Combinatorial Properties of Multivariate Subdivision Scheme with Nonnegative Masks

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Zusammenfassung

Unterteilungsalgorithmen liefern wichtige Techniken zur schnellen Erzeugung von Kurven und Oberflächen. Diese spielen auch eine zentrale Rolle in Wavelets. Ein Unterteilungsalgorithmus ist durch eine Maske definiert. Es ist bekannt, dass die Konvergenz dieser Algorithmen per gemeinsamen Spektralradius charakterisiert werden kann, der durch endlich viele Matrizen definiert ist. Allerdings ist die Berechnung des gemeinsamen Spektralradius im allgemeinen sehr schwierig.

Unser Ziel ist es im multivariaten Fall einfach zu überprüfende Kriterien zu finden, die hinreichend und notwendig für die Konvergenz dieser Algorithmen sind. Die Einfachheit der Kriterien bedeutet, dass sich die Kriterien in polynomialer Zeit bzgl. der Masken, z.B. die Größe des Trägers von Masken, nachprüfen lassen.

Nach einem einleitenden Kapitel 1 und einem grundlegenden Kapitel 2 konzentrieren wir uns daher in drei Schritten auf die Klasse der multivariaten Subdivisions-Schemata mit nichtnegativen Masken. Die Dissertation ist folgendermaßen aufgebaut:

Wir beginnen zunächst in Kapitel 3 und 4 mit einer Demonstration des Zusammenhangs zwischen der Konvergenz des Subdivisions-Schemas und einiger Abbildungen für Gitter. Danach geben wir ein neues hinreichendes und notwendiges Konvergenzkriterium für nichtnegative Subdivisions-Schemata an. Theorem 3.3.1 stellt den zentralen Beitrag dieses Kapitels dar.

Darauffolgend betrachten wir in Kapitel 5 und 6, dass die Konvergenz eines nichtnegativen Subdivisions-Schemas nicht von den Werten der Maske abhängt, sondern lediglich von ihrem Träger. Wir geben die unterschiedlichen Eigenschaften zwischen inneren Punkten und Randpunkten auf ihrem Träger mit Hilfe der weiterer notwendiger Konvergenzbedingung an. Dabei stellt sich heraus, dass der Zusammenhang der Matrix A eine einfache und adäquate Bedingung ist, um diese Eigenschaften zu garantieren.

Im letzten Kaptiel leiten wir nun einfach und schnell zu berechnende hinreichende Konvergenzbedingungen für multivariate Subdivisions-Schemata mit nichtnegativer Maske her, sofern der Träger spezielle Eigenschaften besitzt. Dabei nutzen wir obige Resultate.

Nomenclature

$(x)_\alpha$	the α-coordinate of the vector $x \in \mathbb{R}^N$
$[x]$	the integer part of x
$[\Omega]$	the convex cover of Ω
$\partial[\Omega]$	boundary of the convex cover $[\Omega]$ formed by Ω
Ω_γ	$\Omega_\gamma = \Omega + \gamma$, for any $\gamma \in \mathbb{Z}^s$ with the understanding $\Omega_0 = \Omega$
$[\Omega]^o$	interior of the convex cover of Ω, i.e. $[\Omega] \setminus \partial[\Omega]$
$\dim L$	the dimension of L, where L is an affine space in $\mathbb{R}^s$
$\lvert\Gamma\rvert$	the cardinality of Γ
Γ^k	direct sum, i.e. $\Gamma^k = \Gamma + 2\Gamma + \cdots + 2^{k-1}\Gamma$
$\Gamma(a)$	an admissible set for the mask $\{a(\alpha)\}$
T^c	complement of the set T
$a^k(\alpha)$	the iterated mask of $a(\alpha)$
$c(z)$	the Laurent polynomial associated with the mask $a(\alpha)$
$\mathcal{A}(\lambda)$	the set defined by $\mathcal{A}(\lambda) = \{\alpha \,:\, a(\alpha) \neq 0 \text{ and } \alpha \equiv \lambda \pmod 2\}$
$d = \gcd(\alpha : \alpha \in \Omega)$	a multi-integer $d = (d_1, ..., d_s)$ such that $\gcd((\alpha)_i : \alpha \in \Omega) = d_i$, $i = 1, ..., s$
E^s	the set of extreme points of $[0,1]^s$, i.e., $E^s = \{(\delta_1, ..., \delta_s)^T : \delta_i \in \{0,1\}, i = 1, ..., s\}$
$\lVert \cdot \rVert_\Delta$	the norm, i.e. $\lVert x\rVert_\Delta = \max_{\alpha,\beta\in\Gamma(\alpha)} \lvert x_\alpha - x_\beta\rvert$
$\lVert A\rVert_\Delta$	the norm of a square matrix A, i.e. $\lVert A\rVert_\Delta = \sup_{\lVert x\rVert_\Delta \neq 0} \frac{\lVert Ax\rVert_\Delta}{\lVert x\rVert_\Delta}$
$\mathfrak{M}_s$	the set of $s \times s$ unimodular matrices
$\rho(A)$	spectral radius of a square matrix A
$\rho(A_1, ..., A_N)$	joint spectral radius of $\{A_1, ..., A_N\}$
χ_T	a vector in $\mathbb{R}^N$ such that $(\chi_T)_\alpha = \begin{cases} 1, & \alpha \in T, \\ 0, & \text{otherwise} \end{cases}$
F_B	a mapping for any nonnegative $N \times N$ row-stochastic matrix B by $F_B(T) = \{\alpha \in \Gamma(a) : (B\chi_T)_\alpha = 1\} \subseteq \Gamma(a)$
ψ	an additive mapping defined by $\psi(\emptyset) = \emptyset$ and $\psi(I) \subseteq \Sigma$, $\forall\ I \subseteq \Sigma$, $\Sigma \subseteq \mathbb{Z}^s$ is a finite set

Contents

Chapter 1

Introduction

Subdivision schemes, the iterative methods for producing smooth curves and surfaces with a built-in multiresolution structure, have been becoming one of the most popular methods for generating curves and surfaces in a fast way due to the facts that subdivision algorithms are recursive in nature, numerically stable, easy to implement on a computer, and therefore, have been involved in the following applications: First, they are widely used in surface modeling in computer aided geometric design (CAGD) and the animation industry. Second, these schemes are also intimately connected to wavelet bases and their associated fast bank algorithms [11]. Moreover, these schemes can be used in recursive refinements of given control points whose limit turns to be a desired visually smooth object. Furthermore, subdivision schemes can also be used in wavelet analysis.

Denote $\mathbb{Z}^s$ to be the integer lattice. A subdivision scheme is defined by a fixed finitely supported real sequence (mask) $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$, (for notational simplicity, we use $\{a(\alpha)\}$ in this dissertation). We should denote the support of $\{a(\alpha)\}$ by $\Omega = \{\alpha : a(\alpha) \neq 0\}$ and $[\Omega]$ the convex cover of Ω. The Laurent polynomial associated with this mask is defined as

$$c(z) = \sum_{\alpha} a(\alpha) z^{\alpha}$$

with $z = (z_1, ..., z_s)^T \in \mathbb{R}^s$ and $z^{\alpha} = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ for $\alpha = (\alpha_1, ..., \alpha_s)^T$.
Given an initial finite sequence of data values, $v^0 = \{v^0(\alpha)\}$, a subdivision scheme with

a mask $\{a(\alpha)\}$ defines a sequence of values $v^k(\alpha)$ recursively by the rule

$$v^k(\alpha) = \sum_\beta v^{k-1}(\beta)a(\alpha - 2\beta).$$

This scheme is said to be convergent if for each v^0 there exists a continuous function f such that

$$\lim_{k\to\infty} \sup_\alpha |f(\frac{\alpha}{2^k}) - v^k(\alpha)| = 0 \tag{1.1}$$

and $f \not\equiv 0$ for at least one v^0.

Clearly, the convergence depends only on the properties of the given mask. Moreover, we can show the convergence of the subdivision scheme is equivalent to the uniform convergence by the following argument. Let H be **the hat function** defined by

$$H(y) = \begin{cases} 1 - |y|, & |y| \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $\psi(x) = H(x_1)\cdots H(x_s)$ for $x = (x_1, ..., x_s)^T \in \mathbb{R}^s$. Using $v^k(\alpha)$ we get a "polygon" as follows

$$f^k(x) = \sum_\beta v^k(\beta)\psi(2^k x - \beta).$$

Then it is easy to see that $f^k(\beta/2^k) = v^k(\beta)$ and therefore the convergence of the subdivision scheme is equivalent to the uniform convergence of f^k. On the other hand, set $a^1(\alpha) = a(\alpha)$ and

$$a^k(\alpha) = \sum_\beta a^{k-1}(\beta)a(\alpha - 2\beta).$$

An induction argument gives $v^k(\beta) = \sum_\alpha a^k(\beta - 2^k\alpha)v^0(\alpha)$. Consequently, we have

$$\begin{aligned} f^k(x) &= \sum_\beta v^k(\beta)\psi(2^k x - \beta) \\ &= \sum_\beta \sum_\alpha a^k(\beta - 2^k\alpha)v^0(\alpha)\psi(2^k x - \beta) \\ &= \sum_\alpha v^0(\alpha) \sum_\beta a^k(\beta - 2^k\alpha)\psi(2^k x - \beta). \end{aligned}$$

Substituting β for $\beta - 2^k\alpha$, we conclude

$$\begin{aligned} f^k(x) &= \sum_\alpha v^0(\alpha) \sum_\beta a^k(\beta)\psi(2^k x - (\beta + 2^k\alpha)), \\ &= \sum_\alpha v^0(\alpha) \sum_\beta a^k(\beta)\psi(2^k(x-\alpha) - \beta). \end{aligned}$$

In particular, taking $v^0(\alpha) = \delta_0(\alpha)$, where

$$\delta_0(\alpha) = \begin{cases} 1, & \text{for } \alpha = 0, \\ 0, & \text{otherwise,} \end{cases}$$

one has $f^k(x) = \sum_\beta a^k(\beta)\psi(2^k x - \beta)$. Thus, the question whether for all given $v^0(\alpha)$ the polygon determined by mask $\{a(\alpha)\}$ converges uniformly to a curve or a surface is equivalent to the uniform convergence of

$$\sum_\beta a^k(\beta)\psi(2^k x - \beta). \tag{1.2}$$

Therefore, the convergence of the subdivision scheme is equivalent to the uniform convergence of (1.2).

In what follows, when we say the subdivision scheme converges to φ, we mean (1.2) converges to φ, which is also equivalent to the scheme with δ_0 converges to φ.

A comprehensive discussion of this subject can be found in [3]. The necessary and sufficient conditions of the convergence of the subdivision schemes with the finitely mask are known (see e.g. [9, 10, 27]) and can be summarized as follows:

Theorem 1.0.1. *A subdivision scheme associated with a fixed finitely supported real sequence (mask)* $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ *converges if and only if*

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha + 2\beta) = 1, \quad \forall\, \alpha \in \mathbb{Z}^s \tag{1.3}$$

and

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}^s,\, e\in E^s} |a^k(\alpha) - a^k(\alpha - e)| = 0, \tag{1.4}$$

where $e \in E^s :=$ *extreme points of* $[0,1]^s$, *i.e.,* $E^s = \{(\delta_1, ..., \delta_s)^T : \delta_i \in \{0,1\}, i = 1, ..., s\}$.

The first condition (1.3) is called the sum rule in the literature. It is clear and easy to check. However, the second one is rather difficult to verify. In Chapter 2 we will give two different ways to prove this theorem, which are presented in [10, 27], and more details can be found there. Here we sketch the main idea for the proof. Denote

$$\rho(\Delta a) := \limsup_{k\to\infty} \sup_{\alpha\in\mathbb{Z}^s,\, e\in E^s} |a^k(\alpha) - a^k(\alpha - e)|^{\frac{1}{k}}.$$

We first show that (1.4) is equivalent to $\rho(\Delta a) < 1$, while $\rho(\Delta a)$ is equal to the so-called joint spectral radius of some square matrices (see [3, 5, 10]). We will present some partial results concerning the computation of $\rho(\Delta a)$, which can be found in [2, 9, 30] and the papers cited there. However, as we will show, by a result in [25] the calculation of the joint spectral radius is generally NP-hard and to conquer the challenges, we will introduce some nontrivial classes of masks with which we can simply determinate whether $\rho(\Delta a) < 1$ for the given mask (see [2]).

Chapter 3 investigates subdivision schemes associated with nonnegative finite masks (i.e., $\{a(\alpha) \geq 0, \alpha \in \mathbb{Z}^s\}$) a class of masks with various applications in geometric modeling. Firstly we collect some results from [26] and [29] to establish a relation between the convergence of the multivariate subdivision scheme and some mappings about lattices, which lead to a new characterization of convergent subdivision schemes with nonnegative masks (see Theorem 3.3.1). This chapter is mainly devoted to the proof of Theorem 3.3.1.

In Chapter 4 we will focus on Theorem 3.3.1 and give some applications and extensions. Using the converse-and-negative statement of Theorem 3.3.1, we will present some examples to demonstrate the power and novel applications of our approach. Theorem 3.3.1 will also be applied to the investigation of other characterizations of convergent subdivision schemes.

In Chapter 5 we will continue to study this subject with finite masks (needn't be nonnegative). We are interested in obtaining the necessary conditions of convergent subdivision schemes in the multivariate case, by means of further analysis on the sum rule and the distribution of support. We hope that this study will help us to get some computable properties, which may lead to solve our problem. Knowing that the convergence of subdivision schemes with nonnegative masks relies on the location of its support of the mask, we consider the position of the points in the support and the convex cover of the support. In the last section of this chapter we will demonstrate the different properties between the inner and boundary points in the support of the given mask. The results show that the convergent subdivision scheme satisfies the so-called inner-point principle.

In Chapter 5 we investigate various properties between the inner and boundary points of the support for the mask, provided that the corresponding subdivision scheme converges. However, it is unknown, whether one can use some simple conditions to guarantee these properties. We find out that the so-called connectivity of a matrix A deduced by given mask (see definition in Chapter 6) is the suitable condition. Another reason to study the matrix A is the fact that in the univariate case the connectivity of the matrix A and the sum rule (1.3) ensure the convergence of the nonnegative subdivision schemes (see [31]). The intensive discussion of this matrix A is our main goal in Chapter 6. At the end of this chapter we give an efficient algorithm, which shows that the connectivity of the matrix A may be tested by depth-first search algorithm from graph theory in linear time with respect to the size of A.

We are interested in conditions on the mask to guarantee the convergence of the subdivision scheme. In the last chapter of this thesis, we take full advantage of the results in previous chapters to inspire to study the multivariate subdivision schemes with nonnegative masks. Moreover those conditions can be quickly calculated. We state the sufficient conditions for the convergence and various partial results. Theorem 7.0.1 is one of the peak points of this thesis. We draw our inspiration from Theorem 6.2.2 and conclude Theorem 7.0.1. For the proof we shall take advantage of Theorem 3.3.1. The key is to find out an irreducible (or primitive) mapping and to show the uniqueness of this mapping.

Chapter 2

Characterization of Convergent Subdivision Schemes

We will present the proofs of Theorem 1.0.1 in this chapter as the starting point for our further investigation. There exist several possible approaches to study this problem and we will use two of them: one from [27], that uses the so-called two operators approach and the other from Jia and Han in 1998 (see [10]), that is based on the estimation of the norm of some matrices, which leads to the concept of the joint spectral radius of matrices. Although Theorem 1.0.1 also remains true for L^p (by modifying the condition (1.4) correspondingly) (see e.g. [10], [15] and [27]), we mainly focus on convergence of multivariate subdivision schemes in the $C(\mathbb{R}^s)$ space.

2.1 The Laurent polynomial deduced by mask and the necessity of sum rule

Recall that for a given mask $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ the iteration $a^k(\alpha)$ is defined as

$$a^k(\alpha) = \sum_{\beta} a^{k-1}(\beta) a(\alpha - 2\beta)$$

with the understanding $a^1(\alpha) = a(\alpha)$ and $c(z)$ is the associated Laurent polynomial of $\{a(\alpha)\}$. It is easy to check that $a^k(\alpha)$ are the coefficients of the Laurent polynomial $\prod_{l=0}^{k-1} c(z^{2^l})$ where $z^\mu = z_1^\mu \cdots z_s^\mu$ and $\mu \in \mathbb{R}$. Indeed, by an induction argument on k we have

$$\begin{aligned}\prod_{l=0}^{k-1} c(z^{2^l}) &= c(z)\prod_{l=0}^{k-2} c(z^{2\cdot 2^l}) \qquad (2.1)\\ &= (\sum_{\alpha} a(\alpha)z^{\alpha})\cdot(\sum_{\beta} a^{k-1}(\beta)z^{2\beta})\\ &= \sum_{\alpha,\beta} a(\alpha)a^{k-1}(\beta)z^{\alpha+2\beta}\\ &= \sum_{\tau}(\sum_{\beta} a(\tau-2\beta)a^{k-1}(\beta))z^{\tau}\\ &= \sum_{\alpha} a^k(\alpha)z^{\alpha}.\end{aligned}$$

The other form of this product can be expressed as

$$\prod_{l=0}^{k-1} c(z^{2^l}) = \sum_{\alpha}\sum_{\beta_0+2\beta_1+\cdots+2^{k-1}\beta_{k-1}=\alpha} a(\beta_0)\cdots a(\beta_{k-1})z^{\alpha}. \qquad (2.2)$$

Using (2.2), it is easy to see that the number of non-zero coefficients of the polynomial in (2.1) can be estimated by $C2^{ks}$, where $C > 0$ depends only on the support of the given mask. In fact, the set of nonzero coefficients in $\prod_{l=0}^{k-1} c(z^{2^l})$ is contained in

$$\{\alpha : \alpha = \beta_0 + 2\beta_1 + \cdots + 2^{k-1}\beta_{k-1} \ \text{ and } \ \beta_0, \beta_1, ..., \beta_{k-1} \in \Omega\}.$$

The set is of course a subset of $[(2^k - 1)\Omega] \cap \mathbb{Z}^s$. The number of multi-integers in this set is bounded by $2^{ks}|[\Omega] \cap \mathbb{Z}^s|$.

As presented in the introduction chapter, the sum rule (see the condition (1.3) of Theorem 1.0.1) is also the necessary condition for the convergence of the subdivision scheme [3].

Lemma 2.1.1. *If a subdivision scheme associated with a fixed finitely supported real sequence (mask) $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ converges, then the mask satisfies the sum rule (1.3), that is*

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha + 2\beta) = 1, \quad \forall\, \alpha \in \mathbb{Z}^s.$$

Proof. Note that the sum rule (1.3) is equivalent to the equation

$$\sum_{\beta\in\mathbb{Z}^s} a(e+2\beta)=1, \quad \forall e\in E^s. \tag{2.3}$$

In fact, write $\alpha=e+2\gamma$ for some $\gamma\in\mathbb{Z}^s$ and $e\in E^s$, i.e., $\alpha\equiv e \pmod 2$ and denote $\beta'=\beta+\gamma$, then

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}^s} a(\alpha+2\beta) &= \sum_{\beta\in\mathbb{Z}^s} a(e+2\gamma+2\beta)\\ &= \sum_{\beta'\in\mathbb{Z}^s} a(e+2\beta').\end{aligned}$$

By hypothesis that the subdivision scheme converges to f with $f\not\equiv 0$, there exists some x_0 such that $f(x_0)\neq 0$. Let $\alpha_k'=[2^kx_0]$ be the integer part of 2^kx_0, then $\alpha_k'=2\mu_k+e_k$ for some $\mu_k\in\mathbb{Z}^s$ and $e_k\in E^s$. Set $\alpha_k=2\mu_k+e$, then $2^kx_0=\alpha_k+(e_k-e)+\epsilon_k$ and $\alpha_k\equiv e \pmod 2$, where $\epsilon_k=2^kx_0-[2^kx_0]$. Therefore $\alpha_k/2^k=x_0-(e_k-e+\epsilon_k)/2^k$,

$$\lim_{k\to\infty}\frac{\alpha_k}{2^k}=x_0 \quad\text{and}\quad \lim_{k\to\infty}\frac{e_k-e+\epsilon_k}{2^k}=0.$$

Moreover, in view of (1.1) one has

$$\lim_{k\to\infty}|f(\frac{\alpha_k}{2^k})-v^k(\alpha_k)|=0.$$

Combining with the continuity of f and (1.1), we have

$$\lim_{k\to\infty}|f(x_0)-v^k(\alpha_k)|=0.$$

Now, by the definition of the subdivision scheme, we know

$$v^k(\alpha_k)=\sum_{\beta\in\mathbb{Z}^s} v^{k-1}(\beta)a(\alpha_k-2\beta)$$

and therefore

$$\begin{aligned} f(\frac{\alpha_k}{2^k})-v^k(\alpha_k) &= f(\frac{\alpha_k}{2^k})-\sum_{\beta\in\mathbb{Z}^s} a(\alpha_k-2\beta)v^{k-1}(\beta) \qquad (2.4)\\ = \quad & f(\frac{\alpha_k}{2^k})(1-\sum_{\beta\in\mathbb{Z}^s} a(\alpha_k-2\beta))-\sum_{\beta\in\mathbb{Z}^s} a(\alpha_k-2\beta)(v^{k-1}(\beta)-f(\frac{\alpha_k}{2^k})).\end{aligned}$$

The second term of (2.4) can be written as

$$\sum_{\beta\in\mathbb{Z}^s,|\alpha_k-2\beta|\le C} a(\alpha_k-2\beta)(v^{k-1}(\beta)-f(\frac{\alpha_k}{2^k})),$$

which can be shown tending to zero.
Actually, the convergence of the scheme implies

$$\lim_{k\to\infty}\sup_{\beta}|f(\frac{2\beta}{2^k})-v^{k-1}(\beta)|=0$$

and $a(\alpha_k-2\beta)=0$ for $\alpha_k-2\beta\notin\Omega$ as well. Then

$$\lim_{k\to\infty}|\frac{2\beta-\alpha_k}{2^k}|\le\lim_{k\to\infty}|\frac{C}{2^k}|=0.$$

Moreover, from the continuity of f we have

$$\lim_{k\to\infty}|f(\frac{2\beta}{2^k})-f(\frac{\alpha_k}{2^k})|=0.$$

Therefore,

$$\begin{aligned}
&\sum_{\beta\in\mathbb{Z}^s} a(\alpha_k-2\beta)(v^{k-1}(\beta)-f(\frac{\alpha_k}{2^k}))\\
=&\sum_{\beta\in\mathbb{Z}^s,|\alpha_k-2\beta|\le C} a(\alpha_k-2\beta)(v^{k-1}(\beta)-f(\frac{2\beta}{2^k}))\\
=&\ o(1).
\end{aligned}$$

For the first term of (2.4), since $f(x_0)\neq 0$, for sufficiently large k, $f(\alpha_k/2^k)\neq 0$. Therefore

$$\lim_{k\to\infty}|1-\sum_{\beta\in\mathbb{Z}^s} a(\alpha_k-2\beta)|=0.$$

Since $\alpha_k\equiv e \pmod 2$, the sum in the last display is independent of k. Hence we establish (2.3) and consequently (1.3). □

2.2 Two operators approach

In order to elaborate the so-called two operator approach (see [27]) to establish the second condition in Theorem 1.0.1, we first introduce some new notations. Let $\mathbb{S}$ be the operator given by

$$\mathbb{S}f(\cdot)=\sum_{\alpha} a(\alpha)f(2\cdot-\alpha),$$

and define two operators $\mathbb{A}$ and $\mathbb{E} = (\mathbb{E}_1, ..., \mathbb{E}_s)$ as follows,

$$\mathbb{A}^m f(\cdot) = f(2^m \cdot), \quad m \in \mathbb{Z} \;\text{ and }\; \mathbb{E}^n f(\cdot) = f(\cdot - n), \quad n \in \mathbb{R}^s.$$

By using the Laurent polynomial $c(z)$ the operator $\mathbb{S}$ can be rewritten in the form of those two operators, namely, $\mathbb{S} = \mathbb{A}c(\mathbb{E})$. In fact,

$$\begin{aligned} \mathbb{A}c(\mathbb{E})f(\cdot) &= \sum_\alpha a(\alpha)\mathbb{A}f(\cdot - \alpha) \\ &= \sum_\alpha a(\alpha) f(2 \cdot -\alpha) = \mathbb{S}f(\cdot). \end{aligned}$$

Since the Lourent polynomial for the hat-function $H(\cdot)$ is $c_0(z) = 1/2z^{-1} + 1 + 1/2z$, we have $H = \mathbb{A}c_0(\mathbb{E})H$. Therefore for $\psi(x) = H(x_1) \cdots H(x_s)$, one has $\psi = \mathbb{A}c'(\mathbb{E})\psi$ where $c'(z)$ is another polynomial. Note that $\mathbb{E}^\alpha \mathbb{A}^m f(\cdot) = \mathbb{E}^\alpha(f(2^m \cdot)) = f(2^m(\cdot - \alpha))$ and $\mathbb{A}^m \mathbb{E}^{2^m \alpha} f(\cdot) = \mathbb{A}^m f(\cdot - 2^m \alpha) = f(2^m(\cdot - \alpha))$. Hence $\mathbb{E}^\alpha \mathbb{A}^m = \mathbb{A}^m \mathbb{E}^{2^m \alpha}$ and

$$\mathbb{S}^k = \mathbb{A}^k \prod_{l=0}^{k-1} c(\mathbb{E}^{2^l})$$

with $\mathbb{E}^n = \mathbb{E}_1^n \cdots \mathbb{E}_s^n$ provided $n \in \mathbb{R}$. Therefore, by (2.1) we conclude that

$$\mathbb{S}^k \psi(\cdot) = \sum_\beta a^k(\beta)\psi(2^k \cdot -\beta).$$

Thus, the convergence of a given subdivision scheme is equivalent to the uniform convergence of $\mathbb{S}^k\psi$, when $k \to \infty$. In other words, to show the convergence of subdivision schemes, we only need to find the conditions for the mask $\{a(\alpha)\}$ such that the iteration $\mathbb{S}^k\psi$ has a limit φ. If this is the case, then φ is the fixed point of $\mathbb{S}$, i.e.,

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\varphi(2x - \alpha), \quad x \in \mathbb{R}^s$$

and the curve or the surface generated by v^0 is

$$f(x) = \sum_\alpha v^0(\alpha)\varphi(x - \alpha).$$

The function φ obtained in this way is clearly compactly supported and the support is contained in $[\Omega]$. In fact, from Section 2.1 we know that $a^k(\beta) \neq 0$ implies $\beta =$

$\alpha_0 + \alpha_1 2 + \cdots + \alpha_{k-1} 2^{k-1}$ for some $\alpha_0, ..., \alpha_{k-1} \in \Omega$ (see (2.2)). Hence if $x \notin [\Omega]$ then $2^k x - \beta \notin (-1,1)^s$ in case of $a^k(\beta) \neq 0$. Consequently, $\mathbb{S}^k\psi(x) = 0$ for those x and so $\varphi(x) = 0$.

Suppose the total degree of the Laurent polynomial $c(z)$ deduced by the mask $\{a(\alpha)\}$ as $N/2$, i.e., if $a(\alpha) \neq 0$ then $|\alpha| \leq N/2$. Associated with this N, define

$$\mathbb{R}_N := \{x \in \mathbb{R}^s : \ |x| \leq 2N\}$$

and let $C(\mathbb{R}_N)$ be the set of complex-valued continuous functions with support $\mathbb{R}_N$ equipped with the maximal norm. Then the norm of the iteration of this operator is given by

$$\|\mathbb{S}^k(I - \mathbb{E}_i)\| = \sup_{f \in C(\mathbb{R}_N), \|f\| \neq 0} \frac{\|\mathbb{S}^k(I - \mathbb{E}_i)f\|}{\|f\|}$$

and

$$\rho(\Delta\mathbb{S}) := \limsup_{k\to\infty} \max_{1\leq i\leq s} \|\mathbb{S}^k(I - \mathbb{E}_i)\|^{\frac{1}{k}}.$$

Although this is not the standard definition of the spectral radii for operators, we still refer $\rho(\Delta\mathbb{S})$ as to the spectral radius of $\mathbb{S}$.

For two sequences $\{x_k\}$ and $\{y_k\}$, if there exists a constant $C > 0$, independent of k, such that

$$C^{-1} y_k \leq x_k \leq C y_k, \quad k = 1, 2, ...,$$

we denote $x_k \sim y_k$. For a vector $x = (x_1, ..., x_s)^T \in \mathbb{R}^s$, we denote $(x)_j$ as the j-component of x, i.e., $(x)_j = x_j$.

Now we present the following main result of the two operator approach. The following result was established in [27], which shows the necessary and sufficient conditions such that $\mathbb{S}^k\psi$ converges to φ, in term of the spectral radius $\rho(\Delta\mathbb{S})$.

Theorem 2.2.1. *Suppose $\mathbb{S} = \mathbb{A}c(\mathbb{E})$ with $\sum_\alpha a(\alpha) = 2^s$, where $c(z)$ is the Laurent polynomial with respect to the mask $\{a(\alpha)\}$. Then $\mathbb{S}^k\psi$ converges in the maximal norm to a function $\varphi \in C(\mathbb{R}_N)$ if and only if $\rho(\Delta\mathbb{S}) < 1$.*

In order to prove Theorem 2.2.1, we first claim two Lemmas (see [27]). The first Lemma illustrates the relationship between the norms of $\mathbb{S}^k$ and the corresponding mask $\{a(\alpha)\}$.

Lemma 2.2.2. *Let* $\mathbb{S} = \mathbb{A}c(\mathbb{E})$ *with* $c(1) = 2^s$. *Then*

$$\| \mathbb{S}^k(I - \mathbb{E}_i) \| \sim \{\max_\alpha |\Delta_{e_i} a^k(\alpha)|\}, \tag{2.5}$$

where $e_i \in E^s$ *is the* i*-th unit vector and*

$$\Delta_{e_i} a^k(\alpha) = a^k(\alpha) - a^k(\alpha - e_i).$$

Moreover,

$$\liminf_{k\to\infty} \max_{1\le i\le s} \| \mathbb{S}^k(I - \mathbb{E}_i) \| = 0 \tag{2.6}$$

implies

$$\sum_\alpha a(\beta + 2\alpha) = 1, \quad \forall\ \beta \in \mathbb{Z}^s. \tag{2.7}$$

Proof. For an arbitrary continuous function $f \in C(\mathbb{R}_N)$, we have

$$\begin{aligned}
\mathbb{S}^k(I - \mathbb{E}_i)f(\cdot) &= \mathbb{A}^k \prod_{l=0}^{k-1} c(\mathbb{E}^{2^l})(I - \mathbb{E}_i)f(\cdot) \\
&= \mathbb{A}^k \sum_\alpha a^k(\alpha)\mathbb{E}^\alpha(I - \mathbb{E}_i)f(\cdot) \\
&= \mathbb{A}^k \sum_\alpha a^k(\alpha)\left(f(\cdot - \alpha) - f(\cdot - (\alpha + e_i))\right) \\
&= \mathbb{A}^k \sum_\alpha \left(a^k(\alpha) - a^k(\alpha - e_i)\right) f(\cdot - \alpha) \\
&= \sum_\alpha \{\Delta_{e_i} a^k(\alpha)\} f(2^k \cdot - \alpha).
\end{aligned}$$

Thus,

$$|\mathbb{S}^k(I - \mathbb{E}_i)f(x)| \le \max_\alpha |\Delta_{e_i} a^k(\alpha)| \sum_\alpha |f(2^k x - \alpha)|.$$

It follows from $\text{supp} f \subseteq \mathbb{R}_N$ that if $f(2^k x - \alpha) \neq 0$ then $2^k x - \alpha \in \mathbb{R}_N$. Hence for any x and $\alpha \in \mathbb{Z}^s$ such that $f(2^k x - \alpha) \neq 0$, the number of those α is bounded by some constant C_N, dependent only on N. Therefore, we conclude that

$$\| \mathbb{S}^k(I - \mathbb{E}_i) \| \le C_N\{\max_\alpha |\Delta_{e_i} a^k(\alpha)|\}. \tag{2.8}$$

On the other hand, let $f = \psi$. Then for $\alpha' \in \mathbb{Z}^s$ and $x' = \alpha'/2^k$ we have

$$\psi(2^k x' - \alpha) = \begin{cases} 1, & \alpha = \alpha', \\ 0, & \alpha \neq \alpha'. \end{cases}$$

Hence,

$$\Delta_{e_i} a^k(\alpha') = \Delta_{e_i} a^k(\alpha')\psi(2^k x' - \alpha') = \sum_{\alpha} \{\Delta_{e_i} a^k(\alpha)\}\psi(2^k x' - \alpha).$$

Noticing $\|\psi\| = 1$, we obtain

$$\begin{aligned} \max_{\alpha} |\Delta_{e_i} a^k(\alpha)| &\leq \| \mathbb{S}^k(I - \mathbb{E}_i)\psi \| \\ &\leq \| \mathbb{S}^k(I - \mathbb{E}_i) \| \cdot \|\psi\| \\ &= \| \mathbb{S}^k(I - \mathbb{E}_i) \| . \end{aligned}$$

The assertion (2.5) directly follows from (2.8) and the last inequality.
To prove the second assertion, we use the fact that (2.7) is equivalent to $c(x) = 0$ for all $x \in \{-1,1\}^s$ and $x \neq (1,1,...,1)^T$. In fact, by the definition

$$c(z) = \sum_{\alpha} a(\alpha) z^{\alpha} = \sum_{e \in E^s} z^e \sum_{\beta} a(2\beta + e) z^{2\beta}.$$

Hence, for $x \in \{-1,1\}^s$ and $x \neq (1,1,...,1)^T$, the equality $x^{2\beta} = 1$ always holds and

$$c(x) = \sum_{e \in E^s} x^e \sum_{\beta} a(2\beta + e).$$

If $s = 1$, then $E^1 = \{0,1\}$. Since $c(1) = 2$, we obtain that $c(-1) = 0$ is equivalent to $\sum_{\beta} a(2\beta) = \sum_{\beta} a(2\beta + 1) = 1$.
In the case of $s \geq 2$, let $c(x) = 0$ for all $x \in \{-1,1\}^s$ and $x \neq (1,1,...,1)^T$. Now we set $\eta = (x_1, y)^T$ with $y \in \{-1,1\}^{s-1}$ and $e = (\delta, e')^T$ with $e' \in E^{s-1}$. Then

$$\sum_{e \in E^s} \eta^e \sum_{\beta} a(2\beta + e) = \sum_{e' \in E^{s-1}} x_1^0 y^{e'} \sum_{\beta} a(2\beta + (0,e')^T) + \sum_{e' \in E^{s-1}} x_1 y^{e'} \sum_{\beta} a(2\beta + (1,e')^T).$$

Thus, for $x_1 = -1$ and $y = (1,...,1)^T$ we have $c((-1,y)^T) = 0$ and $c(1) = 2^s$. Therefore,

$$\sum_{e' \in E^{s-1}} \sum_{\beta} a(2\beta + (0,e')^T) = \sum_{e' \in E^{s-1}} \sum_{\beta} a(2\beta + (1,e')^T) = 2^{s-1}.$$

Moreover, by choosing $x = (-1,y)^T$ and $x = (1,y)^T$ with $y \in \{-1,1\}^{s-1} \setminus \{(1,...,1)^T\}$, respectively, we obtain

$$\sum_{e' \in E^{s-1}} y^{e'} \sum_{\beta} a(2\beta + (0,e')^T) = \sum_{e' \in E^{s-1}} y^{e'} \sum_{\beta} a(2\beta + (1,e')^T) = 0.$$

Hence, by an induction argument on s, we get $\sum_\beta a(2\beta + e) = 1$ for all $e \in E^s$, which gives (2.7) and the vice versa.

Now we show that (2.6) implies $c(x) = 0$ for all $x \in \{-1, 1\}^s$ and $x \neq (1, 1, ..., 1)^T$. Let $e_i \in E^s$ such that only the i-th coordinate is 1. Thus for $z = (z_1, ..., z_s)^T$ we have $z^{e_i} = z_i$ and by the Abel's transformation

$$\begin{aligned}(1 - z_i)\prod_{l=0}^{k-1} c(z^{2^l}) &= \sum_\alpha a^k(\alpha)(z^\alpha - z^{\alpha+e_i}) \\ &= \sum_\alpha \Delta_{e_i} a^k(\alpha) z^\alpha.\end{aligned}$$

Moreover, let $x \in \{-1, 1\}^s$ such that $(x)_i = -1$, then $c(x^{2^l}) = 2^s$ for $l = 1, ..., k-1$. In this case, the above identity implies

$$2^{(k-1)s+1} c(x) = \sum_\alpha \Delta_{e_i} a^k(\alpha) x^\alpha.$$

It is easy to see that the number of α in the above sum is bounded by $C2^{ks}$ as mentioned in Section 2.1. Therefore,

$$2^{(k-1)s+1}|c(x)| \leq C2^{ks} \max_\alpha |\Delta_{e_i} a^k(\alpha)|$$

or

$$|c(x)| \leq C2^{s-1} \max_\alpha |\Delta_{e_i} a^k(\alpha)|.$$

Consequently, it follows from (2.5) that, for some constant $C > 0$, which does not depend on k,

$$|c(x)| \leq C \max_{1\leq j\leq s} \|\mathbb{S}^k(I - \mathbb{E}_j)\|.$$

The right hand side is independent on the choice of x and hence tending to 0, as $k \to \infty$ by (2.6). Therefore, $c(x) = 0$ for all $x \in \{-1, 1\}^s$ and $x \neq (1, ..., 1)^T$. $\square$

Next, we estimate the quantity $\rho(\Delta\mathbb{S})$. Although $\rho(\Delta\mathbb{S})$ does not follow the standard definition of the spectral radius of $\mathbb{S}$, the following result shows that $\rho(\Delta\mathbb{S})$ has the similar property as a spectral radius, namely, the constant $\rho^k(\Delta\mathbb{S})$ is a lower bound of $\sup_{1\leq i\leq s} \|\mathbb{S}^k(I - \mathbb{E}_i)\|$.

