Motives for an elliptic curve without complex multiplication

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Abstract

We provide an algebraic model of the idempotent complete rigid tensor subcategory $\mathbf{DMEM}(k, \mathbf{Q})_E$ of Voevodsky's triangulated category of geometric motives $\mathbf{DM}_{gm}(k, \mathbf{Q})$, defined over a base field k of characteristic zero with rational coefficients, generated by the motive of a fixed elliptic curve E without complex multiplication. Our construction relies upon two major ingredients. The first is to study the general properties of commutative differential graded algebras in the category of representations of a reductive algebraic group with a non-trivial central cocharacter. This includes a description of the derived category of differential graded modules over a given algebra and a criterion for the existence of a t-structure on the derived category. The second ingredient is to use Friedlander-Suslin complexes to define a cycle complex for E, which is a commutative differential graded algebra over GL_2. With these ingredients in hand, we apply the general theory to our particular cycle algebra and show the desired equivalence between the full subcategory of the derived category of differential graded modules over the cycle algebra consisting of compact objects and $\mathbf{DMEM}(k, \mathbf{Q})_E$.

Zusammenfassung

Wir konstruieren ein algebraisches Modell der idempotent-vollständigen starren $\otimes$-Unterkategorie $\mathbf{DMEM}(k, \mathbf{Q})_E$ von Voevodskys triangulierter Kategorie von geometrischen Motiven $\mathbf{DM}_{gm}(k, \mathbf{Q})$ über einen Körper k der Charakteristik Null mit rationalen Koeffizienten, erzeugt von dem Motiv einer festen elliptischen Kurve E ohne komplexe Multiplikation. Unsere Konstruktion beruht auf zwei Hauptbestandteilen. Der erste Bestandteil ist die Beschreibung der allgemeinen Eigenschaften der kommutativen, differentiell-graduierten Algebren in der Kategorie der Darstellungen einer reduktiven algebraischen Gruppe mit einem nicht-trivialen zentralen Cocharacter. Dies schließt eine Beschreibung der abgeleiteten Kategorie der differentiell-graduierten Moduln über einer gegebenen Algebra und ein Kriterium für die Existenz einer t-Struktur auf der abgeleiteten Kategorie ein. Der zweite Bestandteil ist die Konstruktion eines Zykluskomplexes für E mit Hilfe von Fridelander-Suslin Komplexen. Dieser ist eine kommutative differentiell-graduierte Algebra über GL_2. Hiermit können wir die allgemeine Theorie auf unsere Zyklus-Algebra anwenden und damit die gewünschte Äquivalenz zwischen der vollen Unterkategorie der kompakten Objekte der abgeleiteten Kategorie der differentiell-graduierten Moduln über der Zyklus-Algebra und $\mathbf{DMEM}(k, \mathbf{Q})_E$ zeigen.

Introduction

1. A quick introduction to motives and their history

In the late sixties, Grothendieck proposed a unified point of view on cohomology theories (for example Betti cohomology, étale cohomolgy, de Rham cohomology) for smooth projective varieties, namely pure motives. The universal property of pure motives makes it easier to grasp the essence of different cohomology theories. Assuming some very deep conjectures, the theory of "pure motives" would produce an elegant proof of Weil conjectures, different from the known proof given by Deligne. Along with the successful development of Deligne's mixed Hodge theory from the late seventies, in [**4**] Beilinson conjectured the existence of an abelian category of mixed motives as the counterpart. Compared to pure motives, the theory of mixed motives reflects the universal properties of cohomology theories for smooth varieties. The existence of such an abelian category of mixed motives would have important consequences for our understanding of smooth varieties. However the question of the existence of such a category is open and mysterious. In [**15**], Voevodsky gave a reasonable definition of a triangulated category of geometric motives[1] $\mathbf{DM}_{gm}(k,\mathbb{Z})$[2]. (We will briefly review one of the constructions of $\mathbf{DM}_{gm}(k,\mathbb{Z})$ in Section 1 of Chapter 3.) His construction satisfies most of the properties predicted in [**4**], although it is unknown if there is a reasonable abelian subcategory of $\mathbf{DM}_{gm}(k,\mathbb{Q})$ that has the desired properties of being the conjectured abelian category of mixed motives. Rather than attempting to study the category $\mathbf{DM}_{gm}(k,\mathbb{Q})$ all at once, a number of authors have considered certain subcategories of $\mathbf{DM}_{gm}(k,\mathbb{Q})$, for instance, the full rigid tensor triangulated subcategory generated by a single object $M(X)$[3] of $\mathbf{DM}_{gm}(k,\mathbb{Q})$. They investigated their structure and constructed concrete models for these subcategories, often as derived categories of an explicit commutative differential graded algebra and in some cases even as the derived category of an abelian category of comodules over an explicit Hopf algebra. They are also interested in explicit conditions for the existence of the abelian category of motives associated to X. The first successful case is the one of mixed Tate motives over number fields. I will give a brief review in next section, since some crucial ideas behind the case of mixed Tate motives will provide our main source of inspiration in the case of motives for an elliptic curve. The aim of this paper is to construct an algebraic model of geometric motives associated to a fixed elliptic curve without complex multiplication.

[1]From now on motives are always in the sense of mixed motives .

[2]The motives are defined over k with coefficients in $\mathbb{Z}$.

[3]Here $M(X)$ mean the motive of X in $\mathbf{DM}_{gm}(k,\mathbb{Q})$ for X a given smooth projective variety over k.

2. Review of the description of mixed Tate motives from the cycle algebra viewpoint

Let us denote the full triangulated subcategory of $\mathbf{DM}_{gm}(k, \mathbb{Q})$ generated by the Tate objects[4] $\mathbb{Q}(n)$ for any $n \in \mathbb{Z}$ by $\mathbf{DMT}(k, \mathbb{Q})$. In [**23**], Levine showed that, if k satisfies the Beilinson-Soulé vanishing conjectures, there is a t-structure defined on $\mathbf{DMT}(k, \mathbb{Q})$, whose heart was called the abelian category $\mathbf{MT}(k, \mathbb{Q})$ of mixed Tate motive. By the work of Borel [**8**], we know that, if k is a number field, then it satisfies the B-S vanishing conjectures.

Bloch and Kriz used the idea of cycle algebras to define an abelian category of mixed Tate motives in [**6**] without assuming the B-S vanishing conjectures. More precisely, they defined the cycle algebra $\mathcal{N}_k = \mathbb{Q} \oplus \bigoplus_{r \geq 1} \mathcal{N}_k(r)$. The r-th component $\mathcal{N}_k(r)$ of $\mathcal{N}_k$ is a shifted, alternating version of Bloch's cycle complex $z^r(k, 2r - *)^{Alt}$. For a non-negative integer p such that $p \leq 2r$, we denote the the $\mathbb{Q}$ vector space generated by codimesnsion r cycles on the "cube" $(\mathbb{P}^1 - \{1\})^{2r-p}$ by $z^r(k, 2r - p)$. Then each individual term $z^r(k, 2r - p)^{Alt}$ of the complex $z^r(k, 2r - *)^{Alt}$ consists of the alternating elements of $z^r(k, 2r - p)$ with respect to the action of the symmetric group Σ_{2r-p} on $(\mathbb{P}^1 - \{1\})^{2r-p}$. The differentials are defined in Section 5 of [**6**]. This additional grading r is called the Adams grading. Then they used the reduced bar construction[5] $\overline{B}(\mathcal{N}_k)$ associated with the cycle algebra $\mathcal{N}_k$ to give an Adams graded Hopf algebra $H^0(\overline{B}(\mathcal{N}_k))$, which corresponds to a pro-group scheme $G_{BK}(k)$. They defined the category $\mathbf{MT}_{BK}(k, \mathbb{Q})$ of Bloch-Kriz mixed Tate motives over k to be the finite dimensional graded $\mathbb{Q}$-representations over $G_{BK}(k)$. However, they did not attempt to relate this category or its derived category to $\mathbf{DM}_{gm}(k, \mathbb{Q})$.

In [**21**], Kriz and May introduced the definitions of an Adams graded commutative differential graded algebra (or cdga for short) $\mathcal{A}$ and Adams graded differential graded (or dg for short) $\mathcal{A}$-modules. They considered the bounded derived category $\mathcal{D}^f_{\mathcal{A}}$ of dg $\mathcal{A}$-modules, which is equipped with a weight filtration. If the Adams cdga $\mathcal{A}$ is cohomologically connected[6], there is a t-structure on $\mathcal{D}^f_{\mathcal{A}}$, whose heart is denoted by $\mathcal{H}^f_{\mathcal{A}}$. Next they showed that, under the assumption that $\mathcal{A}$ is cohomologically connected, there is an exact functor

$$\rho : D^b(gr - rep^f_{\mathbb{Q}}(G_{\mathcal{A}})) \to \mathcal{D}^f_{\mathcal{A}},$$

where $G_{\mathcal{A}} = Spec(H^0(\overline{B}(\mathcal{A})))$ and $D^b(gr - rep^f_{\mathbb{Q}}(G_{\mathcal{A}}))$ is the bounded derived category of finite graded $\mathbb{Q}$-representations over $G_{\mathcal{A}}$. The interesting part is that they showed that ρ induces an equivalence between $gr - rep^f_{\mathbb{Q}}(G_{\mathcal{A}})$ and $\mathcal{H}^f_{\mathcal{A}}$. (In general, ρ may not be an equivalence.)

As an application of Kriz and May's theory, when we take $\mathcal{A} = \mathcal{N}_k$ and assume k satisfies the B-S vanishing conjectures, which is equivalent to saying that $\mathcal{N}_k$ is cohomologically connected, we can identify the heart $\mathcal{H}^f_{\mathcal{N}_k}$ of $\mathcal{D}^f_{\mathcal{N}_k}$ with $\mathbf{MT}_{BK}(k, \mathbb{Q})$.

[4]See Example 3.12.

[5]See Section 8 of Chapter 2.

[6]The cohomologically connectedness condition means the negative cohomology groups of the complex $\mathcal{A}$ vanish and the 0-th cohomology group of $\mathcal{A}$ is isomorphic to $\mathbb{Q}$.

Later Spitzweck defined an equivalence

$$\theta_k : \mathcal{D}^f_{\mathcal{N}_k} \to \mathbf{DMT}(k, \mathbb{Q})$$

for any field k in [**32**]. If k satisfies the B-S vanishing conjectures, then θ_k gives an equivalence between $\mathcal{H}^f_{\mathcal{N}_k}$ and $\mathbf{MT}(k, \mathbb{Q})$. Altogether, if k satisfies the B-S vanishing conjectures, we may identify all these constructions of the abelian category of mixed Tate motives.

Recently Levine generalized the work of Kriz and May to the setting of relative cdgas in [**22**]. Using the relative theory of cdgas (Chapter 2 in [**22**]), Levine extends Spitzweck's equivalence from fields to $S \in \mathbf{Sm}_k$ (Theorem 5.3.2 in [**22**]).

3. The case of an elliptic curve without CM and main results

After handling the case of mixed Tate motives, the next case is to understand the category of motives of an elliptic curve, since the category of mixed Tate motives can be considered as the category of motives associated to the projective line[7]. Then the reader may ask:

Can the methods for describing the category of mixed Tate motives as a derived category of Adams graded dg modules can be applied to the "elliptic" case?

Convention 0.1. Throughout this paper, we restrict ourself to the case that E is an elliptic curve defined over a base field k of characteristic zero without complex multiplication.

In order to answer the above question, one important hint for us is: conjecturally, the category of mixed motives (resp. geometric motives) can be understood as an abelian (resp. triangulated) category graded[8] over the category of numerical motives. (The definition of numerical motives is contained in Chapter 4 of [**2**].) In our case, we already know that the rigid tensor subcategory of the category of numerical motives generated by E is isomorphic to the category of rational representations of GL_2. (Further explanation is included in Example 7.6.4.1 in [**2**].) Then the conjectured abelian category of mixed motives generated by E is believed to be graded over the category of representations of GL_2. Therefore we also expect that *the pseudo-abelian rigid tensor triangulated subcategory generated by the motive of E should be considered as a triangulated category graded over the category of GL_2 representations.*

The modified approach of studying motives of an elliptic curve without CM from the cycle algebra viewpoint will be divided into the following steps.

Step 1: Define a reasonable theory replacing Kriz and May's framework. As explained above, we need to develop a good theory of dg modules over cdgas over GL_2.

Let A be a cdga over GL_2, which is a cdga object in the category of GL_2 representations (Definition 2.2). Then we can consider dg A-modules (Definition 2.3) and their derived category $\mathcal{D}_A^{GL_2}$ (Definition 2.13). In the category of dg A-modules, there is a special kind of

[7]See Example 3.12.

[8]There is a functor from the category of mixed motives to the category of numerical motives. See Section 21.1 in [**2**].

dg A-module, namely *the cell A-modules*. Roughly speaking, a cell A-module is obtained by iterate extensions of some kinds of "free" dg A-modules. The precise definition is contained in Definition 2.7. If we denote the homotopy category of cell modules by $\mathcal{KCM}_A^{GL_2}$, our first result says that any dg A-module can be approximated by cell A-modules:

THEOREM 0.2. *(Theorem 2.21 for $G = GL_2$)*
The functor

$$\mathcal{KCM}_A^{GL_2} \to \mathcal{D}_A^{GL_2}$$

is an equivalence of triangulated categories.

Assume in addition that A is *cohomologically connected*, that is, the negative cohomology groups of A vanish and the 0-th cohomology group of A is isomorphic to the trivial representation (Definition 2.3). Then there exists a t-structure on $\mathcal{D}_A^{GL_2,f}$, the subcategory of $\mathcal{D}_A^{GL_2}$ consisting of *finite objects*[9]. See Theorem 2.60. We denote the heart of $\mathcal{D}_A^{GL_2,f}$ by $\mathcal{H}_A^{GL_2,f}$. Then we have:

THEOREM 0.3. *(Theorem 2.89 for $G = GL_2$)*
Let A be a cohomologically connected cdga over GL_2. Then

- *There is a functor: $\rho : D^b(\mathcal{H}_A^{GL_2,f}) \longrightarrow \mathcal{D}_A^{GL_2,f}$.*
- *The functor ρ constructed above is an equivalence of triangulated categories if and only if A is 1-minimal.*

For the delicate definition of 1-minimal, we refer to Definition 2.54.

REMARK 0.4. More generally, we prove the above theorems for G any reductive group over a base field of characteristic zero with a nontrivial central cocharacter.

Step 2: Construct the cycle algebra for E.
This is the technical heart of the paper.

(1) Let $\mathbf{F}$ be the fundamental representation of GL_2 and *det* the determinant representation. Every rational representation over GL_2 is a direct summand of a rational representation of the form $\mathbf{F}^{\otimes a} \otimes det^{\otimes b}$ for some $a \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}$. In order to construct a cdga over GL_2, one needs to imagine a reasonable "coefficient" associated to the factor $\mathbf{F}^{\otimes a} \otimes det^{\otimes b}$. Roughly speaking, our candidate is the cycle complexes whose cohomology groups are exactly the hom groups between "the motive corresponding to the representation $\mathbf{F}^{\otimes a} \otimes det^{\otimes b}$" and the trivial motive $\mathbb{Q}$ in $\mathbf{DM}_{gm}(k, \mathbb{Q})$. These data constitute one piece of the final cycle algebra, namely $\mathcal{E}_{a,b}^{*}$[10] for $a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}$ (Definition 4.24). Then we show that the direct sum of the objects in the family $\{\mathcal{E}_{a,b}^*\}_{a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}, a \geq b}$ will form a bi-graded algebra. (However this is not the correct one!)
(2) Under the assumption of the elliptic curve E (Convention 0.1), we compute that $\mathcal{E}_{2,1}^*$ is quasi-isomorphic to the trivial GL_2 representation. The multiplicative structure induces an inclusion η from $\mathcal{E}_{a,b}^*$ to $\mathcal{E}_{a+2,b+1}^*$, which is defined in Section 3 of

[9]Finite objects are "dualizable" in a suitable sense. See Definition 2.37.
[10]The superscript $*$ denotes its cohomology degree. Sometimes we will omit this index for simplicity.

Chapter 4. After stabilization with η, we get two versions of the desired cycle algebra for E — $\mathcal{E}^*_{ell}$ and $\mathcal{E}^*$ (Definition 4.24), where $\mathcal{E}^*$ is a sub cdga of $\mathcal{E}^*_{ell}$ over GL_2. In Corollary 4.31 we show that $\mathcal{E}^*$ and $\mathcal{E}^*_{ell}$ are quasi-isomorphic if and only if the following properties (Definition 3.21) hold for E:

a) (The 0-th vanishing property for E):

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i}M_1(E), \mathbb{Q}(i)[j]) \cong 0,$$

for any $j \in \mathbb{Z}_{\leq 0}$ and any $i \in \mathbb{Z}_{>0}$.

b) (The r-th vanishing property for E for r any positive integer):

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E), \mathbb{Q}(i)[j]) \cong 0,$$

for any $j \in \mathbb{Z}$ such that $r + j \leq 0$ and any $i \in \mathbb{Z}_{\geq 0}$.

For the meaning of the symbols in the above isomorphisms, see Section 2 of Chapter 3. We also make a conjecture (Conjecture 3.23) asserting that the r-th vanishing properties for all $r \in \mathbb{Z}_{\geq 0}$ hold for E.

Remark 0.5. For i a positive integer, the analogue of the $2i$-th vanishing properties in the mixed Tate case is the following:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\mathbb{Q}(i), \mathbb{Q}[j]) \cong 0,$$

for any $j \in \mathbb{Z}$, which trivially holds. Therefore, the r-th vanishing properties are a new phenomenon in the elliptic case.

Step 3: Combine the previous two steps.

We denote the full pseudo-abelian rigid tensor triangulated subcategory of $\mathbf{DM}_{gm}(k, \mathbb{Q})$ generated by the motive of E by $\mathbf{DMEM}(k, \mathbb{Q})_E$. Using some general properties of triangulated categories (Section 4 of Chapter 1) together with the results in the previous two steps, we obtain the main theorem of this paper.

Theorem 0.6. *(Theorem 5.16) There is an exact functor*

$$\mathcal{M} : \mathcal{D}^{GL_2}_{\mathcal{E}_{ell}} \to \mathbf{DM}(k, \mathbb{Q}),$$

which is a lax tensor functor. Furthermore, the restriction of $\mathcal{M}$ to

$$\mathcal{M}^c : (\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c \to \mathbf{DM}(k, \mathbb{Q})$$

defines an equivalence of $(\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$ with $\mathbf{DMEM}(k, \mathbb{Q})_E$ as triangulated tensor categories, where $(\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$ is the subcategory of $\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}$ consisting of the compact objects.

Warning: $\mathbf{DM}(k, \mathbb{Q})$ is Voevodsky's big category of motives, which contains the category of geometric motives $\mathbf{DM}_{gm}(k, \mathbb{Q})$ as a full subcategory. $\mathbf{DM}(k, \mathbb{Q})$ is closed under colimits. However, $\mathbf{DM}_{gm}(k, \mathbb{Q})$ isn't. The detailed construction of $\mathbf{DM}(k, \mathbb{Q})$ using motivic symmetric spectra may be found in [**10**]; we omit giving details of the construction here. For a short introduction to $\mathbf{DM}(k, \mathbb{Q})$, we refer to Chapter 3 in [**22**].

We recall that, in the case of mixed Tate motives, the cohomological connectedness condition of the cycle algebra $\mathcal{N}_k$ is equivalent to saying that the base field k satisfies the

B-S vanishing conjectures. This motivates us to propose the generalized Beillison-Soulé vanishing conjectures (Conjecture 5.17)[11].

CONJECTURE 0.7. *An elliptic curve E over a field k without complex multiplication satisfies the conditions:*

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M_1(E)^{\otimes a}, \mathbb{Q}(a-b)[m]) = 0$$

in the following two cases:

A. $a = 0, b < 0, m \leq 0$;
B. $a > 0, a \geq 2b, m \leq 0$.

In our framework, the cohomological connectedness condition of the cycle algebra $\mathcal{E}_{ell}$ for E is equivalent to saying the generalized Beillison-Soulé vanishing conjectures and the r-th vanishing properties for $r \in \mathbb{Z}_{\geq 0}$ hold for E.

Then we show that:

THEOREM 0.8. *(Corollary 5.19) Assume that E is an elliptic curve without complex multiplication, that satisfies the r-th vanishing properties for $r \in \mathbb{Z}_{\geq 0}$ (Definition 3.21) and assume the generalized Beillison-Soulé vanishing conjectures holds for E. Then:*

1. $\mathbf{DMEM}(k, \mathbb{Q})_E$ *has a t-structure which is induced from*

$$\mathcal{M}^f : \mathcal{D}_{\mathcal{E}}^{GL_2,f} \to \mathbf{DMEM}(k, \mathbb{Q})_E,$$

where $\mathcal{M}^f$ is the restriction of the functor $\mathcal{M}$ (Theorem 0.6) to $\mathcal{D}_{\mathcal{E}}^{GL_2,f}$. Denote its heart by $\mathbf{MEM}(k, \mathbb{Q})_E$.

2. $\mathcal{M}^f$ *induces an equivalence of Tannakian categories:*

$$H^0(\mathcal{M}^f) : \mathcal{H}_{\mathcal{E}}^{GL_2,f} \to \mathbf{MEM}(k, \mathbb{Q})_E.$$

REMARK 0.9. Let us give a brief introduction to other related constructions or understanding of motives for an elliptic curve.

- In [**30**], Patashnick constructed a different cycle algebra for an elliptic curve E and defines one candidate for the abelian category of motives for E. Compared to his work, the advantage of our construction is its identification with a full subcategory of $\mathbf{DM}_{gm}(k, \mathbb{Q})$. Another difference between Patashnick's construction and ours is the use of Friedlander-Suslin complexes in our paper rather than Bloch's cycle complexes. This allows a purely functorial construction that should be easy to extend to families of elliptic curves over a smooth base scheme as Levine did in the mixed Tate motive case. We may tackle this problem in the future. This problem is related to a question raised in Section 24.5 of [**17**].
- We also mention that, besides the approach of cycle algebras along the lines of work of Bloch, Kriz, May et al., Kimura and Terasoma in [**19**] developed a theory of relative DGAs and used their theory to define another candidate for an

[11] In Remark 5.18, we discuss the relation between the the generalized Beillison-Soulé vanishing conjectures and the classical Beillison-Soulé vanishing conjectures.

abelian category of mixed elliptic motives. It would be interesting to relate their construction with ours.

At the very end of the paper, we show how the results in the case of mixed Tate motives enter into our setting.

4. Outline of the paper

Chapter 1: Preparations.

- Recap of the classical representation theory of the symmetric group and the reductive group GL_2 over a base field of characteristic zero;
- Review of some basic properties of triangulated categories.

Chapter 2: DG modules over G.

- Generalization of the theory of Adams graded cdgas and Adams graded dg modules to the theory of cdga over G, a reductive group defined over a base field of characteristic zero with a non-trivial central cocharacter;
- Establishment of fundamental properties of the new theory of cdgas over G.

Chapter 3: The category of motives.

- Sketch of the construction of the category of geometric motives $\mathbf{DM}_{gm}(k, \mathbb{Q})$;
- Description of motives of an elliptic curve;
- Relation between the r-th vanishing properties and other well-known conjectures in the theory of motives.

Chapter 4: Cycle algebras.

- Recollection of Susllin-Friedlander complexes and their cubical version;
- Construction of the cycle algebra for an elliptic curve E.
- Computation of cohomology groups of the cycle algebras and hom groups between cell modules over the cycle algebra.

Chapter 5: Connection with $\mathbf{DM}_{gm}(k, \mathbb{Q})$.

- Definition of the "motivic version" of the cycle algebra for E;
- Construction of a functor from the derived category of dg modules over the cycle algebra to Voevodsky's big category of motives;
- Relation with mixed Tate motives.

Notations and Conventions:
$\mathbb{Z}$ = the ring of integers.
$\mathbb{Z}_{\geq 0}$(resp. $\mathbb{Z}_{>0}$) = non-negative (resp. positive) integers.
Let k be a base field.
Let G be a reductive group over a base field of characteristic zero.
$\mathbf{Rep}_G$: the category of linear G-representations.
$D(G)$: the derived category of $\mathbf{Rep}_G$.
$\mathbf{Sch}_k$: the category of separated schemes (of finite type) over k.
$\mathbf{Sm}_k$: the category of smooth varieties over k.
$\mathcal{S}h^{tr}_{Nis}(k)$: the category of Nisnevich sheaves with transfers over k.
Ab: the category of Abelian groups.
For any additive category M, we let $C(M)$ denote the category of unbounded chain complexes over M.

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Contents

CHAPTER 1

PREPARATIONS

1. Representations of symmetric group Σ_n

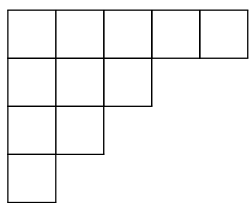
In this section, we recollect some basic facts about the representation of the n-th symmetric group Σ_n over a field of characteristic zero. The main reference is [**14**].

Definition 1.1. Fix a positive integer n. A partition $\lambda = (n_1, \cdots, n_r)$ of n is a sequence of numbers such that:

- $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$;
- $\sum_{i=1}^{r} n_i = n$.

We use $|\lambda| = n$ to denote that λ is a partition of n. △

To any partition, one can associate with a diagram of boxes, which is called the associated Young diagram. For instance, the Young diagram associated to $\lambda = (5, 3, 2, 1)$ is:



The transpose (or conjugate) of λ, which is denoted by λ^t, is defined by interchanging rows and columns in the Young diagram associated to λ.

Define a tableau on a given Young diagram corresponding to a partition λ to be a numbering of boxes by the integers $1, \cdots, n$. For example, the Young tableau of shape $\lambda = (5, 3, 2, 1)$ is:

1	2	3	4	5
6	7	8		
9	10			
11				

We consider two subgroups of the symmetric group Σ_n:

$$P_\lambda = \{\sigma \in \Sigma_n | \sigma \text{ preserves each row}\},$$

and

$$Q_\lambda = \{\sigma \in \Sigma_n | \sigma \text{ preserves each column}\}.$$

Then we define :

$$a_\lambda = \sum_{\sigma \in P_\lambda} e_\sigma \qquad \text{and} \qquad b_\lambda = \sum_{\sigma \in Q_\lambda} sgn(\sigma) e_\sigma$$

in $\mathbb{Z}[\Sigma_n]$. Here e_g stands for the base vector of $\mathbb{Z}[\Sigma_n]$ associated to $g \in \Sigma_n$ and $sgn(\sigma)$ is the sign of the permutation σ. Set:

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{Z}[\Sigma_n], \tag{1}$$

which is called a Young symmetrizer.

Example 1.2. When $\lambda = (n)$, $c_{(n)} = \sum_{\sigma \in \Sigma_n} e_\sigma$. When $\lambda = (1, \cdots, 1)$, $c_{(1,\cdots,1)} = \sum_{\sigma \in \Sigma_n} sgn(\sigma) e_\sigma$. △

In fact, given a field k of characteristic zero, all irreducible k-representations of the symmetric group Σ_n are determined by the Young symmetrizer.

Theorem 1.3. *In the group ring $\mathbb{Q}[\Sigma_n]$, there exists an integer n_λ such that $c_\lambda^2 = n_\lambda c_\lambda$. The image $V_\lambda = \mathbb{Q}[\Sigma_n]c_\lambda$ of c_λ is an irreducible representation of Σ_n. Every irreducible representation of Σ_n can be obtained in this way, i.e., there is a one-to-one correspondence between partitions of n and irreducible representations of Σ_n.*

Proof. See Theorem 4.3 in [**14**]. □

Remark 1.4. Given a partition λ of n, one can define an element

$$e_\lambda = \frac{dim V_\lambda}{n!} c_\lambda. \tag{2}$$

in $\mathbb{Q}[\Sigma_n]$, which is idempotent by Theorem 1.3.

2. Representations of GL_2

In this section, let k be a field of characteristic zero. Consider the linear algebraic group GL_2 defined over k.

Definition 1.5. Let V be a finite dimensional vector space over k. A rational representation of GL_2 in V is a homomorphism of algebraic groups $r : GL_2 \to GL(V)$ defined over k. △

Convention 1.6. Denote the morphism of group schemes sending λ to the scalar matrix $\lambda \cdot Id$ by $w : \mathbb{G}_m(k) \to GL_2(k)$.

Definition 1.7. Let V be a rational GL_2 representation. For any $r \in \mathbb{Z}$, define the weight r part of V to be a subrepresentation of V:

$$V\langle r| = \{x \in V | w(\lambda) \cdot x = \lambda^r x \quad \text{for any} \quad \lambda \in \mathbb{G}_m(k)\}.$$

A rational GL_2 representation V is called pure of weight r if $V\langle r| = V$. △

Let $\mathbf{F}$ denote the 2-dimensional fundamental rational representation of GL_2 over k.

REMARK 1.8. Because GL_2 is a split reductive algebraic group over k, the category of rational representations of GL_2 is isomorphic to the category of complex representations of $GL_2(\mathbb{C})$. See [**18**] chapter 2 in Part II.

A basic fact is that the action of the symmetric group Σ_n on $\mathbf{F}^{\otimes n}$ by permuting the factors commutes with the diagonal action of GL_2 on $\mathbf{F}^{\otimes n}$.

Denote the image of c_λ (see (1)) on $\mathbf{F}^{\otimes n}$ by $S_\lambda\mathbf{F}$:

$$S_\lambda\mathbf{F} = Im(c_\lambda|_{\mathbf{F}^{\otimes n}}).$$

THEOREM 1.9. *Let λ be a partition of n and m_λ be the dimension of the irreducible representation V_λ of Σ_n. Then*

$$\mathbf{F}^{\otimes n} \cong \bigoplus_{|\lambda|=n} S_\lambda\mathbf{F}^{\oplus m_\lambda}.$$

Furthermore, assume $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1)$, then if $k \geq 3$, $S_\lambda\mathbf{F}$ is zero. Otherwise,

$$\dim S_\lambda\mathbf{F} = \lambda_1 - \lambda_2 + 1.$$

PROOF. For the rational representations of $GL_2(\mathbb{C})$, we refer to Theorem 6.3 in [**14**]. Using Remark 1.8, we get the same result for GL_2 over any field k. □

Therefore, we can write $\mathbf{F}^{\otimes n}$ as a representation of $\Sigma_n \times GL_2$ as

$$\mathbf{F}^{\otimes n} \cong \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda\mathbf{F}, \tag{3}$$

where V_λ is the irreducible representation of Σ_n described above and the index set is taken over all the partitions of n with the length small and equal than 2. In addition, $S_\lambda\mathbf{F}$ is an irreducible representation of GL_2.

CONVENTION 1.10. We let $C_{a,b}$ be $\dim V_{(a+b,b)}$. In the rest of the paper, we use *det* (or $(\cdot)$) to denote the determinant representation of GL_2. For a partition $(a+b, b)$, $S_{(a+b,b)}\mathbf{F}$ is the same as the tensor product of the symmetric representation $Sym^a\mathbf{F}$ with the determinant representation $det^{\otimes b}$, i.e., $Sym^a\mathbf{F} \otimes det^{\otimes b}$, simply as denoted by $Sym^a(b)$. More generally, for $a \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}$, we define $Sym^a(b)$ to be the tensor product of the symmetric representation $Sym^a\mathbf{F}$ with the determinant representation[1] $det^{\otimes b}$.

We recall that every irreducible rational GL_2 representation is of form $Sym^a(b)$ for some non-negative integer a and integer b. See Chapter 15.5 in [**14**].

3. Linear representations of GL_2

In this section, we consider some general GL_2 representations over a base field k (not necessarily finite dimensional).

