level ℓ attached to $\mathcal{V}$, meaning

$$\widetilde{\mathcal{H}_\ell}(\mathcal{V}) := U\widehat{\mathfrak{h}}_{\mathcal{L}} \underset{UF^0\widehat{\mathfrak{h}}_{\mathcal{L}}}{\otimes} \mathcal{V}$$

where $F^0\widehat{\mathfrak{h}}_{\mathcal{L}}$ acts on $U\widehat{\mathfrak{h}}_{\mathcal{L}}$ by multiplication on the right and $U\widehat{\mathfrak{h}}_{\mathcal{L}}$ acts on $\widetilde{\mathcal{H}_\ell}(\mathcal{V})$ by left multiplication.

REMARK 2.2.4. Note that when $\mathcal{V}$ is the trivial representation of $\sigma^*\mathfrak{h}$, we obtain that $\widetilde{\mathcal{H}_\ell}(\mathcal{V})$ coincides with $\widetilde{\mathcal{H}_\ell}(0)$ given in Definition 2.2.1.

In the constant case $\sigma^*\mathfrak{h} \cong \mathfrak{g}$, the properties af $\widetilde{\mathcal{H}_\ell}(\mathcal{V})$ have been studied in [**Kac90**, Chapter 7] when $R = k$ and $\mathcal{V}$ an irreducible representation of $\mathfrak{g}$ of level at most ℓ, where it is shown that it has a maximal irreducible quotient $\mathcal{H}_\ell(\mathcal{V})$. From this, one generalizes the construction to families of curves, but still in the constant case $\sigma^*\mathfrak{h} \cong \mathfrak{g}$, which in view of Lemma 1.2.6 means working on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_g^1$. The new step is to descend $\mathcal{H}_\ell(\mathcal{V})$ from $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_g^1$ to $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}$.

We then first of all recall the construction in the constant case in Section 2.2.1 and then show how it descends to $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}$ in Section 2.2.2.

2.2.1. Integrable representations of level ℓ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)^1_g$. The forgetful morphism $\mathbf{Forg}^1_1\colon \overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)^1_g \to \mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1}$ is a finite étale covering, so if we want to define a module on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$, we could first define it on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)^1_g$ and later show that the construction is Γ-equivariant, hence it descends to a module on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$. As already written, the advantage of working on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)^1_g$, is the identification of $\mathfrak{h}_{\mathcal{L}}$ with $\mathfrak{g}\mathcal{L}$, which allows us to use representation theory of $\mathfrak{g}$ and of the affine Lie algebra $\widehat{\mathfrak{g}}\mathcal{L}$ [**Kac90**, Chapter 7].

We recall here some facts about representation theory of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}\mathcal{L}$. Let $R(G,T) = R(\mathfrak{g},\mathfrak{t})$ be the root system of $\mathfrak{g}$ with basis of positive roots Δ. The *dominant weights* of $\mathfrak{g}$ are those element $\lambda \in \mathfrak{t}^*$ such that $\lambda(H_\alpha) \in \mathbb{N}_0$ for all positive roots α. By [**Bou75**, Théorème 1, Chapitre VIII, §7] the set of dominant weights P_+ of $\mathfrak{g}$ is in bijection with the isomorphism classes of irreducible and finite dimensional representations of $\mathfrak{g}$. The representation V_λ associated with λ is characterized by the property of being generated by a highest weight vector v_λ which is annihilated by the elements of $\mathfrak{g}^\alpha$ for every positive root α and such that $H(v_\lambda) = \lambda(H)v_\lambda$ for every $H \in \mathfrak{t}$. Let θ be the highest root and denote by H_θ the highest coroot of $\mathfrak{g}$. Then for every $\ell \in \mathbb{N}$ we set

$$P_\ell := \{\lambda \in P_+ | \lambda(H_\theta) \leq \ell\}.$$

In view of the correspondence between weights and representations, the set P_ℓ collects the equivalence classes of representations of level at most ℓ, meaning those representations V_λ of $\mathfrak{g}$ where $X^{\ell+1}$ acts trivially on V_λ for every nilpotent element $X \in \mathfrak{g}$. In what follows we will use P_ℓ to denote either the weights or the representations of level at most ℓ. Note that the trivial representation corresponds to the trivial weight $\lambda = 0$, so that it belongs to P_ℓ for every ℓ.

REMARK 2.2.5. We note that the action of Γ on $\mathfrak{g}$ induces an action of Γ on P_ℓ in the following way. Let $\rho_\lambda\colon \mathfrak{g} \times V \to V$ be the representation associated to λ, then we define the representation $\rho_{\gamma\lambda}\colon \mathfrak{g} \times V \to V$ as $\rho_{\gamma\lambda}(X,v) := \rho_\lambda(\gamma^{-1}X)v$ for all $X \in \mathfrak{g}$ and $v \in V$. The weight $\gamma\lambda$ belongs to P_ℓ since Γ sends nilpotent elements to nilpotent elements.

Let V_λ be an irreducible and finite dimensional representation of $\mathfrak{g}$ and consider the $U\widehat{\mathfrak{g}}k((t))$-module $\widetilde{\mathcal{H}_\ell}(V_\lambda)$ constructed as in Definition 2.2.3. The properties of $\widetilde{\mathcal{H}_\ell}(V_\lambda)$ are well known and described for example in [**Kac90**], [**KR87**], [**TUY89**] and in [**Bea96**]. The main results are collected in the following proposition.

PROPOSITION 2.2.6. *Let V_λ be an irreducible and finite dimensional representation of $\mathfrak{g}$ of level at most ℓ.*

(1) The module $\widetilde{\mathcal{H}_\ell}(V_\lambda)$ contains a maximal proper $U\widehat{\mathfrak{g}}\mathcal{L}$ submodule $\mathcal{Z}_\lambda$, so that it has a unique maximal irreducible quotient $\mathcal{H}_\ell(V_\lambda) := \widetilde{\mathcal{H}_\ell}(V_\lambda)/\mathcal{Z}_\lambda$.
(2) The natural map $V_\lambda \to \mathcal{H}_\ell(V_\lambda)$ sending v to $1 \otimes v$ identifies V_λ with the submodule of $\mathcal{H}_\ell(V_\lambda)$ annihilated by $UF^1\mathfrak{g}\mathcal{L} = U\mathfrak{g}tk[[t]]$.
(3) The module $\mathcal{H}_\ell(V_\lambda)$ is integrable, i.e. for any $X \in \mathfrak{g}$ nilpotent and every $f(t) \in k((t))$, the element $Xf(t)$ acts locally nilpotently on $\mathcal{H}_\ell(V_\lambda)$. This means that there exists $n \in \mathbb{N}$ such that $Xf(t)^{\circ n}$ acts trivially on $\mathcal{H}_\ell(V_\lambda)$.

It follows that to every $(\widetilde{X} \to X \to \mathrm{Spec}(k), \tau) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)^1_g(\mathrm{Spec}(k))$ and $\lambda \in P_\ell$, we can associate the irreducible $U\widehat{\mathfrak{h}}_{\mathcal{L}}$ module $\mathcal{H}_\ell(V_\lambda)$ realized as quotient of $\widetilde{\mathcal{H}_\ell}(V_\lambda)$.

Let $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \tau) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)^1_g(S)$ and call σ the composition $q\tau$. An isomorphism of $\mathfrak{h}_{\mathcal{L}}$ with $\mathfrak{g}\mathcal{L}$ is fixed by τ, as well as an isomorphism of $\sigma^*\mathfrak{h}$ with $\mathfrak{g} \otimes_k \mathscr{O}_S$. Denote by $\mathcal{V}_\lambda := V_\lambda \otimes_k \mathscr{O}_S$ the extension of scalars of V_λ from k to $\mathscr{O}_S$, so that $\mathcal{V}_\lambda$ is naturally a representation of $\mathfrak{g} \otimes_k \mathscr{O}_S = \sigma^*\mathfrak{h}$. We show how to construct $\mathcal{H}_\ell(\mathcal{V}_\lambda)$ as quotient of $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)$.

Let us assume first that $\widehat{\mathcal{O}} \cong \mathscr{O}_S[[t]]$. This implies that $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)$ is isomorphic to

$$U\mathfrak{g}t^{-1}\mathscr{O}_S[t^{-1}] \otimes_{\mathscr{O}_S} \mathcal{V}_\lambda = U\mathfrak{g}t^{-1}k[t^{-1}] \otimes_k V_\lambda \otimes_k \mathscr{O}_S = \widetilde{\mathcal{H}_\ell}(V_\lambda) \otimes_k \mathscr{O}_S.$$

REMARK 2.2.7. Observe that the isomorphism that we have obtained between $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)$ and $\widetilde{\mathcal{H}_\ell}(V_\lambda) \otimes_k \mathscr{O}_S$ does not depend on the choice of the parameter t.

It follows that $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)$ has a unique maximal $U\widehat{\mathfrak{g}} \otimes_k k((t))$ proper submodule $\mathcal{Z}_S := \mathcal{Z}_\lambda \otimes \mathscr{O}_S$, where $\mathcal{Z}_\lambda$ is the maximal proper submodule of $\mathcal{H}_\ell(V_\lambda)$. We define $\mathcal{H}_\ell(\mathcal{V}_\lambda)$ as the quotient $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)/\mathcal{Z}_S$ or equivalently as $\mathcal{H}_\ell(V_\lambda) \otimes \mathscr{O}_S$. This construction uses a choice of the isomorphism $\mathfrak{h}_{\mathcal{L}} \cong \mathfrak{g}\mathcal{L}$, but since $\mathcal{Z}_\lambda$ and hence $\mathcal{Z}_S$ satisfy a maximality condition, they do not depend on the isomorphism $\mathfrak{h}_{\mathcal{L}} \cong \mathfrak{g}\mathcal{L}$, concluding that $\mathcal{H}_\ell(\mathcal{V}_\lambda)$ is the maximal irreducible quotient of $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)$.

We now drop the assumpion that $\widehat{\mathcal{O}}$ is globally isomorphic to $S[[t]]$. We want however show that Zariski locally on S we can reduce to $\widehat{\mathcal{O}} \cong \mathscr{O}_S[[t]]$ so that we can locally define $\mathcal{H}_\ell(\mathcal{V}_\lambda)$ and then show that this gives rise to a global object. Since $\mathcal{I}_\sigma$ is locally principal, we can find an open covering $\{U_i\}$ of X such that $\mathcal{I}_\sigma|_{U_i}$ is principal. This implies that $\varprojlim_n \mathscr{O}_{U_i}/(\mathcal{I}_\sigma|_{U_i})^n$ is isomorphic to $\mathscr{O}_{S_i}[[t]]$ where $S_i := \sigma^{-1}(U_i)$. Observe that this does not imply that $\widehat{\mathcal{O}} \otimes_S \mathscr{O}_{S_i} \cong \mathscr{O}_{S_i}[[t]]$, but only that $\widehat{\mathcal{O}} \widehat{\otimes}_S \mathscr{O}_{S_i} \cong \mathscr{O}_{S_i}[[t]]$. Consider then the sheaf of Lie algebras $\mathfrak{g}\mathcal{L}_i := \mathfrak{g} \otimes \mathscr{O}_{S_i}((t)) \cong \mathfrak{g}\mathcal{L} \widehat{\otimes} \mathscr{O}_{S_i}$, and construct the $U\widehat{\mathfrak{g}}\mathcal{L}_i$-module $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)_i$.

CLAIM. *The inclusion $\mathfrak{g}\mathcal{L} \otimes \mathscr{O}_{S_i} \to \mathfrak{g}\mathcal{L}_i$ induces an isomorphism of $\mathscr{O}_{S_i}$-modules between $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda) \otimes_S \mathscr{O}_{S_i}$ and $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)_i$.*

PROOF. We need to prove that $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda) \otimes_S \mathscr{O}_{S_i} \to \widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)_i$ is surjective. We use induction on the length of the elements of $U\mathfrak{g}\mathcal{L}_i$, where the length of an element $u \in U\mathfrak{g}\mathcal{L}_i$ is the minimum n such that $u \in \oplus_{j=0}^n \mathfrak{g}\mathcal{L}_i^{\otimes j}$. Let $aX \in \mathfrak{g}\mathcal{L}_i$ with $X \in \mathfrak{g}$ and $a = \sum_{i \geq -N} a_i t^i \in \mathscr{O}_{S_i}((t))$, and take $v \in \mathcal{V}_\lambda$. The class of $aX \otimes v$ in $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)_i$ is the same as the one of $[aX] \otimes v := [a]X \otimes v$, where $[a] = \sum_{i \geq -N}^0 a_i t^i$, which then belongs to $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda) \otimes_S \mathscr{O}_{S_i}$. Let now $Y = Y_1 \circ \cdots \circ Y_n$ be an element of $U\mathfrak{g}\mathcal{L}_i$, and note that in $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)_i$ the element $Y \otimes v$ is equivalent to the class of $([Y_n] \circ \cdots \circ [Y_1] + u) \otimes v$ where u has length lower than n. Using the induction hypothesis we conclude the proof. □

We define the $\mathscr{O}_{S_i}$-module $\mathcal{H}_\ell(\mathcal{V}_\lambda)|_{S_i}$ to be $\mathcal{H}_\ell(\mathcal{V}_\lambda)_i = \widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)_i/\mathcal{Z}_i$. This gives rise to the $\mathscr{O}_S$-module $\mathcal{H}_\ell(\mathcal{V}_\lambda)$ because on the intersection S_{ij} the modules $\mathcal{H}_\ell(\mathcal{V}_\lambda)_i$ and $\mathcal{H}_\ell(\mathcal{V}_\lambda)_j$ are isomorphic via to the transition morphisms defining $\mathcal{I}_\sigma$. Equivalently we could have defined $\mathcal{Z}|_{S_i}$ to be the image of $\mathcal{Z}_i$ in $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)|_{S_i}$ and so $\mathcal{H}_\ell(\mathcal{V}_\lambda)|_{S_i}$ would be the quotient of $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)|_{S_i}$ by $\mathcal{Z}|_{S_i}$. The modules $\mathcal{Z}|_{S_i}$ glue and give rise to a $\widehat{\mathfrak{g}}\mathcal{L}$-module $\mathcal{Z}$ on S, so that $\mathcal{H}_\ell(\mathcal{V}_\lambda)$ is given by $\widetilde{\mathcal{H}_\ell}(\mathcal{V}_\lambda)/\mathcal{Z}$. This construction is invariant under the action of Γ, hence it defines $\widehat{\mathfrak{h}}_{\mathcal{L}}$ as a $U\widehat{\mathfrak{h}}_{\mathcal{L}}$-module.

2.2.2. Integrable representations of level ℓ on $\overline{\mathcal{H}\mathbf{ur}}(\Gamma,\xi)_{g,1}$. We show here how to descend $\mathcal{H}_\ell(\mathcal{V})$ from $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$ to $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1}$, so let consider $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \sigma) \in \mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1}(S)$. The first issue is that, unless we choose an isomorphism between $\sigma^*\mathfrak{h}$ and $\mathfrak{g}$, we are not able to provide a representation of $\sigma^*\mathfrak{h}$ associated to a $\lambda \in P_\ell$. In fact, one obstruction to this, as we noticed in Remark 2.2.5, is that Γ does not in general act trivially on P_ℓ, so it is impossible to identify $\mathcal{V}$ as $\mathcal{V}_\lambda$, with λ independent of the isomorphism between $\mathfrak{h}_\mathcal{L}$ and $\mathfrak{g}\mathcal{L}$.

The conclusion is that it seems unreasonable to associate to $\lambda \in P_\ell$ a module $\mathcal{H}_\ell(\mathcal{V}_\lambda)$ on $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1}$ because P_ℓ contains only local information. The following set is what replaces P_ℓ.

DEFINITION 2.2.8. A representation $\mathcal{V}$ of $\sigma^*\mathfrak{h}$ is said to be of level at most ℓ if for every nilpotent element X of $\sigma^*\mathfrak{h}$, then $X^{\ell+1}$ acts trivially on $\mathcal{V}$. Equivalently this means that locally étale we can identify $\mathcal{V}$ with $V \otimes \mathscr{O}_S$ for a representation $V \in P_\ell$. Define $\mathrm{IrRep}_\ell(\sigma^*\mathfrak{h})$ or by abuse of notation only $\mathrm{IrRep}_\ell(\sigma)$ or IrRep_ℓ to be the set of isomorphism classes of irreducible and finite dimensional representations $\mathcal{V}$ of $\sigma^*\mathfrak{h}$ of level at most ℓ.

The main step towards the definition of the sheaf of conformal blocks attached to $\mathcal{V} \in \mathrm{IrRep}_\ell$ is the following result.

PROPOSITION 2.2.9. *Let $\mathcal{V} \in IrRep_\ell$. Then there exists a unique maximal proper $U\widehat{\mathfrak{h}}_\mathcal{L}$ submodule $\mathcal{Z}$ of $\widetilde{\mathcal{H}_\ell}(\mathcal{V})$.*

PROOF. We show that the maximal proper submodule of $\widetilde{\mathcal{H}_\ell}(\mathcal{V})$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_g^1$ descends along $\mathbf{Forg}_1^1$ to the maximal proper submodule of $\widetilde{\mathcal{H}_\ell}(\mathcal{V})$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$. Recall that since $\sigma(S)$ does not intersect the branch locus of q, we can find an étale covering $S' \to S$ such that the pullback of $(\widetilde{X} \to X \to S, \sigma)$ lies in the image of $\mathbf{Forg}_1^1$. This implies that to give $\mathcal{V} \in \mathrm{IrRep}_\ell$ is equivalent to give an irreducible and finite dimensional representation $\mathcal{V}'$ of $\sigma'^*\mathfrak{h}$ and an isomorphism $\phi\colon p_1^*\mathcal{V}' \to p_2^*\mathcal{V}'$ satisfying the cocycle conditions on S''', where $p_i\colon S'' = S' \times_S S' \to S'$ is the i-th projection.

This tells us moreover that $\widetilde{\mathcal{H}_\ell}(\mathcal{V})$ is obtained by descending $\widetilde{\mathcal{H}_\ell}(\mathcal{V}')$ from S' to S. Observe that up to the choice of an isomorphism $\sigma'^*\mathfrak{h} \cong \mathfrak{g} \otimes_k \mathscr{O}_{S'}$, the representation $\mathcal{V}'$ is of the form $\mathcal{V}_\lambda$, so that $\mathcal{Z}'$ and $\mathcal{H}_\ell(\mathcal{V}')$ are well defined. We construct $\mathcal{H}_\ell(\mathcal{V})$ by descending $\mathcal{Z}'$ to a module $\mathcal{Z}$ on S, so that $\mathcal{H}_\ell(\mathcal{V}) := \widetilde{\mathcal{H}_\ell}(\mathcal{V}) \big/ \mathcal{Z}$.

Since $\mathfrak{h}_\mathcal{L}$ is a module on $\mathscr{O}_S$, we have a canonical isomorphism $\phi_{12}\colon {p_1}^*\mathfrak{h}_\mathcal{L}|_{S'} \to {p_2}^*\mathfrak{h}_\mathcal{L}|_{S'}$ satisfying the cocycle conditions on $S''' := S'' \times_S S'$. Recall moreover that $\mathcal{Z}'$ is the maximal proper $U\widehat{\mathfrak{h}}_\mathcal{L}$ submodule of $\widetilde{\mathcal{H}_\ell}(\mathcal{V}')$, which is then Γ-invariant. This induces an isomorphism between $p_1^*\mathcal{Z}'$ and $p_2^*\mathcal{Z}'$ which satisfied the cocycle condition on S''' and it is independent of the isomorphism $\mathfrak{h}_\mathcal{L} \cong \mathfrak{g}\mathcal{L}$. □

DEFINITION 2.2.10. Let $\mathcal{V} \in \mathrm{IrRep}_\ell$. The maximal irreducible quotient of $\widetilde{\mathcal{H}_\ell}(\mathcal{V})$ is denoted $\mathcal{H}_\ell(\mathcal{V})$ and defines a sheaf

$$\mathcal{H}_\ell(\mathcal{V})_X = \mathfrak{h}_\mathcal{A} \circ \mathcal{H}_\ell(\mathcal{V}) \big\backslash \mathcal{H}_\ell(\mathcal{V})$$

on S which is called the *sheaf of conformal blocks attached to* $\mathcal{V}$. When $\mathcal{V}$ is the trivial representation of $\sigma^*\mathfrak{h}$, we denote $\mathcal{H}_\ell(\mathcal{V})$ by $\mathcal{H}_\ell(0)$ and its quotient $\mathfrak{h}_\mathcal{A} \big\backslash \mathcal{H}_\ell(0)$ is called the *sheaf of covacua.*

The collection $\{\mathcal{H}_\ell(0)_X\}_{\widetilde{X}\to X\to S}$ determines the sheaf of modules $\mathcal{H}_\ell(0)_{X_{univ}}$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$. In similar way, given compatible families $\{\mathcal{V}(\sigma)\}_{\{\widetilde{X}\to X\to S,\sigma\}}$ defining an element $\mathcal{V}$ of $\mathrm{IrRep}_\ell(\sigma_{univ})$, the collection $\mathcal{H}_\ell(\mathcal{V}(\sigma))_X$ defines $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$.

Observe that Proposition 2.2.6 generalizes as follows.

COROLLARY 2.2.11. *Let $\mathcal{V}\in IrRep_\ell(\sigma)$, then:*

(1) The natural map $\mathcal{V}\to\mathcal{H}_\ell(\mathcal{V})$ sending v to $1\otimes v$ identifies $\mathcal{V}$ with the submodule of $\mathcal{H}_\ell(\mathcal{V})$ annihilated by $UF^1\widehat{\mathfrak{h}}_{\mathcal{L}}$.