Lemma 2.2.3. *Let $\mathbb{S} = \mathbb{A}c(\mathbb{E})$ with $\sum_\alpha a(\beta + 2\alpha) = 1$ for all $\beta \in \mathbb{Z}^s$. Then, there exists a constant $C > 0$ such that for all $k \geq 1$,*

$$\rho^k(\Delta\mathbb{S}) \leq C \sup_{1\leq i\leq s} \|\mathbb{S}^k(I - \mathbb{E}_i)\|.$$

Proof. We start from considering a simple case $s = 1$ to illustrate the idea of proof. In this case the condition implies $c(z) = b(z)(1 + z)$ for some polynomial b. Since $\mathbb{E}^k\mathbb{A}^m = \mathbb{A}^m\mathbb{E}^{2^m k}$ and $(I - \mathbb{E})\prod_{j=0}^{k-1}(I + \mathbb{E}^{2^j}) = (I - \mathbb{E}^{2^k})$, we have

$$\begin{aligned}
\mathbb{S}^k(I - \mathbb{E}) &= \mathbb{A}^k \prod_{j=0}^{k-1} b(\mathbb{E}^{2^j})(I + \mathbb{E}^{2^j}) \cdot (I - \mathbb{E}) \\
&= \mathbb{A}^k \prod_{j=0}^{k-1} b(\mathbb{E}^{2^j}) \cdot (I - \mathbb{E}^{2^k}) \\
&= (I - \mathbb{E})\mathbb{T}^k,
\end{aligned}$$

where $\mathbb{T} := \mathbb{A}b(\mathbb{E})$.

Although the proof for $s \geq 2$ is similar, we present the proof here for easy reference. To do that, we need to compare $\mathbb{S}^k(I - \mathbb{E}_i)$ with the iteration of a matrix operator. Using the notations in the proof of Lemma 2.2.2, we have

$$c(z) = \sum_{e\in E^s} z^e \sum_\alpha a(2\alpha + e) z^{2\alpha}.$$

Denote $b_e(z^2) := \sum_\alpha a(2\alpha + e)z^{2\alpha}$, then $b_e(1) = 1$. Thus, the Taylor's formula implies

$$b_e(z^2) = 1 + \sum_{j=1}^{s}(1 - z_j^2)p_{j,e}(z).$$

Furthermore, since

$$\sum_{e\in E^s} z^e = \prod_{j=1}^{s}(1 + z_j),$$

there exist polynomials $q_{i,j}(z)$ such that

$$(1 - z_i)c(z) = (1 - z_i)\sum_{e\in E^s} z^e(1 + \sum_{j=1}^{s}(1 - z_j^2)p_{j,e}(z)) = \sum_{j=1}^{s} q_{i,j}(z)(1 - z_j^2).$$

Following the idea suggested in [3], we denote $Q(z)$ as the matrix $(q_{i,j}(z))_{1\le i,j\le s}$. Then the above calculation implies

$$(1-z_1,...,1-z_s)^T c(z) = Q(z)(1-z_1^2,...,1-z_s^2)^T.$$

Thus

$$\begin{aligned}(1-z_1,...,1-z_s)^T c(z)c(z^2) &= Q(z)(1-z_1^2,...,1-z_s^2)^T c(z^2)\\ &= Q(z)Q(z^2)(1-z_1^{2^2},...,1-z_s^{2^2}).\end{aligned}$$

Consequently,

$$(1-z_1,...,1-z_s)^T \prod_{l=0}^{k-1} c(z^{2^l}) = \prod_{l=0}^{k-1} Q(z^{2^l})(1-z_1^{2^k},...,1-z_s^{2^k})^T.$$

On the other hand, if we define $\mathbb{T} := \mathbb{A}Q(\mathbb{E})$ to be a matrix operator and $\mathcal{I}-\mathcal{E} := (I-\mathbb{E}_1,...,I-\mathbb{E}_s)^T$, then

$$\mathbb{A}^k(I-\mathbb{E}_1,...,I-\mathbb{E}_s)^T \prod_{l=0}^{k-1} c(\mathbb{E}^{2^l}) = \mathbb{A}^k \prod_{l=0}^{k-1} Q(\mathbb{E}^{2^l})(I-\mathbb{E}_1^{2^k},...,I-\mathbb{E}_s^{2^k})^T.$$

That is

$$\mathbb{S}^k(\mathcal{I}-\mathcal{E}) = \mathbb{T}^k(\mathcal{I}-\mathcal{E}^{2^k}).$$

Again by $\mathbb{E}^k\mathbb{A}^m = \mathbb{A}^m\mathbb{E}^{2^m k}$, we obtain

$$\mathbb{S}^k(\mathcal{I}-\mathcal{E}) = (\mathcal{I}-\mathcal{E})\mathbb{T}^k. \tag{2.9}$$

Now we show that the operator $\mathcal{I}-\mathcal{E}$ on the right hand side of (2.9) can be dropped, based on which $\|\mathbb{S}^k(\mathcal{I}-\mathcal{E})\| \sim \|\mathbb{T}^k\|$ and the desired inequality follows from this estimate.

In fact, for $F = (f_1,...,f_s)^T$ with $f_j \in C(\mathbb{R}_N)$ and $j = 1,...,s$, we define in the usual way the norm of $(\mathcal{I}-\mathcal{E})\mathbb{T}^k F$. With this agreement we claim that

$$\|(\mathcal{I}-\mathcal{E})\mathbb{T}^k F\| \sim \|\mathbb{T}^k F\|. \tag{2.10}$$

If we rewrite $\mathbb{T}^k F = (g_1,...,g_s)^T$, then it is clear that $g_j \in C(\mathbb{R}_N)$ by the definition of the $\mathbb{R}_N$. Since

$$\max_{1\le j\le s} |g_j(x) - g_j(x-e_i)| \le C \max_{1\le j\le s} |g_j(x)|,$$

we conclude

$$\|(\mathcal{I}-\mathcal{E})\mathbb{T}^k F\| \le C\|\mathbb{T}^k F\|. \tag{2.11}$$

Let us now show

$$\|\mathbb{T}^k F\| \le C\|(\mathcal{I}-\mathcal{E})\mathbb{T}^k F\|,$$

for a constant C, that does not depend on k. To this end, for a fixed $i = 1, ..., s$, we set $I_\mu := \{x = (x_1, ..., x_s)^T \in \mathbb{R}_N : x_i \in [-2N + \mu - 1, -2N + \mu]\}$, which satisfies

$$\bigcup_{\mu=1}^{4N} I_\mu = \mathbb{R}_N.$$

For $\mu = 1$, we have $I_1 = \{x \in \mathbb{R}_N : x_i \in [-2N, -2N + 1]\}$, then

$$\|g_j\|_{C(I_1)} = \|g_j - \mathbb{E}_i g_j\|_{C(I_1)} = \|(I - \mathbb{E}_i)g_j\|_{C(I_1)} \le \|(I - \mathbb{E}_i)g_j\|,$$

since $\mathbb{E}_i g_j(x) = 0$ for $x \in I_1$.
For $2 \le \mu \le 4N$,

$$\begin{aligned}
\|g_j\|_{C(I_\mu)} &= \|g_j - \mathbb{E}_i g_j + \mathbb{E}_i g_j\|_{C(I_\mu)} \\
&\le \|(I - \mathbb{E}_i)g_j\|_{C(I_\mu)} + \|\mathbb{E}_i g_j\|_{C(I_\mu)} \\
&\le \|(I - \mathbb{E}_i)g_j\| + \|g_j\|_{C(I_{\mu-1})}.
\end{aligned}$$

Repeatedly, we conclude

$$\|g_j\|_{C(I_{\mu-1})} \le \|(I - \mathbb{E}_i)g_j\| + \|g_j\|_{C(I_{\mu-2})}.$$

Therefore, for all $1 \le \mu \le 4N$,

$$\begin{aligned}
\|g_j\|_{C(I_\mu)} &\le 2\|(I - \mathbb{E}_i)g_j\| + \|g_j\|_{C(I_{\mu-2})} \\
&\cdots \\
&\le (\mu - 1)\|(I - \mathbb{E}_i)g_j\| + \|g_j\|_{C(I_1)} \\
&\le \mu\|(I - \mathbb{E}_i)g_j\|.
\end{aligned}$$

According to

$$\bigcup_{\mu=1}^{4N} I_\mu = \mathbb{R}_N,$$

there is a constant $C_N > 0$ independent of g_j such that

$$\|g_j\| \le C_N \|(I - \mathbb{E}_i) g_j\|, \quad i, j = 1, ..., s.$$

Therefore,

$$\|\mathbb{T}^k F\| \le C_N \|(\mathcal{I} - \mathcal{E})\mathbb{T}^k F\|. \tag{2.12}$$

The desired assertion follows from (2.11) and (2.12), i.e.

$$\|(\mathcal{I} - \mathcal{E})\mathbb{T}^k F\| \sim \|\mathbb{T}^k F\|.$$

After proving (2.10), we get by (2.9)

$$\max_{1 \le i \le s} \|\mathbb{S}^k (I - \mathbb{E}_i)\| \sim \|\mathbb{T}^k\|, \tag{2.13}$$

which in turn implies

$$\rho(\Delta\mathbb{S}) = \lim_{k \to \infty} \|\mathbb{T}^k\|^{\frac{1}{k}}.$$

The right side of the above equality is the usual definition for the spectral radius of $\mathbb{T}$. Therefore, one has for all $k \ge 1$ the estimate

$$\lim_{l \to \infty} \|\mathbb{T}^l\|^{\frac{1}{l}} \le \|\mathbb{T}^k\|^{\frac{1}{k}}.$$

From this inequality and (2.13), we conclude that there exists a constant $C > 0$ such that for all $k \ge 1$,

$$\rho^k(\Delta\mathbb{S}) \le C \sup_{1 \le i \le s} \|\mathbb{S}^k (I - \mathbb{E}_i)\|,$$

which gives the desired assertion. □

By Lemmas 2.2.2 and 2.2.3, we obtain

$$\rho(\Delta\mathbb{S}) = \lim_{k \to \infty} \{ \max_{\alpha \in \mathbb{Z}^s, 1 \le i \le s} |\triangle e_i a^k(\alpha)| \}^{\frac{1}{k}}.$$

According to Theorem 2.2.1 the second condition in Theorem 1.0.1 is equivalent to $\rho(\Delta\mathbb{S}) < 1$. We now proceed to the proof of Theorem 2.2.1.

Proof of the Theorem 2.2.1 . We first discuss the necessity of the condition. On one hand, if $\mathbb{S}^k \psi$ converges to a function $\varphi \in C(\mathbb{R}_N)$ in a continuous norm, then, for $i = 1, ..., s$,

$$\lim_{k \to \infty} \|(I - \mathbb{E}_i^{\frac{1}{2^k}})\mathbb{S}^k \psi\| = 0.$$

On the other hand, using $\mathbb{E}^k\mathbb{A}^m = \mathbb{A}^m\mathbb{E}^{2^m k}$ and Lemma 2.2.2, we have for $i = 1, ..., s$,

$$\|(I - \mathbb{E}_i^{\frac{1}{2^k}})\mathbb{S}^k\psi\| = \|\mathbb{S}^k(I - \mathbb{E}_i)\psi\| \sim \|\mathbb{S}^k(I - \mathbb{E}_i)\|.$$

Thus, $\lim_{k\to\infty} \|\mathbb{S}^k(I - \mathbb{E}_i)\| = 0$ and

$$\liminf_{k\to\infty} \max_{1\le i\le s} \|\mathbb{S}^k(I - \mathbb{E}_i)\| = 0.$$

Moreover, by Lemma 2.2.3, we have for any fixed $0 < \varepsilon < 1$ and sufficiently large k ,

$$\rho^k(\Delta\mathbb{S}) \le C \sup_{1\le i\le s} \|\mathbb{S}^k(I - \mathbb{E}_i)\| < \varepsilon,$$

which implies $\rho(\Delta\mathbb{S}) < 1$ as desired.

To show the sufficiency, we suppose $\rho(\Delta\mathbb{S}) < 1$. Note that for ψ there is a Laurent polynomial $c'(z)$ such that $\psi = \mathbb{A}c'(\mathbb{E})\psi$. On the other hand, by the second assertion of Lemma 2.2.2, $\rho(\Delta\mathbb{S}) < 1$ implies $\sum_\alpha a(\beta + 2\alpha) = 1$ for $\beta \in \mathbb{Z}^s$. Since $c(x) - c'(x) = 0$ for $x \in \{-1, 1\}^s$, then the Taylor's formula means that there exist some polynomials p_j satisfying

$$c(z) - c'(z) = \sum_{j=1}^{s} (1 - z_j^2)p_j(z).$$

The detailed proof can be found in Lemma 2.2.3.

Hence, $\mathbb{S}\psi - \psi = \mathbb{A}c(\mathbb{E})\psi - \mathbb{A}c'(\mathbb{E})\psi = \mathbb{A}(c(\mathbb{E}) - c'(\mathbb{E}))\psi$. Using these two identities and $\mathbb{E}^k\mathbb{A}^m = \mathbb{A}^m\mathbb{E}^{2^m k}$, we obtain, for some $C > 0$, $0 < r < 1$ and k large enough,

$$\begin{aligned}
\|\mathbb{S}^{k+1}\psi - \mathbb{S}^k\psi\| &= \|\sum_{j=1}^{s} \mathbb{S}^k\mathbb{A}(I - \mathbb{E}_j^2)p_j(\mathbb{E})\psi\| \\
&= \|\sum_{j=1}^{s} \mathbb{S}^k(I - \mathbb{E}_j)\mathbb{A}p_j(\mathbb{E})\psi\| \\
&\le C \max_{1\le j\le s} \|\mathbb{S}^k(I - \mathbb{E}_j)\| \\
&\le Cr^k.
\end{aligned}$$

Thus, $\mathbb{S}^k\psi$ is a Cauchy sequence, whose limit is a nonzero function φ as mentioned at the beginning of this section. The proof is now complete. □

2.3 Characterization by matrices

In this section we give further characterizations for the convergence of the subdivision scheme by the collection matrices associated with the corresponding mask. First of all, we introduce the concept of an admissible set, based on which the matrices can be defined.

A finite set $\Gamma(a) \subset \mathbb{Z}^s$ is defined to be an **admissible set** for the mask $\{a(\alpha)\}$ provided that if $\alpha \in \Gamma(a)$ and $\beta \notin \Gamma(a)$, there holds $a(2\beta - \alpha + e) = 0$ for every $e \in E^s$. It is known (see [10]) that an admissible set $\Gamma(a)$ can be characterized as follows.

Lemma 2.3.1. *A finite set* $\Gamma(a) \subset \mathbb{Z}^s$ *is an admissible set if and only if*

$$\frac{\Gamma(a) + \Omega - e}{2} \cap \mathbb{Z}^s \subseteq \Gamma(a), \ \forall \ e \in E^s, \tag{2.14}$$

where Ω *is the support of mask* $\{a(\alpha)\}$*.*

Proof. Suppose $\Gamma(a)$ is an admissible set. If (2.14) is not true, there exists $\beta \in (\Gamma(a) + \Omega - e)/2 \cap \mathbb{Z}^s$ such that $\beta \notin \Gamma(a)$. By the definition of the admissible set, we get $a(2\beta - \alpha + e) = 0$, for $\alpha \in \Gamma(a)$. That is $2\beta - \alpha + e \notin \Omega$. But, on the other hand, there is some $\gamma \in \Omega$ and $\alpha \in \Gamma(a)$ such that $\beta = (\alpha + \gamma - e)/2$, or $\gamma = e - \alpha + 2\beta \in \Omega$, i.e., $a(2\beta - \alpha + e) = a(\gamma) \neq 0$, which is a contradiction.

Conversely, suppose (2.14) holds. Then it follows that $2\beta - \alpha + e \notin \Omega$ or $a(2\beta - \alpha + e) = 0$ for any $\alpha \in \Gamma(a)$ and any $\beta \notin \Gamma(a)$. Hence $\Gamma(a)$ is an admissible set. $\square$

In order to understand the concept of admissible sets better, we give a simple example of the construction of admissible sets. If the mask $\{a(\alpha)\}$ has the property that $a(\alpha) = 0$ for $\alpha \notin \Gamma_{k,k'}$, where $k = (k_1, ..., k_s)^T, \ k' = (k'_1, ..., k'_s)^T \in \mathbb{Z}^s$ and

$$\Gamma_{k,k'} = \{\alpha \in \mathbb{Z}^s : \ \alpha = (\alpha_1, ..., \alpha_s)^T, \ k_i \leq \alpha_i \leq k'_i, k'_i - k_i > 1, \ i = 1, ..., s\},$$

then

$$\Gamma_{k,k'-1} = \{\alpha \in \mathbb{Z}^s : \ \alpha = (\alpha_1, ..., \alpha_s)^T, \ k_i \leq \alpha_i \leq k'_i - 1, \ i = 1, ..., s\}$$

is an admissible set for $\{a(\alpha)\}$. In particular, any rectangle Q with side length greater than 1, whose extreme points are in $\mathbb{Z}^s$ and which contains Ω, is an admissible set. Moreover, we may also assume that $\Gamma(a)$ is convex (i.e., $[\Gamma(a)] \cap \mathbb{Z}^s = \Gamma(a)$) and for each $e \in E^s$, there is $\alpha \in \Gamma(a)$ such that $\alpha - e \in \Gamma(a)$. For a set $\Gamma \subset \mathbb{Z}^s$ we denote $|\Gamma|$ to be the cardinality of Γ in what follows.

Note that a square matrix $A = (A(\alpha,\beta))_{1\le\alpha,\beta\le N}$ is called a **row-stochastic matrix** if it satisfies the following two conditions:

$$1)\ \ A(\alpha,\beta) \ge 0, \quad \forall \alpha,\beta = 1,...,N,$$

$$2)\ \ \sum_{\beta=1}^{N} A(\alpha,\beta) = 1, \quad \alpha = 1,...,N.$$

A square matrix A is a **generalized row-stochastic matrix**, if A satisfies the second condition.

Suppose $\Gamma(a)$ is an admissible set. Let $N = |\Gamma(a)|$. Then for each $e \in E^s$ the $N \times N$ matrix A_e is defined by

$$A_e(\alpha,\beta) = a(-\alpha + e + 2\beta), \quad \alpha,\beta \in \Gamma(a).$$

Here, for any $N \times N$ matrix, α stands for the row index, while β accounts for the column index. We assume that the N points from $\Gamma(a)$ have been put into some order (e.g., using lexicographic order), which we assume to prevail also in subsequent formulas where the components of row vectors, or of column vectors, are indexed by pairs $\alpha,\beta \in \Gamma(a)$. Clearly, if the mask $\{a(\alpha)\}$ satisfies the sum rule (1.3), then A_e is a generalized row-stochastic matrix for all $e \in E^s$.

In order to study convergence of the subdivision scheme, we need to analyze the sequence (mask) $\{a^k(\alpha)\}, k = 1,2,...,$ in term of the matrix A_e. The following result explores the connection between the mask and the associated matrices (see [10]).

Lemma 2.3.2. *Suppose* $\alpha = -\alpha_0 + \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k + 2^k\beta_0$*, where* $\delta_1,\delta_2,...,\delta_k \in E^s$ *and* $\alpha_0,\beta_0 \in \Gamma(a)$*, then*

$$a^k(\alpha) = A_{\delta_1}\cdots A_{\delta_k}(\alpha_0,\beta_0).$$

Proof. We use an induction argument on k to prove this lemma. For $k = 1$, suppose $\alpha = -\alpha_0 + \delta_1 + 2\beta_0$, where $\alpha_0,\beta_0 \in \Gamma(a)$ and $\delta_1 \in E^s$. It is easy to see

$$A_{\delta_1}(\alpha_0,\beta_0) = a(-\alpha_0 + \delta_1 + 2\beta_0) = a^1(\alpha).$$

Suppose $k > 1$ and the lemma holds for $k-1$. Then by the hypothesis we have

$$A_{\delta_2}\cdots A_{\delta_k}(\alpha',\beta') = a^{k-1}(\alpha),$$

for any $\alpha', \beta' \in \Gamma(a)$ and $\alpha \in \mathbb{Z}^s$ such that $\alpha = -\alpha' + \delta_2 + 2\delta_3 + \cdots + 2^{k-2}\delta_k + 2^{k-1}\beta'$. Suppose now $\alpha = -\alpha_0 + \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k + 2^k\beta_0$. Because $\Gamma(a)$ is an admissible set and

$$a^k(\alpha) = \sum_\beta a^{k-1}(\beta)a(\alpha - 2\beta),$$

we obtain

$$\begin{aligned}
A_{\delta_1}A_{\delta_2}\cdots A_{\delta_k}(\alpha_0, \beta_0) &= \sum_{\tau\in\Gamma(a)} A_{\delta_1}(\alpha_0, \tau)\cdot A_{\delta_2}\cdots A_{\delta_k}(\tau, \beta_0) \\
&= \sum_{\tau\in\Gamma(a)} a(-\alpha_0 + \delta_1 + 2\tau)\cdot a^{k-1}(-\tau + \delta_2 + \cdots + 2^{k-2}\delta_k + 2^{k-1}\beta_0) \\
&= \sum_{\tau\in\mathbb{Z}^s} a(-\alpha_0 + \delta_1 + 2\tau)\cdot a^{k-1}(-\tau + \delta_2 + \cdots + 2^{k-2}\delta_k + 2^{k-1}\beta_0) \\
&= \sum_{\mu\in\mathbb{Z}^s} a(-\alpha_0 + \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k + 2^k\beta_0 - 2\mu)\cdot a^{k-1}(\mu) \\
&= \sum_{\mu\in\mathbb{Z}^s} a(\alpha - 2\mu)a^{k-1}(\mu) \\
&= a^k(\alpha).
\end{aligned}$$

The induction and the calculation above give the desired assertion. □

Before proceeding further we introduce the following notations. For a given mask $\{a(\alpha)\}$, the set $\Gamma(a)$ is an admissible set with N integers, which will be arranged as $\{\alpha_1, ..., \alpha_N\}$. For $x = (x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_N})^T \in \mathbb{R}^N$, let

$$\|x\|_\Delta = \max_{\alpha,\beta\in\Gamma(\alpha)} |x_\alpha - x_\beta|$$

be the norm in the factor space $\mathbb{R}^N/\{||x||_\Delta = 0 : x \in \mathbb{R}^N\}$ and for any generalized row-stochastic matrix A of size N, we denote

$$\|A\|_\Delta = \sup_{\|x\|_\Delta\neq 0} \frac{\|Ax\|_\Delta}{\|x\|_\Delta}.$$

Then it is easy to verify the following lemma.

Lemma 2.3.3. *Suppose A and B are two generalized row-stochastic matrices of the same size, then*

$$\|AB\|_\Delta \leq \|A\|_\Delta \cdot \|B\|_\Delta.$$

Now we are going to present the second proof of Theorem 1.0.1 by using the matrix norm $||\cdot||_\Delta$ (see also in [10]).

Theorem 2.3.4. *The subdivision scheme associated with the finite mask* $\{a(\alpha):\ \alpha\in\mathbb{Z}^s\}$, *which satisfies the sum rule (1.3), converges if and only if there exists* k_0 *such that for all* $k\geq k_0$ *and all* $\delta_j\in E^s$ *one holds*

$$\|A_{\delta_1}\cdots A_{\delta_k}\|_\Delta<1.$$

Proof. We start with the following assertion:

$$\begin{aligned}\sup_{\gamma\in\mathbb{Z}^s,e\in E^s}|a^k(\gamma)-a^k(\gamma-e)| &\leq \max_{\delta_1,\ldots,\delta_k\in E^s}\|A_{\delta_1}\cdots A_{\delta_k}\|_\Delta \\ &\leq N^2\sup_{\alpha\in\mathbb{Z}^s,e\in E^s}|a^k(\alpha)-a^k(\alpha-e)|,\end{aligned}\tag{2.15}$$

wherever $\{a(\alpha):\ \alpha\in\mathbb{Z}^s\}$ satisfies the sum rule (1.3).
In fact, let us begin with the first inequality of (2.15) and let $\gamma\in\mathbb{Z}^s$. We distinguish between the trivial case when $a^k(\gamma)=0$, $a^k(\gamma-e)=0$ and the more involved cases when $a^k(\gamma)\neq 0$ or $a^k(\gamma-e)\neq 0$.
For the trivial case, we have nothing more to do.
In the case of $a^k(\gamma)\neq 0$, it is clear that $\gamma\in\Omega^k$. Then by the definition of the admissible set (see Section 2.3), there exist $\alpha_0,\alpha_0+e\in\Gamma(a)$ such that for some $\delta_j\in E^s$,

$$\gamma+\alpha_0=\delta_1+2\delta_2+\cdots+2^{k-1}\delta_k+2^k\beta_0.$$

Therefore $\alpha_0=2^k\beta_0+\lambda-\gamma$ with the understanding $\lambda=\delta_1+2\delta_2+\cdots+2^{k-1}\delta_k$, which implies that $\beta_0\in\Gamma(a)$. It follows from Lemma 2.3.2 that

$$a^k(\gamma)=A_{\delta_1}A_{\delta_2}\cdots A_{\delta_k}(\alpha_0,\beta_0)\ \text{ and }\quad a^k(\gamma-e)=A_{\delta_1}A_{\delta_2}\cdots A_{\delta_k}(\alpha_0+e,\beta_0).$$

Assume $x=(x_{\alpha_1},...,x_{\alpha_N})^T\in\mathbb{R}^N$ with $\{\alpha_1,...,\alpha_N\}=\Gamma(a)$ such that $x_{\beta_0}=1$ and $x_\beta=0$, $\beta\neq\beta_0$. Thus,

$$a^k(\gamma)=\sum_{\beta\in\Gamma(a)}A_{\delta_1}A_{\delta_2}\cdots A_{\delta_k}(\alpha_0,\beta)x_\beta\ \text{ and }\ a^k(\gamma-e)=\sum_{\beta\in\Gamma(a)}A_{\delta_1}A_{\delta_2}\cdots A_{\delta_k}(\alpha_0+e,\beta)x_\beta.$$

We conclude by the definition of the norm and $||x||_\Delta=1$ that

$$|a^k(\gamma)-a^k(\gamma-e)|\leq\|A_{\delta_1}\cdots A_{\delta_k}\|_\Delta.$$

This inequality is also valid for the case of $a^k(\gamma - e) \neq 0$. Thus, the first estimate follows from the last inequality.

We now prove the second inequality of (2.15). As $A_{\delta_1} \cdots A_{\delta_k}$ is generalized row-stochastic, we have for any $y \in \mathbb{R}^N$

$$\begin{aligned}
\|A_{\delta_1} \cdots A_{\delta_k} y\|_\Delta &= \max_{\alpha,\alpha' \in \Gamma(a)} \Big| \sum_{\beta \in \Gamma(a)} (A_{\delta_1} \cdots A_{\delta_k}(\alpha,\beta) - A_{\delta_1} \cdots A_{\delta_k}(\alpha',\beta)) \cdot y_\beta \Big| \\
&= \max_{\alpha,\alpha' \in \Gamma(a)} \Big| \sum_{\beta \in \Gamma(a)} (A_{\delta_1} \cdots A_{\delta_k}(\alpha,\beta) - A_{\delta_1} \cdots A_{\delta_k}(\alpha',\beta)) \cdot (y_\beta - y_{\beta'}) \Big| \\
&\leq \max_{\alpha,\alpha' \in \Gamma(a)} \sum_{\beta \in \Gamma(a)} |(A_{\delta_1} \cdots A_{\delta_k}(\alpha,\beta) - A_{\delta_1} \cdots A_{\delta_k}(\alpha',\beta)| \cdot ||y||_\Delta.
\end{aligned}$$

That is

$$\begin{aligned}
\|A_{\delta_1} \cdots A_{\delta_k}\|_\Delta &\leq \max_{\alpha,\alpha' \in \Gamma(a)} \sum_{\beta \in \Gamma(a)} |(A_{\delta_1} \cdots A_{\delta_k}(\alpha,\beta) - A_{\delta_1} \cdots A_{\delta_k}(\alpha',\beta)| \qquad (2.16) \\
&\leq N \max_{\alpha,\alpha',\beta \in \Gamma(a)} |(A_{\delta_1} \cdots A_{\delta_k}(\alpha,\beta) - A_{\delta_1} \cdots A_{\delta_k}(\alpha',\beta)| \\
&\leq N \max_{\alpha,\alpha',\beta \in \Gamma(a)} |a^k(-\alpha + \lambda' + 2^k\beta) - a^k(-\alpha' + \lambda' + 2^k\beta)|,
\end{aligned}$$

where $\lambda' = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k$. Since the number of elements of $\Gamma(a)$ is N, the distance of $\alpha, \alpha' \in \Gamma(a)$ is bounded by N. The same result holds also for $-\alpha + \lambda' + 2^k\beta$ and $-\alpha' + \lambda' + 2^k\beta$. With this in mind we obtain from (2.16) that

$$\|A_{\delta_1} \cdots A_{\delta_k}\|_\Delta \leq N \max_{\alpha,\alpha' \in \mathbb{Z}^s, |\alpha - \alpha'| \leq N} |a^k(\alpha) - a^k(\alpha')|.$$

We may write $\alpha' = \alpha - e_0 - 2e_1 - \cdots - 2^p e_p$ for some $p \leq [\log_2 N]$ and $e_j \in E^s$. Thus,

$$\begin{aligned}
|a^k(\alpha) - a^k(\alpha')| &\leq |a^k(\alpha) - a^k(\alpha - e_0)| + |a^k(\alpha - e_0) - a^k(\alpha - e_0 - e_1)| \\
&+ |a^k(\alpha - e_0 - e_1) - a^k(\alpha - e_0 - e_1 - e_1)| \\
&+ |a^k(\alpha - e_0 - e_1 - e_1) - a^k(\alpha - e_0 - 2e_1 - e_2)| + \cdots \\
&+ |a^k(\alpha - e_0 - \cdots - (2^p - 1)e_p) - a^k(\alpha - e_0 - \cdots - 2^p e_p)| \\
&\leq N \max_{\alpha \in \mathbb{Z}^s, e \in E^s} |a^k(\alpha) - a^k(\alpha - e)|,
\end{aligned}$$

which gives

$$\|A_{\delta_1}\cdots A_{\delta_k}\|_\Delta \le N^2 \sup_{\alpha\in\mathbb{Z}^s, e\in E^s} |a^k(\alpha)-a^k(\alpha-e)|.$$

Moreover, let $e_1, ..., e_s$ be the canonical unit vectors of $\mathbb{Z}^s$. Then there holds

$$\begin{aligned}\sup_{\alpha\in\mathbb{Z}^s, 1\le j\le s} |a^k(\alpha)-a^k(\alpha-e_j)| &\le \sup_{\alpha\in\mathbb{Z}^s, e\in E^s} |a^k(\alpha)-a^k(\alpha-e)| \\ &\le s \sup_{\alpha\in\mathbb{Z}^s, 1\le j\le s} |a^k(\alpha)-a^k(\alpha-e_j)|.\end{aligned}$$

In other words, if the mask satisfies the sum rule (1.3), then

$$\max_{\delta_1,\ldots,\delta_k\in E^s} \|A_{\delta_1}\cdots A_{\delta_k}\|_\Delta \sim \sup_{\alpha\in\mathbb{Z}^s, 1\le j\le s} |a^k(\alpha)-a^k(\alpha-e_j)|.$$

By Lemma 2.2.2 we conclude that

$$\max_{1\le i\le s} ||\mathbb{S}^k(I-\mathbb{E}_i)|| \sim \max_{\delta_1,\ldots,\delta_k\in E^s} \|A_{\delta_1}\cdots A_{\delta_k}\|_\Delta.$$

Consequently,

$$\rho(\Delta\mathbb{S}) = \liminf_{k\to\infty} \max_{1\le i\le s} ||\mathbb{S}^k(I-\mathbb{E}_i)||^{\frac{1}{k}} = \liminf_{k\to\infty} \max_{\delta_1,\ldots,\delta_k\in E^s} \|A_{\delta_1}\cdots A_{\delta_k}\|_\Delta^{\frac{1}{k}}.$$

Then the assertion of this theorem follows from Theorem 2.2.1 □

Now we investigate $||A_{\delta_1}\cdots A_{\delta_k}||_\Delta$ for the case $s=1$. If the mask $\{a(\alpha)\}$ satisfies the sum rule, then the associated Laurent polynomial can be written as $c(z)=(1+z)b(z)$ for some polynomial b. It follows that

$$\prod_{j=0}^{k-1} c(z^{2^j}) = \frac{1-z^{2^k}}{1-z} \prod_{j=0}^{k-1} b(z^{2^j}).$$

Hence (see Section 2.2),

$$||\mathbb{S}^k(I-\mathbb{E})|| \sim ||\mathbb{T}^k|| \sim \max_\alpha |b^k(\alpha)|,$$

where the mask $\{b(\alpha)\}$ is associated with the Laurent polynomial b. Let $B_0(i,j) = b(-i+2j)$ and $B_1(i,j) = b(-i+1+2j)$ be two square matrices with $i,j\in\Gamma(a)$. We conclude from Lemma 2.3.2 that $B_{\delta_1}\cdots B_{\delta_k}(i,j) = b^k(-i+\lambda+2^kj)$ with $\lambda = \delta_1+2\delta_2+\cdots+2^{k-1}\delta_k$ and $\delta_j\in\{0,1\}$. Hence for some matrix norm $||\cdot||$ there holds

$$||A_{\delta_1}\cdots A_{\delta_k}||_\Delta \sim ||B_{\delta_1}\cdots B_{\delta_k}||.$$

We know that the **spectral radius of a square matrix** A is defined by

$$\rho(A) := \overline{\lim_{k\to\infty}} \|A^k\|^{\frac{1}{k}}, \quad \|A\| := \max_{\|v\|=1}\{\| Av \|\},$$

where $\|\cdot\|$ can be any given norm. Let $\{A_1, ..., A_N\}$ be a finite collection of square matrices of the same size and $||\cdot||$ a fixed matrix norm. Then the **joint spectral radius** of $\{A_1, ..., A_N\}$ (see [24]) is defined to be

$$\rho(A_1, ..., A_N) := \overline{\lim_{k\to\infty}} \max_{j_1,...,j_k \in \{1,...,N\}} \|A_{j_k} \cdots A_{j_1}\|^{\frac{1}{k}}.$$

In recent years much progress has been made on the joint spectral radius, and in practice, it can often be computed to satisfactory precision. Moreover, it brings interesting insights into engineering and mathematical problems. The quantity

$$\limsup_{k\to\infty} \max_{\delta_j \in \{0,1\}} ||B_{\delta_1} \cdots B_{\delta_k}||^{\frac{1}{k}}$$

is the joint spectral radius $\rho(B_0, B_1)$ of B_0 and B_1. Theorem 2.3.4 implies that the convergence of subdivision schemes can be characterized by $\rho(B_0, B_1) < 1$, when $s = 1$ (see [10]). Hence, there is a close connection between convergence of the subdivision schemes and the joint spectral radius of the collection matrices associated with the corresponding mask. However, the calculation of joint spectral radii even for those defined by masks, seems very difficult. In fact, the decision problem is in general NP-hard as showed in [25].

2.4 Decision of the joint spectral radius

In this section, we show that the decision of the joint spectral radius is generally NP-hard. In 1997, Tsitsiklis and Blondel have concluded that, unless P=NP, approximating algorithms for the joint spectral radius cannot possibly run in polynomial time. They proved in [25] the following

Theorem 2.4.1. *Let $N \geq 2$. Then to decide whether the joint spectral radius $\rho(A_1, ..., A_N) \geq m$ is NP-hard.*

Proof. Clearly, we only need to show this assertion for $N = 2$. Suppose A_0 and A_1 are square matrices with the same size. Denote

$$\Gamma = \{\{A_0, A_1\} \,:\, \rho(A_0, A_1) \geq m \text{ and } m > 0\}.$$

Since $3SAT$ is an NP-complete problem (see [8]), Γ is NP-hard, whenever $3SAT \leq_p \Gamma$, i.e., $3SAT$ can be reduced in polynomial time according to the size of $3SAT$ to Γ. To this end, let

$$3SAT = \{f(x_1, ..., x_n) : \exists x_1, ..., x_n \in \{0, 1\}, f(x_1, ..., x_n) = 1, n \in \mathbb{N}\},$$

where $f(x_1, ..., x_n)$ has the form $f(x_1, ..., x_n) = L_1 \wedge \cdots \wedge L_l$, and clauses L_i, $i = 1, ..., l$, are defined by

$$L_i = x_{i,1}^{\epsilon_{i,1}} \vee x_{i,2}^{\epsilon_{i,2}} \vee x_{i,3}^{\epsilon_{i,3}}, \;\; x_{i,1}, x_{i,2}, x_{i,3} \in \{x_1, ..., x_n\}$$

with the understanding

$$x^{\epsilon} = \begin{cases} x, & \epsilon = 1, \\ \neg x, & \epsilon = 0 \end{cases}$$

and $\neg x$ is the negation of variable x .

