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MOTIVES AND ALGEBRAIC CYCLES WITH MODULI CONDITIONS

DISSERTATION

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Contents

Abstract

This thesis is dedicated to the study of motives and algebraic cycles subject to certain constraints at infinity, called the modulus condition. Following ideas of Kerz and Saito, we discuss the notion of relative motivic cohomology of a pair $(\overline{X}, D)$ (consisting of a smooth separated scheme over k and an effective non-reduced Cartier divisor on it) in terms of algebraic cycles, modeled on Bloch's cycle complex. We construct a cycle class map from the group of higher zero cycles with modulus to the relative K-groups of the pair $(\overline{X}, D)$ and we prove some vanishing results concerning zero cycles on affine varieties. In the second part of the thesis, we construct and study an unstable motivic homotopy category with modulus $\overline{\mathbf{M}}\mathcal{H}(k)$, extending the Morel-Voevodksy construction from smooth schemes over a field k to certain diagrams of schemes. We present this category as a candidate environment for studying representability problems for non $\mathbb{A}^1$-invariant generalized cohomology theories.

Zusammenfassung der Dissertation

Diese Arbeit ist dem Studium von Motiven und algebraischen Zykeln unter einer bestimmten Bedingung im Unendlichen gewidmet - der sogenannten Modulus-Bedingung. Den Ideen von Kerz und Saito folgend behandeln wir den Begriff von relativer motivischer Kohomologie eines Paares $(\overline{X}, D)$ (bestehend aus einem glatten, separierten Schema über k und einem darauf definierten effektiven, nicht reduzierten Cartier-Divisor) im Bezug auf algebraische Zykeln, die sich an dem Blochschen Zykelnkomplex orientiert. Wir konstruieren eine Zykelklassenabbildung von der Gruppe der höheren Verschwindungszykeln mit Modulus in die relativen K-Gruppen des Paares $(\overline{X}, D)$, und wir beweisen einige Verschwindungsresultate bezüglich Verschwindungszykeln auf affinen Varietäten. In dem zweiten Teil dieser Arbeit konstruieren und studieren wir eine nicht stabile motivische Homotopie-Kategorie mit Modulus $\overline{\mathbf{M}}\mathcal{H}(k)$. Damit verallgemeinern wir die Morel-Voevodsky-Konstruktion von glatten Schemata über k auf gewisse Diagramme von Schemata. Möglicherweise stellt diese Kategorie einen Kandidaten dar für die Lösbarkeit gewisser Darstellbarkeitsprobleme nicht $\mathbb{A}^1$-invarianter verallgemeinerter Kohomologietheorien.

Introduction

1. The pursuit of a geometrically defined "universal" cohomology theory for algebraic varieties, expressing the kinship of all different cohomology theories and playing in algebraic geometry the role of ordinary cohomology for a topological space, has a long history. Its origins date back to the work of Alexander Grothendieck, who with the word "motive" (as he explains in [20]) wanted to suggest that there was a "common motive" or a "common reason" behind the multitude of cohomological invariants attached to algebraic varieties. Since then, progress has been made towards his dream.

In the early 1980s, Alexander Beĭlinson gave in [2] a precise conjectural framework for the expected cohomology theory, in the context of a new theory of *mixed motives*. It became clear at that time that among the many hoped-for properties, there should have been a relationship between *motivic cohomology*, to be defined in some way using algebraic cycles, and Quillen's algebraic K-theory. This relationship should take the form of a convergent Atiyah-Hirzebruch type spectral sequence

$$E_2^{p,q} = \mathrm{H}^{p-q}_{\mathcal{M}}(S, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(S) \tag{1}$$

for any scheme S, arbitrarily singular.

Reducing our ambitions, fix a perfect field k and consider the category $\mathbf{Sch}(k)$ of separated schemes of finite type over k. Although the definition of the right *abelian* category of mixed motives over k is still out of reach, the definition of a *triangulated* category of mixed motives turned out to be a more tractable task.

The constructions of V. Voevodsky [67] and M. Levine [45] of a triangulated category of motives over k (the *derived category of mixed motives*) gave a solid foundation to an otherwise purely conjectural world. For X a smooth and quasi-projective variety, the motivic cohomology groups, defined a priori as Zariski hypercohomology of certain complexes of sheaves, have a concrete description in terms of algebraic cycles via S. Bloch's higher Chow groups [7], and satisfy most of the properties that one expects, including Chern classes and Chern character isomorphism from the higher K-groups. Moreover, as shown in [46, Section 8.10] and [15], there is a convergent spectral sequence to algebraic K-theory of the form (1) for every smooth variety X. In fact, for any scheme X of finite type over a field (but these assumptions can be relaxed), there is G-theory spectral sequence, converging to the homotopy groups of the K-theory spectrum of the category of coherent sheaves on X. For X regular, it coincides with the K-theory of vector bundles.

If the case of smooth varieties is established, this theory of motivic cohomology has one serious deficiency that prevents it from giving the right answer to the conjecture in the general form proposed by Beilinson. To give an illustrative example, consider for a smooth variety X, its m-th thickening $X_m = X \times \mathrm{Spec}(k[t]/(t^m))$. According to the available definitions, we have an isomorphism

$$\mathrm{H}^*_{\mathcal{M}}(X_m, \mathbb{Z}(*)) \xrightarrow{\simeq} \mathrm{H}^*_{\mathcal{M}}(X, \mathbb{Z}(*))$$

for every m, manifesting the fact that, in Voevodsky's triangulated category $\mathbf{DM}(k, \mathbb{Z})$, one has $M(X) = M(X_m)$. On the other hand, the natural map

$$\rho_m \colon K_*(X_m) \to K_*(X)$$

is far from being an isomorphism, as one can see already from the case $X = \mathrm{Spec}(k)$. The difference between the groups is measured by the *relative K groups*

$$K_*(X_m, X)). \tag{2}$$

Notably, the presence of infinitesimal thickenings destroys the homotopy invariant property of K-theory that holds for regular schemes. Since $\mathbf{DM}(k, \mathbb{Z})$ has the idea of $\mathbb{A}^1$-invariance built in from its very foundation, it fails to encompass phenomena of this kind.

2. Many attempts have been made in the past years to construct a general framework that allows to incorporate the K-theory of infinitesimal thickenings of smooth schemes in the picture. In fact, there is a plethora of non-homotopy invariant constructions in algebraic geometry, from wild ramification in characteristic p to the (cohomological) Chow groups of singular varieties, that do not have a right to citizenship in the above-mentioned categories of motives. In order to shed some light on these problems, the quest started again from algebraic cycles.

The first attempt to give a cycle-theoretic interpretation to the K-theory of the ring $k[t]/(t^2)$ was made by S. Bloch and H. Esnault. In [8], they introduced the *additive higher Chow groups* of 0-cycles over a field and provided the first evidences that this newborn theory "with modulus" was going in the right direction. They showed that their groups agree with the absolute differentials $\Omega^n_{k/\mathbb{Z}}$, coherently with Hesselholt-Madsen computations of the K-groups of truncated polynomial algebras. Going further in that direction, K. Rülling extended their results to the case of higher modulus, showing that the additive higher Chow groups of 0-cycles (of a field) are isomorphic to the generalized deRham-Witt complexes of Hesselholt-Madsen.

Additive higher Chow groups for an arbitrary smooth variety X were introduced in [54] by J. Park and further studied by A. Krishna and M. Levine in [37], where many foundational properties were established.

To motivate their definitions, let us reconsider the relative K-groups (2). Using the homotopy invariance of K-theory of a regular scheme, we can rewrite them, with a shift by 1 in degree, as

$$K_n(X_m, X) \xrightarrow{\simeq} K_{n-1}(X \times \mathbb{A}^1_k, X_m). \tag{3}$$

In other words, we can think to the K-theory of infinitesimal thickenings relative to a nilpotent ideal as the K-theory of $X \times \mathbb{A}^1$ relative to the effective, non reduced, Cartier divisor

$$X_m \hookrightarrow X \times \mathbb{A}^1.$$

Additive higher Chow groups of X are constructed out of this idea. They are a modified version of Bloch's higher Chow groups defined by imposing some extra conditions on the behaviour of cycles "at infinity", commonly called the *modulus condition*. Conjecturally, they give a cycle-theoretic description of the relative K-groups (3) and are therefore provide a candidate definition for the motivic cohomology of the pair

$$(X \times \mathbb{A}^1_k, X \times \mathrm{Spec}(k[t]/(t^m))).$$

Motivated and inspired by their work [33] on wild ramification in class field theory for varieties over finite fields, M. Kerz and S. Saito first conceived the idea of considering a cycle theory with modulus encompassing the case of more general pairs of schemes,

$$M = (\overline{X}, D),$$

where $\overline{X}$ is a k-variety and D is an arbitrary, possibly non reduced, effective Cartier divisor on it. This project was carried over by S. Saito and the author in [5]. For any such pair, we defined a cubical abelian subgroup of Bloch's cubical cycle complex,

$$z^r(\overline{X}|D,\bullet) \subset z^r(\overline{X},\bullet)$$

and we called the n-th cohomology groups of the associated complex the *higher Chow groups of $\overline{X}$ with modulus D*,

$$\mathrm{CH}^r(\overline{X}|D,n) = \mathrm{H}^n(z^r(\overline{X}|D,*)).$$

The groups are naturally contravariantly functorial for flat maps and covariantly functorial for proper maps. This makes the complex of presheaves $z^r((-)|D_{(-)},*)$ a complex of sheaves for the étale (and thus, for the Nisnevich and the Zariski) topology on $\overline{X}$. With the usual shift convention, we obtain for every $r \geq 0$ a *motivic complex with modulus*

$$\mathbb{Z}_{\overline{X}|D}(r) \to \mathbb{Z}_{\overline{X}}(r)$$

whose (Zariski or Nisnevich) hypercohomology groups $\mathrm{H}^*_{\mathcal{M}}(\overline{X}|D,\mathbb{Z}(r))$ are ambitiously called the *relative motivic cohomology groups* of the pair $(\overline{X},D)$ (see I.1 for a recollection of Definitions and basic results).

3. The idea of considering groups of algebraic cycles subject to some modulus condition comes from afar. When $\overline{X} = C$ is a smooth projective curve over k and D is an effective divisor on it, the Chow group of 0-cycles with modulus is in fact a classical object. In [59], J-P. Serre introduced and studied the equivalence relation on the set of divisors on C defined by the "modulus" D (this explains the terminology), describing in terms of divisors the relative Picard group $\mathrm{Pic}(C,D)$, that is the group of equivalence classes of pairs $(\mathcal{L},\sigma)$, where $\mathcal{L}$ is a line bundle on C and σ is a fixed trivialization of $\mathcal{L}$ on D. When the base field k is finite and C is geometrically connected, the group

$$\varprojlim_{D} \mathrm{CH}_0(C|D)$$

is isomorphic to the idèle class group of the function field $k(C)$ of C. In [33], M. Kerz and S. Saito introduced Chow groups of 0-cycles with modulus for varieties over finite fields and used it to prove their main theorem on wildly ramified Class Field Theory. If X is smooth over k, take a compactification $X \hookrightarrow \overline{X}$, with $\overline{X}$ integral and proper over k, and a (possibly non reduced) closed subscheme D supported on $\overline{X} - X$. Then the group $\mathrm{CH}_0(\overline{X}|D)$ is defined as the quotient of the group of 0-cycles $z_0(X)$ modulo rational equivalence with modulus D, and it is used to describe the abelian fundamental group $\pi_1^{ab}(X)$. Higher Chow groups with moduli and relative motivic cohomology groups of [5] are a generalization to higher cycles of these insights, combined with the previously known definition of additive higher Chow groups.

The Chow group of 0-cycles $\mathrm{CH}_0(\overline{X}|D)$ is itself a very interesting object even behind the case of varieties over finite fields. As shown by A. Krishna and the author in [4], it is intimately connected to both the relative K_0-group $K_0(\overline{X},D)$ and to certain Chow groups of zero cycles on singular varieties, in the sense of Levine-Weibel [48]. Over an algebraically closed field (for $\overline{X}$ smooth and projective), its "universal regular quotient" is a commutative connected algebraic group of general type, displaying a unipotent part that depends heavily on the multiplicity of D.

4. If the cycle-side of the story is getting a stable foundation and highly non trivial connections to new and old "non-homotopy invariant" invariants are found (thanks notably to the work of Krishna-Park [40] and [39], Rülling-Saito [58] and Kai [31]), we are still lacking a complete description, in the spirit of Voevodksy's work, of a category of "motives with modulus".

The existence of such a category, together with a list of expected properties, was conjectured by M. Kerz [32].

A first answer to Kerz's conjectures, giving a homological approach to the problem based on Voevodksy's construction of $\mathbf{DM}^{\mathrm{eff}}(k,\mathbb{Z})$, was given in the recent work of Kahn-Saito-Yamazaki [30]. A homotopy theoretical-approach, in the spirit of Morel-Voevodsky's construction of $\mathcal{H}(k)$ and $\mathcal{SH}(k)$ is missing.

In this thesis, we present some new results in two directions. We will discuss some applications of the theory of cycles with modulus for the construction of classes in relative K-theory groups $K_n(\overline{X},D)$ for $n\geq 0$ and we prove some vanishing results about torsion 0-cycles with modulus on affine varieties. The two results are representative of ideas and techniques that, we think, might be useful for future developments of the theory. In the second part of the thesis, we try to answer to the question of constructing an (unstable) motivic homotopy category without homotopy invariance, $\overline{\mathbf{M}}\mathcal{H}(k)$, that we present as a candidate environment for representing cohomology theories "with modulus". In doing so, we are forced to generalize many techniques developed by Morel and Voevodsky in the construction of a homotopy category of a site with interval.