(2) The module $\mathcal{H}_\ell(\mathcal{V})$ is integrable.

Inspired by [**Sor96**, Section 2.5] we prove the following statement.

PROPOSITION 2.2.12. *The $\mathcal{O}_S$-module $\mathcal{H}_\ell(\mathcal{V})_X$ is coherent. It follows that $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is a coherent module on $\overline{\mathcal{H}ur}(\Gamma,\xi)_{g,1}$.*

PROOF. This is essentially a consequence of [**Sor96**, Lemma 2.5.2]. As this is a local statement, we can assume that $\mathcal{L}\cong R((t))$ and we can fix an isomorphism $\mathfrak{h}_{\mathcal{L}}\cong\mathfrak{g}\mathcal{L}$. Observe that the quotient $\mathfrak{h}_{\mathcal{A}}\backslash\mathfrak{h}_{\mathcal{L}}/F^0\mathfrak{h}_{\mathcal{L}}$ is a finitely generated R-module as it computes $H^1(X,\mathfrak{h})$ and $\mathfrak{h}$ is locally free over X. This implies that $\mathfrak{h}_{\mathcal{A}}\backslash\widehat{\mathfrak{h}}_{\mathcal{L}}/F^1\mathfrak{h}_{\mathcal{L}}$ is finitely generated too over R and so we can choose finitely many generators $e_1,\dots,e_n$ so that we can write

$$\widehat{\mathfrak{h}}_{\mathcal{L}}=F^1\mathfrak{h}_{\mathcal{L}}+\mathfrak{h}_{\mathcal{A}}+\sum_{i=1}^n Re_i.$$

which in terms of enveloping algebras becomes

$$U\widehat{\mathfrak{h}}_{\mathcal{L}}=\sum_{(N_1,\dots,N_n)\in\mathbb{N}_0{}^n}U(\mathfrak{h}_{\mathcal{A}})\circ e_1{}^{\circ N_1}\circ\cdots\circ e_n{}^{\circ N_n}\circ U(F^1\mathfrak{h}_{\mathcal{L}})$$

thing that can be proven using induction on the length of elements of $U\widehat{\mathfrak{h}}_{\mathcal{L}}$.

We can furthermore assume that the elements e_i acts locally nilpotently on $\mathcal{H}_\ell(\mathcal{V})$, meaning that there exists $M\in\mathbb{N}$ such that $e_i^{\circ M}$ acts trivially on $\mathcal{H}_\ell(\mathcal{V})$. In fact we might use the isomorphism $\mathfrak{h}_{\mathcal{L}}$ with $\mathfrak{g}\mathcal{L}$ and the Cartan decomposition of $\mathfrak{g}=\mathfrak{t}\oplus_{\alpha\in R(\mathfrak{g},\mathfrak{t})}\mathfrak{g}_\alpha$. The algebras $\mathfrak{g}_\alpha$'s are nilpotent and generate $\mathfrak{g}$, so that $\mathfrak{g}\mathcal{L}$ is generated by $\oplus_{\alpha\in R(\mathfrak{g},\mathfrak{t})}\mathfrak{g}_\alpha\mathcal{L}$. This means that also the elements e_i are generated by elements of $\oplus_{\alpha\in R(\mathfrak{g},\mathfrak{t})}\mathfrak{g}^\alpha\mathcal{L}$ so that, up to replace e_i with a choice of nilpotent generators, we can ensure that all the e_i's live in $\oplus_{\alpha\in R(\mathfrak{g},\mathfrak{t})}\mathfrak{g}^\alpha\mathcal{L}$ and so using Corollary 2.2.11 (2) the e_i's will act locally nilpotently on $\mathcal{H}_\ell(\mathcal{V})$.

It follows that

$$\widetilde{\mathcal{H}_\ell}(\mathcal{V})=\sum_{(N_1,\dots,N_n)\in\mathbb{N}_0{}^n}U(\mathfrak{h}_{\mathcal{A}})\circ e_1{}^{\circ N_1}\circ\cdots\circ e_n{}^{\circ N_n}\underset{c=\ell}{\otimes}\mathcal{V}$$

and that

$$\mathcal{H}_\ell(\mathcal{V})=\sum_{(N_1,\dots,N_n)\in\mathbb{N}_0{}^n}U(\mathfrak{h}_{\mathcal{A}})\circ e_1{}^{\circ N_1}\circ\cdots\circ e_n{}^{\circ N_n}\underset{c=\ell}{\otimes}\mathcal{V}/\mathcal{Z}$$

Using induction on n and the fact that the e_i's act locally nilpotently, we can conclude that the sum can be taken over finitely many $(N_1,\dots,N_n)\in\mathbb{N}_0{}^n$, hence that the quotient $\mathfrak{h}_{\mathcal{A}}\backslash\mathcal{H}_\ell(\mathcal{V})=\mathcal{H}_\ell(\mathcal{V})_X$ is finitely generated. □

3 | The projective connection on $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$

We want to prove that the sheaf of conformal blocks $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is a vector bundle on the Hurwitz stack $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$, so that its rank will be constant. Since we already know that $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is coherent, one method to exhibit local freeness is to provide a projectively flat connection on it. In this section we provide a projective action of $\mathcal{T}_{\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}/k}(-\log(\Delta))$ on $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$, showing its freeness when restricted to $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1}$.

3.1. The tangent to $\overline{\mathcal{H}\mathbf{ur}}(\Gamma,\xi)_{g,1}$

Let $(\widetilde{X} \to X,\sigma) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}(\mathrm{Spec}(k))$ and recall that in Remark 1.2.3 we saw that the tangent space of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$ at $(\widetilde{X} \to X,\sigma)$ is isomorphic to the tangent space of $\mathcal{M}_{g,(1+\deg(\xi))}$ at $(X,\sigma\cup\mathcal{R})$. The latter, which is the space of infinitesimal deformations of $(X,\sigma\cup\mathcal{R})$, can be explicitely described as the space $\mathrm{Ext}^1(\Omega_{X/k},\mathscr{O}(-\mathcal{R}-\sigma(S))$ [**ACG11**, Chapter XI] which sits in the short exact sequence

$$0 \to H^1(X,\mathcal{T}_{X/k}(-\mathcal{R}-\sigma(S)) \to \mathrm{Ext}^1(\Omega_{X/k},\mathscr{O}(-\mathcal{R}-\sigma(S)) \to$$
$$\to H^0(X,\mathscr{E}xt^1(\Omega_{X/k},\mathscr{O}(-\mathcal{R}-\sigma(S))) \to 0$$

where the last term is supported on the singular points of X.

We now use the assumption that the curve $(X,\sigma\cup\mathcal{R})$ is stably marked to assume that there exists a versal family $\mathcal{X} \to S$ with a reduced divisor $\sigma_{\mathcal{X}}+\mathcal{R}_{\mathcal{X}}$ deforming it and such that the subscheme of S whose fibres are singular is a normal crossing divisor Δ. Call s_0 the point of S such that $\mathcal{X}|_{s_0}$ is X. The versality condition means that the Kodaira-Spencer map

$$\mathtt{KS}\colon \mathcal{T}_{S/k} \to \mathscr{E}xt^1(\Omega_{\mathcal{X}/S},\mathscr{O}(-\mathcal{R}_{\mathcal{X}}-\sigma_{\mathcal{X}}))$$

is an isomorphism, so that we identify the tangent space of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$ at $(\widetilde{X} \to X,\sigma)$ with the tangent space of S at s_0.

The conclusion is that to provide a projective connection on $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is equivalent to provide an action of $\mathcal{T}_{S/k}$ on $\mathcal{H}_\ell(\mathcal{V})_X$ for every versal family $\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S$. As aforementioned, we will however not be able to provide a projective action of the whole $\mathcal{T}_{S/k}$, but only of the submodule $\mathcal{T}_{S/k}(-\log(\Delta))$, which via the Kodaira-Spencer map is identified with $R^1\pi_*(\mathcal{T}_{X/S}(-\mathcal{R}-\sigma(S)))$.

3.2. Tangent bundles and the action of Γ

In view of the previous observations, we assume that the element $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \sigma)$ of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}(S)$ is a versal family, so that the locus of points s of S such that the fibres X_s (or equivalently $\widetilde{X}_s$) are non smooth is a normal crossing divisor Δ of S. We give in this section a description of $\mathcal{T}_{S/k}(-\log(\Delta))$ and $\mathcal{T}_{S/k}$, by realizing it as a quotient of certain sheaves of derivations. Consider, to begin with, the following situation described by Looienga [**Loo13**, Section 2]. Let $R \in k$-**Alg** and $L = R((t))$, then we can consider the following two modules: $\theta_{L/R}$ consisting of continuous R-linear derivations of L and $\theta_{L,R}$ consisting of continuous k-linear derivations of L which restrict to derivations of R into itself. The quotient $\theta_{L,R}/\theta_{L/R}$ is canonically identified with the module of k-linear derivations of R.

Take $(f \colon \widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \sigma) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}(S)$ as above. We already introduced in Subsection 2.1 the $\mathscr{O}_S$-module $\theta_{\mathcal{L}/S}$ of continuous $\mathscr{O}_S$-linear derivations of $\mathcal{L}$ and we define now $\theta_{\mathcal{L},S}$ as the $\mathscr{O}_S$-module of continuous k-linear derivations of $\mathcal{L}$ which restrict to derivations of $\mathscr{O}_S$. Observe that $\theta_{\mathcal{L}/S}$ and $\theta_{\mathcal{L},S}$ depend only on the marked curve $(X \to S, \sigma)$, so the following well known result belongs to the classical setting.

PROPOSITION 3.2.1. *The sequence of $\mathscr{O}_S$-modules*

$$0 \to \theta_{\mathcal{L}/S} \to \theta_{\mathcal{L},S} \to \mathcal{T}_{S/k} \to 0$$

is exact.

PROOF. As exactness can be checked on formal neighbourhoods, we can assume that $\mathcal{L} \cong R((t))$ so that the result follows from the example presented above. $\square$

In similar fashion we now describe the subsheaf $\mathcal{T}_{S/k}(-\log(\Delta))$ as quotient of appropriate sheaves of derivations.

The sheaves $\theta_{\mathcal{A}/S}(-\mathcal{R})$ and $\theta_{\mathcal{A},S}(-\mathcal{R})$. Following Looijenga's notation, we denote by $\theta_{\mathcal{A}/S}$ the sheaf of derivations $f_*\mathcal{T}_{X^*/S}$. Recall that in Lemma 1.1.2 we have showed that there is an isomorphism between $(q_*\mathcal{T}_{\widetilde{X}/S})^\Gamma$ and $\mathcal{T}_{X/S}(-\mathcal{R})$. This implies that $f_*(\mathcal{T}_{\widetilde{X}^*/S}))^\Gamma \cong \theta_{\mathcal{A}/S} \otimes_S \pi_*\mathscr{O}(-\mathcal{R})$ and by abuse of notation we will denote this sheaf by $\theta_{\mathcal{A}/S}(-\mathcal{R})$. In a similar way we consider the action of Γ on the pushforward to S of $\mathcal{T}_{\widetilde{X}^*,S}$, the sheaf of k-linear derivations of $\mathscr{O}_{\widetilde{X}^*}$ which restrict to derivations of $f^{-1}\mathscr{O}_S$ and we call $\theta_{\mathcal{A},S}(-\mathcal{R})$ the sub module of Γ-invariants.

REMARK 3.2.2. Recall that we defined $\widetilde{\mathcal{L}}$ as $\varinjlim_N f_* \varprojlim_n (q^*\mathcal{I}_\sigma)^{-N}/(q^*\mathcal{I}_\sigma)^n$ and define now

$$\theta_{\widetilde{\mathcal{L}}/S} := \varinjlim_N \pi_* q_* \varprojlim_n \mathcal{T}_{\widetilde{X}/S} \otimes_{\mathscr{O}_{\widetilde{X}}} q^*\mathcal{I}_\sigma^{-N}/q^*\mathcal{I}_\sigma^n$$

or equivalently $\theta_{\widetilde{\mathcal{L}}/S}$ is the module of continuous derivations of $\widetilde{\mathcal{L}}$ which are $\mathscr{O}_S$ linear. Thanks to Lemma 1.1.2, the $\mathscr{O}_S$-submodule of Γ-invariants of $f_*\mathcal{T}_{\widetilde{X}/S}$ is identified with $\varinjlim_N \pi_* \varprojlim_n \mathcal{T}_{X/S}(-\mathcal{R}) \otimes_{\mathscr{O}_X} \mathcal{I}_\sigma^{-N}/\mathcal{I}_\sigma^n$ which equals $\varinjlim_N \pi_* \varprojlim_n \mathcal{T}_{X/S} \otimes_{\mathscr{O}_X} \mathcal{I}_\sigma^{-N}/\mathcal{I}_\sigma^n$ as $\mathcal{R}$ and $\sigma(S)$ are disjoint. The latter is the $\mathscr{O}_S$-module of continuous and $\mathscr{O}_S$-linear derivations of $\mathcal{L}$, which is $\theta_{\mathcal{L}/S}$.

The previous remark implies moreover that $\theta_{\mathcal{A}/S}$ is a submodule of $\theta_{\mathcal{L}/S}$.

REMARK 3.2.3. Observe that the action of $\mathcal{T}_{\widetilde{X}/S}$ (resp. of $\mathcal{T}_{\widetilde{X},S}$) on $\mathfrak{g} \otimes_k \mathscr{O}_{\widetilde{X}}$ by coefficientwise derivation is Γ-equivariant. This implies that $(q_* \mathcal{T}_{\widetilde{X}/S})^{\Gamma}$ (resp. $(q_* \mathcal{T}_{\widetilde{X},S})^{\Gamma}$) acts on $\mathfrak{h}$ and we will say that the action is by coefficientwise derivation. By restricting ourselves to $\widetilde{X}^*$ this implies that $\theta_{\mathcal{A}/S}(-\mathcal{R})$ (resp. $\theta_{\mathcal{A},S}(-\mathcal{R})$) acts on $\mathfrak{h}_{\mathcal{A}}$ by coefficientwise derivation. The same holds for $\theta_{\mathcal{L}/S}$ and $\theta_{\mathcal{L},S}$ acting on $\mathfrak{h}_{\mathcal{L}}$.

EXAMPLE 3.2.4. We recall in this example the local description of the quotient $\theta_{\mathcal{A},S} / \theta_{\mathcal{A}/S}$ at nodal points [**Loo13**, Section 5]. Let R be a local complete k-algebra with maximal ideal $\mathfrak{m}$ and let $A := R[[x,y]]/xy - t$ for some $t \in \mathfrak{m}$. We are interested in the following R-modules: $\theta_{A/R}$, the module of R-linear derivations of A, which lives inside $\theta_{A,R}$, the module of k-linear derivations of A which restrict to derivations of R. We are interested in understanding the quotient $\theta_{A,R}/\theta_{A/R}$ and we claim that this gets identified with $\theta_{R/k}(-\log(t))$, the module of k-linear derivations of R which send the element t to an element of the ideal tR. We in fact note that from the relation $xy = t$, each derivation $D \in \theta_{A,R}$ should send t to a multiple of itself inside R, hence the natural map $\theta_{A,R} \to \theta_{R/k}$, whose kernel is $\theta_{A/R}$, has image landing inside $\theta_{R/k}(-\log(t))$ and what we need to prove is that this is exactly the image. Given in fact a derivation $D \in \theta_{R/k}(-\log(t))$, we claim that it is possible to extend it to a derivation of A, i.e. to define $D(x)$ and $D(y)$ satisfying $xD(y) + yD(x) = D(t)$. Since $D(t) = tr = 2xyr$ for some $r \in R$, it will be enough to set $D(y) = yr$ and $D(x) = xr$.

PROPOSITION 3.2.5. *The sequence*

$$0 \to \theta_{\mathcal{A}/S}(-\mathcal{R}) \to \theta_{\mathcal{A},S}(-\mathcal{R}) \to \mathcal{T}_S(-\log(\Delta)) \to 0$$

is exact.

PROOF. As taking Γ-invariants is an exact functor ($\mathrm{char}(k) = 0$) and Γ acts trivially on $\mathcal{T}_{S/k}(-\log(\Delta))$, it suffices to prove that the sequence

$$0 \to f_* \mathcal{T}_{\widetilde{X}^*/S} \to f_* \mathcal{T}_{\widetilde{X}^*,S} \to \mathcal{T}_S(-\log(\Delta)) \to 0$$

is exact. This statement does not depend on the covering, and appears in [**Sor96**] and [**Loo13**]. We give the proof of it, by starting observing that in the case $\widetilde{X}^* = \mathbb{A}^1_S$, the result follows by using the same argument of the proof of Proposition 3.2.1.

Let $U = S \setminus \Delta$ and denote by $\widetilde{X}_U := \widetilde{X} \times_S U$. As $\widetilde{X}_U$ is smooth over S, there exists an affine covering $\{\mathrm{Spec}(A_i) = \widetilde{X}_i\}$ of $\widetilde{X}_U$ and $\{\mathrm{Spec}(R_i) = U_i\}$ of U such that the map $f_i = f|_{\widetilde{X}_i} \colon \widetilde{X}_i \to S_i$ factors through $\mathbb{A}^1_{R_i}$ via an étale map ϕ_i as in the diagram:

$$\begin{array}{ccccc} \widetilde{X} & \longleftarrow & \widetilde{X}_i & \xrightarrow{\phi_i} & \mathbb{A}^1_{R_i} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_i} & \swarrow{\scriptstyle a} & \\ S & \longleftarrow & U_i & & \end{array}$$

It follows that the sequence $0 \to a_* \mathcal{T}_{\mathbb{A}^1_{R_i}/U_i} \to a_* \mathcal{T}_{\mathbb{A}^1_{R_i},U_i} \to \mathcal{T}_{U_i|k} \to 0$ is exact. The étaleness of ϕ_i provides an isomorphism between $f_{i*} \mathcal{T}_{\widetilde{X}_i/S_i}$ and $a_* \mathcal{T}_{\mathbb{A}^1_{R_i}/U_i}$ and similarly for $f_{i*} \mathcal{T}_{\widetilde{X}_i,U_i}$ and $a_* \mathcal{T}_{\mathbb{A}^1_{R_i},U_i}$, concluding the argument for the smooth points.

Let s be a geometric point of Δ and $\widetilde{x} \in \widetilde{X}_s$ a nodal point. This means that there exists an isomorphism

$$\widehat{\mathscr{O}_{\widetilde{X},\widetilde{x}}} \cong \widehat{\mathscr{O}_{S,s}}[[x,y]]/xy - t$$

for some $t \in \widehat{\mathfrak{m}}_s$. It follows that the Example 3.2.4 represents the local picture of the situation, where the result holds, concluding the proof. $\square$

3.3. The Virasoro algebra of $\mathcal{L}$

Now that we can express $\mathcal{T}_{S/k}(-\log(\Delta))$ as $\theta_{\mathcal{A},S}(-\mathcal{R})\,/\theta_{\mathcal{A}/S}(-\mathcal{R})$, we will define a projective action of $\theta_{\mathcal{A},S}(-\mathcal{R})$ on $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ which factors through that quotient.

In order to achieve this result, we will follow the methods of [**Loo13**] and define as first step the Virasoro algebra $\widehat{\theta}_{\mathcal{L}/S}$ of $\mathcal{L}$ as a central extension of $\theta_{\mathcal{L}/S}$. We report in this section the construction of $\widehat{\theta}_{\mathcal{L}/S}$ as explained in [**Loo13**, Section 2], for which we will use the same notation. As mentioned before, the $\mathscr{O}_S$-module $\theta_{\mathcal{L}/S}$ does not depend on the covering, and the same holds for its central extension $\widehat{\theta}_{\mathcal{L}/S}$. We will see in Section 3.4, how $\widehat{\theta}_{\mathcal{L}/S}$ acts on $\mathcal{H}_\ell(\mathcal{V})$ and how it induces a central extension of $\theta_{\mathcal{L},S}$.

The Lie algebra $\mathfrak{l}$ and its central extension $\widehat{\mathfrak{l}}$. We denote by $\mathfrak{l}$ the sheaf of abelian Lie algebras (over S) whose underlying module is $\mathcal{L}$. The filtration $F^*\mathcal{L}$ gives the filtration $F^*\mathfrak{l}$. Denote by $U\mathfrak{l}$ the universal enveloping algebra, which is isomorphic to $\mathrm{Sym}(\mathfrak{l})$ since $\mathfrak{l}$ is abelian. This algebra is not complete with respect to the filtration $F^*\mathfrak{l}$, so we complete it on the right obtaining

$$\overline{U}\mathfrak{l} := \varprojlim_n U\mathfrak{l}/U\mathfrak{l} \circ F^n\mathfrak{l}.$$

REMARK 3.3.1. Note that in this case the completions on the right $\varprojlim_n U\mathfrak{l}/U\mathfrak{l}\circ F^n\mathfrak{l}$ and on the left $\varprojlim_n F^n\mathfrak{l}\circ U\mathfrak{l}\backslash U\mathfrak{l}$ coincide because $\mathfrak{l}$ is abelian. The element $\sum_{i\in\mathbb{N}} t^{-i}\circ t^i$ belongs to $\overline{U}\mathfrak{l}$, as well as $\sum_{i\in\mathbb{N}} t^{-(i)^m}\circ t^i$ for every $m\in\mathbb{N}$. However $\sum_{i\in\mathbb{N}} t^{-i}\circ t$ is not an element of $\overline{U}\mathfrak{l}$.