The instance of Γ with respect to f is a set of two adjacent matrices of two directed graphs, that depend on f. We begin with the construction of those graphs $G_0(V, E_0)$ and $G_1(V, E_1)$. The set of vertexes V for both graphs are the same. These are given as follows: for each pair $\{L_i, x_j\}$, let $w_{ij} \in V$, $i = 1, ..., l;\; j = 1, ..., n$. The vertex $w_{0j} \in V$ corresponds to x_j, $j = 1, ..., n$. Finally, $w_{i(n+1)} \in V$ is associated with each L_i and $s \in V$ is the start vertex. So we have a total of $r := |V| = (n+1)(l+1)$ vertices. Next, we construct edges in the following way:

i) For $i = 1, ..., l$ and $j = 1, ..., n-1$, let E be the edges in both $G_0(V, E_0)$ and $G_1(V, E_1)$ such that

$$(s, w_{i1}),\;\; (w_{0j}, w_{0(j+1)}) \;\text{ and }\; (w_{0n}, s) \in E.$$

ii) Moreover, let

$$\begin{aligned}
E_{0,1} &= \{(w_{ij}, w_{i(j+1)}):\ \text{if } x_j \text{ appears in } L_i,\ i = 1, ..., l;\ j = 1, ..., n\},\\
E_{0,2} &= \{(w_{ij}, w_{0j}):\ \text{if } \neg x_j \text{ appears in } L_i,\ i = 1, ..., l;\ j = 1, ..., n\},\\
E_{1,1} &= \{(w_{ij}, w_{0j}):\ \text{if } x_j \text{ appears in } L_i,\ i = 1, ..., l;\ j = 1, ..., n\},\\
E_{1,2} &= \{(w_{ij}, w_{i(j+1)}):\ \text{if } \neg x_j \text{ appears in } L_i,\ i = 1, ..., l;\ j = 1, ..., n\},\\
E_3 &= \{(w_{ij}, w_{i(j+1)}):\ \text{if } x_j \text{ or } \neg x_j \text{ does not appears in } L_i,\ i = 1, ..., l;\ j = 1, ..., n\}.
\end{aligned}$$

Now define $E_0 = E_{0,1} \cup E_{0,2} \cup E_3 \cup E$ and $E_1 = E_{1,1} \cup E_{1,2} \cup E_3 \cup E$. Note that for each $i = 1, ..., l$ and each $j = 1, ..., n$ the graphs $G_0(V, E_0)$ and $G_1(V, E_1)$ always have the edge $(w_{ij}, w_{i(j+1)})$. Furthermore, (w_{ij}, w_{0j}) is an edge of $G_0(V, E_0)$ only if $\neg x_j$ appears in L_i. The matrices A_0 and A_1 are adjacent matrices of $G_0(V, E_0)$ and $G_1(V, E_1)$, respectively. We may regard in sometimes that V is arranged as $\{1, ..., r\}$. Hence, $A_0 = (a_{u,v})_{1 \le u,v \le r}$ is defined to be

$$a_{uv} = \begin{cases} 1, & (u, v) \in E_0,\\ 0, & \text{otherwise.} \end{cases}$$

The entries of A_1 are given in the same way. It is easy to see that the construction of A_0 and A_1 can be realized in polynomial time with respect to the size of the instance f.

Next we will prove that the instance f of $3SAT$ is satisfiable if and only if $\rho(A_1, A_2) \ge l^{\frac{1}{n+2}}$, i.e., with $m = l^{\frac{1}{n+2}}$, $f \in 3SAT$, which is equivalent to $\{A_0, A_1\} \in \Gamma$. To this end, let $\alpha \in V$ and the r-dimensional column vector $x(\alpha) = (x_1, ..., x_\alpha, ..., x_r)^T$ such that

$$x_i = \begin{cases} 1, & i = \alpha,\\ 0, & i \ne \alpha. \end{cases}$$

We divide V to be

$$\begin{aligned}
P_1 &:= \{w_{0n}, w_{i(n+1)} : i = 1, ..., l\},\\
P_2 &:= \{w_{0(n-1)}, w_{in} : i = 1, ..., l\},\\
&\cdots\\
P_j &:= \{w_{0(n+1-j)}, w_{i(n+2-j)}\},\\
P_{j+1} &:= \{w_{0(n+1-j-1)}, w_{i(n+2-j-1)}\},\\
&\cdots\\
P_n &:= \{w_{01}, w_{i2} : i = 1, ..., l\},\\
P_{n+1} &:= \{w_{i1} : i = 1, ..., l\} \text{ and}\\
P_{n+2} &:= \{s\}.
\end{aligned}$$

Denote $t(\alpha) = h$, if $\alpha \in P_h$, for $1 \leq h \leq n+2$. By this definition any edge (from $G_0(V, E_0)$ or $G_1(V, E_1)$) leaving from a vertex of partition P_h, goes to a vertex of partition P_{h-1}, i.e., if $w_h \in P_h$, then there exists at least one $w_{h-1} \in P_{h-1}$ such that (w_h, w_{h-1}) is an edge. Furthermore, the unique edge in both $G_0(V, E_0)$ and $G_1(V, E_1)$ from partition P_1 to partition P_{n+2} is (w_{0n}, s). In other words, this partition builds with the edges in a cyclical form:

$$P_{n+2} \to P_{n+1} \to ... \to P_2 \to P_1 \to P_{n+2}.$$

Thus, any path in $G_0(V, E_0)$ and $G_1(V, E_1)$ starting from vertex α, i.e., $\alpha \in P_{t(\alpha)}$, either gets to a vertex $w_{i(n+1)}$, from which there is no outgoing edge, or visits node s after $t(\alpha)$ steps. The transformation of this observation into matrix terms implies the following: let α be any arbitrary vertex and $t(\alpha)$ be its associated partition index. If $h \equiv t(\alpha) \pmod{(n+2)}$ and A is a product of h matrices of A_0 and A_1, then $Ax(\alpha) = \mu x(s)$ for some $\mu > 0$. For example, if $\alpha = w_{i(n+1-j)}$ then $\alpha \in P_{j+1}$. Hence

$$A_0 x(\alpha) = \delta x(w_{0(n+1-j)}) + x(w_{i(n+2-j)}),$$

where

$$\delta = \begin{cases} 1, & (w_{i(n+1-j)}, w_{(i-1)(n+1-j)}) \in E_0,\\ 0, & \text{otherwise;} \end{cases}$$

and $A_1 x(\alpha) = (1+\varepsilon) x(w_{i(n+2-j)})$ with

$$\varepsilon = \begin{cases} 1, & (w_{i(n+1-j)}, w_{i(n+1-j)}) \in E_1, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $A_0 x(w_{0(n+1-j)}) = x(w_{0(n+2-j)})$, $A_1 x(w_{0(n+1-j)}) = x(w_{0(n+2-j)})$ and

$$A(x(w_{0n})) = x(s) \quad \text{for} \quad A = A_0 \quad \text{or} \quad A_1.$$

Next let $\delta_1, ..., \delta_n \in \{0,1\}$ be a truth assignment of f. We consider the product $A_{\delta_n} \cdots A_{\delta_1}$. Thus, with the vector $x(w_{i1})$ there holds

$$A_{\delta_n} \cdots A_{\delta_1} x(w_{i1}) = \begin{cases} x(w_{0n}), & \text{if the clause } L_i \text{ is satisfied,} \\ x(w_{i(n+1)}), & \text{otherwise.} \end{cases}$$

On the other hand, let B be any of A_0 or A_1. Because
1) there are no edges leaving from $w_{i(n+1)}$,
2) there is one edge from w_{0n} to s,
3) there are edges from s to w_{i1}, for $i = 1, ..., l$,
we have $Bx(w_{i(n+1)}) = 0$, $Bx(w_{0n}) = x(s)$ and $Bx(s) = \sum_{i=1}^{l} x(w_{i1})$. We conclude, therefore, that

$$\begin{aligned} BA_{\delta_n} \cdots A_{\delta_1} Bx(s) &= BA_{\delta_n} \cdots A_{\delta_1} \sum_{i=1}^{l} x(w_{i1}) \\ &= B \sum_{i=1}^{l} A_{\delta_n} \cdots A_{\delta_1} x(w_{i1}) \\ &= \lambda x(s), \end{aligned}$$

where λ is equal to the number of clauses that are satisfied by the given truth assignment. We further notice that λ is an eigenvalue of $BA_{\delta_n} \cdots A_{\delta_1} B$ with the eigenvector $x(s)$.

Now we prove the theorem. First assume that the instance f of $3SAT$ is satisfied by the assignment $x_i = \delta_i$ for $\delta_1, ..., \delta_n \in \{0,1\}$ and define A to be $BA_{\delta_n} \cdots A_{\delta_1} B$ with B being any of A_0 or A_1. Since all l clauses of f are satisfied, we have from the above discussion $Ax(s) = lx(s)$. By the definition of the joint spectral radius for two matrices we obtain $\rho(A_0, A_1) \geq l^{\frac{1}{n+2}}$.

On the other hand, assume that the instance f of $3SAT$ is not satisfiable. Suppose $y_t = \sum_{\alpha \in P_t} x(\alpha)$ for $t = 1, ..., n+2$ and let A be a product of $n+2$ matrices of A_0 and A_1. Since the instance f of $3SAT$ is not satisfiable, the number of clauses that are satisfied by any given truth assignment is less than l, i.e., $\lambda < l$. Consequently, we have

$$\|Ay_t\| \leq (l-1)\|y_t\| = l - 1 \text{ for } t = 1, ..., n+2,$$

where $\|\cdot\|$ is the vector maximal norm. Now let $\epsilon = \sum_{t=1}^{n+2} y_t$ and $A\epsilon = \sum_{t=1}^{n+2} Ay_t$. Clearly, the entries of ϵ are all equal to 1. The nonzero entries of Ay_t are at the same place as the nonzero entries of y_t. Hence, together with $Ay_t = \lambda y_t$, we deduce

$$\|A\epsilon\| = \|\sum Ay_t\| = \max_t \|Ay_t\| \leq l - 1.$$

The entries of A are all nonnegative and so $\|A\| = \|A\epsilon\|$ for the maximal row sum matrix norm. Thus we have $\|A\| \leq l - 1$, i.e., $\rho(A_0, A_1) \leq (l-1)^{\frac{1}{n+2}}$. $\square$

According to Theorem 2.4.1, it is usually impractical to calculate the value of the joint spectral radius by the definition. We see also that the direct estimation of this quantity has an exponentially increasing cost, if $P \neq NP$. Therefore, it is useful in practice to find some nontrivial classes of matrices, for that we can simply determinate the value of $\rho(A_1, ..., A_N)$.

2.5 Computability of two 2×2 matrices

In 2000 Bröker and Zhou (see [2]) investigated the joint spectral radius constructed by a four-coefficient mask and obtained a computable condition for the existence of a continuous, compactly supported mask. In this section we will present that, for certain families of 2×2 matrices, this joint spectral radius can be exactly calculated. The following theorem was first proved in [2].

Theorem 2.5.1. *Suppose B_0 and B_1 are two 2×2 matrices. If $\det(B_0) \leq 0$ or $\det(B_1) \leq 0$, then*

$$\rho(B_0, B_1) = \sup_{i+j \geq 1, i,j \geq 0} (\rho(B_0^i B_1^j))^{\frac{1}{i+j}}. \tag{2.17}$$

Proof. First we notice (see [1]) that the joint spectral radius $\rho(A_1,...,A_N)$ can be obtained by

$$\rho(A_1,...,A_N) = \limsup_{k\to\infty} \max_{l_1,...,l_k\in\{1,...,N\}} \rho(A_{l_1}\cdot A_{l_2}\cdots A_{l_k})^{\frac{1}{k}}.$$

Using this result we need to estimate $\rho(B_{d_1}\cdots B_{d_n})$ for all $d_1,...,d_n \in \{0,1\}$ and $n \geq 1$. To this end, denote the value on the right-hand side of (2.17) by ρ. Without loss of generality, we may assume that $\det(B_0) \leq 0$.

We need only to prove this assertion for $\det(B_0) < 0$, because of the continuity of the joint spectral radius with respect to the determinants of B_0 and B_1 (see [12]). Assume λ_1 and λ_2 are the eigenvalues of B_0, then $\det(B_0) = \lambda_1 \cdot \lambda_2 < 0$. The matrix B_0 is similar to a diagonal matrix according to the condition on this matrix. On the other hand, there holds $\rho(B_0,B_1) = \rho(MB_0M^{-1}, MB_1M^{-1})$, where M is any regular 2×2 matrix (see [11]). Therefore, we may suppose that B_0 is a diagonal matrix and write

$$B_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

To prove the assertion with the restriction $\det B_0 < 0$, we will divide the proof into two cases according to $\det(B_1) = 0$ and $\det(B_1) \neq 0$, respectively. Write $\kappa := \max\{\rho(B_0), \rho(B_1)\}$.

Case 1. $\det(B_1) \neq 0$. We notice that the trace of a 2×2 matrix B has the following property,

$$|\,|Tr(B)| - \rho(B)| \leq |\det(B)|^{\frac{1}{2}}.$$

According to this inequality, we conclude that, for any $d_j \in \{0,1\}$,

$$\begin{aligned} |\,|Tr(B_{d_1}B_{d_2}\cdots B_{d_n})| - \rho(B_{d_1}B_{d_2}\cdots B_{d_n})| &\leq |\det(B_{d_1}B_{d_2}\cdots B_{d_n})|^{\frac{1}{2}} \\ &\leq (|\det(B_{d_1})|\cdot|\det(B_{d_2})|\cdots|\det(B_{d_k})|)^{\frac{1}{2}} \\ &\leq \kappa^n. \end{aligned}$$

Furthermore, because the trace is cyclical, i.e.,

$$Tr(A_1\cdots A_n) = Tr(A_i\cdots A_nA_1\cdots A_{i-1}), \tag{2.18}$$

we may write $Tr(B_{d_1}B_{d_2}\cdots B_{d_n})$ as $Tr(B_1^{j_1}B_0^{j_2}\cdots B_1^{j_{m-1}}B_0^{j_m})$ with some $j_\tau > 0$ such that $\sum_{\tau=1}^{m} j_\tau = n$. Assume $j_m > 1$ and

$$B_1^{j_1} B_0^{j_2} \cdots B_1^{j_{m-1}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$B_1^{j_1} B_0^{j_2} \cdots B_1^{j_{m-1}} B_0^{j_m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \lambda_1^{j_m} & 0 \\ 0 & \lambda_2^{j_m} \end{pmatrix} = \begin{pmatrix} a\lambda_1^{j_m} & b\lambda_2^{j_m} \\ c\lambda_1^{j_m} & d\lambda_2^{j_m} \end{pmatrix}.$$

Therefore, if $a \cdot d \leq 0$, then $(a\lambda_1) \cdot (d\lambda_2) \geq 0$ (since $\lambda_1 \cdot \lambda_2 < 0$). We obtain

$$\begin{aligned} |Tr(B_1^{j_1} B_0^{j_2} \cdots B_1^{j_{m-1}} B_0^{j_m})| &= |a\lambda_1^{j_m} + d\lambda_2^{j_m}| \\ &= |(a\lambda_1)\lambda_1^{j_m-1} + (d\lambda_2)\lambda_2^{j_m-1}| \\ &\leq \kappa^{j_m-1}|a\lambda_1 + d\lambda_2| \\ &= \kappa^{j_m-1}|Tr(B_1^{j_1} B_0^{j_2} \cdots B_1^{j_{m-1}} B_0)| \\ &= \kappa^{j_m-1}|Tr(B_1^{j_{m-1}} B_0 B_1^{j_1} B_0^{j_2} \cdots B_1^{j_{m-3}} B_0^{j_{m-2}})|. \end{aligned}$$

If however $a \cdot d > 0$, so a and d have the same sign. We get in the same way

$$|Tr(B_1^{j_1} B_0^{j_2} \cdots B_1^{j_{m-1}} B_0^{j_m})| \leq \kappa^{j_m}|Tr(B_1^{j_{m-1}+j_1} B_0^{j_2} \cdots B_1^{j_{m-3}} B_0^{j_{m-2}})|.$$

Now it is clear that if an exponent of B_0 in the product is greater than 1, then (2.18) and the above consideration imply that we can reduce this exponent to 1 or 0. Finally, we obtain either

$$|Tr(B_{d_1} B_{d_2} \cdots B_{d_n})| \leq \kappa^l |Tr(B_1^{n-l})| \leq 2\kappa^n$$

or

$$|Tr(B_{d_1} B_{d_2} \cdots B_{d_n})| \leq \kappa^\tau |Tr(B_1^{\tau_1} B_0 B_1^{\tau_2} B_0 \cdots B_1^{\tau_p} B_0)| \qquad (2.19)$$

with $\tau + \tau_1 + \tau_2 + ... + \tau_p + p = n$.

Let us treat (2.19). Clearly, if $\det(B_1) < 0$, then the trace on the right hand side of (2.19) can be estimated by

$$|Tr(B_1^{\tau_1} B_0 B_1^{\tau_2} B_0 \cdots B_1^{\tau_p} B_0)| \leq \kappa^n + \kappa^{n-\tau-2l}|Tr((B_0 B_1)^l)|,$$

since both $\det(B_0)$ and $\det(B_1)$ are less than 0. Consequently,

$$|Tr(B_{d_1} B_{d_2} \cdots B_{d_n})| \leq \kappa^{n-2l}(\kappa^{2l} + |Tr((B_1 B_2)^l)|).$$

If however $\det(B_1) > 0$, let $\tau' := \min\{\tau_i : i = 1, ..., p\}$ and $\kappa_1 := \sup_{j\geq 0}(\rho(B_1^j B_0))^{\frac{1}{j+1}}$. In view of the property of the trace we may write for some $l_i \geq 0$,

$$Tr(B_1^{\tau_1} B_0 B_1^{\tau_2} B_0 \cdots B_1^{\tau_p} B_0) = Tr(B_1^{l_1}(B_1^{\tau'} B_0) \cdots B_1^{l_{p-1}}(B_1^{\tau'} B_0)(B_1^{\tau'} B_0)).$$

As $\det(B_1^{\tau'} B_0) < 0$ we may regard $(B_1^{\tau'} B_0)$ as the matrix B_0 in the above process to obtain either

$$|Tr(B_1^{l_1}(B_1^{\tau'} B_0) \cdots B_1^{l_{p-1}}(B_1^{\tau'} B_0)(B_1^{\tau'} B_0))| \leq \kappa_1^{\tau'+1} |Tr(B_1^{l_1}(B_1^{\tau'} B_0) \cdots B_1^{l_{p-1}}(B_1^{\tau'} B_0))|$$

or

$$|Tr(B_1^{l_1}(B_1^{\tau'} B_0) \cdots B_1^{l_{p-1}}(B_1^{\tau'} B_0)(B_1^{\tau'} B_0))| \leq \kappa_1^{2\tau'+2} |Tr(B_1^{l_1}(B_1^{\tau'} B_0) \cdots B_1^{l_{p-1}})|.$$

Hence, in this case the number of B_0 in the product

$$B_1^{\tau_1} B_0 B_1^{\tau_2} B_0 \cdots B_1^{\tau_p} B_0$$

(i.e., $B_1^{l_1}(B_1^{\tau'} B_0) \cdots B_1^{l_{p-1}}(B_1^{\tau'} B_0)(B_1^{\tau'} B_0)$) can be reduced by at least one. Repeatedly, we obtain for (2.19) in case $\det(B_1) > 0$,

$$|Tr(B_1^{\tau_1} B_0 B_1^{\tau_2} B_0 \cdots B_1^{\tau_p} B_0)| \leq \kappa_1^{\tau_1+\tau_2+\cdots+\tau_p+p-j-1} |Tr(B_1^j B_0)| = \kappa_1^{n-\tau-j-1} |Tr(B_1^j B_0)|$$

with some $j \geq 0$. Therefore, as $\kappa \leq \kappa_1 \leq \rho$ we conclude from (2.18) for the case $\det(B_1) \neq 0$ and $\det(B_0) < 0$ that

$$\rho(B_{d_1} B_{d_2} \cdots B_{d_n}) \leq |Tr(B_{d_1} B_{d_2} \cdots B_{d_n})| + \kappa^n \leq 3\rho^n.$$

The relation (see [1])

$$\rho(B_0, B_1) = \limsup_{k\to\infty} \sup_{d_i \in \{0,1\}} (\rho(B_{d_1} B_{d_2} \cdots B_{d_k}))^{\frac{1}{k}}$$

implies now $\rho = \rho(B_0, B_1)$ in case $\det(B_1) \neq 0$ and $\det(B_0) < 0$.

Case 2. $\det(B_1) = 0$. To relax the restriction $\det(B_1) \neq 0$ we note that there exists a sequence e_k, which satisfies $\lim_{k\to\infty} e_k = 0$, such that $\det(B_{1,e_k}) \neq 0$ for $B_{1,e_k} := B_1 + e_k I$, where I is the identity matrix, i.e., $I = \text{diag}(1, 1)$. The continuity of the joint spectral radius (see [12]) tells us that

$$\lim_{k\to\infty} \rho(B_0, B_{1,e_k}) = \rho(B_0, B_1).$$

In view of the above calculations for the case $\det(B_1) \neq 0$, we have for some i_k and j_k

$$\rho(B_0, B_1) = \lim_{k\to\infty} (\rho(B_0^{i_k} B_{1,e_k}^{j_k}))^{\frac{1}{i_k+j_k}}. \tag{2.20}$$

We may assume that one of the i_k and j_k tends to infinity or for a subsequence of k. Otherwise, if both i_k and j_k are bounded, our assertion is already true. Without loss of generality we may assume $j_k \to \infty$, so

$$\rho(B_0^{i_k} B_{1,e_k}^{j_k}) \leq \|B_0^{i_k}\| \cdot \|B_{1,e_k}^{j_k}\|. \tag{2.21}$$

On the other hand, for any $\varepsilon > 0$ there exists n such that

$$\|B_1^m\| \leq (\rho(B_1) + \varepsilon)^m, \quad \forall\, m \geq n.$$

Hence, for some $\eta > 0$ we have

$$\begin{aligned} \|B_{1,e_k}^{j_k}\| = \|(B_1 + e_k I)^{j_k}\| &= \|\sum_{m=0}^{j_k} \binom{j_k}{m} e_k^{j_k-m} B_1^m\| \\ &\leq \sum_{m=0}^{j_k} \binom{j_k}{m} e_k^{j_k-m} (\rho(B_1)+\varepsilon)^m + \eta e_k^{j_k-n} j_k^n \\ &\leq (\rho(B_1)+\varepsilon+e_k)^{j_k} + \eta e_k^{j_k-n} j_k^n. \end{aligned}$$

We conclude that

$$\limsup_{k\to\infty} \|B_{1,e_k}^{j_k}\|^{\frac{1}{j_k}} \leq \rho(B_1) + \varepsilon,$$

which is valid for arbitrary $\varepsilon > 0$. Thus

$$\limsup_{k\to\infty} \|B_{1,e_k}^{j_k}\|^{\frac{1}{j_k}} \leq \rho(B_1).$$

According to (2.20) and (2.21) we get

$$\begin{aligned} \rho(B_0, B_1) = \lim_{k\to\infty} (\rho(B_0^{i_k} B_{1,e_k}^{j_k}))^{\frac{1}{i_k+j_k}} &\leq \lim_{k\to\infty} (\|B_0^{i_k}\|^{\frac{1}{i_k}} \; \|B_{1,e_k}^{j_k}\|^{\frac{1}{j_k}}) \\ &\leq \max\{\rho(B_0), \rho(B_1)\}. \end{aligned}$$

Therefore, $\rho = \rho(B_0, B_1)$. □

From the proof of Theorem 2.5.1, the spectral radii of the following cases can be easily calculated. More precisely, suppose B_0 and B_1 to be two 2×2 matrices, we have
i) If $\det(B_0)\leq 0$ and $\det(B_1)\leq 0$, then

$$\rho(B_0,B_1)=\max\{(\rho(B_0B_1))^{\frac{1}{2}},\ \rho(B_0),\ \rho(B_1)\}.$$

ii) If $\det(B_0)\leq 0$ and $\det(B_1)\geq 0$, then

$$\rho(B_0,B_1)=\sup_{j\geq 0}(\rho(B_1^jB_0))^{\frac{1}{j+1}}.$$

Moreover, in the case (ii), there exists some j' so that

$$\rho(B_0,B_1)=\max\{\max_{0\leq j\leq j'}\big(\rho(B_1^jB_0)\big)^{\frac{1}{j+1}},\ \rho(B_1)\}. \tag{2.22}$$

Indeed, otherwise we would have $\rho(B_0,B_1)>\rho(B_1)$ and for any $j'>1$,

$$\begin{aligned}\rho(B_0,B_1) &= \sup_{j\geq j'}(\rho(B_1^jB_0))^{\frac{1}{j+1}}\\ &\leq \sup_{j\geq j'}(\|B_1^j\|^{\frac{1}{j+1}}\|B_0\|^{\frac{1}{j+1}}),\end{aligned}$$

which, however, implies $\rho(B_0,B_1)\leq\rho(B_1)$.

Chapter 3

Convergent Subdivision Schemes with Nonnegative Masks

In many problems arising from computer-aided geometric design, the mask is nonnegative (see [6] and [21]). Since the first example of B-spline subdivision (whose multivariate counterparts are box spline subdivision) arose, a comprehensive discussion on the particular convergence properties of the subdivision schemes with nonnegative masks are presented. In 2005 the uniform convergence of nonnegative univariate subdivision has been completely characterized in [28]. The result can be described as follows.

Theorem 3.0.1. *Let* $\{a(j) : j = 0, ..., N\}$ *be a nonnegative mask, which satisfies* $a(0), a(N) \neq 0$. *Then the univariate subdivision scheme associated with this mask converges if and only if*
1) $\sum_j a(2j) = \sum_j a(2j+1) = 1$ *and* $0 < a(0), a(N) < 1$, *and*
2) the greatest common divisor of $\{j : a(j) \neq 0\}$ *is 1.*

This result shows that the assumption (1.4) can be replaced by some very simple conditions for the univariate case. Thus, the conjecture raised in [20, 26] is confirmed. In this case, (1.4) can be tested very quickly even in linear time with respect to the size of the mask. In 2012 the same problems are considered in [14, 18] based on the so-called SIA matrices (refering to the properties of being stochastic, indecomposable and aperiodic). However, for the multivariate case, the corresponding problem is still open, i.e., whether (1.4) can be replaced by simply and computable conditions. In our investigation, we will be interested in this problem and hope to get some results, which are as good as possible.

3.1 Results for multivariate subdivision

There are various partial results on convergence of nonnegative subdivision. It is a remarkable fact that the convergence does not rely on the actual values of the mask but rather on the support of the masks, i.e., $\{\alpha : \ a(\alpha) \neq 0\}$ (see [7, 16, 17, 20, 22]). In 1999 Jia and Zhou (see [16]) characterised the convergence of the subdivision scheme by using the products of matrices when the mask is nonnegative (see also [22]). Thus, the problem of the convergence is related to several row-stochastic matrices induced by the mask. In this way, the convergence of the subdivision scheme can be determined within a finite number of steps by checking whether each finite product of those row-stochastic matrices has a positive column. In [16] (see also [22]) it is shown, among others, that the following theorem holds.

Theorem 3.1.1. *The multivariate subdivision scheme with the nonnegative finite mask* $\{a(\alpha)\}$ *converges if and only if*
1) the mask satisfies the sum rule (1.3), and
2) for each $\delta_j \in E^s, j = 1, ..., k, \ k = 2^{N^2}$*, the matrix* $A_{\delta_1} \cdots A_{\delta_k}$ *has a positive column, where* $N = |\Gamma(a)|$ *with* $\Gamma(a)$ *being an admissible set.*

Let us recall the definition of matrix A_δ introduced in Section 2.3. For each $\delta \in E^s$, let the $N \times N$ matrix A_δ be

$$A_\delta(\alpha, \beta) = a(-\alpha + \delta + 2\beta), \quad \alpha, \beta \in \Gamma(a). \tag{3.1}$$

Clearly, if the mask $\{a(\alpha)\}$ is nonnegative and satisfies the sum rule (1.3), then A_δ is row-stochastic matrix for all $\delta \in E^s$. An interesting consequence of this characterization is that the convergence of the subdivision scheme with a nonnegative mask relies only on the location of its positive coefficients. However, it seems difficult to verify the second condition of Theorem 3.1.1 since $2^{s2^{N^2}}$ different matrices need to be checked. Although we may use a result in [23] to reduce the number of matrices by $\mathcal{O}(2^{s3^N})$, it seems still unrealistic to examine so many matrices. We also note that the complexity to build the product $A_{\delta_1} \cdots A_{\delta_k}$ and to check the positivity of those products is $\mathcal{O}(N^3 2^{N^2})$.

Some partial results have been obtained, which simplify the second condition of Theorem 3.1.1. For example, the corresponding problem is solved in [29] when the support Ω of the mask $\{a(\alpha)\}$ is the so-called centered zonotope. If a nonnegative mask satisfies the sum rule (1.3) and its support is a centered zonotope, then the subdivision scheme deduced from this mask is always convergent. Nevertheless, the convergence problem for the multivariate subdivision scheme with nonnegative finite masks supported on non-centered zonotope is unresolved.

We will focus on this subject in this dissertation and present some quickly computable sufficient conditions on the convergence of the subdivision scheme with nonnegative finite masks finally. At the beginning, we will use a new approach to investigate the convergence of the subdivision schemes with nonnegative masks and try to replace the second condition of Theorem 3.1.1 by a simple and easily calculable one.

3.2 Mappings generated by masks

We will make some reductions to transform the problems of convergence of multivariate subdivision schemes into one of combinatorics and number theory in this section. Before doing so we introduce more notations and lemmas, that will be applied in the process of some proofs in the subsequent parts of this thesis.

We begin with the construction of the iterated mask $\{a^k(\alpha)\}$. Let $\Gamma \subset \mathbb{Z}^s$ and the direct sum Γ^k, $k \in \mathbb{N}$, be defined by

$$\Gamma^k = \Gamma + 2\Gamma + \cdots + 2^{k-1}\Gamma. \tag{3.2}$$

Lemma 3.2.1. *Let the finite mask $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ be nonnegative and Ω be the support of $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$. Then $\Omega^k = \{\alpha : a^k(\alpha) \neq 0\}$.*

Proof. By (2.2) there holds

$$a^k(\alpha) = \sum_{\beta_0+2\beta_1+\cdots+2^{k-1}\beta_{k-1}=\alpha} a(\beta_0)\cdots a(\beta_{k-1}), \;\; \forall\, \alpha \in \mathbb{Z}^s.$$

Assume $\beta \in \Omega^k$, so there are $\beta_j \in \Omega$, $j = 0, 1, ..., k-1$, such that $\beta = \beta_0 + 2\beta_1 + \cdots + 2^{k-1}\beta_{k-1}$. Thus $a(\beta_0)\cdots a(\beta_{k-1}) > 0$. Consequently,

$$a^k(\beta) \geq a(\beta_0)\cdots a(\beta_{k-1}) > 0$$

or $\beta \in \{\alpha : \; a^k(\alpha) \neq 0\}$. Conversely, if $\beta \in \{\alpha : \; a^k(\alpha) \neq 0\}$, then

$$a^k(\beta) = \sum_{\beta_0 + 2\beta_1 + \cdots + 2^{k-1}\beta_{k-1} = \beta} a(\beta_0)\cdots a(\beta_{k-1}) > 0.$$

There is at least one $(\beta_0, ..., \beta_{k-1})$ satisfying $a(\beta_0)\cdots a(\beta_{k-1}) > 0$. Hence, $\beta_j \in \Omega, j = 0, ..., k-1$, and $\beta = \beta_0 + 2\beta_1 + \cdots + 2^{k-1}\beta_{k-1} \in \Omega^k$. □

We see from this lemma

$$\alpha \in \Omega^k \quad \Longleftrightarrow \quad \alpha = \sum_{j=0}^{k-1} 2^j \gamma_j \quad \text{for some } \gamma_j \in \Omega, \tag{3.3}$$

which explains that the elements in the support Ω^k of the iterated mask $\{a^k(\alpha)\}$ can be represented by the points in the support Ω of the mask $\{a(\alpha)\}$. We will frequently use this relation in the subsequent chapters. We recall that $(x)_\alpha$ is defined to be the α-coordinate of the vector $x \in \mathbb{R}^N$ (see Section 2.2). From the above lemma we conclude immediately the connection between the support Ω^k with the iterated mask $\{a^k(\alpha)\}$ and the admissible set for the mask $\{a(\alpha)\}$ (see also Lemma 2.3.2).

Lemma 3.2.2. *Let $\{a(\alpha)\}$ be a finite nonnegative mask and satisfy the sum rule (1.3). Let further $\Gamma(a)$ be an admissible set of $\{a(\alpha)\}$ and $\lambda \in \mathbb{Z}^s$ satisfying $0 \leq (\lambda)_j \leq (2^k - 1)$ with $k \in \mathbb{N}$. Then, for any $\alpha \in \Gamma(a)$*

$$a^k(-\alpha + 2^k\beta + \lambda) = 0, \quad \forall\, \beta \notin \Gamma(a).$$

Furthermore, if $\alpha \in \Gamma(a)$ and for some $\beta \in \mathbb{Z}^s$ and $\gamma \in \Omega^k$ there holds

$$\alpha = 2^k\beta + \lambda - \gamma.$$

Then, $\beta \in \Gamma(a)$.

Proof. Without loss of generality, we take $\lambda = \sum_{j=1}^{k} 2^{j-1}\delta_j$ with $\delta_j \in E^s$. We prove the first assertion by induction on k. For $k = 1$, putting $\lambda = \delta \in E^s$ (for aesthetic expressions, we use δ substitute δ_k in the case of $k = 1$) the statement is just the definition of an admissible set. Suppose $k > 1$ and assume that the lemma in the case of $k-1$ has been verified, more precisely, take $\lambda' = \sum_{j=1}^{k-1} 2^{j-1}\delta_j$ with $\delta_j \in E^s$, which satisfies $0 \le (\lambda')_j \le (2^{k-1}-1)$ with $k \in \mathbb{N}$ such that one has $a^{k-1}(-\gamma+2^{k-1}\beta+\lambda') = 0$, for any $\gamma \in \Gamma(a)$ and any $\beta \notin \Gamma(a)$. Let $\lambda = \delta_1 + 2\lambda'$ with $\delta_1 \in E^s$, then for any $\alpha \in \Gamma(a)$ and any $\beta \notin \Gamma(a)$, we have

$$\begin{aligned} a^k(-\alpha + 2^k\beta + \lambda) &= \sum_{\eta} a^{k-1}(\eta)a(-\alpha + 2^k\beta + \lambda - 2\eta) \\ &= \sum_{\gamma} a^{k-1}(-\gamma + 2^{k-1}\beta + \lambda')a(-\alpha + 2\gamma + \delta_1). \end{aligned}$$

Because $\Gamma(a)$ is an admissible set, γ in the last sum can be restricted to $\Gamma(a)$. However, for those γ by the hypothesis of induction $a^{k-1}(-\gamma + 2^{k-1}\beta + \lambda') = 0$, which gives $a^k(-\alpha + 2^k\beta + \lambda) = 0$ and the first assertion is proved.

The second assertion follows from the first one and Lemma 3.2.1. Suppose $\beta \notin \Gamma(a)$, from the first assertion, for $\alpha \in \Gamma(a)$, we have

$$a^k(-\alpha + 2^k\beta + \lambda) = 0.$$

Since

$$\alpha = 2^k\beta + \lambda - \gamma,$$

then

$$\gamma = -\alpha + 2^k\beta + \lambda.$$

It follows that $a^k(\gamma) = 0$, so $\gamma \notin \Omega^k$, a contradiction which implies $\beta \in \Gamma(a)$. □

In order to reduce the convergence problem, we introduce a mapping F_B as follows (see [26]). First, for any $T \subseteq \Gamma(a)$, let χ_T be a vector in $\mathbb{R}^N$ and $N = |\Gamma(a)|$ such that

$$(\chi_T)_\alpha = \begin{cases} 1, & \alpha \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Next we define a mapping for any nonnegative $N \times N$ row-stochastic matrix B by

$$F_B(T) = \{\alpha \in \Gamma(a) : (B\chi_T)_\alpha = 1\} \subseteq \Gamma(a)$$

and for simplicity we use F_δ instead of F_{A_δ}, where A_δ is given by (3.1).
Recall the definition of the norm $\|\cdot\|_\Delta$ (see Section 2.3). So for $x \in \mathbb{R}^N$, we have $\|x\|_\Delta = \max x - \min x$, where $\max x := \max_{\alpha\in\Gamma(a)}(x)_\alpha$ and $\min x := \min_{\alpha\in\Gamma(a)}(x)_\alpha$. Next we shall focus on some peculiar properties of the mapping F_B and afterwards characterize the divergent subdivision scheme with a nonnegative mask by means of the mapping F_B. The following argument is similar as the univariate case in [26] (see also [29]). We have

Lemma 3.2.3. *Let B be a nonnegative row-stochastic matrix of size $N = |\Gamma(a)|$.*
1) $\|Bx\|_\Delta \leq \|x\|_\Delta$ and $F_B(T_1)\bigcap F_B(T_2) = \emptyset$, if $T_1 \bigcap T_2 = \emptyset$.
2) Let B_1, B_2 be two nonnegative row-stochastic matrices with the same size, then

$$F_{B_1B_2}(T) = F_{B_1}(T) \circ F_{B_2}(T),$$

where $F_{B_1}(T) \circ F_{B_2}(T)$ is also written as $F_{B_1}(F_{B_2}(T))$.
3) The subdivision scheme with a nonnegative mask, which satisfies the sum rule (1.3), diverges if and only if there exist disjoint proper subsets T and T' of $\Gamma(a)$, and a sequence $(\delta_1, \delta_2, ..., \delta_m)$, $\delta_l \in E^s$ for some $m > 1$, such that

$$T = F_{\delta_1} \circ \cdots \circ F_{\delta_m}(T) \text{ and } T' = F_{\delta_1} \circ \cdots \circ F_{\delta_m}(T'). \tag{3.4}$$

Proof. 1) Let $x \in \mathbb{R}^N$. For $\alpha \in \Gamma(a)$ we have $(Bx)_\alpha = \sum_{\beta\in\Gamma(a)} B(\alpha,\beta)(x)_\beta$ and

$$\begin{aligned}\|Bx\|_\Delta &= \max_{\alpha\in\Gamma(a)} \sum_{\beta\in\Gamma(a)} B(\alpha,\beta)(x)_\beta - \min_{\alpha\in\Gamma(a)} \sum_{\beta\in\Gamma(a)} B(\alpha,\beta)(x)_\beta \\ &= \max_{\alpha,\gamma\in\Gamma(a)} |\sum_{\beta\in\Gamma(a)} B(\alpha,\beta)(x)_\beta - \sum_{\beta\in\Gamma(a)} B(\gamma,\beta)(x)_\beta|.\end{aligned}$$

Put $(x)_\mu = \min x$. Then, we get

$$\begin{aligned}\|Bx\|_\Delta &= \max_{\alpha,\gamma\in\Gamma(a)} |\sum_{\beta\in\Gamma(a)} B(\alpha,\beta)((x)_\beta - (x)_\mu) - \sum_{\beta\in\Gamma(a)} B(\gamma,\beta)((x)_\beta - (x)_\mu)| \\ &\leq \max_{\alpha\in\Gamma(a)} \sum_{\beta\in\Gamma(a)} B(\alpha,\beta)((x)_\beta - (x)_\mu) \\ &\leq \max_{\alpha\in\Gamma(a)} \sum_{\beta\in\Gamma(a)} B(\alpha,\beta)\|x\|_\Delta \\ &= \|x\|_\Delta.\end{aligned}$$

To prove the rest of 1), suppose to the contrary that $F_B(T_1)\bigcap F_B(T_2) \neq \emptyset$. Let $\alpha \in F_B(T_1)\bigcap F_B(T_2)$, then for $N \times N$ nonnegative matrix B, we obtain $(B\chi_{T_1})_\alpha = 1$ and $(B\chi_{T_2})_\alpha = 1$, which imply that

$$\sum_{j\in\Gamma(a)} B(\alpha, j)(\chi_{T_1})_j = 1 \quad \text{and} \quad \sum_{j\in\Gamma(a)} B(\alpha, j)(\chi_{T_2})_j = 1.$$

As B is row-stochastic, we conclude from those identities that $B(\alpha, j) \neq 0$ implies $(\chi_{T_1})_j = 1$ and $(\chi_{T_2})_j = 1$. Consequently,

$$\sum_{j\in\Gamma(a),(\chi_{T_1})_j=1} B(\alpha, j) = 1 \quad \text{and} \quad \sum_{j\in\Gamma(a),(\chi_{T_2})_j=1} B(\alpha, j) = 1.$$

Hence, $T_1 \cap T_2 \neq \emptyset$. The proof of 1) is complete.

