Regression-based Monte Carlo methods with optimal control variates

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Zusammenfassung

In der vorliegenden Dissertation werden regressionsbasierte Monte-Carlo-Verfahren für diskretisierte Diffusionsprozesse vorgestellt. Diese Verfahren beinhalten die Konstruktion von geeigneten Kontrollvariaten, die zu einer signifikanten Reduktion der Varianz führen. Dadurch kann die Komplexität des Standard-Monte-Carlo-Ansatzes (ε^{-3} für Schemen erster Ordnung und $\varepsilon^{-2.5}$ für Schemen zweiter Ordnung) im besten Fall reduziert werden auf eine Ordnung von $\varepsilon^{-2+\delta}$ für ein beliebiges $\delta \in [0, 0.25)$, wobei ε die zu erzielende Genauigkeit bezeichnet. In der Komplexitätsanalyse werden sowohl die Fehler, die auch beim Standard-Monte-Carlo-Ansatz auftreten (Diskretisierungs- und statistischer Fehler), als auch die aus der Schätzung bedingter Erwartungswerte mittels Regression resultierenden Fehler berücksichtigt. Darüber hinaus werden verschiedene Algorithmen hergeleitet, die zwar zu einer ähnlichen theoretischen Komplexität führen, jedoch numerisch gesehen bei der Regressionsschätzung unterschiedlich stabil und genau sind. Die Effektivität dieser Algorithmen wird anhand von numerischen Beispielen veranschaulicht und mit anderen bekannten Methoden verglichen. Zudem werden geeignete Kontrollvariaten für die Bewertung von Bermuda-Optionen sowie amerikanischen Optionen basierend auf einer dualen Monte-Carlo-Methode hergeleitet. Auch hierbei ergibt sich eine signifikante Komplexitätsreduktion, sofern die zugrunde liegenden Funktionen gewisse Glattheitsannahmen erfüllen.

Contents

Chapter 1

Introduction

Monte Carlo methods belong to the class of algorithms, which use random simulations, and have become quite popular in various applications. In particular, realisations of random variables are generated via a "pseudorandom number generator" to solve specific problems, e.g. the estimation of an integral via averaging those realisations. In terms of a practical implementation, a random number generator is represented by a non-random sequence of real numbers between 0 and 1, initialised by a changing seed. Obviously, simulations on a different set of random numbers lead to different outputs. The more samples are drawn from the distribution, the smaller becomes the *variance* (statistical error) of the estimator. Since Monte Carlo methods are easily implementable, they can provide the evaluation of very complicated quantities in a simple way. The usual measure of "goodness" of a Monte Carlo algorithm, its *numerical complexity*, is defined as the minimal cost, in terms of the desired accuracy ε, needed to achieve *mean squared error* (MSE) at most of order ε^2. Let us consider the example of numerical integration via Monte Carlo simulation. Here the MSE converges to zero when the number of samples tends to infinity. More precisely, the MSE is of order $1/N$, where N is the number of samples. The requirement that the MSE is at most of order ε^2 necessitates that N should be of order ε^{-2}. The cost being of order N is then also of order ε^{-2}. Thus, the complexity here is ε^{-2} and is not affected by the dimensionality of the problem, that describes the number of variables over which we shall compute the integral in this case. In contrast, many deterministic (non-random) approaches, such as the rectangle and trapezoid methods, become too expensive and thus inefficient, as the dimensionality increases. In such approaches the complexity is typically of the form ε^{-cd}, where c is a positive constant and d is the dimension. This phenomenon of the complexity growing exponentially in the dimensionality (with the basis ε^{-1}, where we recall that ε is the desired accuracy) is often called "curse of dimensionality" (see e.g. [21]). Thus, Monte Carlo methods are especially useful in high-dimensional situations, since they do not suffer from the curse of dimensionality.

Typically one does not use Monte Carlo methods in its plain form, but rather considers *variance reduction methods*. The aim of such a variance reduction is to reduce the statistical error of the plain Monte Carlo method, leading to an acceleration of the convergence (see

e.g. [46]). Clearly, a reduction of the variance is desirable, since it shall also reduce the complexity: either, for similar cost one shall achieve a smaller error, or, to achieve similar errors, less cost shall be required.

For a detailed overview of Monte Carlo methods, including several variance reduction techniques, see e.g. [19], [21], [23] and [46].

Main goal. The starting point of this thesis is the problem to compute the expectation $\mathbb{E}[f(X_T)]$, where T is a fixed time horizon, f is a given continuous function on $\mathbb{R}^d$ and $(X_t)_{t\in[0,T]}$ is a d-dimensional diffusion process, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$, by the Itô stochastic differential equation (SDE)

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x_0 \in \mathbb{R}^d, \tag{1.1}$$

with $(W_t)_{t\in[0,T]}$ being a standard m-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion. The functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are assumed to be Lipschitz continuous, such that the SDE (1.1) has a strong solution, and pathwise uniqueness holds. In fact, the expectation $\mathbb{E}[f(X_T)]$ can alternatively be computed by solving the following parabolic Cauchy problem:

$$\frac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{on } [0,T)\times\mathbb{R}^d, \tag{1.2}$$

$$u(T,x) = f(x) \quad \text{for } x \in \mathbb{R}^d. \tag{1.3}$$

Here, $\mathcal{L}$ is the differential operator associated with the equation (1.1)

$$(\mathcal{L}u)(t,x) := \sum_{i=1}^{d} \mu_i(x)\frac{\partial u}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^\top)_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x),$$

where $\sigma^\top$ denotes the transpose of σ. Under appropriate conditions on μ, σ and f, the solution of the Cauchy problem (1.2)–(1.3) is unique in the class of solutions satisfying certain growth conditions, and the following Feynman-Kac stochastic representation (see Section 5.7 in [37], which is based on [16] and [36]) holds

$$u(t,x) = \mathbb{E}[f(X_{t,x}(T))],$$

where $X_{t,x}$ denotes the solution started at time t in point x. Moreover we have (see e.g. Newton [49])

$$u(t,X_t) = \mathbb{E}[f(X_T)|X_t], \quad \text{a.s.}$$

for $t \in [0,T]$, where we simply write X_t rather than $X_{0,x_0}(t)$. That is, for the computation of $\mathbb{E}[f(X_T)] = u(0,x_0)$ one can use both Monte Carlo and deterministic algorithms. However, the deterministic approach related becomes inefficient for high dimensional problems ($d, m \gg 5$).

The *standard Monte Carlo (SMC) approach* for estimating $\mathbb{E}[f(X_T)]$ consists of three steps: first an approximation $\overline{X}_T$ for X_T is constructed via a time discretisation of the

equation (1.1), since the exact distribution of X_T is usually not known. Next N independent copies of the approximation $\overline{X}_T$ are generated and finally a Monte Carlo estimate V_N is defined as an average of the values of f at simulated points

$$V_N := \frac{1}{N}\sum_{n=1}^{N} f\Big(\overline{X}_T^{(n)}\Big). \tag{1.4}$$

As for the complexity of the SMC approach, we would like to have

$$\mathbb{E}\left[\left(V_N - \mathbb{E}\left[f(X_T)\right]\right)^2\right] \lesssim \varepsilon^2,$$

where ε is again the accuracy to be achieved. Rewriting the MSE leads to two types of errors: a discretisation error (below we will rather write "bias") $\mathbb{E}[f(X_T)] - \mathbb{E}[f(\overline{X}_T)]$ and a Monte Carlo (statistical) error, which result from the substitution of $\mathbb{E}[f(\overline{X}_T)]$ with the sample average V_N. More precisely, it holds

$$\begin{aligned}\mathbb{E}\left[\left(V_N - \mathbb{E}\left[f(X_T)\right]\right)^2\right] &= \left(\mathbb{E}\left[V_N - f(X_T)\right]\right)^2 + \mathrm{Var}\left[V_N\right] \\ &= \left(\mathbb{E}\left[f(\overline{X}_T) - f(X_T)\right]\right)^2 + \frac{\mathrm{Var}\left[f(\overline{X}_T)\right]}{N}.\end{aligned}$$

A *discretisation scheme* has (numerically) weak convergence of order $\alpha > 0$ if, for all functions f that belong to a certain class, it holds $\left|\mathbb{E}[f(\overline{X}_T)] - \mathbb{E}[f(X_T)]\right| \leq ch^\alpha$, where h denotes the norm of the partition (in case of an equidistant partition its grid size) of $[0,T]$ on which the discretisation scheme is constructed, and c is a positive constant that does not depend on h (but depends on f). As a consequence, the complexity of the SMC approach is, independently of the dimensionalities d and m, of order

$$\mathcal{C}_{SMC} \asymp \varepsilon^{-2-\frac{1}{\alpha}}. \tag{1.5}$$

If the solution of (1.1) at point T could be computed at cost one, then the complexity would have been of order ε^{-2}. Regarding the weak convergence, in the literature they consider somewhat different classes of functions for different discretisation schemes. For instance, for a popular method of discretisation, the so called *Euler scheme*, the class of functions f is typically the class of four times continuously differentiable functions with partial derivatives of order four having polynomial growth (then the functions themselves have polynomial growth).

From now on we focus on a specific variance reduction approach, namely the method of *control variates*, which are very useful in our setting, since they can be conveniently constructed. In particular, one looks for a random variable ξ with $\mathbb{E}\xi = 0$, that can be simulated, such that the variance of the difference $f(\overline{X}_T) - \xi$ is significantly reduced, i.e.,

$$\mathrm{Var}[f(\overline{X}_T) - \xi] \ll \mathrm{Var}[f(\overline{X}_T)].$$

Then one uses the sample average

$$V_N^{CV} := \frac{1}{N}\sum_{n=1}^{N}\left[f\Big(\overline{X}_T^{(n)}\Big) - \xi^{(n)}\right] \tag{1.6}$$

instead of (1.4) to approximate $\mathbb{E}[f(\overline{X}_T)]$.

As for the construction of potential control variates, it holds (cf. [49])

$$f(X_T) = \mathbb{E}[f(X_T)] + M_T^*, \quad \text{a.s.} \tag{1.7}$$

where

$$M_T^* := \int_0^T \nabla_x u(t, X_t)\, \sigma(X_t)\, dW_t, \tag{1.8}$$

and where $\nabla_x u(t,x)$ denotes the gradient of u w.r.t. x.

The use of control variates for solving (1.1) via Monte Carlo path simulation approach was initiated by Newton [49] and further developed in Milstein and Tretyakov [45]. Since the function $u(t,x)$ is unknown, the control variate M_T^* cannot be directly computed. This is why Milstein and Tretyakov [45] proposed to use regression for getting a preliminary approximation for $u(t,x)$ in a first step. In fact, the construction of the appropriate control variates in the above two papers essentially relies on the identity (1.7) implying that the zero-mean random variable M_T^* can be viewed as an optimal control variate, since

$$\mathrm{Var}[f(X_T) - M_T^*] = \mathrm{Var}[\mathbb{E}f(X_T)] = 0.$$

However, due to the fact that we only can simulate from the distribution of the discretised process and not from the exact one, we will derive proper control variates reducing the variance of $f(\overline{X}_T)$ rather than the one of $f(X_T)$. As a by-product our control variates can be computed in a rather simple and constructive way. More importantly, we are able to get a sufficient convergence order of the resulting variance to zero at low cost, which leads to a significant complexity reduction as compared to the SMC algorithm. Another prominent example of Monte Carlo algorithms with this property is the *multilevel Monte Carlo (MLMC) algorithm* of [18], where one uses a telescoping sum to estimate $\mathbb{E}[f(X_T)]$ at different levels, that is, with different number of time steps. It turned out that the complexity of the MLMC algorithm can at best reduced down to order ε^{-2}. Further interesting approaches that reduce the complexity via deterministic quadrature-based algorithms can be found in [47] and [48]. Our aim is to derive an efficient algorithm with complexity rate better than ε^{-2}. This is achieved by using regression-type algorithms for the construction of control variates. As opposite to the SMC approach, our method takes advantage of the smoothness in μ, σ and f (which is needed for nice convergence properties of regression methods) and hence is especially efficient for smooth problems.

This dissertation is organised in the following way: in Chapter 2 we focus on regression estimates and smoothness conditions on general discretisation schemes. Chapter 3 describes the construction of control variates for schemes with Gaussian increments. The construction of control variates for weak approximation schemes is discussed in Chapter 4, where the schemes of first and second order are analysed. Chapter 5 contains a stratified regression-based variance reduction approach for weak schemes. Control variates for the pricing of early-exercise options are conducted in Chapter 6. Finally, we give an outlook in Chapter 7.

Chapter 2

Setup

In this chapter we derive connections and results which will be very important in the sequel.

In Section 2.1 we focus on Monte Carlo regression estimates. More precisely, we summarise regression error bounds for both a general framework and a specific approach, namely the piecewise polynomial regression. As for a detailed overview of regression approaches, see e.g. [24], [25], [35] and [54].

Section 2.2 presents a technical result in terms of a general class of discretisation schemes. For an overview of various discretisation schemes, see e.g. [4], [29], [30], [32] and [38].

2.1 Regression estimates and its convergence rates

This section is partly based on the paper [7].

We consider a $(d+1)$-dimensional random vector (X, Y) where X is $\mathbb{R}^d$-valued and Y is $\mathbb{R}$-valued. Suppose that we want to find an approximation which is "close to" the $\mathbb{R}$-valued function

$$a(x) := \mathbb{E}\left[Y|X = x\right]. \tag{2.1}$$

Below we present Monte Carlo algorithms which give us regression estimates for a. Moreover, we present L^2-upper bounds of the regression error.

2.1.1 Monte Carlo regression

Let us choose Q real-valued functions $\psi_1, \ldots, \psi_Q$ on $\mathbb{R}^d$ and simulate a big number[1] N_r of samples from the distributions of X and Y. In what follows these N_r samples are denoted by $\mathcal{D}_{N_r}$:

$$\mathcal{D}_{N_r} := \left\{(X^{(n)}, Y^{(n)}) : n = 1, \ldots, N_r\right\}.$$

[1] In the complexity analyses in the chapters below we show how large N_r is required to be in order to provide an estimate within some given tolerance.

Let $\beta = (\beta_1, \dots, \beta_Q)$ be a solution of the following least squares optimisation problem:

$$\operatorname{argmin}_{\beta \in \mathbb{R}^Q} \sum_{n=1}^{N_r} \left[Y^{(n)} - \sum_{k=1}^{Q} \beta_k \psi_k(X^{(n)}) \right]^2 .$$

Define an estimate for the function a via

$$\hat{a}(x) := \hat{a}(x, \mathcal{D}_{N_r}) := \sum_{k=1}^{Q} \beta_k \psi_k(x), \quad x \in \mathbb{R}^d.$$

The intermediate expression $\hat{a}(x, \mathcal{D}_{N_r})$ in the above formula emphasises that the estimates $\hat{a}$ of the functions a are random in that they depend on the simulated samples. The cost of computing β is of order $N_r Q^2$, since β is of the form $\beta = B^{-1} b$ with

$$B_{k,l} := \frac{1}{N_r} \sum_{n=1}^{N_r} \psi_k\big(X^{(n)}\big) \psi_l\big(X^{(n)}\big) \tag{2.2}$$

and

$$b_k := \frac{1}{N_r} \sum_{n=1}^{N_r} \psi_k\big(X^{(n)}\big)\, Y^{(n)},$$

where $k, l \in \{1, \dots, Q\}$.

In what follows, we use the notation $\mathbb{P}_X$ for the distribution of X. In particular, we will work with the corresponding L^2-norm:

$$\|g\|^2_{L^2(\mathbb{P}_X)} := \int_{\mathbb{R}^d} g^2(x)\, \mathbb{P}_X(dx) = \mathbb{E}\left[g^2(X)\right].$$

We assume that, for some positive constants Σ and A, it holds

(A1) $\sup_{x \in \mathbb{R}^d} \operatorname{Var}[Y | X = x] \le \Sigma < \infty$,

(A2) $\sup_{x \in \mathbb{R}^d} |a(x)| \le A < \infty$.

Next we denote by $\tilde{a}$ the truncated regression estimate, which is defined as follows:

$$\tilde{a}(x) := T_A \hat{a}(x) := \begin{cases} \hat{a}(x) & \text{if } |\hat{a}(x)| \le A, \\ A \operatorname{sgn} \hat{a}(x) & \text{otherwise.} \end{cases} \tag{2.3}$$

We again emphasise that, in fact, $\tilde{a}(x) = \tilde{a}(x, \mathcal{D}_{N_r})$, that is, the estimates $\tilde{a}$ of the functions a depend on the simulated samples.

Under (A1)–(A2) we obtain the following L^2-upper bound

$$\mathbb{E}\|\tilde{a} - a\|^2_{L^2(\mathbb{P}_X)} \le \tilde{c}\left(\Sigma + A^2(\log N_r + 1)\right) \frac{Q}{N_r} + 8 \inf_{g \in \Psi_Q} \|a - g\|^2_{L^2(\mathbb{P}_X)}, \tag{2.4}$$

where $\Psi_Q := \operatorname{span}(\{\psi_1, \dots, \psi_Q\})$ and $\tilde{c} > 0$ is a universal constant (cf. Theorem 11.3 in [24]).

Remark 2.1. When applying Theorem 11.3 in [24], we obtain actually

$$\mathbb{E}\|\tilde{a} - a\|^2_{L^2(\mathbb{P}_X)} \le \tilde{c} \max\left\{\Sigma, A^2\right\} (\log N_r + 1) \frac{Q}{N_r} + 8 \inf_{g \in \Psi_Q} \|\tilde{a} - g\|^2_{L^2(\mathbb{P}_X)}. \tag{2.5}$$

However, the maximum in (2.5) is in fact a sum of two terms $A^2(\log N_r + 1)$ and Σ so that the logarithm is only included in one term (see proof of Theorem 11.3 in [24]).

Let us introduce the assumption that the function a can be well approximated by the functions from Ψ_Q in the sense that there are constants $\kappa > 0$ and $D_\kappa > 0$ such that

$$\inf_{g \in \Psi_Q} \|a - g\|^2_{L^2(\mathbb{P}_X)} \leq \frac{D_\kappa}{Q^\kappa}. \tag{2.6}$$

Note that this is a natural condition to be satisfied for good choices of Ψ_Q.

In what follows, we perform a detailed analysis for the specific choice of the basis functions, which leads to the so-called piecewise polynomial partitioning estimates.

2.1.2 Error bounds for piecewise polynomial regression

There are different ways to choose the basis functions $\psi_1, \ldots, \psi_Q$. In this section we describe piecewise polynomial partitioning estimates and present L^2-upper bounds for the estimation error. We fix some $p \in \mathbb{N}$, which will denote the maximal degree of polynomials involved in our basis functions. The piecewise polynomial partitioning estimate of a works as follows: consider some $R > 0$ and an equidistant partition of $[-R, R]^d$ in S^d cubes $K_1, \ldots, K_{S^d}$, where $S \in \mathbb{N}$ denotes the number of equidistant subintervals of $[-R, R]$. Further, consider the basis functions $\psi_{k,1}, \ldots, \psi_{k,c_{p,d}}$ with $k \in \{1, \ldots, S^d\}$ and $c_{p,d} := \binom{p+d}{d}$ such that $\psi_{k,1}(x), \ldots, \psi_{k,c_{p,d}}(x)$ are polynomials with degree less than or equal to p for $x \in K_k$ and $\psi_{k,1}(x) = \ldots = \psi_{k,c_{p,d}}(x) = 0$ for $x \notin K_k$. Then we obtain the least squares regression estimate $\hat{a}(x)$ for $x \in \mathbb{R}^d$ as described in Subsection 2.1.1, based on $Q = S^d c_{p,d} = O(S^d p^d)$ basis functions. In particular, we have $\hat{a}(x) = 0$ for any $x \notin [-R, R]^d$. We note that the cost of computing $\hat{a}$ is $O(N_r S^d p^{2d})$ rather than $O(N_r S^{2d} p^{2d})$ due to a block diagonal matrix structure of B in (2.2). An equivalent approach, which leads to the same estimator $\hat{a}(x)$, is to perform separate regressions for each cube $K_1, \ldots, K_{S^d}$. Here, the number of basis functions at each regression is of order p^d so that the overall cost is of order $N_r S^d p^{2d}$, too.

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $h \in [1, \infty)$, we will use the notations

$$|x|_h := \left(\sum_{i=1}^d |x_i|^h \right)^{1/h}, \quad |x|_\infty := \max_{i=1,\ldots,d} |x_i|.$$

Let us define the operator D^α as follows

$$D^\alpha g(x) := \frac{\partial^{|\alpha|} g(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \tag{2.7}$$

where g is a real-valued function, $\alpha \in \mathbb{N}_0^d$ and $|\cdot|$ means the cardinality of a set. For $s \in \mathbb{N}_0$, $C > 0$ and $h \in [1, \infty]$, we say that a function $g \colon \mathbb{R}^d \to \mathbb{R}$ is $(s+1, C)$-*smooth w.r.t. the norm* $|\cdot|_h$ whenever, for all α with $|\alpha| = \sum_{i=1}^d \alpha_i = s$, we have

$$|D^\alpha g(x) - D^\alpha g(y)| \leq C|x - y|_h, \quad x, y \in \mathbb{R}^d,$$

i.e. the function $D^\alpha g$ is globally Lipschitz with the Lipschitz constant C with respect to the norm $|\cdot|_h$ on $\mathbb{R}^d$ (cf. Definition 3.3 in [24]). We assume that, for some constant $h \in [1, \infty]$ and some positive constants C_h, ν, B_ν, it holds:

(A3) a is $(p+1, C_h)$-smooth w.r.t. the norm $|\cdot|_h$,

(A4) $\mathbb{P}(|X|_\infty > R) \leq B_\nu R^{-\nu}$ for all $R > 0$.

Remark 2.2. Let us note that it is only a matter of convenience which h to choose in (A3) because all norms $|\cdot|_h$ are equivalent.

Let $\hat{a}$ be the piecewise polynomial partitioning estimate of a described in the beginning of this section. By $\tilde{a} = T_A \hat{a}$ we denote again the truncated estimate.

Lemma 2.3. *Under (A1)–(A4), we have*

$$\mathbb{E}\|\tilde{a} - a\|^2_{L^2(\mathbb{P}_X)} \leq \tilde{c}\left(\Sigma + A^2(\log N_r + 1)\right) \frac{\binom{p+d}{d} S^d}{N_r} + \frac{8\, C_h^2}{(p+1)!^2 d^{2-2/h}} \left(\frac{Rd}{S}\right)^{2(p+1)} + 8A^2 B_\nu R^{-\nu}, \tag{2.8}$$

where $\tilde{c} > 0$ is again a universal constant.

Remark 2.4. Notice that the terms in the second line of (2.8) are of order

$$\left(\frac{R}{S}\right)^{2(p+1)} + R^{-\nu}, \tag{2.9}$$

provided that we only track R and S and ignore the remaining parameters, such as p.[2] Let us assume that both terms in (2.9) are of the same order. Then we get $R = O(S^{\frac{2(p+1)}{\nu+2(p+1)}})$ and thus $R^{-\nu} = O(S^{-\frac{2\nu(p+1)}{\nu+2(p+1)}})$. Together with the fact that the overall number of basis functions Q is of order S^d, we have $R^{-\nu} = O(Q^{-\frac{2\nu(p+1)}{d(\nu+2(p+1))}})$. Hence, when in the framework of the piecewise polynomial regression approach the assumptions (A3)–(A4) are satisfied, then the assumption (2.6) in Subsection 2.1.1 is satisfied with

$$\kappa = \frac{2\nu(p+1)}{d(\nu + 2(p+1))}. \tag{2.10}$$

The only difference between the frameworks in Subsections 2.1.1 and 2.1.2 is that the cost in the more general framework in Subsection 2.1.1 is of order $N_r Q^2$, while the cost in case of the piecewise polynomial regression is $O(N_r p^{2d} S^d) = O(N_r p^d Q)$, that is $O(N_r Q)$ for fixed p.

Below we focus on discretisation schemes for the Itô process $(X_t)_{t\in[0,T]}$. In particular, using the notation (2.1), Y is related to a real-valued function which is evaluated at the discretised process at some time t_j, and X is given by the discretised process at some time t_l, where $0 \leq t_l < t_j \leq T$.

2.2 Smoothness theorem for a general setting of discretisation schemes

To begin with, let $J \in \mathbb{N}$ denote the time discretisation parameter, we set $\Delta := \frac{T}{J}$ and consider discretisation schemes defined on the grid $\{j\Delta : j = 0, \ldots, J\}$.

[2] In the complexity analyses in the chapters below we need to have $R, S \to \infty$ to make the statistical error tend to zero, whereas the remaining parameters are fixed.

Let us consider a scheme, where d-dimensional approximations $X_{\Delta,j\Delta}$, $j = 0,\dots,J$, satisfy $X_{\Delta,0} = x_0$ and

$$X_{\Delta,j\Delta} = \Phi_\Delta\left(X_{\Delta,(j-1)\Delta}, \xi_j\right), \quad j = 1,\dots,J, \tag{2.11}$$

for some Borel measurable functions $\Phi_\Delta\colon \mathbb{R}^{d\times\tilde{m}} \to \mathbb{R}^d$, where $\tilde{m} \geq m$, and for $\tilde{m}$-dimensional i.i.d. random vectors $\xi_j = (\xi_j^1,\dots,\xi_j^{\tilde{m}})^\top$ with independent coordinates satisfying $\mathbb{E}\left[\xi_j^i\right] = 0$ and $\mathrm{Var}\left[\xi_j^i\right] = 1$ for all $i = 1,\dots,\tilde{m}$, $j = 1,\dots,J$. Moreover, let $\mathcal{G}_0$ be the trivial σ-field and $\mathcal{G}_j = \sigma(\xi_1,\dots,\xi_j)$, $j = 1,\dots,J$. In the chapters below we will focus on different kinds of discretisation schemes, resulting in different convergence behaviour.

We now define the function $G_{l,j}(x)$ for $J \geq l \geq j \geq 0$, $x \in \mathbb{R}^d$, as follows

$$\begin{aligned} &G_{l,j}(x) \equiv \Phi_{\Delta,l} \circ \Phi_{\Delta,l-1} \circ \ldots \circ \Phi_{\Delta,j+1}(x), \quad l > j, \\ &G_{l,j}(x) \equiv x, \quad l = j, \end{aligned} \tag{2.12}$$

where $\Phi_{\Delta,l}(x) := \Phi_\Delta(x,\xi_l)$ for $l = 1,\dots,J$. By $\Phi_{\Delta,l}^k$, $k \in \{1,\dots,d\}$, we denote the k-th component of the function $\Phi_{\Delta,l}$. Note that it holds

$$q_j(x) := \mathbb{E}\left[f(X_{\Delta,T})\,|X_{\Delta,j\Delta} = x\right] = \mathbb{E}\left[f(G_{J,j}(x))\right] \tag{2.13}$$

with f introduced in Chapter 1.

In the next theorem we present some smoothness conditions on q_j, which will be used several times in the chapters below.

Theorem 2.5. *Let $K \in \{1,2,3\}$. Suppose that f is K times continuously differentiable with bounded partial derivatives up to order K, $\Phi_\Delta(\cdot,\xi)$ is K times continuously differentiable (for any fixed ξ), and that, for any $n \in \mathbb{N}$, $l \geq j$, $k \in \{1,\dots,d\}$, $\alpha \in \mathbb{N}_0^d$ with $1 \leq |\alpha| \leq K$, it holds*

$$\left|\mathbb{E}\left[\left(D^\alpha \Phi_{\Delta,l+1}^k(G_{l,j}(x))\right)^n \middle| \mathcal{G}_l\right]\right| \leq \begin{cases} (1 + A_n\Delta), & |\alpha| = \alpha_k = 1 \\ B_n\Delta, & (|\alpha| > 1) \vee (\alpha_k \neq 1) \end{cases} \tag{2.14}$$

with probability one for some constants $A_n > 0$, $B_n > 0$. Moreover, suppose that for any $n_1, n_2 \in \mathbb{N}$, $\alpha, \tilde{\alpha} \in \mathbb{N}_0^d$, $1 \leq |\alpha| \leq K$, $1 \leq |\tilde{\alpha}| \leq K$, $\alpha \neq \tilde{\alpha}$, it holds

$$\left|\mathbb{E}\left[\left(D^\alpha \Phi_{\Delta,l+1}^k(G_{l,j}(x))\right)^{n_1}\left(D^{\tilde{\alpha}} \Phi_{\Delta,l+1}^k(G_{l,j}(x))\right)^{n_2} \middle| \mathcal{G}_l\right]\right| \leq C_{n_1,n_2}\Delta \tag{2.15}$$

for some constants $C_{n_1,n_2} > 0$. Then we obtain for all $j \in \{0,\dots,J\}$ that q_j is K times continuously differentiable with bounded partial derivatives up to order K.

2.3 Proofs

First of all, we require the following multivariate genereliseation of Lemma 11.1 in [24] to prove Lemma 2.3:

Lemma 2.6. *Let $a\colon [0,1]^d \to \mathbb{R}$ be a $(p+1, C)$-smooth function w.r.t. the norm $|\cdot|_h$, where $d \in \mathbb{N}$, $h \in [1,\infty]$ and $p \in \mathbb{N}_0$. Further, let g be a piecewise polynomial of degree less than or equal to p w.r.t. an equidistant partition of $[0,1]^d$ in S^d cubes. Then it holds*

$$\sup_{x\in[0,1]^d} |a(x) - g(x)| \leq \frac{C}{d^{1-1/h}\,(p+1)!} \left(\frac{d}{2S}\right)^{p+1}. \tag{2.16}$$

Proof. Consider the Taylor expansion of the function a up to the degree p around $z \in (0,1)^d$:

$$a_p(x) = \sum_{n=0}^{p} \frac{1}{n!} \sum_{l_1+\ldots+l_d=n} \binom{n}{l_1,\ldots,l_d} \frac{\partial^n m(z)}{\partial x_1^{l_1}\cdots\partial x_d^{l_d}} \prod_{i=1}^{d} (x_i - z_i)^{l_i},$$

where $\binom{n}{l_1,\ldots,l_d} := \frac{n!}{l_1!\cdots l_d!}$ is the multinomial coefficient. The remainder term has the form

$$a(x) - a_p(x) = \frac{1}{p!} \int_0^1 (1-t)^p \sum_{l_1+\ldots+l_d=p+1} \binom{p+1}{l_1,\ldots,l_d} \frac{\partial^{p+1} a(z + t(x-z))}{\partial x_1^{l_1}\cdots\partial x_d^{l_d}} \prod_{i=1}^{d} (x_i - z_i)^{l_i}\, dt.$$

At first, we will focus on the case $p > 0$. For $g = a_p$ we have

$$\begin{aligned} a(x) - g(x) &= a(x) - a_{p-1}(x) - \frac{1}{p!} \sum_{l_1+\ldots+l_d=p} \binom{p}{l_1,\ldots,l_d} \frac{\partial^p a(z)}{\partial x_1^{l_1}\cdots\partial x_d^{l_d}} \prod_{i=1}^{d} (x_i - z_i)^{l_i} \\ &= \frac{1}{(p-1)!} \int_0^1 \Bigg[(1-t)^{p-1} \sum_{l_1+\ldots+l_d=p} \binom{p}{l_1,\ldots,l_d} \left(\frac{\partial^p a(z + t(x-z))}{\partial x_1^{l_1}\cdots\partial x_d^{l_d}} - \frac{\partial^p a(z)}{\partial x_1^{l_1}\cdots\partial x_d^{l_d}} \right) \\ &\qquad \cdot \prod_{i=1}^{d} (x_i - z_i)^{l_i}\, dt \Bigg]. \end{aligned}$$

Since a is $(p+1, C)$-smooth, we obtain

$$\begin{aligned} |a(x) - g(x)| &\leq \frac{C}{(p-1)!} |x-z|_h \int_0^1 t\,(1-t)^{p-1}\, dt \sum_{l_1+\ldots+l_d=p} \binom{p}{l_1,\ldots,l_d} \prod_{i=1}^{d} |x_i - z_i|^{l_i} \\ &= \frac{C}{(p+1)!} |x-z|_h \left(\sum_{i=1}^{d} |x_i - z_i| \right)^p \leq \frac{C}{(p+1)!} |x-z|_h^{p+1}\, d^{p(1-1/h)}. \end{aligned} \tag{2.17}$$

As for the remaining case $p = 0$, (2.17) also holds due to the $(p+1, C)$-smoothness assumption.

Next, we consider the equidistant partitioning of $[0,1]^d$ into S^d cubes $K_1, \ldots, K_{S^d}$ with $\bigcup_{k=1}^{S^d} K_k = [0,1]^d$. Let y_k be the midpoint of K_k. We then have $\sup_{x\in K_k} |x - y_k|_h = \frac{d^{1/h}}{2S}$ for all $k \in \{1, \ldots, S^d\}$. This finally yields (2.16). □

Proof of Lemma 2.3

The first term in (2.8) comes directly from the first term in (2.4) with Q being replaced by $\binom{p+d}{d} S^d$. Define the set

$$\Psi_{S,p} := \operatorname{span}\left(\{ \psi_{k,1}, \ldots, \psi_{k,c_{p,d}} \colon k \in \{1, \ldots, S^d\} \} \right).$$

We split the integral in (2.5) into two parts:

$$\|a-g\|^2_{L^2(\mathbb{P}_X)} = \int\limits_{[-R,R]^d} \left(a(x)-g(x)\right)^2 \mathbb{P}_X(dx) + \int\limits_{\mathbb{R}^d\setminus[-R,R]^d} a^2(x)\,\mathbb{P}_X(dx), \tag{2.18}$$

since $g(x)=0$ for $x\notin[-R,R]^d$ and $g\in\Psi_{S,p}$. The second integral in (2.18) refers to the case $|X|_\infty > R$, where we simply use Assumptions (A2) and (A4) to get

$$\begin{aligned}\int\limits_{\mathbb{R}^d\setminus[-R,R]^d} a^2\left(x\right)\mathbb{P}_X(dx) &\le \sup_{x\in\mathbb{R}^d}|a(x)|^2\,\mathbb{P}(|X|_\infty > R)\\ &\le A^2 B_\nu R^{-\nu}.\end{aligned}$$

Regarding the first integral in (2.18), we obtain by Lemma 2.6

$$\begin{aligned}\inf_{g\in\Psi_{S,p}}\int\limits_{[-R,R]^d}\left(a\left(x\right)-g(x)\right)^2\mathbb{P}_X(dx) &\le \inf_{g\in\Psi_{S,p}}\sup_{x\in[-R,R]^d}|a(x)-g(x)|^2\\ &\le \frac{C_h^2}{d^{2-2/h}\,(p+1)!^2}\left(\frac{Rd}{S}\right)^{2p+2}.\end{aligned}$$

Notice that, since we consider $[-R,R]^d$ instead of $[0,1]^d$, the expression $\frac{d}{2S}$ in (2.16) is replaced by $\frac{Rd}{S}$ because $\sup_{x\in K_k}|x-y_k|_h = \frac{Rd^{1/h}}{S}$ with y_k being the midpoint of K_k. □

Proof of Theorem 2.5

Let us begin with the case $K=1$. We have for some $k,r\in\{1,\dots,d\}$

$$\frac{\partial}{\partial x_r}G^k_{l+1,j}(x) = \sum_{s=1}^d \frac{\partial}{\partial x_s}\Phi^k_{\Delta,l+1}(G_{l,j}(x))\frac{\partial}{\partial x_r}G^s_{l,j}(x) =: \sum_{s=1}^d \gamma_s$$

and $\frac{\partial}{\partial x_r}G^s_{j+1,j}(x) = \frac{\partial}{\partial x_r}\Phi^s_\Delta\left(x,\xi_{j+1}\right)$, where $G^s_{l+1,j}$ and Φ^s_Δ, $s\in\{1,\dots,d\}$, denote the s-th component of the functions $G_{l+1,j}$ and Φ_Δ. Hence

$$\mathbb{E}\left[\left(\frac{\partial}{\partial x_r}G^k_{l+1,j}(x)\right)^2\right] \le \mathbb{E}\left[\gamma_k^2 + \sum_{s\neq k}\left(2\gamma_k\gamma_s + (d-1)\gamma_s^2\right)\right].$$

Denote

$$\rho^{r,s}_{l+1,n,1} := \mathbb{E}\left[\left(\frac{\partial}{\partial x_r}G^s_{l+1,j}(x)\right)^n\right],$$

then, due to the assumptions (2.14) and (2.15), we get for $l=j+1,\dots,J-1$,

$$\rho^{r,k}_{l+1,2,1} \le (1+A_2\Delta)\rho^{r,k}_{l,2,1} + \sum_{s\neq k}\left(C_{1,1}\Delta(\rho^{r,k}_{l,2,1}+\rho^{r,s}_{l,2,1}) + (d-1)B_2\Delta\rho^{r,s}_{l,2,1}\right).$$

Further, denote

$$\rho^r_{l+1,n,1} := \sum_{s=1}^d \rho^{r,s}_{l+1,n,1},$$

then we get

$$\rho^r_{l+1,2,1} \le (1+A_2\Delta)\rho^r_{l,2,1} + 2(d-1)C_{1,1}\Delta\rho^r_{l,2,1} + (d-1)^2B_2\Delta\rho^r_{l,2,1}.$$

This gives us

$$\rho^r_{l+1,2,1} \leq (1+\kappa_1\Delta)\rho^r_{l,2,1}$$

for some constant $\kappa_1 > 0$, leading to

$$\rho^r_{l,2,1} \leq (1+\kappa_1\Delta)^{l-j-1}\rho^r_{j+1,2,1}, \quad l = j+1,\dots,J-1, \tag{2.19}$$

where

$$\rho^r_{j+1,2,1} = \sum_{s=1}^{d} \mathbb{E}\left[\left(\frac{\partial}{\partial x_r}\Phi^s_\Delta\left(x,\xi_{j+1}\right)\right)^2\right],$$

which is bounded due to (2.14). Together with (2.19) we obtain the boundedness of $\{\rho^r_{J,2,1} : J \in \mathbb{N}\}$ and hence the boundedness of

$$\begin{aligned}\left|\frac{\partial}{\partial x_r}q_j(x)\right| &\leq \sum_{s=1}^{d}\mathbb{E}\left|\frac{\partial}{\partial x_s}f(G_{J,j}(x))\frac{\partial}{\partial x_r}G^s_{J,j}(x)\right| \\ &\leq \sum_{s=1}^{d}\sqrt{\mathbb{E}\left[\left(\frac{\partial}{\partial x_s}f(G_{J,j}(x))\right)^2\right]\rho^{r,s}_{J,2,1}}\end{aligned}$$

for all $r \in \{1,\dots,d\}$, since f is assumed to be continuously differentiable with bounded partial derivatives.

Let us proceed with the case $K = 2$. We have, due to $(\sum_{k=1}^d a_k)^n \leq d^{n-1}\sum_{k=1}^d a_k^n$,

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{\partial}{\partial x_r}G^k_{l+1,j}(x)\right)^4\right] \\ \leq\ &\mathbb{E}\left[\gamma_k^4 + \sum_{s\neq k}\left(4\gamma_k^3\gamma_s + 6(d-1)\gamma_k^2\gamma_s^2 + 4(d-1)^2\gamma_k\gamma_s^3 + (d-1)^3\gamma_s^4\right)\right]\end{aligned}$$

and thus, due to $4a^3b \leq 3a^4 + b^4$ and $2a^2b^2 \leq a^4 + b^4$,

$$\begin{aligned}\rho^{r,k}_{l+1,4,1} \leq\ &(1+A_4\Delta)\rho^{r,k}_{l,4,1} + \sum_{s\neq k}\Big(C_{3,1}\Delta(3\rho^{r,k}_{l,4,1} + \rho^{r,s}_{l,4,1}) + 3(d-1)C_{2,2}\Delta(\rho^{r,k}_{l,4,1} + \rho^{r,s}_{l,4,1}) \\ &+(d-1)^2C_{1,3}\Delta(\rho^{r,k}_{l,4,1} + 3\rho^{r,s}_{l,4,1}) + (d-1)^3B_4\Delta\rho^{r,s}_{l,4,1}\Big).\end{aligned}$$

This gives us

$$\begin{aligned}\rho^r_{l+1,4,1} \leq\ &(1+A_4\Delta)\rho^r_{l,4,1} + 4(d-1)C_{3,1}\Delta\rho^r_{l,4,1} + 6(d-1)^2C_{2,2}\Delta\rho^r_{l,4,1} \\ &+4(d-1)^3C_{1,3}\Delta\rho^r_{l,4,1} + (d-1)^4B_4\Delta\rho^r_{l,4,1}.\end{aligned}$$

Hence, we obtain

$$\rho^r_{l+1,4,1} \leq (1+\kappa_2\Delta)\rho^r_{l,4,1},$$

for some constant $\kappa_2 > 0$, leading to

$$\rho^r_{l,4,1} \leq (1+\kappa_2\Delta)^{l-j-1}\rho^r_{j+1,4,1}, \quad l = j+1,\dots,J-1,$$

where

$$\rho^r_{j+1,4,1} = \sum_{s=1}^{d}\mathbb{E}\left[\left(\frac{\partial}{\partial x_r}\Phi^s_\Delta\left(x,\xi_{j+1}\right)\right)^4\right].$$

Next, we have for some $k, o, r \in \{1, \dots, d\}$

$$\begin{aligned}\frac{\partial^2}{\partial x_r \partial x_o} G^k_{l+1,j}(x) &= \sum_{s=1}^{d} \frac{\partial}{\partial x_s} \Phi^k_{\Delta,l+1}(G_{l,j}(x)) \frac{\partial^2}{\partial x_r \partial x_o} G^s_{l,j}(x) \\ &\quad + \sum_{s,u=1}^{d} \frac{\partial^2}{\partial x_s \partial x_u} \Phi^k_{\Delta,l+1}(G_{l,j}(x)) \frac{\partial}{\partial x_r} G^s_{l,j}(x) \frac{\partial}{\partial x_o} G^u_{l,j}(x) \\ &=: \sum_{s=1}^{d} \eta_{1,s} + \sum_{s,u=1}^{d} \eta_{2,s,u}\end{aligned}$$

and $\frac{\partial^2}{\partial x_r \partial x_o} G^s_{j+1,j}(x) = \frac{\partial^2}{\partial x_r \partial x_o} \Phi^s_\Delta(x, \xi_{j+1})$. Hence

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{\partial^2}{\partial x_r \partial x_o} G^k_{l+1,j}(x)\right)^2\right] \\ \leq\ &\mathbb{E}\left[\eta^2_{1,k} + \sum_{s \neq k} \left(2\eta_{1,k}\eta_{1,s} + (d-1)\eta^2_{1,s}\right) + 2\sum_{s,u,v=1}^{d} \eta_{1,v}\eta_{2,s,u} + d^2 \sum_{s,u=1}^{d} \eta^2_{2,s,u}\right].\end{aligned}$$

Denote

$$\rho^{r,o,s}_{l+1,n,2} = \mathbb{E}\left[\left(\frac{\partial^2}{\partial x_r \partial x_o} G^s_{l+1,j}(x)\right)^n\right],$$

then we get, due to

$$\begin{aligned}2\mathbb{E}\left[XYZ\right] &\leq 2\sqrt{\mathbb{E}\left[X^2\right]} \sqrt[4]{\mathbb{E}\left[Y^4\right]} \sqrt[4]{\mathbb{E}\left[Z^4\right]} \leq \mathbb{E}\left[X^2\right] + \sqrt{\mathbb{E}\left[Y^4\right]} \sqrt{\mathbb{E}\left[Z^4\right]} \\ &\leq \mathbb{E}\left[X^2\right] + \frac{1}{2}\left(\mathbb{E}\left[Y^4\right] + \mathbb{E}\left[Z^4\right]\right),\end{aligned}$$

as well as assumptions (2.14) and (2.15),

$$\begin{aligned}\rho^{r,o,k}_{l+1,2,2} \quad \leq \quad & (1 + A_2\Delta)\rho^{r,o,k}_{l,2,2} + \sum_{s \neq k} \left(C_{1,1}\Delta(\rho^{r,o,k}_{l,2,2} + \rho^{r,o,s}_{l,2,2}) + (d-1)B_2\Delta\rho^{r,o,s}_{l,2,2}\right) \\ & + \sum_{s,u,v=1}^{d} C_{1,1}\Delta\left(\rho^{r,o,v}_{l,2,2} + \frac{1}{2}\left(\rho^{r,s}_{l,4,1} + \rho^{o,u}_{l,4,1}\right)\right) \\ & + d^2 \sum_{s,u=1}^{d} B_2\Delta\frac{1}{2}\left(\rho^{r,s}_{l,4,1} + \rho^{o,u}_{l,4,1}\right).\end{aligned}$$

Further, denote

$$\rho^{r,o}_{l+1,n,2} = \sum_{s=1}^{d} \rho^{r,o,s}_{l+1,n,2},$$

then we get for $l = j+1, \dots, J-1$,

$$\begin{aligned}\rho^{r,o}_{l+1,2,2} \quad \leq \quad & (1 + A_2\Delta)\rho^{r,o}_{l,2,2} + 2(d-1)C_{1,1}\Delta\rho^{r,o}_{l,2,2} + (d-1)^2 B_2\Delta\rho^{r,o}_{l,2,2} \\ & + d^3 C_{1,1}\Delta\left(\rho^{r,o}_{l,2,2} + \frac{1}{2}\left(\rho^{r}_{l,4,1} + \rho^{o}_{l,4,1}\right)\right) + d^4 B_2\Delta\frac{1}{2}\left(\rho^{r}_{l,4,1} + \rho^{o}_{l,4,1}\right).\end{aligned}$$

This gives us

$$\rho^{r,o}_{l+1,2,2} \leq (1 + \kappa_3\Delta)\rho^{r,o}_{l,2,2} + \kappa_4\Delta,$$

for some constants $\kappa_3, \kappa_4 > 0$, leading to

$$\rho^{r,o}_{l,2,2} \leq (1+\kappa_3\Delta)^{l-j-1}\rho^{r,o}_{j+1,2,2} + \kappa_5, \quad l = j+1, \ldots, J-1,$$

where $\kappa_5 > 0$ and

$$\rho^{r,o}_{j+1,2,2} = \sum_{s=1}^{d} \mathbb{E}\left[\left(\frac{\partial^2}{\partial x_r \partial x_o}\Phi^s_\Delta(x, \xi_{j+1})\right)^2\right].$$

Thus, we obtain the boundedness of

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x_r \partial x_o} q_j(x)\right| &\leq \sum_{s=1}^{d} \mathbb{E}\left|\frac{\partial}{\partial x_s} f(G_{J,j}(x)) \frac{\partial^2}{\partial x_r \partial x_o} G^s_{J,j}(x)\right| \\
&\quad + \sum_{s,u=1}^{d} \mathbb{E}\left|\frac{\partial^2}{\partial x_s \partial x_u} f(G_{J,j}(x)) \frac{\partial}{\partial x_r} G^s_{J,j}(x) \frac{\partial}{\partial x_o} G^u_{J,j}(x)\right| \\
&\leq \sum_{s=1}^{d} \sqrt{\mathbb{E}\left[\left(\frac{\partial}{\partial x_s} f(G_{J,j}(x))\right)^2\right] \rho^{r,o,s}_{J,2,2}} \\
&\quad + \sum_{s,u=1}^{d} \sqrt{\mathbb{E}\left[\left(\frac{\partial^2}{\partial x_s \partial x_u} f(G_{J,j}(x))\right)^2\right]} \sqrt[4]{\rho^{r,s}_{J,4,1}\rho^{o,u}_{J,4,1}}
\end{aligned}$$

for all $r, o \in \{1, \ldots, d\}$, since f is assumed to be twice continuously differentiable with bounded partial derivatives up to order 2.