We extend centrally $\mathfrak{l}$ via the residue pairing described in Section 2.1 defining the Lie bracket on $\widehat{\mathfrak{l}} = \mathfrak{l}\oplus\hbar\mathscr{O}_S$ as

$$[f+\hbar r, g+\hbar s] = \hbar\mathrm{Res}(g df)$$

for every $f,g\in\mathfrak{l}$ and $r,s\in\mathscr{O}_S$. The filtration of $\mathfrak{l}$ extends to a filtration of $\widehat{\mathfrak{l}}$ by setting $F^i\widehat{\mathfrak{l}} = F^i\mathfrak{l}$ for $i\geq 0$ and $F^i\widehat{\mathfrak{l}} = F^i\mathfrak{l}\oplus\hbar\mathscr{O}_S$ for $i\leq 0$. The universal enveloping algebra of $\widehat{\mathfrak{l}}$ is denoted by $U\widehat{\mathfrak{l}}$ and $\overline{U\widehat{\mathfrak{l}}}$ denotes its completion on the right with respect to the filtration $F^*\widehat{\mathfrak{l}}$. Note that since $\hbar$ is a central element, we have that $\overline{U\widehat{\mathfrak{l}}}$ is an $\mathscr{O}_S[\hbar]$ algebra so that we will write $\hbar^2$ instead of $\hbar\circ\hbar$ and similarly $\hbar^n$ for every $n\in\mathbb{N}$.

REMARK 3.3.2. Since $\widehat{\mathfrak{l}}$ is no longer abelian, completion on the right and on the left differ. Take for example the element $\sum_{i\in\mathbb{N}} t^i\circ t^{-i}$ which belongs to $\overline{U}\mathfrak{l}$. It does not belong to $\overline{U\widehat{\mathfrak{l}}}$: an element on the completion on the right morally should have zeros of increasing order on the right side, but in this case, in order to "bring the element t^i on the right side", we should use the equality $t^i\circ t^{-i} = t^{-i}\circ t^i + \hbar i$ infinitely many times, which is not allowed.

3.3.1. The Virasoro algebra of $\mathcal{L}$. We use the residue morphism $\mathrm{Res}\colon \omega_{\mathcal{L}/S}\to\mathscr{O}_S$ to view $\theta_{\mathcal{L}/S}$ as an $\mathscr{O}_S$ submodule of $\overline{U\widehat{\mathfrak{l}}}$ and induce from this a central extension. Let $D\in\theta_{\mathcal{L}/S}$, and since $\omega_{\mathcal{L}/S}$ and $\theta_{\mathcal{L}/S}$ are $\mathcal{L}$ dual we identify D with the map

$$\phi_D\colon \omega_{\mathcal{L}/S}\times\omega_{\mathcal{L}/S}\to\mathscr{O}_S,\quad (\alpha,\beta)\mapsto\mathrm{Res}(D(\alpha)\beta).$$

Notice that since $\mathrm{Res}(D(\alpha)\beta) = \mathrm{Res}(D(\beta)\alpha)$, we have that ϕ_D belongs to $(\mathrm{Sym}^2(\omega_{\mathcal{L}/S}))^\vee$. Moreover, since the latter is canonically isomorphic to the closure of $\mathrm{Sym}^2(\mathfrak{l})$ in $\overline{U}\mathfrak{l}$ (see Remark 3.3.3), we will consider ϕ_D as an element of $\overline{U}\mathfrak{l}$. We define $C\colon \theta_{\mathcal{L}/S} \to \overline{U}\mathfrak{l}$ by setting $2C(D) = \phi_D$.

REMARK 3.3.3. The fact that $(\mathrm{Sym}^2(\omega_{\mathcal{L}/S}))^\vee$ is canonically isomorphic to the closure of $\mathrm{Sym}^2(\mathfrak{l})$ in $\overline{U}\mathfrak{l}$ is essentially a consequence of the fact that $\mathcal{L}$ is defined as a limit of finitely generated $\mathscr{O}_S$-modules $\mathcal{I}_\sigma^{-n}/\mathcal{I}_\sigma^{m+1}$. We assume, for simplicity, that $\mathcal{L} = \mathscr{O}_S((t))$. Write $\omega_{\mathcal{L}/S}$ as $\varinjlim_n \varprojlim_m \omega_m^n$ where ω_m^n is the free $\mathscr{O}_S$-module generated by $\{t^{-n}dt, \cdots, t^m dt\}$. The maps defining the projective system are truncation maps, while the ones for the inductive limit are inclusions. As the direct limit is given by unions, we deduce that $(\mathrm{Sym}^2(\omega_{\mathcal{L}/S}))^\vee$ is isomorphic to $\varprojlim_n \varinjlim_m \mathrm{Hom}_{\mathscr{O}_S}(\omega_m^n \circ \omega_m^n, \mathscr{O}_S)$. The residue pairing gives the isomorphism between $\mathrm{Hom}_{\mathscr{O}_S}(\omega_m^n, \mathscr{O}_S)$ and the sub $\mathscr{O}_S$-module of $\mathfrak{l}$ generated freely by $\{t^{-m-1}, \cdots, t^{n-1}\}$ which we denote by $\mathfrak{l}_{n-1}^{m+1}$. As these are free modules of finite dimension we get canonical identification with

$$\varprojlim_n \varinjlim_m \left(\mathfrak{l}_{n-1}^{m+1} \circ \mathfrak{l}_{n-1}^{m+1}\right) = \varprojlim_n \left(t^{n-1}\mathscr{O}_S[t^{-1}] \circ t^{n-1}\mathscr{O}_S[t^{-1}]\right).$$

As the product is symmetric, this is identified with $\varprojlim_n \left(\mathscr{O}_S((t)) \circ t^{n-1}\mathscr{O}_S[t^{-1}]\right)$. By decomposing $\mathscr{O}_S((t))$ as $t^{n-1}\mathscr{O}_S[t^{-1}] \oplus t^n\mathscr{O}_S[[t]]$ this module equals $\varprojlim_n \dfrac{\mathscr{O}_S((t)) \circ \mathscr{O}_S((t))}{\mathscr{O}_S((t)) \circ t^n\mathscr{O}_S[[t]]}$, which is the closure of $\mathrm{Sym}^2(\mathfrak{l})$ in $\overline{U}\mathfrak{l}$.

REMARK 3.3.4. Assume for simplicity that $R = k$ and identify $\mathcal{L}$ with $k((t))$. For every $i \in \mathbb{Z}$ we set $\alpha_i = t^{-i-1}dt$ and $a_i = t^i$ so that $\mathrm{Res}(a_i\alpha_j) = \delta_{ij}$ and $\{\alpha_i\}$ and $\{a_i\}$ are linearly independent generators of $\omega_{\mathcal{L}/S}$ and $\mathcal{L}$. Then we can write explicitly

$$C(D) = \frac{1}{2}\sum_{i\in\mathbb{Z}} D(t^{-i-1}dt) \circ t^i.$$

In general, let $\{\alpha_i\}$ and $\{a_i\}$ be linearly independent generators of $\omega_{\mathcal{L}/S}$ and $\mathcal{L}$ with the property that $\mathrm{Res}(a_i\alpha_j) = \delta_{ij}$. Then we can write

$$C(D) = \frac{1}{2}\sum_{i\in\mathbb{Z}} D(\alpha_i) \circ a_i$$

which is a well defined object of $\overline{U}\mathfrak{l}$ thanks to the previous remark.

As explained in [**Loo13**, Section 2], the central extension $\widehat{\mathfrak{l}}$ of $\mathfrak{l}$ and the inclusion $C\colon \theta_{\mathcal{L}/S} \to \overline{U\widehat{\mathfrak{l}}}$ induce a central extension $\widehat{\theta}_{\mathcal{L}/S}$ of $\theta_{\mathcal{L}/S}$. We recall here how this is achieved. Consider now $\mathfrak{l} \otimes \mathfrak{l}$, and call $\mathfrak{l}_2$ its image in $U\widehat{\mathfrak{l}}$. This means that $\mathfrak{l}_2 = \mathfrak{l} \otimes \mathfrak{l} \oplus \hbar\mathscr{O}_S$ modulo the relation $f \otimes g = g \otimes f + \hbar\mathrm{Res}(gdf)$. Denote by $\overline{\mathfrak{l}_2}$ its closure in $\overline{U\widehat{\mathfrak{l}}}$ and observe the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hbar\mathscr{O}_S & \longrightarrow & \overline{\mathfrak{l}_2} & \xrightarrow{[-]_\hbar} & \overline{\mathrm{Sym}^2(\mathfrak{l})} & \longrightarrow & 0 \\ & & & & & & \big\uparrow{\scriptstyle C} & & \\ & & & & & & \theta_{\mathcal{L}/S} & & \end{array}$$

where $[-]_\hbar$ is the reduction modulo the central element $\hbar\mathscr{O}_S$ so that the short sequence is exact.

DEFINITION 3.3.5. We define $\widehat{\theta}_{\mathcal{L}/S}$ to be the pullback of $\theta_{\mathcal{L}/S}$ along $[-]_\hbar$. Equivalently its elements are pairs $(D,u) \in \theta_{\mathcal{L}/S} \times \overline{\mathfrak{l}_2}$ such that $C(D) = u \mod \hbar\mathscr{O}_S$.

Denote by $\widehat{C}\colon \widehat{\theta}_{\mathcal{L}/S} \to \overline{U}\widehat{\mathfrak{l}}$ the injection $\widehat{C}(D,u) = u$ and we write $[-]^\theta_\hbar$ for the pullback of $[-]_\hbar$ along C so that we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hbar\mathscr{O}_S & \longrightarrow & \overline{\mathfrak{l}_2} & \xrightarrow{[-]_\hbar} & \overline{\mathrm{Sym}^2(\mathfrak{l})} & \longrightarrow & 0 \\ & & \Vert & & \uparrow \widehat{C} & & \uparrow C & & \\ 0 & \longrightarrow & \hbar\mathscr{O}_S & \longrightarrow & \widehat{\theta}_{\mathcal{L}/S} & \xrightarrow{[-]^\theta_\hbar} & \theta_{\mathcal{L}/S} & \longrightarrow & 0. \end{array}$$

Observe, for example using Remark 3.3.4, that the map C is not a Lie algebra morphism, and so $\widehat{\theta}_{\mathcal{L}/S}$ does not arise naturally as a Lie algebra which centrally extends $\theta_{\mathcal{L}/S}$.

We however want to induce a Lie bracket on $\widehat{\theta}_{\mathcal{L}/S}$ from the one of $\overline{U}\widehat{\mathfrak{l}}$ by conveniently modifying $\widehat{C}$. To understand how to do this, local computations are carried out.

DEFINITION 3.3.6. Choose a local parameter t so that locally $\mathcal{L} \cong \mathscr{O}_S((t))$. Define the *normal ordering* $: \star :: \overline{\mathrm{Sym}^2(\mathfrak{l})} \to \overline{\mathfrak{l}_2}$ by setting

$$: t^n \otimes t^m := \begin{cases} t^n \otimes t^m & n \leq m \\ t^m \otimes t^n & n \geq m \end{cases}$$

and extend it by linearity to every element of $\overline{\mathrm{Sym}^2(\mathfrak{l})}$.

The map $: \star :$ defines a section of $[-]_\hbar$, so that $(\mathrm{Id}, : \star : C)$ is a section of $[-]^\theta_\hbar$. Once we make the choice of a local parameter defining the ordering $: \star :$, we will denote by $\widehat{D}$ the element $(D, : C(D) :) \in \widehat{\theta}_{\mathcal{L}/S}$. Consider the following relations which hold in $\overline{U}\widehat{\mathfrak{l}}$ and which are proved in [**Loo13**, Lemma 2.1].

LEMMA 3.3.7. *Let $D \in \theta_{\mathcal{L}/S}$ and $D_i = t^{i+1}d/dt \in \theta_{\mathcal{L}/S}$, Then we have*
(1) $[\widehat{C}(\widehat{D}), f] = -\hbar D(f)$ *for every* $f \in \mathfrak{l} \subset \widehat{\mathfrak{l}}$;
(2) $[\widehat{C}(\widehat{D}_k), \widehat{C}(\widehat{D}_l)] = -\hbar(l-k)\widehat{C}(\widehat{D}_{k+l}) + \dfrac{k^3-k}{12}\hbar^2\delta_{k,-l}.$ □

This suggests to rescale the morphism $\widehat{C}$ and to define

$$T := -\frac{\widehat{C}}{\hbar}\colon \widehat{\theta}_{\mathcal{L}/S} \to \overline{U}\widehat{\mathfrak{l}}\left[\frac{1}{\hbar}\right]$$

which is injective and its image is a Lie subalgebra of the target. Denote by c_0 the element $(0, -\hbar)$ which is sent to 1 by T. By construction we obtain the following result.

PROPOSITION 3.3.8. [**Loo13**, Corollary-Definition 2.2] *The Lie algebra structure induced on $\widehat{\theta}_{\mathcal{L}/S}$ by T is a central extension of the canonical Lie algebra structure on $\theta_{\mathcal{L}/S}$ by $c_0\mathscr{O}_S$. This is called the* Virasoro algebra *of $\mathcal{L}$.*

3.4. Sugawara construction

In this section we generalize to our case, i.e. using $\widehat{\mathfrak{h}}_{\mathcal{L}}$ in place of $\widehat{\mathfrak{g}}\mathcal{L}$, the construction of

$$T_{\mathfrak{g}}\colon \widehat{\theta}_{\mathcal{L}/S} \to \left(\overline{U}\widehat{\mathfrak{g}}\mathcal{L}[(c+\check{h})^{-1}]\right)^{\mathrm{Aut}(\mathfrak{g})}$$

described by Looijenga in [**Loo13**, Corollary 3.2], which essentially represents the local picture of our situation. In the classical case the idea is to use the Casimir element of $\mathfrak{g}$ to induce, from $\widehat{C}$ the map $\widehat{C}_{\mathfrak{g}}\colon \widehat{\theta}_{\mathcal{L}/S} \to \overline{U}\widehat{\mathfrak{g}}\mathcal{L}$ which, in turn, will give the map of Lie algebras $T_{\mathfrak{g}}$. When in place of $\mathfrak{g}\mathcal{L}$ we have $\mathfrak{h}_{\mathcal{L}}$, we can run the same argument using the element Casimir $\mathfrak{c}$ of $\mathfrak{h}_{\mathcal{L}}$. This is the content of this section.

As in Section 2.1, we consider the normalized Killing form defined on $\mathfrak{g}$. Recall that it provides an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^{\vee}$, hence it gives an identification of $\mathfrak{g}\otimes\mathfrak{g}$ with $\mathrm{End}_k(\mathfrak{g}) = \mathfrak{g}\otimes\mathfrak{g}^{\vee}$. Moreover, as $\sigma(S)$ is disjoint from the ramification locus, we also have that $(\,|\,)_{\mathfrak{h}_{\mathcal{L}}}$ provides an isomorphism of $\mathfrak{h}_{\mathcal{L}}$ with ${\mathfrak{h}_{\mathcal{L}}}^{\vee}$, giving in this way an identification of $\mathfrak{h}_{\mathcal{L}}\otimes_{\mathcal{L}}\mathfrak{h}_{\mathcal{L}}$ with $\mathrm{End}_{\mathcal{L}}(\mathfrak{h}_{\mathcal{L}})$. The *Casimir element* of $\mathfrak{h}_{\mathcal{L}}$ with respect to the form $(\,|\,)_{\mathfrak{h}_{\mathcal{L}}}$ is the element in $\mathfrak{h}_{\mathcal{L}}\otimes_{\mathcal{L}}\mathfrak{h}_{\mathcal{L}}$ corresponding to the identity $\mathrm{Id}_{\mathfrak{h}_{\mathcal{L}}}$ via the identification provided by $(\,|\,)_{\mathfrak{h}_{\mathcal{L}}}$. We denote it by $\mathfrak{c}$.

REMARK 3.4.1. We could have defined the Casimir element of $\mathfrak{h}_{\mathcal{L}}$ via the local isomorphism of $\mathfrak{h}_{\mathcal{L}}$ with $\mathfrak{g}\mathcal{L}$. Let $\mathfrak{c}(\mathfrak{g})$ be the Casimir element of $\mathfrak{g}$, and observe that via the inclusion $\mathfrak{g}\to\mathfrak{g}\mathcal{L}$, we can see it as an element of $\mathfrak{g}\mathcal{L}\otimes_{\mathcal{L}}\mathfrak{g}\mathcal{L}$. Since $\mathfrak{c}(\mathfrak{g})$ is invariant under automorphisms, it is invariant under Γ, hence it gives an element $\mathfrak{h}_{\mathcal{L}}\otimes_{\mathcal{L}}\mathfrak{h}_{\mathcal{L}}$ which equals $\mathfrak{c}$.

Since $(\,|\,)_{\mathfrak{h}_{\mathcal{L}}}$ is a symmetric form, we have that also $\mathfrak{c}$ is a symmetric element of $\mathfrak{h}_{\mathcal{L}}\otimes_{\mathcal{L}}\mathfrak{h}_{\mathcal{L}}$ and moreover $\mathfrak{c}$ lies in the centre of $U_{\mathcal{L}}(\mathfrak{h}_{\mathcal{L}})$. As $\mathfrak{h}_{\mathcal{L}}$ is simple over $\mathcal{L}$, this implies that there exists $\check{h}\in k$ such that $\mathrm{ad}(\mathfrak{c})X = 2\check{\mathfrak{h}}X$ for all $X\in\mathfrak{h}_{\mathcal{L}}$, where $\mathrm{ad}(-)$ denotes the adjoint representation of $\mathfrak{h}_{\mathcal{L}}$.

REMARK 3.4.2. Locally, for every bases $\{X_i\}_{i=1}^{\dim(\mathfrak{g})}$ and $\{Y_i\}_{i=1}^{\dim(\mathfrak{g})}$ of $\mathfrak{h}_{\mathcal{L}}$ such that $(X_i|Y_j)_{\mathfrak{h}_{\mathcal{L}}} = \delta_{ij}$ we have the explicit description of $\mathfrak{c}$ as $\sum_{i=1}^{\dim(\mathfrak{g})} X_i\circ Y_i$. It follows that $\check{h}$ is given by the equality $\sum_{i=1}^{\dim(\mathfrak{g})}[X_i,[Y_i,Z]] = 2\check{h}Z$ for every $Z\in\mathfrak{h}_{\mathcal{L}}$.

Let denote by $\overline{U}\widehat{\mathfrak{h}}_{\mathcal{L}}$ the completion on the right of $U\widehat{\mathfrak{h}}_{\mathcal{L}}$ with respect to the filtration $F^*\widehat{\mathfrak{h}}_{\mathcal{L}}$ given by $F^n\mathfrak{h}_{\mathcal{L}}$ for $n\geq 1$. We now construct $\widehat{\gamma}_{\mathfrak{c}}\colon \bar{\mathfrak{l}}\to\overline{U}\widehat{\mathfrak{h}}_{\mathcal{L}}$ which composed with $\widehat{C}$ will give $\widehat{C}_{\mathfrak{h}}\colon\widehat{\theta}_{\mathcal{L}/S}\to\overline{U}\widehat{\mathfrak{h}}_{\mathcal{L}}$.

Let consider the map $\gamma_{\mathfrak{c}}\colon \mathfrak{l}\otimes_{\mathscr{O}_S}\mathfrak{l}\to\mathfrak{h}_{\mathcal{L}}\otimes_{\mathscr{O}_S}\mathfrak{h}_{\mathcal{L}}\subset U\widehat{\mathfrak{h}}_{\mathcal{L}}$ given by tensoring with $\mathfrak{c}$. This map uniquely extends to a map of Lie algebras $\mathfrak{l}_2\to U\widehat{\mathfrak{h}}_{\mathcal{L}}$ as follows. Using local bases as in Remark 3.4.2 and the symmetry of $\mathfrak{c}$ we deduce the following equality

$$\gamma_{\mathfrak{c}}\left(\hbar\mathrm{Res}(gdf)\right) = \gamma_{\mathfrak{c}}(f\circ g - g\circ f) = c\dim(\mathfrak{g})\mathrm{Res}(gdf) + c\sum_{i=1}^{\dim(\mathfrak{g})}\mathrm{Res}(fg(dY_i|X_i))$$

and recalling that Remark 2.1.3 implies that $(dX_i|Y_i) = -(dY_i|X_i)$, we conclude that

$$\gamma_{\mathfrak{c}}\left(\hbar\mathrm{Res}(gdf)\right) = c\dim(\mathfrak{g})\mathrm{Res}(gdf).$$

We then define $\widehat{\gamma}_{\mathfrak{c}}\colon\mathfrak{l}_2\to U\widehat{\mathfrak{h}}_{\mathcal{L}}$ by sending $\hbar$ to $c\dim(\mathfrak{g})$ and acting as $\gamma_{\mathfrak{c}}$ on $\mathfrak{l}\otimes\mathfrak{l}$. Such a map can be extended to the closure of $\mathfrak{l}_2$ in $\overline{U}\widehat{\mathfrak{l}}$ once we extend the target to $\overline{U}\widehat{\mathfrak{h}}_{\mathcal{L}}$, obtaining $\widehat{\gamma}_{\mathfrak{c}}\colon\overline{\mathfrak{l}_2}\to U\widehat{\mathfrak{h}}_{\mathcal{L}}$. We define $\widehat{C}_{\mathfrak{h}}$ as the composition

$$\widehat{\gamma}_{\mathfrak{c}}\widehat{C}:\widehat{\theta}_{\mathcal{L}/S}\to\overline{U}\widehat{\mathfrak{h}}_{\mathcal{L}}.$$

As for $\widehat{C}$, also in this case the morphism $\widehat{C}_{\mathfrak{h}}$ does not preserve the Lie bracket, and thanks to local computation we understand how to solve this issue. Following [**TUY89**] we first of all extend the normal ordering defined in 3.3.6 as as follows.