2) According to the definition of the mapping F_B, we have

$$F_{B_1}(T) = \{\alpha \in \Gamma(a) : (B_1\chi_T)_\alpha = 1\}, \quad F_{B_2}(T) = \{\alpha \in \Gamma(a) : (B_2\chi_T)_\alpha = 1\}$$

and

$$F_{B_1B_2}(T) = \{\alpha \in \Gamma(a) : (B_1B_2\chi_T)_\alpha = 1\}.$$

Moreover, assume $\alpha \in F_{B_1B_2}(T)$, so

$$1 = (B_1B_2\chi_T)_\alpha = \sum_{j\in\Gamma(a)} B_1B_2(\alpha, j)(\chi_T)_j.$$

Since $B_1B_2(\alpha, j) = \sum_{\tau\in\Gamma(a)} B_1(\alpha, \tau)B_2(\tau, j)$, we conclude by the above equation that

$$\begin{aligned} 1 &= \sum_{j\in\Gamma(a)} \sum_{\tau\in\Gamma(a)} B_1(\alpha, \tau)B_2(\tau, j)(\chi_T)_j \\ &= \sum_{\tau\in\Gamma(a)} B_1(\alpha, \tau) \left(\sum_{j\in\Gamma(a)} B_2(\tau, j)(\chi_T)_j \right) = \sum_{\tau\in\Gamma(a)} B_1(\alpha, \tau)(\chi_{F_{B_2}(T)})_\tau. \end{aligned}$$

As B_1 is row-stochastic, we must have that $B_1(\alpha, \tau) \neq 0$ implies $\tau \in F_{B_2}(T)$, so $\alpha \in F_{B_1}(F_{B_2}(T))$.

Let now $\alpha \in F_{B_1}(F_{B_2}(T))$, so $\sum_{\tau\in\Gamma(a)} B_1(\alpha, \tau)(\chi_{F_{B_2}(T)})_\tau = 1$. Thus, as the above, we obtain $(B_1B_2\chi_T)_\alpha = 1$. Hence, $\alpha \in F_{B_1B_2}(T)$.

3) Firstly we give the following claim:
Suppose that there exists a $k_0 \in \mathbb{N}$ such that for all $k > k_0$, $\delta_j \in E^s$ and all $T \subseteq \Gamma(a)$,

$$F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T) = \emptyset \quad \text{or} \quad F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T^c) = \emptyset$$

holds, where $T^c := \Gamma(a) \setminus T$. Then the subdivision scheme with the nonnegative mask converges.

In fact, according to Theorem 2.3.4, we need only to prove that there is $k_0 \in \mathbb{N}$ such that for all $k > k_0$ and all $\delta_j \in E^s$, the inequality $\|A_{\delta_1} \cdots A_{\delta_k} x\|_\Delta < \|x\|_\Delta$ holds. Without loss of generality we may suppose that $\max x = 1$ and $\min x = 0$, where $x \in \mathbb{R}^N$, for we may always normalize it to such form. So $\|x\|_\Delta = 1$. Let $T = \{\alpha \in \Gamma(a) : (x)_\alpha > \min x\}$, then $(x)_\alpha \le (\chi_T)_\alpha$, $\alpha \in \Gamma(a)$. This deduces $(A_{\delta_1} \cdots A_{\delta_k} x)_\alpha \le (A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha$, $\alpha \in \Gamma(a)$. On the other hand, by the definition of mapping F_B and Lemma 3.2.3 (2) we have

$$\{\alpha : (A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha = 1\} = F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T), \tag{3.5}$$

$$\{\alpha : (A_{\delta_1} \cdots A_{\delta_k} \chi_{T^c})_\alpha = 1\} = F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T^c). \tag{3.6}$$

It is clear that (3.6) means for those α

$$\sum_{j \in \Gamma(a)} A_{\delta_1} \cdots A_{\delta_k}(\alpha, j)(\chi_{T^c})_j = 1.$$

As $A_{\delta_1} \cdots A_{\delta_k}$ is row-stochastic, we conclude then that $A_{\delta_1} \cdots A_{\delta_k}(\alpha, j) \ne 0$ implies $(\chi_{T^c})_j = 1$, i.e., $(\chi_T)_j = 0$. So

$$\sum_{j \in \Gamma(a)} A_{\delta_1} \cdots A_{\delta_k}(\alpha, j)(\chi_T)_j = 0.$$

In other words,

$$\{\alpha : (A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha = 0\} = F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T^c). \tag{3.7}$$

Therefore by (3.5), $F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T) = \emptyset$ means that $0 \le (A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha < 1$ for all $\alpha \in \Gamma(a)$. Thus,

$$\|A_{\delta_1} \cdots A_{\delta_k} x\|_\Delta \le \max_{\alpha \in \Gamma(a)} (A_{\delta_1} \cdots A_{\delta_k} x)_\alpha \le \max_{\alpha \in \Gamma(a)} (A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha < 1 = \|x\|_\Delta.$$

While by (3.7) $F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T^c) = \emptyset$ means $0 < (A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha \le 1$ for all $\alpha \in \Gamma(a)$. So for each $\alpha \in \Gamma(a)$ there exists at least one $\beta \in \Gamma(a)$ such that $A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta) \ne 0$

and $(\chi_T)_\beta \neq 0$. Hence, $\beta \in T$ and $(x)_\beta > 0$. We conclude that $(A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha > 0$ implies $(A_{\delta_1} \cdots A_{\delta_k} x)_\alpha > 0$. Consequently, as $0 < (A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha \leq 1$ for all $\alpha \in \Gamma(a)$, one has $0 < (A_{\delta_1} \cdots A_{\delta_k} x)_\alpha \leq 1$ for all those α. We obtain

$$\|A_{\delta_1} \cdots A_{\delta_k} x\|_\Delta \leq \max_{\alpha,\alpha' \in \Gamma(a)} |(A_{\delta_1} \cdots A_{\delta_k} x)_\alpha - (A_{\delta_1} \cdots A_{\delta_k} x)_{\alpha'}| < 1 = \|x\|_\Delta.$$

Thus, $\|A_{\delta_1} \cdots A_{\delta_k}\|_\Delta < 1$, $\delta_j \in E^s$. The proof of this claim is complete.

Now we prove the necessity of 3). If the subdivision scheme diverges, by the above claim, then there exist a sequence $\epsilon_j \in E^s$, $j = 1, ..., n$ with sufficiently large $n > 2^{2N}$ and a proper subset T_0 of $\Gamma(a)$ such that

$$F_{\epsilon_1} \circ \cdots \circ F_{\epsilon_n}(T_0) \neq \emptyset \quad \text{and} \quad F_{\epsilon_1} \circ \cdots \circ F_{\epsilon_n}(T_0^c) \neq \emptyset.$$

Without loss of generality, we take that

$$T_j := F_{\epsilon_{n-j+1}} \circ \cdots \circ F_{\epsilon_n}(T_0)$$

and

$$R_j := F_{\epsilon_{n-j+1}} \circ \cdots \circ F_{\epsilon_n}(T_0^c) \quad \forall\ 0 \leq j \leq n.$$

It is obvious that all T_j and R_j are nonempty. Moreover, $T_j \cap R_j = \emptyset$. Because $\Gamma(a)$ has $2^N - 1$ nonempty subsets and $n > 2^{2N}$, there exists $j_1 < j_2 < \cdots < j_k$ with $k > 2^N$ such that $T_{j_i} = T$ for some nonempty set $T \subset \Gamma(a)$. Now, $k > 2^N$ implies that there exist $j_s < j_t$ such that

$$R_{j_s} = R_{j_t} = T',$$

where T' is also nonempty. Hence

$$F_{\epsilon_{n-j_s+1}} \circ \cdots \circ F_{\epsilon_{n-j_t}}(T) = T \quad \text{and} \quad F_{\epsilon_{n-j_s+1}} \circ \cdots \circ F_{\epsilon_{n-j_t}}(T') = T'.$$

Now $T_j \cap R_j = \emptyset$ for all j by Lemma 3.2.3(1). So in particular $T \cap T' = \emptyset$. The assertion (3.4) follows by setting $(\delta_1, ..., \delta_m) = (\epsilon_{n-j_s+1}, ..., \epsilon_{n-j_t})$.

Finally we prove the sufficiency of 3). Suppose that (3.4) holds, so $(F_{\delta_1} \circ \cdots \circ F_{\delta_m})^n(T) \neq \emptyset$ and $(F_{\delta_1} \circ \cdots \circ F_{\delta_m})^n(T^c) \neq \emptyset$. Clearly $T' \subseteq T^c$. It follows from (3.5) and (3.7) that $\max((A_{\delta_1} \cdots A_{\delta_m})^n \chi_T)_\alpha = 1$ and $\min((A_{\delta_1} \cdots A_{\delta_m})^n \chi_T)_\alpha = 0$, respectively. Then

$$\|(A_{\delta_1} \cdots A_{\delta_m})^n\|_\Delta = 1, \quad n \geq 1.$$

This shows that the subdivision scheme diverges. $\square$

Remark 3.2.4. Lemma 3.2.3(3) holds under the same hypothesis but with (3.4) replaced by the relaxed condition:

$$T \subseteq F_{\delta_1} \circ \cdots \circ F_{\delta_m}(T) \text{ and } T' \subseteq F_{\delta_1} \circ \cdots \circ F_{\delta_m}(T')$$

(see the proof of Theorem 3.3.1 in Section 3.3).

Before proceeding further, we introduce another mapping Ψ, in order to compute mapping F_δ explicitly. To this end, let us first observe the quotient group $\mathbb{Z}^s/(2\mathbb{Z}^s)$. Clearly,

$$\mathbb{Z}^s/(2\mathbb{Z}^s) = \{< \lambda >: \ \lambda \in \mathbb{Z}^s\} \text{ and } \quad < \lambda >= \{\alpha \in \mathbb{Z}^s : \ \alpha \equiv \lambda \pmod 2\}.$$

We should also denote for a given finitely supported real mask $\{a(\alpha)\}$ the set

$$\mathcal{A}(\lambda) = \{\alpha \, : \, a(\alpha) \neq 0 \text{ and } \alpha \equiv \lambda \pmod 2\}, \quad \forall \, \lambda \in \mathbb{Z}^s.$$

Thus, $\mathcal{A}(\lambda)$ is a subset of $< \lambda >$. Let

$$\Psi(T) = \bigcup_{\delta \in E^s} \{ \bigcap_{\beta \in \mathcal{A}(\delta)} (2T - \beta)\}, \ \ \forall \, T \subset \mathbb{Z}^s.$$

For the sake of the demonstration of Lemma 3.2.6, firstly we construct a column index set associated with nonzero entries of some row of matrix A_δ. Given $\delta \in E^s$ and $\alpha \in \Gamma(a)$, denote $I_\alpha = \{\beta : \ A_\delta(\alpha, \beta) \neq 0\}$. Observe this column index set, then we get the following property (see [26]).

Lemma 3.2.5. *For some $\delta' \in E^s$ such that $\alpha \equiv \delta' - \delta \pmod 2$, there holds*

$$I_\alpha = \frac{\mathcal{A}(\delta') - \delta + \alpha}{2} \cap \mathbb{Z}^s.$$

Proof. Let $\beta \in I_\alpha$. By the definition of A_δ, we conclude that $a(\delta + 2\beta - \alpha) \neq 0$. Together with $\delta + 2\beta - \alpha \equiv \delta' \pmod 2$, it yields that $\delta + 2\beta - \alpha \in \mathcal{A}(\delta')$, which implies $\beta \in (\mathcal{A}(\delta') - \delta + \alpha)/2$.

Conversly, suppose $\beta \in (\mathcal{A}(\delta') - \delta + \alpha)/2$ to be a multi-integer. So $\delta + 2\beta - \alpha \in \mathcal{A}(\delta') \subset \Omega$ and $a(\delta + 2\beta - \alpha) \neq 0$. That is $\beta \in I_\alpha$. □

In [26] Wang gave the relationship between mappings F_δ and Ψ for the univariate case, where the support of the mask is contained in $\{0, ..., N\}$ and $\Gamma(\alpha) = \{0, ..., N-1\}$. For the multivariate case, the argument is similar and is generalized in [29]. For convenience, we repeat and explain the proof.

Lemma 3.2.6. *For any $T \subseteq \Gamma(a)$ and any $\delta \in E^s$, we have*

$$F_\delta(T) = (\Psi(T) + \delta) \cap \Gamma(a).$$

Furthermore, for any $\delta_l \in E^s$, $l = 1, ..., k$, there holds for $\lambda = \sum_{j=1}^{k} \delta_j 2^{j-1}$,

$$F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T) = (\Psi^k(T) + \lambda) \cap \Gamma(a).$$

Proof. To show the first assertion, let $\alpha \in F_\delta(T)$. By the definition of mapping $F_\delta(T)$, we have $(A_\delta \chi_T)_\alpha = 1$, i.e.,

$$\sum_{\beta \in I_\alpha} (\chi_T)_\beta a(\delta + 2\beta - \alpha) = 1.$$

Then $I_\alpha \subseteq T$. Moreover, we obtain by Lemma 3.2.5 that

$$\alpha \in \bigcap_{\beta \in \mathcal{A}(\delta')} (2T - \beta + \delta).$$

Consequently,

$$F_\delta(T) \subseteq \bigcup_{\delta' \in E^s} \bigcap_{\beta \in \mathcal{A}(\delta')} (2T - \beta + \delta) = (\Psi(T) + \delta) \cap \Gamma(a).$$

On the other hand, let $\alpha \in (\Psi(T) + \delta) \cap \Gamma(a)$. By the definition of $\Psi(T)$, for some $\delta' \in E^s$, one has $\alpha \in \Gamma(a)$ and $\alpha \in \bigcap_{\beta \in \mathcal{A}(\delta')}(2T - \beta + \delta)$. It in turn implies that $\alpha \in 2T - \beta + \delta$ for all $\beta \in \mathcal{A}(\delta')$, or $\alpha + \mathcal{A}(\delta') - \delta \subseteq 2T$. Thus by Lemma 3.2.5, we have $I_\alpha \subseteq T$ and $(\chi_T)_\beta = 1$ whenver $\beta \in I_\alpha$. Since $I_\alpha := \{\beta : A_\delta(\alpha, \beta) \neq 0\}$ and A_δ is a row-stochastic matrix, we conclude that

$$\sum_{\beta \in I_\alpha} (\chi_T)_\beta a(\delta + 2\beta - \alpha) = 1.$$

Therefore, $\alpha \in F_\delta(T)$ and the first assertion holds.

To prove the second assertion, we use the induction on the number of compositions. We suppose that the second assertion holds for $k - 1 \geq 1$, i.e., for $\lambda' = \sum_{j=1}^{k-1} \delta_{j+1} 2^{j-1}$, we have

$$F_{\delta_2} \circ \cdots \circ F_{\delta_k}(T) = (\Psi^{k-1}(T) + \lambda') \cap \Gamma(a).$$

Then,

$$\begin{aligned} F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T) &= (\Psi(F_{\delta_2}\circ\cdots\circ F_{\delta_k}(T))+\delta_1)\cap\Gamma(a)\\ &= (\Psi\{(\Psi^{k-1}(T)+\lambda')\cap\Gamma(a)\}+\delta_1)\cap\Gamma(a)\\ &= (\Psi^k(T)+2\lambda'+\delta_1)\cap\Gamma(a)\\ &= (\Psi^k(T)+\lambda)\cap\Gamma(a). \end{aligned}$$

This is the second assertion. □

The following lemma allows us to choose k and λ explicitly, which leads to the computation of T for some Ω (see also [29]).

Lemma 3.2.7. *Let $\{a(\alpha)\}$ be a nonnegative mask and let its support be Ω. If the corresponding subdivision scheme with this nonnegative mask, which satisfies the sum rule (1.3), diverges, then there exist disjoint proper subsets T and T' of $\Gamma(a)$, such that*

$$T=(\Psi^k(T)+\lambda)\cap\Gamma(a)\quad\textit{and}\quad T'=(\Psi^k(T')+\lambda)\cap\Gamma(a), \tag{3.8}$$

where $\lambda=\sum_{j=1}^k\delta_j2^{j-1}$ with $\delta_j\in E^s$, $j=1,...,k$. In particular, we can choose $k=k'm$ for some $k'\geq 1$ and any fixed $m\geq m_0\geq 1$ such that $0\leq(\lambda)_j\leq(2^k-1)$, $j=1,2,...,s$. Moreover, if $(\lambda)_j\neq 0,\ \ 2^k-1$, then for some $0<\varepsilon_1<\varepsilon_2<1$, there holds

$$\varepsilon_1 2^k\leq(\lambda)_j\leq\varepsilon_2 2^k.$$

Proof. By Lemma 3.2.3(3) the divergence implies that for some disjoint proper subsets T and T', there holds

$$T=F_{\delta_1}\circ\cdots\circ F_{\delta_{k'}}(T)\ \text{ and }\ T'=F_{\delta_1}\circ\cdots\circ F_{\delta_{k'}}(T'),$$

for some $k'\geq 1$ and $\delta_j\in E^s$, $j=1,2,...,k'$. Denote $L:=F_{\delta_1}\circ\cdots\circ F_{\delta_{k'}}$ and $\lambda=\sum_{j=1}^{k'}\delta_j2^{j-1}$. Then for all $m\geq 1$,

$$T=L^m(T)\ \text{ and }\ T'=L^m(T').$$

It follows from Lemma 3.2.6 that for any $\delta_j\in E^s$ with $j=1,...,k'$,

$$F_{\delta_1}\circ\cdots\circ F_{\delta_{k'}}\circ\cdots\circ F_{\delta_1}\circ\cdots\circ F_{\delta_{k'}}(T)=(\Psi^{mk'}(T)+\lambda')\cap\Gamma(a)\ \text{ and}$$

$$F_{\delta_1} \circ \cdots \circ F_{\delta_{k'}} \circ \cdots \circ F_{\delta_1} \circ \cdots \circ F_{\delta_{k'}}(T') = (\Psi^{mk'}(T') + \lambda') \cap \Gamma(a),$$

where

$$\begin{aligned} \lambda' &= \lambda + \lambda 2^{k'} + \cdots + \lambda 2^{(m-1)k'} \\ &= \lambda \sum_{i=0}^{m-1} 2^{ik} = \frac{\lambda}{2^{k'} - 1}(2^{k'm} - 1). \end{aligned}$$

Then

$$T = (\Psi^{k'm}(T) + \lambda') \cap \Gamma(a) \quad \text{and} \quad T' = (\Psi^{k'm}(T') + \lambda') \cap \Gamma(a). \tag{3.9}$$

We can choose sufficiently large m to meet the restrictions. Finally, substituting $k = k'm$ and defining λ to be λ', we obtain (3.8) from (3.9). □

3.3 New characterization of the convergence

We are now in the position to establish the main result of this chapter, which presents the necessary and sufficient condition on the convergent subdivision scheme with finitely supported nonnegative mask.

Theorem 3.3.1. *The subdivision scheme with a nonnegative mask $\{a(\alpha)\}$, whose support is Ω and which satisfies the sum rule (1.3), converges if and only if for any $k \in \mathbb{N}$, $\delta_1, \ldots, \delta_k \in E^s$ and $\lambda = \sum_{j=1}^{k} 2^{j-1}\delta_j$, the inclusion relations for any nonempty sets T and T' of $\Gamma(a)$*

$$\frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T \quad \textit{and} \quad \frac{T' - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T'$$

imply $T \cap T' \neq \emptyset$.

Proof. We need one more technical claim before we attack the proof of this assertion. We claim that for any $k \in \mathbb{N}$, $\delta_1, \ldots, \delta_k \in E^s$ and $\lambda = \sum_{j=1}^{k} 2^{j-1}\delta_j$, there holds

$$T \subseteq F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T) \quad \Leftrightarrow \quad \frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T. \tag{3.10}$$

Indeed in one direction, since $T \subseteq F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T) = \{\alpha \in \Gamma(a) : (A_{\delta_1} \cdots A_{\delta_k} \chi_T)_\alpha = 1\}$, $A_{\delta_1} \cdots A_{\delta_k}$ restricted to T is row-stochastic. On the other hand, by Lemma 2.3.2,

$$A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta) = a^k(-\alpha + \lambda + 2^k \beta). \tag{3.11}$$

Thus, if

$$\beta \in \frac{T - \lambda + \Omega^k}{2^k},$$

there are $\alpha \in T$ and $r \in \Omega^k$ satisfying $2^k\beta - \alpha + \lambda \in \Omega^k$. Consequently, $a^k(-\alpha + \lambda + 2^k\beta) \neq 0$. As $A_{\delta_1} \cdots A_{\delta_k}$ is row-stochastic on T, we must have $\beta \in T$.
In the opposite direction, because of (3.11), the condition $(T - \lambda + \Omega^k)/2^k \cap \mathbb{Z}^s \subseteq T$ implies that $A_{\delta_1} \cdots A_{\delta_k}$ is row-stochastic on T. Thus, for all $\alpha \in T$

$$1 = \sum_{\beta \in T} A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta) = \sum_{\beta \in \Gamma(a)} A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta)(\chi_T)_\beta.$$

Hence, $\alpha \in F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T)$. So $T \subseteq F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T)$.

We now return to the proof of Theorem 3.3.1 and firstly prove the necessity. Assume $T, T' \subseteq \Gamma(a)$ to be nonempty satisfying

$$\frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T \quad \text{and} \quad \frac{T' - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T'.$$

In view of (3.10) we obtain with $\lambda = \delta_1 + 2\delta_2 + \cdots + 2^{k-1}\delta_k$

$$T \subseteq F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T) \quad \text{and} \quad T' \subseteq F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T').$$

Clearly the matrix $A_{\delta_1} \cdots A_{\delta_k}$ restricted to T and T', respectively, is row-stochastic. If $T \cap T' \neq \emptyset$, we have nothing more to do. Otherwise $T \cap T' = \emptyset$, we claim that

$$F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T) \cap F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T') = \emptyset.$$

In fact, if $\alpha \in F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T) \cap F_{\delta_1} \circ \cdots \circ F_{\delta_k}(T')$, then

$$1 = \sum_{\beta \in T} A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta)(\chi_T)_\beta = \sum_{\beta \in T'} A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta)(\chi_{T'})_\beta.$$

We would have $\beta \in T$ and $\beta' \in T'$ such that

$$A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta) \neq 0 \quad \text{and} \quad A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta') \neq 0.$$

This is however impossible, because

$$\sum_{\beta \in T} A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta) = 1 \quad \text{and} \quad \sum_{\beta \in T'} A_{\delta_1} \cdots A_{\delta_k}(\alpha, \beta) = 1.$$

Now it is easy to see that $A_{\delta_1}\cdots A_{\delta_k}$ restricted to $F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T)$ is again row-stochastic, since for $\alpha\in F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T)$

$$1\geq\sum_{\beta\in F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T)} A_{\delta_1}\cdots A_{\delta_k}(\alpha,\beta)(\chi_{F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T)})_\beta\geq\sum_{\beta\in T}A_{\delta_1}\cdots A_{\delta_k}(\alpha,\beta)(\chi_T)_\beta=1.$$

In other words,

$$F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T)=(F_{\delta_1}\circ\cdots\circ F_{\delta_k})(F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T)).$$

The same holds also for $F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T')$. Consequently, for proper disjoint subsets $T_1=F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T)$ and $T_2=F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T')$, we get

$$T_1=F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T_1)\quad\text{and}\quad T_2=F_{\delta_1}\circ\cdots\circ F_{\delta_k}(T_2).$$

By Lemma 3.2.3(3), the subdivision scheme diverges. This contradiction means $T\cap T'\neq\emptyset$, which leads to the required result.

We finally prove the sufficiency. Suppose to the contrary that the corresponding subdivision scheme diverges. By Lemma 3.2.3(3), there exist disjoint proper subsets T and T' of $\Gamma(a)$, for some $m\geq 1$ and $\delta_j\in E_s,\quad j=1,...,m$,

$$T\subseteq F_{\delta_1}\circ\cdots\circ F_{\delta_m}(T)\quad\text{and}\quad T'\subseteq F_{\delta_1}\circ\cdots\circ F_{\delta_m}(T').$$

So by (3.10), with $\lambda=\delta_1+2\delta_2+\cdots+2^{m-1}\delta_m$

$$\frac{T-\lambda+\Omega^m}{2^m}\cap\mathbb{Z}^s\subseteq T\quad\text{and}\quad\frac{T'-\lambda+\Omega^m}{2^m}\cap\mathbb{Z}^s\subseteq T'.$$

This ends the proof of the sufficiency. □

The sets in Theorem 3.3.1 have a nice property worth presenting before we go on. It will be used to prove the theorems in the later chapters.

Corollary 3.3.2. *Let $k\in\mathbb{N}$, $\Omega\subset\mathbb{Z}^s$, $|\Omega|<\infty$ and $\lambda\in\mathbb{Z}^s$. If $T\subset\mathbb{Z}^s$ satisfies*

$$\frac{T-\lambda+\Omega^k}{2^k}\cap\mathbb{Z}^s=T, \tag{3.12}$$

then $(2^k-1)T+\lambda\subseteq[\Omega^k]\cap\mathbb{Z}^s$.

Proof. (3.12) means

$$\frac{(2^k-1)T+\lambda+(2^k-1)\Omega^k}{2^k}\cap\mathbb{Z}^s=(2^k-1)T+\lambda.$$

Denote $L=(2^k-1)T+\lambda$, then for any $x_0\in L$, there exist $x_i\in L$ and $\eta_i=\sum_{\mu=0}^{k-1}2^\mu r_{i,\mu}\in\Omega^k$, $r_{i,\mu}\in\Omega,\ i=1,2,...$, such that

$$\begin{aligned} x_0 &= \frac{x_1+\eta_1(2^k-1)}{2^k}, \\ x_1 &= \frac{x_2+\eta_2(2^k-1)}{2^k}, \\ x_2 &= \frac{x_3+\eta_3(2^k-1)}{2^k}, \\ &... \\ x_{l-1} &= \frac{x_l+\eta_l(2^k-1)}{2^k}, \\ &.... \end{aligned} \tag{3.13}$$

The second formula of (3.13), which is an expression of variable x_1, will be substituted for the variable x_1 in the right hand side of the first formula of (3.13). Then we get

$$x_0=\frac{\frac{x_2+\eta_2(2^k-1)}{2^k}+\eta_1(2^k-1)}{2^k}=\frac{x_2+\eta_2(2^k-1)+2^k\eta_1(2^k-1)}{2^{2k}}.$$

The variable x_2 in the right hand side will be replaced by the third formula of (3.13). Repeating the above steps, x_0 can be written as:

$$x_0=\sum_{i=1}^{\infty}\frac{\eta_i(2^k-1)}{2^{ki}}=\frac{2^k-1}{2^k}\cdot\sum_{i=1}^{\infty}\frac{\eta_i}{2^{k(i-1)}}.$$

On the other hand, we have

$$\frac{2^k-1}{2^k}\sum_{i=1}^{\infty}\frac{1}{2^{k(i-1)}}=1.$$

Hence, x_0 can be denoted as a convex combination of η_i, which means that $x_0\in[\Omega^k]\cap\mathbb{Z}^s$. The proof is complete. □

Theorem 3.3.1 illustrates further that the convergence of the subdivision scheme does not rely on the actual values of the mask but rather on the support of the mask. In the next chapter, we will use this result to study the convergence of a few subdivision schemes with finitely supported nonnegative mask.

Chapter 4

Applications of Theorem 3.3.1 and Further Reductions

A remarkable fact of the class of nonnegative masks is that the convergence does not rely on the actual values of the mask but rather on the support of the mask. It means that the distribution of the support determines whether the subdivision scheme converges or not. In the previous chapter, we have given the improved necessary and sufficient condition on the convergence of the subdivision schemes with nonnegative finite masks by Theorem 3.3.1. In this chapter we will use this theorem to get some applications and extensions.

In order to understand Theorem 3.3.1 better, we give some examples to demonstrate the power and applicability of our approach, and also introduce some theorems and corollaries associated with our main result in this chapter. First we are concerned with the converse-and-negative statement of Theorem 3.3.1.

Theorem 4.0.1. *The subdivision scheme with a nonnegative mask* $\{a(\alpha)\}$*, whose support is* Ω *and which satisfies the sum rule (1.3), diverges if and only if there exist disjoint proper subsets* T *and* T' *of* $\Gamma(a)$ *and a sequence* $(\delta_1, ..., \delta_k)$ *with* $\delta_j \in E^s$*, for some* $k \geq 1$ *and* $\lambda = \sum_{j=1}^{k} 2^{j-1}\delta_j$ *such that*

$$\frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T \text{ and } \frac{T' - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T'. \tag{4.1}$$

4.1 Some divergent examples

By Theorem 4.0.1, we may easily determine a divergent subdivision scheme with nonnegative finite mask $\{a(\alpha)\}$.

Example 4.1.1. *For $s = 1$, the set $\Omega = \{1,\ 2\} \subseteq \mathbb{Z}$ (see Figure 4.1) is the support of the corresponding univariate subdivision scheme with the nonnegative mask $\{a(j)\}$ and $[\Omega] \cap \mathbb{Z}$ is the same as Ω. The sum rule (1.3) holds, i.e., $a(1) = 1$ and $a(2) = 1$. Take $T_1 = \{1\}$ and $T_2 = \{2\}$, then $T_1 \cap T_2 = \emptyset$. Let $k = 1$ and $\lambda = 0$, then we have*

$$\frac{T_1 + \Omega}{2} \cap \mathbb{Z} = \{\frac{1+1}{2}\} = \{1\} \subseteq T_1 \text{ and } \frac{T_2 + \Omega}{2} \cap \mathbb{Z} = \{\frac{2+2}{2}\} = \{2\} \subseteq T_2.$$

It is clear that the condition of Theorem 4.0.1 holds, so the subdivision scheme diverges.

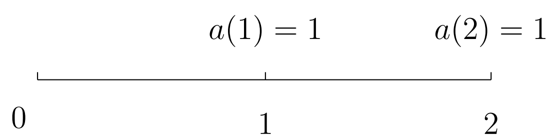

Figure 4.1: $s = 1$

Example 4.1.2. *For $s = 2$, the set $\Omega \subseteq \mathbb{Z}^2$ (see Figure 4.2) is the support of the corresponding bivariate subdivision scheme with nonnegative mask $\{a(\alpha)\}$ and $[\Omega] \cap \mathbb{Z}^2$ is the same as Ω. There are 8 points, in detail $a_1,\ a_2,\ b_1,\ b_2,\ c_1,\ c_2,\ d_1,\ d_2$, in Ω such that*

$$a_1 \equiv a_2 \pmod 2,\ b_1 \equiv b_2 \pmod 2,\ c_1 \equiv c_2 \pmod 2 \text{ and } d_1 \equiv d_2 \pmod 2.$$

Without loss of generality, let $a_1 = (0,0)^T$, $a_2 = (0,2)^T$, $b_1 = (1,0)^T$, $b_2 = (1,3)^T$, $c_1 = (1,1)^T$, $c_2 = (1,3)^T$, $d_1 = (0,1)^T$ and $d_2 = (0,3)^T$. Moreover, we choose $T_1 = \{a_1,\ a_2,\ d_1,\ d_2\}$ and $T_2 = \{b_1,\ b_2,\ c_1,\ c_2\}$. So $T_1 \cap T_2 = \emptyset$. For $k = 1$ and $\lambda = (0,0)^T$, we have

$$\frac{T_1 - \lambda + \Omega}{2} \cap \mathbb{Z}^2 \subseteq T_1 \text{ and } \frac{T_2 - \lambda + \Omega}{2} \cap \mathbb{Z}^2 \subseteq T_2.$$

It means that T_1 and T_2 satisfy (4.1). Therefore, by Theorem 4.0.1, the subdivision scheme with this nonnegative mask, whose support is Ω, diverges.

$\Omega =$

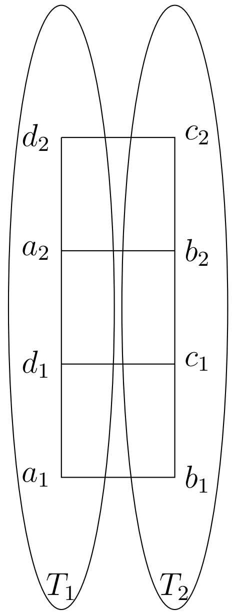

Figure 4.2: $s = 2$

For the 3-dimensional divergent subdivision scheme, the simplest one, we draw three unit cubes with 16 integral points (see Figure 4.3). Then take two parallel planes respectively, where there are 8 multi-integer points in each one. The integral points for each plane are grouped into T_1 and T_2, which indeed have same construction to Example 4.1.2 and have the same results. In order to go beyond this construction, we'll give a more complex example in the case $s = 3$.