Let us proceed with the final case $K = 3$. We have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{\partial}{\partial x_r} G^k_{l+1,j}(x)\right)^6\right] \\
\leq\ &\mathbb{E}\Bigg[\gamma_k^6 + \sum_{s\neq k}\big(6\gamma_k^5\gamma_s + 15(d-1)\gamma_k^4\gamma_s^2 + 20(d-1)^2\gamma_k^3\gamma_s^3 + 15(d-1)^3\gamma_k^2\gamma_s^4 \\
&\qquad + 6(d-1)^4\gamma_k\gamma_s^5 + (d-1)^5\gamma_s^6\big)\Bigg]
\end{aligned}$$

and thus, due to $6a^5b \leq 5a^6 + b^6$, $3a^4b^2 \leq 2a^6 + b^6$ and $2a^3b^3 \leq a^6 + b^6$,

$$\begin{aligned}
\rho^{r,k}_{l+1,6,1} &\leq (1 + A_6\Delta)\rho^{r,k}_{l,6,1} \\
&\quad + \sum_{s\neq k}\Big(C_{5,1}\Delta(5\rho^{r,k}_{l,6,1} + \rho^{r,s}_{l,6,1}) + 5(d-1)C_{4,2}\Delta(2\rho^{r,k}_{l,6,1} + \rho^{r,s}_{l,6,1}) \\
&\qquad + 10(d-1)^2 C_{3,3}\Delta(\rho^{r,k}_{l,6,1} + \rho^{r,s}_{l,6,1}) + 5(d-1)^3 C_{2,4}\Delta(\rho^{r,k}_{l,6,1} + 2\rho^{r,s}_{l,6,1}) \\
&\qquad + (d-1)^4 C_{1,5}\Delta(\rho^{r,k}_{l,6,1} + 5\rho^{r,s}_{l,6,1}) + (d-1)^5 B_6\Delta\rho^{r,s}_{l,6,1}\Big).
\end{aligned}$$

This gives us

$$\begin{aligned}
\rho^{r}_{l+1,6,1} &\leq (1 + A_6\Delta)\rho^{r}_{l,6,1} + 6(d-1)C_{5,1}\Delta\rho^{r}_{l,6,1} + 15(d-1)^2 C_{4,2}\Delta\rho^{r}_{l,6,1} \\
&\quad + 20(d-1)^3 C_{3,3}\Delta\rho^{r}_{l,6,1} + 15(d-1)^4 C_{2,4}\Delta\rho^{r}_{l,6,1} + 6(d-1)^5 C_{1,5}\Delta\rho^{r}_{l,6,1} \\
&\quad + (d-1)^6 B_6\Delta\rho^{r}_{l,6,1}.
\end{aligned}$$

Hence, we obtain

$$\rho^r_{l+1,6,1} \leq (1+\kappa_6\Delta)\rho^r_{l,6,1}$$

for some constant $\kappa_6 > 0$, leading to

$$\rho^r_{l,6,1} \leq (1+\kappa_6\Delta)^{l-j-1}\rho^r_{j+1,6,1}, \quad l = j+1,\ldots,J-1,$$

where

$$\rho^r_{j+1,6,1} = \sum_{s=1}^{d} \mathbb{E}\left[\left(\frac{\partial}{\partial x_r}\Phi^s_\Delta\left(x,\xi_{j+1}\right)\right)^6\right]$$

Moreover, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{\partial}{\partial x_r}G^k_{l+1,j}(x)\right)^8\right] \\
\leq\ &\mathbb{E}\Bigg[\gamma_k^8 + \sum_{s\neq k}\big(8\gamma_k^7\gamma_s + 28(d-1)\gamma_k^6\gamma_s^2 + 56(d-1)^2\gamma_k^5\gamma_s^3 + 70(d-1)^3\gamma_k^4\gamma_s^4 \\
&\qquad +56(d-1)^4\gamma_k^3\gamma_s^5 + 28(d-1)^5\gamma_k^2\gamma_s^6 + 8(d-1)^6\gamma_k\gamma_s^7 + (d-1)^7\gamma_s^8\big)\Bigg]
\end{aligned}$$

and thus, due to $8a^7b \leq 7a^8 + b^8$, $4a^6b^2 \leq 3a^8 + b^8$, $8a^5b^3 \leq 5a^8 + 3b^8$ and $2a^4b^4 \leq a^8 + b^8$,

$$\begin{aligned}
\rho^{r,k}_{l+1,8,1} \leq\ & (1+A_8\Delta)\rho^{r,k}_{l,8,1} \\
&+\sum_{s\neq k}\Big(C_{7,1}\Delta(7\rho^{r,k}_{l,8,1} + \rho^{r,s}_{l,8,1}) + 7(d-1)C_{6,2}\Delta(3\rho^{r,k}_{l,8,1} + \rho^{r,s}_{l,8,1}) \\
&\qquad + 7(d-1)^2C_{5,3}\Delta(5\rho^{r,k}_{l,8,1} + 3\rho^{r,s}_{l,8,1}) + 35(d-1)^3C_{4,4}\Delta(\rho^{r,k}_{l,8,1} + \rho^{r,s}_{l,8,1}) \\
&\qquad + 7(d-1)^4C_{3,5}\Delta(3\rho^{r,k}_{l,8,1} + 5\rho^{r,s}_{l,8,1}) + 7(d-1)^5C_{2,6}\Delta(\rho^{r,k}_{l,8,1} + 3\rho^{r,s}_{l,8,1}) \\
&\qquad +(d-1)^6C_{1,7}\Delta(\rho^{r,k}_{l,8,1} + 7\rho^{r,s}_{l,8,1}) + (d-1)^7B_8\Delta\rho^{r,s}_{l,8,1}\Big).
\end{aligned}$$

This gives us

$$\begin{aligned}
\rho^r_{l+1,8,1} \leq\ & (1+A_8\Delta)\rho^r_{l,8,1} + 8(d-1)C_{7,1}\Delta\rho^r_{l,8,1} + 28(d-1)^2C_{6,2}\Delta\rho^r_{l,8,1} \\
&+56(d-1)^3C_{5,3}\Delta\rho^r_{l,8,1} + 70(d-1)^4C_{4,4}\Delta\rho^r_{l,8,1} \\
&+56(d-1)^5C_{3,5}\Delta\rho^r_{l,8,1} + 28(d-1)^6C_{2,6}\Delta\rho^r_{l,8,1} \\
&+8(d-1)^7C_{1,7}\Delta\rho^r_{l,8,1} + (d-1)^8B_8\Delta\rho^r_{l,8,1}.
\end{aligned}$$

Hence, we obtain

$$\rho^r_{l+1,8,1} \leq (1+\kappa_7\Delta)\rho^r_{l,8,1},$$

for some constant $\kappa_7 > 0$, leading to

$$\rho^r_{l,8,1} \leq (1+\kappa_7\Delta)^{l-j-1}\rho^r_{j+1,8,1}, \quad l = j+1,\ldots,J-1,$$

where

$$\rho^r_{j+1,8,1} = \sum_{s=1}^{d} \mathbb{E}\left[\left(\frac{\partial}{\partial x_r}\Phi^s_\Delta\left(x,\xi_{j+1}\right)\right)^8\right].$$

Moreover, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{\partial^2}{\partial x_r \partial x_o} G^k_{l+1,j}(x)\right)^4\right] \\
\leq\ &\mathbb{E}\Bigg[\eta^4_{1,k} + \sum_{s\neq k}\left(4\eta^3_{1,k}\eta_{1,s} + 6(d-1)\eta^2_{1,k}\eta^2_{1,s} + 4(d-1)^2\eta_{1,k}\eta^3_{1,s} + (d-1)^3\eta^4_{1,s}\right) \\
&\quad + \sum_{s,u,v=1}^{d}\left(4d^2\eta^3_{1,v}\eta_{2,s,u} + 6d^3\eta^2_{1,v}\eta^2_{2,s,u} + 4d^4\eta_{1,v}\eta^3_{2,s,u}\right) + d^6\sum_{s,u}^{d}\eta^4_{2,s,u}\Bigg]
\end{aligned}$$

and thus, due to $4a^3bc \leq 3a^4 + \frac{1}{2}\left(b^8+c^8\right)$, $2a^2b^2c^2 \leq a^4 + \frac{1}{2}\left(b^8+c^8\right)$ and $4ab^3c^3 \leq a^4 + \frac{3}{2}\left(b^8+c^8\right)$,

$$\begin{aligned}
\rho^{r,o,k}_{l+1,4,2} \leq\ & (1+A_4\Delta)\rho^{r,o,k}_{l,4,2} \\
&+\sum_{s\neq k}\Big(C_{3,1}\Delta(3\rho^{r,o,k}_{l,4,2} + \rho^{r,o,s}_{l,4,2}) + 3(d-1)C_{2,2}\Delta(\rho^{r,o,k}_{l,4,2} + \rho^{r,o,s}_{l,4,2}) \\
&\qquad +(d-1)^2C_{1,3}\Delta(\rho^{r,o,k}_{l,4,2} + 3\rho^{r,o,s}_{l,4,2}) + (d-1)^3B_4\Delta\rho^{r,o,s}_{l,4,2}\Big) \\
&+\sum_{s,u,v=1}^{d}\Bigg(d^2C_{3,1}\Delta\left(3\rho^{r,o,v}_{l,4,2} + \frac{1}{2}\left(\rho^{r,s}_{l,8,1} + \rho^{o,u}_{l,8,1}\right)\right) \\
&\qquad + 3d^3C_{2,2}\Delta\left(\rho^{r,o,v}_{l,4,2} + \frac{1}{2}\left(\rho^{r,s}_{l,8,1} + \rho^{o,u}_{l,8,1}\right)\right) \\
&\qquad +d^4C_{1,3}\Delta\left(\rho^{r,o,v}_{l,4,2} + \frac{3}{2}\left(\rho^{r,s}_{l,8,1} + \rho^{o,u}_{l,8,1}\right)\right)\Bigg) \\
&+d^6\sum_{s,u=1}^{d}B_4\Delta\frac{1}{2}\left(\rho^{r,s}_{l,8,1} + \rho^{o,u}_{l,8,1}\right).
\end{aligned}$$

This gives us

$$\begin{aligned}
\rho^{r,o}_{l+1,4,2} \leq\ & (1+A_4\Delta)\rho^{r,o}_{l,4,2} + 4(d-1)C_{3,1}\Delta\rho^{r,o}_{l,4,2} + 6(d-1)^2C_{2,2}\Delta\rho^{r,o}_{l,4,2} \\
&+4(d-1)^3C_{1,3}\Delta\rho^{r}_{l,4,1} + (d-1)^4B_4\Delta\rho^{r,o}_{l,4,2} \\
&+d^5C_{3,1}\Delta\left(3\rho^{r,o}_{l,4,2} + \frac{1}{2}\left(\rho^{r}_{l,8,1} + \rho^{o}_{l,8,1}\right)\right) \\
&+3d^6C_{2,2}\Delta\left(\rho^{r,o}_{l,4,2} + \frac{1}{2}\left(\rho^{r}_{l,8,1} + \rho^{o}_{l,8,1}\right)\right) \\
&+d^7C_{1,3}\Delta\left(\rho^{r,o}_{l,4,2} + \frac{3}{2}\left(\rho^{r}_{l,8,1} + \rho^{o}_{l,8,1}\right)\right) + d^8B_4\Delta\frac{1}{2}\left(\rho^{r}_{l,8,1} + \rho^{o}_{l,8,1}\right).
\end{aligned}$$

Hence, we obtain

$$\rho^{r,o}_{l+1,4,2} \leq (1+\kappa_8\Delta)\rho^{r,o}_{l,4,2} + \kappa_9\Delta,$$

for some constants $\kappa_8, \kappa_9 > 0$, leading to

$$\rho^{r,o}_{l,4,2} \leq (1+\kappa_8\Delta)^{l-j-1}\rho^{r,o}_{j+1,4,2} + \kappa_{10}, \quad l = j+1, \ldots, J-1,$$

where $\kappa_{10} > 0$ and

$$\rho^{r,o}_{j+1,4,2} = \sum_{s=1}^{d} \mathbb{E}\left[\left(\frac{\partial^2}{\partial x_r \partial x_o}\Phi^s_\Delta\left(x, \xi_{j+1}\right)\right)^4\right].$$

Next, we have for some $k, o, r, z \in \{1, \ldots, d\}$

$$\begin{aligned}
&\frac{\partial^3}{\partial x_r \partial x_o \partial x_z} G^k_{l+1,j}(x) \\
=& \sum_{s=1}^{d} \frac{\partial}{\partial x_s}\Phi^k_{\Delta,l+1}(G_{l,j}(x)) \frac{\partial^3}{\partial x_r \partial x_o \partial x_z} G^s_{l,j}(x) \\
&+ \sum_{s,u=1}^{d} \frac{\partial^2}{\partial x_s \partial x_u}\Phi^k_{\Delta,l+1}(G_{l,j}(x)) \left(\frac{\partial^2}{\partial x_r \partial x_o} G^s_{l,j}(x) \frac{\partial}{\partial x_z} G^u_{l,j}(x) + \frac{\partial^2}{\partial x_r \partial x_z} G^s_{l,j}(x) \frac{\partial}{\partial x_o} G^u_{l,j}(x)\right. \\
&\qquad\qquad \left. + \frac{\partial}{\partial x_r} G^s_{l,j}(x) \frac{\partial^2}{\partial x_o \partial x_z} G^u_{l,j}(x)\right) \\
&+ \sum_{s,u,v=1}^{d} \frac{\partial^3}{\partial x_s \partial x_u \partial x_v}\Phi^k_{\Delta,l+1}(G_{l,j}(x)) \frac{\partial}{\partial x_r} G^s_{l,j}(x) \frac{\partial}{\partial x_o} G^u_{l,j}(x) \frac{\partial}{\partial x_z} G^v_{l,j}(x) \\
=:& \sum_{s=1}^{d} \psi_{1,s} + \sum_{s,u=1}^{d} \psi_{2,s,u} + \sum_{s,u,v=1}^{d} \psi_{3,s,u,v}
\end{aligned}$$

and $\frac{\partial^3}{\partial x_r \partial x_o \partial x_z} G^s_{j+1,j}(x) = \frac{\partial^3}{\partial x_r \partial x_o \partial x_z}\Phi^s_\Delta\left(x, \xi_{j+1}\right)$. Hence

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{\partial^3}{\partial x_r \partial x_o \partial x_z} G^k_{l+1,j}(x)\right)^2\right] \\
\leq\ & \mathbb{E}\left[\psi^2_{1,k} + \sum_{s \neq k}\left(2\psi_{1,k}\psi_{1,s} + (d-1)\psi^2_{1,s}\right) + 2\sum_{s,u,v=1}^{d} \psi_{1,v}\psi_{2,s,u}\right. \\
&\left. + 2\sum_{s,u,v,w=1}^{d} \psi_{1,w}\psi_{3,s,u,v} + 2d^2 \sum_{s,u=1}^{d} \psi^2_{2,s,u} + 2d^3 \sum_{s,u,v=1}^{d} \psi^2_{3,s,u,v}\right].
\end{aligned}$$

Denote

$$\rho^{r,o,z,s}_{l+1,n,3} = \mathbb{E}\left[\left(\frac{\partial^3}{\partial x_r \partial x_o \partial x_z} G^s_{l+1,j}(x)\right)^n\right],$$

then we get, due to $3a^2b^2c^2 \leq a^6 + b^6 + c^6$ and

$$\begin{aligned}
2\mathbb{E}\left[XYZU\right] \leq& 2\sqrt{\mathbb{E}\left[X^2\right]}\sqrt[6]{\mathbb{E}\left[Y^6\right]}\sqrt[6]{\mathbb{E}\left[Z^6\right]}\sqrt[6]{\mathbb{E}\left[U^6\right]} \leq \mathbb{E}\left[X^2\right] + \sqrt[3]{\mathbb{E}\left[Y^6\right]}\sqrt[3]{\mathbb{E}\left[Z^6\right]}\sqrt[3]{\mathbb{E}\left[U^6\right]} \\
&\leq \mathbb{E}\left[X^2\right] + \frac{1}{3}\left(\mathbb{E}\left[Y^6\right] + \mathbb{E}\left[Z^6\right] + \mathbb{E}\left[U^6\right]\right),
\end{aligned}$$

as well as the assumptions (2.14) and (2.15),

$$\begin{aligned}
\rho^{r,o,z,k}_{l+1,2,3} \quad \leq \quad & (1+A_2\Delta)\rho^{r,o,z,k}_{l,2,3} + \sum_{s\neq k}\left(C_{1,1}\Delta(\rho^{r,o,z,k}_{l,2,3}+\rho^{r,o,z,s}_{l,2,3}) + (d-1)B_2\Delta\rho^{r,o,z,s}_{l,2,3}\right) \\
& + \sum_{s,u,v=1}^{d} C_{1,1}\Delta\left(\rho^{r,o,z,v}_{l,2,2} + \frac{1}{2}\left(\rho^{r,s}_{l,4,1}+\rho^{o,u}_{l,4,1}+\rho^{z,u}_{l,4,1}+\rho^{r,o,s}_{l,4,2}+\rho^{r,z,s}_{l,4,2}+\rho^{o,z,u}_{l,4,2}\right)\right) \\
& + \sum_{s,u,v,w=1}^{d} C_{1,1}\Delta\left(\rho^{r,o,z,w}_{l,2,2} + \frac{1}{3}\left(\rho^{r,s}_{l,6,1}+\rho^{o,u}_{l,6,1}+\rho^{z,v}_{l,6,1}\right)\right) \\
& + 3d^2\sum_{s,u=1}^{d} B_2\Delta\left(\rho^{r,s}_{l,4,1}+\rho^{o,u}_{l,4,1}+\rho^{z,u}_{l,4,1}+\rho^{r,o,s}_{l,4,2}+\rho^{r,z,s}_{l,4,2}+\rho^{o,z,u}_{l,4,2}\right) \\
& + d^3\sum_{s,u,v=1}^{d} B_2\Delta\frac{1}{3}\left(\rho^{r,s}_{l,6,1}+\rho^{o,u}_{l,6,1}+\rho^{z,v}_{l,6,1}\right).
\end{aligned}$$

Further, denote

$$\rho^{r,o,z}_{l+1,2,3} = \sum_{s=1}^{d}\rho^{r,o,z,s}_{l+1,2,3},$$

then we get

$$\begin{aligned}
\rho^{r,o,z}_{l+1,2,3} \quad \leq \quad & (1+A_2\Delta)\rho^{r,o,z}_{l,2,2} + 2(d-1)C_{1,1}\Delta\rho^{r,o,z}_{l,2,2} + (d-1)^2B_2\Delta\rho^{r,o,z}_{l,2,2} \\
& + d^3C_{1,1}\Delta\left(\rho^{r,o,z}_{l,2,2} + \frac{1}{2}\left(\rho^{r}_{l,4,1}+\rho^{o}_{l,4,1}+\rho^{z}_{l,4,1}+\rho^{r,o}_{l,4,2}+\rho^{r,z}_{l,4,2}+\rho^{o,z}_{l,4,2}\right)\right) \\
& + d^4C_{1,1}\Delta\left(\rho^{r,o,z}_{l,2,2} + \frac{1}{3}\left(\rho^{r}_{l,6,1}+\rho^{o}_{l,6,1}+\rho^{z}_{l,6,1}\right)\right) \\
& + 3d^4B_2\Delta\left(\rho^{r}_{l,4,1}+\rho^{o}_{l,4,1}+\rho^{z}_{l,4,1}+\rho^{r,o}_{l,4,2}+\rho^{r,z}_{l,4,2}+\rho^{o,z}_{l,4,2}\right) \\
& + d^6B_2\Delta\frac{1}{3}\left(\rho^{r}_{l,6,1}+\rho^{o}_{l,6,1}+\rho^{z}_{l,6,1}\right).
\end{aligned}$$

This gives us

$$\rho^{r,o,z}_{l+1,2,3} \leq (1+\kappa_{11}\Delta)\rho^{r,o,z}_{l,2,2} + \kappa_{12}\Delta,$$

for some constants $\kappa_{11}, \kappa_{12} > 0$, leading to

$$\rho^{r,o,z}_{l,2,2} \leq (1+\kappa_{11}\Delta)^{l-j-1}\rho^{r,o,z}_{j+1,2,3} + \kappa_{13}, \quad l = j+1,\ldots,J-1,$$

where $\kappa_{13} > 0$ and

$$\rho^{r,o,z}_{j+1,2,3} = \sum_{s=1}^{d}\mathbb{E}\left[\left(\frac{\partial^3}{\partial x_r\partial x_o\partial x_z}\Phi^s_\Delta\left(x,\xi_{j+1}\right)\right)^2\right].$$

Thus, we obtain the boundedness of

$$
\begin{aligned}
&\left|\frac{\partial^3}{\partial x_r \partial x_o \partial x_z} q_j(x)\right| \\
\leq\ & \sum_{s=1}^{d} \mathbb{E}\left|\frac{\partial}{\partial x_s} f(G_{J,j}(x)) \frac{\partial^3}{\partial x_r \partial x_o \partial x_z} G^s_{J,j}(x)\right| \\
&+ \sum_{s,u=1}^{d} \mathbb{E}\left|\frac{\partial^2}{\partial x_s \partial x_u} f(G_{J,j}(x)) \left(\frac{\partial^2}{\partial x_r \partial x_o} G^s_{J,j}(x) \frac{\partial}{\partial x_z} G^u_{J,j}(x) + \frac{\partial^2}{\partial x_r \partial x_z} G^s_{J,j}(x) \frac{\partial}{\partial x_o} G^u_{J,j}(x)\right.\right. \\
&\qquad\qquad \left.\left. + \frac{\partial}{\partial x_r} G^s_{J,j}(x) \frac{\partial^2}{\partial x_o \partial x_z} G^u_{J,j}(x)\right)\right| \\
&+ \sum_{s,u,v=1}^{d} \mathbb{E}\left|\frac{\partial^3}{\partial x_s \partial x_u \partial x_v} f(G_{J,j}(x)) \frac{\partial}{\partial x_r} G^s_{J,j}(x) \frac{\partial}{\partial x_o} G^u_{J,j}(x) \frac{\partial}{\partial x_z} G^v_{J,j}(x)\right] \\
\leq\ & \sum_{s=1}^{d} \sqrt{\mathbb{E}\left[\left(\frac{\partial}{\partial x_s} f(G_{J,j}(x))\right)^2\right] \rho^{r,o,z,s}_{J,2,3}} \\
&+ \sum_{s,u=1}^{d} \sqrt{\mathbb{E}\left[\left(\frac{\partial^2}{\partial x_s \partial x_u} f(G_{J,j}(x))\right)^2\right]} \left(\sqrt[4]{\rho^{r,o,s}_{J,4,2} \rho^{z,u}_{J,4,1}} + \sqrt[4]{\rho^{r,z,s}_{J,4,2} \rho^{o,u}_{J,4,1}} + \sqrt[4]{\rho^{r,s}_{J,4,1} \rho^{o,z,u}_{J,4,2}}\right) \\
&+ \sum_{s,u,v=1}^{d} \sqrt{\mathbb{E}\left[\left(\frac{\partial^3}{\partial x_s \partial x_u \partial x_v} f(G_{J,j}(x))\right)^2\right]} \sqrt[6]{\rho^{r,s}_{J,6,1} \rho^{o,u}_{J,6,1} \rho^{z,v}_{J,6,1}}
\end{aligned}
$$

for all $r, o, z \in \{1, \ldots, d\}$, since f is assumed to be three times continuously differentiable with bounded partial derivatives up to order 3. □

Chapter 3

Regression-based variance reduction for strong approximation schemes

This chapter is mainly based on the paper [9].

Below we present control variates for schemes with Gaussian increments. In particular, we consider the Euler discretisation scheme[3] and derive truncation errors for specific control variates. Then we perform error and complexity analyses for two regression-based approaches, where the above-mentioned control variates are involved. Finally, we illustrate numerical results, in which we compare the performance of the novel algorithms with well-established ones.

3.1 Construction of control variates

To begin with, let us note that elements of $\mathbb{R}^d$ (resp. $\mathbb{R}^{1\times d}$) are understood throughout as column-vectors (resp. row-vectors). Generally, most vectors in what follows are column-vectors. However, gradients of functions and some vectors defined via them are row-vectors. Next we define $\Delta_j W := W_{j\Delta} - W_{(j-1)\Delta}$ for $j \in \{1,\dots,J\}$, and by $\Delta_j W^i$, $i \in \{1,\dots,m\}$, we denote the i-th component of the vector $\Delta_j W$. Further, for $k \in \mathbb{N}$, $H_k\colon \mathbb{R} \to \mathbb{R}$ stands for the (normalised) k-th Hermite polynomial, i.e.

$$H_k(x) := \frac{(-1)^k}{\sqrt{k!}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

Notice that $H_0 \equiv 1$, $H_1(x) = x$, $H_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$. To motivate a general construction of optimal control variates, let us first look at an example.

[3] abbreviation for Euler-Maruyama discretisation scheme, see [42]

3.1.1 Motivating example

Consider a simple one-dimensional SDE ($d = m = 1$)

$$dX_t = \sigma X_t dW_t, \quad t \in [0, T],$$

with $X_0 = x_0$, and its Euler discretisation $(X_{\Delta,j\Delta})_{j=0,\dots,J}$, where $X_{\Delta,0} = x_0$ and

$$X_{\Delta,j\Delta} = X_{\Delta,(j-1)\Delta}(1 + \sigma\Delta_j W), \quad j = 1, \dots, J.$$

Suppose that $f(x) = x^2$, that is, we would like to approximate $u(0, x_0) = \mathbb{E}[X_T^2]$. It is easy to see that $\mathbb{E}\big[X_{\Delta,T}^2\big] = x_0^2\,(1 + \sigma^2\Delta)^J$ and using a telescoping sum trick, we derive

$$X_{\Delta,T}^2 - \mathbb{E}\left[X_{\Delta,T}^2\right] = \sum_{j=1}^{J} \left(X_{\Delta,j\Delta}^2 (1 + \sigma^2\Delta)^{J-j} - X_{\Delta,(j-1)\Delta}^2 (1 + \sigma^2\Delta)^{J-j+1} \right). \tag{3.1}$$

Since $\Delta_j W = \frac{X_{\Delta,j\Delta} - X_{\Delta,(j-1)\Delta}}{\sigma X_{\Delta,(j-1)\Delta}}$, we get

$$X_{\Delta,j\Delta}^2 - X_{\Delta,(j-1)\Delta}^2 (1 + \Delta\sigma^2) = 2\sigma X_{\Delta,(j-1)\Delta}^2 \Delta_j W + \sigma^2 X_{\Delta,(j-1)\Delta}^2 \left(\Delta_j W^2 - \Delta\right).$$

As a result

$$X_{\Delta,T}^2 - \mathbb{E}\left[X_{\Delta,T}^2\right] = \sum_{j=1}^{J} \left(a_{j,1}(X_{\Delta,(j-1)\Delta}) H_1\left(\frac{\Delta_j W}{\sqrt{\Delta}}\right) + a_{j,2}(X_{\Delta,(j-1)\Delta}) H_2\left(\frac{\Delta_j W}{\sqrt{\Delta}}\right) \right) \tag{3.2}$$

with $a_{j,1}(x) = 2\sigma\sqrt{\Delta}x^2(1 + \sigma^2\Delta)^{J-j}$ and $a_{j,2}(x) = \sqrt{2}\sigma^2\Delta x^2(1 + \sigma^2\Delta)^{J-j}$. Notice that representation (3.2) has a very simple form. Furthermore, the coefficients $a_{j,1}$ and $a_{j,2}$ can be represented as conditional expectations

$$a_{j,k}(x) = \mathbb{E}\left[X_{\Delta,T}^2 \, H_k\left(\frac{\Delta_j W}{\sqrt{\Delta}}\right) \middle| \, X_{\Delta,(j-1)\Delta} = x \right], \quad k = 1, 2.$$

Thus, the control variate

$$M_{\Delta,T} := \sum_{j=1}^{J} \sum_{k=1}^{2} a_{j,k}(X_{\Delta,(j-1)\Delta}) \, H_k\left(\frac{\Delta_j W}{\sqrt{\Delta}}\right), \tag{3.3}$$

is a perfect control variate, as it satisfies $\mathrm{Var}[X_{\Delta,T}^2 - M_{\Delta,T}] = 0$. The above example encourages us to look for control variates of the form (3.3), where the coefficients $a_{j,k}(x)$ have the form of conditional expectations, which in turn can be computed by regression methods. As we will see in the next sections, such perfect control variates can be constructed in the general case.

Remark 3.1. The control variate in (3.3) is a sum over all time steps. At this point it is, therefore, unclear whether the variance reduction achieved in the proposed method outweighs the additional computational work required to implement such a control variate. After the detailed description of our algorithm we will present the complexity analysis, which shows that, given the precision ε to be achieved, implementing such a control variate results in less total computational work, provided several parameters are chosen a proper way.

3.1.2 Series representation

Let us consider a scheme, where d-dimensional approximations $X_{\Delta,j\Delta}$, $j = 0, \dots, J$, satisfy $X_{\Delta,0} = x_0$ and

$$X_{\Delta,j\Delta} = \Phi_\Delta \left(X_{\Delta,(j-1)\Delta}, \frac{\Delta_j W}{\sqrt{\Delta}} \right), \quad j = 1, \dots, J, \tag{3.4}$$

for some Borel measurable functions $\Phi_\Delta \colon \mathbb{R}^{d\times m} \to \mathbb{R}^d$. That is, relating to the framework in Section 2.2, we have $\tilde{m} = m$ and use standard normal distributed increments $\xi_j^i = \frac{\Delta_j W^i}{\sqrt{\Delta}}$, $i = 1, \dots, m$. Clearly, the (strong) Euler scheme with Φ_Δ given by

$$\Phi_\Delta(x, y) = x + \mu(x)\Delta + \sigma(x)\sqrt{\Delta}y. \tag{3.5}$$

is a special case of this setting.

Theorem 3.2. *Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable function such that it holds $\mathbb{E}|f(X_{\Delta,T})|^2 < \infty$. Then we have the representation*

$$f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J} \sum_{k\in\mathbb{N}^m\setminus\{0_m\}} a_{j,k}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{m} H_{k_i}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right), \tag{3.6}$$

where $k = (k_1, \dots, k_m)$ and $0_m := (0, \dots, 0) \in \mathbb{R}^m$ (in the second summation), and the coefficients $a_{j,k} \colon \mathbb{R}^d \to \mathbb{R}$ are given by the formula

$$a_{j,k}(x) = \mathbb{E}\left[f(X_{\Delta,T}) \prod_{i=1}^{m} H_{k_i}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right) \,\middle|\, X_{\Delta,(j-1)\Delta} = x \right], \tag{3.7}$$

for all $j \in \{1, \dots, J\}$ and $k \in \mathbb{N}^m \setminus \{0_m\}$.

Remark 3.3. (i) Representation (3.6) shows that we have a perfect control variate, namely

$$M_{\Delta,T} := \sum_{j=1}^{J} \sum_{k\in\mathbb{N}^m\setminus\{0_m\}} a_{j,k}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{m} H_{k_i}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right), \tag{3.8}$$

for the random variable $f(X_{\Delta,T})$, i.e. $\mathrm{Var}[f(X_{\Delta,T}) - M_{\Delta,T}] = 0$.

(ii) Representation (3.6) can be viewed as a discrete-time analogue of the Clark-Ocone formula. See e.g. [1] (Gaussian increments), [53] (Bernoulli increments) and the references therein for representations of similar types. Our form (3.6) is aimed at constructing control variates via regression methods.

(iii) A comparison of (3.3) and (3.6) gives rise to the question whether our motivating example fits the framework (3.6). The answer is affirmative: a straightforward calculation using the facts that $f(x) = x^2$ in the motivating example and that, for $k \geq 3$, $H_k\left(\frac{\Delta_j W}{\sqrt{\Delta}}\right)$ is orthogonal to all polynomials in $\Delta_j W$ of degree two reveals that $a_{j,k} \equiv 0$ whenever $k \geq 3$ in the situation of our motivating example.

Theorem 3.4. *Alternatively to* (3.6)*, we also have the following representation formula*

$$f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J}\sum_{k=1}^{\infty}\sum_{i=1}^{m} a_{j,k,i}(X_{\Delta,(j-1)\Delta}, (\Delta_j W^r)_{r=1}^{i-1}) H_k\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right), \quad (3.9)$$

where the coefficients $a_{j,k,i}$ *can be computed by the formula*

$$a_{j,k,i}(x,(y_r)_{r=1}^{i-1}) = \mathbb{E}\left[f(X_{\Delta,T}) H_k\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right) \big| X_{\Delta,(j-1)\Delta} = x, (\Delta_j W^r)_{r=1}^{i-1} = (y_r)_{r=1}^{i-1}\right].$$

Remark 3.5. Compared with the control variate based on (3.6), the one based on (3.9) contains a smaller number of coefficients to be computed, but these coefficients are functions of a greater number of variables (provided that we would truncate both sums on k at some level).

From the computational point of view, it is unprofitable to use the control variate based on (3.9), since the matrix B (cf. (2.2)), containing the basis functions for a regression, changes even for one fixed $j \in \{1,\dots,J\}$ due to the additional conditioning on $(\Delta_j W^r)_{r=1}^{i-1} = (y_r)_{r=1}^{i-1}$. This results in a high computational effort, whereas we can use the same matrix for one fixed j when using (3.6) (only conditioning on $X_{\Delta,(j-1)\Delta} = x$).

Let us introduce the following "truncated" control variate

$$M_{\Delta,T}^{ser} := \sum_{j=1}^{J}\sum_{i=1}^{m} a_{j,e_i}(X_{\Delta,(j-1)\Delta}) \frac{\Delta_j W^i}{\sqrt{\Delta}}, \quad (3.10)$$

where e_i denotes the i-th unit vector in $\mathbb{R}^m$ and a_{j,e_i} is given by (cf. (3.7))

$$a_{j,e_i}(x) = \mathbb{E}\left[f(X_{\Delta,T}) \frac{\Delta_j W^i}{\sqrt{\Delta}} \big| X_{\Delta,(j-1)\Delta} = x\right]. \quad (3.11)$$

Note that the superscript "ser" comes from "series".

In the next subsection we will derive another control variate, which is theoretically equivalent to $M_{\Delta,T}^{ser}$.

3.1.3 Integral representation

Below we assume $\mathbb{E}\,|f(X_{\Delta,T})| < \infty$. Let us introduce the function $u_\Delta\colon [0,T]\times\mathbb{R}^{d+m} \to \mathbb{R}$ via

$$u_\Delta(t,x,y) \equiv \mathbb{E}\left[u_\Delta\left(t_j, \Phi_\Delta\left(x, \frac{y + W_{t_j} - W_t}{\sqrt{\Delta}}\right), 0\right)\right], \quad t \in [t_{j-1}, t_j), \quad (3.12)$$
$$u_\Delta(T,x,0) \equiv f(x),$$

where $t_j := \frac{jT}{J}$, $j \in \{0,\dots,J\}$. Note that it holds

$$u_\Delta(t_j,x,0) = \mathbb{E}\left[f(X_{\Delta,T}) \big| X_{\Delta,t_j} = x\right]. \quad (3.13)$$

Theorem 3.6. *The following representation holds*

$$f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J} \int_{t_{j-1}}^{t_j} \nabla_y u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}})\, dW_t, \quad (3.14)$$

where $\nabla_y u_\Delta(t,x,y) \in \mathbb{R}^{1\times m}$ *denotes the gradient of* u_Δ *w.r.t.* y*.*

Let us define the operator (cf. (2.7))

$$D^{\alpha}_{t,y} := \frac{\partial^{|\alpha|}}{\partial t^{\alpha_1} \partial y_1^{\alpha_2} \cdots \partial y_m^{\alpha_{m+1}}}. \tag{3.15}$$

for $\alpha \in \mathbb{N}_0^{m+1}$. Next, we derive a connection between the series and integral representations.

Theorem 3.7. *Provided that it holds*

$$\left| D^{\alpha}_{t,y} \left(\frac{\partial}{\partial y_r} u_\Delta(t,x,y) \right) \right| \leq C^K \tag{3.16}$$

for all $K \in \mathbb{N}$, $|\alpha| = K$, $r \in \{1,\dots,m\}$, $t \in [t_{j-1}, t_j)$, $j \in \{1,\dots,J\}$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^m$ and for some constant $C > 0$, we have

$$f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J} \sum_{l=1}^{\infty} \Delta^{l/2} \sum_{\substack{k \in \mathbb{N}_0^m \\ \sum_{r=1}^m k_r = l}} \frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_1^{k_1} \cdots \partial y_m^{k_m}} \prod_{r=1}^{m} \frac{H_{k_r}\left(\frac{\Delta_j W^r}{\sqrt{\Delta}}\right)}{\sqrt{k_r!}} \tag{3.17}$$

whenever $0 < \Delta < \frac{1}{C^2}$. (The series converge in L^2.) Consequently, we obtain for $l = \sum_{r=1}^m k_r \in \mathbb{N}$

$$\frac{\Delta^{l/2}}{\sqrt{k_1!} \cdots \sqrt{k_m!}} \cdot \frac{\partial^l u_\Delta(t_{j-1}, x, 0)}{\partial y_1^{k_1} \cdots \partial y_m^{k_m}} = a_{j,k}(x). \tag{3.18}$$

Let us recall the control variate $M^{ser}_{\Delta,T}$ in (3.10). We get from Theorem 3.8 that $M^{ser}_{\Delta,T}$ is equivalent to the following control variate

$$M^{int}_{\Delta,T} := \sum_{j=1}^{J} \sum_{i=1}^{m} \frac{\partial u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_i} \Delta_j W^i. \tag{3.19}$$

(Note that the superscript "int" comes from "integral".) Next, we derive that the equivalence of $M^{ser}_{\Delta,T}$ and $M^{int}_{\Delta,T}$ also holds without assumption (3.16).

Theorem 3.8. *We have for $i \in \{1,\dots,m\}$*

$$a_{j,e_i}(x) = \sqrt{\Delta} \frac{\partial}{\partial y_i} u_\Delta(t_{j-1}, x, 0). \tag{3.20}$$

Let us study the order of the truncation error, which arises from replacing the control variate $M_{\Delta,T}$ in (3.8) (including infinite number of terms) by $M^{int}_{\Delta,T}$, respectively $M^{ser}_{\Delta,T}$ (including finite number of terms).

Theorem 3.9. *Provided that the function $u_\Delta(t,x,y)$ has bounded partial derivatives in y of orders 2 and 3, it holds*

$$\mathrm{Var}\left[f(X_{\Delta,T}) - M^{int}_{\Delta,T}\right] = \mathrm{Var}\left[f(X_{\Delta,T}) - M^{ser}_{\Delta,T}\right] \lesssim \Delta. \tag{3.21}$$

Below we focus on the Euler scheme (3.5) and assume that all the functions f, μ_k, σ_{ki}, $k \in \{1,\dots,d\}$, $i \in \{1,\dots,m\}$, are continuously differentiable. Let us define by I_d the identity

matrix of size d and by $\delta^i X^k_{s,x}(t) := \frac{\partial X^k_{s,x}(t)}{\partial x_i}$ the "derivative processes" for $i,k \in \{1,\dots,d\}$ and $s \le t$. Similar to the process X_t, we simply write $\delta^i X^k_t$ rather than $\delta^i X^k_{0,x_0}(t)$ for $t \in [0,T]$ and define the matrix

$$\delta X_t := \begin{pmatrix} \delta^1 X^1_t & \cdots & \delta^d X^1_t \\ \vdots & \ddots & \vdots \\ \delta^1 X^d_t & \cdots & \delta^d X^d_t \end{pmatrix} \in \mathbb{R}^{d\times d}$$

as well as the vectors $\delta^i X_t := \left(\delta^i X^1_t, \dots, \delta^i X^d_t\right)^\top \in \mathbb{R}^d$ for $i \in \{1,\dots,d\}$.

Remark 3.10. Note that $\delta^i X_t$ satisfies the following SDE

$$d\delta^i X_t = \sum_{k=1}^d \delta^i X^k_t \left[\frac{\partial \mu(X_t)}{\partial x_k}\,dt + \frac{\partial \sigma(X_t)}{\partial x_k}\,dW_t\right], \quad \delta^i X^k_0 = \begin{cases} 1, & i = k \\ 0, & i \ne k \end{cases}, \tag{3.22}$$

and it holds (cf. (1.7) and [45])

$$f(X_T) = \mathbb{E}[f(X_T)] + \int_0^T \mathbb{E}\left[\nabla f(X_T)\delta X_T \,|X_t\right] \delta X_t^{-1}\sigma(X_t)\,dW_t. \tag{3.23}$$

Regarding the Euler discretisation $\delta X_{\Delta,j\Delta} = (\delta^i X^k_{\Delta,j\Delta})_{i,k=1,\dots,d}$ of δX_t, we have

$$\delta^i X_{\Delta,t_j} = \delta^i X_{\Delta,t_{j-1}} + \sum_{k=1}^d \delta^i X^k_{\Delta,t_{j-1}} \left[\frac{\partial \mu(X_{\Delta,t_{j-1}})}{\partial x_k}\Delta + \frac{\partial \sigma(X_{\Delta,t_{j-1}})}{\partial x_k}\Delta_j W\right], \tag{3.24}$$

where $\delta X_{\Delta,0} = I_d$ and $\delta^i X_{\Delta,t_j} := \left(\delta^i X^1_{\Delta,t_j}, \dots, \delta^i X^d_{\Delta,t_j}\right)^\top \in \mathbb{R}^d$ for $i \in \{1,\dots,d\}$.

As for Theorem 3.6, we can derive the following result.

Theorem 3.11. *It holds for the Euler scheme*

$$f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \mathbb{E}\left[\nabla f(X_{\Delta,T})\delta X_{\Delta,T}\delta X^{-1}_{\Delta,t_j} \,|\mathcal{F}_t\right] \sigma(X_{\Delta,t_{j-1}})\,dW_t. \tag{3.25}$$

Remark 3.12. For $t = t_{j-1}$, $j \in \{1,\dots,J\}$, we obtain from (3.25) (cf. (3.14))

$$\nabla_y u_\Delta(t_{j-1},x,0) = \mathbb{E}\left[\nabla f(X_{\Delta,T})\delta X_{\Delta,T}\delta X^{-1}_{\Delta,t_j} \,\big|X_{\Delta,t_{j-1}} = x\right]\sigma(x). \tag{3.26}$$

Note that we may write (3.26) by means of expectations conditioned on $X_{\Delta,t_{j-1}}$ instead of $\mathcal{F}_{t_{j-1}}$, since $(X_{\Delta,t_j})_{j=0,\dots,J}$, is a Markov chain.

Let us define the function $g_j\colon \mathbb{R}^d \to \mathbb{R}^{1\times d}$, $j \in \{1,\dots,J\}$, through

$$g_j(x) = (g_{j,1}(x),\dots,g_{j,d}(x)) := \mathbb{E}\left[\nabla f(X_{\Delta,T})\delta X_{\Delta,T}\delta X^{-1}_{\Delta,t_j} \,\big|X_{\Delta,t_{j-1}} = x\right], \tag{3.27}$$

such that we have (cf. (3.26))

$$\nabla_y u_\Delta(t_{j-1},x,0) = g_j(x)\sigma(x). \tag{3.28}$$

Note that it also holds (see the proof of Theorem 3.11)

$$g_j(x) = \mathbb{E}\left[\nabla_x u_\Delta(t_j, X_{\Delta,t_j}, 0)\,\middle|\, X_{\Delta,t_{j-1}} = x\right], \tag{3.29}$$

where $\nabla_x u_\Delta(t,x,y)$ denotes the gradient of u_Δ w.r.t. x. We get that the control variate $M^{int}_{\Delta,T}$ can be expressed by means of function g_j, that is (cf. (3.19) and (3.28))

$$\begin{aligned} M^{int}_{\Delta,T} &= \sum_{j=1}^{J} g_j(X_{\Delta,t_{j-1}})\sigma(X_{\Delta,t_{j-1}})\Delta_j W \\ &= \sum_{j=1}^{J}\sum_{k=1}^{d} g_{j,k}(X_{\Delta,t_{j-1}}) \sum_{i=1}^{m} \sigma_{ki}(X_{\Delta,t_{j-1}})\Delta_j W^i. \end{aligned} \tag{3.30}$$

Remark 3.13. (i) Note that there will appear more precise assumptions on the functions f, μ, σ in Section 3.3 which ensure the assumption on u_Δ in Theorem 3.9.

(ii) Moreover, the control variate $M^{int}_{\Delta,T}$ in (3.30) differs from the one suggested in [45] only in an index concerning the inverted matrix, i.e. we have $\delta X^{-1}_{\Delta,t_j}$ inside of $g_j(X_{\Delta,t_{j-1}})$ rather than the $\mathcal{F}_{t_{j-1}}$-measurable random variable $\delta X^{-1}_{\Delta,t_{j-1}}$. In case of the exact solution, one obtains from a discretisation of the stochastic integral in (3.23)

$$\sum_{j=1}^{J} \mathbb{E}\left[\nabla f(X_T)\delta X_T \,\middle|\, X_{t_{j-1}}\right] \delta X^{-1}_{t_{j-1}}\sigma(X_{t_{j-1}})\Delta_j W$$

as an analogue of $M^{int}_{\Delta,T}$.

(iii) In order to use the control variate $M^{int}_{\Delta,T}$ in practice, we need to estimate the unknown coefficients $g_{j,k}$. Thus, practically implementable control variates $\tilde{M}^{int}_{\Delta,T}$ have the form (3.30) with some estimated functions $\tilde{g}_{j,k}\colon \mathbb{R}^d \to \mathbb{R}$. Notice that they remain valid control variates, i.e. we still have $\mathbb{E}\big[\tilde{M}^{int}_{\Delta,T}\big] = 0$, which is due to the martingale transform structure[4] in (3.30).

(iv) It is natural to expect that, under some additional conditions, a generalisation of Theorem 3.9 has the form

$$\mathrm{Var}\left[f(X_{\Delta,T}) - M^{int,(K)}_{\Delta,T}\right] = \mathrm{Var}\left[f(X_{\Delta,T}) - M^{ser,(K)}_{\Delta,T}\right] \lesssim \Delta^K,$$

where $K \in \mathbb{N}$ and

$$\begin{aligned} M^{int,(K)}_{\Delta,T} &:= \sum_{j=1}^{J}\sum_{l=1}^{K} \Delta^{l/2} \sum_{\substack{k\in\mathbb{N}_0^m \\ \sum_{r=1}^m k_r = l}} \frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_1^{k_1}\cdots\partial y_m^{k_m}} \prod_{r=1}^{m} \frac{H_{k_r}\left(\frac{\Delta_j W^r}{\sqrt{\Delta}}\right)}{\sqrt{k_r!}}, \\ M^{ser,(K)}_{\Delta,T} &:= \sum_{j=1}^{J}\sum_{l=1}^{K} \sum_{\substack{k\in\mathbb{N}_0^m \\ \sum_{r=1}^m k_r = l}} a_{j,k}(X_{\Delta,t_{j-1}}) \prod_{r=1}^{m} H_{k_r}\left(\frac{\Delta_j W^r}{\sqrt{\Delta}}\right). \end{aligned} \tag{3.31}$$

[4] This phrase means that the discrete-time process $\tilde{M} = (\tilde{M}_l)_{l=0,\dots,J}$, where $\tilde{M}_0 = 0$ and $\tilde{M}_l$ is defined like the right-hand side of (3.30) but with $\sum_{j=1}^{J}$ being replaced by $\sum_{j=1}^{l}$ and $g_{j,k}$ by $\tilde{g}_{j,k}$ is a martingale, which is a straightforward calculation.

However, on the one hand we will observe below that control variates based on the direct estimation of $a_{j,k}$ (in particular $M_{\Delta,T}^{ser,(K)}$) are numerically not convincing. On the other hand, for the implementation of an approximation for $M_{\Delta,T}^{int,(K)}$ with $K > 1$, one has to derive conditional expectation formulas for $\frac{\partial^l u_\Delta(t_{j-1},x,0)}{\partial y_1^{k_1}\cdots\partial y_m^{k_m}}$ as for $K = 1$, that is (3.26) (note that $M_{\Delta,T}^{int} = M_{\Delta,T}^{int,(1)}$ and $M_{\Delta,T}^{ser} = M_{\Delta,T}^{ser,(1)}$). In the one-dimensional case $d = m = 1$, we will derive such a representation for $K = 2$ in Subsection 3.1.4.

Regarding the convergence of the Euler scheme, we have the following result (cf. Theorem 2.1 in [44]).

Proposition 3.14. *Assume that μ and σ in* (1.1) *are Lipschitz continuous with components μ_k, $\sigma_{ki}\colon \mathbb{R}^d \to \mathbb{R}$, $k = 1,\dots,d$, $i = 1,\dots,m$, being 4 times continuously differentiable with their partial derivatives of order up to 4 having polynomial growth. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be 4 times continuously differentiable with partial derivatives of order up to 4 having polynomial growth. Then, for the Euler scheme* (3.5)*, we have*

$$|\mathbb{E}\,[f(X_T) - f(X_{\Delta,T})]| \le c\Delta, \tag{3.32}$$

where the constant c does not depend on Δ.

Notice that the assumption that, for sufficiently large $n \in \mathbb{N}$, the expectations $\mathbb{E}|X_{\Delta,j\Delta}|^{2n}$ are uniformly bounded in J and $j = 0,\dots,J$ (cf. Theorem 2.1 in [44]) is automatically satisfied for the Euler scheme because μ and σ, being globally Lipschitz, have at most linear growth.

3.1.4 Second order derivatives in the one-dimensional case

Let us consider the case $d = m = 1$ and define the "second derivative process" $\delta^2 X_{s,x}(t) := \frac{\partial^2 X_{s,x}(t)}{\partial x^2}$. Note that it holds $\frac{\partial^2 X_{s,x}(t)}{\partial x^2} = \frac{\partial \delta X_{s,x}(t)}{\partial x}$, where $\delta X_{s,x}(t)$ is defined in Chapter 1. Below we simply write $\delta^2 X_t$ rather than $\delta^2 X_{0,x_0}(t)$. The following SDE holds for $\delta^2 X_t$ (cf. [45])

$$d\delta^2 X_t = \delta^2 X_t(\mu'(X_t)\,dt + \sigma'(X_t)\,dW_t) + \delta X_t^2(\mu''(X_t)\,dt + \sigma''(X_t)\,dW_t), \quad \delta^2 X_0 = 0.$$

Regarding the Euler discretisation $\delta^2 X_{\Delta,t_j}$ for $\delta^2 X_t$ we have for $j = 1,\dots,J$

$$\begin{aligned}\delta^2 X_{\Delta,t_j} =& \delta^2 X_{\Delta,t_{j-1}}(1 + \mu'(X_{\Delta,t_{j-1}})\Delta + \sigma'(X_{\Delta,t_{j-1}})\Delta_j W)\\ &+ \delta X_{\Delta,t_{j-1}}^2(\mu''(X_{\Delta,t_{j-1}})\Delta + \sigma''(X_{\Delta,t_{j-1}})\Delta_j W).\end{aligned} \tag{3.33}$$

Proposition 3.15. *We obtain for the second derivative of the function u_Δ w.r.t. y*

$$\begin{aligned}\frac{\partial^2 u_\Delta}{\partial y^2}(t_{j-1},x,0) =& \mathbb{E}\left[f'(X_{\Delta,T})\frac{\delta^2 X_{\Delta,T}\delta X_{\Delta,t_j} - \delta^2 X_{\Delta,t_j}\delta X_{\Delta,T}}{\delta X_{\Delta,t_j}^3}\right.\\ &\left. + f''(X_{\Delta,T})\frac{\delta X_{\Delta,T}^2}{\delta X_{\Delta,t_j}^2} \,\middle|\, X_{\Delta,t_{j-1}} = x\right]\sigma^2(x).\end{aligned} \tag{3.34}$$

Consequently, in the one-dimensional case we have derived conditional expectation formulas for the first and second derivatives of u_Δ w.r.t. y and thus could implement an approximation of the control variate

$$M_{\Delta,T}^{int,(2)} = \sum_{j=1}^{J} \left(\frac{\partial}{\partial y}(t_{j-1}, X_{\Delta,t_{j-1}}, 0)\Delta_j W + \frac{\partial^2}{\partial y^2}(t_{j-1}, X_{\Delta,t_{j-1}}, 0)\frac{\Delta_j W^2 - \Delta}{2} \right).$$

However, below we will not consider control variates including second order derivatives of u_Δ (note that this will be justified in Section 3.4).

3.2 Generic regression algorithm

In the previous sections we have given several representations for the control variates. For the sake of clarity, we focus on the control variate given by (3.30), that is, we estimate the functions $g_{j,k}$ in (3.27) via regression.

3.2.1 Summary of the algorithm

The algorithm of the "integral approach" consists of two phases: "training phase" and "testing phase". In the training phase, we simulate N_r independent "training paths"

$$\mathcal{D}_{N_r}^{tr} := \left\{ (X_{\Delta,j\Delta}^{tr,(n)}, \delta X_{\Delta,j\Delta}^{tr,(n)})_{j=0,\dots,J} : n = 1, \dots, N_r \right\}$$

and construct regression estimates $\hat{g}_{j,k}(\cdot, \mathcal{D}_{N_r}^{tr})$ for the coefficients $g_{j,k}(\cdot)$, $k \in \{1, \dots, d\}$ (cf. Subsection 2.1.1). (Note that the superscript "tr" comes from "training".) In the testing phase, independently from $\mathcal{D}_{N_r}^{tr}$ we simulate N independent "testing paths" $(X_{\Delta,j\Delta}^{(n)})_{j=0,\dots,J}$, $n = 1, \dots, N$, and build the Monte Carlo estimator for $\mathbb{E}f(X_T)$ as

$$\mathcal{E} = \frac{1}{N} \sum_{n=1}^{N} \left(f(X_{\Delta,T}^{(n)}) - \hat{M}_{\Delta,T}^{int,(n)} \right), \tag{3.35}$$

where (cf. (3.30))

$$\hat{M}_{\Delta,T}^{int,(n)} := \sum_{j=1}^{J} \sum_{k=1}^{d} \hat{g}_{j,k}(X_{\Delta,t_{j-1}}^{(n)}) \sum_{i=1}^{m} \sigma_{ki}(X_{\Delta,t_{j-1}}^{(n)}) \Delta_j W^{i,(n)}. \tag{3.36}$$

Due to the martingale transform structure in (3.36) (recall footnote 4 on page 27), we have $\mathbb{E}\left[\hat{M}_{\Delta,T}^{int,(n)} | D_{N_r}^{tr} \right] = 0$, hence $\mathbb{E}[\mathcal{E}|\mathcal{D}_{N_r}^{tr}] = \mathbb{E}[f(X_{\Delta,T}^{(n)}) - \hat{M}_{\Delta,T}^{int,(n)}|\mathcal{D}_{N_r}^{tr}] = \mathbb{E}[f(X_{\Delta,T})]$, and we obtain

$$\begin{aligned} \mathrm{Var}[\mathcal{E}] &= \mathbb{E}[\mathrm{Var}(\mathcal{E}|\mathcal{D}_{N_r}^{tr})] + \mathrm{Var}[\mathbb{E}(\mathcal{E}|\mathcal{D}_{N_r}^{tr})] = \mathbb{E}[\mathrm{Var}(\mathcal{E}|\mathcal{D}_{N_r}^{tr})] \\ &= \frac{1}{N}\mathbb{E}\left[\mathrm{Var}\left[f(X_{\Delta,T}^{(1)}) - \hat{M}_{\Delta,T}^{int,(1)} | \mathcal{D}_{N_r}^{tr} \right] \right] = \frac{1}{N} \mathrm{Var}\left[f(X_{\Delta,T}^{(1)}) - \hat{M}_{\Delta,T}^{int,(1)} \right]. \end{aligned}$$

Summarising, we have

$$\mathbb{E}[\mathcal{E}] = \mathbb{E}[f(X_{\Delta,T})], \tag{3.37}$$

$$\mathrm{Var}[\mathcal{E}] = \frac{1}{N} \mathrm{Var}\left[f(X_{\Delta,T}^{(1)}) - \hat{M}_{\Delta,T}^{int,(1)} \right]. \tag{3.38}$$

Notice that the result of (3.38) indeed requires the computations above and cannot be stated right from the outset because the summands in (3.35) are dependent (through $\mathcal{D}^{tr}_{N_r}$).

This concludes the description of the generic regression algorithm for constructing the control variate. Further details, such as bounds for the right-hand side of (3.38), depend on a particular implementation, i.e. on the quality of the chosen basis functions. In what follows, we focus on the piecewise polynomial partitioning estimates.

3.3 Error bounds for piecewise polynomial regression

Below we use the notation $\mathbb{P}_{\Delta,j-1}$ for the distribution of $X_{\Delta,(j-1)\Delta}$. In particular, we will work with the corresponding L^2-norm:

$$\|g\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} := \int_{\mathbb{R}^d} g^2(x)\,\mathbb{P}_{\Delta,j-1}(dx) = \mathbb{E}\left[g^2\left(X_{\Delta,(j-1)\Delta}\right)\right].$$

We now define $\zeta_{J,j,k}$ as the k-th component of the vector

$$\zeta_{J,j} := (\zeta_{J,j,1},\dots,\zeta_{J,j,d}) := \nabla f(X_{\Delta,T})\delta X_{\Delta,T}\delta X^{-1}_{\Delta,j\Delta}$$

and remark that $g_{j,k}(x) = \mathbb{E}[\zeta_{J,j,k}|X_{\Delta,(j-1)\Delta} = x]$. In what follows, we consider the following assumptions: there exist $h \in [1,\infty]$ and positive constants $\Sigma, A, C_h, \nu, B_\nu$ such that, for all $J \in \mathbb{N}$, $j \in \{1,\dots,J\}$ and $k \in \{1,\dots,d\}$, it holds:

(A1) $\sup_{x\in\mathbb{R}^d} \mathrm{Var}\left[\zeta_{J,j,k}|X_{\Delta,(j-1)\Delta} = x\right] \le \Sigma < \infty$,

(A2) $\sup_{x\in\mathbb{R}^d} |g_{j,k}(x)| \le A < \infty$,

(A3) $g_{j,k}$ is $(p+1, C_h)$-smooth w.r.t. the norm $|\cdot|_h$,

(A4) $\mathbb{P}(|X_{\Delta,(j-1)\Delta}|_\infty > R) \le B_\nu R^{-\nu}$ for all $R > 0$.

Remark 3.16. As we mentioned in Subsection 2.1.2, it is only a matter of convenience which h to choose in (A3) because all norms $|\cdot|_h$ are equivalent. Moreover, since μ and σ are assumed to be globally Lipschitz, hence have linear growth, then, given any $\nu > 0$, (A4) is satisfied with a sufficiently large $B_\nu > 0$. In other words, (A4) is needed only to introduce the constant B_ν, which appears in the formulations below.

In the next theorem we present, based on Theorem 2.5, sufficient conditions in terms of the functions f, μ and σ that imply the preceding assumptions for the Euler scheme.

Theorem 3.17. *(i) Let all functions f, μ_k, σ_{ki}, $k \in \{1,\dots,d\}$, $i \in \{1,\dots,m\}$, be continuously differentiable with bounded partial derivatives. Then (A1) and (A2) hold.*

(ii) Moreover, if all functions σ_{ki} are bounded and all functions f, μ_k, σ_{ki} are 3 times continuously differentiable with bounded partial derivatives up to order 3, then the function $u_\Delta(t,x,y)$ has bounded partial derivatives in y up to order 3. In particular, (3.21) *holds true.*

Remark 3.18. As a generalisation of Theorem 3.17, it is natural to expect that assumption (A3) is satisfied with a sufficiently large constant $C_h > 0$ if all the functions f, μ_k, σ_{ki} are $(p+2)$ times continuously differentiable with bounded partial derivatives up to order $p+2$ (might be proven by di Bruno's formula, see e.g. [17]).

Let $\hat{g}_{j,k}$ be the piecewise polynomial partitioning estimate of $g_{j,k}$ described in Subsection 2.1.2. By $\tilde{g}_{j,k} = T_A \hat{g}_{j,k}$ we denote the truncated estimate (see (2.3)).

Under (A1)–(A4), we have due to Theorem 2.3

$$\mathbb{E}\|\tilde{g}_{j,k} - g_{j,k}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \leq \tilde{c}\left(\Sigma + A^2(\log N_r + 1)\right) \frac{\binom{p+d}{d} S^d}{N_r} + \frac{8\, C_h^2}{(p+1)!^2 d^{2-2/h}} \left(\frac{Rd}{S}\right)^{2p+2} + 8A^2 B_\nu R^{-\nu}, \tag{3.39}$$

where $\tilde{c}$ is a universal constant. It is worth noting that the expectation in the left-hand side of (3.39) accounts for the averaging over the randomness in $\mathcal{D}^{tr}_{N_r}$. To explain this in more detail, let $(X_{\Delta,j\Delta})_{j=0,\dots,J}$ be a "testing path" which is independent of the training paths $\mathcal{D}^{tr}_{N_r}$. Then it holds

$$\begin{aligned}\|\tilde{g}_{j,k} - g_{j,k}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} &\equiv \|\tilde{g}_{j,k}(\cdot, \mathcal{D}^{tr}_{N_r}) - g_{j,k}(\cdot)\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \\ &= \mathbb{E}\left[\left(\tilde{g}_{j,k}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) - g_{j,k}(X_{\Delta,(j-1)\Delta})\right)^2 \mid \mathcal{D}^{tr}_{N_r}\right],\end{aligned}$$

hence,

$$\mathbb{E}\|\tilde{g}_{j,k} - g_{j,k}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} = \mathbb{E}\left[\left(\tilde{g}_{j,k}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) - g_{j,k}(X_{\Delta,(j-1)\Delta})\right)^2\right], \tag{3.40}$$

which provides an alternative form for the expression in the left-hand side of (3.39).

Next we estimate the variance of the random variable $f(X_{\Delta,T}) - \tilde{M}^{int}_{\Delta,T}$, where

$$\tilde{M}^{int}_{\Delta,T} := \sum_{j=1}^{J} \sum_{k=1}^{d} \tilde{g}_{j,k}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) \sum_{i=1}^{m} \sigma_{ki}(X_{\Delta,(j-1)\Delta}) \Delta_j W^i. \tag{3.41}$$

Theorem 3.19. *Let us assume* $\sup_{x\in\mathbb{R}^d} |\sigma_{ki}(x)| \leq \sigma_{\max} < \infty$ *for all* $k \in \{1,\dots,d\}$ *and* $i \in \{1,\dots,m\}$. *Then we have under (A1)–(A4)*

$$\begin{aligned}\mathrm{Var}[f(X_{\Delta,T}) - \tilde{M}^{int}_{\Delta,T}] \;\lesssim\; &\frac{1}{J} + d^2 T m \sigma^2_{\max} \left\{ \tilde{c}\left(\Sigma + A^2(\log N_r + 1)\right) \frac{\binom{p+d}{d} S^d}{N_r} \right. \\ &\left. + \frac{8\, C_h^2}{(p+1)!^2 d^{2-2/h}} \left(\frac{Rd}{S}\right)^{2p+2} + 8A^2 B_\nu R^{-\nu} \right\}. \end{aligned} \tag{3.42}$$

In the case of piecewise polynomial regression, the estimator $\mathcal{E}$ given in (3.35) with "hat" replaced by "tilde" is an unbiased estimator of $\mathbb{E}[f(X_{\Delta,T})]$, and, by (3.38), the upper bound for its variance is $\frac{1}{N}$ times the last expression in (3.42).

3.4 Complexity analysis

3.4.1 Integral approach

Below we present a complexity analysis which explains how we can asymptotically approach the complexity order $\varepsilon^{-2}\sqrt{|\log(\varepsilon)|}$ with ε being the precision to be achieved.[5]

For the integral approach we perform d regressions in the training phase and d evaluations of $\tilde{g}_{j,k}$ in the testing phase (using the regression coefficients from the training phase) at each time step. Therefore, the overall cost is of order

$$JS^d dc_{p,d} \max\left\{c_{p,d}N_r, N\right\}, \tag{3.43}$$

where $c_{p,d} := \binom{p+d}{p}$. We have the following constraints

$$\max\left\{\frac{1}{J^2}, \frac{1}{JN}, \frac{S^d d^2 m c_{p,d}\log(N_r)}{N_r N}, \frac{d^2 m}{(p+1)!^2 N}\left(\frac{Rd}{S}\right)^{2(p+1)}, \frac{d^2 m B_\nu}{NR^\nu}\right\} \lesssim \varepsilon^2, \tag{3.44}$$

to ensure a MSE of order ε^2. Note that the first term in (3.44) comes from the squared bias of the estimator (due to (3.32) and $\mathbb{E}[\tilde{M}^{int}_{\Delta,T}] = 0$) and the remaining four ones come from the variance of the estimator (see (3.42) and (3.35)).