DEFINITION 3.4.3. Let fix an isomorphism between $\mathcal{L}$ and $R((t))$ for a local parameter $t \in \mathcal{I}_\sigma$. Let Xt^n and Yt^m be elements of $\mathfrak{g}\mathcal{L} = \mathfrak{g} \otimes R((t))$. Then we set

$$\,^\circ_\circ\, Xt^n \otimes Yt^m \,^\circ_\circ = \begin{cases} Xt^n \otimes Yt^m & n < m \\ \frac{1}{2}(Xt^n \otimes Yt^m + Yt^m \otimes Xt^n) & n = m \\ Yt^m \otimes Xt^n & n > m. \end{cases}$$

This definition is Γ-equivariant, hence defines a normal ordering on $\mathfrak{h}_\mathcal{L} \otimes \mathfrak{h}_\mathcal{L}$.

This defines a section from the image of γ_c to the image of $\widehat{\gamma}_c$ which makes the diagram to commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c\mathscr{O}_S & \longrightarrow & \operatorname{Im}(\widehat{\gamma}_c) & \xrightarrow{[-]_c} & \operatorname{Im}(\gamma_c) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \widehat{\gamma}_c} & {\scriptstyle \,^\circ_\circ\star^\circ_\circ} & \uparrow{\scriptstyle \gamma_c} & & \\ 0 & \longrightarrow & \hbar\mathscr{O}_S & \longrightarrow & \overline{\mathfrak{l}_2} & \xrightarrow{[-]_\hbar} & \overline{\operatorname{Sym}^2(\mathfrak{l})} & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle \widehat{C}} & {\scriptstyle :\star:} & \uparrow{\scriptstyle C} & & \\ & & & & \widehat{\theta}_{\mathcal{L}/S} & \longrightarrow & \theta_{\mathcal{L}/S}. & & \end{array}$$

For any $D \in \theta_{\mathcal{L}/S}$, we write $\widehat{C}_\mathfrak{h}(\widehat{D})$ to denote the element $\,^\circ_\circ\, \gamma_\mathfrak{c} C(D) \,^\circ_\circ = \widehat{\gamma}_\mathfrak{c} : C(D) :$.

REMARK 3.4.4. As we have done in Remark 3.3.4 we can write locally the element $\widehat{C}_\mathfrak{h}(\widehat{D})$ in a more explicit way. Consider the morphism $1 \otimes (\,|\,)\colon \omega_{\mathcal{L}/S} \otimes \mathfrak{h}_\mathcal{L} \to \omega_{\mathcal{L}/S} \otimes {\mathfrak{h}_\mathcal{L}}^\vee$ and, after tensoring it with $\mathfrak{h}_\mathcal{L}$, compose it with $\operatorname{Res}_\mathfrak{h}$ to obtain the pairing $\operatorname{Res}_{(\,|\,)}\colon \omega_{\mathcal{L}/S} \otimes_\mathcal{L} \mathfrak{h}_\mathcal{L} \times \mathfrak{h}_\mathcal{L} \to \mathscr{O}_S$. Let $\{A_i\}$ and $\{B_i\}$ be orthonormal bases of $\omega_{\mathcal{L}/S} \otimes \mathfrak{h}_\mathcal{L}$ and $\mathfrak{h}_\mathcal{L}$ with respect to $\operatorname{Res}_{(\,|\,)}$. Then for every $D \in \theta_{\mathcal{L}/S}$ we have

$$\widehat{C}_\mathfrak{h}(\widehat{D}) = \frac{1}{2} \sum \,^\circ_\circ D(A_i) \circ B_i \,^\circ_\circ$$

where we see D as a linear map $\omega_{\mathcal{L}/S} \to \mathcal{L}$, so that $D(A_i) \in \mathfrak{h}_\mathcal{L}$.

As in [**Loo13**, Lemma 3.1] we have the following result.

LEMMA 3.4.5. *The following equalities hold true in* $\overline{U}\widehat{\mathfrak{h}}_\mathcal{L}$*:*

(1) $[\widehat{C}_\mathfrak{h}(\widehat{D}), X] = -(c + \check{h})D(X)$ *for all* $X \in \mathfrak{h}_\mathcal{L}$ *and* $D \in \theta_{\mathcal{L}/S}$*;*

(2) $[\widehat{C}_\mathfrak{h}(\widehat{D}_k), \widehat{C}_\mathfrak{h}(\widehat{D}_l)] = (c + \check{h})(k - l)\widehat{C}_\mathfrak{h}(\widehat{D}_{k+l}) + c\dim(\mathfrak{g})(c + \check{h})\frac{k^3 - k}{12}\delta_{k,-l}$ *where* $D_i = t^{i+1}d/dt$.

As Lemma 3.3.7 also Lemma 3.4.5 suggests to rescale $\widehat{C}_\mathfrak{h}$ and consider instead the map

$$T_\mathfrak{h} := -\frac{\widehat{C}_\mathfrak{h}}{c + \check{h}} \colon \widehat{\theta}_{\mathcal{L}/S} \to \overline{U}\widehat{\mathfrak{h}}_\mathcal{L}\left[\frac{1}{c + \check{h}}\right]$$

which is compatible with the Lie brackets of $\widehat{\theta}_{\mathcal{L}/S}$ and $\overline{U}\widehat{\mathfrak{h}}_\mathcal{L}[(c + \check{h})^{-1}]$, proving the following statement.

PROPOSITION 3.4.6. *The map* $T_\mathfrak{h}$ *is a homomorphism of Lie algebras which sends the central element* $c_0 = (0, -\hbar)$ *to* $(c\dim(\mathfrak{g}))/(c + \check{h})$*. We call* $T_\mathfrak{h}$ *the* Sugawara representation *of* $\widehat{\theta}_{\mathcal{L}/S}$.

3.4.1. Fock representation. We induce the representation $T_{\mathfrak{h}}$ to the quotient $\mathcal{F}^+(\mathfrak{h}_{\mathcal{L}})$ of $\overline{U\widehat{\mathfrak{h}}_{\mathcal{L}}}$ defined as

$$\mathcal{F}^+(\mathfrak{h}_{\mathcal{L}}) := \left(U\widehat{\mathfrak{h}}_{\mathcal{L}} / U\widehat{\mathfrak{h}}_{\mathcal{L}} \circ F^1\widehat{\mathfrak{h}}_{\mathcal{L}}\right)\left[\frac{1}{c+\check{h}}\right] = \left(\overline{U\widehat{\mathfrak{h}}_{\mathcal{L}}} / \overline{U\widehat{\mathfrak{h}}_{\mathcal{L}}} \circ F^1\widehat{\mathfrak{h}}_{\mathcal{L}}\right)\left[\frac{1}{c+\check{h}}\right]$$

By abuse of notation call $T_{\mathfrak{h}}$ the composition of $T_{\mathfrak{h}}$ with the projection of $\overline{U\widehat{\mathfrak{h}}_{\mathcal{L}}}$ to $\mathcal{F}^+(\mathfrak{h}_{\mathcal{L}})$, so that $\mathcal{F}^+(\mathfrak{h}_{\mathcal{L}})$ is a representation of $\widehat{\theta}_{\mathcal{L}/S}$. We can depict the result as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_S \cdot \mathrm{Id} & \longrightarrow & \mathrm{End}(\mathcal{F}^+(\mathfrak{h}_{\mathcal{L}})) & \longrightarrow & \mathbb{P}\mathrm{End}(\mathcal{F}^+(\mathfrak{h}_{\mathcal{L}})) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle T_{\mathfrak{h}}} & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S c_0 & \longrightarrow & \widehat{\theta}_{\mathcal{L}/S} & \longrightarrow & \theta_{\mathcal{L}/S} & \longrightarrow & 0
\end{array}$$

where the first vertical arrow maps c_0 to $c\dim(\mathfrak{g})(\check{h}+c)^{-1}\cdot \mathrm{Id}$ and by abuse of notation we wrote $T_{\mathfrak{h}}$ instead of $T_{\mathfrak{h}}(-)\circ$.

REMARK 3.4.7. We give a local description of how the action looks like. Choose for this purpose a local parameter t of $\mathcal{I}_\sigma$ so that we can associate to $D \in \theta_{\mathcal{L}/S}$ the element $\widehat{D} \in \widehat{\theta}_{\mathcal{L}/S}$. Let $X_r \circ \cdots \circ X_1$ be representatives of an element of $\mathcal{F}^+(\mathfrak{h}_{\mathcal{L}})$ whith $X_i \in \mathfrak{h}_{\mathcal{L}}$. Then the action of $\theta_{\mathcal{L}/S}$ is described as follows

$$T_{\mathfrak{h}}(\widehat{D}) \circ X_r \circ \cdots \circ X_1 = \sum_{i=1}^{r} X_r \circ \cdots \circ D(X_i) \circ \cdots \circ X_1 + X_r \circ \cdots \circ X_1 \circ T_{\mathfrak{h}}(\widehat{D})$$

where $D(X_i)$ denotes the image of X_i under coefficientwise derivation by D (Remark 3.2.3).

3.4.2. Projective representation of $\theta_{\mathcal{L},S}$. We want to define in this section a map of Lie algebras $\mathbb{P}T_{\mathfrak{h},S}\colon \theta_{L,S} \to \mathbb{P}\mathrm{End}(\mathcal{H}_\ell(\mathcal{V}))$ which is induced by $T_{\mathfrak{h}}$ and which will lead, in a second time, to a projective connection on the sheaves of conformal blocks. The construction of $\mathbb{P}T_{\mathfrak{h},S}$ in the classical case is the content of [**Loo13**, Corollary 3.3].

Let $F^0\theta_{\mathcal{L}/S}$ be the subsheaf of $\theta_{\mathcal{L}/S}$ given by those derivations D such that $D(F^1\mathcal{L}) \in F^1\mathcal{L}$, and similarly we set $F^0\theta_{\mathcal{L},S}$ to be the subsheaf of $\theta_{\mathcal{L},S}$ whose elements D satisfy $D(F^1\mathcal{L}) \in F^1\mathcal{L}$.

REMARK 3.4.8. Assume that $\mathcal{L} = \mathcal{O}_S((t))$ so that every element of $F^0\theta_{\mathcal{L}/S}$ is written as $D = \sum_{i\geq 0} a_i t^i d/dt$. The element $T_{\mathfrak{h}}(\widehat{D})$ acts on $\mathcal{V}$ as the operator $\dfrac{a_0}{-2(\ell+\check{h})}\mathfrak{c}$ where $\mathfrak{c}$ is the Casimir element of $\mathfrak{h}$, hence the action is by scalar multiplication. Combining this with Remark 3.4.7, we obtain that $F^0(\theta_{\mathcal{L}/S})$ acts on $\mathcal{H}_\ell(\mathcal{V})$ by coefficientwise derivation up to scalars.

As in the classical case, also in our context this observation is the key input to define $\mathbb{P}T_{\mathfrak{h},S}$. In fact we let $F^0\theta_{\mathcal{L},S}$ act on $\mathcal{H}_\ell(\mathcal{V})$ by coefficientwise derivation so that we obtain a map

$$F^0\theta_{\mathcal{L},S} \times \theta_{L/S} \to \mathbb{P}\mathrm{End}(\mathcal{H}_\ell(\mathcal{V}))$$

which uniquely defines the Lie algebra homomorphism

$$\mathbb{P}T_{\mathfrak{h},S}\colon \theta_{\mathcal{L},S} \to \mathbb{P}\mathrm{End}(\mathcal{H}_\ell(\mathcal{V}))$$

and hence the central extension $\widehat{\theta}_{\mathcal{L},S} \to \theta_{\mathcal{L},S}$ and the map $T_{\mathfrak{h},S}\colon \widehat{\theta}_{\mathcal{L},S} \to \mathrm{End}(\mathcal{H}_\ell(\mathcal{V}))$.

PROOF. We only have to prove that the Lie algebra generated by $F^0\theta_{\mathcal{L},S}$ and $\theta_{L/S}$ is $\theta_{\mathcal{L},S}$. This can be checked locally, where the choice of a local parameter t allows us to split the exact sequence

$$0 \to \theta_{\mathcal{L}/S} \to \theta_{\mathcal{L},S} \to \mathcal{T}_{S/k} \to 0,$$

hence to write $\theta_{\mathcal{L},S}$ as $\theta_{\mathcal{L}/S} \oplus \mathcal{T}_{S/k}$. We can in fact decompose every element $D \in \theta_{\mathcal{L},S}$ as $D_{ver} \oplus D_{hor}$ which are uniquely determined by the conditions

$$(\star) \quad D_{ver} \in \theta_{\mathcal{L}/S}, D_{hor}(t) = 0 \quad \text{and} \quad D_{hor}(s) = D(s) \text{ for all } s \in \mathscr{O}_S.$$

This implies that $F^0\theta_{\mathcal{L},S} = F^0\theta_{\mathcal{L}/S} \oplus \mathcal{T}_{S/k}$, concluding the argument. □

REMARK 3.4.9. Assume that $\mathcal{L} = R((t))$ so that we can write every element $D \in \theta_{\mathcal{L},S}$ as $D = D_{ver} + D_{hor}$ satisfying $(\star)$. Then Remark 3.4.7 tells us that the action of D on $\mathcal{H}_\ell(\mathcal{V})$ is given by componentwise derivation by D plus right multiplication by $T(\widehat{D}_{ver})$.

REMARK 3.4.10. We want to remark that in the case in which $\mathcal{V}$ is the trivial representation, then the central extension $\widehat{\theta}_{\mathcal{L},S}$ is isomorphic to $\widehat{\theta}_{\mathcal{L}/S} \oplus \mathcal{T}_{S/k}$, viewed as a Lie subalgebra of $\mathfrak{gl}(\mathcal{H}_\ell(0))$, where the action of $\mathcal{T}_{S/k}$ is by coefficientwise derivation. In fact in the previous proof we saw that locally on S, and up to the choice of a local parameter this is the case. By looking at Remark 3.4.8 and the previous proof, we note that the obstruction to deduce this statement globally lies in the action of the Casimir element $\mathfrak{c}$ on $\mathcal{V}$. When $\mathcal{V}$ is the trivial representation $\mathfrak{c}$ acts as multiplication by zero, hence there is no obstruction. In particular, the central charge $c_0 = (0, -\hbar) \in \widehat{\theta}_{\mathcal{L},S}$ acts by multiplication by $\dim(\mathfrak{g})\ell/(\ell + \check{h})$.

3.5. The projective connection on $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$

The aim of this section is to induce, from $\mathbb{P}T_{\mathfrak{h},S}$, the projectively flat connection $\nabla\colon \mathcal{T}_{S/k}(-\log(\Delta)) \to \mathbb{P}\mathrm{End}(\mathcal{H}_\ell(\mathcal{V})_X)$. In Proposition 3.2.5 we realised $\mathcal{T}_{S/k}(-\log(\Delta))$ as the quotient $\theta_{\mathcal{A},S}(-\mathcal{R})/\theta_{\mathcal{A}/S}(-\mathcal{R})$, so that the content of this section can be summarized in the following statement.

THEOREM 3.5.1. *The actions of $\theta_{\mathcal{A},S}$ and of $\theta_{\mathcal{A}/S}(-\mathcal{R})$ on $\mathcal{H}_\ell(\mathcal{V})$ induce a projective action of $\mathcal{T}_S(-\log(\Delta))$ on $\mathcal{H}_\ell(0)_X$. In particular $\mathcal{H}_\ell(\mathcal{V})_X$ is locally free if restricted to $S \setminus \Delta$.*

As a consequence of it, we obtain that $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is locally free on $\mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,1}$.

COROLLARY 3.5.2. *The sheaf $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ on $\overline{\mathcal{H}ur}(\Gamma, \xi)_{g,1}$ is equipped with a projective connection with logarithmic singularities along the boundary Δ_{univ}. In particular $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is locally free on $\mathcal{H}ur(\Gamma, \xi)_{g,1}$.*

PROOF. As pointed out in Subsection 3.1 the tangent space of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}$ at a versal covering $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \sigma)$ is identified via the Kodaira-Spencer map with the tangent bundle $\mathcal{T}_{S/k}(-\log(\Delta))$. The previous theorem gives the projective action of the latter on $\mathcal{H}_\ell(\mathcal{V})_X$, concluding in this way the argument. □

Proof of Theorem 3.5.1. We first of all prove that the action of $\theta_{\mathcal{A},S}(-\mathcal{R})$ on $\mathcal{H}_\ell(\mathcal{V})$ descends to $\mathcal{H}_\ell(\mathcal{V})_X$.

PROPOSITION 3.5.3. *The projective action of $\theta_{\mathcal{A},S}(-\mathcal{R})$ on $\mathcal{H}_\ell(\mathcal{V})$ preserves the space $\mathfrak{h}_{\mathcal{A}} \circ \mathcal{H}_\ell(\mathcal{V})$, hence induces a projective action on $\mathcal{H}_\ell(\mathcal{V})_X$.*

PROOF. By the local description of the action of $\theta_{\mathcal{A},S}(-\mathcal{R}) \subset \theta_{\mathcal{L},S}$ explained in Remark 3.4.9, it suffices to show that the action of $\theta_{\mathcal{A},S}(-\mathcal{R})$ on $\mathfrak{h}_{\mathcal{A}}$ by coefficientwise derivation is well defined. This follows from Remark 3.2.3. □

We denote by $\mathbb{P}T_{\mathfrak{h}_{\mathcal{A}},S}$ the morphism $\theta_{\mathcal{A},S}(-\mathcal{R}) \to \mathcal{H}_\ell(\mathcal{V})_X$ induced by $\mathbb{P}T_{\mathfrak{h},S}$. To conclude the proof of Theorem 3.5.1 we are left to show the following proposition.

PROPOSITION 3.5.4. *The morphism* $\mathbb{P}T_{\mathfrak{h}_{\mathcal{A}},S}\colon \theta_{\mathcal{A},S}(-\mathcal{R}) \to \mathbb{P}End(\mathcal{H}_\ell(\mathcal{V})_X)$ *factorizes through*

$$\mathbb{P}T_{\mathfrak{h}_{\mathcal{A}},S}\colon \mathcal{T}_S(-\log(\Delta)) = \theta_{\mathcal{A},S}(-\mathcal{R})/\theta_{\mathcal{A}/S}(-\mathcal{R}) \to \mathbb{P}End(\mathcal{H}_\ell(\mathcal{V})_X).$$

PROOF. We need to prove that $\theta_{\mathcal{A}/S}(-\mathcal{R})$ acts on $\mathcal{H}_\ell(\mathcal{V})_X$ by scalar multiplication. As this can be checked locally, we can assume to have a local parameter, so that we can associate to $D \in \theta_{\mathcal{A}/S}$ the element $\widehat{D} \in \widehat{\theta}_{\mathcal{L}/S}$. We need to prove that, up to scalars, $T_{\mathfrak{h}}(\widehat{D})$ lies in the closure of $\mathfrak{h}_{\mathcal{A}} \circ \mathfrak{h}_{\mathcal{L}}$ in $\overline{U\widehat{\mathfrak{h}}_{\mathcal{L}}}[(c+\check{h})^{-1}]$.

For this purpose we use the description of $\widehat{C}_{\mathfrak{h}}(\widehat{D})$ provided in Remark 3.4.4. Let consider the orthonormal bases with respect to $\mathrm{Res}_{(\,|\,)}$ given by elements $\{\alpha_i, \beta_j\}$ and $\{a_i, b_j\}$ of $\omega_{\mathcal{L}/S} \otimes \mathfrak{h}_{\mathcal{L}}$ and $\mathfrak{h}_{\mathcal{L}}$ and with $a_i \in \mathfrak{h}_{\mathcal{A}}$. From Remark 3.4.4 we can write

$$T_{\mathfrak{h}}(\widehat{D}) = \sum {}^\circ_\circ D(\alpha_i) \circ a_i {}^\circ_\circ + \sum {}^\circ_\circ D(\beta_j) \circ b_j {}^\circ_\circ.$$

Observe that up to an element in $c\mathcal{O}_S$ we have the equality $\sum {}^\circ_\circ D(\alpha_i) \circ a_i {}^\circ_\circ = \sum a_i \circ D(\alpha_i)$, so that to conclude it is enough to show that $D(\beta_j) \in \mathfrak{h}_{\mathcal{A}}$. To do this, we first need to identify where β_j's live. Since the basis is $\mathrm{Res}_{(\,|\,)}$-orthonormal we know that $(1 \otimes (\,|\,)_{\mathfrak{h}_{\mathcal{L}}})(\beta_j) \in \omega_{\mathcal{A}} \otimes \mathfrak{h}_{\mathcal{A}}{}^\vee$. Recall that in Lemma 1.1.6, we decomposed $\mathfrak{h}$ as $\oplus g^{\zeta^{-i}} \otimes \mathscr{E}_i$. Using this decomposition, and the fact that $(\,|\,)_{\mathfrak{h}}$ provides an isomorphism between $\mathfrak{g}^{\zeta^{-i}} \otimes_k \mathscr{E}_i$ and $(\mathfrak{g}^{\zeta^i} \otimes_k \mathscr{E}_{p-i}(\mathcal{R}))^\vee$ for $i \neq 0$, we deduce that

$$\beta_j \in \left(\mathfrak{g}^\Gamma \pi_* \mathscr{O}_{X^*} \oplus \bigoplus_{i=1}^{p-1} \left(\mathfrak{g}^{\zeta^i} \otimes_k \pi_* \mathscr{E}_{p-i}(\mathcal{R})|_{X^*} \right) \right) \otimes \omega_{\mathcal{A}}$$

It follows that

$$D(\beta_j) \in \mathfrak{g}^\Gamma \pi_* \mathscr{O}_{X^*}(-\mathcal{R}) \oplus \bigoplus_{i=1}^{p-1} (\mathfrak{g}^{\zeta^i} \otimes_k \pi_* \mathscr{E}_{p-i}|_{X^*})$$

and hence is contained in $\mathfrak{h}_{\mathcal{A}}$. □

3.6. The semi local case

We extend the notions introduced so far to the stack $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$ with $n \geq 1$. In fact, as in the classical case one needs to work with curves with many marked points, also in our context we will need to fix more sections of the covered curve. This is explained in the classical context in the last paragraphs of [**Loo13**, Section 3].