Example 4.1.3. *For $s = 3$, let $\Omega \subseteq \mathbb{Z}^3$ (see Figure 4.4) be the support of the corresponding three-variate subdivision scheme with the nonnegative mask $\{a(\alpha)\}$. There are 16 points, in detail $a_i,\ b_i,\ c_i,\ d_i,\ e_i,\ f_i,\ g_i,\ h_i$ with $i = 1, 2$ in Ω such that*

$$a_i \equiv c_i \pmod 2,\ b_i \equiv e_i \pmod 2,\ h_i \equiv d_i \pmod 2 \text{ and } f_i \equiv g_i \pmod 2.$$

Without loss of generality, let $a_1 = (0,1,2)^T,\ a_2 = (1,1,2)^T,\ b_1 = (0,2,2)^T,\ b_2 = (1,2,2)^T,\ c_1 = (0,3,2)^T,\ c_2 = (1,3,2)^T,\ d_1 = (0,1,3)^T,\ d_2 = (1,1,3)^T,\ e_1 = (0,0,0)^T,\ e_2 = (1,0,0)^T,\ f_1 = (0,2,1)^T,\ f_2 = (1,2,1)^T,\ h_1 = (0,1,1)^T,\ h_2 = (1,1,1)^T,\ g_1 = (2,2,1)^T$ and $g_2 = (-1,2,1)^T$.
We note that

i) there are $a_1, b_1, c_1, d_1, e_1, f_1$ and h_1 in the plane S^0 where $S^0 := \{(x,y,z) : x = 0\}$,

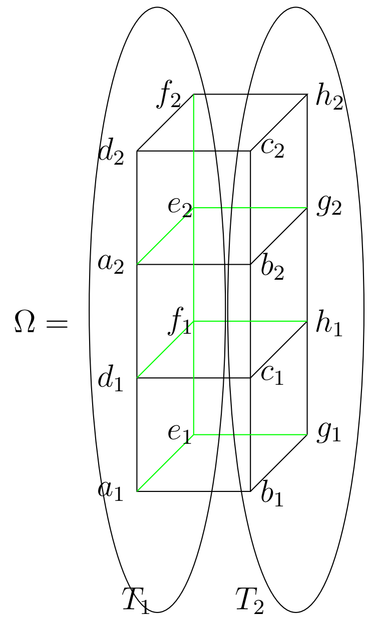

Figure 4.3: $s = 3$ (*the simplest case*)

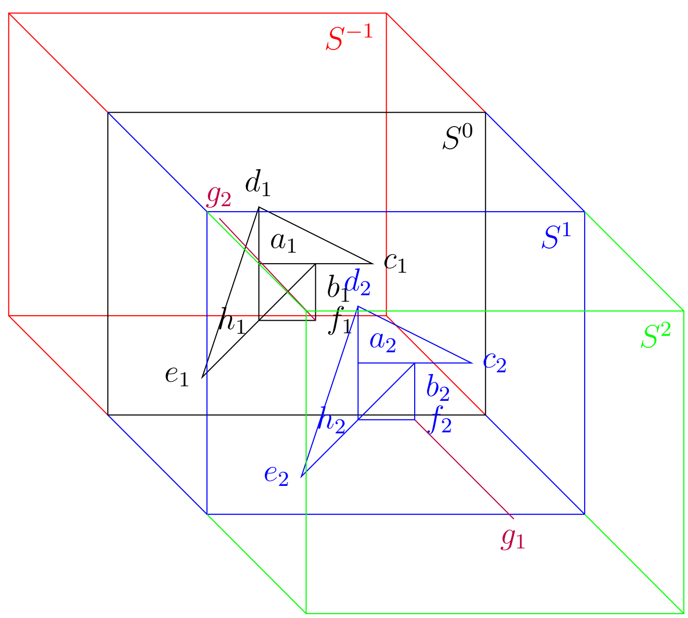

Figure 4.4: $s = 3$

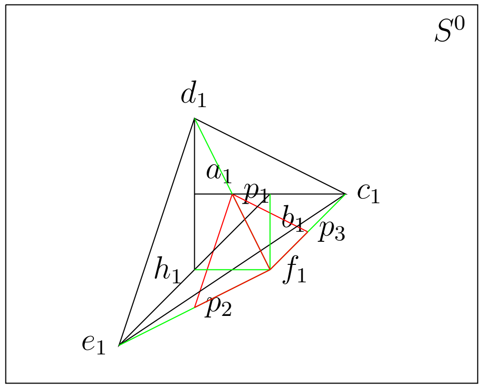

Figure 4.5: Projection on S^0

ii) there are $a_2, b_2, c_2, d_2, e_2, f_2$ and h_2 in the plane S^1 where $S^1 := \{(x, y, z) : x = 1\}$,

iii) there are g_2 in the plane S^{-1} where $S^{-1} := \{(x, y, z) : x = -1\}$ and

iv) there are g_1 in the plane S^2 where the plane $S^2 := \{(x, y, z) : x = 2\}$.
Let

$$T_1 = \{a_1,\ b_1,\ c_1,\ d_1,\ e_1,\ h_1\} \quad \textit{and}$$

$$T_2 = \{a_2,\ b_2,\ c_2,\ d_2,\ e_2,\ h_2,\ f_1,\ f_2,\ g_1,\ g_2\}.$$

It is easy to see that T_1 is a tringle $c_1d_1e_1$ on S^0 with 6 integers $a_1,\ b_1,\ c_1,\ d_1,\ e_1,\ h_1$ (see Figure 4.5 with color black) and T_2 is a polyhedron whose projection on S^0 is $p_1p_2f_1p_3$ (see Figure 4.5 with color red). It shows that $T_1 \cap T_2 = \emptyset$.
For $k = 1$ and $\lambda = (0, 0, 0)^T$ we have

$$\frac{T_1 - \lambda + \Omega}{2} \cap \mathbb{Z}^3 \subseteq T_1 \ \textit{and} \ \frac{T_2 - \lambda + \Omega}{2} \cap \mathbb{Z}^3 \subseteq T_2.$$

It means that T_1 and T_2 satisfy (4.1). By Theorem 4.0.1, the three-variate subdivision scheme with the nonnegative mask $\{a(\alpha)\}$, whose support is Ω, diverges.
According to this example, we also get an observation that $\Omega = \Omega_1 \cup \Omega_2$ and $\alpha \not\equiv \beta \pmod 2$ for arbitrary $\alpha \in [\Omega_1] \cap \mathbb{Z}^3$ and arbitrary $\beta \in [\Omega_2] \cap \mathbb{Z}^3$ in full similarity with Example 4.1.2.

4.2 Irreducible mapping

Comparing these three observations (see Examples 4.1.1, 4.1.2 and 4.1.3) from the supports in the different dimension, we claim that the supports of the divergent multivariate subdivision scheme with nonnegative mask $\{a(\alpha)\}$ have the similar property. In order to demonstrate this property, we recall that the concept of 'irreducible', which is used in several ways in mathematics, such as irreducible (algebraic) set (if it is not the union of two proper algebraic subsets), irreducible polynomial over field F (if its coefficients belong to F and it cannot be factored into the product of two polynomials with coefficients in F), irreducible matrix (if it is not similar via a permutation to a block upper triangular matrix), and so on. A good understanding of the 'irreducible' is important in the representation of the irreducible support Ω (actually set) associated with the nonnegative mask $\{a(\alpha)\}$ as follows.

Definition 4.2.1. A set $\Omega \subseteq \mathbb{Z}^s$ is called reducible if there exist two disjoint subsets Ω_1 and Ω_2 satisfying $\Omega = \Omega_1 \cup \Omega_2$ such that $\alpha \not\equiv \beta \pmod 2$ for any $\alpha \in [\Omega_1] \cap \mathbb{Z}^s$ and $\beta \in [\Omega_2] \cap \mathbb{Z}^s$. A set Ω that is not reducible is said to be irreducible .

Remark 4.2.2. Since uniform convergence of non-negative univariate subdivision has been full characterized, we assume in this section that $s > 1$.

Let us revisit Examples 4.1.2 and 4.1.3, then we conclude that the support Ω in these two examples are both reducible according to Definition 4.2.1. Furthermore, it is easy to get the following result.

Corollary 4.2.3. *If the support Ω of the multivariate subdivision scheme with the nonnegative mask $\{a(\alpha)\}$ is reducible, then the corresponding multivariate subdivision scheme diverges.*

Proof. There exist two disjoint subsets Ω_1 and Ω_2 satisfying $\Omega = \Omega_1 \cup \Omega_2$ such that $\alpha \not\equiv \beta \pmod 2$ for any $\alpha \in [\Omega_1] \cap \mathbb{Z}^s$ and $\beta \in [\Omega_2] \cap \mathbb{Z}^s$. If we put $T_1 = [\Omega_1] \cap \mathbb{Z}^s$ and $T_2 = [\Omega_2] \cap \mathbb{Z}^s$, then it is clear that $T_1 \cap T_2 = \emptyset$. Take $k = 1$ and $\lambda = 0$, then we have

$$\frac{T_1 + \Omega_1}{2} \cap \mathbb{Z}^s \subseteq T_1 \text{ and } \frac{T_2 + \Omega_2}{2} \cap \mathbb{Z}^s \subseteq T_2,$$

by the definition of the support Ω and $[\Omega]$ being the convex cover of Ω. Together with the Theorem 4.0.1, it yields that the corresponding multivariate subdivision scheme diverges. $\square$

Now we give one criterion for checking the necessary condition on a convergent multivariate subdivision scheme with the nonnegative mask $\{a(\alpha)\}$. It is easy to see that the condition is necessary but not sufficient.

Corollary 4.2.4. *If the multivariate subdivision scheme with the nonnegative mask* $\{a(\alpha)\}$, *whose support is* Ω, *converges , then the corresponding support* Ω *is irreducible.*

For simplicity in the discussion of Chapter 7, we will go on introducing in this section the property of being reducible, irreducible and primitive for a mapping among set of integers (see [19]).

Definition 4.2.5. Let $\Sigma \subseteq \mathbb{Z}^s$ be a finite set. Let further ψ be an additive mapping defined in the following way:

$$\psi(\emptyset) = \emptyset \quad \text{and} \quad \psi(I) \subseteq \Sigma, \quad \forall\ I \subseteq \Sigma.$$

We say that ψ is **reducible**, if there exists a nonempty proper subset I of Σ satisfying $\psi(I) \subseteq I$; otherwise ψ is **irreducible**.

This definition leads to the following concept that ψ is **primitive** if for all $l \in \mathbb{N}$, ψ^l is irreducible. It is known (see [19]) that, if ψ is irreducible, there exist $k \geq 1$ and $I_1, I_2, ..., I_k \subseteq \Sigma$ such that $I_i \cap I_j = \emptyset$ with $i \neq j$ and $I = I_1 \cup \cdots \cup I_k$,

$$\psi(I_i) = I_{i+1},\ \ i = 1, ..., k$$

with the understanding $I_{k+1} = I_1$ and ψ^k restricted to I_i, $i = 1, ..., k$, is primitive. Moreover, there is $\tau \in \mathbb{N}$ such that for all $\alpha \in I_i$

$$\psi^{k\tau}(\alpha) = I_i, \quad i = 1, ..., k.$$

For us the additive mapping ψ has the form:

$$\psi(T) := \frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s,$$

where $T \subseteq \mathbb{Z}^s$, $\lambda \in \mathbb{Z}^s$ and $\Omega \subseteq \mathbb{Z}^s$ are finite sets. Instead of ψ, we may simply say, T is irreducible with respect to Ω^k and λ, if

$$\frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s = T \tag{4.2}$$

and the additive mapping restricted to the power set of T is irreducible. We call T is primitive with respect to λ and Ω^k, if for all $l \in \mathbb{N}$,

$$\frac{T - \lambda - 2^k\lambda - \cdots - 2^{k(l-1)}\lambda + \Omega^{kl}}{2^{kl}} \cap \mathbb{Z}^s = T \tag{4.3}$$

and T is irreducible with respect to $\lambda^* := \lambda + 2^k\lambda + \cdots + 2^{k(l-1)}\lambda$ and Ω^{kl}.
We come back to Example 4.1.1. It's easy to see that T_1 and T_2 corresponding to this divergent scheme are not only irreducible but also primitive if $\lambda = 0$. Furthermore, T_1 and T_2 are disjoint. The arguments for 2-dimension and 3-dimension are similar (see Examples 4.1.2 and 4.1.3).

4.3 Further reductions

In [29], a sufficient condition of the convergent subdivision scheme with the nonnegative mask $\{a(\alpha)\}$ is given. Here we will show that the condition in [29] is indeed also necessary. It can be carried out by Theorem 4.0.1.
We need more notations. Let $\Gamma(a)$ be a finite set of $\mathbb{Z}^s$ ($\Gamma(a)$ may not be an admissible set). For $k \in \mathbb{N}$ and $\lambda = \sum_{j=1}^k \delta_j 2^{j-1}$ from Theorem 3.3.1 let us define $\mathcal{B}_0 = \{\alpha\}$, where $\alpha \in \Gamma(a)$, and for $l = 0, 1, ...,$

$$\mathcal{B}_{l+1} = \left\{ 2^k y + \lambda - r_k \in \Gamma(a) : \ r_k \in \Omega^k, \quad y \in \bigcup_{0 \le j \le l} \mathcal{B}_j \right\}.$$

Finally, denote

$$B(\Gamma(a), \alpha, k, \lambda) = \bigcup_{l \ge 0} \mathcal{B}_l.$$

It follows from Lemma 3.2.6 that for $k \ge 1$ and $0 \le (\lambda)_j \le 2^k - 1$, $j = 1, ..., s$ there holds

$$B(\Gamma(a), \alpha, mk, \lambda \sum_{i=0}^{m-1} 2^{ki}) \subseteq B(\Gamma(a), \alpha, k, \lambda), \ \ \forall \ m \ge 1. \tag{4.4}$$

Indeed $x \in B(\Gamma(a), \alpha, k, \lambda)$ means that for some $\mu \in \mathbb{N}$,

$$x = 2^{\mu k}\alpha + \lambda \sum_{j=0}^{\mu-1} 2^{kj} - \sum_{j=0}^{\mu-1} 2^{kj}\gamma_{k,j}, \quad \gamma_{k,j} \in \Omega^k, \tag{4.5}$$

where the last sum is an element of $\Omega^{\mu k}$ (see (3.3)). On the other hand, by the definition of $B(\Gamma(a),\alpha,k,\lambda)$ any number from $\Gamma(a)$ which can be expressed as the sum of the right hand side of (4.5) belongs to $B(\Gamma(a),\alpha,k,\lambda)$. Next, suppose $y \in B(\Gamma(a),\alpha,mk,\lambda\sum_{i=0}^{m-1}2^{ki})$. So for some $\gamma_{mk,j}\in\Omega^{mk}$, y can be written as

$$y = 2^{pmk}\alpha + (\lambda\sum_{i=0}^{m-1}2^{ki})(\sum_{j=0}^{p-1}2^{mkj}) - \sum_{j=0}^{p-1}2^{mkj}\gamma_{mk,j},$$

which yields that

$$y = 2^{pmk}\alpha + (\lambda\sum_{j=0}^{pm-1}2^{kj}) - \sum_{j=0}^{p-1}2^{mkj}\gamma_{mk,j}. \tag{4.6}$$

Again by (3.3), one has $\sum_{j=0}^{p-1}2^{mkj}\gamma_{mk,j}\in\Omega^{pmk}$. Setting $\mu = pm$ we see that the right hand side of (4.6) has the same form as (4.5). Hence, $y\in B(\Gamma(a),\alpha,k,\lambda)$ as desired.

The partial result of the following theorem first appeared in the article [29]. Here, we use Theorem 3.3.1 for a simpler proof and for an improvement of the result given there.

Theorem 4.3.1. *Let $\{a(\alpha)\}$ be a nonnegative finite mask and $\Gamma(a)$ an admissible set of $\{a(\alpha)\}$. If $\{a(\alpha)\}$ satisfies the sum rule (1.3), then the corresponding subdivision scheme is convergent if and only if for any given $k\in\mathbb{N}$ and $0\le(\lambda)_j\le 2^k-1$, $j=1,\ldots,s$, there exists $\alpha\in\Gamma(a)$ such that $B(\Gamma(a),\alpha,k,\lambda)=\Gamma(a)$.*

Proof. Sufficiency. Suppose to the contrary that the subdivision scheme is divergent. Thus, according to Theorem 4.0.1 there exist disjoint proper subsets T and T' of $\Gamma(a)$ satisfying (4.1), i.e., there exist disjoint proper subsets T and T' of $\Gamma(a)$, and a sequence $(\delta_1,\ldots,\delta_k)$ with $\delta_j\in E^s$, for some $k\ge 1$ and $\lambda=\sum_{j=1}^{k}2^{j-1}\delta_j$, such that

$$\frac{T-\lambda+\Omega^k}{2^k}\cap\mathbb{Z}^s\subseteq T \text{ and } \frac{T'-\lambda+\Omega^k}{2^k}\cap\mathbb{Z}^s\subseteq T'.$$

Moreover, without loss of generality, suppose $\alpha\notin T$, i.e., $\mathcal{B}_0\subseteq\Gamma(a)\setminus T$, then we have

$$2^k\alpha+\lambda-\gamma_k\notin T,\quad \forall\ r_k\in\Omega^k.$$

It follows from the definition of $B(\Gamma(a),\alpha,k,\lambda)$ that $\mathcal{B}_1\subseteq\Gamma(a)\setminus T$, ..., $\mathcal{B}_j\subseteq\Gamma(a)\setminus T$ with $j=0,1,\ldots$. Thus, we conclude $\mathcal{B}_0\cup\cdots\cup\mathcal{B}_j\subseteq\Gamma(a)\setminus T$. In other words, $B(\Gamma(a),\alpha,k,\lambda)\subseteq\Gamma(a)\setminus T$. Together with the condition $B(\Gamma(a),\alpha,k,\lambda)=\Gamma(a)$, we conclude that $\Gamma(a) \subseteq \Gamma(a) \setminus T$, which implies that $T = \emptyset$. It means that the set T or T' are not contained in $B(\Gamma(a), \alpha, k, \lambda)$ whenever α does not belong to T or T'. Hence, T or T' must be empty. Consequently, the subdivision scheme converges.

Necessity. Let the subdivision scheme be convergent. So it follows from Theorem 3.1.1 that there is $k' \geq 1$ such that for any $l \geq k'$ and for any l-tuple $(\epsilon_1, ..., \epsilon_l)$ with $\epsilon_j \in E^s$ the matrix $A_{\epsilon_1} \cdots A_{\epsilon_l}$ has a positive column. On the other hand, by (4.4) we need only to show that for any given $k \geq k'$ and $0 \leq (\lambda)_j \leq 2^k - 1$ there is an $\alpha \in \Gamma(a)$ such that

$$B(\Gamma(a), \alpha, k, \lambda) = \Gamma(a).$$

To this end, by Lemma 3.2.1 we notice that for $\eta = \epsilon_1 + 2\epsilon_2 + \cdots + 2^{k-1}\epsilon_k + 2^k\alpha = \lambda + 2^k\alpha$

$$A_{\epsilon_1} \cdots A_{\epsilon_k}(\beta, \alpha) = a^k(\eta - \beta) = a^k(-\beta + \lambda + 2^k\alpha).$$

Thus, we can choose $\alpha \in \Gamma(a)$ such that for all $\beta \in \Gamma(a)$

$$A_{\epsilon_1} \cdots A_{\epsilon_k}(\beta, \alpha) > 0.$$

Clearly, there holds for some $\gamma_k \in \Omega^k$,

$$\eta - \beta = \lambda + 2^k\alpha - \beta = \gamma_k.$$

In other words,

$$\beta = 2^k\alpha + \lambda - \gamma_k,$$

or $B(\Gamma(a), \alpha, k, \lambda) = \Gamma(a)$. This relation holds also for $1 \leq k < k'$ because of (4.4). $\square$

Later we will frequently use the following formula (see Lemma 2.3.2): for $\lambda = \epsilon_1 + 2\epsilon_2 + \cdots + 2^{k-1}\epsilon_k$ and $\epsilon_j \in E^s$

$$A_{\epsilon_1} \cdots A_{\epsilon_k}(\alpha, \beta) = a^k(-\alpha + \lambda + 2^k\beta), \ \ \forall\, \alpha, \beta \in \Gamma(a). \tag{4.7}$$

We need also the concept of connected matrices. Let $B = \{B(\alpha, \beta)\}_{\alpha,\beta\in\Gamma}$ be a square matrix. According to this matrix we should define a directed graph $G(\Gamma, K)$, whose set of edges is given by

$$K = \{(\beta, \alpha) : \ \beta, \alpha \in \Gamma, \ B(\alpha, \beta) \neq 0\}.$$

Definition 4.3.2. Let $B = \{B(\alpha,\beta)\}_{\alpha,\beta\in\Gamma}$ be a square matrix and $G(\Gamma, K)$ be the associated directed graph. B is **connected** if the directed graph $G(\Gamma, K)$ is so, i.e., for some vertex $\beta' \in \Gamma$ and any $\alpha \in \Gamma \setminus \{\beta'\}$ there exists a directed path from β' to α. We call B is strongly connected if the directed graph $G(\Gamma, K)$ is so, i.e., for any points $\alpha, \beta \in \Gamma$ there is a directed path from α to β.

It is now clear that $B(\Gamma, \alpha, k, \lambda) = \Gamma$ means that $A_{\epsilon_1} \cdots A_{\epsilon_k}$ is connected with α, where $\lambda = \epsilon_1 + 2\epsilon_2 + \cdots + 2^{k-1}\epsilon_k$. In fact, the associated graph $G = (\Gamma, K)$ is given by $\Gamma = \Gamma(a)$ and $B = A_{\epsilon_1} \cdots A_{\epsilon_l}$. Let $\beta \in B(\Gamma(a), \alpha, k, \lambda)$, so there is l and $\beta_1 \in \mathcal{B}_l$ such that for some $\gamma_k \in \Omega^k$

$$\beta = 2^k\beta_1 + \lambda - \gamma_k.$$

We conclude from Lemma 3.2.1 and (4.7) that $A_{\epsilon_1} \cdots A_{\epsilon_k}(\beta, \beta_1) = a^k(-\beta + \lambda + 2^k\beta_1) > 0$. By the definition of the graph $G(\Gamma, K)$ one has $(\beta_1, \beta) \in K$. Recursively, K contains

$$(\alpha, \beta_\nu), (\beta_\nu, \beta_{\nu-1}), ..., (\beta_2, \beta_1), (\beta_1, \beta).$$

We obtain a directed path from α to β.
It follows from this observation and Theorem 4.3.1 that

Corollary 4.3.3. *Let $\{a(\alpha)\}$ be a nonnegative finite mask satisfying the sum rule (1.3). Let further A_ϵ be given by (3.1), $\epsilon \in E^s$. Then $A_{\epsilon_1}A_{\epsilon_2} \cdots A_{\epsilon_k}$, $\epsilon_j \in E^s$ with $j = 1, ..., k$, is connected if and only if for some $\alpha \in \Gamma(a)$*

$$B(\Gamma(a), \alpha, k, \lambda) = \Gamma(a),$$

where $\lambda = \epsilon_1 + 2\epsilon_2 + \cdots + 2^{k-1}\epsilon_k$ and $\epsilon_j \in E^s$. Furthermore, the corresponding subdivision scheme converges if and only if any product $A_{\epsilon_1}A_{\epsilon_2} \cdots A_{\epsilon_k}$, $\epsilon_j \in E^s$ with $j = 1, ..., k$ is connected. Moreover, it follows from (4.4) that the connectivity of $(A_{\epsilon_1}A_{\epsilon_2} \cdots A_{\epsilon_k})^l$ for some $l \geq 1$ implies the connectivity of $A_{\epsilon_1}A_{\epsilon_2} \cdots A_{\epsilon_k}$.

Chapter 5

Necessary Conditions for the Convergence

In previous chapters, we focus on the necessary and sufficient conditions on the convergent subdivision schemes. The characterization has combinatorial nature. However, this characterization is still unsatisfactory and seems rather difficult to calculate. How can we simplify those conditions? We begin in this chapter with the investigation of the necessary conditions of convergent subdivision schemes in the multivariate case. We hope that this study will help us to get some computable properties, which may lead to solve our problem. Knowing that the convergence of subdivision schemes with nonnegative masks relies on the location of its support of the mask, we consider the positions of the points in the support and the convex cover of the support. In the last section we will demonstrate the different properties between the inner and boundary points of the support, that will be applied for the study of the matrix $A(\alpha, \beta) = a(-\alpha + 2\beta),\ \ \alpha, \beta \in [\Omega] \cap \mathbb{Z}^s$ in the next chapter and that may help us to design convergent subdivision schemes.

5.1 Unimodular matrices

In the following research, we will often use the concept of unimodular matrices. A **unimodular matrix** M is a square matrix with integer entries having determinant 1 or -1. Let $\mathfrak{M}_s$ be the set of $s \times s$ unimodular matrices, namely,

$$\mathfrak{M}_s = \{M : \ M \text{ is an } s \times s \, \text{matrix with integer entries and} \, |\det M| = 1\}.$$

Equivalently, it is an integer matrix that is invertible over the integers: there is an integer matrix M^{-1} which is its inverse (these are equivalent under Cramer's rule). Thus every equation $M\alpha = \beta$, where M is unimodular, α, β are vectors and β is an integer, has an integral solution. Clearly, $\mathfrak{M}_s$ is a group under the matrix multiplication, which has for-reaching applications in arithmetic and geometry. In particular, identity matrix, the inverse of a unimodular matrix and product of two unimodular matrices are again unimodular. Moreover, invertibility of unimodular matrices is in general more numerically stable. According to the properties of unimodular matrices, we find that the transformation of masks under a unimodular matrix does not affect the convergence and the divergence of the corresponding subdivision scheme. As a result, we have

Lemma 5.1.1. *Let $\{a(\alpha) : \ \alpha \in \mathbb{Z}^s\}$ be a finite mask in $\mathbb{R}^s$ and satisfy the sum rule (1.3). Let further $b(\alpha) = a(M\alpha)$ for any given $M \in \mathfrak{M}_s$. Then, $\{b(\alpha)\}$ satisfies the sum rule. Moreover, the convergence behavior of the subdivision schemes associated with $\{a(\alpha)\}$ and $\{b(\alpha)\}$, respectively, are the same.*

Proof. By the definition of the set of unimodular matrices $\mathfrak{M}_s$ one has

$$\sum_{\beta} a(M(\alpha + 2\beta)) = \sum_{\beta} a(M\alpha + 2M\beta).$$

It yields from the sum rule (1.3) that

$$\sum_{\beta} a(M\alpha + 2M\beta) = 1.$$

Therefore, we have

$$\sum_{\beta} b(\alpha + 2\beta) = \sum_{\beta} a(M(\alpha + 2\beta)) = 1,$$

since $b(\alpha) = a(M\alpha)$ for any given $M \in \mathfrak{M}_s$.
To show the second assertion we need only to verify

$$b^l(\alpha) = a^l(M\alpha), \quad l = 1, 2, ... \tag{5.1}$$

We use induction to do this. Clearly, for $l = 1$, $b(\alpha) = a(M\alpha)$. Suppose for $l = k$, the identity (5.1) is true, i.e., $b^k(\alpha) = a^k(M\alpha)$. Then for $l = k+1$, we have by the iteration formula (see Chapter 1)

$$\begin{aligned} b^{k+1}(\alpha) &= \sum_\beta b^k(\beta) b(\alpha - 2\beta) \\ &= \sum_\beta a^k(M\beta) a(M\alpha - 2M\beta) \\ &= \sum_\gamma a^k(\gamma) a(M\alpha - 2\gamma) = a^{k+1}(M\alpha). \end{aligned}$$

Therefore, if (1.4) in Theorem 1.0.1 holds for $\{a(\alpha)\}$, so does $\{b(\alpha)\}$, and vice versa. □

5.2 Translation of masks

For any $\gamma \in \mathbb{Z}^s$, let $\Omega_\gamma = \Omega + \gamma$ with the understanding $\Omega_0 = \Omega$. The translational mask is denoted as $\{b(\alpha) : b(\alpha) = a(\alpha + \gamma)\}$, where $\{a(\alpha) : \ \alpha \in \mathbb{Z}^s\}$ is a finite mask in $\mathbb{R}^s$. Note that the translation of mask does not affect the convergence and the divergence of the corresponding subdivision scheme. We have

Lemma 5.2.1. *Let $\{a(\alpha) : \ \alpha \in \mathbb{Z}^s\}$ be a finite mask in $\mathbb{R}^s$ and satisfy the sum rule (1.3). Let further $b(\alpha) = a(\alpha+\gamma)$ for any given $\gamma \in \mathbb{Z}^s$. Then, $\{b(\alpha)\}$ satisfies the sum rule. Moreover, the convergence behavior of the subdivision schemes associated with $\{a(\alpha)\}$ and $\{b(\alpha)\}$ respectively are the same.*

Proof. As we have already seen by the proof of Lemma 2.1.1, the sum rule (1.3) is equivalent to

$$\sum_{\beta \in \mathbb{Z}^s} a(e + 2\beta) = 1, \quad \forall\ e \in E^s.$$

It yields that with $e \equiv \alpha + \gamma \pmod 2$,

$$\sum_\beta b(\alpha + 2\beta) = \sum_\beta a((\alpha + \gamma) + 2\beta) = \sum_\beta a(e + 2\beta) = 1.$$

So $\{b(\alpha)\}$ satisfies the sum rule as well.

To show the second assertion we need only to verify

$$b^l(\alpha) = a^l(\alpha + \sum_{i=0}^{l-1} 2^i\gamma), \quad l = 1, 2, ... \tag{5.2}$$

As $b(\alpha) = a(\alpha + \gamma)$, (5.2) is true for $l = 1$. Suppose (5.2) is true, for $l = k$, i.e., $b^k(\alpha) = a^k(\alpha + \sum_{i=0}^{k-1} 2^i\gamma)$, then for $l = k + 1$, we have

$$\begin{aligned}
b^{k+1}(\alpha) &= \sum_\beta b^k(\beta)b(\alpha - 2\beta) \\
&= \sum_\beta a^k(\beta + \sum_{i=0}^{k-1} 2^i\gamma))a(\alpha + \gamma - 2\beta) \\
&= \sum_\eta a^k(\eta)a(\alpha + \gamma - 2(\eta - \sum_{i=0}^{k-1} 2^i\gamma)) \\
&= \sum_\eta a^k(\eta)a(\alpha + \sum_{i=0}^{k} 2^i\gamma - 2\eta) \\
&= a^{k+1}(\alpha + \sum_{i=0}^{k} 2^i\gamma).
\end{aligned}$$

Therefore, if (1.4) in Theorem 1.0.1 holds for $\{a(\alpha)\}$, so does $\{b(\alpha)\}$, and vice versa. $\square$

5.3 Compression of subdivision schemes

In order to present more properties of convergent subdivision schemes, we introduce in this section the concept of **compression**, which explores a family of subdivision methods obtained from 'compressing' of a given subdivision scheme into one defined on a space of lower dimension (see also [3]).

Begin with the $s \times n$ matrix X whose columns are given by $x^1, ..., x^n \in \mathbb{Z}^s$ and $s \leq n$ such that $X\mathbb{Z}^n = \mathbb{Z}^s$. We observe the so-called compressed mask $\{b(\beta) : \beta \in \mathbb{Z}^s\}$ defined by

$$b(\beta) = 2^{s-n} \sum_{X\alpha=\beta, \alpha\in\mathbb{Z}^n} a(\alpha), \tag{5.3}$$

which is compressed by the corresponding mask $\{a(\alpha) : \alpha \in \mathbb{Z}^n\}$.

For the iterated mask we get the connection between the original mask $\{a^k(\alpha)\}$ and the compressed mask $\{b^k(\beta)\}$. That is

$$b^k(\beta) = 2^{k(s-n)} \sum_{X\alpha=\beta,\alpha\in\mathbb{Z}^n} a^k(\alpha).$$

To this end, we recall that $\{a^k(\alpha)\}$ and $\{b^k(\beta)\}$ are generated inductively by the equations

$$a^1(\alpha) = a(\alpha), \quad a^k(\alpha) = \sum_{\gamma\in\mathbb{Z}^n} a^{k-1}(\gamma)a(\alpha-2\gamma) \quad \text{for} \quad k \geq 2$$

and

$$b^1(\beta) = b(\beta), \quad b^k(\beta) = \sum_{\mu\in\mathbb{Z}^s} b^{k-1}(\mu)b(\beta-2\mu) \quad \text{for} \quad k \geq 2.$$

Assume that $b^{k-1}(\beta) = 2^{(k-1)(s-n)} \sum_{X\alpha=\beta} a^{k-1}(\alpha)$. Then by $X\mathbb{Z}^n = \mathbb{Z}^s$, we have

$$\begin{aligned} b^k(\beta) &= \sum_{\mu\in\mathbb{Z}^s} b^{k-1}(\mu)b(\beta-2\mu) \\ &= \sum_{\mu\in\mathbb{Z}^s} \left(2^{(k-1)(s-n)} \sum_{X\gamma=\mu,\gamma\in\mathbb{Z}^n} a^{k-1}(\gamma)\right) \cdot \left(2^{s-n} \sum_{X\alpha=\beta-2\mu,\alpha\in\mathbb{Z}^n} a(\alpha)\right). \end{aligned}$$

Simple computation yields

$$\begin{aligned} b^k(\beta) &= 2^{k(s-n)} \sum_{\mu\in\mathbb{Z}^s} \left(\sum_{X\gamma=\mu,\gamma\in\mathbb{Z}^n} a^{k-1}(\gamma) \cdot \sum_{X(\alpha-2\gamma)=\beta-2\mu,\alpha\in\mathbb{Z}^n} a(\alpha-2\gamma)\right) \\ &= 2^{k(s-n)} \sum_{\mu\in\mathbb{Z}^s} \sum_{X\gamma=\mu} \sum_{X(\alpha-2\gamma)=\beta-2\mu} a^{k-1}(\gamma)a(\alpha-2\gamma) \\ &= 2^{k(s-n)} \sum_{X\alpha=\beta} \sum_{\gamma\in\mathbb{Z}^n} a^{k-1}(\gamma)a(\alpha-2\gamma) \\ &= 2^{k(s-n)} \sum_{X\alpha=\beta} a^k(\alpha). \end{aligned}$$

We are now in the position to state the relationship between the original subdivision scheme and the one formed by compression (see [3]).

Lemma 5.3.1. *Let $X = \{x^1, ..., x^n\} \subseteq \mathbb{Z}^s$ be an $s\times n$ matrix with $\{X\alpha : \alpha \in \mathbb{Z}^n\} = \mathbb{Z}^s$. Suppose the subdivision scheme, which is determined by the mask $\{a(\alpha) : \alpha \in \mathbb{Z}^n\}$, converges. Then the subdivision scheme determined by the compressed mask $\{b(\beta) : \beta \in \mathbb{Z}^s\}$ as given in (5.3) converges.*

Proof. By hypothesis on the convergent subdivision scheme determined by the mask $\{a(\alpha) : \alpha \in \mathbb{Z}^n\}$, it follows from Theorem 1.0.1 and Lemma 2.1.1 that for any $e \in E^n$,

$$\sum_{\beta\in\mathbb{Z}^n} a(2\beta + e) = 1$$

and

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}^n,\, e\in E^n} |a^k(\alpha) - a^k(\alpha - e)| = 0.$$

We note that we may extend X to a unimodular matrix by adding $n - s$ suitable rows (say Y). Thus, $M = \binom{X}{Y} \in \mathfrak{M}_n$ and $Y\mathbb{Z}^n = \mathbb{Z}^{n-s}$. Denote $c(M\alpha) = a(M^{-1}M\alpha)$, for $\alpha \in \mathbb{Z}^n$. Then according to (5.3), we have for $\beta \in \mathbb{Z}^s$

$$\begin{aligned} b(\beta) &= 2^{s-n} \sum_{X\alpha=\beta} a(\alpha) = 2^{s-n} \sum_{X\alpha=\beta} c(M\alpha) \\ &= 2^{s-n} \sum_{X\alpha=\beta,\alpha\in\mathbb{Z}^n} c\begin{pmatrix}\beta \\ Y\alpha\end{pmatrix} = 2^{s-n} \sum_{\mu\in\mathbb{Z}^{n-s}} c\begin{pmatrix}\beta \\ \mu\end{pmatrix}. \end{aligned} \tag{5.4}$$

On one hand, it follows from (5.4) that for $\epsilon \in E^s$

$$\begin{aligned} \sum_{\beta\in\mathbb{Z}^s} b(\epsilon + 2\beta) &= 2^{s-n} \sum_{\beta\in\mathbb{Z}^s} \sum_{\mu\in\mathbb{Z}^{n-s},\delta\in E^{n-s}} c\left(2\begin{pmatrix}\beta \\ \mu\end{pmatrix} + \begin{pmatrix}\epsilon \\ \delta\end{pmatrix}\right) \\ &= 2^{s-n} \sum_{\delta\in E^{n-s}} \sum_{\binom{\beta}{\mu}\in\mathbb{Z}^n} c\left(2\begin{pmatrix}\beta \\ \mu\end{pmatrix} + \begin{pmatrix}\epsilon \\ \delta\end{pmatrix}\right). \end{aligned}$$

Set $\binom{\beta}{\mu} = M\alpha$. Since $M\mathbb{Z}^n = \mathbb{Z}^n$, we deduce

$$\begin{aligned} \sum_{\beta\in\mathbb{Z}^s} b(\epsilon + 2\beta) &= 2^{s-n} \sum_{\delta\in E^{n-s}} \sum_{\alpha\in\mathbb{Z}^n} c(M(2\alpha + M^{-1}\begin{pmatrix}\epsilon \\ \delta\end{pmatrix})) \\ &= 2^{s-n} \sum_{\delta\in E^{n-s}} \sum_{\alpha\in\mathbb{Z}^n} a(2\alpha + M^{-1}\begin{pmatrix}\epsilon \\ \delta\end{pmatrix}) = 1. \end{aligned}$$

On the other hand, it follows from (5.4) and the proof of Lemma 5.1.1 that

$$\begin{aligned} |b^k(\beta) - b^k(\beta - \epsilon)| &= \left| 2^{k(s-n)} (\sum_{X\alpha=\beta} a^k(\alpha) - \sum_{X\alpha=\beta-\epsilon} a^k(\alpha)) \right| \\ &= 2^{k(s-n)} \left| \sum_{\mu\in\mathbb{Z}^{n-s}} c^k\begin{pmatrix}\beta \\ \mu\end{pmatrix} - \sum_{\mu\in\mathbb{Z}^{n-s}} c^k\begin{pmatrix}\beta - \epsilon \\ \mu\end{pmatrix} \right| \\ &\leq 2^{k(s-n)} \cdot C_N \cdot 2^{k(n-s)} \max_{\alpha\in\mathbb{Z}^n, e\in E_n} |a^k(\alpha) - a^k(\alpha - e)|, \end{aligned}$$

where the constant C_N is dependent only on the size of the mask $\{a(\alpha)\}$. It tells us that the subdivision scheme determined by the compressed mask $\{b(\beta) : \beta \in \mathbb{Z}^s\}$ satisfies the second condition of Theorem 1.0.1. The proof is complete. □

Now we look at an application of the compression. The following lemma gives us more information about the convergent subdivision scheme (see [28]).

Theorem 5.3.2. *Let $\{a(\alpha)\}$ be a nonnegative finite mask in $\mathbb{R}^n$. Assume that the subdivision scheme associated with $\{a(\alpha) : \alpha \in \mathbb{Z}^n\}$ converges, then there holds*

$$\gcd(\alpha : \quad \alpha \in \Omega_\gamma) = (1, ..., 1)^T, \quad \forall\ \gamma \in \mathbb{Z}^n,$$

where $\gcd(\alpha : \alpha \in \Omega)$ *is a multi-integer $d = (d_1, ..., d_s)$ such that* $\gcd((\alpha)_i : \alpha \in \Omega) = d_i, i = 1, ..., s.$

Proof. In view of Lemma 5.2.1 we may assume $\gamma = 0$. By hypothesis on the convergent subdivision scheme determined by the mask $\{a(\alpha) : \alpha \in \mathbb{Z}^n\}$, it follows from Theorem 5.3.1 that the compressed subdivision scheme determined by $\{b(\beta) : \beta \in \mathbb{Z}^s\}$ also converges. In particular, we choose the $(1 \times n)$-matrix X_i with 1 in the i-th column and 0 in the other columns, i.e.,

$$X_i = (0, ..., 0, 1, 0, ..., 0),$$

which is compressed matrices for the i-th coordinate of the corresponding mask $\{a(\alpha)\}$. Then we get a compressed mask

$$b(j) = 2^{1-n} \sum_{X_i \alpha = j} a(\alpha).$$

It's clear that $b(j) \neq 0$ if and only if there is at least one α such that $(\alpha)_i = j$ and $a(\alpha) \neq 0$. We have compressed the n-dimensional subdivision scheme into a univariate subdivision scheme. Then, together with Theorem 3.0.1, the result follows. □

5.4 j-dimensional faces

In order to study the boundary points of Ω better, we recall the concept of faces. The boundary of the convex cover $[\Omega]$ formed by Ω will be denoted by $\partial[\Omega]$. Thus, $[\Omega]$

is a polytope. For $0 \leq j < s$, a j-dimensional face S_j is a j-dimensional polytope. Moreover, an $(s-1)$-dimensional face S_{s-1} is a facet of $[\Omega]$. If $0 \leq j < s-1$, then a j-dimensional face S_j is a facet of a $(j+1)-$dimensional face. For $0 \leq j < s$ let us observe the j-dimensional face S_j of $[\Omega]$. For example, if a polytope is s-dimensional, then

1) each extreme point (or vertex) is 0-dimensional face,
2) each edge is 1-dimensional face, and so on,
3) each facet is $(s-1)$-dimensional face.