Theorem 3.20. *We obtain the following solution for the integral approach (provided that*[6] *it holds* $2(p+1) > d$ *and* $\nu > \frac{2d(p+1)}{2(p+1)-d}$*)*

$$\begin{aligned} &J \asymp \varepsilon^{-1}, \quad S \asymp C_S \cdot \varepsilon^{-\frac{2\nu+4(p+1)}{d\nu+2(p+1)(d+2\nu)}}, \quad R \asymp C_R \cdot \varepsilon^{-\frac{4(p+1)}{d\nu+2(p+1)(d+2\nu)}}, \\ &N_r \asymp C_{N_r} \cdot \varepsilon^{-\frac{2d\nu+4(p+1)(d+\nu)}{d\nu+2(p+1)(d+2\nu)}}\sqrt{|\log(\varepsilon)|}, \quad N \asymp C_N \cdot \varepsilon^{-\frac{2d\nu+4(p+1)(d+\nu)}{d\nu+2(p+1)(d+2\nu)}}\sqrt{|\log(\varepsilon)|} \asymp N_r, \end{aligned} \tag{3.45}$$

where

$$\begin{aligned} C_S &:= \left[\frac{B_\nu^{4(p+1)} d^{2\nu+4(p+1)(\nu+1)} m^{\nu+2(p+1)}}{c_{p,d}^{2\nu+4(p+1)}(p+1)!^{4\nu}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}}, \\ C_R &:= \left[\frac{B_\nu^{d+4(p+1)}(p+1)!^{2d} m^{2(p+1)}}{c_{p,d}^{4(p+1)} d^{2(p+1)(d-2)}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}}, \\ C_{N_r} &:= c_{d,\nu,p}\left[\frac{B_\nu^{2d(p+1)} d^{2d\nu+2(p+1)(d\nu+2d+2\nu)} m^{d\nu+2(p+1)(d+\nu)}}{c_{p,d}^{d\nu+2d(p+1)}(p+1)!^{2d\nu}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}}, \\ C_N &:= c_{d,\nu,p}\left[\frac{B_\nu^{2d(p+1)} c_{p,d}^{4\nu(p+1)} d^{2d\nu+2(p+1)(d\nu+2d+2\nu)} m^{d\nu+2(p+1)(d+\nu)}}{(p+1)!^{2d\nu}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}} = C_{N_r} c_{p,d}, \end{aligned}$$

[5] Notice that the multilevel Monte Carlo (MLMC) algorithm can at best achieve the complexity of order ε^{-2}.

[6] When deriving the solution via Lagrange multipliers (cf. proof of Theorem 3.20) one can see that these parameter values are *not* optimal if $p \leq \frac{d-2}{2}$ or $\nu \leq \frac{2d(p+1)}{2(p+1)-d}$ (a Lagrange multiplier corresponding to a "≤ 0" constraint is negative). Therefore, the recommendation is to choose $p \in \mathbb{N}$ and $\nu > 0$ according to $p > \frac{d-2}{2}$ and $\nu > \frac{2d(p+1)}{2(p+1)-d}$. The opposite choice is allowed as well (the method converges), but theoretical complexity of the method would be then worse than that of the SMC.

and $c_{d,\nu,p} := \sqrt{\frac{2d\nu+4(p+1)(d+\nu)}{d\nu+2(p+1)(d+2\nu)}}$. *Thus, we have for the complexity*

$$\mathcal{C}_{int} \asymp JS^d N_r \asymp JS^d N \asymp C_{\mathcal{C}} \cdot \varepsilon^{-\frac{5d\nu+2(p+1)(5d+4\nu)}{d\nu+2(p+1)(d+2\nu)}} \sqrt{|\log(\varepsilon)|}, \tag{3.46}$$

where

$$C_{\mathcal{C}} := c_{d,\nu,p} \left[\frac{B_\nu^{6d(p+1)} c_{p,d}^{2(p+1)(4\nu-d)-d\nu} d^{5d\nu+2(p+1)(3d\nu+5d+4\nu)} m^{3d\nu+6(p+1)(d+\nu)}}{(p+1)!^{6d\nu}} \right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}}$$
$$= dc_{p,d}^2 C_S^d C_{N_r} = dc_{p,d} C_S^d C_N.$$

Remark 3.21. For the sake of comparison with the SMC and MLMC approaches, we recall at this point that their complexities are

$$\mathcal{C}_{SMC} \asymp \varepsilon^{-3} \quad \text{and} \quad \mathcal{C}_{MLMC} \asymp \varepsilon^{-2}$$

at best (referring to (1.5), we have $\alpha = 1$ here).

(i) Complexity estimate (3.46) shows that one can asymptotically approach the complexity order $\varepsilon^{-2}\sqrt{|\log(\varepsilon)|}$, when $p, \nu \to \infty$, i.e. if the coefficients $g_{j,k}$ are smooth enough and the solution X of SDE (1.1) lives in a compact set.

(ii) For all p, ν with $2(p+1) > d$ and $\nu > \frac{2d(p+1)}{2(p+1)-d}$, $\mathcal{C}_{int}$ gives us a better complexity order compared to $\mathcal{C}_{smc}$, since we have in this case for the exponent of ε in (3.46)

$$\frac{5d\nu + 2(p+1)(5d+4\nu)}{d\nu + 2(p+1)(d+2\nu)} = 3 - \frac{2(\nu(2(p+1)-d) - 2d(p+1))}{d\nu + 2(p+1)(d+2\nu)} < 3.$$

(iii) We would have obtained the same complexity even when the variance in (3.21) were of order Δ^K with $K > 1$ (in particular using control variates of the form (3.31)). This is due to the fact that the second constraint in (3.44) is the only inactive one and this would still hold if the condition were $\frac{1}{J^K N} \lesssim \varepsilon^2$. Hence, it is not useful to derive a control variate with a higher variance order for the Euler scheme.

(iv) Since the exponent of the base ε in (3.46) converges when $d \to \infty$ for all p, ν, the integral approach does not suffer from the curse of dimensionality[7], which is mentioned in Chapter 1.

(v) Let us also note that the constant $C_{\mathcal{C}}$ in (3.46) tends to infinity (with exponential growth in d) when $p, \nu \to \infty$. Indeed, even if we ignore the constant B_ν, we get

$$C_{\mathcal{C}} \overset{p,\nu\to\infty}{\propto} \left[\frac{c_{p,d}^{2(p+1)(4\nu-d)-d\nu}}{(p+1)!^{6d\nu}} \right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}} \propto \left[\frac{p^{8dp\nu}}{p^{6pd\nu}} \right]^{\frac{1}{4p\nu}} = p^{\frac{d}{2}}.$$

This is a negative by-product of the piecewise polynomial regression. In Chapter 5 we will present a complexity analysis under a more general framework, similar to Subsection 2.1.1. Then we will find out criteria for which the constant converges when the corresponding parameter tends to infinity.

[7] This statement also holds true for each of the novel algorithms presented below.

3.4.2 Series approach

Below we present a complexity analysis for the series representation, defined in Subsection 3.1.2. Again we focus on the Euler scheme (3.5). Then we compare the resulting complexity with the one in (3.46). Similar to Section 3.3, we define $\zeta_{J,j,i}$ as the i-th component of the vector $\zeta_{J,j} = (\zeta_{J,j,1}, \dots, \zeta_{J,j,m})^\top := f(X_{\Delta,T})\frac{\Delta_j W}{\sqrt{\Delta}}$ and remark that $a_{j,e_i}(x) = \mathbb{E}[\zeta_{J,j,i}|X_{\Delta,(j-1)\Delta} = x]$. We will work under the following assumptions: there exist $h \in [1,\infty]$ and positive constants Σ, A, C_h such that, for all $J \in \mathbb{N}$, $j \in \{1,\dots,J\}$ and $i \in \{1,\dots,m\}$, it holds:

(B1) $\sup_{x\in\mathbb{R}^d} \operatorname{Var}\left[\zeta_{j,i}|X_{\Delta,(j-1)\Delta} = x\right] \le \Sigma < \infty$,

(B2) $\sup_{x\in\mathbb{R}^d} |a_{j,e_i}(x)| \le A\sqrt{\Delta} < \infty$,

(B3) a_{j,e_i} is $(p+1, C_h)$-smooth w.r.t. the norm $|\cdot|_h$.

Remark 3.22. (i) Note the difference between (B2) and (A2) of Section 3.3, while (B1) has the same form as (A1). This is due to (3.18), hence the additional factor $\sqrt{\Delta}$ in (B2).

(ii) A sufficient condition to ensure assumption (B1) is given by f being bounded (which is not required for the integral approach), since

$$\begin{aligned}\operatorname{Var}\left[\zeta_{J,j,i}|X_{\Delta,(j-1)\Delta} = x\right] &\le \mathbb{E}\left[f^2(X_{\Delta,T})\frac{(\Delta_j W^i)^2}{\Delta}\,\middle|\,X_{\Delta,(j-1)\Delta} = x\right]\\ &\le \sqrt{3\cdot\mathbb{E}\left[f^4(X_{\Delta,T})|X_{\Delta,(j-1)\Delta} = x\right]}.\end{aligned}$$

(iii) Further, we get, by means of Theorem 2.5, (3.18) and (3.28), that assumption (B2) is satisfied if all functions σ_{ki} are bounded and all functions f, μ_k, σ_{ki}, $k \in \{1,\dots,d\}$, $i \in \{1,\dots,m\}$, are continuously differentiable with bounded partial derivatives.

In what follows the N_r training paths are denoted by

$$\mathcal{D}^{tr}_{N_r} := \left\{(X^{tr,(n)}_{\Delta,j\Delta})_{j=0,\dots,J} : n = 1,\dots,N_r\right\}, \tag{3.47}$$

that is, we do not need to simulate paths for the discretised derivative processes $\delta X_{\Delta,j\Delta}$. Let $\hat{a}_{j,e_i}$ be the piecewise polynomial partitioning estimate of a_{j,e_i} described in Section 3.3. By $\tilde{a}_{j,e_i}$ we denote the truncated estimate, which is defined as follows:

$$\tilde{a}_{j,e_i}(x) = T_{A\sqrt{\Delta}}\hat{a}_{j,e_i}(x) = \begin{cases}\hat{a}_{j,e_i}(x) & \text{if } |\hat{a}_{j,e_i}(x)| \le A\sqrt{\Delta},\\ A\sqrt{\Delta}\operatorname{sgn}\hat{a}_{j,e_i}(x) & \text{otherwise.}\end{cases} \tag{3.48}$$

Under (B1)–(B3) and (A4), we have due to Lemma 2.3

$$\begin{aligned}\mathbb{E}\|\tilde{a}_{j,e_i} - a_{j,e_i}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \le \tilde{c}\left(\Sigma + A^2\Delta(\log N_r + 1)\right)\frac{c_{p,d}S^d}{N_r}\\ + \frac{8\,C_h^2}{(p+1)!^2 d^{2-\frac{2}{h}}}\left(\frac{R}{S}\right)^{2p+2} + 8A^2\Delta B_\nu R^{-\nu},\end{aligned} \tag{3.49}$$

where $\tilde{c}$ is a universal constant. Let us now estimate the variance of the random variable $f(X_{\Delta,T}) - \tilde{M}^{ser}_{\Delta,T}$, where

$$\tilde{M}^{ser}_{\Delta,T} := \sum_{j=1}^{J} \sum_{i=1}^{m} \tilde{a}_{j,e_i}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) \frac{\Delta_j W^i}{\sqrt{\Delta}}. \tag{3.50}$$

Theorem 3.23. *Under (B1)–(B3) and (A4), we have*

$$\begin{aligned} \mathrm{Var}[f(X_{\Delta,T}) - \tilde{M}^{ser}_{\Delta,T}] \quad \lesssim \quad & \frac{1}{J} + Jm \left\{ \tilde{c} \left(\Sigma + A^2 \Delta (\log N_r + 1) \right) \frac{c_{p,d} S^d}{N_r} \right. \\ & \left. + \frac{8\, C_h^2}{(p+1)!^2 d^{2-\frac{2}{h}}} \left(\frac{R}{S} \right)^{2p+2} + 8A^2 \Delta B_\nu R^{-\nu} \right\}. \end{aligned} \tag{3.51}$$

Let us study the complexity of the following "series approach": in the training phase, we simulate N_r independent training paths $\mathcal{D}^{tr}_{N_r}$ and construct regression estimates $\tilde{a}_{j,e_i}(\cdot, \mathcal{D}^{tr}_{N_r})$ for the coefficients $a_{j,e_i}(\cdot)$, $i \in \{1, \dots, m\}$. In the testing phase, independently from $\mathcal{D}^{tr}_{N_r}$ we simulate N independent testing paths $(X^{(n)}_{\Delta, j\Delta})_{j=0,\dots,J}$, $n = 1, \dots, N$, and build the Monte Carlo estimator for $\mathbb{E}f(X_T)$ as

$$\frac{1}{N} \sum_{n=1}^{N} \left(f(X^{(n)}_{\Delta,T}) - \tilde{M}^{ser,(n)}_{\Delta,T} \right). \tag{3.52}$$

Therefore, the overall cost is of order

$$JS^d m c_{p,d} \max\left\{ c_{p,d} N_r, N \right\}. \tag{3.53}$$

The expectation of the estimator in (3.52) equals $\mathbb{E}f(X_{\Delta,T})$, and the upper bound for the variance is $\frac{1}{N}$ times the expression in (3.51). Hence, we have the following constraints

$$\max\left\{ \frac{1}{J^2}, \frac{1}{JN}, \frac{JS^d m c_{p,d}}{N_r N}, \frac{Jm}{(p+1)!^2 N} \left(\frac{Rd}{S} \right)^{2(p+1)}, \frac{m B_\nu}{N R^\nu} \right\} \lesssim \varepsilon^2, \tag{3.54}$$

to ensure a MSE of order ε^2 (due to $\mathbb{E}[M^{ser}_{\Delta,T}] = 0$ as well as (3.51) and (3.52)). Note that there is no longer a log-term in (3.54). This is due to the factor Δ in (3.51) such that Σ is of a higher order, compared to $A^2\Delta(\log N_r + 1)$.

Theorem 3.24. *We obtain the following solution for the series approach (provided that*[8] *it holds* $2(p+1) > d$ *and* $\nu > \frac{2(p+1)}{2(p+1)-d}$*)*

$$\begin{aligned} & J \asymp \varepsilon^{-1}, \quad S \asymp C_S \cdot \varepsilon^{-\frac{3\nu+2(p+1)}{d\nu+2(p+1)(d+2\nu)}}, \quad R \asymp C_R \cdot \varepsilon^{-\frac{2(p+1)-d}{d\nu+2(p+1)(d+2\nu)}}, \\ & N_r \asymp C_{N_r} \cdot \varepsilon^{-\frac{3d\nu+2(p+1)(2d+3\nu)}{d\nu+2(p+1)(d+2\nu)}}, \quad N \asymp C_N \cdot \varepsilon^{-\frac{3d\nu+2(p+1)(2d+3\nu)}{d\nu+2(p+1)(d+2\nu)}} \asymp N_r, \end{aligned} \tag{3.55}$$

[8] Footnote 6 on page 32 applies.

where

$$C_S := \left[\frac{B_\nu^{4(p+1)} d^{4\nu(p+1)} m^{\nu+2(p+1)}}{c_{p,d}^{2\nu+4(p+1)}(p+1)!^{4\nu}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}},$$

$$C_R := \left[\frac{B_\nu^{d+4(p+1)}(p+1)!^{2d} m^{2(p+1)}}{c_{p,d}^{4(p+1)} d^{2d(p+1)}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}},$$

$$C_{N_r} := \left[\frac{B_\nu^{2d(p+1)} d^{2d\nu(p+1)} m^{d\nu+2(p+1)(d+\nu)}}{c_{p,d}^{d\nu+2d(p+1)}(p+1)!^{2d\nu}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}},$$

$$C_N := \left[\frac{B_\nu^{2d(p+1)} c_{p,d}^{4\nu(p+1)} d^{2d\nu(p+1)} m^{d\nu+2(p+1)(d+\nu)}}{(p+1)!^{2d\nu}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}} = C_{N_r} c_{p,d}.$$

Thus, we have for the complexity

$$\mathcal{C}_{ser} \asymp JS^d N_r \asymp JS^d N \asymp C_{\mathcal{C}} \cdot \varepsilon^{-\frac{7d\nu+2(p+1)(4d+5\nu)}{d\nu+2(p+1)(d+2\nu)}}, \tag{3.56}$$

where

$$C_{\mathcal{C}} := c_{d,\nu,p}\left[\frac{B_\nu^{6d(p+1)} c_{p,d}^{2(p+1)(4\nu-d)-d\nu} d^{6d\nu(p+1)} m^{3d\nu+6(p+1)(d+\nu)}}{(p+1)!^{6d\nu}}\right]^{\frac{1}{d\nu+2(p+1)(d+2\nu)}}$$
$$= m c_{p,d}^2 C_S^d C_{N_r} = m c_{p,d} C_S^d C_N.$$

3.4.3 Discussion

(i) Complexity estimate (3.56) shows that one cannot go beyond the complexity order $\varepsilon^{-2.5}$ in this case, no matter how large p, ν are. This is mainly due to the factor J within the third constraint in (3.54) which does not arise in (3.44).

(ii) To get a better complexity order for $\mathcal{C}_{ser}$ compared to $\mathcal{C}_{SMC}$, we need to choose $p+1 > 2d$ and $\nu > \frac{d(p+1)}{p+1-2d}$, since we have for the exponent of ε in (3.56)

$$\frac{7d\nu + 2(p+1)(4d+5\nu)}{d\nu + 2(p+1)(d+2\nu)} = 3 - \frac{2(\nu(p+1-2d) - d(p+1))}{d\nu + 2(p+1)(d+2\nu)}.$$

(iii) When comparing (3.56) with (3.46), one clearly sees that (3.46) always achieves a better complexity for $\nu > \frac{2(p+1)}{2(p+1)-d}$ (in terms of ε).

(iv) Similar to Section 3.4.1, we would have obtained the same complexity even when we used a control variate with a higher variance order Δ^K for some $K > 1$.

(v) Furthermore, from the computational point of view it is preferable to consider the integral approach rather than the series approach, even though the control variates $M_{\Delta,T}^{ser}$ and $M_{\Delta,T}^{int}$ are theoretically equivalent (cf. (3.20)). This is mainly due to the factor $\Delta_j W^i$ in a_{j,e_i} (see (3.11)), which is independent of $X_{\Delta,(j-1)\Delta}$ and has zero expectation and thus may lead to poor regression results. Regarding the integral approach, such a destabilising factor is not present in $g_{j,k}$.

3.5 Numerical results

In this section, we consider the Euler scheme and compare the numerical performance of the SMC, MLMC, series and integral approaches. For simplicity we implemented a global regression (i.e. the one without truncation and partitioning, as a part of the general description in Section 3.2). Regarding the choice of basis functions, we use in both series and integral approaches the same polynomials $\psi(x) = \prod_{i=1}^{d} x_i^{l_i}$, where $l_1, \dots l_d \in \{0, 1, \dots, p\}$ and $\sum_{l=1}^{d} l_i \leq p$. In addition to the polynomials, we consider the function f as a basis function for the series approach and $\frac{\partial f}{\partial x_k}$, $k \in \{1, \dots, d\}$, as a basis function for the estimation of $g_{j,k}(x)$, that is, for the integral approach (since $g_{j,k}$ is "close to" $\frac{\partial f}{\partial x_k}$ for larger j, cf. (3.27)). Hence, we have overall $c_{p,d} + 1$ basis functions in each regression for both approaches.

The following results are based on program codes written and vectorised in MATLAB and running on a Linux 64-bit operating system.

3.5.1 One-dimensional example

Here $d = m = 1$. We consider the following SDE

$$dX_t = -\frac{1}{2}\tanh(X_t)\operatorname{sech}^2(X_t)\,dt + \operatorname{sech}(X_t)\,dW_t, \quad X_0 = 0, \tag{3.57}$$

for $t \in [0, 1]$, where $\operatorname{sech}(x) := \frac{1}{\cosh(x)}$. This SDE has an exact solution $X_t = \operatorname{arsinh}(W_t)$. Furthermore, we consider the function $f(x) = \operatorname{sech}(x) + 15\arctan(x)$, that is, we have

$$\mathbb{E}[f(X_1)] = \mathbb{E}[\operatorname{sech}(\operatorname{arsinh}(W_1))] = \mathbb{E}\left[\frac{1}{\sqrt{1+W_1^2}}\right] \approx 0.789640. \tag{3.58}$$

We choose $p = 3$ (that is, 5 basis functions) and, for each $\varepsilon = 2^{-i}$, $i \in \{2, 3, 4, 5, 6\}$, we set the parameters J, N_r and N as follows (compare with the formulas in Section 3.4 for the "limiting" case $\nu \to \infty$ and ignore the constant B_ν as well as the log-terms for the integral approach):

$$J = \varepsilon^{-1}, \quad N_r = 256 \cdot \begin{cases} \lceil 0.6526 \cdot \varepsilon^{-1.0588} \rceil & \text{integral approach,} \\ \lceil 0.6342 \cdot \varepsilon^{-1.5882} \rceil & \text{series approach,} \end{cases}$$

$$N = 256 \cdot \begin{cases} \lceil 2.6102 \cdot \varepsilon^{-1.0588} \rceil & \text{integral approach,} \\ \lceil 2.5367 \cdot \varepsilon^{-1.5882} \rceil & \text{series approach.} \end{cases}$$

Regarding the SMC approach, the number of paths is set $N = 256 \cdot \varepsilon^{-2}$. The factor 256 is here for stability purposes. As for the MLMC approach, we set the initial number of paths for the first level ($l = 0$) equal to 10^3 as well as the "discretisation parameter" equal to 4 (leading to time steps of size $\frac{1}{4^l}$ at level l) and use the algorithm described in [18]. Next we compute the numerical root mean squared errors (RMSE) (the exact value is known, see (3.58)) by means of 100 independent repetitions of the algorithm. As can be seen from the first plot in Figure 3.1, the estimated numerical complexity is about $\text{RMSE}^{-1.82}$ for the integral approach, $\text{RMSE}^{-2.43}$ for the series approach, $\text{RMSE}^{-1.99}$ for the MLMC approach and $\text{RMSE}^{-3.02}$ for the SMC approach, which we get by regressing the log-time

(logarithmic computing time of the whole algorithm in seconds) vs. log-RMSE. Thus, the complexity reduction works best with the integral approach.

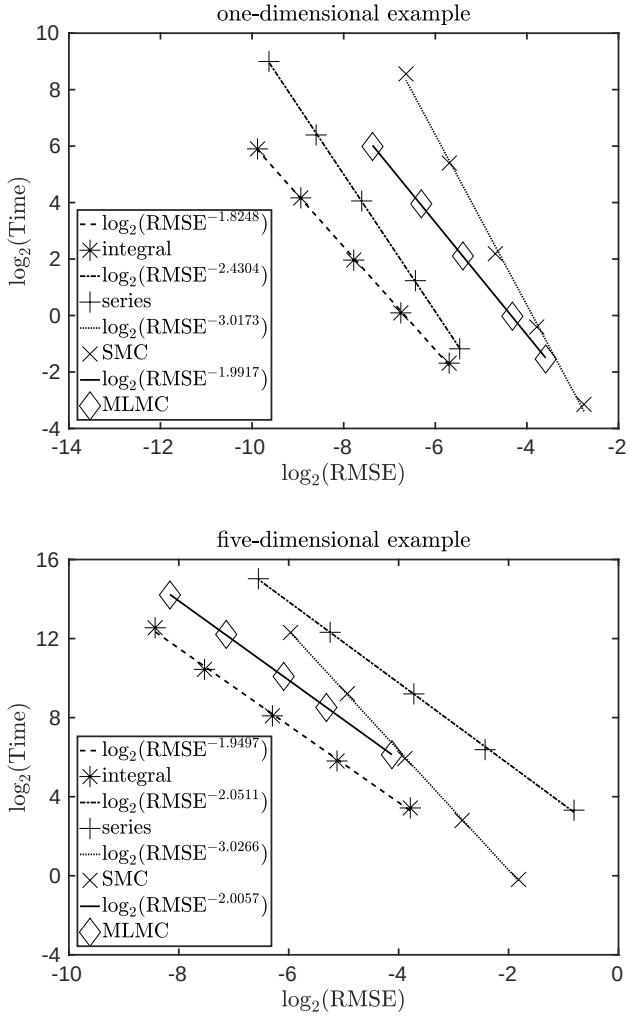

Figure 3.1. Numerical complexities of the integral, series, SMC and MLMC approaches in the one- and five-dimensional case.

3.5.2 Five-dimensional example

Here $d = m = 5$. We consider the SDE

$$\begin{aligned} dX_t^i &= -\sin\left(X_t^i\right)\cos^3\left(X_t^i\right)dt + \cos^2\left(X_t^i\right)dW_t^i, \quad X_0^i = 0, \quad i \in \{1,2,3,4\}, \\ dX_t^5 &= \sum_{i=1}^{4}\left[-\frac{1}{2}\sin\left(X_t^i\right)\cos^2\left(X_t^i\right)dt + \cos\left(X_t^i\right)dW_t^i\right] + dW_t^5, \quad X_0^5 = 0. \end{aligned} \tag{3.59}$$

The solution of (3.59) is given by

$$X_t^i = \arctan\left(W_t^i\right), \quad i \in \{1,2,3,4\},$$
$$X_t^5 = \sum_{i=1}^{4} \operatorname{arsinh}\left(W_t^i\right) + W_t^5,$$

for $t \in [0,1]$. Further, we consider the function

$$f(x) = \cos\left(\sum_{i=1}^{5} x_i\right) - 20\sum_{i=1}^{4} \sin\left(x_i\right),$$

that is, we have

$$\mathbb{E}\left[f\left(X_1\right)\right] = \left(\mathbb{E}\left[\cos\left(\arctan\left(W_1^1\right) + \operatorname{arsinh}\left(W_1^1\right)\right)\right]\right)^4 \mathbb{E}\left[\cos\left(W_1^5\right)\right] \approx 0.002069.$$

We again choose $p = 3$ (this now results in 57 basis functions) and consider the same values of ε as above. Moreover, we set (compare again with the formulas in Section 3.4 for $\nu \to \infty$ and ignore the the parameter B_ν as well as the log-terms for the integral approach):

$$J = \varepsilon^{-1}, \quad N_r = \begin{cases} \lceil 40.0274 \cdot \varepsilon^{-1.2381} \rceil & \text{integral approach,} \\ 4 \cdot \lceil 4.9044 \cdot \varepsilon^{-1.8571} \rceil & \text{series approach,} \end{cases}$$
$$N = \begin{cases} \lceil 2241.5320 \cdot \varepsilon^{-1.2381} \rceil & \text{integral approach,} \\ 4 \cdot \lceil 274.6480 \cdot \varepsilon^{-1.8571} \rceil & \text{series approach.} \end{cases}$$

The number of paths for the SMC approach is set $N = 256 \cdot \varepsilon^{-2}$, as in the previous example. Regarding the MLMC approach, we again use time steps of size $\frac{1}{4^l}$ at level l, but the initial number of paths in the first level is increased to 10^4. As in the one-dimensional case, we compute the numerical RMSE by means of 100 independent repetitions of the algorithm. Our empirical findings are illustrated in the second plot in Figure 3.1. We observe the numerical complexity $\mathrm{RMSE}^{-1.95}$ for the integral approach, $\mathrm{RMSE}^{-2.05}$ for the series approach, $\mathrm{RMSE}^{-2.01}$ for the MLMC approach and $\mathrm{RMSE}^{-3.03}$ for the SMC approach. Even though the complexity order of the series approach is better than that of the SMC approach and close to that of MLMC approach, the series approach is practically outperformed by the other approaches (see Figure 3.1; the multiplicative constant influencing the computing time is obviously very big). However, the integral approach remains numerically the best one also in this five-dimensional example.

3.6 Proofs

First of all, we require the following Lemma to prove Proposition 3.15:

Lemma 3.25. *In case of the Euler scheme, it holds for $J \geq l \geq j \geq 0$ and $d = m = 1$*

$$\delta^2 X_{t_j, X_{\Delta,t_j}}(\Delta, t_l) = \frac{\delta^2 X_{\Delta,t_l} \delta X_{\Delta,t_j} - \delta^2 X_{\Delta,t_j} \delta X_{\Delta,t_l}}{\delta X_{\Delta,t_j}^3}, \tag{3.60}$$

where $\delta^2 X_{t_j,x}(\Delta,t_l) := \frac{\partial^2 X_{t_j,x}(\Delta,t_l)}{\partial x^2}$, $\delta X_{t_j,x}(\Delta,t_l) := \frac{\partial X_{t_j,x}(\Delta,t_l)}{\partial x}$ *and* $X_{t_j,x}(\Delta,t_l)$ *denotes the Euler discretisation, starting at time* t_j *in* x *(analogous to* $X_{s,x}(t)$ *for the exact solution). (That is, while the present time point* t_l *of the discretised process,* $l \in \{0,\dots,J\}$*, is used by subscripts in the notation* X_{Δ,t_l}*, it is used in brackets in the notation* $X_{\Delta,t_j}(\Delta,t_l)$*,* $j \le l$*).*

Proof. We use the mathematical induction technique. First, for $l = j$ we clearly have $\delta^2 X_{t_j,X_{\Delta,t_j}}(\Delta,t_l) = 0$. Regarding the inductive step, we assume (3.60) and further use (3.33) as well as $\delta X_{\Delta,X_{\Delta,t_j}}(\Delta,t_l) = \frac{\delta X_{\Delta,t_l}}{\delta X_{\Delta,t_j}}$ (see proof of Theorem 3.11) to derive

$$\begin{aligned}\delta^2 X_{\Delta,X_{\Delta,t_j}}(\Delta,t_{l+1}) &= \delta^2 X_{\Delta,X_{\Delta,t_j}}(\Delta,t_l)A_{l+1} + \delta X^2_{\Delta,X_{\Delta,t_j}}(\Delta,t_l)B_{l+1}\\ &= \frac{\delta^2 X_{\Delta,t_l}\delta X_{\Delta,t_j} - \delta^2 X_{\Delta,t_j}\delta X_{\Delta,t_l}}{\delta X^3_{\Delta,t_j}}A_{l+1} + \frac{\delta X^2_{\Delta,t_l}}{\delta X^2_{\Delta,t_j}}B_{l+1},\end{aligned}$$

where

$$\begin{aligned}A_j &:= \frac{\partial}{\partial x}\Phi_\Delta\left(X_{\Delta,t_{j-1}}, \frac{\Delta_j W}{\sqrt{\Delta}}\right) = 1 + \mu'(X_{\Delta,t_{j-1}})\Delta + \sigma'(X_{\Delta,t_{j-1}})\Delta_j W,\\ B_j &:= \frac{\partial^2}{\partial x^2}\Phi_\Delta\left(X_{\Delta,t_{j-1}}, \frac{\Delta_j W}{\sqrt{\Delta}}\right) = \mu''(X_{\Delta,t_{j-1}})\Delta + \sigma''(X_{\Delta,t_{j-1}})\Delta_j W,\end{aligned}$$

for $j = 1,\dots,J$ (cf. (3.33)). Next we use

$$\begin{aligned}A_{l+1} &= \frac{\delta X_{\Delta,t_{l+1}}}{\delta X_{\Delta,t_l}},\\ B_{l+1} &= \frac{\delta^2 X_{\Delta,t_{l+1}} - \delta^2 X_{\Delta,t_l}A_{l+1}}{\delta X^2_{\Delta,t_l}} = \frac{\delta^2 X_{\Delta,t_{l+1}}\delta X_{\Delta,t_l} - \delta^2 X_{\Delta,t_l}\delta X_{\Delta,t_{l+1}}}{\delta X^3_{\Delta,t_l}},\end{aligned}$$

which gives us finally

$$\begin{aligned}\delta^2 X_{\Delta,X_{\Delta,t_j}}(\Delta,t_{l+1}) =& \frac{\delta^2 X_{\Delta,t_l}\delta X_{\Delta,t_j} - \delta^2 X_{\Delta,t_j}\delta X_{\Delta,t_l}}{\delta X^3_{\Delta,t_j}} \cdot \frac{\delta X_{\Delta,t_{l+1}}}{\delta X_{\Delta,t_l}}\\ &+ \frac{\delta X^2_{\Delta,t_l}}{\delta X^2_{\Delta,t_j}} \cdot \frac{\delta^2 X_{\Delta,t_{l+1}}\delta X_{\Delta,t_l} - \delta^2 X_{\Delta,t_l}\delta X_{\Delta,t_{l+1}}}{\delta X^3_{\Delta,t_l}}\\ =& \frac{\delta^2 X_{\Delta,t_{l+1}}\delta X_{\Delta,t_j} - \delta^2 X_{\Delta,t_j}\delta X_{\Delta,t_{l+1}}}{\delta X^3_{\Delta,t_j}}.\end{aligned}$$

Note that it holds analogously for the non-discretised process $\delta^2 X$ (cf. formula (3.15) in [45], which is of a similar form)

$$\delta^2 X_{s,X_s}(t) = \frac{\delta^2 X_t \delta X_s - \delta^2 X_s \delta X_t}{\delta X^3_s},$$

where $0 \le s \le t \le T$. □

Proof of Theorem 3.2

The proof uses the well-known fact that the system

$$\left\{\prod_{j=1}^{J}\prod_{r=1}^{m} H_{k_{j,r}}\left(\frac{\Delta_j W^r}{\sqrt{\Delta}}\right) : k = (k_{j,r}) \in \mathbb{N}_0^{J\times m}\right\}$$

is an orthonormal basis in $L^2(\mathcal{G}_J)$, where the σ-field $\mathcal{G}_J = \sigma(\Delta_j W : j = 1, \dots, J)$ is generated by the Gaussian increments, and goes along the lines of the proof of Theorem 4.11. Note that we rather prove Theorem 4.11, since we will achieve a better complexity based on the representation there. □

Proof of Theorem 3.4

For $p \in \mathbb{N}_0^{J\times m}$ we define

$$c_p = \mathbb{E}\left[f(X_{\Delta,T}) \prod_{j=1}^{J} \prod_{i=1}^{m} H_{p_{j,i}}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right)\right].$$

Further, for $j \in \{1, \dots, J\}$, $i \in \{1, \dots, m\}$, and $k \in \mathbb{N}$, we define

$$I_{j,i,k} = \left\{ p \in \mathbb{N}_0^{J\times m} \colon p_{j,i} = k,\, p_{j,r} = 0\, \forall r > i,\, p_{l,r} = 0\, \forall l > j,\, \forall r \in \{1, \dots, m\} \right\}.$$

The Wiener chaos expansion of $f(X_{\Delta,T})$ with respect to $(\Delta_j W)_{j=1}^J$ is given by (see e.g. [52])

$$f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{k=1}^{\infty} \sum_{\substack{p\in\mathbb{N}_0^{J\times m} \\ |p|=k}} c_p \prod_{j=1}^{J} \prod_{i=1}^{m} H_{p_{j,i}}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right). \tag{3.61}$$

Rearranging of the terms in (3.61) leads to (cf. proof of Theorem 4.11)

$$\begin{aligned} f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J} \sum_{i=1}^{m} \sum_{k=1}^{\infty} H_k\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right) \sum_{p\in I_{j,i,k}} \Bigg[c_p \left(\prod_{l=1}^{j-1} \prod_{r=1}^{m} H_{p_{l,r}}\left(\frac{\Delta_l W^r}{\sqrt{\Delta}}\right)\right) \\ \cdot \left(\prod_{r=1}^{i-1} H_{p_{j,r}}\left(\frac{\Delta_j W^r}{\sqrt{\Delta}}\right)\right)\Bigg]. \end{aligned}$$

Finally, we get for all $j \in \{1, \dots, J\}$, $i \in \{1, \dots, m\}$, and $k \in \mathbb{N}$

$$\begin{aligned} &\sum_{p\in I_{j,i,k}} c_p \left(\prod_{l=1}^{j-1} \prod_{r=1}^{m} H_{p_{l,r}}\left(\frac{\Delta_l W^r}{\sqrt{\Delta}}\right)\right) \left(\prod_{r=1}^{i-1} H_{p_{j,r}}\left(\frac{\Delta_l W^r}{\sqrt{\Delta}}\right)\right) \\ =&\mathbb{E}\left[f(X_{\Delta,T}) H_k\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right) \middle| (\Delta_l W)_{l=1}^{j-1}, (\Delta_j W^r)_{r=1}^{i-1}\right] \\ =&\mathbb{E}\left[f(X_{\Delta,T}) H_k\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right) \middle| X_{\Delta,(j-1)\Delta}, (\Delta_j W^r)_{r=1}^{i-1}\right] = a_{j,k,i}\left(X_{\Delta,(j-1)\Delta}, (\Delta_j W^r)_{r=1}^{i-1}\right), \end{aligned}$$

which completes the proof.

Proof of Theorem 3.6

First of all, we derive

$$\begin{aligned} &\lim_{t\nearrow t_j} u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}) \\ =&\lim_{t\nearrow t_j} \mathbb{E}\left[u_\Delta(t_j, \Phi_\Delta\left(x, \frac{y + W_{t_j} - W_t}{\sqrt{\Delta}}\right), 0)\right] \Big|_{x=X_{\Delta,(j-1)\Delta},\, y=W_t - W_{t_{j-1}}} \\ =&u_\Delta\left(t_j, \Phi_\Delta\left(X_{\Delta,(j-1)\Delta}, \frac{\Delta_j W}{\sqrt{\Delta}}\right), 0\right) = u_\Delta(t_j, X_{\Delta,t_j}, 0). \end{aligned} \tag{3.62}$$

By means of Itô's Lemma and the fact that u_Δ satisfies the heat equation

$$\sum_{i=1}^{m} \frac{\partial^2 u_\Delta}{\partial y_i^2} = -\frac{1}{2}\frac{\partial u_\Delta}{\partial t} \tag{3.63}$$

due to its relation to the normal distribution, we then obtain

$$\begin{aligned} f(X_{\Delta,T}) - \mathbb{E}[f(X_{\Delta,T})] =& u_\Delta(T, X_{\Delta,T}, 0) - u_\Delta(0, x_0, 0) \\ =& \sum_{j=1}^{J} \big(u_\Delta(t_j, X_{\Delta,t_j}, 0) - u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)\big) \\ =& \sum_{j=1}^{J} \lim_{t \nearrow t_j} \big(u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}) - u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)\big) \\ =& \sum_{j=1}^{J}\sum_{i=1}^{m} \lim_{t \nearrow t_j} \int_{t_{j-1}}^{t} \frac{\partial u_\Delta}{\partial y_i}(s, X_{\Delta,t_{j-1}}, W_s - W_{t_{j-1}})\, dW_s^i \\ =& \sum_{j=1}^{J} \int_{t_{j-1}}^{t_j} \nabla_y u_\Delta(s, X_{\Delta,t_{j-1}}, W_s - W_{t_{j-1}})\, dW_s. \end{aligned} \tag{3.64}$$

□

Proof of Theorem 3.7

Below we simply write $u_{\Delta,t_{j-1}}$ rather than $u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)$. Let us consider the Taylor expansion for $\frac{\partial}{\partial y_r} u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}})$ of order $K \in \mathbb{N}$ around $(t_{j-1}, X_{\Delta,t_{j-1}}, 0)$, with $r \in \{1, \dots, m\}$, that is, for $t \in [t_{j-1}, t_j)$, we set

$$T_{j,r}^K(t) := \sum_{|\alpha| \le K} \frac{D_{t,y}^\alpha \left(\frac{\partial}{\partial y_r} u_{\Delta,t_{j-1}}\right)}{\alpha_1! \cdots \alpha_{m+1}!} (t - t_{j-1})^{\alpha_1} (W_t^1 - W_{t_{j-1}}^1)^{\alpha_2} \cdots (W_t^m - W_{t_{j-1}}^m)^{\alpha_{m+1}}, \tag{3.65}$$

where $\alpha \in \mathbb{N}_0^{m+1}$ and $D_{t,y}^\alpha$ is defined in (3.15). Via Taylor's theorem we obtain

$$\begin{aligned} &\frac{\partial}{\partial y_r} u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}) - T_{j,r}^K(t) \\ &= \sum_{|\alpha| = K+1} \left[\int_0^1 (1-z)^K D_{t,y}^\alpha \left(\frac{\partial}{\partial y_r} u_\Delta(t_{j-1}(1-z) + tz, X_{\Delta,t_{j-1}}, z(W_t - W_{t_{j-1}})) \right) dz \right. \\ &\qquad \left. \cdot \frac{(K+1)!}{\alpha_1! \cdots \alpha_{m+1}!} (t - t_{j-1})^{\alpha_1} (W_t^1 - W_{t_{j-1}}^1)^{\alpha_2} \cdots (W_t^m - W_{t_{j-1}}^m)^{\alpha_{m+1}} \right]. \end{aligned}$$

Provided that (3.16) holds, we get

$$
\begin{aligned}
&\operatorname{Var}\left[\sum_{j=1}^{J}\sum_{r=1}^{m}\int_{t_{j-1}}^{t_j}\left(\frac{\partial}{\partial y_r}u_\Delta(t,X_{\Delta,t_{j-1}},W_t-W_{t_{j-1}})-T_{j,r}^K(t)\right)dW_t^r\right]\\
=&\sum_{j=1}^{J}\sum_{r=1}^{m}\int_{t_{j-1}}^{t_j}\mathbb{E}\left[\left(\frac{\partial}{\partial y_r}u_\Delta(t,X_{\Delta,t_{j-1}},W_t-W_{t_{j-1}})-T_{j,r}^K(t)\right)^2\right]dt\\
\lesssim& C^{2(K+1)}\sum_{j=1}^{J}\sum_{|\alpha|=K+1}\int_{t_{j-1}}^{t_j}\mathbb{E}\left[(t-t_{j-1})^{2\alpha_1}(W_t^1-W_{t_{j-1}}^1)^{2\alpha_2}\cdots(W_t^m-W_{t_{j-1}}^m)^{2\alpha_{m+1}}\right]dt\\
\lesssim&(C^2\Delta)^{K+1}\overset{K\to\infty}{\longrightarrow}0,
\end{aligned}
$$

and thus $T_{j,r}^K$ converges for $K\to\infty$ in $L^2(\Omega\times[0,T])$ to $\frac{\partial u_\Delta}{\partial y_r}(t,X_{\Delta,t_{j-1}},W_t-W_{t_{j-1}})$. Moreover, due to (3.63), the limit of $T_{j,r}^K$ simplifies to (cf. (3.65))

$$
\begin{aligned}
&\frac{\partial u_{\Delta,t_{j-1}}}{\partial y_r}+\sum_{i=1}^{m}\frac{\partial^2 u_{\Delta,t_{j-1}}}{\partial y_r\partial y_i}(W_t^i-W_{t_{j-1}}^i)\\
&+\frac{1}{2}\sum_{i=1}^{m}\frac{\partial^3 u_{\Delta,t_{j-1}}}{\partial y_r\partial y_i^2}((W_t^i-W_{t_{j-1}}^i)^2-(t-t_{j-1}))\\
&+\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{m}\frac{\partial^3 u_{\Delta,t_{j-1}}}{\partial y_r\partial y_{i_1}\partial y_{i_2}}(W_t^{i_1}-W_{t_{j-1}}^{i_1})(W_t^{i_2}-W_{t_{j-1}}^{i_2})\\
&+\Bigg[\frac{1}{6}\sum_{i=1}^{m}\frac{\partial^4 u_{\Delta,t_{j-1}}}{\partial y_r\partial y_i^3}((W_t^i-W_{t_{j-1}}^i)^3-3(W_t^i-W_{t_{j-1}}^i)(t-t_{j-1}))\\
&\quad+\frac{1}{2}\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{m}\frac{\partial^4 u_{\Delta,t_{j-1}}}{\partial y_r\partial y_{i_1}^2\partial y_{i_2}}((W_t^{i_1}-W_{t_{j-1}}^{i_1})^2-(t-t_{j-1}))(W_t^{i_2}-W_{t_{j-1}}^{i_2})\\
&\quad+\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{m}\frac{\partial^4 u_{\Delta,t_{j-1}}}{\partial y_r\partial y_{i_1}\partial y_{i_2}\partial y_{i_3}}(W_t^{i_1}-W_{t_{j-1}}^{i_1})(W_t^{i_2}-W_{t_{j-1}}^{i_2})(W_t^{i_3}-W_{t_{j-1}}^{i_3})\Bigg]\\
&+\ldots\\
=&\sum_{l=1}^{\infty}(t-t_{j-1})^{\frac{l-1}{2}}\sum_{\substack{k\in\mathbb{N}_0^m\\ \sum_{i=1}^m k_i=l-1}}\frac{\partial^l u_{\Delta,t_{j-1}}}{\partial y_r\partial y_1^{k_1}\cdots\partial y_m^{k_m}}\prod_{i=1}^{m}\frac{H_{k_i}\left(\frac{W_t^i-W_{t_{j-1}}^i}{\sqrt{t-t_{j-1}}}\right)}{\sqrt{k_i!}}.
\end{aligned}
$$

To compute the stochastic integral

$$
\begin{aligned}
&\int_{t_{j-1}}^{t_j}\nabla_y u_\Delta(t,X_{\Delta,t_{j-1}},W_t-W_{t_{j-1}})\,dW_t\\
=&\sum_{l=1}^{\infty}\sum_{r=1}^{m}\int_{t_{j-1}}^{t_j}(t-t_{j-1})^{\frac{l-1}{2}}\sum_{\substack{k\in\mathbb{N}_0^m\\ \sum_{i=1}^m k_i=l-1}}\frac{\partial^l u_{\Delta,t_{j-1}}}{\partial y_r\partial y_1^{k_1}\cdots\partial y_m^{k_m}}\prod_{i=1}^{m}\frac{H_{k_i}\left(\frac{W_t^i-W_{t_{j-1}}^i}{\sqrt{t-t_{j-1}}}\right)}{\sqrt{k_i!}}\,dW_t^r,
\end{aligned}
$$

we apply Itô's Lemma w.r.t. the functions $F_k(t, y_1, \dots, y_m) := t^{l/2} \prod_{i=1}^m \frac{H_{k_i}\left(\frac{y_i}{\sqrt{t}}\right)}{\sqrt{k_i!}}$, where $\sum_{i=1}^m k_i = l$. Thus, we obtain

$$\begin{aligned} &dF_k(t - t_{j-1}, W_t^1 - W_{t_{j-1}}^1, \dots, W_t^m - W_{t_{j-1}}^m) \\ =&(t - t_{j-1})^{\frac{l-1}{2}} \sum_{r=1}^m \frac{H_{k_r-1}\left(\frac{W_t^r - W_{t_{j-1}}^r}{\sqrt{t-t_{j-1}}}\right)}{\sqrt{(k_r-1)!}} \prod_{\substack{i=1 \\ i\neq r}}^m \frac{H_{k_i}\left(\frac{W_t^i - W_{t_{j-1}}^i}{\sqrt{t-t_{j-1}}}\right)}{\sqrt{k_i!}}\, dW_t^r. \end{aligned} \tag{3.66}$$

This gives us finally

$$\begin{aligned} &\int_{t_{j-1}}^{t_j} \nabla_y u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}})\, dW_t \\ =&\sum_{l=1}^\infty \Delta^{l/2} \sum_{\substack{k\in\mathbb{N}_0^m \\ \sum_{i=1}^m k_i = l}} \frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_1^{k_1} \cdots \partial y_m^{k_m}} \prod_{i=1}^m \frac{H_{k_i}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right)}{\sqrt{k_i!}}. \end{aligned}$$ □

Proof of Theorem 3.8

Let us define the function $G_{l,j}(x)$ for $J \geq l \geq j \geq 0$, $x \in \mathbb{R}^d$, as follows (cf. (2.12))

$$\begin{aligned} &G_{l,j}(x) \equiv \Phi_{\Delta,l} \circ \Phi_{\Delta,l-1} \circ \ldots \circ \Phi_{\Delta,j+1}(x), \quad l > j, \\ &G_{l,j}(x) \equiv x, \quad l = j, \end{aligned} \tag{3.67}$$

where $\Phi_{\Delta,l}(x) := \Phi_\Delta\left(x, \frac{\Delta_l W}{\sqrt{\Delta}}\right)$ for $l = 1, \dots, J$. Note that it holds (cf. (2.13) and (3.13))

$$u_\Delta(t_j, x, 0) = \mathbb{E}\left[f(G_{J,j}(x))\right]. \tag{3.68}$$

Similar to G we define the function $\tilde{G}_j(x, z)$, $0 \leq j < J$, $x \in \mathbb{R}^d$, $z := (z_1, \dots, z_{J-j}) \in \mathbb{R}^{m\times(J-j)}$, $z_l := (z_l^1, \dots, z_l^m)^\top \in \mathbb{R}^m$ for $l = 1, \dots, J-j$, as follows

$$\tilde{G}_j(x, z) := \tilde{\Phi}_{\Delta,z_{J-j}} \circ \ldots \circ \tilde{\Phi}_{\Delta,z_1}(x),$$

where $\tilde{\Phi}_{\Delta,z_l}(x) := \Phi_\Delta(x, z_l)$. Note that G and $\tilde{G}$ have the following relation

$$G_{J,j}(x) = \tilde{G}_j\left(x, \frac{1}{\sqrt{\Delta}}\left(\Delta_{j+1}W, \Delta_{j+2}W, \dots, \Delta_J W\right)\right), \quad j < J. \tag{3.69}$$

Let us represent $\sqrt{\Delta}\frac{\partial}{\partial y_i} u_\Delta(t_{j-1}, x, 0)$, where $j \in \{1, \dots, J\}$ and $i \in \{1, \dots, m\}$, as a $(J-j+1)m$-dimensional integral, that is (cf. (3.69))

$$\begin{aligned} &\sqrt{\Delta}\frac{\partial}{\partial y_i} u_\Delta(t_{j-1}, x, 0) = \sqrt{\Delta}\frac{\partial}{\partial y_i}\mathbb{E}\left[f\left(G_{J,j}\left(\Phi_\Delta\left(x, \frac{\Delta_j W + y}{\sqrt{\Delta}}\right)\right)\right)\right]\,|_{y=0_m} \\ =&\int_{\mathbb{R}^{(J-j+1)m}} \sqrt{\Delta}\frac{\partial}{\partial y_i} f\left(\tilde{G}_{j-1}\left(x, \left(z_1 + \frac{y}{\sqrt{\Delta}}, z_2, \dots, z_{J-j+1}\right)\right)\right) \varphi_{(J-j+1)m}(z)\, dz\,|_{y=0_m}\,, \end{aligned}$$

where $\varphi_{(J-j+1)m}$ denotes the $(J-j+1)m$-dimensional standard normal density function. Since it holds

$$\begin{aligned}&\sqrt{\Delta}\frac{\partial}{\partial y_i} f\left(\tilde{G}_{j-1}\left(x,\left(z_1+\frac{y}{\sqrt{\Delta}},z_2,\dots,z_{J-j+1}\right)\right)\right)\\ =&\frac{\partial}{\partial z_1^i} f\left(\tilde{G}_{j-1}\left(x,\left(z_1+\frac{y}{\sqrt{\Delta}},z_2,\dots,z_{J-j+1}\right)\right)\right),\end{aligned}$$

we obtain via integration by parts

$$\begin{aligned}&\sqrt{\Delta}\frac{\partial}{\partial y_i} u_\Delta(t_{j-1},x,0)\\ =&\int\limits_{\mathbb{R}^{(J-j+1)m}} \frac{\partial}{\partial z_1^i} f\left(\tilde{G}_{j-1}\left(x,z\right)\right)\varphi_{(J-j+1)m}(z)\,dz\\ =&-\int\limits_{\mathbb{R}^{(J-j+1)m}} f\left(\tilde{G}_{j-1}\left(x,z\right)\right)\frac{\partial}{\partial z_1^i}\varphi_{(J-j+1)m}(z)\,dz\\ =&\int\limits_{\mathbb{R}^{(J-j+1)m}} f\left(\tilde{G}_{j-1}\left(x,z\right)\right)z_1^i\varphi_{(J-j+1)m}(z)\,dz\\ =&\mathbb{E}\left[f(G_{J,j-1}(x))\frac{\Delta_j W^i}{\sqrt{\Delta}}\right]=\mathbb{E}\left[f(X_{\Delta,T})\frac{\Delta_j W^i}{\sqrt{\Delta}}\,\middle|X_{\Delta,(j-1)\Delta}=x\right]=a_{j,e_i}(x).\end{aligned}$$ □

Proof of Theorem 3.9

Via Taylor's theorem we get

$$\begin{aligned}&\frac{\partial u_\Delta(t,X_{\Delta,t_{j-1}},W_t-W_{t_{j-1}})}{\partial y_i}\\ =&\frac{\partial u_\Delta(t_{j-1},X_{\Delta,t_{j-1}},0)}{\partial y_i}+(t-t_{j-1})\int\limits_0^1\frac{\partial^2 u_\Delta(t_{j-1}(1-z)+tz,X_{\Delta,t_{j-1}},z(W_t-W_{t_{j-1}}))}{\partial y_i\partial t}\,dz\\ &+\sum_{r=1}^m(W_t^r-W_{t_{j-1}}^r)\int\limits_0^1\frac{\partial^2 u_\Delta(t_{j-1}(1-z)+tz,X_{\Delta,t_{j-1}},z(W_t-W_{t_{j-1}}))}{\partial y_i\partial y_r}\,dz.\end{aligned}\tag{3.70}$$

Due to (3.63), (3.70) simplifies to

$$\begin{aligned}&\frac{\partial u_\Delta(t,X_{\Delta,t_{j-1}},W_t-W_{t_{j-1}})}{\partial y_i}\\ =&\frac{\partial u_\Delta(t_{j-1},X_{\Delta,t_{j-1}},0)}{\partial y_i}\\ &-2(t-t_{j-1})\int\limits_0^1\sum_{r=1}^m\frac{\partial^3 u_\Delta(t_{j-1}(1-z)+tz,X_{\Delta,t_{j-1}},z(W_t-W_{t_{j-1}}))}{\partial y_i\partial y_r^2}\,dz\\ &+\sum_{r=1}^m(W_t^r-W_{t_{j-1}}^r)\int\limits_0^1\frac{\partial^2 u_\Delta(t_{j-1}(1-z)+tz,X_{\Delta,t_{j-1}},z(W_t-W_{t_{j-1}}))}{\partial y_i\partial y_r}\,dz.\end{aligned}$$

Provided that u_Δ has bounded partial derivatives in y of orders 2 and 3, we have

$$\begin{aligned}
&\mathrm{Var}\left[\int_{t_{j-1}}^{t_j} \frac{\partial u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}})}{\partial y_i}\, dW_t^i - \frac{\partial u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_i}\Delta_j W^i\right] \\
&= \int_{t_{j-1}}^{t_j} \mathbb{E}\left[\left(\int_0^1 \sum_{r=1}^m \Big((W_t^r - W_{t_{j-1}}^r)\frac{\partial^2 u_\Delta(t_{j-1}(1-z)+tz, X_{\Delta,t_{j-1}}, z(W_t - W_{t_{j-1}}))}{\partial y_i \partial y_r}\right.\right. \\
&\qquad\qquad \left.\left. -2(t-t_{j-1})\frac{\partial^3 u_\Delta(t_{j-1}(1-z)+tz, X_{\Delta,t_{j-1}}, z(W_t - W_{t_{j-1}}))}{\partial y_i \partial y_r^2}\Big)\, dz\right)^2\right] dt \\
&\lesssim \sum_{r=1}^m \int_{t_{j-1}}^{t_j} \mathbb{E}\left[(W_t^r - W_{t_{j-1}}^r)^2 + 4(t-t_{j-1})^2\right] dt \lesssim \Delta^2.
\end{aligned}$$

Thus, we finally obtain

$$\begin{aligned}
&\mathrm{Var}\left[f(X_{\Delta,T}) - M_{\Delta,T}^{int}\right] \\
&= \sum_{j=1}^J \mathrm{Var}\left[\int_{t_{j-1}}^{t_j} \frac{\partial u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}})}{\partial y_i}\, dW_t^i - \frac{\partial u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_i}\Delta_j W^i\right] \lesssim \Delta.
\end{aligned}$$

□

Proof of Theorem 3.11

Due to Theorem 3.6, it is sufficient to show

$$\nabla_y u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}) = \mathbb{E}\left[\nabla f(X_{\Delta,T})\delta X_{\Delta,T}\delta X_{\Delta,t_j}^{-1} \,|\mathcal{F}_t\right]\sigma(X_{\Delta,t_{j-1}})$$

for $t \in [t_{j-1}, t_j)$.