Let V be a vector space over k. We let GL_V denote the functor sending a k-algebra R to $Aut(V_R)$, where $Aut(V_R)$ is the group of R-linear automorphisms of V_R.

[1]If b is negative, $det^{\otimes b}$ is the dual representation of $det^{\otimes -b}$.

Definition 1.11. A linear representation of GL_2 is a homomorphism $r : GL_2 \to GL_V$ of group valued functors. For the definition of group valued functors and homomorphism between group valued functors, we refer to Chapter 2 in [**18**]. We denote the category of linear representations of GL_2 by $\mathbf{Rep}_{GL_2}$. △

Definition 1.12. An algebraic group is linearly reductive if every finite dimensional representation is semisimple. △

Remark 1.13. When the characteristic of the base field k is zero, by Remark 1.8, we know that GL_2 is linearly reductive. One important property for a linearly reductive group G is that every representation of G (not necessarily finite-dimensional) is a direct sum of simple representations. The proof is contained in Remark 14.49 of [**28**]. Therefore $\mathbf{Rep}_{GL_2}$ is a semisimple abelian category.

In the rest of this section, we assume that k has characteristic zero. As a consequence of the above remark, the simple objects in $\mathbf{Rep}_{GL_2}$ are just irreducible rational representations of GL_2, i.e., one element of $\{Sym^a(b) | a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}\}$.

Remark 1.14. $\mathbf{Rep}_{GL_2}$ admits a categorical direct sum.

Convention 1.15. The homomorphism in $\mathbf{Rep}_{GL_2}$ is simply denoted by $Hom_{GL_2}(\cdot, \cdot)$.

Definition 1.16. Given T a linear representation of GL_2, we define

$$V_{a,b}^T = Hom_{GL_2}(Sym^a(b), T).$$

Then we have an isomorphism between linear GL_2 representations:

$$T \cong \bigoplus_{a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}} V_{a,b}^T \otimes Sym^a(b).$$

A linear representation of GL_2 is called finite type if the index set in the above decomposition is finite. △

Remark 1.17. A linear representation T of GL_2 is rational if it is finite type and its multiplicity spaces $V_{a,b}^T$ are all finite dimensional.

Remark 1.18. The tensor structure for rational representations of GL_2 will extend to $\mathbf{Rep}_{GL_2}$. Similarly, the definition of the weight can be extended for every linear representation.

Definition 1.19. Let $\{M_i\}_{i \in I}$ be a family of linear representations of GL_2, where the index set I can be infinite. We define the product of $\{M_i\}_{i \in I}$ to be the following linear representation of GL_2:

$$\bigoplus_{a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}} \left(\prod_{i \in I} V_{a,b}^{M_i}\right) \otimes Sym^a(b).$$

△

Lemma 1.20. *The product defined above is a categorical product in the category of linear representations of GL_2.*

PROOF. The proof is straightforward. □

DEFINITION 1.21. Let M be a linear representation of GL_2. We write it as:

$$\bigoplus_{a\in\mathbb{Z}_{\geq 0}, b\in\mathbb{Z}} V^M_{a,b}\otimes Sym^a(b).$$

Then we define its dual $M^\vee$ to be

$$\bigoplus_{a\in\mathbb{Z}_{\geq 0}, b\in\mathbb{Z}} (V^M_{a,b})^\vee\otimes Sym^a(-a-b).$$

Here $(V^M_{a,b})^\vee$ is the dual vector space of $V^M_{a,b}$. △

DEFINITION 1.22. Given M, N two linear representations of GL_2, we define their internal hom $\mathcal{H}om(M,N)$ to be:

$$\bigoplus_{a,c\in\mathbb{Z}_{\geq 0}, b,d\in\mathbb{Z}} Hom_k(V^M_{a,b}, V^N_{c,d})\otimes (Sym^a(-a-b)\otimes Sym^c(d)).$$

△

REMARK 1.23. From Definition 1.22, it's easy to see that $\mathcal{H}om(M,k)\cong M^\vee$.

It is easy to verify the following two lemmas.

LEMMA 1.24. *Let* $\{M_i\}_{i\in I}$ *be a family of linear representations and* N *be a linear representation. Then we have:*

$$\mathcal{H}om(\bigoplus_{i\in I} M_i, N)\cong \prod_{i\in I}\mathcal{H}om(M_i,N).$$

LEMMA 1.25. *Given* T, M, N *three linear representations of* GL_2*, then there is an isomorphism:*

$$Hom_{GL_2}(T\otimes M, N)\cong Hom_{GL_2}(T, \mathcal{H}om(M,N)).$$

REMARK 1.26. Let G be a reductive group over a field of characteristic zero. If one replace GL_2 by G, all the definitions and statements in Section 2, 3 work. The key point is any reductive group over a base field of characteristic zero is linearly reductive. See Theorem 21.138 of [**28**].

4. Some properties of triangulated categories

In this section, we recall some useful definitions about (tensor) triangulated categories. For the basic definition of triangulated categories, we refer to Chapter 10 in [**35**].

Firstly we recall definitions about different kinds of generators of a triangulated category. See [**7**] or [**31**] for example. Every subcategory $\mathcal{U}$ of a triangulated category $\mathcal{T}$ we consider is strict, which means that, each object of $\mathcal{T}$, which is isomorphic to an object of $\mathcal{U}$, is an object of $\mathcal{U}$.

DEFINITION 1.27. Given $\mathfrak{S}$ a set of objects in a triangulated category $\mathcal{T}$, then we denote $\langle\mathfrak{S}\rangle$ to be the smallest strict full subcategory containing $\mathfrak{S}$ and closed under finite direct sums, direct summands and shifts. △

Definition 1.28. Give $\mathcal{A}, \mathcal{B}$ two subcategories of a triangulated category $\mathcal{T}$. We define:

- $\mathcal{A} \star \mathcal{B}$ is the full subcategory of $\mathcal{T}$ consisting of objects X which can be fit into a triangle
$$A \to X \to B \to A[1],$$
where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
- $\mathcal{A} \diamond \mathcal{B} = \langle \mathcal{A} \star \mathcal{B} \rangle$.
- $\langle \mathcal{A} \rangle_0 = 0$ and $\langle \mathcal{A} \rangle_n = \langle \mathcal{A} \rangle_{n-1} \diamond \langle \mathcal{A} \rangle$ inductively.
- Set $\langle \mathcal{A} \rangle_\infty = \bigcup_{n \geq 0} \langle \mathcal{A} \rangle_n$.

△

Definition 1.29. Let $\mathfrak{S}$ be a set of objects in a triangulated category $\mathcal{T}$. Then

- $\mathfrak{S}$ classically generates $\mathcal{T}$ if the smallest thick (i.e. closed under isomorphisms and direct summands) subcategory of $\mathcal{T}$ containing $\mathfrak{S}$ is $\mathcal{T}$ itself. Equivalently, $\mathcal{T} = \langle \mathfrak{S} \rangle_\infty$.
- $\mathfrak{S}$ generates $\mathcal{T}$ if, given an object $A \in \mathcal{T}$ such that
$$Hom_{\mathcal{T}}(S, A[n]) = 0$$
for all $S \in \mathfrak{S}$ and any $n \in \mathbb{Z}$, implies that $A = 0$.

△

Definition 1.30. Let $\mathcal{T}$ be a triangulated category admitting arbitrary direct sums. An object B in $\mathcal{T}$ is called compact if $Hom(B, \cdot)$ commutes with direct sums. Let $\mathcal{T}^c$ be the full subcategory of $\mathcal{T}$ consisting of compact objects. $\mathcal{T}$ is compactly generated if $\mathcal{T}$ is generated by $\mathcal{T}^c$. △

Recall the following result of Ravenel and Neeman in [**29**]:

Theorem 1.31. *Assume that $\mathcal{T}$ is compactly generated. Then a set of objects $\mathfrak{S} \subset \mathcal{T}^c$ classically generates $\mathcal{T}^c$ if and only if it generates $\mathcal{T}$.*

Now we want to build up a criterion for equivalences between triangulated categories.

Lemma 1.32. *Give $\mathcal{T}_1, \mathcal{T}_2$ two triangulated categories and $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ a triangulated functor. Let $\mathfrak{S}$ be a set of objects in $\mathcal{T}_1$, which classically generates $\mathcal{T}_1$ and is closed under shifts. Assume:*

1. *The set of the images of $\mathfrak{S}$ under ϕ classically generates $\mathcal{T}_2$;*
2. *ϕ restricted to $\mathfrak{S}$, which is viewed as a full subcategory of $\mathcal{T}_1$, is fully faithful.*

Then ϕ induce an equivalence between $\mathcal{T}_1$ and $\mathcal{T}_2$.

Proof. We denote the image of $\mathfrak{S}$ by $\phi(\mathfrak{S})$. It's enough to show that:
ϕ induces an equivalence between $\langle \mathfrak{S} \rangle_n$ and $\langle \phi(\mathfrak{S}) \rangle_n$ for any $n \in \mathbb{Z}_{\geq 0}$.
The case $n = 0$ is obvious.
Assume $n = 1$. Every object in $\langle \phi(\mathfrak{S}) \rangle$ is finite direct sums, direct summands and shifts of some objects in $\phi(\mathfrak{S})$. Since ϕ is a triangulated functor, it commutes with shifts and direct sums. Because ϕ is fully faithful restricting on $\mathfrak{S}$, the direct summands of an

object $\phi(A)$ in $\phi(\mathfrak{S})$ is one-to-one corresponding to the direct summands of $A \in \mathfrak{S}$. This implies that:

$$\phi : \langle \mathfrak{S} \rangle_1 \to \langle \phi(\mathfrak{S}) \rangle_1$$

is essential surjective. Furthermore ϕ is clearly fully faithful, which implies that ϕ is an equivalence.

Assume ϕ induces an equivalence between $\langle \mathfrak{S} \rangle_n$ and $\langle \phi(\mathfrak{S}) \rangle_n$. Let us prove the case $n+1$.

Take an element B_{n+1} in $\langle \phi(\mathfrak{S}) \rangle_n \star \langle \phi(\mathfrak{S}) \rangle_1$, which implies that there exists a distinguished triangle:

$$B_n \to B_{n+1} \to B_1 \to B_n[1],$$

where $B_i \in \langle \phi(\mathfrak{S}) \rangle_n$. By induction, we know that: there exist $A_1 \in \langle \mathfrak{S} \rangle_1$ and $A_n \in \langle \mathfrak{S} \rangle_n$ such that: $B_n = \phi(A_n), B_1 = \phi(A_1)$.

Therefore, we have $A_{n+1} \in \langle \mathfrak{S} \rangle_{n+1}$, such that:

$$A_n \to A_{n+1} \to A_1 \to A_n[1]$$

is a distinguished triangle in $\mathcal{T}_1$. Applying ϕ to this triangle, we get an isomorphism $\phi(A_{n+1}) \cong B_{n+1}$. After a suitable choice of the isomorphism class of A_{n+1}, we can find a preimage of B_{n+1}.

In other words, we have shown that:

$$\phi : \langle \mathfrak{S} \rangle_n \star \langle \mathfrak{S} \rangle_1 \to \langle \phi(\mathfrak{S}) \rangle_n \star \langle \phi(\mathfrak{S}) \rangle_1$$

is essentially surjective.

Next, let us check that the above functor is fully faithful. Given $A, \tilde{A} \in \langle \mathfrak{S} \rangle_n \star \langle \mathfrak{S} \rangle_1$, then we can assume that there exist two distinguished triangles:

$$A_n \to A \to A_1 \to A_n[1] \tag{4}$$

and

$$\tilde{A}_n \to \tilde{A} \to \tilde{A}_1 \to \tilde{A}_n[1]. \tag{5}$$

Then applying $Hom(A_n, \cdot)$ to the triangle (5), we get a long exact sequence:

$$Hom(A_n, \tilde{A}_n) \to Hom(A_n, \tilde{A}) \to Hom(A_n, \tilde{A}_1) \to Hom(A_n, \tilde{A}_n[1]) \to \cdots$$

After compared to the image of the above long exact sequence under ϕ, and by induction on n and the five lemma, we get that:

$$Hom(A_n, \tilde{A}[*]) \cong Hom(\phi(A_n), \phi(\tilde{A})[*]).$$

Similarly, we have $Hom(A_1, \tilde{A}[*]) \cong Hom(\phi(A_1), \phi(\tilde{A})[*])$.

Next applying $Hom(\cdot, \tilde{A})$ to the triangle (5), we get another long exact sequence:

$$Hom(A_n[1], \tilde{A}) \to Hom(A_1, \tilde{A}) \to Hom(A, \tilde{A}) \to Hom(A_n, \tilde{A}) \to \cdots .$$

Compared to its image under ϕ and isomorphisms above, we get:

$$Hom(A, \tilde{A}[*]) \cong Hom(\phi(A), \phi(\tilde{A})[*]).$$

Now, we have shown that:

$$\phi : \langle \mathfrak{S} \rangle_n \star \langle \mathfrak{S} \rangle_1 \to \langle \phi(\mathfrak{S}) \rangle_n \star \langle \phi(\mathfrak{S}) \rangle_1$$

is an equivalence.

Recall that ϕ commutes with shifts and finite direct sums, and maps the idempotent in $End(A)$ to the idempotent in $End(\phi(A))$ for any $A \in \langle \mathfrak{S} \rangle_n \star \langle \mathfrak{S} \rangle_1$. This implies that:

$$\phi : \langle \mathfrak{S} \rangle_n \diamond \langle \mathfrak{S} \rangle_1 \to \langle \phi(\mathfrak{S}) \rangle_n \diamond \langle \phi(\mathfrak{S}) \rangle_1$$

is an equivalence. □

In order to study motives for an elliptic curve later, let us recall some construction of Schur functors very briefly. The main references are [**11, 26**].

The basic ideas are applying the constructions of representations of symmetric groups Σ_n into more general context.

Definition 1.33. Let $\mathfrak{C}$ be a category. $\mathfrak{C}$ is called symmetric monoidal if there exists a bifunctor $\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ (tensor functor) and an identity object which satisfy the associativity and the commutativity constraints (see [**25**], p. 161 and p.184). △

Definition 1.34. A $\mathbb{Q}$-linear symmetric monoidal category $\mathfrak{C}$ is a category satisfying:

- $\mathfrak{C}$ is symmetric monoidal;
- $\mathfrak{C}$ is an additive category, and every hom group is endowed with the structure of a $\mathbb{Q}$-vector space such that the composition is $\mathbb{Q}$-bilinear;
- the tensor product is $\mathbb{Q}$-bilinear.

△

Definition 1.35. A tensor category is a symmetric monoidal category which is additive, idempotent complete, and the tensor product is bilinear. A $\mathbb{Q}$-linear tensor category is a $\mathbb{Q}$-linear symmetric monoidal cateogry which is idempotent complete and where the tensor product is $\mathbb{Q}$-bilinear. We denote the unit object by $\mathbb{Q}$. △

Remark 1.36. Let $\mathcal{A}$ be a $\mathbb{Q}$-linear tensor category. The symmetric group Σ_n acts on $X^{\otimes n}$ for $X \in \mathcal{A}$. For every partition λ of n, we define

$$S_\lambda(X) = Im(e_\lambda|_{X^{\otimes n}}),$$

where e_λ is defined in Remark 1.4.

Convention 1.37. $Sym^n X = S_{(n)}(X)$, $Sym^0 X = \mathbb{Q}$, $\wedge^n X = S_{(1,\cdots,1)}(X)$.

Remark 1.38. In fact, this assignment makes S_λ into a functor, which is called the Schur functor of λ. We have the following decomposition as the case of GL_2-representations (Theorem 1.9):

$$X^{\otimes n} \cong \bigoplus_\lambda V_\lambda \otimes S_\lambda(X). \tag{6}$$

Here the index set runs through all the partitions of n.

REMARK 1.39. Recall the following fact:

Let X be an object of a $\mathbb{Q}$-linear tensor category, then we have:

$$S_\mu(X) \otimes S_\nu(X) \cong \bigoplus_\lambda N_{\mu\nu\lambda} S_\lambda(X), \tag{7}$$

where λ satisfies $|\lambda| = |\mu| + |\nu|$. Here numbers $N_{\mu\nu\lambda}$ can calculated by the Littlewood-Richardson rule. See [**11, 14, 26**].

CONVENTION 1.40. For the later use, we denote the multiplicity of

$$Sym^{a+b-2i}(X) \otimes det^i(X)$$

in $Sym^a(X) \otimes Sym^b(X)$ by $D^i_{a,b}$.

CHAPTER 2

DG MODULES OVER G

Let G be a reductive algebraic group over $\mathbb{Q}$ and $w : \mathbb{G}_m \to G$ is a central cocharacter – that is, the image of w is contained in the center of G. Furthermore, we assume that w is nontrivial i.e., injective. Using the map w, we can define the weight of representations of G as in Definition 1.7. Fix a finite dimensional faithful representation $\mathbf{F}$ of G with positive weights[1]. In this chapter, we want to generalize the results of [**21, 22**], replacing graded $\mathbb{Q}$-vector spaces by the category of linear representations of G over $\mathbb{Q}$. $\otimes$ means the tensor product of two G representations if not specified otherwise.

1. Basic definitions

Convention 2.1. The Adams degree for a pure weight r representation W of G over $\mathbb{Q}$ is defined to be $-r$. Given a complex of G linear representations A^*, the Adams degree r part of A^* is denoted by $A^*|r\rangle$. We call the category of linear G representations over $\mathbb{Q}$ simply as the category of G representations.

Definition 2.2. A cdga $(A^*, d, \cdot)$ over G consists of a complex (A^*, d) in the category of G representations, where $d = \oplus_n d^n : A^n \to A^{n+1}$ is a homomorphism between G representations, satisfying:

- there exists a homomorphism of complexes of G representations: $\cdot : A^* \otimes A^* \to A^*$, which is unital, graded commutative and associative.
- $d^{n+m}(a \cdot b) = d^n a \cdot b + (-1)^n a \cdot d^m b$, where $a \in A^n, b \in A^m$.
- the Adams grading gives a decomposition of A^* into subcomplexes $A^* = \oplus_{r\in\mathbb{Z}} A^*|r\rangle$ and $\mathbb{Q}$ (the trivial G representation) is a direct summand of $A^*|0\rangle$.

A^* is called Adams connected if the Adams decomposition satisfies $A^* = \oplus_{r\geq 0} A^*|r\rangle$ and $A^*|0\rangle = \mathbb{Q}$. Furthermore, A^* is called connected (resp. cohomologically connected) if $A^n = 0$ for $n < 0$ and $A^0 = \mathbb{Q}$ (resp. $H^n(A^*) = 0$ for $n < 0$ and $H^0(A^*) = \mathbb{Q}$).

For $x \in A^n(r)$, we call n the cohomological degree of x, denoted by $n = deg(x)$, and r the Adams degree of x, denoted by $r = |x|$. △

Definition 2.3. Let A be a cdga over G. A dg A-module (M^*, d) over G consists of a complex M^* of G representations with the differential d, together with a map $A^* \otimes M^* \to M^*, a \otimes m \to a \cdot m$, which makes M^* into a A^*-module, and satisfies the Leibniz rule

$$d(a \cdot m) = da \cdot m + (-1)^{dega} a \cdot dm; a \in A^*, m \in M^*.$$

△

[1]For the existence of such $\mathbf{F}$, we refer to Corollary 2.5 in [**12**].

Remark 2.4. By definition, there exists a decomposition of M^* into subcomplexes $M^* = \oplus_s M^*|s\rangle$ satisfying $A^*|r\rangle \cdot M^*|s\rangle \subset M^*|r+s\rangle$, which is called the Adams decomposition.

Definition 2.5. Let M and N be two dg A modules. A morphism f between M and N is a morphism between the underlying complexes of G-representations of M and N such that $a \cdot f(m) = f(a \cdot m)$ for any $a \in A$ and $m \in M$. △

Example 2.6. Let $A[n]$ denote the A^*-module which is A^{m+n} in degree m, with a natural action of A^* by multiplication. Given A^* a cdga over G, we let $A\langle r\rangle[n]$ be A^*-module which is $\bigoplus_{t\in\mathbb{Z}} A^{m+n}|t\rangle \otimes \mathbf{F}^{\otimes r}|s-t\rangle$ in bi-degree (m,s), with the action given by multiplication. More generally, given any G-representation W, $A[n]\otimes W$, with $\bigoplus_{t\in\mathbb{Z}} A^{m+n}|t\rangle \otimes W|s-t\rangle$ in degree (m,s), is also a dg A-module over G. When W is a rational representation of G, $A[n]\otimes W$ is called the generalized sphere A-modules for any $n \in \mathbb{Z}$.

Definition 2.7. A dg A-module M is a cell module if

(1) There is an isomorphism of A-modules in the category of G representations:

$$\oplus_{j\in J} A[-n_j] \otimes V_j \to M,$$

where all the V_j are rational representations of G and all n_j are integers.

(2) There is a filtration on the index set J:

$$J_{-1} = \emptyset \subset J_0 \subset J_1 \cdots \subset J$$

such that $J = \bigcup_{n=0}^{\infty} J_n$ and for $j \in J_n$,

$$db_j = \sum_{i\in J_{n-1}} a_{ij} b_i,$$

where b_j is in the cohomological degree n_j part of the complex $Hom_G(V_j, M)$ and a_{ij} is in A.

A finite cell module is a cell module with finite index set J. △

Remark 2.8. Given M a cell module, using the condition 1 and 2, one can construct a filtration of sub cell modules M_n, where M_n is isomorphic to $\oplus_{j\in J_n} A[-n_j]\otimes V_j$ as complexes of G representations[2]. $\{M_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is called the sequential filtration of M.

Definition 2.9. Assume that the dimension of the fixed faithful representation $\mathbf{F}$ is n and denote $\wedge^n\mathbf{F}$ by det. A cell module is called Tate-type if all the generalized sphere modules appearing in the first condition of Definition 2.7 are of the form $A[-n] \otimes det^{\otimes r}$ for some $r, n \in \mathbb{Z}$. △

Definition 2.10. A rational representation V of G has non-positive Adams degrees if $V|r\rangle = 0$ for any $r > 0$. △

[2]In other words, a cell A-module M can be considered as the union of an expanding sequence of sub A-modules M_n such that $M_0 = 0$ and M_{n+1} is the cofiber of a map $\phi_n : F_n \to M_n$, where all the F_n are generalized sphere A-modules.

Definition 2.11. A cell module is called effective if all the generalized sphere modules appearing in the definition are of the form $A[-n] \otimes V_j$ with V_j a direct summand of $\mathbf{F}^{\otimes i_j}$ for some $i_j \in \mathbb{Z}_{\geq 0}$. △

We denote the category of dg A-modules over G by $\mathcal{M}_A^G$, the category of cell A-modules by $\mathcal{CM}_A^G$ and the category of finite cell modules by $\mathcal{CM}_A^{G,f}$. Furthermore, we denote the category of effective cell A-modules by $\mathcal{CM}_A^{G,eff}$ and the category of finite effective cell modules by $\mathcal{CM}_A^{G,eff,f}$. Finally we denote the full subcategory of cell A-modules of Tate-type by $\mathcal{CM}_A^{\mathbb{G}_m}$.

2. The derived category of DG modules

Let A be a cdga over G and let M and N be dg A-modules. Let $\mathcal{H}om_A(M,N)$ be the dg A-module over G with $\mathcal{H}om_A(M,N)^n$ consisting of linear maps $f : M \to N$ with $f(M^a) \subset N^{a+n}$ and with the differential d defined by $df(m) = d(f(m)) - (-1)^n f(dm)$ for $f \in \mathcal{H}om_A(M,N)^n$. Let $Hom_A(M,N)$ be the dg $\mathbb{Q}$-module over G with $Hom_A(M,N)^n$ consisting of linear maps $f : M \to N$ with $f(M^a) \subset N^{a+n}$, $f(am) = (-1)^{np} af(m)$ for $a \in A^p$ and $m \in M^a$, and with the differential d defined by $df(m) = d(f(m)) - (-1)^n f(dm)$ for $f \in Hom_A(M,N)^n$.

Definition 2.12. For $f : M \to N$ a morphism of dg A-modules, we let $Cone(f)$ be the dg A-module with:

$$Cone(f)^n(r) = N^n(r) \oplus M^{n+1}(r)$$

and the differential is given by $d(n,m) = (dn + f(m), -dm)$. △

Given M a dg A-module, we let $M[1]$ denote a dg-A module such that $M[1]^n = M^{n+1}$ with the differential $-d$, where d is the differential of M. Then we have the following sequence:

$$M \xrightarrow{f} N \xrightarrow{i} Cone(f) \to M[1],$$

which is called a cone sequence.

Definition 2.13. We let $\mathcal{K}_A^G$ denote the homotopy category of the category of dg A-modules over G. The objects are the same as $\mathcal{M}_A^G$ and

$$Hom_{\mathcal{K}_A^G}(M,N) = Hom_G(\mathbb{Q}, H^0(Hom_A(M,N))).$$

The derived category $\mathcal{D}_A^G$ of dg A-modules over G is the localization of $\mathcal{K}_A^G$ with respect to quasi-isomorphisms between dg A-modules, which are defined as morphisms $M \to N$ being quasi-isomorphic on the underlying complexes of $\mathbb{Q}$-vector spaces. △

Given $M, N \in \mathcal{M}_A^G$ (resp. $\mathcal{CM}_A^G$), we can define their direct sum to be the direct sum $M \oplus N$ of the chain complexes of GL_2-representations which is equipped with a natural A-module structure (resp. cell A-module structure). Furthermore, the infinite direct sum exists in both $\mathcal{M}_A^G$ and $\mathcal{CM}_A^G$.

Lemma 2.14. *The infinite direct sums defined above is the categorical sum in* $\mathcal{K}_A^G$.

Proof. By definition of the categorical sum, given an index set I and a family of dg cell A-modules $\{M_i\}_{i\in I}$ and N, we need to show that:

$$Hom_{\mathcal{K}_A^G}(\bigoplus_{i\in I} M_i, N) \cong \prod_{i\in I} Hom_{\mathcal{K}_A^G}(M_i, N).$$

Notice that there is a natural isomorphism between dg-$\mathbb{Q}$ modules $Hom_A(\bigoplus_{i\in I} M_i, N)$ and $\prod_{i\in I} Hom_A(M_i, N)$. This is because that, in each cohomological degree n, there is a natural isomorphism of G representations between $Hom_A(\bigoplus_{i\in I} M_i, N)^n$ and $\prod_{i\in I} Hom_A(M_i, N)^n$. Furthermore, the differentials of two dg A-modules are compatible under these isomorphisms between each cohomological degree. Then by our definition of hom, we have:

$$\begin{aligned}
&Hom_{\mathcal{K}_A^G}(\bigoplus_{i\in I} M_i, N) = Hom_G(Q, H^0(Hom_A(\bigoplus_{i\in I} M_i, N)))\\
&\cong Hom_G(Q, H^0(\prod_{i\in I} Hom_A(M_i, N)))\\
&\cong Hom_G(Q, \prod_{i\in I} H^0(Hom_A(M_i, N)))\\
&\cong \prod_{i\in I} Hom_G(Q, H^0(Hom_A(M_i, N)) = \prod_{i\in I} Hom_{\mathcal{K}_A^G}(M_i, N).
\end{aligned}$$

Here we use the product is exact in the category of complexes of G representations. □

Convention 2.15. Let I be the complex

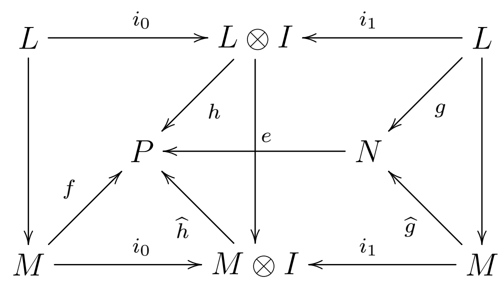
$$\mathbb{Q} \xrightarrow{\delta} \mathbb{Q} \oplus \mathbb{Q}$$

with a free $\mathbb{Q}$ generator $[I]$ in degree -1, two free $\mathbb{Q}$ generators $[0], [1]$ in degree 0 and $\delta[I] = [0] - [1]$. We have two inclusions $i_0, i_1 : \mathbb{Q} \to I$ sending 1 to $[0], [1]$ respectively.

For M a dg A-module, we let $CM = Cone(id_M)$. Notice that the cone CM is the quotient module $M \otimes (I/\mathbb{Q}[1])$.

Using the same idea of proof as in [**21**], we can show the following theorems.

Theorem 2.16. *(HELP) Let L be a cell A-submodule of a cell A-module M. Let $e : N \to P$ be a quasi-isomorphism of dg A-modules. Then given maps $f : M \to P, g : L \to N$, and $h : L \otimes I \to P$ such that $f|_L = h \circ i_0$ and $e \circ g = h \circ i_1$, there are maps $\widehat{g}, \widehat{h}$ that make the following diagram commute.*



PROOF. By induction on the filtration on the index set J and pullback along cells not in L, we may assume that $M \cong C(A[n] \otimes W)$ and $L \cong A[n] \otimes W$. Then using the semi-simplicity of the category of G representations, we can further assume that W is an irreducible G representation. Let's denote the generator of W by w^n.

Let $u = w^n \otimes [0]$ and $v = w^n \otimes [I]$ be the generators of $C(A[n] \otimes W)$. By definition, we have $d(v) = (-1)^n u$. We also have: $e \circ g(w^n) = h(w^n \otimes [1])$ and $f(u) = h(u)$. Therefore

$$\begin{aligned} &d(h(w^n \otimes [I]) - f(v)) = hd(w^n \otimes [I]) - f(dv) \\ =&h(d(w^n) \otimes [I] + (-1)^n h(w^n \otimes ([0] - [1]))) - (-1)^n f(u) \\ =&(-1)^n h(w^n \otimes [0]) + (-1)^{n+1} h(w^n \otimes [1]) - (-1)^n f(u) \\ =&(-1)^{n+1} h(w^n \otimes [1]) = (-1)^{n+1} e \circ g(w^n). \end{aligned}$$

Because $e \circ g(w^n)$ is a coboundary and e induces a quasi-isomorphism, we know that $g(w^n)$ is also a coboundary, i.e., there exist $\tilde{n} \in N^{n-1}$ such that $d(\tilde{n}) = g(w^n)$. Then $p = e(\tilde{n}) + h(w^n \otimes [I]) - f(v)$ is a cocycle. Then using the quasi-isomorphism at $n-1$, there exist a cocycle $n \in N$ and a chain $q \in P$ such that $d(q) = p - e(n)$.

We define $\widehat{g}(j) = (-1)^n(\tilde{n} - n)$ and $\widehat{h}(j \otimes [I]) = q$. □

THEOREM 2.17. *(Whitehead) If M is a cell A-module and $e : N \to P$ is a quasi-isomorphism of A-modules, then*

$$e_* : Hom_{\mathcal{K}_A^G}(M, N) \to Hom_{\mathcal{K}_A^G}(M, P)$$

is an isomorphism. So a quasi-isomorphism between cell A-modules is a homotopy equivalence.

