The Supersingular Locus of Unitary Shimura Varieties with Exotic Good Reduction

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Chapter 1

Introduction

We are interested in the geometry of the basic loci of Shimura varieties, which may have important applications in the Langlands program and Kudla program, for example, see the work of M. Harris & R. Taylor in [HT01] and S. Kudla & M. Rapoport in [KR11] & [KR14]. The basic locus of a Shimura variety is the unique closed and, in some sense, the most interesting Newton stratum. However, usually the geometric structure of the basic locus cannot be described explicitly, for example, we even do not know the dimension of the basic locus in the Siegel moduli spaces with Iwahori level structure in the odd case (cf. [GY12, Theorem 1.1]). Many mathematicians contributed to the general problem of giving a concrete description of basic loci of Shimura varieties. For the work in this area before 2005, we refer to the introduction of [Vol10]. Let us review the work after 2005.

- I. Vollaard & T. Wedhorn study the supersingular locus of the reduction of the Shimura variety for $\mathrm{GU}(1, n-1)$ at an inert prime p in [VW11].

- U. Görtz & C.-F. Yu study the supersingular locus of the Siegel modular varieties with Iwahori level structure $\mathcal{A}_{g,I}$ in [GY12]. They show that if g is even, the dimension of the supersingular locus is $g^2/2$. If g is odd, they give an estimate of the dimension of the supersingular locus. And in any case, the supersingular locus is not equidimensional if $g \geq 2$.

- M. Rapoport, U. Terstiege & S. Wilson study the supersingular locus of the Shimura variety for $\mathrm{GU}(1, n-1)$ over a ramified prime with the parahoric level structure given by a selfdual lattice in [RTW14].

- B. Howard & G. Pappas study the supersingular locus of the Shimura variety for $\mathrm{GU}(2, 2)$ at an inert prime in [HP14].

- U. Görtz & X. He in [GH15] claim that the supersingular locus of the Shimura variety for $\mathrm{GU}(2, 2)$ at a split prime can be written down similarly to [HP14].

In all the above cases except the Görtz-Yu case, the supersingular locus is a union of Ekedahl-Oort strata and admits a stratification by classical Deligne-Lusztig varieties, and the index set and the closure relations between strata can be described in terms of the Bruhat-Tits building of a certain inner form of the underlying group. Such Shimura varieties are called of Coxeter type in [GH15]. U. Görtz & X. He study the analogous problem in the equi-characteristic case, i.e. the basic affine Deligne-Lusztig varieties of Coxeter type. They give a (finite) complete list of ADLV of Coxeter type (cf. [GH15, Theorem 5.1.2]). In the mixed characteristic case, the affine Deligne-Lusztig "variety" is only a set. X. Zhu shows that the perfection of the special fiber of a Rapoport-Zink space (whose underlying group is unramified) is canonically isomorphic to an affine Deligne-Lusztig variety in his mixed affine Grassmannian in [Zhu, Proposition 0.4].

Recently, M. Chen & E. Viehmann claim that they can give a complete description of the Shimura variety for $\mathrm{GU}(2, n-2)$ at an inert prime in [CV].

This paper is a contribution to the program of giving a concrete description of the basic loci of the Shimura varieties of Coxeter type. We study the supersingular locus of the unitary Shimura varieties for $\mathrm{GU}(1, n-1)$ at a ramified prime with special parahoric level structure. More precisely, let E be an imaginary quadratic field extension of $\mathbb{Q}$ together with a ramified rational prime $p \geq 3$. Let (W, φ) be a hermitian space of signature $(1, n-1)$, $\mathbb{G}$ the corresponding unitary similitude group. Let C_p be the special parahoric subgroup corresponding to the 0-th vertex of the local Dynkin diagram (2.3.16) and (2.3.24), C^p a sufficiently small open compact subgroup of $\mathbb{G}(\mathbb{A}_f^p)$. Let $\mathcal{A}$ be the integral model of the Shimura variety $\mathrm{Sh}_{C^p}(\mathbb{G}, h)$, then $\mathcal{A}$ is smooth by [Arz09, Proposition 4.16]. The smoothness of $\mathcal{A}$ is unexpected because p is ramified, so we use the terminology "exotic good reduction".

The supersingular locus of the special fiber of $\mathcal{A}$ can be studied using Rapoport-Zink's p-adic uniformization theorem. Now let us consider the corresponding Rapoport-Zink spaces.

Let F be a ramified quadratic field extension of $\mathbb{Q}_p$, together with the unique non-trivial automorphism $^- \in \mathrm{Gal}(F/\mathbb{Q}_p)$ and the uniformizer π such that $\bar{\pi} = -\pi$. We denote L the completion of the maximal unramified field extension of $\mathbb{Q}$ and let $\breve{F} := F \otimes_{\mathbb{Q}_p} L$. Let $\mathbb{F}$ denote the algebraic closed field $\overline{\mathbb{F}}_p$.

For an $\mathbb{F}$-scheme S, a unitary p-divisible group of signature $(1, n-1)$ over S (cf. [RSZ, 3.1]) is a triple (X, ι_X, λ_X), where ι_X is an $\mathcal{O}_F$-action satisfying the Kottwitz condition, the Wedge condition and the extra Spin condition if n is even. The polarization λ_X satisfies the condition that the Rosati involution on $\mathrm{End}(X)$ attached to λ_X induces the non-trivial automorphism on $\mathcal{O}_F$ over $\mathbb{Q}_p$. Furthermore, the periodicity condition is assumed: if n is even, $\ker(\lambda_X) = X[\iota_X(\pi)]$; if n is odd, $\ker(\lambda_X) \subset X[\iota_X(\pi)]$ is of height $n-1$.

We fix a supersingular unitary p-divisible group $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of signature $(1, n-1)$ over $\mathbb{F}$, and consider the moduli functor $\mathcal{N}^e$:

$$\begin{aligned} (\mathbb{F}\text{-schemes}) &\longrightarrow (Sets), \\ S &\longmapsto \{(X, \iota_X, \lambda_X, \rho_X)/\cong\}, \end{aligned} \tag{1.0.1}$$

where (X, ι_X, λ_X) is a unitary p-divisible group and ρ_X is an $\mathcal{O}_F$-linear quasi-isogeny such that $\rho^*(\lambda_{\mathbb{X}})$ and λ_X differ locally on $\overline{S}$ by a scalar in $\mathbb{Q}_p^\times$. Then $\mathcal{N}^e$ is of relative dimension $n-1$ and has the same underlying topological space with the honest Rapoport-Zink space (cf. Proposition 3.2.2). Let $\mathcal{N}$ be the honest Rapoport-Zink space, then we have $\mathcal{N}_{\mathrm{red}} = \mathcal{N}^e_{\mathrm{red}}$.

Let $G = \mathrm{GU}(N, \varphi)$ be the unitary similitude group of signature $(1, n-1)$ where $(N, b\sigma)$ is the isocrystal given by the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and φ is the hermitian form corresponding to the polarization $\lambda_{\mathbb{X}}$. Let $K = \mathrm{Stab}(\mathbb{M})$ be the special parahoric subgroup corresponding to the 0-th vertex of the local Dynkin diagram of G (see (2.3.16) and (2.3.24)) and μ the geometric minuscule cocharacter $(1, 0^{n-1}; 1)$. Then, via Dieudonné theory, we have a bijection

$$\Phi\colon X(\mu, b)_K \longrightarrow \mathcal{N}^e(\mathbb{F}), \tag{1.0.2}$$

$$g \longmapsto g\mathbb{M}, \tag{1.0.3}$$

where $X(\mu, b)_K$ is a union of affine Deligne-Lusztig varieties. Then the map Φ induces a scheme structure on the left hand side. Let $X(\mu, b)'_K$ (resp. $\mathcal{S}$) be the connected component with trivial Kottwitz invariant of $X(\mu, b)_K$ (resp. $\mathcal{N}^e$). In [GH15] Görtz-He show that the affine Deligne-Lusztig variety is a disjoint union of fine affine Deligne-Lusztig varieties (aka. Ekedahl-Oort strata)

$$X(\mu, b)'_K = \biguplus_{w \in \mathrm{EO}_{\mathrm{cox}}} X^f_w(b), \tag{1.0.4}$$

and each Ekedahl-Oort stratum is a disjoint union of classical Deligne-Lusztig varieties

$$X^f_w(b) \cong \coprod_{j \in \mathbb{J}/\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma}} j \cdot Y_{\Sigma^\sharp}(w), \tag{1.0.5}$$

where

$$Y_{\Sigma^\sharp}(w) = \{g \in P_{\tilde{\mathbb{S}}-\Sigma}/P_{\Sigma^\sharp} : g^{-1} b_{\mathrm{ad}} \sigma(g) \in P_{\Sigma^\sharp} w P_{\Sigma^\sharp}\}. \tag{1.0.6}$$

For the framing object, we associate to it a hermitian space C. A lattice Λ in C is called a vertex lattice if $\Lambda \subset \Lambda^\sharp \subset \pi^{-1}\Lambda$, where $\Lambda^\sharp$ is the dual of Λ. The dimension of the $\mathbb{F}_p$-vector space $\Lambda/\pi\Lambda^\sharp$ is called the type of the lattice, denoted by $t(\Lambda)$. Let $\mathcal{B}$ be the set of vertex lattices. Via the crucial lemma (cf. Lemma 4.3.1), each basic EO element $w \in \mathrm{EO}_{\mathrm{cox}}$ is attached to a vertex lattice Λ. And we can show that the map Φ induces an isomorphism from the closure of the Deligne-Lusztig variety $Y_{\Sigma^\sharp(w)}$ to a closed subscheme $\mathcal{S}_\Lambda$ of $\mathcal{S}$.

Using these group-theoretic results and Smithling's result in [Smi15], via the map Φ, we have the main theorem.

Theorem 1.

1. *There is a stratification, which is called the Bruhat-Tits stratification, of $\mathcal{S}$ by locally closed subschemes*

$$\mathcal{S} = \biguplus_{\Lambda \in \mathcal{B}} \mathcal{S}^\circ_\Lambda, \tag{1.0.7}$$

and each stratum is isomorphic to the Deligne-Lusztig variety associated to the orthogonal group $\mathrm{SO}(\mathbb{B}_\Lambda)$ and a σ-Coxeter element. The closure of each stratum $\mathcal{S}^\circ_\Lambda$ in $\mathcal{S}$ is given by

$$\overline{\mathcal{S}^\circ_\Lambda} = \biguplus_{\Lambda' \subset \Lambda} \mathcal{S}^\circ_{\Lambda'} = \mathcal{S}_\Lambda. \tag{1.0.8}$$

2. *The scheme $\mathcal{S}$ is geometrically connected of pure dimension $[\frac{n-1}{2}]$. The irreducible components of $\mathcal{S}$ are those $\mathcal{S}_\Lambda$ with $t(\Lambda) = n$.*

Then, using the p-adic uniformization theorem, we have the description of the supersingular locus of $\mathcal{A} \otimes \mathbb{F}$.

Theorem 2. *The supersingular locus $\mathcal{A}_{\mathbb{F}}^{\mathrm{ss}}$ is of pure dimension $[\frac{n-1}{2}]$. We have natural bijections*

$$\{\textit{irreducible components of } \mathcal{A}_{\mathbb{F}}^{\mathrm{ss}}\} \xrightarrow{1:1} \mathbb{I}(\mathbb{Q})\backslash(J(\mathbb{Q}_p)/K_{\max} \times \mathbb{G}(\mathbb{A}_f^p)/C^p), \quad (1.0.9)$$

and

$$\{\textit{connected components of } \mathcal{A}_{\mathbb{F}}^{\mathrm{ss}}\} \xrightarrow{1:1} \mathbb{I}(\mathbb{Q})\backslash(J(\mathbb{Q}_p)/J^0 \times \mathbb{G}(\mathbb{A}_f^p)/C^p). \quad (1.0.10)$$

where J^0 is the subgroup of $J(\mathbb{Q}_p)$ consisting of those j such that $c(j)$ is a unit and $K_{\max}$ is the stabilizer of some maximal-type vertex lattice in $J(\mathbb{Q}_p)$.

This paper is structured as follows. In Chapter 2 we collect some group data from the literature. In Chapter 3 we establish the bijection between the Rapoport-Zink space and the affine Deligne-Lusztig variety. In Chapter 4 we describe the set-theoretic structure of the Rapoport-Zink space using Görtz-He's group-theoretic result. In Chapter 5 we establish the Bruhat-Tits stratification scheme-theoretically. In Chapter 6 using the p-adic uniformization theorem we describe the supersingular locus.

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Chapter 2

Preliminaries

2.1 Notations

We list some notations which would be used through the whole paper. Let p be an odd prime number, F a ramified quadratic field extension of $\mathbb{Q}_p$. We denote $\bar{}\in \mathrm{Gal}(F/\mathbb{Q}_p)$ the non-trivial automorphism. Let π be a uniformizer of F such that $\bar{\pi} = -\pi$ and $\pi^2 = \varpi$, where $\varpi = \epsilon p$ is a uniformizer of $\mathbb{Q}_p$ and ϵ is a unit in $\mathbb{Z}_p$. We denote L the completion of the maximal unramified field extension of $\mathbb{Q}_p$ and $\breve{F} = F \otimes_{\mathbb{Q}_p} L$. Let σ be the Frobenius automorphism of $L/\mathbb{Q}_p$. Let $\Gamma = \mathrm{Gal}(\overline{L}/L)$.

2.2 Hermitian forms over local fields

Let E/E_0 be a quadratic extension of local fields of mixed characteristic $(0, p)$, W an n-dimensional vector space over E together with a non-degenerate hermitian form

$$\varphi\colon W \times W \to E$$

with respect to the quadratic extension E/E_0, i.e. φ is E-linear in the first factor and *-linear in the second factor, where $^* \in \mathrm{Gal}(E/E_0)$ is the non-trivial automorphism. The pair (W, φ) is called a *hermitian space*.

Remark 2.2.1. To give a hermitian space (W, φ) of dimension n with respect to a ramified quadratic field extension E/E_0 is equivalent to give a vector space W of dimension $2n$ over E_0, together with a homomorphism $\iota\colon E \to \mathrm{End}(W)$ and an E_0-linear alternating non-degenerate form

$$\langle\, ,\, \rangle\colon W \times W \to E_0,$$

such that

$$\langle \iota(\gamma)x, y\rangle = \langle x, \iota(\gamma^*)y\rangle, \tag{2.2.1}$$

where γ is a uniformizer of E satisfying $\gamma^* = -\gamma$. To see this, given a hermitian form φ, define

$$\langle x, y\rangle := \frac{1}{2}\,\mathrm{Tr}_{E/E_0}(\gamma^{-1}\cdot \varphi(x, y)).$$

Conversely, given an alternating form $\langle\, ,\, \rangle$ on W satisfying condition (2.2.1), define a structure of E-vector space on W using ι and let

$$\varphi(x,y) \coloneqq \langle \iota(\gamma)x, y\rangle + \iota(\gamma)\langle x, y\rangle.$$

The isomorphism classes of hermitian forms can be determined by their discriminants in the group $E_0^\times / \mathrm{N}_{E/E_0} E^\times$.

Proposition 2.2.2 ([Jac62, Th 3.1])**.** *Let V and W be n-dimensional hermitian spaces over E, then V is isomorphic to W if and only if their discriminants coincide in $E_0^\times / \mathrm{N}_{E/E_0} E^\times$.*

Definition 2.2.3. A hermitian space (W,φ) is called *split* if it has trivial discriminant, i.e. the image of $(-1)^{n(n-1)/2} \det W$ in the group $E_0^\times / \mathrm{N}_{E/E_0} E^\times$ is trivial, otherwise (W,φ) is called *non-split*.

Remark 2.2.4. The local class field theory shows the group $E_0^\times / \mathrm{N}_{E/E_0} E^\times$ is of order 2. If the field extension E/E_0 is ramified, the group $E_0^\times / \mathrm{N}_{E/E_0} E^\times$ is generated by the units in $\mathcal{O}_{E_0}$. Therefore, in the ramified case, when n is odd, there is only one similarity class of hermitian forms; when n is even, there are two similarity classes of hermitian forms. Here the similar class of a hermitian form φ is the set of all $a \cdot \varphi$ for all $a \in E_0^\times$.

Remark 2.2.5. For an n-dimensional hermitian space (W,φ) with respect to E/E_0, we have the *Witt decomposition*:

$$W = H_1 \oplus \cdots \oplus H_q \oplus W_0, \tag{2.2.2}$$

where H_i is a hyperbolic plane for all i, W_0 is anisotropic of at most dimension 2 by [O'M00, 63:19]. When n is odd, W_0 is a line; when n is even, W is split if and only if $W_0 = 0$.

Proposition 2.2.6. *For an n-dimensional hermitian space (W,φ) with respect to E/E_0, let $\mathrm{SU}(W,\varphi)$ be the special unitary group over E_0. Then when n is odd, $\mathrm{SU}(W,\varphi)$ is always quasi-split; when n is even, $\mathrm{SU}(W,\varphi)$ is quasi-split if and only if the hermitian form φ is split.*

Proof. We have the Witt decomposition (2.2.2)

$$W = H_1 \oplus \cdots \oplus H_q \oplus W_0. \tag{2.2.3}$$

Let S be the maximal E_0-split torus with respect to the decomposition (2.2.3). Then, by definition, $\mathrm{SU}(W,\varphi)$ is quasi-split if and only if the centralizer of S is a maximal torus, which is equivalent to the condition that the E_0-rank of S is $[\frac{n}{2}]$, i.e. W_0 is a line when n is odd and $W_0 = 0$ when n is even. □

We are interested in lattices in hermitian spaces. A lattice M in (W,φ) is called *γ-modular* if $M^\vee = \gamma^{-1}M$, where $M^\vee$ is the dual lattice of M with respect to φ and γ is a uniformizer of E; M is called *nearly γ-modular*[1] if $M \subset M^\vee \overset{1}{\subset} \gamma^{-1}M$, where the symbol $\overset{k}{\subset}$ means that the quotient of the inclusion is of dimension k over the residue field of E.

[1] Here we adopt the terminology in [RSZ].

Lemma 2.2.7. *Let (W, φ) be an n-dimensional hermitian space with respect to the ramified field extension $\breve{F}/L$. Then when n is odd, φ is similar to a split hermitian form; when n is even, φ is split if and only if it contains a π-modular lattice.*

Proof. If n is odd, there exists $a \in L^\times$ such that $a\varphi$ has trivial discriminant because the group $L^\times / \mathrm{N}_{\breve{F}/L} \breve{F}^\times$ is generated by the units $\mathcal{O}_L^\times$. If $n = 2m$ is even and (W, φ) is split, by Remark 2.2.5, we can choose a basis $e_0, \dots, e_n$ such that $\varphi(e_i, e_j) = \delta_{i,n+1-j}$, then the lattice

$$\mathrm{Span}_{\mathcal{O}_L}\{e_1, \dots, e_m, \pi e_{m+1}, \dots, \pi e_n\}$$

is π-modular. If $n = 2m$ is even and W contains a π-modular lattice M, then [Jac62, Proposition 8.1 (b)] shows that there exists a basis $e_1, \dots, e_n$ of M such that $\varphi(e_i, e_j) = \pi\delta_{i,n+1-j}$, in particular, W is split. □

2.3 Group data

Let (V, ϕ) be an n-dimensional split hermitian space over F, $(e_1, \dots, e_n)$ a basis such that $\phi(e_i, e_j) = \delta_{i,n+1-j}$. Let $G = \mathrm{GU}(V, \phi)$ be the general unitary group defined over $\mathbb{Q}_p$, i.e. for each $\mathbb{Q}_p$-algebra R,

$$G(R) = \left\{ g \in \mathrm{GL}_{F\otimes_{\mathbb{Q}_p} R}(V \otimes_{\mathbb{Q}_p} R) \;\middle|\; \begin{array}{l} \phi(gv, gw) = c(g)\phi(v, w) \\ \text{for some } c(g) \in R^\times \\ \text{and for any } v, w \in V. \end{array} \right\}. \tag{2.3.1}$$

The algebraic group G is a reductive group over $\mathbb{Q}_p$, and its derived group $G_{\mathrm{der}} = \mathrm{SU}(V, \phi)$ is semisimple and simply connected. We have the following exact sequence of linear algebraic groups

$$1 \to G_{\mathrm{der}} \to G \to D \to 1,$$

where D is the torus G/G_{der}. We identify $\pi_1(G) = \pi_1(D) = X_*(D)$.

Let $S \subset G$ be the maximal L-split torus consisting of diagonal matrices defined over $\mathbb{Q}_p$, T its centralizer, N its normalizer. Then T is a maximal torus of G because G is quasi-split. Over $\breve{F}$, we have the following isomorphism:

$$\begin{aligned} G_{\breve{F}} &\simeq \mathrm{GL}_{n,\breve{F}} \times \mathbb{G}_{\mathrm{m},\breve{F}}, \\ g &\mapsto (g_0, c(g)), \end{aligned} \tag{2.3.2}$$

where g_0 is the image of g under the map $x \otimes y \mapsto xy$ on entries of matrices. More precisely, if we write $g = (g_{i,j}^{(0)} \otimes g_{i,j}^{(1)})_{i,j}$ with $g_{i,j}^{(0)} \in F$ and $g_{i,j}^{(1)} \in \breve{F}$, then $g_0 = (g_{i,j}^0 \cdot g_{i,j}^1)_{i,j}$. Then via the identification (2.3.2), the action of the non-trivial automorphism $^- = {}^- \otimes \mathrm{id}_L \in \mathrm{Gal}(\breve{F}/L)$ on RHS is given by the map $(g_0, c) \mapsto (\overline{c\phi^{-1}\, {}^t g_0{}^{-1} \phi}, \bar{c})$.

In this section, we collect some group data from [Tit79] [HR08] [PR08] [PR09] [Smi11] [Smi14] [Bou02].

2.3.1 Affine root systems and Iwahori-Weyl groups

First of all, we will compute the relative root system $(X^*, X_*, \Phi, \Phi^\vee)$ of G and its Iwahori-Weyl group.

(a) odd case.

We write $n = 2m+1$. Then

$$S(L) = \{\mathrm{diag}(s_1, \dots, s_n) : s_i \in L^\times \text{ and } s_1 s_n = \cdots = s_m s_{m+2} = s_{m+1}^2\},$$
$$T(L) = \{\mathrm{diag}(t_1, \dots, t_n) : t_i \in \breve{F}^\times \text{ and } t_1 \bar{t}_n = \cdots = t_m \bar{t}_{m+2} = t_{m+1}\bar{t}_{m+1}\}.$$

Under the identification (2.3.2), $X_*(T)$ can be identified with $\mathbb{Z}^n \times \mathbb{Z}$. The action of Γ on $X_*(T)$ factors through $\mathrm{Gal}(\breve{F}/L)$, the non-trivial automorphism $\bar{\ }$ acts on $X_*(T)$ by

$$(x_1, \dots, x_n; y) \mapsto (y - x_n, \dots, y - x_1; y). \tag{2.3.3}$$

So $X_*(S)$ can be identified with the subgroup of $\mathbb{Z}^n \times \mathbb{Z}$:

$$\{(x_1, \dots, x_n; y) \in \mathbb{Z}^n \times \mathbb{Z} : x_1 + x_n = \cdots = x_m + x_{m+2} = 2x_{m+1} = y\}. \tag{2.3.4}$$

And $X_*(T)_\Gamma$ is identified with $\mathbb{Z}^m \times \mathbb{Z}$ under the following map:

$$\begin{aligned} X_*(T) &\longrightarrow X_*(T)_\Gamma, \\ (x_1, \dots, x_n; y) &\longmapsto (x_1 - x_n, \dots, x_m - x_{m+2}, y). \end{aligned} \tag{2.3.5}$$

Let $X_* = X_*(T)_\Gamma \otimes \mathbb{R} = \mathbb{R}^m \times \mathbb{R}$, then we identify $X_*(S)$ with its image $2X_*(T)_\Gamma$ in X_* under the above map. It is easy to check that the following diagram is commutative

$$\begin{array}{ccc} T(L) & \xrightarrow{\quad\nu\quad} & X_* \\ & {\searrow}^{\kappa_T} \quad \nearrow & \\ & X_*(T)_\Gamma, & \end{array} \tag{2.3.6}$$

where ν is the (positive) *Tits' map* (cf. [Tit79, 1.2(1)] [Lan96, Lemma 1.1]) given by

$$\mathrm{diag}(t_1, \dots, t_n) \mapsto (\mathrm{val}(\frac{t_1}{t_n}), \dots, \mathrm{val}(\frac{t_m}{t_{m+2}}); \mathrm{val}(c(t))), \tag{2.3.7}$$

and κ_T is the *Kottwitz map* of T (cf. [Kot97, 7.2]).

Similarly, the non-trivial automorphism $\bar{\ }$ acts on $X^*(T) = \mathbb{Z}^n \times \mathbb{Z}$ by

$$(x_1, \dots, x_n; y) \mapsto (-x_n, \dots, -x_1; y + \sum_{i=1}^n x_i). \tag{2.3.8}$$

So we may identify $X^*(S)$ with $\mathbb{Z}^m \times \mathbb{Z}$ under the following map:

$$\begin{aligned} X^*(T) &\longrightarrow X^*(T)_{\mathrm{Gal}(\breve{F}/L)}/\text{torsion} = X^*(S), \\ (x_1, \dots, x_n; y) &\longmapsto (x_1 - x_n, \dots, x_m - x_{m+2}; y - \sum_{i=m+2}^n x_i). \end{aligned} \tag{2.3.9}$$

Let $X^* = X^*(S) \otimes \mathbb{R}$. Then the set of roots Φ is just the image of the absolute roots $\Phi(T, G)$, which is of type A_{n-1} by (2.3.2), under the natural map (2.3.9). Let $\epsilon_i \in X^*$ be the function on X_* given by

$$(x_1, \dots, x_m; y) \mapsto x_i$$

for any $i \in \{1, 2, \ldots, m\}$. Then

$$\Phi = \left\{ \begin{array}{ll} \pm\epsilon_i, & 1 \le i \le m, \\ \pm 2\epsilon_i, & 1 \le i \le m, \\ \pm\epsilon_i \pm \epsilon_j, & 1 \le i < j \le m \end{array} \right\}. \tag{2.3.10}$$

So Φ is non-reduced, it belongs to two reduced root systems:

$$\Phi_{B_m} = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j\}, \tag{2.3.11}$$
$$\Phi_{C_m} = \{\pm 2\epsilon_i, \pm\epsilon_i \pm \epsilon_j\}. \tag{2.3.12}$$

in the sense of [Bor66, 6.2], i.e. the root system Φ_{B_m} is obtained by removing the longer multiple of α for each $\alpha \in \Phi$, and Φ_{C_m} is obtained by removing the shorter ones.

Let's look at the set of affine roots Φ_a, by [PR09, Proposition 2.2],

$$\Phi_a = \left\{ \begin{array}{ll} \pm\epsilon_i + \frac{1}{2}\mathbb{Z}, & 1 \le i \le m, \\ \pm 2\epsilon_i + \frac{1}{2} + \mathbb{Z}, & 1 \le i \le m, \\ \pm\epsilon_i \pm \epsilon_j + \frac{1}{2}\mathbb{Z}, & 1 \le i < j \le m \end{array} \right\}. \tag{2.3.13}$$

So the affine hyperplanes associated to Φ_a can be viewed as the zero loci of the affine functions

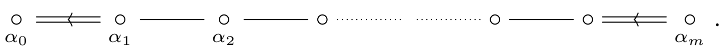
$$\left\{ \begin{array}{l} \pm 2\epsilon_i + \frac{1}{2}\mathbb{Z}, \\ \pm\epsilon_i \pm \epsilon_j + \frac{1}{2}\mathbb{Z}, \end{array} \right. \tag{2.3.14}$$

which can be viewed as an affine root system of type C_m. Let $W_0 = N(L)/T(L)$ be the Weyl group, which is isomorphic to $\mathfrak{S}_m \rtimes \{\pm 1\}^m$ in the spirit of (2.3.14). The affine Weyl group $W_a = X_*(T^{\mathrm{sc}})_\Gamma \rtimes W_0 \cong \mathbb{Z}^m \rtimes W_0$, where T^{sc} is $T \cap \mathrm{SU}(V, \phi)$, and the Iwahori-Weyl group $\tilde{W} = X_*(T)_\Gamma \rtimes W_0$ is isomorphic to $W_a \rtimes \pi_1(G)_\Gamma$, where $\pi_1(G)_\Gamma$ is isomorphic to $X_*(T)_\Gamma / X_*(T^{\mathrm{sc}})_\Gamma = \mathbb{Z}$.

Following [Tit79, 1.8], we choose a basis of Φ_a

$$\left\{ \begin{array}{ll} \alpha_i = \epsilon_{m+1-i} - \epsilon_{m-i}, & 1 \le i \le m-1, \\ \alpha_m = 2\epsilon_1, & \\ \alpha_0 = \frac{1}{2} - \epsilon_m, & \end{array} \right. \tag{2.3.15}$$

then we get the local Dynkin diagram of type C-BC_m.