Leitfaden

5. In Chapter I, we start by recalling some generalities on the construction and the definitions of the higher Chow groups with modulus and the relative motivic cohomology groups. Sections I.1 and I.2 are a sample of some of the results of [5] (though our computations in weight 1 are slightly different then the one in *loc.cit.*). We explain the connection, for (quasi)affine varieties, between the relative Picard group and the codimension 1 Chow group with modulus and we show that the groups $\mathrm{CH}^1(\overline{X}|D,n)$ are trivial for $n\geq 2$. This is compatible with the classical computation by Bloch.

Going back to the case of 0-cycles, in Section I.3 we discuss a Rojtman-style result for affine varieties "with modulus". This result appeared in the joint paper [4] with a significantly different proof. Still, our approach is representative of the vast source of inspiration that the world of singular varieties gives to the cycle theory with modulus. A great intellectual debt goes to [42], as the reader will notice. Here's the formulation of our result.

Theorem (see Theorem I.3.4.7)**.** *Let X be a smooth affine k-variety of dimension at least 2 and D an effective Cartier divisor on it. Then the Chow group $\mathrm{CH}_0(X|D)$ of zero 0-cycles on X with modulus D, is torsion free, except possibly for p-torsion if the characteristic of k is $p>0$.*

The proof makes use of some classical moving arguments for 0-cycles, as well as a form of rigidity for cycles with modulus as established in [3]. Turning to "higher" 0-cycles, we discuss in I.4 the construction of a cycle class map.

Theorem (see Theorem I.4.4.10)**.** *Let $\overline{X}$ be a smooth quasi-projective variety of dimension d over a field k and let D be an effective Cartier divisor on it. Assume that the support $|D|_{\mathrm{red}}$ is a strict normal crossing divisor on $\overline{X}$. Then, there exists a cycle class map*

$$\mathrm{cyc}^{d+n}\colon \mathrm{CH}^{d+n}(\overline{X}|D,n)_{M_{\mathrm{ssup}},\mathbb{Q}}\to K_n(\overline{X};D)^{(d+n)}$$

from the higher Chow group of 0-cycles with modulus to the n-th relative K-group. For $n=0$, the map is injective if k is algebraically closed and X is affine or if k is any perfect field and $\dim(\overline{X})\leq 2$.

The M_{ssup} subscript refers to a competing definition of the modulus condition, the *strong sup modulus condition*. We refer the reader to Section I.1.2 for a discussion on this. We limit ourselves to notice here that there is no difference for $n = 0$ between this condition and the more classical condition introduced by Kerz-Saito in [33].

In the proof of this result, we have two main ingredients. First, adapting techniques developed by Levine in [43], we construct a model for a delooping of the relative K-theory spectrum $\mathbf{K}(\overline{X}, D)$. In doing this, we replace the $\mathbb{A}^1$-invariance for K-theory of regular schemes with an appropriate application of the $\mathbb{P}^1$-bundle formula. This insight is also the starting point of the construction of our motivic homotopy category, as we will explain below. It is easy at this point to get a first map

$$z^{d+n}(\overline{X}|D, n)_{\mathbb{Q}} \xrightarrow{\mathrm{cyc}^{d+n}} K_n(\overline{X}; D)^{(d+n)}, \tag{4}$$

from the group of points on $\overline{X} \times \square^n$ away from $D \times \square^n$. A more difficult task is to show that the map (4) factors through the Chow group. Here comes the second ingredient. As we will see in I.4.4, the modulus condition plays a non-trivial role, and turns out to be essential in the construction of relations in relative K_0-groups. Since understanding relative K-groups is one of the sources of motivations for the whole theory, it is in fact interesting to see how the *a priori* arbitrary relations among cycles given by the modulus condition manifest in this cycle class construction.

The injectivity part of the statement is proved in [4, Theorem 11.6 and Theorem 12.2] (in fact, for $n = 0$ we don't need to assume that the divisor has normal crossing support to get the map, see Theorem I.4.5.5).

6. In Chapter II, we start developing a machinery in the spirit of Morel-Voevodsky $\mathbb{A}^1$-homotopy theory of schemes as described in [51]. The name that we choose, *additive homotopy theory of schemes*, is reminiscent of the additive higher Chow groups of Bloch-Esnault (and then of Park, Rülling, Krishna and Levine and others), where additive refers to the additive group $\mathbb{G}_a$.

The lighthouse that guides our construction is the behaviour of K-theory for possibly singular schemes. Homotopy invariance is lost, but the projective bundle formula as formulated in [63] is still available with only mild finiteness assumptions. For any scheme S quasi-compact and quasi-separated, the K-theory spectrum $\mathbf{K}(S \times \mathbb{P}^1)$ decomposes, as module over $\mathbf{K}(S)$, as follows

$$\mathbf{K}(S \times \mathbb{P}^1) \simeq \mathbf{K}(S)[H]/(H^2), \quad H = [\mathcal{O}(-1)], H^0 = [\mathcal{O}] \in K_0(\mathbb{P}^1_{\mathbb{Z}}),$$

where the summand $[H]\mathbf{K}(S)$ is supported on the hyperplane at infinity of $S \times \mathbb{P}^1$. The pullback along the projection $\pi \colon S \times \mathbb{P}^1 \to S$ induces then an isomorphism (in the homotopy category of spectra)

$$\pi^* \colon \mathbf{K}(S) \xrightarrow{\simeq} \mathrm{hocof}(\mathbf{K}(S) \xrightarrow{\iota_{\infty,*}} \mathbf{K}(S \times \mathbb{P}^1)).$$

The homotopy cofiber computes the K-theory of $S \times \mathbb{P}^1$ modulo the K-theory of $S \simeq S \times \{\infty\}$. If we think of $\mathbb{P}^1$ as being the compactification of $\mathbb{A}^1$, with the point at infinity as boundary, we can imagine that such cofiber works as a replacement for the K-theory of the open complement $S \times \mathbb{A}^1$ (in fact, it *is* equivalent to the K-theory of the complement when S is regular). In a way, the lost homotopy invariance is found again in a form of invariance with respect to the pair $(\mathbb{P}^1, \infty)$, this time without regularity assumptions.

It looked therefore reasonable to conceive a theory where the basic objects are not *schemes* but rather *pairs of schemes*, and where the role played by $\mathbb{A}^1$ in Voevodsky's theory is played by the *closed box*[1]

$$\overline{\square} = (\mathbb{P}^1, \infty).$$

This is, with some simplifications, the point of view adopted by Kahn-Saito-Yamazaki in [30] (although with a different starting point), following original insights of Kerz-Saito. Objects in the category $\underline{\mathbf{M}}\mathbf{Sm}(k)$ of modulus pairs (see [30, Definition 1.1] or Definition 1.2.1 of Chapter II) are "partial compactifications" $\overline{X}$ (not necessarily proper) of a smooth and separated k-scheme X with the specified datum of an effective, possibly non reduced, divisor "at infinity" X^∞ with support on $\overline{X} \setminus X$. The morphisms are given by certain finite correspondences between two compactifications $\overline{X}$ and $\overline{Y}$ satisfying suitable admissibility and properness conditions, and restricting to actual morphisms of schemes between the open complements X and Y.

7. This picture, however, does not fit completely well with our initial goals. We used the word *pair* above to designate the datum $(\overline{X}, D)$ consisting of a smooth and separated k-variety and an effective Cartier divisor D on it. The relative motivic cohomology groups $\mathrm{H}^*_{\mathcal{M}}(\overline{X}|D, \mathbf{Q}(*))$ are conjectured to describe rationally the relative K-groups $K_*(\overline{X}, D)$, defined as homotopy groups of the homotopy *fiber*

$$\mathbf{K}(\overline{X}, D) = \mathrm{hofib}(\mathbf{K}(\overline{X}) \to \mathbf{K}(D))$$

and are related to other invariants, such as relative Deligne cohomology and relative deRham cohomology as defined in [5, Sections 6 and 7]. The functoriality of these invariants is for morphisms of pairs in the topological sense, i.e., morphisms $f\colon \overline{X} \to \overline{Y}$ that restrict to morphisms of the closed subschemes $f_D\colon D_X \to D_Y$. This is contrasting with the idea of compactification with boundary divisor that was coming from the analysis of K-theory of singular schemes.

We are then in front of two forces pulling in opposite directions. On one side, in the hope of finding an easy replacement for homotopy invariance, we would like to have a divisor ∂X representing the boundary of an abstract compactification, such as $(\mathbb{P}^1, \infty)$. On the other hand, we would like to have a relative theory for a divisor D, that we think as effective subscheme of $\overline{X}$, to which we attach relative invariants such as relative K-theory, Chow groups with modulus and other generalized relative cohomologies. All phenomena that are arising from fiber constructions, rather then cofiber constructions, and that have different functoriality constraints. Our solution is to incorporate both aspects in one category.

Instead of working with the category $\mathbf{Sm}(k)$ of smooth and separated schemes over a field k (Voevodksy's model) or with the category of smooth modulus pairs $\underline{\mathbf{M}}\mathbf{Sm}(k)$ over k built out from the insights of Kahn-Saito-Yamazaki, we introduce a category of *modulus data*, $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. Objects of $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ are triples

$$M = (\overline{M}, \partial M, D_M),$$

where $\overline{M}$ is a smooth and separated k-schemes and ∂M and D_M are effective Cartier divisors on $\overline{M}$. The different roles are reflected in the notation that we have chosen. The divisor ∂M is assumed to be a strict normal crossing divisor on $\overline{M}$, that we think of as a log-compactification of $\overline{M} \setminus M$. Insisting on our interpretation of the divisor at infinity as a boundary (and unlike [30]), we assume it to be always reduced. The non-reduced piece of information will then come from the second divisor D_M. A simpler category of schemes with compactifications, $\mathbf{Sm}_{\log}(k)$, is identified with the full subcategory of $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ of modulus data with empty "modulus divisor" D_M,

$$u\colon \mathbf{Sm}_{\log}(k) \hookrightarrow \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k).$$

[1]Calling a projective line a "box" might sound awkward. It is, however, a terminology in widespread use due to Kahn-Saito-Yamazaki. We see no reason to change that.

Both categories come equipped with a symmetric monoidal structure, modelled on the tensor structure of $\underline{\mathbf{M}}\mathbf{Sm}(k)$ of [30], and there are natural cd-structures inherited from the standard Zariski and Nisnevich cd-structures on the underlying categories of schemes over k (see Section II.2).

Although much of our construction can be carried out without this assumption, we will furthermore ask that $\partial M + |D_M|_{\mathrm{red}}$ forms a strict normal crossing divisor on $\overline{M}$. This restriction turns out to be useful in the definition of a K-theory realization of a modulus datum (see II.5.2.6), and is consistent with the assumptions of [5].

Morphisms in $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ are modelled on the two constraints of the "compactification divisor" and of the "relative divisor" that we mentioned before. We refer the reader to Definition II.1.4.1 for details, but we remark here that we follow the opposite convention of [30] (whence the overline notation instead of the underline notation). Out of $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ we build, in parallel with Morel-Voevodsky construction of (unstable) motivic homotopy categories in [51], a homotopy category with "modulus". We briefly sketch what are the main difficulties that we have encountered.

8. There is a first evident difficulty in trying to extend Voevodsky's formalism of sites with interval. The multiplication map

$$\mu\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$$

does not extend to a morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, but only to a rational map. It is, however, defined as correspondence, and as such satisfies a suitable admissibility condition. This is the path followed by Kahn-Saito-Yamazaki in [30], that eventually leads to the definition of the (homological) triangulated category $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}(k,\mathbb{Z})$, built as suitable localization of the derived category of complexes of presheaves on $\underline{\mathbf{M}}\mathbf{Cor}(k)$, the category of *modulus correspondences*, an enlargement of the category $\underline{\mathbf{M}}\mathbf{Sm}(k)$ introduced above.

If considering presheaves on categories of correspondences is the basic input in the construction of "derived " categories of motives, the intrinsically *linear* nature of them does not fit well with the desire of putting homotopy theory in the picture. We abandon then the idea of extending directly the set of admissible morphisms and we turn to a different approach.

The key observation is the following. There is no direct way of making the closed box $\overline{\square}$ into an interval object in the category $s\mathbf{Psh}(\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k))$ of simplicial presheaves on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, nor in the category $s\mathbf{Psh}(\mathbf{Sm}_{\log}(k))$ of simplicial presheaves over $\mathbf{Sm}_{\log}(k)$. Instead, we consider an auxiliary category $\mathbf{BSm}_{\log}(k)$, built as localization of $\mathbf{Sm}_{\log}(k)$ with respect to a suitable class of admissible blow-ups. For making sense of this, we need our base field k to admit strong resolution of singularities (see Section II.1.3).

In $\mathbf{BSm}_{\log}(k)$, the multiplication map $\mu\colon \overline{\square}\otimes\overline{\square} \to \overline{\square}$ is an acceptable morphism, and $\overline{\square}$ is naturally an interval object. Starting from it, we build a non-representable simplicial presheaf I in $s\mathbf{Psh}(\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k))$, that is a $\otimes$-interval object with respect to the convolution product of presheaves (see Sections II.1.6-II.1.8 and II.3.3). It comes naturally equipped with a map

$$\overline{\square} \xrightarrow{\eta} I,$$

and we define the *motivic model structure with modulus* $\overline{\mathbf{M}}\mathcal{M}(k)$ to be the (left) Bousfield localization of the standard (Nisnevich-local) model structure on $s\mathbf{Psh}(\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k))$ (the category of motivic spaces with modulus) to the class of maps $\mathscr{X}\otimes I \to \mathscr{X}$. The resulting homotopy category $\overline{\mathbf{M}}\mathcal{H}(k)$ is called the *unstable (unpointed) motivic homotopy category with modulus* (see II.4.7).