To understand the concept of j-dimensional face S_j of $[\Omega]$ easily, we give an example.

Example 5.4.1. *(see Figur 5.1) Consider a 3-dimensional convex polytope $[\Omega]$, where Ω is a cube, starting from the origin, with edges parallel to the axes and the length of 2, i.e.,*

$$\Omega := \{(x, y, z) \ : \ 0 \leq x \leq 2, \ 0 \leq y \leq 2 \ \ and \ \ 0 \leq z \leq 2\} \cap \mathbb{Z}^s.$$

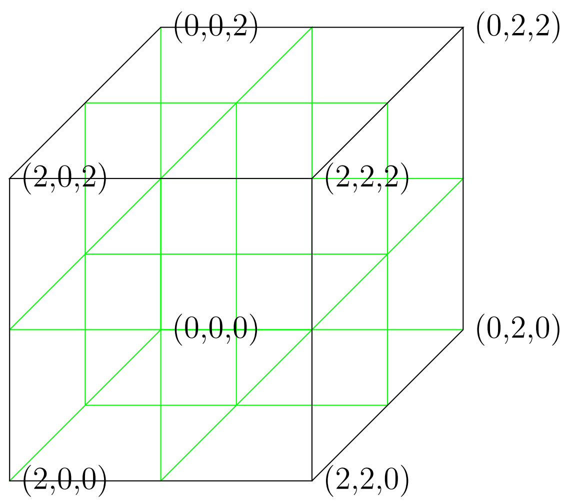

Figure 5.1: Cube

Now we have: S_0 is any point of 8 extreme points (or vertexes) of the cube $[\Omega]$, i.e., $(0,0,0)^T$, $(0,2,0)^T$, $(2,2,0)^T$, $(2,0,0)^T$, $(0,0,2)^T$, $(0,2,2)^T$, $(2,0,2)^T$ and $(2,2,2)^T$; S_1 is any edge of 12 edges of the cube $[\Omega]$; S_2 is any face of 6 square faces of the cube $[\Omega]$.

5.5 New results concerning with the necessary conditions

It was shown in Theorem 3.0.1 that if the univariate subdivision scheme with the nonnegative mask converges, then the boundary points of the corresponding support have the behaviour, i.e., $0 < a(0) < 1$ and $0 < a(N) < 1$. Furthermore, $|a(0)| < 1$ and $|a(N)| < 1$ without the restriction of the positivity. Thus, it is of some interest to consider whether the boundary points of the support have the similar behavior for the multivariate case. Indeed we get more involved results.

We are now in a position to present some necessary conditions, which show the relationship between the convergence of a subdivision scheme and the position of points in the support.

Let us recall the definition of the set $\mathcal{A}(\lambda)$ introduced in Section 3.2. We denote for a given finitely supported real mask $\{a(\alpha)\}$ the set

$$\mathcal{A}(\lambda) = \{\alpha : a(\alpha) \neq 0 \text{ and } \alpha \equiv \lambda \pmod 2\}, \quad \forall\, \lambda \in \mathbb{Z}^s.$$

Denote $|\mathcal{A}(\lambda)|$ to be the number of the elements in the set $\mathcal{A}(\lambda)$. The recursion formula and the sum rule (1.3) tell us that for each $\lambda \in \mathbb{Z}^s$ the set $\mathcal{A}(\lambda)$ has at least one element or $|\mathcal{A}(\lambda)| \geq 1$. Moreover, the sum rule (1.3) implies

$$\sum_{\alpha \in \mathcal{A}(\lambda)} a(\alpha) = 1, \quad \forall\, \lambda \in \mathbb{Z}^s, \quad m = 1, 2, ... \tag{5.5}$$

We state the necessary condition of the convergence on the subdivision schemes, which describes the behavior of the points in the support of the mask as follows.

Theorem 5.5.1. *Let $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ be a finite mask in $\mathbb{R}^s$. Assume that the subdivision scheme associated with $\{a(\alpha)\}$ converges to a continuous function φ. If $|\mathcal{A}(\lambda)| = 1$ for some $\lambda \in \mathbb{Z}^s$, then the only one element of $\mathcal{A}(\lambda)$ (say α') belongs to $\Omega \setminus \partial[\Omega]$ and $\varphi(\alpha') = 1$. Furthermore, for any j-dimensional face S_j of the polytope $[\Omega]$ and $0 \leq j < s$ there holds*

$$|\sum_{\alpha \in S_j \cap \Omega} a(\alpha)| < 2^j. \tag{5.6}$$

If, in addition, the mask $\{a(\alpha)\}$ is nonnegative, then there is at most one $\beta \in \mathbb{Z}^s$ satisfying $\varphi(\beta) = 1$ and

$$0 < \sum_{\alpha \in S_j \cap \Omega} a(\alpha) < 2^j, \quad 0 \le j < s. \tag{5.7}$$

Proof. Suppose the subdivision scheme with mask $\{a(\alpha)\}$ converges, then by Theorem 1.0.1, one has the sum rule (1.3)

$$\sum_{\beta} a(\alpha + 2\beta) = 1, \quad \forall\, \alpha \in \mathbb{Z}^s.$$

Together with the condition of $|\mathcal{A}(\lambda)| = 1$ for some $\lambda \in E^s$, i.e., there is only one element α' in set $\mathcal{A}(\lambda)$. It follows from (5.5) that $a(\alpha') = 1$.
On the one hand, by the definition of the convergence of the subdivision scheme, we have for the continuous function φ,

$$\lim_{k \to \infty} \sup_{\alpha} |\varphi(\frac{\alpha}{2^k}) - a^k(\alpha)| = 0. \tag{5.8}$$

It follows from the properties of the hat function and $\lim_{k\to\infty} f^k(x) = \varphi(x)$ that

$$\varphi(x) = 0, \quad \forall\, x \in \partial[\Omega]. \tag{5.9}$$

On the other hand, since $a^k(\alpha) = \sum_\beta a^{k-1}(\beta) a(\alpha - 2\beta)$, one has

$$\begin{aligned} a^k((2^k-1)\alpha') &= \sum_{\beta} a^{k-1}(\beta) a((2^k-1)\alpha' - 2\beta) \\ &= a(\alpha') a^{k-1}((2^{k-1}-1)\alpha'), \end{aligned}$$

since there is only one element $\beta = (2^{k-1} - 1)\alpha'$ such that $a((2^k - 1)\alpha' - 2\beta) \neq 0$. Recursively,

$$\begin{aligned} a^k((2^k-1)\alpha') &= a(\alpha') a^{k-1}((2^{k-1}-1)\alpha') \\ &= (a(\alpha'))^2 a^{k-2}(\alpha'(2^{k-2}-1)) \\ &\cdots \\ &= (a(\alpha'))^k, \end{aligned}$$

which yields $a^k((2^k-1)\alpha')=1$. Furthermore, by (5.8) one has

$$\lim_{k\to\infty}|\varphi(\frac{(2^k-1)\alpha'}{2^k})-a^k((2^k-1)\alpha')|=0,$$

i.e., $\lim_{k\to\infty}\varphi((2^k-1)\alpha'/2^k)=1$. It follows from the continuity of φ that

$$\begin{aligned}\lim_{k\to\infty}\varphi(\frac{(2^k-1)\alpha'}{2^k}) &= \varphi(\lim_{k\to\infty}\frac{(2^k-1)\alpha'}{2^k})\\ &= \varphi(\alpha').\end{aligned}$$

Therefore $\varphi(\alpha')=1$, which yields that α' must be an inner point of $[\Omega]$ according to (5.9).

Now let us prove (5.6). It follows from Lemmas 5.1.1 and 5.2.1 that the subdivision scheme associated with $\{a'(\alpha)\}=\{a(M^{-1}\alpha-\beta)\}$ for any fixed $\beta\in\mathbb{Z}^s$ and $M\in\mathfrak{M}_s$ is also convergent. On the other hand, the support of $\{a'(\alpha)\}$ is $M(\beta+\Omega)$ where Ω is the support of $\{a(\alpha)\}$.

Let S_j be a j-dimensional face of $[\Omega]$. So $M(S_j+\beta)$ is a j-dimensional face of $[M(\Omega+\beta)]$. It is well-known (see e.g. [13] Chapter 14) for any j vectors $x_i\in\mathbb{Z}^s, i=1,...,j$, there is a unimodular matrix $M'\in\mathfrak{M}_s$ such that the first $s-j$ components of $M'x_i$, $i=1,...,j$, are zero. More precisely, there holds

$$M'(x_1,...,x_j)=(\eta_1,...,\eta_j)$$

with $\eta_i=(0,...,0,\eta_{s-j+1,i},...,\eta_{s,i})^T$, $i=1,...,j$. Therefore, for a fixed $0\le j'<s$ we can choose a unimodular matrix $M\in\mathfrak{M}_s$ and a vector $\beta\in\mathbb{Z}^s$ such that $0\in M(S_{j'}+\beta)\cap M(\Omega+\beta)$ and $M(S_{j'}+\beta)$ is orthogonal to the space $\mathbb{Z}^{s-j'}$ embedded in $\mathbb{Z}^s$, i.e. $\mathbb{Z}^{s-j'}$ is the subspace of $\mathbb{Z}^s$ whose last j' components are zero.

Next by the definition of the compressed mask (see (5.3)), we write out the mask in $\mathbb{Z}^{s-j'}$ as follows,

$$b(\alpha)=\frac{1}{2^{j'}}\sum_{u|_{\mathbb{Z}^{s-j'}}=\alpha}a'(u).$$

According to the choice of M and β, there holds $0\in M(S_{j'}+\beta)\cap M(\Omega+\beta)$. Consequently, we have

$$\begin{aligned} b(0) &= \frac{1}{2^{j'}} \sum_{u|_{\mathbb{Z}^{s-j'}}=0} a'(u) \\ &= \frac{1}{2^{j'}} \sum_{u \in M(S_{j'}+\beta)\cap M(\Omega+\beta)} a(M^{-1}u - \beta) \\ &= \frac{1}{2^{j'}} \sum_{u \in S_{j'} \cap \Omega} a(u), \end{aligned}$$

which tells us that in order to get the inequality (5.6) we need only to verify $|b(0)| < 1$. It's clear that $|b(0)| < 1$, if $b(0) = 0$. It remains to show $|b(0)| < 1$ for $b(0) \neq 0$. The fact that $S_{j'}$ is a j'-dimensional face implies that 0 is an extreme point of $[\{\alpha : b(\alpha) \neq 0\}]$. Furthermore, Lemma 5.3.1 means that the subdivision scheme determined by the compressed mask $\{b(\alpha)\}$ converges. Without loss the generality we suppose that this subdivision scheme converges to φ_b, so the support of φ_b is contained in the polytope $[\{\alpha : b(\alpha) \neq 0\}]$. We know also that 0 is an extreme point of this polytope, which yields $\varphi_b(0) = 0$. On the other hand, by the definition, we have

$$\lim_{k\to\infty} |\varphi_b(0) - b^k(0)| = 0.$$

However, by (2.2) we get with the choice $\beta = 0$ that

$$b^k(0) = \sum_{\beta_0+2\beta_1+\cdots+2^{k-1}\beta_{k-1}=0} b(\beta_0)\cdots b(\beta_{k-1})$$

for all $\beta_0, ..., \beta_{k-1} \in \{\alpha : b(\alpha) \neq 0\}$. As 0 is an extreme point one must have $\beta_i = 0$, $i = 0, ..., k-1$. Hence, $b^k(0) = (b(0))^k$. Noticing $|b^k(0)| = o(1)$, so $|b(0)| < 1$.

Now we show the assertions for the nonnegative mask. By hypothesis that $\{a(\alpha)\}$ is nonnegative, it follows from the definition of a compressed mask that $\{b(\alpha)\}$ is also nonnegative and $b(0) > 0$. Then (5.7) follows from (5.6).

It remains to show that there is at most one $\beta \in \mathbb{Z}^s$ satisfying $\varphi(\beta) = 1$. Suppose that there exist α_1 and α_2 with $\alpha_1 \neq \alpha_2$ such that

$$\varphi(\alpha_1) = 1 \quad \text{and} \quad \varphi(\alpha_2) = 1.$$

However, since φ is nonnegative, we obtain from identity (see [3]) $\sum_\alpha \varphi(x-\alpha) = 1$ that

$$1 = \sum_\alpha \varphi(\alpha_1 + \alpha_2 - \alpha) = \varphi(\alpha_1) + \varphi(\alpha_2) + \sum_{\alpha \neq \alpha_1, \alpha_2} \varphi(\alpha_1 + \alpha_2 - \alpha) \geq 2.$$

Thus, there is at most one $\beta \in \mathbb{Z}^s$ such that $\varphi(\beta) = 1$. □

For convenience of expression, we define the property of the convergent subdivision schemes in Theorem 5.5.1 as **inner-point principle**. More precisely, it possesses the following two properties.

1). If $|\mathcal{A}(\lambda)| = 1$ for some $\lambda \in \mathbb{Z}^s$, then the only one element of $\mathcal{A}(\lambda)$ (say α') belongs to $\Omega \setminus \partial[\Omega]$.

2). $|\mathcal{A}(\lambda)| \geq 1$, $\forall \lambda \in \mathbb{Z}^s$. There is at most one set $\mathcal{A}(\lambda), \lambda \in \mathbb{Z}^s$, with $|\mathcal{A}(\lambda)| = 1$.

The following necessary condition on the convergent subdivision scheme with a finite nonnegative mask is inspired from Theorem 5.5.1 and Example 4.1.2 in Section 4.1. For an affine space L in $\mathbb{R}^s$, we denote $\dim L$ to be the dimension of L.

Corollary 5.5.2. *Let $\{a(\alpha)\}$ be a finite nonnegative mask and $\Omega \subset \mathbb{Z}^s$ be the support. Assume that the corresponding subdivision scheme converges. If there exist two affine spaces L_1 and L_2 with $0 \leq \dim L_1, \dim L_2 \leq s$ such that*

$$\sum_{\alpha \in L_1 \cap \Omega} a(\alpha) = 2^{\dim L_1} \quad \textit{and} \quad \sum_{\alpha \in L_2 \cap \Omega} a(\alpha) = 2^{\dim L_2}, \tag{5.10}$$

then $L_1 \cap L_2 \neq \emptyset$ and

$$\sum_{\alpha \in L_1 \cap L_2 \cap \Omega} a(\alpha) = 2^{\dim L_1 \cap L_2}. \tag{5.11}$$

Proof. Assume $l = \dim L_1$. Then there are 2^l integer points

$$\alpha_1, \alpha_2, ..., \alpha_{2^l} \in L_1 \cap \mathbb{Z}^s$$

such that

$$\alpha_i \not\equiv \alpha_j \pmod 2, \quad \text{for } i, j = 1, ..., 2^l \text{ and } i \neq j.$$

We know that the subdivision scheme converges, which implies (5.5), i.e.,

$$\sum_{\alpha \in \mathcal{A}(\alpha_j)} a(\alpha) = 1, \quad j = 1, ..., 2^l.$$

It follows from (5.10) and (5.5) that $\mathcal{A}(\alpha_j) \subseteq L_1$, $j = 1, 2, ..., 2^l$. Assume $\beta \in L_1 \cap \mathbb{Z}^s$. So there exists a number $\gamma \in \Omega$ with $\beta \equiv \gamma \pmod 2$, which belongs to L_1. In other words,

$$\frac{\beta + \gamma}{2} \in L_1 \cap \mathbb{Z}^s$$

or

$$\frac{(L_1 \cap \mathbb{Z}^s) + \Omega}{2} \cap \mathbb{Z}^s \subseteq L_1 \cap \mathbb{Z}^s.$$

Consequently, according to the definition of the irreducible mapping (see (4.2)), we obtain $T \subseteq L_1 \cap \mathbb{Z}^s$, which is irreducible with respec to Ω and $\lambda = 0$, i.e.

$$\frac{T + \Omega}{2} \cap \mathbb{Z}^s = T.$$

On the other hand, by the hypothesis that the subdivision scheme converges, we conclude from Theorem 3.3.1 and the definition of the irreducible mapping (4.2) (in detail T is minimal) that T is unique. The same argument used above can also be applied to derive the same assertion for T by L_2 instead of L_1. To be specific, $T \subseteq L_2 \cap \mathbb{Z}^s$. We find, therefore, that $T \subseteq L_1 \cap L_2 \cap \mathbb{Z}^s$, which means $L_1 \cap L_2 \neq \emptyset$.

Next we show that (5.11) is valid. We know that $L_1 \cap L_2$ is again an affine space. Take $\alpha \in L_1 \cap L_2 \cap \mathbb{Z}^s$. Then all $\gamma \in \Omega$, which satisfies $\gamma \equiv \alpha \pmod 2$, must belong to $L_1 \cap L_2$. It follows from (5.5) that

$$\sum_{\alpha \in L_1 \cap L_2 \cap \Omega} a(\alpha) = 2^{\dim(L_1 \cap L_2)}.$$

The proof is complete. □

Chapter 6

Connectivity of a Matrix

In Chapter 5 we have investigated the different properties between the inner and boundary points of the support for the mask, when the corresponding subdivision scheme converges. However, it is unknown, whether one can use some simple conditions to guarantee those properties. We find out that the so-called connectivity of a matrix A deduced by a given mask (see the definition below) is the suitable condition. Another motivation to discuss the behaviour of the matrix A is that we believe firmly that the convergence of the subdivision scheme with a nonnegative mask can be described by the matrix A. For the univariate case it is already known that the connectivity of the matrix A and the sum rule imply the convergence (see [31]), although for the bivariate case the connectivity of the matrix A is not enough, as the example in Section 6.1 shows. The study of this matrix A is our main focus in this chapter. To this end, let A be the square matrix given by

$$A(\alpha, \beta) = a(-\alpha + 2\beta), \ \ \alpha, \beta \in [\Omega] \cap \mathbb{Z}^s. \tag{6.1}$$

So A is a row-stochastic matrix if the mask $\{a(\alpha)\}$ is nonnegative and the sum rule (1.3) is satisfied. Indeed, let $\alpha \in [\Omega] \cap \mathbb{Z}^s$. Hence, $a(-\alpha + 2\beta) \neq 0$ means $-\alpha + 2\beta = \gamma$ for some $\gamma \in \Omega$, which in turn implies $\beta = (\alpha + \gamma)/2 \in [\Omega] \cap \mathbb{Z}^s$. In other words, if $\alpha \in [\Omega] \cap \mathbb{Z}^s$ and $\beta \notin [\Omega] \cap \mathbb{Z}^s$ then $a(-\alpha + 2\beta) = 0$. We get therefore for any $\alpha \in [\Omega] \cap \mathbb{Z}^s$,

$$\sum_{\beta \in [\Omega] \cap \mathbb{Z}^s} a(-\alpha + 2\beta) = \sum_{\beta \in \mathbb{Z}^s} a(-\alpha + 2\beta) = 1$$

or

$$\sum_{\beta\in[\Omega]\cap\mathbb{Z}^s} A(\alpha,\beta) = 1.$$

In this chapter, we present some properties concerning with the matrix A. Moreover, we try to make use of the related methods from graph theory to consider the convergence of the subdivision schemes. Furthermore we give an efficient algorithm, which shows that the connectivity of the matrix A may be tested by depth-first search algorithm from graph theory in linear time with respect to the size of A.

6.1 Convergence and connectivity of A

The connectivity of a square matrix is defined by Definition 4.3.2 in Section 4.3. The following theorem shows that the connectivity of A^l, where A is given by (6.1), is necessary for the convergence of the subdivision scheme with nonnegative finite masks.

Theorem 6.1.1. *If the subdivision scheme with a nonnegative finite mask* $\{a(\alpha)\}$ *converges, then* $\{a(\alpha)\}$ *satisfies the sum rule (1.3) and* A^l *,* $l = 1, ..., N$*, is connected, where* A *is given by (6.1) and* $N = |[\Omega]\cap\mathbb{Z}^s|$*.*

Proof. According to Theorem 3.1.1, if the subdivision scheme with the nonnegative mask $\{a(\alpha)\}$ converges, then the sum rule (1.3) is certainly fulfilled and we need to prove the second assertion. Again from Theorem 3.1.1 we conclude that for the admissible set $\Gamma(a)$ of $\{a(\alpha)\}$ with $[\Omega]\cap\mathbb{Z}^s \subseteq \Gamma(a)$, there is an $\alpha_0 \in \Gamma(a)$ and $k \geq 1$ such that the α_0-column of A_0^k is positive, where A_0 is the $|\Gamma(a)| \times |\Gamma(a)|$ matrix defined by (3.1) with $\delta = 0$, i.e.,

$$A_0(\alpha,\beta) = a(-\alpha + 2\beta), \quad \alpha,\beta \in \Gamma(a).$$

In other words, we choose $\alpha_0 \in \Gamma(a)$ and for all $\alpha \in \Gamma(a)$, $a^k(-\alpha + 2^k\alpha_0) > 0$, which yields $-\alpha + 2^k\alpha_0 \in \Omega^k$. This holds in particular for all $\alpha \in [\Omega]\cap\mathbb{Z}^s$. It follows from Lemma 3.2.1 and (3.3) that for some $\gamma_j \in \Omega$ we have

$$\alpha = 2^k\alpha_0 - \sum_{j=0}^{k-1} 2^j\gamma_j \quad \textit{or} \quad \alpha_0 = \frac{1}{2^k}\left(\alpha + \sum_{j=0}^{k-1} 2^j\gamma_j\right) \in [\Omega].$$

On the other hand, we notice that for $\alpha,\beta \in [\Omega]\cap\mathbb{Z}^s$ there holds $A^k(\alpha,\beta) = A_0^k(\alpha,\beta)$. Therefore the α_0-column of A^k is also positive. It follows from Corollary 4.3.3 that A^l, $l = 1, ..., N$, is connected. □

We note from the above proof that α_0 lies in $[\Omega] \setminus \partial[\Omega]$. To see this we choose $\alpha \in [\Omega] \setminus \partial[\Omega]$, then as a convex combination of integers α and $\gamma_j \in \Omega$ the number α_0 must belong to $[\Omega] \setminus \partial[\Omega]$. Later we need this fact.

In the remainder of this section, we focus on the discussion how (or whether) the connectivity of the matrix A influences the convergence of the subdivision scheme with nonnegative finite mask $\{a(\alpha)\}$.

Lemma 6.1.2. *Let the nonnegative finite mask $\{a(\alpha)\}$ satisfy the sum rule (1.3). If A^l is connected, then for any $M \in \mathfrak{M}_s$ and any finite convex set $\Gamma \subset \mathbb{Z}^s$, i.e., $[\Gamma] \cap \mathbb{Z}^s = \Gamma$, such that $M([\Omega] \cap \mathbb{Z}^s) \subseteq \Gamma$, the matrix B defined by*

$$B(\alpha, \beta) = a^l(-M^{-1}\alpha + 2^l M^{-1}\beta), \quad \alpha,\ \beta \in \Gamma$$

is row-stochastic and connected with some $\beta \in ([M\Omega] \cap \mathbb{Z}^s) \setminus \partial[M\Omega]$.

Proof. By Lemma 3.2.1 we know that the support of the mask $\{a^l(\alpha)\}$ is Ω^l. So the support of the mask $\{a^l(M^{-1}\alpha)\}$ is $M\Omega^l$, which may also be referred to as $(M\Omega)^l$. On one hand, since the nonnegative mask $\{a(\alpha)\}$ satisfies the sum rule (1.3), A is a row-stochastic matrix and

$$\sum_{\beta \in [\Omega] \cap \mathbb{Z}^s} A(\alpha, \beta) = 1, \quad \forall \alpha \in [\Omega] \cap \mathbb{Z}^s.$$

Then

$$\sum_{\beta \in [\Omega] \cap \mathbb{Z}^s} A^l(\alpha, \beta) = 1, \quad i.e., \quad \sum_{\beta \in [\Omega] \cap \mathbb{Z}^s} a^l(-\alpha + 2^l\beta) = 1, \quad \forall \alpha \in [\Omega] \cap \mathbb{Z}^s.$$

It follows from Lemmas 5.1.1 and 5.2.1 that for all $\alpha \in [\Omega] \cap \mathbb{Z}^s$

$$\sum_{\beta \in [\Omega] \cap \mathbb{Z}^s} B(\alpha, \beta) = 1, \quad i.e., \quad \sum_{\beta \in [\Omega] \cap \mathbb{Z}^s} a^l(-M^{-1}\alpha + 2^l M^{-1}\beta) = 1.$$

On the other hand, let $\alpha \in \Gamma$, $\gamma \in M\Omega^l$ satisfying $(\alpha + \gamma)/2^l \in \mathbb{Z}^s$. Denote $\beta = (\alpha + \gamma)/2^l$. Then by the hypothesis that the set Γ is convex and $M[\Omega] \cap \mathbb{Z}^s \subseteq \Gamma$, we conclude $\beta \in \Gamma$. So $\gamma = -\alpha + 2^l\beta \notin M\Omega^l$, if $\beta \notin \Gamma$ and $\alpha \in \Gamma$. In other words, $a^l(-M^{-1}\gamma) = a^l(-M^{-1}\alpha + 2^l M^{-1}\beta) = 0$. Hence, for fixed $\alpha \in \Gamma$ the number

$a^l(-M^{-1}\alpha + 2^l M^{-1}\beta) \neq 0$ implies $\beta \in \Gamma$. It follows from the sum rule (1.3) that for any $\alpha \in \Gamma$,

$$\sum_{\beta\in\Gamma} a^l(-M^{-1}\alpha + 2^l M^{-1}\beta) = \sum_{\beta\in\mathbb{Z}^s} a^l(-M^{-1}\alpha + 2^l M^{-1}\beta) = 1,$$

which means that B is row-stochastic.

We now show that the matrix B is connected with some $\beta \in ([M\Omega] \cap \mathbb{Z}^s) \setminus \partial[M\Omega]$. Let $\alpha \in \Gamma$ and assume that A^l is connected with $M^{-1}\beta \in ([\Omega] \cap \mathbb{Z}^s) \setminus \partial[\Omega]$. To carry out the assertion, we recall the definition of graph $G([\Omega] \cap \mathbb{Z}^s, K)$ according to a square matrix A^l, whose set of edges is given by

$$K = \{(\nu, \mu) : \, \nu, \mu \in [\Omega] \cap \mathbb{Z}^s, \, A^l(\mu, \nu) \neq 0\}$$

and the notation of graph $G(\Gamma, K')$ according to the square matrix B, whose set of edges is given by $K' = \{(v, u) : \, u, v \in \Gamma, \, B(u, v) \neq 0\}$, where

$$B(u, v) = a^l(-M^{-1}u + 2^l M^{-1}v), \quad u, \; v \in \Gamma.$$

Then we divide the proof into following two cases.

Case 1. We prove that B restricted to $M([\Omega] \cap \mathbb{Z}^s)$ is connected. Let $M^{-1}\alpha \in [\Omega] \cap \mathbb{Z}^s$. Since A^l is connected with a vertex $M^{-1}\beta \in ([\Omega] \cap \mathbb{Z}^s) \setminus \partial[\Omega]$, we have a path from $M^{-1}\beta$ to $M^{-1}\alpha$ in $G([\Omega] \cap \mathbb{Z}^s, K)$. Hence, we have a path from β to α in $G(M[\Omega] \cap \mathbb{Z}^s, K'')$, where K'' is a subset of K' restricted to $M[\Omega] \cap \mathbb{Z}^s$. In other words, the matrix B restricted to $M([\Omega] \cap \mathbb{Z}^s)$ is connected with $\beta \in ([M\Omega] \cap \mathbb{Z}^s) \setminus \partial[M\Omega]$.

Case 2. We now treat the case $\alpha \notin M[\Omega] \cap \mathbb{Z}^s$, i.e., $\alpha \in \Gamma \setminus (M[\Omega] \cap \mathbb{Z}^s)$, then due to the row-stochastic property of B there exists $\alpha_1 \in \Gamma$ such that $a^l(-M^{-1}\alpha + 2^l M^{-1}\alpha_1) \neq 0$, i.e., for some $\gamma_0 \in M\Omega^l$ one has

$$\alpha = 2^l \alpha_1 - \gamma_0.$$

If $\alpha_1 \in M[\Omega] \cap \mathbb{Z}^s$, we have nothing more to do, because there is a path from β to α_1, as we have proved in Case 1. Otherwise, we obtain recursively $\alpha_1, ..., \alpha_m$ in Γ such that $\gamma_j = -\alpha_j + 2^l \alpha_{j+1}, \; j = 0, ..., m-1$. We conclude for some $\gamma_\nu \in M\Omega^l$ and $\nu = 0, 1, 2, ..., m-1$

$$\begin{aligned} \alpha &= 2^l \alpha_1 - \gamma_0 = 2^l(2^l \alpha_2 - \gamma_1) - \gamma_0 = 2^{2l}\alpha_2 - 2^l \gamma_1 - \gamma_0 \\ &= 2^{ml}\alpha_m - \sum_{\nu=0}^{m-1} 2^{\nu l}\gamma_\nu. \end{aligned}$$

On the other hand, because $|\Gamma| < \infty$, we obtain in this way a subsequence $m_i \in \mathbb{N}$ with $\alpha_0 = \lim_{i\to\infty} \alpha_{m_i}$. We know $\gamma_\nu/(2^l - 1) \in M[\Omega]$ for all ν. So we get

$$\begin{aligned} \alpha_0 &= \lim_{i\to\infty} \alpha_{m_i} \\ &= \lim_{i\to\infty} \frac{\alpha + \sum_{\nu=0}^{m_i-1} 2^{\nu l}\gamma_\nu}{2^{m_i l}} \\ &\in \lim_{i\to\infty} \frac{\alpha + (2^{0l} + 2^{1l} + \cdots + 2^{(m_i-1)l})M[\Omega^l]}{2^{m_i l}}. \end{aligned}$$

Therefore

$$\begin{aligned} \alpha_0 &\in \lim_{i\to\infty} \frac{\alpha + (\frac{1-2^{m_i l}}{1-2^l})M[\Omega^l]}{2^{m_i l}} \\ &\subseteq \lim_{i\to\infty} \left(\frac{\alpha}{2^{m_i l}} + \frac{(2^{m_i l} - 1)M[\Omega]}{2^{m_i l}} \right) \\ &\subseteq M[\Omega]. \end{aligned}$$

Again as $|\Gamma| < \infty$, there is a path from α_0 to α. But we know from Case 1 that there is also a path from β to α_0. Because of the arbitrariness of $\alpha \in \Gamma \setminus (M[\Omega] \cap \mathbb{Z}^s)$, we obtain finally that B is connected with some $\beta \in (M[\Omega] \cap \mathbb{Z}^s) \setminus \partial[M\Omega]$. □

The connectivity of A and the sum rule (1.3) imply also the inner-point principle mentioned in Section 5.5 (see Theorem 5.5.1), which will be confirmed by the following

Theorem 6.1.3. *Let the nonnegative finite mask $\{a(\alpha)\}$ satisfy the sum rule (1.3). Then the connectivity of A implies $\gcd\{\alpha : \alpha \in \Omega_\beta\} = (1, ..., 1)^T$ for all $\beta \in \mathbb{Z}^s$ and that there is at most one set $\mathcal{A}(\delta), \delta \in E^s$, with $|\mathcal{A}(\delta)| = 1$. Moreover, for any $0 \le j < s$ and any j-dimensional face S_j of $[\Omega]$ one has*

$$0 < \sum_{\alpha \in S_j \cap \Omega} a(\alpha) < 2^j. \tag{6.2}$$

Proof. We note $\Omega_\beta = \Omega + \beta$ for $\beta \in \mathbb{Z}^s$ (see Section 5.2). Denote $\gcd\{\alpha : \alpha \in \Omega_\beta\} = d$, with $d = (d_1, ..., d_s)^T$. The sum rule (1.3) implies that every component of d is odd. Hence, if $2^l\alpha \equiv 0 \pmod{d}$ for some $\alpha \in \mathbb{Z}^s$ and $l \in \mathbb{N}$ then $\alpha \equiv 0 \pmod{d}$. On the other hand, the connectivity of A means that one has $\alpha_0 \in [\Omega] \cap \mathbb{Z}^s$ such that for any $\alpha \in ([\Omega] \cap \mathbb{Z}^s) \setminus \{\alpha_0\}$ there is $j \ge 1$ so that for some $\gamma_l \in \Omega$

$$\alpha = 2^j\alpha_0 - \sum_{l=0}^{j-1} 2^l\gamma_l, \tag{6.3}$$

that is,

$$\alpha+\beta = 2^j(\alpha_0+\beta) - \sum_{l=0}^{j-1} 2^l(\gamma_l+\beta).$$

We get by choosing $\alpha \in \Omega$ that $2^j(\alpha_0+\beta) \equiv 0 \pmod{d}$. Hence, $\alpha_0+\beta \equiv 0 \pmod{d}$, which gives $\alpha+\beta \equiv 0 \pmod{d}$ for all $\alpha \in [\Omega]\cap\mathbb{Z}^s$. By Lemma 6.1.2 this relation holds also for any convex set $\Gamma \supseteq [\Omega]\cap\mathbb{Z}^s$ instead of $[\Omega]\cap\mathbb{Z}^s$. We may therefore let $\Gamma \supseteq E^s$. Thus, $\alpha+\beta \equiv 0 \pmod{d}$ for all $\alpha \in E^s$, which yields $d = (1, ..., 1)^T$.

To show that there is at most one set $\mathcal{A}(\delta)$ for some $\delta \in E^s$ with $|\mathcal{A}(\delta)| = 1$, we assume $\mathcal{A}(\delta') = \{r\}$ for some $\delta' \in E^s$. Clearly, $r \in [\Omega]\cap\mathbb{Z}^s$. The connectivity of A implies

$$r = 2^j\alpha_0 - \sum_{l=0}^{j-1} 2^l\gamma_l.$$

It is easy to see $r \equiv \gamma_0 \pmod{2}$. Hence, together with the definition of $\mathcal{A}(\delta')$ we have $\gamma_0 = r$ and

$$r = 2^{j-1}\alpha_0 - \sum_{l=0}^{j-2} 2^l\gamma_{l+1},$$

which in turn implies $\gamma_1 = r$. Recursively, $\gamma_l = r$, $l = 0, 1, ..., j-1$, or $\alpha_0 = r$. Therefore, there is at most one such set.

To verify the inequalities (6.2) we begin with the observation of the case $j = 0$. We notice that $S_0 \cap \Omega$ contains only one extreme point of $[\Omega]$ (say α_1). It is clear $0 < a(\alpha_1) \leq 1$ and now we need to prove $a(\alpha_1) \neq 1$. To see this, we let $a(\alpha_1) = 1$. On one hand, it follows from Theorem 5.5.1 that the set $\mathcal{A}(\delta)$, which contains α_1, has only one element. Then we have from the above proof, that A is connected with respect to α_1, i.e., for any $\alpha \in [\Omega]\cap\mathbb{Z}^s \setminus \{\alpha_1\}$, there are $j \in \mathbb{N}$ and $\gamma_l \in \Omega$ such that

$$\alpha = 2^j\alpha_1 - \sum_{l=0}^{j-1} 2^l\gamma_l.$$

However, by Lemma 6.1.2, the element α_1 must be an inner point of $[\Omega]$. This contraction means that (6.2) is true for $j = 0$.

Next we observe (6.2) for the case $1 \leq j < s$. Let S_j be a $j-$dimensional face. We may assume that $0 \in S_j \cap \Omega$. Indeed, if $0 \notin S_j \cap \Omega$, we can just shift the mask $\{a(\alpha)\}$ with some $\beta' \in \mathbb{Z}^s$ so that $0 \in (S_j+\beta')\cap\Omega_{\beta'}$ and $\{a'(\alpha)\} = \{a(\alpha-\beta')\}$, which satisfies also the sum rule (1.3). Moreover, the matrix A with entries $a'(-\alpha+2\beta)$, $\alpha,\beta \in [\Omega_{\beta'}]\cap\mathbb{Z}^s$, is also connected.