(2.3.16)

Note that α_0 and α_m are special vertices.

(b) even case.

We write $n = 2m$. Following the same procedure as in the odd case, the root system $(X^*, X_*, \Phi, \Phi^\vee)$ can be computed similarly.

Similarly, $X_*(T) = \mathbb{Z}^n \times \mathbb{Z}$, so $X_*(T)_\Gamma$ can be identified with $\mathbb{Z}^m \times \mathbb{Z}$ under the following map

$$\begin{aligned} X_*(T) &\longrightarrow X_*(T)_\Gamma, \\ (x_1, \ldots, x_n; y) &\longmapsto (x_1 - x_n, \ldots, x_m - x_{m+1}, y). \end{aligned} \tag{2.3.17}$$

Then $X_*(S)$ consists of those $(x_1, \dots, x_n; y) \in X_*(T)$ satisfying $x_1 + x_n = \cdots = x_m + x_{m+1} = y$, we identify $X_*(S) \otimes \mathbb{R} \cong X_*(T)_\Gamma \otimes \mathbb{R} = X_*$. Furthermore, $X^*(S)$ can be identified with $\mathbb{Z}^m \times \mathbb{Z}$ under the following map

$$\begin{aligned} X^*(T) &\longrightarrow X^*(T)_\Gamma/\text{torsion} = X^*(S), \\ (x_1, \dots, x_n; y) &\longmapsto (x_1 - x_n, \dots, x_m - x_{m+1}; y + \sum_{i=m+1}^{n} x_i). \end{aligned} \tag{2.3.18}$$

So the relative roots are

$$\Phi = \left\{ \begin{array}{ll} \pm 2\epsilon_i, & 1 \leq i \leq m, \\ \pm\epsilon_i \pm \epsilon_j, & i \neq j \end{array} \right\}. \tag{2.3.19}$$

Then the affine roots are

$$\Phi_a = \left\{ \begin{array}{ll} \pm 2\epsilon_i + \mathbb{Z}, & 1 \leq i \leq m, \\ \pm\epsilon_i \pm \epsilon_j + \frac{1}{2}\mathbb{Z}, & i \neq j \end{array} \right\}. \tag{2.3.20}$$

The affine hyperplanes can be viewed as zero loci of the affine functions:

$$\left\{ \begin{array}{ll} \pm\epsilon_i + \frac{1}{2}\mathbb{Z}, & 1 \leq i \leq m, \\ \pm\epsilon_i \pm \epsilon_j + \frac{1}{2}\mathbb{Z}, & i \neq j, \end{array} \right. \tag{2.3.21}$$

which are, as affine root hyperplanes, of type B_m. So the Weyl group is $W_0 = \mathfrak{S}_m \rtimes \{\pm 1\}^m$, the Iwahori-Weyl group is $\tilde{W} = X_*(T)_\Gamma \rtimes W_0$, and the affine Weyl group is $W_a = X_*(T^{\text{sc}})_\Gamma \rtimes W_0$, where

$$X_*(T^{\text{sc}})_\Gamma = \{(x_1, \dots, x_m) \in \mathbb{Z}^m : \sum_{i=1}^{m} x_i \equiv 0 \mod 2\}. \tag{2.3.22}$$

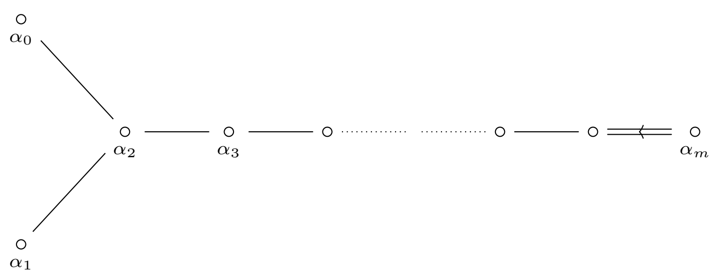
So $\pi_1(G)_\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$.

We choose a basis of Φ_a

$$\left\{ \begin{array}{ll} \alpha_i = \epsilon_i - \epsilon_{i+1}, & 1 \leq i \leq m-1, \\ \alpha_m = 2\epsilon_m, & \\ \alpha_0 = \epsilon_1 + \epsilon_2 - \frac{1}{2}, & \end{array} \right. \tag{2.3.23}$$

then we get the local Dynkin diagram of type B-C_m.

(2.3.24)



Note that α_0 and α_1 are special vertices.

2.3.2 Bruhat decomposition and μ-admissible set

Let $\mathfrak{B}$ be the Bruhat-Tits building of G_{ad}, then $G(L)$ acts on $\mathfrak{B}$ via the canonical map $G \to G_{\mathrm{ad}}$. Following [HR08], a subgroup $P \subset G(L)$ is called *parahoric* if

$$P = \mathrm{Stab}_{G(L)}(\mathfrak{F}) \cap \ker(\kappa_G),$$

where $\mathfrak{F}$ is a facet of $\mathfrak{B}$, and $\kappa_G \colon G(L) \to \pi_1(G)_\Gamma$ is the Kottwitz map. *Iwahori subgroups* of $G(L)$ are those parahoric subgroups associated to an alcove. We call the apartment associated with the torus S the *standard apartment.*

Proposition 2.3.1 ([HR08, Proposition 8]). *For any parahoric subgroups P and Q, whose corresponding facets are contained in the standard apartment, we have*

$$W_P\backslash\tilde{W}/W_Q \cong P\backslash G(L)/Q,$$

where W_P (resp. W_Q) is defined as $(N(L)\cap P)/T(L)_1$ (resp. $(N(L)\cap Q)/T(L)_1$), and $T(L)_1$ is the kernel of the Kottwitz map κ_T.

Let $\mu \in X_*(T)$ be a minuscule cocharacter, λ its image in $X_*(T)_\Gamma$, in [Rap05] the *admissible subset* of $\tilde{W}$ is defined as

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W} : w \leq t^{w_0(\lambda)} \text{ for some } w_0 \in W_0\}. \tag{2.3.25}$$

In the spirit of Proposition 2.3.1, we are interested in the image, denoted by $\mathrm{Adm}^0(\mu)$, of $W_0 \cdot \mathrm{Adm}(\mu)$ in $W_0\backslash\tilde{W}/W_0$. Note that all elements in $\mathrm{Adm}(\mu)$ have the same image in $\pi_1(G)_\Gamma$. Because once a special vertex is chosen, we may write $\tilde{W} = X_*(T)_\Gamma \rtimes W_0$, $\mathrm{Adm}^0(\mu)$ is completely determined by the dominance order on $X_*(T)_\Gamma$ induced by the Bruhat order on $\tilde{W}$.

From now on, $\mu = (1, (0)^{n-1}; 1) \in X_*(T) = \mathbb{Z}^n \times \mathbb{Z}$; for $s = 0, 1$, $\lambda_s = (1^s, 0^{m-s}; 1) \in X_*(T)_\Gamma = \mathbb{Z}^m \times \mathbb{Z}$ in both odd and even cases. Then, as in [PR09, 2.4.1 & 2.4.2],

$$\mathrm{Adm}^0(\mu) = \begin{cases} \{\lambda_1, \lambda_0\} & n \text{ odd}, \\ \{\lambda_1\} & n \text{ even}. \end{cases} \tag{2.3.26}$$

For convenience of computation, we choose representative(s) μ_1 (and μ_0 in the odd case) of $\mathrm{Adm}^0(\mu)$ in $T(L)$ under the Kottwitz map (2.3.6) as follows

$$\mu_1 = \begin{pmatrix} \pi^2 & & & & \\ & \pi & & & \\ & & \ddots & & \\ & & & \pi & \\ & & & & -1 \end{pmatrix} \tag{2.3.27}$$

in both odd and even case, and

$$\mu_0 = \begin{pmatrix} \pi & & & & \\ & \pi & & & \\ & & \ddots & & \\ & & & \pi & \\ & & & & \pi \end{pmatrix} \tag{2.3.28}$$

in odd case. Then, if K is the special parahoric subgroup of $G(L)$ corresponding to the 0-th vertex of the local Dynkin diagram (2.3.16) and (2.3.24),

$$\bigcup_{w\in \mathrm{Adm}(\mu)} KwK = \begin{cases} K\mu_1 K \cup K\mu_0 K & \text{odd case,} \\ K\mu_1 K & \text{even case.} \end{cases} \tag{2.3.29}$$

2.3.3 Lattice models for Bruhat-Tits buildings and parahoric subgroups

Recall that $G = \mathrm{GU}(V, \phi)$, now we describe parahoric subgroups of $G(L)$ in terms of lattices, following [PR08] [PR09].

For $i = 0, \dots, n-1$, let

$$\Lambda_i = \mathrm{span}_{\mathcal{O}_L}\{\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n\}. \tag{2.3.30}$$

More generally, for $j = kn + i$, $\Lambda_j := \pi^{-k}\Lambda_i$. Let $\mathcal{L}_I$ be the lattice chain $\{\Lambda_j : j \in n\mathbb{Z} \pm I\}$ for any non-empty subset $I \subset \{0, 1, \dots, m\}$. For simplicity, we write $\mathcal{L}_i := \mathcal{L}_{\{i\}}$. Note that for each minimal lattice chain $\mathcal{L}_i$, there exists a unique lattice $M \in \mathcal{L}_i$ such that $M \subset M^\vee \subset \pi^{-1}M$, such M is called the *standard representative* of $\mathcal{L}_i$ (see [AN02, 6.1]). It is easy to see that $\Lambda_i^\vee$ is the standard representative of $\mathcal{L}_i$.

Note that for $g \in \mathrm{GL}_{n,\breve{F}}$, $(g\Lambda_0)^\vee = {}^t\phi^{-1} \cdot {}^t\bar{g}^{-1} \cdot \Lambda_0$. So if $g \in G(L)$, for any lattice M, we have $(gM)^\vee = c(g)^{-1}gM^\vee$.

(a) odd case.

In this case, the Kottwitz map is given by

$$\begin{aligned} \kappa_G \colon G(L) &\to \pi_1(G)_\Gamma = \mathbb{Z}, \\ g &\mapsto \mathrm{val}(c(g)). \end{aligned} \tag{2.3.31}$$

Let I as before, P_I the stabilizer of $\mathcal{L}_I$, i.e.

$$P_I = \{g \in G(L) : gM = M \text{ for any } M \in \mathcal{L}_I\}. \tag{2.3.32}$$

To see an element $g \in P_I$ has trivial Kottwitz invariant, consider a lattice M which is the standard representative of some minimal sub-chain of $\mathcal{L}_I$, then

$$c(g)M^\vee = c(g)(gM)^\vee = gM^\vee = M^\vee. \tag{2.3.33}$$

Proposition 2.3.2 ([PR09, 1.2.3.(a)]). *The subgroup P_I is a parahoric subgroup of $G(L)$, and any parahoric subgroup is conjugate to P_I for some subset I. The sets $I = \{0\}$ and $I = \{m\}$ correspond to special maximal parahoric subgroups.*

Note that for maximal parahoric subgroups, we have

$$P_{\{i\}} = \mathrm{Stab}_{G(L)}(M \subset M^\vee \subset \pi^{-1}M), \tag{2.3.34}$$

where M is the standard representative of $\mathcal{L}_i$.

For the special unitary group $\mathrm{SU}(V, \phi)$, we have a similar result, i.e., $P_I \cap \mathrm{SU}(V, \phi)$ is a parahoric sub-group of $\mathrm{SU}(V, \phi)$, and any parahoric subgroup of $\mathrm{SU}(V, \phi)$ is conjugate to $P_I \cap \mathrm{SU}(V, \phi)$ for some subset I.

Remark 2.3.3. Note that the maximal parahoric subgroup $P_{\{i\}}$ for some $i \in \{0, 1, 2, \dots, m\}$ corresponds to the $(m-i)$-th vertex of the local Dynkin diagram (2.3.16), in particular the special parahoric subgroup $P_{\{m\}}$ corresponds to the 0-th vertex.

(b) even case.

In this case, the Kottwitz map is given by

$$\begin{aligned}\kappa_G \colon G(L) &\to \pi_1(G)_\Gamma = \mathbb{Z} \times \{\pm 1\}, \\ g &\mapsto (\mathrm{val}(c(g)), (-1)^{\mathrm{val}(b)}),\end{aligned} \tag{2.3.35}$$

where $b \in \breve{F}^\times$ such that $b/\bar{b} = \det(g) \cdot c(g)^{-m}$ by Hilbert's Satz 90.

To describe parahoric subgroups, we need to consider the lattice

$$\Lambda_{m'} \coloneqq \mathrm{span}_{\mathcal{O}_L}\{\pi^{-1}e_1, \dots, \pi^{-1}e_{m-1}, e_m, \pi^{-1}e_{m+1}, e_{m+2}, \dots, e_m\}. \tag{2.3.36}$$

For $J \subset \{0, 1, 2, \dots, m-2, m', m\}$, Q_J is defined as the stabilizer of the lattices $\{\Lambda_i\}_{i\in J}$, let $Q_J^0 \coloneqq Q_J \cap \ker(\kappa_G)$. Then Q_J^0 is a parahoric subgroup of $G(L)$, and each parahoric subgroup of $G(L)$ is conjugate to Q_J^0 for some J. However, usually such lattices $\{\Lambda_i\}_{i\in J}$ can not form a lattice chain. Note that $\Lambda_{m-1} = \Lambda_m \cap \Lambda_{m'}$. Let $I \subset \{0, 1, 2, \dots, m-2, m-1, m\}$, P_I is defined as the stabilizer of the lattice chain $\mathcal{L}_I$. If both m and m' lie in J, then $Q_J = P_I$, where $I \coloneqq (J - \{m'\}) \cup \{m-1\}$. If $m' \in J$ but $m' \notin J$, let τ be the unitary isomorphism exchanging e_m and e_{m+1}, but fixing all the other e_i's. It is easy to see that $\tau\Lambda_{m'} = \Lambda_m$ and we have $Q_J = {}^{\tau}P_I$, where I is obtained from J by replacing m' by m. Let $P_I^0 \coloneqq P_I \cap \ker \kappa_G$. In summary, we have the following result.

Proposition 2.3.4 ([PR09, 1.2.3(b)])**.** *The subgroup P_I^0 is a parahoric subgroup of $G(L)$, and any parahoric subgroup of $G(L)$ is conjugate to P_I^0 for a unique subset I with the property that if $m-1 \in I$, then $m \in I$. For such a subset I, $P_I^0 = P_I$ if and only if $m \in I$. The set $I = \{m\}$ corresponds to a special maximal parahoric subgroup.*

For the special unitary group $\mathrm{SU}(V, \phi)$, we have a similar description for parahoric subgroups, however, the Kottwitz invariant needs not to be considered because it is always trivial in the semisimple and simply connected case.

Remark 2.3.5. Note that the maximal parahoric subgroup $P_{\{i\}}^0$ for some $i \in \{0, 1, \dots, m\}$ corresponds to the $(m-i)$-th vertex of the local Dynkin diagram (2.3.24), in particular the special parahoric subgroup $P_{\{m\}}$ corresponds to the 0-th vertex.

Chapter 3

Rapoport-Zink spaces and affine Deligne-Lusztig varieties

3.1 Unitary p-divisible groups

Let $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$ be the category of $\mathcal{O}_{\breve{F}}$-schemes S such that π is locally nilpotent on S. For $S \in \mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$, a *unitary p-divisible group of signature* $(1, n-1)$ over S, following [RSZ, 3.1], consists of the following data:

1. a p-divisible group X over S,
2. an $\mathcal{O}_F$-action on X
$$\iota_X : \mathcal{O}_F \to \mathrm{End}_S(X),$$
3. a polarization $\lambda_X : X \to X^\vee$ such that the Rosati involution on $\mathrm{End}_S(X)$ attached to λ_X induces the non-trivial automorphism on $\mathcal{O}_F$ over $\mathbb{Q}_p$,

satisfying the following conditions:

1. *Kottwitz condition*:
$$\mathrm{charpol}(\iota_X(\pi)|\,\mathrm{Lie}(X)) = (T-\pi)(T+\pi)^{n-1} \in \mathcal{O}_S[T], \tag{3.1.1}$$
2. *Wedge condition*:
$$\bigwedge^n (\iota(\pi) - \pi|\,\mathrm{Lie}(X)) = 0, \tag{3.1.2}$$
$$\bigwedge^2 (\iota(\pi) + \pi|\,\mathrm{Lie}(X)) = 0 \text{ if } n \geq 3, \tag{3.1.3}$$
3. when n is even, the extra *Spin condition* is assumed: $\iota_X(\pi)|\,\mathrm{Lie}(X_s)$ non-vanishing for any $s \in S$,
4. *Periodicity condition*: if n is even, $\ker(\lambda_X) = X[\iota_X(\pi)]$; if n is odd, $\ker(\lambda_X) \subset X[\iota_X(\pi)]$ is of height $n-1$.

Remark 3.1.1. Our definition of unitary p-divisible groups is slightly different from the one in [RSZ, 3.1]. In our context, the Spin condition is not assumed in the odd case, because later we will see that the corresponding Rapoport-Zink space is has the same underlying topological space with the honest Rapoport-Zink space (see Proposition 3.2.2) which is enough for our purposes because in the p-adic uniformization theorem (see Theorem 6.2.1), the underlying reduced scheme structure is required.

3.2 Moduli space of p-divisible groups

From now on, the sign $\mathbb{F}$ denotes the algebraic closure of $\mathbb{F}_p$. To define the *Rapoport-Zink space*, we fix a supersingular unitary p-divisible group $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of signature $(1, n-1)$ over $\mathbb{F}$ as the framing object henceforth. Note that [RSZ, Proposition 3.1] shows that such a framing object exists and is unique up to a quasi-isogeny.

There are two ways to define the *honest Rapoport-Zink space*. Let $\mathcal{M}^{\mathrm{naive}}$ be the *naive Rapoport-Zink space*, i.e. the formal scheme representing the functor

$$\begin{aligned} \mathrm{Nilp}_{\mathcal{O}_{\breve{F}}} &\longrightarrow \mathrm{Sets}, \\ S &\longmapsto (X, \iota_X, \lambda_X, \rho_X)/\cong, \end{aligned} \tag{3.2.1}$$

where

- (X, ι_X, λ_X) is a *naive unitary p-divisible group over S*, i.e. defined in the same way as a unitary p-divisible group but without the wedge condition and the extra spin condition;
- $\rho_X \colon X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}} \overline{S}$ is an $\mathcal{O}_F$-linear quasi-isogeny (of any height) such that $\rho^*(\lambda_{\mathbb{X}})$ and λ_X differ locally on $\overline{S}$ by a scalar in $\mathbb{Q}_p^\times$.

Two quadruples $(X, \iota_X, \lambda_X, \rho_X)$ and $(Y, \iota_Y, \lambda_Y, \rho_Y)$ are isomorphic if there exists an $\mathcal{O}_F$-linear isomorphism of p-divisible groups $\alpha\colon X \to Y$ such that the diagram is commutative

$$\begin{array}{ccc} X \times_S \overline{S} & & \\ {\scriptstyle\alpha}\downarrow & \overset{\rho_X}{\searrow} & \\ Y \times_S \overline{S} & \xrightarrow{\rho_Y} & \mathbb{X} \times_{\mathbb{F}} \overline{S} \end{array}$$

and $\alpha^*(\lambda_Y)$ and λ_X differ locally on $\overline{S}$ by a scalar in $\mathbb{Q}_p^\times$.

The formal scheme $\mathcal{M}^{\mathrm{naive}}$ is formally locally of finite type over $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ (cf. [RZ96, Theorem 3.25]). Unfortunately, $\mathcal{M}^{\mathrm{naive}}$ is not flat over $\mathcal{O}_{\breve{F}}$ (cf. [Pap00, Proposition 3.8]) because dimensions of the special and generic fiber of $\mathcal{M}^{\mathrm{naive}}$ may differ. Let $\mathcal{M}$ be the flat closure in $\mathcal{M}^{\mathrm{naive}}$ of its generic fiber. Then $\mathcal{M}$ is called the *honest Rapoport-Zink space*. However, it's not clear whether $\mathcal{M}$ has a moduli description. Another way to define Rapoport-Zink spaces is to add some extra conditions on the p-divisible groups and get a moduli space of p-divisible groups with extra conditions.

Now we associate to $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ a set-valued functor $\mathcal{M}^e$ on the category $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$. The superscript e stands for "exotic".

Definition 3.2.1. For any $S \in \mathrm{Nilp}_{\mathbb{F}}$, $\mathcal{M}^e(S)$ is the set of isomorphism classes of $(X, \iota_X, \lambda_X, \rho_X)$, where

- (X, ι_X, λ_X) is a unitary p-divisible group of signature $(1, n-1)$ over S;
- $\rho_X : X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}} \overline{S}$ is an $\mathcal{O}_F$-linear quasi-isogeny (of any height) such that $\rho^*(\lambda_{\mathbb{X}})$ and λ_X differ locally on $\overline{S}$ by a scalar in $\mathbb{Q}_p^\times$.

Proposition 3.2.2 (Smithling)**.** *The functor $\mathcal{M}^e$ is represented by a separated formal scheme over $\mathrm{Spf}(\mathcal{O}_{\breve{F}})$, which is locally formally of finite type, of relative formal dimension $n-1$ over $\mathcal{O}_{\breve{F}}$ and has the same underlying topological space with $\mathcal{M}$. Furthermore, if n is even, $\mathcal{M}^e$ is flat over $\mathcal{O}_{\breve{F}}$.*

Let us explain the standard procedure to reduce the proof of the proposition to a result on local models (which is established by Smithling in [Smi11] [Smi14] and Rapoport-Smithling-Zhang in [RSZ]).

Let $\mathbb{M}$ be the Dieudonné module of $\mathbb{X}$, which is a free $\mathcal{O}_L$-module of rank $2n$, N its isocrystal with Frobenius $\mathcal{F}$ and Verschiebung $\mathcal{V}$. Then the $\mathcal{O}_F$-action on $\mathbb{X}$ induces an F-action on N such that $\mathbb{M}$ is stable under the F-action. Let π denote the operator $\iota(\pi)$ on N by abuse of notation. So we may view N as a vector space over $\breve{F}$, and $\mathbb{M}$ as an $\mathcal{O}_{\breve{F}}$-module of rank n. The polarization $\lambda_{\mathbb{X}}$ induces an alternating L-bilinear non-degenerate form on N

$$\langle\ ,\ \rangle \colon N \times N \to L, \tag{3.2.2}$$

such that

$$\langle \mathcal{F}x, y\rangle = \langle x, \mathcal{V}y\rangle^\sigma \tag{3.2.3}$$

for any $x, y \in N$, and

$$\langle \pi x, y\rangle = \langle x, \bar{\pi} y\rangle. \tag{3.2.4}$$

By Remark 2.2.1, this is equivalent to giving a hermitian form φ on N such that

$$\langle x, y\rangle = \frac{1}{2}\operatorname{Tr}_{\breve{F}/L}(\pi^{-1}\varphi(x,y)). \tag{3.2.5}$$

Then the periodicity condition for $\mathbb{X}$ means that $\mathbb{M}$ is a nearly π-modular lattice in the odd case, and a π-modular lattice in the even case. Note that by (3.2.5), the dual of $\mathbb{M}$ with respect to φ is the same as the dual with respect to $\langle\ ,\ \rangle$. So by Lemma 2.2.7, we can choose a $\breve{F}$-basis $\{e_1, \ldots, e_n\}$ of N such that $\varphi(e_i, e_j) = \delta_{i,n+1-j}$. We borrow the notation from (2.3.30) to denote the "standard" lattices, so $\mathbb{M} = \Lambda_m^\vee$.

Let us define a functor $\mathbf{M}^{\mathrm{naive}}$ on the category of $\mathcal{O}_{\breve{F}}$-schemes.

Definition 3.2.3 ([RZ96, Definition 3.27])**.** For each $\mathcal{O}_{\breve{F}}$-scheme S, $\mathbf{M}^{\mathrm{naive}}(S)$ consists of the following data:

- a functor from the category $\mathcal{L}_m$ to the category of $\mathcal{O}_{\breve{F}} \otimes_{\mathcal{O}_L} \mathcal{O}_S$-modules on S:

$$\begin{aligned} \mathcal{L}_m &\longrightarrow (\mathcal{O}_{\breve{F}} \otimes_{\mathcal{O}_L} \mathcal{O}_S\text{-modules}), \\ \Lambda_i &\longmapsto \mathcal{T}_i; \end{aligned} \tag{3.2.6}$$

- an inclusion

$$\mathcal{T}_i \subset \Lambda_i \otimes_{\mathcal{O}_L} \mathcal{O}_S \tag{3.2.7}$$

for each $i \in \{n\mathbb{Z} \pm m\}$ and functorial in Λ_i;

satisfying the following conditions:

1. the inclusion $\mathcal{T}_i \subset \mathcal{O}_{\breve{F}} \otimes_{\mathcal{O}_L} \mathcal{O}_S$ is a Zariski locally direct $\mathcal{O}_S$-summand of rank n;
2. the isomorphism

$$\pi \otimes 1 \colon \Lambda_i \otimes_{\mathcal{O}_L} \mathcal{O}_S \longrightarrow \Lambda_{i-n} \otimes_{\mathcal{O}_L} \mathcal{O}_S \tag{3.2.8}$$

identifies $\mathcal{T}_i$ with $\mathcal{T}_{i-n}$;

3. the perfect $\mathcal{O}_S$-bilinear paring

$$\langle\, ,\, \rangle \otimes \mathcal{O}_S \colon \Lambda_i \otimes_{\mathcal{O}_L} \mathcal{O}_S \;\times\; \Lambda_{-i} \otimes_{\mathcal{O}_L} \mathcal{O}_S \longrightarrow \mathcal{O}_S \tag{3.2.9}$$

identifies $\mathcal{T}_i^\vee$ with $\mathcal{T}_{-i}$;

4. (*Kottwitz condition*) $\pi \otimes 1$ acts on $\mathcal{T}_i$ as an $\mathcal{O}_S$-linear endomorphism with characteristic polynomial

$$\mathrm{charpol}(\pi \otimes 1 | \mathcal{T}_i) = (T - \pi)(T + \pi)^{n-1} \in \mathcal{O}_S[T]. \tag{3.2.10}$$

Clearly $\mathbf{M}^{\mathrm{naive}}$ is represented by a closed subscheme of a product of Grassmannians. Local models and Rapoport-Zink spaces are related by the *local model diagram* ([RZ96, Proposition 3.33]). Let $\widetilde{\mathcal{M}}^{\mathrm{naive}}$ be the set-valued functor on $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$ sending S to the set of isomorphism classes of (X, ρ_X, γ_X), where $(X, \rho_X) \in \mathcal{M}^{\mathrm{naive}}(S)$ is a naive unitary p-divisible group and an isomorphism of polarized multichains $\gamma_{\Lambda_i} \colon E_{\Lambda_i} \to \Lambda_i \otimes \mathcal{O}_S$ in the sense of [RZ96, Definition 3.21], where E_{Λ_i} is the Lie algebra of the universal vector extension of X_{Λ_i} for each $\Lambda_i \in \mathcal{L}_m$. Let $\mathcal{G}$ be the automorphism group of the polarized chain $\{\Lambda_i \otimes \mathcal{O}_S\}_{\Lambda_i \in \mathcal{L}_m}$. We have the local model diagram:

$$\begin{array}{ccc} & \widetilde{\mathcal{M}}^{\mathrm{naive}} & \\ \swarrow & & \searrow \\ \mathcal{M}^{\mathrm{naive}} & & \mathbf{M}^{\mathrm{naive}}, \end{array} \tag{3.2.11}$$

where $\widetilde{\mathcal{M}}^{\mathrm{naive}} \to \mathcal{M}^{\mathrm{naive}}$ is the natural projection, which is smooth, and a $\mathcal{G}$-torsor over $\mathcal{M}^{\mathrm{naive}}$; the morphism $\widetilde{\mathcal{M}}^{\mathrm{naive}} \to \mathbf{M}^{\mathrm{naive}}$ is given by

$$(X_\Lambda, \rho_\Lambda, \gamma_\Lambda) \longmapsto (\Lambda \otimes \mathcal{O}_S \xrightarrow{\gamma^{-1}} E_\Lambda \longrightarrow \mathrm{Lie}\, X_\Lambda), \tag{3.2.12}$$

which is formally smooth. In other words, there is a smooth morphism of algebraic stacks to the quotient stack

$$\mathcal{M}^{\mathrm{naive}} \longrightarrow [\mathbf{M}^{\mathrm{naive}}/\mathcal{G}]. \tag{3.2.13}$$

Let $\mathbf{M}^{\mathrm{loc}}$ be the flat closure of $\mathbf{M}^{\mathrm{naive}}$ in its generic fiber, then the honest Rapoport-Zink space $\mathcal{M}$ sits inside the cartesian diagram

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & [\mathbf{M}^{\mathrm{loc}}/\mathcal{G}] \\ \downarrow & & \downarrow \\ \mathcal{M}^{\mathrm{naive}} & \longrightarrow & [\mathbf{M}^{\mathrm{naive}}/\mathcal{G}]. \end{array} \tag{3.2.14}$$

Let $\mathbf{M}^e$ be the sub-functor of $\mathbf{M}^{\mathrm{naive}}$ whose S-points consist of those $\mathcal{T}_i$ satisfying the following additional conditions:

5. (*Wedge condition*)

$$\bigwedge^{n}(\pi \otimes 1 - 1 \otimes \pi | \mathcal{T}_i) = 0, \tag{3.2.15}$$

$$\bigwedge^{2}(\pi \otimes 1 + 1 \otimes \pi | \mathcal{T}_i) = 0 \text{ if } n \geq 3; \tag{3.2.16}$$

6. (*Spin condition*) if n is even, then $\pi \otimes 1 | \mathcal{T}_i \otimes_{\mathcal{O}_S} \kappa(s)$ is nonvanishing for all $s \in S$.