Even though the map η is not a weak equivalence (and does not become such after localization), every (Nisnevich-local fibrant) $\overline{\square}$-invariant simplicial presheaf $\mathscr{X}$ is I-local. Thus, the category $\overline{\mathbf{M}}\mathcal{H}(k)$ can serve the purpose of representing $\overline{\square}$-invariant theories equally well. We investigate the precise relationship between the $\overline{\square}$-theory, the I-theory and the $\mathbb{A}^1$-theory in II.3.3 and in II.4.4.

9. In constructing the I-model structure on $\overline{\mathbf{M}}\mathcal{M}(k)$, we go through a certain amount of technical work for generalizing Voevodksy's homotopy theory for a site with interval. We generalize his results in two directions. First, we deal with a (closed) monoidal structure on simplicial presheaves that is not the Cartesian product but is induced via Day convolution by the monoidal structure on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. Second, the interval object we consider is not, as remarked above, representable. Nevertheless, we draw inspiration from the work of Morel-Voevodksy [51] and of Jardine [27] to construct a replacement for the $\mathbb{A}^1$-singular functor $\mathrm{Sing}^{\mathbb{A}^1}$, that we denote

$$\mathrm{Sing}_I^{\otimes}(-)\colon \overline{\mathbf{M}}\mathcal{M}(k) \to \overline{\mathbf{M}}\mathcal{M}(k),$$

and that allows us to reprove in this generality the Properness Theorem [51, Theorem 3.2].

Theorem (see Theorem II.4.6.4 and II.4.5.1)**.** *The I-$\otimes$-localization $\overline{\mathbf{M}}\mathcal{M}(k) = \overline{\mathbf{M}}\mathcal{M}(k)^{I-loc}_{\mathrm{inj}}$ of the category of motivic spaces with modulus (over k) equipped with the local injective (for the Nisnevich topology) model structure is a proper cellular simplicial monoidal (for the Cartesian product) model category.*

The proofs are quite long and cover Sections II.4.5 and II.4.6. We conclude with a characterization of simplicial presheaves satisfying the Brown-Gersten (B.G.) property in our context (II.4.7) and with the following formal representability result.

Theorem (see Theorem II.4.9.3)**.** *Let $\mathscr{X}$ be a pointed motivic space with modulus that is Nisnevich excisive (Proposition II.4.7.4) and $\overline{\square}$-$\otimes$-invariant (in particular, I-$\otimes$-invariant). Then, for any $n \geq 0$ and any modulus datum M, we have a natural isomorphism*

$$\pi_n(\mathscr{X}(M)) \simeq [S^n \wedge (M)_+, \mathscr{X}]_{\overline{\mathbf{M}}\mathcal{H}_\bullet(k)}.$$

After having established the categorical foundations of the theory, we turn to a potential application. In II.5 we present a construction of a K-theory functor

$$\mathbf{K}\colon \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)^{\mathrm{op}} \to \mathbf{HoSpt}_{S^1}$$

from the category of modulus data to the homotopy category of S^1-spectra. For a modulus datum of the form $M = (\overline{X}, \emptyset, D_X)$, it agrees with the usual relative K-theory $\mathbf{K}(\overline{X}, D)$, and is naturally $\overline{\square}$-invariant. We don't investigate in this work the problem of rectifying our construction to obtain a strict model $\mathbf{K}(-) \in \mathbf{Spt}_{S^1}(\overline{\mathbf{M}}\mathcal{M}(k))$. We remark, however, that with such model one could conclude formally, using our machinery, the representability of relative K-theory in our category. We leave this application to a future work.

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This thesis is dedicated to my grandparents. To my grandfathers who are no longer with us, and to my grandmothers, with whom I share the sweetest memories of my childhood.

Ai miei nonni.

CHAPTER I

Algebraic cycles with moduli conditions

Notations. Throughout this Chapter, we fix a perfect field k. We denote by $\mathbf{Sch}(k)$ the category of separated schemes of finite type over k. We write $\mathbf{Sm}(k)$ for the subcategory of smooth quasi-projective k-schemes. All schemes are assumed to be separated.

1. Relative cycle complexes

1.1. A recollection on cubical objects in a category. For the reader's convenience, we fix here some notation and recall some basic facts about cubical and cocubical objects. Let **Cube** be the cubical category, i.e. the subcategory of the category of finite sets **Set**, having for objects the sets $\mathbb{1}^n = \{0,1\}^n$, for $n \in \mathbb{N}$, and generated by the following sets of morphisms:

i) $d^n_{i,\varepsilon}\colon \mathbb{1}^n \to \mathbb{1}^{n+1}$ with
$$d^n_{i,\varepsilon}(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \varepsilon, t_i, \ldots, t_n),$$
for $n \in \mathbb{N}, 1 \leq i \leq n+1, \varepsilon \in \{0,1\}$, called *faces*.

ii) $p^n_i\colon \mathbb{1}^n \to \mathbb{1}^{n-1}$ with
$$p^n_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n),$$
for $n \in \mathbb{N} \setminus \{0\}, 1 \leq i \leq n$, called *projections*.

Let $\mathcal{C}$ be a category. We call *cocubical (resp. cubical) object* any covariant functor $Q^\bullet\colon \mathbf{Cube} \to \mathcal{C}$ (resp. $Q^\bullet\colon \mathbf{Cube}^{\mathrm{op}} \to \mathcal{C}$).

1.1.1. Let $\mathcal{A}$ be an abelian category and let Q be a cubical object of $\mathcal{A}$. We attach to Q the chain complex $\hat{C}_*(Q)$, with $\hat{C}_n(Q) = Q(\mathbb{1}^n)$ for $n \in \mathbb{N}$ and differentials $d_n\colon \hat{C}_n(Q) \to \hat{C}_{n-1}(Q)$ given by $d_n = \sum_{i=1}^n (-1)^i (d^*_{i,1} - d^*_{i,0})$. We denote by $C_*(Q)$ the subcomplex of $\hat{C}_*(Q)$ with $C_n(Q) = \bigcap_{i=1}^n \mathrm{Ker}(d^*_{i,1})$. The chain complex $\hat{C}_*(Q)$ has another canonical subcomplex that we better ignore when computing homology, according to the following Lemma.

Lemma 1.1.2. *The complex $\hat{C}_*(Q)$ can be functorially decomposed as direct sum*
$$\hat{C}_*(Q) = C^{degn}_*(Q) \oplus C_*(Q)$$
having the following properties:

(1) The compositions $C^{degn}_n(Q) \to \hat{C}_n(Q) \xrightarrow{d_n} \hat{C}_{n-1}(Q)$ and $C_n(Q) \to \hat{C}_n(Q) \xrightarrow{d^*_{i,0}} \hat{C}_{n-1}(Q)$ are zero for every $n \in \mathbb{N}, 1 \leq i \leq n$.

(2) The projection morphism $p^*_i\colon \hat{C}_{n-1}(Q) \to \hat{C}_n$ factorizes through $C^{degn}_n(Q)$ for every $n \in \mathbb{N}, 1 \leq i \leq n$.

Definition 1.1.3. *Let Q be a cubical object in an abelian category $\mathcal{A}$. The complex $C_*(Q)$ of 1.1.2 is called the reduced complex associated to Q. For every $n \geq 0$, the differential $d_n\colon C_n(Q) \to C_{n-1}(Q)$ of the reduced complex is given by the alternating sum $\sum_{i=1}^n (-1)^i d^*_{i,0}$.*

1.1.4. We denote by **ECube** the category of *extended cubes* defined in [47]. This is the subcategory of **Set** that contains **Cube** and the additional morphism $\mu\colon [2] \to [1]$ that sends (x, y) to xy. Specifically, continuing with the numbering of 1.1, we include the following morphisms:

iii) $\mu_i^n\colon \mathbb{1}^n \to \mathbb{1}^{n-1}$ with $\mu_i^n(t_1,\dots,t_n) = (t_1,\dots,t_{i-1},t_it_{i+1},t_{i+2},\dots,t_n)$, for $n\in\mathbb{N}, n\neq 0,1$ and $1\le i\le n-1$, called *multiplications*.

We call *extended cocubical* (resp. extended cubical) object any covariant functor $Q\colon \mathbf{ECube}\to\mathcal{C}$ (resp. $Q\colon \mathbf{ECube}^{\mathrm{op}}\to\mathcal{C}$).

1.1.5. Let $\mathcal{A}$ be an Abelian category and let Q be an extended cubical object of $\mathcal{A}$. In presence of the extra multiplication structure, there is another canonically defined subcomplex of $\hat{C}_*(Q)$ that computes its homology. Let $N(Q)$ be the subobject of Q defined by the assignment

$$\mathbb{1}^n \mapsto \bigcap_{i=2}^{n}\mathrm{Ker}(d_{i,0}^{n-1,*})\cap\bigcap_{i=1}^{n}\mathrm{Ker}(d_{i,1}^{n-1,*}).$$

This is called the *normalized cubical object* associated to Q. The corresponding chain complex $N_*(Q)$ is a subcomplex of $\hat{C}_*(Q)$. Note that $N_*(Q)$ is actually a subcomplex of $C_*(Q)$.

Lemma 1.1.6 ([47], 1.6)**.** *Let $Q\colon \mathbf{ECube}^{op}\to\mathcal{A}$ be an extended cubical object in a (pseudo)-abelian category $\mathcal{A}$. Then the inclusion $N_*(Q)\to C_*(Q)$ is a chain homotopy equivalence.*

Remark 1.1.7. By Lemma 1.1.6, given an extended cubical object $Q\colon \mathbf{ECube}^{\mathrm{op}}\to\mathcal{A}$ in $\mathcal{A}$ we have the following description of the homology groups:

$$H_n(C_*(Q))\simeq H_n(N_*(Q)) = \frac{\bigcap_{i=1}^{n}\mathrm{Ker}(d_{i,0}^{n-1,*})\cap\bigcap_{i=1}^{n}\mathrm{Ker}(d_{i,1}^{n-1,*})}{d_{1,0}^{n,*}\left(\bigcap_{i=2}^{n+1}\mathrm{Ker}(d_{i,0}^{n,*})\cap\bigcap_{i=1}^{n+1}\mathrm{Ker}(d_{i,1}^{n,*})\right)}$$

The main cocubical object that we have in mind for this Section is the following.

Definition 1.1.8. *Let $\mathbb{P}^1_k = \mathrm{Proj}(k[Y_0,Y_1])$ be the projective line over k and let $y = Y_1/Y_0$ be the standard rational coordinate function on it. For $n\in\mathbb{N}\setminus\{0\}, 1\le i\le n$, let $p_i^n\colon (\mathbb{P}^1)^n\to(\mathbb{P}^1)^{n-1}$ be the projection onto the i-th component. We use on $(\mathbb{P}^1)^n$ the rational coordinate system $(y_1,\dots,y_n)$, where $y_i = y\circ p_i$. Let*

$$\square^n = (\mathbb{P}^1_k\setminus\{1\})^n$$

be the open (n-dimensional) box and let $\iota^n_{i,\varepsilon}\colon\square^n\to\square^{n+1}$ be the closed embedding

$$\iota^n_{i,\varepsilon}(y_1,\dots,y_n) = (y_1,\dots,y_{i-1},\varepsilon,y_i,\dots,y_n),\ \text{for } n\in\mathbb{N}, 1\le i\le n+1, \varepsilon\in\{0,\infty\},$$

of the codimension one face given by $y_i=\varepsilon$, for $\varepsilon\in\{0,\infty\}$. The assignment $n\mapsto\square^n$ defines an extended cocubical object $\square^\bullet$. We conventionally set $\square^0 = \mathrm{Spec}(k)$.

A face of the open box $\square^n$ is a closed subscheme F defined by equations of the form

$$y_{i_1}=\varepsilon_1,\dots,y_{i_r}=\varepsilon_r\ ;\ \ \varepsilon_j\in\{0,\infty\}.$$

For a face F, we write $\iota_F\colon F\hookrightarrow\square^n$ for the inclusion. We write $F_i^n\subset(\mathbb{P}^1)^n$ for the Cartier divisor on $(\mathbb{P}^1)^n$ defined by $\{y_i=1\}$ and put $F_n = \sum_{1\le i\le n}F_i^n$. Finally, we write $\overline{\square}^n$ for $(\mathbb{P}^1)^n$ and we call it the closed box.

Remark 1.1.9. A word of warning on the notation. The pair $(\overline{\square},1) = (\mathbb{P}^1,\{1\})$ should really be thought as a compactification of the affine line $\mathbb{A}^1 = \mathbb{P}^1\setminus\{1\}$, so that the modulus conditions on cycles, that we will introduce in the next Section, has to be interpreted as a condition to be satisfied "at infinity". The choice of notation that we are making here, that agrees with the one made in [5] and elsewhere, conflicts with the notation in Chapter II, where we remove the point ∞ insted of 1 from $\mathbb{P}^1$. We ask the reader to be forgiving.