Now we have j linearly independent vectors $\gamma_1,...,\gamma_j \in \Omega$ such that each element x in S_j is a linear combination of $\gamma_1,...,\gamma_j$. We have just as in the proof of Lemma 5.5.1 a unimodular matrix $M \in \mathfrak{M}_s$ such that the first $s-j$ components of $M\gamma_i$ are zero for $i=1,...,j$ (see e.g. [13] Chapter 14). More precisely, there holds

$$M(\gamma_1,...,\gamma_j) = (\eta_1,...,\eta_j)$$

with $\eta_i = (0,...,0,\eta_{s-j+1,i},...,\eta_{s,i})^T$, $i=1,...,j$. Thus, under the mapping M the relation (6.3) can be written as for any $\alpha \in M([\Omega]\cap\mathbb{Z}^s)$ there is $\tau \geq 1$ such that for some $\gamma_l \in \Omega$

$$\alpha = 2^\tau M\alpha_0 - \sum_{l=0}^{\tau-1} 2^l M\gamma_l. \tag{6.4}$$

Hence, the matrix B given by $B(\alpha,\beta) = a'(-\alpha+2\beta)$, $\alpha,\beta \in M([\Omega]\cap\mathbb{Z}^s)$, is again connected, where $a'(\alpha) = a(M^{-1}\alpha)$. Clearly, $\{a'(\alpha)\}$ also satisfies the sum rule (1.3). In what follows we should project $M\Omega$ on the space $\mathbb{Z}^{s-j}$ deduced by the first $s-j$ vectors of $\mathbb{Z}^s$. Obviously, as $0 \in S_j$ and the first $s-j$ components of MS_j are zero, the projector of MS_j on $\mathbb{Z}^{s-j}$ is the single point zero, i.e. $MS_j|_{\mathbb{Z}^{s-j}} = 0$. Furthermore,

$$b(\alpha) = \frac{1}{2^j}\sum_{\beta|_{\mathbb{Z}^{s-j}}=\alpha} a'(\beta)$$

is also a nonnegative mask in $\mathbb{Z}^{s-j}$. It is easy to see that $\{\alpha : b(\alpha)\neq 0\} = M\Omega|_{\mathbb{Z}^{s-j}}$. Moreover, since $\{a'(\alpha)\}$ satisfies the sum rule (1.3) we have for any $\delta \in E^{s-j}$

$$\begin{aligned}
\sum_\alpha b(2\alpha+\delta) &= \frac{1}{2^j}\sum_{\alpha\in\mathbb{Z}^{s-j}}\sum_{\beta|_{\mathbb{Z}^{s-j}}=2\alpha+\delta} a'(\beta) = \frac{1}{2^j}\sum_{\alpha\in\mathbb{Z}^{s-j}}\sum_{\beta\in\mathbb{Z}^j} a'((2\alpha+\delta,\beta)^T) \\
&= \frac{1}{2^j}\sum_{\alpha\in\mathbb{Z}^{s-j}}\sum_{\eta\in E^j}\sum_{\beta\in\mathbb{Z}^j} a'((2\alpha+\delta,2\beta+\eta)^T) \\
&= \frac{1}{2^j}\sum_{\eta\in E^j}\left(\sum_{\alpha\in\mathbb{Z}^{s-j}}\sum_{\beta\in\mathbb{Z}^j} a'((2\alpha+\delta,2\beta+\eta)^T)\right) = \frac{1}{2^j}\sum_{\eta\in E^j} 1 = 1.
\end{aligned}$$

Hence, the mask $\{b(\alpha)\}$ satisfies the sum rule (1.3).

But (6.4) tells us

$$\alpha|_{\mathbb{Z}^{s-j}} = 2^\tau (M\alpha_0)|_{\mathbb{Z}^{s-j}} - \sum_{l=0}^{\tau-1} 2^l (M\gamma_l)|_{\mathbb{Z}^{s-j}}.$$

Hence the matrix A defined by $\{b(\alpha)\}$ is also connected. On the other hand, as MS_j is a face of $[M\Omega]$, the zero is an extreme point of $[M\Omega]|_{\mathbb{Z}^{s-j}}$. Thus, the assertion (6.2) for $j = 0$ implies $0 < b(0) < 1$. However, since

$$\begin{aligned} b(0) &= \frac{1}{2^j} \sum_{\beta|_{\mathbb{Z}^{s-j}}=0} a'(\beta) = \frac{1}{2^j} \sum_{\beta \in M(S_j \cap \Omega)} a'(\beta) \\ &= \frac{1}{2^j} \sum_{M^{-1}\beta \in S_j \cap \Omega} a(M^{-1}\beta) \\ &= \frac{1}{2^j} \sum_{\beta \in S_j \cap \Omega} a(\beta), \end{aligned}$$

we conclude that (6.2) is also true for $1 \le j \le s-1$. □

Now let us look at the case of $s = 1$. From Theorem 6.1.3 it is easy to see that the connectivity of A and the sum rule (1.3) imply 1) and 2) of Theorem 3.0.1. In other words, we have that the subdivision scheme with a nonnegative finite mask in this case converges if and only if the matrix A is connected and the sum rule (1.3) is fulfilled. However, in the case of $s \ge 2$ these two conditions (the connectivity of A and the sum rule (1.3)) cannot sufficiently ensure the convergence of the subdivision scheme with a nonnegative finite mask. We illustrate this fact by the following example (see Figure 6.1).

Example 6.1.1. *Let $s = 2$, we observe the subdivision scheme with the mask*

$$\begin{aligned} a((0,0)^T) &= a((0,1)^T) = a((1,2)^T) = a((2,2)^T) = \frac{1}{2}, \\ a((3,1)^T) &= a((3,0)^T) = a((2,-1)^T) = a((1,-1)^T) = \frac{1}{2} \end{aligned}$$

and

$$a((1,0)^T) = a((2,0)^T) = a((1,1)^T) = a((2,1)^T) = 0.$$

It's easy to check that the sum rule (1.3) is fulfilled and the matrix A is connected. However A^2 is not connected. Hence, according to Theorem 6.1.1 this subdivision scheme is not convergent.

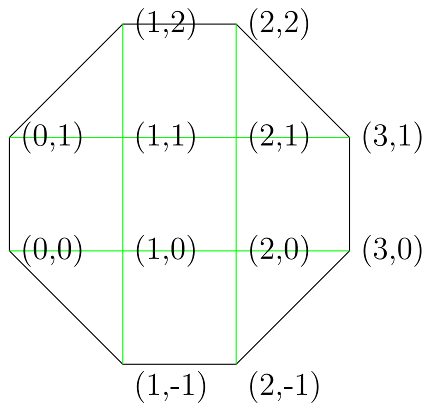

Figure 6.1: Octagon

Theorems 6.1.3 and 5.5.1 provide a property of the convergent subdivision schemes, i.e., the so-called inner-point principle. In the following section, we will continue to study the related properties.

Our first application of the inner-point principle gives the special expression of λ under certain conditions.

Corollary 6.1.4. *Let $\{a(\alpha)\}$ be a finite nonnegative mask and $\Omega \subset \mathbb{Z}^s$ be the support. Assume that Ω satisfies the inner-point principle. If there exists $T \subseteq \mathbb{Z}^s$ with $|T| = 1$ such that for some $k \in \mathbb{N}$ and $\lambda = \delta_0 + 2\delta_1 + \cdots + 2^{k-1}\delta_{k-1}$ with $\delta_j \in E^s$,*

$$\frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s = T,$$

then $\lambda = \delta(2^k - 1)$ for some $\delta \in E^s$.

Proof. Without loss of generality, we may write $\{\alpha_0\} = T_0 = T$ and

$$\begin{aligned} &\frac{T_0 - \delta_0 + \Omega}{2} \cap \mathbb{Z}^s =: T_1, \\ &\frac{T_1 - \delta_1 + \Omega}{2} \cap \mathbb{Z}^s =: T_2, \\ &\cdots \\ &\frac{T_{k-1} - \delta_{k-1} + \Omega}{2} \cap \mathbb{Z}^s = T_0. \end{aligned} \tag{6.5}$$

Then we conclude $|T_{k-1}| = 1$. Indeed, if $|T_{k-1}| \neq 1$, i.e., there are at least two elements $\alpha_{k-1}, \alpha'_{k-1}$ in T_{k-1}. We may assume first that $\alpha_{k-1} \not\equiv \alpha'_{k-1} \pmod 2$. Suppose $|\mathcal{A}(\alpha'_{k-1} -$

$\delta_{k-1})| = 1$. Then since Ω satisfies the inner-point principle, for the other element $\alpha_{k-1} - \delta_{k-1}$, we have $|\mathcal{A}(\alpha_{k-1} - \delta_{k-1})| \geq 2$. Let $r, r' \in \mathcal{A}(\alpha_{k-1} - \delta_{k-1})$. It follows that

$$\alpha_0 = \frac{\alpha_{k-1} - \delta_{k-1} + r}{2} \in T_0 \text{ and } \alpha_0 = \frac{\alpha_{k-1} - \delta_{k-1} + r'}{2} \in T_0,$$

which shows $r = r'$ and so $a(r) = 1$. This is a contradiction to the fact that Ω satisfies the inner-point principle. Consequently, it follows from (6.5) that $|T_j| = 1$, $j = 0, 1, ..., k-1$. Furthermore, there exists only one $r \in \Omega$ such that $\alpha_{k-1} - \delta_{k-1} \equiv r \pmod 2$. We obtain, therefore, that $\alpha_j - \delta_j \equiv r \pmod 2$, $j = 0, 1, ..., k-1$. Or again by recursion (6.5), we have $\alpha_0 = 2^k \alpha_0 + \lambda - r(2^k - 1)$, which yields that $\lambda = (r - \alpha_0) \cdot (2^k - 1)$. Thus $(2^k - 1) | (\lambda)_\tau$, $\tau = 1, ..., s$. The restriction $0 \leq (\lambda)_\tau \leq 2^k - 1$ implies $(\lambda)_\tau = 0$ or $2^k - 1$, $\tau = 1, ..., s$. In other words, there is $\delta \in E^s$ such that $\lambda = \delta(2^k - 1)$.
The case $\alpha_{k-1} \equiv \alpha'_{k-1} \pmod 2$ can be treated in the same way. □

6.2 More about the connectivity

We continue to study the matrices A defined by (6.1). The following result is a consequence about the irreducible mapping (see Section 4.2). Indeed similar results are obtained also by [14] and [18].

Lemma 6.2.1. *Let the nonnegative finite mask $\{a(\alpha)\}$ satisfy the sum rule (1.3). Let A defined by (6.1) be connected with β_0 and $G([\Omega] \cap \mathbb{Z}^s, K)$ be the directed graph generated by A. Further denote Γ_1 to be a strongly connected component of $G([\Omega] \cap \mathbb{Z}^s, K)$, which contains β_0, and B to be the submatrix of A given by*

$$B(\alpha, \beta) = A(\alpha, \beta), \quad \forall \alpha,\ \beta \in \Gamma_1.$$

Then B is row-stochastic. Moreover either there exists an $L \geq 1$ such that all entries of B^L are positive or for some $1 < J \leq |\Gamma_1|$ there is a decomposition U_1, U_2,...,U_J of Γ_1 such that $B|_{U_{i+1} \times U_i}$, $i = 1, ..., J$, is row-stochastic and all other entries of B are zero, where $U_{J+1} = U_1$.

Proof. It is easy to see that B is row-stochastic. To show the other assertions, we define the mapping ψ in the following way: for any nonempty set $T \subseteq [\Omega] \cap \mathbb{Z}^s$, let

$$\psi(T) = \frac{T + \Omega}{2} \cap \mathbb{Z}^s \subseteq [\Omega] \cap \mathbb{Z}^s.$$

Then ψ restricted to Γ_1 is irreducible. According to [19] either ψ on Γ_1 is primitive, or $\Gamma_1 = U_1 \cup U_2 \cup \cdots \cup U_J$ such that

$$\psi(U_j) = U_{j+1}, \quad j = 1, ..., J.$$

In the first case, there is L such that

$$\Gamma_1 = \psi^L(\{\alpha\}), \quad \forall\ \alpha \in \Gamma_1.$$

The assertions of this lemma follow if we translate the above into B. □

Now we use the connectivity of the matrix A to develop some useful property on the convergent subdivision scheme. It follows from Lemmas 6.1.2 and 6.2.1 that

Theorem 6.2.2. *Suppose the nonnegative finite mask* $\{a(\alpha)\}$ *satisfy the sum rule (1.3). If* A^l*,* $l \geq 1$*, is connected, where* A *is defined by (6.1), then* A^k *has a positive column whenever* $k \geq 2^{N^2}$*. Furthermore, if* $\Gamma_1 \subset \mathbb{Z}^s$ *is a convex and finite set such that* $[\Omega] \cap \mathbb{Z}^s \subseteq \Gamma_1$*, then the above condition on* A *implies that for* $\tau \geq 2^{|\Gamma_1|^2}$ *the matrix* B^τ *has a positive column, where*

$$B(\alpha, \beta) = a(-\alpha + 2\beta), \quad \forall\ \alpha,\ \beta \in \Gamma_1.$$

.

Proof. In order to facilitate the expression, we write at first $\Gamma = [\Omega] \cap \mathbb{Z}^s$. By hypothesis on the connectivity of A and Lemma 6.1.2, for some $\alpha_1 \in \Gamma \setminus \partial\Gamma$ and any $\alpha \in \Gamma_1 \setminus \{\alpha_1\}$ there is $j \geq 1$ so that for some $\gamma_i \in \Omega$

$$\alpha = 2^j \alpha_1 - \sum_{i=0}^{j-1} 2^i \gamma_i. \tag{6.6}$$

So the matrix B is connected. Next, we need to prove the matrix B^τ has a positive column for $\tau \geq 2^{|\Gamma_1|^2}$. To this end, we need the concept of $\mathcal{A}(\lambda)$ introduced in Section 3.2. More precisely, for a given finitely supported real mask $\{a(\alpha)\}$

$$\mathcal{A}(\lambda) = \{\alpha\ :\ a(\alpha) \neq 0 \text{ and } \alpha \equiv \lambda \pmod 2\}, \quad \forall\, \lambda \in \mathbb{Z}^s$$

and the set $\mathcal{A}(\lambda)$ hat at least one element, because of the sum rule (1.3). We should divide the proof into two cases according to whether $|\mathcal{A}(\delta)| = 1$ for some $\delta \in E^s$ or not.

Case 1. Suppose that there is $\delta' \in E^s$ such that $|\mathcal{A}(\delta')| = 1$. It follows from Lemma 6.1.3 that for all $\delta \in E^s \setminus \{\delta'\}$ the set $\mathcal{A}(\delta)$ has at least two elements. Moreover, denote $\mathcal{A}(\delta') = \{r\}$, then $r \in \Omega$. We conclude from the proof of Lemma 6.1.3 that $\alpha_1 = r$. Then, $\alpha_1 = 2\alpha_1 - r$ and for any $h \in \mathbb{N}$ and $\gamma_i = r$

$$\alpha_1 = 2^h \alpha_1 - \sum_{i=0}^{h-1} 2^i \gamma_i. \tag{6.7}$$

For each $\alpha \in \Gamma_1 \setminus \{\alpha_1\}$ let $j(\alpha)$ be the smallest satisfying (6.6) and J be the maximum among all such $j(\alpha)$, i.e., $J = \max\{j(\alpha) : \alpha \in \Gamma_1 \setminus \{\alpha_1\}\}$. Clearly $|J| \le |\Gamma_1| \le 2^{|\Gamma_1|}$. Thus, for any $1 \le j \le J$ we have $h \ge 0$ such that $j + h = J$. Combining the last identity 6.7 with (6.6) we conclude that j in (6.6) can always be replaced by J, which means that $B^J(\alpha, \alpha_1) > 0$, and in other words, the α_1-column of B^J is positive.

Case 2. Suppose $|\mathcal{A}(\delta)| \ge 2$ for all $\delta \in E^s$. Denote $\Gamma_1 = \Gamma' \cup \Gamma''$ with Γ' being the strongly connected component of $G(\Gamma_1, K)$ deduce by B and $\alpha_1 \in \Gamma'$. According to (6.6) we conclude that the parents of each vertex in Γ' also belong to Γ' . Let C be the restriction of A given by $C(\alpha, \beta) = B(\alpha, \beta)$, $\alpha, \beta \in \Gamma'$. So C is again a row-stochastic matrix. We can therefore arrange B as

$$B = \begin{pmatrix} C & 0 \\ D_1 & F \end{pmatrix} \quad \text{and} \quad B^l = \begin{pmatrix} C^l & 0 \\ D_l & F^l \end{pmatrix}.$$

It follows from the connectivity of the matrix B^l that C^l is also connected. Thus, for any $M_1, M_2 \in \mathfrak{M}_{|\Gamma'|}$ the matrix $M_1 C M_2$ cannot be block diagonal with more than one block. It follows from Lemma 6.2.1 that for some $k \ge 1$ all entries of C^k are positive. It remains to show that for some $\tau \ge 1$ there is at least one positive column of B^τ. Clearly (D_1, F) is row-stochastic and $D_1 \ne 0$, since B is connected. Thus, assume $D_1(\alpha', \beta) > 0$ for one $\beta \in \Gamma'$. It follows from the positivity of C^k that $D_{k+1}(\alpha', \beta) > 0$ for all $\beta \in \Gamma'$. Write

$$D_1 = \begin{pmatrix} D_1^1 \\ D_1^2 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$$

such that F_1 and F_4 are square matrices and D_1^2 is zero, the number of the row of D_1^1 and F_1 is the same and the number of the row of D_1^2 and F_4 is the same. Note that $F_3 \neq 0$, since B is connected, then without loss the generality, we assume that the first row of F_3 is not all zeros.

Adding this nonzero row of F_3 to F_1 and making corresponding changes for the other matrices we obtain that D_{k+2}^1 is positive. Repeating this process we conclude that there is a $m \geq 1$ such that the Γ' columns of B^{k+m} are positive. Finally, since the signs of the entries have total $2^{|\Gamma_1|^2}$ possibilities, so among B^j, $j = 1, ..., 2^{|\Gamma_1|^2}$ we have $1 \leq j_1 < j_2 \leq 2^{|\Gamma_1|^2}$ such that $\text{sgn}(B^{j_1}) = \text{sgn}(B^{j_2})$, where the sign matrix of a given nonnegative matrix B is defined by

$$\text{sgn}(B)(\alpha, \beta) := \begin{cases} 1, & \text{if } B(\alpha, \beta) > 0, \\ 0, & \text{if } B(\alpha, \beta) = 0. \end{cases}$$

We may choose $k + m = 2^{|\Gamma_1|^2}$. Consequently, B^k has a positive column whenever $k \geq 2^{|\Gamma_1|^2}$. □

We are in a position to present the relation between the connection of the square matrix A and the eigenvalues of A. The matrix A possesses the so-called **1-condition**, if $1, r_2, ..., r_N$ are the eigenvalues of the matrix A and $|r_j| < 1$, $j = 2, ..., N$.

Let $1, r_2, ..., r_N$ be the eigenvalues of the matrix A. From Lemma 6.2.1 and Theorem 6.2.2 it is now clear that the conditions on A in Theorem 6.2.2 imply that the eigenvalues of A satisfy $|r_j| < 1$, $j = 2, ..., N$. Conversely, A^j must be connected and the decomposition in the sense of Lemma 6.2.1 does not exist. So from the proof of Theorem 6.2.2 there is a k such that A^k has a positive column, which yields the connectivity of A^j, $j = 1, 2, ...$ For matrix A given by (6.1) the following conditions are equivalent:

1. $|r_j| < 1$, $j = 2, ..., N$, i.e., A possesses the 1-condition.
2. A^j is connected for $j = 1, 2, ...$
3. A^k has a positive column for some $k \geq 1$.

Using Theorems 3.1.1 and 3.3.1 in the case of $k = 1$ and $\delta_0 = 0$, it is not hard to show that the following corollary holds.

Corollary 6.2.3. *Let the finite mask $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ be nonnegative and Ω be the support of the mask $\{a(\alpha)\}$. The matrix A denoted by (6.1) possesses the* 1*-condition ,* *if and only if, for* $T,\ T' \subseteq [\Omega] \cap \mathbb{Z}^s$*, the inclusion relations*

$$\frac{T+\Omega}{2} \cap \mathbb{Z}^s \subseteq T \quad \textit{and} \quad \frac{T'+\Omega}{2} \cap \mathbb{Z}^s \subseteq T'$$

imply $T \cap T' \neq \emptyset$*.*

In this chapter we have investigated the connectivity of the matrix A. But how can we examine whether a given row-stochastic matrix is connected? The following algorithm shows that the connectivity of the matrix A may be tested by depth-first search algorithm from graph theory in linear time with respect to the size of A (see [4]). As the connectivity does not depend on the actual values of A we may assume $\mathrm{sgn}(A) = A$. Our algorithm is as follows.

Algorithm CHECK (A):
(1) *build the graph* $G(\Gamma, K)$ *from* A*;*
(2) *calculate all strongly connected components of* $G(\Gamma, K)$*, say* $\Gamma_1,\ \Gamma_2, \ldots,\ \Gamma_m$*;*
(3) *build a new graph* $G'(V, E)$*, where* $V = \{1, \ldots, m\}$ *and*

$$E = \{(i,j)\ :\ i,j \in V,\ \textit{if there is an edge from } \Gamma_i \textit{ to } \Gamma_j\};$$

(4) *if there are at least two vertexes of* V *whose in-degree is zero, return false, otherwise return true.*

A strongly connected component of a directed graph $G(\Gamma, K)$ is the maximal set of vertexes $\Gamma' \subseteq \Gamma$ such that every pair of vertexes u and v in Γ' are reachable from each other. The in-degree of a vertex is the number of edges which is incident to this vertex. Let the size of A be $N \times N$, so to build $G(\Gamma, K)$ in the form of adjacency list one needs the complexity $\mathcal{O}(N^2)$. To find all strongly connected components of $G(\Gamma, K)$ and to construct the new graph $G'(V, E)$ one needs $\mathcal{O}(|\Gamma| + |K|) = \mathcal{O}(N^2)$ as shown in [4]. Finally, finding the vertex in $G'(V, E)$ with zero in-degree costs $\mathcal{O}(|\Gamma| + |K|)$. Thus, the complexity of CHECK (A) is $\mathcal{O}(N^2)$.

The matrix A is connected if and only if the output of CHECK (A) is true. Indeed, if $G'(V, E)$ has more than one vertex with zero in-degree then $G(\Gamma, K)$ cannot be connected. To see this let a and b two vertexes of $G'(V, E)$ with zero in-degree. So there is no path in $G'(V, E)$ between a and b, which in turn implies that there is no paths among the corresponding strongly connected components of $G(\Gamma, K)$. On the other hand, as $G'(V, E)$ is acyclic, there is at least one vertex with in-degree zero. If there is only one vertex $v \in V$ with zero in-degree then for any $u \in V \setminus \{v\}$ one can always find a path from v to u. Assume the strongly component corresponding to v is Γ_1, hence, any vertex from Γ_j, $j \neq 1$ can be reached from the vertexes of Γ_1, in particular from a vertex of Γ_1. In other words, A is connected.

Chapter 7

Sufficient Conditions for the Convergence

In the previous chapters, we have studied the necessary and sufficient conditions on the convergent subdivision schemes with the finite nonnegative mask $\{a(\alpha)\}$. Those results may help us to understand the distribution of the supports of those masks, for which the subdivision schemes converge. Unfortunately, it is still rather difficult from those results to obtain quickly computable criteria for the convergence. In the last chapter of this thesis, we take full advantage of the results in previous to study the multivariate subdivision scheme with nonnegative masks, whose support possesses some special properties.

We draw our inspiration from Theorem 6.2.2 and conclude the following result as well.

Theorem 7.0.1. *Let the nonnegative finite mask $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ satisfy the sum rule (1.3) and $\Omega = \{\alpha : a(\alpha) \neq 0\}$ the support of this mask. If for some $l \geq 2$ the set Ω^l is convex and $[\Omega]^o \cap \mathbb{Z}^s \neq \emptyset$, then the subdivision scheme associated with $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ converges.*

Obviously, the condition $[\Omega]^o \cap \mathbb{Z}^s \neq \emptyset$ is necessary. In fact the mask given by $a(\alpha) = 1$ for $\alpha \in E^s$ satisfies the sum rule and $(E^s)^i$ is convex for any $i \geq 1$. However, the subdivision scheme defined by this mask does not converge in the continuous norm. Moreover, Example 4.1.3 shows that in general $l \geq 2$ in Theorem 7.0.1 cannot be replaced by $l \geq 1$. It turns out that the substance of this result consists in the investigation of the properties of the convex sets. Therefore in the first section of this chapter, we discuss some properties of the convex set Ω^l in detail. In the second section, we will focus on the proof of the above result.

7.1 On the convex set Ω^l

We begin in this section with the investigation of Ω^l. The behaviour of Ω^l provides some useful information concerning the primitive sets (see Section 4.2), that will lead to the proof of Theorem 7.0.1.

Lemma 7.1.1. *Let $\Omega \subseteq \mathbb{Z}^s$. If for some $l \geq 2$ the set Ω^l is convex, so is $\Omega^{\tau l}$ for all $\tau \in \mathbb{N}$.*

Proof. By hypothesis the integer set Ω^l is convex with some $l \geq 2$. We prove firstly that, for any $r_1, ..., r_{2^l-1} \in \Omega$,

$$\sum_{j=1}^{2^l-1} r_j \in \Omega^l. \tag{7.1}$$

Indeed, if $r \in \Omega$, then in view of (3.3) and the definition of Ω^l (see (3.2)),

$$(2^l - 1)r = r + 2r + \cdots + 2^{l-1}r \in \Omega^l.$$

Hence, $(2^l - 1)r_j \in \Omega^l$, $j = 1, ..., 2^l - 1$. Because Ω^l is convex, the convex combination

$$\sum_{j=1}^{2^l-1} r_j = \frac{1}{2^l - 1}((2^l - 1)r_1 + \cdots + (2^l - 1)r_{2^l-1})$$

belongs to Ω^l.

To show the assertion of this lemma let $x,\ y \in \Omega^{l\tau}$ and $x \equiv y \pmod 2$. Then there are $r_j,\ r'_j \in \Omega$ satisfying

$$x = \sum_{j=0}^{l\tau-1} 2^j r_j \quad \text{and} \quad y = \sum_{j=0}^{l\tau-1} 2^j r'_j.$$

Write

$$\begin{aligned}\sum_{j=0}^{l\tau-1} 2^j r_j &= \sum_{j=0}^{l-1} 2^j r_j + \sum_{j=l}^{2l-1} 2^j r_j + \sum_{j=2l}^{3l-1} 2^j r_j + \cdots + \sum_{j=(\tau-1)l}^{\tau l-1} 2^j r_j \\ &= \sum_{\mu=0}^{\tau-1} 2^{\mu l} \sum_{j=0}^{l-1} 2^j r_{\mu l+j}.\end{aligned}$$

Similarly,

$$\sum_{j=0}^{l\tau-1} 2^j r'_j = \sum_{\mu=0}^{\tau-1} 2^{\mu l} \sum_{j=0}^{l-1} 2^j r'_{\mu l+j}.$$

Thus, we have

$$\frac{x+y}{2} = \frac{1}{2}\sum_{j=0}^{l-1} 2^j (r_j + r'_j) + \sum_{\mu=1}^{\tau-1} 2^{\mu l-1} \sum_{j=0}^{l-1} 2^j (r_{\mu l+j} + r'_{\mu l+j}). \tag{7.2}$$

Clearly, $\sum_{j=0}^{l-1} 2^j r_j \equiv \sum_{j=0}^{l-1} 2^j r'_j \pmod 2$, because $x \equiv y \pmod 2$. As Ω^l is convex, the first term in (7.2) belongs to Ω^l. The definition of Ω^l (see (3.2)) and (3.3) tell us that there are $\eta_j \in \Omega$, $j = 0, ..., l-1$ such that

$$\frac{1}{2}\sum_{j=0}^{l-1} 2^j (r_j + r'_j) = \sum_{j=0}^{l-1} 2^j \eta_j.$$

So (7.2) can be rewritten as

$$\frac{x+y}{2} = \sum_{j=0}^{l-1} 2^j \eta_j + \sum_{\mu=1}^{\tau-1} 2^{\mu l-1} \sum_{j=0}^{l-1} 2^j (r_{\mu l+j} + r'_{\mu l+j}). \tag{7.3}$$

Next we prove that for $\xi, \xi_i, \xi'_i \in \Omega$, $i = 0, 1, ..., l-1$, there are $\eta, \eta_i \in \Omega$, $i = 0, 1, ..., l-1$, so that

$$\xi + \sum_{i=0}^{l-1} 2^i (\xi_i + \xi'_i) = \eta + 2\sum_{i=0}^{l-1} 2^i \eta_i. \tag{7.4}$$

Indeed, we may regard

$$\xi + \sum_{i=0}^{l-2} 2^i (\xi_i + \xi'_i)$$

as a sum of total $2^l - 1$ members of Ω. Thus, by (7.1) there are $\eta_i \in \Omega$, $i = 0, ..., l-1$, so that

$$\xi + \sum_{i=0}^{l-2} 2^i (\xi_i + \xi'_i) = \sum_{i=0}^{l-1} 2^i \eta_i.$$

Consequently,

$$\begin{aligned} \xi + \sum_{i=0}^{l-1} 2^i (\xi_i + \xi'_i) &= \sum_{i=0}^{l-1} 2^i \eta_i + 2^{l-1}(\xi_{l-1} + \xi'_{l-1}) \\ &= \eta_0 + 2\left(\sum_{j=1}^{l-1} 2^{j-1}\eta_j + 2^{l-2}(\xi_{l-1} + \xi'_{l-1})\right). \end{aligned}$$

We may again regard

$$\sum_{j=1}^{l-1} 2^{j-1}\eta_j + 2^{l-2}(\xi_{l-1} + \xi'_{l-1})$$

as the sum of total $2^l - 1$ members of Ω. So there exist $\varsigma_j \in \Omega$, $j = 0, ..., l-1$, satisfying

$$\sum_{j=1}^{l-1} 2^{j-1}\eta_j + 2^{l-2}(\xi_{l-1} + \xi'_{l-1}) = \sum_{j=0}^{l-1} 2^j \varsigma_j.$$

That is

$$\xi + \sum_{i=0}^{l-1} 2^i(\xi_i + \xi'_i) = \eta_0 + 2\sum_{j=0}^{l-1} 2^j \varsigma_j.$$

The last equation is (7.4), if η_0 is replaced by η and ς_j by η_j, $j = 0, ..., l-1$.
Applying (7.4) to the sum of $2^{l-1}\eta_{l-1}$ and the second term in (7.3) with $\mu = 1$, we obtain (perhaps with a new $\eta_{l-1} \in \Omega$)

$$\frac{x+y}{2} = \sum_{j=0}^{l-1} 2^j \eta_j + 2^l \sum_{j=0}^{l-1} 2^j \eta_{l+j} + \sum_{\mu=2}^{\tau-1} 2^{\mu l-1} \sum_{j=0}^{l-1} 2^j (r_{\mu l+j} + r'_{\mu l+j}).$$

Similarly, $2^{2l-1}\eta_{2l-1}$ together with the sum of $\mu = 2$ can be rewritten as

$$2^{2l-1}\left(\eta_{2l-1} + \sum_{j=0}^{l-1} 2^j (r_{2l+j} + r'_{2l+j})\right) = 2^{2l-1}\left(\eta_{2l-1} + 2\sum_{j=0}^{l-1} 2^j \eta_{2l+j}\right),$$

where the $\eta_{2l-1} \in \Omega$ on the right hand side may not be the same as $\eta_{2l-1} \in \Omega$ on the left hand side. Repeating this process, we obtain finally

$$\frac{x+y}{2} = \sum_{\mu=0}^{\tau-1} 2^{\mu l} \sum_{j=0}^{l-1} 2^j \eta_{\mu l+j} = \sum_{\mu=0}^{\tau-1} \sum_{j=\mu l}^{(\mu+1)l-1} 2^j \eta_j = \sum_{j=0}^{l\tau-1} 2^j \eta_j \in \Omega^{l\tau}.$$

We have already seen that for any $x, y \in \Omega^{l\tau}$ with $x \equiv y \pmod 2$, then the integer $(x+y)/2$ belongs to $\Omega^{l\tau}$. To show the convexity of $\Omega^{l\tau}$ we have to prove that if $x, y \in \Omega^{l\tau}$ and all integers, which can be expressed as a sum of $\delta x + (1-\delta)y =: z$ for some $0 \le \delta \le 1$, belong also to $\Omega^{l\tau}$. To this end, let $x = a_0 + 2^l a_1 + \cdots + 2^{l(\tau-1)}a_{\tau-1}$ and $y = b_0 + 2^l b_1 + \cdots + 2^{l(\tau-1)}b_{\tau-1}$, for some $a_i, b_i \in \Omega^l$, $\ i = 0, ..., \tau-1$.
We may assume $0 \in \Omega^l$. So the integer z belongs to the convex cover Σ deduced by $\{0, pa_0, ..., pa_{\tau-1}, pb_0, ..., pb_{\tau-1}\}$, where $p = (2^{l\tau}-1)/(2^l-1)$. We prove that $\Sigma \cap \mathbb{Z}^s \subseteq$

$\Omega^{l\tau}$. In fact, let $c_0, c_{j,i} \in \{0, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$. Clearly all integers of the form $c_0 + \sum_{\mu=1}^{\tau-1} 2^{l\mu} c_{\mu,i}$ belongs to $\Omega^{l\tau}$. Thus, for fixed c_0 the middle point of any two of these integers belongs to $\Omega^{l\tau}$. That is

$$c_0 + 2^{l-1}(c_{1,i} + c_{1,j}) + \sum_{\mu=2}^{\tau-1} 2^{l\mu-1}(c_{\mu,i} + c_{\mu,j}) \in \Omega^{l\tau}.$$

Consequently,

$$c_0 + \sum_{i=1}^{2^l} c_{1,i} + \sum_{\mu=2}^{\tau-1} 2^{l(\mu-1)} \sum_{i=1}^{2^l} c_{\mu,i} \in \Omega^{l\tau}.$$

Repeatedly, any sum of $q(\leq p)$ integers from $\{0, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$ is a member of $\Omega^{l\tau}$. Thus in Σ the integers belongs to $\Omega^{l\tau}$, which can be written as a sum of p integers from $\{0, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$.

Let Σ' be the convex cover of $\{0, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$. So $\Sigma' \cap \mathbb{Z}^s \subseteq \Omega^l$ because Ω^l is convex. If in Σ there exist other integers which cannot be written as a sum of p integers from $\{0, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$, then there exist integers in Σ', which are different to $\{0, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$. Let $x \in \Sigma' \cap \mathbb{Z}^s$ be different to $\{0, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$. So $x \in \Omega^l$ and

$$x + 2^l c_{1,i} + \cdots + 2^{l(\tau-1)} c_{\tau-1,i} \in \Omega^{l\tau}.$$

Our consideration implies that any sum of x and $q(\leq p-1)$ integers from

$$\{0, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$$

is a member of $\Omega^{l\tau}$. That is that any sum of p intergers from $\{0, x, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$ belongs to $\Omega^{l\tau}$. If there is again integer in Σ, that cannot be expressed as a sum of p members from $\{0, x, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$, so we have $y \in \Sigma' \cap \mathbb{Z}^s$ be different to $\{0, x, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$ and any sum of p members from

$$\{0, x, y, a_0, ..., a_{\tau-1}, b_0, ..., b_{\tau-1}\}$$

is in $\Omega^{l\tau}$.

In this way, any sum of $q(\leq p)$ members of Ω^l belongs to $\Omega^{l\tau}$. Consequently, $\Sigma \cap \mathbb{Z}^s \subseteq \Omega^{l\tau}$ and $\Omega^{l\tau}$ is convex. $\square$

To prove Theorem 7.0.1, we also need some technical lemmas. The next one shows the relation between the irreducibility and the primitivity with respect to the convex set.

Lemma 7.1.2. *Let the set $\Omega \subseteq \mathbb{Z}^s$ be finite and further Ω^l be convex for some $l \geq 2$. If T is irreducible with respect to Ω^p and $\lambda \in \mathbb{Z}^s$, i.e.,*

$$\frac{T - \lambda + \Omega^p}{2^p} \cap \mathbb{Z}^s = T,$$

and $l|p$, then T is primitive.

Proof. If T is not primitive, then there exist $k \geq 2$ and a disjoint partition of T (say T_1, T_2,...,T_k) such that

$$\frac{T_i - \lambda + \Omega^p}{2^p} \cap \mathbb{Z}^s = T_{i+1}, \quad i = 1, ..., k$$

with $T_{k+1} = T_1$ (see Section 4.2 for the details). Thus, if we can prove that for any $x \in T$ there is $r \in \Omega^p$ satisfying

$$\frac{x - \lambda + r}{2^p} = x,$$

then $k = 1$, which shows the primitivity of T. To this end, let $x \in T$. Lemma 7.1.1 tells us that Ω^p is convex. Hence, by Corollary 3.3.2 we have $(2^p - 1)x + \lambda \in \Omega^p$. Consequently, write $r = (2^p - 1)x + \lambda$ and it follows that

$$\frac{x - \lambda + r}{2^p} = \frac{x - \lambda + (2^p - 1)x + \lambda}{2^p} = x.$$

In other words, $k = 1$. This completes the proof of the lemma. □

We will propose the properties of the primitive sets with respect to the convex set Ω^l.

Lemma 7.1.3. *Let T_1 and T_2 be primitive with respect to the convex set Ω^l for some $l \geq 2$ and $\lambda \in \mathbb{Z}^s$. If there are $x_1, ..., x_w \in T_1$ and $x_{w+1}, ..., x_{w+t} \in T_2$ with $w, t \geq 1$ such that*

$$a = \sum_{i=1}^{w+t} \alpha_i x_i \in \mathbb{Z}^s, \quad \textit{for} \ \ 0 < \alpha_i < 1 \ \ \textit{and} \ \ \sum_{i=1}^{w+t} \alpha_i = 1,$$

then $T_1 = T_2$.