Let $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \sigma_1, \ldots, \sigma_n)$ be an $S = \mathrm{Spec}(R)$ point of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$. For all $i \in \{1, \ldots, n\}$ we denote by S_i the divisor of X defined by σ_i and by $\mathcal{I}_i$ its ideal of definition. We denote by X^* the open complement of $S_1 \cup \cdots \cup S_n$ in X and we denote by $\mathfrak{h}_{\mathcal{A}}$ the pushforward to S of $\mathfrak{h}$ restricted to X^*, in other words $\mathfrak{h}_{\mathcal{A}} := \pi_*(\mathfrak{h} \otimes_{\mathscr{O}_X} \mathscr{O}_{X^*})$. As in the case $n = 1$, we assume that $X^* \to S$ is affine.

In the same way as we defined $\widehat{\mathcal{O}}$ in the case $n = 1$, we set now $\widehat{\mathcal{O}}_i$ to be the formal completion of $\mathscr{O}_X$ at S_i, i.e. $\widehat{\mathcal{O}}_i = \pi_* \varprojlim_n \mathscr{O}_X/(\mathcal{I}_i)^n$. We set $\mathcal{L}_i = \varinjlim_N \pi_* \varprojlim_n \mathcal{I}_i^{-N}/\mathcal{I}_i^n$ and

$$\mathfrak{h}_{\mathcal{L}_i} := \varinjlim_N \pi_* \varprojlim_n \mathcal{I}_i^{-N}/\mathcal{I}_i^n \otimes_{\mathscr{O}_X} \mathfrak{h}$$

for all $i \in \{1, \dots, n\}$. The direct sum $\mathfrak{h}_{\mathcal{L}_1} \oplus \cdots \oplus \mathfrak{h}_{\mathcal{L}_n}$ is denoted by $\mathfrak{h}_{\mathcal{L}}$ and $\mathcal{L} = \oplus \mathcal{L}_i$.

We extend centrally $\mathfrak{h}_{\mathcal{L}_i}$ in the same way as we did in the case $n = 1$ obtaining $\widehat{\mathfrak{h}}_{\mathcal{L}_i}$ with central element c_i. We denote by $\widehat{\mathfrak{h}}_{\mathcal{L}}$ the direct sum of $\widehat{\mathfrak{h}}_{\mathcal{L}_i}$ modulo the relation that identifies all the central elements c_i's so that

$$0 \to c\mathscr{O}_S \to \widehat{\mathfrak{h}}_{\mathcal{L}} \to \mathfrak{h}_{\mathcal{L}} \to 0$$

is exact. The Lie algebra $\mathfrak{h}_{\mathcal{A}}$ is still a sub Lie algebra of $\widehat{\mathfrak{h}}_{\mathcal{L}}$.

3.6.1. Sheaves of conformal blocks. Let $i \in \{1, \dots, n\}$. We denote by $\mathrm{IrRep}_\ell(i)$ the set of irreducible and finite dimensional representations of $\sigma_i{}^*\mathfrak{h}$ of level at most ℓ. As we have done in Section 2.2.2 we attach to any $\mathcal{V}_i \in \mathrm{IrRep}_\ell(i)$ the irreducible $U\widehat{\mathfrak{h}}_{\mathcal{L}_i}$-module $\mathcal{H}_\ell(\mathcal{V}_i)$. Taking their tensor product we obtain

$$\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) := \mathcal{H}_\ell(\mathcal{V}_i) \otimes \cdots \otimes \mathcal{H}_\ell(\mathcal{V}_n)$$

which then is an irreducible $U\widehat{\mathfrak{h}}_{\mathcal{L}}$-module with central charge c acting by multiplication by ℓ. Since $\mathfrak{h}_{\mathcal{A}}$ is a sub Lie algebra of $\widehat{\mathfrak{h}}_{\mathcal{L}}$, it acts on the left and we are interested in the sheaf of coinvariants.

DEFINITION 3.6.1. The *sheaf of conformal blocks attached to* $(\mathcal{V}_i)_{i=1}^n$ is the $\mathscr{O}_S$-module

$$\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_X := \mathfrak{h}_{\mathcal{A}} \circ \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \backslash \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n).$$

For every $i \in \{1, \dots, n\}$, we consider $\mathcal{V}_i$ as a representation of $\sigma_{i,un}{}^*\mathfrak{h}$ defined by a compatible family $\{\mathcal{V}_i(\sigma_i)\}_{\{\widetilde{X} \to X \to S, \{\sigma_j\}\}}$ of representations of $\sigma_i{}^*\mathfrak{h}$. The collection of $\mathcal{H}_\ell(\mathcal{V}_1(\sigma_1), \dots, \mathcal{V}_n(\sigma_n))_X$ defines $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_{X_{univ}}$, the *universal sheaf of conformal blocks attached to* $\{\mathcal{V}_i\}$.

3.6.2. The projectively flat connection on $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_{X_{univ}}$. Also the construction of projectively flat connection extends to the semilocal case. Observe first of all that the identification of the tangent space of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$ at a versal covering $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \{\sigma_i\})$ with $\mathcal{T}_{S/k}(-\log(\Delta))$ still holds. Since Proposition 3.2.5 still holds, this implies that we are allowed to provide the projective connection on $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_X$ in terms of a projective action of $\theta_{\mathcal{A},S}(-\mathcal{R})$ on $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_X$.

We denote by $\theta_{\mathcal{L}/S}$ the direct sum of $\theta_{\mathcal{L}_i/S}$, and we obtain a central extension $\widehat{\theta}_{\mathcal{L}/S}$ thereof as the quotient of the direct sum of $\widehat{\theta}_{\mathcal{L}_i/S}$ which identifies $(0, \hbar_i) \in \widehat{\theta}_{\mathcal{L}_i/S}$ with $(0, \hbar_j) \in \widehat{\theta}_{\mathcal{L}_j/S}$. The Sugawara representation $T_{\mathfrak{h}} \colon \widehat{\theta}_{\mathcal{L}/S} \to \overline{U}\widehat{\mathfrak{h}}_{\mathcal{L}}[(c + \check{h})^{-1}]$ is induced from the Sugawara representations of $\widehat{\theta}_{\mathcal{L}_i/S}$ and gives the projective action of $\theta_{\mathcal{L},S}$ on $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)$.

Combining all these elements with the case $n = 1$ we obtain the following generalization of Theorem 3.5.1 and Corollary 3.5.2.

COROLLARY 3.6.2. *For every $i \in \{1, \dots, n\}$ let $\mathcal{V}_i \in IrRep_\ell(\sigma_{i,un}{}^*\mathfrak{h})$. The module $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_{X_{univ}}$ is a coherent module over $\overline{\mathcal{H}ur}(\Gamma, \xi)_{g,n}$ which is equipped with a projective action of $\mathcal{T}_{\overline{\mathcal{H}ur}(\Gamma,\xi)_{g,n}}(-\log(\Delta))$. In particular it is locally free over $\mathcal{H}ur(\Gamma, \xi)_{g,n}$.*

4 | Factorization rules and propagation of vacua

In this chapter we prove the properties of the sheaves of conformal blocks mentioned in the introduction. More precisely we show that the sheaf $\mathcal{H}_\ell(0)_{X_{univ}}$ descends to $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_g$ by means of the propagation of vacua, and we provide the factorization rules which compare the fibre of $\mathcal{H}_\ell(0)_{X_{univ}}$ over a nodal curve X with the fibres of the sheaves $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ on its normalization X_N. We will proceed following the approach of [**Loo13**, Section 4].

4.1. Independence of number of sections

In this section we want to show that the sheaf $\mathcal{H}_\ell(0)_{X_{univ}}$ actually descends to a vector bundle on $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_g$ as a consequence of Proposition 4.1.1. Following [**Bea96**, Proposition 2.3] we state and prove the aforementioned proposition, called also *propagation of vacua*, because it shows that we can modify the sheaf of conformal blocks by adding as many sections as we want to which we attach the trivial representation to obtain a sheaf isomorphic to the one we started with.

Setting and notation. In this section we fix the following objects.

- Let $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S = \mathrm{Spec}(R), \sigma_1, \dots, \sigma_n, \sigma_{n+1}, \dots, \sigma_{n+m})$ be an element of $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,n+m}(S)$.
- Denote by $\mathcal{B} := \mathscr{O}_{X\setminus\{S_1,\dots,S_n\}}$ and by $\mathcal{A} := \mathscr{O}_{X\setminus\{S_1,\dots,S_n,S_{n+1},\dots,S_{n+m}\}}$ and set $\mathfrak{h}_\mathcal{B} := \pi_*(\mathfrak{h}\otimes\mathcal{B})$ which is contained in $\mathfrak{h}_\mathcal{A} := \pi_*(\mathfrak{h}\otimes\mathcal{A})$.
- For every $i \in \{1,\dots,n\}$ fix $\mathcal{V}_i \in \mathrm{IrRep}_\ell(i) := \mathrm{IrRep}_\ell(\sigma_i^*\mathfrak{h})$ and for every $j \in \{1,\dots,m\}$ we fix $\mathcal{W}_j \in \mathrm{IrRep}_\ell(n+j)$.

Under these conditions we notice that $\mathfrak{h}_\mathcal{B}$ acts on each $\mathcal{W}_j$ since ${\sigma_{n+j}}^*\pi^*\mathfrak{h}_\mathcal{B}$ maps naturally to ${\sigma_{n+j}}^*\mathfrak{h}$ and the latter acts on $\mathcal{W}_j$ by definition.

Proposition 4.1.1. *The inclusions $\mathcal{W}_j \to \mathcal{H}_\ell(\mathcal{W}_j)$ induce an isomorphism*

$$\mathfrak{h}_\mathcal{B} \setminus \left(\mathcal{H}_\ell(\mathcal{V}_1,\dots,\mathcal{V}_n) \otimes \bigotimes_{j=1}^m \mathcal{W}_j \right) \cong \mathfrak{h}_\mathcal{A} \setminus \mathcal{H}_\ell(\mathcal{V}_1,\dots,\mathcal{V}_n,\mathcal{W}_1,\dots,\mathcal{W}_m)$$

of $\mathscr{O}_S$-modules.

PROOF. We sketch here the main ideas of the proof, using the same techniques of the original one [**Bea96**, Proof of Proposition 2.3]. By induction it is enough to prove the assertion for $m = 1$, so that we need to prove that the inclusion $\mathcal{W} \to \mathcal{H}_\ell(\mathcal{W})$ induces an isomorphism

$$\phi\colon \mathfrak{h}_{\mathcal{B}} \setminus (\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \otimes \mathcal{W}) \xrightarrow{\cong} \mathfrak{h}_{\mathcal{A}} \setminus \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n, \mathcal{W}).$$

The morphism is well defined on the quotients as the inclusion of $\mathfrak{h}_{\mathcal{B}}$ in $\mathfrak{h}_{\mathcal{L}n+1}$ factors through $\mathfrak{h}_{\mathcal{B}} \to \mathfrak{h}_{\mathcal{A}}$.

Since the inclusion $\mathcal{W} \to \mathcal{H}_\ell(\mathcal{W})$ factors through $\widetilde{\mathcal{H}_\ell}(\mathcal{W})$, we prove the proposition in two steps.

Claim 1. *The inclusion $\mathcal{W} \to \widetilde{\mathcal{H}_\ell}(\mathcal{W})$ induces an isomorphism*

$$\widetilde{\phi}\colon \mathfrak{h}_{\mathcal{B}} \setminus (\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \otimes \mathcal{W}) \longrightarrow \mathfrak{h}_{\mathcal{A}} \setminus \left(\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \otimes \widetilde{\mathcal{H}_\ell}(\mathcal{W})\right).$$

Claim 2. *The projection map*

$$\mathfrak{h}_{\mathcal{A}} \setminus \left(\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \otimes \widetilde{\mathcal{H}_\ell}(\mathcal{W})\right) \longrightarrow \mathfrak{h}_{\mathcal{A}} \setminus (\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n, \mathcal{W}))$$

is an isomorphism.

We give the proof of **Claim 1**, as for **Claim 2** one can refer to [**Bea96**, (3.4)]. We just remark that in the proof of Claim 2 it is used that the level of $\mathcal{W}$ is bounded by ℓ.

Since checking that $\widetilde{\phi}$ is an isomorphism can be done locally on S, there is no loss in generality in assuming that the $\mathcal{I}_i$'s are principal so that $\widehat{\mathcal{O}} \cong \bigoplus R[[t_i]]$ and that there are isomorphisms $\mathfrak{h}_{\mathcal{L}_i} \cong \mathfrak{g}\mathcal{L}_i$. Observe that the exact sequence of R-modules

$$0 \to \mathfrak{h}_{\mathcal{B}} \to \mathfrak{h}_{\mathcal{A}} \to \mathfrak{h}_{\mathcal{A}}/\mathfrak{h}_{\mathcal{B}} \to 0$$

splits because the quotient $\mathfrak{h}^- := \mathfrak{h}_{\mathcal{A}}/\mathfrak{h}_{\mathcal{B}}$ is isomorphic to $\mathfrak{h}_{\mathcal{L}_{n+1}}/\mathfrak{h}_{\widehat{\mathcal{O}}_{n+1}}$ which can be identified with $\mathfrak{g} \otimes_k R[t_{n+1}^{-1}]t_{n+1}^{-1}$. We then are left to prove that

$$\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \otimes \mathcal{W} \longrightarrow \mathfrak{h}^- \setminus \left(\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \otimes \widetilde{\mathcal{H}_\ell}(\mathcal{W})\right)$$

is an isomorphism. Observe that this statement no longer depends on the covering $\widetilde{X} \to X$, so once we choose isomorphisms $\mathfrak{h}_{\mathcal{L}_i} \cong \mathfrak{g}\mathcal{L}_i$, this follows from the classical case. □

As previously announced, this proposition has important corollaries.

NOTATION. Let $X_{univ} \xrightarrow{\pi} \mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n}$ be the universal curve over $\mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,n}$ with sections $\sigma_1, \dots, \sigma_n$. For every $i \in \{1, \dots, n\}$ we denote by $X_{univ,i}$ the open curve $X_{univ} \setminus \{\sigma_1, \dots, \sigma_i\}$ and whenever $\mathfrak{h}_{\mathcal{A}_i} := \pi_*(\mathfrak{h} \otimes \mathcal{O}_{X_{univ,i}})$ acts on a module M, we denote the quotient $\mathfrak{h}_{\mathcal{A}_i} \setminus M$ by $M_{X_{univ,i}}$. In particular, for $i = n$ we can use the notation $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_{X_{univ,n}}$ instead of $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_{X_{univ}}$ to stress that the action takes into account all the sections.

COROLLARY 4.1.2. *For all n and $m \in \mathbb{N}$ there is a natural isomorphism*

$$\left(\left(\boldsymbol{Forg}_{n+m,n}\right)^* (\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)) \otimes \bigotimes_{j=1}^{m} \mathcal{W}_j\right)_{X_{univ,n}} \cong \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n, \mathcal{W}_1, \dots \mathcal{W}_m)_{X_{univ,n+m}}$$

of vector bundles on $\mathcal{H}ur(\Gamma, \xi)_{g,n+m}$. □

In particular if we assume that the $\mathcal{W}_j$'s are trivial representations, we obtain the so called *propagation of vacua*.

COROLLARY 4.1.3. *For all n and $m \in \mathbb{N}$ there is a natural isomorphism*

$$(\textbf{\textit{Forg}}_{n+m,n})^* \left(\mathcal{H}_\ell(\mathcal{V}_1,\dots,\mathcal{V}_n)_{X_{univ,n}}\right) \cong \mathcal{H}_\ell(\mathcal{V}_1,\dots,\mathcal{V}_n,0,\dots 0)_{X_{univ,n+m}}$$

of vector bundles on $\mathcal{H}ur(\Gamma,\xi)_{g,n+m}$. □

Which leads to the following result.

COROLLARY 4.1.4. *The vector bundle $\mathcal{H}_\ell(0)$ defined on $\mathcal{H}ur(\Gamma,\xi)_{g,1}$ descends to a vector bundle on $\mathcal{H}ur(\Gamma,\xi)_g$.*

PROOF. We can construct the sequence

$$\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,3} \Rrightarrow \mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,2} \underset{f_1}{\overset{f_2}{\rightrightarrows}} \mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1} \longrightarrow \mathcal{H}\mathrm{ur}(\Gamma,\xi)_g$$

where the horizontal morphisms, which are faithfully flat, are given by forgetting one of the sections. The vector bundle $\mathcal{H}_\ell(0)_{X_{univ,1}}$ on $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1}$ then descends from $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,1}$ to $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_g$ because Corollary 4.1.3 provides a canonical isomorphism ϕ_{12} between $f_1^*\mathcal{H}_\ell(0)_{X_{univ,1}}$ and $f_2^*\mathcal{H}_\ell(0)_{X_{univ,1}}$. The compatibility of the isomorphisms ϕ_{ij} on $\mathcal{H}\mathrm{ur}(\Gamma,\xi)_{g,3}$ holds by construction. □

REMARK 4.1.5. When we defined $\mathcal{H}_\ell(\mathcal{V})_X$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}$, we assumed that $X \setminus \sigma$ was affine. Corollary 4.1.3 allows us to remove this assumption: in fact if this is not the case, we can add finitely many sections, say M, to which we attach the trivial representation and set $\mathcal{H}_\ell(\mathcal{V})_{X,1}$ to be $\mathcal{H}_\ell(\mathcal{V},0,\dots,0)_{X,M+1}$. The same holds for $\mathcal{H}_\ell(\mathcal{V}_1,\dots,\mathcal{V}_n)_X$ on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,n}$.

4.2. Nodal degeneration and fusion rules

In this section we want to compare the sheaf of covacua $\mathcal{H}_\ell(0)_X$ attached to a covering of nodal curves $\widetilde{X} \to X$, to the sheaves of the form $\mathcal{H}_\ell(\mathcal{V})_{X_N}$, attached to the normalization $\widetilde{X}_N \to X_N$ of the covering we started with.

SETTING AND NOTATION. We will consider the following objects.

- Let $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} \mathrm{Spec}(k), \mathfrak{p}_1,\dots,\mathfrak{p}_n) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,n}(\mathrm{Spec}(k))$ and assume that X is irreducible and has only one double point $\mathfrak{p} \in X(k)$.
- Let X_N be the normalization of X and set $q_N \colon \widetilde{X}_N := \widetilde{X} \times_X X_N \to X_N$. The points of X_N mapping to $\mathfrak{p}$ are denoted $\mathfrak{p}_+$ and $\mathfrak{p}_-$.

REMARK 4.2.1. Observe that q_N is a Γ-covering with the action of Γ induced by the one on $\widetilde{X}$, so that it is ramified only over $\mathcal{R}$. The Lie algebra $\mathfrak{h}_N := \mathfrak{h} \times_X X_N$ is then isomorphic to the Lie algebra of Γ-invariants of $\mathfrak{g} \otimes_k q_{N*}\mathscr{O}_{\widetilde{X}_N}$. Furthermore, the normalization provides an isomorphism between the k-Lie algebra $\mathfrak{h}|_\mathfrak{p}$ and $\mathfrak{h}_N|_{\mathfrak{p}_\pm}$.

Let $\pi_N \colon X_N \to \mathrm{Spec}(k)$ be the structural morphism and consider the Lie algebras

$$\mathfrak{h}_{\mathcal{A}_N} := \pi_{N*}(\mathfrak{h}_N \otimes \mathscr{O}_{X_N \setminus \{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}})$$

and

$$\mathfrak{h}_{\mathcal{L}_N} := \bigoplus_{i=1}^{n} \varinjlim_{N} \pi_* \varprojlim_{m} \mathfrak{h}_N \otimes \mathcal{I}_i^{-N}/\mathcal{I}_i^m$$

which are the analogues of $\mathfrak{h}_{\mathcal{A}}$ and $\mathfrak{h}_{\mathcal{L}}$ for the marked covering $(\widetilde{X}_N \to X_N, \{\mathfrak{p}_i\})$. Observe that since X and X_N are isomorphic outside of $\mathfrak{p}$, the Lie algebras $\mathfrak{h}_{\mathcal{L}_N}$ and $\mathfrak{h}_{\mathcal{L}}$ are isomorphic.