1.2. Moduli conditions on higher cycles. There are a couple of useful technical Lemmas that we need to establish.

Lemma 1.2.1. *Let X be a Noetherian normal scheme, D_1, D_2 effective Cartier divisors on X such that $D_1\ge D_2$ as Weil divisors. Let $\iota\colon Y\to X$ be a closed subscheme of X such that* $\mathrm{Supp}(D_i)\cap$

$\mathrm{Ass}(\mathcal{O}_Y) = \emptyset$ *for* $i = 1, 2$*. Let* $\varphi_Y\colon Y^N \to X$ *be the composition of the normalization morphism* $Y^N \to Y_{red}$ *with the inclusion. Then* $\varphi_Y^*(D_1) \geq \varphi_Y^*(D_2)$*.*

Proof. This is a tiny generalization of [38], Lemma 2.1. By localizing at the generic points of D_1, we may assume $X = \mathrm{Spec}(A)$, where A is a discrete valuation ring. For $i = 1, 2$, let $\mathcal{O}_X(D_i)$ be the ideal sheaf of D_i. It's a free $\mathcal{O}_X$-module of rank 1. Notice that we have $D_1 \geq D_2$ if and only if $\mathcal{O}_X(D_1) \subseteq \mathcal{O}_X(D_2)$. By the assumption, $\mathrm{Supp}(D_i) \cap \mathrm{Ass}(\mathcal{O}_Y) = \emptyset$ and we have canonical isomorphisms

$$\mathcal{O}_X(D_i)_{|Y} \simeq \mathcal{O}_Y(D_{i|Y}) \text{ for } i = 1, 2$$

and so $\mathcal{O}_Y(D_{1|Y}) \subseteq \mathcal{O}_Y(D_{2|Y})$. Since the natural morphisms

$$Y^N \to Y_{red} \to Y$$

is dominant and Y^N is reduced, with finitely many irreducible components, each of these dominating an irreducible component of Y, $\varphi_Y^*(D_i)$ is an effective Cartier divisor on Y^N for $i = 1, 2$. Moreover,

$$\mathcal{O}_{Y^N}(\varphi_Y^* D_1) = \varphi_Y^* \mathcal{O}_Y(D_{1|Y}) \subseteq \varphi_Y^* \mathcal{O}_Y(D_{2|Y}) = \mathcal{O}_{Y^N}(\varphi_Y^* D_2).$$

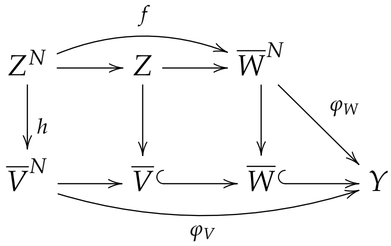
Hence $\varphi_Y^*(D_1) \geq \varphi_Y^*(D_2)$ as required. □

Lemma 1.2.2. *Let* $f\colon Y \to X$ *be a dominant morphism of normal* k*-schemes of finite type. Let* D *be a Cartier divisor on* X *and assume that every generic point of* D *is in the image of* Y*. If* $f^*(D) \geq 0$ *on* Y*, then* $D \geq 0$ *on* X*.*

Proof. See [37], Lemma 3.2. By localizing at the generic points of D, we can assume that X is the spectrum of a discrete valuation ring R. Let η be the generic point of X, x its closed point. Let $f \in R$ be a local equation for D. By choosing an uniformizer π for R, we can write $f = u\pi^n$ for $u \in R^\times$ and $n \in \mathbb{Z}$. The statement of the lemma is then reduced to show that $n \geq 0$. Since f is surjective on the set of generic points of D, the closed fibre Y_x of f is not empty. Let $y \in Y_x$ and consider $\mathrm{Spec}(\mathcal{O}_{Y,y})$. As Y is a domain, the canonical morphism $\mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$ is flat. Let $f_y\colon \mathrm{Spec}(\mathcal{O}_{Y,y}) \to X$ be the composition morphism. Notice that f_y it's also dominant, so that the image of π in $\mathcal{O}_{Y,y}$ via the induced map $R \to \mathcal{O}_{Y,y}$ it's a non zero element ϖ of the maximal ideal $\mathfrak{m}_y$ of $\mathcal{O}_{Y,y}$. By assumption, $f^*(D) \geq 0$, so $f_y^*(D)$ is also effective. Since $f_y^*(D)$ is defined, as closed subscheme of $\mathrm{Spec}(\mathcal{O}_{Y,y})$, by the image of $u\pi^n$, the condition $f_y^*(D) \geq 0$ implies that $\varpi^n \in \mathcal{O}_{Y,y}$, so that $n \geq 0$ as required. □

Lemma 1.2.3. *Let* Y *be a scheme of finite type over* k*, equidimensional over* k*,* D *and* F *two effective Cartier divisors on* Y*. Assume that* D *and* F *have no common components. Let* X *be the open complement* $X = Y - (F + D)$*. Let* W *be an integral closed subscheme of* X *and let* $V \subset W$ *be an integral closed subscheme of* W*. Let* $\overline{W}$ *(resp.* $\overline{V}$*) be the closure of* W *(resp. of* V*) in* Y*. Let* $\varphi_W\colon \overline{W}^N \to Y$ *(resp.* $\varphi_V\colon \overline{V}^N \to Y$*) be the normalization morphism. Then the inequality* $\varphi_W^*(D) \leq \varphi_W^*(F)$ *as Cartier divisors on* $\overline{W}^N$ *implies the inequality* $\varphi_V^*(D) \leq \varphi_V^*(F)$ *as Cartier divisors on* $\overline{V}^N$*.*

Proof. We use the same argument of [38, Proposition 2.4]. Let $Z = \overline{W}^N \times_{\overline{W}} \overline{V} \hookrightarrow \overline{W}^N$ and let Z^N be its normalization. By the universal property of the normalization, there exists a unique surjective morphism h making the diagram



commutative. Note that all the schemes are of finite type over the base field k, so the normalization morphisms are finite and hence h is finite too. By assumptions, we have $\varphi_W^*(D) \leq \varphi_W^*(F)$ as Cartier divisors on $\overline{W}^N$. We can apply Lemma 1.2.1 to get the inequality

$$f^*\varphi_W^*(D) \leq f^*\varphi_W^*(F) \text{ on } Z^N,$$

which is equivalent to the inequality $h^*(\varphi_V^*(F) - \varphi_V^*(D)) \geq 0$. Since h is a finite surjective morphism between normal varieties, we can apply Lemma 1.2.2 to get $\varphi_V^*(F) - \varphi_V^*(D) \geq 0$, proving the statement. □

1.2.4. Let $\overline{X}$ be a (reduced) equidimensional scheme of finite type over k, and let D be an effective Cartier divisor on $\overline{X}$. Let X be the open complement of D in $\overline{X}$. Let $n \geq$ and suppose that W is an integral closed subscheme of $X \times \square^n$. Let $\overline{W}$ denote the closure of W in $\overline{X} \times \overline{\square}^n$ and $\overline{W}^N$ denote its normalization. Write $\varphi_{\overline{W}} \colon \overline{W}^N \to \overline{X} \times \overline{\square}^n$ for the composition of the natural map $\overline{W}^N \to \overline{W}$ with the closed immersion $\overline{W} \to \overline{X} \times \overline{\square}^n$. Suppose $(D \times \overline{\square}^n) \cap \overline{W} \neq \emptyset$. We say that

i) W satisfies the $M_{\sum}$ modulus condition (the *sum-modulus condition*) if we have the inequality
$$\varphi_{\overline{W}}^*(D \times \overline{\square}^n) \leq \varphi_{\overline{W}}^*(X \times F_n).$$
ii) W satisfies the M_{ssup} modulus condition (the *strong* sup-*modulus condition*) if there exists an integer $1 \leq i \leq n$ such that we have the inequality
$$\varphi_{\overline{W}}^*(D \times \overline{\square}^n) \leq \varphi_{\overline{W}}^*(X \times F_{i,1}^n).$$

The above definitions are generalizations of [38]. The sum-modulus condition is by far the most used in literature and it's the point of view adopted in [5]. If we don't specify otherwise, we will refer to it simply as the *modulus condition*. Note that the strong sup condition is strictly stronger then the sum condition (though there are "conjectures" about the resulting cycle complexes to be quasi-isomorphic, see again [38]).

Remark 1.2.5. There are other possible *moduli conditions* on cycles that are reminiscent of older stages of the theory. For example, in [38] (generalizing the original definition of Bloch-Esnault [8]) one can find a *sup-modulus condition*, where the relevant inequality of divisors is checked on the supremum over i of $\varphi_{\overline{W}}^*(X \times F_{i,1}^n)$.

Definition 1.2.6. *Let $M \in \{M_{\sum}, M_{ssup}\}$. We write $C^r(\overline{X}|D,n)_M$ for the set of all integral closed subschemes V of codimension r in $X \times \square^n$ satisfying the following conditions:*

(1) V has proper intersection with $X \times F$ for all faces F of $\square^n$.
(2) For $n = 0$, $C^r(\overline{X}|D,0) = C^r(\overline{X})_D$ is the set of integral closed subschemes of $\overline{X}$ not intersecting D.
(3) For $n > 0$, let $\overline{V}$ be the closure of V in $\overline{X} \times \overline{\square}^n$. If $(D \times \overline{\square}^n) \cap \overline{V} \neq \emptyset$, then V satisfies the M-modulus condition.

An element in $C^r(\overline{X}|D,n)_M$ is called an admissible cycle with M-modulus D of codimension r, or simply an admissible cycle with modulus D if $M = M_{\sum}$.

Remark 1.2.7. Any of the above modulus conditions M imply that $\overline{V} \cap (D \times \overline{\square}^n) \subset \overline{X} \times F_n$ as closed subsets of $\overline{X} \times \overline{\square}^n$. Hence $\overline{V} \cap (D \times \square^n) = \emptyset$ and V is closed in $\overline{X} \times \square^n$. This implies that $C^r(\overline{X}|D,n)_M$ can be viewed as a subset of the set of all integral closed subschemes W of codimension r on $\overline{X} \times \square^n$ which intersects properly with $\overline{X} \times F$ for all faces F of $\square^n$.

Let $V \subset W$ be integral closed subschemes of $\overline{X} \times \square^n$ and assume that W satisfies one of the modulus conditions $M \in \{M_{\sum}, M_{\mathrm{ssup}}\}$. Lemma 1.2.3 shows that the same is true for V. This, together with the good position assumption on admissible cycles, proves the following Lemma.

Lemma 1.2.8. *Let $V \in C^r(\overline{X}|D,n)_M$ and let F be a face of dimension m of $\square^n$. Then the cycle $(\mathrm{id}_X \times \iota_F)^*(V)$ on $X \times F \simeq X \times \square^m$ belongs to $C^r(\overline{X}|D,m)$.*

Definition 1.2.9. *We denote by $\underline{z}^r(\overline{X}|D,n)_M$ the free abelian group on the set $C^r(\overline{X}|D,n)_M$. The standard cubical structure on the box $\square^\bullet$ of Definition 1.1.8 gives rise to an extended cubical object in the category of abelian groups*

$$\mathbb{1}^n \mapsto \underline{z}^r(\overline{X}|D,n)_M, \quad n \geq 0.$$

In particular, the groups $\underline{z}^r(\overline{X}|D,n)_M$ define a chain complex with boundary

$$\partial = \sum_{1 \leq i \leq n} (-1)^i (\partial_i^\infty - \partial_i^0),$$

where $\partial_i^\epsilon \colon \underline{z}^r(\overline{X}|D,n)_M \to \underline{z}^r(\overline{X}|D,n-1)_M$ is the pullback along $(\mathrm{id}_{\overline{X}} \times \iota_{i,\epsilon}^{n-1})^*$ for $\epsilon \in \{0,\infty\}$. We call the associated non-degenerate complex $z^r(\overline{X}|D,*)_M$ the cycle complex of $\overline{X}$ with (M)-modulus D. Its homology groups are denoted by

$$\mathrm{CH}^r(\overline{X}|D,n)_M = \mathrm{H}_n(z^r(\overline{X}|D,*)_M)$$

and called higher Chow groups of $\overline{X}$ with (M)-modulus D. As before, if M is not specified then is tacitly assumed to be M_Σ. Note that we have a natural inclusion of cycle complexes

$$z^r(\overline{X}|D,*)_{M_{\mathrm{ssup}}} \subset z^r(\overline{X}|D,*)_{M_\Sigma},$$

and therefore natural homomorphisms

$$\mathrm{CH}^r(\overline{X}|D,n)_{M_{\mathrm{ssup}}} \to \mathrm{CH}^r(\overline{X}|D,n)_{M_\Sigma} = \mathrm{CH}^r(\overline{X}|D,n).$$

Remark 1.2.10. Every admissible cycle with modulus $V \in C^r(\overline{X}|D,n)_M$ is closed in $\overline{X} \times \square^n$ as noticed in Remark 1.2.7. In particular, we can naturally view the complex $z^r(\overline{X}|D,*)$ as a subcomplex of the (non-degenerate) cubical cycle complex $z^r(\overline{X},*)$ of Bloch (see [43, Section 3] for a proof that the cubical version coincides with the original simplicial version of [7]). This gives a map

$$\mathrm{CH}^r(\overline{X}|D,n) \to \mathrm{CH}^r(\overline{X},n)$$

from higher Chow groups with modulus to Bloch's higher Chow group. Of course, when $D = \emptyset$, there is no modulus condition to check and our definition recovers the usual cubical higher Chow groups.