Proof. We may assume $2^l \alpha_i > 1$, $i = 1, ..., w+t$. Otherwise, we consider $\Omega^{\tau l}$ with $\tau \in \mathbb{N}$ instead of Ω^l. Lemma 7.1.2 tells us T_1 and T_2 are still primitive with respect to $\Omega^{\tau l}$ and

$\lambda^* = \lambda + 2^l\lambda + \cdots + 2^{l(\tau-1)}\lambda$ (see also (4.3)). We can therefore take τ large enough such that $2^{\tau l}\alpha_i > 1$. Let now

$$\mu = \sum_{i=1}^{w+t} 2^l \alpha_i x_i - x_1 + \lambda = (2^l\alpha_1 - 1)x_1 + \sum_{i=2}^{w+t} 2^l\alpha_i x_i + \lambda \in \mathbb{Z}^s.$$

From Corollary 3.3.2 we know that

$$(2^l - 1)x_i + \lambda \in \Omega^l, \quad i = 1, ..., w+t.$$

Thus μ can be rewritten as the convex combination of $(2^l-1)x_i + \lambda$ with $i = 1, ..., w+t$, i.e.,

$$\mu = \frac{2^l\alpha_1 - 1}{2^l - 1}\big((2^l-1)x_1 + \lambda\big) + \sum_{i=2}^{w+t} \frac{2^l\alpha_i}{2^l - 1}\big((2^l-1)x_i + \lambda\big) \in \Omega^l.$$

It follows from $x_1 \in T_1$ and the definition of the primitivity that

$$\frac{x_1 - \lambda + \mu}{2^l} = \sum_{i=1}^{w+t} \alpha_i x_i = a \in T_1.$$

Similarly, for $x_{w+t} \in T_2$, we have

$$\nu = \sum_{i=1}^{w+t} 2^l\alpha_i x_i - x_{w+t} + \lambda \in \Omega^l,$$

which implies that

$$\frac{x_{w+t} - \lambda + \nu}{2^l} = a \in T_2.$$

Thus $T_1 \cap T_2 \neq \emptyset$. But T_1 and T_2 are primitive. It follows from the definition of the primitivity that $T_1 = T_2$. □

The following lemma gives us more information about the primitive set with respect to the convex set Ω^l.

Lemma 7.1.4. *Let $\Omega \subseteq \mathbb{Z}^s$ and $|\Omega| < \infty$. Let further $\lambda \in \mathbb{Z}^s$, $k \in \mathbb{N}$ and $T \subseteq \mathbb{Z}^s$ satisfying*

$$\frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s = T$$

be primitive. If Ω^l is convex for some $l \geq 2$, so is T.

Proof. Denote $\lambda^* = \lambda + 2^k\lambda + \cdots + 2^{k(l\tau-1)}\lambda$. It follows from the property of the primitivity of T (see (4.3)) that

$$\frac{T - \lambda^* + \Omega^{kl\tau}}{2^{kl\tau}} \cap \mathbb{Z}^s = T$$

For large enough τ we have

$$-\alpha + \lambda^* + 2^{kl\tau}\beta \in \Omega^{kl\tau}, \quad \forall\, \alpha, \beta \in T.$$

Let $x, y \in T$ such that $\delta x + (1-\delta)y \in \mathbb{Z}^s$ for some $0 < \delta < 1$. So $-x + \lambda^* + 2^{kl\tau}x \in \Omega^{kl\tau}$ and $-x + \lambda^* + 2^{kl\tau}y \in \Omega^{kl\tau}$. We obtain that

$$\delta(-x + \lambda^* + 2^{kl\tau}x) + (1-\delta)(-x + \lambda^* + 2^{kl\tau}y) = -x + \lambda^* + 2^{kl\tau}(\delta x + (1-\delta)y)$$

is an integer and belongs to $\Omega^{kl\tau}$. Thus for some $r \in \Omega^{kl\tau}$,

$$\delta x + (1-\delta)y = \frac{x - \lambda^* + r}{2^{kl\tau}} \in T.$$

Hence T is convex. □

7.2 Proof of Theorem 7.0.1

We are now ready to prove Theorem 7.0.1. In what follows, we shall take advantage of Theorem 3.3.1 (see Section 3.3 for details). The key is now for given λ and k to choose the irreducible (or primitive) mapping and to show the uniqueness of this mapping.

Proof of Theorem 7.0.1. Using Theorem 3.3.1 to prove Theorem 7.0.1 we have to show that for any $k \in \mathbb{N}$, $\lambda \in \mathbb{Z}^s$ with $0 \le (\lambda)_j \le 2^k - 1$, $j = 1, ..., s$, if

$$\frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T \quad \text{and} \quad \frac{T' - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s \subseteq T'$$

then $T \cap T' \neq \emptyset$. It is clear that any such set T contains a subset T_0, which is irreducible and satisfies

$$\frac{T_0 - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s = T_0.$$

Therefore, we need only show that for any $k \in \mathbb{N}$, $\lambda \in \mathbb{Z}^s$ with $0 \le (\lambda)_j \le 2^k - 1$, $j = 1, ..., s$, there is only one irreducible T_0. Clearly, if T_0 is not primitive, then there is a partition of $T_0 = T_1 \cup \cdots \cup T_p$ with $p \ge 2$ such that

$$\frac{T_j - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s = T_{j+1}$$

with the understanding $T_{p+1} = T_1$ (see Section 4.2 for the details). Moreover, T_j, $j = 1, ..., p$, is primitive with respect to $\Omega^{kp\tau}$ and $\lambda^*(\tau) = \lambda(2^{kp\tau} - 1)/(2^k - 1)$ for all $\tau \geq 1$. In other words, for the primitive set T_j, $j = 1, ..., p$, there holds

$$\frac{T_j - \lambda^*(\tau) + \Omega^{kp\tau}}{2^{kp\tau}} \cap \mathbb{Z}^s = T_j, \quad \tau = 1, 2, ...$$

On the other hand, if we define the matrix B_j as

$$B_j(\alpha, \beta) = a^{kp}(-\alpha + \lambda^*(1) + 2^{kp}\beta), \ \forall \ \alpha, \beta \in T_j,$$

then it follows from the sum rule (1.3) of the nonnegative mask $\{a(\alpha)\}$ that B_j is row-stochastic. The primitivity of T_j, $j = 1, ..., p$, implies that for some $\tau' \geq 1$ all entries of $B_j^{\tau'}$, $j = 1, ..., p$, are positive. By the construction of B_j we obtain

$$B_j^{\tau'}(\alpha, \beta) = a^{kp\tau'}(-\alpha + \lambda^*(\tau') + 2^{kp\tau'}\beta) > 0, \quad \forall \ \alpha, \beta \in T_j, \ \ j = 1, ..., p.$$

Thus, we conclude that for all $\tau \geq \tau'$

$$-\alpha + \lambda^*(\tau) + 2^{kp\tau}\beta \in \Omega^{kp\tau}, \ \forall \ \alpha, \beta \in T_j, \ \ j = 1, ..., p. \tag{7.5}$$

The arbitrariness of $\tau \geq \tau'$ allows us to choose $\tau \geq \tau'$ so that $l|(kp\tau)$. The above discussion tells us that to prove the desired assertion we need only to verify: for any given $k \in \mathbb{N}$ such that $l|k$ and $\lambda \in \mathbb{Z}^s$ with $0 \leq (\lambda)_j \leq 2^k - 1$ there exists only one primitive or irreducible T (see Lemma 7.1.2) with respect to Ω^k and λ, i.e.,

$$\frac{T - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s = T$$

and the set T is primitive and $l|k$.

By the way the sum rule implies that the volume of $[\Omega]$ is greater than 0. Hence, if $x \in [\Omega]^o$, then there is $\epsilon_0 > 0$ such that $x + \epsilon \in [\Omega]$ for $\epsilon \in \mathbb{R}^s$ and $|\epsilon| \leq \epsilon_0$. We will divide the proof into two cases according to different values of λ.

Case 1. $\lambda = (2^k - 1)\delta$, $\delta \in E^s$ and $l|k$.

We observe firstly $\delta = 0$. Let T be primitive with respect to Ω^k and $\lambda = 0$. Then

$$\frac{T + \Omega^k}{2^k} \cap \mathbb{Z}^s = T.$$

We claim that T is also primitive with respect to Ω and $\lambda = 0$. Otherwise, denote $T_0 = T$, we have

$$\frac{T_0 + \Omega}{2} \cap \mathbb{Z}^s = T_1,$$
$$\frac{T_1 + \Omega}{2} \cap \mathbb{Z}^s = T_2,$$
$$\cdots$$
$$\frac{T_{k-1} + \Omega}{2} \cap \mathbb{Z}^s = T_0.$$

Then, $T_1 \subseteq T_0$. To see this, let $x_1 \in T_1$. So there are $x_0 \in T_0$ and $r \in \Omega$ satisfying $x_1 = (x_0 + r)/2$. Let $\beta = (2^{k-1} - 1)x_0 + 2^{k-1}r$. Since $l|k$, the set Ω^k is convex (see Lemma 7.1.1). Corollary 3.3.2 tells us $(2^k - 1)x_0, (2^k - 1)r \in \Omega^k$. Finally the number β can be represented as a convex combination of $(2^k - 1)x_0$ and $(2^k - 1)r$:

$$\beta = \frac{2^{k-1} - 1}{2^k - 1}(2^k - 1)x_0 + \frac{2^{k-1}}{2^k - 1}(2^k - 1)r \in \Omega^k.$$

Now we obtain

$$\frac{x_0 + \beta}{2^k} = \frac{x_0 + r}{2} = x_1.$$

Consequently, $x_1 \in T_0$ or $T_1 \subseteq T_0$. Recursively we get

$$T_0 \subseteq T_{k-1} \subseteq \cdots \subseteq T_1 \subseteq T_0.$$

Therefore, $T_j = T$, $j = 1, ..., k-1$ and

$$\frac{T + \Omega}{2} \cap \mathbb{Z}^s = T.$$

Obviously, T is primitive with respect to Ω and $\lambda = 0$.

Next we will apply the conditions of this theorem to construct a primitive set T and to show that T is unique. By assumption we can find a multi-integer $x \in [\Omega]^o \cap \mathbb{Z}^s$. Define

$$T_1 := \frac{x + \Omega}{2} \cap \mathbb{Z}^s, \ T_2 := \frac{T_1 + \Omega}{2} \cap \mathbb{Z}^s, ..., T_{i+1} := \frac{T_i + \Omega}{2} \cap \mathbb{Z}^s, ...$$

Then $T_i \subseteq [\Omega]^o \cap \mathbb{Z}^s$. The sum rule (1.3) implies $T_i \neq \emptyset$. Because $[\Omega] \cap \mathbb{Z}^s$ is finite, there exist i and μ such that $T_i = T_{i+\mu}$. In other words,

$$\frac{T_i + \Omega^\mu}{2^\mu} \cap \mathbb{Z}^s = T_i.$$

T_i contains a subset T', that is irreducible with respect to Ω^μ and $\lambda = 0$. If T' is not primitive, then we have p and $T \subset T'$ so that T is primitive with respect to $\Omega^{\mu p \tau}$ and $\lambda = 0$ for all $\tau = 1, 2, ...$ Taking $\tau = l$, we conclude that T is also primitive with respect to Ω and $\lambda = 0$ as mentioned above. Moreover $T \subseteq [\Omega]^o \cap \mathbb{Z}^s$. The number of elements in T is finite, hence, for some $\epsilon_0 > 0$ there holds

$$T + \epsilon \subseteq [\Omega], \quad \forall \ \epsilon \in \mathbb{R}^s, \ \ |\epsilon| \leq \epsilon_0.$$

Clearly, $[\Omega] = [\Omega^\tau/(2^\tau - 1)]$ for any $\tau \geq 1$ (see Corollary 3.3.2). Let $\beta \in T$ be fixed. Then any $y \in [\Omega] \cap \mathbb{Z}^s$ can be written as $y = \beta + b$. Moreover, for sufficiently large $\tau \in \mathbb{N}$ all those b satisfy $|b|/(2^\tau - 1) \leq \epsilon_0$. We may choose τ with $l|\tau$. So, by Lemma 7.1.1 and the conditions of this theorem, Ω^τ is convex. Consequently, as $\beta - b/(2^\tau - 1) \in [\Omega]$, we obtain $(2^\tau - 1)\beta - b \in \Omega^\tau$ for all those b. Now we have $\{(\beta + b) + (2^\tau - 1)\beta - b\}/2^\tau = \beta$. In other words, for any $y \in [\Omega] \cap \mathbb{Z}^s$ there is $r \in \Omega^\tau$ satisfying $(y + r)/2^\tau = \beta$. Let L be an another primitive set with respect to Ω^k and $\lambda = 0$. Then, $L \subseteq [\Omega] \cap \mathbb{Z}^s$ and

$$\beta \in \frac{L + \Omega^\tau}{2^\tau} \cap \mathbb{Z}^s = L.$$

By Lemma 7.1.3 both T and L are the same. Therefore, there is only one primitive set with respect to Ω^k and $\lambda = 0$.

The assertion for $\delta \neq 0$ follows from the following relation:

$$\frac{L - (2^k - 1)\delta + \Omega^k}{2^k} \cap \mathbb{Z}^s = L \quad \Longleftrightarrow \quad \frac{L + \delta + \Omega^k}{2^k} \cap \mathbb{Z}^s = L + \delta.$$

Therefore the first case is as announced.

Case 2. $\lambda \neq (2^k - 1)\delta, \delta \in E^s, l|k.$

The proof for this case is much more involved, although the idea is the same as in Case 1. In order to pave the way for the proof we first make some simplifies. Let T be primitive with respect to Ω^k and λ. Suppose $\lambda = (\eta_1, ..., \eta_s)^T$ and $\gcd(\eta_1, ..., \eta_s) = p$. It follows that $0 \leq p \leq 2^k - 1$. If $p = 0$ or $p = 2^k - 1$, we have nothing more to do, because $\lambda = (2^k - 1)\delta$ with $\delta \in E^s$ (see Case 1). If $1 \leq p < 2^k - 1$, we have an $s \times s$ unimodular matrix $M \in \mathfrak{M}_s$ (see Section 5.1) satisfying $M\lambda = pe_1$ with $e_1 = (1, 0, ..., 0)^T \in E^s$. Using this M we get

$$\frac{MT - M\lambda + (M\Omega)^k}{2^k} \cap \mathbb{Z}^s = MT.$$

Clearly, MT is also primitive with respect to $(M\Omega)^k$ and $M\lambda$. If we write $p = \sum_{j=0}^{k-1} 2^j \epsilon_j$ with $\epsilon_j \in \{0, 1\}$, then there exists j_0 such that $\epsilon_{j_0} = 0$. Next we write

$$\lambda' = \sum_{i=0}^{k-1} 2^i e_1 \epsilon_{j_0+i} = e_1 \epsilon_{j_0} + 2e_1 \epsilon_{j_0+1} + \cdots + 2^{k-1} e_1 \epsilon_{j_0+k-1},$$

where $j_0 + i$ is cyclic, i.e.,

$$j_0 + i = \begin{cases} j_0 + i, & \text{if } j_0 + i \le k-1, \\ j_0 + i - k, & \text{if } j_0 + i > k-1. \end{cases}$$

Clearly, $\lambda' \equiv 0 \pmod 2$. On the other hand, denote $T_0 = MT$ we obtain from the above

$$\begin{aligned} &\frac{T_{j_0} - e_1\epsilon_{j_0} + M\Omega}{2} \cap \mathbb{Z}^s = \frac{T_{j_0} + M\Omega}{2} \cap \mathbb{Z}^s = T_{j_0+1}, \\ &\frac{T_{j_0+1} - e_1\epsilon_{j_0+1} + M\Omega}{2} \cap \mathbb{Z}^s = T_{j_0+2}, \\ &\cdots \\ &\frac{T_{j_0+k-1} - e_1\epsilon_{j_0+k-1} + M\Omega}{2} \cap \mathbb{Z}^s = T_{j_0}. \end{aligned}$$

Obviously, T_{j_0} is primitive with respect to $(M\Omega)^k$ and λ'. Therefore in the following proof we may assume, without loss of generality, that our λ has already this property, in particular $\lambda \equiv 0 \pmod 2$.

The key is to find a primitive set T with respect to Ω^k and λ such that $(2^k - 1)T + \lambda \subseteq [\Omega^k]^o$. Let us begin with the discussion of the uniqueness of such set T. If we have such a set T, then for any $\tau \ge 1$ the set T is also primitive with respect to $\Omega^{\tau k}$ and $\lambda^* = \lambda(2^{k\tau} - 1)/(2^k - 1)$. Moreover, $(2^{\tau k} - 1)T + \lambda^* \subseteq [\Omega^{\tau k}]^o$ and with $u = \lambda/(2^k - 1)$ the set $T + u$ is a subset of $[\Omega]^o$. The finiteness of T implies that there exists an $\epsilon_0 > 0$ satisfying

$$T + u + \epsilon \subseteq [\Omega], \quad \forall\ \epsilon \in \mathbb{R}^s,\ |\epsilon| \le \epsilon_0.$$

Thus, for the integer b that can be written as $(2^{\tau k} - 1)\epsilon$ with $|\epsilon| \le \epsilon_0$, we get

$$(2^{\tau k} - 1)T + b + \lambda^* \subseteq \Omega^{\tau k}.$$

This relation implies that for any $\beta \in T$ there is $r := (2^{\tau k} - 1)\beta + b + \lambda^* \in \Omega^{\tau k}$ such that

$$\frac{(2^{\tau k} - 1)\beta + b + \lambda^* + (2^{\tau k} - 1)r}{2^{\tau k}} = (2^{\tau k} - 1)\beta + b + \lambda^*.$$

Hence, for any $\beta \in T$ and any such number b there is $r \in \Omega^{\tau k}$ satisfying

$$\frac{(\beta - b) - \lambda^* + r}{2^{\tau k}} = \beta$$

or

$$\frac{(\beta - b) - \lambda^* + r}{2^{\tau k}} \in T.$$

Let τ be large enough. Then any multi-integer α from the admissible set $\Gamma(a)$ can be expressed as $\beta - b$ for some $\beta \in T$ and $b = (2^{\tau k} - 1)\epsilon$ with $|\epsilon| \leq \epsilon_0$. Hence, for any $\alpha \in \Gamma(a)$ there is $r \in \Omega^{\tau k}$ such that

$$\frac{\alpha - \lambda^* + r}{2^{\tau k}} \in T.$$

Clearly, any primitive set is a subset of $\Gamma(a)$. Let T' be another primitive set with respect to Ω^k and λ. Thus, T' is a subset of $\Gamma(a)$. Moreover, T' is also primitive with respect to $\Omega^{\tau k}$ and λ^*. The above relation shows however that for $\alpha \in T'$ and some $r \in \Omega^{\tau k}$, we have

$$\frac{\alpha - \lambda^* + r}{2^{\tau k}} \in T \cap T'.$$

Therefore, by Lemma 7.1.3, $T = T'$.

We have seen that $(2^k - 1)T + \lambda \subseteq [\Omega^k]^o$ implies the uniqueness of T.

We now turn to the existence of such a set T, if there is $\alpha \in \mathbb{Z}^s$ satisfying $(2^k - 1)\alpha + \lambda \in [\Omega^k]^o$. To see this, we define

$$T_0 := \frac{\alpha - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s,\ T_1 := \frac{T_0 - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s, ..., T_{j+1} := \frac{T_j - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s, ...$$

Clearly, for $j = 0, 1, ...$ the set $(2^k - 1)T_j + \lambda$ is a subset of $[\Omega^k]^o$. The finiteness of T_j means that there are $\tau \geq 1$ and j such that $T_j = T_{j+\tau}$. That is with $\lambda^* = \lambda(2^{k\tau} - 1)/(2^k - 1)$

$$\frac{T_j - \lambda^* + \Omega^{\tau k}}{2^{\tau k}} \cap \mathbb{Z}^s = T_j.$$

From this T_j we can get a primitive set $T \subseteq T_j$ with respect to $\Omega^{\tau k}$ and λ^*. Obviously, $(2^k - 1)T + \lambda \subseteq [\Omega^k]^o$.

Our task is now to show that there is either a primitive set T with respect to Ω^k and λ satisfying $(2^k - 1)T + \lambda \subseteq [\Omega^k]^o$ or an $\alpha \in \mathbb{Z}^s$ such that $(2^k - 1)\alpha + \lambda \in [\Omega^k]^o$.

Let T be a primitive set with respect to Ω^k and λ. We remember

$$\frac{(2^k - 1)T + \lambda + (2^k - 1)\Omega^k}{2^k} \cap \mathbb{Z}^s = (2^k - 1)T + \lambda. \tag{7.6}$$

We know also that T is primitive with respect to $\Omega^{k\tau}$ and $\lambda^*(\tau) = \lambda(2^{k\tau} - 1)/(2^k - 1)$ for all $\tau \geq 1$. On the other hand, by (7.5) for sufficiently large τ

$$-\alpha + \lambda^*(\tau) + 2^{k\tau}\beta \in \Omega^{k\tau}, \ \ \forall\, \alpha, \beta \in T.$$

In the following proof we may therefore assume that our k instead of $k\tau$ already satisfies this relation. Moreover, we may also assume that our k is large enough.
From Lemma 7.1.4 the T is convex and from Corollary 6.1.4 the set T contains at least 2 members.
To prove $(2^k - 1)T + \lambda \subseteq [\Omega^k]^o$, suppose to the contrary that $((2^k - 1)T + \lambda) \cap \partial[\Omega^k] \neq \emptyset$. Then there is a j-dimensional face S_j^k of $[\Omega^k]$ satisfying $((2^k - 1)T + \lambda) \cap S_j^k \neq \emptyset$. We know that S_j^k is again a polytope and whose faces are still faces of $[\Omega^k]$. We may therefore choose the dimension of S_j^k to be minimal. Let $t \in ((2^k - 1)T + \lambda) \cap S_j^k$. So $t = (2^k - 1)\beta + \lambda$ for some $\beta \in T$. In view of (7.5) our choice of k means

$$-\alpha + \lambda + 2^k\beta \in \Omega^k, \ \ \forall\, \alpha, \beta \in T.$$

Multiplying both sides by $(2^k - 1)$ we obtain

$$-((2^k - 1)\alpha + \lambda) + 2^k((2^k - 1)\beta + \lambda) \in (2^k - 1)\Omega^k, \ \ \ \forall\ \alpha, \beta \in T.$$

Thus, for any given $x \in (2^k - 1)T + \lambda$ there is $r \in \Omega^k$ such that

$$\frac{x + (2^k - 1)r}{2^k} = t$$

or

$$(2^k - 1)T + \lambda = \{x \in (2^k - 1)T + \lambda : \ -x + 2^k t \in (2^k - 1)\Omega^k\}. \tag{7.7}$$

If $j = 0$, then t is an extreme point of Ω^k. So as a convex combination of x and r we must have $t = x = r$. This is however impossible because $|T| \geq 2$. Hence, $1 \leq j \leq s - 1$ and t is an inner point of S_j^k, i.e., $t \in (S_j^k)^o$. Recalling (7.6), we have $x_1 \in (2^k - 1)T + \lambda$ and $r \in \Omega^k$ so that $(x_1 + (2^k - 1)r)/2^k = t$.
In order to prove that for $1 \leq j \leq s - 1$,

$$(2^k - 1)T + \lambda \subseteq (S_j^k)^o, \tag{7.8}$$

we distinguish between a trivial case when $j = s - 1$ and the more involved case when $j < s - 1$. In the first case that $j = s - 1$, both x_1 and r must lie on the same side of S_j^k. But t is the convex combination of x_1 and r. So $x_1, r \in S_j^k$. The minimality of j implies that $x_1 \in (S_j^k)^o$. Hence, for all $x \in (2^k-1)T+\lambda$ and $r \in \Omega^k$ such that $(x+(2^k-1)r)/2^k = t$ we have $x \in (S_j^k)^o$. In other words,

$$\{x \in (2^k-1)T+\lambda : \ -x+2^k t \in (2^k-1)\Omega^k\} \subseteq (S_j^k)^o.$$

It follows from this relation and (7.7) that the inclusion (7.8) holds for $j = s-1$.

In the second case that $j < s-1$, there exist μ-dimensional faces S_μ^k of $[\Omega^k]$, $\mu = j, j+1, ..., s-1$, such that $S_j^k \subset S_{j+1}^k \subset \cdots \subset S_{s-1}^k$. Regard t as a point of S_{s-1}^k, so $x_1 \in S_{s-1}^k$. However, $t \in S_{s-2}^k$, hence $x_1 \in S_{s-2}^k$. We conclude recursively that $x_1 \in S_\nu^k$, $\nu = s-1, s-2, ..., j$, and therefore $x_1 \in S_j^k$. The minimality of j implies again $x_1 \in (S_j^k)^o$, i.e., x_1 is an inner point of S_j^k. Consequently, the above inclusion (7.8) is also valid for $j < s-1$.

On the other hand, since $[\Omega^k/(2^k-1)] = [\Omega]$, the set $S_j := [S_j^k/(2^k-1)]$ is a j-dimensional face of $[\Omega]$. Clearly, the set S_j does not depend on k. Let $u = \lambda/(2^k-1)$. We obtain from (7.8) that $T + u \subseteq (S_j)^o$.

Before moving further, let us provide with some notations. Denote L_j to be a j-dimensional affine space that contains S_j and $\mathbf{n}_{j+1}, ..., \mathbf{n}_s \in \mathbb{Z}^s$ the unit normal vectors of L_j, where $\mathbf{n}_{j+1}, ..., \mathbf{n}_s$ are chosen in the following way: first there are μ-dimensional faces S_μ, $\mu = j+1, ..., s-1$ such that $S_j \subset S_{j+1} \subset \cdots \subset S_{s-1}$. Let L_μ be μ-dimensional affine space, which contains S_μ. Finally, if we regard L_μ as a μ-dimensional affine space in $(\mu+1)$-dimensional space, which is imbedded in $\mathbb{R}^s$, then the vector $\mathbf{n}_{\mu+1}$ belongs to this $(\mu+1)$-dimensional space and is orthogonal to L_μ, $\mu = s-1, ..., j+1$. More precisely, $L_j \subset L_{j+1} \subset \cdots \subset L_{s-1}$ and $\mathbf{n}_{i+1}, ..., \mathbf{n}_s$ are orthogonal to L_i, $i = j, ..., s-1$. Since $S_i \subset L_i$, we have also that $\mathbf{n}_{i+1}, ..., \mathbf{n}_s$ are orthogonal to S_i, $i = j, ..., s-1$.

If necessary we may also regard T as a primitive set with respect to $\Omega^{k\nu}$ and $\lambda(\nu) = \lambda(2^{k\nu}-1)/(2^k-1)$ for $\nu = 1, 2, ...$ Hence, $(2^{k\nu}-1)T + \lambda(\nu) \subseteq (S_j^{k\nu})^o$ and $S_j^{k\nu}$ is a j-dimensional face of $\Omega^{k\nu}$. Moreover $\mathbf{n}_{j+1}, ..., \mathbf{n}_s$ are the normal vectors of $S_j^{k\nu}$.

With this in mind let us observe that $[\Omega]^o \cap \mathbb{Z}^s \neq \emptyset$. So there is an integer $\alpha \in [\Omega]^o$.

Consequently, we define

$$T_0 := \frac{\alpha - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s, \tag{7.9}$$

$$T_1 := \frac{T_0 - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s,$$

$$...,$$

$$T_{\mu+1} := \frac{T_\mu - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s,$$

$$....$$

There are only two possible cases: either for some $\mu \geq 0$ the set $(2^k - 1)T_\mu + \lambda$ has at least one inner point of $[\Omega^k]$ or for all $\mu \geq 0$ the set $(2^k - 1)T_\mu + \lambda$ contains no inner points of $[\Omega^k]$.

By the first case we have an $x \in \mathbb{Z}^s$. The number $(2^k - 1)x + \lambda$ is an inner point of $[\Omega^k]$, which implies the uniqueness of the primitive set T and $(2^k - 1)T + \lambda \subseteq [\Omega^k]^o$. This however contradicts our assumption.

In the second case, because of the finiteness of T_μ we have μ and ν such that $T_\mu = T_{\mu+\nu}$ and with $\lambda(\nu) = \lambda(2^{k\nu} - 1)/(2^k - 1)$

$$\frac{T_\mu - \lambda(\nu) + \Omega^{k\nu}}{2^{k\nu}} \cap \mathbb{Z}^s = T_\mu.$$

From this equation and Lemma 7.1.2 we obtain a primitive set $T' \subseteq T_\mu$ with respect to $\Omega^{k\nu}$ and $\lambda(\nu)$. Moreover, the set $(2^{k\nu} - 1)T' + \lambda(\nu)$ contains no inner points of $[\Omega^{k\nu}]$ and $(2^{k\nu} - 1)T' + \lambda(\nu) \subseteq [\Omega^{k\nu}]$ by Corollary 3.3.2. So for some $1 \leq i < s$ there is a i-dimensional face $S_i^{k\nu}$ of $[\Omega^{k\nu}]$ such that $(2^{k\nu} - 1)T' + \lambda(\nu) \subseteq (S_i^{k\nu})^o$. In view of (7.9) any $b \in (2^{k\nu} - 1)T' + \lambda(\nu) \subseteq (S_i^{k\nu})^o$ is a convex combination of $(2^{k\nu} - 1)\alpha + \lambda(\nu)$ and the members from $\Omega^{k\nu}$, so there are $c_i > 0, i = 1, ..., m$, satisfying $c_1 + \cdots + c_m = 1$ and $b = c_1((2^{k\nu} - 1)\alpha + \lambda(\nu)) + c_2 r_2 + \cdots + c_m r_m$ for some $r_\tau \in \Omega^{k\nu}$. The number (regarded as vector) $\lambda(\nu)$ cannot be orthogonal to the normal vectors of $S_i^{k\nu}$. For otherwise we will have $b - c_1\lambda(\nu) \in L_i^{k\nu}$, which is an i-dimensional affine space and contains $S_i^{k\nu}$. We observe the number

$$b - c_1\lambda(\nu) = c_1(2^{k\nu} - 1)\alpha + c_2 r_2 + \cdots + c_m r_m.$$

The right hand side is a convex combination of the numbers from $[\Omega^{k\nu}]$ since $(2^{k\nu}-1)\alpha \in$

$[\Omega^{k\nu}]^o \cap \mathbb{Z}^s$. So $b - c_1\lambda(\nu)$ belongs to $[\Omega^{k\nu}]$. However, as $b - c_1\lambda(\nu) \in L_i^{k\nu} \cap [\Omega^{k\nu}] = S_i^{k\nu}$, this is impossible. So $\lambda(\nu)$ is not orthogonal to normal vectors of $S_i^{k\nu}$.
On the other hand, Case 1 tells us that the matrix A defined by (6.1) is connected. Hence, by Theorem 6.1.3 for any j-dimensional face S_j of Ω, $0 \le j \le s-1$

$$0 < \sum_{\alpha \in S_j \cap \Omega} a(\alpha) < 2^j. \tag{7.10}$$

To finish the proof we need also the following fact: under some unimodular matrix $M \in \mathfrak{M}_s$ the set $M\Omega$ contains E^s. To see this let without loss of generality 0 be an extreme point of $[\Omega]$, i.e. 0-dimensional face. Let $0, \alpha \in [\Omega] \cap \mathbb{Z}^s$ belong to a 1-dimensional face, such that the segment $[0, \alpha]$ contains no any other integers. We have $\alpha \in \Omega$. Indeed, as Ω^l is convex, $\alpha \in \Omega^l$. So there are some $\gamma_j \in \Omega$ satisfying $\alpha = \gamma_0 + 2\gamma_1 + \cdots + 2^{l-1}\gamma_{l-1}$. Thus all γ_j belong to this 1-dimensional face. Since in the segment $[0, \alpha]$ there are no any other integers, we must have $a = \gamma_0$. Moreover, let integers 0, α and β belong to 1-dimensional faces of a 2-dimensional face, such that the segment $[0, \alpha, \beta, \alpha + \beta]$ contains no other integers. We prove $\alpha + \beta \in \Omega$. In fact, the above proof tells us $\alpha, \beta \in \Omega$. The convexity of Ω^l implies $2\alpha, 2\beta \in \Omega^l$ and as well as $\alpha + \beta = (2\alpha + 2\beta)/2 \in \Omega^l$. On the other hand, $\alpha + \beta$ belongs to a 2-dimensional space, that contains this 2-dimensional face. We have again $\gamma_j \in \Omega$ satisfying $\alpha + \beta = \gamma_0 + 2\gamma_1 + \cdots + 2^{l-1}\gamma_{l-1}$. Consequently, $\alpha + \beta \in \Omega$. Therefore under some unimodular matrix $M \in \mathfrak{M}_s$ the set $M\{0, \alpha, \beta, \alpha + \beta\}$ is E^2 imbedded in $\mathbb{Z}^s$ and belongs to $M\Omega$. In the same way, if for those α, β there is $0 < \delta < 1$ such that $\delta\alpha + (1-\delta)\beta$ is an integer, then $\delta\alpha + (1-\delta)\beta \in \Omega$ for the largest δ and some unimodular matrix $M \in \mathfrak{M}_s$ the set $M\{0, \alpha, \delta\alpha + (1-\delta)\beta, (1+\delta)\alpha + (1-\delta)\beta\}$ is E^2 imbedded in $\mathbb{Z}^s$. Inductive arguments provide the desired assertion.
Let us go back to our proof. We may assume that 0 is an extreme point of Ω. Thus, we have already seen that under some unimodular matrix $M \in \mathfrak{M}_s$, Ω contains ME^s. We notice $u = \lambda/(2^k - 1) = \lambda(\nu)/(2^{k\nu} - 1)$. Denote $J := T'$. For simplification purposes, we should write k and λ instead of $k\nu$ and $\lambda(\nu)$, respectively. Thus, J is primitive with respect to λ and Ω^k. In particular, $J + u \subseteq (S_i)^o$, that is contained in the i-dimensional affine space L_i, and

$$\frac{J - \lambda + \Omega^k}{2^k} \cap \mathbb{Z}^s = J.$$

It follows from Theorem 6.1.3 that for any $r \in S_i \cap \Omega$ the set $\mathcal{A}(r)$ has at least two elements. Let $x \in J$. The last display means that for some $\tau \geq 1$ and $\gamma_j \in \Omega$

$$x = 2^{k\tau}x + \lambda(\tau) - \sum_{j=0}^{k\tau-1} 2^j \gamma_j.$$

As $(2^{k\tau} - 1)x + \lambda(\tau) \in S_i^{k\tau}$, we obtain $\gamma_j \in S_i$, $j = 0, ..., k\tau - 1$. In other words, $\mathcal{A}(x) \subseteq S_i \cap \Omega$ has at least two elements. Thus,

$$\bigcup_{x \in J} \mathcal{A}(x) \subseteq S_i \cap \Omega.$$

Consequently, for any those J and S_i we have always $S_i \cap ME^s \neq \emptyset$ and 0 belongs to S_i.

Let S_{s-1} be an $(s-1)$-dimensional face and L_{s-1} affine space satisfying $S_i \subset L_{i-1}$ and $S_{s-1} \subset L_{s-1}$, respectively. Then $J \subset L_{s-1} - u$. The inequality (7.10) guarantees that there are an integer $y \in L_{s-1} - u$ and $r' \in \Omega$ satisfying $y \equiv r' \pmod 2$ and $r' \notin S_{s-1}$. We may choose $y \in L_{s-1} - u$ such that either the segment $[r', y + u]$ cuts through an $(s-1)$-dimensional face of $[\Omega]$, that does not contain 0, or r' belongs to this face. Noticing $\lambda = \delta_0 + 2\delta_1 + \cdots + 2^{k-1}\delta_{k-1}$ with $\delta_0 = 0$, we define $x_1 = (y + r')/2$. Let $r_j \in \Omega$, $j = 1, ..., k-1$ and $x_2, x_3, ..., x_k \in \mathbb{Z}^s$ such that $x_j - \delta_j \equiv r_j \pmod 2$ and $x_{j+1} = (x_j - \delta_j + r_j)/2$, $j = 1, ..., k-1$. Hence for $r^* = r' + 2r_1 + \cdots + 2^{k-1}r_k$ we have $(y - \lambda + r^*)/2^k = x_k$.

Replacing α by x_k in (7.9) we obtain in this way a primitive set J' with respect to $\Omega^{k\mu}$ and $\lambda(\mu)$ for some $\mu \in \mathbb{N}$. $(2^{k\mu} - 1)J' + \lambda(\mu)$ lies again in some face of $[\Omega^{k\mu}]$. Let $x \in J'$. So for some $\tau \in \mathbb{N}$ and $\gamma_v \in \Omega^k$ the integer x_k can be expressed as

$$x_k = 2^{k\tau}x + \lambda(\tau) - \sum_{j=0}^{\tau-1} 2^{kj}\gamma_j.$$

But $y = 2^k x_k + \lambda - r^*$ we conclude

$$\begin{aligned} y &= 2^k(2^{k\tau}x + \lambda(\tau) - \sum_{j=0}^{\tau-1} 2^{kj}\gamma_j) + \lambda - r^* \\ &= 2^{k(\tau+1)}x + \lambda(\tau+1) - r^* - 2^k \sum_{j=0}^{\tau-1} 2^{kj}\gamma_j. \end{aligned}$$

Noticing

$$\lambda(\tau + 1) = \lambda \cdot (\frac{2^{k(\tau+1)} - 1}{2^k - 1}) = u \cdot (2^{k(\tau+1)} - 1).$$

We obtain $y + u = 2^{k(\tau+1)}(x + u) - r^* - 2^k \sum_{j=0}^{\tau-1} 2^{kj}\gamma_j$ or

$$x + u = \frac{(y + u) + r^* + 2^k \sum_{j=0}^{\tau-1} 2^{kj}\gamma_j}{2^{k(\tau+1)}}. \tag{7.11}$$

The number $x + u$ belongs to some face of $[\Omega]$ and is an inner point of this face. We may regard that this face is i'-dimensional. Noticing $r^* = r' + 2r_1 + \cdots + 2^{k-1}r_k$.
Since $y+u \in L_{s-1}$, all $y+u$, γ_j, r', r_1, r_2,...,r_k lie on the same side of L_{s-1}. As $r' \notin S_{s-1}$, this i'-dimensional face cannot be contained in S_{s-1}. Moreover, our choice of y implies that this face does not contain 0.
Thus, all cases that we have considered lead to the contradiction. That means finally that our assumption at the beginning was wrong. Therefore, the primitive set T has the property $(2^k - 1)T + \lambda \subseteq [\Omega^k]^o$. Theorem 7.0.1 is now fully proved.

□

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