PROOF. The surjectivity is coming from Theorem 2.16, when we take $L = 0$. The injectivity can be checked when we replace M and L by $M \otimes_{\mathbb{Q}} I$ and $M \otimes_{\mathbb{Q}} (\partial I)$ respectively. When N, P are both cell A-modules, taking $M = P$, we get a map $f : P \to N$ which corresponds to id_P. From the functoriality, f is the homotopy inverse of e. □

COROLLARY 2.18. *Let M, N be two dg A-modules, and $f : M \to N$ be a morphism between dg A-modules. Let $\widehat{M}$ and $\widehat{N}$ be two cell-A modules such that $\widehat{M} \xrightarrow{r_M} M$ and $\widehat{N} \xrightarrow{r_N} N$ are quasi-isomphisms. Then there exists a morphism between cell A-modules $\widehat{f} : \widehat{M} \to \widehat{N}$ lifting f.*

PROOF. From Theorem 2.17, we know that

$$r_{N*} : Hom_{\mathcal{K}_A^G}(\widehat{M}, \widehat{N}) \to Hom_{\mathcal{K}_A^G}(\widehat{M}, N)$$

is an isomorphism. Therefore $f \circ r_M \in \mathcal{K}_A(\widehat{M}, N)$ have a preimage, which is just $\widehat{f}$. □

REMARK 2.19. From the above proof, we also know that: Given a cell module M and an arbitrary dg A-module N, we have:

$$Hom_{\mathcal{D}_A^G}(M, N) \cong Hom_{\mathcal{K}_A^G}(M, \widehat{N}) \cong Hom_{\mathcal{K}_A^G}(M, N),$$

where $\widehat{N}$ is a cell module and $\widehat{N} \to N$ is a quasi-isomorphism between dg A-modules.

THEOREM 2.20. *(Approximation by cell modules) For any dg A-module M, there is a cell A-module N and a quasi-isomorphism $e : N \to M$.*

PROOF. We will construct a sequential filtration N_n and compatible maps $e_n : N_n \to M$ inductively. More precisely, we need to construct cell modules N_n, whose index set is denoted by J_n, satisfy the condition 2 in the definition of cell modules. For every pair (q, r), we decompose $H^q(M)|r\rangle \cong \oplus_i V_i$ as the direct sum of irreducible G representations V_i with the Adams degree r. Choosing a splitting of $Ker(M^q|r\rangle \xrightarrow{d} M^{q+1}|r\rangle) \twoheadrightarrow H^q(M)|r\rangle$, we think V_i as sub G representations in $M^q|r\rangle$, because of the semi-simplicity of the category of G representations. Then we take $N_1 = \oplus_{(q,r)} \oplus_i A[-q] \otimes V_i$ with trivial differential. There is a morphism between dg A-modules: $N_1 \to M$, which is epimorphism on the cohomologies. Inductively, assume that $e_n : N_n \to M$ has been constructed. Consider the set of the pair of cocycles consisting the pairs of unequal cohomology classes on N_n and mapping under $(e_n)^*$ to the same element of $H^*(M)$. Choose a pair $W_1^q|r\rangle$ and $W_2^q|r\rangle$ that live in the bidegree (q, r) satisfies above condition, i.e., we can view $W_1^q|r\rangle \oplus W_2^q|r\rangle$ as the kernel of the morphism e_{n*} on the cohomology of bidegree (q, r). (Here one need to take a sign for the second component.) Simply denote $W_1^q|r\rangle \oplus W_2^q|r\rangle$ by W_1. There is a morphism between dg A-modules $A[-q] \otimes W_1$ to N_n extending the map between G representations $W_1 \to H^q(N)(r)$. Take N_{n+1} to be the pushout of N_n and $A[-q] \otimes W_1 \oplus A[-q] \otimes W_1[1]$ over $A[-q] \otimes W_1$. Then we have $0 \to W_1 \to H^q(N_n)(r) \to H^q(N_{n+1})(r) \to 0$. We get N_{n+1} by attaching N_n with a generalized sphere dg A-module $A[-q] \otimes W_1[1]$, which implies N_{n+1} is a cell A-module. It is easy to see the differentials on N_{n+1} satisfy the condition 2 in the definition of cell modules. Now we have a distinguish triangle of dg A-modules:

$$A[-q] \otimes W_1 \xrightarrow{i} N_n \to N_{n+1} \to (A[-q] \otimes W_1)[1].$$

Notice that:

$$Hom_A(A[-q] \otimes W_1, M) \cong Hom_{D(G)}(W_1, M[q]) \cong Hom_G(W_1, H^q(M)).$$

Therefore we have:

$$\begin{aligned} &Hom_A(N_{n+1}, M) \to Hom_A(N_n, M) \\ &\xrightarrow{i} Hom_A(A[-q] \otimes W_1, M) \cong Hom_G(W_1, H^q(M)). \end{aligned}$$

Because W_1 as a G representation maps to zero in the cohomology group of $H^q(M)|r\rangle$, which implies $i(e_n) = 0$ in $Hom_G(W_1, H^q(M))$, one may find $e_{n+1} \in Hom_A(N_{n+1}, M)$, which extends e_n.

Let N be the direct limit of the N_n. Then N is a cell module and the morphism $N \to M$ is a quasi-isomorphism by the construction. □

Putting together with all previous results, we get:

THEOREM 2.21. *Let A be a cdga over G. Then the functor*

$$\mathcal{KCM}_A^G \to \mathcal{D}_A^G$$

is an equivalence of triangulated categories.

Definition 2.22. We define $\mathcal{D}_A^{G,f}$ to be the full subcategory of $\mathcal{D}_A^G$ whose objects are quasi-isomorphic to some finite cell A-module in $\mathcal{D}_A^G$. △

Remark 2.23. From the above proposition, we know that $\mathcal{KCM}_A^{G,f} \to \mathcal{D}_A^{G,f}$ is an equivalence of triangulated categories.

Example 2.24. Let $A = \mathbb{Q}$, then $\mathcal{KCM}_A^{G,f}$ is just the bounded derived category of the category of rational representations of G, denoted by $D^b(G)$.

Definition 2.25. The full triangulated subcategory of $\mathcal{D}_A^G$ generated by effective cell modules is denoted by $\mathcal{D}_A^{G,eff}$. The full triangulated subcategory of $\mathcal{D}_A^G$ generated by a family of objects $\{A\langle r\rangle[n] | r \geq 0, n \in \mathbb{Z}\}$ is denoted by $\mathcal{T}_A^G$. △

Recall the definition of the idempotent completion of a dg category. Given $\mathcal{C}$ a dg category, then its idempotent completion $\mathcal{C}^\natural$ has the objects (M, p) with $p : M \to M$ an idempotent endomorphism in $Z^0\mathcal{C}$ and the hom complex given by

$$\mathcal{H}om_{\mathcal{C}^\natural}((M,p),(N,q))^* = p^* q_* \mathcal{H}om_{\mathcal{C}}(M,N).$$

Remark 2.26. In [**3**], Balmer and Schlichting have shown that, for $\mathcal{A}$ a triangulated category, $\mathcal{A}^\natural$ has a canonical structure of a triangulated category which makes the natural functor $\mathcal{A} \to \mathcal{A}^\natural$ exact. The same holds for the triangulated tensor categories.

Example 2.27. Let us consider $A = \mathbb{Q}$ and $G = GL_2$. Then $\mathcal{D}_A^{GL_2,eff,f}$ is a subcategory of the bounded derived category of the category of rational representations of GL_2 generated by $Sym^a(b)$ for $a, b \in \mathbb{Z}_{\geq 0}$ and denoted by $D^{eff,b}(GL_2)$.

3. The weight filtration for dg modules

In this section, we assume that A is an Adams connected cdga over G. (Definition 2.2.)

Definition 2.28. A dg A-module M is called almost free, if there exists a family of irreducible G representations $\{V_j\}_{j\in J}$ and morphisms of dg A-modules $\phi_j : A \otimes V_j \to M$, such that the induced morphism:

$$\oplus_{j\in J} A \otimes V_j \xrightarrow{\oplus \phi_j} M$$

is an isomorphism of graded A-modules, which means that, forgetting the differentials, this is an isomorphism between G representations. We call such $\{V_j, \phi_j\}_{j\in J}$ the generating data for M[3]. △

Example 2.29. All cell A-modules are almost free. Conversely, any cell A-module is obtained from the generating data together with suitable differentials.

The reason for introducing the notion of almost free is that we can define the weight filtration on these data. Assume that $d(\phi_j(V_j)) \subset \oplus_{i\in I} \phi_i(A \otimes V_i)$. Here we restrict ϕ_j to $A^*|0\rangle \otimes V_j \cong V_j$. The left hand side has the Adams degree $|V_j|$ and the Adams degree of

[3]This looks like a "basis" of M.

right hand side is larger than or equal to $|V_i|$. So we get $|V_i| \leq |V_j|$ if $d\phi_j \neq 0$. Hence we have the subcomplex

$$W_n^J M = \oplus_{\{j,|V_j| \leq n\}} \phi_j(A \otimes V_j)$$

of M.

REMARK 2.30. The subcomplex of $W_n^J M$ is independent of the choice of the family $\{\phi_j\}_{j \in J}$. This is because if we choose another family $\{\phi_{j'}\}$, then the same process as above shows that $\phi_{j'}(V_{j'}) \in W_n^J M$ and hence $W_n^{J'} M \subset W_n^J M$. By symmetry, we get the result. So we delete the J in the definition.

This gives us the increasing filtration as a dg A-module

$$W_* M : \cdots \subset W_n M \subset W_{n+1} M \subset \cdots \subset M$$

with $M = \cup_n W_n M$.

In the same way, we can define $W_{n/n'} M$ as the cokernel of the inclusion $W_{n'} M \to W_n M$ for $n \geq n'$. Write gr_n^W for $W_{n/n-1}$ and $W^{>n}$ for $W_{\infty/n}$.

W_n defines an endofunctor in $\mathcal{CM}_A^G$. Furthermore, $\{W_n\}_{n \in \mathbb{Z}}$ form a functorial tower of endofunctors on $\mathcal{KCM}_A^G$:

$$\cdots \to W_n \to W_{n+1} \to \cdots \to id.$$

REMARK 2.31.
- The endofunctor W_n is exact for all n.
- For $m \leq n \leq \infty$, the sequence of endofunctors $W_m \to W_n \to W_{n/m}$ can extend to a distinguish triangle of endofunctors, i.e., for any $M \in \mathcal{KCM}_A^G$, we have a distinguish triangle $W_m M \to W_n M \to W_{n/m} M \to$ in $\mathcal{KCM}_A^G$.

REMARK 2.32. Using the isomorphism of categories between $\mathcal{KCM}_A^G$ and $\mathcal{D}_A^G$, we could define the tower of exact endofunctors on $\mathcal{D}_A^G$

$$\cdots \to W_n \to W_{n+1} \to \cdots \to id.$$

Similarly we define $W_{n/n'}$, gr_n^W and $W^{>n}$ on $\mathcal{D}_A^G$.

The existence of the weight filtration provides a powerful tool for showing a lot of properties of dg A-modules.

4. Tensor structure

Recall that the Hom functor $\mathcal{H}om_A(M, N)$ defines a bi-exact bi-functor:

$$\mathcal{H}om_A : (\mathcal{KCM}_A^G)^{op} \otimes \mathcal{KCM}_A^G \to \mathcal{D}_A^G,$$

which gives a well-defined derived functor of $\mathcal{H}om_A$ between the derived categories of dg A-modules (also the derived categories of finite cell modules) by Proposition 2.21:

$$R\mathcal{H}om_A : (\mathcal{D}_A^G)^{op} \otimes \mathcal{D}_A^G \to \mathcal{D}_A^G.$$

In this section, we use these constructions to define the tensor structure on $\mathcal{D}_A^G$.

Firstly, we define a tensor structure on $\mathcal{T}_A^G$. It is defined on the generator $\{A\langle r\rangle[n]\}$ by:

$$A\langle r\rangle[n] \otimes_A A\langle s\rangle[m] = A\langle r+s\rangle[m+n].$$

Using the approximation theorem, we get a derived tensor product $\otimes^{\mathbb{L}}$ on $\mathcal{T}_A^G$. Then it will induce a tensor structure on $(\mathcal{T}_A^G)^\natural$. More precisely, take $(M,p),(N,q) \in (\mathcal{T}_A^G)^\natural$, then

$$(M,p) \otimes_A^{\mathbb{L}} (N,q) = (M \otimes_A^{\mathbb{L}} N, p \otimes q).$$

From above construction, we have a functor:

$$\otimes_A^{\mathbb{L}} : (\mathcal{T}_A^G)^\natural \otimes (\mathcal{T}_A^G)^\natural \to (\mathcal{T}_A^G)^\natural.$$

Next we want to extend the above tensor structure on $(\mathcal{T}_A^G)^\natural$ to $\mathcal{D}_A^G$.

Notice that given a generalized sphere modules $A \otimes W$, where W is a rational G representation, there exists a positive integer n big enough, such that $A \otimes W(n)$ is in $(\mathcal{T}_A^G)^\natural$. Here (1) means the 1 dimensional G - representation $\wedge^n \mathbf{F}$ and $n = \dim \mathbf{F}$.

Definition 2.33. Given two generalized sphere modules $A \otimes W_i, i = 1,2$, and $n_i \in \mathbb{Z}_{\geq 0}$, such that $A \otimes W_i(n_i)$ is effective, then we define:

$$(A \otimes W_1) \otimes_A^{\mathbb{L}} (A \otimes W_2) = \mathcal{H}om_A(A(n_1+n_2), (A \otimes W_1(n_1)) \otimes_A^{\mathbb{L}} (A \otimes W_2(n_2))).$$

By the definition of the internal hom, it's easy to see that the definition is independent of the choice of n_i. △

By Theorem 2.20, we get a well-defined derived functor of $\otimes_A$:

$$\otimes_A^{\mathbb{L}} : \mathcal{D}_A^G \otimes \mathcal{D}_A^G \to \mathcal{D}_A^G.$$

Remark 2.34. Two formal properties are listed below.

- These bi-functors are adjoint, i.e.,
$$R\mathcal{H}om_A(M \otimes_A^{\mathbb{L}} N, K) \cong R\mathcal{H}om_A(M, R\mathcal{H}om_A(N,K)).$$
- The derived tensor product makes $\mathcal{D}_A^G$ into a triangulated tensor category with unit A and $\mathcal{D}_A^{G,f}$ are triangulated tensor subcategories.

These properties allow us to apply the category duality theory in [**24**].

Convention 2.35. Denote $M^\vee = R\mathcal{H}om_A(M, A)$.

Definition 2.36. An object $M \in \mathcal{D}_A^G$ is called rigid, if there exists an $N \in \mathcal{D}_A^G$ and morphisms $\delta : A \to M \otimes_A^{\mathbb{L}} N$ and $\epsilon : N \otimes_A^{\mathbb{L}} M \to A$ such that:

$$(id_M \otimes \epsilon) \circ (\delta \otimes id_M) = id_M$$
$$(id_N \otimes \delta) \circ (\epsilon \otimes id_N) = id_N$$

△

Definition 2.37. An object $M \in \mathcal{D}_A^G$ is finite if there exists a coevaluation map $\tilde{\eta} : A \to M \otimes^{\mathbb{L}} M^\vee$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\eta}} & M \otimes^{\mathbb{L}} M^{\vee} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \gamma} \\ R\mathcal{H}om_A(M,M) & \xleftarrow{\mu} & M^{\vee} \otimes^{\mathbb{L}} M \end{array}$$

commutes. Here η and μ are given by the adjunction. γ changes the places of these two modules. △

REMARK 2.38. By Theorem 1.6 of [**24**], M is rigid if and only if the function

$$\epsilon_* : Hom_{\mathcal{D}_A^G}(W, Z \otimes_A^{\mathbb{L}} N) \to Hom_{\mathcal{D}_A^G}(W \otimes_A^{\mathbb{L}} M, Z)$$

is a bijection for all W and Z, where $\epsilon_*(f)$ is the composite

$$W \otimes_A^{\mathbb{L}} M \xrightarrow{f \otimes 1} Z \otimes_A^{\mathbb{L}} N \otimes_A^{\mathbb{L}} M \xrightarrow{1 \otimes \epsilon} Z \otimes_A^{\mathbb{L}} A \cong Z.$$

These conditions are also equivalent to saying that M is finite.

In the following, we will discuss the relations between finite objects in $\mathcal{D}_A^G$ and finite cell modules.

DEFINITION 2.39. We say that a cell module N is a summand of a cell module M in $\mathcal{D}_A^G$ if there is a homotopy equivalence of A-modules between M and $N \oplus N'$ for some cell A-module N'. △

Following the same proof as Theorem 5.7 in Part III of [**21**], we can get:

LEMMA 2.40. *A cell module M is rigid if and only if it is a summand of a finite cell module in $\mathcal{D}_A^G$.*

REMARK 2.41. Let $\mathcal{FCM}_A^G$ be the full subcategory of $\mathcal{CM}_A^G$ whose objects are the direct summands up to homotopy of finite cell A-modules. Then the homotopy category $\mathcal{KFCM}_A^G$ is the idempotent completion of $\mathcal{D}_A^{G,f}$. The above lemma implies that $\mathcal{KFCM}_A^G$ is the largest rigid tensor subcategory of the derived category $\mathcal{D}_A^G$. See section 5 in Part III of [**21**]. In particular, $\mathcal{D}_A^{G,f}$ is a rigid tensor subcategory of $\mathcal{KFCM}_A^G$.

THEOREM 2.42. *Let A be an Adams connected cdga over G. Then $M \in \mathcal{D}_A^G$ is rigid if and only if $M \in \mathcal{D}_A^{G,f}$, which implies that there is an equivalence between $\mathcal{D}_A^{G,f}$ and $\mathcal{KFCM}_A^G$.*

Proof. It depends on the following lemma.

LEMMA 2.43. *Assume that A is an Adams connected cdga over G. Let M be a finite A-cell module. Suppose N is a summand of M in $\mathcal{D}_A^G$. Then there is a finite A-cell module M' with $N \cong M'$ in $\mathcal{D}_A^G$.*

PROOF. By Theorem 2.20, we can assume that N is a cell module. By our assumption, we have $M = N \oplus N'$ in $\mathcal{KCM}_A^G$. Since M is finite, there is a minimal n such that $W_n M \neq 0$.

Thus $W_{n-1}N$ is homotopy equivalent to zero. We may assume that $W_{n-1}N = 0$ in $\mathcal{CM}_A^G$. Similarly, we may assume that $M = W_{n+r}M$ and $N = W_{n+r}N$ in $\mathcal{CM}_A^G$ for some $r \geq 0$. Then we proceed by induction on r.

Choose generating data $\{V_j, \phi_j\}_{j\in J}$ for W_nM. Let us prove that $W_nM = A \otimes V$ for a finite complex of G representations V. Indeed, by the definition of the weight functor and $W_{n-1}M = 0$, we can get an isomorphism:

$$W_nM = \oplus_{|V_j|=n}\phi_j(A \otimes V_j).$$

Notice that $d(\phi_j(V_j)) \subset \oplus_i\phi_i(A \otimes V_i)$ and all these $|V_i|$'s have the same value. Using $A^*|0\rangle = \mathbb{Q}$, we get $d(\phi_j(V_j)) \subset \oplus\phi_i(V_i)$. Set $V = \oplus_{j\in J}\phi_j(V_j)$, which is a complex of G representations. So we have $W_nM = A \otimes V$ as dg A-modules. Because the category of G representations is semisimple, we can assume that all differentials of V are zero.

Let $p : M \to M$ be the composition of the projection $M \to N$ and the inclusion $N \to M$. Then we can see $W_np = id \otimes q$, where $q : V \to V$ is an idempotent of V. V is a direct sum of G representations with some shifts. Thus $W_nN \cong A \otimes im(q)$. We finish the case of $r = 0$.

Using the distinguished triangle

$$W_nN \to N \to W_{n+r/n}N \to W_nN[1],$$

we can replace N with the shifted cone of the map $W_{n+r/n}N \to A \otimes im(q)[1]$. Since $W_{n+r/n}N$ is a summand of $W_{n+r/n}M$, by induction, we get that $W_{n+r/n}$ is homotopy equivalent to a finite cell module. So the cone of $W_{n+r/n}N \to A \otimes im(q)$ is also homotopy equivalent to a finite cell module. □

Corollary 2.44. *Assume A is an Adams connected cdga over G. Then $\mathcal{D}_A^{G,f}$ is idempotent complete.*

5. Base change

Lemma 2.45. *Let N be a cell module. Then the functor $M \otimes_A N$ preserves exact sequences and quasi-isomorphisms in the variable M.*

Proof. Because N is a cell module, N has a sequential filtration $\{N_n\}$, and N_{n+1} is given by the extension of N_n by the generalized sphere modules. Notice that Lemma 2.45 is true for generalized sphere modules. By the induction of the filtration and passage to the colimits, we get the result for the general case. □

If $\phi : A \to B$ is a homomorphism of cdgas over G, we have the functor

$$\otimes_A B : \mathcal{M}_A^G \to \mathcal{M}_B^G$$

which induces a functor on cell modules and the homotopy category

$$\phi_* : \mathcal{KCM}_A^G \to \mathcal{KCM}_B^G.$$

So we have a base change functor on the derived categories level,

$$\phi_* : \mathcal{D}_A^G \to \mathcal{D}_B^G.$$

REMARK 2.46. The restriction of ϕ_* on finite objects gives the functor on the bounded case.

PROPOSITION 2.47. *If ϕ is a quasi-isomorphism, then ϕ_* is an equivalence of tensor triangulated categories.*

PROOF. Firstly, there is an isomorphism:

$$Hom_{\mathcal{M}_B^G}(B \otimes_A M, N) \cong Hom_{\mathcal{M}_A^G}(M, \phi^* N),$$

for $M \in \mathcal{M}_A^G$ and $N \in \mathcal{M}_B^G$. Here ϕ^* is the pullback functor, which means that, for a given dg B-module, there is a natural dg A-module structure.

Then we have:

$$Hom_{\mathcal{K}_B^G}(B \otimes_A M, N) \cong Hom_{\mathcal{K}_A^G}(M, \phi^* N).$$

Using Remark 2.19, we get:

$$\begin{aligned} Hom_{\mathcal{D}_B^G}(B \otimes_A M, N) &\cong Hom_{\mathcal{K}_B^G}(B \otimes_A \widehat{M}, N) \\ \cong Hom_{\mathcal{K}_A^G}(B \otimes_A \widehat{M}, \phi^* N) &\cong Hom_{\mathcal{D}_A^G}(B \otimes_A M, \phi^* N), \end{aligned}$$

where $\widehat{M}$ is a cell A-module quasi-isomorphic to M.

Next, we will check that the unit of the adjunction and the counit are both quasi-isomorphisms.

For the unit of the adjunction, if M is a cell dg-A module, then

$$\phi \otimes Id : M \cong A \otimes_A M \to \phi^*(B \otimes_A M)$$

is a quasi-isomorphism of A-modules. Firstly, assume that $M = B$. By assumption, we know that $\phi^* B$ is quasi-isomorphic to A as a dg A-module. Then assume $M = A[n] \otimes W$ for W a G representation. $\phi^*(B \otimes_A A[n] \otimes W)$ is the same as $\phi^*(B[n] \otimes W)$. The latter is naturally quasi-isomorphic to $A[n] \otimes W$. If M is cell module, using the induction on the length of its sequential filtration, we can get the desired quasi-isomorphism.

For the counit part, given N a dg B-module, and choosing a quasi-isomorphim of dg B-module $\widehat{N} \to N$, where $\widehat{N}$ is cell B-module, then we have:

$$B \otimes_A \widehat{N} \to B \otimes_B N \cong N,$$

which is also a quasi-isomorphism.

□

COROLLARY 2.48. *Assume that A and B are Adams connected cdgas over G. If ϕ is a quasi-isomorphism, then*

$$\phi_* : \mathcal{D}_A^{G,f} \to \mathcal{D}_B^{G,f}$$

is an equivalence of triangulated tensor categories.

PROOF. Notice that an equivalence between tensor triangulated categories induces an equivalence on the subcategories of rigid objects. By Proposition 2.47, we know that $\mathcal{D}_A^G$ and $\mathcal{D}_B^G$ are equivalent. Then by Theorem 2.42, we know that ϕ induces an equivalence between $\mathcal{D}_A^{G,f}$ and $\mathcal{D}_B^{G,f}$. □

REMARK 2.49. For any cdga A over G, we have a morphism $\delta : \mathbb{Q} \to A$, which sends $A^*|0\rangle$ to A. Then, for any $M \in \mathcal{M}_{\mathbb{Q}}^G$ and $N \in \mathcal{M}_A^G$, we have:

$$Hom_{\mathcal{D}_A^G}(A \otimes M, N) \cong Hom_{\mathcal{D}_{\mathbb{Q}}^G}(M, \delta^* N).$$

Here δ^* is the forgetful functor, which forgets the A-module structure. We will omit δ^* in the computation later.

6. Minimal models

In the rest of this chapter, we always assume that the cdgas are Adams connected.

DEFINITION 2.50. A cdga A over G is said to be generalized nilpotent if:

- A is a free commutative graded algebra over G, i.e, $A = Sym^* E$ for some $\mathbb{Z}_{>0}$-graded G representations E. (Or a complex of G representations concentrated in position degrees and with zero differentials).
- For $n \geq 0$, let $A\langle n\rangle \subset A$ be the subalgebra generated by the elements of degree $\leq n$. Set $A\langle n+1, 0\rangle = A\langle n\rangle$ and for $q \geq 0$ define $A\langle n+1, q+1\rangle$ inductively as the subalgebra generated by $A\langle n\rangle$ and

  $$A\langle n+1, q+1\rangle^{n+1} = \{x \in A\langle n+1\rangle | dx \in A\langle n+1, q\rangle\}.$$

  Then for all $n \geq 0$, $A\langle n+1\rangle = \cup_{q\geq 0} A\langle n+1, q\rangle$.

A cdga A over G is called nilpotent, if for each $n \geq 1$, there is a $q_n \in \mathbb{Z}_{\geq 0}$ such that $A\langle n\rangle = A\langle n, q_n\rangle$ in the second condition above. △

DEFINITION 2.51. A connected cdga A over G is minimal if it is a free commutative graded algebra over G with decomposable differential: $d(A) \subset (IA)^2$. IA is the fundamental ideal, i.e., $IA = Ker(A \to \mathbb{Q} \cong A^0|0\rangle)$. △

CONVENTION 2.52. For a cdga A over G, we let QA be $IA/(IA \cdot IA)$.

PROPOSITION 2.53. *If a connected cdga A over G is generalized nilpotent, then it is minimal. Conversely, if A is a minimal connected cdga over G and $A^q|r\rangle = 0$ unless $2r \geq q$, then A is generalized nilpotent.*

PROOF. The proof is the same as Proposition 2.3 in Part of IV of [**21**]. If A is generalized nilpotent, then $d(A) \subset (IA)^2$ is the consequence of the double induction on n and q. Assume A is minimal. Suppose that A is not generalized nilpotent and let n be minimal such that there is an element a which does not belong to any $A\langle n, q\rangle$. Assume that a is also the element which has the minimal Adams degree. Consider any summand bc of the decomposable element da. One can assume that $0 < deg(b) \leq deg(c)$. Then $bc \in A\langle n-1\rangle$ or $deg(b) = 1$. Since $A^q|r\rangle = 0$ unless $2r \geq q$, b and c have strictly lower Adams grading than a. So b, c are in some $A\langle n, q\rangle$. Therefore da is in some $A\langle n, q\rangle$, so is a. □

DEFINITION 2.54. Let A be a cdga over G. Given a positive integer n, an n-minimal model[4] of A over G is a map of cdgas over G:

$$s : A\{n\} \longrightarrow A,$$

with $A\{n\}$ generalized nilpotent and generated as an algebra in degrees $\leq n$, such that s induces an isomorphism on H^m for $1 \leq m \leq n$ and an injection on H^{n+1}. △

PROPOSITION 2.55. *Let A be a cohomologically connected cdga A over G. Then for each $n = 1, 2, \cdots, \infty$, there is an n-minimal model $A\{n\}$ over G: $A\{n\} \to A$.*

PROOF. Follow the idea of Proposition 2.4.9 in [**22**]. Because A is cohomologically connected, we have a canonical decomposition $A = \mathbb{Q} \oplus IA$. Let $E_{10}(1) \subset I^1|1\rangle$ be the G representation $H^1(I)|1\rangle$, and we need to think it as a sub-module of $I^1|1\rangle$. We give it cohomological degree 1 and Adams degree 1. Then we have a natural inclusion $E_{10}(1) \to A$, which extends to $Sym^* E_{10}(1) \to A$ using the algebra structure of A. In fact, this is a map between cdgas over G and induces an isomorphism on $H^1(-)|1\rangle$.

Then one can adjoint elements in cohomological degree 1 and Adams degree 1 to kill elements in the kernel of the map on $H^2(-)|1\rangle$. So we have a $\mathbb{Z}$-graded G representations $E_1(1)$, of Adams degree 1 and cohomological degree 1, a generalized nilpotent cdga $A_{1,1} = Sym^* E_1(1)$ over G and a map of cdgas over G: $A_{1,1} \to A$, which induces an isomorphism on $H^1(-)|1\rangle$ and an injection on $H^2(-)|1\rangle$.

We also have a canonical decomposition of $A_{1,1} = \mathbb{Q} \oplus I_{1,1}$.

Notice that $H^p(I_{1,1}|r\rangle) = 0$ for $r > 1, p \leq 1$. This is because that the lowest degree of cohomology of $I_{1,1}|r\rangle$ is coming from $Sym^r E_1(1)$ and all the elements of $E_1(1)$ have cohomological degree 1. Iterating this process, one can construct the Adams degree ≤ 1 part of the n-minimal model in case $n > 1$. This gives us a generalized nilpotent cdga over G,

$$A_{1,n} = Sym^* E_n(1),$$

with $E_n(1)$ in Adams degree 1 and cohomological degrees $1, 2, \cdots, n$ together with a map over G: $A_{1,n} \to A$, which induces an isomorphism on $H^i(-)|1\rangle$ for $1 \leq i \leq n$ and an injection for $i = n+1$. In addition, letting $A_{1,n} = \mathbb{Q} \oplus I_{1,n}$, we have $H^p(I_{1,n}|r\rangle) = 0$ for $r > 1, p \leq 1$.

Suppose we have constructed $\mathbb{Z}$-graded G representations:

$$E_n(1) \subset E_n(2) \subset \cdots \subset E_n(m)$$

where $E_n(j)$ have Adams degrees $1, \cdots, j$ and cohomological degrees $1, \cdots, n$, a differential on $A_{n,m} = Sym^* E_n(m)$ making $A_{m,n}$ a generalized nilpotent cdga over G, and a map $A_{m,n} \to A$ of cdgas over G that is an isomorphism on $H^i(-)|j\rangle$ for $1 \leq i \leq n, j \leq m$, and an injection for $i = n+1, j \leq m$.

If $A_{m,n} = \mathbb{Q} \oplus I_{m,n}$, then $H^p(I_{m,n}|r\rangle) = 0$ for $r > m, p \leq 1$. Extending $E_n(m)$ to $E_n(m+1)$ by repeating the construction for $E_n(1)$ above. Then one can check the above condition sill hold. The induction goes through.

[4]n is also allowed to be ∞. In this case, we mean that there is a map of cdgas over G: $A\{\infty\} \xrightarrow{s} A$ with $A\{\infty\}$ generalized nilpotent and s a quasi-isomorphism.

Taking $E_n = \cup_m E_n(m)$, we have a differential on $A\{n\} = Sym^* E_n$ making $A\{n\}$ a generalized nilpotent cdga over G, and a map $A\{n\} \to A$ of cdgas over G that is an isomorphism on $H^i(-)$ for $1 \le i \le n$ and an injection for $i = n+1$. □

REMARK 2.56. If $f : A \to B$ is a quasi-isomorphism of cdgas over G, and $s : A\{n\} \to A$, $t : B\{n\} \to B$ are n-minimal models, then there is an isomorphism of cdgas over G: $g : A\{n\} \to B\{n\}$ such that $g \circ s$ is homotopic to $t \circ f$. The proof is the same as the usual case in Chapter 4 of [**9**].

7. The t-structure of $\mathcal{D}_A^{G,f}$

The aim of this section is to define a t-structure on $\mathcal{D}_A^{G,f}$ for A which is a cohomologically connected cdga over G. We recall the definition of the t-structure.