Remark 1.2.11. Higher Chow groups with moduli conditions are a generalization of the *additive higher Chow groups* introduced by Bloch-Esnault [8] and subsequently studied by Park [54], Rülling [57], Krishna-Levine [37] and others. For $\overline{X} = Y \times \mathbb{A}^1_k$ with Y an integral scheme of finite type over k and $D = m \cdot Y \times \{0\} \subset Y \times \mathbb{A}^1_k$ for some $m > 0$, the groups $\mathrm{CH}^r(\overline{X}|D,n)$ coincide with $\mathrm{TCH}^r(Y, n+1; m)$.

The Definition proposed above, for the sum-modulus condition, was initially conceived by Kerz and Saito as generalization to higher cycles of the Chow group of zero cycles with modulus used in [33] to study wildly ramified class field theory for varieties over finite fields. It first appeared in written form in [5, Definition 2.2].

1.3. Relative Chow groups. For $n = 0$, the groups $\mathrm{CH}^r(\overline{X}|D, n)$ introduced above admit a description in more classical terms using divisors of functions and rational equivalence. We present here our candidate definition of relative cycle groups for a pair $(\overline{X}, D)$, where $\overline{X}$ denotes as before a reduced equidimensional (though the main interest is in the case $\overline{X}$ integral) scheme of finite type over k and D is an effective Cartier divisor on $\overline{X}$. This definition was first presented in [5, Definition 3.1], and generalizes the 0-cycles case of [33, Definition 1.6].

Definition 1.3.1. *Let $(\overline{X}, D)$ be a pair as above and fix $r \geq 0$. We denote by $C^r(\overline{X})_D$ the set of integral closed subschemes W of $\overline{X}$ of codimension r such that $W \cap D = \emptyset$. Write X for the open complement $\overline{X} \setminus D$. We denote by $C^r(X)$ the set of integral closed subschemes W' of X of codimension r. Finally, we write $z^r(\overline{X}|D)$ for the free abelian group on the set $C^r(\overline{X})_D$ and $z^r(X)$ for the free abelian group on the set $C^r(X)$. For an integral scheme V and a proper closed subscheme E on it, we set*

$$G(V, E) = \varinjlim_{U \supset |E|} \Gamma(U, \mathrm{Ker}(\mathcal{O}_U^\times \to \mathcal{O}_E^\times)) \subset k(V)^\times$$

where U runs over the set of open subschemes of V containing $|E|$. We say that a rational function $f \in G(V, E)$ satisfies the modulus condition with respect to E. Note in particular that $G(V, E) = k(V)^\times$ if $|E| = \emptyset$.

1.3.2. Let $W \in C^{r-1}(X)$ and write $\overline{W} \hookrightarrow \overline{X}$ for the closure of W in $\overline{X}$. Let $\overline{W}^N \to \overline{W}$ be the normalization morphism and write $\gamma_W \colon \overline{W}^N \to \overline{X}$ for the natural map. It is clear by construction that the image of γ_W is not contained in the divisor D, and that the pullback $\gamma_W^*(D)$ gives an effective Cartier divisor on $\overline{W}^N$. The push-forward of cycles composed with the divisor map on $\overline{W}^N$ gives a group homomorphism

$$\delta_W \colon G(\overline{W}^N, \gamma_W^*(D)) \to z^r(\overline{X}|D), \quad f \mapsto (\gamma_W)_*(\mathrm{div}_{\overline{W}^N}(f)).$$

Since the function $f \in k(\overline{W}^N)^\times = k(W)^\times$ on $\overline{W}$ is a unit along $\gamma_W^*(D)$, the support of the r-cycle $(\gamma_W)_*(\mathrm{div}_{\overline{W}^N}(f)$ misses Z and the morphism δ_W is well defined.

Definition 1.3.3. *In the notations of Definition 1.3.1 and 1.3.2, we set*

$$\Phi^r(\overline{X}, D) = \bigoplus_{W \in C^{r-1}(X)} G(\overline{W}^N, \gamma_W^*(D)).$$

We define the codimension r relative Chow group of $(\overline{X}, D)$, or the codimension r Chow group of $\overline{X}$ with modulus D to be the cokernel

$$\mathrm{CH}^r(\overline{X}|D) = \mathrm{Coker}(\Phi^r(\overline{X}, D) \xrightarrow{\delta} z^r(\overline{X}|D))$$

where δ is induced by the maps δ_W defined above.

Remark 1.3.4. It has been remarked to the author that restricting to the case where the subscheme D is an effective Cartier divisor on $\overline{X}$, produces a pretty boring definition in many cases of "global" nature. For example, when $\overline{X}$ is a projective variety and D is a very ample divisor, then there are no positive dimensional closed subschemes of $\overline{X}$ missing D. Thus, the above definition would simply give a trivial group apart from the (interesting) case of 0-cycles. To remedy this situation, we proposed in [5] to consider instead of the naive relative Chow group, the hypercohomology groups of the relative motivic complex $\mathbb{Z}_{\overline{X}|D}(r)$ (see Section 1.5).

We will go back to this point later. For the moment, we limit ourselves to note that there are generalizations of Definition 1.3.3 to give a notion of relative Chow group for a pair $(\overline{X}, Z)$ where Z is a proper closed subscheme of $\overline{X}$, but we don't discuss them in this text.

The relation between the groups $\mathrm{CH}^r(\overline{X}|D)$ and the groups $\mathrm{CH}^r(\overline{X}|D, 0)$ of Definition 1.2.9 is the content of the following theorem.

Theorem 1.3.5 ([5], Theorem 3.3)**.** *There is a natural isomorphism*

$$\mathrm{CH}^r(\overline{X}|D, 0) \xrightarrow{\simeq} \mathrm{CH}^r(\overline{X}|D).$$

1.4. Easy functorialities. Let $(\overline{X}, D)$ and $(\overline{Y}, E)$ be two pairs consisting of reduced equidimensional schemes $\overline{X}$ and $\overline{Y}$ of finite type over k and effective Cartier divisors D and E respectively on them. When $\overline{X}$ and $\overline{Y}$ are smooth over k, we call the pairs $(\overline{X}, D)$ and $(\overline{Y}, E)$ *modulus pairs*. They form a nice category $\overline{\mathbf{M}}\mathbf{Sm}(k)$, that we study in details in Section 1 of Chapter II. Let $f\colon Y \to X$ be a morphism in $\mathbf{Sch}(k)$ and assume that the pull-back $f^*(D)$ is defined as effective Cartier divisor on Y. We say that f is *admissible* (resp. *coadmissible*) if there is an inequality $f^*(D) \geq E$ (resp. if there is an inequality $f^*(D) \leq E$).

Lemma 1.4.1. *Let $f\colon (\overline{Y}, E) \to (\overline{X}, D)$ be a flat admissible morphism of pairs. The flat pull-back of cycles induces a morphism of complexes*

$$f^*\colon z^r(X|D, *)_M \to z^r(Y|E, *)_M$$

compatible with composition in the sense that $f^*g^* = (g \circ f)^*$ for composable admissible flat morphisms f and g.

Proof. The proof does not really depend on the choice of the modulus condition M, and we write it, for example, for $M = M_{\mathrm{ssup}}$. Let $V \subset X \times \square^n$ be an admissible integral cycle with modulus D. By flatness, it is straightforward to check that $f^*([V]) = [f^{-1}(V)]$ is in good position with respect to the faces of $Y \times \square^n$ and that the cycle $f^*([V])$ has the right codimension, so that the only thing to do is to check that the modulus condition is respected. In doing this, we can assume that $f^{-1}(V)$ is a non-empty irreducible subscheme W of $Y \times \square^n$ and that $n > 0$, as the modulus condition for a cycle $\sum n_i V_i$ is checked on its integral components V_i. Write f for the induced morphism $\overline{Y} \times \overline{\square}^n \to \overline{X} \times \overline{\square}^n$ and write $\overline{W}$ for the closure of W in $\overline{Y} \times \overline{\square}^n$ (resp. $\overline{V}$ for the closure of V in $\overline{X} \times \overline{\square}^n$). Note that the induced morphism $\overline{W} \to \overline{V}$ is dominant (and equidimensional). By the universal property of the normalization, there exists a unique dominant morphism $h\colon \overline{W}^N \to \overline{V}^N$ making the diagram

(1.4.1.1)
$$\begin{array}{ccc} \overline{W}^N & \xrightarrow{\varphi_{\overline{W}}} & \overline{Y} \times \overline{\square}^n \\ \downarrow h & & \downarrow \\ \overline{V}^N & \xrightarrow{\varphi_{\overline{Z}}} & \overline{X} \times \overline{\square}^n \end{array}$$

commutative. As V satisfies by assumption the modulus condition with respect to D, there is an integer $1 \leq i \leq n$ such that $\varphi_{\overline{Z}}^*(D \times \overline{\square}^n) \leq \varphi_{\overline{Z}}^*(\overline{X} \times F_i^n)$. Pulling back via h^* and using (1.4.1.1), we deduce the inequality $\varphi_{\overline{W}}^*(E \times \overline{\square}^n) \leq \varphi_{\overline{W}}^*(f^*(D) \times \overline{\square}^n) \leq \varphi_{\overline{W}}^*(\overline{Y} \times F_i^n)$ as required. □

Remark 1.4.2. The choice of calling a morphism of pairs admissible when the inequality $f^*(D) \geq E$ is satisfied is, in a way, justified by the previous Lemma. This direction is also compatible with the natural functoriality of relative K-theory, as explained in Chapter II, but constrasts with the choice of [30]. We will discuss more about this discrepancy in Section 1 of Chapter II.

Lemma 1.4.3. *Let $f\colon (\overline{Y}, E) \to (\overline{X}, D)$ be a proper coadmissible morphism of pairs. Then there is a well defined push-forward map of cycles*

$$f_*\colon z^{r+\dim X-\dim Y}(Y|E,*)_M \to z^r(\overline{X}|D,*)_M.$$

Proof. As above, we simply give the proof in the M_{ssup} case and we omit the subscript M in what follows. Let Z be an integral relative cycle, $[Z] \in z^{r+\dim X-\dim Y}(Y|E,*)$ and consider the image $f(Z)$ as a closed integral subscheme of X. The push forward cycle $f_*(Z)$ is classically given by

$$\deg(Z/f(Z))[f(Z)] \in z^r(X \times \square^n),$$

where $\deg(Z/f(Z))$ is equal to 0 if $\dim(f(Z)) < \dim(Z)$ and is given by the degree of the field extension $R(f(Z)) \subset R(Z)$ if $\dim(f(Z)) = \dim(Z)$. Suppose then that $\dim(f(Z)) = \dim(Z)$. We have to check that $f(Z) \in z^r(\overline{X}|D,n)$. The good position condition for $f(Z)$ is checked as in [7, Proposition 1.3]. As for the modulus condition, let $\overline{Z}$ be again the closure of Z in $\overline{Y} \times \overline{\square}^n$, $\overline{Z}^N$ its normalization and $\varphi_{\overline{Z}}\colon \overline{Z} \to \overline{Y} \times \overline{\square}^n$ the canonical map. Similarly, write $\overline{f(Z)}$ for the closure of $f(Z)$ $\overline{f(Z)}^N$ for its normalization and $\varphi_{\overline{f(Z)}}$ for the corresponding map to $\overline{X} \times \overline{\square}^n$. Since f is proper, we have the equality $f(\overline{Z}) = \overline{f(Z)}$ and the composite morphism $\overline{Z}^N \to \overline{f(Z)}$ is a surjective. By the universal property of the normalization, there exists a unique surjective morphism $h\colon \overline{Z}^N \to \overline{f(Z)}^N$ making the diagram

$$\begin{array}{ccc}
\overline{Z}^N & \xrightarrow{\varphi_{\overline{Z}}} & \overline{Y} \times \overline{\square}^n \\
\downarrow & & \downarrow \\
\overline{f(Z)}^N & \xrightarrow{\varphi_{\overline{f(Z)}}} & \overline{X} \times \overline{\square}^n
\end{array}$$

commutative. As Z satisfies the modulus condition M_{ssup} with respect to E, there exists an integer $1 \le i \le n$ such that

$$\begin{aligned}
h^*(\varphi^*_{\overline{f(Z)}}(D \times \overline{\square}^n)) &= \varphi^*_{\overline{Z}}(f \times \mathrm{id})^*(D \times \overline{\square}^n) = \varphi^*_{\overline{Z}}(f^*D \times \overline{\square}^n) \\
&\le \varphi^*_{\overline{Z}}(E \times \overline{\square}^n) \le \varphi^*_{\overline{Z}}(X \times F^n_i) \\
&= \varphi^*_{\overline{Z}}((f \times \mathrm{id})^*(\overline{X} \times F^n_i)) = h^*(\varphi^*_{\overline{f(Z)}}(\overline{X} \times F^n_i)).
\end{aligned}$$

Again, we get the result by Lemma 1.2.2. □

Remark 1.4.4. Lemmas 1.4.1 and 1.4.3 represent how far we can go without much technology. More serious functorialities are known only in specific situations, and require non-trivial moving lemmas.

1.5. Relative motivic cohomology. Consider again a pair $(\overline{X}, D)$ consisting of an equidimensional scheme X over k and an effective Cartier divisor D on it. For $U \to \overline{X}$ flat, write D_U for the divisor $D \times_{\overline{X}} U$ on U. By Lemma 1.4.1, the assignment

$$(U \xrightarrow{\text{étale}} \overline{X}) \mapsto z^r(U|D_U,*)$$

defines a complex of sheaves on the étale site over $\overline{X}$, and therefore on the small Nisnevich and Zariski site of $\overline{X}$ (we suppress the subscript M and we ignore the choice of the modulus condition). For τ any of these topologies and A an abelian group, we define

$$A_{\overline{X}|D}(r)_\tau = (z^r(-|D_{(-)},*)_\tau \otimes A)[-2r]$$

and call it the *relative motivic complex* of the pair $(\overline{X}, D)$ (see [5, 2.1.3]). The complex $A_{\overline{X}|D}(r)_\tau$ is unbounded below.