Then $\mathbf{M}^e$ is represented by a closed subscheme of $\mathbf{M}^{\mathrm{naive}}$ and sits inside the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{M}^e & \longrightarrow & [\mathbf{M}^e/\mathcal{G}] \\ \downarrow & & \downarrow \\ \mathcal{M}^{\mathrm{naive}} & \longrightarrow & [\mathbf{M}^{\mathrm{naive}}/\mathcal{G}]. \end{array} \tag{3.2.17}$$

In both odd and even case, we have the closed immersion $\mathbf{M}^{\mathrm{loc}} \subset \mathbf{M}^e$ and

$$\begin{array}{ccccc} & & \widetilde{\mathcal{M}}^{\mathrm{naive}} & & \\ & \swarrow & \uparrow & \searrow & \\ \mathcal{M}^{\mathrm{naive}} & & \widetilde{\mathcal{M}}^e & & \mathbf{M}^{\mathrm{naive}} \\ \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\ \mathcal{M}^e & & \widetilde{\mathcal{M}} & & \mathbf{M}^e \\ \uparrow & \swarrow & & \searrow & \uparrow \\ \mathcal{M} & & & & \mathbf{M}^{\mathrm{loc}}, \end{array} \tag{3.2.18}$$

where $\widetilde{\mathcal{M}}^e$ and $\widetilde{\mathcal{M}}$ are the $\mathcal{G}$-torsors corresponding to the smooth morphisms to the quotient stacks respectively.

Proof of Proposition 3.2.2. The topological flatness of $\mathbf{M}^e$ follows from [Smi11, Corollary 5.6.3] and [Smi14, Theorem 1.3], so $\mathcal{M}^e$ has the same underlying topological space with $\mathcal{M}$ by the diagram (3.2.18). The flatness of $\mathbf{M}^e$ in the even case follows from [RSZ, Proposition 3.10], so by the diagram (3.2.18), $\mathcal{M}^e$ is flat, i.e. $\mathcal{M} = \mathcal{M}^e$. $\square$

Let $\mathcal{N}, \mathcal{N}^e, \mathcal{N}^{\mathrm{naive}}$ be the special fibers of $\mathcal{M}, \mathcal{M}^e, \mathcal{M}^{\mathrm{naive}}$ respectively, we are interested in the geometric structure of $\mathcal{N}$. Because of the topological flatness of $\mathcal{N}^e$, we have $\mathcal{N}_{\mathrm{red}} = \mathcal{N}^e_{\mathrm{red}}$. The $\mathbb{F}$-valued points of $\mathcal{N}^e$ have a simple description: the Kottwitz condition means that X is of dimension n and of height $2n$; the Wedge condition (3.1.2) is trivial, (3.1.3) means

$$\bigwedge^2(\iota(\pi)|\operatorname{Lie}(X)) = 0, \tag{3.2.19}$$

i.e. the rank of the operator $\iota(\pi)|\operatorname{Lie}(X)$ is less than or equal to 1; in the even case the spin condition means the rank of the operator $\iota(\pi)|\operatorname{Lie}(X)$ is 1.

Proposition 3.2.4. *Via Dieudonné theory, $\mathcal{N}(\mathbb{F}) = \mathcal{N}^e(\mathbb{F})$ can be identified with the set of $\mathcal{O}_{\breve{F}}$-lattices M in N satisfying the following conditions:*

1. *M is stable under $\mathcal{F}$ and $\mathcal{V}$;*
2. *$M \overset{n-1}{\subset} p^h M^\vee \overset{1}{\subset} \pi^{-1} M$ if n is odd, and $p^h M^\vee = \pi^{-1} M$ if n is even for some $h \in \mathbb{Z}$;*
3. *$pM \overset{n}{\subset} \mathcal{V}M \overset{n}{\subset} M$;*
4. *$\mathcal{V}M \overset{\leq 1}{\subset} \mathcal{V}M + \pi M$;*
5. *if n is even, $\mathcal{V}M \overset{1}{\subset} \mathcal{V}M + \pi M$.*

Proof. Via Dieudonné theory, the Lie algebra of a p-divisible group X can be identified with $M/\mathcal{V}M$, where M is its Dieudonné module. So condition 3 is just the Kottwitz condition, condition 4 is the wedge condition and condition 5 is the extra Spin condition. □

3.3 Local PEL datum

Let G be the algebraic group $\mathrm{GU}(N, \varphi)$ over L. We write $\mathcal{F} = b \cdot \mathrm{id}_F \otimes \sigma$ for some $b \in \mathrm{GL}_{\breve{F}}(N)$ in terms of the basis $\{e_1, \dots, e_n\}$, by (3.2.3) and (3.2.5), we have

$$\varphi(\mathcal{F}x, \mathcal{F}y) = p \cdot \varphi(x^\sigma, y^\sigma) \tag{3.3.1}$$

which implies that $b \in G(L)$ with $\mathrm{val}(c(b)) = 1$. Let $[b] \in B(G)$ be the $G(L)$-conjugacy classes of b, i.e. the set $\{g^{-1} b \sigma(g) : g \in G(L)\}$. We use the notation from Section 2.3, i.e. S is the maximal L-split torus of G, T is the centralizer of S with $X_*(T) \simeq \mathbb{Z}^n \times \mathbb{Z}$, and $\mu \in X_*(T)$ is the geometric minuscule cocharacter $(1, 0^{n-1}; 1)$. By the assumption of supersingularity and Kottwitz condition on $\mathbb{X}$, we have

$$[b] \in B(G, \{\mu\})_b, \tag{3.3.2}$$

where $\{\mu\}$ is the geometric conjugacy classes of μ and the well-known set $B(G, \{\mu\})$ is the subset of $B(G)$ consisting of neutral acceptable elements (cf. [Kot97, 6.2] [RV14, Definition 2.3]).

In summary, $(\breve{F}, N, \varphi, \{\mu\}, [b], \mathbb{M})$ forms a *simple integral Rapoport-Zink PEL-datum* in the sense of [RV14, 4.1] (cf. [RZ96, Definition 3.18]).

Another important group is the algebraic group J consisting of automorphisms of the unitary isocrystal N, i.e.

$$J(R) = \left\{ g \in \mathrm{GL}_{\breve{F}\otimes R}(N \otimes_{\mathbb{Q}_p} R) \;\middle|\; \begin{array}{l} g\mathcal{F} = \mathcal{F}g, \varphi(gx, gy) = c(g)\varphi(x, y) \\ \text{for some } c(g) \in (L \otimes R)^\times \end{array} \right\} \tag{3.3.3}$$

for any $\mathbb{Q}_p$-algebra R. The group J acts on $\mathcal{N}^e$: for $g \in J$, the action is given by sending $(X, \iota_X, \lambda_X, \rho_X) \in \mathcal{N}^e$ to $(X, \iota_X, \lambda_X, g \circ \lambda_X)$. By [Kot85, 5.2], J is an inner form of G because $[b]$ is basic.

The group J is closely related to a hermitian space, namely C, with respect to $F/\mathbb{Q}_p$ as discussed in [RTW14]. Recall that $\pi^2 = \varpi = \epsilon p$, let $\eta, \delta \in \mathcal{O}_L^\times$ such that $\eta^2 = \epsilon^{-1}$ and $\delta^\sigma = -\delta$ respectively. Then all slopes of the $\mathrm{id} \otimes\sigma$-linear operator $\chi := \eta\pi\mathcal{V}^{-1} \colon N \to N$ are zero. Let C be the set of points in N fixed by χ, then C is a vector space over F and the isomorphism

$$C \otimes_{\mathbb{Q}_p} L \simeq N \tag{3.3.4}$$

identifies $\mathrm{id}_C \otimes\sigma$ with χ. Let $\psi(x, y) := \delta\varphi(x, y)$ for $x, y \in C$, then by (3.3.1), we have

$$\psi(x, y) = \psi(x, y)^\sigma. \tag{3.3.5}$$

So ψ takes values in F and hence (C, ψ) becomes a hermitian space with respect to $F/\mathbb{Q}_p$.

Lemma 3.3.1 ([RTW14, Lemma 2.3])**.** *The group J is isomorphic to the general unitary group* $\mathrm{GU}(C, \psi)$.

Lemma 3.3.2 ([RSZ, Lemma 3.3])**.** *The hermitian space (C, ψ) is split if n is odd, non-split if n is even.*

Remark 3.3.3. In [Smi15], when n is odd, the moduli description of $\mathbf{M}^{\mathrm{loc}}$, hence of $\mathcal{N}$, is formulated by proposing a further refinement of the spin condition, which is unfortunately very complicated. For the purpose of studying basic loci of Shimura varieties in this paper, for us it is enough to work with $\mathcal{N}^e$ since $\mathcal{N}^e_{\mathrm{red}} = \mathcal{N}_{\mathrm{red}}$.

3.4 Kottwitz invariants of quasi-isogenies

In this section, we will define a morphism

$$\kappa \colon \mathcal{N}^e \to \pi_1(G)_\Gamma. \tag{3.4.1}$$

Let $X \in \mathcal{N}^e(\mathbb{F})$ be a unitary p-divisible group with a quasi-isogeny $\rho \colon X \to \mathbb{X}$, let M be its corresponding Dieudonné lattice in N, recall that the height of ρ is defined as

$$\mathrm{ht}(\rho) := \mathrm{ht}(p^s\rho) - \mathrm{ht}(p^s), \tag{3.4.2}$$

where s is an integer such that $p^s\rho$ is an honest isogeny.

Proposition 3.4.1. *If M satisfies $M \subset p^h M^\vee \subset \pi^{-1}M$ for some integer h, then* $\mathrm{ht}(\rho) = nh$.

Proof. For two lattices M_1, M_2 in N, we define

$$[M_1 : M_2] \coloneqq \dim_{\mathbb{F}} M_1/p^s M_2 - \dim_{\mathbb{F}} M_2/p^s M_1 \tag{3.4.3}$$

for some large enough integer s. Then by (3.4.2),

$$\begin{aligned} \mathrm{ht}(\rho) &= [\mathbb{M} : M] \\ &= [M^\vee : \mathbb{M}^\vee] \\ &= [M^\vee : p^{-h}M] + [p^{-h}M : p^{-h}\mathbb{M}] + [p^{-h}\mathbb{M} : \mathbb{M}] \\ &\quad + [\mathbb{M} : \mathbb{M}^\vee]. \end{aligned} \tag{3.4.4}$$

Note that, in both odd and even cases, $[M^\vee : p^{-h}M] = [\mathbb{M}^\vee : \mathbb{M}]$, so we get the desired result. □

For a unitary p-divisible group $X \in \mathcal{N}^e(S)$ with a quasi-isogeny ρ, the height is locally constant on S, so we get a morphism

$$\begin{aligned} \kappa_1 \colon \mathcal{N}^e &\longrightarrow \mathbb{Z}, \\ (X, \rho) &\longmapsto \frac{1}{n}\,\mathrm{ht}(\rho). \end{aligned} \tag{3.4.5}$$

By abuse of notation, we denote by κ_1 the composite morphism $\mathcal{N} \subset \mathcal{N}^e \to \mathbb{Z}$. Let $\mathcal{N}_h$ (resp. $\mathcal{N}^e_h$) be the fiber $\kappa_1^{-1}(h)$ for $h \in \mathbb{Z}$, then $\mathcal{N}_h$ (resp. $\mathcal{N}^e_h$) is an open and closed subscheme of $\mathcal{N}$ (resp. $\mathcal{N}^e$), we have a decomposition

$$\mathcal{N} = \coprod_{h \in \mathbb{Z}} \mathcal{N}_h \quad \text{and} \quad \mathcal{N}^e = \coprod_{h \in \mathbb{Z}} \mathcal{N}^e_h. \tag{3.4.6}$$

In the even case, $\mathcal{N} = \mathcal{N}^e$, there is an extra invariant of quasi-isogenies, which has been discussed in [RSZ, Lemma 3.2]. Let $(\tilde{X}, \rho_1, \rho_2)$ be the *minimal cover* of the quasi-isogeny ρ in the following sense: $\tilde{X}$ is a p-divisible group together with isogenies ρ_1, ρ_2 making the following diagram commutative,

$$\begin{array}{ccc} & \tilde{X} & \\ {\scriptstyle \rho_1}\swarrow & & \searrow{\scriptstyle \rho_2} \\ X & \xrightarrow{\quad \rho \quad} & \mathbb{X}, \end{array} \tag{3.4.7}$$

such that for any p-divisible group Y with isogenies α_1, α_2 satisfying $\rho \circ \alpha_1 = \alpha_2$, there exists a unique isogeny $\beta \colon Y \to \tilde{X}$ making the following diagram commutative

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle \alpha_1}\swarrow & \downarrow & \searrow{\scriptstyle \alpha_2} & \\ & & \tilde{X} & & \\ & {\scriptstyle \rho_1}\swarrow & & \searrow{\scriptstyle \rho_2} & \\ X & & \xrightarrow{\quad \rho \quad} & & \mathbb{X}. \end{array} \tag{3.4.8}$$

Note that, via Dieudonné theory, $\tilde{X}$ corresponds to the lattice $M \cap \mathbb{M}$. By (3.4.2) and Proposition 3.4.1, we have

$$nh = \mathrm{ht}(\rho) = \mathrm{ht}(\rho_2) - \mathrm{ht}(\rho_1). \tag{3.4.9}$$

Because n is even,

$$\mathrm{ht}(\rho_2) \equiv \mathrm{ht}(\rho_1) \mod 2. \tag{3.4.10}$$

Hence we get a morphism

$$\begin{aligned}(\kappa_1, \kappa_2) \colon \mathcal{N} &\longrightarrow \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \\ (X, \rho) &\longmapsto (\frac{1}{n}\,\mathrm{ht}(\rho),\, \mathrm{ht}(\rho_1) \mod 2).\end{aligned} \tag{3.4.11}$$

In summary, we have the *Kottwitz morphism*

$$\kappa \colon \mathcal{N} \longrightarrow \pi_1(G)_\Gamma, \tag{3.4.12}$$

when n is odd, $\kappa = \kappa_1$, when n is even, $\kappa = (\kappa_1, \kappa_2)$. We have the decomposition

$$\mathcal{N} = \coprod_{\kappa \in \pi_1(G)_\Gamma} \mathcal{N}_{(\kappa)}, \tag{3.4.13}$$

where $\mathcal{N}_{(\kappa)}$ consists of those quasi-isogenies with Kottwitz invariants $\kappa \in \pi_1(G)_\Gamma$. For any κ, κ', let $g \in J$ such that $\kappa(g) = \kappa' - \kappa$, then g defines an isomorphism

$$\begin{aligned}\mathcal{N}_{(\kappa)} &\longrightarrow \mathcal{N}_{(\kappa')}, \\ (X, \rho) &\longmapsto (X, g \circ \rho).\end{aligned} \tag{3.4.14}$$

Via Dieudonné theory, we have, in the odd case

$$\mathcal{N}_{(\kappa)}(\mathbb{F}) = \{M \in \mathcal{N}(\mathbb{F}) : M \subset p^{\kappa} M^\vee \subset \pi^{-1} M\}. \tag{3.4.15}$$

In the even case for $\kappa = (\kappa_1, \kappa_2)$

$$\mathcal{N}_{(\kappa)} = \left\{M \in \mathcal{N}(\mathbb{F}) \;\middle|\; \begin{array}{c} p^{\kappa_1} M^\vee = \pi^{-1} M, \\ \dim_{\mathbb{F}}(M + \mathbb{M}/M) \equiv \kappa_2 \mod 2 \end{array} \right\}. \tag{3.4.16}$$

Example 3.4.2. When n is even, the lattice $\Lambda_{m'}^\vee$ (see (2.3.36)) lies in $\mathcal{N}(\mathbb{F})$ and has non-trivial Kottwitz invariant, i.e.

$$\dim_{\mathbb{F}}(\Lambda_{m'}^\vee + \mathbb{M}/\Lambda_{m'}^\vee) = 1. \tag{3.4.17}$$

3.5 Affine Deligne-Lusztig varieties

Recall that for the local PEL-datum $(\breve{F}, N, \varphi, \{\mu\}, [b], \mathbb{M})$ in Section 3.3, we may associate to it the *generalized affine Deligne-Lusztig variety* (cf. [Rap05, Definition 4.1])

$$X(\mu, b)_K \coloneqq \{g \in G(L)/K : g^{-1} b \sigma(g) \in \bigcup_{w \in \mathrm{Adm}(\mu)} KwK\}, \tag{3.5.1}$$

where $K = \mathrm{Stab}_{G(L)}(\mathbb{M} \subset \mathbb{M}^\vee \subset \pi^{-1}\mathbb{M})$ which is the special parahoric subgroup $P_{\{m\}}$ of $G(L)$ corresponding to the 0-th vertex of the local Dynkin diagram in both odd and even cases by Proposition 2.3.2 and 2.3.4, because $\mathbb{M} = \Lambda_m^\vee$. Note that the group J also acts on $X(\mu, b)_K$ because J is just the σ-centralizer of b in $G(L)$.

Proposition 3.5.1. *The map*

$$\begin{aligned} \Phi\colon X(\mu,b)_K &\longrightarrow \mathcal{N}(\mathbb{F}), \\ g &\longmapsto g\mathbb{M}, \end{aligned} \tag{3.5.2}$$

is bijective.

Proof. For $g \in X(\mu,b)_K$, we will check that $g\mathbb{M} \in \mathcal{N}(\mathbb{F})$, i.e. it satisfies the conditions in Proposition 3.2.4. Recall that we choose representative(s) μ_1 (and μ_0 in odd case) of $\mathrm{Adm}^0(\mu)$ in $T(L)$ in the subsection 2.3.2.

1. The condition $g\mathbb{M}$ is stable under $\mathcal{F} = b\cdot \mathrm{id}\otimes\sigma$ is equivalent to the condition
$$g^{-1}b\sigma(g)\mathbb{M} \subset \mathbb{M}, \tag{3.5.3}$$
so by (2.3.29), it is enough to check $\mu_1\mathbb{M} \subset \mathbb{M}$ (and $\mu_0\mathbb{M} \subset \mathbb{M}$ in odd case). By the choice of μ_1 (and μ_0) in (2.3.27) and (2.3.28), it is easy to see $g\mathbb{M}$ is $\mathcal{F}$-stable, and similarly, $\mathcal{V}$-stable.

2. Recall that $(g\mathbb{M})^\vee = c(g)^{-1}g\mathbb{M}^\vee$, so $g\mathbb{M}$ sits inside the chain
$$g\mathbb{M} \subset c(g)(g\mathbb{M})^\vee \subset \pi^{-1}(g\mathbb{M}). \tag{3.5.4}$$

3. The condition $pg\mathbb{M} \overset{n}{\subset} \mathcal{V}g\mathbb{M} \overset{n}{\subset} g\mathbb{M}$ is equivalent to
$$p\mathbb{M} \overset{n}{\subset} g^{-1}b\sigma(g)\mathbb{M} \overset{n}{\subset} \mathbb{M}, \tag{3.5.5}$$
i.e. $p\mathbb{M} \overset{n}{\subset} \mu_1\mathbb{M} \overset{n}{\subset} \mathbb{M}$ (and similarly for μ_0 in odd case) which is true by (2.3.27) and (2.3.28).

4. The condition $\mathcal{V}g\mathbb{M} \overset{\leq 1}{\subset} \mathcal{V}g\mathbb{M} + \pi g\mathbb{M}$ is equivalent to
$$\pi\mathbb{M} \overset{\leq 1}{\subset} \pi\mathbb{M} + g^{-1}b\sigma(g)\mathbb{M}. \tag{3.5.6}$$
By (2.3.27) and (2.3.28), the dimension of the quotient space is 1 if and only if $g^{-1}b\sigma(g) \in K\mu_1K$, and is 0 if and only if $g^{-1}b\sigma(g) \in K\mu_0K$ in the odd case.

It's very easy to see that Φ is injective. For the surjectivity of Φ, we use the *Görtz local model diagram* (cf. [GY10, 5.2]):

$$\begin{array}{ccccc} & & G(L) & & \\ & \overset{\mathrm{pr}}{\swarrow} & & \overset{\mathrm{pr}^\sigma}{\searrow} & \\ G(L)/K & & & & G(L)/K \\ \uparrow & & & & \uparrow \\ \mathcal{N}^{\mathrm{naive}}(\mathbf{F}) & & & & \mathbf{M}^{\mathrm{naive}}_{\mathbb{F}} \\ \uparrow & & & & \uparrow \\ \mathcal{N}^e(\mathbb{F}) & & & & \mathbf{M}^e_{\mathbb{F}}, \end{array} \tag{3.5.7}$$

where pr is the natural projection, pr^σ is the composite of the Lang map $g \mapsto g^{-1}b\sigma(g)$, with the projection pr, and all vertical arrows are injective. Then $\mathrm{pr}^{-1}(\mathcal{N}^e(\mathbf{F})) = (\mathrm{pr}^\sigma)^{-1}(\mathbf{M}^e_{\mathbb{F}})$. Note that

$$\mathbf{M}^e_{\mathbb{F}} = \bigcup_{w \in \mathrm{Adm}(\mu)} KwK/K \tag{3.5.8}$$

in $G(L)/K$ by [Smi11, Corollary 5.6.2] and [Smi14, Theorem 1.4]. So

$$\mathrm{pr}^{-1}(X(\mu, b)_K) = (\mathrm{pr}^\sigma)^{-1}(\mathbf{M}^e_{\mathbb{F}}), \tag{3.5.9}$$

and the injectivity of Φ implies that Φ is surjective. □

Remark 3.5.2. For $g \in X(\mu, b)_K$, the Kottwitz invariant is well defined by the definition of parahoric subgroups. We have the following commutative diagram:

$$\begin{array}{ccc} X(\mu, b)_K & \xrightarrow{\Phi} & \mathcal{N}(\mathbb{F}) \\ & {}_{\kappa}\searrow \quad \swarrow_{\kappa} & \\ & \pi_1(G)_\Gamma. & \end{array} \tag{3.5.10}$$

So $X(\mu, b)_K$ can be decomposed into a disjoint union of some subsets indexed by Kottwitz invariants. In the odd case, for any $\kappa \in \pi_1(G)_\Gamma \simeq \mathbb{Z}$, $\mathcal{N}_{(\kappa)}(\mathbb{F})$ can be identified with a generalized affine Deligne-Lusztig variety $X(\tilde{\mu}, \tilde{b})_{K'}$ associated to the derived group of G, i.e. the special unitary group $\mathrm{SU}(V, \varphi)$, where $K' = K \cap \mathrm{SU}(V, \varphi)$. Because in this case, there exists a central element ζ such that $\mu = \zeta\tilde{\mu}$ and $b = \tilde{b}\zeta$. Then the map

$$\begin{aligned} X(\tilde{\mu}, \tilde{b})_{K'} &\longrightarrow \mathcal{N}_{(1)}(\mathbb{F}), \\ g &\longmapsto g\zeta\mathbb{M}, \end{aligned} \tag{3.5.11}$$

gives the desired identification. However, this is no longer true in the even case, because $\mu = (1, 0^{(m-1)}; 1) \in X_*(T)_\Gamma$ and $\tilde{\mu} = (2, 0^{(m-1)}) \in X_*(T^{\mathrm{sc}})$ differ in a non-central element in Ω. We will work with the corresponding *semisimple group of adjoint type* G_{ad}, i.e. the quotient of G by its center.

Let $b_{\mathrm{ad}}, \mu_{\mathrm{ad}}, K_{\mathrm{ad}}$ be the images in $G_{\mathrm{ad}}(L)$ of b, μ, K respectively. Similarly to (3.5.1), we define

$$X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}} := \{g \in G_{\mathrm{ad}}(L)/K_{\mathrm{ad}} : g^{-1}b_{\mathrm{ad}}\sigma(g) \in \bigcup_{w \in \mathrm{Adm}(\mu_{\mathrm{ad}})} K_{\mathrm{ad}}wK_{\mathrm{ad}}\}, \tag{3.5.12}$$

where $\mathrm{Adm}(\mu_{ad})$ is the μ_{ad}-admissible subset of the Iwahori-Weyl group $\tilde{W}_{\mathrm{ad}}$ of G_{ad}, which is bijective to $\mathrm{Adm}(\mu)$ under the canonical map $\tilde{W} \to \tilde{W}_{\mathrm{ad}}$. However, there is no reasonable map from $X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}}$ to $\mathcal{N}(\mathbb{F})$, because for $g \in X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}}$, $M \in \mathcal{N}(\mathbb{F})$, the notation gM doesn't make sense. gM is no longer a lattice, but a homothety class of lattices. However, by [PR08, 6.a], (see also [GHN15, 2.2]) the natural map

$$G(L)/K \longrightarrow G_{\mathrm{ad}}(L)/K_{\mathrm{ad}} \tag{3.5.13}$$

induces a bijection

$$(G(L)/K)_\kappa \xrightarrow{\cong} (G_{\mathrm{ad}}(L)/K_{\mathrm{ad}})_{\kappa_{\mathrm{ad}}}, \tag{3.5.14}$$

where $\kappa \in \pi_1(G)_\Gamma$, κ_{ad} is the image of κ in $\pi_1(G_{\mathrm{ad}})_\Gamma$, the notation $(\cdot)_\kappa$ stands for the fiber of corresponding Kottwitz maps. Immediately, we have

$$(X(\mu, b)_K)_\kappa \xrightarrow{\cong} (X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}})_{\kappa_{\mathrm{ad}}}. \tag{3.5.15}$$

In particular, $X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}}$ can be identified with the moduli space of quasi-isogenies of height 0, i.e.

$$\begin{aligned} \Phi_{\mathrm{ad}} \colon X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}} &\xrightarrow{\cong} \mathcal{N}_0(\mathbb{F}), \\ gK_{\mathrm{ad}} &\longmapsto \pi^{-\operatorname{val}(\dot{g})}\dot{g}\mathbb{M}, \end{aligned} \tag{3.5.16}$$

where $\dot{g}K$ is a lifting of gK_{ad} under the map (3.5.13).

Let $G_{\mathrm{ad}}(L)'$ be the subgroup of $G_{\mathrm{ad}}(L)$ generated by all the parahoric subgroups of $G_{\mathrm{ad}}(L)$. Let

$$X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})'_{K_{\mathrm{ad}}} := \{g \in G_{\mathrm{ad}}(L)'/K_{\mathrm{ad}} : g^{-1}b_{\mathrm{ad}}\sigma(g) \in \bigcup_{w \in \mathrm{Adm}(\mu_{\mathrm{ad}})} K_{\mathrm{ad}}wK_{\mathrm{ad}}\}. \tag{3.5.17}$$

Note that $G_{\mathrm{ad}}(L)' = G_{\mathrm{ad}}(L)_1$ by [HR08, Lemma 17], in other words, the kernel of Kottwitz map is generated by all the parahoric subgroups. When n is odd, nothing is new because $X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})'_{K_{\mathrm{ad}}} = X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}}$; when n is even, the map Φ_{ad} in (3.5.16) induces the following isomorphism:

$$\Phi_{\mathrm{ad}} \colon X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})'_{K_{\mathrm{ad}}} \xrightarrow{\cong} \mathcal{N}_{(0,0)}(\mathbb{F}). \tag{3.5.18}$$

From now on, let

$$\mathcal{S} := \begin{cases} \mathcal{N}_0 & \text{if } n \text{ is odd}, \\ \mathcal{N}_{(0,0)} & \text{if } n \text{ is even}. \end{cases} \tag{3.5.19}$$

Remark 3.5.3. When n is even, by the definition of $\mathcal{S}$, $\mathcal{S}(\mathbb{F})$ is a single $G_{\mathrm{ad}}(L)'$-orbit of $\mathbb{M}$, and the intersection of the $G_{\mathrm{ad}}(L)'$-orbit of $\Lambda^\vee_{m'}$ and $\mathcal{S}(\mathbb{F})$ is empty.