DEFINITION 2.57. A t-structure on a triangulated category $\mathcal{D}$ consists of essentially full subcategories $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ of $\mathcal{D}$ such that:

- $\mathcal{D}^{\le 0}[1] \subset \mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0}[-1] \subset \mathcal{D}^{\ge 0}$;
- $Hom_{\mathcal{D}}(M, N[-1]) = 0$ for $M \in \mathcal{D}^{\le 0}$ and $N \in \mathcal{D}^{\ge 0}$;
- For every $M \in \mathcal{D}$, there is a distinguished triangle
$$M^{\le 0} \to M \to M^{>0} \to M^{\le 0}[1]$$
with $M^{\le 0} \in \mathcal{D}^{\le 0}$ and $M^{>0} \in \mathcal{D}^{\ge 0}[-1]$.

Write $\mathcal{D}^{\le n}$ for $\mathcal{D}^{\le 0}[-n]$ and $\mathcal{D}^{\ge n}$ for $\mathcal{D}^{\ge 0}[-n]$.

A t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ is non-degenerate if
$$A \in \bigcap_{n \le 0} \mathcal{D}^{\le n} \quad \text{and} \quad B \in \bigcap_{n \ge 0} \mathcal{D}^{\ge n} \quad \text{imply} \quad A \cong B \cong 0.$$
△

There is a canonical augmentation $\epsilon : A \to \mathbb{Q}$, given by the projection onto $A^0|0\rangle = \mathbb{Q}$. So we have a functor:
$$q = \epsilon_* : \mathcal{CM}_A^G \to \mathcal{M}_{\mathbb{Q}}^G, \qquad q(M) = M \otimes_A \mathbb{Q}$$
and an exact tensor functor:
$$q : \mathcal{D}_A^G \to \mathcal{D}_{\mathbb{Q}}^G.$$

REMARK 2.58. We recall that $\mathcal{D}_{\mathbb{Q}}^{G,f}$ is the derived category of finite dimensional G-representations. There is a canonical t-structure for $\mathcal{D}_{\mathbb{Q}}^{G,f}$. We want to use q to get the induced t-structure for $\mathcal{D}_A^{G,f}$ when A is a cohomologically connected cdga over G. This is reasonable because the following general fact:

Let $\phi : A \to B$ be a map of cohomologically connected cdgas over G. Then $\phi_* : \mathcal{D}_A^{G,f} \to \mathcal{D}_B^{G,f}$ is conservative, i.e., $\phi_*(M) \cong 0$ implies $M \cong 0$.

PROOF. Take a non-zero object $M \in \mathcal{D}_A^{G,f}$. Then we can find a cell module P and a quasi-isomorphism $P \to M$ such that $W_{n-1}P = 0$, but $W_n P$ is not acyclic. We choose generating data $\{V_j, \phi_j\}_{j \in J}$ for P, such that $|V_j| \ge n$ for $j \in J$. Because n is the minimal integer of the possible Adams degree, (like the proof of Lemma 2.43), we can know that $W_nP \otimes_A \mathbb{Q}$ is not acyclic. Notice that $W_n(P \otimes_A B) = W_nP \otimes_A B$ and $W_nP \otimes_A \mathbb{Q} = (W_nP \otimes_A B) \otimes_B \mathbb{Q}$. Therefore $P \otimes_A B$ is not isomorphic to zero in $\mathcal{KCM}_B^G$ and $\phi_* M$ is non-zero in $\mathcal{D}_B^{G,f}$. □

REMARK 2.59. The inclusion $\mathbb{Q} \to A$ splits ϵ. Then the functor q defined above can be identified with the functor $gr_*^W = \prod_{n\in\mathbb{Z}} gr_n^W$. One can prove it by using the decomposition of the differential $d = d^0 + d^+$, which is described in Lemma 2.61 below and comparing two functors directly.

Define full subcategories $\mathcal{D}_A^{G,f,\leq 0}, \mathcal{D}_A^{G,f,\geq 0}$ and $\mathcal{H}_A^{G,f}$ of $\mathcal{D}_A^{G,f}$.

$$\begin{aligned} \mathcal{D}_A^{G,f,\leq 0} &= \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \; for \; n > 0\} \\ \mathcal{D}_A^{G,f,\geq 0} &= \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \; for \; n < 0\} \\ \mathcal{H}_A^{G,f} &= \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \; for \; n \neq 0\}. \end{aligned}$$

Then we have the following theorem as in [**21, 22**].

THEOREM 2.60. *Suppose A is cohomologically connected. Then*

$$(\mathcal{D}_A^{G,f,\leq 0}, \mathcal{D}_A^{G,f,\geq 0})$$

is a non-degenerate t-structure on $\mathcal{D}_A^{G,f}$ with heart $\mathcal{H}_A^{G,f}$.

Proof. We can assume that A is connected after replacing by its minimal model. The proof will divide into the following lemmas.

LEMMA 2.61. *Suppose that A is connected. Let $M \in \mathcal{D}_A^{G,f,\leq 0}$ (resp. $M \in \mathcal{D}_A^{G,f,\geq 0}$). Then there is an A-cell module $P \in \mathcal{CM}_A^{G,f}$ with generating data $\{V_j, \phi_j\}_{j\in J}$ such that $deg(\phi_j) \leq 0$ for all $j \in J$(resp. $deg(\phi_j) \geq 0$ for all $j \in J$), and a quasi-isomorphism $P \to M$.*

PROOF. We prove the case $M \in \mathcal{D}_A^{G,f,\leq 0}$ only.

We choose a quasi-isomorphism $Q \to M$ with $Q \in \mathcal{CM}_A^{G,f}$. Let $\{V_j, \phi_j\}_{j\in J}$ be generating data for Q. We can decompose d_Q with two parts d_Q^0 and d_Q^+, where d_Q^0 maps $\phi_j(V_j)$ to the submodule whose generating data (ϕ_i, V_i) have the Adams degree $|V_j|$ and d_Q^+ map to the complement part. After choosing suitable generating data, we may assume the collection S_0 of (V_j, ϕ_j) with $deg(\phi_j) = 0$ and $d_Q^0(\phi_j(V_j)) = 0$ forms a basis of

$$ker(d^0 : \oplus_{deg(\phi_j)=0}\phi_j(V_j) \to \oplus_{deg(\phi_i)=1}\phi_i(V_i)).$$

Let $\tau^{\leq 0}Q$ be the A sub-module of Q with the generating data of $S = \{(V_j, \phi_j) | deg(\phi_j) < 0\} \bigcup S_0$.

Claim: $\tau^{\leq 0}Q$ is a subcomplex of Q. Consider

$$d_Q(\phi_\alpha(V_\alpha)) = d_Q^0(\phi_\alpha(V_\alpha)) \oplus d_Q^+(\phi_\alpha(V_\alpha)).$$

Using the connected condition of A, we can know that:

1. If $\phi_\beta(V_\beta) \subset d_Q^+(\phi_\alpha(V_\alpha))$, then $deg(\phi_\beta) \leq deg(\phi_\alpha)$.

Or 2. If $\phi_\beta(V_\beta) \subset d_Q^0(\phi_\alpha(V_\alpha))$, then $deg(\phi_\beta) = deg(\phi_\alpha) + 1$.

Consider $(V_\alpha, \phi_\alpha) \in S$ with $deg(\phi_\alpha) \leq -1$. Because $(d_Q^0)^2 = 0$, every summand of $d_Q^0(\phi_\alpha(V_\alpha))$ lies in S_0. We only need to consider elements in S_0. Let $(V_\alpha, \phi_\alpha) \in S_0$. Then we have:

$$d_Q(\phi_\alpha(V_\alpha)) \subset \bigoplus_{deg(\phi_\beta)=0} \phi_\beta(A \otimes V_\beta) \oplus \bigoplus_{deg(\phi_\gamma)\leq -1} \phi_\gamma(A \otimes V_\gamma).$$

Using $d_Q^2(\phi_\alpha(V_\alpha)) = 0$, we can know that $d^0(\phi_\beta(V_\beta)) = 0$ for $deg(\phi_\beta) = 0$. This means that $d_Q(\phi_\alpha(V_\alpha)) \subset \tau^{\leq 0}Q$.

Next we can see that $\tau^{\leq 0}Q \to Q$ is a quasi-isomorphism. Using Remark 2.58 , we need only to check that $q\tau^{\leq 0}Q \to qQ$ is a quasi-isomorphism. This is clear because of $qQ \cong qM$ and $M \in \mathcal{D}_A^{f,\leq 0}$.

For the case $M \in \mathcal{D}_A^{G,f,\geq 0}$, we need to check the proof of Theorem 2.20 carefully, where we can add the extra conditions on the degrees of the generating datum. See Lemma 1.6.2 in [**22**]. □

Lemma 2.62. *Suppose that A is connected. Then $Hom_{\mathcal{D}_A^{G,f}}(M, N[-1]) = 0$ for $M \in \mathcal{D}_A^{G,f,\leq 0}$ and $N \in \mathcal{D}_A^{G,f,\geq 0}$.*

Proof. By Lemma 2.61, we may assume that M and $N[-1]$ are A-cell modules with the generating datum $\{(V_\alpha, \phi_\alpha)\}_{deg(\phi_\alpha)\leq 0}$ and $\{(V_\beta, \phi_\beta)\}_{deg(f_\beta)\geq 1}$. Recall we have:

$$Hom_{\mathcal{D}_A^{G,f}}(M, N[-1]) = Hom_{\mathcal{KCM}_A^{G,f}}(M, N[-1]).$$

If $\phi : M \to N[-1]$, then $deg(\phi) = 0$, which is impossible when we compute the degrees of both sides. □

Lemma 2.63. *Suppose that A is connected. For $M \in \mathcal{D}_A^{G,f}$, there is a distinguished triangle*

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1]$$

with $M^{\leq 0} \in \mathcal{D}^{G,f,\leq 0}$ and $M^{>0} \in \mathcal{D}^{G,f,\geq 0}[-1]$.

Proof. Following the same proof of Lemma 2.61, we can get a sub A-cell module $\tau^{\leq 0}M$ of M such that:

- $\tau^{\leq 0}M$ have generating data $\{(V_\alpha, \phi_\alpha)\}_{deg(\phi_\alpha)\leq 0}$.
- The map $q\tau^{\leq 0}M \to qM$ gives an isomorphism on H^n for $n \leq 0$.

Let $M^{\leq 0} = \tau^{\leq 0}M$ and let $M^{>0}$ be the cone of $\tau^{\leq 0}M \to M$. This gives us the distinguished triangle in $\mathcal{D}_A^G$:

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1].$$

Because $M \in \mathcal{D}_A^{G,f}$, then $gr_n^W M \in D_{\mathbb{Q}}^{G,f}$ for all n and is isomorphic to zero for all but finitely many n. This means $M^{\leq 0}$ and $M^{>0}$ all satisfy these two conditions by using the long exact sequence. Recall the distinguished triangle given by the weight filtration $gr_n^W M \to M \to M^{>n} \to gr_n^W M[1]$. By induction, we can show that $M^{\leq 0}$ and $M^{>0}$ are all in $\mathcal{D}_A^{G,f}$. After applying the functor q, we can know that $M^{\leq 0} \in \mathcal{D}^{G,f,\leq 0}$ and $M^{>0} \in \mathcal{D}^{G,f,\geq 0}[-1]$.

The only thing need to check is non-degenerate for the t-structure. If we take $M \in \bigcap_{n\leq 0}\mathcal{D}^{\leq n}$, then $H^n(qM)=0$ for all n, i.e., $qM\cong 0$ in $\mathcal{D}^{G,f}_{\mathbb{Q}}$. By the conservative property of the functor q, we know that $A\cong 0$ in $\mathcal{D}^{G,f}_A$. Another case is similar. □

Remark 2.64. The key point for the proof is that, for $M\in\mathcal{D}^{G,f}_A$, we have a lowest bound for the Adams degree.

We recall the definition of the neutral Tannakian category.

Definition 2.65. A neutral Tannkian category over k is a rigid abelian tensor category $(C,\otimes)$ such that $k=End(1)$ for which there exists an exact faithful k-linear tensor functor $\omega: C\to Vec_k$. Here 1 means the identity object under tensor product. △

Proposition 2.66. *$\mathcal{H}^{G,f}_A$ is a neutral Tannakian category over $\mathbb{Q}$.*

Proof. The derived tensor product makes $\mathcal{H}^{G,f}_A$ into an abelian tensor category. First we give a description about $\mathcal{H}^{G,f}_A$.

Lemma 2.67. *$\mathcal{H}^{G,f}_A$ is the smallest abelian subcategory of $\mathcal{H}^{G,f}_A$ containing the objects $A\otimes V$, where V is any rational G representation, and closed under extensions in $\mathcal{H}^{G,f}_A$.*

Proof. (Induction on the weight filtration.) Let $\mathcal{H}^{G,T}_A$ be the full abelian subcategory containing all the objects $A\otimes V$, where V is any rational G representation, and closed under extensions in $\mathcal{H}^{G,f}_A$. Let $M\in\mathcal{H}^{G,f}_A$ and $N=min\{n|W_nM\neq 0\}$. Then we have an exact sequence:

$$0\to gr^W_N M\to M\to W^{>N}M\to 0.$$

By Lemma 2.43, we have $gr^W_N M\cong A\otimes C$, where C is in $D^b(G)$. Because the category of representations of G is semisimple, we can think C as a direct sum of G rational representations with some shifts. Assume there exists a summand $W[i]$ of C with shift $i\neq 0$. Then applying q, we get that $0\neq H^i(q(gr^W_N M))\subset H^i(qM)$, which is a contradiction of our choice of $M\in\mathcal{H}^{G,f}_A$. This implies that $gr^W_N M\in\mathcal{H}^{G,T}_A$. By induction on the length of the weight filtration, $W^{>N}M$ is in $\mathcal{H}^{G,T}_A$. So $M\in\mathcal{H}^{G,T}_A$ and $\mathcal{H}^{G,T}_A=\mathcal{H}^{G,f}_A$. □

Since $(A\otimes V)^\vee = A\otimes V^\vee$, where $V^\vee$ is the dual of V in the category of rational G representations, it follows from the above description that $M\to M^\vee$ restricts from $\mathcal{D}^{G,f}_A$ to an exact involution on $\mathcal{H}^{G,f}_A$. $\mathcal{H}^{G,f}_A$ is rigid because $\mathcal{D}^{G,f}_A$ is rigid.

The identity for the tensor product is A and $\mathcal{H}^{G,f}_A$ is $\mathbb{Q}$-linear.

We have a rigid tensor functor $q:\mathcal{H}^{G,f}_A\to\mathcal{H}^{G,f}_{\mathbb{Q}}$. Notice that $\mathcal{H}^{G,f}_{\mathbb{Q}}$ is equivalent to the category of rational representations of G. So there is a faithful forgetful functor $w:\mathcal{H}^{G,f}_{\mathbb{Q}}\to Vec_{\mathbb{Q}}$. We need only to see that q is faithful.

Recall we can identify q with $gr^W_*=\oplus gr^W_n$. Let $f:M\to N$ be a map in $\mathcal{H}^f_A$ such that $gr^W_n(f)=0$ for all n. We need to show that $f=0$. Again do the induction on the length of the weight filtration. We may assume that $W^nf=0$, where n is the minimal integer such that $W_nM\oplus W_nN\neq 0$. Thus f is given by a map

$$\tilde{f}: W^{>n}M\to gr^W_n N.$$

Claim: $\tilde{f} = 0$.

Using the induction on the weight filtration, we need to show the following statement:

Given V and W pure weight rational G representations such that $|V| > |W|$, then we have:

$$Hom_{\mathcal{H}_A^{G,f}}(A \otimes V, A \otimes W) \cong 0.$$

Firstly we have:

$$Hom_{\mathcal{H}_A^{G,f}}(A \otimes V, A \otimes W) \cong Hom_{\mathcal{D}_A^{G,f}}(A \otimes V, A \otimes W)$$
$$\cong Hom_{D(G)}(V, A \otimes W) \cong Hom_G(\mathbb{Q}, H^0(A \otimes W \otimes V^\vee)).$$

Because A is connected, $H^0(A \otimes W \otimes V^\vee) \cong W \otimes V^\vee$, which is a rational representation of G with Adams degree strictly smaller than zero. This implies that:

$$Hom_G(\mathbb{Q}, H^0(A \otimes W \otimes V^\vee)) \cong 0.$$

Therefore we get that q is faithful. □

8. The bar construction

Let A be a cdga over G and let M, N be two dg-A modules. Then we define:

$$T^G(N, A, M) = N \otimes T(A) \otimes M,$$

where $T(A) = \mathbb{Q} \oplus A \oplus (A \otimes A) \oplus \cdots = \oplus_{r \geq 0} T^r(A)$ is the tensor algebra. It is spanned by the elements of the form $n[a_1|\cdots|a_r]m$. Notice that $T^G(N, A, M)$ is a simplicial graded abelian group with $N \otimes T^r(A) \otimes M$ in degree r, whose face maps are:

$$\delta_0(n[a_1|\cdots|a_r]m) = na_1[a_2|\cdots|a_r]m,$$
$$\delta_i(n[a_1|\cdots|a_r]m) = na_1[a_2|\cdots|a_ia_{i+1}|\cdots|a_r]m, \ 1 \leq i \leq r-1$$
$$\delta_r(n[a_1|\cdots|a_{r-1}]a_rm) = n[a_1|\cdots|a_{r-1}]a_rm,$$

and degeneracies are:

$$s_i(n[a_1|\cdots|a_r]m) = n[a_1|\cdots|a_{i-1}|1|a_i|\cdots|a_r]m.$$

Define:

$$\delta = \sum_{0 \leq i \leq r} (-1)^i \delta_i : N \otimes T^r(A) \otimes M \to N \otimes T^{r-1}(A) \otimes M$$

Let $D^G(N, A, M)$ be the degenerate elements, those elements are spanned by the images of the s_i for every i.

Definition 2.68. Define the bar complex of M and N to be:

$$B^G(N, A, M) = T^G(N, A, M)/D^G(N, A, M).$$

△

Note that $B^G(N, A, M)$ is a bicomplex. The total differential is defined by

$$d(n[a_1|\cdots|a_r]m)$$
$$= \partial(n[a_1|\cdots|a_r]m)) + (-1)^{deg(n)+deg(m)+\sum deg(a_i)}\delta(n[a_1|\cdots|a_r]m),$$

where ∂ denotes the usual differential of A. We will consider the following special case that $M = N = \mathbb{Q}$, which is denoted by $\bar{B}^G(A)$, called the reduced bar construction. We collect formal properties of $\bar{B}^G(A)$.

- (Shuffle product) $\cup : \bar{B}^G(A) \otimes \bar{B}^G(A) \to \bar{B}^G(A)$
$$[a_1|\cdots|a_p] \cup [a_{p+1}|\cdots|a_{p+q}] = \sum sgn(\sigma)[x_{\sigma(1)}|\cdots|x_{\sigma(p+q)}]$$
where the sum is over all (p,q) shuffles $\sigma \in \Sigma_{p+q}$ (recall Σ_{p+q} is the symmetric group on $p+q$ letters) and the sign of σ is taking into account the degree of a_i.
- (Coproduct) $\Delta : \bar{B}^G(A) \to \bar{B}^G(A) \otimes \bar{B}^G(A)$
$$\Delta([a_1|\cdots|a_n]) = \sum_{i=0}^{n}(-1)^{i(deg(a_{i+1})+\cdots+deg(a_n))}[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n].$$
- (involution) $\iota : \bar{B}^G(A) \to \bar{B}^G(A)$
$$\iota([a_1|\cdots|a_n]) = (-1)^m[a_n|a_{n-1}|\cdots|a_1], m = \sum_{1\le i<j\le n} deg(a_i)deg(a_j).$$

These properties make $\bar{B}^G(A)$ a graded-commutative differential graded Hopf algebra in the category of G representations. So $H^0(\bar{B}^G(A))$ is a commutative Hopf algebra over G. If consider the Adams grading structure, we can see that $\bar{B}^G(A)$ also has the Adams grading structure, and $\chi_A = H^0(\bar{B}^G(A))$ is an Adams graded Hopf algebra over G (or a graded Hopf algebra object in $\mathbf{Rep}_G$).

Definition 2.69. Define $\gamma_A = I_{\chi_A}/(I_{\chi_A})^2$, where I_{χ_A} is the augmentation ideal of χ_A. $\triangle$

Lemma 2.70. *γ_A determines a structure of a cdga over G.*

Remark 2.71. Recall the definition of co-Lie algebras firstly. A co-Lie algebra is a k-module γ with a cobracket map $\gamma \to \gamma \otimes \gamma$ such that the dual $\gamma^\vee$ is a Lie algebra via the dual homomorphism. Sullivan showed the following statement (Lemma 2.7 in [**21**] or p.279 in [**33**]):

A co-Lie algebra γ determines and is determined by a structure of DGA on $\wedge(\gamma[-1])$.

Proof. The coproduct Δ of $\bar{B}^G(A)$ induce a coproduct on χ_A, denoted also by Δ, satisfying:
$$\Delta(x) = x \otimes 1 + 1 \otimes x$$
for $x \in I_{\chi_A}$. (And also the involution.) Then $\Delta - \iota\Delta$ gives the cobracket on χ_A. So is γ_A and the quotient map $I_{\chi_A} \to \gamma_A$ is a map of co-Lie algebras. By the remark above, we know γ_A determines a cdga structure $\wedge(\gamma_A[-1])$. This structure is compatible with the G representation structure. Notice that I_{χ_A} does not have the Adams degree 0 part. Then $\wedge(\gamma_A[-1])|0\rangle = \mathbb{Q}$. So $\wedge(\gamma_A[-1])$ is a cdga over G. □

Lemma 2.72. *Let A be a cdga over G. Then $H^*(\bar{B}^G(A))$ and χ_A is functorial in A and is a quasi-isomorphism invariant in A.*

Proof. Use the Eilenberg-Moore spectral sequence. See Lemma 2.21 in [**6**]. □

THEOREM 2.73. *Let A be a cohomologically connected cdga over G. Then the 1-minimal model $A\{1\}$ of A is isomorphic to $\wedge(\gamma_A[-1])$.*

PROOF. Follow the proof of [**6**]. From Lemma 2.72, we can assume that A is a generalized nilpotent cdga over G. A generalize nilpotent cdga A over G is a direct limit (A_α) of nilpotent cdga's. So we can assume that A is a nilpotent cdga over G with a free generator E which is a complex of G representations. We need to use the following lemma, whose proof is totally the same as Lemma 2.32 in [**6**].

LEMMA 2.74. *Let A be as above with free generator G-representation V. Fix an integer $s > 0$. Consider the decreasing filtration on $\bar{B}^G(A)$,*

$$F^k\bar{B}^G(A) = \langle x_{11}\cdots x_{1n_1}\otimes\cdots\otimes x_{m1}\cdots x_{mn_m}|$$

$$s\sum degx_{ij} + s\sum n_j - (2s-1)m \geq k\rangle.$$

Then, for a sufficiently large s depending on A, the resulting spectral sequence satisfies

$$\wedge(V[1]) \cong E_{2s} \cong E_\infty \cong Gr_F H^*(\bar{B}^G(A)).$$

Assuming this lemma, the projection map $\bar{B}^G(A) \to A$ induces a map

$$\phi : QH^*(\bar{B}^G(A)) \to (QA)[1].$$

This is because that the boundaries and decomposable elements map to decomposable elements. In fact, boundaries are the sum of the form $\partial(a)$ and $a_1 \cdot a_2$. Using that the generalized nilpotence implies the minimal property (Proposition 2.53), both of these elements are decomposable. By the above lemma, this is an isomorphism. Furthermore, ϕ is an isomorphism between co-Lie algebras.

Restricting ϕ to the degree 0 part, we know that

$$\wedge(\gamma_A[-1]) = \wedge(QH^0(\bar{B}^G(A))[-1])$$

is isomorphic to 1-minimal model of A. □

9. Alternative identifications of the category $\mathcal{H}_A^f$

In Proposition 2.53, we show that the connected generalized nilpotent cdga over G can be recognized as a connected minimal cdga over G. Similarly we can also define the minimal cell A-module.

DEFINITION 2.75. An Adams degree bounded below cell A-module is minimal if it is almost free and $d(M) \subset (IA)M$. △

DEFINITION 2.76. Let M be an Adams degree bounded below A-module. We define the nilpotent filtration $\{F_tM\}$ by letting $F_0M = 0$ and inductively letting F_tM be the sub A-module generated by $F_{t-1}M \cup \{m|dm \in F_{t-1}M\}$. △

REMARK 2.77. The minimal cell modules have the similar properties as the connected minimal cdgas. We can also define the generalized nilpotent A-modules. Because the proof of the following properties is the same as Part IV, section 3 in [**21**], we only list the main properties.

- A bounded below A-module M is generalized nilpotent if and only if it is a minimal cell A module.
- Let N be a dg A-module. Then there is a quasi-isomorphism $e : M \to N$, where M is a minimal A-module. This is unique up to the homotopy.

Next, we want to use another way to describe cell A-modules, which is called the connection matrix. See [**21**](namely the twisting matrix) or [**22**].

Definition 2.78. Let (M, d_M) be a complex of G representations. An A-connection for M is a map

$$\Gamma : M \to IA \otimes M$$

of G representations and cohomological degree 1. We say Γ is flat if

$$d\Gamma + \Gamma^2 = 0.$$

Here $d\Gamma = d_{IA\otimes M} \circ \Gamma + \Gamma \circ d_M$ and we extend Γ to

$$\Gamma : IA \otimes M \to IA \otimes M$$

by the Leibniz rule. △

Remark 2.79. Given a connection $\Gamma : M \to IA \otimes M$, we define

$$d_0 : M \to A \otimes M = M \oplus IA \otimes M, m \to d_M m \oplus \Gamma m$$

and extend d_0 to $d_\Gamma : A \otimes M \to A \otimes M$ by the Leibniz rule. The above equation is equivalent to saying that $d_\Gamma^2 = 0$.

Definition 2.80. We call an A-connection Γ for M nilpotent if M admits a filtration by complexes of G representations:

$$0 = M_{-1} \subset M_0 \subset \cdots \subset M_n \subset \cdots \subset M$$

such that $M = \cup_n M_n$ and such that $d_M(M_n) \subset M_{n-1}$ and $\Gamma(M_n) \subset IA \otimes M_{n-1}$ for every $n \geq 0$. △

Remark 2.81. Let $\Gamma : M \to IA \otimes M$ be a flat nilpotent connection. Then the dg A-module $(A \otimes M, d_\Gamma)$ is a cell module.

Lemma 2.82. *Let $\Gamma : M \to IA \otimes M$ be a flat connection. Suppose there is an integer r_0 such that $|m| \geq r_0$ for all $m \in M$. Then Γ is nilpotent.*

Proof. The proof is the same as Lemma 1.13.3 in [**22**]. □

Definition 2.83. A morphism $f : (M, d_M, \Gamma_M) \to (N, d_N, \Gamma_N)$ is a map of complexes of G representations:

$$f = f_0 + f^+ : M \to A \otimes N = N \oplus IA \otimes N$$

such that $d_{\Gamma_N} f = f d_{\Gamma_M}$. △

Definition 2.84. We denote the category of flat nilpotent connections over A by $Conn_A^G$ and denote the full subcategory of flat nilpotent connections on M with M a bounded complex of rational G-representations by $Conn_A^{G,f}$. △

We can define a tensor operation on $Conn_A$ by

$$(M, \Gamma) \otimes (M^{'}, \Gamma^{'}) = (M \otimes M^{'}, \Gamma \otimes id + id \otimes \Gamma^{'}).$$

Complexes of $\mathbb{Q}$-vector spaces act on $Conn_A$ by:

$$(M, \Gamma) \otimes K = (M, \Gamma) \otimes (K, 0).$$

We recall that I is the complex

$$\mathbb{Q} \xrightarrow{\delta} \mathbb{Q} \oplus \mathbb{Q}$$

with $\mathbb{Q}$ in degree -1 and with connection 0. We have the two inclusions $i_0, i_1 : \mathbb{Q} \to I$.

Definition 2.85. Two maps $f, g : (M, \Gamma) \to (M^{'}, \Gamma^{'})$ are homotopic if there is a map $h : (M, \Gamma) \otimes I \to (M^{'}, \Gamma^{'})$ satisfying $f = h \circ (id \otimes i_0), g = h \circ (id \otimes i_1)$. △

Definition 2.86. Denote the homotopy category of $Conn_A^G$ by $\mathcal{H}Conn_A^G$, which has the same objects as $Conn_A^G$ and morphisms are homotopy classes of maps in $Conn_A^G$. △

Remark 2.87. When we pass to homotopy classes and given a cell A-module M, it is totally determined by the underlying G representation M_0, i.e., $M_0 = M \otimes_A \mathbb{Q}$.

We list the main properties of flat connections, and the proof is the same as in Section 1.14 in [**22**].

- The category of A-cell modules is equivalent to the category of flat nilpotent A-connections.
- The above equivalence passes to an equivalence of $\mathcal{H}Conn_A^G$ with the homotopy category $\mathcal{KCM}_A^G$ as triangulated tensor categories.
- Suppose that A is connected. The second equivalence defines an equivalence of Tannakian categories $\mathcal{H}_A^{G,f}$ and the category of flat connections on G representations $Conn_A^{G,f}$.

Given A a cohomologically connected cdga over G, by Theorem 2.73, we know that:

$$\gamma_A = A\{1\}^1.$$

Assume that A is a generalized nilpotent dga over G, which implies that $A\{1\} \cong A$. Then the co-Lie algebra structure of γ_A is given by the restriction of d to A^1. Notice that, by the minimal property (Proposition 2.53), d factors through:

$$d : A^1 \to \wedge^2 A^1 \subset A^2.$$

Let M be a complex of rational G representations and $\Gamma : M \to IA \otimes M$ is a flat connection. In fact, Γ is a map

$$\Gamma ; M \to A^1 \otimes M$$

and the flatness is just saying that Γ makes M into an Adams graded co-module for the co-Lie algebra γ_A over G. In fact, we have equivalences between categories:

$$\mathcal{H}_A^{G,f} \cong Conn_A^{G,f} \cong co - rep^{G,f}(\gamma_A).$$

10. The main theorem

LEMMA 2.88. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. We denote its heart by $\mathcal{H}$. Assume that there is a triangulated functor $\rho : D^b(\mathcal{H}) \to \mathcal{D}$ such that:*

- *$\rho|_{\mathcal{H}[i]}$ is an inclusion for any $i \in \mathbb{Z}$;*
- *$\mathcal{D}$ is bounded, i.e., for any $M \in \mathcal{D}$, there exist $a \leq b \in \mathbb{Z}$ satisfying $M \in \mathcal{D}^{[a,b]} = \mathcal{D}_{\geq a} \cap \mathcal{D}_{\leq b}$.*
- *For any $M, N \in \mathcal{H}$ and $n \in \mathbb{Z}$, ρ induces an isomorphism*

$$Hom_{D^b(\mathcal{H})}(M, N[n]) \xrightarrow{\sim} Hom_{\mathcal{D}}(\rho(M), \rho(N)[n]).$$

Then ρ is an equivalence between triangulated categories.