Let us derive a relation between $\nabla_y u_\Delta$ and $\nabla_x u_\Delta$. We have for $t \in [t_{j-1}, t_j)$

$$\begin{aligned}
\nabla_y u_\Delta(t,x,y) &= \nabla_y \mathbb{E}\left[u_\Delta\left(t_j, \Phi_\Delta\left(x, \frac{y + W_{t_j} - W_t}{\sqrt{\Delta}}\right), 0\right)\right] \\
&= \nabla_x \mathbb{E}\left[u_\Delta\left(t_j, \Phi_\Delta\left(x, \frac{y + W_{t_j} - W_t}{\sqrt{\Delta}}\right), 0\right)\right]\sigma(x).
\end{aligned} \tag{3.71}$$

Thus, the term $\nabla_y u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}})$ in (3.64) takes the form

$$\nabla_y u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}) = \mathbb{E}[\nabla_x u_\Delta(t_j, X_{\Delta,t_j}, 0)\,|\mathcal{F}_t]\sigma(X_{\Delta,t_{j-1}}). \tag{3.72}$$

Note that it holds

$$u_\Delta(t_j, x, 0) = \mathbb{E}[f(X_{t_j,x}(\Delta, T))], \tag{3.73}$$

where $X_{t_j,x}(\Delta, t_l)$ denotes the Euler discretisation, starting at time t_j in x (cf. Lemma 3.25). Hence, we have for $\nabla_x u_\Delta$

$$\nabla_x u_\Delta(t_j, x, 0) = \mathbb{E}[\nabla f(X_{t_j,x}(\Delta, T))\delta X_{t_j,x}(\Delta, T)],$$

respectively

$$\nabla_x u_\Delta(t_j, X_{\Delta,t_j}, 0) = \mathbb{E}\left[\nabla f(X_{t_j,X_{\Delta,t_j}}(\Delta,T))\delta X_{t_j,X_{\Delta,t_j}}(\Delta,T)\,|\mathcal{F}_{t_j}\right],$$

where $\delta^i X^k_{t_j,x}(\Delta,t_l) := \frac{\partial X^k_{t_j,x}(\Delta,t_l)}{\partial x_i}$ with $l \geq j$ and $i,k \in \{1,\dots,d\}$. Below we simply write X_{Δ,t_l} and $\delta X_{\Delta,t_l}$ rather than $X_{0,x_0}(\Delta,t_l)$ and $\delta X_{0,x_0}(\Delta,t_l)$.

Let us denote by

$$\mathcal{J}_{\Phi_\Delta}(x,y) := \begin{pmatrix} \frac{\partial}{\partial x^1}\Phi^1_\Delta(x,y) & \cdots & \frac{\partial}{\partial x^d}\Phi^1_\Delta(x,y) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x^1}\Phi^d_\Delta(x,y) & \cdots & \frac{\partial}{\partial x^d}\Phi^d_\Delta(x,y) \end{pmatrix} \tag{3.74}$$

the Jacobian matrix of the function $\Phi_\Delta(x,y)$ w.r.t. the variable x. Regarding the discretisation scheme for $\delta X_{\Delta,j\Delta}$, we can use, alternatively to (3.24), the matrix form

$$\delta X_{\Delta,j\Delta} = A_j \delta X_{\Delta,(j-1)\Delta} = A_j A_{j-1}\cdots A_1, \tag{3.75}$$

where

$$A_j := \mathcal{J}_{\Phi_\Delta}\left(X_{\Delta,t_{j-1}}, \frac{\Delta_j W}{\sqrt{\Delta}}\right).$$

for $j = 1,\dots,J$. This gives us for $l \geq j$

$$\delta X_{t_j,X_{\Delta,t_j}}(\Delta,t_l) = A_l A_{l-1}\cdots A_{j+1} = A_l A_{l-1}\cdots A_1 \left(A_j A_{j-1}\cdots A_1\right)^{-1} = \delta X_{\Delta,t_l}\delta X^{-1}_{\Delta,t_j}.$$

Let us recall the function G in (3.67). We can derive

$$X_{t_j,X_{\Delta,t_j}}(\Delta,t_l) = G_{l,j}(X_{\Delta,t_j}) = G_{l,j}(G_{j,0}(x_0)) = G_{l,0}(x_0) = X_{\Delta,t_l}, \tag{3.76}$$

where again $l \geq j$. Finally we have for $t \in [t_{j-1},t_j)$

$$\nabla_x u_\Delta(t_j, X_{\Delta,t_j}, 0) = \mathbb{E}\left[\nabla f(X_{\Delta,T})\delta X_{\Delta,T}\delta X^{-1}_{\Delta,t_j}\,|\mathcal{F}_{t_j}\right],$$

which gives us

$$\begin{aligned}\nabla_y u_\Delta(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}) &= \mathbb{E}\left[\mathbb{E}\left[\nabla f(X_{\Delta,T})\delta X_{\Delta,T}\delta X^{-1}_{\Delta,t_j}\,|\mathcal{F}_{t_j}\right]|\mathcal{F}_t\right]\sigma(X_{\Delta,t_{j-1}}) \\ &= \mathbb{E}\left[\nabla f(X_{\Delta,T})\delta X_{\Delta,T}\delta X^{-1}_{\Delta,t_j}\,|\mathcal{F}_t\right]\sigma(X_{\Delta,t_{j-1}}).\end{aligned}$$

Note that it holds analogously for the non-discretised processes X and δX (cf. [45])

$$\begin{aligned}X_{s,X_s}(t) &= X_t, \\ \delta X_{s,X_s}(t) &= \frac{\delta X_t}{\delta X_s},\end{aligned}$$

where $0 \leq s \leq t \leq T$. □

Proof of Proposition 3.15

Since it holds for $j = 0, 1, \ldots, J$ (cf. (3.73))

$$u_\Delta(t_j, x, 0) = \mathbb{E}[f(X_{t_j,x}(\Delta, T))],$$

we have for the second derivative of u_Δ w.r.t. x

$$\frac{\partial^2}{\partial x^2} u_\Delta(t_j, x, 0) = \mathbb{E}[f''(X_{t_j,x}(\Delta, T))(\delta X_{t_j,x}(\Delta, T))^2 + f'(X_{t_j,x}(\Delta, T))\delta^2 X_{t_j,x}(\Delta, T)].$$

Next, we have the following relation (similar to (3.72) for the first derivative)

$$\frac{\partial^2}{\partial y^2} u_\Delta(t_{j-1}, x, 0) = \mathbb{E}\left[\frac{\partial^2}{\partial x^2} u_\Delta(t_j, X_{\Delta,t_j}, 0) \,\middle|\, X_{\Delta,t_{j-1}} = x\right] \sigma^2(x).$$

Hence, we get for $\frac{\partial^2}{\partial y^2} u_\Delta(t_{j-1}, x, 0)$

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} u_\Delta(t_{j-1}, x, 0) =& \mathbb{E}\Big[f''(X_{t_j,X_{\Delta,t_j}}(\Delta, T))(\delta X_{t_j,X_{\Delta,t_j}}(\Delta, T))^2 \\
&+ f'(X_{t_j,X_{\Delta,t_j}}(\Delta, T))\delta^2 X_{t_j,X_{\Delta,t_j}}(\Delta, T) \,\Big|\, X_{\Delta,t_{j-1}} = x\Big] \sigma^2(x) \\
=& \mathbb{E}\left[f''(X_{\Delta,T}) \left(\frac{\delta X_{\Delta,T}}{\delta X_{\Delta,t_j}}\right)^2 \right. \\
&\left. + f'(X_{\Delta,T})\delta^2 X_{t_j,X_{\Delta,t_j}}(\Delta, T) \,\middle|\, X_{\Delta,t_{j-1}} = x\right] \sigma^2(x).
\end{aligned} \tag{3.77}$$

We complete the proof 3.15 by inserting (3.60) (from Lemma 3.25) into (3.77) for $l = J$. □

Proof of Theorem 3.17

First of all, let us note that we may apply Theorem 2.5 here, since $u_\Delta(t_j, x, 0)$ is an analogue of the function $q_j(x)$, defined in (2.13) (cf. (3.13)). One can easily verify assumptions (2.14) and (2.15) in Theorem 2.5 for $K \in \{1, 2, 3\}$ when all the functions f, μ_k, σ_{ki}, $k \in \{1, \ldots, d\}$, $i \in \{1, \ldots, m\}$, are K times continuously differentiable with bounded partial derivatives up to order K (cf. (3.5)).

(i) Assumption (A2) follows straightforwardly from (3.29) for $K = 1$. As for assumption (A1), we use (3.76) to get for $j \in \{0, 1, \ldots, J\}$

$$\mathcal{J}_{G_{J,j}}(X_{\Delta,t_j}) = \delta X_{\Delta,t_j}(\Delta, T) = \delta X_{\Delta,T} \delta X_{\Delta,t_j}^{-1},$$

where $\mathcal{J}_{G_{J,j}}(x)$ denotes the Jacobian matrix of the function $G_{J,j}$ (cf. (3.74)). This gives us

$$\nabla\left(f \circ G_{J,j}\right)(X_{\Delta,t_j}) = \nabla f(G_{J,j}(X_{\Delta,t_j}))\mathcal{J}_{G_{J,j}}(X_{\Delta,t_j}) = \zeta_{J,j},$$

where $\zeta_{J,j}$ is defined on page 30. Then we obtain for $k \in \{1,\dots,d\}$ and $j \in \{1,\dots,J\}$

$$\begin{aligned}
& \operatorname{Var}\left[\zeta_{J,j,k} \mid X_{\Delta,t_{j-1}} = x\right] \\
\leq\ & \mathbb{E}\left[\zeta_{J,j,k}^2 \mid X_{\Delta,t_{j-1}} = x\right] \\
=\ & \mathbb{E}\left[\left(\sum_{s=1}^{d} \frac{\partial}{\partial x_s} f(G_{J,j}(X_{\Delta,t_j})) \frac{\partial}{\partial x_k} G_{J,j}^s(X_{\Delta,t_j})\right)^2 \mid X_{\Delta,t_{j-1}} = x\right] \\
\leq\ & d \sum_{s=1}^{d} \mathbb{E}\left[\left(\frac{\partial}{\partial x_s} f(G_{J,j-1}(x)) \frac{\partial}{\partial x_k} G_{J,j-1}^s(x)\right)^2\right] \\
\leq\ & d \sum_{s=1}^{d} \sqrt{\mathbb{E}\left[\left(\frac{\partial}{\partial x_s} f(G_{J,j-1}(x))\right)^4\right] \mathbb{E}\left[\left(\frac{\partial}{\partial x_k} G_{J,j-1}^s(x)\right)^4\right]}.
\end{aligned}$$

Hence, assumption (A1) is satisfied under the above assumptions for $K = 1$ (cf. $\rho_{J,4,1}^{k,s}$ in the proof of Theorem 2.5).

(ii) As an extension of (3.71), we get

$$\begin{aligned}
& \frac{\partial^2}{\partial y_i \partial y_r} u_\Delta(t,x,y) \\
=\ & \sum_{s,u=1}^{d} \mathbb{E}\left[\frac{\partial^2}{\partial x_s \partial x_u} u_\Delta\left(t_j, \Phi_\Delta\left(x, \frac{y + W_{t_j} - W_t}{\sqrt{\Delta}}\right), 0\right)\right] \sigma_{si}(x)\sigma_{ur}(x), \\
& \frac{\partial^3}{\partial y_i \partial y_r \partial y_o} u_\Delta(t,x,y) \\
=\ & \sum_{s,u,v=1}^{d} \mathbb{E}\left[\frac{\partial^3}{\partial x_s \partial x_u \partial x_v} u_\Delta\left(t_j, \Phi_\Delta\left(x, \frac{y + W_{t_j} - W_t}{\sqrt{\Delta}}\right), 0\right)\right] \sigma_{si}(x)\sigma_{ur}(x)\sigma_{vo}(x),
\end{aligned}$$

where $j \in \{1,\dots,J\}$, $t \in [t_{j-1}, t_j)$ and $i,r,o \in \{1,\dots,m\}$. Applying Theorem 2.5 and considering that all functions $\sigma_{k,i}$, $k \in \{1,\dots,d\}$ $i \in \{1,\dots,m\}$, are bounded, we get that the partial derivatives in y up to order 3 are bounded, too. □

Proof of Theorem 3.19

Using the martingale transform structure in (3.30) and (3.41) (recall footnote 4 on page 27) together with the orthogonality of the system $\Delta_j W^i$, we get by (3.39)

$$\begin{aligned}
\operatorname{Var}[f(X_{\Delta,T}) - \tilde{M}_{\Delta,T}^{int}] &= \operatorname{Var}[f(X_{\Delta,T}) - M_{\Delta,T}^{int}] + \operatorname{Var}[M_{\Delta,T}^{int} - \tilde{M}_{\Delta,T}^{int}] \\
&\lesssim \frac{1}{J} + \Delta \sum_{j=1}^{J} \sum_{i=1}^{m} \mathbb{E} \| \sum_{k=1}^{d} (\tilde{g}_{j,k} - g_{j,k}) \sigma_{ki} \|_{L^2(\mathbb{P}_{\Delta,j-1})}^2 \\
&\leq \frac{1}{J} + d\Delta \sum_{j=1}^{J} \sum_{i=1}^{m} \sum_{k=1}^{d} \mathbb{E} \| (\tilde{g}_{j,k} - g_{j,k}) \sigma_{ki} \|_{L^2(\mathbb{P}_{\Delta,j-1})}^2 \\
&\leq \frac{1}{J} + d^2 T m \sigma_{\max}^2 \left\{ \tilde{c} \left(\Sigma + A^2(\log N_r + 1)\right) \frac{\binom{p+d}{d} S^d}{N_r} \right. \\
&\quad \left. + \frac{8\, C_h^2}{(p+1)!^2 d^{2-2/h}} \left(\frac{Rd}{S}\right)^{2p+2} + 8A^2 B_\nu R^{-\nu} \right\}.
\end{aligned}$$

□

Proof of Theorem 3.20

Let us, for simplicity, first ignore the $\log(N_r)$-term in (3.44) and consider only the terms w.r.t. the variables J, N_r, N, S, R which shall be optimised, since the constants $d, m, c_{p,d}$, $(p+1)!$, B_ν do not affect the terms on ε. Further, we consider the log-cost and log-constraints rather than (3.43) and (3.44). Let us subdivide the optimisation problem into two cases:

(i) $N_r \lesssim N$. This gives us the Lagrange function

$$\begin{aligned} & L_{\lambda_1,\dots,\lambda_6}(J, N_r, N, S, R) \\ :=& \log(J) + \log(N) + d\log(S) + \lambda_1(-2\log(J) - 2\log(\varepsilon)) \\ & + \lambda_2(-\log(J) - \log(N) - 2\log(\varepsilon)) \\ & + \lambda_3(d\log(S) - \log(N_r) - \log(N) - 2\log(\varepsilon)) \\ & + \lambda_4(2(p+1)(\log(R) - \log(S)) - \log(N) - 2\log(\varepsilon)) \\ & + \lambda_5(-\nu\log(R) - \log(N) - 2\log(\varepsilon)) + \lambda_6(\log(N_r) - \log(N)), \end{aligned} \tag{3.78}$$

where $\lambda_1, \dots, \lambda_6 \geq 0$. Thus, considering of the conditions $\frac{\partial L}{\partial J} = \frac{\partial L}{\partial N_r} = \frac{\partial L}{\partial N} = \frac{\partial L}{\partial S} = \frac{\partial L}{\partial R} \stackrel{!}{=} 0$ gives us the following relations

$$\begin{aligned} \lambda_1 &= \frac{1-\lambda_2}{2}, \\ \lambda_3 &= \frac{2(p+1)(\nu(1-\lambda_2) - d) - d\nu}{d\nu + 2(p+1)(d+2\nu)} = \lambda_6, \\ \lambda_4 &= \frac{d\nu(3-\lambda_2)}{d\nu + 2(p+1)(d+2\nu)}, \\ \lambda_5 &= \frac{2d(p+1)(3-\lambda_2)}{d\nu + 2(p+1)(d+2\nu)}. \end{aligned}$$

The case $\lambda_1, \dots, \lambda_6 > 0$ is not feasible, since all constraints in (3.78) cannot be active, that is, they cannot become zero simultaneously because of six (linearly independent) equalities on five unknowns. Hence, we derive the solutions under $\lambda_i = 0$ for different i and observe which one is actually optimal.

a) $\lambda_1 = 0 \;\Rightarrow\; \lambda_3 = \lambda_6 = -\frac{d(2(p+1)+\nu)}{d\nu+2(p+1)(d+2\nu)} < 0$. Due to negative λ_3, λ_6, this case is not optimal.

b) $\lambda_2 = 0 \;\Rightarrow\; \lambda_1, \lambda_4, \lambda_5 > 0$, $\lambda_3 = \lambda_6 = \frac{2(p+1)(\nu-d)-d\nu}{d\nu+2(p+1)(d+2\nu)}$. Again, we make a case distinction:

   I. $\lambda_3 = \lambda_6 = 0$. From this condition, we get $\nu = \frac{2d(p+1)}{2(p+1)-d}$ and $2(p+1) > d$. (The latter guarantees that ν is positive). This gives us, due to $\lambda_1, \lambda_4, \lambda_5 > 0$,

$$\begin{aligned} J &\asymp \varepsilon^{-1}, \\ S &\asymp \left[\frac{1}{N\varepsilon^2}\right]^{\frac{1}{d}}. \end{aligned}$$

   Hence, the complexity $JS^dN \asymp \varepsilon^{-3}$ is no improvement compared to the SMC approach.

II. $\lambda_3 = \lambda_6 > 0$. From this condition, we get $\nu > \frac{2d(p+1)}{2(p+1)-d}$ and $2(p+1) > d$. (Again, the second condition guarantees that ν is positive.) In this case, all constraints apart from the second one in (3.78), corresponding to λ_2, are active. Then we obtain

$$\begin{aligned} J &\asymp \varepsilon^{-1}, \\ S &\asymp \varepsilon^{-\frac{2\nu+4(p+1)}{d\nu+2(p+1)(d+2\nu)}}, \\ N &\asymp \varepsilon^{-\frac{2d\nu+4(p+1)(d+\nu)}{d\nu+2(p+1)(d+2\nu)}}. \end{aligned}$$

Here, the complexity $JS^dN \asymp \varepsilon^{-\frac{5d\nu+2(p+1)(5d+4\nu)}{d\nu+2(p+1)(d+2\nu)}}$ is a better solution than the previous one. Moreover, the remaining constraint $\frac{1}{JN} \lesssim \varepsilon^2$ is also satisfied under this solution.

c) $\lambda_3 = \lambda_6 = 0 \Rightarrow \lambda_1, \lambda_4, \lambda_5 > 0$, $\lambda_2 = \frac{2(p+1)(\nu-d)-d\nu}{2(p+1)\nu}$. The case $\lambda_2 = 0$ is the same as the last but one and thus gives us the complexity $JS^dN \asymp \varepsilon^{-3}$. The case $\lambda_2 > 0$ leads to four active constraints in (3.78), namely the ones corresponding to $\lambda_1, \lambda_2, \lambda_4, \lambda_5$, such that

$$\begin{aligned} J &\asymp \varepsilon^{-1}, \\ S &\asymp \varepsilon^{-\frac{\nu+2(p+1)}{2\nu(p+1)}}, \\ N &\asymp \varepsilon^{-1}. \end{aligned}$$

The complexity $JS^dN \asymp \varepsilon^{-\frac{d\nu+2(p+1)(d+2\nu)}{2\nu(p+1)}}$ seems to be nice at the first moment. However, it does not satisfy both constraints corresponding to λ_3, λ_6. On the one hand, we have for the third constraint $N_r \gtrsim \varepsilon^{-1-\frac{d\nu+2d(p+1)}{2\nu(p+1)}}$. On the other hand, we have for the sixth constraint $N_r \lesssim \varepsilon^{-1}$. Hence, this is not an admissible solution.

d) $\lambda_4 = 0 \Rightarrow \lambda_1 = -1$. Since λ_1 is negative, this case is not optimal.

e) $\lambda_5 = 0 \Rightarrow \lambda_1 = -1$. As for the previous one, this case is not optimal.

(ii) $N_r \gtrsim N$. This gives us the Lagrange function

$$\begin{aligned} &\tilde{L}_{\lambda_1,\dots,\lambda_6}(J, N_r, N, S, R) \\ &:= \log(J) + \log(N_r) + d\log(S) + \lambda_1(-2\log(J) - 2\log(\varepsilon)) \\ &\quad + \lambda_2(-\log(J) - \log(N) - 2\log(\varepsilon)) \\ &\quad + \lambda_3(d\log(S) - \log(N_r) - \log(N) - 2\log(\varepsilon)) \\ &\quad + \lambda_4(2(p+1)(\log(R) - \log(S)) - \log(N) - 2\log(\varepsilon)) \\ &\quad + \lambda_5(-\nu\log(R) - \log(N) - 2\log(\varepsilon)) + \lambda_6(\log(N) - \log(N_r)). \end{aligned}$$

Analogously to the procedure above we get the same optimal solution, that is

$$\begin{aligned} J &\asymp \varepsilon^{-1}, \\ S &\asymp \varepsilon^{-\frac{2\nu+4(p+1)}{d\nu+2(p+1)(d+2\nu)}}, \\ N_r &\asymp \varepsilon^{-\frac{2d\nu+4(p+1)(d+\nu)}{d\nu+2(p+1)(d+2\nu)}}. \end{aligned}$$

Thus, we have again for the complexity $JS^dN_r \asymp \varepsilon^{-\frac{5d\nu+2(p+1)(5d+4\nu)}{d\nu+2(p+1)(d+2\nu)}}$.

Next we consider also the remaining terms $c_{p,d}$, $(p+1)!$, B_ν and arrive at (3.45)–(3.46) via equalising all constraints in (3.44) apart from the second one as well as considering $N \asymp N_r$ (provided that $p > \frac{d-2}{2}$ and $\nu > \frac{2d(p+1)}{2(p+1)-d}$). Finally, we add the log-term concerning ε in the parameters N_r, N to ensure that all constraints are really satisfied.

Let us additionally prove the statement in footnote 6 on page 32, i.e. that the complexity of the integral approach would be worse than that of the SMC whenever at least one of the above inequalities is violated. More precisely, the statement we are going to prove sounds as follows. If either $p \leq \frac{d-2}{2}$ (recall that $p \in \mathbb{N}_0$) or $\nu \leq \frac{2(p+1)d}{2(p+1)-d}$ (recall that $\nu > 0$), then the cost $\mathcal{C}$ of the integral algorithm given in (3.43) is worse than ε^{-3} regardless of the choice of J, S, R, N and N_r such that (3.44) holds true.

We first remark that any choice of J, S, R, N, N_r such that R does not tend to infinity as $\varepsilon \searrow 0$ results in $\mathcal{C} \gtrsim \varepsilon^{-3}$. Indeed, in this case we see from the first and the fifth terms in (3.44) that $J \gtrsim \varepsilon^{-1}$ and $N \gtrsim \varepsilon^{-2}$, hence $\mathcal{C} \gtrsim JN \gtrsim \varepsilon^{-3}$. Therefore, below we consider without loss of generality only such choices of J, S, R, N_r, N, where R tends to infinity as $\varepsilon \searrow 0$, and discuss the following two cases.

Let $p \leq \frac{d-2}{2}$, that is, $2(p+1) \leq d$. Then we obtain from the fourth term in (3.44)

$$S^dN \gtrsim S^{2(p+1)}N \gtrsim \varepsilon^{-2}R^{2(p+1)} \gtrsim \varepsilon^{-2}$$

and hence, together with $J \gtrsim \varepsilon^{-1}$ (see the first term in (3.44)), we have for the cost

$$\mathcal{C} \gtrsim JS^dN \gtrsim \varepsilon^{-2}J \gtrsim \varepsilon^{-3}.$$

Next, let $p > \frac{d-2}{2}$, that is, $2(p+1) > d$, and $0 < \nu \leq \frac{2(p+1)d}{2(p+1)-d}$. Then we get from the fourth and the fifth terms in (3.44)

$$R^{2(p+1)} \lesssim S^{2(p+1)}N\varepsilon^2,$$
$$R^{\frac{2(p+1)d}{2(p+1)-d}} \gtrsim R^\nu \gtrsim N^{-1}\varepsilon^{-2}.$$

Therefore,

$$S^{\frac{2(p+1)d}{2(p+1)-d}}N^{\frac{d}{2(p+1)-d}}\varepsilon^{\frac{2d}{2(p+1)-d}} \gtrsim N^{-1}\varepsilon^{-2}.$$

This yields

$$S^{\frac{2(p+1)d}{2(p+1)-d}}N^{\frac{2(p+1)}{2(p+1)-d}} \gtrsim \varepsilon^{-\frac{4(p+1)}{2(p+1)-d}},$$

and we deduce

$$S^dN \gtrsim \varepsilon^{-2}.$$

Together with $J \gtrsim \varepsilon^{-1}$, we obtain for the cost

$$\mathcal{C} \gtrsim JS^dN \gtrsim J\varepsilon^{-2} \gtrsim \varepsilon^{-3},$$

which completes the proof. □

Proof of Theorem 3.23

The proof is similar to the one of Theorem 3.19. □

Proof of Theorem 3.24

The proof is similar to the one of Theorem 3.20. □

Chapter 4

Optimal control variates for weak approximation schemes

This chapter is mainly based on the paper [7].

Below, we derive the analogue of representation (3.6) for so-called weak approximation schemes, i.e. the schemes, where simple random variables are used in place of Gaussian increments. We observe that optimal control variates contain a finite sum in this case, in contrast to strong approximation schemes. Moreover, we present a recursive algorithm, which shall lead to accurate regression estimates.

In recent years weak approximation schemes became quite popular. The weak Euler scheme is a first order scheme with weak order of convergence $\alpha = 1$, and has been studied by many researchers. Milstein [43] showed the first order convergence of the weak Euler scheme. The fact that the same weak convergence rate of the Euler scheme also holds for certain irregular functions under a Hörmander type condition was proved by Bally and Talay [5] using Malliavin calculus. The Itô-Taylor (weak Taylor) high-order scheme is a natural extension of the weak Euler scheme. In the diffusion case, some new discretisation schemes (also called Kusuoka type schemes) which are of order $\alpha \geq 2$ without the Romberg extrapolation have been introduced by Kusuoka [39], Lyons and Victoir [41], Ninomiya and Victoir [51], and Ninomiya and Ninomiya [50]. A general class of weak approximation methods, comprising many well-known discretisation schemes, was constructed in Kohatsu-Higa and Tanaka [57]. The main advantage of the weak approximation schemes is that simple discrete random variables can be used to approximate multiple Wiener integrals arising in higher order schemes.

In this chapter we focus on the weak schemes of first and second order.

4.1 First order schemes

In this subsection we treat weak schemes of order 1. Let us consider a scheme, where d-dimensional approximations $X_{\Delta,j\Delta}$, $j = 0, \dots, J$, satisfy $X_{\Delta,0} = x_0$ and

$$X_{\Delta,j\Delta} = \Phi_\Delta(X_{\Delta,(j-1)\Delta}, \xi_j), \quad j = 1, \dots, J, \tag{4.1}$$

for some functions $\Phi_\Delta \colon \mathbb{R}^{d+m} \to \mathbb{R}^d$, with $\xi_j = (\xi_j^1, \dots, \xi_j^m)^\top$, $j = 1, \dots, J$, being m-dimensional i.i.d. random vectors with i.i.d. coordinates such that

$$\mathbb{P}\left(\xi_j^k = \pm 1\right) = \frac{1}{2}, \quad k = 1, \dots, m.$$

A particular case is the weak Euler scheme (also called the *simplified weak Euler scheme* in [38, Section 14.1]) of order 1, which is given by (3.5).

Theorem 4.1. *The following representation holds*

$$f(X_{\Delta,T}) = \mathbb{E}f(X_{\Delta,T}) + \sum_{j=1}^{J} \sum_{r=1}^{m} \sum_{1 \le s_1 < \dots < s_r \le m} a_{j,r,s}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{r} \xi_j^{s_i}, \tag{4.2}$$

where we use the notation $s = (s_1, \dots, s_r)$. Moreover, the coefficients $a_{j,r,s} \colon \mathbb{R}^d \to \mathbb{R}$ can be computed by the formula

$$a_{j,r,s}(x) = \mathbb{E}\left[f(X_{\Delta,T}) \prod_{i=1}^{r} \xi_j^{s_i} \,\middle|\, X_{\Delta,(j-1)\Delta} = x \right] \tag{4.3}$$

for all j, r, and s as in (4.2).

Example 4.2. Let us recall the example in Subsection 3.1.1. In case of the simplified weak Euler scheme it holds $X_{\Delta,j\Delta} = X_{\Delta,(j-1)\Delta}(1 + \sigma\sqrt{\Delta}\xi_j)$ and thus we have again the representation (3.1). However, due to $\mathbb{P}(\xi_j^2 = 1) = 1$ for all $j \in \{1, \dots, J\}$, we obtain

$$X_{\Delta,j\Delta}^2 - X_{\Delta,(j-1)\Delta}^2(1+\sigma^2\Delta) = \sigma\sqrt{\Delta}X_{\Delta,(j-1)\Delta}^2\left(2\xi_j + \sigma\sqrt{\Delta}(\xi_j^2 - 1)\right) = 2\sigma\sqrt{\Delta}X_{\Delta,(j-1)\Delta}^2\xi_j.$$

This gives us a simplified formula compared to (3.2), namely (cf. (4.2))

$$X_{\Delta,T}^2 - \mathbb{E}\left[X_{\Delta,T}^2\right] = \sum_{j=1}^{J} a_{j,1,1}(X_{\Delta,(j-1)\Delta})\xi_j$$

with $a_{j,1,1}(x) = 2\sigma\sqrt{\Delta}x^2(1+\sigma^2\Delta)^{J-j}$.

Next proposition shows the properties of the simplified Euler scheme combined with the control variate

$$M_{\Delta,T}^{(1)} := \sum_{j=1}^{J} \sum_{r=1}^{m} \sum_{1 \le s_1 < \dots < s_r \le m} a_{j,r,s}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{r} \xi_j^{s_i}, \tag{4.4}$$

where the coefficients $a_{j,r,s}(x)$ are given by (4.3). It is a combination of the above Theorem 4.1 together with Theorem 2.1 in [44].

Proposition 4.3. *Assume that μ and σ in* (1.1) *are Lipschitz continuous with components μ_k, $\sigma_{ki}\colon \mathbb{R}^d \to \mathbb{R}$, $k = 1, \dots, d$, $i = 1, \dots, m$, being* 4 *times continuously differentiable with their partial derivatives of order up to* 4 *having polynomial growth. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be* 4 *times continuously differentiable with partial derivatives of order up to* 4 *having polynomial growth. Provided that* (3.5) *holds and that, for sufficiently large $n \in \mathbb{N}$, the expectations $\mathbb{E}|X_{\Delta,j\Delta}|^{2n}$ are uniformly bounded in J and $j = 0, \dots, J$, we have for this "simplified weak Euler scheme" (cf. Proposition 3.14)*

$$|\mathbb{E}\left[f(X_T) - f(X_{\Delta,T})\right]| \leq c\Delta,$$

where the constant c does not depend on Δ. Moreover, it holds

$$\operatorname{Var}\left[f(X_{\Delta,T}) - M^{(1)}_{\Delta,T}\right] = 0. \tag{4.5}$$

Remark 4.4. (i) In order to use the control variate $M^{(1)}_{\Delta,T}$ in practice, we need to estimate the unknown coefficients $a_{j,r,s}$. Thus, practically implementable control variates $\tilde{M}^{(1)}_{\Delta,T}$ have the form (4.4) with some estimated functions $\tilde{a}_{j,r,s}\colon \mathbb{R}^d \to \mathbb{R}$ (similar to the control variates in Chapter 3).

(ii) Notice that the weak Euler scheme is, in contrast to the (strong) Euler scheme with Gaussian increments, not strongly convergent.[9] However, the assumption on strong convergence is not required in our setting. That is, we only need weak convergence and here we have the same convergence order as for the Euler scheme (cf. Proposition 3.14).

4.1.1 Computation of coefficients

Coefficients (4.3) can be directly computed using various regression algorithms as discussed in Subsection 2.1.1. From a computational point of view it is sometimes advantageous to look for another representation which only involves a regression over one time step (note that in (4.3) regression should be performed over $J - j + 1$ time steps). To this end, we recall the functions (cf. (2.13))

$$q_j(x) = \mathbb{E}[f(X_{\Delta,T})|X_{\Delta,j\Delta} = x]. \tag{4.6}$$

The next proposition contains backward recursion formulas for the functions q_j as well as the expressions for the coefficients (4.3) in terms of q_j, $j = 1, \dots, J$.

Proposition 4.5. *We have $q_J \equiv f$ and for each $j \in \{1, \dots, J\}$,*

$$q_{j-1}(x) = \mathbb{E}\big[q_j(X_{\Delta,j\Delta})|X_{\Delta,(j-1)\Delta} = x\big] = \frac{1}{2^m} \sum_{y=(y_1,\dots,y_m)\in\{-1,1\}^m} q_j(\Phi_\Delta(x,y)). \tag{4.7}$$

Moreover, the coefficients (4.3) *can be expressed in terms of the functions q_j, $j = 1, \dots, J$, as*

$$a_{j,r,s}(x) = \frac{1}{2^m} \sum_{y=(y_1,\dots,y_m)\in\{-1,1\}^m} \left[\prod_{i=1}^{r} y_{s_i}\right] q_j(\Phi_\Delta(x,y)) \tag{4.8}$$

for all j, r and $s = (s_1, \dots, s_r)$ as in (4.2).

[9] One usually speaks about strong convergence when the approximations $(X_{\Delta,j\Delta})$ are defined on the same space as the solution (X_t) and we have the convergence in some L^p-space, e.g. $\lim_{\Delta\searrow 0} \mathbb{E}|X_{\Delta,T} - X_T|^2 = 0$.

Remark 4.6. The advantage of the representation (4.8) over the original one consists in the fact that all functions q_j, $j = 1, \dots, J$, can be recursively computed using regressions over one time step (based on the first equality in (4.7)) and without involvement of the independent of $X_{\Delta,(j-1)\Delta}$ centred random variables $\xi_j^{s_i}$ (cf. (4.3)), rendering the estimates for q_j more stable. If q_j is approximated as a linear combination of Q basis functions, then the cost of computing the coefficients in this combination by least squares regression on N_r paths is of order $N_r Q^2$. Once q_j is approximated, the cost of estimating $a_{j,r,s}(x)$ in a given point x via (4.8) is of order $2^m(c_1 + c_2 Q)$, where the constant c_1 describes the cost of computing $\Phi_\Delta(x, y)$ for given points x and y (this is dm in case of (3.5)), and the constant c_2 describes the cost of computing the value of a basis function at a point in $\mathbb{R}^d$ (this is typically d).

4.1.2 Additional representation formula

Theorem 4.7. *Alternatively to* (4.2)*, we also have the following representation formula*

$$f(X_{\Delta,T}) = \mathbb{E} f(X_{\Delta,T}) + \sum_{j=1}^{J} \sum_{r=1}^{m} a_{j,r}(X_{\Delta,(j-1)\Delta}, (\xi_j^i)_{i=1}^{r-1}) \xi_j^r, \tag{4.9}$$

where the coefficients $a_{j,r} \colon \mathbb{R}^{d+r-1} \to \mathbb{R}$ *can be computed by the formula*

$$a_{j,r}(x, (y_i)_{i=1}^{r-1}) = \mathbb{E}\left[f(X_{\Delta,T}) \xi_j^r \,\middle|\, X_{\Delta,(j-1)\Delta} = x, (\xi_j^i)_{i=1}^{r-1} = (y_i)_{i=1}^{r-1} \right] \tag{4.10}$$

for $j = 1, \dots, J$ *and* $r = 1, \dots, m$.

Remark 4.8. Theorems 4.1 and 4.7 suggest two perfect control variates for scheme (4.1). Compared with the control variate based on (4.2), the control variate based on (4.9) contains a smaller number of coefficients to be computed, but these coefficients are functions of a greater number of variables and thus it is numerically advantageous to use the control variate (4.3) for implementation (cf. Remark 3.5 in case of strong schemes).

Proposition 4.9. *An equivalent form of* (4.10) *is*

$$a_{j,r}\big(x, (y_i)_{i=1}^{r-1}\big) = \frac{1}{2^{m-r+1}} \sum_{(y_{r+1}, \dots, y_m) \in \{-1,1\}^{m-r}} \left[q_j(\Phi_\Delta(x, \overline{y})) - q_j(\Phi_\Delta(x, \underline{y})) \right], \tag{4.11}$$

where

$$\overline{y} := (y_1, \dots, y_{r-1}, 1, y_{r+1}, \dots, y_m), \tag{4.12}$$

$$\underline{y} := (y_1, \dots, y_{r-1}, -1, y_{r+1}, \dots, y_m) \tag{4.13}$$

for all $j \in \{1, \dots, J\}$ *and* $r \in \{1, \dots, m\}$.

4.2 Second order schemes

Next we treat weak schemes of order 2. We consider a scheme, where d-dimensional approximations $X_{\Delta,j\Delta}$, $j = 0, \dots, J$, satisfy $X_{\Delta,0} = x_0$ and

$$X_{\Delta,j\Delta} = \Phi_\Delta(X_{\Delta,(j-1)\Delta}, \xi_j, V_j), \quad j = 1, \dots, J, \tag{4.14}$$

for some functions $\Phi_\Delta\colon \mathbb{R}^{d+m+m\times m} \to \mathbb{R}^d$. Here,

(S1) $\xi_j = (\xi_j^k)_{k=1}^m$ are m-dimensional random vectors,

(S2) $V_j = (V_j^{kl})_{k,l=1}^m$ are random $m \times m$-matrices,

(S3) the pairs (ξ_j, V_j), $j = 1, \dots, J$, are i.i.d.,

(S4) for each j, the random elements ξ_j and V_j are independent,

(S5) for each j, the random variables ξ_j^k, $k = 1, \dots, m$, are i.i.d. with

$$\mathbb{P}\left(\xi_j^k = \pm\sqrt{3}\right) = \frac{1}{6}, \quad \mathbb{P}\left(\xi_j^k = 0\right) = \frac{2}{3},$$

(S6) for each j, the random variables V_j^{kl}, $1 \le k < l \le m$, are i.i.d. with

$$\mathbb{P}\left(V_j^{kl} = \pm 1\right) = \frac{1}{2},$$

(S7) $V_j^{lk} = -V_j^{kl}$, $1 \le k < l \le m$, $j = 1, \dots, J$,

(S8) $V_j^{kk} = -1$, $k = 1, \dots, m$, $j = 1, \dots, J$.

Hence, the matrices V_j can be generated by means of $\frac{m(m-1)}{2}$ i.i.d. random variables. That is, relating to the framework in Section 2.2, we have $\tilde{m}$-dimensional random vectors $\tilde{\xi}_j := ((\xi_j^i)_{i=1,\dots,m}, (V_j^{il})_{1\le i<l\le m})$ with $\tilde{m} = m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$. Notice that the random vectors in Section 4.1 are not useful here, since they do not satisfy specific moment conditions (cf. Section 14.2 in [38]). In contrast, the random vectors ξ_j, introduced in this section, agree in the first five moments with the standard normal distributed ones, which is sufficient.

Remark 4.10. In order to obtain an order 2 weak scheme in the multidimensional case, we need to incorporate additional random elements V_j into the structure of the scheme. This is the reason why we now consider (4.14) instead of (4.1). For instance, to get the *simplified order 2 weak Taylor scheme* of [38, Section 14.2] in the multidimensional case, we need to define the functions $\Phi_\Delta(x, y, z)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times m}$, as explained below. First we define the function $\Sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ by the formula

$$\Sigma(x) = \sigma(x)\sigma(x)^\top \tag{4.15}$$

and remark that the coordinates of Σ and Φ_Δ are denoted by $\Sigma_{kl}(x)$ and $\Phi_\Delta^k(x, y, z)$ for $k, l = 1, \dots, d$. Let us introduce the operators $\mathcal{L}^r$, $r = 0, \dots, m$, that act on sufficiently smooth functions $g\colon \mathbb{R}^d \to \mathbb{R}$ as follows:

$$\begin{aligned}
\mathcal{L}^0 g(x) &:= \sum_{k=1}^d \mu_k(x) \frac{\partial g}{\partial x_k}(x) + \frac{1}{2} \sum_{k,l=1}^d \Sigma_{kl}(x) \frac{\partial^2 g}{\partial x_l \partial x_k}(x), \\
\mathcal{L}^r g(x) &:= \sum_{k=1}^d \sigma_{kr}(x) \frac{\partial g}{\partial x_k}(x), \quad r = 1, \dots, m.
\end{aligned} \tag{4.16}$$

The r-th coordinate Φ^r_Δ, $r = 1, \dots, d$, in the simplified order 2 weak Taylor scheme of [38, Section 14.2] is now given by the formula

$$\begin{aligned}\Phi^r_\Delta(x, y, z) = x_r &+ \sum_{k=1}^m \sigma_{rk}(x)\, y_k \sqrt{\Delta} \\ &+ \left[\mu_r(x) + \frac{1}{2}\sum_{k,l=1}^m \mathcal{L}^k \sigma_{rl}(x)(y_k y_l + z_{kl})\right]\Delta \\ &+ \frac{1}{2}\sum_{k=1}^m \left[\mathcal{L}^0 \sigma_{rk}(x) + \mathcal{L}^k \mu_r(x)\right] y_k\, \Delta^{3/2} + \frac{1}{2}\mathcal{L}^0 \mu_r(x)\, \Delta^2,\end{aligned} \tag{4.17}$$

provided the coefficients μ and σ of (1.1) are sufficiently smooth. We will need to work explicitly with (4.17) at some point, but all results in this subsection assume structure (4.14) only.

Let us define the index sets

$$\mathcal{I}_1 = \{1, \dots, m\}, \quad \mathcal{I}_2 = \left\{(k, l) \in \mathcal{I}_1^2 : k < l\right\}$$

and the system

$$\mathcal{A} = \left\{(U_1, U_2) \in \mathcal{P}(\mathcal{I}_1) \times \mathcal{P}(\mathcal{I}_2) : U_1 \cup U_2 \neq \emptyset\right\},$$

where $\mathcal{P}(\mathcal{I})$ denotes the set of all subsets of a set $\mathcal{I}$. For any $U_1 \subseteq \mathcal{I}_1$ and $o \in \{1, 2\}^{U_1}$, we write o as $o = (o_r)_{r \in U_1}$. Below we use the convention that a product over the empty set is always one.

Theorem 4.11. *It holds*

$$f(X_{\Delta,T}) = \mathbb{E} f(X_{\Delta,T}) + \sum_{j=1}^J \sum_{(U_1,U_2)\in\mathcal{A}} \sum_{o\in\{1,2\}^{U_1}} a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) \prod_{r\in U_1} H_{o_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl}, \tag{4.18}$$

where the coefficients $a_{j,o,U_1,U_2} \colon \mathbb{R}^d \to \mathbb{R}$ can be computed by the formula

$$a_{j,o,U_1,U_2}(x) = \mathbb{E}\left[f(X_{\Delta,T}) \prod_{r\in U_1} H_{o_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} \,\middle|\, X_{\Delta,(j-1)\Delta} = x \right]. \tag{4.19}$$

Combining Theorem 4.11 with Theorem 2.1 in [44] we obtain the following result, which provides a bound for the discretisation error and a perfect control variate for the discretised quantity.

Proposition 4.12. *Assume, that μ and σ in* (1.1) *are Lipschitz continuous with components μ_k, $\sigma_{ki} \colon \mathbb{R}^d \to \mathbb{R}$, $k = 1, \dots, d$, $i = 1, \dots, m$, being* 6 *times continuously differentiable with their partial derivatives of order up to* 6 *having polynomial growth. Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be* 6 *times continuously differentiable with partial derivatives of order up to* 6 *having polynomial growth. Provided that* (4.17) *holds and that, for sufficiently large $n \in \mathbb{N}$, the expectations*

$\mathbb{E}|X_{\Delta,j\Delta}|^{2n}$ *are uniformly bounded in* J *and* $j = 0, \dots, J$, *we have for this "simplified second order weak Taylor scheme"*

$$|\mathbb{E}\left[f(X_T) - f(X_{\Delta,T})\right]| \leq c\Delta^2,$$

where the constant c *does not depend on* Δ. *Moreover, we have* $\mathrm{Var}\left[f(X_{\Delta,T}) - M^{(2)}_{\Delta,T}\right] = 0$ *for the control variate*

$$M^{(2)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{(U_1,U_2)\in\mathcal{A}} \sum_{o\in\{1,2\}^{U_1}} a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) \prod_{r\in U_1} H_{o_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl}, \tag{4.20}$$

where the coefficients $a_{j,o,U_1,U_2}(x)$ *are defined in* (4.19).

4.2.1 Computation of coefficients

Similar to the case of first order schemes, one can derive an alternative representation for the coefficients (4.19) making their computation more efficient and stable. The next result contains backward recursions for the functions q_j of (4.6) and for a_{j,o,U_1,U_2} of (4.19).

Proposition 4.13. *We have* $q_J \equiv f$ *and, for each* $j \in \{1, \dots, J\}$,

$$\begin{aligned} q_{j-1}(x) &= \mathbb{E}[q_j(X_{\Delta,j\Delta})|X_{\Delta,(j-1)\Delta} = x] \qquad (4.21)\\ &= \frac{1}{2^{\frac{m(m-1)}{2}}} \frac{1}{6^m} \sum_{(y_1,\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^m} \sum_{(z_{uv})_{1\leq u<v\leq m}\in\{-1,1\}^{\frac{m(m-1)}{2}}} 4^{\sum_{i=1}^m I(y_i=0)} q_j(\Phi_\Delta(x,y,z)), \end{aligned}$$

and, for all $j \in \{1, \dots, J\}$, $(U_1, U_2) \in \mathcal{A}$ *and* $o \in \{1, 2\}^{U_1}$, *it holds*

$$\begin{aligned} a_{j,o,U_1,U_2}(x) = \frac{1}{2^{\frac{m(m-1)}{2}}} \frac{1}{6^m} &\sum_{(y_1,\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^m} \sum_{(z_{uv})_{1\leq u<v\leq m}\in\{-1,1\}^{\frac{m(m-1)}{2}}} \qquad (4.22)\\ &\cdot 4^{\sum_{i=1}^m I(y_i=0)} \prod_{r\in U_1} H_{o_r}(y_r) \prod_{(k,l)\in U_2} z_{kl}\, q_j(\Phi_\Delta(x,y,z)), \end{aligned}$$

where $y = (y_1, \dots, y_m)$ *and* $z = (z_{uv})$ *is the* $m\times m$*-matrix with* $z_{vu} = -z_{uv}$, $u < v$, $z_{uu} = -1$.

4.2.2 Additional representation formula

We now introduce the following ordering on $\mathcal{I}_2$: for $(k_i, l_i) \in \mathcal{I}_2$, $i = 1, 2$, we say

$$(k_1, l_1) \prec (k_2, l_2) \quad \Longleftrightarrow \quad k_1 < k_2 \text{ or } (k_1 = k_2 \text{ and } l_1 < l_2).$$

Theorem 4.14. *We have the following representation*

$$\begin{aligned} f(X_{\Delta,T}) = \mathbb{E}f(X_{\Delta,T}) + \sum_{j=1}^{J} \Bigg[&\sum_{k=1}^{2} \sum_{r=1}^{m} a_{j,k,r}(X_{\Delta,(j-1)\Delta}, (\xi_j^i)_{i=1}^{r-1}) H_k(\xi_j^r) \qquad (4.23)\\ &+ \sum_{(k,l)\in\mathcal{I}_2} b_{j,k,l}(X_{\Delta,(j-1)\Delta}, \xi_j, (V_j^{rs})_{(r,s)\in\mathcal{I}_2,(r,s)\prec(k,l)}) V_j^{kl} \Bigg], \end{aligned}$$

where the coefficients $a_{j,k,r}$ can be computed by the formula

$$a_{j,k,r}(x,(y_i)_{i=1}^{r-1}) = \mathbb{E}\left[f(X_{\Delta,T})H_k(\xi_j^r)\middle| X_{\Delta,(j-1)\Delta}=x, (\xi_j^i)_{i=1}^{r-1}=(y_i)_{i=1}^{r-1}\right], \qquad (4.24)$$
$$x\in\mathbb{R}^d, \quad y_i\in\mathbb{R},$$

for $j=1,\dots,J$, $k=1,2$, $r=1,\dots,m$, and the coefficients $b_{j,k,l}$ are given by the formula

$$b_{j,k,l}(x,y,(z_{rs})_{(r,s)\in\mathcal{I}_2,(r,s)\prec(k,l)}) = \mathbb{E}\Big[f(X_{\Delta,T})V_j^{kl}\,\Big|\, X_{\Delta,(j-1)\Delta}=x, \xi_j=y, \qquad (4.25)$$
$$(V_j^{rs})_{(r,s)\in\mathcal{I}_2,(r,s)\prec(k,l)} = (z_{rs})_{(r,s)\in\mathcal{I}_2,(r,s)\prec(k,l)}\Big],$$
$$x\in\mathbb{R}^d, \quad y\in\mathbb{R}^m, \quad z_{rs}\in\mathbb{R},$$

for $j=1,\dots,J$ and $(k,l)\in\mathcal{I}_2$.

Again, representations (4.18) and (4.23) suggest two perfect control variates for scheme (4.14).

Proposition 4.15. *Let us fix $j\in\{1,\dots,J\}$ and $r\in\{1,\dots,m\}$. An equivalent form of (4.24) is*

$$a_{j,1,r}(x,(y_i)_{i=1}^{r-1}) = \frac{1}{2^{\frac{m(m-1)}{2}}}\frac{\sqrt{3}}{6^{m-r+1}} \sum_{(y_{r+1},\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^{m-r}} \sum_{(z_{kl})_{1\le k<l\le m}\in\{-1,1\}^{\frac{m(m-1)}{2}}}$$
$$\cdot 4^{\sum_{i=r+1}^m I(y_i=0)}\left[q_j(\Phi_\Delta(x,\overline{y},z)) - q_j(\Phi_\Delta(x,\underline{y},z))\right],$$
$$a_{j,2,r}(x,(y_i)_{i=1}^{r-1}) = \frac{1}{2^{\frac{m(m-1)}{2}}}\frac{\sqrt{2}}{6^{m-r+1}} \sum_{(y_{r+1},\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^{m-r}} \sum_{(z_{kl})_{1\le k<l\le m}\in\{-1,1\}^{\frac{m(m-1)}{2}}}$$
$$\cdot 4^{\sum_{i=r+1}^m I(y_i=0)}\left[q_j(\Phi_\Delta(x,\overline{y},z)) + q_j(\Phi_\Delta(x,\underline{y},z)) - 2q_j(\Phi_\Delta(x,y_\circ,z))\right],$$

where

$$\overline{y} = (y_1,\dots,y_{r-1},\sqrt{3},y_{r+1},\dots,y_m),$$
$$\underline{y} = (y_1,\dots,y_{r-1},-\sqrt{3},y_{r+1},\dots,y_m),$$
$$y_\circ = (y_1,\dots,y_{r-1},0,y_{r+1},\dots,y_m)$$

and $z=(z_{kl})$ is the $m\times m$-matrix with $z_{lk}=-z_{kl}$, $k<l$, $z_{kk}=-1$.

Let us now fix $j\in\{1,\dots,J\}$ and $(k,l)\in\mathcal{I}_2$. Denote by $c_{k,l}$ the cardinality of the set $\mathcal{C}_{k,l}=\{(r,s)\in\mathcal{I}_2 : (k,l)\prec(r,s)\}$. An equivalent form of (4.25) is

$$b_{j,k,l}(x,y,(z_{rs})_{(r,s)\in\mathcal{I}_2,(r,s)\prec(k,l)}) = \frac{1}{2^{c_{k,l}+1}}\sum_{(z_{r's'})_{(r',s')\in\mathcal{C}_{k,l}}}\left[q_j(\Phi_\Delta(x,y,\overline{z})) - q_j(\Phi_\Delta(x,y,\underline{z}))\right],$$

where $\overline{z}=(\overline{z}^{rs})$ and $\underline{z}=(\underline{z}^{rs})$ are the $m\times m$-matrices with the elements z_{rs}, $(r,s)\in\mathcal{I}_2$, $(r,s)\neq(k,l)$, $\overline{z}^{kl}=1$, $\underline{z}^{kl}=-1$, and with $z_{sr}=-z_{rs}$, $z_{rr}=-1$.

4.3 Error bounds for piecewise polynomial regression

In the previous sections we have given several representations for perfect control variates. For the sake of clarity, we focus on second order schemes and representation (4.18) with coefficients given by (4.19). As in Section 3.3, we focus on the piecewise polynomial partitioning estimates.

4.3.1 Summary of the algorithm

The algorithm consists of a training and a testing phase (cf. algorithms in Chapter 3). In the training phase, we simulate N_r independent training paths

$$\mathcal{D}^{tr}_{N_r} = \left\{ (X^{tr,(n)}_{\Delta,j\Delta})_{j=0,\dots,J} : n = 1,\dots,N_r \right\}$$

and construct regression estimates $\hat{a}_{j,o,U_1,U_2}(\cdot, \mathcal{D}^{tr}_{N_r})$ for the coefficients $a_{j,o,U_1,U_2}(\cdot)$. In the testing phase, we simulate N independent testing paths $(X^{(n)}_{\Delta,j\Delta})_{j=0,\dots,J}$, $n = 1,\dots,N$, independently from $\mathcal{D}^{tr}_{N_r}$, and build the Monte Carlo estimator for $\mathbb{E}[f(X_T)]$ as

$$\mathcal{E} = \frac{1}{N} \sum_{n=1}^{N} \left(f(X^{(n)}_{\Delta,T}) - \hat{M}^{(2),(n)}_{\Delta,T} \right), \tag{4.26}$$

where (cf. (4.20))

$$\hat{M}^{(2),(n)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{(U_1,U_2)\in\mathcal{A}} \sum_{o\in\{1,2\}^{U_1}} \hat{a}_{j,o,U_1,U_2}(X^{(n)}_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) \prod_{r\in U_1} H_{o_r}(\xi^{r,(n)}_j) \prod_{(k,l)\in U_2} V^{kl,(n)}_j. \tag{4.27}$$

This gives us

$$\mathbb{E}[\mathcal{E}] = \mathbb{E}[f(X_{\Delta,T})], \tag{4.28}$$

$$\mathrm{Var}[\mathcal{E}] = \frac{1}{N} \mathrm{Var}\left[f(X^{(1)}_{\Delta,T}) - \hat{M}^{(2),(1)}_{\Delta,T} \right]. \tag{4.29}$$

Let us fix some $j \in \{1,\dots,J\}$, $(U_1,U_2) \in \mathcal{A}$, $o \in \{1,2\}^{U_1}$, set

$$\zeta_{J,j,o,U_1,U_2} := f(X_{\Delta,T}) \prod_{r\in U_1} H_{o_r}(\xi^r_j) \prod_{(k,l)\in U_2} V^{kl}_j$$

and remark that $a_{j,o,U_1,U_2}(x) = \mathbb{E}[\zeta_{J,j,o,U_1,U_2} | X_{\Delta,(j-1)\Delta} = x]$. We assume that, for some constant $h \in [1,\infty]$ and some positive constants $\Sigma, A, C_h, \nu, B_\nu$, it holds:

(A1) $\sup_{x\in\mathbb{R}^d} \mathrm{Var}[\zeta_{J,j,o,U_1,U_2} | X_{\Delta,(j-1)\Delta} = x] \le \Sigma < \infty$,

(A2) $\sup_{x\in\mathbb{R}^d} |a_{j,o,U_1,U_2}(x)| \le A\sqrt{\Delta} < \infty$,

(A3) a_{j,o,U_1,U_2} can be extended to $\mathbb{R}^d$ in a $(p+1, C_h)$-smooth way w.r.t. the norm $|\cdot|_h$,

(A4) $\mathbb{P}(|X_{\Delta,(j-1)\Delta}|_\infty > R) \le B_\nu R^{-\nu}$ for all $R > 0$.