Definition 1.5.1. *The motivic cohomology of the pair $(\overline{X}, D)$ or the motivic cohomology of $\overline{X}$ with modulus D (with coefficients in A) is defined as the hypercohomology of the complex of sheaves $A_{\overline{X}|D}(r)_\tau$,*

$$\mathrm{H}^n_{\mathcal{M},\tau}(\overline{X}|D, A(r)) = \mathbb{H}^n_\tau(\overline{X}, A_{\overline{X}|D}(r)_\tau).$$

1.5.2. When $D = \emptyset$, the complex of presheaves $U \mapsto z^r(U|\emptyset, *) = z^r(U,*)$ on $\overline{X}_{\mathrm{Zar}}$ satisfies the Mayer-Vietoris property (see [7, Section 3] for the statement and [46] for the proofs) and therefore has Zariski descent, in the sense that the natural maps

$$\mathrm{CH}^r(\overline{X}, 2r-n) = \mathrm{H}^n(z^r(\overline{X},*)[-2r]) \xrightarrow{\simeq} \mathbb{H}^n_{\mathrm{Zar}}(\overline{X}, \mathbb{Z}_{\overline{X}}(r)_{\mathrm{Zar}})$$

are isomorphisms. When $D \neq \emptyset$, the situation is considerably more intricate. The natural map

$$\mathrm{CH}^r(\overline{X}|D, 2r-n) = \mathrm{H}^n(z^r(\overline{X}|D,*)[-2r]) \to \mathbb{H}^n_{\mathrm{Zar}}(\overline{X}, \mathbb{Z}_{\overline{X}|D}(r)_{\mathrm{Zar}}) = \mathrm{H}^n_{\mathcal{M},\mathrm{Zar}}(\overline{X}|D, \mathbb{Z}(r))$$

has been object of several speculations and, in general, is not expected to be an isomorphism. An evident example is the case where $\overline{X}$ is a smooth projective variety and D is a very ample divisor on it. For $n = 0$, there are simply no cycles missing D, so that the group $\mathrm{CH}^r(\overline{X}|D, 0) = 0$ for $r < \dim \overline{X}$ (see also Remark 1.3.4), while, in general, the groups $\mathrm{H}^{2r}_{\mathcal{M},\mathrm{Zar}}(\overline{X}|D, \mathbb{Z}(r))$ have no reason to be zero (we discuss the case $r = 1$ in the next Section).

To get a more serious example, which illustrates the rather pathological nature of the "naive" cycle groups with modulus (i.e., the actual homology groups of $z^r(\overline{X}|D,*)$ and not the relative motivic cohomology groups defined above), we mention the following result, communicated to the author by Amalendu Krishna (and recently published in [36]).

Proposition 1.5.3 (A. Krishna)**.** *Let k be an algebraically closed field of characteristic zero with infinite transcendence degree over the field of rational numbers. Let Y be a connected projective curve over k of genus $g \geq 1$. For $m \geq 2$, let $D_m = \mathrm{Spec}(k[t]/(t^m)) \hookrightarrow \mathbb{A}^1_k$. Then for any inclusion $i\colon \{P\} \hookrightarrow Y$ of a closed point, the localization sequence*

$$\mathrm{CH}_0(\{P\} \times \mathbb{A}^1 | \{P\} \times D_m) \xrightarrow{i_*} \mathrm{CH}_0(Y \times \mathbb{A}^1_k | Y \times D_m) \xrightarrow{j^*} \mathrm{CH}_0(Y \setminus \{P\} \times \mathbb{A}^1_k | Y \setminus \{P\} \times D_m) \to 0$$

fails to be exact at $\mathrm{CH}_0(Y \times \mathbb{A}^1_k | Y \times D_m)$.

On the bright side, we mention the following important result recently obtained by Wataru Kai (see [31, Theorem 1.4]).

Theorem 1.5.4 (W. Kai)**.** *Let $(\overline{X}, D)$ and $(\overline{Y}, E)$ be pairs of equidimensional schemes of finite type over k and effective divisors on them. Assume that $\overline{Y} \setminus E$ is smooth. Let $f\colon (\overline{X}, D) \to (\overline{Y}, E)$ be an admissible morphism of pairs. Then there are natural maps of abelian groups*

$$f^*\colon \mathrm{H}^n_{\mathcal{M},\mathrm{Nis}}(\overline{Y}|E, \mathbb{Z}(r)) \to \mathrm{H}^n_{\mathcal{M},\mathrm{Nis}}(\overline{X}|D, \mathbb{Z}(r)).$$

This makes Nisnevich motivic cohomology with modulus controvariantly functorial for any map of smooth schemes with effective Cartier divisors.

2. Some computations in weight 1

The purpose of this section is to compute the groups of $\mathrm{CH}^1(\overline{X}|D; n)$, under some assumptions. The computations follow the lines of [5, Section 4], with some differences in the arguments. The result in *loc.cit.* is a consequence of a local computation, and gives the following Theorem.

Theorem 2.0.1 ([5], see Theorem 4.3)**.** *Let $\overline{X}$ be a regular connected k-variety. Then there is a quasi-isomorphism of complexes of Zariski sheaves on $\overline{X}$*

$$\mathbb{Z}_{\overline{X}|D}(1) \simeq \mathcal{O}^\times_{\overline{X}|D}[-1] = \mathrm{Ker}(\mathcal{O}^\times_{\overline{X}} \to \mathcal{O}^\times_D)[-1].$$

We prove here essentially the same result, with an explicit comparison in the normal quasi-affine case between $\mathrm{CH}^1(\overline{X}|D,0)$ and the relative Picard group of the pair $(\overline{X},D)$. Theorem 2.0.1 is the analogue for motivic cohomology with modulus of Bloch's computation of motivic cohomology in weight 1, $\mathbb{Z}_{\overline{X}}(1) \simeq \mathcal{O}_{\overline{X}}^{\times}[-1]$, and proves one of the first expected properties of our motivic cohomology groups.

2.1. Relative Picard and cycles with modulus. In view of Theorem 1.3.5, for $n=0$ we denote $\mathrm{CH}^1(\overline{X}|D,0)$ simply by $\mathrm{CH}^1(\overline{X}|D)$. We want to compare this group with two other objects, namely the group of relative Cartier divisors and the relative Picard group, extending the isomorphisms

$$\mathrm{Pic}(X) \xleftarrow{\sim} \mathrm{Div}(X) \xrightarrow{\sim} \mathrm{CH}^1(X)$$

for X smooth. We start with recalling some basic definitions and properties.

Definition 2.1.1. *Let X be a scheme and let Y be a closed subscheme of X. We denote by* $\mathrm{Pic}(X,Y)$ *the group of isomorphism classes of pairs $(\mathcal{L},t)$, where $\mathcal{L}$ is a line bundle on X and $t\colon \mathcal{L}_{|Y} \xrightarrow{\sim} \mathcal{O}_Y$ is a trivialization of $\mathcal{L}$ over Y. The operation is given by the tensor product.*

Let $j\colon Y \hookrightarrow X$ be the closed embedding and consider the sequence of sheaves

$$1 \to (1+\mathcal{I}_Y)^{\times} \to \mathcal{O}_X^{\times} \to j_*\mathcal{O}_Y^{\times} \to 1.$$

defining $(1+\mathcal{I}_Y)^{\times}$ as kernel of the restriction map. Note that the sequence is exact both in the étale and in the Zariski topology. By [62], Lemma 2.1, we have

$$\mathrm{H}^1_{Zar}(X,(1+\mathcal{I}_Y)^{\times}) \simeq \mathrm{H}^1_{\text{ét}}(X,(1+\mathcal{I}_Y)^{\times}) \simeq \mathrm{Pic}(X,Y)$$

and the evident exact sequence

$$\Gamma(X,\mathcal{O}_X^{\times}) \to \Gamma(Y,\mathcal{O}_Y^{\times}) \to \mathrm{Pic}(X,Y) \to \mathrm{Pic}(X) \to \mathrm{Pic}(Y).$$

2.1.2. Let $\mathcal{K}_X$ be the sheaf of stalks of meromorphic functions on X and let $D=(U_\alpha,f_\alpha) \in \mathrm{H}^0(X,\mathcal{K}_X^{\times}/\mathcal{O}_X^{\times})$ be a Cartier divisor on X. Recall that the support of D, denoted $\mathrm{Supp}(D)$, is defined to be the set of points $x\in X$ such that $D_x \neq 1$, i.e. the union of all subvarieties Z of X such that a local equation for D in the local ring $\mathcal{O}_{X,Z}$ is not a unit. The set $\mathrm{Supp}(D)$ is a closed subset of X. A relative Cartier divisor on X with respect to Y is a Cartier divisor D on X such that $\mathrm{Supp}(D)\cap Y=\emptyset$. We denote by $\mathrm{Div}(X,Y)$ the group of relative Cartier divisors.

We have a natural group homomorphism $\mathrm{Div}(X,Y)\to \mathrm{Pic}(X,Y)$. Indeed, let $\mathcal{O}_X(D)$ be the line bundle on X determined by the Cartier divisor D. Let U be the complement of $\mathrm{Supp}(D)$ in X. Then there is a canonical, nowhere vanishing, section of $\mathcal{O}_X(D)$ over U, which we denote by s_D. The assignment $D\mapsto (\mathcal{O}_X(D),s_D)$ gives the required homomorphism between the group of relative divisors and the relative Picard group.

Remark 2.1.3. Suppose X integral. Then the image of this morphism is given by the isomorphism classes of pairs $(\mathcal{L},t)$ such that t admits an extension t_U to some open subset U of X containing Y (see [16], 2.2.1 and 2.2). When Y has an affine open neighbourhood U in X, every trivialization $t\colon \mathcal{L}_{|Y} \xrightarrow{\sim} \mathcal{O}_Y$ extends to an open subset $Y\subset V\subset U$. Therefore the morphism $\mathrm{Div}(X,Y)\to \mathrm{Pic}(X,Y)$ is surjective.

2.1.4. We go back to the notations of the previous sections. Let $(\overline{X},D)$ be a pair consisting of a normal integral scheme $\overline{X}$ and an effective Cartier divisor D. As in 1.3.1, write $G(\overline{X},D)$ for the subgroup of $k(\overline{X})^{\times}$ consisting of rational functions that are regular in a neighborhood of D and congruent to 1 at every point of D. Suppose that D has an affine open neighbourhood in $\overline{X}$ (essentially, this forces $\overline{X}$ to be quasi-affine as long as $\dim(\overline{X})>1$). Then we have an exact sequence (see, for example, [62, Lemma 2.3])

$$G(\overline{X}, D) \xrightarrow{\text{div}} \mathrm{Div}(\overline{X}, D) \to \mathrm{Pic}(\overline{X}, D) \to 0$$

giving the isomorphism $\mathrm{Div}(\overline{X}, D)/G(\overline{X}, D) \xrightarrow{\sim} \mathrm{Pic}(\overline{X}, D)$.

The natural morphism

$$\mathrm{H}^0(\overline{X}, \mathcal{K}^\times_{\overline{X}}/\mathcal{O}^\times_{\overline{X}}) \to z^1(\overline{X}) \quad E \mapsto \sum_{x \in \overline{X}^{(1)}} \mathrm{mult}_x(E)[\overline{\{x\}}]$$

restricts to a morphism

$$\text{(2.1.4.1)} \qquad \mathrm{cl}\colon \mathrm{Div}(\overline{X}, D) \to z^1(\overline{X}|D, 0) = z^1(\overline{X}|D).$$

Indeed, let $E = (U_\alpha, f_\alpha) \in \mathrm{H}^0(\overline{X}, \mathcal{K}^\times_{\overline{X}}/\mathcal{O}^\times_{\overline{X}})$ be a relative Cartier divisor on $\overline{X}$ with respect to D. Being $\overline{X}$ regular in codimension 1, for every $x \in \overline{X}^{(1)}$ we have $\mathrm{mult}_x(E) = v_{\mathcal{O}_{\overline{X},x}}(f_\alpha)$ for $x \in U_\alpha$, and this integer does not depend on α. For $y \in D$ and $y \in U_\alpha$, we have $(f_\alpha)_y \in \mathcal{O}^\times_{\overline{X},y}$ and thus for every $x \in \overline{X}^{(1)}$ such that $D \cap \overline{\{x\}} \neq \emptyset$, we have $v_{\mathcal{O}_{\overline{X},x}}(f_\alpha) = 0$, showing that the morphism (2.1.4.1) is well defined.