PROOF. We do the induction on the length of the object. Given an object A in $\mathcal{D}$, there exist the minimal a and maximal b such that $A \in \mathcal{D}_{\geq a} \cap \mathcal{D}_{\leq b}$. Then we define the length of A to be $b - a$. Firstly, we prove the following:

For any $A, B \in D^b(\mathcal{H})$ and $n \in \mathbb{Z}$, we have:

$$Hom_{D^b(\mathcal{H})}(M, N[n]) \xrightarrow{\sim} Hom_{\mathcal{D}}(\rho(M), \rho(N)[n]). \tag{8}$$

By induction, we assume that, for any $A \in D^b(\mathcal{H})^{a,b}$, $B \in D^b(\mathcal{H})^{c,d}$ and $max\{b-a, d-c\} \leq m-1$, the above is true. Take any A with length smaller than m, and B with length $m = b - a$. There is a distinguished triangle:

$$\tau_{\geq a}\tau_{\leq b-1}B \to B \to \tau_{\geq b}\tau_{\leq b}B \to \tau_{\geq a}\tau_{\leq b-1}B[1] \to .$$

Then we have a long exact sequence:

$$Hom(A, \tau_{\geq a}\tau_{\leq b-1}B) \to Hom(A, B) \to Hom(A, \tau_{\geq b}\tau_{\leq b}B)$$
$$\to Hom(A, \tau_{\geq a}\tau_{\leq b-1}B[1]) \to \cdots .$$

Using functoriality to compare the above sequence with:

$$Hom(\rho(A), \rho(\tau_{\geq a}\tau_{\leq b-1}B)) \to Hom(\rho(A), \rho(B)) \to Hom(\rho(A), \rho(\tau_{\geq b}\tau_{\leq b}B))$$
$$\to Hom(\rho(A), \rho(\tau_{\geq a}\tau_{\leq b-1}B[1])) \to \cdots .$$

We know (8) is holding for A, B by the five lemma and induction. Then we assume both A and B have length m. Using the similar method and induction again, we can know that (8) is holding, i.e., ρ is fully faithful.

Next we want to use induction to show that ρ is essentially surjective. It is enough to show that, for any object $B \in \mathcal{D}$, there exists $A \in D^b(\mathcal{H})$ such that $\rho(A) \cong B$. Take any element $B \in \mathcal{D}$ with length m. Then we have:

$$\tau_{\geq a}\tau_{\leq b-1}B \to B \to \tau_{\geq b}\tau_{\leq b}B \to \tau_{\geq a}\tau_{\leq b-1}B[1] \to .$$

In other words, we have:

$$\tau_{\geq b}\tau_{\leq b}B[-1] \xrightarrow{f} \tau_{\geq a}\tau_{\leq b-1}B \to B \to \tau_{\geq b}\tau_{\leq b}B \to .$$

By assumption, we have A_1 and $A_2 \in D^b(\mathcal{H})$ map to $\tau_{\geq b}\tau_{\leq b}B[-1]$ and $\tau_{\geq a}\tau_{\leq b-1}B$ respectively. By (8), we know that there exists a map g from A_1 to A_2, whose image under ρ is just f. We take $A = cone(g)$. Then by the axiom of triangulated categories, there exists a map $\rho(A) \to B$. By the five lemma and Yoneda lemma, applying the functor of type $Hom(\tilde{B},)$, where $\tilde{B} \in \mathcal{D}$, we know that $\rho(A) \cong B$. □

THEOREM 2.89. *Let A be a cohomologically connected cdga over G. Then*

- *There is a functor:*
$$\rho : D^b(\mathcal{H}_A^{G,f}) \longrightarrow \mathcal{D}_A^{G,f}.$$
- *The functor ρ constructed above is an equivalence of triangulated categories if and only if A is 1-minimal.*

PROOF. We first need to construct a functor ρ. Consider a bounded complex
$$M^* = \{M^n, \delta^n : M^n \to M^{n+1}\}$$
in $\mathcal{H}_A^{G,f}$. Assume that each M^n is minimal. Furthermore, we assume that it is given by the generating data $\{V_{j^n}, \phi_{j^n}\}_{j^n \in J^n}$ and the connection matrix Γ^n. Then we define ρM^* with its generating data $\{V_{j^n}, \phi_{j^n}[n]\}_{j^n \in J^n}$ and its differential given by:
$$d|_{\phi_{j^n}[n](V_{j^n})} = \Gamma^n[n] + \delta^n[n].$$
If $f^* : M^* \to N^*$ is a quasi-isomorphism of chain complexes, then $\rho(f^*)$ is a quasi-isomorphism of A-modules.

Let's prove the second statement now. We assume that A is 1-minimal, i.e., $A \cong \wedge^*(\gamma[-1])$, where γ is the co-Lie algebra consisted by the indecomposable elements of $H^0(\bar{B}^G(A))$.

In order to apply the above result to our case ($\mathcal{D} = \mathcal{D}_A^{G,f}$ and $\mathcal{H} = \mathcal{H}_A^{G,f}$), we need to check the conditions in Lemma 2.88. The first and second condition are automatic. Let's check the third condition.

Notice that $\mathcal{H}_A^{G,f}$ can be identified with the category of co-representations of γ in the category of G representations. In fact, given a finite dimensional co-representation V, we can associate it with a cell module $A \otimes V$.

We recall the following basic facts. (Lemma 23.1, Example 1(p.315) and p.319, 320 in[**13**].) Given a differential graded Lie algebra L, we have:
$$Ext_L^n(\mathbb{Q}, \mathbb{Q}) \cong Ext_{UL}^n(\mathbb{Q}, \mathbb{Q}) \cong H^n((\wedge^*(L[-1]))^\vee),$$
and
$$Ext_L^n(\mathbb{Q}, V) \cong Ext_{UL}^n(\mathbb{Q}, V) \cong H^n((\wedge^*(L[-1]))^\vee \otimes V),$$
where $\vee$ means taking the dual, UL is the universal enveloping Lie algbera of L and V is any L-module. $L[-1]_k = L_{k-1}$.

Applying to the co-Lie algbebra γ, we get:
$$Ext_\gamma^n(\mathbb{Q}, \mathbb{Q}) \cong H^n(\wedge^*(\gamma[-1])).$$
In fact, the proof of this isomorphism can be extended to the following case.

Given a co-Lie algebra γ over G and a γ co-representation V, we have:
$$Ext_\gamma^n(\mathbb{Q}, V) \cong H^n(\wedge^*(\gamma[-1]) \otimes V)|0\rangle.$$

Notice that the left hand side computes the extension groups in the category of γ-representations. We have:

$$H^n(\wedge^*(\gamma[-1])\otimes V)|0\rangle = Hom_G(\mathbb{Q}, H^n(\wedge^*(\gamma[-1])\otimes V)).$$

Therefore we get:

$$\begin{aligned}
&Hom_{D^b(\mathcal{H}_A^{G,f})}(\wedge^*(\gamma[-1])\otimes V, \wedge^*\gamma[-1]\otimes W[n])\\
\cong &Ext^n_{\mathcal{H}_A^{G,f}}(\wedge^*(\gamma[-1])\otimes V, \wedge^*(\gamma[-1])\otimes W)\\
\cong &Ext^n_\gamma(\mathbb{Q}, V^\vee\otimes W)\\
\cong &H^n(\wedge^*(\gamma[-1])\otimes V^\vee\otimes W)|0\rangle\\
\cong &H^0(\wedge^*(\gamma[-1])\otimes V^\vee\otimes W[n])|0\rangle\\
\cong &Hom_{\mathcal{D}_A^{G,f}}(\mathbb{Q}, \wedge^*(\gamma[-1])\otimes V^\vee\otimes W[n])\\
\cong &Hom_{\mathcal{D}_A^{G,f}}(\wedge^*(\gamma[-1])\otimes V, \wedge^*(\gamma[-1])\otimes W[n]).
\end{aligned}$$

It's easy to check that the composition of these isomorphisms is given by $\rho : D^b(\mathcal{H}_A^{G,f}) \longrightarrow \mathcal{D}_A^{G,f}$.

Conversely, we assume that ρ is an equivalence. Without loss of generality, we assume that A is generalized nilpotent. The above computation tell us that $H^n(A\otimes V)|0\rangle \cong Ext^n_\gamma(\mathbb{Q}, V)$, where γ is the co-Lie algebra consisting of the indecomposable elements in $H^0(\bar{B}(A))$. Let us consider the map $A\to \wedge^*(\gamma[-1])$, Applying the functor $Hom_G(V[n], \cdot)$ for any $n\in\mathbb{Z}$ and any G-representation V, we get:

$$Hom_G(V[n], A)\cong H^n(A\otimes V)|0\rangle \cong Ext^n_\gamma(\mathbb{Q}, V)\cong Hom_G(V[n], \wedge^*(\gamma[-1])).$$

This implies that, under the level of the G representations, the above map is a quasi-isomorphism. Therefore A is 1-minimal. □

Corollary 2.90. *Let A be a cohomologically connected cdga over G. Then*

- *There is a functor:*

$$\rho : D^b(co - rep_{\mathbb{Q}}^{G,f}(\chi_A)) \longrightarrow \mathcal{D}_A^{G,f}.$$

  Furthermore, ρ induces a functor on the hearts:

$$\mathcal{H}(\rho) : co - rep^{G,f}(\chi_A) \to \mathcal{H}_A^{G,f},$$

  which is an equivalence of Tannakian categories.
- *The functor ρ is an equivalence of triangulated categories if and only if A is 1-minimal.*

11. Concluding Remarks

The results in the first two sections work for arbitrary reductive group over a base field of characteristic zero. Only after assuming the existence of a non-trivial central cocharacter, we can define the weight filtration as in Section 3 of Chapter 2. If we take $G = \mathbb{G}_m$ and $w(\lambda) = \lambda^2$ for $\lambda\in\mathbb{G}_m(\mathbb{Q})$, our constructions and properties for cdgas over $\mathbb{G}_m$, dg modules over $\mathbb{G}_m$ are the same as the constructions and properties of the case of Adams cdgas considered in [**21**], [**22**]. For our later use, we take $G = GL_2$ and w defined in Section 2 of Chapter 1. In addition, we take $\mathbf{F}$ to be the fundamental representation of GL_2. The construction for the theory of cdgas over G is closely related with weighted completion of profinite groups. See [**16**] for the definition of weighted completion. We may discuss their relationship somewhere in the future.

CHAPTER 3

THE CATEGORY OF MOTIVES

1. Construction of $\mathbf{DM}_{gm}(k,\mathbb{Z})$

In this section, we will briefly recall the construction of $\mathbf{DM}_{gm}(k,\mathbb{Z})$. For the details about the construction of $\mathbf{DM}_{gm}(k,\mathbb{Z})$, we refer to [**15**] or [**27**]. We assume that k is a perfect field in this section.

Definition 3.1. Let X be a smooth connected scheme over k and Y a separated scheme over k. We define an elementary correspondence from X to Y to be an irreducible closed subset of $X \times Y$ whose associated integral subscheme is finite and surjective over X. When X is non-connected, an elementary correspondence is defined from one connected component of X to Y. We use $\mathbf{Cor}(X,Y)$ to denote the free abelian group generated by the elementary correspondences from X to Y. The elements in $\mathbf{Cor}(X,Y)$ are called finite correspondences. △

Remark 3.2. One may define an associative and bilinear composition for finite correspondences. More precisely, given $V \in \mathbf{Cor}(X,Y), W \in \mathbf{Cor}(Y,Z)$, we define $W \circ V$ to be the push-forward of the intersection product $(V \times Z) \cdot (X \times W)$ along the projection $X \times Y \times Z \to X \times Z$. However we need to check this gives a well-defined elementary correspondence in $\mathbf{Cor}(X,W)$. See Lemma 1.4 in [**27**].

Definition 3.3. We define $\mathbf{Cor}_k$ to be the category whose objects are smooth separated schemes over k and whose morphisms from X to Y are finite correspondences. Notice that $\mathbf{Cor}_k$ is an additive category with zero object as $\emptyset$ and disjoint union as coproduct. △

Definition 3.4. A presheaf with transfers is a contravariant additive funtor $F : (\mathbf{Cor}_k)^{op} \to \mathbf{Ab}$. Similarly, a $\mathbf{Rep}_{GL_2}$-valued presheaf with transfers is a contravariant additive functor $F : (\mathbf{Cor}_k)^{op} \to \mathbf{Rep}_{GL_2}$. We denote the category whose objects are presheaves with transfers and whose morphisms are natural transformations by $\mathbf{PST}$. △

Remark 3.5. The basic property for $\mathbf{PST}$ is the following: It is an abelian category with enough injectives and projectives. See Theorem 2.3 in [**27**]. A typical example of presheaves with transfers is given by the representable functors. Let X be a smooth scheme over k. Then we use $\mathbb{Z}_{tr}(X)$ to denote the prsheaf with transfers represented by X, i.e. $\mathbb{Z}_{tr}(X)(U) = \mathbf{Cor}_k(U,X)$ for $U \in \mathbf{Sm}_k$.

Definition 3.6. A ($\mathbf{Rep}_{GL_2}$-valued) Nisnevich sheaf with transfers is a ($\mathbf{Rep}_{GL_2}$-valued) presheaf with transfers whose underlying presheaf is a Nisnevich sheaf on $\mathbf{Sm}_k$.

For the definition of Nisnevich sheaves, we refer to Lecture 12 in [**27**]. We denote the category of Nisnevich sheaves with transfers by $\mathcal{S}h^{tr}_{Nis}(k)$ and the category of $\mathbf{Rep}_{GL_2}$-valued Nisnevich sheaves with transfers by $\mathcal{S}h^{tr}_{Nis}(k) \otimes \mathbf{Rep}_{GL_2}$. △

CONVENTION 3.7. We denote the derived category of bounded above complexes of Nisnevich sheaves with transfers by $\mathbf{D}^-(\mathcal{S}h^{tr}_{Nis}(k))$.

DEFINITION 3.8. Let $\mathbb{E}_{\mathbb{A}}$ be the smallest thick subcategory of $\mathbf{D}^-(\mathcal{S}h^{tr}_{Nis}(k))$ satisfies:

- it contains the cone of $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$ for any $X \in \mathbf{Sm}_k$;
- it is closed under any direct sum that exists in $\mathbf{D}^-(\mathcal{S}h^{tr}_{Nis}(k))$.

△

DEFINITION 3.9. The triangulated category of effective motives over k is defined to be the quotient triangulated category $\mathbf{DM}^{eff,-}_{Nis}(k,\mathbb{Z}) = \mathbf{D}^-(\mathcal{S}h^{tr}_{Nis}(k))/\mathbb{E}_{\mathbb{A}}$. △

For the definition of the quotient triangulated category, one may find in lecture 9 in [**27**].

DEFINITION 3.10. Let X be a smooth scheme over k. We define the motive of X to be the class of $\mathbb{Z}_{tr}(X)$ in $\mathbf{DM}^{eff,-}_{Nis}(k,\mathbb{Z})$ and denote it by $M(X)$. △

DEFINITION 3.11. We define $\mathbf{DM}^{eff}_{gm}(k,\mathbb{Z})$ to be the thick subcategory of $\mathbf{DM}^{eff,-}_{Nis}(k,\mathbb{Z})$ generated by the motives $M(X)$ for $X \in \mathbf{Sm}_k$. The objects in $\mathbf{DM}^{eff}_{gm}(k,\mathbb{Z})$ are called effective geometric motives. △

EXAMPLE 3.12. (Tate motives) The motive of $\mathbb{P}^1$ has the following decomposition:

$$M(\mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}(1)[2],$$

where $\mathbb{Z}$ is the motive of $Spec(k)$. In fact, after choosing a rational point in $\mathbb{P}^1$, we get a map:

$$\mathbb{Z} \cong M(Spec(k)) \to M(\mathbb{P}^1).$$

On the other hand, the structure map of $\mathbb{P}^1$ will give us a map:

$$M(\mathbb{P}^1) \to M(Spec(k)).$$

From the functoriality of the construction of motives, we know that the composition:

$$M(Spec(k)) \to M(\mathbb{P}^1) \to M(Spec(k))$$

is the identity map. Because $\mathbf{DM}^{eff}_{gm}(k,\mathbb{Z})$ is idempotent complete, $\mathbb{Z} \cong M(Spec(k))$ is a direct summand of $M(\mathbb{P}^1)$. The complementary part is denoted by $\mathbb{Z}(1)[2]$. More generally, we define $\mathbb{Z}(n)$ by $\mathbb{Z}^{\otimes n}$ for n a positive integer. △

REMARK 3.13. $\mathbf{DM}^{eff,-}_{Nis}(k,\mathbb{Z})$ is a tensor triangulated category. For the definition of tensor triangulated category and the proof of this fact, we both refer to [**27**].

DEFINITION 3.14. The category of geometric motives, which is denoted by $\mathbf{DM}_{gm}(k,\mathbb{Z})$, is obtained from $\mathbf{DM}^{eff}_{gm}(k,\mathbb{Z})$ by inverting the Tate twist operation:

$$\mathbf{DM}^{eff}_{gm}(k,\mathbb{Z}) \xrightarrow{\otimes\mathbb{Z}(1)} \mathbf{DM}^{eff}_{gm}(k,\mathbb{Z}).$$

△

REMARK 3.15. In [**27**], Mazza, Voevodsky and Weibel also define the category of $\mathbf{DM}^{-}_{Nis}(k,\mathbb{Z})$, which is obtained from $\mathbf{DM}^{eff,-}_{Nis}(k,\mathbb{Z})$ by inverting the Tate twist operation $M \to M(1)$. However, this category is not closed under arbitrary direct sum. There is another construction of motives $\mathbf{DM}(k,\mathbb{Z})$, which is called "Voevodsky's big category of motives over k" by using symmetric spectra. For the details about this construction, we refer to Chapter 3 in [**22**] and [**10**][1].

DEFINITION 3.16. Let $\mathbf{Chow}(k,\mathbb{Z})$ denote the category of chow motives over k, whose objects are (X,α,n), where X is a smooth projective variety over k, $\alpha \in CH_{\dim X}(X \times X)$ such that $\alpha \circ \alpha = \alpha^2$ and $n \in \mathbb{Z}$. The morphism between two objects in $\mathbf{Chow}(k,\mathbb{Z})$ is given by:

$$Hom_{\mathbf{Chow}(k,\mathbb{Z})}((X,\alpha,n),(Y,\beta,m)) = \beta \circ CH_{\dim X+n-m}(X \times Y) \circ \alpha.$$

△

We recall basic properties of $\mathbf{DM}_{gm}(k,\mathbb{Z})$, which are proved in [**27**].

- $\mathbf{DM}_{gm}(k,\mathbb{Z})$ is a rigid tensor triangulated category.
- The functor $(\mathbf{Chow}(k,\mathbb{Z}))^{op} \xrightarrow{\Phi} \mathbf{DM}_{gm}(k,\mathbb{Z})$, which sends (X,α,n) to $\mathbf{im}\ \alpha$ on $M(X)(n)[2n]$ is a fully faithful embedding.
- $\mathbf{DM}_{gm}(k,\mathbb{Z})$ is a full subcategory of compact objects in $\mathbf{DM}^{-}_{Nis}(k,\mathbb{Z})$.

REMARK 3.17. The above construction can also apply to arbitrary coefficient commutative ring R. See [**27**]. In the rest of the paper, we only consider $\mathbf{DM}_{gm}(k,\mathbb{Q})$.

REMARK 3.18. It is shown in Lecture 19 of [**27**] that: for a given smooth variety X, and $p \in \mathbb{Z}, q \in \mathbb{Z}_{\geq 0}$, we have:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M(X),\mathbb{Q}(q)[p]) = CH^q(X, 2q-p).$$

2. Motives for an elliptic curve

In the rest of this chapter, we assume our base field k with characteristic zero. In $\mathbf{DM}_{gm}(k,\mathbb{Q})$, the motive of E will decompose into:

$$M(E) = \mathbb{Q} \oplus M_1(E)[1] \oplus \mathbb{Q}(1)[2].$$

Recall that $\mathbf{DM}^{eff}_{gm}(k,\mathbb{Q})$ is a $\mathbb{Q}$-linear tensor category. Therefore, using Remark 1.38, we have the following decomposition of $M_1(E)^{\otimes n}$ (also in the category of Chow motives):

$$M_1(E)^{\otimes n} \cong \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(M_1(E)). \tag{9}$$

[1]In [**10**] Cisinski and Déglise define triangulated category of mixed motives over a quite general scheme, not only a field.

[2]$\circ$ means the composition of cycles

LEMMA 3.19. *Give E an elliptic curve over k. Then we have $S_\lambda(M_1(E)) = 0$ if $\lambda = (n_1, n_2, \cdots, n_r)$ with $r \geq 3$ and $\wedge^2 M_1(E) = \mathbb{Q}(1)$. In other words, equality (9) can be written as:*

$$M_1(E)^{\otimes n} \cong \bigoplus_{\lambda=(a+b,b),a+2b=n} V_\lambda \otimes Sym^a(M_1(E))(b). \tag{10}$$

PROOF. By Proposition 20.1 in [**27**], we know that the category of effective chow motives embeds contravariantly into $\mathbf{DM}_{gm}^{eff}(k,\mathbb{Q})$. (Recall we denote this functor by Φ.) In the category of chow motives, we have the following decomposition of the chow motive of E:

$$h(E) = h^0(E) \oplus h^1(E) \oplus h^2(E).$$

Notice that the image of $h^1(E)$ under Φ is $M_1(E)[1]$. Using Theorem 4.2 in [**20**], we get:

$$Sym^i h^1(E) = 0 \ \text{ if } \quad i \geq 3.$$

and

$$Sym^2 h^1(E) = \mathbb{L}.$$

Here $\mathbb{L}$ is the Lefschetz motive in the category of chow motives. Recall that the image of $\mathbb{L}$ under Φ is $\mathbb{Q}(1)[2]$(Remark 20.2 in [**27**]). Because Φ is a tensor functor, using commutative constraint in $\mathbf{DM}_{gm}^{eff}(k,\mathbb{Q})$, we have:

$$Sym^i h^1(E) = Sym^i(M_1(E)[1]) = (\wedge^i M_1(E))[i]$$

This implies that:

$$\wedge^i M_1(E) = 0 \quad \text{if} \quad i \geq 3,$$

and

$$\wedge^2 M_1(E) = \mathbb{Q}(1).$$

Given $\lambda = (n_1, n_2, \cdots, n_r)$, by the definition of Young symmetrizer, we know that: $S_\lambda(M_1(E))$ is a direct summand of $\wedge^{m_1} M_1(E) \otimes \cdots \otimes \wedge^{m_s} M_1(E)$, where $(m_1, \cdots, m_s) = \lambda^t$. When $r \geq 3$, then we have $m_1 \geq 3$. By the above computation, we obtain that $S_\lambda(M_1(E)) = 0$. □

DEFINITION 3.20. Given an elliptic curve E, the full idempotent complete rigid tensor triangulated subcategory of $\mathbf{DM}_{gm}(k,\mathbb{Q})$ generated by $M(E)$ is denoted by $\mathbf{DMEM}(k,\mathbb{Q})_E$. △

DEFINITION 3.21. 1) We call the 0-th vanishing property holds for E if :

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i} M_1(E), \mathbb{Q}(i)[j]) \cong 0,$$

for any $j \in \mathbb{Z}_{\leq 0}$ and any $i \in \mathbb{Z}_{>0}$.

2) Let r be a positive integer. We call the r-th vanishing property holds for E if

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r} M_1(E), \mathbb{Q}(i)[j]) \cong 0,$$

for any $j \in \mathbb{Z}$ such that $r + j \leq 0$ and any $i \in \mathbb{Z}_{\geq 0}$.

△

REMARK 3.22. Let r be a non-negative integer. For any $j \in \mathbb{Z}$ such that $r+j>0$, we have:

$$\begin{aligned}
&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E), \mathbb{Q}(i)[j]) \\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E)[2i+r], \mathbb{Q}(i)[2i+r+j]) \\
\subset &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M(E^{2i+r}), \mathbb{Q}(i)[2i+r+j]) \\
\cong &CH^i(E^{2i+r}, -r-j) = 0.
\end{aligned}$$

CONJECTURE 3.23. *If E be an elliptic curve over k without complex multiplication, then E has the r-th vanishing property for any non-negative integer r.*

EXAMPLE 3.24. Assume that E is an elliptic curve without complex multiplication, then we have:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^2 M_1(E), \mathbb{Q}(1)[*]) \cong 0.$$

PROOF. Notice that:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^2 M_1(E), \mathbb{Q}(1)[i]) \cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^2 M_1(E)[2], \mathbb{Q}(1)[i+2])$$

is a direct summand of $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E)[1])^{\otimes 2}, \mathbb{Q}(1)[i+2])$, therefore a direct summand of

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M(E\times E), \mathbb{Q}(1)[i+2]).$$

It's well known that, (for example, chapter 3 in [**27**]):

- When $i=0$, $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M(E\times E), \mathbb{Q}(1)[2]) \cong Pic(E\times E)$;
- When $i=-1$, $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M(E\times E), \mathbb{Q}(1)[1]) \cong k^*$;
- Otherwise, $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M(E\times E), \mathbb{Q}(1)[2+i]) \cong 0$.

Notice that $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}(1)[1]) \cong k^*$ is a direct summand of $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M(E\times E), \mathbb{Q}(1)[1])$, which implies that:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E)[1])^{\otimes 2}, \mathbb{Q}(1)[i]) \cong 0 \quad \text{if} \quad i \neq 0.$$

Then

$$\begin{aligned}
&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M_1(E^2), \mathbb{Q}(1)) \\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^2 M_1(E), \mathbb{Q}(1)) \oplus Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\mathbb{Q}(1), \mathbb{Q}(1)) \qquad (11)\\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^2 M_1(E), \mathbb{Q}(1)) \oplus \mathbb{Q}
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M_1(E^2), \mathbb{Q}(1)) \\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M_1(E), M_1(E)) \qquad (12)\\
\cong &Hom_{Ab_{\mathbb{Q}}}(E, E),
\end{aligned}$$

where $Ab_{\mathbb{Q}}$ is in the category of abelian varieties up to isogeny. For the first isomorphism, we use the facts that the dual motive of $M_1(E)$ is $M_1(E)(-1)$ and the properties of internal hom in $\mathbf{DM}_{gm}(k,\mathbb{Q})$. For the second one, we use the fact that the category of abelian varieties up to isogeny fully embeds into $\mathbf{DM}^{eff}(k,\mathbb{Q})$, for example, Proposition 2.2.1 in [**1**].

Putting together (11) and (12), we get:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^2M_1(E),\mathbb{Q}(1))\oplus\mathbb{Q}\cong Hom_{Ab_{\mathbb{Q}}}(E,E).$$

If E is an elliptic curve without complex multiplication, then we have: $Hom_{Ab_{\mathbb{Q}}}(E,E)\cong\mathbb{Q}$, which implies that:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^2M_1(E),\mathbb{Q}(1))\cong 0.$$

□

△

3. The r-th vanishing properties

In this section, we will discuss the relation between the r-th vanishing properties and other various well-known conjectures such as Bloch-Beilinson-Murre conjectures, the conjecture of the existence of motivic t-structure.

At the beginning, we briefly recall Bloch-Beilinson-Murre conjectures and the conjecture of the existence of motivic t-structure.

For the relevant definitions of B-B-M conjectures, we refer to [**2**]. Fix a Weil cohomology theory H^*. (Defined in Chapter 3 in [**2**].)

CONJECTURE 3.25. ***(Bloch-Beilinson)*** *For every smooth projective variety X over k,*

a) The Künneth projector π_X^i of X associated with H^* are algebraic.
b) For every integer $r\geq 0$, there exists a filtration $(F^\nu CH^r(X)_{\mathbb{Q}})_{\nu\geq 0}$ of $CH^r(X)_{\mathbb{Q}}$ such that:
  1) $F^0CH^r(X)_{\mathbb{Q}}=CH^r(X)_{\mathbb{Q}}$,
  $F^1CH^r(X)_{\mathbb{Q}}=Ker(CH^r(X)_{\mathbb{Q}}\to Z^r_{hom}(X)_{\mathbb{Q}})$.
  2) $F^\mu CH^r(X)_{\mathbb{Q}}\cdot F^\nu CH^r(X)_{\mathbb{Q}}\subset F^{\mu+\nu}CH^r(X)_{\mathbb{Q}}$.
  3) F^ν is stable under direct image and inverse image.
  4) $(\pi_X^i)_{|Gr^\mu_F CH^r(X)_{\mathbb{Q}}}=Id$ if $i=2r-\nu$ and 0 otherwise.
  5) $F^\nu CH^r(X)_{\mathbb{Q}}=0$ for $\nu\gg 0$.

CONJECTURE 3.26. ***(Murre)*** *For every smooth projective variety X over k,*

a) The Künneth projector π_X^i of X associated with H^* are algebraic.
b) There exists a lift Π_X^i of π_X^i in $CH^{\dim X}(X\times X)_{\mathbb{Q}}$ such that:
  1) These Π_X^i are an orthogonal set of idempotents satisfying

$$\sum_i \Pi_X^i=\Delta_X\in CH^{\dim X}(X\times X)_{\mathbb{Q}}.$$

  2) Π_X^i acts as zero on $CH^r(X)_{\mathbb{Q}}$ if $i>2r$.
  3) Let $F^\nu CH^r(X)_{\mathbb{Q}}=\bigcap_{i>2r-\nu}Ker\Pi_X^i\subset CH^r(X)_{\mathbb{Q}}$. Then the filtration $F^\cdot$ is independent of the choices of these lifts Π_X^i.
  4) $F^1CH^r(X)_{\mathbb{Q}}=Ker(CH^r(X)_{\mathbb{Q}}\to Z^r_{hom}(X)_{\mathbb{Q}})$.

REMARK 3.27. Jannsen shows that these two conjectures are equivalent. In addition, two conjectured filtrations coincide. For the status of B-B-M conjectures, we refer to Chapter 11 of [**2**].

Assume B-B-M conjectures are true. Then the chow motive $h(X)$, which is defined by $(X, \Delta, 0)$, decomposes as:

$$h(X) = \bigoplus_i h^i(X),$$

where $h^i(X) = (X, \Pi^i_X, 0)$.

DEFINITION 3.28. A chow motive is called weight i if it is isomorphic to a chow motive of the form $eh^{2r+i}(X)(r)$, where e is an idempotent. △

We cite an important corollary of B-B-M conjectures wihout proof. See Proposition 11.3.5.2 in [**2**].

PROPOSITION 3.29. *Let M, N be two chow motives with the same weight i. Then:*

$$Hom_{\mathbf{Chow}(k,\mathbb{Q})}(M,N) \cong Hom_{\mathbf{M}_{hom}(k,\mathbb{Q})}(M,N),$$

*where $\mathbf{M}_{hom}(k,\mathbb{Q})$ is the category of homological motive. (see Chapter 4 in [**2**]). Let L be another chow motive with the weight larger than i. Then:*

$$Hom_{\mathbf{Chow}(k,\mathbb{Q})}(M,L) = 0.$$

REMARK 3.30. It is well known that, for any abelian variety X over a field of characteristic zero, the homological equivalence is the same as the numerical equivalence. See Chapter 5 in [**2**]. As a corollary, for two chow motives M, N lying the smallst idempotent complete, tensor full subcategory of $\mathbf{Chow}(k,\mathbb{Q})$ generating by the motives of abelian varieties, we have:

$$Hom_{\mathbf{M}_{hom}(k,\mathbb{Q})}(M,N) \cong Hom_{\mathbf{NM}(k,\mathbb{Q})}(M,N),$$

where $\mathbf{NM}(k,\mathbb{Q})$ is the category of numerical motives with rational coefficients. For the definition of numerical motives, we refer to Chapter 4 in [**2**].

PROPOSITION 3.31. *Let E be an elliptic curve without complex multiplication over a field of characterstic zero. Assume that B-B-M conjectures hold for $E, E^2, E^3, \cdots$. Then we have:*

a) $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i}M_1(E), \mathbb{Q}(i)) \cong 0$ when $r = 0, i \in \mathbb{Z}_{>0}$;
b) $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E), \mathbb{Q}(i)[-r]) \cong 0$ when $r \in \mathbb{Z}_{>0}, i \in \mathbb{Z}_{\geq 0}$

PROOF. We recall that $\Phi : (\mathbf{Chow}(k,\mathbb{Q}))^{op} \to \mathbf{DM}_{gm}(k,\mathbb{Q})$ is a fully faithful embedding, which sends $h^1(E)$ to $M_1(E)[1]$ and $\mathbb{L}$ (Lefschetz object) to $\mathbb{Q}(1)[2]$.