Remark 3.5.4. An equivalent way to identify $X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}}$ with a reasonable Rapoport-Zink space is to define the *adjoint Rapoport-Zink space*

$$\mathcal{N}_{\mathrm{ad}} := \mathcal{N}/\mathbb{G}_m(F).$$

Because in both odd and even cases, the action of π on $\mathcal{N}$ via $\iota_{\mathbb{X}} \colon F \to \mathrm{End}(\mathbb{X})$ gives an isomorphism

$$\mathcal{N}_h \longrightarrow \mathcal{N}_{h+1}. \tag{3.5.20}$$

The set $\mathcal{N}_{\mathrm{ad}}(\mathbb{F})$ can be described as a set of homothety classes of lattices satisfying the Kottwitz, wedge and the extra spin conditions, so that for $g \in X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}}$, the notation $g\mathbb{M}$ makes sense as a homothety class of lattices, i.e. the following diagram is commutative

$$\begin{array}{ccc} X(\mu, b)_K & \xrightarrow{\cong} & \mathcal{N}(\mathbb{F}) \\ \downarrow & & \downarrow \\ X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})_{K_{\mathrm{ad}}} & \xrightarrow{\cong} & \mathcal{N}_{\mathrm{ad}}(\mathbb{F}). \end{array} \tag{3.5.21}$$

Chapter 4

Set structure of $\mathcal{N}$

4.1 Deligne-Lusztig varieties

We need some results about classical Deligne-Lusztig varieties. Let H_0 be a reductive group over $\mathbb{F}_q$. We fix a maximal torus T_0 and Borel subgroup B_0 over $\mathbb{F}_q$. Let H be the reductive group $H_0 \otimes \overline{\mathbb{F}}_q$ over $\overline{\mathbb{F}}_q$, $B \coloneqq B_0 \otimes \overline{\mathbb{F}}_q$.

, with a Frobenius action σ. We fix a σ-stable maximal torus T and Borel subgroup B. Let $W = W_H$ be the Weyl group of H. The *(classical) Deligne-Lusztig variety* (cf. [DL76, Definition 1.4]) $X(w)$ is defined as

$$X(w) \coloneqq \{g \in H/B : g^{-1}\sigma(g) \in BwB\}, \tag{4.1.1}$$

for each $w \in W_H$. We also say that g and h are in *relative position* w if $g^{-1}h \in BwB$ for $g, h \in G/B$ and $w \in W_H$.

Let X be the variety of Borel subgroups of H, consider the diagonal action

$$\begin{aligned} H \times (X \times X) &\longrightarrow X \times X, \\ (h, (B_1, B_2)) &\longmapsto ({}^hB_1, {}^hB_2). \end{aligned} \tag{4.1.2}$$

Let $\mathcal{O}(w)$ be the H-orbit of $(B, {}^wB)$. Then the Bruhat decomposition is equivalent to the fact that $X \times X = \cup_{w \in W_H} \mathcal{O}(w)$. Two Borel subgroups B_1 and B_2 are in relative position w if and only if $(B_1, B_2) \in \mathcal{O}(w)$.

Proposition 4.1.1 (cf. [DL76], see also [Gör10, Proposition 4.4])**.** *For $w \in W_H$.*

1. *The Deligne-Lusztig variety $X(w)$ is smooth and of pure dimension $\ell(w)$, where $\ell(w)$ is the length of w.*

2. *The flag variety H/B is the disjoint union of all Deligne-Lusztig varieties, i.e.*

$$H/B = \bigcup_{w \in W_H} X(w). \tag{4.1.3}$$

The closure $\overline{X(w)}$ of $X(w)$ in the flag variety H/B is normal, and

$$\overline{X(w)} = \bigcup_{w' \leq w} X(w'), \tag{4.1.4}$$

where $\leq$ denotes the Bruhat order in W_H. Furthermore, if w is a Coxeter element, $\overline{X(w)}$ is smooth.

3. *The Deligne-Lusztig variety $X(w)$ is irreducible if and only if w is not contained in any σ-stable standard parabolic subgroup of W_H.*

Example 4.1.2 ([DL76, 2.2])**.** Let $H = \mathrm{GL}(V)$, where V is an $\mathbb{F}$-vector space defined over $\mathbb{F}_q$ of dimension l. Let $\{e_1, \ldots, e_l\}$ be a basis of V. In this case the Weyl group $W = S_l$. Let s_i be the transposition $(i, i+1)$ for $1 \leq i \leq l-1$, $\mathbb{S} = \{s_1, \ldots, s_{l-1}\}$. Let B be the standard Borel subgroup. For any two Borel subgroups B_1 and B_2 with their corresponding flags

$$\begin{aligned} B_1 &: 0 \subset V_1 \subset \cdots \subset V_{l-1} \subset V, \\ B_2 &: 0 \subset V_1' \subset \cdots \subset V_{l-1}' \subset V, \end{aligned} \tag{4.1.5}$$

B_1 and B_2 are in relative position w if and only if $(B_1, B_2) \in \mathcal{O}(w)$, if and only if $(B_1, B_2) = h \cdot (B, {}^wB)$ for some $h \in H$. The last condition means there exists a basis $\{f_1, \ldots, f_l\}$ of V, such that

$$\begin{aligned} V_i &= \mathrm{span}\{f_1, \ldots, f_i\}, \\ V_i' &= \mathrm{span}\{f_{w(1)}, \ldots, f_{w(i)}\}. \end{aligned} \tag{4.1.6}$$

Let w be the Coxeter element $(k, k-1, \ldots, 1, k+1, k+2, \ldots, l-k+1)$, $r = w(1) - 1$, then for a flag $V.$, the flags $V.$ and $\sigma(V.)$ are in relative position w if and only if $V.$ is of the form:

$$\begin{aligned} V_{r-i} &= V_r \cap \sigma(V_r) \cap \cdots \cap \sigma^i(V_r), & 1 \leq i \leq r-1, \\ V_{r+i} &= V_r + \sigma(V_r) + \cdots + \sigma^i(V_r), & 1 \leq i \leq l-r. \end{aligned} \tag{4.1.7}$$

Let P_r be the standard parabolic subgroup corresponding to $\mathbb{S} - \{s_r\}$, then we have the following commutative diagram:

$$\begin{array}{ccc} X(w) & \hookrightarrow & H/B \\ & \overset{\phi}{\searrow} & \downarrow \\ & & H/P_r, \end{array} \tag{4.1.8}$$

where $\phi \colon V. \mapsto V_r$ is injective. And $\mathrm{im}(\phi)$ is the subvariety of H/P_r parameterizing all the r-dimensional subspaces V_r of V such that

$$\begin{aligned} \dim(V_r \cap \sigma(V_r) \cap \cdots \cap \sigma^i(V_r)) &= r - i, & 1 \leq i \leq r, \\ \dim(V_r + \sigma(V_r) + \cdots + \sigma^i(V_r)) &= r + i, & 1 \leq i \leq l-r. \end{aligned} \tag{4.1.9}$$

Example 4.1.3 (The split odd orthogonal group)**.** Let V be an l-dimensional vector space over $\mathbb{F}_q$, where $l = 2d+1$ is odd, together with "the" split non-degenerate symmetric form $\langle\, , \rangle$. Let $\mathrm{SO}(V)_0$ be the (split) special orthogonal group over $\mathbb{F}_q$. We fix a Borel subgroup B_0 over $\mathbb{F}_q$. Let $\mathrm{SO}(V) := \mathrm{SO}(V)_0 \otimes \overline{\mathbb{F}}_q$ and $B := B_0 \otimes \overline{\mathbb{F}}_q$. Note that a Borel subgroup of $\mathrm{SO}(V)$ can be described as the stabilizer of a complete isotropic flag:

$$0 \subset V_1 \subset \cdots \subset V_d \subset V_d^{\perp} \subset \cdots \subset V_1^{\perp} \subset V_{\mathbb{F}}. \tag{4.1.10}$$

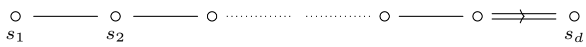
The (absolute) Weyl group W can be identified with a subgroup of S_l:

$$W = \{w \in S_l : w(i) + w(l+1-i) = l+1\}. \tag{4.1.11}$$

Let $\mathbb{S}$ be the set of simple reflections $\{s_i, 1 \leq i \leq d\}$, where

$$s_i = \begin{cases} (i,i+1)(l-i,l-i+1), & \text{if } 1 \leq i \leq d-1, \\ (d,d+2), & \text{if } i = d. \end{cases} \tag{4.1.12}$$

The Dynkin diagram of type B_d is



(4.1.13)

Let $w = s_d s_{d-1} \cdots s_1 = (d, d+2)(d, d-1, \ldots, 1)(l, l-1, \ldots, d+2)$, then, using the same trick as Example 4.1.2, for a flag $V.$, the $V.$ and $\sigma(V.)$ are in relative position w if and only if the flag $V.$ is of the form:

$$V_{d-i} = V_d \cap \sigma(V_d) \cap \cdots \cap \sigma^i(V_d), \tag{4.1.14}$$

for $1 \leq i \leq d$. Let P be the standard parabolic subgroup corresponding to $\mathbb{S} - \{s_d\}$, then we have the following commutative diagram:

$$\begin{array}{ccc} X(w) & \hookrightarrow & \mathrm{SO(V)}/B \\ & \overset{\phi}{\searrow} & \downarrow \\ & & \mathrm{SO}(V)/P, \end{array} \tag{4.1.15}$$

where ϕ: $V. \mapsto V_d$ is injective. And $\mathrm{im}(\phi)$ is the subvariety of $\mathrm{SO}(V)/P$ parameterizing all the d-dimensional isotropic subspaces V_d such that

$$\dim(V_d \cap \sigma(V_d) \cap \cdots \cap \sigma^i(V_d)) = d - i, \tag{4.1.16}$$

for $1 \leq i \leq d$.

Example 4.1.4 (The non-split even orthogonal group)**.** Let V be a vector space of even dimension $l = 2d$ over $\mathbb{F}_q$, together with “the” non-split non-degenerate symmetric form $\langle\, , \rangle$. We assume that $d \geq 2$. Let $\mathrm{SO}(V)_0$ be the special orthogonal group which is a quasi-split but non-split reductive group over $\mathbb{F}_q$. We have the Witt decomposition:

$$V = H_1 \oplus \cdots \oplus H_{d-1} \oplus Q_0, \tag{4.1.17}$$

where H_i is “the” hyperbolic plane for all i, and Q_0 is “the” anisotropic plane (cf. [O'M00, 62:1b]). We can choose a basis $\{e_1, \ldots, e_d, f_1, \ldots, f_d\}$ of V such that

$$\begin{aligned} V_0 &= \mathrm{span}_{\mathbb{F}_q}\{e_d, f_d\}, \\ H_i &= \mathrm{span}_{\mathbb{F}_q}\{e_i, f_i\}, \end{aligned} \tag{4.1.18}$$

for $1 \leq i \leq d-1$. We define the standard isotropic flag $V.$ as follows

$$V_i \coloneqq \mathrm{span}_{\mathbb{F}_q}\{e_1, \ldots, e_i\}, \tag{4.1.19}$$

for $1 \le i \le d-1$. Let B_0 be the standard Borel subgroup over $\mathbb{F}_q$, i.e. $B_0 := \mathrm{Stab}(V.)$.

Now consider the group $\mathrm{SO}(V) := \mathrm{SO}(V)_0 \otimes \overline{\mathbb{F}}_q$. Let $B := B_0 \otimes \overline{\mathbb{F}}_q$. Over $\mathbb{F}$, a non-degenerate symmetric form is always split, so we can choose a $\mathbb{F}$-basis $\{g, h\}$ of $Q_0 \otimes \mathbb{F}$ such that $\langle g, h\rangle = 1$ and $\langle g, g\rangle = \langle h, h\rangle = 0$. We define the standard flag $W.$ as follows:

$$W.\colon 0 \subset W_1 \subset \cdots \subset W_{d-2} \subset (W_d \text{ and } W_{d'}) \subset W_{d-2}^{\perp} \subset \cdots \subset W_1^{\perp} \subset V, \tag{4.1.20}$$

where

$$\begin{aligned} W_i &= V_i \otimes \mathbb{F}, & &\text{for } 1 \le i \le d-2, \\ W_d &= \mathrm{span}_{\mathbb{F}}(e_1, \dots, e_{d-1}, g), \\ W_{d'} &= \mathrm{span}_{\mathbb{F}}(e_1, \dots, e_{d-1}, h). \end{aligned} \tag{4.1.21}$$

Let $W_{d-1} = V_{d-1} \otimes \mathbb{F}$, we claim that

$$\begin{aligned} \mathrm{Stab}(W.) &= \mathrm{Stab}(W_1 \subset \cdots \subset W_{d-2} \subset W_{d-1} \subset W_d), \\ &= \mathrm{Stab}(W_1 \subset \cdots \subset W_{d-2} \subset W_{d-1} \subset W_{d'}). \end{aligned} \tag{4.1.22}$$

If $g \in \mathrm{Stab}(W.)$, then g stabilizes W_{d-1} because $W_{d-1} = W_d \cap W_{d'}$. Conversely, if g stabilizes W_{d-1} (and $W_{d-1}^{\perp}$), then $gW_d = W_d$ and $gW_{d'} = W_{d'}$ automatically, because there are exactly two isotropic lines in the hyperbolic plane $W_{d-1}^{\perp}/W_{d-1}$, the element g would have negative determinant if $gW_d = W_{d'}$. So we prove the claim. Therefore we have $B = \mathrm{Stab}(W.)$.

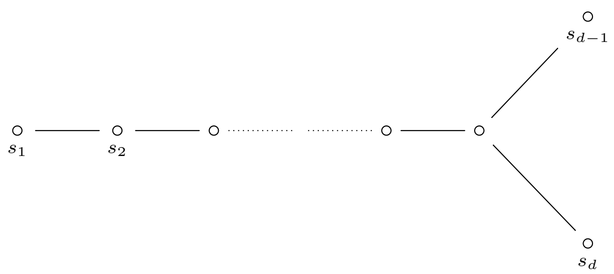
The absolute Weyl group W can be identified as the subgroup of S_l:

$$W = \left\{ w \in S_l \;\middle|\; \begin{array}{l} w(i) + w(l+1-i) = l+1, \\ \#\{i, 1 \le i \le d : w(i) > d\} \text{ is even} \end{array} \right\}. \tag{4.1.23}$$

Let $\mathbb{S} = \{s_i : 1 \le i \le d\}$ be the set of simple reflections, where

$$s_i = \begin{cases} (i, i+1)(l-i, l-i+1), & 1 \le i \le d-1, \\ (d, d+2)(d-1, d+1), & i = d. \end{cases} \tag{4.1.24}$$

The Dynkin diagram of type D_d is



(4.1.25)

Note that the standard parabolic subgroup $P_{d'}$ corresponding to $\mathbb{S} - \{s_{d-1}\}$ is the stabilizer of $W_{d'}$, and the standard parabolic subgroup P_d corresponding to $\mathbb{S} - \{s_d\}$ is the stabilizer of W_d. The Frobenius δ acts on the Weyl group W by exchanging s_{d-1} and s_d, but fixing all the other s_i's.

Using the same trick as Example 4.1.2, two Borel subgroups $B_1 = \mathrm{Stab}(U_\bullet)$ and $B_2 = \mathrm{Stab}(U'_\bullet)$ are in relative position w if and only if there exists a $\mathbb{F}$-basis $\{f_1, \ldots, f_l\}$ of $V_{\mathbb{F}}$, such that

$$\begin{aligned} U_i &= \mathrm{span}(f_1, \ldots, f_i), \\ U_d &= \mathrm{span}(f_1, \ldots, f_{d-1}, f_d), \\ U_{d'} &= \mathrm{span}(f_1, \ldots, f_{d-1}, f_{d+1}), \end{aligned} \tag{4.1.26}$$

and

$$\begin{aligned} U'_i &= \mathrm{span}(f_{w(1)}, \ldots, f_{w(i)}), \\ U'_d &= \mathrm{span}(f_{w(1)}, \ldots, f_{w(d-1)}, f_{w(d)}), \\ U'_{d'} &= \mathrm{span}(f_{w(1)}, \ldots, f_{w(d-1)}, f_{w(d+1)}), \end{aligned} \tag{4.1.27}$$

and U_d and U'_d lie in the same $\mathrm{SO}(V)$-orbit in the set of all maximal isotropic subspaces of $V_{\mathbb{F}}$. Note that the Frobenius δ exchanges U_d and $U_{d'}$, hence $U_{d-1} = U_d \cap \delta(U_d) = U_{d'} \cap \delta(U_{d'})$.

Let $w_1 = s_{d-1} \cdots s_2 s_1 = (d, \ldots, 1)(d+1, \ldots, l)$, then a flag $U_\bullet$ lying in the Deligne-Lusztig variety $X(w_1)$ is of the form:

$$\begin{aligned} U_{d-i} &= U_{d'} \cap \delta(U_{d'}) \cap \cdots \cap \delta^i(U_{d'}), \\ &= U_d \cap \delta(U_d) \cap \cdots \cap \delta^i(U_d), \end{aligned} \tag{4.1.28}$$

for $1 \leq i \leq d$. Therefore we have a commutative diagram

$$\begin{array}{ccc} X(w_1) & \hookrightarrow & \mathrm{SO}(V)/B \\ & \overset{\phi}{\searrow} & \downarrow \\ & & \mathrm{SO}(V)/P, \end{array} \tag{4.1.29}$$

where P is P_d (resp. $P_{d'}$), ϕ is an injection sending $U_\bullet$ to U_d (resp. $U_{d'}$). And $\mathrm{im}(\phi)$ is the subvariety of $\mathrm{SO}(V)/P$ parameterizing all the maximal isotropic subspaces U of $V_{\mathbb{F}}$ such that U lies in the $\mathrm{SO}(V)$-orbit of W_d (resp. $W_{d'}$) and

$$\dim(U \cap \delta(U) \cap \cdots \cap \delta^i(U)) = d - i, \tag{4.1.30}$$

for $1 \leq i \leq d$.

Let $w_2 = s_d s_{d-2} \ldots s_2 s_1$, by the same procedure as above we can show that a flag $U_\bullet \in X(w_2)$ is completely determined by U_d (or equivalently, by $U_{d'}$):

$$U_{d-i} = U_d \cap \delta(U_d) \cap \cdots \cap \delta^{i-1}(U_d), \tag{4.1.31}$$

where $1 \leq i \leq d$. We have the following commutative diagram:

$$\begin{array}{ccc} X(w_2) & \hookrightarrow & \mathrm{SO}(V)/B \\ & \overset{\phi}{\searrow} & \downarrow \\ & & \mathrm{SO}(V)/P_d, \end{array} \tag{4.1.32}$$

where $\phi\colon U_\bullet \mapsto U_d$ is injective.

Returning to the general case, let P_I be the standard parabolic subgroup of H, where I is a subset of the set of simple reflections $\mathbb{S}$ of W_H. Let W_I be the subgroup of W_H generated by simple reflections in I, W^I (resp. ${}^I W$) the set of minimal length representatives of the cosets in W_H/W_I (resp. $W_I \backslash W_H$). Let ${}^I W^J$ denote ${}^I W \cap W^J$. Then we can define the generalized Deligne-Lusztig varieties.

Definition 4.1.5. For each $w \in W_H$, the *generalized Deligne-Lusztig variety* $X_{P_I}(w)$ is defined as

$$X_{P_I}(w) := \{g \in H/P_I : g^{-1}\sigma(g) \in P_I w P_{\sigma(I)}\}. \tag{4.1.33}$$

Proposition 4.1.6 ([Hoe10, Lemma 2.1.3])**.** *For $w \in {}^I W^{\sigma(I)}$, the Deligne-Lusztig variety $X_{P_I}(w)$ is smooth of dimension $\ell(w) + \ell(W_{\sigma(I)}) - \ell(W_{I\cap {}^w\sigma(I)})$, where $\ell(W_J)$ denotes the maximal length of elements in W_J for $J \subset \mathbb{S}$.*

The partial flag variety H/P_I can be written as the disjoint union of all such Deligne-Lusztig varieties:

$$H/P_I = \bigcup_{w \in {}^I W^{\sigma(I)}} X_{P_I}(w). \tag{4.1.34}$$

However, this partition is very coarse.

Example 4.1.7. Let us look at Example 4.1.2 again, now $H/P_1 = \mathbb{P}(V)$, and we have the partition

$$\mathbb{P}(V) = X_{P_1}(\mathrm{id}) \cup X_{P_1}(s_1), \tag{4.1.35}$$

where $X_{P_1}(\mathrm{id})$ is the set of all rational lines of V, $X_{P_1}(s_1)$ is the set of all unrational lines. And obviously, the image of $X(w)$, where $w = (1, 2, \ldots, l)$, in $\mathbb{P}(V)$ is much smaller than $X_{P_1}(w)$. Recall that the image of $X(w)$ in $\mathbb{P}(V)$ is the set of lines L such that $L \oplus \sigma(L) \oplus \sigma^2(L) \oplus \cdots \oplus \sigma^{l-1}(L) = V$ by Example 4.1.2.

To get finer partition of the partial flag variety H/P_I, the refinement of parabolic subgroups is used. For two parabolic subgroups P and Q, we define *the refinement of P with respect to Q*

$$P^Q := (P \cap Q) \cdot U_P, \tag{4.1.36}$$

which is parabolic again (cf. [Bor91, Proposition 14.22]). If P is of type $I \subset \mathbb{S}$, Q is of type $J \subset \mathbb{S}$, then P^Q is of type $I \cap {}^w J$, where $w \in {}^I W^J$ is the relative position of P and Q (cf. [Béd85, Theorem 4]). The idea to define the fine Deligne-Lusztig varieties is to consider not only the relative position of P and $\sigma(P)$, but also their refinements. To state the definition of fine Deligne-Lusztig varieties, we need some combinatorial results.

Definition 4.1.8 (Lusztig-Bédard sequence)**.** For $I \subset \mathbb{S}$, let $\mathcal{T}(I, \sigma)$ be the set of sequences $(I_i, w_i)_{i \geq 0}$ such that

- $I_0 = I$, $w_0 \in {}^I W^{\sigma(I)}$,
- $I_{i+1} = I_i \cap {}^{w_i}\sigma(I_i)$, for $i \geq 0$,
- $w_i \in {}^{I_i} W^{\sigma(I_i)}$ and $w_{i+1} \in W_{I_{i+1}} w_i W_{\sigma(I_i)}$, for $i \geq 1$.

Let $I_\infty := I_i$, $w_\infty := w_i$ for $i \gg 0$. Then we have a bijection $\mathcal{T}(I,\sigma) \to {}^IW$ given by $(I_i, w_i)_{i\geq 0} \mapsto w_\infty$ (cf. [Béd85, Proposition 9]).

For a parabolic subgroup P of type I, we associate a sequence of parabolic subgroups:

$$\begin{aligned} P^0 &:= P, \\ P^{i+1} &:= (P^i)^{\sigma(P^i)}, \end{aligned} \tag{4.1.37}$$

for any integer $i \geq 0$, a sequence $w_0, w_1, w_2, \dots$ in W, where w_i is the relative position of P^i and $\sigma(P^i)$, a sequence $I_0 \supset I_1 \supset \cdots$ of subsets of $\mathbb{S}$, where I_i is the type of P^i. Then the sequence $(I_i, w_i)_{i\geq 0} \in \mathcal{T}(I,\sigma)$. Let $P^\infty := P^i$ for $i \gg 0$.

Definition 4.1.9. For $w \in {}^IW$, let $\mathbf{t} := (I_i, w_i)$ be the corresponding Lusztig-Bédard sequence, we define the *fine Deligne-Lusztig variety* $\mathcal{P}_{I,w}$ as the locally closed subscheme of H/P_I

$$\mathcal{P}_{I,w} := \{P \in H/P_I : P^i \text{ and } \sigma(P^i) \text{ are in relative position } w_i, \forall i \geq 0\}. \tag{4.1.38}$$

We also use the notation $\mathcal{P}_I^{\mathbf{t}}$ to denote the fine Deligne-Lusztig variety $\mathcal{P}_{I,w}$.

Let $\mathbf{t} := (I_i, w_i)$ be a Lusztig-Bédard sequence in $\mathcal{T}(I,\sigma)$, for $n \geq 1$, we define the n-truncated sequence $\mathbf{t}_n := (I'_i, w'_i)$ as follows:

$$I'_i := I_{i+n},\ w'_i := w'_{i+n}. \tag{4.1.39}$$

Then $\mathbf{t}_n \in \mathcal{T}(I_n,\sigma)$. Let $\mathbf{t}_\infty := (I_\infty, w_\infty)$ be the constant sequence, in this case, $\mathcal{P}_{I_\infty}^{\mathbf{t}_\infty}$ is the generalized Deligne-Lusztig variety $X_{P^\infty}(w_\infty)$.

Proposition 4.1.10 ([Lus07, 4.2])**.** *The map*

$$\begin{aligned} \mathcal{P}_I^{\mathbf{t}} &\longrightarrow \mathcal{P}_{I_1}^{\mathbf{t}_1}, \\ P &\longmapsto P^1, \end{aligned} \tag{4.1.40}$$

defines an isomorphism between fine Deligne-Lusztig varieties, whose inverse map is the natural projection. Furthermore, the map $P^1 \mapsto P^\infty$ defines an isomorphism:

$$\mathcal{P}_{I,w} \longrightarrow \mathcal{P}_{I_\infty,w_\infty} = X_{P^\infty}(w_\infty). \tag{4.1.41}$$

Proposition 4.1.11 ([He09, Theorem 3.1])**.** *Let* $\mathrm{pr}\colon H/B \to H/P_I$ *be the natural projection, then for* $w \in {}^IW$, $\mathrm{pr}(X(w)) = \mathcal{P}_{I,w}$ *and we have*

$$\overline{\mathcal{P}_{I,w}} = \bigcup_{v \leq_{I,\sigma} w} \mathcal{P}_{I,v}, \tag{4.1.42}$$

where $\leq_{I,w}$ is the natural partial order on ${}^IW$ (cf. [He09, 1.4]).

Example 4.1.12. Let $X(w)$ be the Deligne-Lusztig variety in Example 4.1.3, the map ϕ is an isomorphism from $X(w)$ to the fine Deligne-Lusztig variety $\mathcal{P}_{I,w}$, whose inverse map is $P \mapsto P^\infty$. The Lusztig-Bédard sequence is $(I_i, w_i)_{i\geq 0}$, where

$$\begin{aligned} w_i &= s_d \cdots s_{d-i}, \\ I_i &= \{s_1, \dots, s_{d-i-1}\}, \end{aligned} \tag{4.1.43}$$

for $0 \leq i \leq d-1$. And $w_\infty = w$, $I_\infty = \emptyset$. The even orthogonal case can be computed similarly.

Lemma 4.1.13 ([Hoe10, Proposition 2.1.9])**.** *For $w \in {}^IW$, the fine Deligne-Lusztig variety $\mathcal{P}_{I,w}$ consists of all $P \in H/P_I$ such that P and P_∞ are in relative position w. In other words,*

$$\mathcal{P}_{I,w} = \{g \in H/P_I : g^{-1}\sigma(g) \in P_I w P_{\sigma(I_\infty)}\}. \tag{4.1.44}$$

Remark 4.1.14. By Lemma 4.1.13, consider the Görtz local model diagram,

$$\begin{array}{ccc} & H & \\ {\scriptstyle L_{\mathrm{opp}}} \swarrow & & \searrow {\scriptstyle \mathrm{pr}} \\ H/P_I & & H/P_I, \end{array} \tag{4.1.45}$$

where pr is the natural projection and L_{opp} is the opposite Lang map $g \mapsto (g^{-1}\sigma(g))^{-1}$. Then L_{opp} and pr are smooth morphisms with the same relative dimension, and

$$L_{\mathrm{opp}}^{-1}(P_{\sigma(I_\infty)} w^{-1} P_I/P_I) = \{g \in H : g^{-1}\sigma(g) \in P_I w P_{\sigma(I_\infty)}\} = \mathrm{pr}^{-1}(\mathcal{P}_{I,w}), \tag{4.1.46}$$

by Lemma 4.1.13. We see that the fine Deligne-Lusztig variety $\mathcal{P}_{I,w}$ is smoothly equivalent to the (generalized) Schubert cell $P_{\sigma(I_\infty)} w^{-1} P_I/P_I$ in the partial flag variety H/P_I. However, by Proposition 4.1.11, the closure $\overline{\mathcal{P}_{I,w}}$ is smoothly equivalent to

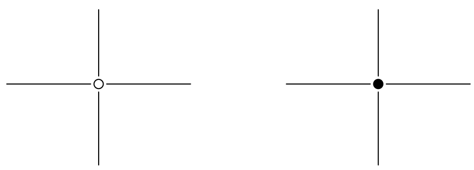
$$\bigcup_{v \le_{I,\sigma} w} P_{\sigma(I_\infty^v)} v^{-1} P_I/P_I, \tag{4.1.47}$$

where I_∞^v is the limit of the Lusztig-Bédard sequence associated to v. In general, (4.1.47) is not a Schubert variety. In fact, the closure of a fine Deligne-Lusztig variety is not normal in general. For example, consider the case GL_3, then we have the decomposition into fine Deligne-Lusztig varieties

$$\mathbb{P}(V) = \mathcal{P}_{\mathrm{id}} \cup \mathcal{P}_{s_1} \cup \mathcal{P}_{s_1 s_2}. \tag{4.1.48}$$

Note that $\mathcal{P}_{s_1}$ is not irreducible and its closure is $\mathcal{P}_{\mathrm{id}} \cup \mathcal{P}_{s_1}$ which is not normal. See the picture.