Remark 4.16. Due to representation (4.22), the smoothness of the coefficient functions a_{j,o,U_1,U_2} is related to the smoothness of the one step conditional distribution of $X_{\Delta,j\Delta}$, given $X_{\Delta,(j-1)\Delta} = x$, for any $j = 1,\dots,J$ (recall the first equality in (4.21)), and to the smoothness in x of the mapping Φ_Δ from (4.14). In the case when the mapping Φ_Δ is given by (4.17), its smoothness in x is related to the smoothness of the coefficients μ and σ.

Let $\hat{a}_{j,o,U_1,U_2}$ be the piecewise polynomial partitioning estimate of a_{j,o,U_1,U_2} described in the beginning of this section. By $\tilde{a}_{j,o,U_1,U_2} = T_{A\sqrt{\Delta}} \hat{a}_{j,o,U_1,U_2}(x)$ we denote the truncated

estimate (cf. (3.48)). We again emphasise that, in fact, $\tilde{a}_{j,o,U_1,U_2}(x) = \tilde{a}_{j,o,U_1,U_2}(x, \mathcal{D}^{tr}_{N_r})$, that is, the estimates $\tilde{a}_{j,o,U_1,U_2}$ of the functions a_{j,o,U_1,U_2} depend on the simulated training paths.

Under (A1)–(A4), we have, due to Lemma 2.3,

$$\begin{aligned}\mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \le \tilde{c}\left(\Sigma + A^2\,\Delta(\log N_r + 1)\right)\frac{\binom{p+d}{d} S^d}{N_r} \\ + \frac{8\,C_h^2}{(p+1)!^2 d^{2-2/h}}\left(\frac{Rd}{S}\right)^{2p+2} + 8A^2\,\Delta B_\nu R^{-\nu},\end{aligned} \tag{4.30}$$

where $\tilde{c}$ is a universal constant and $\mathbb{P}_{\Delta,j-1}$ denotes the distribution of $X_{\Delta,(j-1)\Delta}$.

Similar to the L^2-errors in Chapter 3, the expectation in the left-hand side of (4.30) accounts for the averaging over the randomness in $\mathcal{D}^{tr}_{N_r}$, hence,

$$\begin{aligned}&\mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \\ =&\mathbb{E}\left[\left(\tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) - a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta})\right)^2\right],\end{aligned} \tag{4.31}$$

which provides an alternative form for the expression in the left-hand side of (4.30).

We now estimate the variance of the random variable $f(X_{\Delta,T}) - \tilde{M}^{(2)}_{\Delta,T}$, where

$$\tilde{M}^{(2)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{(U_1,U_2)\in\mathcal{A}} \sum_{o\in\{1,2\}^{U_1}} \tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) \prod_{r\in U_1} H_{o_r}(\xi^r_j) \prod_{(k,l)\in U_2} V^{kl}_j. \tag{4.32}$$

Using the martingale transform structure in (4.20) and (4.32) together with the orthonormality (in L^2) of the system $\prod_{r\in U_1} H_{o_r}(\xi^r_j) \prod_{(k,l)\in U_2} V^{kl}_j$, we get by (4.30)

$$\begin{aligned}\mathrm{Var}[f(X_{\Delta,T}) - \tilde{M}^{(2)}_{\Delta,T}] &= \mathrm{Var}[M^{(2)}_{\Delta,T} - \tilde{M}^{(2)}_{\Delta,T}] \\ &= \sum_{j=1}^{J} \sum_{(U_1,U_2)\in\mathcal{A}} \sum_{o\in\{1,2\}^{U_1}} \mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \\ &\le J\left(3^m 2^{\frac{m(m-1)}{2}} - 1\right)\left\{\tilde{c}\left(\Sigma + A^2\,\Delta(\log N_r + 1)\right)\frac{\binom{p+d}{d} S^d}{N_r}\right. \\ &\quad \left. + \frac{8\,C_h^2}{(p+1)!^2 d^{2-2/h}}\left(\frac{Rd}{S}\right)^{2p+2} + 8A^2\,\Delta B_\nu R^{-\nu}\right\}.\end{aligned} \tag{4.33}$$

Similar to Chapter 3, the estimator $\mathcal{E}$ given in (4.26) with "hat" replaced by "tilde" is an unbiased estimator of $\mathbb{E}[f(X_{\Delta,T})]$, and, by (4.29), the upper bound for its variance is $\frac{1}{N}$ times the last expression in (4.33).

4.4 Complexity analysis

Below we present a complexity analysis, which explains how we can go beyond the complexity order ε^{-2} with ε being the precision to be achieved.[10]

We will consider two variants of the Monte Carlo approach with regression-based control variate. The first algorithm, which is abbreviated below as *RCV approach* ("RCV" stands

[10] Recall that the MLMC algorithm can at best achieve the complexity of order ε^{-2}.

for "Regression-based Control Variate"), is the algorithm described in detail in Section 4.3. Here the estimates $\tilde{a}_{j,o,U_1,U_2}$ needed in (4.32) are constructed via regressions based on (4.19). In the second algorithm, which we call *recursive RCV (RRCV) approach*, we construct in the training phase regression-based estimates $\tilde{q}_j$ of the functions q_j backwards in time via regressions based on the first equality in (4.21). Given the approximations $\tilde{q}_j(\cdot, \mathcal{D}^{tr}_{N_r})$ of the functions $q_j(\cdot)$, we construct in the testing phase the approximations of the values $\tilde{a}_{j,o,U_1,U_2}(X^{(n)}_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r})$ on the testing paths via (4.22) with $q_j(\cdot)$ replaced by $\tilde{q}_j(\cdot, \mathcal{D}^{tr}_{N_r})$. Then, again, the values of the control variate on the testing paths are computed via (4.32), and the Monte Carlo estimator for $\mathbb{E}f(X_T)$ is computed as in (4.26).

4.4.1 Complexity analysis of the RCV approach

The overall cost of the algorithm (training and testing phase) is of order

$$\mathcal{C} \asymp JS^d \max\{N_r, N\}, \tag{4.34}$$

provided that we only track the parameters J, N_r, N, S that tend to infinity when $\epsilon \searrow 0$. Further, we have the following constraints

$$\max\left\{\frac{1}{J^4}, \frac{JS^d}{N_r N}, \frac{J}{N}\left(\frac{R}{S}\right)^{2(p+1)}, \frac{1}{R^\nu N}\right\} \lesssim \varepsilon^2, \tag{4.35}$$

provided that we, in addition to J, N_r, N, S, track the parameter R, which also tends to infinity when $\epsilon \searrow 0$. Note that the first term in (4.35) comes from the squared bias of the estimator and the remaining three ones come from the variance of the estimator (see (4.33) and (4.26)).

Theorem 4.17. *We obtain the following solution*

$$J \asymp \varepsilon^{-\frac{1}{2}}, \quad S \asymp \varepsilon^{-\frac{5\nu+6(p+1)}{2d\nu+4(p+1)(2\nu+d)}}, \quad R \asymp \varepsilon^{-\frac{6(p+1)-d}{2d\nu+4(p+1)(2\nu+d)}}, \quad N_r \asymp N \asymp \varepsilon^{-\frac{5d\nu+2(p+1)(5\nu+4d)}{2d\nu+4(p+1)(2\nu+d)}}, \tag{4.36}$$

provided that $p > \frac{d-2}{2}$ *and* $\nu > \frac{2d(p+1)}{2(p+1)-d}$.[11] *As a result the complexity order is given by*

$$\mathcal{C}_{RCV} \asymp JS^d N_r \asymp JS^d N \asymp \varepsilon^{-\frac{11d\nu+2(p+1)(7\nu+8d)}{2d\nu+4(p+1)(2\nu+d)}}. \tag{4.37}$$

4.4.2 Complexity of the RRCV approach

In the training phase, the cost of approximating all functions q_j is of order $N_r JS^d$. In the testing phase, the coefficients $\tilde{a}_{j,o,U_1,U_2}$ are computed via direct summation in (4.22) (with q_j replaced by their approximations $\tilde{q}_j$) at a cost of order NJS^d, and, finally, the control variate is computed via (4.32) on all testing paths at a cost of order NJ. Therefore, the overall cost is of order $JS^d \max\{N_r, N\}$, which is the same as for the RCV approach. (In the latter formula we ignore the cost constituents of smaller orders.)

[11] Compare with footnote 3.20 on page 32.

We now establish the constraints that are pertinent to the RRCV approach. The regressions are now performed for the functions q_j. Pertinent assumptions are in the spirit of (A1)–(A4) with different bounds in (A1) and (A2): the conditional variance in such regressions over one time step is typically of order Δ, that is

$$\sup_{x\in\mathbb{R}^d} \mathrm{Var}\left[q_j(X_{\Delta,j\Delta}) \,|\, X_{\Delta,(j-1)\Delta} = x\right] = O(\Delta)$$

for $j \in \{1,\dots,J\}$. Hence we require the bound $\Sigma\Delta$ in the analogue of (A1); while in the analogue of (A2) and for the truncated estimate we require only the constant bound A, that is

$$\sup_{x\in\mathbb{R}^d} |q_{j-1}(x)| = O(1), \quad j \in \{1,\dots,J\}\,.$$

As for the regression error, instead of (4.30) we get

$$\begin{aligned}\mathbb{E}\|\tilde{q}_j - q_j\|^2_{L^2(\mathbb{P}_{\Delta,j})} \le \tilde{c}\left(\Sigma\Delta + A^2(\log N_r + 1)\right)\frac{\binom{p+d}{d}S^d}{N_r} \\ + \frac{8\,C_h^2}{(p+1)!^2 d^{2-2/h}}\left(\frac{Rd}{S}\right)^{2p+2} + 8A^2B_\nu R^{-\nu},\end{aligned} \tag{4.38}$$

where $\tilde{c}$ is a universal constant.

The following theorem shows a connection between the above assumptions (A1) and (A2) and assumptions on the functions f, μ, σ for the second order weak scheme.

Theorem 4.18. *(i) Suppose that the function f is bounded. Then we obtain the boundedness of* $\mathrm{Var}[\zeta_{J,j,o,U_1,U_2}|X_{\Delta,(j-1)\Delta} = x]$ *and $q_{j-1}(x)$ for all $x \in \mathbb{R}^d$ and $j \in \{1,\dots,J\}$. In particular, assumption (A1) for the RCV approach and assumption (A2) for the RRCV approach are satisfied.*

(ii) Suppose that

a) f is continuously differentiable with bounded partial derivatives,

b) all functions μ_k, σ_{ki}, $k \in \{1,\dots,d\}$, $i \in \{1,\dots,m\}$, are bounded and three times continuously differentiable with bounded partial derivatives up to order 3.

Then we get that $a_{j,o,U_1,U_2}(x)$ is of order $\sqrt{\Delta}$ and $\mathrm{Var}\left[q_j(X_{\Delta,j\Delta}) \,|\, X_{\Delta,(j-1)\Delta} = x\right]$ *is of order Δ. In particular, assumption (A2) for the RCV approach and assumption (A1) for the RRCV approach are satisfied.*

Remark 4.19. (i) For the weak Euler scheme with Φ_Δ given by (3.5) we can derive similar results as in Theorem 4.18. More precisely, it is sufficient to have the following assumptions (cf. Theorem 3.17):

c) f is bounded and continuously differentiable with bounded partial derivatives,

d) all functions σ_{ki}, $k \in \{1,\dots,d\}$, $i \in \{1,\dots,m\}$, are bounded and continuously differentiable with bounded partial derivatives,

e) all functions μ_k, $k \in \{1,\dots,d\}$, are continuously differentiable with bounded partial derivatives.

That is, we require less smoothness on σ_{ki} and μ_k. In addition, we do not need to have bounded functions μ_k.

(ii) As a generalisation of Theorem 4.18, it is natural to expect that assumption (A3) (see page 63) is satisfied with a sufficiently large constant $C_h > 0$ if all the functions μ_k, σ_{ki} are bounded and all the functions f, μ_k, σ_{ki} are $(p+4)$ times continuously differentiable with bounded partial derivatives up to order $p+4$. In case of the weak Euler scheme, we expect to require boundedness only on σ_{ki} and smoothness as well as bounded partial derivatives only up to order $p+2$ (cf. Remark 3.18).

Regarding the RRCV approach, it turns out that

$$\mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \leq \mathbb{E}\|\tilde{q}_j - q_j\|^2_{L^2(\mathbb{P}_{\Delta,j})}, \tag{4.39}$$

for all j, o, U_1 and U_2. To prove (4.39), we use (4.31) and the similar formula involving q_j and $\tilde{q}_j$. As in (4.31), we consider a testing path $(X_{\Delta,j\Delta})_{j=0,\dots,J}$ which is independent of $\mathcal{D}^{tr}_{N_r}$. Since $\tilde{a}_{j,o,U_1,U_2}(\cdot, \mathcal{D}^{tr}_{N_r})$ is given by (4.22) with $q_j(\cdot)$ replaced by $\tilde{q}_j(\cdot, \mathcal{D}^{tr}_{N_r})$, it holds

$$\tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) = \mathbb{E}\left[\tilde{q}_j(X_{\Delta,j\Delta}, \mathcal{D}^{tr}_{N_r}) \prod_{r\in U_1} H_{o_r}(\xi^r_j) \prod_{(k,l)\in U_2} V^{kl}_j \,\middle|\, X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}\right].$$

Furthermore, we have

$$a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) = \mathbb{E}\left[q_j(X_{\Delta,j\Delta}) \prod_{r\in U_1} H_{o_r}(\xi^r_j) \prod_{(k,l)\in U_2} V^{kl}_j \,\middle|\, X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}\right].$$

The latter formula remains true also without conditioning on $\mathcal{D}^{tr}_{N_r}$, but this (seemingly superfluous) conditioning is helpful in the following calculation:

$$\begin{aligned}
&\left(\tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) - a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta})\right)^2 \\
\leq &\mathbb{E}\left[\left(\tilde{q}_j(X_{\Delta,j\Delta}, \mathcal{D}^{tr}_{N_r}) - q_j(X_{\Delta,j\Delta})\right)^2 \middle| X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}\right] \\
&\cdot \mathbb{E}\left[\left(\prod_{r\in U_1} H_{o_r}(\xi^r_j) \prod_{(k,l)\in U_2} V^{kl}_j\right)^2 \middle| X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}\right] \\
= &\mathbb{E}\left[\left(\tilde{q}_j(X_{\Delta,j\Delta}, \mathcal{D}^{tr}_{N_r}) - q_j(X_{\Delta,j\Delta})\right)^2 \middle| X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}\right].
\end{aligned} \tag{4.40}$$

We arrive at (4.39) by taking expectations in (4.40) and using (4.31) together with the similar formula for q_j and $\tilde{q}_j$. Finally, we get an upper bound for the variance in the RRCV approach by the same calculation as in (4.33) using (4.38) and (4.39) (instead of (4.30)), and the resulting upper bound is the same as in (4.33) except for that $A^2\Delta$ is replaced by A^2, while Σ is replaced by $\Sigma\Delta$. Thus, in the case of the RRCV approach, our constraints are

$$\max\left\{\frac{1}{J^4}, \frac{JS^d \log N_r}{N_r N}, \frac{J}{N}\left(\frac{R}{S}\right)^{2(p+1)}, \frac{J}{R^\nu N}\right\} \lesssim \varepsilon^2, \tag{4.41}$$

where we again only track the parameters J, N_r, N, S, R.

Theorem 4.20. *We obtain the following solution*

$$J \asymp \varepsilon^{-\frac{1}{2}}, \quad S \asymp \varepsilon^{-\frac{5\nu+10(p+1)}{2d\nu+4(p+1)(2\nu+d)}}, \quad R \asymp \varepsilon^{-\frac{5(p+1)}{2d\nu+4(p+1)(2\nu+d)}},$$
$$N_r \asymp N \asymp \varepsilon^{-\frac{5d\nu+10(p+1)(\nu+d)}{2d\nu+4(p+1)(2\nu+d)}} \sqrt{|\log(\varepsilon)|},$$

provided that $p > \frac{d-2}{2}$ *and* $\nu > \frac{2d(p+1)}{2(p+1)-d}$. *Thus, we have for the complexity*

$$\mathcal{C}_{RRCV} \asymp JS^d N_r \asymp JS^d N \asymp \varepsilon^{-\frac{11d\nu+2(p+1)(7\nu+11d)}{2d\nu+4(p+1)(2\nu+d)}} \sqrt{|\log(\varepsilon)|}. \tag{4.42}$$

4.4.3 Discussion

For the sake of comparison with the SMC and MLMC approaches, we recall at this point that their complexities are

$$\mathcal{C}_{SMC} \asymp \varepsilon^{-2.5} \quad \text{and} \quad \mathcal{C}_{MLMC} \asymp \varepsilon^{-2}$$

at best (we are considering the second order scheme). Complexity estimates (4.37) and (4.42) show that one can go beyond the complexity order ε^{-2}, provided that

$$p > \frac{7d-2}{2}, \quad \nu > \frac{8d(p+1)}{2(p+1)-7d}$$

in case of the RCV approach and

$$p > \frac{7d-2}{2}, \quad \nu > \frac{14d(p+1)}{2(p+1)-7d}$$

in case of the RRCV approach. Both in (4.37) and (4.42) the power of ε converges to -1.75 as $p, \nu \to \infty$ (the log-term is ignored). Notice that, while d and m are fixed, p and ν are free parameters in our algorithms, which can be chosen large, provided the smoothness in μ, σ and f allows that. Therefore, whenever it is possible to take arbitrarily large p and ν, the complexity of our scheme can be reduced to $\varepsilon^{-1.75-\delta}$ for arbitrarily small $\delta > 0$.

Let us remark that we obtain such a complexity for piecewise polynomial regression with the second order weak scheme. A natural question is to perform a similar complexity analysis also for the weak Euler scheme. We then get the complexities (cf. complexity (3.56) of the series approach in Chapter 3)

$$O\left(\varepsilon^{-\frac{7d\nu+2(p+1)(4d+5\nu)}{d\nu+2(p+1)(d+2\nu)}}\right)$$

for the RCV approach and

$$O\left(\varepsilon^{-\frac{7d\nu+2(p+1)(7d+5\nu)}{d\nu+2(p+1)(d+2\nu)}} \sqrt{|\log(\varepsilon)|}\right) \tag{4.43}$$

for the RRCV approach, which tend to order $\varepsilon^{-2.5}$ in the limit as $p, \nu \to \infty$ (provided that we ignore the log-term for the RRCV approach). That is, both the RCV and the RRCV approaches with the weak Euler scheme cannot outperform the MLMC approach as well as the SMC approach with the second order scheme (but they still outperform the SMC approach with the Euler or the weak Euler scheme because the complexity of the latter

is ε^{-3}). Still, both the RCV and the RRCV approaches might be useful also with the weak Euler scheme (and also the series approach with the Euler scheme), provided we choose basis functions other than those in piecewise polynomial regression (recall the last paragraph in Section 3.2).

Obviously, the complexity estimate (4.37) of the RCV approach gives us a better order compared to the one of the RRCV approach (4.42) (due to the factor J which arises in the last expression of the maximum term (4.41) but not in (4.35)). However, the larger is the parameter ν, the closer are both complexities to each other (provided that we ignore the log-term). As we mentioned in Sections 4.1.1 and 4.2.1, from the computational point of view it is preferable to consider the RRCV approach rather than the RCV one, since we perform regressions over only one time step in RRCV. In addition, in case of the RCV approach, there are destabilising factors $\prod_{r\in U_1} H_{o_r}(\xi_j^r)\prod_{(k,l)\in U_2} V_j^{kl}$ in the estimation of a_{j,o,U_1,U_2}, which are independent of $X_{\Delta,(j-1)\Delta}$ and have zero expectation and thus may lead to poor regression results. Regarding the RRCV approach, such destabilising factors are not present in the regression for q_j.

4.5 Numerical results

In this section, we consider weak schemes of second order and compare the numerical performance of the SMC, MLMC, RCV and RRCV approaches. As in Section 3.5, we implemented a global regression. In what follows it is convenient to have notations for the following constant $c_m = 3^m 2^{\frac{m(m-1)}{2}}$. Regarding the choice of basis functions, we use in both RCV and RRCV approaches the same polynomials $\psi(x) = \prod_{i=1}^d x_i^{l_i}$, where $l_1, \dots l_d \in \{0, 1, \dots, p\}$ and $\sum_{l=1}^d l_i \le p$. In addition to the polynomials, we consider the function f as a basis function. Hence, we have again overall $\tilde{c}_{p,d}$ basis functions in each regression, where $\tilde{c}_{p,d} := c_{p,d} + 1$ (recall that $c_{p,d} = \binom{p+d}{d}$). As for the MLMC approach, we use the same simulation results as in Section 3.5.

4.5.1 One-dimensional example

Here $d = m = 1$. We consider the example from Subsection 3.5.1. Again, we choose $p = 3$ (that is, 5 basis functions) and, for each $\varepsilon = 2^{-i}$, $i \in \{2, 3, 4, 5, 6\}$, we set the parameters J, N_r and N as follows (compare with the formulas in Section 4.4 for the "limiting" case $\nu \to \infty$ and ignore the log-terms for the RRCV approach):

$$J = \lceil \varepsilon^{-0.5} \rceil, \quad N_r = c_{N_r} \cdot \lceil \varepsilon^{-1.3235} \rceil, \qquad c_{N_r} = \begin{cases} 64 & \text{RRCV} \\ 32 & \text{RCV} \end{cases}, \quad N = 128 \cdot \lceil \varepsilon^{-1.3235} \rceil.$$

Regarding the SMC approach, the number of paths is set $N = 32 \cdot \varepsilon^{-2}$. The factors 32, 64 and 128 are here for stability purposes. We use different constants for the training and testing paths due the fact that, if we also track the constants $\tilde{c}_{p,d}$ and c_m, we will have the cost of order $J\tilde{c}_{p,d}(c_m - 1)\max\{N_r\tilde{c}_{p,d}, N\}$ for the RCV approach and $J\tilde{c}_{p,d}\max\{N_r\tilde{c}_{p,d}, Nc_m\}$ for the RRCV approach (cf. (4.34)). Since we get from Theorems 4.17 and 4.20 that both

components in the maximum term are of the same order in the optimal solution, we choose the constants such that $N_r \tilde{c}_{p,d} \approx N$ in case of the RCV approach and $N_r \tilde{c}_{p,d} \approx N c_m$ in case of the RRCV approach. Next we compute the numerical RMSE by means of 100 independent repetitions of the algorithm. As can be seen from the first plot in Figure 4.1, the estimated numerical complexity is about $\text{RMSE}^{-1.41}$ for the RRCV approach, $\text{RMSE}^{-1.66}$ for the RCV approach, $\text{RMSE}^{-1.99}$ for the MLMC approach and $\text{RMSE}^{-2.53}$ for the SMC approach, which we get again by regressing the log-time vs. log-RMSE. Thus, the complexity reduction works best with the RRCV approach.

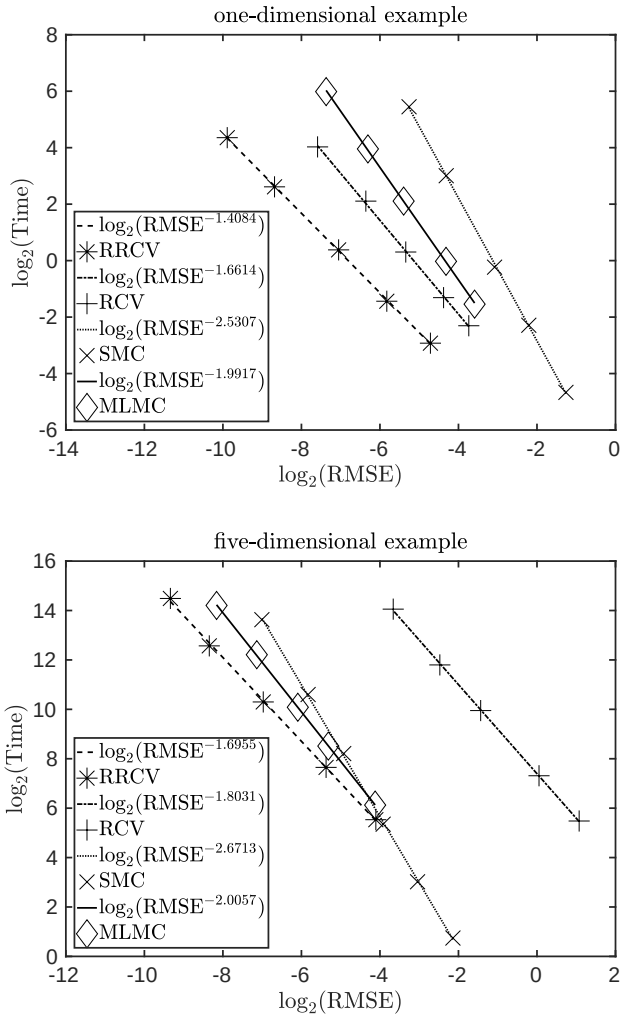

Figure 4.1. Numerical complexities of the RRCV, RCV, SMC and MLMC approaches in the one- and five-dimensional case.

4.5.2 Five-dimensional example

Here $d = m = 5$. We consider the example from Subsection 3.5.2. Note that we do not need to consider random variables V_j^{kl} in the second order weak scheme, since $\mathcal{L}^k \sigma_{rl}(x) = 0$ for

$k \neq l$ (see (4.17) and (3.59)). This gives us a smaller constant $\tilde{c}_m := 3^m = 243$ compared to $c_m = 248832$ and hence a smaller number of terms for the control variate (the factor $2^{\frac{m(m-1)}{2}} \equiv 1024$ is no longer present). We again choose $p = 3$ (this now results in 57 basis functions), consider the same values of ε as above (and, in addition, consider the value $\varepsilon = 2^{-7}$ for the SMC approach to obtain a similar computing time as for the RCV, RRCV and MLMC approaches). Moreover, we set

$$J = \lceil \varepsilon^{-0.5} \rceil, \quad N_r = c_{N_r} \cdot \lceil \varepsilon^{-1.5476} \rceil, \quad c_{N_r} = \begin{cases} 512 & \text{RRCV} \\ 32 & \text{RCV} \end{cases},$$
$$N = c_N \cdot \lceil \varepsilon^{-1.5476} \rceil, \quad c_N = \begin{cases} 128 & \text{RRCV} \\ 1024 & \text{RCV} \end{cases}$$

(similar to the previous example we consider the limiting case $\nu \to \infty$, ignore the log-terms for the RRCV approach and consider the relations $N_r \tilde{c}_{p,d} \approx N$ in case of the RCV approach and $N_r \tilde{c}_{p,d} \approx N \tilde{c}_m$ in case of the RRCV approach). The number of paths for the SMC approach is set $N = 512 \cdot \varepsilon^{-2}$. Since the estimated variance of $f(X_{\Delta,T})$ is much higher than in the previous example, we use a higher constant here for the SMC approach. This is due to the fact that we get $N \gtrsim \mathrm{Var}\left[f(X_{\Delta,T})\right] \varepsilon^{-2}$ from the condition $\mathrm{Var}\left[\frac{1}{N}\sum_{n=1}^{N} f(X_{\Delta,T}^{(n)})\right] = \frac{\mathrm{Var}\left[f(X_{\Delta,T})\right]}{N} \lesssim \varepsilon^2$. As in the one-dimensional case, we compute the numerical RMSE by means of 100 independent repetitions of the algorithm. Our empirical findings are illustrated in the second plot in Figure 4.1. We observe the numerical complexities $\mathrm{RMSE}^{-1.70}$ for the RRCV approach, $\mathrm{RMSE}^{-1.80}$ for the RCV approach, $\mathrm{RMSE}^{-2.01}$ for the MLMC approach and $\mathrm{RMSE}^{-2.67}$ for the SMC approach. Even though the complexity order of the RCV approach is better than those of the MLMC and SMC approaches, the RCV approach is practically outperformed by the other approaches (see Figure 4.1; the multiplicative constant influencing the computing time is obviously very big). However, the RRCV approach remains numerically the best one also in this five-dimensional example.

4.6 Proofs

Proof of Theorem 4.1

The proof is similar to the one of Theorem 4.11. □

Proof of Proposition 4.5

Let $\mathcal{G}_0$ be the trivial σ-field and $\mathcal{G}_j = \sigma(\xi_1, \ldots, \xi_j)$, $j = 1, \ldots, J$. It follows from (4.1) that the process $(X_{\Delta,j\Delta})_{j=0}^{J}$ is Markov with respect to $(\mathcal{G}_j)_{j=0}^{J}$. By the Markov property, we have

$$q_j(X_{\Delta,j\Delta}) \equiv \mathbb{E}[f(X_{\Delta,T})|X_{\Delta,j\Delta}] = \mathbb{E}[f(X_{\Delta,T})|\mathcal{G}_j],$$

hence, by the tower property of conditional expectation,

$$q_{j-1}(x) = \mathbb{E}[q_j(\Phi_\Delta(X_{\Delta,(j-1)\Delta}, \xi_j))|X_{\Delta,(j-1)\Delta} = x] = \frac{1}{2^m} \sum_{y=(y_1,\ldots,y_m)\in\{-1,1\}^m} q_j(\Phi_\Delta(x,y)),$$

where in the last equality we use independence between $X_{\Delta,(j-1)\Delta}$ and ξ_j. This proves (4.7). We now apply intermediate conditioning with respect to $\mathcal{G}_j$ in (4.3) and arrive at

$$a_{j,r,s}(x) = \mathbb{E}\left[q_j(\Phi_\Delta(X_{\Delta,(j-1)\Delta}, \xi_j)) \prod_{i=1}^{r} \xi_j^{s_i} \,\middle|\, X_{\Delta,(j-1)\Delta} = x \right],$$

which implies (4.8) due to the independence between $X_{\Delta,(j-1)\Delta}$ and ξ_j. □

Proofs of Theorems 4.7 and 4.14

The proofs are similar to the one of Theorem 3.4. □

Proof of Proposition 4.9

As in the proof of Proposition 4.5, $\mathcal{G}_0$ denotes the trivial σ-field and $\mathcal{G}_j = \sigma(\xi_1, \ldots, \xi_j)$, $j = 1, \ldots, J$. By conditioning with respect to $\mathcal{G}_j$ in (4.10), we have

$$\begin{aligned} a_{j,r}(x, (y_i)_{i=1}^{r-1}) &= \mathbb{E}\left[q_j(X_{\Delta,j\Delta})\xi_j^r \,\middle|\, X_{\Delta,(j-1)\Delta} = x, (\xi_j^i)_{i=1}^{r-1} = (y_i)_{i=1}^{r-1} \right] \\ &= \mathbb{E}\left[q_j\left(\Phi_\Delta\left(x, \left(y_1, \ldots, y_{r-1}, \xi_j^r, \xi_j^{r+1}, \ldots, \xi_j^m\right)^\top \right) \right) \xi_j^r \right], \end{aligned}$$

which implies (4.11). □

Proof of Theorem 4.11

Let $\mathcal{G}_0$ denote trivial σ-field, and, for $j = 1, \ldots, J$, define the σ-field $\mathcal{G}_j = \sigma(\xi_1, V_1, \ldots, \xi_j, V_j)$. Since each of the random variables ξ_j^r, $j = 1, \ldots, J$, $r \in \mathcal{I}_1$ can take 3 different values, each of the random variables V_j^{kl}, $(k,l) \in \mathcal{I}_2$, can take 2 different values and $|\mathcal{I}_1| = m$, $|\mathcal{I}_2| = \frac{m(m-1)}{2}$, $L^2(\mathcal{G}_J)$ is a $(3^m 2^{\frac{m(m-1)}{2}})^J$-dimensional vector space. A simple calculation reveals that, for any fixed $j = 1, \ldots, J$, the system $\{\prod_{r\in\mathcal{I}_1} H_{o_j^r}(\xi_j^r) \prod_{(k,l)\in\mathcal{I}_2} (V_j^{kl})^{s_j^{kl}} : o_j^r \in \{0,1,2\}, s_j^{kl} \in \{0,1\}\}$ is orthonormal in $L^2(\mathcal{G}_J)$. Due to independence of $\xi_1, V_1 \ldots, \xi_J, V_J$, the system

$$\left\{ \prod_{j=1}^{J} \prod_{r\in\mathcal{I}_1} H_{o_j^r}(\xi_j^r) \prod_{(k,l)\in\mathcal{I}_2} (V_j^{kl})^{s_j^{kl}} : o_j^r \in \{0,1,2\}, s_j^{kl} \in \{0,1\} \right\} \tag{4.44}$$

is orthonormal in $L^2(\mathcal{G}_J)$, and therefore, linear independent. The cardinality of system (4.44) is $(3^m 2^{\frac{m(m-1)}{2}})^J$, i.e. equals the dimension of $L^2(\mathcal{G}_J)$. Hence, linear independent system (4.44) is an orthonormal basis in $L^2(\mathcal{G}_J)$. We have $\mathbb{E}|f(X_{\Delta,T})|^2 < \infty$ because $X_{\Delta,T}$ takes finitely many values. Therefore, $f(X_{\Delta,T})$ belongs to $L^2(\mathcal{G}_J)$ and can be written

$$f(X_{\Delta,T}) = \sum_{\bar{o}\in\{0,1,2\}^{mJ}} \sum_{\bar{s}\in\{0,1\}^{\frac{m(m-1)}{2}J}} c_{\bar{o}\bar{s}} \prod_{j=1}^{J} \prod_{r\in\mathcal{I}_1} H_{o_j^r}(\xi_j^r) \prod_{(k,l)\in\mathcal{I}_2} (V_j^{kl})^{s_j^{kl}},$$

where $\bar{o} = (o_1^1, \dots, o_J^1, \dots, o_1^m, \dots, o_J^m)$, $\bar{s} = (s_1^{12}, \dots, s_J^{12}, s_1^{13}, \dots, s_J^{13}, \dots, s_1^{(m-1)m}, \dots, s_J^{(m-1)m})$. Note that $c_{\bar{o}\bar{s}} = \mathbb{E}[f(X_{\Delta,T}) \prod_{j=1}^J \prod_{r\in\mathcal{I}_1} H_{o_j^r}(\xi_j^r) \prod_{(k,l)\in\mathcal{I}_2} (V_j^{kl})^{s_j^{kl}}]$, in particular, $c_{\bar{0}\bar{0}} = \mathbb{E}f(X_{\Delta,T})$. Rearranging the terms in the expression for $f(X_{\Delta,T})$ we rewrite it as

$$f(X_{\Delta,T}) = \mathbb{E}f(X_{\Delta,T}) + \sum_{j=1}^J \sum_{(U_1,U_2)\in\mathcal{A}} \sum_{p\in\{1,2\}^{U_1}} A_{j,p,U_1,U_2} \prod_{r\in U_1} H_{p_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} \quad (4.45)$$

with $\mathcal{G}_{j-1}$-measurable random variables A_{j,p,U_1,U_2}. Let us now multiply both sides of the last equality by $\prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl}$, with some $j^0 \in \{1,\dots,J\}$, $(U_1^0, U_2^0) \in \mathcal{A}$, $p^0 \in \{1,2\}^{U_1^0}$ and calculate conditional expectations of the resulting expressions given $\mathcal{G}_{j_0-1}$. Notice that, with $j^h < j^0$ and $j^g > j^0$, we have

$$\mathbb{E}\left[\prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl} | \mathcal{G}_{j^0-1}\right] = \mathbb{E}\left[\prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl}\right] = 0,$$

$$\mathbb{E}\left[A_{j^h,p,U_1,U_2} \prod_{r\in U_1} H_{p_r}(\xi_{j^h}^r) \prod_{(k,l)\in U_2} V_{j^h}^{kl} \prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl} | \mathcal{G}_{j^0-1}\right]$$

$$= A_{j^h,p,U_1,U_2} \prod_{r\in U_1} H_{p_r}(\xi_{j^h}^r) \prod_{(k,l)\in U_2} V_{j^h}^{kl} \cdot \mathbb{E}\left[\prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl} | \mathcal{G}_{j^0-1}\right] = 0,$$

$$\mathbb{E}\left[A_{j^g,p,U_1,U_2} \prod_{r\in U_1} H_{p_r}(\xi_{j^g}^r) \prod_{(k,l)\in U_2} V_{j^g}^{kl} \prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl} | \mathcal{G}_{j^0-1}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[A_{j^g,p,U_1,U_2} \prod_{r\in U_1} H_{p_r}(\xi_{j^g}^r) \prod_{(k,l)\in U_2} V_{j^g}^{kl} \prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl} \mid \mathcal{G}_{j^g-1}\right] \mid \mathcal{G}_{j^0-1}\right] = 0,$$

$$\mathbb{E}\left[A_{j^0,p,U_1,U_2} \prod_{r\in U_1} H_{p_r}(\xi_{j^0}^r) \prod_{(k,l)\in U_2} V_{j^0}^{kl} \prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl} \mid \mathcal{G}_{j^0-1}\right]$$

$$= A_{j^0,p,U_1,U_2} \mathbb{E}\left[\prod_{r\in U_1} H_{p_r}(\xi_{j^0}^r) \prod_{(k,l)\in U_2} V_{j^0}^{kl} \prod_{r\in U_1^0} H_{p_r^0}(\xi_{j^0}^r) \prod_{(k,l)\in U_2^0} V_{j^0}^{kl}\right]$$

$$= A_{j^0,p,U_1,U_2} \delta_{p,p^0} \delta_{U_1,U_1^0} \delta_{U_2,U_2^0},$$

where $\delta_{\cdot,\cdot}$ is the Kronecker delta. Thus, the coefficients A_{j,p,U_1,U_2} in (4.45) are given by

$$A_{j,p,U_1,U_2} = \mathbb{E}[f(X_{\Delta,T}) \prod_{r\in U_1} H_{p_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} | \mathcal{G}_{j-1}]. \quad (4.46)$$

Let us now prove that

$$\mathbb{E}[f(X_{\Delta,T}) \prod_{r\in U_1} H_{p_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} | \mathcal{G}_{j-1}] = \mathbb{E}[f(X_{\Delta,T}) \prod_{r\in U_1} H_{p_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} | X_{\Delta,(j-1)\Delta}]. \quad (4.47)$$

In what follows we use the functions q_j from (4.6) and notice that, by the Markov property of $(X_{\Delta,j\Delta})_{j=0,\dots,J}$ with respect to $(\mathcal{G}_j)$, which is due to (4.14), we also have

$$q_j(X_{\Delta,j\Delta}) = \mathbb{E}[f(X_{\Delta,T}) | \mathcal{G}_j]. \quad (4.48)$$

Let us set

$$h(X_{\Delta,(j-1)\Delta}, \xi_j, V_j) = \prod_{r\in U_1} H_{p_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} q_j(X_{\Delta,j\Delta}) \tag{4.49}$$

and notice that, due to (4.14), this is indeed a function of $X_{\Delta,(j-1)\Delta}$, ξ_j and V_j only. Further, let us set

$$g(x) = \mathbb{E}\left[h(x, \xi_j, V_j)\right]. \tag{4.50}$$

Using the tower property of conditional expectations together with (4.48), (4.49) and (4.50), we get

$$\begin{aligned}\mathbb{E}[f(X_{\Delta,T}) \prod_{r\in U_1} H_{p_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} | \mathcal{G}_{j-1}] &= \mathbb{E}[\prod_{r\in U_1} H_{p_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} \mathbb{E}[f(X_{\Delta,T})|\mathcal{G}_j] \,|\mathcal{G}_{j-1}] \\ &= \mathbb{E}[h(X_{\Delta,(j-1)\Delta}, \xi_j, V_j)|\mathcal{G}_{j-1}] = g(X_{\Delta,(j-1)\Delta}),\end{aligned} \tag{4.51}$$

where the last equality is due to the facts that $X_{\Delta,(j-1)\Delta}$ is $\mathcal{G}_{j-1}$-measurable and the pair (ξ_j, V_j) is independent of $\mathcal{G}_{j-1}$. Moreover, applying (4.51), we also obtain

$$\mathbb{E}[f(X_{\Delta,T}) \prod_{r\in U_1} H_{p_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} | X_{\Delta,(j-1)\Delta}] = \mathbb{E}[g(X_{\Delta,(j-1)\Delta})|X_{\Delta,(j-1)\Delta}] = g(X_{\Delta,(j-1)\Delta}). \tag{4.52}$$

Comparing (4.51) and (4.52), we arrive at (4.47). Together with (4.46) and (4.45), this proves (4.18) and (4.19). □

Proof of Proposition 4.13

The proof is similar to the one of Proposition 4.5. □

Proof of Proposition 4.15

The proof is similar to the one of Proposition 4.9. □

Proofs of Theorems 4.17 and 4.20

The proofs are similar to the one of Theorem 3.20. □

Proof of Theorem 4.18

(i) The boundedness of

$$q_{j-1}(x) = \mathbb{E}\left[f(X_{\Delta,T}) \,|\, X_{\Delta,(j-1)\Delta} = x\right]$$

and

$$\mathrm{Var}\left[\zeta_{J,j,o,U_1,U_2} \,|\, X_{\Delta,(j-1)\Delta} = x\right] = \mathrm{Var}\left[f(X_{\Delta,T}) \prod_{r\in U_1} H_{o_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl} \,|\, X_{\Delta,(j-1)\Delta} = x\right]$$

follows straightforwardly, when f is bounded.

(ii) Let us first focus on assumption (A2) for the RCV approach and denote

$$p_m(y) := \frac{4^{\sum_{i=1}^m I(y_i=0)}}{2^{\frac{m(m-1)}{2}} 6^m},$$
$$g_{o,U_1,U_2}(y,z) := \prod_{r\in U_1} H_{o_r}(y_r) \prod_{(k,l)\in U_2} z_{kl}.$$

Then we have for a_{j,o,U_1,U_2} (see (4.22))

$$a_{j,o,U_1,U_2}(x) = \sum_{\substack{(y_1,\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ (z_{uv})_{1\le u<v\le m}\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y) g_{o,U_1,U_2}(y,z) q_j(\Phi_\Delta(x,y,z)), \tag{4.53}$$

Let us denote

$$\tilde{\Phi}_\Delta(x,y,z) := \Phi_\Delta(x,y,z) - \mu_\Delta(x),$$
$$\mu_\Delta(x) := x + \mu(x)\Delta + \frac{1}{2}\mathcal{L}^0\mu(x)\,\Delta^2,$$

where $\mathcal{L}^0$ is defined in (4.16). Consider the Taylor expansion of the function $q_j(\Phi_\Delta(x,y,z))$ around $\mu_\Delta(x)$, that is

$$q_j(\Phi_\Delta(x,y,z)) = q_j(\mu_\Delta(x)) + \sum_{k=1}^d \tilde{\Phi}^k_\Delta(x,y,z) \int_0^1 \frac{\partial}{\partial x_k} q_j(\mu_\Delta(x) + t\tilde{\Phi}_\Delta(x,y,z))\,dt. \tag{4.54}$$

Inserting (4.54) into (4.53) gives us

$$a_{j,o,U_1,U_2}(x) \tag{4.55}$$
$$= \sum_{\substack{(y_1,\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ (z_{uv})_{1\le u<v\le m}\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y) g_{o,U_1,U_2}(y,z) \sum_{k=1}^d \tilde{\Phi}^k_\Delta(x,y,z) \int_0^1 \frac{\partial}{\partial x_k} q_j(\mu_\Delta(x) + t\tilde{\Phi}_\Delta(x,y,z))\,dt,$$

due to

$$\sum_{\substack{(y_1,\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ (z_{uv})_{1\le u<v\le m}\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y) g_{o,U_1,U_2}(y,z) = \mathbb{E}\left[\prod_{r\in U_1} H_{o_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl}\right] = 0.$$

Obviously, $\tilde{\Phi}^k_\Delta(x,y,z)$ is of order $\sqrt{\Delta}$ under the assumptions that (cf. (4.17))

f) all functions σ_{ki}, $k\in\{1,\dots,d\}$, $i\in\{1,\dots,m\}$, are bounded and twice continuously differentiable with bounded partial derivatives up to order 2,

g) all functions μ_k, $k\in\{1,\dots,d\}$, are bounded and continuously differentiable with bounded partial derivatives.

Note that the assumptions a) and b) in Theorem 4.18 contain the above assumptions f) and g). Next we apply Theorem 2.5 and get that q_j is continuously differentiable with bounded partial derivatives for all $j\in\{1,\dots,J\}$, when all functions Φ^k_Δ, $k\in\{1,\dots,d\}$, are

continuously differentiable with bounded partial derivatives. It turns out that this conditions on Φ_Δ^k are satisfied under assumptions a) and b) (cf. (4.17)). This gives us that a_{j,o,U_1,U_2} is of order $\sqrt{\Delta}$ for all j, o, U_1, U_2.

Let us proceed with assumption (A1) for the RRCV approach. We have (cf. (4.21))

$$\begin{aligned}
&\operatorname{Var}\left[q_j(X_{\Delta,j\Delta}) \mid X_{\Delta,(j-1)\Delta} = x\right] \\
= \;& \mathbb{E}\left[q_j^2(X_{\Delta,j\Delta}) \mid X_{\Delta,(j-1)\Delta} = x\right] - \left(\mathbb{E}\left[q_j(X_{\Delta,j\Delta}) \mid X_{\Delta,(j-1)\Delta} = x\right]\right)^2 \\
= \;& \sum_{\substack{(y_1,\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ (z_{uv})_{1\le u<v\le m}\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y) q_j^2(\Phi_\Delta(x,y,z)) \\
& - \left(\sum_{\substack{(y_1,\dots,y_m)\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ (z_{uv})_{1\le u<v\le m}\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y) q_j(\Phi_\Delta(x,y,z)) \right)^2 \\
= \;& \sum_{\substack{y\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y)(1-p_m(y)) q_j^2(\Phi_\Delta(x,y,z)) \\
& - \sum_{\substack{y,\tilde{y}\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z,\tilde{z}\in\{-1,1\}^{\frac{m(m-1)}{2}} \\ y\neq\tilde{y}, z\neq\tilde{z}}} p_m(y) p_m(\tilde{y}) q_j(\Phi_\Delta(x,y,z)) q_j(\Phi_\Delta(x,\tilde{y},\tilde{z})).
\end{aligned} \tag{4.56}$$

In (4.54) we have derived that $q_j(\Phi_\Delta(x,y,z))$ has the form

$$q_j(\Phi_\Delta(x,y,z)) = \mu_\Delta(x) + \sqrt{\Delta} h_\Delta(x,y,z),$$

where $h_\Delta(x,y,z)$ is $O(1)$ for all $y \in \{-\sqrt{3},0,\sqrt{3}\}^m, z \in \{-1,1\}^{\frac{m(m-1)}{2}}$ under the assumptions a) and b). Hence, we get

$$\begin{aligned}
&\operatorname{Var}\left[q_j(X_{\Delta,j\Delta}) \mid X_{\Delta,(j-1)\Delta} = x\right] \\
= \;& \sum_{\substack{y\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y)(1-p_m(y))(\mu_\Delta(x) + \sqrt{\Delta} h_\Delta(x,y,z))^2 \\
& - \sum_{\substack{y,\tilde{y}\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z,\tilde{z}\in\{-1,1\}^{\frac{m(m-1)}{2}} \\ y\neq\tilde{y}, z\neq\tilde{z}}} p_m(y) p_m(\tilde{y}) (\mu_\Delta(x) + \sqrt{\Delta} h_\Delta(x,y,z))(\mu_\Delta(x) + \sqrt{\Delta} h_\Delta(x,\tilde{y},\tilde{z})),
\end{aligned}$$

such that it is sufficient to show

$$\begin{aligned}
&\sum_{\substack{y\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z\in\{-1,1\}^{\frac{m(m-1)}{2}}}} 2 p_m(y)(1-p_m(y)) h_\Delta(x,y,z) \\
=\;& \sum_{\substack{y,\tilde{y}\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z,\tilde{z}\in\{-1,1\}^{\frac{m(m-1)}{2}} \\ y\neq\tilde{y}, z\neq\tilde{z}}} p_m(y) p_m(\tilde{y}) (h_\Delta(x,y,z) + h_\Delta(x,\tilde{y},\tilde{z}))
\end{aligned}$$

to obtain $\mathrm{Var}\left[q_j(X_{\Delta,j\Delta}) \,|\, X_{\Delta,(j-1)\Delta} = x\right] = O(\Delta)$. Notice that it clearly holds

$$\sum_{\substack{y\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y)(1-p_m(y)) - \sum_{\substack{y,\tilde{y}\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z,\tilde{z}\in\{-1,1\}^{\frac{m(m-1)}{2}} \\ y\neq\tilde{y},z\neq\tilde{z}}} p_m(y)p_m(\tilde{y}) = 0.$$

(For instance, replace $q_j(x)$ in (4.56) by 1 which gives us zero variance.) We have

$$\begin{aligned}
&\sum_{\substack{y,\tilde{y}\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z,\tilde{z}\in\{-1,1\}^{\frac{m(m-1)}{2}} \\ y\neq\tilde{y},z\neq\tilde{z}}} p_m(y)p_m(\tilde{y})h_\Delta(x,y,z) \\
&= \sum_{\substack{y\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y)h_\Delta(x,y,z) \sum_{\substack{\tilde{y}\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ \tilde{z}\in\{-1,1\}^{\frac{m(m-1)}{2}} \\ \tilde{y}\neq y,\tilde{z}\neq z}} p_m(\tilde{y}) \\
&= \sum_{\substack{y\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y)h_\Delta(x,y,z)\left(1-\mathbb{P}\left(\xi_j = y, (V_j^{il})_{1\le i<l\le m} = (z_{il})_{1\le i<l\le m}\right)\right) \\
&= \sum_{\substack{y\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y)h_\Delta(x,y,z)(1-p_m(y)).
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
\sum_{\substack{y,\tilde{y}\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z,\tilde{z}\in\{-1,1\}^{\frac{m(m-1)}{2}} \\ y\neq\tilde{y},z\neq\tilde{z}}} p_m(y)p_m(\tilde{y})h_\Delta(x,\tilde{y},\tilde{z}) &= \sum_{\substack{\tilde{y}\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ \tilde{z}\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(\tilde{y})h_\Delta(x,\tilde{y},\tilde{z})(1-p_m(\tilde{y})) \\
&= \sum_{\substack{y\in\{-\sqrt{3},0,\sqrt{3}\}^m \\ z\in\{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y)h_\Delta(x,y,z)(1-p_m(y)),
\end{aligned}$$

which completes the proof. □

Chapter 5

Stratified regression for weak approximation schemes

This chapter is mainly based on the paper [10].

Below, we further enhance the performance of the RRCV algorithm by combining it with *stratification* (see e.g. [21]). The idea of the resulting *stratified RCV (SRCV)* algorithm is based on partitioning of the state space into a collection of sets $\mathcal{A}_1, \dots, \mathcal{A}_K$ and then performing conditional regressions separately on each set. It turns out that by choosing $\mathcal{A}_1, \dots, \mathcal{A}_K$ to be the level sets of the discrete-valued random variables used in the weak approximation scheme, we can achieve a further variance reduction effect as compared to the original approach in Chapter 4.

5.1 SRCV approach and its differences with RCV and RRCV ones

In Subsection 5.1.1 we consider weak schemes of order 1. In this setting we introduce the SRCV algorithm and explain how it compares to the RRCV one. In Subsection 5.1.2 we briefly discuss the case of weak schemes of order 2.

5.1.1 SRCV algorithm for first order schemes

First of all, we derive an equivalent reformulation of Theorem 4.1.

Theorem 5.1. *It holds*

$$f(X_{\Delta,T}) = \mathbb{E}f(X_{\Delta,T}) + \sum_{j=1}^{J} \sum_{k \in \{0,1\}^m \setminus \{0_m\}} a_{j,k}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{m} (\xi_j^i)^{k_i}, \tag{5.1}$$

where $k = (k_1, \dots, k_m)$. *Moreover, the coefficients* $a_{j,k} \colon \mathbb{R}^d \to \mathbb{R}$ *can be computed by the*

formula

$$a_{j,k}(x) = \mathbb{E}\left[f(X_{\Delta,T}) \prod_{i=1}^{m} (\xi_j^i)^{k_i} \,\middle|\, X_{\Delta,(j-1)\Delta} = x \right] \tag{5.2}$$

for all j and k as in (5.1).

Hence, the optimal control $M^{(1)}_{\Delta,T}$, introduced in Chapter 4, can also be representated through the functions $a_{j,k}$, that is

$$M^{(1)}_{\Delta,T} = \sum_{j=1}^{J} \sum_{k \in \{0,1\}^m \setminus \{0_m\}} a_{j,k}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{m} (\xi_j^i)^{k_i}. \tag{5.3}$$

The next result is similar to Proposition 4.5.

Proposition 5.2. *We have $q_J \equiv f$ and, for each $j \in \{2, \dots, J\}$,*

$$q_{j-1}(x) = \mathbb{E}\big[q_j(X_{\Delta,j\Delta}) | X_{\Delta,(j-1)\Delta} = x\big] = \frac{1}{2^m} \sum_{y \in \{-1,1\}^m} q_j(\Phi_\Delta(x,y)). \tag{5.4}$$

Moreover, for all $j \in \{1, \dots, J\}$ and $k = (k_i) \in \{0,1\}^m \setminus \{0_m\}$, the functions $a_{j,k}(x)$ in (5.2) *can be expressed in terms of the functions $q_j(x)$ as follows:*

$$a_{j,k}(x) = \frac{1}{2^m} \sum_{y=(y_1,\dots,y_m) \in \{-1,1\}^m} \left[\prod_{i=1}^{m} y_i^{k_i} \right] q_j(\Phi_\Delta(x,y)). \tag{5.5}$$

The first equality in (5.4) shows that we can recursively approximate the functions $q_j(x)$ via regressions over one time step only (the regression for approximating $a_{j,k}$ in case of the RCV approach is performed over $J - j + 1$ time steps). This gives the RRCV algorithm in Chapter 4: first compute regression-based approximations $\tilde{q}_j(x)$ of the functions $q_j(x)$ (via regressions over one time step based on the first equality in (5.4)), then obtain approximations $\tilde{a}_{j,k}(x)$ of the functions $a_{j,k}(x)$ via (5.5) with q_j being replaced by $\tilde{q}_j$, and, finally, construct the control variate $\tilde{M}^{(1)}_{\Delta,T}$ using (5.3) with $a_{j,k}(x)$ being replaced by $\tilde{a}_{j,k}(x)$.

To introduce the SRCV algorithm, we first define functions $h_{j,y}$, for all $j \in \{1, \dots, J\}$ and $y \in \{-1,1\}^m$, by the formula

$$h_{j,y}(x) := q_j(\Phi_\Delta(x,y)) = \mathbb{E}[q_j(X_{\Delta,j\Delta}) | X_{\Delta,(j-1)\Delta} = x, \xi_j = y] \tag{5.6}$$

(the second equality is straightforward) and observe that the knowledge of these functions for some j and all y provides us with the functions q_{j-1} and $a_{j,k}$, $k \in \{0,1\}^m \setminus \{0_m\}$, via the second equality in (5.4) and via (5.5). Inspired by this observation together with the second equality in (5.6), we arrive at the idea of the stratified regression: approximate each function $h_{j,y}(x)$ via its projection on a given set of basis functions $\psi_1(x), \dots, \psi_Q(x)$. In detail, the SRCV algorithm consists a training and a testing phase.