For case a), we have:

$$\begin{aligned}
&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i}M_1(E),\mathbb{Q}(i))\\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E)[2i],\mathbb{Q}(i)[2i])\\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\Phi(\wedge^{2i}h^1(E)),\Phi(\mathbb{L}^{\otimes i}))\\
\cong &Hom_{(Chow(k,\mathbb{Q}))^{op}}(\wedge^{2i}h^1(E),\mathbb{L}^{\otimes i})\\
\cong &Hom_{Chow(k,\mathbb{Q})}(\mathbb{L}^{\otimes i},\wedge^{2i}h^1(E))\\
\cong &Hom_{M_{hom}(k,\mathbb{Q})}(\mathbb{L}^{\otimes i},\wedge^{2i}h^1(E)) \quad \textbf{Proposition } 3.29\\
\cong &Hom_{NM(k,\mathbb{Q})}(\mathbb{L}^{\otimes i},\wedge^{2i}h^1(E)) \quad \textbf{Remark } 3.30
\end{aligned}$$

Notice that the motivic Galois group of an elliptic curve without complex multiplication is GL_2. See Example 7.6.4.1 in Chapter 7 of [**2**]. Roughly speaking, this means that $\mathbf{NM}(k,\mathbb{Q})$, after changing the commutative constraint, is the same as the category of rational representations of GL_2. Under this identification, $\mathbb{L}^{\otimes i}$ (resp. $\wedge^{2i}h^1(E)$) corresponds to the GL_2 representation det^i (resp. $Sym^{2i}\mathbf{F}$), which implies that:

$$Hom_{NM(k,\mathbb{Q})}(\mathbb{L}^{\otimes i},\wedge^{2i}h^1(E)) \cong Hom_{GL_2}(det^i, Sym^{2i}\mathbf{F}) \cong 0.$$

For case b), we have:

$$\begin{aligned}
&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E),\mathbb{Q}(i)[-r])\\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E)[2i+r],\mathbb{Q}(i)[2i])\\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\Phi(\wedge^{2i+r}h^1(E)),\Phi(\mathbb{L}^{\otimes i}))\\
\cong &Hom_{(\mathbf{Chow}(k,\mathbb{Q}))^{op}}(\wedge^{2i+r}h^1(E),\mathbb{L}^{\otimes i})\\
\cong &Hom_{\mathbf{Chow}(k,\mathbb{Q})}(\mathbb{L}^{\otimes i},\wedge^{2i+r}h^1(E))\\
\cong &Hom_{\mathbf{Chow}(k,\mathbb{Q})}(\mathbb{Q},\wedge^{2i+r}h^1(E)(i)).
\end{aligned}$$

Using Proposition 3.29 and $r \in \mathbb{Z}_{>0}$, we obtain that:

$$Hom_{\mathbf{Chow}(k,\mathbb{Q})}(\mathbb{Q},\wedge^{2i+r}h^1(E)(i)) \cong 0.$$

□

REMARK 3.32. The above proof of case b) works for any elliptic curve over a base field of characteristic zero.

CONJECTURE 3.33. ***(The existence of motivic** t**-structure)*** *There is a suitable* t*-structure over* $\mathbf{DM}_{gm}(k,\mathbb{Q})$*, which is called motivic* t*-structure, such that the following diagram commutes:*

$$\begin{array}{ccc}
\mathbf{Chow}(k,\mathbb{Q}) & \xrightarrow{\Phi} & \mathbf{DM}_{gm}(k,\mathbb{Q})\\
\downarrow & & \downarrow\\
\mathbf{NM}(k,\mathbb{Q}) & \longrightarrow & \mathbf{MM}(k,\mathbb{Q})
\end{array}$$

where $\mathbf{MM}(k, \mathbb{Q})$ *denote the heart of motivic* t*-structure.*

A special case is the following:

CONJECTURE 3.34. ***(The existence of motivic*** t***-structure for*** $\mathbf{DMEM}(k, \mathbb{Q})_E$***)*** *Given an elliptic curve curve* E *over* k*, there is a suitable* t*-structure over* $\mathbf{DMEM}(k, \mathbb{Q})_E$*, whose heart is denoted by* $\mathbf{MEM}(k, \mathbb{Q})_E$*.*

REMARK 3.35. For more information about the conjecture of the existence of motivic t-structure, we refer to Chapter 21 in [**2**]. We only mention that the conjecture of the existence of motivic t-structure implies B-B-M conjectures. See Section 21.3 in [**2**].

LEMMA 3.36. *Let* E *be an elliptic curve over* k*. Assume that the conjecture of the existence of motivic* t*-structure for* $\mathbf{DMEM}(k, \mathbb{Q})_E$ *is true. Then we have:*

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E), \mathbb{Q}(i)[j]) \cong 0$$

for any $i, r \in \mathbb{Z}_{\geq 0}$ *and* $j \in \mathbb{Z}$ *such that* $j + r < 0$*.*

PROOF. Firstly, notice that $Sym^{2i+r}M_1(E)$ and $\mathbb{Q}(i)$ are in the heart of motivic t-structure. Because there is no negative extension in abelian categories and $j < -r \leq 0$, we have:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{2i+r}M_1(E), \mathbb{Q}(i)[j]) \cong 0.$$

□

COROLLARY 3.37. *Let* E *be an elliptic curve without complex multiplication over* k*. The conjecture of the existence of motivic* t*-structure implies that all the* r*-th vanishing properties hold for* E *for any non-negative integer* r*, i.e., Conjecture 3.23 holds.*

PROOF. This is a combination of Lemma 3.31, Lemma 3.36 and Remark 3.35. □

CHAPTER 4

CYCLE ALGEBRAS

1. Suslin - Friedlander complexes

Let k be any field in the first two sections.

DEFINITION 4.1. Take Y in $\mathbf{Sm}_k$ and X in $\mathbf{Sch}_k$. Let $z_{q.fin}(X)(Y)$ be the subgroup of $z_*(Y \times_k X)$ generated by integral closed subschemes $W \subset Y \times_k X$ such that $p_1 : W \longrightarrow Y$ is quasi-finite and dominant over an irreducible component of Y. $\triangle$

REMARK 4.2. $z_{q.fin}(X)$ is a Nisnevich sheaf with transfers. See Lecture 16 in [**27**].

DEFINITION 4.3. Define the cosimplicial scheme Δ^* over k by:

$$\Delta^n = \mathbf{Spec}\ k[x_0, \cdots, x_n]/(\sum_{i=0}^{n} x_i - 1),$$

and whose j-th face map $\partial_j : \Delta^n \to \Delta^{n+1}$ is given by the equation $x_j = 0$. $\triangle$

DEFINITION 4.4. Let $\mathcal{F}$ be a presheaf of abelian groups on $\mathbf{Sm}_k$. For $Y \in \mathbf{Sm}_k$. We define a simplical presheaf $C_\bullet(\mathcal{F})$ sending Y to the simplical abelian groups $\mathcal{F}(Y \times \Delta^*)$. Let $C_*(\mathcal{F})$ be the complex of presheaves with transfer associated to $C_\bullet(\mathcal{F})$. $\triangle$

REMARK 4.5. If F is a presheaf with transfers, then $C_*(\mathcal{F})$ is a complex of presheaves with transfers. See lecture 2 in [**27**].

DEFINITION 4.6. For any $i \in \mathbb{Z}$, we define the Suslin-Fridlander complexes $\mathbb{Z}^{SF}(i)$ by:

$$\mathbb{Z}^{SF}(i) = C_* z_{q.fin}(\mathbb{A}^i)[-2i].$$

This is a complex of Nisnevich sheaves with transfers on $\mathbf{Sm}_k$. $\triangle$

We recall an important result, which is a special case of Lemma 19.4 in the Lecture 19 of [**27**].

LEMMA 4.7. *For all $X \in \mathbf{Sch}_k$, we have an isomorphism in $D(\mathbf{Ab})$:*

$$\mathbb{Z}^{SF}(i)[2i](X) \cong z^i(X, *),$$

where $z^i(X, *)$ is Bloch's higher Chow group complex.

2. Cubical version

Let $(\square^1, \partial\square^1)$ denote the pair $(\mathbb{A}^1, \{0, 1\})$. Let $(\square^n, \partial\square^n)$ denote the pair of $\mathbb{A}^n$ together with a divisor $\partial\square^n$ of $\mathbb{A}^n$ which is given by $\sum_{i=1}^n (x_i = 0) + \sum_{i=1}^n (x_i = 1)$, where $x_1 \cdots x_n$

are the coordinates on $\mathbb{A}^n$. A face of $\square^n$ is a subscheme defined by equations of the form $x_{i_1} = \delta_1, \cdots, x_{i_s} = \delta_s$ with all the $\delta_j \in \{0, 1\}$.

Definition 4.8. Let X be in $\mathbf{Sch}_k$. Let $\underline{z}^q(X, n)^{cb}$ be the free abelian group on the codimension q subvarieties $W \subset X \times \square^n$ such that $W \cap X \times F$ has codimension q for every face $F \subset \square^n$. △

For $\varepsilon \in \{0, 1\}$ and $j \in \{1, \cdots, n\}$, we let $\iota_{j,\varepsilon} : \square^{n-1} \longrightarrow \square^n$ be the closed embedding defined by inserting an ε in the j-th coordinate. Let $\pi_j : \square^n \longrightarrow \square^{n-1}$ be the projection which omits the j-th coordinate. Then the pull-back maps $\iota_{j,\varepsilon}^* : \underline{z}^q(X, n)^{cb} \longrightarrow \underline{z}^q(X, n-1)^{cb}$ and $\pi_j^* : \underline{z}^q(X, n-1)^{cb} \longrightarrow \underline{z}^q(X, n)^{cb}$ are well-defined.

Recall the definition of the cubical version of Bloch's cycle complex.

Definition 4.9. Let $z^q(X, n)^{cb}$ be the quotient of $\underline{z}^q(X, n)^{cb}$ by the degenerate cycles. Explicitly,

$$z^q(X, n)^{cb} = \underline{z}^q(X, n)^{cb} / \sum_{j=1}^{n} \pi_j^*(\underline{z}^q(X, n-1)^{cb}).$$

△

Note the symmetric group Σ_n acts on $\square^n$ by permutation.

Definition 4.10.

$$z^q(X, n)^{Alt} = \{x \in z^q(X, n)^{cb} \otimes \mathbb{Q} |\ (id \times \sigma)^*(x) = sgn(\sigma) \cdot x \text{ for all } \sigma \in \Sigma_n\}.$$

△

In fact, we can also define $C_*^{cb}(\mathcal{F})$ and $C_*^{Alt}(\mathcal{F})$ for $\mathcal{F}$ a presheaf over $\mathbf{Sm}_k$.

Definition 4.11. Let $X \in \mathbf{Sm}_k$ and $\mathcal{F}$ as above. Let $C_n^{cb}(\mathcal{F})$ be the presheaf

$$C_n^{cb}(\mathcal{F})(X) = \mathcal{F}(X \times \square^n) / \sum_{j=1}^{n} \pi_j^*(\mathcal{F}(X \times \square^{n-1})).$$

and the differential is given by:

$$d_n = \sum_{j=1}^{n} (-1)^{j-1} \mathcal{F}(\iota_{j,1}) - \sum_{j=1}^{n} (-1)^{j-1} \mathcal{F}(\iota_{j,0}).$$

△

If $\mathcal{F}$ is a Nisnevich prsheaf (sheaf, with transfers), then $C_*^{cb}(\mathcal{F})$ is a complex of Nisnevich prsheaves (sheaves, with transfers). One can extend the construction of complexes of sheaves (with transfers) by taking the total complex. We can define $C_*^{Alt}(\mathcal{F})$ as a subcomplex of $C_*^{cb}(\mathcal{F}) \otimes \mathbb{Q}$ by taking the alternating element in $C_*^{cb}(\mathcal{F})(Y)$ as in Definition 4.10 for every $Y \in \mathbf{Sm}_k$.

Remark 4.12. There is another definition of the alternating complex without taking the quotient by the degenerate cycles. See [**22**].

Some comparison results can be found in [**22**], which is:

THEOREM 4.13. *Let $\mathcal{F}$ be a complex of presheaves on $\mathbf{Sm}_k$.*

- *There is a natural isomorphism $C_*(\mathcal{F}) \cong C_*^{cb}(\mathcal{F})$ in the derived category of presheaves on $\mathbf{Sm}_k$. If $\mathcal{F}$ is a complex of presheaves with transfer, there is also an isomorphism $C_*(\mathcal{F}) \cong C_*^{cb}(\mathcal{F})$ in the derived category of presheaves with transfers $D(\mathbf{PST})$.*
- *The inclusion $C_*^{Alt}(\mathcal{F})(Y) \subset C_*^{cb}(\mathcal{F})_{\mathbb{Q}}(Y)$ is a quasi-isomorphism for all $Y \in \mathbf{Sm}_k$.*

3. The cycle algebra for an elliptic curve

Let a be a positive integer. We denote the a-th power of E by E^a. Recall E is an elliptic curve defined over a base field k of characteristic zero without complex multiplication.

DEFINITION 4.14. The sign character $\mathbf{sgn} : \mathbb{Z}/2\mathbb{Z} \to \{\pm 1\}$ extends to the map

$$\rho : (\mathbb{Z}/2\mathbb{Z})^a \to \{\pm 1\}^a$$

by

$$\rho(\eta_1, \cdots, \eta_a) = \{\mathbf{sgn}(\eta_1), \cdots, \mathbf{sgn}(\eta_a)\}$$

for $(\eta_1, \cdots, \eta_a) \in (\mathbb{Z}/2\mathbb{Z})^a$. △

The group $\Gamma_n = (\mathbb{Z}/2\mathbb{Z})^a \rtimes \Sigma_a$ act on E^a in the following way: Σ_a permutes the components of E^a and the i-th generator $(0, \cdots, 1, \cdots, 0)$ in $(\mathbb{Z}/2\mathbb{Z})^a$ acts on the i-th component E of E^a by the inversion, i.e., $x \xrightarrow{\sigma_i} -x$.

For $i \in \mathbb{Z}$, we define a subgroup of $C_i^{Alt}(z_{q.fin}(\mathbb{A}^b))(E^a)$:

$$\begin{aligned} & C_i^{Alt,-}(z_{q.fin}(\mathbb{A}^b))(E^a) \\ = \ & \{Z \in C_i^{Alt}(z_{q.fin}(\mathbb{A}^b))(E^a) | g(Z) = \rho(g) Z \ \forall g \in (\mathbb{Z}/2\mathbb{Z})^a\}. \end{aligned}$$

We denote the corresponding cycle complex by $C_*^{Alt,-}(z_{q.fin}(\mathbb{A}^b))(E^a)$.

Given $\sigma \in \mathbb{Q}[\Sigma_a]$, define

$$Z \bullet \sigma = sgn(\sigma)\sigma^{-1}(Z)$$

for $Z \in C_i^{Alt,-}(z_{q.fin}(\mathbb{A}^b))(E^a)$. This makes $C_i^{Alt,-}(z_{q.fin}(\mathbb{A}^b))(E^a)$ into a right $\mathbb{Q}[\Sigma_a]$-module. We also have the action of the symmetric group Σ_b on $C_i^{Alt,-}(z_{q.fin}(\mathbb{A}^b))(E^a)$, by permuting the coordinates of $\mathbb{A}^b$. Taking the symmetric sections with respect to the action of Σ_b, we get a sub-complex $\widetilde{C}_i^{Alt,-}(z_{q.fin}(\mathbb{A}^b))(E^a)$ of $C_i^{Alt,-}(z_{q.fin}(\mathbb{A}^b))(E^a)$.

Notation: For $i < 0$, $V^{\otimes i} = (V^\vee)^{\otimes -i}$, where $V^\vee$ is the dual representation of V. We also use this notation for motives.

DEFINITION 4.15. Let a, b be integers such that $a \geq b, a \geq 0$. For $i \in \mathbb{Z}$, we define:

$$\mathcal{E}_{a,b}^i = \widetilde{C}_{a-2b-i}^{Alt,-}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a) \otimes_{\mathbb{Q}[\Sigma_a]} \mathbf{F}^{\otimes a}(b-a).$$

Here $\mathbf{F}$ is the fundamental representation of GL_2 and $\mathbf{F}^{\otimes a}(b-a) = \mathbf{F}^{\otimes a} \otimes det^{\otimes b-a}$. △

REMARK 4.16. We collect some facts for the later use.

(1) Using the external product of cycles, we define a map:

$$\widetilde{C}^{Alt,-}_{a-2b-i_1}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a) \otimes_{\mathbb{Q}} \widetilde{C}^{Alt,-}_{c-2d-i_2}(z_{q.fin}(\mathbb{A}^{c-d}))(E^c)$$
$$\xrightarrow{\boxtimes} \widetilde{C}^{Alt,-}_{a-2b+c-2d-i_1-i_2}(z_{q.fin}(\mathbb{A}^{a-b+c-d}))(E^{a+c}),$$

which sends $Z_1 \otimes Z_2$ to $(-1)^{c(a-2b-i_1)} m(Z_1 \times Z_2)$. Here m is the map

$$E^a \times \mathbb{A}^{a-b} \times \square^{a-2b-i_1} \times E^c \times \mathbb{A}^{c-d} \times \square^{c-2d-i_2}$$
$$\to E^{a+c} \times \mathbb{A}^{a-b+c-d} \times \square^{a-2b+c-2d-i_1-i_2},$$

changing the positions of the factors.

(2) We have the map of GL_2 representations: $\mathbf{F}^a \otimes \mathbf{F}^c \to \mathbf{F}^{a+c}$.

In the following, we wnat to define a product map $\mathcal{E}^*_{a,b} \otimes \mathcal{E}^*_{c,d} \to \mathcal{E}^*_{a+c,b+d}$. For simplicity, we denote $\widetilde{C}^{Alt,-}_{a-2b-i}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a)$ by $C^i_{a,b}$.

Then we have:

$$(C^i_{a,b} \otimes_{\mathbb{Q}[\Sigma_a]} \mathbf{F}^a(b-a)) \otimes_{\mathbb{Q}} (C^j_{c,d} \otimes_{\mathbb{Q}[\Sigma_c]} \mathbf{F}^c(d-c))$$
$$= (C^i_{a,b} \otimes_{\mathbb{Q}} C^j_{c,d}) \otimes_{\mathbb{Q}[\Sigma_a \times \Sigma_c]} (\mathbf{F}^a(b-a) \otimes_{\mathbb{Q}} \mathbf{F}^c(d-c)).$$

Using the external product of cycles and GL_2-representations (see Remark 4.16), we have a map:

$$(C^i_{a,b} \otimes_{\mathbb{Q}} C^j_{c,d}) \otimes_{\mathbb{Q}[\Sigma_a \times \Sigma_c]} (\mathbf{F}^a(b-a) \otimes_{\mathbb{Q}} \mathbf{F}^c(d-c))$$
$$\to C^{i+j}_{a+c,b+d} \otimes_{\mathbb{Q}[\Sigma_a \times \Sigma_c]} \mathbf{F}^{a+c}(b-a+d-c).$$

The injection of groups $\Sigma_a \times \Sigma_c \to \Sigma_{a+c}$ induces a map $\mathbb{Q}[\Sigma_a \times \Sigma_c] \to \mathbb{Q}[\Sigma_{a+c}]$. Note that both $C^{i+j}_{a+c,b+d}$ and $\mathbf{F}^{a+c}(b-a+d-c)$ are $\mathbb{Q}[\Sigma_{a+c}]$ modules, and their $\mathbb{Q}[\Sigma_{a+c}]$ modules structures are compatible with their $\mathbb{Q}[\Sigma_a \times \Sigma_c]$ module structure coming from the respective external products. This tells us that there is a map:

$$C^{i+j}_{a+c,b+d} \otimes_{\mathbb{Q}[\Sigma_a \times \Sigma_c]} \mathbf{F}^{a+c}(b-a+d-c) \to C^{i+j}_{a+c,b+d} \otimes_{\mathbb{Q}[\Sigma_{a+c}]} \mathbf{F}^{a+c}(b-a+d-c).$$

Putting these maps together, we get a map:

$$(C^i_{a,b} \otimes_{\mathbb{Q}[\Sigma_a]} \mathbf{F}^a(b-a)) \otimes_{\mathbb{Q}} (C^j_{c,d} \otimes_{\mathbb{Q}[\Sigma_c]} \mathbf{F}^c(d-c))$$
$$\xrightarrow{\mu^{a,b}_{c,d}} C^{i+j}_{a+c,b+d} \otimes_{\mathbb{Q}[\Sigma_{a+c}]} \mathbf{F}^{a+c}(b-a+d-c). \tag{13}$$

REMARK 4.17. We will use the following identification in the next lemma.

Let G be a finite group and V (resp. W) be a right (resp. left) $\mathbb{Q}[G]$ module, then:

$$V \otimes_{\mathbb{Q}[G]} W = (V \otimes_{\mathbb{Q}} W)_G,$$

The right hand means the following: The right $\mathbb{Q}[G]$ module V can be considered as a left module, $g \bullet v \doteq v \bullet g^{-1}$. $V \otimes_{\mathbb{Q}} W$ is considered as a left module. Then take the G co-invariant part.

LEMMA 4.18. *The product structure defined in (13) is associative and graded commutative. More precisely,* $(-1)^{ij}\mu^{c,d}_{a,b} \circ \tau = \mu^{a,b}_{c,d}$, *where* τ *is the map* $\mathcal{E}^i_{a,b} \otimes \mathcal{E}^j_{c,d} \xrightarrow{\tau} \mathcal{E}^j_{c,d} \otimes \mathcal{E}^i_{a,b}$ *changing two factors.*

PROOF. For the associativity part, it's the direct result of the associativity of external products of cycles and representations and compatibility of actions of symmetric groups.

For the commutativity part, we need to check that $(-1)^{ij}\mu_{a,b}^{c,d}\circ\tau = \mu_{c,d}^{a,b}$. Take $Z_1 \otimes W_1$, where $Z_1 \in C_{a,b}^i$, $W_1 \in V^a(b-a)$. Similarly take $Z_2 \otimes W_2$, where $Z_2 \in C_{c,d}^j$, $W_2 \in V^c(d-c)$. Set:

$$\sigma = \begin{pmatrix} 1 & \cdots & a & a+1 & \cdots & a+c \\ a+1 & \cdots & a+c & 1 & \cdots & a \end{pmatrix} \in \mathbb{Q}[\Sigma_{a+c}].$$

Let δ act on $\mathbb{A}^{a-b}\times\square^{a-2b-i}\times\mathbb{A}^{c-d}\times\square^{c-2d-j}$ by permuting $\mathbb{A}^{a-b}$ and $\mathbb{A}^{c-d}$, and permuting $\square^{a-2b-i}$ and $\square^{c-2d-j}$.

Also use $\boxtimes$ to denote the external product of modules.

Then, in $C_{a+c,b+d}^{i+j}\otimes_{\mathbb{Q}}\mathbf{F}^{a+c}(b-a+d-c)$, we have:

$$\begin{aligned}
&\delta\sigma((Z_1\boxtimes Z_2)\otimes_{\mathbb{Q}}(W_1\boxtimes W_2))\\
&= \delta(\sigma(Z_1\boxtimes Z_2))\otimes_{\mathbb{Q}}(\sigma(W_1\boxtimes W_2))\\
&= (-1)^{c(a-i)+a(c-j)+(a-i)(c-j)}(Z_2\boxtimes Z_1)\otimes_{\mathbb{Q}}(W_2\boxtimes W_1)\\
&= (-1)^{c(a-i)+a(c-j)+(a-i)(c-j)}(Z_2\boxtimes Z_1)\otimes_{\mathbb{Q}}(W_2\boxtimes W_1)\\
&= (-1)^{ac+ij}(Z_2\boxtimes Z_1)\otimes_{\mathbb{Q}}(W_2\boxtimes W_1).
\end{aligned}$$

Here we use $\delta(Z) = sgn(\delta)Z$. This implies that the image of

$$\mu_{c,d}^{a,b}((Z_1\otimes W_1)\otimes_{\mathbb{Q}}(Z_2\otimes W_2)) - (-1)^{ij}(\mu_{a,b}^{c,d}\circ\tau)((Z_1\otimes W_1)\otimes_{\mathbb{Q}}(Z_2\otimes W_2))$$

in $\mathcal{E}_{a+c,b+d}^{i+j}$ is the same as

$$\mu_{c,d}^{a,b}((Z_1\otimes W_1)\otimes_{\mathbb{Q}}(Z_2\otimes W_2)) - (-1)^{ac}\sigma(\mu_{a,b}^{c,d}((Z_1\otimes W_1)\otimes_{\mathbb{Q}}(Z_2\otimes W_2))),$$

i.e.,

$$\mu_{c,d}^{a,b}((Z_1\otimes W_1)\otimes_{\mathbb{Q}}(Z_2\otimes W_2)) - \sigma\bullet(\mu_{a,b}^{c,d}((Z_1\otimes W_1)\otimes_{\mathbb{Q}}(Z_2\otimes W_2))),$$

which is zero in $\mathcal{E}_{a+c,b+d}^{i+j}$. This implies the graded commutativity. □

For simplicity, we denote the multiplication $\mu_{c,d}^{a,b}$ by $\bullet$.

LEMMA 4.19. *Given* $Z_1\otimes W_1 \in C_{a,b}^i\otimes_{\mathbb{Q}[\Sigma_a]}\mathbf{F}^a(b-a)$, $Z_2\otimes W_2\in C_{c,d}^j\otimes_{\mathbb{Q}[\Sigma_b]}\mathbf{F}^c(d-c)$, *then we have:*

$$\begin{aligned}
&d_{a+c,b+d}^{i+j}((Z_1\otimes W_1)\bullet(Z_2\otimes W_2))\\
&= (d_{a,b}^i(Z_1\otimes W_1))\bullet(Z_2\otimes W_2) + (-1)^i(Z_1\otimes W_1)\bullet(d_{c,d}^j(Z_2\otimes W_2)),
\end{aligned}$$

where $d_{a,b}^i$ *is the map* $C_{a,b}^i\otimes_{\mathbb{Q}[\Sigma_a]}\mathbf{F}^a(b-a)\xrightarrow{d\otimes id}C_{a,b}^{i+1}\otimes_{\mathbb{Q}[\Sigma_a]}\mathbf{F}^a(b-a)$.

PROOF. By the definition of $\bullet$ and differentials, we only need to check the above equality for cycle parts, i.e,

$$d_{a+c,b+d}^{i+j}(Z_1\boxtimes Z_2) = d_{a,b}^i(Z_1)\boxtimes Z_2 + (-1)^i Z_1\boxtimes d_{c,d}^j(Z_2).$$

We use $\cap$ to denote the cycle intersection.

$$\begin{aligned}
&d_{i+j}^{a+c,b+d}(Z_1 \boxtimes Z_2) \\
=&\sum_{k=1}^{i+j}(-1)^{k-1}(Z_1 \boxtimes Z_2) \cap (E^{a+c} \times \mathbb{A}^{a-b+c-d} \times \square \times \cdots \times \{1\} \cdots \times \square) \\
-&\sum_{k=1}^{i+j}(-1)^{k-1}(Z_1 \boxtimes Z_2) \cap (E^{a+c} \times \mathbb{A}^{a-b+c-d} \times \square \times \cdots \times \{0\} \cdots \times \square)
\end{aligned}$$

module degenerate cycles, where 0 and 1 is the point in $\square$ putting in the k-th place. Similarly,

$$\begin{aligned}
&d_{a,b}^{i}(Z_1) \boxtimes Z_2 + (-1)^i Z_1 \boxtimes d_{c,d}^{j}(Z_2) \\
=&\sum_{l=1}^{i}(-1)^{l-1}(Z_1 \cap (E^a \times \mathbb{A}^{a-b} \times \square \times \cdots \times \{1\} \cdots \times \square)) \boxtimes Z_2 \\
-&\sum_{l=1}^{i}(-1)^{l-1}(Z_1 \cap (E^a \times \mathbb{A}^{a-b} \times \square \times \cdots \times \{0\} \cdots \times \square)) \boxtimes Z_2 \\
+&(-1)^i\sum_{m=1}^{j}(-1)^{m-1}(Z_1 \boxtimes (Z_2 \cap E^c \times \mathbb{A}^{c-d} \times \square \times \cdots \times \{1\} \cdots \times \square)) \\
-&(-1)^i\sum_{m=1}^{j}(-1)^{m-1}(Z_1 \boxtimes (Z_2 \cap E^c \times \mathbb{A}^{c-d} \times \square \times \cdots \times \{0\} \cdots \times \square)) \\
=&\sum_{l=1}^{i}(-1)^{l-1}(Z_1 \cap (E^a \times \mathbb{A}^{a-b} \times \square \times \cdots \times \{1\} \cdots \times \square)) \boxtimes Z_2 \\
+&\sum_{m=1}^{j}(-1)^{i+m-1}(Z_1 \boxtimes (Z_2 \cap E^c \times \mathbb{A}^{c-d} \times \square \times \cdots \times \{1\} \cdots \times \square)) \\
-&\sum_{l=1}^{i}(-1)^{l-1}(Z_1 \cap (E^a \times \mathbb{A}^{a-b} \times \square \times \cdots \times \{0\} \cdots \times \square)) \boxtimes Z_2 \\
-&\sum_{m=1}^{j}(-1)^{i+m-1}(Z_1 \boxtimes (Z_2 \cap E^c \times \mathbb{A}^{c-d} \times \square \times \cdots \times \{0\} \cdots \times \square)) \\
=&\sum_{k=1}^{i+j}(-1)^{k-1}(Z_1 \boxtimes Z_2) \cap (E^{a+c} \times \mathbb{A}^{a-b+c-d} \times \square \times \cdots \times \{1\} \cdots \times \square) \\
-&\sum_{k=1}^{i+j}(-1)^{k-1}(Z_1 \boxtimes Z_2) \cap (E^{a+c} \times \mathbb{A}^{a-b+c-d} \times \square \times \cdots \times \{0\} \cdots \times \square)
\end{aligned}$$

module degenerate cycles. Therefore, we obtain that:

$$d^{i+j}_{a+c,b+d}(Z_1 \boxtimes Z_2) = d^i_{a,b}(Z_1) \boxtimes Z_2 + (-1)^i Z_1 \boxtimes d^j_{c,d}(Z_2).$$

□

By lemma 4.18 and lemma 4.19, our products

$$\mathcal{E}^*_{a,b} \otimes \mathcal{E}^*_{c,d} \to \mathcal{E}^*_{a+c,b+d}$$

give $\oplus_{a\geq b\geq 0}\mathcal{E}^*_{a,b}$ the structure of a bi-graded commutative differential graded algebra in GL_2-representations.

EXAMPLE 4.20. Let us use Example 3.24 to compute $\mathcal{E}^*_{2,1}$. By our definition, we have:

$$\mathcal{E}^*_{2,1} = \widetilde{C}^{Alt,-}_{-*}(z_{q.fin}(\mathbb{A}^1))(E^2) \otimes_{\mathbb{Q}[\Sigma_2]} \mathbf{F}^{\otimes 2}(-1).$$

Notice that as a GL_2 representation, $V^{\otimes 2}(-1)$ decomposes as the direct sum of $Sym^2(-1)$ and $\mathbb{Q}$, both factors with multiplicity one. Computing the corresponding cycle complexes, we get:

$$\mathcal{E}^*_{2,1} = (\widetilde{C}^{Alt,-}_{-*}(z_{q.fin}(\mathbb{A}^1))(E^2))^{sym} \otimes_{\mathbb{Q}} Sym^2\mathbf{F}(-1)$$
$$\oplus(\widetilde{C}^{Alt,-}_{-*}(z_{q.fin}(\mathbb{A}^1))(E^2))^{alt} \otimes_{\mathbb{Q}} \mathbb{Q}.$$

Using Example 3.24, we obtain that the first term of right hand side is quasi-isomorphic to zero. Similarly the second term is quasi-isomorphic to the trivial GL_2 representation, generated by a cycle of codimension one in E^2. If we denote the diagonal (resp. anti-diagonal) of $E \times E$ by Δ^+ (resp. Δ^-), then we can take this generator to be the cycle $\frac{1}{2}(\Delta^+ - \Delta^-)$.