As observed in the previous remark, since $\mathfrak{h}|_{\mathfrak{p}}$ and $\mathfrak{h}_N|_{\mathfrak{p}_\pm}$ are isomorphic, every representation $\mathcal{W}$ of $\mathfrak{h}|_{\mathfrak{p}}$, is also a representation of $\mathfrak{h}_N|_{\mathfrak{p}_\pm}$. Let denote by $\mathcal{W}^*$ the dual of $\mathcal{W}$ and view $\mathcal{W} \otimes_k \mathcal{W}^*$ as a representation of $\mathfrak{h}_N|_{\mathfrak{p}_+} \oplus \mathfrak{h}_N|_{\mathfrak{p}_-}$, with $\mathfrak{h}_N|_{\mathfrak{p}_+}$ acting on $\mathcal{W}$ and $\mathfrak{h}_N|_{\mathfrak{p}_-}$ on $\mathcal{W}^*$. This induces an action of $\mathfrak{h}_{\mathcal{A}_N}$ on $\mathcal{W} \otimes_k \mathcal{W}^*$ as

$$\alpha * (w \otimes \phi) = [X]_{\mathfrak{p}_+} w \otimes \phi + w \otimes [X]_{\mathfrak{p}_-}\phi$$

where $[\star]_{\mathfrak{p}_\pm}$ denotes the reduction modulo the ideal defining $\mathfrak{p}_\pm$. Let $b_{\mathcal{W}}$ denote the trace morphism $\mathrm{End}(\mathcal{W}) = \mathcal{W} \otimes \mathcal{W}^* \to k$ which is compatible with the action of $\mathfrak{h}|_{\mathfrak{p}}$. We can formulate the *fusion rules* controlling the nodal degeneration as follows.

PROPOSITION 4.2.2. *The morphisms $\{b_{\mathcal{W}}\}$ induce an isomorphism*

$$\bigoplus_{\mathcal{W} \in IrRep_\ell(\mathfrak{h}|_{\mathfrak{p}})} \mathfrak{h}_{\mathcal{A}_N} \backslash (\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \otimes (\mathcal{W} \otimes \mathcal{W}^*)) \to \mathfrak{h}_{\mathcal{A}} \backslash \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)$$

The proof of this result is a mild generalization of the proof of [**Loo13**, Proposition 6.1], which in turn is a consequence of Schur's Lemma. We give an overview of it.

PROOF. Fix an isomorphism between $\mathfrak{h}|_{\mathfrak{p}}$ and $\mathfrak{g}$ so that $\mathrm{IrRep}_\ell(\mathfrak{h}|_{\mathfrak{p}})$ is identified with P_ℓ. Denote by $\mathrm{Spec}(\mathcal{A}) = X \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and $\mathrm{Spec}(\mathcal{A}_N) = X_N \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and let $I_{\mathfrak{p}} \subset \mathcal{A}$ be the ideal defining $\mathfrak{p}$, so that the normalization gives the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{\mathfrak{p}} & \longrightarrow & \mathcal{A} & \longrightarrow & k & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \Delta & & \\ 0 & \longrightarrow & I_{\mathfrak{p}} & \longrightarrow & \mathcal{A}_N & \longrightarrow & k \oplus k & \longrightarrow & 0 \end{array}$$

whose rows are exact. As in the classical case, we consider a similar diagram of Lie algebras. Define $\mathfrak{h}_I$ as the tensor product $I_{\mathfrak{p}} \otimes_{\mathcal{A}} \mathfrak{h}_{\mathcal{A}}$, and observe that the quotient $\mathfrak{h}_{\mathcal{A}}/\mathfrak{h}_{I_{\mathfrak{p}}}$ is $\mathfrak{h}|_{\mathfrak{p}}$, which is then isomorphic to $\mathfrak{g}$. Repeating the construction on X_N, we obtain the commutative diagram of k-Lie algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h}_{I_{\mathfrak{p}}} & \longrightarrow & \mathfrak{h}_{\mathcal{A}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \Delta & & \\ 0 & \longrightarrow & \mathfrak{h}_{I_{\mathfrak{p}}} & \longrightarrow & \mathfrak{h}_{\mathcal{A}_N} & \longrightarrow & \mathfrak{g} \oplus \mathfrak{g} & \longrightarrow & 0. \end{array}$$

Let consider the quotient $M := \mathfrak{h}_{I_{\mathfrak{p}}} \backslash \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)$ of the $\mathfrak{h}_{\mathcal{A}_N}$-module $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)$ and observe that it is a finite dimensional representation of $\mathfrak{g} \oplus \mathfrak{g}$, because the quotient $\mathfrak{h}_{\mathcal{A}} \backslash \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)$ is finite dimensional and the quotient $\mathfrak{h}_{\mathcal{A}}/\mathfrak{h}_{I_{\mathfrak{p}}}$ is one dimensional. It is moreover a representation of $\mathfrak{g} \oplus \mathfrak{g}$ of level less or equal to ℓ relative to each factors.

Since $\mathfrak{h}_{I_{\mathfrak{p}}}$ acts trivially on $\mathcal{W} \otimes \mathcal{W}^*$, the maps $\{b_{\mathcal{W}}\}$ induce the morphism

$$\bigoplus_{W \in P_\ell} \mathfrak{h}_I \backslash \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n) \otimes (\mathcal{W} \otimes \mathcal{W}^*) \to \mathfrak{h}_I \backslash \mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)$$

Observe that if we consider M as a $\mathfrak{g}$-module via the diagonal action, and we denote this $\mathfrak{g}$-representation by M^Δ, then $\mathfrak{g}\backslash M^\Delta$ is exactly $\mathfrak{h}_\mathcal{A}\backslash\mathcal{H}_\ell(\mathcal{V}_1,\ldots,\mathcal{V}_n)$. After these considerations, the proof of the proposition boils down to showing that if M is a finite dimensional representation of $\mathfrak{g}\oplus\mathfrak{g}$ of level at most ℓ, then the morphisms $\{b_\mathcal{W}\}$ induce the isomorphism

$$\bigoplus_{W\in P_\ell} \mathfrak{g}\oplus\mathfrak{g}\backslash(M\otimes(W\otimes W^*))\to\mathfrak{g}\backslash M^\Delta.$$

Schur's Lemma ensures that the set of morphisms between irreducible Lie algebra representations is a skew field, and since without loss of generality we might assume M to be an irreducible $\mathfrak{g}\oplus\mathfrak{g}$ representation of the form $V_1\otimes V_2$ for $V_i\in P_\ell$, we conclude. □

Denote by $\mathfrak{h}_{\mathcal{A}_N{}^*}$ the Lie algebra $\pi_{N*}(\mathfrak{h}_N\otimes\mathscr{O}_{X_N\setminus\{\mathfrak{p}_1,\ldots,\mathfrak{p}_n,\mathfrak{p}_+,\mathfrak{p}_-\}})$. Then Proposition 4.1.1 allows us to rewrite the previous proposition as an isomorphism

$$\bigoplus_{\mathcal{W}\in\mathrm{IrRep}_\ell(\mathfrak{h}|_\mathfrak{p})} \mathfrak{h}_{\mathcal{A}_N{}^*}\backslash\mathcal{H}_\ell(\mathcal{V}_1,\ldots,\mathcal{V}_n,\mathcal{W},\mathcal{W}^*)\to\mathfrak{h}_\mathcal{A}\backslash\mathcal{H}_\ell(\mathcal{V}_1,\ldots,\mathcal{V}_n)$$

and in particular implies the isomorphism

$$\bigoplus_{\mathcal{W}\in\mathrm{IrRep}_\ell(\mathfrak{h}|_\mathfrak{p})} \mathcal{H}_\ell(\mathcal{W},\mathcal{W}^*)_{X_N}\to\mathcal{H}_\ell(0)_X$$

where we see X_N naturally marked by $\mathfrak{p}_+$ and $\mathfrak{p}_-$.

REMARK 4.2.3. We assumed, at the beginning of this section, that X is an in irreducible curve with a unique nodal point $\mathfrak{p}$. The irreducibility conditions ensures that the curve $X\setminus\{\mathfrak{p}_1,\ldots,\mathfrak{p}_n\}$ is affine and, as we have seen in in Remark 4.1.5, we can drop this assumption in view of the Propagation of Vacua. Moreover, also the assumption that $\mathfrak{p}$ is the only nodal point of X is not necessary. In the case in which X has other nodes, then it will be enough to replace X_N with the partial normalization of X at the point $\mathfrak{p}$.

REMARK 4.2.4. Let $(\widetilde{X}\to X\to S,\sigma)\in\overline{\mathcal{H}\mathrm{ur}}(\Gamma,\xi)_{g,1}(S)$ and assume that it is possible to normalize the family (for example assuming that the nodes of X are given by a section $\varsigma\colon X\to S$). Then Proposition 4.2.2 still holds by replacing the index set $\mathrm{IrRep}_\ell(\mathfrak{h}|_\mathfrak{p})$ with $\mathrm{IrRep}_\ell(\varsigma^*\mathfrak{h})$.

5 | Locally freeness of the sheaf of conformal blocks

In this chapter we prove that the sheaves of conformal blocks $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_{X_{univ}}$ are locally free also on the boundary of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,n}$. For simplicity only we will assume $n = 1$.

5.1. Canonical smoothing

As previously stated, we want to prove that $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is a locally free sheaf on $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}$. For this purpose we describe here a procedure to realise a covering of nodal curves as the special fibre of a family of coverings which is generically smooth. The idea is to induce a deformation of the covering $\widetilde{X} \to X$ from the canonical smoothing of the base curve X provided in [**Loo13**]. As already noted in Remark 1.2.3, it is essential that the branch locus $\mathcal{R}$ of the covering $q\colon \widetilde{X} \to X$ is contained in the smooth locus of X.

Let $(\widetilde{X} \overset{q_0}{\to} X \overset{\pi_0}{\to} \mathrm{Spec}(k), \sigma_0) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}(\mathrm{Spec}(k))$ with $\mathfrak{p} \in X(k)$ the unique nodal point of X. The goal of this section is to construct a family $\widetilde{\mathcal{X}} \to \mathcal{X}$ belonging to $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}(\mathrm{Spec}(k[[\tau]]))$ which deforms $(\widetilde{X} \to X)$ and whose generic fibre is smooth, i.e. it lies in $\mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,1}(\mathrm{Spec}(k((\tau))))$.

5.1.1. The intuitive idea. The idea which is explained in [**Loo13**], is to find a deformation $\mathcal{X}$ of X which replaces the formal neighbourhood $k[[t_+, t_-]]/t_+t_-$ of the nodal point $\mathfrak{p}$ with the $k[[\tau]]$-algebra $k[[t_+, t_-, \tau]]/t_+t_- = \tau$. This can be achieved with the following geometric construction. We first normalize the curve X obtaining the curve X_N with two points $\mathfrak{p}_+$ and $\mathfrak{p}_-$ above $\mathfrak{p}$. We blow up the trivial deformation $X_N[[\tau]]$ of X_N at the points $\mathfrak{p}_+$ and $\mathfrak{p}_-$ and note that the formal coordinate rings at $\mathfrak{p}_\pm$ in the strict transform are of the form $k[[t_\pm, \tau/t_\pm]]$. We then obtain the neighbourhood $k[[t_+, t_-, \tau]]/t_+t_- = \tau$ by identifying t_+ with τ/t_-. The deformation $\widetilde{\mathcal{X}}$ of $\widetilde{X}$ is induced from the one of X because the singular point $\mathfrak{p}$ does not lie in $\mathcal{R}$.

5.1.2. Construction of $\widetilde{\mathcal{X}} \to \mathcal{X}$. We will realise the canonical smoothing of $\widetilde{X} \to X$ it by constructing compatible families

$$(\widetilde{\mathcal{X}}^n \overset{q_n}{\to} \mathcal{X}^n \overset{\pi_n}{\to} \mathrm{Spec}(k[\tau]_n), \sigma_n) \in \overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1}(\mathrm{Spec}(k[\tau]_n))$$

where $k[\tau]_n := k[\tau]/(\tau^{n+1})$ for $n \in \mathbb{N}_0$. As these are infinitesimal deformations, we only need to change the structure sheaf. As we have previously done, we normalize $\widetilde{X}$ and X obtaining $(\widetilde{X}_N \xrightarrow{q} X_N \xrightarrow{\pi} \mathrm{Spec}(k), \sigma_0, \mathfrak{p}_+, \mathfrak{p}_-) \in \mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,3}(\mathrm{Spec}(k))$. We fix furthermore local coordinates t_+ and t_- at the points $\mathfrak{p}_+$ and $\mathfrak{p}_-$.

Let U be an open subset of X and $n \in \mathbb{N}_0$. If U does not contain $\mathfrak{p}$ we set $\mathscr{O}_{\mathcal{X}^n}(U) := \mathscr{O}_X(U)[\tau]/\tau^{n+1}$. Otherwise, if $\mathfrak{p} \in U$, we set

$$\mathscr{O}_{\mathcal{X}^n}(U) := \ker\left(\frac{k[[t_+,t_-]][\tau]}{t_+t_- = \tau, \tau^{n+1}} \oplus \mathscr{O}_{\mathcal{X}^n}(U \setminus \{\mathfrak{p}\}) \xrightarrow{\alpha_n - \beta_n} \frac{k((t_+))[\tau]}{\tau^{n+1}} \oplus \frac{k((t_-))[\tau]}{\tau^{n+1}}\right)$$

where

$$\alpha_n \colon \frac{k[[t_+,t_-]][\tau]}{t_+t_- = \tau, \tau^{n+1}} \longrightarrow \frac{k((t_+))[\tau]}{\tau^{n+1}} \oplus \frac{k((t_-))[\tau]}{\tau^{n+1}}$$

is the $k[\tau]_n$-linear morphism given by $t_+ \mapsto (t_+, (t_-)^{-1}\tau)$ and $t_- \mapsto ((t_+)^{-1}\tau, t_-)$, and

$$\beta_n \colon \mathscr{O}_{\mathcal{X}^n}(U \setminus \{\mathfrak{p}\}) \longrightarrow \frac{k((t_+))[\tau]}{\tau^{n+1}} \oplus \frac{k((t_-))[\tau]}{\tau^{n+1}}$$

sends $\psi \in \mathscr{O}_{\mathcal{X}^n}(U \setminus \{\mathfrak{p}\})$ to (ψ_+, ψ_-) where $\psi_\pm$ is the expansion of ψ at the point $\mathfrak{p}_\pm$ using the identifications

$$\mathscr{O}_{\mathcal{X}^n}(U \setminus \{\mathfrak{p}\}) = \mathscr{O}_X(U \setminus \{\mathfrak{p}\})[\tau]/\tau^{n+1} = \mathscr{O}_{X_N}(U_N \setminus \{\mathfrak{p}_+, \mathfrak{p}_-\})[\tau]/\tau^{n+1}.$$

REMARK 5.1.1. Observe that the completion of $\mathscr{O}_{\mathcal{X}^n}$ at the point $\mathfrak{p}$ is isomorphic to $k[[t_+,t_-]][\tau]/(t_+t_- = \tau, \tau^{n+1}) = k[[t_+,t_-]]/(t_+t_-)^{n+1}$. In fact note that once we take the completion of $\mathscr{O}_{\mathcal{X}^n}(U \setminus \mathfrak{p})$ at the point $\mathfrak{p}$ we obtain exactly $\frac{k((t_+))[\tau]}{\tau^{n+1}} \oplus \frac{k((t_-))[\tau]}{\tau^{n+1}}$, the map β_n becoming the identity. The kernel of $\alpha_n - \beta_n$ is then identified with $k[[t_+,t_-]][\tau]/(t_+t_- = \tau, \tau^{n+1})$ as claimed.

Observe furthermore that once we take the limit for $n \to \infty$, then the formal neighbourhood of $\mathfrak{p}$ will be $k[[t_+,t_-,\tau]]/t_+t_- = \tau$ as asserted in the subsection 5.1.1. The map α_n describes the process of glueing the formal charts around $\mathfrak{p}_+$ and $\mathfrak{p}_-$.

Note moreover that for all $n \in \mathbb{N}$ there are natural maps $g^n \colon \mathcal{X}^{n-1} \to \mathcal{X}^n$ induced by the identity on topological spaces and by the projection $k[\tau]_n \to k[\tau]_{n-1}$ on the structure sheaves.

LEMMA 5.1.2. *For every $n \in \mathbb{N}_0$ the family $\mathcal{X}^n$ is a curve over Spec$(k[\tau]_n)$ which deforms X.*

PROOF. We need to prove that $\mathscr{O}_{\mathcal{X}^n}$ is flat and proper over $\mathrm{Spec}(k[\tau]_n)$. Once we show that the $\mathcal{X}^n$ is of finite type, we can use the valuative criterion to deduce that $\mathcal{X}^n$ is proper over $k[\tau]_n$. Observe that the kernel of the map $g^{n*} \colon \mathscr{O}_{\mathcal{X}^n} \to \mathscr{O}_{\mathcal{X}^{n-1}} \to 0$ is $\tau^n \mathscr{O}_X$. Outside $\mathfrak{p}$, as the deformation is trivial, this is true. On an open U containing $\mathfrak{p}$, the snake lemma tells us that this is the kernel of

$$\tau^n \frac{k[[t_+,t_-]]}{t_+t_-} \oplus \tau^n \mathscr{O}_X(U \setminus \{\mathfrak{p}\}) \xrightarrow{\alpha_n - \beta_n} \tau^n k((t_+)) \oplus \tau^n k((t_-))$$

where α_n and β_n are the gluing functions defining $\tau^n \mathscr{O}_X$. We can conclude that $\mathscr{O}_{\mathcal{X}^n}$ is of finite type by using induction on n and observing that $\mathcal{X}^0 = X$. This moreover shows the flatness of the family. ☐

This deformation of X induces a deformation of $\widetilde{X}$ which is, in rough terms, obtained as the trivial deformation outside the points $\mathfrak{p}_1, \dots, \mathfrak{p}_p$ lying above $\widetilde{X}$ over $\mathfrak{p}$, and for every $j \in \{1, \dots, p\}$, the formal neigbourhood of $\widetilde{\mathcal{X}}_n$ around $\mathfrak{p}_i$ will be isomorphic to the formal neighbourhood of $\mathcal{X}_n$ around $\mathfrak{p}$.

To do this, let denote by $\mathfrak{p}_{j,+}$ and $\mathfrak{p}_{j,-}$ the two points of $\widetilde{X}_N$ mapping to $\mathfrak{p}_j$. We fix local coordinates $t_{j,+}$ at $\mathfrak{p}_{j,\pm}$ so that

$$\widetilde{X}_N \times_{X_N} \mathrm{Spec}(k[[t_\pm]]) \cong \mathrm{Spec}\left(\oplus_{j=1}^{p} k[[t_{j,\pm}]]\right)$$

and let consider U an open subset of $\widetilde{X}$. If U is disjoint from $q^{-1}(\mathfrak{p}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_p\}$, we set $\mathscr{O}_{\widetilde{\mathcal{X}}^n}(U) = \mathscr{O}_{\widetilde{X}}(U)[\tau]/\tau^{n+1}$. Let $i \in \{1, \dots, p\}$ and if U contains $\mathfrak{p}_i$ but not $\mathfrak{p}_j$ for all $j \neq i$ we set

$$\mathscr{O}_{\widetilde{\mathcal{X}}^n}(U) := \ker\left(\frac{k[[t_{i,+}, t_{i,-}]][\tau]}{t_{i,+}t_{i,-} = \tau, \tau^{n+1}} \oplus \mathscr{O}_{\widetilde{\mathcal{X}}^n}(U \setminus \{\mathfrak{p}_i\}) \xrightarrow{\widetilde{\alpha}_n - \widetilde{\beta}_n} \frac{k((t_{i,+}))[\tau]}{\tau^{n+1}} \oplus \frac{k((t_{i,-}))[\tau]}{\tau^{n+1}}\right)$$

where the maps $\widetilde{\alpha}_n$ and $\widetilde{\beta}_n$ are defined as in the case of $\mathcal{X}^n$.

REMARK 5.1.3. As was shown for $\mathcal{X}^n$, also $\widetilde{\mathcal{X}}^n$ is a curve over $\mathrm{Spec}(k[\tau]_n)$ deforming $\mathcal{X}$.

Let denote by $\mathcal{R}_n$ the trivial deformation of the branch locus $\mathcal{R}$ inside $\mathcal{X}^n$. The natural map $q_n \colon \widetilde{\mathcal{X}}^n \to \mathcal{X}^n$ which extends $q_0 \colon \widetilde{X} \to X$ realizes $\widetilde{\mathcal{X}}^n \to \mathcal{X}^n$ as a Γ-covering which is étale exactly outside $\mathcal{R}_n$ since the map q_n is étale on $\mathfrak{p}$ by construction. Furthermore, as σ_0 is disjoint from the singular locus, it follows that for every $n \in \mathbb{N}_0$ we can set σ_n to be the trivial deformation of σ_0.

By taking the direct limit of this family of deformations we obtain the Γ-covering of formal schemes $\widetilde{\mathcal{X}}^\infty \to \mathcal{X}^\infty$ over $\mathrm{Spf}(k[[t]])$. To prove that $\widetilde{\mathcal{X}}^\infty \to \mathcal{X}^\infty$ is algebraizable, i.e. that is comes from an algebraic object $\widetilde{\mathcal{X}} \to \mathcal{X} \to \mathrm{Spec}(k[[\tau]]])$, we can invoke Grothendieck's existence theorem ([**Gro63**, Théorème 5.4.5]) so that we are left to prove that the family $(\widetilde{\mathcal{X}}^n \to \mathcal{X}^n)_n$ is equipped with a compatible family of very ample line bundles. This is true because given a smooth point P of X which is not in $\mathcal{R}$ and m sufficiently big, we know that $\mathscr{O}(mP)$ is a very ample line bundle on X whose pullback to $\mathscr{O}_{\widetilde{X}}$ is also very ample. Since P lies in the smooth locus of X these line bundles extend naturally to very ample line bundles on $\mathcal{X}^n$ and on $\widetilde{\mathcal{X}}^n$, providing the wanted family of very ample line bundles.