(4.1.49)



$\mathcal{P}_{s_1}$ $\overline{\mathcal{P}_{s_1}}$

Example 4.1.15 (The odd orthogonal group)**.** Notations are the same as in Example 4.1.3. We will show that the closure of the fine Deligne-Lusztig variety $\mathcal{P}_w := \phi(X(w))$ is normal using the same method as [GH15, Proposition 7.3.2]. Consider the inclusion of closures

$$\overline{\mathcal{P}_w} \subset \overline{X_P(w)}. \tag{4.1.50}$$

The variety $X_P(w)$ is irreducible, so the inclusion (4.1.50) is an equality if and only if $\ell(w) = \dim(X_P(w))$. In this case, let $I = \{s_1, s_2, \ldots, s_{d-1}\}$ be the type of P, $w_{\min} = s_d$ the minimal representative of w in ${}^I W^{\sigma(I)}$. Note that σ acts on the Dynkin diagram trivially. Then $I \cap {}^{w_{\min}}\sigma(I) = \{s_1, s_2, \ldots, s_{d-2}\}$. Therefore

$$\dim(X_P(w)) = 1 + \frac{d(d-1)}{2} - \frac{(d-2)(d-1)}{2} = \ell(w). \tag{4.1.51}$$

Then the closure $\overline{\mathcal{P}_w}$ is normal, and has isolated singularities by [GH15, Proposition 7.3.2]. The closure $\overline{\mathcal{P}_w}$ can be described as the subvariety of $\mathrm{SO}(V)/P$ parameterizing all the d-dimensional isotropic subspaces V_d such that

$$\dim(V_d \cap \sigma(V_d)) \geq d - 1. \tag{4.1.52}$$

Example 4.1.16. Notations are the same as Example 4.1.4. Let $\mathcal{P}_{w_1} := \phi(X(w_1))$ the fine Deligne-Lusztig variety. Let $I = \{s_1, s_2, \ldots, s_{d-2}, s_d\}$ be the type of P_{w_1}. Note that the Frobenius δ exchanges s_{d-1} and s_d, and fixes all the other s_i's. Let $w_{\min} = 1$ the minimal representative of w_1 in ${}^I W^{\delta(I)}$. Then $I \cap {}^{w_{\min}}\delta(W) = \{s_1, s_2, \ldots, s_{d-2}\}$. We have

$$\dim(X_{P_{d'}}(w_1)) = \frac{d(d-1)}{2} - \frac{(d-2)(d-1)}{2} = \ell(w_1). \tag{4.1.53}$$

Therefore the closure $\overline{\mathcal{P}_{w_1}}$ is normal. Furthermore, $\overline{\mathcal{P}_{w_1}}$ is smooth because $\delta(P_{d'}) = P_d$. The closure $\overline{\mathcal{P}_{w_1}} = X_{P_{d'}}(\mathrm{id})$ can be described as the subvariety of $\mathrm{SO}(V)/P_{d'}$ parameterizing all the maximal isotropic subspaces U of V such that U lies in the $\mathrm{SO}(V)$-orbit of W_d and

$$\dim(U \cap \delta(U)) = d - 1. \tag{4.1.54}$$

Remark 4.1.17. The odd orthogonal case has been listed in [GH15, Proposition 7.3.2], which corresponds to the triple $(\tilde{C}_d, \omega_1^\vee, \mathbb{S}, \mathrm{id})$. However, the even orthogonal case is not in the list, which corresponds to the triple $(\tilde{B}_m, \omega_1^\vee, \mathbb{S}, \mathrm{id})$. Furthermore, by the same procedure, it is easy to check that the triples $(\tilde{D}_l, \omega_1^\vee, \mathbb{S}, \mathrm{id})$ and $(\tilde{D}_l, \omega_1^\vee, \mathbb{S}, \sigma_0)$ should be also included in the list of [GH15, Proposition 7.3.2]. The extra three smooth cases make the list complete.

4.2 The group-theoretic approach

In this section, we collect some results from [GH15]. There is no harm to look at only one connected component of the Rapoport-Zink space, because all connected components are isomorphic to each other by (3.4.14). By (3.5.18), we may work with the group $G_{\mathrm{ad}}(L)'$ instead of $G_{\mathrm{ad}}(L)$. All notations are the same as in previous sections.

Let $\tilde{\mathbb{S}} = \{s_0, s_1, \ldots, s_m\}$ be the set of affine simple reflections in $\tilde{W}_{\mathrm{ad}}$. For $Z \subset \tilde{\mathbb{S}}$, we denote P_Z the corresponding standard parahoric subgroup of $G_{\mathrm{ad}}(L)$. We will write $\mathrm{EO}_{\mathrm{cox}}$ instead of $\mathrm{EO}^{\mathbb{S}}_{\sigma,\mathrm{cox}}(\mu_{\mathrm{ad}})$ in [GH15, 5.1] to lighten the notations because in our case σ acts on the affine Dynkin diagram trivially. Let $\mathbb{J} = J_{\mathrm{ad}}(\mathbb{Q}_p)'$.

Proposition 4.2.1.

$$X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})'_{K_{\mathrm{ad}}} = \biguplus_{\Sigma \in \mathscr{J}} \ \biguplus_{j \in \mathbb{J}/\mathbb{J} \cap P_{\tilde{\mathbb{S}} \setminus \Sigma}} j \cdot Y(w_\Sigma) \tag{4.2.1}$$

Instead of reinventing the wheel, we will explain the meaning of this proposition in terms of a series of lemmas.

For any *affine Deligne-Lusztig variety*

$$X_w(b) \coloneqq \{g \in G_{\mathrm{ad}}(L)'/I : g^{-1}b\sigma(g) \in IwI\} \tag{4.2.2}$$

(cf. [Rap05, Definition 4.1]), its image

$$X_w^f(b) \coloneqq \{g \in G_{\mathrm{ad}}(L)'/K_{\mathrm{ad}} : g^{-1}b\sigma(g) \in K_{\mathrm{ad}} \cdot_\sigma IwI\} \tag{4.2.3}$$

under the map $G_{\mathrm{ad}}(L)'/I \to G_{\mathrm{ad}}(L)'/K_{\mathrm{ad}}$ is called the *fine affine Deligne-Lusztig variety*, where the superscript f stands for "fine".

The first step is to show that $X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})'_{K_{\mathrm{ad}}}$ is a disjoint union of fine affine Deligne-Lusztig varieties.

Lemma 4.2.2 ([GH15, Theorem 5.1.2])**.**

$$X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})'_{K_{\mathrm{ad}}} = \biguplus_{w \in \mathrm{EO}_{\mathrm{cox}}} X_w^f(b_{\mathrm{ad}}) \tag{4.2.4}$$

Let $\mathrm{EO}(\mu)$ be the set $\mathrm{Adm}^\circ(\mu) \cap {}^J\tilde{W}_{\mathrm{ad}}$. Let $\mathrm{EO}_{\mathrm{cox}}$ be the subset of $\mathrm{EO}(\mu)$ consisting of those w such that $\mathrm{supp}_\sigma(w)$ is a proper subset of $\tilde{\mathbb{S}}$ and w is a σ-Coxeter element of $W_{\mathrm{supp}_\sigma(w)}$. Then to prove Lemma 4.2.2 is just to show that the triple $(G_{\mathrm{ad}}, \mu_{\mathrm{ad}}, \mathbb{S})$ is of Coxeter type, which has been completely listed in [GH15, Theorem 5.1.2].

Let τ be the image of b_{ad} in Ω. For $v \in \tilde{\mathbb{S}}$ let $d(v)$ be the minimal distance between the $\tau\sigma$-orbit containing v and the vertex outside $\mathbb{S}$. Let $\mathscr{J}$ be the set of subsets Σ of $\tilde{\mathbb{S}}$, that is $\tau\sigma$-stable and $d(v) = d(v')$ for any $v, v' \in \Sigma$. For $\Sigma \in \mathscr{J}$ let $d(\Sigma) \coloneqq d(v)$ for some $v \in \Sigma$, $\Sigma^\flat$ the union of all the $\tau\sigma$-orbits Σ' that is not contained in Σ and $d(\Sigma') \leq d(\Sigma)$, $\Sigma^\sharp$ the union of all the $\tau\sigma$-orbits Σ' such that $d(\Sigma') > d(\Sigma)$.

Lemma 4.2.3 ([GH15, Proposition 7.1.1])**.** *The map*

$$\mathscr{J} \longrightarrow \mathrm{EO}_{\mathrm{cox}}, \tag{4.2.5}$$
$$\Sigma \longmapsto w_\Sigma, \tag{4.2.6}$$

is bijective, where w_Σ *is the unique element in* $\mathrm{EO}_{\mathrm{cox}}$ *such that* $\mathrm{supp}_\sigma(w_\Sigma) = \Sigma^\flat$. *We have* $\ell(w_\Sigma) = d(\Sigma)$.

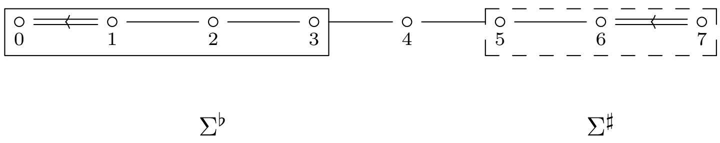
(a) odd case

In this case, τ is identity.

$$\mathrm{EO}_{\mathrm{cox}} = \{1, s_0, s_0s_1, \ldots, s_0s_1\ldots s_{m-1}\}. \tag{4.2.7}$$
$$\mathscr{J} = \{\{s_0\}, \{s_1\}, \ldots, \{s_m\}\}. \tag{4.2.8}$$

If $\Sigma = \{s_i\} \in \mathscr{J}$ for some i, then $\Sigma^\flat = \{s_0, \ldots, s_{i-1}\}$ if $i > 0$ or empty otherwise; $\Sigma^\sharp = \{s_{i+1}, \ldots, s_m\}$ if $i < m$ or empty otherwise; $w_\Sigma = s_0s_1\ldots s_{i-1}$ if $i > 0$ or 1 otherwise.

For example, if $m = 7$, $\Sigma = \{s_4\}$, then $\Sigma^\flat = \{s_0, s_1, s_2, s_3\}$ and $\Sigma^\sharp = \{s_5, s_6, s_7\}$. See the diagram (4.2.9), where $\Sigma^\flat$ is surrounded by the solid frame, and $\Sigma^\sharp$ is surrounded by the dashed frame.



(4.2.9)

(b) even case

In this case, τ switches between s_0 and s_1, and fixes all the other vertices.

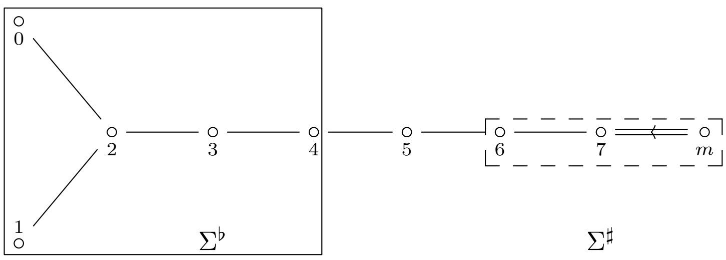
$$\mathrm{EO}_{\mathrm{cox}} = \{\tau, s_0\tau, s_0s_2\tau, \dots, s_0s_2\dots s_{m-1}\tau\}. \tag{4.2.10}$$

$$\mathscr{J} = \{\{s_0, s_1\}, \{s_2\}, \dots, \{s_m\}\}. \tag{4.2.11}$$

For $\Sigma \in \mathscr{J}$, if $\Sigma = \{s_0, s_1\}$, then $\Sigma^\flat = \emptyset$, $\Sigma^\sharp = \{s_2, \dots, s_m\}$, $w_\Sigma = \tau$; if $\Sigma = \{s_i\}$ for some $i > 1$, then $\Sigma^\flat = \{s_0, s_1, \dots, s_{i-1}\}$, $\Sigma^\sharp = \{s_{i+1}, \dots, s_m\}$ if $i < m$ or empty otherwise $w_\Sigma = s_0s_2\dots s_{i-1}\tau$.

For example, if $m = 8$, $\Sigma = \{s_5\}$, then $\Sigma^\flat = \{s_0, s_1, \dots, s_4\}$ and $\Sigma^\sharp = \{s_6, s_7, s_8\}$. See the diagram (4.2.12), where $\Sigma^\flat$ is surrounded by the solid frame, and $\Sigma^\sharp$ is surrounded by the dashed frame.

(4.2.12)



Lemma 4.2.4 ([GH15, Proposition 2.2.1])**.** *For each $w \in \mathrm{EO}_{\mathrm{cox}}$, we have*

$$X_w(b_{\mathrm{ad}}) = \coprod_{j \in \mathbb{J}/\mathbb{J}\cap P_{\Sigma^\flat}} j \cdot Y(w), \tag{4.2.13}$$

where

$$Y(w) = \{g \in P_{\Sigma^\flat}/I : g^{-1}b_{\mathrm{ad}}\sigma(g) \in IwI\} \tag{4.2.14}$$

is a Deligne-Lusztig variety in $P_{\Sigma^\flat}/I$.

The key step of proving Lemma 4.2.4 is, by Lang's theorem ([He14, Lemma 3.2]) and [He14, Proposition 4.5], for each $g \in X_w(b_{\mathrm{ad}})$, there exists some $x \in P_{\mathrm{supp}_\sigma(w)}$ such that $x^{-1}g^{-1}b_{\mathrm{ad}}\sigma(g)\sigma(x) = b_{\mathrm{ad}}$, i.e. $g \in \mathbb{J}P_{\mathrm{supp}_\sigma(w)}$.

Remark 4.2.5. Note that the reductive quotient $\bar{P}_{\Sigma^\flat}$ of $P_{\Sigma^\flat}$ has the Dynkin diagram $\Sigma^\flat$, i.e. when $n = 2m+1$ is odd, $\bar{P}_{\Sigma^\flat}$ is an algebraic group of type B_m; when $n = 2m$, $\bar{P}_{\Sigma^\flat}$ is of type D_m.

Remark 4.2.6. When $n = 2m + 1$ is odd, w_Σ is a Coxeter element of $W_{\Sigma^\flat}$. However, when $n = 2m$, w_Σ is not a Coxeter element, but a σ-Coxeter element.

Let $w' := w\tau^{-1}$, $\delta := \tau\sigma\tau^{-1}$, then δ is the twisted Frobenius on the reductive quotient $\bar{P}_{\Sigma^\flat}$, and w' is a δ-twisted Coxeter element. We have

$$Y(w) = \{g \in \bar{P}_{\Sigma^\flat}/\bar{I} : g^{-1}\delta(g) \in \bar{I}w'\bar{I}\}, \tag{4.2.15}$$

where $\bar{I}$ is the image of I in $\bar{P}_{\Sigma^\flat}$. Note that $\bar{I}$ is a δ-stable Borel subgroup of $\bar{P}_{\Sigma^\flat}$.

Lemma 4.2.7 ([GH15, Theorem 4.1.2 (1)])**.** *For each $w = w_\Sigma \in \mathrm{EO}_{\mathrm{cox}}$, we have*

$$X_w^f(b_{\mathrm{ad}}) \cong \{g \in G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp} : g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp}wP_{\Sigma^\sharp}\}. \tag{4.2.16}$$

Let P and Q be two parahoric subgroups of $G_{\mathrm{ad}}(L)'$, similarly to the parabolic case, we define the refinement of P with respect to Q as

$$P^Q := (P \cap Q) \cdot U_P, \tag{4.2.17}$$

where U_P is the pro-unipotent radical of P. The group P^Q is a parahoric subgroup of $G_{\mathrm{ad}}(L)'$ again, and its pro-unipotent radical is $(P \cap U_Q) \cdot U_P$, which can be proved analogously. Görtz-He generalize the Lusztig-Bédard sequence to the case of affine Weyl groups.

Definition 4.2.8 (Lusztig-Bédard sequence)**.** For $J \subset \tilde{\mathbb{S}}$, let $\mathcal{T}(J, \tau\sigma)$ be the set of sequences $(J_i, w_i)_{i \geq 0}$ such that

(a) $J_0 = J$, $w_0 \in {}^J W_a{}^{\tau\sigma(J)}$,

(b) $J_{i+1} = J_i \cap {}^{w_i}(\tau\sigma(J_i))$, for $i \geq 0$,

(c) $w_i \in {}^{J_i}W_a{}^{\tau\sigma(J_i)}$ and $w_{i+1} \in W_{J_{i+1}}w_iW_{\tau\sigma(J_i)}$, for $i \geq 1$.

Let $J_\infty := J_i$, $w_\infty := w_i$, for $i \gg 0$, then the map $(J_i, w_i) \mapsto w_\infty$ defines a bijection $\mathcal{T} \to {}^J W_a$. For each parahoric subgroup P of type J, we associate a sequence of parahoric subgroups

$$P^0 := P, \quad P^{i+1} := (P^i)^{b_{\mathrm{ad}}\sigma(P^i)b_{\mathrm{ad}}^{-1}} \text{ for } i \geq 0, \tag{4.2.18}$$

and a sequence (J_i, w_i), where J_i is the type of P^i and w_i is the relative position of P^i and $b_{\mathrm{ad}}\sigma(P^i)b_{\mathrm{ad}}^{-1}$. Then (J_i, w_i) is a Lusztig-Bédard sequence. Let P^∞ denote P^m for $m \gg 0$. Then P^∞ is of type $J_\infty := J_m$ for $m \gg 0$.

Now for each $g \in X_w^f(b_{\mathrm{ad}})$, let $P := {}^gK_{\mathrm{ad}}$ be the corresponding parahoric subgroup of $G_{\mathrm{ad}}(L)'$. We write $w = x\tau$ for some $x \in {}^J W_a$ and $\tau \in \Omega$. Recall that $w = w_\Sigma$. Then the type of P is $J_0 = \mathbb{S}$. By the same procedure as in previous paragraph, we get the Lusztig-Bédard sequence (J_n, x_n), and by [He07, Lemma 1.4], $J_\infty = \Sigma^\sharp$. Then, by [Lus07, 4.2 (c)(d)], the map ${}^gK_{\mathrm{ad}} \mapsto ({}^gK_{\mathrm{ad}})^\infty$ gives the isomorphism:

$$\theta\colon X_w^f(b_{\mathrm{ad}}) \xrightarrow{\cong} \{g \in G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp} : g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp}wP_{\Sigma^\sharp}\}, \tag{4.2.19}$$

whose inverse map is the natural projection map, i.e. the following diagram is commutative:

$$\begin{array}{ccc} X_w(b_{\mathrm{ad}}) & \hookrightarrow & G_{\mathrm{ad}}(L)'/I \\ \downarrow & & \downarrow \\ \{g \in G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp} : g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp}wP_{\Sigma^\sharp}\} & \hookrightarrow & G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp} \\ \downarrow \cong & & \downarrow \\ X_w^f(b_{\mathrm{ad}}) & \hookrightarrow & G_{\mathrm{ad}}(L)'/K_{\mathrm{ad}}. \end{array} \tag{4.2.20}$$

Remark 4.2.9. Using the same trick as Example 4.1.3 and Example 4.1.4, we can describe the map θ in (4.2.19) in terms of lattices. Let $g \in G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp}$ such that $g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp}wP_{\Sigma^\sharp}$.

1. When $n = 2m+1$ is odd, let $\Sigma = \{s_i\}$ for some $0 \leq i \leq m$, then $P_{\Sigma^\sharp}$ is the stabilizer of the lattice chain

$$\Lambda_{m-i} \subset \cdots \subset \Lambda_{m-1} \subset \Lambda_m, \tag{4.2.21}$$

Let $\Delta_j := g \cdot \Lambda_j$, then ${}^gP_{\Sigma^\sharp}$ is the stabilizer of the lattice chain

$$\Delta_{m-i} \subset \cdots \subset \Delta_{m-1} \subset \Delta_m. \tag{4.2.22}$$

The condition $g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp}wP_{\Sigma^\sharp}$ is equivalent to the condition that the pair $({}^gP_{\Sigma^\sharp}, b_{\mathrm{ad}}\sigma({}^gP_{\Sigma^\sharp}))$ lies in the $G_{\mathrm{ad}}(L)'$-orbit of $(P_{\Sigma^\sharp}, wP_{\Sigma^\sharp})$. So the lattice chain $\Delta_\bullet$ is of the form:

$$\Delta_{m-j} = \Delta_m \cap (b_{\mathrm{ad}}\sigma)(\Delta_m) \cap \cdots \cap (b_{\mathrm{ad}}\sigma)^j(\Delta_m), \tag{4.2.23}$$

for $1 \leq j \leq i$ and $\Delta_{m-i} = (b_{\mathrm{ad}}\sigma)(\Delta_{m-i})$. Therefore, the map θ is

$$\Delta_m \longmapsto \Delta_\bullet, \tag{4.2.24}$$

such that $\Delta_\bullet$ satisfies condition (4.2.23).

2. When $n = 2m$ is even, let $\Sigma = \{s_i\}$ for some $2 \leq i \leq m$, then $P_{\Sigma^\sharp}$ is the stabilizer of the lattices

$$\Lambda_{m-i} \subset \cdots \subset \Lambda_{m-2} \subset (\Lambda_m \text{ and } \Lambda_{m'}) \subset \Lambda_{m-2}^\vee \subset \cdots \subset \Lambda_{m-i}^\vee. \tag{4.2.25}$$

Note that $\Lambda_m \cap \Lambda_{m'} = \Lambda_{m-1}$, and the lattice $\Lambda_{m'}$ is uniquely determined by Λ_{m-1} and Λ_m, because there are exactly two isotropic lines in the hyperbolic plane $\Lambda_{m-1}^\vee/\Lambda_{m-1}$. We also use the notation $\Lambda_m^\dagger$ to denote the unique isotropic line (hence the lattice) determined by Λ_m and Λ_{m-1}. Let $\Delta_j := g \cdot \Lambda_j$ for $m - i \leq j \leq m$ and $j = m'$. Then the condition $g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp}wP_{\Sigma^\sharp}$ implies that $\Delta_\bullet$ is of the form:

$$\Delta_{m'} = b_{\mathrm{ad}}\sigma(\Delta_m) \tag{4.2.26}$$
$$\Delta_{m-j} = \Delta_m \cap (b_{\mathrm{ad}}\sigma)(\Delta_m) \cap \cdots \cap (b_{\mathrm{ad}}\sigma)^j(\Delta_m)$$

for $1 \leq j \leq i$ and $\Delta_{m-i} = (b_{\mathrm{ad}}\sigma)(\Delta_{m-i})$. Therefore, the map θ is

$$\Delta_m \longmapsto \Delta_\bullet, \tag{4.2.27}$$

such that $\Delta.$ satisfies the condition described above.

For $\Sigma = \{s_0, s_1\}$, $P_{\Sigma^\sharp}$ is the stabilizer of the lattice chain:

$$\Lambda_{m-1} \subset \Lambda_m. \tag{4.2.28}$$

Note that $P_{\Sigma^\sharp}$ is also the stabilizer of the lattices

$$\Lambda_{m-1} \subset (\Lambda_m \text{ and } \Lambda_{m'}) \subset \Lambda_{m-1}^\vee. \tag{4.2.29}$$

Therefore, the map θ is

$$\Delta \longmapsto \Delta., \tag{4.2.30}$$

where $\Delta. = \{\Lambda_{m-1} \subset (\Lambda_m \text{ and } \Lambda_{m'}) \subset \Lambda_{m-1}^\vee\}$ satisfying $\Lambda_{m'} = b_{\mathrm{ad}}\sigma(\Delta_m)$.

Similarly to Lemma 4.2.4, the set $\{g \in G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp} : g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp} w P_{\Sigma^\sharp}\}$ can be decomposed into a disjoint union of some classical Deligne-Lusztig varieties, we have the following lemma:

Lemma 4.2.10 ([GH15, Corollary 4.6.2])**.** *For each $w \in \mathrm{EO}_{\mathrm{cox}}$, the fine Deligne-Lusztig variety*

$$X_w^f(b_{\mathrm{ad}}) \cong \coprod_{j \in \mathbb{J}/\mathbb{J}\cap P_{\tilde{\mathbb{S}}-\Sigma}} j \cdot Y_{\Sigma^\sharp}(w), \tag{4.2.31}$$

where

$$Y_{\Sigma^\sharp}(w) = \{g \in P_{\tilde{\mathbb{S}}-\Sigma}/P_{\Sigma^\sharp} : g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp} w P_{\Sigma^\sharp}\}. \tag{4.2.32}$$

Moreover, the natural projection $G_{\mathrm{ad}}(L)'/I \to G_{\mathrm{ad}}(L)'/K_{\mathrm{ad}}$ induces an isomorphism from $Y(w)$ in $P_{\Sigma^\flat}/I$ to $Y_{\Sigma^\sharp}(w)$, i.e. we have the following commutative diagram

$$\begin{array}{ccccc}
Y(w) & \hookrightarrow & X_w(b_{\mathrm{ad}}) & \hookrightarrow & G_{\mathrm{ad}}(L)'/I \\
\downarrow \cong & & \downarrow & & \downarrow \\
Y_{\Sigma^\sharp}(w) & \hookrightarrow & \{gP_{\Sigma^\sharp} : g^{-1}b_{\mathrm{ad}}\sigma(g) \in P_{\Sigma^\sharp} w P_{\Sigma^\sharp}\} & \hookrightarrow & G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp} \\
 & & \downarrow \cong & & \downarrow \\
 & & X_w^f(b_{\mathrm{ad}}) & \hookrightarrow & G_{\mathrm{ad}}(L)'/K_{\mathrm{ad}}.
\end{array} \tag{4.2.33}$$

Now let us consider the closure of $Y(w)$ in the partial flag variety $P_{\Sigma^\flat}/P_{\Sigma^\flat \cap \mathbb{S}}$.

Proposition 4.2.11. *For each $w = w_\Sigma \in \mathrm{EO}_{\mathrm{cox}}$, we have*

$$\overline{Y(w)} = \coprod_{(\Sigma')^\flat \subset \Sigma^\flat} \coprod_{\substack{(\mathbb{J}\cap P_{\tilde{\mathbb{S}}-\Sigma}) \cap {}^j(\mathbb{J}\cap P_{\tilde{\mathbb{S}}-\Sigma'}) \neq \emptyset, \\ j \in \mathbb{J}/(\mathbb{J}\cap P_{\tilde{\mathbb{S}}-\Sigma'})}} jY(w_{\Sigma'}). \tag{4.2.34}$$

Proof. Let $Q = P_{\Sigma^\flat \cap \mathbb{S}}$, $\mathrm{pr}\colon P_{\Sigma^\flat}/I \to P_{\Sigma^\flat}/Q$ the natural projection which is proper. Then

$$\overline{Y(w)} = \bigcup_{v \leq w} \mathrm{pr}(Y(v)). \tag{4.2.35}$$

The rest of the proof is exactly the same as [GH15, Theorem 7.2.1], so we omit it. ∎

Remark 4.2.12. For $i \in \mathbb{J}$, $i \cdot Y_{\Sigma^\sharp}(w)$ is a Deligne-Lusztig variety in the partial flag variety ${}^iP_{\tilde{\mathbb{S}}-\Sigma}/{}^iP_{\Sigma^\sharp}$, more precisely,

$$i \cdot Y_{\Sigma^\sharp}(w) = \{x \in {}^iP_{\tilde{\mathbb{S}}-\Sigma}/{}^iP_{\Sigma^\sharp} : x^{-1}b_{\mathrm{ad}}\sigma(x) \in {}^iP_{\Sigma^\sharp} iw\sigma(i)^{-1}({}^{\sigma(i)}P_{\Sigma^\sharp}), \quad (4.2.36)$$

which is isomorphic to $Y_{\Sigma^\sharp}(w)$. Using the same method as the proof of Proposition 4.2.11, it is easy to show that

$$\overline{i \cdot Y_{\Sigma^\sharp}(w)} = \coprod_{(\Sigma')^\flat \subset \Sigma^\flat} \coprod_{\substack{i \cap j \neq \emptyset, \\ j \in \mathbb{J}/(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma'})}} j \cdot Y(w_{\Sigma'}). \quad (4.2.37)$$

The closure relations can be described by the rational Bruhat-Tits building of $\mathbb{J}$.