Proposition 2.1.5. *Let $\overline{X}$ be a normal integral k-variety and D an effective Cartier divisor in $\overline{X}$. Assume that D has an affine open neighborhood in $\overline{X}$. Then the morphism* (2.1.4.1) *induces an isomorphism*

$$\text{(2.1.5.1)} \qquad \mathrm{Pic}(\overline{X}, D) \xleftarrow{\simeq} \mathrm{Div}(\overline{X}, D)/G(\overline{X}, D) \xrightarrow{\simeq} \mathrm{CH}^1(\overline{X}|D)$$

Proof. We first show that the map is well defined. The proof is elementary: for $n = 1$ the groups $z^1(\overline{X}|D, 1)_{M_\Sigma}$ and $z^1(\overline{X}|D, 1)_{M_{\mathrm{ssup}}}$ are the same, and we denote them simply by $z^1(\overline{X}|D, 1)$. For every $f \in G(\overline{X}, D)$, we have a rational map $f\colon \overline{X} \dashrightarrow \mathbb{P}^1$. If U denotes the domain of definition of f, $\mathrm{codim}_{\overline{X}} \overline{X} \setminus U \geq 2$, and since f is regular in a neighbourhood of D, we have $D \subset U$. Replacing $\overline{X}$ by U we can assume that f determines a morphism $f\colon \overline{X} \to \mathbb{P}^1$. Let $j\colon \Gamma_f \to \overline{X} \times \mathbb{P}^1$ be the closed embedding of the graph of f in $\overline{X} \times \mathbb{P}^1$ and let $\Gamma'_f = \Gamma_f \setminus (\Gamma_f \cap \overline{X} \times F^1_1)$. Note that Γ_f is a cycle of codimension 1 in $\overline{X} \times \square^1$, in good position with respect to the faces (as $f \in G(\overline{X}, D)$ implies that f cannot be constantly 0) and

$$[\mathrm{div}(f)] = \sum_{x \in X^{(1)}} v_x(f)[\overline{\{x\}}] = [f^{-1}(0)] - [f^{-1}(\infty)] = (\partial^0 - \partial^\infty)(\Gamma'_f) = (\partial^0 - \partial^\infty)(\Gamma_f),$$

so that $[\mathrm{div}(f)] = 0$ in $\mathrm{CH}^1(X|D)$ if and only if $\Gamma'_f \in z^1(\overline{X}|D, 1)$, i.e. if it satisfies the modulus condition. Since the projection p_1 on the first component induces an isomorphism between Γ_f and $\overline{X}$, the graph of f is a normal subscheme of $X \times \mathbb{P}^1$ and we are left to check the inequality $\Gamma_f \cdot (D \times \mathbb{P}^1) \leq \Gamma_f \cdot (\overline{X} \times F_1)$ of Weil divisors on Γ_f. Since every point of codimension 1 in $\Gamma_f \cap (D \times \mathbb{P}^1)$ is the restriction to Γ_f of the pullback $p_1^*(\eta) = \eta \times \mathbb{P}^1$ of a generic point of D, it's enough to check that the previous holds at every generic point η of D_{red}.

Up to localization at η, we can assume that $\overline{X}$ is the spectrum of a discrete valuation ring A. Let t be a generator for the maximal ideal $\mathfrak{m}$ of A and write $D = \mathrm{Spec}(A/(t^n))$. Then every $f \in G = \mathrm{Ker}(A^\times \to (A/(t^n))^\times)$ is of the form $f = 1 + t^n g$ for some $g \in A$. Let y be a rational coordinate on $\mathbb{P}^1$ and $y - 1 - t^n g$ be the defining equation of the graph of f in $\overline{X} \times \mathbb{P}^1 \setminus \{\infty\}$.

By construction, the ring $A[y]/(y - 1 - t^n g)$ is a discrete valuation ring, with uniformizer t. From this description we can immediately see that the divisor $D \times \mathbb{P}^1$ meets the graph of f uniquely in the point $D \times \{y = 1\}$, with multiplicity $m \geq n$, and that the graph of f intersects the divisor $ol X \times F_1$ in that point with the same multiplicity, showing that the modulus condition is satisfied. The morphism 2.1.5.1 is then well defined.

Since every generator of $\mathrm{CH}^1(\overline{X}|D)$ is already in the image of $\mathrm{Div}(\overline{X},D) \to z^1(\overline{X}|D)$, we are left to show the injectivity of (2.1.5.1). For W an integral cycle in $z^1(X|D,1)$, we actually show that W has the same image in $z^1(X|D,0)$ as the graph of a rational function on $\overline{X}$ that takes value 1 in D and that is regular in a neighbourhood of D (i.e. a function in G). Let $\overline{W}$ be the closure of W in $\overline{X}\times\mathbb{P}^1$, $\pi\colon \overline{W}\to\overline{X}$ be composition of the closed embedding $j\colon \overline{W}\to\overline{X}\times\mathbb{P}^1$ with the first projection p_1. We can assume that W is a non degenerate cycle and that the map π is generically finite.

Let $k(\overline{W})^\times$ be the function field of W and let $N = N_{\overline{W}/\overline{X}}\colon k(\overline{W})^\times \to k(\overline{X})^\times$ be the norm. On $\overline{W}$ we have a canonical rational function y, namely the function corresponding to the morphism $\overline{W}\to\mathbb{P}^1$, obtained by composition of j with the second projection p_2. Note that we have

$$[\mathrm{div}(N(y))] = \pi_*[\mathrm{div}(y)] = \pi_*([y^{-1}(0)]-[y^{-1}(\infty)]) = (\partial^0-\partial^\infty)(W)$$

(see [16], Prop. 1.4). We argue as above to show that the graph of $N(y)$ is in $z^1(\overline{X}|D,1)$. Let η be a generic point of D_{red}, $A=\mathcal{O}_{\overline{X},\eta}$ the local ring of $\overline{X}$ at η, t a generator for the maximal ideal $\mathfrak{m}$ of A. By replacing $\overline{X}$ with $\mathrm{Spec}(A)$ we can assume $D=\mathrm{Spec}(A/(t^n))$, $D_{\mathrm{red}}=\mathrm{Spec}(A/(t))$. By removing the point at infinity of $\mathbb{P}^1$ we can further replace $\overline{W}$ by $\mathrm{Spec}(A[y]/(f))$, for $f\in A[y]$. The canonical rational function on W is therefore given by the class of y in $R=A[y]/(f)$. Note that R is a one dimensional semilocal ring, finite over A.

Write R^N for its normalization. The divisor $D\times\mathbb{P}^1\setminus\{\infty\}$ (resp. $\overline{X}\times F_1$) restricts to the zero locus of the ideal of R^N generated by t^n (resp. of the ideal of R^N generated by $y-1$). Let $\pi_1,\dots,\pi_r$ be the maximal ideals of R^N, with generators $z_1,\dots,z_r$ respectively and corresponding valuations $v_1,\dots,v_r$. The modulus condition on W implies then that there exist $g_i,\dots,g_r\in k(R^N)^\times$ and non negative integers $m_1,\dots,m_r$ such that $y-1=g_iz_i^{m_in}$, where $v_i(t)=m_i$. Hence we can find $g\in R^N$ such that $y-1=gt^n$ in R^N, i.e. $y\equiv 1 \mod t^n$ in R^N. To complete the proof we are left to show that the norm $N(y)$ as element of A is congruent to 1 modulo t^n, i.e. $N(y)\in G$. But this is clear as $N(y)=\det(I+t^nM_g)$, where M_g is the matrix of multiplication by g. □

2.2. An interlude: 0-cycles on curves. The only case of a pair $(\overline{X},D)$ with $\overline{X}$ proper over k and D a divisor admitting an affine open neighborhood is 1-dimensional. For curves, the Chow group $\mathrm{CH}_0(\overline{X}|D)$ (or $\mathrm{CH}^1(\overline{X}|D)$) is really not a new object, and its history can be traced back to the construction of generalized Jacobian varieties of curves, as presented by Serre in [59]. We discuss here briefly how related is this group with relative K-theory by constructing a cycle map from the group $\mathrm{CH}_0(\overline{X}|D)$ of a smooth curve $\overline{X}$ with modulus D to the relative K_0 group $K_0(\overline{X},D)$. This is a very easy and special instance of a general construction that we discuss in [4].

2.2.1. Let $\overline{X}$ be a smooth connected curve over k, P a closed point of $\overline{X}$, $n\in\mathbb{Z}_{>0}$ and $j\colon D=nP\hookrightarrow\overline{X}$. Let S be a finite set of closed points of $\overline{X}$ not in D. For a quasi-projective k-scheme Y, we denote by $K(Y)$ the K-theory space of Y (we discuss this in detail in Chapter II). We have the diagram

(2.2.1.1)
$$\begin{array}{ccccc} K(\overline{X},D) & \longrightarrow & K(\overline{X}) & \xrightarrow{j^*} & K(D) \\ \downarrow & & \downarrow & & \| \\ K(\overline{X}\setminus S,D) & \longrightarrow & K(\overline{X}\setminus S) & \longrightarrow & K(D) \end{array}$$

with lines homotopy fibre sequences. Let V be the set of all closed points of X different from P. By taking the colimit on S in (2.2.1.1) and passing to the long exact sequence of homotopy groups we get the commutative diagram

(2.2.1.2)
$$\begin{array}{ccccccc} K_1(\overline{X}\setminus V, D) & \xrightarrow{\partial'} & \coprod_{x\in V} K_0(k(x)) & \longrightarrow & K_0(\overline{X}, D) & \longrightarrow & K_0(\overline{X}\setminus V, D) \to 0 \\ \downarrow & & \| & & \downarrow & & \downarrow \\ K_1(\mathcal{O}_{\overline{X},P}) & \xrightarrow{\partial} & \coprod_{x\in V} K_0(k(x)) & \longrightarrow & K_0(\overline{X}) & \longrightarrow & K_0(\overline{X}\setminus V) \to 0 \end{array}$$

as well as the exact sequence

$$K_1(X\setminus V; D) \to K_1(\mathcal{O}_{X,P}) \to K_1(D) \to K_0(X\setminus V; D).$$

Since $\mathcal{O}_{X,P}$ is local, the morphism $\mathcal{O}_{X,P}^\times = K_1(\mathcal{O}_{X,P}) \to K_1(D) = \mathcal{O}_{D,P}^\times$ is surjective and from the exact sequence

$$1 \to (1+\mathfrak{m}_P^n)^\times \to \mathcal{O}_{X,P}^\times \to \mathcal{O}_{D,P}^\times \to 1$$

(where $\mathfrak{m}_P$ denotes the maximal ideal of $\mathcal{O}_{X,P}$) we get that the map $K_1(X\setminus V, D) \to K_1(\mathcal{O}_{X,P})$ factors through $(1+\mathfrak{m}_P^n)^\times$.

By (2.2.1.2) we have then the exact sequence

$$(1+\mathfrak{m}_P^n)^\times \xrightarrow{\partial} \coprod_{x\in V} K_0(k(x)) \simeq z_0(X|D) \to K_0(X, D) \to K_0(X\setminus V; D) \to 0 \qquad (2.2.1.3)$$

where the composite morphism $(1+\mathfrak{m}_P^n)^\times \to z_0(X|D)$ coincides with the divisor map.

By Proposition 2.1.5, we get then an injective morphism

$$cyc\colon \mathrm{CH}_0(X|D) = \mathrm{CH}^1(X|D) \to K_0(X, D) \qquad (2.2.1.4)$$

those cokernel is $K_0(X\setminus V, D)$. On the other hand, we have already noticed that the sequence

$$0 \to K_0(\mathrm{Spec}(\mathcal{O}_{X,P}); D) = K_0(X\setminus V; D) \to K_0(\mathcal{O}_{X,P}) \to K_0(D) \to 0$$

is exact. But since $\mathcal{O}_{X,P}$ is local, $K_0(\mathcal{O}_{X,P}) \xrightarrow{\sim} K_0(D)$ and therefore $K_0(\mathrm{Spec}(\mathcal{O}_{X,P}); D) = 0$, showing that the cycle map (2.2.1.4) is an isomorphism.

Remark 2.2.2. Let X be a smooth connected curve over k. We have the classical isomorphism

$$K_0(X) \xrightarrow{\sim} \mathrm{CH}^0(X) \oplus \mathrm{CH}^1(X) \xrightarrow{\sim} \mathbb{Z} \oplus \mathrm{Pic}(X).$$

The previous computation shows that we have an analogous result

$$K_0(X, D) \xrightarrow{\sim} \mathrm{CH}^0(X|D) \oplus \mathrm{CH}^1(X|D) \xrightarrow{\sim} \mathrm{Pic}(X; D)$$

by noticing that $\mathrm{CH}^0(X|D) = 0$ by definition when X is connected. This is not an accident that happens only in dimension 1. We will discuss more about the relationship between Chow groups with moduli and relative K-theory in Section 4.

2.3. A description of relative cycles and a vanishing result. It is convenient to give an alternative description of the modulus conditions for cycles (see [5, Section 3]). For $1 \le i \le n$, let $\overline{\square}_i^n$ be the product $\square \times \ldots \times \mathbb{P}^1 \times \ldots \times \square$ with $\mathbb{P}^1$ in the i-th position, i.e., the scheme obtained by removing from $(\mathbb{P}^1)^n$ the divisor $\sum_{i\neq j} F_{j,i}^n$. We will refer to $\overline{\square}_i^n$ as the closure of the cube $\square^n$ in the i-th direction. For simplicity, we will denote by F_i^n the face $F_i^n \cap \overline{\square}_i^n$. Let p_i denote the projection $p_i\colon \overline{\square}_i^n \to \square^{n-1}$.

Lemma 2.3.1. *Let $V \in C^r(X\times\square^n)$ be an integral cycle, and $\overline{V}$ the closure of V in $\overline{X}\times(\mathbb{P}^1)^n$. For $1\le i\le n$, let $\overline{V}_i$ be the closure of V in $\overline{X}\times\overline{\square}_i^n$, $\overline{V}_i^N$ be its normalization. Let $\varphi_{\overline{V}_i}\colon \overline{V}_i^N \to \overline{X}\times\overline{\square}_i^n$ be the natural map. Then the the condition (3) of Definition 1.2.6 for $M = M_\Sigma$ implies the following condition:*

(3)′ *The following inequality as Cartier divisors holds for all $1 \leq i \leq n$:*

$$\varphi^*_{\overline{V}_i}(D \times \overline{\square}^n_i) \leq \varphi^*_{\overline{V}_i}(\overline{X} \times F^n_i). \tag{2.3.1.1}$$

The converse implication holds if either $n = 1$ or none of the components of $\overline{V} \cap (D \times (\mathbb{P}^1)^n)$ is contained in $\bigcap_{1\leq i\leq n} \overline{X} \times F^n_i$.