Define a map:

$$\mathcal{E}^*_{a,b} \xrightarrow{\eta} \mathcal{E}^*_{a+2,b+1} \tag{14}$$

by mapping $Z\otimes_{\mathbb{Q}} W \in C^i_{a,b}\otimes_{\Sigma_a}\mathbf{F}^a(b-a)$ to $(Z\times\frac{1}{2}(\Delta^+ - \Delta^-))\otimes_{\mathbb{Q}}\phi(W)$, where ϕ is the composition of maps between GL_2 representations $\mathbf{F}^a(b-a) \to \mathbf{F}^a(b-a)\otimes_{\mathbb{Q}}\mathbf{F}^2(-1) \xrightarrow{\cong} \mathbf{F}^{a+2}(b-a-1)$. The first map is defined in the following way. Because $\mathbf{F}^2(-1) \cong Sym^2\mathbf{F}(-1) \oplus \mathbb{Q}$ as GL_2 representations, we have a natural injective map:

$$\mathbf{F}^a(b-a) \cong \mathbf{F}^a(b-a) \otimes \mathbb{Q} \to \mathbf{F}^a(b-a) \otimes \mathbf{F}^2(-1),$$

sending $1 \in \mathbb{Q}$ to $1 \in \mathbb{Q} \subset \mathbf{F}^2(-1)$.

REMARK 4.21. Using the computation in Example 4.20, the above definition is just the composition of maps:

$$\mathcal{E}^*_{a,b} \to \mathcal{E}^*_{a,b} \otimes \mathcal{E}^*_{2,1} \xrightarrow{\bullet} \mathcal{E}^*_{a+2,b+1}.$$

DEFINITION 4.22. For $a \in \mathbb{Z}$, we define:

$$\mathcal{E}^*_a = \varinjlim_{i\geq -a} \mathcal{E}^*_{-a+2i,-a+i}.$$

△

REMARK 4.23. As GL_2 representations, every term of the complex $\mathcal{E}^*_a$ has pure Adams weight a. The reason for the process of taking colimit is to kill the infinite repeated information. Notice each irreducible GL_2 representation appear infinite times for the representation part of $\{\mathcal{E}^{a,b}\}$, which take the same cycle complexes. We will see these facts later in Corollary 4.32.

DEFINITION 4.24. Define:

$$\mathcal{E}^* = \mathbb{Q} \oplus \bigoplus_{a \geq 1} \mathcal{E}^*_a.$$

and

$$\mathcal{E}^*_{ell} = \bigoplus_{a \in \mathbb{Z}} \mathcal{E}^*_a.$$

△

REMARK 4.25. The products on $\mathcal{E}^*_{a,b}$ descend to products on $\mathcal{E}^*$ and $\mathcal{E}^*_{ell}$. By the construction of the multiplication map, we have:

$$\mathcal{E}^*_{a+2i,\ i} \otimes \mathcal{E}^*_{b+2j,\ j} \to \mathcal{E}^*_{a+b+2i+2j,\ i+j},$$

and the commutative diagram

$$\begin{array}{ccc} \mathcal{E}^*_{a+2i,\ i} \otimes \mathcal{E}^*_{b+2j,\ j} & \longrightarrow & \mathcal{E}^*_{a+b+2i+2j,\ i+j} \\ \downarrow{\scriptstyle \eta \otimes id} & & \downarrow{\scriptstyle \eta} \\ \mathcal{E}^*_{a+2i+2,\ i+1} \otimes \mathcal{E}^*_{b+2j,\ j} & \longrightarrow & \mathcal{E}^*_{a+b+2i+2j+2,\ i+j+1} \end{array}$$

Fix b, j. Using these diagrams for i varying, we get a map:

$$\mathcal{E}^*_a \otimes \mathcal{E}^*_{b+2j,\ j} \to \mathcal{E}^*_{a+b}.$$

We also have the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{E}^*_a \otimes \mathcal{E}^*_{b+2j,\ j} & \longrightarrow & \mathcal{E}^*_b \\ \downarrow{\scriptstyle id \otimes \eta} & \nearrow & \\ \mathcal{E}^*_a \otimes \mathcal{E}^*_{b+2j+2,\ j+1} & & \end{array}$$

and

$$\begin{array}{ccc} \mathcal{E}^*_{a+2i,\ i} \otimes \mathcal{E}^*_{2b+j,\ b} & \longrightarrow & \mathcal{E}^*_{a+b+2i+2j,\ i+j} \\ \downarrow{\scriptstyle id \otimes \eta} & & \downarrow{\scriptstyle \eta} \\ \mathcal{E}^*_{a+2i,\ i} \otimes \mathcal{E}^*_{2b+j+2,\ b+1} & \longrightarrow & \mathcal{E}^*_{a+b+2i+2j+2,\ i+j+1} \end{array}$$

Then for $i, j \in \mathbb{Z}$, we get $\mathcal{E}^*_i \otimes \mathcal{E}^*_j \to \mathcal{E}^*_{i+j}$, which induce product structures on $\mathcal{E}$ and $\mathcal{E}^*_{ell}$. By Lemma 4.18 and Lemma 4.19, these give $\mathcal{E}$ and $\mathcal{E}^*_{ell}$ the structures of commutative differential graded algebra objects in the category of GL_2 representations.

4. Computations

LEMMA 4.26. *There are isomorphisms:*

$$H^i(\widetilde{C}^{Alt,-}_{a-2b-*}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a)) \cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a}, \mathbb{Q}(a-b)[i]),$$

for $i \in \mathbb{Z}$.

PROOF. By results in Chapter 4 of [**15**] and Theorem 4.13, we have:

$$H^i(\widetilde{C}^{Alt}_{a-2b-*}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a)) \cong CH^{a-b}(E^a, 2a-2b-i).$$

Using Theorem 19.1 in [**27**], we have:

$$CH^{a-b}(E^a, 2a-2b-i) \cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M(E^a)(b), \mathbb{Q}(a)[i]).$$

Notice that the action of $(\mathbb{Z}/2)^a$ on these groups are compatible under the above isomorphisms. Considering the $-$ part, we get:

$$H^i(\widetilde{C}^{Alt,-}_{a-2b-*}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a)) \cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a}, \mathbb{Q}(a-b)[i]).$$

□

LEMMA 4.27. *The external product, which is defined on the cohomology groups of the cycle complex* $\widetilde{C}^{Alt,-}_{a-2b-*}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a)$*, is compatible with the external product on Hom groups* $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a}, \mathbb{Q}(a-b)[i])$*, defined in the following way:*

$$\begin{aligned}
&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a}, \mathbb{Q}(a-b)[i]) \otimes Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes c}, \mathbb{Q}(c-d)[j]) \\
&\to Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a} \otimes (M_1(E))^{\otimes c}, \mathbb{Q}(a+c-b-d)[i+j]) \qquad (15)\\
&\to Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a+c}, \mathbb{Q}(a+c-b-d)[i+j]),
\end{aligned}$$

where the first map is taking the external product in $\mathbf{DM}_{gm}(k,\mathbb{Q})$.

PROOF. By Lemma 4.26, $Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a}, \mathbb{Q}(a-b)[i])$ can be identified as the cohomology group of a subcomplex $\widetilde{C}^{Alt,-}_{a-2b-*}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a)$ of $C_{a-2b-i}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a)$. The external product of

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a}, \mathbb{Q}(a-b)[i])$$

defined as above is just induced by the external product defined on cohomology groups of

$$C_{a-2b-i}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a).$$

On the other hand, notice that the external product on the cohomology groups of the cycle complex

$$\widetilde{C}^{Alt,-}_{a-2b-*}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a)$$

is given by the external product on the cohomology groups of

$$C_{a-2b-i}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a).$$

□

LEMMA 4.28. *The cohomologies of $\mathcal{E}^*_{a,b}$ are canonically isomorphic to the cohomologies of the following complex of GL_2 representations*

$$\bigoplus_{c+2d=a,c,d\geq 0} Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^c M_1(E)(d), \mathbb{Q}(a-b)[*]) \otimes Sym^c\mathbf{F}(d+b-a),$$

where we view its differential maps as zero.

PROOF. By Lemma 4.26, we have the following isomorphism between GL_2 representations:

$$H^i(\mathcal{E}^*_{a,b}) \cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a}, \mathbb{Q}(a-b)[i]) \otimes_{\mathbb{Q}[\Sigma_a]} \mathbf{F}^a(b-a).$$

Then by Lemma 3.19, we know that:

$$\begin{aligned}
&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes a}, \mathbb{Q}(a-b)[i]) \otimes_{\mathbb{Q}[\Sigma_a]} \mathbf{F}^a(b-a)\\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\oplus_{c+2d=a,c,d\geq 0} V_{(c+d,d)} \otimes Sym^c(M_1(E))(d), \mathbb{Q}(a-b)[i])\\
&\otimes_{\mathbb{Q}[\Sigma_a]} (\oplus_{e+2f=a,e,f\geq 0} V_{(e+f,f)} \otimes Sym^e\mathbf{F}(f+b-a))\\
\cong &\oplus_{c+2d=e+2f=a,c,d,e,f\geq 0} Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^c(M_1(E))(d), \mathbb{Q}(a-b)[i])\\
&\otimes (V^\vee_{(c+d,d)} \otimes_{\mathbb{Q}[\Sigma_a]} V_{(e+f,f)}) \otimes Sym^e\mathbf{F}(f+b-a)\\
\cong &\oplus_{c+2d=a,c,d\geq 0} Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^c M_1(E)(d), \mathbb{Q}(a-b)[i])\\
&\otimes Sym^c\mathbf{F}(d+b-a).
\end{aligned}$$

Notice that given two irreducible representations V, W of a finite group G over $\mathbb{Q}$, then $V \otimes_{\mathbb{Q}[G]} W = \mathbb{Q}$ if $V \cong W$. Otherwise, it's zero. □

COROLLARY 4.29. *If the 0-vanishing property holds for an elliptic curve E, then for any $a > 0$, the cohomolgies of $\mathcal{E}^*_{2a,a}$ are all isomorphic to the trivial GL_2-representation $\mathbb{Q}$ concentrated in degree zero.*

PROOF. By Lemma 4.28, we have:

$$H^*(\mathcal{E}^*_{2a,a}) \cong$$
$$\bigoplus_{c+2d=2a,c,d\geq 0} Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^c M_1(E), \mathbb{Q}(a-d)[*]) \otimes Sym^c\mathbf{F}(d-a). \tag{16}$$

From Definition 3.21, we know that:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^c M_1(E), \mathbb{Q}(a-d)[*]) \cong 0 \quad \text{if} \quad c \geq 1.$$

Therefore, we have:

$$H^n(\mathcal{E}^*_{2a,a}) \cong \begin{cases} 0 & \text{if } n \neq 0; \\ \mathbb{Q} & \text{if } n = 0, \end{cases}$$

where $\mathbb{Q}$ is the trivial GL_2 representation. □

Recall in the previous section, we have defined η in equality (13). In the next lemma, we want to give a description of η under the identification in Lemma 4.28.

LEMMA 4.30. *Via the identification of Lemma 4.28, the map:*

$$\eta : \mathcal{E}^*_{a,b} \to \mathcal{E}^*_{a+2,b+1},$$

induces the following map on cohomology groups:

$$\begin{aligned}&(Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^cM_1(E)(d),\mathbb{Q}(a-b)[*])\otimes Sym^c\mathbf{F}(d+b-a))\\&\otimes(Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\mathbb{Q}(1),\mathbb{Q}(1))\otimes\mathbb{Q})\\\to&(Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^cM_1(E)(d+1),\mathbb{Q}(a-b+1)[*])\\&\otimes Sym^c\mathbf{F}(d+b-a).\end{aligned}$$

Moreover, the map on cohomology groups induces by η is a monomorphism in the category of GL_2 representations.

Proof. By Example 3.24, we have a simple description of $\mathcal{E}^*_{2,1}$:

$$H^*(\mathcal{E}^*_{2,1})\cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\mathbb{Q}(1),\mathbb{Q}(1))\otimes\mathbb{Q}.$$

Using Lemma 4.27 and Lemma 4.28, we can identify η as sending the piece

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^cM_1(E)(d),\mathbb{Q}(a-b)[*])\otimes Sym^c\mathbf{F}(d+b-a)$$

to

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^cM_1(E)(d+1),\mathbb{Q}(a-b+1)[*])\otimes Sym^c\mathbf{F}(d+b-a)$$

By Voevodsky's cancellation theorem in [**34**], on each piece of $\mathcal{E}^*_{a,b}$, η is an isomorphism, which implies that η is an injection. ☐

Corollary 4.31. *If an elliptic curve E satisfies the r-th vanishing property for all positive integer r, then all the $H^*(\mathcal{E}^*_{-r})$ are zero. Furthermore, if the elliptic curve E satisfies the r-th vanishing property for all non-negative integer r, then we have $H^*(\mathcal{E}^*)=H^*(\mathcal{E}^*_{ell})$.*

Proof. By Lemma 4.28, we have a quasi-isomorphism:

$$\begin{aligned}&H^*(\mathcal{E}^*_{r+2i,r+i})\cong\\&\bigoplus_{c+2d=r+2i,c,d\geq0}Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^cM_1(E)(d),\mathbb{Q}(i)[*])\otimes Sym^c\mathbf{F}(d-i).\end{aligned}$$

If E satisfies the r-th vanishing property for $r\in\mathbb{Z}_{>0}$, we have:

$$\begin{aligned}&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^cM_1(E),\mathbb{Q}(i-d)[*])\\\cong&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{r+2(i-d)}M_1(E),\mathbb{Q}(i-d)[*])\cong0.\end{aligned}$$

Therefore, $H^*(\mathcal{E}^*_{r+2i,r+i})\cong0$ for any $r\in\mathbb{Z}_{>0}$ and any $i\in\mathbb{Z}_{\geq0}$, which implies that $H^*(\mathcal{E}^*_{-r})=0$.

Furthermore, if E also satisfies the 0-th vanishing property, then by Corollary 4.29, we know that $H^*(\mathcal{E}_{2a,a})\cong\mathbb{Q}$. Also, from Lemma 4.30, we know the connecting map η is the identity. Therefore we obtain that $H^*(\mathcal{E}^*)=H^*(\mathcal{E}^*_{ell})$. ☐

COROLLARY 4.32. *Let a be any integer. In the category of the derived category of GL_2 representations, we have the following isomorphisms:*

$$H^*(\mathcal{E}_a^*) \cong \bigoplus_{i\geq 0, a\equiv i(2)} Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^i M_1(E), \mathbb{Q}(\frac{a+i}{2})[*]) \otimes Sym^i \mathbf{F}(-\frac{a+i}{2}).$$

PROOF. Using Lemma 4.28, we have:

$$H^*(\mathcal{E}_{a,b}^*) \cong \bigoplus_{c+2d=a, c,d\geq 0} Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^c M_1(E)(d), \mathbb{Q}(a-b)[*]) \otimes Sym^c \mathbf{F}(d+b-a). \tag{17}$$

By Lemma 4.30, the connecting map

$$\eta : \mathcal{E}_{a,b}^* \to \mathcal{E}_{a+2,b+1}^*$$

sends the summand

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^c M_1(E)(d), \mathbb{Q}(a-b)[*]) \otimes Sym^c \mathbf{F}(d+b-a),$$

of $H^*(\mathcal{E}_{a,b}^*)$ to the same direct summand in $H^*(\mathcal{E}_{a+2,b+1}^*)$ by the identity map. Therefore taking the direct limit, we will get the direct sum of all the pieces of the form

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^c M_1(E)(d), \mathbb{Q}(a-b)[*]) \otimes Sym^c \mathbf{F}(d+b-a).$$

Rewriting the index set, one obtains the desired presentations. □

Next we want to compute the hom-groups between some special dg $\mathcal{E}_{ell}$-modules.

We let $\mathcal{T}_{\mathcal{E}_{ell}}^{GL_2}$ be the full triangulated subcategory of the derived category of dg $\mathcal{E}_{ell}$-module generated by the dg $\mathcal{E}_{ell}$-module of the form $\{\mathcal{E}_{ell} \otimes \mathbf{F}^{-a}(b)[n]\}_{a,b,n\in\mathbb{Z}}$. Simply denote these elements by $\mathcal{E}_{ell}\langle a,b\rangle[n]$.

For convenience, we use the index $\mathcal{E}_{ell}$ to denote the hom group in $\mathcal{T}_{\mathcal{E}_{ell}}^{GL_2}$ and use GL_2 to stand for the derived category of GL_2 representations in next lemma.

$$\begin{aligned} &Hom_{\mathcal{E}_{ell}}(\mathcal{E}_{ell}\langle a,b\rangle[n], \mathcal{E}_{ell}\langle c,d\rangle[m]) \\ =&Hom_{\mathcal{E}_{ell}}(\mathcal{E}_{ell} \otimes \mathbf{F}^{\otimes -a}(b)[n], \mathcal{E}_{ell} \otimes \mathbf{F}^{\otimes -c}(d)[m]) \\ =&Hom_{GL_2}(\mathbf{F}^{\otimes -a}(b), \mathcal{E}_{ell} \otimes \mathbf{F}^{\otimes -c}(d)[m-n]) \\ =&H^{m-n}(Hom_{GL_2}(\mathbb{Q}, \mathcal{E}_{ell} \otimes \mathbf{F}^{\otimes -c} \otimes \mathbf{F}^{\otimes a}(-b+d))). \end{aligned} \tag{18}$$

LEMMA 4.33. *For $a,b,c,d,i \in \mathbb{Z}$, we have:*

$$\begin{aligned} &H^i(Hom_{GL_2}(\mathbb{Q}, \mathcal{E}_{ell} \otimes \mathbf{F}^{\otimes -c} \otimes \mathbf{F}^{\otimes a}(-b+d))) \\ \cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes -a}(b), (M_1(E))^{\otimes -c}(d)[i]). \end{aligned}$$

PROOF. The isomorphism $\mathbf{F}^\vee \cong \mathbf{F}(-1)$ gives us the isomorphism $\mathbf{F}^{-a} \cong \mathbf{F}^a(-a)$ in the category of GL_2-representations. Similarly using $M_1(E)^\vee \cong M_1(E)(-1)$ gives us an isomorphism between geometric motives $M_1(E)^{-a} \cong M_1^{\otimes a}(E)(-a)$. Without loss of generality, we can assume $a, c \geq 0$. By Voevodsky's Cancellation theorem, we can also assume that $b = 0$. For simplicity, we only prove the case $d = 0$.

$\mathbf{F}^{\otimes -c} \otimes \mathbf{F}^{\otimes a}$ is the direct sum of $Sym^{a-2n}\mathbf{F}(n) \otimes Sym^{-c-2m}\mathbf{F}(m)$, where $0 \leq 2n \leq a, c + 2m \leq 0, c + m \geq 0$, with multiplicities $C_{a-2n,n} \times C_{-c-2m,c+m}$.

Furthermore, we can decompose $\mathbf{F}^{\otimes -c} \otimes \mathbf{F}^{\otimes a}$ as the direct sum of irreducible GL_2 representations of the form $Sym^{a-2n-c-2m-2l}\mathbf{F}(m + n + l)$, where the index set μ satisfies $0 \leq 2n \leq a, c + 2m \leq 0, c + m \geq 0, 0 \leq 2l \leq a - c - 2(m + n)$, with multiplicity $C_{a-2n,n} \times C_{-c-2m,c+m} \times D^l_{a-2n,-c-2m}$. (Notations are defined in Convention 1.10 and 1.40). From this decomposition, we get $m+c+n+l \geq 0$, which implies that $a-2m-c-2n-2l \leq a + c$.

For each irreducible representation $Sym^{a-2n-c-2m-2l}\mathbf{F}(m + n + l)$, we have:

$$
\begin{aligned}
&H^i(Hom_{GL_2}(\mathbb{Q}, \mathcal{E}^*_{ell} \otimes Sym^{a-2n-c-2m-2l}\mathbf{F}(m + n + l)))\\
\cong &H^i(Hom_{GL_2}(\mathbb{Q}, \mathcal{E}^*_{c-a} \otimes Sym^{a-2n-c-2m-2l}\mathbf{F}(m + n + l)))\\
\cong &H^i(Hom_{GL_2}((Sym^{a-2n-c-2m-2l}\mathbf{F}(m + n + l))^*, \mathcal{E}^*_{c-a}))\\
\cong &H^i(Hom_{GL_2}(Sym^{a-2n-c-2m-2l}\mathbf{F}(m + n + l - a + c), \mathcal{E}^*_{c-a}))\\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{a-2n-c-2m-2l}M_1(E), \mathbb{Q}(a - c - m - n - l)[i]),
\end{aligned}
\tag{19}
$$

For the last isomorphism, we use Corollary 4.32.

On the other hand, let us compute the hom-groups between motives.

$$
\begin{aligned}
&Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes -a}, (M_1(E))^{\otimes -c}[i])\\
\cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M_1(E)^{\otimes a} \otimes M_1(E)^{\otimes c}, \mathbb{Q}(a)[i])\\
\cong &\oplus_{0\leq 2s\leq a, 0\leq 2t\leq c} (C_{a-2s,s} \times C_{c-2t,t})\\
&\cdot Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{a-2s}M_1(E)(s) \otimes Sym^{c-2t}M_1(E)(t), \mathbb{Q}(a)[i])\\
\cong &\oplus_{0\leq 2s\leq a, 0\leq 2t\leq c, 0\leq 2r\leq a+c-2s-2t} (C_{a-2s,s} \times C_{c-2t,t} \times D^r_{a-2s,c-2t})\\
&\cdot Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{a+c-2s-2t-2r}M_1(E)(r), \mathbb{Q}(a - s - t)[i])
\end{aligned}
\tag{20}
$$

Rewrite the index set, and let $s = n, t = c + m, r = l$. Then this index set is the same as μ. Notice that the multiplicities of the term

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(Sym^{a-2n-c-2m-2l}M_1(E), \mathbb{Q}(a - c - m - n - l)[i])$$

in

$$H^i(Hom_{GL_2}(\mathbb{Q}, \mathcal{E}^*_{ell} \otimes Sym^{a-2n-c-2m-2l}\mathbf{F}(m + n + l)))$$

and

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}((M_1(E))^{\otimes -a}, (M_1(E))^{\otimes -c}[i])$$

are the same. Both are $C_{a-2n,n} \times C_{-c-2m,c+m} \times D^l_{a-2n,-c-2m}$.

□

CHAPTER 5

CONNECTION WITH $\mathbf{DM}_{gm}(k, \mathbb{Q})$

1. The cycle cdga over GL_2 in $\mathbf{DM}(k, \mathbb{Q})$

Let E be an elliptic curve without complex multiplication as before.

Definition 5.1. Let a, b be integers such that $a \geq b, a \geq 0$. For $i \in \mathbb{Z}$ and $T \in \mathbf{Sm}_k$, we define:

$$\mathfrak{E}^i_{a,b}(T) = \widetilde{C}^{Alt,-}_{a-2b-i}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a \times T) \otimes_{\mathbb{Q}[\Sigma_a]} \mathbf{F}^{\otimes a}(b-a).$$

Then $\mathfrak{E}^i_{a,b}$[1] is a $\mathbf{Rep}_{GL_2}$-valued Nisnevich sheaf with transfers, i.e.,

$$\mathfrak{E}^*_{a,b} \in C(\mathcal{S}h^{tr}_{Nis}(k) \otimes \mathbf{Rep}_{GL_2}).$$

△

Remark 5.2. From the definition, we have $\mathfrak{E}^*_{a,b}(k) = \mathcal{E}^*_{a,b}$. In fact, by computations similar in Section 4 of Chapter 4, one can get the following isomorphism in $\mathbf{DM}_{gm}(k, \mathbb{Q})$:

$$\mathfrak{E}^*_{a,b} \cong \underline{RHom}(M_1(E)^{\otimes a}, \mathbb{Q}(a-b)) \otimes_{\mathbb{Q}[\Sigma_a]} \mathbf{F}^{\otimes a}(b-a).$$

Here $\underline{RHom}$ is defined in Remark 14.12 in [**27**].

Remark 5.3. $\{\mathfrak{E}^*_{a,b}\}$ is a cdga over GL_2 object in $C(\mathcal{S}h^{tr}_{Nis}(k))_{\mathbb{Q}}$. More precisely, for $S, T \in \mathbf{Sm}_k$, the external product of correspondences gives the following product map:

$$\begin{aligned} &C_{a-2b-i}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a \times S) \otimes C_{c-2d-i}(z_{q.fin}(\mathbb{A}^{c-d}))(E^c \times S) \\ \longrightarrow &C_{a+c-2b-2d-i}(z_{q.fin}(\mathbb{A}^{a+c-b-d}))(E^a \times S \times E^c \times T) \end{aligned} \tag{21}$$

Taking the alternating projection with respect to the component $\square$, $-$ part with respect to the component E and symmetric projection with respect to the component $\mathbb{A}$ yields:

$$\begin{aligned} &\widetilde{C}^{Alt,-}_{a-2b-i}(z_{q.fin}(\mathbb{A}^{a-b}))(E^a \times S) \otimes \widetilde{C}^{Alt,-}_{c-2d-i}(z_{q.fin}(\mathbb{A}^{c-d}))(E^c \times S) \\ \longrightarrow &\widetilde{C}^{Alt,-}_{a+c-2b-2d-i}(z_{q.fin}(\mathbb{A}^{a+c-b-d}))(E^a \times S \times E^c \times T) \end{aligned} \tag{22}$$

Then we get the map as in (13):

$$\cdot : \mathfrak{E}^*_{a,b} \otimes \mathfrak{E}^*_{c,d} \to \mathfrak{E}^*_{a+c,b+d}. \tag{23}$$

As before, one may check this map is associative and graded commutative.

[1]This could be thought as the "motivic version" of the cycle algebra $\mathcal{E}^i_{a,b}$.

REMARK 5.4. A key observation is:

$$\begin{aligned}H^*(\mathfrak{E}^*_{2,1}) &\cong \underline{RHom}(M_1(E)^{\otimes 2},\mathbb{Q}(1))\otimes_{\mathbb{Q}[\Sigma_2]}\mathbf{F}^{\otimes 2}(-1)\\ &\cong \underline{RHom}(Sym^2(M_1(E)),\mathbb{Q}(1))\otimes Sym^2\mathbf{F}(-1)\oplus\mathbb{Q}\cong\mathbb{Q}\in DM^{eff}(k).\end{aligned}$$

This computation relies on Proposition 13.7 in [**27**] and the fact that:

$$\underline{RHom}(Sym^2(M_1(E)),\mathbb{Q}(1))(K)\cong 0,$$

whose proof is the same in Example 3.24.

Similarly using the multiplicative structure, we have:

$$\eta:\mathfrak{E}^*_{a,b}\to\mathfrak{E}^*_{a+2,b+1}.$$

We now define $\mathfrak{E}^*$ as in Definition 4.22.

DEFINITION 5.5. For $a\in\mathbb{Z}$, we define:

$$\mathfrak{E}^*_a=\varinjlim_{i\geq -a}\mathfrak{E}^*_{-a+2i,-a+i}. \tag{24}$$

Then we denote:

$$\mathfrak{E}^*=\mathbb{Q}\oplus\bigoplus_{a\geq 1}\mathfrak{E}^*_a.$$

and

$$\mathfrak{E}^*_{ell}=\bigoplus_{a\in\mathbb{Z}}\mathfrak{E}^*_a.$$

△

PROPOSITION 5.6. *$\mathfrak{E}^*$ and $\mathfrak{E}^*_{ell}$ are cdga objects in the category of complexes of $\mathbf{Rep}_{GL_2}$-valued Nisnevich sheaf with transfers.*

PROOF. The proof can be found in Section 4.3 of [**22**]. □

REMARK 5.7. Following the same proofs as Lemma 4.28, Lemma 4.30 and Corollary 4.32, we obtain the following properties of $\mathfrak{E}^*_{a,b}$.

(a). The cohomologies of $\mathfrak{E}^*_{a,b}$ are canonically isomorphic to the cohomologies of the following complex of GL_2 representations

$$\bigoplus_{c+2d=a,c,d\geq 0}\underline{RHom}(Sym^cM_1(E)(d),\mathbb{Q}(a-b)[*])\otimes Sym^c\mathbf{F}(d+b-a),$$

where we view the differentials as zero.

(b). Via the identification of property(a), the map:

$$\eta:\mathfrak{E}^*_{a,b}\to\mathfrak{E}^*_{a+2,b+1},$$

is compatible with the following map:

$$\begin{aligned}&(\underline{RHom}(Sym^cM_1(E)(d),\mathbb{Q}(a-b)[*])\otimes Sym^c\mathbf{F}(d+b-a))\\ &\otimes(\underline{RHom}(\mathbb{Q}(1),\mathbb{Q}(1))\otimes\mathbb{Q})\\ \to&(\underline{RHom}(Sym^cM_1(E)(d+1),\mathbb{Q}(a-b+1)[*])\\ &\otimes Sym^c\mathbf{F}(d+b-a).\end{aligned}$$

Moreover, the maps on cohomologies induced by η are injective in the category of GL_2 representations.

(c). Let a be any non negative integer. In the derived category of GL_2 representations, we have the following isomorphisms:

$$H^*(\mathfrak{E}^*_a) \cong$$
$$\bigoplus_{i\geq 0, a\equiv i(2)} \underline{RHom}(Sym^i M_1(E), \mathbb{Q}(\frac{a+i}{2})[*]) \otimes Sym^i \mathbf{F}(-\frac{a+i}{2}).$$

2. DG modules and motives for an elliptic curve

Fix $r \in \mathbb{Z}_{\geq 0}$. Given $M \in \mathcal{CM}^{GL_2}_{\mathcal{E}^*_{ell}}$, its Adams graded r summand is defined as:

$$M(r) = Hom_{GL_2}(det^{\otimes -r}, \mathfrak{E}^*_{ell} \otimes_{\mathcal{E}^*_{ell}} M[2r]).$$

Here $[2r]$ means the shift of the complex. $Hom_{GL_2}(\cdot,\cdot)$ is the usual hom complex in $C(\mathbf{Rep}_{GL_2})$. In fact, this defines a dg functor:

$$\mathcal{M}(r)^{dg} : \mathcal{CM}_{\mathcal{E}^*_{ell}} \to C(\mathcal{S}h^{tr}_{Nis}(k))$$

and also an exact functor :

$$\mathcal{M}(r) : \mathcal{KCM}_{\mathcal{E}^*_{ell}} \to D(\mathcal{S}h^{tr}_{Nis}(k)).$$

Definition 5.8. Let T^{tr} be the presheaf with transfers:

$$T^{tr} = coker(\mathbb{Q}_{tr}(Spec(k)) \xrightarrow{i_{\infty *}} \mathbb{Q}_{tr}(\mathbb{P}^1)),$$

where i_∞ is the inclusion of ∞ into $\mathbb{P}^1$. △

In fact, T^{tr} is a Nisnevich sheaf with transfers.