We refer to the covering $(q \colon \widetilde{\mathcal{X}} \to \mathcal{X}, \sigma)$ that we have just constructed as the *canonical smoothing* of $(q_0 \colon \widetilde{X} \to X, \sigma_0)$. Observe that the generic fibre of $\widetilde{\mathcal{X}} \to \mathcal{X}$ is a covering of smooth curves over $k((\tau))$ because, as one can deduce from Remark 5.1.1, the formal neighbourhood of $\mathfrak{p}$ is given by $k[[t_+, t_-]]((\tau))/t_+t_- = \tau$.

5.2. Local freeness

The aim of this section is to show that, in the setting of the previous section, $\mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}}$ is a locally free $k[[\tau]]$-module. We can depict the situation that we described in the previous section in the following diagram

$$\begin{array}{ccccc}
\widetilde{X} = \widetilde{\mathcal{X}}_0 & \longrightarrow & \widetilde{\mathcal{X}} & \longleftarrow & \widetilde{\mathcal{X}}_\eta \\
\downarrow q_0 & & \downarrow q & & \downarrow q_\eta \\
X = \mathcal{X}_0 & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_\eta \\
\sigma_0 \uparrow \downarrow \pi_0 & & \sigma \uparrow \downarrow \pi & & \sigma_\eta \uparrow \downarrow \pi_\eta \\
\mathrm{Spec}(k) & \xrightarrow{0} & \mathrm{Spec}(k[[\tau]]) & \xleftarrow{\eta} & \mathrm{Spec}(k((\tau)))
\end{array}$$

where the covering $\widetilde{\mathcal{X}}_\eta \to \mathcal{X}_\eta$ is a Γ-covering of smooth curves and we denote by $\mathfrak{p}$ the nodal point of X. Let $\mathcal{V} \in \mathrm{IrRep}_\ell(\sigma)$ so that $\mathcal{H}_\ell(\mathcal{V})$ is a $k[[\tau]]$-module. We use the subscript 0 to denote the pullback along 0, i.e. the restriction to the special fibre, so that $\mathcal{V}_0$ denotes the induced representation of ${\sigma_0}^*\mathfrak{h}$.

REMARK 5.2.1. Observe that there is a canonical injection of $k[[\tau]]$-modules $\mathcal{H}_\ell(\mathcal{V}) \to \mathcal{H}_\ell(\mathcal{V}_0)[[\tau]]$ which is an isomorphism modulo τ^n for every $n \in \mathbb{N}$. Moreover we have by construction that $(\mathfrak{h}_{\mathcal{A}})_0$ is isomorphic to $\mathfrak{h}_{\mathcal{A}_0} = \mathfrak{h}_A$ and so $(\mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}})_0$ is isomorphic to $\mathcal{H}_\ell(\mathcal{V}_0)_X$.

The main result is that $\mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}}$ is the trivial deformation of $\mathcal{H}_\ell(\mathcal{V}_0)_X$ as stated in the following theorem.

THEOREM 5.2.2. *There is an isomorphism*

$$\mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}} \cong \mathcal{H}_\ell(\mathcal{V}_0)_X[[\tau]]$$

of $k[[\tau]]$-modules. In particular $\mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}}$ is a free $k[[\tau]]$-module.

5.2.1. Proof of Theorem 5.2.2.

NOTATION. In what follows we denote by $\widehat{\mathcal{O}}_N := k[[t_+]] \oplus k[[t_-]]$ the k-algebra which is the coordinate ring of the disjoint union of the formal neighbourhoods at the points $\mathfrak{p}_\pm$ in X_N. Similarly $\mathcal{L}_N := k((t_+)) \oplus k((t_-))$ represents the disjoint union of the punctured formal neighbourhoods at the points $\mathfrak{p}_\pm$ in X_N. Moreover we will write $k[[t_+, t_-]]$ in place of $k[[\tau, t_+, t_-]] / t_+ t_- = \tau$. Recall that this is the completion of $\mathscr{O}_{\mathcal{X}}$ at the point $\mathfrak{p}$.

LEMMA 5.2.3. *The canonical smoothing identifies $k[[t_+, t_-]]$ with the subalgebra of $\mathcal{L}_N[[\tau]]$ consisting of elements*

$$\widehat{\mathcal{O}}_{\mathfrak{p}} := \left\{ \sum_{i,j \geq 0} a_{ij} \left(t_+^{i-j}\tau^j \, , \, t_-^{j-i}\tau^i \right) \mid a_{ij} \in k \right\}$$

via the map sending t_+ to $(t_+, t_-^{-1}\tau)$ and t_- to $(t_+^{-1}\tau, t_-)$.

PROOF. Taking the limit of the definition of $\mathscr{O}_{\mathcal{X}^n}$ we identify the formal neighbourhood of $\mathcal{X}^\infty$ at $\mathfrak{p}$ with

$$\ker \left(k[[t_+, t_-]] \oplus \mathcal{L}_N[[\tau]] \xrightarrow{\alpha - Id} \mathcal{L}_N[[\tau]] \right)$$

where $\alpha(t_+) = (t_+, t_-^{-1}\tau)$ and $\alpha(t_-) = (t_+^{-1}\tau, t_-)$. $\square$

In view of Proposition 4.2.2 we identify $\mathcal{H}_\ell(\mathcal{V}_0)_X[[\tau]]$ with

$$\bigoplus_{W\in \mathrm{IrRep}_\ell(\mathfrak{h}|_{\mathfrak{p}})} \mathfrak{h}_{A_N} \setminus \left(\mathcal{H}_\ell(\mathcal{V}_0)\otimes (W\otimes W^*)\right)[[\tau]]$$

or equivalently with

$$\bigoplus_{W\in \mathrm{IrRep}_\ell(\mathfrak{h}|_{\mathfrak{p}})} \mathfrak{h}_{A_N{}^*} \setminus \mathcal{H}_\ell(\mathcal{V}_0, W, W^*)[[\tau]].$$

Recall that $\widehat{\mathfrak{h}}_{\mathcal{L}}$ is a filtered Lie algebra, hence this induces a filtration on $U\widehat{\mathfrak{h}}_{\mathcal{L}}$ and by consequence on $\mathcal{F}^+(\widehat{\mathfrak{h}}_{\mathcal{L}})$. Since for every $W \in \mathrm{IrRep}_\ell(\mathfrak{h}|_{\mathfrak{p}})$ the k-vector space $\mathcal{H}_\ell(W)$ is a quotient of $\mathcal{F}^+(\widehat{\mathfrak{h}}_{\mathcal{L}})$, also the latter is equipped with a filtration $F^*\mathcal{H}_\ell(W)$ inducing the associated decomposition $\mathcal{H}_\ell(W) = \bigoplus_{d\leq 0}\mathcal{H}_\ell(W)(d)$ where $\mathcal{H}_\ell(W)(d) = F^d\mathcal{H}_\ell(W)\,/F^{d-1}\mathcal{H}_\ell(W)$.

REMARK 5.2.4. Once we choose local coordinates and an isomorphism between $\mathfrak{h}_{\mathcal{L}}$ and $\mathfrak{g}\mathcal{L}$ we observed that the elements of $\mathcal{F}^+(\widehat{\mathfrak{h}}_{\mathcal{L}})$ are $k[(c+\hbar)^{-1}]$-linear combinations of elements $X_r t^{-k_r}\circ\cdots\circ X_1 t^{-k_1}\circ e_0$ with $k_r \geq \cdots \geq k_1 \geq 0$ and $r \geq 0$, where e_0 stands for $1 \in k$. We can explicitly write the graded pieces of $\mathcal{F}^+\widehat{\mathfrak{h}}_{\mathcal{L}}$ as

$$\mathcal{F}^+(\widehat{\mathfrak{h}}_{\mathcal{L}})(-d) = \left\langle X_r t^{-k_r}\circ\cdots\circ X_1 t^{-k_1}\circ e_0 \mid \sum_{i=1}^r k_i = d\right\rangle$$

so that it is not zero only for $d \leq 0$ and in particular $\mathcal{F}^+(\widehat{\mathfrak{h}}_{\mathcal{L}})(0) = k$ which shows that $\mathcal{H}_\ell(W)(0) = W$.

The key ingredient to provide a morphism between $\mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}}$ and $\mathcal{H}_\ell(V)_X[[\tau]]$ lies in the construction of the element $\epsilon(W)$ given by the following Proposition which we can see as a consequence of [**Loo13**, Lemma 6.5].

PROPOSITION 5.2.5. *Let $W \in \mathrm{IrRep}_\ell(\mathfrak{h}|_{\mathfrak{p}})$ and $b^0_W\colon W\otimes W^* \to k$ be the trace morphism. Then there exists an element*

$$\epsilon(W) = \sum_{d\geq 0}\epsilon(W)_d\cdot \tau^d \in \left(\mathcal{H}_\ell(W)\otimes\mathcal{H}_\ell(W^*)\right)[[\tau]]$$

satisfying the following conditions:

- *(a) the constant term $\epsilon(W)_0 \in W\otimes W^*$ is the dual of b^0_W and for every $d\in\mathbb{Z}_{\geq 0}$ we have $\epsilon(W)_d \in \mathcal{H}_\ell(W)(-d)\otimes\mathcal{H}_\ell(W^*)(-d)$;*
- *(b) $\epsilon(W)$ is annihilated by the image of $\mathfrak{h}_{k[[t_+,t_-]]}$ in $\overline{U\widehat{\mathfrak{h}}}_{\mathcal{L}_N}[[\tau]]$.*

PROOF. We choose an isomorphism between $\mathfrak{h}_{\mathcal{L}_N}$ and $\mathfrak{g}\otimes\mathcal{L}_N$, as well as an isomorphism between $\mathfrak{h}_{\widehat{\mathcal{O}}_{\mathfrak{p}}}$ and $\mathfrak{g}\otimes\widehat{\mathcal{O}}_p$. The construction of $\epsilon(W)$ essentially lies in showing that the pairing $b^{(0)}_W\colon W\otimes W^*\to k$ extends to a unique pairing

$$b_W\colon \mathcal{H}_\ell(W)\otimes\mathcal{H}_\ell(W^*)\to k$$

such that for all $(u,v)\in\mathcal{H}_\ell(W)\otimes\mathcal{H}_\ell(W^*)$ we have

$$b_W(Xt_+^m u, v) + b_W(u, Xt_-^{-m}v) = 0 \tag{1}$$

for all $m\in\mathbb{Z}$ and $X\in\mathfrak{g}$ and that b_W is identically zero when restricted to $\mathcal{H}_\ell(W)(d)\otimes\mathcal{H}_\ell(W^*)(d')$ if $d\neq d'$. This is essentially [**TUY89**, Claim 1 of the proof of Proposition 6.2.1] and we report here the proof for completeness.

Since we have that $\mathcal{H}_\ell(W) = \varinjlim_{d\in\mathbb{N}_0} F^{-d}\mathcal{H}_\ell(W)$, it is enough to show that b^0_W extends uniquely to $b_W^{(-d)} \colon F^{-d}\mathcal{H}_\ell(W) \otimes F^{-d}\mathcal{H}_\ell(W^*) \to k$ satisfying the above conditions. By induction hypothesis, assume that $b_W^{(-j)}$ is already defined for every $j \leq d$, and we show how to extend it to $b_W^{(-d-1)}$. Let $u \in F^{-d}\mathcal{H}_\ell(W)$ and assume that $Xt_+^{-m}u \in F^{-d-1}\mathcal{H}_\ell(W)$ for some $m \geq 1$. Let $v \in F^{-d-1}\mathcal{H}_\ell(W^*)$ and set

$$b_W^{(-d-1)}(Xt_+^{-m}u, v) := -b_W^{(-d)}(u, Xt_-^m v)$$

which is well defined as $Xt_-^m v \in F^{-d-1+m}\mathcal{H}_\ell(W^*)$. As every element of $F^{-d-1}\mathcal{H}_\ell(W)$ is obtained as linear combinations of elements of the above type, this defines uniquely the form $b_W^{(-d-1)}$ and hence b_W. It follows by construction that b_W is identically zero when restricted $\mathcal{H}_\ell(W)(-d) \otimes \mathcal{H}_\ell(W^*)(-d')$ if $d \neq d'$ and it is a perfect pairing when $d = d'$.

We define $\epsilon(W)_d$ as the dual of $b_W^{-d} \colon \mathcal{H}_\ell(W)(-d) \otimes \mathcal{H}_\ell(W^*)(-d) \to k$.

We prove now the points of the proposition and for simplicity of notation we will write ϵ instead of $\epsilon(W)$ throughout the rest of the proof.

(a) This is true by definition.

(b) Since b_W is characterized by the property (1) we have that this implies that

$$(Xt_+^m, 0)\epsilon_{m+d} + (0, Xt_-^{-m})\epsilon_d = 0$$

for every $m \in \mathbb{Z}$ and $d \in \mathbb{N}_0$. This means that $(Xt_+^m, Xt_-^{-m}\tau^m)$ annihilates ϵ, which by Lemma 5.2.3, is exactly the image of $\mathfrak{g} \otimes_k k[[t_+, t_-]]$ in $\overline{U}\widehat{\mathfrak{g}}_{\mathcal{L}_N}[[\tau]]$.

□

We saw how to attach to any representation W the element $\epsilon(W)$: we now use these elements to obtain the isomorphism map between $\mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}}$ and $\mathcal{H}_\ell(\mathcal{V}_0)_X[[\tau]]$. The following statement, combined with Proposition 4.2.2 implies Theorem 5.2.2.

PROPOSITION 5.2.6. *The $k[[\tau]]$ linear map*

$$E \colon \mathcal{H}_\ell(\mathcal{V}) \subset \mathcal{H}_\ell(\mathcal{V}_0)[[\tau]] \to \bigoplus_{W \in IrRep_\ell(\mathfrak{h}|_\mathfrak{p})} \mathcal{H}_\ell(\mathcal{V}_0 \otimes W \otimes W^*)[[\tau]]$$

$$u = \sum_{i\geq 0} u_i\tau^i \mapsto (u \otimes \epsilon(W))_{W\in IrRep_\ell(\mathfrak{h}|_\mathfrak{p})} = \left(\sum_{i,d\geq 0} u_i \otimes \epsilon(W)_d \tau^{i+d}\right)_{W\in IrRep_\ell(\mathfrak{h}|_\mathfrak{p})}$$

induces the isomorphism

$$E_{\mathfrak{h}_\mathcal{A}} \colon \mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}} \to \bigoplus_{W \in IrRep_\ell(\mathfrak{h}|_\mathfrak{p})} \mathcal{H}_\ell(\mathcal{V}_0 \otimes W \otimes W^*)_{X_N}[[\tau]].$$

of $k[[\tau]]$-modules.

PROOF. In order to prove that $E_{\mathfrak{h}_\mathcal{A}}$ is an isomorphism we first mod out by τ and using the identifications observed in Remark 5.2.1 we get the map

$$[E_{\mathfrak{h}_\mathcal{A}}]_{\tau=0} \colon \mathfrak{h}_A \backslash \mathcal{H}_\ell(\mathcal{V}_0) \to \bigoplus_{W\in \mathrm{IrRep}_\ell(\mathfrak{h}|_\mathfrak{p})} \mathfrak{h}_{A_{N^*}} \backslash \mathcal{H}_\ell(\mathcal{V}_0 \otimes W \otimes W^*)$$

which sends the class of u to $(u_0 \otimes \epsilon(W)_0)_{W\in\mathrm{IrRep}_\ell(\mathfrak{h}|_\mathfrak{p})}$. Property (a) of $\epsilon(\lambda)$ tells us that $[E_{\mathfrak{h}_\mathcal{A}}]_{\tau=0}$ is, up to some invertible factors, the inverse of the morphism induced by the $\{b_W\}$, which we showed to be an isomorphism in Proposition 4.2.2. Since $\mathfrak{h}_\mathcal{A} \backslash \mathcal{H}_\ell(\mathcal{V})$ is finitely generated, Nakayama's lemma guarantees that $E_{\mathfrak{h}_\mathcal{A}}$ is an isomorphism. □

REMARK 5.2.7. The argument we used run similarly if instead of starting with a covering of curves over $\operatorname{Spec}(k)$, we would have considered a family of coverings $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \sigma)$ where the singular locus of X is given by one (or more) sections of π and whose normalization is a covering of versal pointed smooth curves. Using these assumptions we are able to construct the canonical smoothing $\widetilde{\mathcal{X}} \to \mathcal{X}$ of $\widetilde{X} \to X$ over $S[[\tau]]$ which is moreover a versal deformation of $(\widetilde{X} \xrightarrow{q} X \xrightarrow{\pi} S, \sigma)$. Once we have this construction, the analogue of Theorem 5.2.2 follows.

COROLLARY 5.2.8. *The sheaves of conformal blocks $\mathcal{H}_\ell(\mathcal{V}_1, \dots, \mathcal{V}_n)_{X_{univ}}$ are locally free on $\overline{\mathcal{H}ur}(\Gamma, \xi)_{g,n}$.*

PROOF. We consider only the case $n = 1$. Let $(\widetilde{X} \xrightarrow{q} X \to \operatorname{Spec}(k), \sigma_0)$ be a k-point of $\overline{\mathcal{H}\mathrm{ur}}(\Gamma, \xi)_{g,1} \setminus \mathcal{H}\mathrm{ur}(\Gamma, \xi)_{g,1}$. We are left to show that $\mathcal{H}_\ell(\mathcal{V})_{X_{univ}}$ is locally free on a neighbourhood of $(\widetilde{X} \xrightarrow{q} X \to \operatorname{Spec}(k), \sigma_0)$, i.e. that for one (hence any) versal deformation $(\widetilde{\mathcal{X}} \to \mathcal{X} \to S, \sigma)$ of $(\widetilde{X} \to X \to \operatorname{Spec}(k), \sigma_0)$, the $\mathscr{O}_S$-module $\mathcal{H}_\ell(\mathcal{V})_{\mathcal{X}}$ is locally free. Assume, for simplicity only, that $\mathfrak{p} \in X(k)$ is the only nodal point of X. Consider the normalization $(\widetilde{X}_N \to X_N, \sigma_0, \mathfrak{p}_+, \mathfrak{p}_-)$ of $(\widetilde{X} \to X, \sigma_0)$ and denote by $(\widetilde{\mathcal{X}}_N \to \mathcal{X}_N \to S, \sigma, \mathfrak{P}_+, \mathfrak{P}_-)$ its universal deformation. Since we can see $\widetilde{X} \to X$ as a fibre of the covering obtained from $\widetilde{\mathcal{X}}_N \to \mathcal{X}_N$ by identifying $\mathfrak{P}_-$ and $\mathfrak{P}_+$, the previous remark allows us to conclude. $\square$

A | The equivalence $\mathrm{Bun}_{\mathcal{H}_{\mathcal{P}}} \cong \mathrm{Bun}^{\mathcal{P}}_{\Gamma,G}$

This appendix provides a generalization of some of the results of [**BS15**] from the case in which ρ is a homomorphism $\Gamma \to G$, to the case in which $\rho \colon \Gamma \to \mathrm{Aut}(G)$ can detect also outer automorphisms of G. Along the way we clarify an issue in [**BS15**, Lemma 4.1.5] by refining the notion of local type of a (Γ, G)-bundle.

In this section we relax the assumptions on k, Γ and G as follows.

Setting and notation. Throughout this appendix we fix the following objects.

- A finite group Γ;
- A field k whose characteristic does not divide the order of Γ;
- An algebraic group G over k.
- A group homomorphism $\rho \colon \Gamma \to \mathrm{Aut}(G)$.
- A (ramified) Galois Γ-covering $\pi \colon \widetilde{Y} \to Y$ of locally Noetherian schemes, i.e.
    - π is a finite flat morphism;
    - the group of automorphisms of $\widetilde{Y}$ over Y is isomorphic to Γ:
    - $\widetilde{Y}$ is a generically étale Γ-torsor over Y via π.

The *ramification locus* of π is the subscheme of $\widetilde{Y}$ which is the support of the sheaf of relative differentials $\Omega_{\widetilde{Y}/Y}$. Its image in Y is denoted by $\mathcal{R}$ and called, by analogy with the case of the curves, the *reduced branch locus* of π.

A.0.1. (Γ, G)**-bundles.** Given a G-bundle $\mathcal{P}$ on $\widetilde{Y}$ we denote by $\mathcal{G}_{\mathcal{P}}$ the automorphisms group scheme $\mathscr{I}so_G(\mathcal{P}, \mathcal{P})$. For any other G-bundle $\mathcal{P}'$ on $\widetilde{Y}$, the scheme $\mathscr{I}_{\mathcal{P}}(\mathcal{P}') := \mathscr{I}so_G(\mathcal{P}, \mathcal{P}')$ is a $\mathcal{G}_{\mathcal{P}}$-bundle.

The following statement is a version of [**BS15**, Lemma 4.1.4].