Training phase of the SRCV algorithm: First, simulate a sufficient number N_r of (independent) training paths of the discretised diffusion. Let us denote the set of these N_r paths by (cf. (3.47))

$$\mathcal{D}^{tr}_{N_r} = \left\{ (X^{tr,(n)}_{\Delta,j\Delta})_{j=0,\dots,J} : n = 1, \dots, N_r \right\}. \tag{5.7}$$

Next, proceed as follows.

Step 1. Set $j = J$, $\tilde{q}_j = f$. Compute the values $\tilde{q}_j(X^{tr,(n)}_{\Delta,j\Delta})$ on all training paths $(n = 1, \dots, N_r)$.

Step 2. For all $y \in \{-1, 1\}^m$, construct regression-based approximations $\tilde{h}_{j,y}$ of the functions $h_{j,y}$ (via regressions over one time step based on the second equality in (5.6) with q_j being replaced by $\tilde{q}_j$). In fact, only training paths with $\xi_j = y$ are used to construct $\tilde{h}_{j,y}$.

Step 3. Using the approximations $\tilde{h}_{j,y}$ for all $y \in \{-1, 1\}^m$, via (5.5) compute the coefficients $\beta_1, \dots, \beta_Q$ in the representations $\sum_{i=1}^Q \beta_i \psi_i$ for the approximations $\tilde{a}_{j,k}$, $k \in \{0, 1\}^m \backslash \{0_m\}$. Note that the cost of computing each $\tilde{a}_{j,k}(x)$ at any point x will be of order Q. Furthermore, again using $\tilde{h}_{j,y}$ for all $y \in \{-1, 1\}^m$, compute the values $\tilde{q}_{j-1}(X^{tr,(n)}_{\Delta,(j-1)\Delta})$ on all training paths $(n = 1, \dots, N_r)$ via the second equality in (5.4).

Step 4. If $j > 1$, set $j = j - 1$ and go to step 2.

Thus, after the training phase is completed, we have the approximations $\tilde{a}_{j,k}(x)$ of $a_{j,k}(x)$ for all $j \in \{1, \dots, J\}$ and $k \in \{0, 1\}^m \setminus \{0_m\}$. Let us emphasise that, in fact,

$$\tilde{a}_{j,k}(x) = \tilde{a}_{j,k}(x, \mathcal{D}^{tr}_{N_r}), \tag{5.8}$$

that is, our approximations are random and depend on the simulated training paths.

Testing phase of the SRCV algorithm: Simulate N testing paths $(X^{(n)}_{\Delta,j\Delta})_{j=0,\dots,J}$, $n = 1, \dots, N$, that are independent from each other and from the training paths and construct the Monte Carlo estimate

$$\frac{1}{N} \sum_{n=1}^{N} \left[f\Big(X^{(n)}_{\Delta,T}\Big) - \tilde{M}^{(1),(n)}_{\Delta,T} \right], \tag{5.9}$$

where $\tilde{M}^{(1),(n)}_{\Delta,T}$ is given by (cf. (5.3)).

$$\tilde{M}^{(1),(n)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{k \in \{0,1\}^m \backslash \{0_m\}} \tilde{a}_{j,k}(X^{(n)}_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r}) \prod_{i=1}^{m} (\xi^{i,(n)}_j)^{k_i}. \tag{5.10}$$

Remark 5.3. Let us briefly discuss the main differences between the RRCV and SRCV algorithms. In the training phase of the RRCV algorithm the functions q_j, $j \in \{1, \dots, J\}$, are approximated recursively via regressions using the first equality in (5.4) (the second equality in (5.4) is not used at all), and the approximations are linear combinations of Q basis functions $\psi_1, \dots, \psi_Q$. This allows to get the control variate in the testing phase via the formula like (5.10) with the coefficients $\tilde{a}_{j,k}$ constructed *on the testing paths* via (5.5) with approximated in the training phase functions q_j. On the contrary, in the training phase of the SRCV algorithm regressions are based on the second equality in (5.6), and we get approximations for all functions $h_{j,y}$ $(\equiv q_j(\Phi_\Delta(\cdot, y)))$, $j \in \{1, \dots, J\}$, $y \in \{-1, 1\}^m$, where the approximations $\tilde{h}_{j,y}$ are again linear combinations of Q basis functions $\psi_1, \dots, \psi_Q$ (notice that what we now need from (5.4) is the second equality but not the first one). Having the approximations $\tilde{h}_{j,y}$, we get the approximations of the functions $\tilde{a}_{j,k}$ via (5.5) as linear combinations of $\psi_1, \dots, \psi_Q$ already in the training phase, while the testing phase is completely described by (5.9)–(5.10). Let us compare the computational costs of the

RRCV and SRCV algorithms. For the sake of simplicity we restrict our attention to the case of "large" parameters[12] J, Q, N_r and N as well as at the "big" constant[13] $c_m := 2^m$ ignoring other constants such as e.g. d or m. As for the RRCV algorithm, J regressions with N_r training paths and Q basis functions result in the cost of order JQ^2N_r, while the cost of the testing phase is of order[14] JQc_mN, which results in the overall cost of order $JQ\max\{QN_r, c_mN\}$. Regarding the SRCV algorithm, we perform Jc_m regressions with Q basis functions in the training phase, but have in average $N_r\mathbb{P}(\xi_j = y)$ $(\equiv N_r/c_m)$, $y \in \{-1,1\}^m$, training paths in each regression, which again results in the cost of order JQ^2N_r, while in the testing phase we now have the cost of order $JQ(c_m-1)N$. This gives us the overall cost of order $JQ\max\{QN_r, (c_m-1)N\}$, which is the same order as for the RRCV algorithm. Finally, regarding the quality of the regressions in the RRCV and SRCV approaches, it is to expect that the regressions in the SRCV algorithm, which are based on the second equality in (5.6), achieve better approximations than the regressions in the RRCV algorithm, provided there are enough training paths and the basis functions are chosen properly, because we have

$$\mathrm{Var}[q_j(X_{\Delta,j\Delta})|X_{\Delta,(j-1)\Delta} = x, \xi_j = y] = \mathrm{Var}[q_j(\Phi_\Delta(x,y))] = 0. \tag{5.11}$$

The latter property implies the absence of the statistical error while approximating $h_{j,y}$. This is well illustrated by the plots in Figure 5.2 on page 91 (the plots are performed for the example of Subsection 5.3.1).

5.1.2 SRCV algorithm for second order schemes

Let us recall the index set

$$\mathcal{I}_2 = \big\{(k,l) \in \{1,\dots,m\}^2 : k < l\big\}$$

and use the notation

$$\mathcal{U} = \big\{(o,r) \in \{0,1,2\}^m \times \{0,1\}^{\mathcal{I}_2} : o_i \neq 0 \text{ for some } i \text{ or } r_{kl} \neq 0 \text{ for some } k,\ l\big\},$$

where o_i, $i = 1,\dots,m$ (resp. r_{kl}, $(k,l) \in \mathcal{I}_2$), denote the coordinates of o (resp. r). The following result is an equivalent reformulation of Theorem 4.11.

[12] We need to have $J \to \infty$, $Q \to \infty$, $N_r \to \infty$, $N \to \infty$ in order to make both the discretisation and the statistical error tend to zero (see Section 5.2 for more details).

[13] In contrast to J, Q, N_r and N, the value $c_m := 2^m$ is fixed, but can be relatively big (compared to other involved constants such as e.g. d or m). Notice that c_m is the number of scenarios that the random variables ξ_j can take, and it comes into play via formulas like (5.10) ($J(c_m-1)$ summands) or (5.5) (c_m summands).

[14] Naive implementation of the testing phase in the RRCV algorithm via (5.5) and (4.4) gives the cost order $JQc_m(c_m-1)N$. To get JQc_mN, one should implement (5.5) on the testing paths in two steps: first, for all $n \in \{1,\dots,N\}$, $j \in \{1,\dots,J\}$ and $y \in \{-1,1\}^m$, compute the values $\tilde{q}_j(\Phi_\Delta(X^{(n)}_{\Delta,(j-1)\Delta}, y))$ (the cost is NJc_mQ); then, using these values, for all $n \in \{1,\dots,N\}$, $j \in \{1,\dots,J\}$ and $k \in \{0,1\}^m \setminus \{0_m\}$, compute $\tilde{a}_{j,k}(X^{(n)}_{\Delta,(j-1)\Delta})$ via (5.5) (the cost is $NJ(c_m-1)c_m$). In this way, the maximal cost order is JQc_mN.

Theorem 5.4. *The following representation holds*

$$f(X_{\Delta,T}) = \mathbb{E}f(X_{\Delta,T}) + \sum_{j=1}^{J} \sum_{(o,r)\in\mathcal{U}} a_{j,o,r}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{m} H_{o_i}(\xi_j^i) \prod_{(k,l)\in\mathcal{I}} (V_j^{kl})^{r_{kl}}, \tag{5.12}$$

the coefficients $a_{j,o,r}\colon \mathbb{R}^d \to \mathbb{R}$ *are given by the formula*

$$a_{j,o,r}(x) = \mathbb{E}\left[f(X_{\Delta,T}) \prod_{i=1}^{m} H_{o_i}(\xi_j^i) \prod_{(k,l)\in\mathcal{I}_2} (V_j^{kl})^{r_{kl}} \,\middle|\, X_{\Delta,(j-1)\Delta} = x \right] \tag{5.13}$$

for all $j \in \{1,\dots,J\}$ *and* $(o,r) \in \mathcal{U}$.

Thus, with

$$M^{(2)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{(o,r)\in\mathcal{U}} a_{j,o,r}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{m} H_{o_i}(\xi_j^i) \prod_{(k,l)\in\mathcal{I}_2} (V_j^{kl})^{r_{kl}}, \tag{5.14}$$

we have $\mathbb{E}\left[M^{(2)}_{\Delta,T}\right] = 0$ and $\mathrm{Var}\left[f(X_{\Delta,T}) - M^{(2)}_{\Delta,T}\right] = 0$ in the case of second order schemes. Next, we set (cf. proof of Theorem 4.18)

$$p_m(y) = \frac{4^{\sum_{i=1}^{m} I(y_i=0)}}{6^m 2^{\frac{m(m-1)}{2}}}. \tag{5.15}$$

Notice that $p_m(y) = \mathbb{P}(\xi_j = y, V_j = z)$ for all $z \in \{-1,1\}^{\mathcal{I}_2}$. The next result is similar to Proposition 4.13.

Proposition 5.5. *We have* $q_J \equiv f$ *and, for each* $j \in \{2,\dots,J\}$,

$$q_{j-1}(x) = \mathbb{E}\big[q_j(X_{\Delta,j\Delta}) | X_{\Delta,(j-1)\Delta} = x\big] = \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} \sum_{z\in\{-1,1\}^{\mathcal{I}_2}} p_m(y)\, q_j(\Phi_\Delta(x,y,z)). \tag{5.16}$$

Moreover, for all $j \in \{1,\dots,J\}$ *and* $(o,r) \in \mathcal{U}$, *the functions* $a_{j,o,r}(x)$ *of* (5.13) *can be expressed in terms of the functions* $q_j(x)$ *as*

$$a_{j,o,r}(x) = \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} \sum_{z\in\{-1,1\}^{\mathcal{I}_2}} \left[p_m(y) \prod_{i=1}^{m} H_{o_i}(y_i) \prod_{(k,l)\in\mathcal{I}_2} z_{kl}^{r_{kl}} \right] q_j(\Phi_\Delta(x,y,z)), \tag{5.17}$$

where o_i *and* y_i, $i = 1,\dots,m$, *denote the coordinates of* o *and* y, *while* r_{kl} *and* z_{kl}, $(k,l) \in \mathcal{I}_2$, *are the coordinates of* r *and* z.

Similar to (5.6), we define functions $h_{j,y,z}$, for all $j \in \{1,\dots,J\}$, $y \in \{-\sqrt{3},0,\sqrt{3}\}^m$ and $z \in \{-1,1\}^{\mathcal{I}_2}$, by the formula

$$h_{j,y,z}(x) := q_j(\Phi_\Delta(x,y,z)) = \mathbb{E}[q_j(X_{\Delta,j\Delta}) | X_{\Delta,(j-1)\Delta} = x, \xi_j = y, V_j = z]. \tag{5.18}$$

The SRCV algorithm for second order schemes now relies on Proposition 5.5 and on (5.18) in the same way as the one for first order schemes relies on Proposition 5.2 and on (5.6). The whole discussion in the end of Subsection 5.1.1, and, in particular, the formula

$$JQ \max\{QN_r, (c_m - 1)N\}$$

for the overall cost order of the SRCV algorithm, apply also in the case of second order schemes, where we only need to change the value of c_m: here $c_m := 3^m 2^{m(m-1)/2}$.

5.2 Complexity analysis

In this section we extend the complexity analysis presented in Chapter 4 to the case of the stratified regression algorithm. Below we only sketch the main results for the second order schemes. We make the following assumptions:

(A1) All functions $h_{j,y,z}(x)$ of (5.18) are uniformly bounded, i.e. there is a constant $A > 0$ such that $\sup_{x\in\mathbb{R}^d} |h_{j,y,z}(x)| \leq A < \infty$.

(A2) The functions $h_{j,y,z}(x)$ can be well approximated by the basis functions $\psi_1, \ldots, \psi_Q$, in the sense that there are constants $\kappa > 0$ and $D_\kappa > 0$ such that

$$\inf_{g\in\Psi_Q} \int_{\mathbb{R}^d} \left(h_{j,y,z}(x) - g(x)\right)^2 \mathbb{P}_{\Delta,j-1}(dx) \leq \frac{D_\kappa}{Q^\kappa},$$

where $\mathbb{P}_{\Delta,j-1}$ denotes the distribution of $X_{\Delta,(j-1)\Delta}$ and $\Psi_Q := \operatorname{span}(\{\psi_1, \ldots, \psi_Q\})$ (cf. (2.6)).

Remark 5.6. A sufficient condition for assumption (A1) is boundedness of f. Moreover, in the case of *piecewise polynomial regression*, (A2) is satisfied with $\kappa = \frac{2\nu(p+1)}{d(\nu+2(p+1))}$ (see Remark 2.4).

Below we present an L^2-upper bound for the estimation error on step 2 of the training phase of the SRCV algorithm (see page 81). To this end, we need to describe more precisely, how exactly the regression-based approximations $\tilde{h}_{j,y,z}$ are constructed:

(A3) Let functions $\hat{h}_{j,y,z}(x)$ be obtained by regression (based on the second equality in (5.18)) onto the set of basis functions $\{\psi_1, \ldots, \psi_Q\}$, while the approximations $\tilde{h}_{j,y,z}(x)$ on step 2 of the training phase of the SRCV algorithm be the truncated estimates, which are defined as truncated estimates $\tilde{h}_{j,y,z}(x) = T_A\hat{h}_{j,y,z}(x)$ (A is the constant from (A1)).

Under (A1)–(A3), we have (cf. (2.4))

$$\mathbb{E}\|\tilde{h}_{j,y,z} - h_{j,y,z}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \leq \tilde{c}\, A^2(\log N_r + 1)\frac{Q}{N_r p_m(y)} + \frac{8\, D_\kappa}{Q^\kappa}, \tag{5.19}$$

where $\tilde{c}$ is a universal constant and $p_m(y)$ is given in (5.15). Note that it holds $\log p_m(y) < 0$, what is also used to obtain (5.19) from (2.4). As in the previous chapters, the expectation in the left-hand side of (5.19) means averaging over the randomness in $\mathcal{D}^{tr}_{N_r}$.

The next step is to provide an upper bound for the regression-based estimates of the coefficients $a_{j,o,r}$, which are constructed on step 3 of the training phase of the SRCV algorithm.

Lemma 5.7. *Under (A1)–(A3), we have*

$$\mathbb{E}\|\tilde{a}_{j,o,r} - a_{j,o,r}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \leq c_m\tilde{c}\, A^2(\log N_r + 1)\frac{Q}{N_r} + \frac{8\, D_\kappa}{Q^\kappa}C_{m,o}, \tag{5.20}$$

where $C_{m,o} := \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} c_m 2^{\frac{m(m-1)}{2}} \left[p_m(y)\prod_{i=1}^m H_{o_i}(y_i)\right]^2$.

Let $(X_{\Delta,j\Delta})_{j=0,\dots,J}$ be a testing path, which is independent of the training paths $\mathcal{D}_{N_r}^{tr}$. We now define

$$\tilde{M}_{\Delta,T}^{(2)} := \sum_{j=1}^{J} \sum_{(o,r)\in\mathcal{U}} \tilde{a}_{j,o,r}(X_{\Delta,(j-1)\Delta}, \mathcal{D}_{N_r}^{tr}) \prod_{i=1}^{m} H_{o_i}(\xi_j^i) \prod_{(k,l)\in\mathcal{I}_2} (V_j^{kl})^{r_{kl}} \tag{5.21}$$

(cf. (5.14)) and bound the variance $\mathrm{Var}[f(X_{\Delta,T}) - \tilde{M}_{\Delta,T}^{(2)}]$ from above.[15] With the help of (5.19) and Lemma 5.7 we now derive the main result of this section:

Theorem 5.8. *Under (A1)–(A3), it holds*

$$\mathrm{Var}[f(X_{\Delta,T}) - \tilde{M}_{\Delta,T}^{(2)}] \le J\left((c_m-1)\,c_m\tilde{c}\,A^2(\log N_r+1)\frac{Q}{N_r} + \frac{8\,D_\kappa}{Q^\kappa}\tilde{c}_m\right),$$

where $\tilde{c}_m = c_m - \left(\frac{3}{2}\right)^m$.

5.2.1 Complexity of the SRCV approach

Let us study the complexity of the SRCV approach. The overall cost is of order

$$JQ\max\{N_rQ, (c_m-1)N\}.$$

We have the following constraints

$$\max\left\{\frac{1}{J^4}, \frac{JQ\log(N_r)c_m(c_m-1)}{N_rN}, \frac{JD_\kappa\tilde{c}_m}{Q^\kappa N}\right\} \lesssim \varepsilon^2,$$

where the first term comes from the squared bias of the estimator and the remaining two ones come from the variance of the estimator (see Theorem 5.8 as well as footnote 15 on page 85).

Theorem 5.9. *Provided that*[16] $\kappa > 1$*, we obtain the following solution*

$$J \asymp \varepsilon^{-\frac{1}{2}}, \quad Q \asymp \left[\frac{\tilde{c}_m^2 D_\kappa^2}{c_m}\right]^{\frac{1}{2(\kappa+1)}} \varepsilon^{-\frac{5}{4(\kappa+1)}}, \quad N_r \asymp (c_m-1)\sqrt{c_m}\varepsilon^{-\frac{5}{4}}\sqrt{|\log(\varepsilon)|},$$
$$N \asymp \frac{N_rQ}{c_m-1} \asymp \left[c_m^\kappa \tilde{c}_m^2 D_\kappa^2\right]^{\frac{1}{2(\kappa+1)}} \varepsilon^{-\frac{5\kappa+10}{4(\kappa+1)}}\sqrt{|\log(\varepsilon)|}.$$

Thus, we have for the complexity

$$\mathcal{C} \asymp JN_rQ^2 \asymp JNQ(c_m-1) \asymp \left[(c_m-1)^{2(\kappa+1)}c_m^{\kappa-1}\tilde{c}_m^4 D_\kappa^4\right]^{\frac{1}{2(\kappa+1)}} \varepsilon^{-\frac{7\kappa+17}{4(\kappa+1)}}\sqrt{|\log(\varepsilon)|}. \tag{5.22}$$

Remark 5.10. (i) Complexity estimate (5.22) shows that one can go beyond the complexity order ε^{-2}, provided that $\kappa > 9$, and that we can, similar to the RCV and RRCV approaches, achieve the complexity order $\varepsilon^{-1.75-\delta}$, for arbitrarily small $\delta > 0$, provided κ is large enough.[17]

---

[15] Notice that the variance of the SRCV estimate $\frac{1}{N}\sum_{n=1}^{N}\left[f\left(X_{\Delta,T}^{(n)}\right) - \tilde{M}_{\Delta,T}^{(2),(n)}\right]$ with N testing paths is $\frac{1}{N}\mathrm{Var}[f(X_{\Delta,T}) - \tilde{M}_{\Delta,T}^{(2)}]$.

[16] Recall that in the case of piecewise polynomial regression we have $\kappa = \frac{2\nu(p+1)}{2d(p+1)+d\nu}$ (see Remark 5.6). Let us note that in the previous chapters it is required to choose the parameters p and ν according to $p > \frac{d-2}{2}$ and $\nu > \frac{2d(p+1)}{2(p+1)-d}$, which implies that $\kappa > 1$, for κ expressed via p and ν by the above formula.

[17] Here, we can also derive a connection to the piecewise polynomial regression, that is, when $p, \nu \to \infty$, then $\kappa \to \infty$, too.

(ii) Let us recall Remark 3.21, where we observed that the constant within the complexity in case of piecewise polynomial regression tends to infinity. Here the analogue constant in (5.22) is $\left[(c_m-1)^{2(\kappa+1)}c_m^{\kappa-1}\tilde{c}_m^4 D_\kappa^4\right]^{\frac{1}{2(\kappa+1)}}$. Suppose that D_κ grows exponentially in κ with base d, i.e., there exists some positive constant γ, such that

$$D_\kappa \asymp d^{\gamma\kappa} \quad \text{as } \kappa \to \infty. \tag{5.23}$$

(For the piecewise polynomial regression (5.23) is not satisfied.) Then we get

$$\lim_{\kappa\to\infty}\left[(c_m-1)^{2(\kappa+1)}c_m^{\kappa-1}\tilde{c}_m^4 D_\kappa^4\right]^{\frac{1}{2(\kappa+1)}} \asymp c_m^{3/2}d^{2\gamma},$$

that is, we have a finite constant in the limiting case. However, in terms of m this limit is of exponential growth, hence the problem of *exponentially growing constants in dimensionality (EGCD)* still arises. Note that this property is also present for the RCV and RRCV approaches, since the control variates contain (c_m-1) terms. As for the series and integral approaches (from Chapter 3) as well as the truncated RCV and truncated SRCV approaches, which will be explained in Subsection 7.2.1, the EGCD problem does not arise under the assumption (5.23), since the control variates are truncated to only m, respectively $O(m^2)$ terms.

5.3 Numerical results

In this section, we present several numerical examples showing the efficiency of the SRCV approach. It turns that even the weak Euler scheme (3.5) already shows the advantage of the new methodology over the standard Monte Carlo (SMC) as well as over the original RCV and RRCV approaches in terms of variance reduction effect. Regarding the choice of basis functions, we use for the RCV, RRCV and SRCV approaches polynomials of degree $\leq p$, that is, $\psi_l(x)=\prod_{i=1}^d x_i^{l_i}$, where $l=(l_1,\dots l_d)\in\{0,1,\dots,p\}^d$ and $\sum_{l=1}^d l_i \leq p$. In addition to the polynomials, we consider the function f as a basis function. We choose $J=100$, $N_r=10^5$, $N=10^7$, $p=1$ in all examples. Hence, we have overall $Q=\binom{p+d}{d}+1=d+2$ basis functions in each regression. Then we compute the estimated variances for the SMC, RCV, RRCV and SRCV approaches. More precisely, when speaking about "variance" below (e.g. in Tables 5.1, 5.2 and 5.3) we mean sample variance of one summand $f(X_{\Delta,T}^{(n)})-\tilde{M}_{\Delta,T}^{(1),(n)}$ (see (5.9)) in the case of RCV, RRCV and SRCV, while, in the case of SMC, the sample variance of $f(X_{\Delta,T}^{(n)})$ is meant. Thus, we analyse the variance reduction effect only, since the bias is the same for all these methods. To measure the numerical performance of a variance reduction method, we look at the ratio of variance vs. computational time, i.e., for the SRCV, we look at

$$\theta_{\mathrm{SRCV}} := \frac{\mathrm{Var}_{\mathrm{SRCV}}}{\mathrm{Var}_{\mathrm{SMC}}}\cdot\frac{\mathrm{Time}_{\mathrm{SRCV}}}{\mathrm{Time}_{\mathrm{SMC}}},$$

where $\mathrm{Var}_{\mathrm{SRCV}}$ and $\mathrm{Time}_{\mathrm{SRCV}}$ denote the variance and the overall computational time of the SRCV approach ($\mathrm{Var}_{\mathrm{SMC}}$ and $\mathrm{Time}_{\mathrm{SMC}}$ have the similar meaning). The smaller θ_{SRCV} is, the more profitable is the SRCV algorithm compared to the SMC one. We similarly

define θ_{RCV} and θ_{RRCV} (each of the regression-based algorithms is compared with the SMC approach).

5.3.1 Geometric Brownian motion (GBM) with high volatility

Here $d = m = 1$ ($Q = 3$). We consider the following SDE

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad X_0 = 1, \tag{5.24}$$

for $t \in [0, 1]$, where $r = -1$ and $\sigma = 4$. Furthermore, we consider the function $f(x) = x^2$. In the following, we plot the empirical cumulative distribution function (ECDF) of the "log-scaled sample", which is

$$\log(1 + f_i - f_{\min}) - \log(1 + \bar{f} - f_{\min})$$

for the SMC approach, and

$$\log(1 + u_i - u_{\min}) - \log(1 + \bar{u} - u_{\min})$$

for the RCV, and RRCV and SRCV approaches, where

$$f_n := f(X^{(n)}_{\Delta,T}), \quad u_n := f_n - \tilde{M}^{(1),(n)}_{\Delta,T}, \quad n \in \{1, \dots, N\},$$

$$f_{\min} := \min_{n=1,\dots,N} f_n, \quad u_{\min} := \min_{n=1,\dots,N} u_n, \quad \bar{f} := \frac{1}{N}\sum_{n=1}^{N} f_n, \quad \bar{u} := \frac{1}{N}\sum_{n=1}^{N} u_n.$$

The results for such a log-scaled sample are illustrated in Table 5.1. As can be also seen from the first plot in Figure 5.1 (ECDFs of the SRCV and SMC), the variance reduction works absolutely fine for SRCV. Most of the sample values produced by SMC are much smaller than the corresponding mean value, whereas the deviation w.r.t. the mean $\bar{u}$ is very small for the SRCV approach. The main problem of the SMC approach in this case is that almost all paths tend to zero so that the small number of outliers is not sufficient to reach the (large) expectation $\mathbb{E}[f(X_{\Delta,T})]$, i.e. N has to be increased a lot to approach the expectation. In contrast, for the SRCV approach all paths (paths close to zero as well as outliers) are "shifted" close to the expectation and thus we obtain a very small variance. We only plot the ECDFs of the SRCV and SMC in Figure 5.1, since the ECDFs of the RCV and RRCV look visually very similar to that for SRCV. The difference is, however, revealed in the "Min" and "Max" columns of the Table 5.1. That is, the RCV and RRCV algorithms produce several outliers which result in that the RCV and RRCV do not give us any variance reduction effect! One reason for this significant difference between the algorithms is given by the plots in Figure 5.2, where we illustrate the regression results for the RCV, RRCV and SRCV algorithms at the last time point, which means the first regression task. Here, we have accurate estimates only for the SRCV (cf. the discussion around (5.11)).

5.3.2 High-dimensional geometric Brownian motion

We consider the following SDE for $d = m = 10$ ($Q = 12$):

$$dX^i_t = rX^i_t dt + \sigma^i X^i_t A^i dW_t, \quad t \in [0, 1], \quad i = 1, \dots, 10,$$

Approach	Min	Max	Variance	Time (sec)	θ
SRCV	-0.5	0.2	$6.3 \cdot 10^{-8}$	30.5	$1.32 \cdot 10^{-23}$
RRCV	-25.4	1.7	$2.7 \cdot 10^{16}$	65.3	12.38
RCV	-27.8	0.1	$1.4 \cdot 10^{17}$	30.0	28.57
SMC	-10.6	15.9	$9.6 \cdot 10^{15}$	15.1	1

Table 5.1. Results of the algorithms for a quadratic function f under a GBM model.

where $X_0^i = 1$, $\sigma^i = 2\,\forall i$, $r = 0.05$ and $A^i := \left(A^{i,1} \cdots A^{i,10}\right)$, $AA^T = (\rho_{ik})_{i,k=1,\ldots,10}$ with $\rho_{ik} = \rho_{ki} \in [-1,1]$ and $\rho_{ik} = 1$ for $i = k$ (that is, A^iW, $i = 1,\ldots,10$, are correlated Brownian motions). For $i < k$ we choose

$$\rho_{ik} = \begin{cases} 0.9 & \text{if } i=1,\, k=2, \\ 0.5 & \text{if } i=5,\, k=6, \\ 0.8 & \text{if } i=9,\, k=10, \end{cases} \qquad \begin{cases} -0.95 & \text{if } i=3,\, k=4, \\ -0.9 & \text{if } i=7,\, k=8, \\ 0 & \text{otherwise.} \end{cases}$$

In this example, we illustrate the performances of the algorithms by means of the function $f(x) = \max\left\{\max_{i\in\{1,\ldots,10\}} x^i - 1, 0\right\}$. For saving a lot of computing time, we use the "simplified control variate"

$$\tilde{\tilde{M}}^{(1)}_{\Delta,T} := \sum_{j=1}^{J}\sum_{i=1}^{m} \tilde{a}_{j,e_i}(X_{\Delta,(j-1)\Delta}, \mathcal{D}^{tr}_{N_r})\xi^i_j$$

rather than $\tilde{M}^{(1)}_{\Delta,T}$ for RCV and SRCV (cf. (3.10) in case of the strong schemes). This simplification already takes much of the variance reduction power into account, while significantly reduces the number of summands needed to construct the control variate ($m = 10$ vs. $c_m - 1 = 2^m - 1 = 1023$ summands in the second sum above). For the SRCV algorithm, this results in the cost order $NJmQ$ instead of $NJ(c_m - 1)Q$ in the testing phase (10^{11} vs. 10^{13} in this example). Such a reduction in computational time due to using $\tilde{\tilde{M}}^{(1)}_{\Delta,T}$ applies also to the RCV algorithm, but does not apply to the RRCV algorithm. Namely, with $\tilde{\tilde{M}}^{(1)}_{\Delta,T}$ the testing phase of the RRCV algorithm would now cost $NJc_mQ + NJmc_m$ (in the second summand we now have the factor m instead of $c_m - 1$, cf. footnote 14 on page 82), which is still of order 10^{13} in the present example. Therefore, we do not consider the RRCV approach in this example. The results for the log-scaled sample are illustrated in Table 5.2. Again, the SRCV approach achieves a much smaller variance compared to the SMC and RCV (see the second plot in Figure 5.1).

Approach	Min	Max	Variance	Time (sec)	θ
SRCV	-5.8	2.0	14.6	573.9	0.13
RCV	-10.4	0.7	11271.0	288.2	51.50
SMC	-1.9	7.2	448.9	140.5	1

Table 5.2. Results of the algorithms for a Call-on-max-option under a high-dimensional GBM.

5.3.3 High-dimensional Heston model

We consider the following SDE for $d = m = 9$ ($Q = 11$) (cf. [28]):

$$dX_t^i = rX_t^i dt + \sigma^i X_t^i \sqrt{X_t^9} A^i dW_t, \quad i = 1, \dots, 8,$$
$$dX_t^9 = \lambda\left(\bar{v} - X_t^9\right) dt + \eta\sqrt{X_t^9} A^9 dW_t,$$

where $t \in [0,1]$, $X_0^i = 1$, $\sigma^i = 1$ for $i = 1, \dots, 8$ as well as $X_0^9 = 4$, $r = 0.05$, $\lambda = 0.1$, $\bar{v} = 4$, $\eta = 1$ and $A^i := \left(A^{i,1} \cdots A^{i,9}\right)$, $AA^T = (\rho_{ik})_{i,k=1,\dots,9}$. Here, for $i < k$ we choose

$$\rho_{ik} = \begin{cases} 0.9 & \text{if } i = 1,\ k = 2, \\ 0.5 & \text{if } i = 5,\ k = 6, \\ -0.2 & \text{if } i \in \{1,2,3,5,6,7\},\ k = 9, \\ 0 & \text{otherwise.} \end{cases} \qquad \begin{cases} -0.95 & \text{if } i = 3,\ k = 4, \\ -0.9 & \text{if } i = 7,\ k = 8, \\ 0.2 & \text{if } i \in \{4,8\},\ k = 9, \end{cases}$$

One might think about $X^1, \dots, X^8$ as price process of 8 stocks, while the CIR process X^9 is their common stochastic volatility. Notice that Feller's condition for X^9 is not satisfied ($\frac{2\lambda\bar{v}}{\eta^2} = 0.8 < 1$), that is, 0 is accessible boundary point for X^9 (with reflecting boundary behaviour). The discretised process $(X^9_{\Delta,j\Delta})_{j=0,\dots,J}$ can become negative. We, therefore, use the following discretisation scheme (see e.g. [2])

$$X^i_{\Delta,j\Delta} = X^i_{\Delta,(j-1)\Delta}\left(1 + r\Delta + \sigma^i\sqrt{\left(X^9_{\Delta,(j-1)\Delta}\right)^+} A^i\sqrt{\Delta}\xi_j\right),$$
$$X^9_{\Delta,j\Delta} = X^9_{\Delta,(j-1)\Delta} + \lambda\left(\bar{v} - \left(X^9_{\Delta,(j-1)\Delta}\right)^+\right)\Delta + \eta\sqrt{\left(X^9_{\Delta,(j-1)\Delta}\right)^+} A^9\sqrt{\Delta}\xi_j,$$

where $i \in \{1, \dots, 8\}$ and $x^+ := \max\{x, 0\}$. Here, we consider the function $f(x) = \max\left\{\max_{i\in\{1,\dots,8\}} x^i - 1, 0\right\}$ and, as in Subsection 5.3.2, use the simplified control variate $\tilde{\tilde{M}}^{(1)}_{\Delta,T}$ (we again exclude the RRCV approach). The results for the log-scaled sample are illustrated in Table 5.3. We get that the ECDF for the SRCV approach has a similar form as the one from Subsection 5.3.2 (see the third plot in Figure 5.1). Notice that the values of the estimators lie in all cases around 4.6 (SMC: 4.62, RCV: 4.59, SRCV: 4.60). Nevertheless, in the case of the SRCV approach 75.5% of the paths are located within the interval $(3, 6)$, whereas in case of the SMC approach this holds for only 13.0% of the paths and in case of the RCV approach for only 9.9%. This is a further indication of a better numerical performance of the SRCV approach.

Approach	Min	Max	Variance	Time (sec)	θ
SRCV	-6.4	2.6	50.1	444.7	0.09
RCV	-10.2	1.0	3208.8	328.6	4.33
SMC	-1.7	9.8	1478.8	164.5	1

Table 5.3. Results of the algorithms for a Call-on-max-option in a high-dimensional Heston model.

Below we illustrate the results of Subsections 5.3.1–5.3.3.

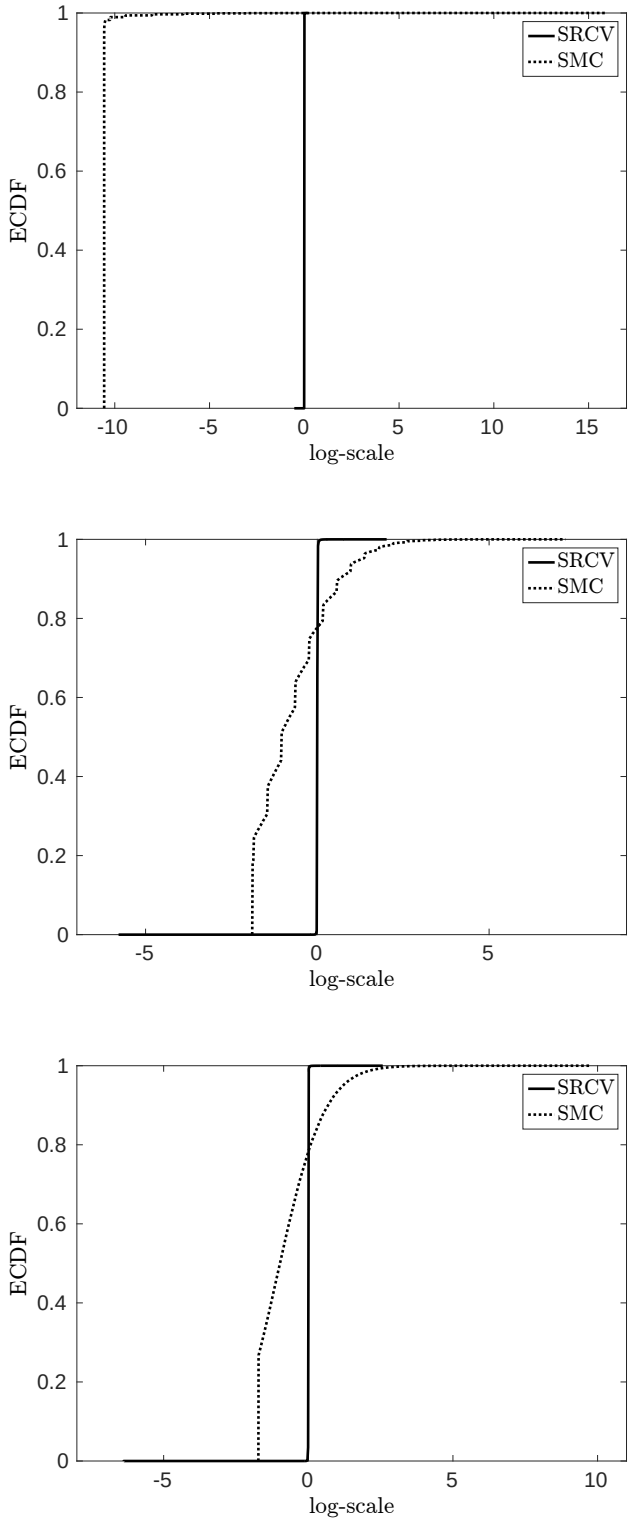

Figure 5.1. top: Subsection 5.3.1, centre: Subsection 5.3.2, bottom: Subsection 5.3.3 (each referring to the ECDF of the log-scaled sample for SRCV and SMC).

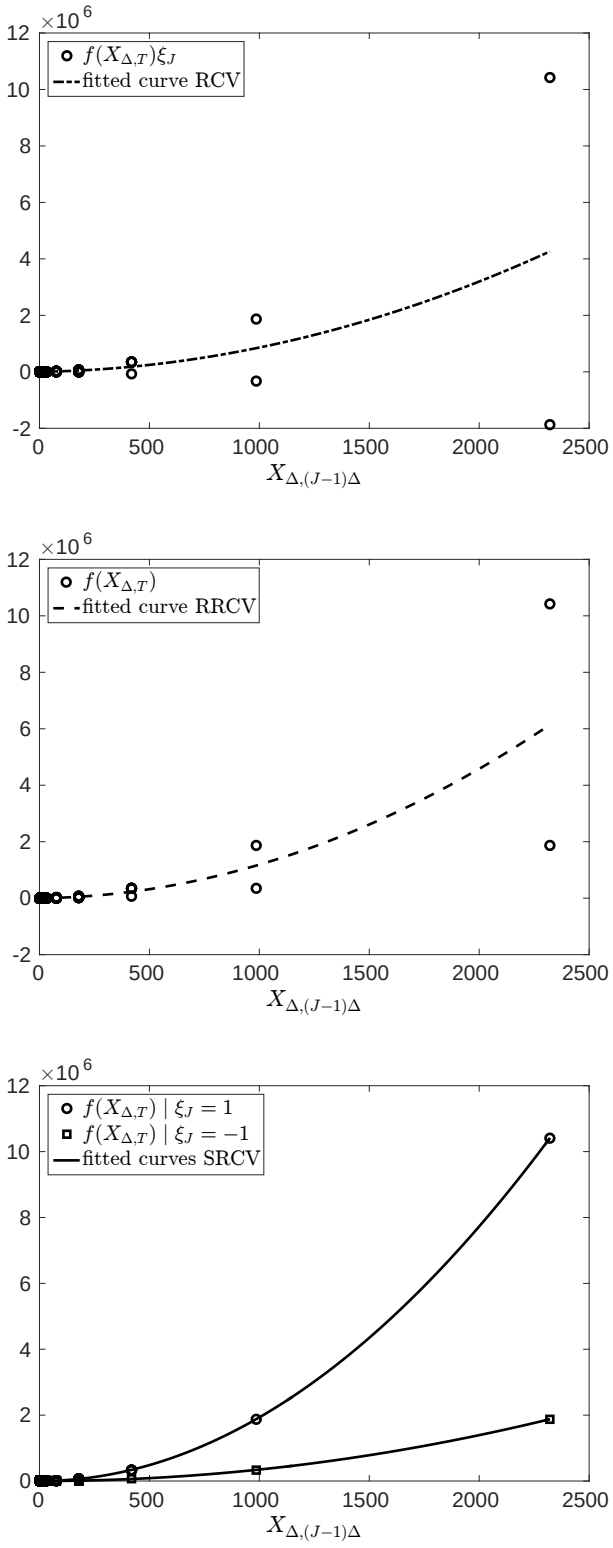

Figure 5.2. top: RCV, centre: RRCV, bottom: SRCV (each referring to the first regression task in Subsection 5.3.1).

5.4 Proofs

Proof of Lemma 5.7

Let us first recall that $2^{m(m-1)/2} p_m(y) = \mathbb{P}(\xi_j = y) = \prod_{i=1}^m \mathbb{P}(\xi_j^i = y_i)$ (cf. (5.15)). Formulas (5.17) and (5.19) as well as $(\sum_{i=1}^{c_m} b_i)^2 \le c_m \sum_{i=1}^{c_m} b_i^2$ yields

$$
\begin{aligned}
&\mathbb{E}\|\tilde{a}_{j,o,r} - a_{j,o,r}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \\
&\le c_m \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} \sum_{z\in\{-1,1\}^{\mathcal{I}_2}} \left[p_m(y) \prod_{i=1}^m H_{o_i}(y_i) \prod_{(k,l)\in\mathcal{I}_2} z_{kl}^{r_{kl}} \right]^2 \mathbb{E}\|\tilde{h}_{j,y,z} - h_{j,y,z}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \\
&\le c_m 2^{\frac{m(m-1)}{2}} \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} \left[\prod_{i=1}^m H_{o_i}(y_i) \right]^2 \left(\tilde{c}\, A^2 (\log N_r + 1) \frac{Q p_m(y)}{N_r} + \frac{8\, D_\kappa p_m(y)^2}{Q^\kappa} \right) \\
&= c_m \tilde{c}\, A^2 (\log N_r + 1) \frac{Q}{N_r} \mathbb{E}\left[\prod_{i=1}^m H_{o_i}(\xi_j^i) \right]^2 + \frac{8\, D_\kappa}{Q^\kappa} \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} c_m 2^{\frac{m(m-1)}{2}} \left[p_m(y) \prod_{i=1}^m H_{o_i}(y_i) \right]^2 \\
&= c_m \tilde{c}\, A^2 (\log N_r + 1) \frac{Q}{N_r} + \frac{8\, D_\kappa}{Q^\kappa} C_{m,o},
\end{aligned}
$$

where in the last equality we used that $\xi_j^1, \dots, \xi_j^m$ are independent and all $H_{o_i}(\xi_j^i)$ have unit L^2-norm. □

Proof of Theorem 5.8

It holds

$$
\begin{aligned}
\mathrm{Var}[f(X_{\Delta,T}) - \tilde{M}^{(2)}_{\Delta,T}] &= \mathrm{Var}[M^{(2)}_{\Delta,T} - \tilde{M}^{(2)}_{\Delta,T}] \\
&= \mathbb{E}\,\mathrm{Var}[M^{(2)}_{\Delta,T} - \tilde{M}^{(2)}_{\Delta,T} | \mathcal{D}^{tr}_{N_r}] + \mathrm{Var}\,\mathbb{E}[M^{(2)}_{\Delta,T} - \tilde{M}^{(2)}_{\Delta,T} | \mathcal{D}^{tr}_{N_r}].
\end{aligned}
$$

Due to the martingale transform structure in (5.14) and (5.21), we have

$$
\mathbb{E}[M^{(2)}_{\Delta,T} - \tilde{M}^{(2)}_{\Delta,T} | \mathcal{D}^{tr}_{N_r}] = 0.
$$

Together with the fact that the system $\left\{ \prod_{i=1}^m H_{o_i}(\xi_j^i) \prod_{(k,l)\in\mathcal{I}_2} (V_j^{kl})^{r_{kl}} : (o,r) \in \mathcal{U} \right\}$ is orthonormal in L^2, we get

$$
\mathrm{Var}[f(X_{\Delta,T}) - \tilde{M}^{(2)}_{\Delta,T}] = \sum_{j=1}^J \sum_{(o,r)\in\mathcal{U}} \mathbb{E}\|\tilde{a}_{j,o,r} - a_{j,o,r}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})}. \tag{5.25}
$$

With the expression $C_{m,o}$ of Lemma 5.7 we compute

$$
\begin{aligned}
\sum_{(o,r)\in\mathcal{U}} C_{m,o} &= \sum_{o\in\{0,1,2\}^m} \sum_{r\in\{0,1\}^{\mathcal{I}_2}} C_{m,o} - \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} c_m 2^{\frac{m(m-1)}{2}} p_m(y)^2 \\
&= \sum_{o\in\{0,1,2\}^m} 2^{\frac{m(m-1)}{2}} C_{m,o} - \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} c_m 2^{\frac{m(m-1)}{2}} p_m(y)^2 \\
&:= \alpha - \beta,
\end{aligned}
$$

where α (resp. β) denotes the first (resp. second) big sum in the above expression. Let us compute α and β. Recalling that

$$2^{m(m-1)/2} p_m(y) = \mathbb{P}(\xi_j = y) = \prod_{i=1}^{m} \mathbb{P}(\xi_j^i = y_i),$$

we get

$$\begin{aligned}\alpha &= c_m \sum_{o\in\{0,1,2\}^m} \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} \prod_{i=1}^{m} \left[\mathbb{P}(\xi_j^i = y_i) H_{o_i}(y_i)\right]^2 \\ &= c_m \left(\sum_{o_1\in\{0,1,2\}} \sum_{y_1\in\{-\sqrt{3},0,\sqrt{3}\}} \left[\mathbb{P}(\xi_j^1 = y_1) H_{o_1}(y_1)\right]^2 \right)^m = c_m,\end{aligned}$$

where the last equality follows by a direct calculation. Recalling that $c_m = 3^m 2^{m(m-1)/2}$ (we consider second order schemes), we obtain

$$\beta = 3^m \sum_{y\in\{-\sqrt{3},0,\sqrt{3}\}^m} \prod_{i=1}^{m} \mathbb{P}(\xi_j^i = y_i)^2 = 3^m \left(\sum_{y_1\in\{-\sqrt{3},0,\sqrt{3}\}} \mathbb{P}(\xi_j^1 = y_1)^2 \right)^m = \left(\frac{3}{2}\right)^m.$$

Thus,

$$\sum_{(o,r)\in\mathcal{U}} C_{m,o} = c_m - \left(\frac{3}{2}\right)^m = \tilde{c}_m.$$

The last expression together with Lemma 5.7 and (5.25) yields the result. □

Proof of Theorem 5.9

The proof is similar to the one of Theorem 3.20. □

Chapter 6

Complexity reduction of the dual nested Monte Carlo methods

This chapter is mainly based on the paper [8].

Next we focus on the pricing of Bermudan options, respectively American options, via dual nested Monte Carlo methods.

6.1 Setup

In contrast to European options that may be exercised only at a fixed date, an American option grants its holder the right to select the time at which to exercise the option. A general class of American option pricing problems can be formulated through an $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-Markov process $(X_t)_{0\leq t\leq T}$ with a deterministic starting point $X_0 = x_0 \in \mathbb{R}^d$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. Let us recall that each $\mathcal{F}_t$ is a σ-algebra of subsets of Ω, and $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \leq t$. We first consider options admitting a finite set of exercise opportunities $0 = t_0 < t_1 < t_2 < \ldots < t_J = T$, called Bermudan options, with corresponding Markov chain

$$X_j := X_{t_j}, \quad j = 0, \ldots, J.$$

This option pays $g_j(X_j)$, if exercised at time t_j, $j = 0, \ldots, J$, for some known functions $g_0, \ldots, g_J$ mapping $\mathbb{R}^d$ into $[0, \infty)$. Below we assume that $g_j(X_j) \in L^2$ for all j. Let $\mathcal{T}_j$ denote the set of stopping times taking values in $\{j, j+1, \ldots, J\}$. As a standard result in the theory of contingent claims, the equilibrium price $v_j^*(x)$ of the Bermudan option at time t_j in state x, given that the option was not exercised prior to t_j, is its value under the optimal exercise policy

$$v_j^*(x) = \sup_{\tau \in \mathcal{T}_j} \mathbb{E}[g_\tau(X_\tau)|X_j = x], \quad x \in \mathbb{R}^d.$$

Clearly, any given stopping rules $\tau_j \in \mathcal{T}_j$ are generally suboptimal and give us lower bounds

$$v_j(x) := \mathbb{E}[g_\tau(X_\tau)|X_j = x] \leq v_j^*(x), \quad j = 0, \ldots, J.$$

An interval on the true price would be completed if an upper bound could be generated from any given exercise policy. The dual approach proposed in Rogers [55] and Haugh and Kogan [27] is based on the following simple observation. For any $0 \leq i \leq J$ and any supermartingale $(Y_j)_{i \leq j \leq J}$ with $Y_i = 0$, it holds

$$\begin{aligned} v_i^*(X_i) = \sup_{\tau \in \mathcal{T}_i} \mathbb{E}\left[g_\tau(X_\tau)|\mathcal{F}_i\right] &\leq \sup_{\tau \in \mathcal{T}_i} \mathbb{E}\left[g_\tau(X_\tau) - Y_\tau|\mathcal{F}_i\right] \\ &\leq \mathbb{E}\left[\max_{i \leq j \leq J} \left(g_j(X_j) - Y_j\right)|\mathcal{F}_i\right] \end{aligned} \tag{6.1}$$

(we now use the shorthand $\mathcal{F}_j := \mathcal{F}_{t_j}$). Therefore the right-hand side of (6.1) provides an upper bound for $v_i^*(X_i)$. It can be derived that both inequalities in (6.1) are equalities for the martingale part of the Doob-Meyer decomposition of the price process $(v_j^*(X_j))_{i \leq j \leq J}$

$$Y_i^* = 0, \quad Y_j^* = \sum_{l=i+1}^{j} \left(v_l^*(X_l) - \mathbb{E}\left[v_l^*(X_l)|\mathcal{F}_{l-1}\right]\right), \quad j = i+1, \ldots, J.$$

In fact, Y^* satisfies the following even stronger almost sure identity

$$v_i^*(X_i) = \max_{i \leq j \leq J} \left(g_j(X_j) - Y_j^*\right), \quad a.s. \tag{6.2}$$

(also see [56]). The duality representation provides a simple way to estimate the Snell envelope from above, using approximations $(v_i(X_i))$ for the value functions $(v_i^*(X_i))$. Let Y be a martingale defined via

$$Y_0 = 0, \quad Y_j = \sum_{l=1}^{j} \left(v_l(X_l) - \mathbb{E}\left[v_l(X_l)|\mathcal{F}_{l-1}\right]\right), \quad j = 1, \ldots, J. \tag{6.3}$$

Then, for $i = 0$, we get that

$$V_0 := v_0(x_0) = \mathbb{E}\left[\max_{0 \leq j \leq J} \left(g_j(X_j) - Y_j\right)\right] \tag{6.4}$$

is an upper bound for $V_0^* := v_0^*(x_0)$. The properties of the dual upper bound were thoroughly studied in Chen and Glasserman [13] and Belomestny et al [12]. Throughout we are going to use nonparametric regression algorithms to construct some computationally efficient approximations for the conditional expectations involved in (6.3). Nonparametric regression algorithms like that of Longstaff and Schwartz (see e.g. [40]) have become among the most successful and widely used methods for approximating the values of American-style (Bermudan) options, in particular for high-dimensional problems. Due to their popularity, the analysis of the convergence properties of these types of Monte Carlo algorithms is a problem of fundamental importance in applied probability and mathematical finance, see e.g. Clément, Lamberton and Protter [14], Zanger [61] and references therein. Here we rigorously analyse the convergence properties of the proposed regression algorithm and derive its complexity. Moreover we illustrate its performance in the case of the max-call Bermudan options.

6.2 Nested simulations approach

The nested simulation approach for computing V_0 relies on the approximation of the inner expectations in (6.4) via (conditional) Monte Carlo. This approach was first proposed in Andersen and Broadie [3] for the computation of the dual upper bound (6.4). Let us describe this method in more details. Fix some natural numbers N_d, N and consider the estimate

$$V_{N,N_d} = \frac{1}{N}\sum_{n=1}^{N}\left[\max_{0\le j\le J}\left(g_j(X_j^{(n)}) - Y_{j,n,N_d}\right)\right],$$

where

$$Y_{j,n,N_d} = \sum_{l=1}^{j}\left(v_l(X_l^{(n)}) - \frac{1}{N_d}\sum_{n_d=1}^{N_d} v_l(X_l^{(n_d,n)})\right), \quad j = 0,\dots,J,$$

($\sum_1^0 := 0$), $(X_l^{(1)},\dots,X_l^{(N)})$ is a sample from the distribution of X_l and for any fixed n, the sample $X_l^{(1,n)},\dots,X_l^{(N_d,n)}$ is drawn from the conditional distribution of X_l given $X_{l-1} = X_{l-1}^{(n)}$.

Theorem 6.1. *We have for the estimator V_{N,N_d}*

$$\mathbb{E}V_{N,N_d} \ge V_0,$$

i.e. it gives us an upper bound for V_0 and hence for V_0^*. Moreover it holds

$$\mathbb{E}\left[|V_{N,N_d} - V_0|^2\right] \le 4\sum_{l=1}^{J}\left(\frac{\mathbb{E}[\mathrm{Var}\left[v_l(X_l)|X_{l-1}\right]]}{N_d}\left(1+\frac{1}{N}\right) + \frac{\mathbb{E}\left[|v_l^*(X_l) - v_l(X_l)|^2\right]}{N}\right). \tag{6.5}$$

The bound (6.5) is very informative, as it not only gives an error estimate for V_{N,N_d}, but also shows ways to improve it. While the second term on the r.h.s. of (6.5) can be reduced by making the bound v_l closer to v_l^*, the first can be made smaller by reducing the magnitude of the conditional variances $\mathrm{Var}\left[v_l(X_l)|X_{l-1}\right]$. Since the cost of computing V_{N,N_d} is of order NN_d (recall that J is fixed for now), the overall complexity of the estimate V_{N,N_d} is of order ε^{-4}.

Let us mention two relevant modifications of the nested dual algorithm proposed in the literature. Firstly, in Belomestny et al [6] an algorithm not involving sub-simulation was suggested, where an approximation for the Doob martingale was constructed using the martingale representation theorem and some approximation of the true price process. However, this method requires an additional discretisation of stochastic integrals and suffers from some instability for small discretisation steps. Secondly, a multilevel-type algorithm was developed in Belomestny et al [12], which has a similar performance, in terms of complexity, as the algorithm described here, but works under very different conditions (e.g. the algorithm in [12] does not take advantage of the smoothness properties of the involved conditional expectations).

In the next section we present a regression-based approach, which will result in a significant reduction of the complexity (see the discussion right after (6.26)).