Lemma 5.9. *We have a natural injectve map in* $C(\mathcal{S}h^{tr}_{Nis}(k))$*:*

$$T^{tr} \to H^0(GL_2, \mathfrak{E}^*_{ell} \otimes det[2]).$$

Proof. By the definition of $\mathfrak{E}^*$, its det^{-1} isotypical part is given by

$$\lim_{i\geq 0} \mathfrak{E}^*_{2i,i-1}.$$

Notice that

$$\mathfrak{E}^*_{0,-1} \cong \widetilde{C}^{Alt,-}_{2-*}(z_{q.fin}(\mathbb{A}^1)) \cong T^{tr}[-2].$$

So there is a natural injective map:

$$T^{tr} \to H^0(GL_2, \mathfrak{E}^*_{ell} \otimes det[2]).$$

□

For $M \in \mathcal{CM}^{GL_2}_{\mathcal{E}^*_{ell}}$, from the above lemma, we have the following composition of maps:

$$\begin{aligned}
&T^{tr} \otimes^{tr} \mathcal{M}(r)^{dg}(M) = T^{tr} \otimes^{tr} Hom_{GL_2}(det^{\otimes -r}, \mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} M[2r]) \\
&\longrightarrow Hom_{GL_2}(det^{-1}, \mathfrak{E}_{ell}[2]) \otimes^{tr} Hom_{GL_2}(det^{\otimes -r}, \mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} M[2r]) \\
&\longrightarrow Hom_{GL_2}(det^{\otimes -r-1}, \mathfrak{E}_{ell} \otimes \mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} M[2r+2]) \\
&\longrightarrow Hom_{GL_2}(det^{\otimes -r-1}, \mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} M[2r+2]) = \mathcal{M}(r+1)^{dg}(M).
\end{aligned} \tag{25}$$

For the last arrow, we use the multiplicative structure of $\mathcal{E}_{ell}$. Denote the composition of these maps by $\epsilon^*_r(M)$.

In order to construct a functor from the homotopy category of cell modules to the category of motives, we need to use Voevodsky's big category of motives $\mathbf{DM}(k,\mathbb{Q})$, which is defined by the symmetric spectra. Roughly speaking, one needs to define a model category $\mathbf{Spt}^{\Sigma}_{T^{tr}}(k,\mathbb{Q})$ of symmetric T^{tr} spectra in $C(\mathcal{S}h^{tr}_{Nis}(k))$ with "a suitable model structure", and then $\mathbf{DM}(k,\mathbb{Q})$ is defined to be the homotopy category of $\mathbf{Spt}^{\Sigma}_{T^{tr}}(k,\mathbb{Q})$. For this approach, we refer to section 3.2, 3.3 and 3.4 in [**22**].

Then sending $M \in \mathcal{CM}^{GL_2}_{\mathcal{E}^*_{ell}}$ to the sequence:

$$\mathcal{M}^{dg}_*(M) = (\mathcal{M}^{dg}(0)(M), \mathcal{M}^{dg}(1)(M), \cdots)$$

with the bonding map $\epsilon^*_r(M)$ defines a dg functor:

$$\mathcal{M}^{dg}_* : \mathcal{CM}^{GL_2}_{\mathcal{E}^*_{ell}} \to \mathbf{Spt}^{\Sigma}_{T^{tr}}(k,\mathbb{Q}),$$

and also an exact functor on their homotopy categories

$$\mathcal{M}_* : \mathcal{KCM}^{GL_2}_{\mathcal{E}^*_{ell}} \to \mathbf{DM}(k,\mathbb{Q}).$$

Here the n-th term in the spectrum is equipped with a trivial Σ_n action.

Lemma 5.10. *We have the following isomorphisms in* $\mathbf{DM}(k,\mathbb{Q})$*:*

(1) $\mathcal{M}(r)(\mathcal{E}_{ell}) \cong \mathbb{Q}(r)[2r]$.
(2) Given $a, b \in \mathbb{Z}$*, for any* $r \in \mathbb{Z}$ *such that* $b + r \geq 0$*, we have:*

$$\mathcal{M}(r)(\mathcal{E}_{ell} \otimes \mathbf{F}^{\otimes a}(b)) \cong M_1(E)^{\otimes a}(b+r)[2r].$$

Proof. Because $\mathcal{M}(r)(\mathcal{E}_{ell}) = Hom_{GL_2}(det^{\otimes -r}, \mathfrak{E}_{ell}[2r])$, the only non-trivial part is coming from weight $-2r$(or Adams degree $2r$) part in $\mathfrak{E}_{ell}$. Using Remark 5.7, we have the following quasi-isomorphism:

$$\mathfrak{E}^*_{2r} \cong \bigoplus_{i \geq 0} \underline{RHom}(Sym^i M_1(E), \mathbb{Q}(\frac{2r+i}{2})) \otimes Sym^i \mathbf{F}(-\frac{2r+i}{2}).$$

Then by definition of $\mathcal{M}(r)(\mathcal{E}_{ell})$, we have:

$$\begin{aligned}\mathcal{M}(r)(\mathcal{E}_{ell}) &= Hom_{GL_2}(det^{\otimes -r}, \mathfrak{E}_{ell}[2r])\\ \cong& Hom_{GL_2}(det^{\otimes -r}, \bigoplus_{i\geq 0}\underline{RHom}(Sym^i M_1(E), \mathbb{Q}(\frac{2r+i}{2}))\\ &\otimes Sym^i\mathbf{F}(-\frac{2r+i}{2})[2r])\\ \cong& Hom_{GL_2}(det^{\otimes -r}, \underline{RHom}(\mathbb{Q}, \mathbb{Q}(r))\otimes det^{\otimes -r}[2r])\\ \cong& \mathcal{H}om_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\mathbb{Q}, \mathbb{Q}(r))[2r] \cong \mathbb{Q}(r)[2r].\end{aligned}$$

For the second isomorphism, we need to compute weight $-2r-1$ part in $\mathfrak{E}_{ell}$. We have:

$$\mathfrak{E}^*_{2r+1} \cong \bigoplus_{i\geq 0}\underline{RHom}(Sym^i M_1(E), \mathbb{Q}(\frac{2r+1+i}{2}))\otimes Sym^i\mathbf{F}(-\frac{2r+1+i}{2}).$$

Therefore:

$$\begin{aligned}\mathcal{M}(r)(\mathcal{E}_{ell}\otimes \mathbf{F}^{\otimes a}(b)) &= Hom_{GL_2}(det^{\otimes -r}, \mathfrak{E}_{ell}\otimes \mathbf{F}^{\otimes a}(b)[2r])\\ \cong& Hom_{GL_2}(\mathbf{F}^{\otimes -a}\otimes det^{\otimes -b-r}, \mathfrak{E}_{ell}[2r])\\ \cong& Hom_{GL_2}(\mathbf{F}^{\otimes -a}\otimes det^{\otimes -b-r}, \mathfrak{E}_{2b+2r+a}[2r])\\ \cong& Hom_{GL_2}(\mathbf{F}^{\otimes -a}\otimes det^{\otimes -b-r}, \bigoplus_{i\geq 0}\underline{RHom}(Sym^i M_1(E), \mathbb{Q}(\frac{2b+2r+a+i}{2}))\\ &\otimes Sym^i\mathbf{F}(-\frac{2b+2r+a+i}{2})[2r])\\ \cong& Hom_{GL_2}(\bigoplus_{0\leq j\leq a} C_j\otimes Sym^j\mathbf{F}(-\frac{2b+2r+a+j}{2}), \bigoplus_{i\geq 0}\underline{RHom}(Sym^i M_1(E),\\ &\mathbb{Q}(\frac{2b+2r+a+i}{2}))\otimes Sym^i\mathbf{F}(-\frac{2b+2r+a+i}{2})[2r])\\ \cong& \mathcal{H}om_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(\bigoplus_{0\leq j\leq a} C_j\otimes Sym^j M_1(E), \mathbb{Q}(\frac{2b+2r+a+j}{2}))[2r]\\ \cong& M_1(E)^{\otimes a}(b+r)[2r].\end{aligned}$$

Here C_j is the multiplicity of $Sym^j\mathbf{F}(-\frac{2b+2r+a+j}{2})$ in $\mathbf{F}^{\otimes -a}\otimes det^{\otimes -b-r}$. □

LEMMA 5.11. *$\{\mathcal{E}_{ell}\otimes \mathbf{F}^a(b) | a, b \in \mathbb{Z}\}$ generate $\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}$.*

PROOF. Let $M \in \mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}$ be a dg module satisfying

$$Hom_{\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}}(\mathcal{E}_{ell}\otimes \mathbf{F}^a(b), M[i]) \cong 0$$

for any $\mathbf{F}^a(b) \in \mathbf{Rep}_{GL_2}$ and $a, b, i \in \mathbb{Z}$. Without loss of generality, we assume that M is a cell module. Using Remark 2.49, we obtain that:

$$Hom_{\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}}(\mathcal{E}_{ell}\otimes \mathbf{F}^a(b), M[i]) \cong H^i(Hom_{GL_2}(\mathbf{F}^a(b), M)) \cong 0,$$

which implies that M is quasi-isomorphic to 0 as a complex of GL_2 representations. $\square$

COROLLARY 5.12. *$\{\mathcal{E}_{ell} \otimes \mathbf{F}^a(b) | a, b \in \mathbb{Z}\}$ classically generates $(\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$.*

PROOF. First we want to show that $\mathcal{E}_{ell} \otimes \mathbf{F}^a(b)$ is a compact object in $\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}$ for any $a, b \in \mathbb{Z}$. Let $\{M_i\}_{i\in I}$ be a family of cell A-modules. By Remark 2.49, we have:

$$\begin{aligned}
&Hom_{\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}}(\mathcal{E}_{ell} \otimes \mathbf{F}^a(b), \bigoplus_{i\in I} M_i) = Hom_{\mathcal{KCM}^{GL_2}_{\mathcal{E}_{ell}}}(\mathcal{E}_{ell} \otimes \mathbf{F}^a(b), \bigoplus_{i\in I} M_i) \\
\cong &Hom_{\mathcal{KCM}^{GL_2}_{\mathbb{Q}}}(\mathbf{F}^a(b), \bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} Hom_{\mathcal{KCM}^{GL_2}_{\mathbb{Q}}}(\mathbf{F}^a(b), M_i) \\
\cong &\bigoplus_{i\in I} Hom_{\mathcal{KCM}^{GL_2}_{\mathcal{E}_{ell}}}(\mathcal{E}_{ell} \otimes \mathbf{F}^a(b), M_i),
\end{aligned}$$

which implies that $\mathcal{E}_{ell} \otimes \mathbf{F}^a(b)$ is compact. Here we use that $\mathbf{F}^a(b)$ is a compact object in $\mathcal{D}^{GL_2}_{\mathbb{Q}}$.

Together with Lemma 5.11, we know that $\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}$, as a compactly generated triangulated category, is generated by $\{\mathcal{E}_{ell} \otimes \mathbf{F}^a(b) | a, b \in \mathbb{Z}\}$. Then using Theorem 1.31, we know that $\{\mathcal{E}_{ell} \otimes \mathbf{F}^a(b) | a, b \in \mathbb{Z}\}$ classically generate $(\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$. $\square$

REMARK 5.13. Recall in Remark 2.41, we have:

$$(\mathcal{KCM}^{GL_2,f}_{\mathcal{E}_{ell}})^{\natural} \subset \mathcal{KFCM}^{GL_2}_{\mathcal{E}_{ell}} \subset (\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c.$$

Using Corollary 5.12, we know that $(\mathcal{KCM}^{GL_2,f}_{\mathcal{E}_{ell}})^{\natural} \cong (\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$. Therefore, we have:

$$(\mathcal{KCM}^{GL_2,f}_{\mathcal{E}_{ell}})^{\natural} \cong \mathcal{KFCM}^{GL_2}_{A} \cong (\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c.$$

LEMMA 5.14. *The restriction of $\mathcal{M}$ to $\mathcal{KCM}^{GL_2}_{\mathcal{E}_{ell}}$ is a lax tensor functor.*

PROOF. Given $M, N \in \mathcal{KCM}^{GL_2}_{\mathcal{E}_{ell}}$, we have the following maps:

$$\begin{aligned}
&(\mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} M) \otimes^{tr} (\mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} N) \longrightarrow (\mathfrak{E}_{ell} \otimes^{tr} \mathfrak{E}_{ell}) \otimes_{\mathcal{E}_{ell}} (M \otimes_{\mathcal{E}_{ell}} N) \\
&\longrightarrow \mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} (M \otimes_{\mathcal{E}_{ell}} N),
\end{aligned}$$

where the last map is obtained by using the multiplicative structure of $\mathfrak{E}_{ell}$ as a cdga over GL_2 in $\mathbf{DM}(k, \mathbb{Q})$ (Proposition 5.6). On the corresponding Adams graded summand, this induces:

$$(\mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} M)(r) \otimes^{tr} (\mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} N)(s) \longrightarrow (\mathfrak{E}_{ell} \otimes_{\mathcal{E}_{ell}} (M \otimes_{\mathcal{E}_{ell}} N))(r+s).$$

And these maps are compatible with bonding maps, giving us the natural transformation:

$$\rho_{M,N} : \mathcal{M}^{dg}(M) \otimes \mathcal{M}^{dg}(N) \to \mathcal{M}^{dg}(M \otimes N)$$

in $\mathbf{Spt}^{\Sigma}_{T^{tr}}(k, \mathbb{Q})$. Passing to homotopy categories, we obtain that $\mathcal{M}$ is a lax tensor functor. $\square$

LEMMA 5.15. *The restriction of $\mathcal{M}$ to $(\mathcal{KCM}^{GL_2,f}_{\mathcal{E}})^{\natural}$ is a tensor functor.*

PROOF. By lemma 5.14, we only need to show that $\rho_{M,N}$ is an isomorphism in the homotopy category. Using induction on the length of the weight filtration, it's enough to show that this is an isomorphism when we take M and N two generalized sphere $\mathcal{E}_{ell}$ modules. Notice that any generalized sphere module can be realized as some idempotent of the dg module of the form $\mathcal{E}_{ell}\otimes\mathbf{F}^a(b)$ for some $a, b \in \mathbb{Z}$. We assume that $M = p(\mathcal{E}_{ell}\otimes\mathbf{F}^a(b))$ and $N = q(\mathcal{E}_{ell}\otimes\mathbf{F}^c(d))$, where p, q are idempotents in the respective endo-groups. Applying Lemma 4.33, we obtain that the idempotents of $\mathcal{E}_{ell} \otimes \mathbf{F}^a(b)$ is one-to-one corresponding to the idempotents of $M_1(E)^{\otimes a}(b)$, i.e.,

$$\mathcal{M}(M) = \mathcal{M}(p(\mathcal{E}_{ell} \otimes \mathbf{F}^a(b))) = \mathcal{M}(p)(M_1(E)^{\otimes a}(b)),$$

where $\mathcal{M}(p)$ is the image of p under $\mathcal{M}$ in the idempotent endomorphism of $M_1(E)^{\otimes a}(b)$. Then $\rho_{M,N}$ can be identify as the morphism:

$$\mathcal{M}(p)(M_1(E)^{\otimes a}(b)) \otimes^{tr} \mathcal{M}(q)(M_1(E)^{\otimes c}(d)) \to \mathcal{M}(p \otimes q)(M_1(E)^{\otimes a+c}(b+d)),$$

which is an isomorphism in $\mathbf{DM}_{gm}(k, \mathbb{Q})$.

□

THEOREM 5.16. *There is an exact functor*

$$\mathcal{M} : \mathcal{D}^{GL_2}_{\mathcal{E}_{ell}} \to \mathbf{DM}(k, \mathbb{Q}),$$

which is a lax tensor functor. Furthermore, the restriction of $\mathcal{M}$ to

$$\mathcal{M}^c : (\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c \to \mathbf{DM}(k, \mathbb{Q})$$

defines an equivalence of $(\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$ with $\mathbf{DMEM}(k, \mathbb{Q})_E$ as triangulated tensor categories, where $(\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$ is the full subcategory of $\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}}$ consisting of compact objects.

PROOF. By Lemma 5.15 and Lemma 5.10, we know that the restriction of $\mathcal{M}$ to $(\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$ is a tensor functor with $\mathcal{M}(\mathcal{E}_{ell} \otimes \mathbf{F}^a(b)) \cong M_1(E)^{\otimes a}(b)$.

From Lemma 4.33, we have:

$$\begin{aligned}&Hom_{\mathcal{KCM}^{GL_2}_{\mathcal{E}_{ell}}}(\mathcal{E}_{ell} \otimes \mathbf{F}^{\otimes a}(b), \mathcal{E}_{ell} \otimes \mathbf{F}^{\otimes c}(d)[i])\\ \cong &Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M_1(E)^{\otimes a}(b), M_1(E)^{\otimes c}(d)[i]).\end{aligned}$$

One can check this isomorphim is induced by the functor $\mathcal{M}$. Using Lemma 1.32 and Corollary 5.12, we obtain that $\mathcal{M}^c$ gives an equivalence between $(\mathcal{D}^{GL_2}_{\mathcal{E}_{ell}})^c$ and $\mathbf{DMEM}(k, \mathbb{Q})_E$.

□

CONJECTURE 5.17. ***(The generalized Beilison-Soulé vanishing conjectures for an elliptic curve without CM)***

An elliptic curve E over a field k without complex multiplication satisfies the conditions:

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Q})}(M_1(E)^{\otimes a}, \mathbb{Q}(a-b)[m]) = 0$$

in the following two cases:

A. $a = 0, b < 0, m \leq 0$;
B. $a > 0, a \geq 2b, m \leq 0$.

REMARK 5.18. In fact, Part (A) of Conjecture 5.17 is the classical Beilison-Soulé vanishing conjectures. See [**23**] for example.

COROLLARY 5.19. *Assume that E is an elliptic curve without complex multiplication, satisfies the r-th vanishing properties for $r \geq 0$ and the generalized Beilison-Soulé vanishing conjectures, then:*

1. $\mathbf{DMEM}(k,\mathbb{Q})_E$ *has a t-structure which is induced from*
$$\mathcal{M}^f : \mathcal{D}_{\mathcal{E}}^{GL_2,f} \to \mathbf{DMEM}(k,\mathbb{Q})_E,$$
where $\mathcal{M}^f$ is the restriction of the functor $\mathcal{M}$ (Theorem 5.16) to $\mathcal{D}_{\mathcal{E}}^{GL_2,f}$. Denote its heart by $\mathbf{MEM}(k,\mathbb{Q})_E$.
2. $\mathcal{M}^f$ *induces an equivalence of Tannakian categories:*
$$H^0(\mathcal{M}^f) : \mathcal{H}_{\mathcal{E}}^{GL_2,f} \to \mathbf{MEM}(k,\mathbb{Q})_E.$$

PROOF. First, it follows from our assumptions and Theorem 5.16 that $\mathcal{E}_{ell} \cong \mathcal{E}$ is a cohomologically connected cdga over GL_2. Then by Theorem 3.33, we have a t-structure on $\mathcal{D}_{\mathcal{E}}^{GL_2,f}$. Therefore the equivalence of Theorem 5.16 gives us an induced t-structure on $\mathbf{DMEM}(k,\mathbb{Q})_E$, which satisfies the desired properties. □

Motivating by Corollary 2.90, we have the following definition.

DEFINITION 5.20. For an elliptic curve E without complex multiplication, the abelian cateogry of motives for E is defined to be the category $co-rep^{GL_2,f}(\chi_{\mathcal{E}})$ of finite co-representations over $\chi_{\mathcal{E}}$ which is a Hopf algebra over GL_2 . Recall that $\chi_{\mathcal{E}} = H^0(\bar{B}^{GL_2}(\mathcal{E}))$. △

3. Relation with mixed Tate motives

As mentioned in Section 11 of Chapter 2, we may put the constructions of the Adams cycle algebra for mixed Tate motives into our setting. Firstly we recall the definitions in Chapter 4 of [**22**] .

DEFINITION 5.21. We let $\mathbb{Z}_{tr}((\mathbb{P}^1/\infty)^q))$ be defined by the cokernel of the map:
$$\oplus_{j=1}^{r}\mathbb{Z}_{tr}((\mathbb{P}^1/\infty)^{q-1})) \xrightarrow{\sum_j i_{j,\infty*}} \mathbb{Z}_{tr}((\mathbb{P}^1/\infty)^q))$$
where $i_{j,\infty} : \mathbb{P}^{q-1} \to \mathbb{P}^q$ inserts ∞ in the j-th place. △

DEFINITION 5.22. The Adams cycle algebra for mixed Tate motives is defined by:
$$\mathcal{N} = \mathbb{Q} \oplus \bigoplus_{q\geq 1} \mathcal{N}(q),$$
where $\mathcal{N}(q) \subset C_*^{Alt}(\mathbb{Z}_{tr}((\mathbb{P}^1/\infty)^q))$ be the subsheaf of symmetric sections with respect to the action of symmetric group Σ_q by permuting the coordinates in $(\mathbb{P}^1)^q$. △

REMARK 5.23. One can show that the homotopy category of finite cell $\mathcal{N}$-modules can be identified as the triangulated category of mixed Tate motives $\mathbf{DMT}(k,\mathbb{Q})$, which is a full rigid tensor subcategory of $\mathbf{DM}_{gm}(k,\mathbb{Q})$ generated by Tate objects (Example 3.12). The proof can be found in Section 5.3 in [**22**]. In fact, one of the main results in [**22**] is to show this equivalence can be generalized to mixed Tate motives over a base scheme that is separated, smooth and essentially of finite type over a field. Along with the strategy in [**22**], we also want to generalize our results into mixed elliptic motives over a general base scheme in the future.

Definition 5.24. We define the modified Adams cycle algebra for mixed Tate motives by:

$$\widehat{\mathcal{N}} = \mathbb{Q} \oplus \bigoplus_{t \geq 1, t \in \mathbb{Z}} \widehat{\mathcal{N}}_{2t},$$

where $\widehat{\mathcal{N}}_{2t} = \mathcal{N}(t) \otimes det^{\otimes -t}$. △

Remark 5.25. By Definition 4.15, we know that: $\mathcal{E}^*_{0,b} = \mathcal{N}(-b) \otimes det^{\otimes b}$ for any $b \in \mathbb{Z}_{\leq 0}$. This implies that $\widehat{\mathcal{N}}_{2t} \subset \mathcal{E}_{2t}$. Using the algebra structure of $\mathcal{N}$ (Section 4.2 in [**22**]) and the tensor structure of determinant representations (viewed as GL_2 representations), we know that $\widehat{\mathcal{N}}$ is sub-algebra of $\mathcal{E}_{ell}$ as a cdga over GL_2.

Remark 5.26. Notice that our Adams grading is different from Adams grading defined in [**22**]. More precisely, Adams degree r in the sense of [**22**] is Adams degree $2r$ in our sense.

Convention 5.27.
(1) Denote the category of cell modules (resp. finite cell modules) over N defined in section 1.4 of [**22**] by $\mathcal{CM}_{\mathcal{N}}$ (resp. $\mathcal{CM}^f_{\mathcal{N}}$).
(2) Denote the derived category of Adams graded dg $\mathcal{N}$-module by $\mathcal{D}_{\mathcal{N}}$, which is defined in section 1.4 of [**22**].
(3) Denote the full subcategory with objects isomorphic in $\mathcal{D}_{\mathcal{N}}$ to a finite cell module by $\mathcal{D}^f_{\mathcal{N}}$.

Recall in Section 1 of Chapter 2, we have defined $\mathcal{CM}^{\mathbb{G}_m}_A$ to be the category of cell modules of Tate-type for a cdga A over GL_2, which is a full subcategory of $\mathcal{CM}^{GL_2}_A$.

Remark 5.28. There is a natural functor:

$$\Psi_1 : \mathcal{CM}_{\mathcal{N}} \to \mathcal{CM}^{\mathbb{G}_m}_{\widehat{\mathcal{N}}},$$

which sends the cell module $\mathcal{N}\langle n \rangle$, defined in Example 1.4.5 of [**22**], to the cell module $\widehat{\mathcal{N}} \otimes det^{\otimes n}$. Ψ_1 induces a functor between their associated homotopy categories, even homotopy categories of finite cell modules. For simplicity, we denote both of these functors by Ψ_1. In particular, we have:

$$\Psi_1 : \mathcal{D}^f_{\mathcal{N}} \to \mathcal{D}^{\mathbb{G}_m, f}_{\widehat{\mathcal{N}}}.$$

Notice that the inclusion: $\mathcal{CM}^{\mathbb{G}_m}_{\widehat{\mathcal{N}}} \to \mathcal{CM}^{GL_2}_{\widehat{\mathcal{N}}}$ induces a functor

$$\Psi_2 : \mathcal{D}^{\mathbb{G}_m}_{\widehat{\mathcal{N}}} \to \mathcal{D}^{GL_2}_{\widehat{\mathcal{N}}}.$$

Similarly, on the level of homotopy category of finite cell modules, we have:

$$\Psi_2 : \mathcal{D}^{\mathbb{G}_m, f}_{\widehat{\mathcal{N}}} \to \mathcal{D}^{GL_2, f}_{\widehat{\mathcal{N}}}.$$

REMARK 5.29. Because $\widehat{\mathcal{N}}$ is Adams connected (Definition 2.2), by the discussion in Section 4 of Chapter 2, we have:

$$\mathcal{D}_{\widehat{\mathcal{N}}}^{GL_2,f} \cong (\mathcal{D}_{\widehat{\mathcal{N}}}^{GL_2})^c.$$

Using Remark 5.25, we have a map between cdgas over GL_2: $\widehat{N} \xrightarrow{i} \mathcal{E}_{ell}$. This induces a functor:

$$\Psi_3 : \mathcal{CM}_{\widehat{\mathcal{N}}}^{GL_2} \to \mathcal{CM}_{\mathcal{E}_{ell}}^{GL_2},$$

which sends M to $M \otimes_{\widehat{\mathcal{N}}} \mathcal{E}_{ell}$. Furthermore, we have:

$$\Psi_3 : \mathcal{D}_{\widehat{\mathcal{N}}}^{GL_2} \to \mathcal{D}_{\mathcal{E}_{ell}}^{GL_2}$$

and

$$\Psi_3 : (\mathcal{D}_{\widehat{\mathcal{N}}}^{GL_2})^c \to (\mathcal{D}_{\mathcal{E}_{ell}}^{GL_2})^c.$$

From our constructions of $\Psi_i, i = 1, 2, 3$, we have the following statement.

PROPOSITION 5.30. *We have the following commutative diagram:*

$$\begin{array}{ccccccccc}
\mathcal{D}_{\mathcal{N}}^{f} & \xrightarrow{\Psi_1} & \mathcal{D}_{\widehat{\mathcal{N}}}^{\mathbb{G}_m,f} & \xrightarrow{\Psi_2} & \mathcal{D}_{\widehat{\mathcal{N}}}^{GL_2,f} & \xrightarrow{\cong} & (\mathcal{D}_{\widehat{\mathcal{N}}}^{GL_2})^c & \xrightarrow{\Psi_3} & (\mathcal{D}_{\mathcal{E}_{ell}}^{GL_2})^c \\
\downarrow{\scriptstyle \mathcal{M}} & & & & & & & & \downarrow{\scriptstyle \mathcal{M}^c} \\
\mathbf{DMT}(k,\mathbb{Q}) & & & \longrightarrow & & & & & \mathbf{DMEM}(k,\mathbb{Q})_E
\end{array}$$

where the left vertical map $\mathcal{M}$ is defined in Section 5.3 of [**22**] *and the right vertical map $\mathcal{M}^c$ is defined in Section 2 of Chapter 5. In particular, the composition of top arrows is fully faithful.*

Bibliography

[1] G. Ancona, S. Enright-Ward, and A. Huber: On the motive of a commutative algebraic group. 2016. arXiv: 1312.4171v2.

[2] Y. André: Une introduction aux motifs: Motifs purs, motifs mixtes, périodes. SMF. Panoramas et Synthèses. No. 17. 2004.

[3] P. Balmer, M. Schlichting: Idempotent completion of triangulated categories. J. Algebra 236. No. 2, 2001.

[4] A. Beilinson: Height pairing between algebraic cycles. In K-Theory, Arithmetic and Geometry. LNM1289. 1987.

[5] S. Bloch: Algebraic cycles and the Lie algebra of mixed Tate motives. J. AMS 4, No. 4. 1991.

[6] S. Bloch, I. Kriz: Mixed Tate motives. Ann of Math. 140, no.3. 1994. 557-605.

[7] A. Bondal, M. Van den Bergh: Generators and representability of functors in commutative and noncommutative geometry. Moscow Math. J. 3. 2003. 1-36.

[8] A. Borel: Stable real cohomology of arithmetic groups. Ann. Sci. Éc. Norm. Sup. 7. 1974. 235-272.

[9] A. Bousfield, V. Guggenheim: On PL de Rham theory and rational homotopy type. Memoirs of the A.M.S. 179. 1976.

[10] D. C. Cisinski, F. Déglise: Triangulated categories of mixed motives. 2009. arXiv: 0912.21110v3.

[11] P. Deligne: Catégories tensorielles. Mosc. Math. J. 2. 2002. 227-248.

[12] P.Deligne, J. S. Milne: Tannakian Categories. in Hodge cycles, Motives, and Shimura Varieties. LNM 900. 1982. 101-228.

[13] Y. Félix, S. Halperin, and J. C. Thomas: Rational homtopy theory. GTM 205. Springer. 2001.

[14] W. Fulton, J. Harris: Representation theory. A first course. GTM 129. Springer. 1999.

[15] E. Friedlander, A. Suslin and V.Voevodsky: Cycles, transfer and motivic homology theories. Annals of Math.Studies 143, Princeton Univ. Press. 2000.

[16] R. Hain, M. Matsumoto: Tannakian Fundamental Groups Associated to Galois Groups, Galois groups and Fundamental Groups. MSRI Publications, vol. 41. Cambridge University Press. 2003. 183-216.

[17] R. Hain, M. Matsumoto: Universal mixed elliptic motives. 2015. arXiv:1512.03975.

[18] J. C. Jantzen: Representations of algebraic group. Acad. Press. 1987.

[19] K. Kimura, T. Terasoma: Relative DGA and mixed elliptic motives. 2010. arXiv:1010.3791v3.

[20] S.-I. Kimura: Chow groups are finite-dimensional, in some sense. Math. Ann. 331, 2005. 173-201.

[21] I. Kriz, J. P. May: Operads, algebras, modules and motives. Astérisque. No. 233. 1995.

[22] M. Levine: Tate motives and the fundamental group. Cycles, Motives and Shimura Varieties. Tata Institute of Fundamental Research, Mumbai, India, August 2010. 265-392.

[23] M. Levine: Tate motives and the vanishing conjectures for algebraic K-theory. In Algbebraic K-Theory and Algebraic Topology. NATO ASI Series C. Vol. 407. 1993. 167-188.

[24] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure): Equivariant stable homotopy theory. Springer Lecture Notes. Vol 1213. 1986.

[25] S. Mac Lane: Categories for the working mathematician, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, 1998.

[26] C. Mazza: Schur functors and motives. Ph.D thesis. 2004.

[27] C. Mazza, V. Voevodsky and C. Weibel: Lecture notes on motivic cohomology. American Mathematical Society. 2006.

[28] J.S. Milne: Algebraic Groups: algebraic group schemes over fields. Course note. 2015.

[29] A. Neeman: The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. Ann. Sci. École. Norm. Sup. (4) 25 no. 5.1992. 547-566.

[30] O. Patashnick: A Candidate for the abelian category of mixed elliptic motives. Journal of K-Theory. 12(3). 2013. 569-600.

[31] R. Rouquier: Dimensions of triangulated categories. Journal of K Theory, 1. 2008.1-36

[32] M. Spitzweck: Operads, algebras and modules in model categories and motives. Ph.D thesis. 2001.

[33] D. Sullivan: Infinitesimal computations in topology. *Publications Mathématiques, Institut des Hautes Études Scientifiques*, 47.1978. 269-331.

[34] V. Voevodsky: Cancellation theorem. Doc. Math. (Extra volume: Andrei A. Suslin sixtieth birthday). 2010. 671-685.

[35] C. A. Weibel: An introduction to homological algebra. Cambridge Studies in Advanced Mathematics 38. 1995.