Lemma A.0.1. *Let $\mathcal{P}'$ be a G-bundle over $\widetilde{Y}$, then $\pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}')$ is a $\pi_*\mathcal{G}_{\mathcal{P}}$-bundle.*

Proof. It is clear from the theory of Weil restriction (see for instance [**BLR90**, Section 7.6]) that $\pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}')$ and $\pi_*\mathcal{G}_{\mathcal{P}}$ are smooth schemes over X. Since fibred product and Weil restriction commute $\pi_*\mathcal{G}_{\mathcal{P}}$ still acts on $\pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}')$. Similarly we have that $\pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}') \times_Y \pi_*\mathcal{G}_{\mathcal{P}} \cong \pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}') \times_Y \pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}')$ via the canonical map $(f, g) \mapsto (f, fg)$, so we are left to prove that for every point $y \in Y(\bar{k})$ there exists an étale neighbourhood U of y such that $(\pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}'))(U) \neq \emptyset$. Since π is finite we know that $\pi^{-1}\{y\}$ is a finite scheme over $\mathrm{Spec}(\bar{k})$ over which both $\mathcal{P}$ and $\mathcal{P}'$ are trivial. It follows that the map

$q\colon \pi_*\mathscr{I}(\mathcal{P}) \to Y$ is surjective. We conclude that q is smooth and surjective, so applying [**Gro67**, Corollaire 17.16.3] for every $y \in Y$ there exists an étale neighbourhood U of y such that $(\pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}'))(U) \neq \varnothing$. □

DEFINITION A.0.2. A (Γ, G, ρ)-*bundle* or simply a (Γ, G)-*bundle*, on $\widetilde{Y}$ is a G-bundle $\mathcal{P}$ together with an action of Γ on its total space lifting the action of Γ on $\widetilde{Y}$ and which is compatible with the action of Γ on G given by ρ, i.e. for any $\gamma \in \Gamma$ we require

$$\gamma_{\mathcal{P}}(pg) = \gamma_{\mathcal{P}}(p)\rho(\gamma)(g)$$

for all $p \in \mathcal{P}$ and $g \in G$.

To every (Γ, G) bundle $\mathcal{P}$, we attach a group scheme $\mathcal{H}_{\mathcal{P}}$ on Y as follows. Let $\gamma_{\mathcal{P}}$ be the automorphism of the total space of $\mathcal{P}$ induced by γ. Then we define the action of Γ on $\mathcal{G}_{\mathcal{P}}$ via the map $\rho_{\mathcal{P}}\colon \Gamma \to \mathrm{Aut}(\mathcal{G}_{\mathcal{P}})$ given by

$$\rho_{\mathcal{P}}(\gamma)(\phi) := \gamma_{\mathcal{P}}\phi\gamma_{\mathcal{P}}^{-1}$$

for all $\gamma \in \Gamma$ and $\phi \in \mathcal{G}_{\mathcal{P}}$. The group $\mathcal{H}_{\mathcal{P}}$ is defined as $(\pi_*(\mathcal{G}_{\mathcal{P}}))^{\Gamma}$, and by [**Edi92**, Proposition 3.4], we know that it is a smooth group over Y.

Observe that for any (Γ, G)-bundle $\mathcal{P}'$, the scheme $\mathscr{I}_{\mathcal{P}}(\mathcal{P}')$ is a $(\Gamma, \mathcal{G}_{\mathcal{P}}, \rho_{\mathcal{P}})$-bundle where the action of Γ is given by

$$(\gamma, \phi) \mapsto \gamma_{\mathcal{P}}\phi\gamma_0^{-1}$$

for all $\gamma \in \Gamma$ and $\phi \in \mathscr{I}_{\mathcal{P}}(\mathcal{P}')$. It is natural to wonder whether $(\pi_*\mathscr{I}_{\mathcal{P}}(\mathcal{P}'))^{\Gamma}$ is an $\mathcal{H}_{\mathcal{P}}$-bundle. Before providing the answer (Proposition A.0.6), we first give an example showing that this is not always the case.

EXAMPLE A.0.3. Let $\Gamma = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ and $G = \mathfrak{S}_4$, the symmetric group on four elements with ρ given by $\rho(-1)(\alpha) = (34)(12)\alpha(12)(34)$. Let $\mathcal{P}_0$ be the trivial G bundle with $\rho_{\mathcal{P}_0} = \rho$ and let $\mathcal{P}$ be the (Γ, G)-bundle which is trivial as a G-bundle, but with Γ acting by $(-1)(\alpha) = (12)\rho(\gamma)(\alpha)$. Assume that $y \in Y$ is a ramification point and U a neighbourhood of y. Then we see that

$$(\pi_*\mathcal{P})^{\Gamma}(U) = \{\alpha \in \mathfrak{S}_4 \mid \alpha = (34)\alpha(12)(34)\} = \varnothing$$

but

$$(\pi_*G)^{\Gamma}(U) = \{\alpha \in \mathfrak{S}_4 \mid \alpha = (34)(12)\alpha(12)(34)\} \neq \varnothing.$$

which then tells us that $(\pi_*\mathcal{P})^{\Gamma}$ cannot be locally isomorphic to $(\pi_*G)^{\Gamma}$, hence cannot be a $(\pi_*G)^{\Gamma}$-bundle.

A.0.2. Local type of a (Γ, G)-bundle. The failure is essentially due to the fact that the required compatibility of the actions of Γ and G on $\mathcal{P}$ does not imply that $\mathcal{P}$ is locally isomorphic to G as a (Γ, G)-bundle. This shows that [**BS15**, Lemma 4.1.5] does not hold in general. To correct this problem we will refine the concept of local type.

DEFINITION A.0.4. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two (Γ, G)-bundles on $\widetilde{Y}$. Then they have the *same local type* at $y \in Y(\bar{k})$ if one of the equivalent conditions holds

(1) $\mathscr{I}so_G\left(\mathcal{P}_1 \times \pi^{-1}\{y\}, \mathcal{P}_2 \times \pi^{-1}\{y\}\right)^{\Gamma}$ is not empty;
(2) $\mathscr{I}so_G\left(\mathcal{P}_1 \times (\pi^{-1}\{y\})_{red}, \mathcal{P}_2 \times (\pi^{-1}\{y\})_{red}\right)^{\Gamma}$ is not empty.

We say that $\mathcal{P}_1$ and $\mathcal{P}_2$ have the *same local type*, and we write $\mathcal{P}_1 \sim \mathcal{P}_2$, if they have the same local type at any geometric point of Y.

We need to prove that the two conditions are equivalent, so that the notion of local type is well defined.

PROOF. We only have to prove that (2) implies (1). Let $\pi^{-1}\{y\} = \mathrm{Spec}(A)$ where A is a finite Artin k-algebra. Let $\mathfrak{m}$ be its maximal nilpotent ideal, so that $\left(\pi^{-1}\{y\}\right)_{red} = \mathrm{Spec}(A/\mathfrak{m})$. Let $\mathrm{Spec}(B) = \mathscr{I}so_G(\mathcal{P}_1, \mathcal{P}_2)$ and by assumption there exists $\varphi_0 \colon B \to A/\mathfrak{m}$ which is Γ-invariant and makes the diagram commute:

$$\begin{array}{ccc} & & B \\ & \swarrow^{\varphi_0} & \uparrow \\ A/\mathfrak{m} & \longleftarrow & A \end{array}$$

The aim is to lift φ_0 to a Γ-equivariant morphism $B \to A$. We construct this lift by induction, showing first how to find a Γ-equivariant lift $\varphi_1 \colon B \to A/\mathfrak{m}^2$ and then repeating this procedure finitely many times we obtain a Γ-invariant map $\varphi_n \colon B \to A/\mathfrak{m}^n = A$.

We reduce in this way to consider only the case $\mathfrak{m}^2 = 0$. Since B is smooth over A we know that φ_0 admits a lift $\varphi \colon B \to A$. For any $\gamma \in \Gamma$ the element $\gamma(\varphi)$ is another lift of φ_1, so the association $\gamma \mapsto \varphi - \gamma(\varphi)$ defines a map $h \colon \Gamma \to \mathrm{Der}_A(B, \mathfrak{m})$. Since h satisfies the cocyle condition, we have that $h \in H^1(\Gamma, \mathrm{Der}_A(B, \mathfrak{m}))$, which is zero because the characteristic of k does not divide the order of Γ. This means that there exists a derivation $\partial \in \mathrm{Der}_A(B, \mathfrak{m})$ such that $h(\gamma) = \gamma(\partial) - \partial$ for every $\sigma \in \Gamma$. This implies that the lift $\varphi_1 := \varphi + \partial$ is a Γ-invariant lift of φ_0 and concludes the proof. □

LEMMA A.0.5. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two (Γ, G)-bundles on $\widetilde{Y}$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same local type if and only if they have the same local type at any geometric point of $\mathcal{R}$.*

PROOF. It is sufficient to show that any two (Γ, G) bundles have the same local type on $Y \setminus \mathcal{R}$. Thus it is sufficient to prove that for every (Γ, G)-bundle $\mathcal{P}$ and for any open $U \subseteq Y$ disjoint from $\mathcal{R}$, there exists an étale covering V of U such that $\left(\pi_* \mathscr{I}so_G(G, \mathcal{P})\right)^{\Gamma}(V) \neq \emptyset$. We can moreover assume that $\mathcal{P}$ is the trivial G-bundle.

As π is étale on $Y \setminus \mathcal{R}$ we can chose $V \to U$ such that $\pi^{-1}(V) = \coprod_{\gamma \in \Gamma} V$, where the action of Γ permutes the points on the different components. We need to show that there always exists an element

$$\alpha \in \mathscr{I}so_G\left(\coprod_{\gamma \in \Gamma} G \times V, \coprod_{\gamma \in \Gamma} G \times V\right)$$

which is Γ-equivariant, where the action on γ on the source is given by $\rho(\gamma)$ and on the target by $\gamma_{\mathcal{P}}$. Giving α is equivalent to give maps $\alpha_\gamma \in \mathscr{I}so_G\left(G \times V, G \times V\right) = G(V)$ for all $\gamma \in \Gamma$, and α is Γ-invariant when $\alpha_{\gamma\sigma} \cdot (\rho(\gamma))(g) = \gamma(\alpha_\sigma \cdot g)$ for all $\gamma, \sigma \in \Gamma$. The map α defined by $\alpha_\gamma := \gamma_{\mathcal{P}}(1)$ does the job. □

PROPOSITION A.0.6. *Let $\mathcal{P}$ be a (Γ, G)-bundle over $\widetilde{Y}$. Then the sheaf $\left(\pi_* \mathscr{I}_{\mathcal{P}}(\mathcal{P}')\right)^{\Gamma}$ is an $\mathcal{H}_{\mathcal{P}}$-bundle if and only if $\mathcal{P}'$ has the same local type as $\mathcal{P}$.*

PROOF. We have already proved in Lemma A.0.1 that $\pi_* \mathscr{I}(\mathcal{P})$ is a $\pi_* \mathcal{G}_{\mathcal{P}}$-bundle, so that we have the isomorphism

$$\alpha \colon \pi_* \mathscr{I}(\mathcal{P}) \times_Y \pi_* \mathcal{G}_{\mathcal{P}} \cong \pi_* \mathscr{I}_{\mathcal{P}}(\mathcal{P}')\,(\mathcal{P}) \times_Y \pi_* \mathscr{I}_{\mathcal{P}}(\mathcal{P}')\,(\mathcal{P})$$

induced from $\mathscr{I}_\mathcal{P}(\mathcal{P}')\,(\mathcal{P}) \times_{\widetilde{Y}} \mathcal{G}_\mathcal{P} \cong \mathscr{I}_\mathcal{P}(\mathcal{P}')\,(\mathcal{P}) \times_{\widetilde{Y}} \mathscr{I}_\mathcal{P}(\mathcal{P}')\,(\mathcal{P})$. This is Γ-equivariant, hence it induces an isomorphism

$$\alpha^\Gamma \colon \left(\pi_*\mathscr{I}_\mathcal{P}(\mathcal{P}')\right)^\Gamma \times_Y \mathcal{H}_\mathcal{P} \cong \left(\pi_*\mathscr{I}_\mathcal{P}(\mathcal{P}')\right)^\Gamma \times_Y \left(\pi_*\mathscr{I}_\mathcal{P}(\mathcal{P}')\right)^\Gamma .$$

In order to finish we need to check that $\left(\pi_*\mathscr{I}_\mathcal{P}(\mathcal{P}')\right)^\Gamma$ is locally non trivial if and only if $\mathcal{P}'$ has the same local type as $\mathcal{P}$. Suppose that for every point $y \in Y$ there exists an étale neighbourhood $f\colon (u, U) \to (y, Y)$ of y such that there exists $\phi \in \left(\pi_*\mathscr{I}_\mathcal{P}(\mathcal{P}')\right)^\Gamma (U)$. This implies in particualr that the composition ϕu is an element of $\left(\pi_*\mathscr{I}_\mathcal{P}(\mathcal{P}')\right)^\Gamma (y)$ which means that $\mathcal{P}$ and $\mathcal{P}'$ have the same local type.

Conversely, assume that $\mathcal{P}'$ and $\mathcal{P}$ have the same local type. By definition this means that $\pi_*\mathscr{I}(\mathcal{P})^\Gamma (y) \neq \emptyset$ for every geometric point y. It follows that the map $q\colon \left(\pi_*\mathscr{I}(\mathcal{P})\right)^\Gamma \to Y$ is surjective on geometric points and since it is smooth, then q it is surjective. Invoking [**Gro67**, Corollaire 17.16.3] we can then conclude that for every $y \in Y$, the map q admits a section on an étale neighbourhood U of y, and so $\left(\pi_*\mathscr{I}_\mathcal{P}(\mathcal{P}')\right)^\Gamma (U) \neq \emptyset$. □

A.0.3. The equivalence $\mathbf{Bun}^{\mathcal{P}}_{(\Gamma,G)} \cong \mathbf{Bun}_{\mathcal{H}_\mathcal{P}}$. Let $\mathrm{Bun}^{\mathcal{P}}_{(\Gamma,G)}$ be the stack over Y parametrizing (Γ, G)-bundles on $\widetilde{Y}$ which have the same local type as $\mathcal{P}$ and let $\mathrm{Bun}_{\mathcal{H}_\mathcal{P}}$ be the stack parametrizing $\mathcal{H}_\mathcal{P}$-bundles over Y. The above proposition just showed that the map

$$\pi_*\mathscr{I}_\mathcal{P}(-)^\Gamma \colon \mathrm{Bun}^{\mathcal{P}}_{(\Gamma,G)} \to \mathrm{Bun}_{\mathcal{H}_\mathcal{P}}$$

is well defined. The following theorem generalizes [**BS15**, Theorem 4.1.6].

THEOREM A.0.7. *The map $\pi_*\mathscr{I}_\mathcal{P}(-)^\Gamma \colon Bun^{\mathcal{P}}_{(\Gamma,G)} \to Bun_{\mathcal{H}_\mathcal{P}}$ is an equivalence of stacks.*

PROOF. As in [**BS15**, Theorem 4.1.6], we construct the inverse to $\pi_*\mathscr{I}_\mathcal{P}(-)^\Gamma$ as

$$\pi^*(-) \times^{\pi^*\mathcal{H}_\mathcal{P}} \mathcal{P} \colon \mathrm{Bun}_{\mathcal{H}_\mathcal{P}} \to \mathrm{Bun}^{\mathcal{P}}_{(\Gamma,G)}$$

where $\pi^*\mathcal{H}_\mathcal{P}$ acts on $\mathcal{P}$ via the map $\pi^*\mathcal{H}_\mathcal{P} \to \mathcal{G}_\mathcal{P}$, provided by adjunction from the inclusion $\mathcal{H}_\mathcal{P} \to \pi_*\mathcal{G}_\mathcal{P}$.

To simplify notation we will give our definitions for $\mathcal{H}_\mathcal{P}$-bundles over Y instead of families of bundles.

First we show that for any $\mathcal{H}_\mathcal{P}$-bundle $\mathcal{F}$, the scheme $\mathcal{F}_\mathcal{P} := \pi^*(\mathcal{F}) \times^{\mathcal{H}_\mathcal{P}} \mathcal{P}$ is a (Γ, G)-bundle. Observe that it has a natural right action of G and a left action of Γ induced by the ones on $\mathcal{P}$. Let $\gamma \in \Gamma$ and $g \in G$ and consider $(f, p) \in \mathcal{F}_\mathcal{P}$. The equalities

$$\gamma((f,p)g)) = \gamma(f,pg) = (f, \gamma_0(pg)) = (f, \gamma_0(p)\rho(\gamma)(g)) = (\gamma(f,p))\,\rho(\gamma)(g)$$

tell us that $\mathcal{F}_\mathcal{P}$ is a (Γ, G)-bundle on $\widetilde{Y}$. We check that $\mathcal{F}_\mathcal{P}$ has the same local type as $\mathcal{P}$. Let y be a geometric point of Y, then then the isomorphism

$$\begin{aligned}
\mathcal{F}_\mathcal{P} \times_{\widetilde{Y}} \pi^{-1}\{y\} &= \left(\pi^*\mathcal{F} \times_{\widetilde{Y}} \pi^{-1}\{y\}\right) \times^{\pi^*\mathcal{H}_\mathcal{P} \times \pi^{-1}\{y\}} \left(\mathcal{P} \times_{\widetilde{Y}} \pi^{-1}\{y\}\right) = \\
&= \pi^*\left(\mathcal{F} \times_T \{y\}\right) \times^{\pi^*(\mathcal{H}_\mathcal{P} \times_T \{y\})} \left(\mathcal{P} \times_{\widetilde{Y}} \pi^{-1}\{y\}\right) \cong \\
&\cong \pi^*\left(\mathcal{H}_\mathcal{P} \times_T \{y\}\right) \times^{\pi^*(\mathcal{H}_\mathcal{P} \times_T \{y\})} \left(\mathcal{P} \times_{\widetilde{Y}} \pi^{-1}\{y\}\right) = \\
&= \mathcal{P} \times_{\widetilde{Y}} \pi^{-1}\{y\}
\end{aligned}$$

is Γ-invariant because it is induced by the isomorphism between the sheaves $\mathcal{F}|_y$ and $\mathcal{H}_\ell(V)_\mathcal{P}|_y$ on which Γ acts trivially. It follows that $\mathcal{F}_\mathcal{P} \sim \mathcal{P}$.

We now show that this construction provides the inverse of the map $\pi_* \mathscr{I}_\mathcal{P}(-)^\Gamma$. The assignment $f \mapsto [\phi_f \colon q \mapsto (f, p)]$ defines a morphism from $\pi^* \mathcal{F}$ to $\mathscr{I}_\mathcal{P}(\mathcal{F}_\mathcal{P})$. By pushing it down to Y and taking Γ invariants we obtain a map

$$\mathcal{F} \to \pi_* \mathscr{I}_\mathcal{P}(\mathcal{F}_\mathcal{P})^\Gamma = \pi_* \mathscr{I}_\mathcal{P}\left(\pi^* \mathcal{F} \times^{\pi^* \mathcal{H}_\mathcal{P}} \mathcal{P}\right)^\Gamma.$$

Since $\mathcal{F}$ is locally trivial, this map is $\mathcal{H}_\mathcal{P}$-equivariant, hence it is an isomorphism.

Conversely, take a (Γ, G)-bundle $\mathcal{P}'$ with the same local type as $\mathcal{P}$. Applying first $(\pi_* \mathscr{I}_\mathcal{P}(-))^\Gamma$ and then $\pi^*(-) \times^{\pi^* \mathcal{H}_\mathcal{P}} \mathcal{P}$ to $\mathcal{P}'$ we obtain

$$\pi^* \left(\pi_*(\mathscr{I}_\mathcal{P}(\mathcal{P}'))^\Gamma\right) \times^{\pi^* \mathcal{H}_\mathcal{P}} \mathcal{P}.$$

The inclusion $(\pi_*(\mathscr{I}_\mathcal{P}(\mathcal{P}'))^\Gamma \subseteq \pi_*(\mathscr{I}_\mathcal{P}(\mathcal{P}'))$ induces by adjunction the map of $\pi^* \mathcal{H}_\mathcal{P}$-bundles

$$\pi^* \left(\pi_*(\mathscr{I}_\mathcal{P}(\mathcal{P}'))^\Gamma\right) \to \mathscr{I}_\mathcal{P}(\mathcal{P}'), \quad f \mapsto \alpha_f$$

which extends to

$$\alpha \colon \pi^* \left(\pi_*(\mathscr{I}_\mathcal{P}(\mathcal{P}'))^\Gamma\right) \times^{\pi^* \mathcal{H}_\mathcal{P}} \mathcal{P} \to \mathscr{I}_\mathcal{P}(\mathcal{P}') \times^{\pi^* \mathcal{H}_\mathcal{P}} \mathcal{P}, \quad (f, p) \mapsto (\alpha_f, p).$$

The map $\beta \colon \mathscr{I}_\mathcal{P}(\mathcal{P}') \times^{\pi^* \mathcal{H}_\mathcal{P}} \mathcal{P} \to \mathcal{P}$ given by evaluation, $\beta(\phi, p) = \phi(p)$ allows us to obtain the morphism

$$\beta\alpha \colon \pi^* \left(\pi_*(\mathscr{I}_\mathcal{P}(\mathcal{P}'))^\Gamma\right) \times^{\pi^* \mathcal{H}_\mathcal{P}} \mathcal{P} \to \mathcal{P}'.$$

which we are left to show to be equivariant with respect to the actions of Γ and G. Since both α and β are G-equivariant, also their composition is. The Γ-invariance translates in showing that $\alpha_f(\gamma_\mathcal{P}(p))$ and $\gamma_{\mathcal{P}'}(\alpha_f(p))$ coincide, which holds because α_f is Γ-equivariant. □

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