6.3 Variance reduction via regression

Suppose that for some $\Delta > 0$, the time approximations $X_{\Delta,l\Delta}$, $l = 1, \dots, L$, with $L = \lfloor T/\Delta \rfloor \geq J$ satisfy the following recurrence relations (cf. (2.11))

$$X_{\Delta,l\Delta} = \Phi_l(X_{\Delta,(l-1)\Delta}, \xi_l), \quad l = 1, \dots, L, \tag{6.6}$$

for some i.i.d. random vectors $\xi_l \in \mathbb{R}^m$ with distribution μ and some Borel-measurable functions $\Phi_l \colon \mathbb{R}^{d+m} \to \mathbb{R}^d$. By $(\mathcal{G}_l)_{l=0,\dots,L}$ we denote the filtration with $\mathcal{G}_0 = \text{triv}$ generated by $(\xi_l)_{l=1,\dots,L}$. Let $(\phi_k)_{k\in\mathbb{Z}_+}$ be a complete orthonormal system in $L^2(\mathbb{R}^m, \mu)$ with $\phi_0 \equiv 1$, i.e.,

$$\mathbb{E}[\phi_i(\xi)\phi_j(\xi)] = \delta_{ij}, \quad i, j \in \mathbb{Z}_+.$$

In particular, the random variables $\phi_k(\xi)$, $k \geq 1$, are centered. Notice that the (normalised) Hermite polynomials H_k are a special case of such orthonormal systems. Let us fix some $j < p$ in $\{0, 1, \dots, L\}$.

Theorem 6.2. *It holds for any function f with $\mathbb{E}[|f(X_{\Delta,p\Delta})|^2] < \infty$*

$$f(X_{\Delta,p\Delta}) = \mathbb{E}\left[f(X_{\Delta,p\Delta}) \middle| X_{\Delta,j\Delta} \right] + \sum_{k\geq 1} \sum_{l=j+1}^{p} a_{p,l,k}(X_{\Delta,(l-1)\Delta})\phi_k(\xi_l), \tag{6.7}$$

where the series on the r.h.s. converges in L^2 sense. The coefficients in (6.7) can be computed via

$$a_{p,l,k}(x) = \mathbb{E}\left[f(X_{\Delta,p\Delta})\phi_k(\xi_l) \middle| X_{\Delta,(l-1)\Delta} = x \right]$$

for $l = 1, \dots, L$ and $k \in \mathbb{N}$.

Identity (6.7) implies that

$$\text{Var}\left[f(X_{\Delta,p\Delta}) - M_{j,p} | X_{\Delta,j\Delta} \right] = 0, \quad \mathbb{E}[M_{j,p}|\mathcal{G}_j] = 0, \quad a.s.$$

with

$$M_{j,p} = \sum_{k\geq 1} \sum_{l=j+1}^{p} a_{p,l,k}(X_{\Delta,(l-1)\Delta})\phi_k(\xi_l) \tag{6.8}$$

and hence $M_{j,p}$ is an optimal control variate. In order to use control variate $M_{j,p}$, we need to compute the coefficients $a_{p,l,k}$. This can be done by using regression in the following way: first we generate N_r discretised paths $X^{(n)}_{\Delta,1\Delta}, \dots, X^{(n)}_{\Delta,L\Delta}$, $n = N+1, \dots, N+N_r$, of the process X and then solve the least squares optimisation problems

$$\hat{a}_{p,l,k} = \underset{\psi\in\text{span}(\psi_1,\dots,\psi_Q)}{\arg\min} \sum_{n=N+1}^{N+N_r} \left| f(X^{(n)}_{\Delta,p\Delta})\phi_k(\xi^{(n)}_j) - \psi(X^{(n)}_{\Delta,(j-1)\Delta}) \right|^2,$$

for $l = j+1, \dots, p$, where $\psi_1, \dots, \psi_Q$ is a set of basis functions on $\mathbb{R}^d$. Furthermore, we truncate the summation in (6.8) to get an implementable version of the control variate $M_{j,p}$

$$\hat{M}_{j,p,K} = \sum_{k=1}^{K} \sum_{l=j+1}^{p} \hat{a}_{p,l,k}(X_{\Delta,(l-1)\Delta})\phi_k(\xi_l). \tag{6.9}$$

To make clear how to understand (6.9), we remark that the random vectors ξ_l, $l = 1, \dots, L$, in (6.9) are independent of the N_r training paths $(X^{(n)}_{\Delta,l\Delta})$ used to obtain the regression-based estimates $\hat{a}_{p,l,k}$, while the testing path $(X_{\Delta,l\Delta})$ in the argument of $\hat{a}_{p,l,k}$ in (6.9) is constructed via those random vectors ξ_l according to (6.6) (and hence is independent of the training paths).

Let us note that $\mathbb{E}\hat{M}_{j,p,K} = 0$ due to the martingale transform structure in (6.9) (recall that $\mathbb{E}\phi_k(\xi_l) = 0$ for $k \geq 1$), i.e. $\hat{M}_{j,p,K}$ is indeed a valid control variate in that it does not introduce any bias. The properties of such a control variate are summarised in the following theorem.

Theorem 6.3. *Consider some $j < p$ in $\{0, 1, \dots, L\}$. Suppose that the function f is uniformly bounded by a constant F. By $\tilde{a}_{p,l,k}$ we denote the truncated at the level F estimate (cf.* (2.3)*)*

$$\tilde{a}_{p,l,k}(x) = T_F\hat{a}_{p,l,k}(x) = \begin{cases} \hat{a}_{p,l,k}(x) & \text{if } |\hat{a}_{p,l,k}(x)| \leq F, \\ F \operatorname{sgn} \hat{a}_{p,l,k}(x) & \text{otherwise,} \end{cases} \tag{6.10}$$

and by $\tilde{M}_{j,p,K}$ the control variate defined like in (6.9) *but with $\hat{a}_{p,l,k}$ replaced by $\tilde{a}_{p,l,k}$. Furthermore, assume that for some $\beta \geq 0$ and $B_\beta > 0$*

$$\sum_{k=1}^{\infty} k^\beta \sum_{l=j+1}^{p} \mathbb{E}[a^2_{p,l,k}(X_{\Delta,(l-1)\Delta})] \leq B_\beta \tag{6.11}$$

and the set of basis functions $\psi_1, \dots, \psi_Q$ is chosen in such a way that

$$\sum_{l=j+1}^{p} \inf_{\psi \in \operatorname{span}(\psi_1,\dots,\psi_Q)} \mathbb{E}\left[\left|a_{p,l,k}(X_{\Delta,(l-1)\Delta}) - \psi(X_{\Delta,(l-1)\Delta})\right|^2\right] \leq D_\kappa Q^{-\kappa}$$

for some constants $\kappa \geq 0$ and $D_\kappa > 0$ (cf. (2.6) *in Subsection 2.1.1). Then*

$$\begin{aligned} \mathbb{E}\left[\operatorname{Var}\left[f(X_{\Delta,p\Delta}) - \tilde{M}_{j,p,K}\middle| X_{\Delta,j\Delta}\right]\right] \leq \tilde{c}F^2(p-j)K\frac{Q(\log(N_r)+1)}{N_r} \\ + B_\beta K^{-\beta} + 8D_\kappa K Q^{-\kappa} \end{aligned}$$

with some universal constant $\tilde{c}$, where

$$\tilde{M}_{j,p,K} = \sum_{k=1}^{K}\sum_{l=j+1}^{p} \tilde{a}_{p,l,k}(X_{\Delta,(l-1)\Delta})\phi_k(\xi_l).$$

6.4 Dual upper bounds with reduced complexity

Next we apply the results of the previous section to the nested simulations of dual upper bounds. For the sake of clarity assume that the exercise times coincide with the discretisation time grid for some $\Delta > 0$, i.e. $L = J$. Instead of V_0, which is constructed in (6.4) via the exact process, we are now going to estimate its analogue $V_{\Delta,0}$ constructed via the discretised process

$$V_{\Delta,0} = \mathbb{E}\left[\max_{0\leq j\leq J}\left(g_j(X_{\Delta,j\Delta}) - Y_{\Delta,j\Delta}\right)\right] \tag{6.12}$$

with $Y_{\Delta,j\Delta} = \sum_{l=1}^{j} \left(v_l(X_{\Delta,l\Delta}) - \mathbb{E}\left[v_l(X_{\Delta,l\Delta})|X_{\Delta,(l-1)\Delta}\right]\right)$. For any $j = 1,\dots,J$, we need to compute the conditional expectations $\mathbb{E}\left[v_j(X_{\Delta,j\Delta})|X_{\Delta,(j-1)\Delta}\right]$. Due to (6.7) we have the following representation

$$v_j(X_{\Delta,j\Delta}) = \mathbb{E}\left[v_j(X_{\Delta,j\Delta})|\, X_{\Delta,(j-1)\Delta}\right] + \sum_{k\geq 1} a_{j,k}(X_{\Delta,(j-1)\Delta})\phi_k(\xi_j), \tag{6.13}$$

where

$$a_{j,k}(x) = \mathbb{E}\left[v_j(X_{\Delta,j\Delta})\phi_k\left(\xi_j\right)|\, X_{\Delta,(j-1)\Delta} = x\right], \tag{6.14}$$

provided $\mathbb{E}\left[v_j^2(X_{\Delta,j\Delta})\right] < \infty$. Representation (6.13) implies that

$$\mathrm{Var}[v_j(X_{\Delta,j\Delta}) - M_j|X_{\Delta,(j-1)\Delta}] = 0 \quad a.s. \tag{6.15}$$

for

$$M_j = \sum_{k\geq 1} a_{j,k}(X_{\Delta,(j-1)\Delta})\phi_k(\xi_j), \tag{6.16}$$

and hence the first term on the r.h.s. of (6.5) is zero. Of course, the control variates $M_1,\dots,M_J$ cannot be used directly, since the coefficients $a_{l,k}$ are unknown. So first we estimate the coefficients $a_{l,k}$ by a preliminary regression using N_r discretised paths of the process X and Q basis functions (see Section 6.3). In this way we construct the estimated and truncated version of the control variate M_l given by

$$\hat{M}_{l,K} = \sum_{k=1}^{K} \hat{a}_{l,k}(X_{\Delta,(l-1)\Delta})\phi_k(\xi_l). \tag{6.17}$$

Now fix some natural numbers N_d, N and consider the dual estimate

$$\hat{V}_{N,N_d,K} = \frac{1}{N}\sum_{n=1}^{N}\left[\max_{0\leq j\leq J}\left(g_j(X^{(n)}_{\Delta,j\Delta}) - \hat{Y}_{j,n,N_d,K}\right)\right], \tag{6.18}$$

where

$$\hat{Y}_{j,n,N_d,K} = \sum_{l=1}^{j}\left(v_l(X^{(n)}_{\Delta,l\Delta}) - \frac{1}{N_d}\sum_{n_d=1}^{N_d}\left(v_l(X^{(n_d,n)}_{\Delta,l\Delta}) - \hat{M}^{(n_d,n)}_{l,K}\right)\right) \tag{6.19}$$

with

$$\hat{M}^{(n_d,n)}_{l,K} = \sum_{k=1}^{K}\hat{a}_{l,k}(X^{(n)}_{\Delta,(l-1)\Delta})\phi_k(\xi^{(n_d,n)}_l). \tag{6.20}$$

We now can prove the following result.

Proposition 6.4. *Assume that all functions v_j, $j = 1,\dots,J$, are uniformly bounded by a constant F. By $\tilde{a}_{j,k}$ we denote the truncated at the level F estimate defined as in* (6.10)*, and by $\tilde{M}_{l,K}$ (resp. $\tilde{M}^{(n_d,n)}_{l,K}$, $\tilde{Y}_{j,n,N_d,K}$, $\tilde{V}_{N,N_d,K}$) the quantities defined like in* (6.17) *(resp.* (6.20)*,* (6.19)*,* (6.18)*) but with "hats" replaced by "tildes". Suppose that the coefficients $(a_{j,k})$ defined in* (6.14) *satisfy, for all $j = 1,\dots,J$,*

$$\sum_{k=1}^{\infty} k^{\beta}\,\mathbb{E}[a^2_{j,k}(X_{\Delta,(j-1)\Delta})] \leq B_\beta \tag{6.21}$$

with some $\beta \geq 0$ and $B_\beta > 0$ and that the basis functions $\psi_1, \ldots, \psi_Q$ are chosen in such a way that, for all $j = 1, \ldots, J$ and $k \in \mathbb{N}$,

$$\inf_{\psi \in \mathrm{span}(\psi_1,\ldots,\psi_Q)} \mathbb{E}\left[\left|a_{j,k}(X_{\Delta,(j-1)\Delta}) - \psi(X_{\Delta,(j-1)\Delta})\right|^2\right] \leq D_\kappa Q^{-\kappa} \tag{6.22}$$

with some $\kappa \geq 0$ and $D_\kappa > 0$. Then it holds

$$\begin{aligned}\mathbb{E}\left[|\tilde{V}_{N,N_d,K} - V_{\Delta,0}|^2\right] \leq \frac{4J}{N_d}\left(1+\frac{1}{N}\right)\left[\tilde{c}F^2K\frac{Q(\log(N_r)+1)}{N_r} + B_\beta K^{-\beta} + 8D_\kappa K Q^{-\kappa}\right] \\ + \frac{4}{N}\sum_{l=1}^{J}\mathbb{E}\left[|v_l^*(X_{\Delta,l\Delta}) - v_l(X_{\Delta,l\Delta})|^2\right]\end{aligned} \tag{6.23}$$

with some universal constant $\tilde{c}$.

Remark 6.5. Notice that in case of weak approximation schemes we do not need to introduce the parameter β in (6.21), since the optimal control variate M_j in (6.16) is a finite sum. Thus, in the complexity analyses below we can find a relation to weak schemes by considering the case $\beta \to \infty$.

6.4.1 Complexity analysis for Bermudan options

Proposition 6.4 allows us to carry out complexity analysis of our algorithm. First note that the overall cost of computing the estimator $\tilde{V}_{N,N_d,K}$ is of order

$$JK \max\left\{N_r Q^2, NQ, NN_d\right\}, \tag{6.24}$$

where the first term in (6.24) comes from the computation of the regression coefficients, the second one from the computation of $\tilde{a}_{l,k}(X^{(n)}_{\Delta,(l-1)\Delta})$ and the last one from the computation of $\tilde{M}^{(n_d,n)}_{l,K}$ (other terms involved in the computation are dominated by one of these quantities). Given $\beta > 0$ and $\kappa > 0$ as in Proposition 6.4, we have the following constraints

$$\max\left\{\frac{JKQ\log(N_r)}{N_r N_d}, \frac{JB_\beta}{K^\beta N_d}, \frac{JD_\kappa K}{Q^\kappa N_d}, \frac{J}{N}\right\} \lesssim \varepsilon^2 \tag{6.25}$$

to ensure the condition $\mathbb{E}\left[|\tilde{V}_{N,N_d,K} - V_{\Delta,0}|^2\right] \lesssim \varepsilon^2$.

Notice that we are interested in getting the order of complexity in ε as $\varepsilon \searrow 0$. To this end, we need to determine the parameters N, N_r, N_d, K and Q via ε in such a way that the order of complexity of $\tilde{V}_{N,N_d,K}$ (given by (6.24)) is minimal under the constraint (6.25). Since B_β, D_κ and J are constants, they can be dropped from (6.24) and (6.25). Straightforward but lengthy calculations now show that the overall complexity of $\tilde{V}_{N,N_d,K}$ is bounded from above by

$$C_{J,\beta,\kappa}\, \varepsilon^{-\frac{4(\beta+1)(\kappa+3)+4\kappa}{(\beta+1)(\kappa+3)+\beta\kappa}} \sqrt{|\log\varepsilon|}, \tag{6.26}$$

where the constant $C_{J,\beta,\kappa}$ does not depend on ε. Moreover, the dependence structure in $C_{J,\beta,\kappa}$ on the parameters β, κ and J is given by the formula $C_{J,\beta,\kappa} = cJ^2 B_\beta^{3/(1+\beta)} D_\kappa^{3/(3+\kappa)}$ with some universal constant c. We, finally, discuss the complexity estimate (6.26):

Remark 6.6. (i) We require to choose $\beta > 1$ in order to be better than the standard nested simulations approach discussed in Section 6.2 because $\frac{4(\beta+1)(\kappa+3)+4\kappa}{(\beta+1)(\kappa+3)+\beta\kappa} < 4$ whenever $\beta > 1$.

(ii) We can achieve the complexity order $\varepsilon^{-2-\delta}$, for arbitrarily small $\delta > 0$, whenever the parameters β and κ are sufficiently large.

(iii) In the limiting case $\kappa = 0$, i.e., if the approximation error in (6.22) does not converge to 0 (e.g. due to an inappropriate choice of basis functions), we end up with the complexity of the standard nested approach of order ε^{-4}.

(iv) While the control variates in the previous chapters lead to a variance reduction in comparison with $\mathrm{Var}\left[f(X_{\Delta,T})\right]$ (and the remaining MSE term, namely the squared bias $\left(\mathbb{E}\left[f(X_T) - f(X_{\Delta,T})\right]\right)^2$, is of the same order as for the SMC approach), the present control variates $\tilde{M}_{l,K}^{(n_d,n)}$ (cf. (6.20)) with $n_d \in \{1,\dots,N_d\}$, $n \in \{1,\dots,N\}$, $l \in \{1,\dots,j\}$, $j \in \{1,\dots,J\}$, lead to a reduction in comparison with the terms

$$\mathbb{E}\left[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta})|X_{\Delta,(l-1)\Delta}\right]\right], \tag{6.27}$$

which are a part of the overall error for the standard approach, but mainly affect the bias (cf. proof of Theorem 6.1). More precisely, the better the estimation of the control variates works, the smaller are the terms (cf. proof of Proposition 6.4)

$$\mathbb{E}\left[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta}) - \tilde{M}_{l,K}|X_{\Delta,(l-1)\Delta}\right]\right]$$

compared to (6.27). This leads to a reduction of both the bias and the variance. However, while the bias tends to zero for optimal control variates (cf. (6.15) and (6.41)), the variance is only slightly reduced (cf. (6.42)) and still depends on the remaining error terms, namely

$$\mathbb{E}\left[\left|v_l^*(X_{\Delta,l\Delta}) - v_l(X_{\Delta,l\Delta})\right|^2\right],$$

which are not affected by the control variates (see (6.5) and (6.23)).

In the next subsection we present the complexity analysis for the case of an increasing number of exercise dates $J \to \infty$. We also take the discretisation error into account, which is the order (in J, $J \to \infty$) of the difference between the upper bound V_0 for the (continuous time) American option price and the upper bounds $V_{\Delta,0}$ for the Bermudan option prices with $\Delta = T/J$.

6.4.2 Complexity analysis for American options

To approximate an upper bound V_0 for a true American (rather than Bermudan) option, we now let J tend to infinity. We shall compare the complexities of the *standard approach* (the one of Section 6.2 applied to the discretised process) and of the *regression-based approach* (the one described in the beginning of Section 6.4).

Standard approach: Set $\Delta = T/J$, then the estimate for $V_{\Delta,0}$ of (6.12) is

$$V_{\Delta,N,N_d} = \frac{1}{N}\sum_{n=1}^{N}\left[\max_{0\le j\le J}\left(g_j(X_{\Delta,j\Delta}^{(n)}) - Y_{\Delta,j\Delta,n,N_d}\right)\right],$$

where

$$Y_{\Delta,j\Delta,n,N_d} = \sum_{l=1}^{j} \left(v_l(X^{(n)}_{\Delta,l\Delta}) - \frac{1}{N_d} \sum_{n_d=1}^{N_d} v_l(X^{(n_d,n)}_{\Delta,l\Delta}) \right), \quad j = 0, \dots, J.$$

The analogue of (6.5) takes the form

$$\begin{aligned} \mathbb{E}\left[|V_{\Delta,N,N_d} - V_{\Delta,0}|^2\right] \leq & \frac{4\sum_{l=1}^{J} \mathbb{E}[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta})|X_{\Delta,(l-1)\Delta}\right]]}{N_d} \left(1 + \frac{1}{N}\right) \\ & + \frac{4\sum_{l=1}^{J} \mathbb{E}\left[|v_l^*(X_{\Delta,l\Delta}) - v_l(X_{\Delta,l\Delta})|^2\right]}{N}. \end{aligned} \tag{6.28}$$

Since we are considering American options in this section, the estimate V_{Δ,N,N_d} can be viewed as an estimate for V_0 rather than for $V_{\Delta,0}$, i.e. this is $\mathbb{E}\left[|V_{\Delta,N,N_d} - V_0|^2\right]$ that should be of order ε^2 in the complexity analysis. Therefore, we need an assumption about the order of the discretisation error $V_{\Delta,0} - V_0$. It seems reasonably general to assume that it is of order $\frac{1}{\sqrt{J}}$. However, the discretisation error might be of a different order in specific situations (see [15]). That is why we impose a more general assumption:

(A1) $V_{\Delta,0} - V_0$ is of order $J^{-\alpha}$ as $J \to \infty$ with some $\alpha > 0$.

We also need an assumption on the order of the second term on the right-hand side of (6.28) (which is also present in (6.23)):

(A2) $\sum_{l=1}^{J} \mathbb{E}\left[|v_l^*(X_{\Delta,l\Delta}) - v_l(X_{\Delta,l\Delta})|^2\right]$ is of order J^q as $J \to \infty$ with some $q \in [0, 1]$.

A typical-to-expect situation here is $q = 1$. Another interesting variant is $q = 0$: here the strategy is to use better and better approximations v_l for v_l^* at each time point $l = 1, \dots, J$, as J grows (see, e.g., Zanger [61] for bounds on $\mathbb{E}\left[\|v_l^* - v_l\|^2\right]$.) Finally, as for the first term on the right-hand side of (6.28) it is reasonable to assume only that

(A3) $\sum_{l=1}^{J} \mathbb{E}[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta})|X_{\Delta,(l-1)\Delta}\right]]$ is of order J as $J \to \infty$.

The overall cost of computing the estimate V_{Δ,N,N_d} is of order JN_dN. Thus, we need to minimise this cost order under the constraint

$$\max\left\{\frac{1}{J^{2\alpha}}, \frac{J}{N_d}, \frac{J^q}{N}\right\} \lesssim \varepsilon^2,$$

which ensures that $\mathbb{E}\left[|V_{\Delta,N,N_d} - V_0|^2\right] \lesssim \varepsilon^2$ (see (6.28) and (A1)–(A3)). This leads to the complexity of V_{Δ,N,N_d} of order $\varepsilon^{-4-\frac{2+q}{\alpha}}$. For instance, in the case $\alpha = 1/2$, $q = 1$ (resp. $\alpha = 1/2$, $q = 0$) we get the complexity $O(\varepsilon^{-10})$ (resp. $O(\varepsilon^{-8})$).

Regression-based approach: We suppose that the assumptions of Proposition 6.4 are satisfied uniformly in $J \in \mathbb{N}$ and again assume (A1) and (A2) (as for (A3), we do not need it here). The cost of computing $\tilde{V}_{N,N_d,K}$ is of order

$$JK \max\left\{N_r Q^2, NQ, NN_d\right\}.$$

We need to minimise this under the constraints

$$\max\left\{\frac{1}{J^{2\alpha}}, \frac{JKQ\log(N_r)}{N_rN_d}, \frac{JB_\beta}{K^\beta N_d}, \frac{JD_\kappa K}{Q^\kappa N_d}, \frac{J^q}{N}\right\} \lesssim \varepsilon^2,$$

which ensures that $\mathbb{E}\left[|\tilde{V}_{N,N_d,K} - V_0|^2\right] \lesssim \varepsilon^2$ (see (6.23) and (A1)–(A2)). Straightforward but lengthy calculations show that the overall complexity of $\tilde{V}_{N,N_d,K}$ is bounded from above by

$$C_{\beta,\kappa}\,\varepsilon^{-\frac{(4\alpha+2+q)(\beta+1)(\kappa+3)+(\beta+4\alpha+1+q)\kappa}{\alpha(\beta+1)(\kappa+3)+\alpha\beta\kappa}}\sqrt{|\log\varepsilon|}, \tag{6.29}$$

where the constant $C_{\beta,\kappa}$ does not depend on ε. Moreover, the dependence on β and κ is described by the formula $C_{\beta,\kappa} = cB_\beta^{3/(1+\beta)}D_\kappa^{3/(3+\kappa)}$ with some universal constant c. We, finally, discuss the complexity estimate (6.29):

Remark 6.7. (i) We again require to choose $\beta > 1$ in order to be better than the standard approach discussed above, because, as a straightforward calculation shows,

$$\frac{(4\alpha+2+q)(\beta+1)(\kappa+3)+(\beta+4\alpha+1+q)\kappa}{\alpha(\beta+1)(\kappa+3)+\alpha\beta\kappa} < 4 + \frac{2+q}{\alpha}$$

whenever $\beta > 1$.

(ii) We can achieve the complexity order $\varepsilon^{-2-\frac{3+q}{2\alpha}-\delta}$, for arbitrarily small $\delta > 0$, whenever the parameters β and κ are sufficiently large. In particular, this gives us $O(\varepsilon^{-6-\delta})$ (resp. $O(\varepsilon^{-5-\delta})$) when $\alpha = 1/2$, $q = 1$ (resp. $\alpha = 1/2$, $q = 0$), which is to be compared with $O(\varepsilon^{-10})$ (resp. $O(\varepsilon^{-8})$) in the case of the standard approach.

6.5 Examples and discussion of conditions

Suppose that the process $(X_t)_{t\in[0,T]}$ solves the SDE (1.1) in the one-dimensional case $d = m = 1$. Consider the Euler discretisation scheme, which is of the form

$$X_{\Delta,j\Delta} = X_{\Delta,(j-1)\Delta} + \mu(X_{\Delta,(j-1)\Delta})\,\Delta + \sigma(X_{\Delta,(j-1)\Delta})\,\xi_j\,\sqrt{\Delta}, \quad j = 1,\dots,J,$$

where $\xi_1,\dots,\xi_J$ are independent $N(0,1)$ random variables. In this case, we have Φ_Δ given by (3.5) and the orthonormal system $(\phi_k)_{k\in\mathbb{Z}_+}$ in $L^2(\mathbb{R}, N(0,1))$ can be chosen to be the system of normalised Hermite polynomials $\phi_k = H_k$. Then the coefficients $a_{j,k}$ are given by formula

$$a_{j,k}(x) = \frac{1}{\sqrt{k!}}\mathbb{E}\left[v_j\left(x + \mu(x)\,\Delta + \sigma(x)\,\xi\,\sqrt{\Delta}\right)H_k(\xi)\right] \tag{6.30}$$

with $\xi \sim N(0,1)$. To get more insight into the behaviour of $a_{j,k}$ in k, we need to know the structure of the approximations v_j. While we did not assume anything on their structure until now, in practice one often models v_j as linear combinations of some basis functions, e.g. polynomials (for instance, in the Longstaff-Schwartz algorithm with a polynomial basis). Let us now verify the assumption (6.21) in a couple of particular examples.

Example 6.8. Let

$$v_j(y) = \sum_{i=0}^{p}\alpha_{j,i}y^i, \quad j = 1,\dots,J$$

(think of polynomial basis functions). Since, with $\xi \sim N(0,1)$, $H_k(\xi)$ is orthogonal in L^2 to all polynomials in ξ of degree less than k, it follows from (6.30) that

$$a_{j,k} \equiv 0 \quad \text{whenever } k \geq p+1.$$

Then, for any $\beta > 0$, there is an appropriate constant $B_\beta > 0$ such that, for all $j = 1, \ldots, J$,

$$\sum_{k=1}^{\infty} k^\beta \, \mathbb{E}[a_{j,k}^2(X_{\Delta,(j-1)\Delta})] = \sum_{k=1}^{p} k^\beta \, \mathbb{E}[a_{j,k}^2(X_{\Delta,(j-1)\Delta})] \leq B_\beta.$$

(Notice that, since the coefficients μ and σ of the SDE are globally Lipschitz, all polynomial moments of the Euler discretisation are finite, hence all $\mathbb{E}\left[a_{j,k}^2(X_{\Delta,(j-1)\Delta})\right]$ are finite.) Thus, assumption (6.21) is satisfied and, moreover, we can take arbitrarily large $\beta > 0$ (at a cost of possibly getting large B_β).

Example 6.9. Let now

$$v_j(y) = \sum_{l=-p}^{p} \alpha_{j,l} \exp\{ihly\}, \quad j = 1, \ldots, J,$$

that is, at each time step $j = 1, \ldots, J$ our approximations v_j are trigonometric polynomials with period $2\pi/h$, for some given $h > 0$. With $\xi \sim N(0,1)$ we have

$$\begin{aligned} a_{j,k}(x) &= \frac{1}{\sqrt{k!}} \mathbb{E}\left[v_j\left(x + \mu(x)\Delta + \sigma(x)\sqrt{\Delta}\,\xi\right) H_k(\xi)\right] \\ &= \frac{1}{\sqrt{k!}} \sum_{l=-p}^{p} \alpha_{j,l} \exp\left\{ihl(x+\mu(x)\Delta)\right\} \mathbb{E}\left[\exp\left\{ihl\sigma(x)\sqrt{\Delta}\,\xi\right\} H_k(\xi)\right]. \end{aligned}$$

Using the definition of the Hermite polynomials and integrating by parts k times, we compute

$$\mathbb{E}\left[\exp\{ia\xi\} H_k(\xi)\right] = (ia)^k \exp\left\{-\frac{a^2}{2}\right\}.$$

Hence,

$$|a_{j,k}(x)| \leq \frac{h^k \Delta^{k/2}}{\sqrt{k!}} \sum_{l=-p}^{p} |\alpha_{j,l}| \, l^k \, |\sigma(x)|^k \exp\left\{-\frac{h^2 l^2 \sigma^2(x) \Delta}{2}\right\}.$$

Assuming for simplicity that σ is bounded, we get

$$|a_{j,k}(x)| \leq \sqrt{\frac{K_j \, C^k}{k!}}$$

with some positive constants K_j and C. Hence, for any $\beta > 0$ and for all $j = 1, \ldots, J$,

$$\sum_{k=1}^{\infty} k^\beta \, \mathbb{E}[a_{j,k}^2(X_{\Delta,(j-1)\Delta})] \leq \left[\max_{j=1,\ldots,J} K_j\right] \sum_{k=1}^{\infty} \frac{k^\beta \, C^k}{k!} =: B_\beta < \infty.$$

Thus, provided σ is bounded, for arbitrarily large $\beta > 0$, there exists an appropriate $B_\beta > 0$ such that assumption (6.21) is satisfied.

6.6 Numerical results

As can be easily seen, the optimal solution for the parameter N is of the same order (w.r.t. ε) both in the standard and in the regression-based approaches. Therefore, let us ignore the error term

$$\frac{4}{N}\sum_{l=1}^{J}\mathbb{E}\left[|v_l^*(X_{\Delta,l\Delta})-v_l(X_{\Delta,l\Delta})|^2\right] \tag{6.31}$$

in (6.23) and (6.28). Hence, we are interested in the remaining "variance terms"

$$\mathbb{E}[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta})|X_{\Delta,(l-1)\Delta}\right]] \tag{6.32}$$

and

$$\mathbb{E}\left[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta})-\tilde{M}_{l,K}|X_{\Delta,(l-1)\Delta}\right]\right], \tag{6.33}$$

for $l=1,\dots,J$, respectively. In terms of the numerical implementation, we will choose N large enough so that (6.31) does not really affect the overall error. That is, we now consider J and N as fixed parameters.

Standard approach with fixed J and N: We recall that the overall cost of computing the estimator V_{Δ,N,N_d} is of order JN_dN. Since we consider only the variance terms, we set $N_d \asymp \varepsilon^{-2}$ to ensure that (see (6.28))

$$\frac{4}{N_d}\left(1+\frac{1}{N}\right)\sum_{l=1}^{J}\mathbb{E}[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta})|X_{\Delta,(l-1)\Delta}\right]] \lesssim \varepsilon^2. \tag{6.34}$$

Thus, we have for the complexity

$$\mathcal{C}_{standard} \asymp JN_dN \asymp \varepsilon^{-2}. \tag{6.35}$$

Regression-based approach with fixed J and N: The overall cost of computing the estimator $\tilde{V}_{N_d,N,K}$ is of order

$$JK\max\left\{N_rQ^2, NN_d\right\}. \tag{6.36}$$

Notice that, since N is considered to be fixed, the term NQ (cf. (6.24)) is dominated by N_rQ^2. We have the constraints

$$\max\left\{\frac{JKQ\log(N_r)}{N_rN_d}, \frac{JB_\beta}{K^\beta N_d}, \frac{JD_\kappa K}{Q^\kappa N_d}\right\} \lesssim \varepsilon^2 \tag{6.37}$$

to ensure the condition

$$\frac{4}{N_d}\left(1+\frac{1}{N}\right)\sum_{l=1}^{J}\mathbb{E}\left[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta})-\tilde{M}_{l,K}|X_{\Delta,(l-1)\Delta}\right]\right] \lesssim \varepsilon^2. \tag{6.38}$$

Then, the resulting complexity bound is given by

$$\mathcal{C}_{regression} \lesssim C_{J,N,\beta,\kappa}\,\varepsilon^{-\frac{2(\beta+1)(\kappa+3)+2\kappa}{(\beta+1)(\kappa+3)+\beta\kappa}}\sqrt{|\log\varepsilon|}, \tag{6.39}$$

where $C_{J,N,\beta,\kappa} = cJ^{3/2}N^{1/2}B_\beta^{3/(1+\beta)}D_\kappa^{3/(3+\kappa)}$ with some universal constant c. Notice that the complexity in (6.39) is better than that in (6.35) whenever $\beta>1$. Moreover, we can achieve the complexity order $\varepsilon^{-1-\delta}$ in (6.39), for arbitrarily small $\delta>0$, whenever the parameters β and κ are sufficiently large.

Remark 6.10. Clearly, for every finite N, we also have the error term (6.31) in both approaches. However, since our control variates $\tilde{M}_{l,K}$ are aimed at reducing only the other error term (from (6.32) to (6.33)), it is useful in the numerical experiments to fix a sufficiently large N in order to make the error (6.31) very small. This allows us to focus on the reduction effect of the variance terms only (and thus mainly on the reduction effect of the bias, see Remark 6.6).

In constructing the numerical experiments below, for the regression-based approach, we need to choose several values of ε and the values of N_r, N_d, K and Q for each value of ε. To this end, we use the "limiting formulas" as $\beta, \kappa \to \infty$. Ignoring the remaining constants as well as the log-term for N_r, those "limiting formulas" give us $N_r = O(\varepsilon^{-1})$, $N_d = O(\varepsilon^{-1})$, $K = O(1)$ and $Q = O(1)$. In more detail, we choose the parameters for each $\varepsilon = 2^{-i}$, $i \in \{2, 3, 4, 5, 6\}$, as follows:

$$N = 5 \cdot 10^4, \quad K = 1, \quad Q = d + 2, \quad N_d = 8 \cdot \varepsilon^{-1}, \quad N_r = 256 \cdot \varepsilon^{-1}.$$

As for the basis functions, we use polynomials of d variables up to degree 1 as well as the function f (altogether $Q = d + 2$ basis functions). Regarding the standard approach, we choose for each $\varepsilon = 2^{-i}$, $i \in \{2, 3, 4, 5\}$, the parameters via

$$N = 5 \cdot 10^4, \quad N_d = 2 \cdot \varepsilon^{-2}.$$

Notice that we use less values for ε in case of the standard approach, since the computing time for $\varepsilon = 2^{-5}$ in the standard approach is already much higher than that in the regression-based approach for $\varepsilon = 2^{-6}$, with comparable values of the estimated root mean squared errors $\sqrt{\mathbb{E}\left[|V_{\Delta,N,N_d} - V_{\Delta,0}|^2\right]}$ and $\sqrt{\mathbb{E}\left[|\tilde{V}_{N,N_d,K} - V_{\Delta,0}|^2\right]}$. In addition, we implement the multilevel approach from [12] in the following way: set $L = -\log_2(\varepsilon) - 2$ for $\varepsilon = 2^{-i}$, $i \in \{2, 3, 4, 5\}$ and choose $(N_d)_l = 48 \cdot 4^l$ and $N_l = 2^{16-l}$ for $l = 0, \ldots, L$. Run the multilevel algorithm until the level L is reached. Thus, the cost is of order $\sum_{l=0}^{L} (N_d)_l N_l = O(2^L) = O(\varepsilon^{-1})$, similar to the one of the regression-based approach.

Below, we compute the numerical complexities, given 500 independent simulations, and compare it with the theoretical ones, namely, $O(\varepsilon^{-2})$ for the standard approach and $O(\varepsilon^{-1})$ for the multilevel and regression-based approaches ("limiting formulas" as $\beta, \kappa \to \infty$). Note that we compute the regression estimates for $v_j(x)$ by means of the algorithm of Tsitsiklis and Van Roy (see [58] and [59]), given $5 \cdot 10^4$ independent paths and $\frac{(d+1)(d+2)}{2} + 1$ basis functions (polynomials of d variables up to degree 2 as well as the function f) for all the standard, regression-based and multilevel approaches. Further, due to practical purposes, we do not allow to exercise at time $t = 0$, which gives us a modified price, namely

$$V_{\Delta,0} = \mathbb{E}\left[\max_{1 \le j \le J} \left(g_j(X_{\Delta,j\Delta}) - Y_{\Delta,j\Delta}\right)\right].$$

6.6.1 Two-dimensional example

We consider the following SDE for $d = m = 2$ ($Q = 4$)

$$dX_t^i = (r - \delta^i) X_t^i dt + \sigma^i X_t^i dW_t^i, \quad t \in [0, 1], \quad i = 1, 2,$$

where $r = 0$ and $x_0^i = 100$, $\sigma^i = 0.2$, $\delta^i = 0.02$, for $i = 1, 2$. Hence, X_t^1 and X_t^2 are two independent geometric Brownian motions. Further, we consider the Bermudan max-call option with strike price 100 and 20 exercise opportunities ($J = 20$), that is, $g_j(x) = \max\{\max\{x_1, x_2\} - 100, 0\}$, $x = (x_1, x_2)$, for all j. The "true" upper bound $V_{\Delta,0} \approx 12.57$ is estimated as the mean value of 100 independent computations of V_{Δ,N,N_d} with $N = N_d = 5 \cdot 10^4$.

As can be seen from the first plot in Figure 6.1, the estimated numerical complexity is about $\text{RMSE}^{-0.84}$ for the regression-based approach, $\text{RMSE}^{-1.31}$ for the standard approach and $\text{RMSE}^{-0.94}$ for the multilevel approach. (We speak about numerically estimated RMSEs here.) The reason for the somewhat unexpected slope 1.31 in the standard approach is that, in this numerical example, the numerical MSE turned out to be strictly smaller than the left-hand side of (6.34), which is of course possible in specific examples. (Indeed, from the plot corresponding to the standard approach we get $\text{RMSE} \asymp \varepsilon^{2/1.31}$, that is, $\text{MSE} \asymp \varepsilon^{4/1.31}$, which is smaller than $\text{const}/N_d \asymp \varepsilon^2$.) We see that the regression-based approach works nicely, and we can save much computing time as compared to the standard and multilevel approaches to obtain similar accuracies.

6.6.2 Five-dimensional example

We consider the following SDE for $d = m = 5$ ($Q = 7$)

$$dX_t^i = (r - \delta^i)X_t^i dt + \sigma^i X_t^i A^i dW_t, \quad t \in [0, 1], \quad i = 1, \ldots, 5,$$

where $r = 0$, $x_0^i = 100$, $\sigma^i = 0.2$, $\delta^i = 0.02$ $\forall i$, and $A^i := \left(A^{i,1} \cdots A^{i,5}\right)$, $AA^T = (\rho_{ik})_{i,k=1,\ldots,5}$ with $\rho_{ik} = \rho_{ki} \in [-1, 1]$ and $\rho_{ik} = 1$ for $i = k$ (that is, $A^i W$, $i = 1, \ldots, 5$, are correlated Brownian motions). For $i < k$ we choose

$$\rho_{ik} = \begin{cases} 0.9 & \text{if } i = 1,\ k = 2, \qquad -0.5 \quad \text{if } i = 3,\ k = 4, \\ 0.2 & \text{if } i \in \{1, 2, 3\},\ k = 5, \qquad -0.2 \quad \text{if } i = 4,\ k = 5, \\ 0 & \text{otherwise.} \end{cases}$$

Again, we consider the Bermudan max-call option with strike price 100, but with only 10 exercise opportunities ($J = 10$), that is, $g_j(x) = \max\left\{\max_{i \in \{1,\ldots,5\}} x_i - 100, 0\right\}$, for all j, and estimate the upper bound $V_{\Delta,0} \approx 21.07$ via 100 independent simulations of V_{Δ,N,N_d} with $N = N_d = 5 \cdot 10^4$.

Our empirical findings are illustrated in the second plot in Figure 6.1. We observe the numerical complexities of order $\text{RMSE}^{-0.76}$ for the regression-based approach, $\text{RMSE}^{-1.22}$ for the standard approach and $\text{RMSE}^{-0.79}$ for the multilevel approach. Even though the numerical complexities of the regression-based and multilevel approaches are close to each other, we observe that the computing time in case of the regression-based approach is much smaller than the multilevel one, whereas the RMSEs are in a similar region. As in the previous example, the regression-based approach shows a significant complexity reduction effect and outperforms the standard and multilevel approaches numerically.

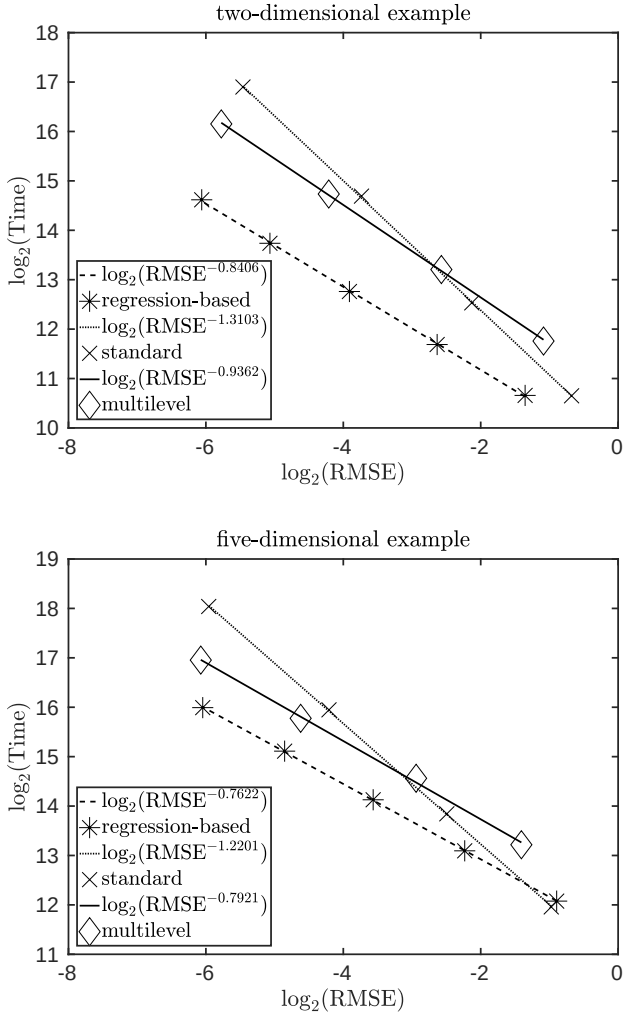

Figure 6.1. Numerical complexities of the regression-based, standard and multilevel approaches in the two- and five-dimensional case.

6.7 Proofs

Proof of Theorem 6.1

In what follows, conditioning on $X_{\cdot}^{(n)}$ is a shorthand for conditioning on $\sigma(X_j^{(n)}, 0 \leq j \leq J)$. We set

$$Y_j^{(n)} := \mathbb{E}\left[Y_{j,n,N_d} | X_{\cdot}^{(n)}\right]$$

and observe that

$$Y_j^{(n)} = \sum_{l=1}^{j} \left(v_l(X_l^{(n)}) - \mathbb{E}[v_l(X_l^{(n)}) | X_{l-1}^{(n)}] \right), \quad j = 0, \ldots, J,$$

in particular, the process $(Y_j^{(n)})$ has the same distribution as (Y_j). Further, we have

$$\begin{aligned}\mathbb{E}\left[V_{N,N_d}|X_\cdot^{(n)}\right] &= \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\max_{0\le j\le J}\left(g_j(X_j^{(n)}) - Y_{j,n,N_d}\right)|X_\cdot^{(n)}\right] \\ &\ge \frac{1}{N}\sum_{n=1}^{N}\max_{0\le j\le J}\mathbb{E}\left[g_j(X_j^{(n)}) - Y_{j,n,N_d}|X_\cdot^{(n)}\right] \\ &= \frac{1}{N}\sum_{n=1}^{N}\max_{0\le j\le J}\left(g_j(X_j^{(n)}) - Y_j^{(n)}\right),\end{aligned}$$

which implies the required inequality $\mathbb{E}V_{N,N_d} \ge V_0$ by taking expectations of both sides. We now introduce the filtrations $\mathcal{F}_j^{(n)} = \sigma(X_1^{(n)},\dots,X_j^{(n)})$ ($\mathcal{F}_0^{(n)} = \text{triv}$) and $\overline{\mathcal{F}}_j^{(n)} = \sigma(X_1^{(n)},\dots,X_j^{(n)},X_1^{(m,n)},\dots,X_j^{(m,n)}, m=1,\dots,M)$ ($\overline{\mathcal{F}}_0^{(n)} = \text{triv}$). Next we have

$$\begin{aligned}\mathbb{E}\left[(V_{N,N_d}-V_0)^2\right] &= \left(\mathbb{E}\left[V_{N,N_d}-V_0\right]\right)^2 + \operatorname{Var}\left[V_{N,N_d}\right] \\ &= \left(\mathbb{E}\left[V_{N,N_d} - \frac{1}{N}\sum_{n=1}^{N}\max_{0\le j\le J}\left(g_j(X_j^{(n)}) - Y_j^{(n)}\right)\right]\right)^2 + \operatorname{Var}\left[V_{N,N_d}\right].\end{aligned} \tag{6.40}$$

For the first term in (6.40), that is the squared bias, we obtain

$$\begin{aligned}&\left(\mathbb{E}\left[V_{N,N_d}-V_0\right]\right)^2 \\ \le&\mathbb{E}\left[\left(V_{N,N_d} - \frac{1}{N}\sum_{n=1}^{N}\max_{0\le j\le J}\left(g_j(X_j^{(n)}) - Y_j^{(n)}\right)\right)^2\right] \\ \le&\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(\max_{0\le j\le J}\left(g_j(X_j^{(n)}) - Y_{j,n,N_d}\right) - \max_{0\le j\le J}\left(g_j(X_j^{(n)}) - Y_j^{(n)}\right)\right)^2\right] \\ \le&\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\max_{0\le j\le J}\left[\left(Y_{j,n,N_d} - Y_j^{(n)}\right)^2\right],\end{aligned}$$

where we used $(\frac{1}{N}\sum_{n=1}^{N}a_n)^2 \le \frac{1}{N}\sum_{n=1}^{N}a_n^2$ in the first inequality and

$$|\max_j a_j - \max_j b_j| \le \max_j |a_j - b_j|$$

in the second one. Since $(Y_{j,n,N_d} - Y_j^{(n)})$ is an $(\overline{\mathcal{F}}_j^{(n)})$-martingale, Doob's L^2 inequality yields

$$\mathbb{E}\max_{0\le j\le J}[(Y_{j,n,N_d} - Y_j^{(n)})^2] \le 4\mathbb{E}[(Y_{J,n,N_d} - Y_J^{(n)})^2],$$

such that we get

$$\left(\mathbb{E}\left[V_{N,N_d}-V_0\right]\right)^2 \le \frac{4}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(Y_{J,n,N_d} - Y_J^{(n)}\right)^2\right].$$

Proceeding as follows

$$\begin{aligned}\mathbb{E}\left[\left(Y_{J,n,N_d} - Y_J^{(n)}\right)^2\right] &= \mathbb{E}\left[\mathrm{Var}\left[Y_{J,n,N_d}|X_\cdot^{(n)}\right]\right] \\ &= \mathbb{E}\left[\mathrm{Var}\left[\sum_{l=1}^{J}\frac{1}{N_d}\sum_{n_d=1}^{N_d} v_l(X_l^{(n_d,n)})\middle| X_\cdot^{(n)}\right]\right] \\ &= \mathbb{E}\left[\sum_{l=1}^{J}\frac{1}{N_d}\mathrm{Var}\left[v_l(X_l^{(n)})|X_{l-1}^{(n)}\right]\right] \\ &= \frac{1}{N_d}\sum_{l=1}^{J}\mathbb{E}\left[\mathrm{Var}\left[v_l(X_l)|X_{l-1}\right]\right],\end{aligned}$$

we obtain the upper bound

$$\left(\mathbb{E}\left[V_{N,N_d} - V_0\right]\right)^2 \leq \frac{4}{N_d}\sum_{l=1}^{J}\mathbb{E}\left[\mathrm{Var}\left[v_l(X_l)|X_{l-1}\right]\right]. \tag{6.41}$$

Next, due to the almost sure property of the Doob martingale

$$Y_j^{*,(1)} := \sum_{l=1}^{j}\left(v_l^*(X_l^{(1)}) - \mathbb{E}\left[v_l^*(X_l^{(1)})|X_{l-1}^{(1)}\right]\right),$$

we derive

$$\begin{aligned}\mathrm{Var}\left[V_{N,N_d}\right] &= \frac{1}{N}\mathrm{Var}\left[\max_{0\leq j\leq J}\left(g_j(X_j^{(1)}) - Y_{j,1,N_d}\right)\right] \\ &= \frac{1}{N}\mathrm{Var}\left[\max_{0\leq j\leq J}\left(g_j(X_j^{(1)}) - Y_{j,1,N_d}\right) - \max_{0\leq j\leq J}\left(g_j(X_j^{(1)}) - Y_j^{*,(1)}\right)\right] \\ &\leq \frac{1}{N}\mathbb{E}\max_{0\leq j\leq J}\left[\left(Y_j^{*,(1)} - Y_{j,1,N_d}\right)^2\right],\end{aligned}$$

for the second term in (6.40). Again using Doob's L^2 inequality together with the fact that martingale differences are uncorrelated, we get

$$\begin{aligned}&\mathrm{Var}\left[V_{N,N_d}\right] \leq \frac{4}{N}\mathbb{E}\left[\left(Y_J^{*,(1)} - Y_{J,1,N_d}\right)^2\right] = \frac{4}{N}\mathrm{Var}\left[Y_J^{*,(1)} - Y_{J,1,N_d}\right] \\ =&\frac{4}{N}\sum_{l=1}^{J}\mathrm{Var}\left[v_l^*(X_l^{(1)}) - v_l(X_l^{(1)}) - \mathbb{E}\left[v_l^*(X_l^{(1)})|X_{l-1}^{(1)}\right] + \frac{1}{N_d}\sum_{n_d=1}^{N_d} v_l(X_l^{(n_d,1)})\right] \\ =&\frac{4}{N}\sum_{l=1}^{J}\mathbb{E}\left[\mathrm{Var}\left[v_l^*(X_l^{(1)}) - v_l(X_l^{(1)}) + \frac{1}{N_d}\sum_{n_d=1}^{N_d} v_l(X_l^{(n_d,1)})|X_{l-1}^{(1)}\right]\right] \\ =&\frac{4}{N}\sum_{l=1}^{J}\left(\mathbb{E}\left[\mathrm{Var}\left[v_l^*(X_l^{(1)}) - v_l(X_l^{(1)})|X_{l-1}^{(1)}\right]\right] + \mathbb{E}\left[\mathrm{Var}\left[\frac{1}{N_d}\sum_{n_d=1}^{N_d} v_l(X_l^{(n_d,1)})|X_{l-1}^{(1)}\right]\right]\right) \\ =&\frac{4}{N}\sum_{l=1}^{J}\left(\mathbb{E}\left[\mathrm{Var}\left[v_l^*(X_l) - v_l(X_l)|X_{l-1}\right]\right] + \frac{1}{N_d}\mathbb{E}\left[\mathrm{Var}\left[v_l(X_l)|X_{l-1}\right]\right]\right).\end{aligned}$$

Finally, we use

$$\mathbb{E}\left[\mathrm{Var}\left[v_l^*(X_l) - v_l(X_l)|X_{l-1}\right]\right] \leq \mathbb{E}\left[\left(v_l^*(X_l) - v_l(X_l)\right)^2\right],$$

which gives us

$$\mathrm{Var}\left[V_{N,N_d}\right] \leq \frac{4}{N}\sum_{l=1}^{J}\left(\mathbb{E}\left[\left(v_l^*(X_l)-v_l(X_l)\right)^2\right]+\frac{1}{N_d}\mathbb{E}\left[\mathrm{Var}\left[v_l(X_l)|X_{l-1}\right]\right]\right). \tag{6.42}$$

Proof of Theorem 6.2

The expansion obviously holds for $p=1$ and $j=0$. Indeed, due to the orthonormality and completeness of the system (ϕ_k), we have

$$f(X_{\Delta,\Delta}) = \mathbb{E}\left[f(X_{\Delta,\Delta})\right] + \sum_{k\geq 1} a_{1,1,k}(x_0)\phi_k(\xi_1)$$

with

$$a_{1,1,k}(x_0) = \mathbb{E}\left[f(X_{\Delta,\Delta})\phi_k\left(\xi_1\right)\right],$$

provided $\mathbb{E}\left[\left|f(X_{\Delta,\Delta})\right|^2\right] < \infty$. Denote $\mathcal{G}_j = \sigma(\xi_1,\ldots,\xi_j)$, $j=1,2,\ldots$, and set $\mathcal{G}_0 = \mathrm{triv}$. Suppose that (6.7) holds for $p=q$ and all $j<q$. Let us prove it for $p=q+1$. Again due to the orthonormality and completeness of the system (ϕ_k), we get by conditioning on $\mathcal{G}_q$,

$$f(X_{\Delta,p\Delta}) = \mathbb{E}\left[f(X_{\Delta,p\Delta})|\, X_{\Delta,q\Delta}\right] + \sum_{k\geq 1} a_{p,q+1,k}(X_{\Delta,q\Delta})\phi_k(\xi_{q+1}), \tag{6.43}$$

where

$$a_{p,q+1,k}(x) = \mathbb{E}\left[f(X_{\Delta,p\Delta})\phi_k(\xi_{q+1})|\, X_{\Delta,q\Delta}=x\right],$$

which is the required statement in the case $j=q$. Notice that we used

$$\mathbb{E}\left[f(X_{\Delta,p\Delta})\phi_k(\xi_{q+1})|\,\mathcal{G}_q\right] = \mathbb{E}\left[f(X_{\Delta,p\Delta})\phi_k(\xi_{q+1})|\, X_{\Delta,q\Delta}\right], \quad k\in\mathbb{N}_0,$$

in (6.43), which can be shown analogously as in the proof of Theorem 4.11. Now the r.v. $\mathbb{E}\left[f(X_{\Delta,p\Delta})|\, X_{\Delta,q\Delta}\right]$ is $\mathcal{G}_q$-measurable and square integrable. Hence

$$\mathbb{E}\left[f(X_{\Delta,p\Delta})|\, X_{\Delta,q\Delta}\right] = \mathbb{E}\left[f(X_{\Delta,p\Delta})|\, X_{\Delta,j\Delta}\right] + \sum_{k\geq 1}\sum_{l=j+1}^{q} a_{p,l,k}(X_{\Delta,(l-1)\Delta})\phi_k(\xi_l)$$

for $j<q$ with

$$\begin{aligned} a_{p,l,k}(X_{\Delta,(l-1)\Delta}) &= \mathbb{E}\left[\mathbb{E}\left[f(X_{\Delta,p\Delta})|\,\mathcal{G}_q\right]\phi_k(\xi_l)|\,\mathcal{G}_{l-1}\right] = \mathbb{E}\left[f(X_{\Delta,p\Delta})\phi_k(\xi_l)|\,\mathcal{G}_{l-1}\right] \\ &= \mathbb{E}\left[f(X_{\Delta,p\Delta})\phi_k(\xi_l)|\, X_{\Delta,(l-1)\Delta}\right]. \end{aligned}$$

□

Proof of Theorem 6.3

It holds

$$\mathbb{E}\left[\mathrm{Var}\left[f(X_{\Delta,p\Delta})-\tilde{M}_{j,p,K}\middle|\, X_{\Delta,j\Delta}\right]\right] = \mathbb{E}\left[\left|M_{j,p}-\tilde{M}_{j,p,K}\right|^2\right].$$

We have

$$\begin{aligned}\mathbb{E}\left[\left|M_{j,p}-\tilde{M}_{j,p,K}\right|^2\right] &= \mathbb{E}\left[\left|\sum_{k=K+1}^{\infty}\sum_{l=j+1}^{p} a_{p,l,k}(X_{\Delta,(l-1)\Delta})\phi_k(\xi_l)\right|^2\right] \\ &\quad + \mathbb{E}\left[\left|\sum_{k=1}^{K}\sum_{l=j+1}^{p}\left(a_{p,l,k}(X_{\Delta,(l-1)\Delta})-\tilde{a}_{p,l,k}(X_{\Delta,(l-1)\Delta})\right)\phi_k(\xi_l)\right|^2\right] \\ &= \sum_{k=K+1}^{\infty}\sum_{l=j+1}^{p}\mathbb{E}\left[a_{p,l,k}^2(X_{\Delta,(l-1)\Delta})\right] \\ &\quad + \sum_{k=1}^{K}\sum_{l=j+1}^{p}\mathbb{E}\left[\left(a_{p,l,k}(X_{\Delta,(l-1)\Delta})-\tilde{a}_{p,l,k}(X_{\Delta,(l-1)\Delta})\right)^2\right].\end{aligned}$$

It follows from (2.4)

$$\begin{aligned}&\mathbb{E}\left[\left(a_{p,l,k}(X_{\Delta,(l-1)\Delta})-\tilde{a}_{p,l,k}(X_{\Delta,(l-1)\Delta})\right)^2\right] \\ \leq&\tilde{c}F^2\frac{Q(\log(N_r)+1)}{N_r}+8\inf_{\psi\in\mathrm{span}(\psi_1,\ldots,\psi_Q)}\mathbb{E}\left[\left|a_{p,l,k}(X_{\Delta,(l-1)\Delta})-\psi(X_{\Delta,(l-1)\Delta})\right|\right]^2\end{aligned}$$

for some universal constant $\tilde{c}$, since

$$\mathrm{Var}\left[f(X_{\Delta,p\Delta})\phi_k(\xi_l)\middle|X_{\Delta,(l-1)\Delta}=x\right]\leq F^2$$

and

$$\left|\mathbb{E}\left[f(X_{\Delta,p\Delta})\phi_k(\xi_l)\middle|X_{\Delta,(l-1)\Delta}=x\right]\right|\leq F. \qquad \square$$

Proof of Proposition 6.4

By the same calculation as the one leading to (6.5) (see the proof of Theorem 6.1), we get

$$\begin{aligned}\mathbb{E}\left[|\tilde{V}_{N,N_d,K}-V_{\Delta,0}|^2\right] \leq& \frac{4}{N_d}\left(1+\frac{1}{N}\right)\sum_{l=1}^{J}\mathbb{E}\left[\mathrm{Var}\left[v_l(X_{\Delta,l\Delta})-\tilde{M}_{l,K}|X_{\Delta,(l-1)\Delta}\right]\right] \\ &+\frac{4}{N}\sum_{l=1}^{J}\mathbb{E}\left[\left|v_l^*(X_{\Delta,l\Delta})-v_l(X_{\Delta,l\Delta})\right|^2\right]. \qquad (6.44)\end{aligned}$$

It remains to apply Theorem 6.3 to the first term on the right-hand side. $\square$

Chapter 7

Outlook

The approaches described in the previous chapters motivate the question if it is possible to obtain a further reduction of the complexity (in particular a better complexity order than $\varepsilon^{-1.75}$). In the following sections we specify some ideas how the algorithms for strong and weak schemes can be generalised or enhanced.