Proposition 4.2.13 ([GH15, Proposition 7.2.2])**.** *Let $i, j \in \mathbb{J}$, $\Sigma, \Sigma' \in \mathscr{J}$, the following are equivalent*[1]*:*

1. *$i(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma}) \cap j(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma'}) \neq \emptyset$,*
2. *${}^i(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma}) \cap {}^j(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma'})$ contains an Iwahori subgroup of $\mathbb{J}$,*
3. *The faces in the building of $\mathbb{J}$ corresponding to ${}^i(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma})$ and ${}^j(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma'})$ are neighbors.*

4.3 Crucial lemma

Recall $\chi := \eta\pi^{-1}\mathcal{F}$ in section 3.3. For each $M \in \mathcal{S}(\mathbb{F})$ and $r \in \mathbb{Z}_{\geq 1}$, we define the lattices

$$\Xi_r(M) := M + \chi(M) + \cdots + \chi^r(M). \quad (4.3.1)$$

By [RZ96, Proposition 2.17], $\Xi_{n-1}(M)$ is invariant under χ. Note that when n is even, $M \overset{1}{\subset} M + \chi(M)$ by Proposition 3.2.4.

Lemma 4.3.1. *Let d be the minimal number such that $\Xi_d(M)$ is χ-stable, then $0 \leq d \leq n/2$ and we have the following long lattice chain*

$$M \overset{1}{\subset} \Xi_1(M) \overset{1}{\subset} \cdots \overset{1}{\subset} \Xi_d(M) \subset \Xi_d(M)^\vee \overset{1}{\subset} \cdots \overset{1}{\subset} \Xi_1(M)^\vee \subset M^\vee. \quad (4.3.2)$$

Furthermore, if n is even, $1 \leq d \leq n/2$.

Proof. Recall that we have a bijection

$$\Phi_{\mathrm{ad}} \colon X(\mu_{\mathrm{ad}}, b_{\mathrm{ad}})'_{K_{\mathrm{ad}}} \longrightarrow \mathcal{S}(\mathbb{F}). \quad (4.3.3)$$

Let gK_{ad} be the pre-image of M for some $g \in G_{\mathrm{ad}}(L)'$, then gK_{ad} corresponds to the parahoric subgroup

$$\mathrm{Stab}_{G_{\mathrm{ad}}}(M \subset M^\vee \subset \pi^{-1}M), \quad (4.3.4)$$

[1]The proposition in loc cit is not correct, we should assume i and j have the same Kottwitz invariants. However in our case this is true because we are working with the groups $G_{\mathrm{ad}}(L)'$ and $\mathbb{J} = J_{\mathrm{ad}}(\mathbb{Q}_p)'$.

which is equal to ${}^gK_{\mathrm{ad}}$. Then, by Lemma 4.2.2, there exists a unique $w \in \mathrm{EO}_{\mathrm{cox}}$ such that $gK_{\mathrm{ad}} \in X_w^f(b_{\mathrm{ad}})$. And by Lemma 4.2.3, w is of the form $w = w_\Sigma$ for some $\Sigma \in \mathscr{J}$. So $({}^gK_{\mathrm{ad}})^\infty$ is of type $\Sigma^\sharp$ and the natural projection $G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp} \to G_{\mathrm{ad}}(L)'/K_{\mathrm{ad}}$ sending $({}^gK_{\mathrm{ad}})^\infty$ to ${}^gK_{\mathrm{ad}}$ by Lemma 4.2.7. In other words, the lattice M sits inside a long lattice chain whose connected stabilizer is the parahoric subgroup $({}^gK_{\mathrm{ad}})^\infty$. The lattice chain corresponding to $({}^gK_{\mathrm{ad}})^\infty$ is

$$M \overset{1}{\subset} \Xi_1(M) \overset{1}{\subset} \cdots \overset{1}{\subset} \Xi_d(M) \subset \Xi_d(M)^\vee \overset{1}{\subset} \cdots \overset{1}{\subset} \Xi_1(M)^\vee \subset M^\vee, \qquad (4.3.5)$$

where $\Xi_d(M)$ is χ-stable and $\Xi_d(M) = \Xi_{d+1}(M) = \cdots$ by Remark 4.2.9. Furthermore, we have

$$d = \begin{cases} \ell(w_\Sigma) = |\Sigma^\flat|, & \text{if } n \text{ is odd,} \\ \ell(w_\Sigma) - 1 = |\Sigma^\flat|, & \text{if } n \text{ is even and } \Sigma = \{s_i\} \text{ for } 2 \le i \le m, \\ \ell(w_\Sigma) = 1, & \text{if } n \text{ is even and } \Sigma = \{s_0, s_1\}, \end{cases} \qquad (4.3.6)$$

by the calculations of $\mathscr{J}$ in (4.2.8) and (4.2.11). □

Remark 4.3.2. The terminology "crucial lemma" is inherited from [Vol10, Lemma 2.1], and the lemma does play a "crucial" role in the theory of Bruhat-Tits stratification. Since the work of Vollaard [Vol10] and Vollaard-Wedhorn [VW11], Rapoport-Terstiege-Wilson [RTW14] and Howard-Pappas [HP14] adopt almost the same approach to the Bruhat-Tits stratification, i.e. proving some variant of "crucial lemma", see [RTW14, Proposition 4.1] and [HP14, Proposition 2.19]. However, the proof of crucial lemmas in all the mentioned literature is elementary and not conceptual so that one can only prove them case by case. Thanks to Lusztig's work in [Lus07], we give the "crucial lemma" a conceptual proof using a group-theoretic method.

Let $\Xi_\infty(M) := \Xi_m(M)$ for $m \gg 0$. Via the identification (3.3.4), the χ-invariant lattice $\Xi_\infty(M)$ can be viewed as an $\mathcal{O}_F$-lattice in the vector space C. And we have $\pi \cdot \Xi_\infty(M)^\vee \subset \Xi_\infty(M) \subset \Xi_\infty(M)^\vee$.

Definition 4.3.3. An $\mathcal{O}_F$-lattice Λ in C is called a vertex lattice if $\Lambda \subset \Lambda^\sharp \subset \pi^{-1}\Lambda$, where $\Lambda^\sharp$ is the dual of Λ with respect to the hermitian form ψ in Section 3.3. The dimension of the $\mathbb{F}_p$-vector space $\Lambda/\pi\Lambda^\sharp$ is called the type of the lattice, denoted by $t(\Lambda)$.

For $M \in \mathcal{S}(\mathbb{F})$, it is easy to see that the lattice $\Xi_\infty(M)$ is a vertex lattice and its type t is

$$t = \begin{cases} 2d + 1, & \text{if } n \text{ is odd,} \\ 2d, & \text{if } n \text{ is even.} \end{cases} \qquad (4.3.7)$$

Remark 4.3.4. Our definition of vertex lattices is slightly different from the one in [RTW14, Definition 3.1], an $\mathcal{O}_F$-lattice Δ is a vertex lattice in loc. cit. if and only if $\Delta^\sharp$ is a vertex lattice in our sense.

Proposition 4.3.5 (Properties of vertex lattices)**.** *Let Λ, Λ' be two vertex lattices.*

1. *The type of Λ has the same parity as n.*

2. *The inclusion* $\Lambda \subset \Lambda'$ *implies* $t(\Lambda) \leq t(\Lambda')$, *and in this case, the equality holds if and only if* $\Lambda = \Lambda'$.

3. *If* $t(\Lambda) = t(\Lambda')$, *then either* $\Lambda = \Lambda'$ *or* $\Lambda \not\subset \Lambda'$ *and* $\Lambda' \not\subset \Lambda$.

4. *The intersection* $\Lambda \cap \Lambda'$ *is a vertex lattice if and only if* $\Lambda^\sharp \subset \pi^{-1}\Lambda'$.

5. *When* n *is odd, for each odd number* t *satisfying* $1 \leq t \leq n$, *there exists a vertex lattice of type* t.

6. *When* n *is even, for each even number* t *satisfying* $2 \leq t \leq n$, *there exists a vertex lattice of type* t, *but there is no vertex lattice of type* 0.

Proof.

1. By Remark 4.3.4, the lattice $\Lambda^\sharp$ is a vertex lattice in the sense of [RTW14, Definition 3.1], then the dimension of the $\mathbb{F}_p$-vector space $\Lambda^\sharp/\Lambda$ is an even number by Lemma 3.2 in loc. cit. Hence the type of Λ has the same parity as n.

2. The inclusion $\Lambda \subset \Lambda'$ implies
$$\pi(\Lambda')^\sharp \subset \pi(\Lambda)^\sharp \subset \Lambda \subset \Lambda' \subset (\Lambda')^\sharp \subset \Lambda^\sharp, \tag{4.3.8}$$
so we have $t(\Lambda) \leq t(\Lambda')$.

3. Trivial.

4. Note that $(\Lambda \cap \Lambda')^\sharp = \Lambda^\sharp + (\Lambda')^\sharp$. Then $\Lambda^\sharp \subset \pi^{-1}\Lambda'$ if and only if $(\Lambda')^\sharp \subset \pi^{-1}\Lambda$ if and only if $\Lambda^\sharp + (\Lambda')^\sharp \subset \pi^{-1}(\Lambda \cap \Lambda')$.

5. When $n = 2m + 1$ is odd, by Lemma 3.3.2, the hermitian space (C, ψ) is split, so we can choose a basis $\{v_1, \ldots, v_n\}$ such that $\psi(v_i, v_j) = \delta_{i,n+1-i}$. We define lattices
$$\Delta_i = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}v_1, \ldots, \pi^{-1}v_i, v_{i+1}, \ldots, v_n\}, \tag{4.3.9}$$
then the lattice $\Delta_t^\sharp = \pi\Delta_{n-t}$ is of type $n - 2t$ for $0 \leq t \leq m$.

6. When $n = 2m$ is even, by Lemma 3.3.2, the hermitian space (C, ψ) is non-split. So there is no vertex lattice of type 0, because a lattice is of type 0 if and only if it is a π-modular lattice, which exists if and only if C is split by Lemma 2.2.7. We may assume C is the direct product of an $(n-2)$ dimensional split hermitian space with the unique non-split 2 dimensional hermitian space. Note that every lattice in the non-split 2 dimensional hermitian space is self-dual. Then similarly to the construction in the odd case, for each even number t satisfying $2 \leq t \leq n$, there exists a lattice Δ of type t.

□

Remark 4.3.6. When n is even, the fact that there does not exist vertex lattice of type 0 means no lattice in $\mathcal{S}(\mathbb{F})$ is χ-stable.

Let $\mathcal{B}$ be the set of vertex lattices in C. Two vertex lattices Λ and Λ' are called neighbors if $\Lambda \subset \Lambda'$ or $\Lambda \subset \Lambda'$. A d-simplex is a vertex lattice chain:

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_d \subset \pi^{-1}\Lambda_0. \tag{4.3.10}$$

Then $\mathcal{B}$ forms a simplicial complex which is connected and isomorphic to the (rational) Bruhat-Tits building of $\mathbb{J}$ by [RTW14, Proposition 3.4].

4.4 The set structure of Bruhat-Tits stratification

Definition 4.4.1. For each vertex lattice Λ,

$$\mathcal{S}_\Lambda(\mathbb{F}) := \{M \in \mathcal{S}(\mathbb{F}) : M \subset \Lambda\}. \tag{4.4.1}$$

Proposition 4.4.2.

1. *$\mathcal{S}(\mathbb{F}) = \bigcup_{\Lambda \in \mathcal{B}} \mathcal{S}_\Lambda(\mathbb{F})$.*
2. *Let Λ, Λ' be two vertex lattices, then the inclusion $\Lambda \subset \Lambda'$ implies that $\mathcal{S}_\Lambda(\mathbb{F}) \subset \mathcal{S}_{\Lambda'}(\mathbb{F})$.*
3. *Let Λ, Λ' be two vertex lattices, then*

$$\mathcal{S}_\Lambda(\mathbb{F}) \cap \mathcal{S}_{\Lambda'}(\mathbb{F}) = \begin{cases} \mathcal{S}_{\Lambda\cap\Lambda'}(\mathbb{F}), & \text{if } \Lambda \cap \Lambda' \text{ is a vertex lattice,} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{4.4.2}$$

Proof.

1. For each $M \in \mathcal{S}(\mathbb{F})$, by Lemma 4.3.1, the lattice $\Xi_\infty(M)$ is a vertex lattice, i.e. we have
$$M \in \mathcal{S}_{\Xi_\infty(M)}(\mathbb{F}). \tag{4.4.3}$$
2. Trivial.
3. If $\Lambda \cap \Lambda'$ is a vertex lattice, then $M \in \mathcal{S}_\Lambda(\mathbb{F}) \cap \mathcal{S}_{\Lambda'}(\mathbb{F})$ implies that $M \subset \Lambda \cap \Lambda'$, in other words, $M \in \mathcal{S}_{\Lambda\cap\Lambda'}(\mathbb{F})$. If $\Lambda \cap \Lambda'$ is not a vertex lattice, and $\mathcal{S}_\Lambda(\mathbb{F}) \cap \mathcal{S}_{\Lambda'}(\mathbb{F})$ is non-empty, we take $M \in \mathcal{S}_\Lambda(\mathbb{F}) \cap \mathcal{S}_{\Lambda'}(\mathbb{F})$, then we have
$$\pi(\Lambda)^\sharp \subset M \subset \Lambda \subset \Lambda^\sharp, \tag{4.4.4}$$
$$\pi(\Lambda')^\sharp \subset M \subset \Lambda' \subset (\Lambda')^\sharp. \tag{4.4.5}$$
In particular, we have
$$\pi(\Lambda^\sharp + (\Lambda')^\sharp) \subset M \subset \Lambda \cap \Lambda' \subset \Lambda^\sharp + (\Lambda')^\sharp, \tag{4.4.6}$$
which implies that $\Lambda \cap \Lambda'$ is a vertex lattice, contrary to the assumption.

□

Definition 4.4.3. For each vertex lattice Λ,

$$\mathcal{S}^\circ_\Lambda(\mathbb{F}) := \{M \in \mathcal{S}(\mathbb{F}) : \Xi_\infty(M) = \Lambda\}. \tag{4.4.7}$$

Lemma 4.4.4. *For any $M_1, M_2 \in \mathcal{S}^\circ_\Lambda(\mathbb{F})$, we have $\Xi_\infty(M_1) = \Xi_\infty(M_2) = \Lambda$.*

The proof is trivial.

Proposition 4.4.5.

1. $\mathcal{S}^\circ_\Lambda(\mathbb{F}) = \mathcal{S}_\Lambda(\mathbb{F}) \setminus \bigcup_{\Lambda' \subsetneq \Lambda} \mathcal{S}_{\Lambda'}(\mathbb{F})$.
2. $\mathcal{S}(\mathbb{F}) = \biguplus_{\Lambda \in \mathcal{B}} \mathcal{S}^\circ_\Lambda(\mathbb{F})$ *and* $\mathcal{S}_\Lambda(\mathbb{F}) = \biguplus_{\Lambda' \subset \Lambda} \mathcal{S}^\circ_{\Lambda'}(\mathbb{F})$.

Proof.

1. Let $M \in \mathcal{S}_\Lambda(\mathbb{F})$ but $M \notin \mathcal{S}_{\Lambda'}(\mathbb{F})$ for any $\Lambda' \subsetneq \Lambda$. Note that $M \subset \Lambda$ implies $\Xi_\infty \subset \Lambda$, i.e. $M \in \mathcal{S}_{\Xi_\infty}(\mathbb{F}) \subset \mathcal{S}_\Lambda(\mathbb{F})$. Thus $\mathcal{S}_{\Xi_\infty(M)}(\mathbb{F}) = \mathcal{S}_\Lambda(\mathbb{F})$.
2. For each $M \in \mathcal{S}(\mathbb{F})$, we have $M \in \mathcal{S}^\circ_{\Xi_\infty(M)}(\mathbb{F})$.

□

For a vertex lattice Λ, let $\mathbb{B}_\Lambda$ be the $\mathbb{F}_p$-vector space $\Lambda/\pi\Lambda^\sharp$ of dimension $t(\Lambda)$. The form ψ in (3.3.5) induces a $\mathbb{F}_p$-valued bilinear symmetric form $\bar{\psi}$ on $\mathbb{B}_\Lambda$ (because $\Lambda \subset \Lambda^\sharp$) defined by

$$\bar{\psi}(x, y) := \overline{\psi(x, y)}, \tag{4.4.8}$$

where the overline denotes the reduction modulo π. Let $\mathbb{B}_\Lambda$ denote the orthogonal space $(\Lambda/\pi\Lambda^\sharp, \bar{\psi})$ by abuse of notation.

Via the identification (3.3.4), we identify $\psi \otimes L$ with the twisted form $\delta\varphi$ in section 3.3, viewing Λ as a lattice in N, then $\Lambda^\sharp = \Lambda^\vee$.

Lemma 4.4.6. *The symmetric form $\bar{\psi}$ is non-degenerate.*

Proof. If $x \in \mathbb{B}_\Lambda$ such that for any $y \in \mathbb{B}_\Lambda$, $\bar{\psi}(x, y) = 0$. Then by definition of $\bar{\psi}$, $\psi(\dot{x}, \dot{y}) \in \pi\mathcal{O}_F$, where the dot denotes liftings in Λ, in other words, $\dot{x} \in \pi\Lambda^\sharp$. Therefore $x = 0$. □

Let $\mathrm{SO}(\mathbb{B}_\Lambda)$ be the special orthogonal group with respect to the orthogonal space $\mathbb{B}_{\Lambda,\mathbb{F}} := \mathbb{B}_\Lambda \otimes \mathbb{F}$ defined over $\mathbb{F}_p$. Recall that via the identification (3.3.4), $\chi = \mathrm{id} \otimes \mathrm{Frob}_{\mathbb{F}/\mathbb{F}_p}$. Let B be a fixed χ-stable Borel subgroup of $\mathrm{SO}(\mathbb{B}_\Lambda)$.

For each $M \in \mathcal{S}(\mathbb{F})$, let $\Lambda = \Xi_\infty(M)$ and $\bar{M} := M/\pi\Lambda^\sharp$, then $\bar{M}^\perp = \pi M^\vee/\pi\Lambda^\sharp$ and thus $\bar{M}^\perp$ is a maximal isotropic subspace in $\mathbb{B}_{\Lambda,\mathbb{F}}$ of dimension $[\frac{t(\Lambda)}{2}]$. Note that by Remark 3.5.3, every M lies in the same $G_{\mathrm{ad}}(L)'$-orbit, and hence every $\bar{M}$ lies in the same $\mathrm{SO}(\mathbb{B}_\Lambda)$-orbit. Let Q' be the standard maximal parabolic subgroup corresponding to the $\mathrm{SO}(\mathbb{B}_\Lambda)$-orbit of some (or equivalently any) $M \in \mathcal{S}(\mathbb{F})$.

Lemma 4.4.7. *The map*

$$\begin{aligned} \mathcal{S}^\circ_\Lambda(\mathbb{F}) &\longrightarrow \mathrm{SO}(\mathbb{B}_\Lambda)/Q', \\ M &\longmapsto \mathrm{Stab}(\bar{M}), \end{aligned} \tag{4.4.9}$$

is injective.

Proof. Let $M_1, M_2 \in \mathcal{S}^\circ_\Lambda(\mathbb{F})$ such that $\bar{M}_1 = \bar{M}_2$. Then $\pi\Lambda^\sharp \subset M_1 \cap M_2 \subset \Lambda$. Modulo $\pi\Lambda^\sharp$ we get $\overline{M_1 \cap M_2} = \bar{M}_1 \cap \bar{M}_2$, hence $M_1 = M_2$. □

Remark 4.4.8. The proof of Lemma 4.4.7 also shows that the map

$$\begin{aligned} \mathcal{S}_\Lambda(\mathbb{F}) &\longrightarrow \mathrm{SO}(\mathbb{B}_\Lambda)/Q', \\ M &\longmapsto \mathrm{Stab}(\bar{M}), \end{aligned} \tag{4.4.10}$$

is injective.

Proposition 4.4.9. *The map* Φ_{ad} *induces a bijection*

$$j \cdot Y_{\Sigma^\sharp}(w) \longrightarrow \mathcal{S}^\circ_\Lambda(\mathbb{F}), \tag{4.4.11}$$

for each $j \in \mathbb{J}$ *and* $w = w_\Sigma \in \mathrm{EO}_{\mathrm{cox}}$, *where* $\Lambda = j \cdot \Xi_\infty(M)$ *for some (or equivalently any)* $M \in Y_{\Sigma^\sharp}(w)$.

Proof. Assume $j = 1$ firstly. Let $g \in Y_{\Sigma^\sharp}(w)$, $\dot{g}$ a lifting of g in $G(L)'$, $M = \dot{g}\mathbb{M}$. Let $\Lambda := \Xi_\infty(M)$.

Recall that from diagram (4.2.33), we have the following diagram

$$\begin{array}{ccccc} Y(w) & \hookrightarrow & P_{\Sigma^\flat}/I & \hookrightarrow & G_{\mathrm{ad}}(L)'/I \\ \downarrow \cong & & \downarrow \cong & & \downarrow \\ Y_{\Sigma^\sharp}(w) & \hookrightarrow & P_{\tilde{\mathbb{S}}-\Sigma}/P_{\Sigma^\sharp} & \hookrightarrow & G_{\mathrm{ad}}(L)'/P_{\Sigma^\sharp} \\ & \searrow^{\phi} & \downarrow \mathrm{pr} & & \downarrow \\ & & P_{\Sigma^\flat}/P_{\Sigma^\flat \cap \mathbb{S}} & \hookrightarrow & G_{\mathrm{ad}}(L)'/K_{\mathrm{ad}}. \end{array} \tag{4.4.12}$$

The condition $g \in \phi(Y_{\Sigma^\sharp}(w))$ implies that $({}^gK_{\mathrm{ad}})^\infty \in P_{\tilde{\mathbb{S}}-\Sigma}/P_{\Sigma^\sharp}$, which implies that $\Xi_\infty(M) = \Lambda^\vee_{m-i}$ if $\Sigma = \{s_i\}$ by Remark 4.2.9 and Lemma 4.3.1, where Λ_{m-i} is the $(m-i)$-th standard lattice in the subsection 2.3.3. In other words, for any $g_1, g_2 \in Y_{\Sigma^\sharp(w)}$, we get the same vertex lattice $\Xi_\infty(g_1\mathbb{M}) = \Xi_\infty(g_2\mathbb{M})$. Therefore the map Φ_{ad} takes $Y_{\Sigma^\sharp}(w)$ into $\mathcal{S}^\circ_\Lambda(\mathbb{F})$.

We write $Q := P_{\Sigma^\flat \cap \mathbb{S}}$. Then $\Sigma^\flat \cap \mathbb{S} = \Sigma^\flat - \{s_0\}$ and the image $\bar{Q}$ of Q in the reductive quotient $\bar{P}_{\Sigma^\flat}$ is a maximal parahoric subgroup if $\Sigma^\flat$ is non-empty, otherwise $P_{\Sigma^\flat} = Q = I$. By Remark 4.2.5, the reductive quotient $\bar{P}_{\Sigma^\flat}$ has the Dynkin diagram $\Sigma^\flat$ which is the same as $\mathrm{SO}(\mathbb{B}_\Lambda)$ in both odd and even cases. So we have the same (partial) flag varieties

$$P_{\Sigma^\flat}/I = \mathrm{SO}(\mathbb{B}_\Lambda)/B, \quad P_{\Sigma^\flat}/Q = \mathrm{SO}(\mathbb{B}_\Lambda)/Q'. \tag{4.4.13}$$

We have the following commutative diagram

$$\begin{array}{ccc} Y_{\Sigma^\sharp}(w) & \xrightarrow{\Phi_{\mathrm{ad}}} & \mathcal{S}^\circ_\Lambda(\mathbb{F}) \\ & \searrow^{\phi} \quad \swarrow & \\ & P_{\Sigma^\flat}/Q. & \end{array} \tag{4.4.14}$$

For $M \in \mathcal{S}^\circ_\Lambda(\mathbb{F})$, by Lemma 4.3.1, its image $\bar{M}$ in the partial flag variety $P_{\Sigma^\flat}/Q$ satisfies

$$\bar{M} \overset{1}{\subset} \bar{M} + \chi(\bar{M}) \overset{1}{\subset} \cdots \overset{1}{\subset} \bar{M} + \chi(\bar{M}) + \cdots \chi^i(\bar{M}) = \mathbb{B}_{\Lambda,\mathbb{F}}. \tag{4.4.15}$$

By the description of the fine Deligne-Lusztig varieties, i.e. the image of ϕ, in Example 4.1.3 and 4.1.4 and taking dual of (4.4.15), we can see that $\bar{M}$ lies in $\operatorname{im}(\phi)$. Hence Φ_{ad} is bijective.

For general j, if $g \in j \cdot Y_{\Sigma^\sharp}(w)$, then $j^{-1}g \in Y_{\Sigma^\sharp}(w)$. Let Λ' be the lattice such that $Y_{\Sigma^\sharp}(w) \cong \mathcal{S}^\circ_{\Lambda'}$. Let $\Lambda \coloneqq j \cdot \Lambda'$, then $j \cdot Y_{\Sigma^\sharp}(w) \cong \mathcal{S}^\circ_\Lambda$ because $j \cdot \Xi_\infty(M) = \Xi_\infty(j \cdot M)$. □

Corollary 4.4.10. *The map Φ_{ad} induces a bijection*

$$\overline{j \cdot Y_{\Sigma^\sharp}(w)} \longrightarrow \mathcal{S}_\Lambda(\mathbb{F}), \tag{4.4.16}$$

for each $j \in \mathbb{J}$, $w = w_\Sigma \in \mathrm{EO}_{\mathrm{cox}}$ and the vertex lattice Λ corresponding to $j \cdot Y_{\Sigma^\sharp}(w)$ via Proposition 4.4.9.

Proof. Let Λ be the vertex lattice such that $\Phi_{\mathrm{ad}}(j \cdot Y_{\Sigma^\sharp}(w)) = \mathcal{S}^\circ_\Lambda(\mathbb{F})$. Then by Proposition 4.2.13, $i(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma'}) \cap j(\mathbb{J} \cap P_{\tilde{\mathbb{S}}-\Sigma}) \neq \emptyset$ if and only if Λ and Λ' are neighbors, where Λ' is the vertex lattice corresponding to $i \cdot Y_{(\Sigma')^\sharp}(w_{\Sigma'})$ via Proposition 4.4.9. And $(\Sigma')^\flat \subset \Sigma^\flat$ if and only if $(\Sigma')^\sharp \supset \Sigma^\sharp$, if and only if $\Lambda' \subset \Lambda$. So we have

$$\Phi_{\mathrm{ad}}(\overline{j \cdot Y_{\Sigma^\sharp}(w)}) = \bigcup_{\Lambda' \subset \Lambda} \mathcal{S}^\circ_{\Lambda'}(\mathbb{F}). \tag{4.4.17}$$

Then by Proposition 4.4.5 we get the desired result. □

Corollary 4.4.11. *Let Λ, Λ' be two vertex lattices, then $\Lambda \subset \Lambda'$ if and only if $\mathcal{S}_\Lambda(\mathbb{F}) \subset \mathcal{S}_{\Lambda'}(\mathbb{F})$.*

Proof. If $\mathcal{S}_\Lambda(\mathbb{F}) \subset \mathcal{S}_{\Lambda'}(\mathbb{F})$, then $\mathcal{S}^\circ_\Lambda(\mathbb{F}) \subset \mathcal{S}_{\Lambda'}(\mathbb{F})$. By Proposition 4.4.9, there is a bijection between $\mathcal{S}^\circ_\Lambda(\mathbb{F})$ and a Deligne-Lusztig variety, in particular, $\mathcal{S}^\circ_\Lambda(\mathbb{F})$ is non-empty. Take $M \in \mathcal{S}^\circ_\Lambda(\mathbb{F}) \subset \mathcal{S}_{\Lambda'}(\mathbb{F})$, then $M \subset \Lambda'$ and by Lemma 4.4.6 we have $\Lambda = \Xi_\infty(M) \subset \Lambda'$. □

Remark 4.4.12. The notations $\mathcal{S}_\Lambda(\mathbb{F})$ and $\mathcal{S}^\circ_\Lambda(\mathbb{F})$ imply that they are the $\mathbb{F}$-points of the schemes $\mathcal{S}_\Lambda$ and $\mathcal{S}^\circ_\Lambda$ which will be defined in section 5.

Remark 4.4.13. For each algebraic closed field extension k of $\mathbb{F}$, replacing $\mathbb{F}$ by k, all results in Chapter 3 and Chapter 4 are true because by the set-up of [GH15], we may work with any algebraic closed field extension k of $\mathbb{F}$.