7.1 Scheme with Gaussian increments and weak convergence of second order

Consider a $\tilde{m}$-dimensional Brownian motion $(W_t)_{t\in[0,T]}$, where $\tilde{m} := \frac{m(m+1)}{2}$, and define $\Delta_j W = W_{j\Delta} - W_{(j-1)\Delta}$, $j = 1, \dots, J$. Further, define

$$\Delta_j W^y := \left(\Delta_j W^1, \dots, \Delta_j W^m\right)^\top \in \mathbb{R}^m,$$
$$\Delta_j W^z := \left(\Delta_j W^{m+1}, \dots, \Delta_j W^{\tilde{m}}\right)^\top \in \mathbb{R}^{\frac{m(m-1)}{2}},$$

as well as the $m \times m$-matrix $\tilde{W}_j = (\tilde{W}_j^{il})_{i,l=1,\dots,m}$ with entries $\tilde{W}_j^{ii} = -\sqrt{\Delta}$ for $i = 1, \dots, m$, $\tilde{W}_j^{12} = \Delta_j W^{m+1}, \tilde{W}_j^{13} = \Delta_j W^{m+2}, \dots, \tilde{W}_j^{(m-1)m} = \Delta_j W^{\tilde{m}}$ and $\tilde{W}_j^{li} = -\tilde{W}_j^{il}$ for $l > i$. Regarding the discretisation, we use the function $\Phi_\Delta(x, y, z)$ from (4.17), but in this case with Gaussian instead of discrete increments, that is

$$X_{\Delta,j\Delta} = \Phi_\Delta\left(X_{\Delta,(j-1)\Delta}, \frac{\Delta_j W^y}{\sqrt{\Delta}}, \frac{\tilde{W}_j}{\sqrt{\Delta}}\right). \tag{7.1}$$

Below we simply write

$$X_{\Delta,j\Delta} = \Phi_\Delta\left(X_{\Delta,(j-1)\Delta}, \frac{\Delta_j W^y}{\sqrt{\Delta}}, \frac{\Delta_j W^z}{\sqrt{\Delta}}\right) =: \Phi_\Delta^{(2)}\left(X_{\Delta,(j-1)\Delta}, \frac{\Delta_j W}{\sqrt{\Delta}}\right) \tag{7.2}$$

rather than (7.1), since we even know all entries of $\tilde{W}_j$, when $\Delta_j W^z$ is known. Our aim is to find out criteria, by means of an example, that justify a weak convergence of order 2 for this scheme, analogously to the second order weak scheme.

Example 7.1. Consider the following SDE for $d = 1$, $m = 2$ and $t \in [0, 1]$

$$dX_t = \sin(X_t)\, dW_t^1 + \cos(X_t)\, dW_t^2, \quad X_0 = 0.$$

Further, consider the function $f(x) = x^2$. Via Itô's formula, we can derive the expectation

$$\mathbb{E}\left[f(X_T)\right] = \int_0^T \mathbb{E}\left[\sin^2(X_t) + \cos^2(X_t)\right]\, dt = T = 1.$$

The function Φ_Δ in (7.2) has the following form in this example

$$\Phi_\Delta(x,y,z) = x + \sqrt{\Delta}(\sin(x)y_1 + \cos(x)y_2)\left(1 - \frac{\Delta}{4} + \frac{\sqrt{\Delta}}{2}(\cos(x)y_1 - \sin(x)y_2)\right) - \frac{\Delta z}{2}. \tag{7.3}$$

Hence, in contrast to previous examples with $m > 1$ (see e.g. Section 3.5.1), the variable $z \in \mathbb{R}^{\frac{m(m-1)}{2}}$ in $\Phi_\Delta(x,y,z)$ is present here. Notice that z is introduced for the second order weak scheme, since we need to approximate the integrals

$$Y_{\Delta,j}^{12} := \int_{t_{j-1}}^{t_j} W_s^1\, dW_s^2,$$
$$Y_{\Delta,j}^{21} := \int_{t_{j-1}}^{t_j} W_s^2\, dW_s^1,$$

which arise from the Itô-Taylor expansion (cf. [38]) of the Itô process

$$X_T = X_0 + \int_0^T \mu(X_t)\, dt + \int_0^T \sigma(X_t)\, dW_t = X_0 + \sum_{j=1}^{J}\left(\int_{t_{j-1}}^{t_j} \mu(X_t)\, dt + \int_{t_{j-1}}^{t_j} \sigma(X_t)\, dW_t\right).$$

In case of the second order weak scheme, the approximations for $Y_{\Delta,j}^{12}$ and $Y_{\Delta,j}^{21}$ are given through (cf. (4.14) and (4.17))

$$\tilde{Y}_{\Delta,j}^{12} := \frac{\Delta}{2}(\xi_j^1\xi_j^2 + V_j^{12}),$$
$$\tilde{Y}_{\Delta,j}^{21} := \frac{\Delta}{2}(\xi_j^1\xi_j^2 - V_j^{12}),$$

where ξ_j and V_j are defined in Section 4.2. Notice that it is not sufficient to approximative $Y_{\Delta,j}^{12}$ and $Y_{\Delta,j}^{21}$ by means of ξ_j^1, ξ_j^2 only (see e.g. the moment conditions below). Therefore, we need to include V_j as an additional random variable which is independent of ξ_j^1, ξ_j^2. Regarding our setting in this section, we have the approximations

$$\tilde{\tilde{Y}}_{\Delta,j}^{12} := \frac{1}{2}(\Delta_j W^1 \Delta_j W^2 + \Delta_j W^3\sqrt{\Delta}),$$
$$\tilde{\tilde{Y}}_{\Delta,j}^{21} := \frac{1}{2}(\Delta_j W^1 \Delta_j W^2 - \Delta_j W^3\sqrt{\Delta}).$$

One can easily show that the following relations hold for the first three moments

$$\mathbb{E}\left[Y_{\Delta,j}^{il}\right] = \mathbb{E}\left[\tilde{Y}_{\Delta,j}^{il}\right] = \mathbb{E}\left[\tilde{\tilde{Y}}_{\Delta,j}^{il}\right] = 0,$$
$$\mathbb{E}\left[\left(Y_{\Delta,j}^{il}\right)^2\right] = \mathbb{E}\left[\left(\tilde{Y}_{\Delta,j}^{il}\right)^2\right] = \mathbb{E}\left[\left(\tilde{\tilde{Y}}_{\Delta,j}^{il}\right)^2\right] = \frac{\Delta^2}{2},$$
$$\mathbb{E}\left[\left(Y_{\Delta,j}^{il}\right)^3\right] = \mathbb{E}\left[\left(\tilde{Y}_{\Delta,j}^{il}\right)^3\right] = \mathbb{E}\left[\left(\tilde{\tilde{Y}}_{\Delta,j}^{il}\right)^3\right] = 0,$$

where $i, l \in \{1, 2\}$, $i \neq l$. Moreover, it holds for the covariances

$$\operatorname{Cov}\left[Y^{12}_{\Delta,j}, Y^{21}_{\Delta,j}\right] = \operatorname{Cov}\left[\tilde{Y}^{12}_{\Delta,j}, \tilde{Y}^{21}_{\Delta,j}\right] = \operatorname{Cov}\left[\tilde{\tilde{Y}}^{12}_{\Delta,j}, \tilde{\tilde{Y}}^{21}_{\Delta,j}\right] = 0.$$

For the fourth moments, we have

$$\mathbb{E}\left[\left(Y^{il}_{\Delta,j}\right)^4\right] = \frac{7\Delta^4}{4} > \mathbb{E}\left[\left(\tilde{\tilde{Y}}^{il}_{\Delta,j}\right)^4\right] = \frac{9\Delta^4}{8} > \mathbb{E}\left[\left(\tilde{Y}^{il}_{\Delta,j}\right)^4\right] = \Delta^4.$$

That is, for both the approximation with Gaussian increments and the approximation with discrete random variables, we have a deviation to the "true" fourth moment. Since we cannot see a reason why it should only work for the second order weak scheme, we expect that the scheme described at the beginning of this section will give us a weak convergence of second order, too (provided that the functions f, μ, σ are smooth enough, cf. Proposition 4.12). Below we estimate the weak convergence order in this example numerically via simulation. For each $\Delta = 2^{-i}$ (and thus $J = 2^i$), $i = 0, 1, 2, 3$, simulate $N = 10^7$ independent paths via (7.2) and (7.3). Based on 500 independent repetitions of the SMC approach, compute the estimator for $\mathbb{E}\left[f(X_{\Delta,T})\right]$. Then we have an estimator for the bias, that is $\mathbb{E}\left[f(X_T) - f(X_{\Delta,T})\right]$, and get the estimated weak convergence order by regressing $\log|\text{Bias}|$ vs. $\log(\Delta)$. To illustrate the importance of including the random variables $\Delta_j W^3$, which we insert for the variable z in $\Phi_\Delta(x, y, z)$, we additionally run a similar simulation with a modified function

$$\tilde{\Phi}_\Delta(x, y) := x + \sqrt{\Delta}(\sin(x)y_1 + \cos(x)y_2)\left(1 - \frac{\Delta}{4}\right) + \frac{\Delta}{2}\sin(x)\cos(x)(y_1^2 - y_2^2).$$

Note that it holds (cf. (4.17))

$$\begin{aligned}\Phi_\Delta(x, y, z) - \tilde{\Phi}_\Delta(x, y) =& \frac{\Delta}{2}(\mathcal{L}^1\sigma_{12}(x)(y_1y_2 + z) + \mathcal{L}^2\sigma_{11}(x)(y_1y_2 - z))\\ =& \frac{\Delta}{2}\left(-\sin^2(x)(y_1y_2 + z) + \cos^2(x)(y_1y_2 - z)\right),\end{aligned}$$

where $\sigma_{11}(x) = \sin(x)$, $\sigma_{12}(x) = \cos(x)$ and the operators $\mathcal{L}^1, \mathcal{L}^2$ are defined in (4.16). Hence, in $\tilde{\Phi}_\Delta$ we ignore the terms of Φ_Δ, where the variable z and thus the random variable $\Delta_j W^3$ is used. As can be seen from Figure 7.1, we observe numerical weak convergence orders of 2.02 for the complete scheme with Φ_Δ (what we expected) and 1.06 for the incomplete scheme with $\tilde{\Phi}_\Delta$ (actually no better order compared to the Euler scheme (3.5)).

Next, we want to derive control variates based on the scheme (7.2). As in Chapter 3, we focus on series and integral approaches and assume that we actually achieve weak convergence of second order for this scheme.

7.1.1 Series approach

Since the results in Subsection 3.1.2 are not restricted to the Euler scheme and only require the structure (3.4), we can apply our above setting there. More precisely, we use the function $\Phi^{(2)}_\Delta(x, y)$, where $y \in \mathbb{R}^{\tilde{m}}$ (see (7.2)), to obtain

$$f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J}\sum_{k\in\mathbb{N}^{\tilde{m}}\setminus\{0_{\tilde{m}}\}} a_{j,k}(X_{\Delta,(j-1)\Delta})\prod_{i=1}^{\tilde{m}} H_{k_i}\left(\frac{\Delta_j W^r}{\sqrt{\Delta}}\right).$$

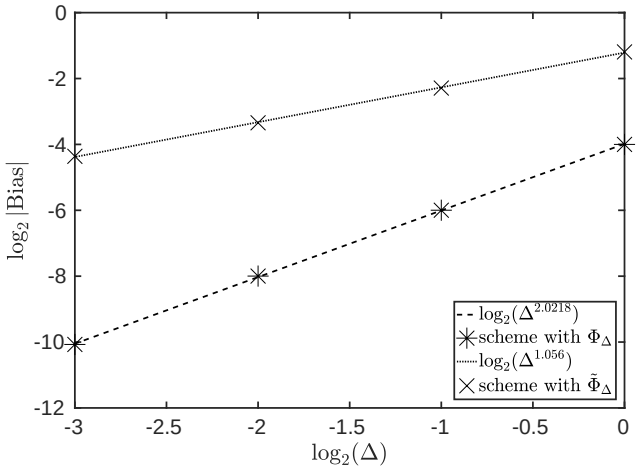

Figure 7.1. Comparison of schemes with Gaussian increments.

Hence, we apply the results in Subsection 3.1.2 for a $\tilde{m}$-dimensional Brownian motion rather than a m-dimensional one. Note that it will not be sufficient to derive a control variate which gives us a variance of order Δ as in (3.21). If this was the case, we would have got the condition under N testing paths $\frac{1}{NJ} \lesssim \varepsilon^2$ and thus for the cost $\mathcal{C} \gtrsim NJ \gtrsim \varepsilon^{-2}$ which is no improvement compared to the integral approach. Let us recall Remark 3.13. Similar to the control variates defined in (3.31) we assume to obtain under some assumptions on f, μ, σ for the variance

$$\operatorname{Var}\left[f(X_{\Delta,T}) - M_{\Delta,T}^{ser,(2)}\right] \lesssim \Delta^2, \tag{7.4}$$

where

$$M_{\Delta,T}^{ser,(2)} := \sum_{j=1}^{J} \sum_{l=1}^{2} \sum_{\substack{k \in \mathbb{N}_0^{\tilde{m}} \\ \sum_{i=1}^{\tilde{m}} k_i = l}} a_{j,k}(X_{\Delta,t_{j-1}}) \prod_{i=1}^{\tilde{m}} H_{k_i}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right). \tag{7.5}$$

Notice that the number of terms for computing the control variate $M_{\Delta,T}^{ser,(2)}$ is $O(J\tilde{m}^2) = O(Jm^4)$. For the series approach in Chapter 3 which is based on the Euler scheme, we have $O(Jm)$ terms.

When performing a complexity analysis, we get the same complexity as for the RCV approach (for second order weak schemes) in Chapter 4, that is

$$\mathcal{C} \asymp \varepsilon^{-\frac{11d\nu+2(p+1)(7\nu+8d)}{2d\nu+4(p+1)(2\nu+d)}}$$

in the case of piecewise polynomial regression. This is due to the fact that we have the same constraints as in (4.35) (apart from the additional one $\frac{1}{J^2N} \lesssim \varepsilon^2$, which comes from (7.4), and is the only inactive one, similar to the condition $\frac{1}{JN} \lesssim \varepsilon^2$ in Chapter 3, cf. Remark 3.21 on page 33). Hence, on the one hand we can achieve a better complexity order than the integral approach in Chapter 3, because we can go beyond the complexity order ε^{-2} for

sufficient large p, ν. On the other hand the achieved complexity is not better than for the RCV and RRCV approaches (for each of the three approaches we get a complexity of order $\varepsilon^{-1.75}$ when $p, \nu \to \infty$). Moreover, we will have the same numerical problems as for the RCV and original series approaches (cf. Subsection 3.4.3). Consequently, there is no reason why one should prefer this series approach based on the control variate $M_{\Delta,T}^{ser,(2)}$ to the numerically more stable RRCV approach (and also SRCV approach).

7.1.2 Integral approach

In this subsection we will not go so much into detail as in Section 3.1.3. Below we just give a description how one could theoretically achieve a complexity of order $\varepsilon^{-1.5-\delta}$ for arbitrary small $\delta > 0$. Consider again the scheme (7.2) with function $\Phi_\Delta^{(2)}(x, y)$. Let us now introduce the function $u_\Delta \colon [0, T] \times \mathbb{R}^{d+\tilde{m}} \to \mathbb{R}$ via

$$\begin{aligned} u_\Delta(t, x, y) &\equiv \mathbb{E}\left[u_\Delta\left(t_j, \Phi_\Delta^{(2)}\left(x, \frac{y + W_{t_j} - W_t}{\sqrt{\Delta}}\right), 0\right)\right], \quad t \in [t_{j-1}, t_j), \\ u_\Delta(T, x, 0) &\equiv f(x). \end{aligned}$$

Suppose that we can obtain a similar representation formula as in (3.17), that is

$$f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J}\sum_{l=1}^{\infty} \Delta^{l/2} \sum_{\substack{k \in \mathbb{N}_0^{\tilde{m}} \\ \sum_{i=1}^{\tilde{m}} k_i = l}} \frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_1^{k_1} \cdots \partial y_{\tilde{m}}^{k_{\tilde{m}}}} \prod_{i=1}^{\tilde{m}} \frac{H_{k_i}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right)}{\sqrt{k_i!}}$$

under some assumptions on f, μ, σ. Moreover, suppose that it holds for the variance

$$\mathrm{Var}\left[f(X_{\Delta,T}) - M_{\Delta,T}^{int,(2)}\right] \lesssim \Delta^2,$$

where the control variate

$$M_{\Delta,T}^{int,(2)} := \sum_{j=1}^{J}\sum_{l=1}^{2} \Delta^{l/2} \sum_{\substack{k \in \mathbb{N}_0^{\tilde{m}} \\ \sum_{i=1}^{\tilde{m}} k_i = l}} \frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_1^{k_1} \cdots \partial y_{\tilde{m}}^{k_{\tilde{m}}}} \prod_{i=1}^{\tilde{m}} \frac{H_{k_i}\left(\frac{\Delta_j W^i}{\sqrt{\Delta}}\right)}{\sqrt{k_i!}}$$

is under some conditions equivalent to $M_{\Delta,T}^{ser,(2)}$ in (7.5). In contrast to the series approach in Subsection 7.1.1 we cannot straightforwardly implement $M_{\Delta,T}^{int,(2)}$. While we have conditional expectation formulas for the functions $a_{j,k}$, we do not have for the derivatives $\frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_1^{k_1} \cdots \partial y_{\tilde{m}}^{k_{\tilde{m}}}}$. Note that the conditional expectation formulas for the integral approach in Chapter 3 are all based on the Euler scheme. Since we need to consider the second derivatives of u_Δ w.r.t. y, we also have to simulate the second derivative processes $\delta^2 X$ (cf. the one-dimensional case in Subsection 3.1.4) so that the dimension of the simulated processes is $d + d^2 + d^3$. In Chapter 3 we only have to simulate X and δX for the integral approach which leads to dimension $d + d^2$. It is natural to expect that the discretisation of δX has the form (cf. (3.24))

$$\delta^i X_{\Delta,t_j} = \sum_{k=1}^{d} \delta^i X_{\Delta,t_{j-1}}^k \frac{\partial}{\partial x_k} \Phi_\Delta^{(2)}\left(X_{\Delta,t_{j-1}}, \frac{\Delta_j W}{\sqrt{\Delta}}\right),$$

where $i \in \{1,\dots,d\}$, $j \in \{1,\dots,J\}$, $\delta^i X_{\Delta,t_j} \in \mathbb{R}^d$ and $\delta X_{\Delta,0} = I_d$. Regarding the one-dimensional case $d = m = 1$, we expect the following discretisation for $\delta^2 X$ (cf. (3.33))

$$\delta^2 X_{\Delta,t_j} = \delta^2 X_{\Delta,t_{j-1}} \frac{\partial}{\partial x} \Phi_\Delta^{(2)} \left(X_{\Delta,t_{j-1}}, \frac{\Delta_j W}{\sqrt{\Delta}} \right) + \delta X_{\Delta,t_{j-1}}^2 \frac{\partial^2}{\partial x^2} \Phi_\Delta^{(2)} \left(X_{\Delta,t_{j-1}}, \frac{\Delta_j W}{\sqrt{\Delta}} \right).$$

However, to implement $M_{\Delta,T}^{int,(2)}$, one still has to derive conditional expectation formulas for $\frac{\partial^l u_\Delta(t_{j-1},X_{\Delta,t_{j-1}},0)}{\partial y_1^{k_1} \cdots \partial y_{\tilde{m}}^{k_{\tilde{m}}}}$, where only the processes $X, \delta X, \delta^2 X$ are involved. Provided that those were available, we would have obtained the complexity of order

$$\mathcal{C} \asymp \varepsilon^{-\frac{9d\nu+2(p+1)(9d+6\nu)}{2d\nu+4(p+1)(d+2\nu)}} \sqrt{|\log(\varepsilon)|} \tag{7.6}$$

for piecewise polynomial regression. That is, for $p, \nu \to \infty$ the complexity tends to the order $\varepsilon^{-1.5}$ (the log-term is ignored). Similar to the integral approach in Chapter 3 we would have assumptions that the conditional variances and conditional expectations are bounded by some constants (cf. assumptions (A1)–(A2) in Subsection 3.4.1). Then we would get, due to

$$\mathbb{E}\left[\left(\Delta^{l/2} \prod_{i=1}^{\tilde{m}} H_{k_i} \left(\frac{\Delta_j W^i}{\sqrt{\Delta}} \right) \right)^2 \right] = \Delta^l,$$

where $\sum_{i=1}^{\tilde{m}} k_i = l \in \{1,2\}$, that the variable J, which comes from the number of terms for the control variate $M_{\Delta,T}^{int,(2)}$, is not present in the variance (cf. the proof of Theorem 3.19 on page 31)

$$\operatorname{Var}\left[M_{\Delta,T}^{int,(2)} - \tilde{M}_{\Delta,T}^{int,(2)} \right].$$

Here $\tilde{M}_{\Delta,T}^{int,(2)}$ denotes the estimated control variate. This would lead to the following constraints (cf. (3.44))

$$\max\left\{ \frac{1}{J^4}, \frac{1}{J^2 N}, \frac{S^d \log(N_r)}{N_r N}, \frac{1}{N} \left(\frac{R}{S} \right)^{2(p+1)}, \frac{1}{N R^\nu} \right\} \lesssim \varepsilon^2,$$

from which we would get the complexity order (7.6). (Note that the cost of the algorithm is still $O(JS^d \max\{N_r, N\})$.) In contrast, for the series approach in Subsection 7.1.1 we are not able to avoid J completely. This is due to the assumption that the conditional variance is bounded (cf. assumption (B1) in Subsection 3.4.2)

$$\sup_{x \in \mathbb{R}^d} \operatorname{Var}\left[f(X_{\Delta,T}) \prod_{i=1}^{\tilde{m}} H_{k_i} \left(\frac{\Delta_j W^i}{\sqrt{\Delta}} \right) \big| X_{\Delta,t_{j-1}} = x \right] \le \Sigma, \tag{7.7}$$

where $\Sigma > 0$, which gives us the constraints (cf. (3.54))

$$\max\left\{ \frac{1}{J^4}, \frac{1}{J^2 N}, \frac{J S^d}{N_r N}, \frac{J}{N} \left(\frac{R}{S} \right)^{2(p+1)}, \frac{1}{N R^\nu} \right\} \lesssim \varepsilon^2. \tag{7.8}$$

If the bound in (7.7) was of the form $\Sigma\Delta$, we would have obtained the same limiting complexity $\varepsilon^{-1.5}$ for $p, \nu \to \infty$. (Note that the factor J is only of importance in the third constraint in (7.8) for the limiting case $p, \nu \to \infty$.) However, an upper bound of order Δ is not realistic so that we can only assume the boundedness by a constant Σ. (Counterexample for smooth

and bounded functions: $d = m = 1$, $J = 1$, $\Delta = T$, $\mu(x) = 0$, $\sigma(x) = 1$, $f(x) = \cos(x)$, hence $X_T = X_{\Delta,T} = W_T$ and $\mathrm{Var}\left[f(X_{\Delta,T})\frac{W_T}{\sqrt{\Delta}}\right] = \mathbb{E}\left[\cos^2(W_T)\frac{W_T^2}{\Delta}\right] = \frac{1}{2}(1 + e^{-2\Delta}(1 - 4\Delta)) = O(1)$.) Indeed, we also have such a condition for the integral approach (similar to assumption (A1) in Subsection 3.4.1). But in this case we estimate other functions, namely $\frac{\partial^l u_\Delta(t_{j-1},x,0)}{\partial y_1^{k_1}\cdots\partial y_{\tilde{m}}^{k_{\tilde{m}}}}$ instead of $a_{j,k}(x)$. When comparing this functions, the factor $\Delta^{l/2}$ (see (3.18)) eliminates the factor J as mentioned above.

7.2 Extensions for weak schemes

7.2.1 Computational enhancements

A comparison of the control variates in Chapters 3–5 points out that the number of terms in the control variates for weak schemes is higher than for strong schemes. In particular, we have $O(Jm)$ terms for the series approach, which is related to the Euler scheme (cf. (3.10)), and $O(J(2^m - 1))$ terms for the weak Euler scheme (cf. (4.4)). This is due to the fact that we use an optimal control variate for the weak Euler scheme, resulting in zero variance (cf. (4.5)), while for the Euler scheme we truncate the series at some level, resulting in a variance of order Δ (cf. Theorem 3.9).

It is possible to derive truncated control variates with less terms for weak schemes, too. The idea is as follows: show that it holds for the weak Euler scheme, under some assumptions on the functions f, μ, σ,

$$\sup_{x\in\mathbb{R}^d} |a_{j,r,s}(x)| \lesssim \Delta, \quad r \geq 2, \tag{7.9}$$

where $a_{j,r,s}$ is defined in (4.3). We deduce (7.9) by considering higher order Taylor expansion for $q_j(\Phi_\Delta(x, y))$, when $r \geq 2$ (cf. proof of Theorem 4.18, where we used an expansion of zero order for the second order weak scheme). That is, we have for any $y \in \{-1, 1\}^m$

$$\begin{aligned} &q_j(\Phi_\Delta(x,y)) \qquad (7.10)\\ =&q_j(x+\mu(x)\Delta) + \sqrt{\Delta}\sum_{k=1}^{d}\frac{\partial}{\partial x_k}q_j(x+\mu(x)\Delta)\sum_{i=1}^{m}\sigma_{ki}(x)y_i\\ &+\Delta\sum_{k,l=1}^{d}(2-\delta_{k,l})\int_0^1(1-t)\frac{\partial^2}{\partial x_k\partial x_l}q_j(x+\mu(x)\Delta+t\sqrt{\Delta}\sigma(x)y)\,dt\sum_{i=1}^{m}\sigma_{ki}(x)y_i\sum_{i=1}^{m}\sigma_{li}(x)y_i, \end{aligned}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker delta. This gives us for $r \geq 2$ (cf. (4.8))

$$\begin{aligned} a_{j,r,s}(x) &= \frac{1}{2^m}\sum_{y\in\{-1,1\}^m} q_j(\Phi_\Delta(x,y))\prod_{o=1}^{r}y_{s_o}\\ &= \frac{\Delta}{2^m}\sum_{k,l=1}^{d}\left[(2-\delta_{k,l})\sum_{y\in\{-1,1\}^m}\left(\sum_{i=1}^{m}\sigma_{ki}(x)y_i\sum_{i=1}^{m}\sigma_{li}(x)y_i\prod_{o=1}^{r}y_{s_o}\right.\right.\\ &\qquad\left.\left.\cdot\int_0^1(1-t)\frac{\partial^2}{\partial x_k\partial x_l}q_j(x+\mu(x)\Delta+t\sqrt{\Delta}\sigma(x)y)\,dt\right)\right], \end{aligned}$$

due to (cf. (7.10))

$$\frac{1}{2^m}\sum_{y\in\{-1,1\}^m} y_i \prod_{o=1}^{r} y_{s_o} = \mathbb{E}\left[\xi_j^i \prod_{o=1}^{r} \xi_j^{s_o}\right] = 0 \tag{7.11}$$

for all $i \in \{1,\dots,m\}$. (Note that (7.11) does not hold for $r = 1$.) Applying Theorem 2.5, we obtain (7.9), provided that all functions σ_{ki}, $k \in \{1,\dots,d\}$, $i \in \{1,\dots,m\}$, are bounded and all functions f, μ_k, σ_{ki} are twice continuously differentiable with bounded partial derivatives up to order 2.

In Chapter 4 we derived upper bounds of order $\sqrt{\Delta}$ for all $r \in \{1,\dots,m\}$ (see assumption (A2) on page 63 for the second order weak scheme), which was sufficient to get a complexity reduction to order $\varepsilon^{-2.5}$ at best for the weak Euler scheme (see Subsection 4.4.3). However, when applying the following control variate with only $O(Jm)$ terms (cf. (4.4) and (3.10))

$$M_{\Delta,T}^{(1),trunc} := \sum_{j=1}^{J}\sum_{i=1}^{m} a_{j,1,e_i}(X_{\Delta,(j-1)\Delta})\xi_j^i, \tag{7.12}$$

where the superscript "trunc" comes from "truncated", we get (cf. (3.21))

$$\mathrm{Var}\left[f(X_{\Delta,T}) - M_{\Delta,T}^{(1),trunc}\right] = \mathrm{Var}\left[\sum_{j=1}^{J}\sum_{r=2}^{m}\sum_{1\le s_1<\ldots<s_r\le m} a_{j,r,s}(X_{\Delta,(j-1)\Delta})\prod_{i=1}^{r}\xi_j^{s_i}\right] \lesssim \Delta. \tag{7.13}$$

Note that the assumption (A2) on page 63 would have led to a variance of order 1 for such a control variate (i.e. the resulting complexity was worse then the one for SMC, namely $O(\varepsilon^{-3})$). As for the stricter condition (7.9) and thus a variance of order Δ in (7.13), we obtain the same complexity order in terms of ε as for the control variate (4.4) with $O(J(2^m-1))$ terms. That is, theoretically we have neither an improvement, nor a detriment. Nevertheless, from numerical point of view, we can save a lot of computing time for large m when using (7.12), without losing too much variance reduction effect. As a generalisation of (7.9), it is natural to expect that it holds, under additional smoothness conditions on f, μ, σ,

$$\sup_{x\in\mathbb{R}^d} |a_{j,r,s}(x)| \lesssim \Delta^{r/2}, \quad r \in \{1,\dots,m\}.$$

However, we would still obtain a variance of order Δ, as in (7.13).

As for the second order weak scheme, we expect to get

$$\sup_{x\in\mathbb{R}^d} |a_{j,o,U_1,U_2}(x)| \le \Delta^{|U_2|+|\mathcal{K}_2|+\frac{1}{2}|\mathcal{K}_1|}, \tag{7.14}$$

where a_{j,o,U_1,U_2} is defined in (4.19) and $\mathcal{K}_1 := \{r \in U_1 : o_r = 1\}$, $\mathcal{K}_2 := \{r \in U_1 : o_r = 2\}$. Here, a reasonable truncated control variate, including $O(Jm(m+1)) = O(Jm^2)$ terms, would have the form (cf. (4.20))

$$M_{\Delta,T}^{(2),trunc} := \sum_{j=1}^{J} \sum_{\substack{(U_1,U_2)\in\mathcal{A}\\ |U_2|+|\mathcal{K}_2|+\frac{1}{2}|\mathcal{K}_1|\le 1}} \sum_{o\in\{1,2\}^{U_1}} a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) \prod_{r\in U_1} H_{o_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl}, \tag{7.15}$$

such that

$$\begin{aligned}&\mathrm{Var}\left[f(X_{\Delta,T}) - M_{\Delta,T}^{(2),trunc}\right] \qquad (7.16)\\ &= \mathrm{Var}\left[\sum_{j=1}^{J} \sum_{\substack{(U_1,U_2)\in\mathcal{A}\\ |U_2|+|\mathcal{K}_2|+\frac{1}{2}|\mathcal{K}_1|>1}} \sum_{o\in\{1,2\}^{U_1}} a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) \prod_{r\in U_1} H_{o_r}(\xi_j^r) \prod_{(k,l)\in U_2} V_j^{kl}\right] \lesssim \Delta^2.\end{aligned}$$

Again, we expect to get a control variate, that is (7.15), which saves a lot of computing time without getting a worse complexity order. Hence, we can also achieve a complexity of order $\varepsilon^{-1.75}$ at best by using the control variate (7.15), provided that (7.14) holds. Let us note that a variance of order Δ instead of Δ^2 in (7.16) would result in a worse complexity than for the control variate (4.20) with $O(J(3^m 2^{\frac{m(m-1)}{2}} - 1))$ terms, namely worse than $O(\varepsilon^{-2})$. We also notice that a control variate similar to (7.15) with $O(Jm^2)$ terms should also result in the case of Gaussian increments, where we mentioned in (7.5) and (7.15) "more conservative" control variates with $O(Jm^4)$ terms, that is, under some less strict assumptions.

Let us remark that the above control variates $M_{\Delta,T}^{(1),trunc}$ (weak Euler scheme) and $M_{\Delta,T}^{(2),trunc}$ (second order weak scheme) will significantly reduce the computing time for the RCV and SRCV approaches, while the reduction is not so big for the RRCV approach (see the discussion in Subsection 5.3.2, where we intuitively also used a truncated control variate). Regarding the SRCV approach, the truncation leads to cost of order $JQ\max\left\{N_r Q, Nm^2\right\}$ (cf. Remark 5.3). Even though in the training phase there is another cost term of order $JQc_m m^2$, the factor c_m (2^m in case of the weak Euler scheme and $3^m 2^{\frac{m(m-1)}{2}}$ in case of the second order weak scheme) is no longer present in the highest order cost terms (in contrast to the truncated RRCV approach, where the cost is still of order $JQ\max\{N_r Q, Nc_m\}$). As a consequence, the EGCD problem, mentioned in Remark 5.10, cannot be avoided in the truncated RRCV approach (as for the RCV and SRCV approaches, it may be theoretically possible, see Section 7.4).

To illustrate the truncation effect on the truncated RCV approach, we recall the example in Subsection 4.5.2 with $d = m = 5$. Again we perform global regressions, but in this case we set $p = 2$, that is, we have $c_{p,d} + 1 = 22$ basis functions in each regression. Under the same ε-values as in Subsection 4.5.2 we choose (compare with the formulas in Subsection 4.4.1 for the "limiting" case $\nu \to \infty$)

$$J = \lceil \varepsilon^{-0.5} \rceil, \quad N_r = 128 \cdot \lceil \varepsilon^{-1.6176} \rceil, \quad N = 2048 \cdot \lceil \varepsilon^{-1.6176} \rceil.$$

The constants in N_r and N are chosen such that they nearly satisfy the relation $N_r(c_{p,d}+1) \approx N$. Similar to Subsection 4.5.2 we estimate the numerical complexity for the RCV approach (based on the "complete control variate" (4.20)) by means of 100 independent simulations and compare it with the ones of the SMC and MLMC approaches, for which we use the same output as before. In addition, we perform analogous simulations for the RCV approach based on the truncated control variate (7.15). Note that we again do not consider the random vectors $V_j \in \mathbb{R}^{\frac{m(m-1)}{2}}$ in both cases, since they do not affect the discretisation scheme (4.17). This gives us only $\frac{m(m+3)}{2} = 20$ terms in (7.15) compared to $(3^m - 1) = 242$

terms in (4.20). As can be seen from Figure 7.2, the estimated numerical complexity is about $\mathrm{RMSE}^{-1.91}$ for the RCV approach with complete control variate and $\mathrm{RMSE}^{-1.82}$ for the RCV approach with truncated control variate. Beyond the numerical complexities we observe that the truncation effect for the RCV approach is huge. While we have again poor results for the complete control variate (i.e. in this region of ε-values the RCV approach is numerically outperformed by the SMC and MLMC ones), the approach with truncated control variate works best (even better than the SMC and MLMC approaches).

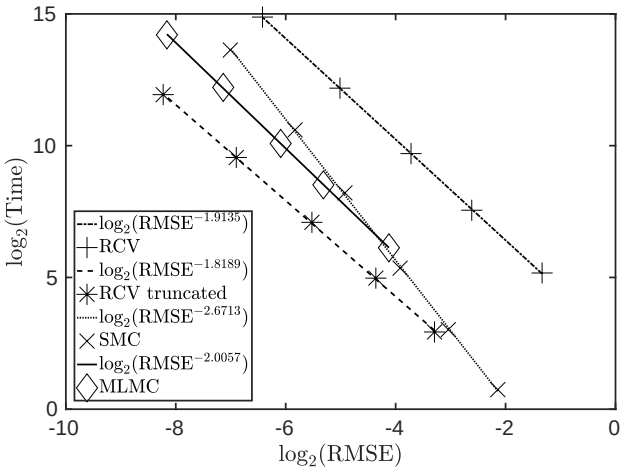

Figure 7.2. Numerical complexities of the RCV, SMC and MLMC approaches with and without truncation.

More details of the above truncation results can be found in [11].

7.2.2 Further complexity reduction for the RRCV approach

Regarding the complexity analysis of the RRCV approach for the second order weak scheme in Section 4.4.2, we used the inequality (4.39) to obtain an upper bound for $\mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})}$ by means of $\mathbb{E}\|\tilde{q}_j - q_j\|^2_{L^2(\mathbb{P}_{\Delta,j})}$. Note that we perform regressions for the estimation of functions q_j in case of the RRCV approach. By means of (4.39), we obtain (4.40), that is

$$\mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \leq \tilde{c}\left(\Sigma\Delta + A^2(\log N_r + 1)\right)\frac{\binom{p+d}{d}S^d}{N_r} + \frac{8\,C_h^2}{(p+1)!^2 d^{2-2/h}}\left(\frac{Rd}{S}\right)^{2p+2} + 8A^2B_\nu R^{-\nu}, \tag{7.17}$$

where $\tilde{c}$ is a universal constant. When comparing (7.17) with the upper bound for the RCV approach, namely (4.30), we observe that the first term, including the constant Σ, is of a better order in case of the RRCV approach (due to assumption (A1)) and the second term, including the constant A^2, is of a better order in case of the RCV approach (due to assumption (A2)). Let us for simplicity first assume that the set of chosen basis functions

$\psi_1, \dots, \psi_Q$ is optimal, such that

$$\inf_{g \in \Psi_Q} \|q_j - g\|^2_{L^2(\mathbb{P}_{\Delta,j})} = 0,$$

where $\Psi_Q = \operatorname{span}(\{\psi_1, \dots, \psi_Q\})$. Then the upper bound (7.17) simplifies to (cf. (2.5))

$$\mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \leq \tilde{c}\left(\Sigma\Delta + A^2(\log N_r + 1)\right) \frac{\binom{p+d}{d} S^d}{N_r} \tag{7.18}$$

for the RRCV approach. That is, even though we have optimal basis functions, the upper bound in (7.18) is $O(1)$.

Let us consider the Taylor expansion for the estimated function $\tilde{a}_{j,o,U_1,U_2}(x)$ in case of the RRCV approach. Similar to the expansion for a_{j,o,U_1,U_2} (see (4.55)), we get (by means of (4.22), with a being replaced by $\tilde{a}$ and q being replaced by $\tilde{q}$)

$$\begin{aligned} &\tilde{a}_{j,o,U_1,U_2}(x) \qquad (7.19)\\ &= \sum_{\substack{y \in \{-\sqrt{3},0,\sqrt{3}\}^m \\ z \in \{-1,1\}^{\frac{m(m-1)}{2}}}} p_m(y) g_{o,U_1,U_2}(y,z) \sum_{k=1}^{d} \tilde{\Phi}^k_\Delta(x,y,z) \int_0^1 \frac{\partial}{\partial x_k} \tilde{q}_j(\mu_\Delta(x) + t\tilde{\Phi}_\Delta(x,y,z))\, dt, \end{aligned}$$

where g_{o,U_1,U_2}, $p_m(y)$, $\tilde{\Phi}_\Delta(x,y,z)$ and μ_Δ are defined on page 75. As in the proof of Theorem 4.18, we get that the function $\tilde{\Phi}_\Delta$ is of order $\sqrt{\Delta}$, provided that all functions σ_{ki}, $k \in \{1, \dots, d\}$, $i \in \{1, \dots, m\}$, are bounded and twice continuously differentiable with bounded partial derivatives up to order 2, and provided that all functions μ_k, $k \in \{1, \dots, d\}$, are bounded and continuously differentiable with bounded partial derivatives. That is, under the additional assumption that all basis functions $\psi_1, \dots, \psi_Q$ (and thus $\tilde{q}_j$, too) are continuously differentiable with bounded partial derivatives, we obtain that $\tilde{a}_{j,o,U_1,U_2}$ is of order $\sqrt{\Delta}$ (similar to a_{j,o,U_1,U_2}). Note that the above condition on the basis functions is satisfied for the piecewise polynomial partitioning approach, since we perform regressions on compact sets, on which the polynomials are both smooth and bounded.

Apparently, the inequality (4.39) is too strict when deriving an upper bound for the RRCV approach. There is evidence that $\mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})}$ should be of order Δ, given perfect basis functions. Indeed, the estimated function $\tilde{q}_j$ is only $O(1)$, but due to the structure within $\tilde{a}_{j,o,U_1,U_2}$ (see again (4.22)), the $O(1)$ terms will be removed such that there remain only $O(\sqrt{\Delta})$ terms (under the above assumptions). Note that this argumentation does not necessarily apply to the SRCV approach, since the estimates of the functions $q_j(\Phi_\Delta(x,y,z))$ result from different paths for different y, z. Hence, it is not clear how far the $O(1)$ terms will be removed in this case. As for the RCV approach, it is reasonable that the upper bound in (4.30) is $O(1)$ (also under optimal basis functions), due to the variance assumption (A1) on page 63.

As an generalisation, we expect intuitively an upper bound of the form (cf. (7.17))

$$\begin{aligned} \mathbb{E}\|\tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} &\leq \tilde{c}\left(\Sigma\Delta + A^2\Delta(\log N_r + 1)\right) \frac{\binom{p+d}{d} S^d}{N_r} \qquad (7.20)\\ &\quad + \frac{8\, C_h^2}{(p+1)!^2 d^{2-2/h}} \left(\frac{Rd}{S}\right)^{2p+2} + 8A^2\Delta B_\nu R^{-\nu} \end{aligned}$$

for the RRCV approach, if the set of basis functions is not optimal.

Provided that (7.20) holds, we would derive a complexity of order (cf. (7.6))

$$\varepsilon^{-\frac{6d\nu+(p+1)(6\nu+9d)}{d\nu+2(p+1)(2\nu+d)}}\sqrt{|\log(\varepsilon)|}, \tag{7.21}$$

which tends to $\varepsilon^{-1.5}$ when $p,\nu\to\infty$ (the log-term is ignored). Compared to the complexity in (4.42) (derived under the inequality (4.39)), the one in (7.21) would give us indeed further complexity reduction.

As for the weak Euler scheme, we would obtain a complexity of order (cf. complexity (3.46) of the integral approach in Chapter 3)

$$\varepsilon^{-\frac{8d\nu+2(p+1)(4\nu+5d)}{d\nu+2(p+1)(2\nu+d)}}\sqrt{|\log(\varepsilon)|},$$

which tends to ε^{-2} when $p,\nu\to\infty$ and when we ignore the log-term again. Similar to the second order weak scheme, we have an improvement compared to the complexity under the inequality (4.39), which cannot go beyond $O(\varepsilon^{-2.5})$ (see (4.43)).

7.3 Nonhomogeneous stochastic differential equations

Consider the following (nonhomogeneous) Itô stochastic differential equation (cf. (1.1))

$$dX_t=\mu(t,X_t)\,dt+\sigma(t,X_t)\,dW_t,\quad X_0=x_0\in\mathbb{R}^d, \tag{7.22}$$

where the functions $\mu\colon\,[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma\colon\,[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times m}$ also depend on the variable $t\in[0,T]$. Note that the articles [45] and [49], mentioned in Chapter 1 concerning the (homogeneous) SDE (1.1), have already dealt with this subject. It turned out that the optimal control variate has a similar form as in (1.8), namely

$$M_T^*:=\int_0^T\nabla_x u(t,X_t)\,\sigma(t,X_t)\,dW_t, \tag{7.23}$$

where

$$\nabla_x u(t,x)=\mathbb{E}\left[\nabla f(X_T)\delta X_T\,|X_t=x\right]\delta X_t^{-1},$$

and the derivative processes $(\delta X_t^i)_{t\in[0,T]}\in\mathbb{R}^d$, $i\in\{1,\ldots,d\}$, come from the SDEs (cf. [45])

$$d\delta^i X_t=\sum_{k=1}^d\delta^i X_t^k\left[\frac{\partial\mu(t,X_t)}{\partial x_k}\,dt+\frac{\partial\sigma(t,X_t)}{\partial x_k}\,dW_t\right],\quad \delta^i X_0^k=\begin{cases}1, & i=k\\ 0, & i\neq k\end{cases}. \tag{7.24}$$

Below we describe which form the above mentioned discretisation schemes have in case of nonhomogeneous SDEs.

7.3.1 (Weak) Euler scheme

As in Chapter 3, we consider independent Gaussian increments $\Delta_j W^i$, $i=1,\ldots,m$, $j=1,\ldots,J$, for the Euler scheme such that

$$X_{\Delta,t_j}=\Phi_\Delta\left(t_{j-1},X_{\Delta,t_{j-1}},\frac{\Delta_j W}{\sqrt{\Delta}}\right),\quad j=1,\ldots,J. \tag{7.25}$$

The function $\Phi_\Delta\colon [0,T]\times\mathbb{R}^{d+m}\to\mathbb{R}^d$ is given as follows

$$\Phi_\Delta(t,x,y) = x + \mu(t,x)\Delta + \sigma(t,x)\sqrt{\Delta}y. \tag{7.26}$$

As for the integral approach, the discretisation scheme for the derivative processes is given through (cf. (3.24))

$$\delta^i X_{\Delta,t_j} = \delta^i X_{\Delta,t_{j-1}} + \sum_{k=1}^{d} \delta^i X^k_{\Delta,t_{j-1}} \left[\frac{\partial\mu(t_{j-1},X_{\Delta,t_{j-1}})}{\partial x_k}\Delta + \frac{\partial\sigma(t_{j-1},X_{\Delta,t_{j-1}})}{\partial x_k}\Delta_j W \right]. \tag{7.27}$$

Hence, the Euler scheme works similarly to the one for homogeneous SDEs. It is natural to expect representations of the same form as in (3.6) and (3.17), which result in control variates as in (3.10) (series approach) and (3.30) (integral approach), where $\sigma(X_{\Delta,t_{j-1}})$ is replaced by $\sigma(t_{j-1},X_{\Delta,t_{j-1}})$. Thus, under additional assumptions, i.e. smoothness on the functions μ,σ in the variable t, we should obtain similar complexity reduction effects.

In case of the weak Euler scheme, we have

$$X_{\Delta,t_j} = \Phi_\Delta(t_{j-1},X_{\Delta,t_{j-1}},\xi_j), \quad j=1,\dots,J, \tag{7.28}$$

with Φ_Δ given by (7.26) and the i.i.d. random vectors ξ_j follow the same distribution as in Section 4.1. Analogously, we expect to obtain a control variate as in (4.4).

7.3.2 Second order weak scheme

The weak scheme of second order has the following form

$$X_{\Delta,t_j} = \Phi_\Delta(t_{j-1},X_{\Delta,t_{j-1}},\xi_j,V_j), \quad j=1,\dots,J, \tag{7.29}$$

where the independent random variables ξ^i_j, $i\in\{1,\dots,m\}$, and V^{il}_j, $1\le i<l\le m$, follow the same distribution as in Subsection 4.2. Let us denote by $\Sigma\colon [0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ the following function (cf. (4.15))

$$\Sigma(t,x) = \sigma(t,x)\sigma(t,x)^\top,$$

which also depends on the time t in contrast to the homogeneous case. Moreover, we define (cf. (4.16) and [38, Section 5.3])

$$\mathcal{L}^0 g(t,x) := \frac{\partial}{\partial t}g(t,x) + \sum_{k=1}^{d}\mu_k(t,x)\frac{\partial g}{\partial x_k}(t,x) + \frac{1}{2}\sum_{k,l=1}^{d}\Sigma_{kl}(t,x)\frac{\partial^2 g}{\partial x_l\partial x_k}(t,x), \tag{7.30}$$

$$\mathcal{L}^r g(t,x) := \sum_{k=1}^{d}\sigma_{kr}(t,x)\frac{\partial g}{\partial x_k}(t,x), \quad r=1,\dots,m,$$

i.e. the operator $\mathcal{L}^0$ also contains the derivative w.r.t. the variable t. The r-th coordinate Φ^r_Δ, $r=1,\dots,d$, in the simplified order 2 weak Taylor scheme of is now given by the formula

(cf. (4.17))

$$\begin{aligned}\Phi_\Delta^r(t,x,y,z) = x_r &+ \sum_{k=1}^m \sigma_{rk}(t,x)\, y_k \sqrt{\Delta} \\ &+ \left[\mu_r(t,x) + \frac{1}{2}\sum_{k,l=1}^m \mathcal{L}^k \sigma_{rl}(t,x)(y_k y_l + z_{kl})\right]\Delta \\ &+ \frac{1}{2}\sum_{k=1}^m \left[\mathcal{L}^0\sigma_{rk}(t,x) + \mathcal{L}^k\mu_r(t,x)\right] y_k\, \Delta^{3/2} + \frac{1}{2}\mathcal{L}^0\mu_r(t,x)\,\Delta^2.\end{aligned} \tag{7.31}$$

Similar to the (weak) Euler scheme we expect to obtain, under additional assumptions on the functions f, μ and σ, a control variate of the form (4.20).

7.4 Open questions

Beyond the ideas mentioned above, there are some open questions concerning the previous chapters, which may be of interest for future research:

(i) Error propagation for the RRCV and SRCV approaches.
Strictly speaking, we did not entirely consider the implemented algorithms in case of the RRCV and SRCV approaches for the complexity analysis. Apart from the fact that we implemented a global regression instead of piecewise polynomial partitioning connected with truncation, the recursion and its error propagation is not present. That is, the estimation at time t_{j-1}, $j \in \{1,\dots,J\}$, depends on the regression estimates at time t_j. For instance, assumption (A1) for the RRCV approach, which is given as follows

$$\sup_{x\in\mathbb{R}^d} \mathrm{Var}\left[q_j(X_{\Delta,j\Delta}) \,|\, X_{\Delta,(j-1)\Delta} = x\right] = O(\Delta)$$

in Chapter 4, should have the form

$$\sup_{x\in\mathbb{R}^d} \mathrm{Var}\left[\tilde{q}_j(X_{\Delta,j\Delta}) \,|\, X_{\Delta,(j-1)\Delta} = x\right] = O(\Delta),$$

where $\tilde{q}_j$ is the estimate of the function q_j. Hence, it is an interesting goal to include the overall error propagation in the complexity analyses for the RRCV and SRCV approaches. Is it then possible to derive as good complexities as in Chapters 4 and 5?

(ii) EGCD problem.
Which algorithms will, in contrast to piecewise polynomial partitioning, satisfy assumptions (2.6) and (5.23), such that the integral, series, truncated RCV and truncated SRCV approaches do not suffer from the EGCD problem (similar to the SMC and MLMC approaches)?

(iii) More efficient algorithms for the pricing of Bermudan options.
In Chapter 6 we have implemented an analogue of the series approach in Chapter 3. Since regressions for approaches of that types (i.e. with factors of independent and zero-expectation random variables within the conditional expectation) may become instable, it is natural to

expect that there are algorithms which are numerically more stable and convincing for the dual nested Monte Carlo method, too (even though the results in Section 6.6 are satisfying).

(iv) Extension to further path dependent options.
To what extent can optimal control variates be derived for Barrier options (e.g. based on [20] and [22]), Asian options, etc.? For a detailed overview of financial derivatives, see e.g. [26], [31] and [60].

(v) Control variates for "tamed" Euler schemes.
Will our derived control variates also work for tamed Euler schemes (see [32] and [34]), which are useful in cases where the Euler scheme does not converge (cf. [33])?

(vi) When to use which algorithm?
In theory, the complexity analysis leads to a solution, where both parts of the MSE, namely the squared bias and the variance, are of the same order ε^2. Numerically, not only the order, but also the value itself, is of interest. That is, there are examples, where the MSE is dominated by the squared bias (and the variance is much smaller) and vice versa. In our experiments we observed (for all methods apart from the dual nested Monte Carlo one, where mainly the bias is reduced, see Remark 6.6) that our variance reduction approaches based on control variates (in particular the RRCV, SRCV and integral ones) work mostly fine compared to the SMC and MLMC approaches, when the MSE is dominated by the variance (that is, the variance is much bigger than the squared bias). In this case, the variance reduction leads to a significant reduction of the MSE. For the contrary situation, when the squared bias is much bigger, variance reduction is still nice, but does not really affect the overall error (MSE) and thus our approaches might perform less efficiently than the SMC and MLMC ones. Note that the bias cannot be estimated directly in general (as in the chosen examples in this work), since the "true" value $\mathbb{E}[f(X_T)]$ is not known (otherwise, we would not need to run simulations). Hence, it would be very interesting to find out criteria, in which situations it is advantageous to use our approaches and when should one rather prefer the MLMC or SMC approaches.

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