Chapter 5

Scheme-theoretic structure of $\mathcal{N}$

5.1 The closed and open Bruhat-Tits strata

Let Λ be a vertex lattice, we define

$$\Lambda^+ := \Lambda, \tag{5.1.1}$$

$$\Lambda^- := \pi\Lambda^\vee. \tag{5.1.2}$$

It is easy to see $\Lambda^\pm$ are Dieudonné modules in N (recall that N is the rational Dieudonné module of $\mathbb{X}$). Let $X_{\Lambda^\pm}$ be the p-divisible $\mathcal{O}_{\breve{F}}$-modules over $\mathbb{F}$ corresponding to $\Lambda^\pm$, together with $\mathcal{O}_F$-linear quasi-isogenies $\rho_{\Lambda^\pm}: X_{\Lambda^\pm} \to \mathbb{X}$ and polarizations $\lambda_{\Lambda^\pm}$. Note that the form $\pi^{-1}\langle\, ,\, \rangle$ induces a perfect paring between Λ^+ and Λ^-, in other words, we have an isomorphism between X_{Λ^+} and ${}^tX_{\Lambda^-}$ such that the following diagram commutes,

$$\begin{array}{ccc} X_{\Lambda^+} & \xrightarrow{\sim} & {}^tX_{\Lambda^-} \\ \Big\downarrow{\scriptstyle \rho_{\Lambda^+}} & & \Big\downarrow{\scriptstyle {}^t\rho_{\Lambda^-}} \\ \mathbb{X} & \xrightarrow[\lambda_{\mathbb{X}}]{} & {}^t\mathbb{X}. \end{array} \tag{5.1.3}$$

For any $\mathbb{F}$-scheme S and any unitary p-divisible group $(X, \rho_X) \in \mathcal{S}(S)$, we define quasi-isogenies:

$$\rho_{X,\Lambda^+}: X_{\bar{S}} \xrightarrow{\rho_X} \mathbb{X}_{\bar{S}} \xrightarrow{\rho_{\Lambda^+}^{-1}} (X_{\Lambda^+})_{\bar{S}}, \tag{5.1.4}$$

$$\rho_{\Lambda^-,X}: (X_{\Lambda^-})_{\bar{S}} \xrightarrow{\rho_{\Lambda^-}} \mathbb{X}_{\bar{S}} \xrightarrow{\rho_X^{-1}} X_{\bar{S}}. \tag{5.1.5}$$

By the same reasoning as in Proposition 3.4.1, we have

$$\mathrm{ht}(\rho_{X,\Lambda^+}) = [\frac{t(\Lambda)}{2}] \tag{5.1.6}$$

$$\mathrm{ht}(\rho_{\Lambda^-,X}) = [\frac{t(\Lambda)+1}{2}] \tag{5.1.7}$$

Definition 5.1.1. The subfunctor $\tilde{\mathcal{S}}_\Lambda$ is defined as

$$\tilde{\mathcal{S}}_\Lambda(S) := \{(X, \rho_X) \in \mathcal{S}(S) : \rho_{X,\Lambda^+} \text{ is an isogeny}\}, \tag{5.1.8}$$

for each vertex lattice Λ and $\mathbb{F}$-scheme S.

Note that ρ_{X,Λ^+} is an isogeny if and only if $\rho_{\Lambda^{-1},X}$ is an isogeny.

Lemma 5.1.2. *The subfunctor $\tilde{\mathcal{S}}_\Lambda$ is represented by a projective scheme over $\mathbb{F}$ and the monomorphism $\tilde{\mathcal{S}}_\Lambda \hookrightarrow \mathcal{S}$ is a closed immersion.*

Proof. The proof is exactly the same as [VW11, Lemma 3.2]. □

Definition 5.1.3. Let $\mathcal{S}_\Lambda := (\tilde{\mathcal{S}}_\Lambda)_{\mathrm{red}}$, we call $\mathcal{S}_\Lambda$ the *closed Bruhat-Tits stratum* associated to Λ.

Remark 5.1.4. The definition of $\mathcal{S}_\Lambda$ coincides with Definition 4.4.1 on k-points, in the spirit of Remark 4.4.13, for any algebraic closed field extension k of $\mathbb{F}$.

If Λ, Λ' are two vertex lattices such that $\Lambda \subset \Lambda'$, by Dieudonné theory, the corresponding quasi-isogeny $X_\Lambda \to X_{\Lambda'}$ is an isogeny, so we have $\mathcal{S}_\Lambda \subset \mathcal{S}_{\Lambda'}$.

Definition 5.1.5. The locally closed subscheme $\mathcal{S}_\Lambda^\circ$ is defined as

$$\mathcal{S}_\Lambda^\circ := \mathcal{S}_\Lambda \setminus \bigcup_{\Lambda' \subsetneq \Lambda} \mathcal{S}_{\Lambda'}, \tag{5.1.9}$$

for each vertex lattice Λ. Then $\mathcal{S}_\Lambda^\circ$ is an open subscheme of $\mathcal{S}_\Lambda$. We call $\mathcal{S}_\Lambda^\circ$ the *open Bruhat-Tits stratum* associated to Λ.

By definition, we have

$$\mathcal{S}_\Lambda = \biguplus_{\Lambda' \subset \Lambda} \mathcal{S}_{\Lambda'}^\circ. \tag{5.1.10}$$

Remark 5.1.6. The definition of $\mathcal{S}_\Lambda^\circ$ coincides with Definition 4.4.3 on k-valued points, in the spirit of Remark 4.4.13, for any algebraic closed field extension k of $\mathbb{F}$ by Proposition 4.4.5.

5.2 An A-windows-theory interlude

Let us recall some basic facts about the Zink's windows theory for formal p-divisible groups in [Zin01].

We fix an odd prime p and a base ring R such that p is nilpotent in R.

Definition 5.2.1 ([Zin01, Definition 1])**.** A triple (A, J, σ) is called a *frame* over R if

- A is a p-adic ring together with a surjective homomorphism $A \to R$;
- as an abelian group, A is torsion free;
- the kernel $J := A \to R$ is an ideal with divided powers;
- σ is an endormorphism of A that lifts the Frobenius on A/pA.

Definition 5.2.2 ([Zin01, Definition 2 & 3])**.** Let (A, J, σ) is a frame over R, the triple $(M, M_1, \$\Upsilon)$ is called an *A-window* if

- M is a finitely generated projective A-module together with a σ-linear map $\Upsilon\colon M \to M$;
- M_1 is a submodule of M such that M/M_1 is a projective R-module;
- $JM \subset M_1$ and $\Upsilon M_1 \subset pM$;
- as an A-module, M is generated by ΥM and $\frac{1}{p}\Upsilon M_1$;
- Let $\Pi\colon M \to M \otimes_{A,\sigma} M$ be the unique A-linear map with the property that $\Pi(\frac{1}{p}\Upsilon(m_1)) = 1 \otimes m_1$ for $m_1 \in M_1$ and $\Pi(\Upsilon(m)) = p \otimes m$ for $m \in M$. Then there exists an integer r such that $\Pi^r(M) \subset M \otimes_{A,\sigma} J$.

A morphism between two A-windows (M, M_1, Υ) and (M', M_1', Υ') is an A-homomorphism $\gamma\colon M \to M'$ which maps M_1 into M_1' and makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & M' \\ \downarrow{\scriptstyle \Upsilon} & & \downarrow{\scriptstyle \Upsilon'} \\ M & \xrightarrow{\gamma} & M' \end{array} \tag{5.2.1}$$

commutes. Hence we get the category of A-windows.

Definition 5.2.3. A morphism from a frame (A_1, J_1, σ_1) over R_1 to a frame (A_2, J_2, σ_2) over R_2 consists of the following data:

- a ring homomorphism $\alpha\colon R_1 \to R_2$ and a ring homomorphism $\beta\colon A_1 \to A_2$ such that the following diagram is commutative

$$\begin{array}{ccc} A_1 & \xrightarrow{\beta} & A_2 \\ \downarrow & & \downarrow \\ R_1 & \xrightarrow{\alpha} & R_2; \end{array} \tag{5.2.2}$$

- the homomorphism β is compatible with the Frobenius, i.e. the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\beta} & A_2 \\ \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} \\ A_1 & \xrightarrow{\beta} & A_2 \end{array} \tag{5.2.3}$$

  is commutative.

Definition 5.2.4. Given a morphism from a frame (A_1, J_1, σ) over R_1 to a frame (A_2, J_2, σ_2) over R_2 and an A_1-window (M, M_1, Υ), we define the *base change functor*

$$\begin{aligned} (A_1\text{-windows}) &\longrightarrow (A_2\text{-windows}), \\ (M, M_1, \Upsilon) &\longmapsto (M', M_1', \Upsilon'), \end{aligned} \tag{5.2.4}$$

where $M' := M \otimes_{A_1} A_2$, $M_1' := \ker(M \otimes_{A_1} A_2 \to M/M_1 \otimes_{R_1} R_2)$, and $\Upsilon' := \Upsilon \otimes \sigma_2$. Note that M_1' is generated by $M_1 \otimes_{A_1} A_2$ and $M \otimes_{A_1} J_2$.

Proposition 5.2.5 ([Zin01, Theorem 4]). *Let R be an excellent ring and (A, J, σ) a frame over R. Then the functor BT_R (cf. [Zin01, p. 500])*

$$\mathrm{BT}_R\colon (A\text{-windows}) \longrightarrow (\text{formal } p\text{-divisible groups over } R) \tag{5.2.5}$$

is an equivalence of categories. If X is the p-divisible group corresponding to an A-window (M, M_1, Υ), then we have $\mathrm{Lie}(X) = M/M_1$. The inclusion of A-windows corresponds to the isogeny of formal p-divisible groups. Furthermore, the functor BT is commutative with the base change functor, more precisely, given a morphism from a frame (A_1, J_1, σ_1) over R_1 to a frame (A_2, J_2, σ_2) over R_2, the diagram

$$\begin{array}{ccc} (A_1\text{-windows}) & \xrightarrow{\mathrm{BT}} & (\text{formal } p\text{-divisible groups over } R_1) \\ {\scriptstyle \text{base change}}\big\downarrow & & \big\downarrow{\scriptstyle \text{base change}} \\ (A_2\text{-windows}) & \xrightarrow[\mathrm{BT}]{} & (\text{formal } p\text{-divisible groups over } R_2) \end{array} \tag{5.2.6}$$

is commutative.

Definition 5.2.6 (cp. [Zin02, Definition 18]). Let (A, J, σ) be a frame over R, (M, M_1, Υ) and (M', M_1', Υ') two A-windows. A *bilinear form of A-windows*

$$\varphi\colon (M, M_1, \Upsilon) \times (M', M_1', \Upsilon') \longrightarrow (A, J, \sigma), \tag{5.2.7}$$

is an A-bilinear form of A-modules

$$\varphi\colon M \times M' \longrightarrow A, \tag{5.2.8}$$

such that $\varphi(M_1, M_1') \subset J$ and

$$\varphi(\Upsilon(x), \Upsilon'(y)) = \varphi(x, y)^\sigma, \tag{5.2.9}$$

for any $x \in M_1$ and $y \in M_1'$.

From now on let us restrict ourselves to the case that the base ring R is a field. Let k be a field of characteristic p, A the Cohen subring of $W(k)$ (cf. [Bou06, IX §2 Definition 2]). Then the triple (A, pA, σ) is a frame over k. Note that $\sigma\colon A \to A$ is injective, but in general not surjective except that k is perfect. For an A-module M and a σ-linear map $f\colon M \to M$, we denote by $f^\sharp\colon \sigma^*M \to M$ its linearization, where $\sigma^*M := M \otimes_{A,\sigma} A$.

Let (M, M_1, Υ) be an A-window. By Definition 5.2.2, M is generated by $\Upsilon(M)$ and $\frac{1}{p}\Upsilon(M_1)$, however the condition $pM \subset M_1$ implies that $\Upsilon(M) \subset \frac{1}{p}\Upsilon(M_1)$, i.e. M is generated by $\frac{1}{p}\Upsilon(M_1)$ or equivalently pM is generated by $\Upsilon(M_1)$ as an A-module. Then it is easy to see that $pM \subset \Upsilon(M)$. Therefore the linearization $\Upsilon^\sharp\colon \sigma^*M \to M$ is injective because inverting p the cokernel of $\Upsilon^\sharp$ is 0.

Lemma 5.2.7 (cp. [Zin02, Lemma 10]). *For an A-window (M, M_1, Υ), there exists a unique A-linear map $\Pi^\sharp\colon M \to \sigma^*M$ such that $\Upsilon^\sharp \circ \Pi^\sharp = \mathrm{id}_M$ and $\Pi^\sharp \circ \Upsilon^\sharp = \mathrm{id}_{\sigma^*M}$.*

Proof. Because pM is generated by $\Upsilon(M_1)$ as an A-module, the linearization $\Upsilon^\sharp$ induces an isomorphism $\Upsilon^\sharp\colon \sigma^*M_1 \cong pM$. More precisely, we have the following commutative diagram:

$$\begin{array}{ccc} \sigma^*M & \xrightarrow{\Upsilon^\sharp} & M \\ \uparrow & & \uparrow \\ \sigma^*M_1 & \underset{\Upsilon^\sharp}{\xrightarrow{\cong}} & pM \end{array} \tag{5.2.10}$$

Then for any $x \in M$, we define $\Pi^\sharp(x) := (\Upsilon^\sharp|_{\sigma^*M_1})^{-1}(px)$. Then it is easy to check that $\Pi^\sharp$ satisfies the desired equality. Furthermore, $\Pi^\sharp$ is trivially unique. □

Definition 5.2.8. For an A-window (M, M_1, Υ), the map $\Pi^\sharp$ determined by Lemma 5.2.7 is called the *A-Verschiebung* associated to the window (M, M_1, Υ).

Proposition 5.2.9. *Let (M, M_1, Υ) be an A-window, $\Pi^\sharp$ the associated A-Verschiebung, then we have $\sigma^*M_1 = \Pi^\sharp M$.*

Proof. By the diagram (5.2.10), we have $\Upsilon^\sharp(\sigma^*M_1) = pM$, so by Lemma 5.2.7, we have $\sigma^*M_1 = \Pi^\sharp M$. □

Remark 5.2.10. Note that we have the canonical injective $M_1 \to \sigma^*M_1$, but in general this is not surjective. So given Υ and $\Pi^\sharp$, we can only determine σ^*M_1.

Lemma 5.2.11. *Let Y be a p-divisible group over k of height $2d$ and dimension d, $(M_Y, M_{Y,1}, \Upsilon_Y)$ its A-window. Then giving a p-divisible group X over k of height $2d$ and dimension d, together with an isogeny $\rho\colon X \to Y$, is equivalent to giving an A-submodule M of M_Y such that M is Υ_Y-stable and $pM \overset{2d}{\subset} M$.*

Proof. By the assumption on Y, we have

$$pM_Y \overset{d}{\subset} M_{Y,1} \overset{d}{\subset} M_Y. \tag{5.2.11}$$

Let X be a p-divisible group over k of height $2d$ and dimension d together with an isogeny $\rho\colon X \to Y$, (M, M_1, Υ_X) its A-window. Then the following diagram is commutative:

$$\begin{array}{ccc} pM & \hookrightarrow & pM_Y \\ \downarrow & & \downarrow \\ M_1 & \hookrightarrow & M_{Y,1} \\ \downarrow & & \downarrow \\ M & \hookrightarrow & M_Y \end{array} \tag{5.2.12}$$

Then the diagram (5.2.12) shows that the injection

$$\mathrm{Lie}(X) = M/M_1 \hookrightarrow \mathrm{Lie}(Y) = M_Y/M_{Y,1} \tag{5.2.13}$$

is an isomorphism because they have the same dimension. Furthermore, Υ_Y induces Υ_X on M, i.e. M is Υ_Y-stable.

Now let M be an A-submodule of M_Y that is Υ_Y-stable and $pM \overset{2d}{\subset} M$. We define $M_1 := \ker(M \to \mathrm{Lie}(Y))$. Then we claim that (M, M_1, Υ_Y) is an A-window. Clearly we have the commutative diagram:

$$\begin{array}{ccc} pM & \hookrightarrow & pM_Y \\ \downarrow & & \downarrow \\ M_1 & \hookrightarrow & M_{Y,1} \\ \downarrow & & \downarrow \\ M & \hookrightarrow & M_Y, \end{array} \tag{5.2.14}$$

which implies that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1/pM & \longrightarrow & M/pM & \longrightarrow & M/M_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & M_{Y,1}/pM_Y & \longrightarrow & M_Y/pM_Y & \longrightarrow & \mathrm{Lie}(Y) & \longrightarrow & 0 \end{array} \tag{5.2.15}$$

is commutative with exact rows. The injectivity of f_2 implies that f_2 is an isomorphism because they have the same dimension. Then f_3 is surjective and hence is an isomorphism. Therefore f_1 is an isomorphism and we have the inclusion

$$pM \overset{d}{\subset} M_1 \overset{d}{\subset} M. \tag{5.2.16}$$

The operator Υ_Y induces an action on M_Y/pM_Y which kills the subspace $M_{Y,1}/pM_Y$ and hence also kills M_1/pM, in other words $\Upsilon_Y(M_1) \subset pM$. Therefore the triple (M, M_1, Υ_Y) is an A-window which gives rise to a p-divisible group X over k of height $2d$ and dimension d. The inclusion of A-windows induces an isogeny $X \to Y$. $\square$

Now let us consider the A-windows associated to unitary p-divisible groups. Let $k \supset \mathbb{F}$ be a field extension, A the Cohen subring of $W(k)$ which is also an $\mathcal{O}_L$-algebra. Then $(\mathcal{O}_L, p\mathcal{O}_L, \sigma)$ is a frame over $\mathbb{F}$ and (A, pA, σ_A) is a frame over k. The inclusion $\mathcal{O}_L \subset A$ induces a morphism of frames

$$(\mathcal{O}_L, p\mathcal{O}_L, \sigma) \longrightarrow (A, pA, \sigma_A). \tag{5.2.17}$$

By abuse of notation, let σ denote σ_A. Recall that we fix a supersingular unitary p-divisible group $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\mathbb{F}$ of signature $(1, n-1)$ in Section 3.2, then the A-window of the underlying p-divisible group $\mathbb{X}$ over k is the base change of the Dieudonné module $(\mathbb{M}, \mathcal{F}, \mathcal{V})$ via the morphism of frames (5.2.17). More precisely, let $(\mathbb{M}_A, \mathbb{M}_{A,1}, \Upsilon)$ be the A-window of $\mathbb{X} \otimes k$, then by Proposition 5.2.5 and Definition 5.2.4, we have $\mathbb{M}_A = \mathbb{M} \otimes_{\mathcal{O}_L} A$, $\mathbb{M}_{A,1}$ is the submodule of $\mathbb{M}_A$ generated by $\mathcal{V}\mathbb{M} \otimes_{\mathcal{O}_L} A$ and $\mathbb{M} \otimes_{\mathcal{O}_L} pA$, and $\Upsilon = \mathcal{F} \otimes \sigma_A$. Note that $\mathbb{M} \otimes_{\mathcal{O}_L} pA = p\mathbb{M} \otimes_{\mathcal{O}_L} A \subset \mathcal{V}\mathbb{M} \otimes_{\mathcal{O}_L} A$, so we have $\mathbb{M}_{A,1} = \mathcal{V}\mathbb{M} \otimes_{\mathcal{O}_L} A$. Let (N_A, Υ) be the rational A-window, i.e. $N_A = \mathbb{M}_A \otimes_{\mathcal{O}_L} \mathrm{Frac}(A)$, together with the $\mathcal{O}_F$-action $\iota_{\mathbb{X}}$ and the non-degenerate alternating form $\langle\, ,\, \rangle$ induced by the polarization $\lambda_{\mathbb{X}} \otimes k$. For any $x, y \in N_A$, we have

$$\langle \Upsilon(x), \Upsilon(y) \rangle = \langle x, y \rangle^{\sigma}, \tag{5.2.18}$$

and

$$\langle \iota(\pi)x, y \rangle = \langle x, \iota(\bar{\pi})y \rangle. \tag{5.2.19}$$

Henceforth, we write π instead of $\iota(\pi)$ to lighten the notations. The π-action defines an $A[\pi]$-module structure on $\mathbb{M} \otimes_{\mathcal{O}_L} A$. Let Λ be a vertex lattice. Then $(\Lambda^{\pm} \otimes A, \mathcal{V}\Lambda^{\pm} \otimes A, \Upsilon)$ are the A-windows of the p-divisible groups $X_{\Lambda^{\pm}} \otimes k$. We will write $\Lambda^{\pm}_A$ instead of $\Lambda^{\pm} \otimes A$, and write $\mathcal{V}\Lambda^{\pm}_A$ instead of $\mathcal{V}\Lambda^{\pm} \otimes A$ for short.

Proposition 5.2.12. *Via A-windows theory, $\mathcal{S}_\Lambda(k)$ can be identified with the set of $A[\pi]$-lattices M in N_A satisfying the following conditions:*

1. *M is Υ-stable;*
2. *$M \overset{n-1}{\subset} M^\vee \overset{1}{\subset} \pi^{-1}M$ if n is odd, and $M^\vee = \pi^{-1}M$ if n is even;*
3. *$pM \overset{n}{\subset} M_1 \overset{n}{\subset} M$;*
4. *$M_1 \overset{\leq 1}{\subset} M_1 + \pi M$;*
5. *if n is even, $M_1 \overset{1}{\subset} M_1 + \pi M$;*
6. *$M \subset \Lambda^+_A$;*

where $M_1 := \ker(M \to \Lambda^+/\mathcal{V}\Lambda^+_A)$.

Proof. Let $(X, \rho_X) \in \mathcal{S}_\Lambda(k)$ be a unitary p-divisible group together with a quasi-isogeny ρ_X. By the definition of $\mathcal{S}_\Lambda$, we know that ρ_{X,Λ^+} is an isogeny. Then by Lemma 5.2.11, X corresponds to an Υ-stable A-submodule M of Λ^+_A and the triple (M, M_1, Υ) is the A-window of X over k, where $M_1 := \ker(M \to \mathrm{Lie}(X_{\Lambda^+}) \otimes k)$. By the definition of $\mathcal{S}_\Lambda$, the π-action on N_A induces the π-action on M, hence M is an $A[\pi]$-lattice in N_A. Note that $\mathrm{Lie}(X) \cong M/M_1$ canonically. So via A-windows theory, the periodicity condtion on the p-divisible group can be translated into condition 2, the Kottwitz condition is translated into condition 3, the Wedge condition is translated into condition 4 and the extra Spin condition is translated into condition 5. Vice versa, an $A[\pi]$-lattice in N_A satisfying all the above conditions gives rise to a unitary p-divisible group $(X, \rho_X) \in \mathcal{S}_\Lambda(k)$. □

5.3 The Bruhat-Tits strata as Deligne-Lusztig varieties

Let T be a scheme over $\mathbb{F}$, $(X, \rho) \in \mathcal{S}(T)$ a unitary p-divisible group. Let $D(X)$ be the Lie algebra of the universal vector extension of X (cf. [Mes72, Chapter IV, Definition 1.12]), then the functor

$$\begin{aligned} (p\text{-divisible groups over } T) &\longrightarrow (\text{locally free } \mathcal{O}_T\text{-modules}), \\ X &\longmapsto D(X), \end{aligned} \tag{5.3.1}$$

commutes with an arbitrary base change $T' \to T$. When $T = \mathrm{Spec}(k)$ for an algebraic closed field extension k of $\mathbb{F}$, we have $D(X) \cong M(X)/pM(X)$ canonically, where $M(X)$ is the Dieudonné module of X.

Lemma 5.3.1. *Let $\rho_i\colon X \to Y_i$, for $i = 1, 2$, be two isogenies of naive unitary p-divisible groups (of any signature) over T, such that $\ker(\rho_1) \subset \ker(\rho_2) \subset X[\pi]$, then both $\ker(D(\rho_1))$ and $\ker(D(\rho_2))$ are locally free $\mathcal{O}_T$-modules and $\ker(D(\rho_1))$ is a locally direct summand of $\ker(D(\rho_2))$.*

Proof. Note that by definition X is endowed with an $\mathcal{O}_F$-action, hence the proof is exactly the same as [VW11, Corollary 3.7] replacing p by π. $\square$

Using Lemma 5.3.1, we can construct a morphism from $\tilde{\mathcal{S}}_\Lambda$ to the partial flag variety $\mathrm{SO}(\mathbb{B}_\Lambda)/Q'$ defined in section 4.4. Let $(X, \rho) \in \tilde{\mathcal{S}}_\Lambda(R)$ for an $\mathbb{F}$-algebra R and a vertex lattice Λ, we have isogenies

$$(X_{\Lambda^-})_{\bar{R}} \xrightarrow{\rho_{\Lambda^-}} X_{\bar{R}} \xrightarrow{\rho_{\Lambda^+}} (X_{\Lambda^+})_{\bar{R}}. \tag{5.3.2}$$

where $\rho_{\Lambda^-} = \rho_{\Lambda^-,X} \otimes \mathrm{id}_R$ by abuse of notation and similarly for ρ_{Λ^+}. The composition $\rho_\Lambda \coloneqq \rho_{\Lambda^+} \circ \rho_{\Lambda^-}$ corresponds to the isogeny $(X_{\Lambda^-})_{\bar{R}} \to (X_{\Lambda^+})_{\bar{R}}$ induced by the inclusion $\Lambda^- \subset \Lambda^+$. Then we have $\ker(\rho_{\Lambda^-}) \subset \ker(\rho_\Lambda) \subset X_{\Lambda^-}[\pi]$. Note that $\ker(D(\rho_\Lambda)) = \mathbb{B}_{\Lambda,R} \coloneqq \mathbb{B}_\Lambda \otimes R$, and when $R = \mathrm{Spec}(k)$ for an algebraic closed field k, $\ker(D(\rho_{\Lambda^-})) = M(X)/\pi\Lambda^\vee$.

Recall that for any $\mathbb{F}$-algebra R, the partial flag variety $\mathrm{SO}(\mathbb{B}_\Lambda)/Q'$ has the following description as a functor

$$(\mathrm{SO}(\mathbb{B}_\Lambda)/Q')(R) = \left\{ \begin{array}{l} U \subset \mathbb{B}_{\Lambda,R} \\ \text{a direct summand} \end{array} \;\middle|\; \begin{array}{l} U \subset U^\perp, \\ \mathrm{rank}_R(U) = [\frac{t(\Lambda)}{2}], \\ U \text{ lies in the } \mathrm{SO}(\mathbb{B}_\Lambda)\text{-orbit} \\ \text{corresponding to } Q' \end{array} \right\}. \tag{5.3.3}$$

For the orthogonal Grassmannian $\mathrm{Grass}(\mathbb{B}_\Lambda)$, we have

$$\mathrm{Grass}(\mathbb{B}_\Lambda)(R) = \left\{ \begin{array}{l} U \subset \mathbb{B}_{\Lambda,R} \\ \text{a direct summand} \end{array} \;\middle|\; \begin{array}{l} U \subset U^\perp, \\ \mathrm{rank}_R(U) = [\frac{t(\Lambda)}{2}] \end{array} \right\}. \tag{5.3.4}$$

Let $E(X) \coloneqq \ker(D(\rho_{\Lambda^-}))$ which is of rank $\mathrm{ht}(\rho_{\Lambda^-}) = [\frac{t(\Lambda)+1}{2}]$, then sending (X, ρ) to $E(X)^\perp$ defines a map

$$\begin{aligned} \tilde{f}\colon \tilde{\mathcal{S}}_\Lambda(R) &\longrightarrow \mathrm{Grass}(\mathbb{B}_\Lambda)(R), \\ (X, \rho) &\longmapsto E(X)^\perp. \end{aligned} \tag{5.3.5}$$

In summary we have a morphism $\tilde{\mathcal{S}}_\Lambda \to \mathrm{Grass}(\mathbb{B}_\Lambda)$, which induces a morphism

$$f\colon \mathcal{S}_\Lambda \longrightarrow \mathrm{Grass}(\mathbb{B}_\Lambda). \tag{5.3.6}$$

Note that by Remark 4.4.8, for any algebraic field extension k of $\mathbb{F}$, we have

$$\mathcal{S}_\Lambda(k) \hookrightarrow (\mathrm{SO}(\mathbb{B}_\Lambda)/Q')(k) \subset \mathrm{Grass}(\mathbb{B}_\Lambda)(k), \tag{5.3.7}$$

i.e. the image of $\mathcal{S}_\Lambda$ lies in $\mathrm{SO}(\mathbb{B}_\Lambda)/Q'$ because $\mathcal{S}_\Lambda$ is reduced.

Lemma 5.3.2. *The morphism $f\colon \mathcal{S}_\Lambda \to \mathrm{SO}(\mathbb{B}_\Lambda)/Q'$ is a closed immersion. In particular, taking closure of $\mathcal{S}_\Lambda^\circ$ in $\mathcal{S}$ is the same as taking closure in $\mathrm{SO}(\mathbb{B}_\Lambda)/Q'$.*

Proof. The scheme $\mathcal{S}_\Lambda$ is proper, $\mathrm{SO}(\mathbb{B}_\Lambda)/Q'$ is separated, so f is proper. Trivially f is a monomorphism by the definition of $\tilde{f}$, therefore f is a closed immersion. $\square$

Lemma 5.3.3. *The morphism f induces a morphism $f\colon \mathcal{S}_\Lambda \to \overline{j\cdot Y_{\Sigma^\sharp(w)}}$, where $j \in \mathbb{J}$ and $w = w_\Sigma \in \mathrm{EO}_{\mathrm{cox}}$ corresponding to Λ via Proposition 4.4.9.*

Proof. For any algebraic closed field extension k of $\mathbb{F}$, we have $f\colon \mathcal{S}_\Lambda(k) \to (\overline{j\cdot Y_{\Sigma^\sharp}(w)})(k)$ by Corollary 4.4.10, since $\mathcal{S}_\Lambda$ is reduced, we prove the claim. □

Lemma 5.3.4. *Let k be a field extension of $\mathbb{F}$ (not necessarily algebraic closed), then the morphism f induces a bijection*

$$\mathcal{S}_\Lambda(k) \longrightarrow \overline{j\cdot Y_{\Sigma^\sharp}(w)}(k). \tag{5.3.8}$$

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_\Lambda(k) & \xrightarrow{f(k)} & \overline{j\cdot Y_{\Sigma^\sharp}(w)}(k) \\ \big\downarrow & & \big\downarrow \\ \mathcal{S}_\Lambda(\bar{k}) & \xrightarrow[f(\bar{k})]{} & \overline{j\cdot Y_{\Sigma^\sharp}(w)}(\bar{k}), \end{array} \tag{5.3.9}$$

where $\bar{k}$ is an algebraic closure of k. So the injectivity of $f(k)$ follows from that $f(\bar{k})$ is bijective by Corollary 4.4.10 and Remark 4.4.13.

Let us prove the surjectivity of $f(k)$. Let A be the Cohen subring of $W(k)$, then $\mathcal{S}_\Lambda(k)$ can be described as the set of all the $A[\pi]$-lattices in N_A satisfying all the conditions in Proposition 5.2.12. Let $U_0 \in \overline{j\cdot Y_{\Sigma^\sharp}(w)}(k)$ be a maximal isotropic subspace of $\mathbb{B}_{\Lambda,k}$, then $U_0^\perp$ gives rise to an $A[\pi]$-module M such that

$$\Lambda_A^- \subset \pi M^\vee \subset M \subset \Lambda_A^+. \tag{5.3.10}$$

To prove the surjectivity of $f(k)$, it only needs to show that the $A[\pi]$-module M lies in $\mathcal{S}_\Lambda(k)$, i.e. M satisfies all the conditions in Proposition 5.2.12. Let M_1 be the kernel of $M \to \mathrm{Lie}(X_{\Lambda^+}) \otimes k$. Here is the routine check:

1. Recall that Λ^+ is $\pi^{-1}\mathcal{F}$-invariant, hence Λ_A^+ is $\pi^{-1}\Upsilon$-stable but not $\pi^{-1}\Upsilon$-invariant in general. Then

$$\Upsilon(\Lambda_A^+) \subset \pi\Lambda_A^+ \subset \Lambda_A^-. \tag{5.3.11}$$

   Hence Υ induces the 0 map on $\mathbb{B}_{\Lambda,k}$, in particular, $\Upsilon(M) \subset \Lambda_A^- \subset M$.

2. By the lattice chain (5.3.10), we have $M \subset M^\vee \overset{1}{\subset} \pi^{-1}M$ if n is odd, and $M^\vee = \pi^{-1}M$ if n is even. Furthermore, we have the following commutative diagram:

$$\begin{array}{ccccc} \pi\Lambda_A^- & \hookrightarrow & \pi M & \hookrightarrow & \pi\Lambda_A^+ \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \Lambda_A^- & \hookrightarrow & M & \hookrightarrow & \Lambda_A^+, \end{array} \tag{5.3.12}$$

   which implies that $\pi M \overset{n}{\subset} M$ and in particular $M \overset{n-1}{\subset} M^\vee$ if n is odd.

3. The commutative diagram:

$$\begin{array}{ccc} pM & \hookrightarrow & p\Lambda_A^+ \\ \downarrow & & \downarrow \\ M_1 & \hookrightarrow & \mathcal{V}\Lambda_A^+ \\ \downarrow & & \downarrow \\ M & \hookrightarrow & \Lambda_A^+ \end{array} \tag{5.3.13}$$

implies that $pM \overset{n}{\subset} M_1 \overset{n}{\subset} M$ by the same procedure as in the proof of Proposition 5.2.12.

4. If n is odd, by Example 4.1.15, the closure $\overline{j \cdot Y_{\Sigma^\sharp}(w)}(k)$ can be described as the set of all the d-dimensional isotropic subspaces U of $\mathbb{B}_{\Lambda,k}$ such that $\dim(U \cap \chi(U)) \geq d-1$, where $\dim_k(\mathbb{B}_{\Lambda,k}) = 2d+1$. Note that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V}\Lambda_A^- & \longrightarrow & M_1 & \longrightarrow & M_1/\mathcal{V}\Lambda_A^- & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Lambda_A^- & \longrightarrow & M & \longrightarrow & M/\Lambda_A^- & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathrm{Lie}(X_{\Lambda^-}) \otimes k & \xrightarrow{\cong} & M/M_1. & & & & \end{array} \tag{5.3.14}$$

Then by the Snake Lemma we have $M_1/\mathcal{V}\Lambda_A^- = M/\Lambda_A^- = U_0$. Hence the condition $\dim(U_0 \cap \chi(U_0)) \geq d-1$ can be translated into the condition

$$M_1 \overset{\leq 1}{\subset} M_1 + \pi^{-1}\Upsilon(M_1). \tag{5.3.15}$$

By the definition of A-windows, M is generated by $\frac{1}{p}\Upsilon(M_1)$ as an A-module, hence πM is generated by $\pi^{-1}\Upsilon(M_1)$ which implies that

$$M_1 + \pi^{-1}\Upsilon(M_1) = M_1 + \pi M. \tag{5.3.16}$$

Therefore we have

$$M_1 \overset{\leq 1}{\subset} M_1 + \pi M. \tag{5.3.17}$$

5. If n is even, by Example 4.1.16, the closure $\overline{j \cdot Y_{\Sigma^\sharp}(w)}(k)$ can be described as the set of all the maximal isotropic subspaces U of $\mathbb{B}_{\Lambda,k}$ which lie in the same $\mathrm{SO}(V)$-orbit of W_d such that $\dim(U \cap \chi(U)) = d-1$, where $\dim_k(\mathbb{B}_{\Lambda,k}) = 2d$. Similarly to the odd case, we have

$$M_1 \overset{1}{\subset} M_1 + \pi^{-1}\Upsilon(M_1). \tag{5.3.18}$$

Then by the same procedure as in the odd case, we have

$$M_1 \overset{1}{\subset} M_1 + \pi M. \tag{5.3.19}$$

6. Trivially we have $M \subset \Lambda_A^+$.

□

Proposition 5.3.5. *The morphism $f\colon \mathcal{S}_\Lambda \to \overline{j \cdot Y_{\Sigma^\sharp}(w)}$ is an isomorphism.*

Proof. Note that f is proper and quasi-finite, hence f is finite. By Corollary 4.4.10, f induces a bijection on k-valued points for each algebraic closed field extension k of $\mathbb{F}$, so f is universally bijective by [Gro60, 3.5.5] & [Gro64, Proposition 1.3.7]. Therefore f is a universal homeomorphism by [Gro66, Proposition 8.11.6]. In particular $\mathcal{S}_\Lambda$ is irreducible and therefore integral. Now by Lemma 5.3.4, f is bijective on k-valued points. Hence f is birational. By Example 4.1.15 and 4.1.16, the closure $\overline{j \cdot Y_{\Sigma^\sharp}(w)}$ is normal. Altogether, f is an integral birational morphism between integral schemes, with the target being normal, so it is an isomorphism[1].

□

Corollary 5.3.6. *The morphism f induces an isomorphism $\mathcal{S}_\Lambda^\circ \to j \cdot Y_{\Sigma^\sharp}(w)$. In particular, the locally closed subscheme $\mathcal{S}_\Lambda^\circ$ is smooth of dimension $\ell(w) = [\frac{t(\Lambda)-1}{2}]$.*

Corollary 5.3.7. *The closure $\overline{\mathcal{S}_\Lambda^\circ}$ of $\mathcal{S}_\Lambda^\circ$ in $\mathcal{S}$ is $\mathcal{S}_\Lambda$.*

Proof. By Lemma 5.3.2 and Corollary 5.3.6, we have $\overline{\mathcal{S}_\Lambda^\circ} \cong \overline{j \cdot Y_{\Sigma^\sharp}(w)}$. Then by Proposition 5.3.5, $\overline{\mathcal{S}_\Lambda^\circ} = \mathcal{S}_\Lambda$. □

By Example 4.1.15 and 4.1.16, we have the following corollary.

Corollary 5.3.8. *The closed subscheme $\mathcal{S}_\Lambda$ of $\mathcal{S}$ is projective and normal of dimension $\ell(w_\Sigma) = [\frac{t(\Lambda)-1}{2}]$. When n is odd, $\mathcal{S}_\Lambda$ has isolated singularities; when n is even, $\mathcal{S}_\Lambda$ is smooth.*

5.4 The Bruhat-Tits stratification

Theorem 5.4.1. *Let Λ and Λ' be two vertex lattices.*

1. *We have $\Lambda \subset \Lambda'$ if and only if $\mathcal{S}_\Lambda \subset \mathcal{S}_{\Lambda'}$.*
2. *We have*

$$\mathcal{S}_\Lambda \cap \mathcal{S}_{\Lambda'} = \begin{cases} \mathcal{S}_{\Lambda\cap\Lambda'}, & \text{if } \Lambda \cap \Lambda' \text{ is a vertex lattice again,} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{5.4.1}$$

3. *Recall that $\mathcal{B}$ is the set of vertex lattices, then we have*

$$\mathcal{S} = \bigcup_{\Lambda \in \mathcal{B}} \mathcal{S}_\Lambda, \tag{5.4.2}$$

and each closed Bruhat-Tits stratum $\mathcal{S}_\Lambda$ is projective and normal of dimension $[\frac{t(\Lambda)-1}{2}]$, with isolated singularities when n is odd, is smooth when n is even.

[1]Our proof is exactly the same as [VW11, Theorem 4.8], except that we didn't compute the dimension of the tangent space of $\mathcal{S}_\Lambda$ at every k-valued point, which seems not necessary.

Proof. It follows from Proposition 4.4.2 and Corollary 4.4.11 that part 1 and 2 are true. Part 3 follows from Corollary 5.3.8. □

Theorem 5.4.2.

1. *There is a stratification, which is called the Bruhat-Tits stratification, of $\mathcal{S}$ by locally closed subschemes*

$$\mathcal{S} = \biguplus_{\Lambda \in \mathcal{B}} \mathcal{S}_{\Lambda}^{\circ}, \tag{5.4.3}$$

and each stratum is isomorphic to the Deligne-Lusztig variety associated to the orthogonal group $\mathrm{SO}(\mathbb{B}_{\Lambda})$ and a σ-Coxeter element. The closure of each stratum $\mathcal{S}_{\Lambda}^{\circ}$ in $\mathcal{S}$ is given by

$$\overline{\mathcal{S}_{\Lambda}^{\circ}} = \biguplus_{\Lambda' \subset \Lambda} \mathcal{S}_{\Lambda'}^{\circ} = \mathcal{S}_{\Lambda}. \tag{5.4.4}$$

2. *The scheme $\mathcal{S}$ is geometrically connected of pure dimension $[\frac{n-1}{2}]$. The irreducible components of $\mathcal{S}$ are those $\mathcal{S}_{\Lambda}$ with $t(\Lambda) = n$.*

Proof.

1. The stratification follows from (5.1.10) and part 3 of Theorem 5.4.1.

2. For a vertex lattice, the form ψ in section 3.3 defines a non-degenerate symplectic form on the quotient space $\Lambda^{\sharp}/\Lambda$ (cf. [RTW14, Lemma 6.4]). Then a vertex lattices Λ' such that $\Lambda' \supset \Lambda$ corresponds to an isotropic subspace of $\Lambda^{\sharp}/\Lambda$. In particular, Λ is contained in a maximal type vertex lattice. By the part 1 of Theorem 5.4.1, $\mathcal{S}_{\Lambda}$ is an irreducible component of $\mathcal{S}$ if $t(\Lambda) = n$. The simplicial complex $\mathcal{B}$ is connected, hence $\mathcal{S}$ is connected of pure dimension $[\frac{n-1}{2}]$.

□

Chapter 6

The supersingular locus of the unitary Shimura varieties

6.1 The integral model

We start with the ramified unitary PEL datum of signature $(1, n-1)$ (cf. [PR09, 1.1] or [Har15, 5.1]). For the definition of the general PEL datum, we refer to [Har15, 2.1].

Let E be an imaginary quadratic field extension of $\mathbb{Q}$ with a fixed embedding $\gamma_0 \colon E \hookrightarrow \mathbb{C}$. Let $^- \in \mathrm{Gal}(E/\mathbb{Q})$ be the unique non-trivial automorphism. Then γ_0 and $\gamma_1 := \gamma_0 \circ {}^-$ give rise to all the embeddings of E into $\mathbb{C}$. Let $W = E^n$ be an n-dimensional vector space over E, where $n \geq 3$, together with a hermitian form φ. We fix an element $\epsilon \in E$ such that $\bar{\epsilon} = -\epsilon$, then the form $\epsilon \cdot \varphi$ is a skew hermitian form on W. Furthermore, we assume that the hermitian form φ is of signature $(1, n-1)$ in the following sense: there exists a $\mathbb{C}$-basis of $W \otimes_{E,\gamma_0} \mathbb{C}$ such that the matrix of φ is

$$H := \mathrm{diag}(-1, 1, \dots, 1). \tag{6.1.1}$$

Note that we have an $\mathbb{R}$-isomorphism $W \otimes_{\mathbb{Q}} \mathbb{R} \cong W \otimes_{E,\gamma_0} \mathbb{C}$. Therefore the matrix $\sqrt{-1} \cdot H$ defines an $\mathbb{R}$-endomorphism of $W \otimes \mathbb{R}$ satisfying $(\sqrt{-1} \cdot H)^2 = -\mathrm{id}$ and hence a complex structure of $W \otimes \mathbb{R}$.

The hermitian form φ defines a $\mathbb{Q}$-linear symplectic form $\langle\, , \rangle \colon W \times W \to \mathbb{Q}$ by $\langle\, , \rangle := \mathrm{Tr}_{E/\mathbb{Q}}(\epsilon \cdot \varphi(\, ,))$. The form $\langle v, \sqrt{-1} \cdot Hw \rangle$, for $v, w \in W \otimes \mathbb{R}$, is $\mathbb{R}$-symmetric, and if it is not positive definite, we replace ϵ by $-\epsilon$ which will guarantee the positive definiteness.

Let p be an odd prime which ramifies in E. Let v be the place above p, E_v the completion of E at v with the ring of integers $\mathcal{O}_v$. Let π be a uniformizer of E_v such that $\bar{\pi} = -\pi$. We assume that the hermitian space $(W \otimes_E E_v, \varphi)$ is split. We can define standard lattices $\{\Lambda_i\}_{i \in \mathbb{Z}}$ in the same manner as in 2.3.3.

In summary, we have the following PEL datum.

Definition 6.1.1 (*The ramified unitary PEL datum of signature* $(1, n-1)$)**.**

1. An imaginary quadratic extension E of $\mathbb{Q}$ with the non-trivial automorphism $^{-} \in \mathrm{Gal}(E/\mathbb{Q})$.
2. An E-vector space W of dimension n, together with a non-degenerate hermitian form φ of signature $(1, n-1)$.
3. The $\mathbb{Q}$-linear symplectic form $\langle\,,\,\rangle\colon W \times W \to \mathbb{Q}$ satisfying $\langle bv, w\rangle = \langle v, \bar{b}w\rangle$ for any $v, w \in W$ and $b \in E$.
4. The endomorphism $\sqrt{-1} \cdot H \in \mathrm{End}_{\mathbb{R}}(W \otimes_{\mathbb{Q}} \mathbb{R})$.

Furthermore, we have the self-dual lattice chain $\mathcal{L} = \{\Lambda_i\}_{i\in\mathbb{Z}}$ in the hermitian space $(W \otimes_E E_v, \varphi)$.

For any $\mathbb{Q}$-algebra R, let

$$\mathbb{G}(R) := \left\{ g \in \mathrm{GL}_{E\otimes R}(W \otimes R) \;\middle|\; \begin{array}{l} \exists c = c(g) \text{ such that } \forall v, w \in W \\ \langle g(v), g(w)\rangle = c\langle v, w\rangle \end{array} \right\}. \tag{6.1.2}$$

Then $\mathbb{G}$ is a reductive group over $\mathbb{Q}$. Sending $\sqrt{-1}$ to $\sqrt{-1} \cdot H$ defines a homomorphism

$$h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \longrightarrow \mathbb{G}_{\mathbb{R}}. \tag{6.1.3}$$

Then the $\mathbb{Q}$-reductive group $\mathbb{G}$ and the $\mathbb{G}(\mathbb{R})$-conjugacy class X of h define a Shimura datum, hence the Shimura variety $\mathrm{Sh}(\mathbb{G}, h)$ over the reflex field E. Let $C = \prod_w C_w$ be an open compact subgroup of $\mathbb{G}(\mathbb{A}_f)$ with $C_w \subset \mathbb{G}(\mathbb{Q}_w)$. Then the Shimura variety $\mathrm{Sh}_C(\mathbb{G}, h)$ is a quasi-projective variety over E whose $\mathbb{C}$-valued points can be identified with

$$\mathbb{G}(\mathbb{Q})\backslash(X \times (\mathbb{G}(\mathbb{A}_f)/C)). \tag{6.1.4}$$

We assume that the subgroup $C^p := \prod_{w\neq p} C_w \subset \mathbb{G}(\mathbb{A}_f^p)$ is sufficiently small, i.e. the subgroup C^p is contained in the principal congruence subgroup of level $N \geq 3$, where N is coprime to the discriminant of E. We also assume that C_p is the parahoric subgroup of $\mathbb{G}(\mathbb{Q}_p)$ stabilizing the lattice Λ_m.

Now we define the integral model of $\mathrm{Sh}_C(\mathbb{G}, h)$ over E_v following [RZ96, Chapter 6]. For a fixed base scheme S, let $\mathrm{AV}(S)$ be the category of abelian $\mathcal{O}_E$-varieties up to isogeny of order prime to p over S (cf. [RZ96, 6.3]). The objects in $\mathrm{AV}(S)$ are pairs (A, ι), where A is an n-dimensional abelian scheme over S together with an $\mathcal{O}_E$-action

$$\iota\colon \mathcal{O}_E \otimes \mathbb{Z}_{(p)} \longrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}. \tag{6.1.5}$$

The morphisms between (A_1, ι_1) and (A_2, ι_2) are the homomorphisms between abelian schemes compatible with the $\mathcal{O}_E$-actions, tensored with $\mathbb{Z}_{(p)}$.

$$\mathrm{Hom}_{\mathrm{AV}}(A_1, A_2) := \mathrm{Hom}_{\mathcal{O}_E}(A_1, A_2) \otimes \mathbb{Z}_{(p)}. \tag{6.1.6}$$

Definition 6.1.2. The naive moduli functor $\mathcal{A}_{C^p}^{\mathrm{naive}}$ over $\mathcal{O}_{E_v}$ is a set-valued functor:

$$\begin{aligned} (\mathcal{O}_{E_v})\text{-schemes} &\longrightarrow (\text{Sets}), \\ S &\longmapsto \text{isomorphism classes of } (A, \iota, \bar{\lambda}, \bar{\eta}), \end{aligned} \tag{6.1.7}$$

where $(A, \iota) \in \mathrm{AV}(S)$, $\bar{\lambda}$ is a $\mathbb{Q}$-homogeneous polarization of (A, ι) which contains a polarization $\lambda\colon A \to A^\vee$ such that

- if n is odd, $\ker(\lambda) \subset A[\iota(\pi)]$ is of height $n-1$,
- if n is even, $\ker(\lambda) = A[\iota(\pi)]$;

and $\bar{\eta}$ is a C^p-level structure

$$\bar{\eta}\colon \mathrm{H}_1(A, \mathbb{A}_f^p) \cong W \otimes \mathbb{A}_f^p \mod C^p. \tag{6.1.8}$$

Furthermore, the pair (A, ι) is required to satisfy the determinant condition:

$$\det_{\mathcal{O}_S}(\iota(a)|\mathrm{Lie}_S(A)) = (T_0 + T_1\pi)(T_0 + T_1\bar{\pi})^{n-1} \in \mathcal{O}_{E_v}[T_0, T_1] \tag{6.1.9}$$

for all $a \in \mathcal{O}_{E_v}$. Then the functor $\mathcal{A}_{C^p}^{\mathrm{naive}}$ is represented by a quasi-projective scheme over $\mathcal{O}_{E_v}$, which is denoted by $\mathcal{A}_{C^p}^{\mathrm{naive}}$ by abuse of notation.

The scheme $\mathcal{A}_{C^p}^{\mathrm{naive}}$ is not flat by [Pap00, Proposition 3.8].

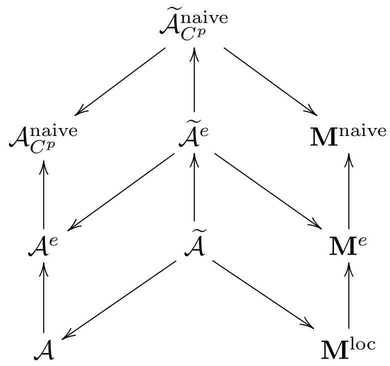
Definition 6.1.3. The subfunctor $\mathcal{A}^e$ of $\mathcal{A}_{C^p}^{\mathrm{naive}}$ is defined by requiring that the quadruple $(A, \iota, \bar{\lambda}, \bar{\eta}) \in \mathcal{A}^e(S)$ satisfy the following condition(s):

1. (Wedge condition.) For each $a \in \mathcal{O}_{E_v}$, the homomorphisms
$$\wedge^n(\iota(a) - a)\colon \wedge^n \mathrm{Lie}(A) \longrightarrow \wedge^n \mathrm{Lie}(A), \tag{6.1.10}$$
$$\wedge^2(\iota(a) - \bar{a})\colon \wedge^2 \mathrm{Lie}(A) \longrightarrow \wedge^2 \mathrm{Lie}(A), \tag{6.1.11}$$
are both equal to zero.
2. When n is even, the extra Spin condition is assumed: $\iota(\pi)|\mathrm{Lie}(A_s)$ non-vanishing for all $s \in S$.

Definition 6.1.4. The *honest integral model* $\mathcal{A}$ is defined as the flat closure of $\mathcal{A}_{C^p}^{\mathrm{naive}}$ in its generic fiber.

Proposition 6.1.5 (Smithling)**.** *The functor $\mathcal{A}^e$ is represented by a closed subscheme of $\mathcal{A}_{C^p}^{\mathrm{naive}}$ over $\mathcal{O}_{E_v}$, which is topologically flat and of dimension $n-1$. Furthermore, when n is even, $\mathcal{A}^e$ is flat over $\mathcal{O}_{E_v}$, in other words, $\mathcal{A}^e = \mathcal{A}$.*

Proof. Note that $\mathcal{A}_{C^p}^{\mathrm{naive}}$, $\mathcal{A}^e$ and $\mathcal{A}$ sit inside the local model diagram:



(6.1.12)

by [Pap00, Theorem 2.2], see the definition of $\widetilde{\mathcal{A}}_{C^p}^{\mathrm{naive}}$ in loc. cit. Then similar to the proof of Proposition 3.2.2, the proposition follows from the property of the local model. $\square$

Remark 6.1.6. By the local model diagram (6.1.12), we can see that the honest integral model is smooth, because the local model $\mathbf{M}^{\mathrm{loc}}$ is smooth (cf. [Arz09, Proposition 4.16] & [Ric13, Remark 0.7]).

Let $\mathcal{A}_{\mathbb{F}}$ (resp. $\mathcal{A}^e_{\mathbb{F}}$) be the special fiber of $\mathcal{A}$ (resp. $\mathcal{A}^e$), then by Proposition 6.1.5, we have $\mathcal{A}_{\mathbb{F},\mathrm{red}} = \mathcal{A}^e_{\mathbb{F},red}$.

6.2 The supersingular locus

Let $\mathcal{A}^{e,\mathrm{ss}}_{\mathbb{F}}$ (resp. $\mathcal{A}^{\mathrm{ss}}_{\mathbb{F}}$) be the supersingular locus of $\mathcal{A}^e_{\mathbb{F}}$ (resp. $\mathcal{A}^{\mathrm{ss}}_{\mathbb{F}}$), then $\mathcal{A}^{\mathrm{ss}}_{\mathbb{F}} = \mathcal{A}^{e,\mathrm{ss}}_{\mathbb{F}}$ because by definition the supersingular locus is endowed with the closed reduced subscheme structure (cf. [RZ96, Theorem 6.27]).

Similarly to the naive case, we have the p-adic uniformization theorem.

Theorem 6.2.1 ([RZ96, Theorem 6.30] & [VW11, 6.4])**.** *Let $(A_0, \iota_0, \bar{\lambda}_0, \bar{\eta}_0) \in \mathcal{A}^e(\mathbb{F})$ be a supersingular abelian variety, together with its corresponding Rapoport-Zink space $\mathcal{N}^e$. Then the uniformization morphism given by $(A_0, \iota_0, \bar{\lambda}_0, \bar{\eta}_0)$*

$$\Theta\colon \mathbb{I}(\mathbb{Q})\backslash\mathcal{N}^e_{\mathrm{red}} \times \mathbb{G}(\mathbb{A}^p_f)/C^p \longrightarrow \mathcal{A}^{\mathrm{ss}}_{\mathbb{F}} \tag{6.2.1}$$

is an isomorphism, $\mathbb{I}$ is the group of $\mathcal{O}_{E_v}$-linear quasi-isogenies in $\mathrm{End}(A_0) \otimes \mathbb{Q}$ *which respect the polarizations $\bar{\lambda}_0$. And the source of the uniformization morphism is a finite disjoint sum*

$$\coprod_{i=1}^{m} \Gamma_i\backslash\mathcal{N}^e_{\mathrm{red}}, \tag{6.2.2}$$

where $\Gamma_i = \mathbb{I}(\mathbb{Q}) \cap g_i C^p g_i^{-1} \subset J(\mathbb{Q}_p)$ which is discrete and cocompact modulo center, and $g_1, \ldots, g_m$ are representatives of the finitely many double cosets in $\mathbb{I}(Q)\backslash\mathbb{G}(\mathbb{A}^p_f)/C^p$. Furthermore, the induced surjective morphism

$$\tilde{\Theta}\colon \coprod_{i=1}^{m} \mathcal{N}^e_{\mathrm{red}} \longrightarrow \mathcal{A}^{\mathrm{ss}}_{\mathbb{F}}, \tag{6.2.3}$$

is a local isomorphism and the restriction of $\tilde{\Theta}$ to any closed quasi-compact subscheme of $\mathcal{N}^e_{\mathrm{red}}$ is finite.

Theorem 6.2.2. *The supersingular locus $\mathcal{A}^{\mathrm{ss}}_{\mathbb{F}}$ is of pure dimension $[\frac{n-1}{2}]$. We have natural bijections*

$$\{\textit{irreducible components of } \mathcal{A}^{\mathrm{ss}}_{\mathbb{F}}\} \xrightarrow{1:1} \mathbb{I}(\mathbb{Q})\backslash(J(\mathbb{Q}_p)/K_{\max} \times \mathbb{G}(\mathbb{A}^p_f)/C^p), \tag{6.2.4}$$

and

$$\{\textit{connected components of } \mathcal{A}^{\mathrm{ss}}_{\mathbb{F}}\} \xrightarrow{1:1} \mathbb{I}(\mathbb{Q})\backslash(J(\mathbb{Q}_p)/J^0 \times \mathbb{G}(\mathbb{A}^p_f)/C^p). \tag{6.2.5}$$

where J^0 is the subgroup of $J(\mathbb{Q}_p)$ consisting of those j such that $c(j)$ is a unit and $K_{\max}$ is the stabilizer of some maximal-type vertex lattice in $J(\mathbb{Q}_p)$.

Proof. the proof is the same as [VW11, 6.5]. □

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