Proof. The fact that the modulus condition implies the displayed condition (3)′ follows by pulling back the cycle along the open embedding $\iota_i \colon X \times \overline{\square}^n_i \to X \times \overline{\square}^n$. The converse implication is also clear. □

The same argument gives the following statement for the M_{ssup} condition.

Lemma 2.3.2. *Let the notation be as in Lemma 2.3.1. Then V satisfies the modulus condition M_{ssup} only if there exists $1 \leq i \leq n$ such that the inequality*

$$\varphi^*_{\overline{V}_i}(D \times \overline{\square}^n_i) \leq \varphi^*_{\overline{V}_i}(X \times F^n_{i,1}) \tag{2.3.2.1}$$

holds as divisors on $\overline{V}^N_i$ and such that for every $j \neq i$, $1 \leq j \leq n$, the divisors $\varphi^*_{\overline{V}_j}(D \times \overline{\square}^n_j)$ and $\varphi^*_{\overline{W}_j}(X \times F^n_{j,1})$ on $\overline{W}^N_j$ are 0. The converse implication holds if either $n = 1$ or none of the components of $\overline{V} \cap (D \times (\mathbb{P}^1)^n)$ is contained in $\bigcap_{1\leq i\leq n} \overline{X} \times F^n_i$.

The following Proposition, that we prove in the M_{ssup} case but that holds true in the M_Σ case as well, is the relative version of the vanishing of Bloch's higher Chow groups $\mathrm{CH}^1(X, n)$ for $n \geq 2$.

Proposition 2.3.3. *Let $\overline{X}$ be a regular connected k-variety and D an effective Cartier divisor on $\overline{X}$. Then for $n \geq 2$ we have $CH^1(\overline{X}|D, n)_{M_{ssup}} = 0$.*

Proof. We omit the subscript M_{ssup}. As already observed in Definition 1.2.9, the objects $z^1(\overline{X}|D, \bullet)$ are extended cubical objects and we can work with the normalized chain complex to compute the groups $\mathrm{CH}^1(\overline{X}|D, n)$. Let

$$W \in N(z^1(\overline{X}|D, n)) = \bigcap_{i=2}^{n} \mathrm{Ker}(d^*_{i,0}) \cap \bigcap_{i=1}^{n} \mathrm{Ker}(d^*_{i,\infty})$$

such that $d^*_{1,0}(W) = 0$, i.e. W is a cycle of codimension 1 in $\overline{X} \times \square^n$ that has trivial intersection with every face. As $\overline{X} \times \square^n$ is regular, we can write W as Cartier divisor on $\overline{X} \times \square^n$. Hence

$$W = (f_\alpha, U_\alpha \times \square^n)_\alpha,$$

where $\cup_\alpha U_\alpha$ is an open covering of $\overline{X}$, and f_α is a rational function on $U_\alpha \times \square^n$ giving a local equation for W. For every $1 \leq i \leq n$ and $\varepsilon \in \{0, \infty\}$, the restriction $f_{\alpha|U_\alpha \times F^n_{i,\varepsilon}}$ of f_α is a local equation for the divisor

$$W \cap U_\alpha \times F^n_{i,\varepsilon}.$$

Since the intersection of W with every face is empty,

$$f_{\alpha|U_\alpha \times F^n_{i,\varepsilon}} \in \mathrm{H}^0(U_\alpha \times F^n_{i,\varepsilon}, \mathcal{O}^\times_{U_\alpha \times F^n_{i,\varepsilon}}).$$

Since every $F^n_{i,\varepsilon}$ is an affine space, the homotopy invariance of the group of units gives that

$$f_{\alpha|U_\alpha \times F^n_{i,\varepsilon}} = \pi^*_{i,\varepsilon} g_{\alpha,i,\varepsilon} \quad \text{for } g_{\alpha,i,\varepsilon} \in \mathrm{H}^0(U_\alpha, \mathcal{O}^\times_{U_\alpha}),$$

where $\pi_{i,\varepsilon} \colon U_\alpha \times F^n_{i,\varepsilon} \to U_\alpha$ is the projection to the first component.

As $n \geq 2$, the boundary of the cube $\square^n$ is connected. Therefore the unit $g_{\alpha,i,\varepsilon}$ does not depend on i and ε and we write simply $g_\alpha = f_{\alpha|U_\alpha \times \partial\square^n}$, where $\partial\square^n$ is the union of all the faces

$F^n_{i,\epsilon}$ (the boundary of the cube). By replacing f_α with $\frac{f_\alpha}{g_\alpha}$ we can assume $g_\alpha = 1$ without changing W, i.e. we can assume $f_{\alpha|U_\alpha\times\partial\square^n} = 1$.

As a global section of the sheaf $\mathcal{K}^\times_{\overline{X}\times\square^n}/\mathcal{O}^\times_{\overline{X}\times\square^n}$, the local equations for W satisfy the gluing condition $f_{\alpha|U_{\alpha\beta}} = \psi_{\alpha\beta}f_{\beta|U_{\alpha\beta}}$, where $U_{\alpha\beta} = U_\alpha\cap U_\beta$ and $\psi_{\alpha\beta}$ is a global unit on $U_{\alpha\beta}\times\square^n$. By homotopy invariance $\psi_{\alpha\beta}$ is a constant and it restricts to 1 on the boundary of the cube, therefore $\psi_{\alpha\beta} = 1$, i.e. $f_{\alpha|U_{\alpha\beta}} = f_{\beta|U_{\alpha\beta}}$, so that W is the divisor of a rational function $f \in \mathrm{H}^0(\overline{X}\times\square^n, \mathcal{K}^\times_{\overline{X}\times\square^n})$.

We fix a rational coordinate system $(t_1,\dots,t_n)$ on $(\mathbb{P}^1)^n$, so that we have the affine coordinate system $(\frac{t_1}{1-t_1},\dots,\frac{t_n}{1-t_n})$ on $\square^n$. We can thus identify $\mathrm{H}^0(\overline{X}\times\square^n, \mathcal{K}^\times_{\overline{X}\times\square^n})$ with

$$k(\overline{X})(\frac{t_1}{1-t_1},\dots,\frac{t_n}{1-t_n}).$$

Let $F(f)$ be the rational function on $\overline{X}\times\square^{n+1}$ defined by

$$F(f) = \frac{f(t_2,\dots,t_{n+1})}{1-t_1} - \frac{t_1}{1-t_1}.$$

We can easily check that the divisor $\mathrm{div}(F(f))$ on $X\times\square^{n+1}$ has trivial intersection with all the faces of the $n+1$-dimensional cube except with $\overline{X}\times F^{n+1}_{1,0} = \overline{X}\times\square^n$, where it is by definition equal to $\mathrm{div}(f) = W$. We are then reduced to show that the modulus condition satisfied by W implies the same condition on $\mathrm{div}(F(f))$, so that $\mathrm{div}(F(f)) \in N(z^1(\overline{X}|D, n+1))$, completing the proof.

By Lemma 2.3.2, there exists $1\le i\le n$ such that $\varphi^*_{\overline{W}_i}(D\times\overline{\square}^n_i) \le \varphi^*_{\overline{W}_i}(\overline{X}\times F^n_{i,1})$. Up to permutation of the factors, we can assume that $i = n$, so that $\overline{\square}^n_i = X\times\square^{n-1}\times\mathbb{P}^1$, with coordinate system $(t_1,\dots,t_{n-1},y)$. As in the proof of 2.1.5, we can localize to the generic points of D and therefore assume that $\overline{X} = \mathrm{Spec}(A)$ is the spectrum of a discrete valuation ring R. Let x be a generator of the maximal ideal of A, $D_{\mathrm{red}} = \mathrm{Spec}(A/(x))$, $D = \mathrm{Spec}(A/(x^l))$. We may assume that $W = \mathrm{div}(f)$ is effective. Then f is of the form

$$f = \frac{(-1)^m y^m + 1 + \sum_{i=1}^{m-1} a_i y^i}{(1-y)^m} \tag{2.3.3.1}$$

with $a_i \in A[t_1,\dots,t_{n-1}]$. The closure of W in the n-th direction is then given on $\overline{X}\times\square^{n-1}\times\mathbb{P}^1\setminus\{\infty\}$ by the divisor of the function $g = (1-y)^m f = (-1)^m y^m + 1 + \sum_{i=1}^{m-1} a_i y^i$. Then

$$g = (1-y)^m + \sum_{k=1}^{m} b_k(1-y)^{m-k}$$

with $b_m = 1 + (-1)^m + \sum_{i=1}^{m-1} a_i$. Suppose first that $\overline{W}_n$ is normal. Then the modulus condition on W is equivalent to the requirement that $b_m \in (x^{lm})$ (resp. $b_k \in (x^l)$ for $1\le k\le m-1$), where (x^{lm}) (resp. (x^k)) here denotes the principal ideal generated by x^{lm} (resp. x^l) in $A[t_1,\dots,t_{n-1}]$. We have

$$F(f) = \frac{(1-t_{n+1})^m + \sum_{k=1}^m b_k(1-t_{n+1})^{m-k}}{(1-t_{n+1})^m(1-t_1)} - \frac{t_1}{1-t_1} \tag{2.3.3.2}$$

and by multiplying (2.3.3.2) by

$$(1-t_{n+1})^m(1-t_1)$$

we get

$$(1-t_{n+1})^m(1-t_1) + \sum_{k=1}^m b_k(1-t_{n+1})^{m-k}.$$

Then $\mathrm{div}(F(f))$ satisfies the same modulus condition of $\mathrm{div}(f)$, completing the proof in the normal case. The general case follows from the same computation (taking norms), after noticing that the corresponding normalized cycle W^N can be written (locally) as divisors of functions of the form (2.3.3.1). □

3. Torsion 0-cycles with modulus on affine varieties

We fix an algebraically closed field k. The main result of this Section, Theorem 3.4.7, can be seen as an affine Rojtman-style result for the group of 0-cycles with modulus. To explain the analogy, recall the following Theorem (essentially due to Rojtman in characteristic 0 and to Milne in positive characteristic — see also [60, Section 7]).

Theorem 3.0.1 ([56] and [50])**.** *Let A be a smooth (finitely generated) algebra over an algebraically closed field k of Krull dimension d. Then the group $\mathrm{CH}^d(\mathrm{Spec}(A))$ is a torsion-free, divisible abelian group.*

When $V = \mathrm{Spec}(A)$ is no longer regular, a generalization of Theorem 3.0.1 is due to Levine.

Theorem 3.0.2 ([42], Theorem 2.6)**.** *Let V be an affine variety over an algebraically closed field k of Krull dimension d. Then the cohomological Chow group of 0-cycles $\mathrm{CH}^d(V)^{LW}$ is torsion-free, except possibly for p-torsion in characteristic p.*

The *cohomological Chow group* of 0-cycles, whose definition is due to Levine and Weibel and recalled below, is the "correct" analogue of the classical Chow group of 0-cycles on a smooth variety. In this Section, we follow the strategy on [42] to prove the following Theorem

Theorem 3.0.3 (see 3.4.7 below)**.** *Let X be a smooth affine k-variety of dimension at least 2, D an effective Cartier divisor on it. Then the Chow group of zero 0-cycles on X with modulus D, $\mathrm{CH}_0(X|D)$, is torsion free, except possibly for p-torsion if the characteristic of k is $p > 0$.*

Independently from the proof of this Theorem, the author and Amalendu Krishna re-proved the same result in [4, Theorem 6.4]. In *loc.cit.*, this is a byproduct of a factorization result for the Levine-Weibel Chow group of 0-cycles of the "double variety" $S(X, D)$, obtained by glueing two copies of X along the given divisor D (see [4], Theorem 1.8). The double construction, together with results of Krishna and Levine on torsion cycles on singular affine varieties, allows one to prove a stronger version of this vanishing result, encompassing p-torsion as well.

3.1. Cycles with modulus and cycles on singular varieties. We give a definition of *relative Chow group of 0-cycles with modulus* that is modeled on Levine-Weibel's cohomological Chow group, as defined in [48]. This will be used to combine the two sets of information, that is the modulus condition along D and the singularities contained in a fixed closed subscheme Y. We start by recalling the following definition from *loc. cit.*

Definition 3.1.1. *Let X be an integral quasi-projective k-variety and let Y be a closed subset of X. A Cartier curve of X relative to Y is a closed subscheme C of X such that*

(1) C is pure of dimension one and no component of C is contained in Y;
(2) if $y \in C \cap Y$, then the ideal of C in $\mathcal{O}_{X,y}$ is generated by a regular sequence.

Let C be a Cartier curve of X relative to Y and let $\eta_1, \ldots, \eta_r$ be the generic points of C. Put $Y_C = Y \cap C$ and $S = Y_C \cup \{\eta_1, \ldots, \eta_r\}$. Then we define $k(C, Y_C)^\times$ to be the image of the natural map

$$\mathcal{O}_{C,S}^\times \to \bigoplus_{i=1}^r \mathcal{O}_{C,\eta_i}^\times.$$

Thus, elements of $k(C, Y_C)^\times$ are meromorphic functions on C that are units in $\mathcal{O}_{C,x}$ at each point $x \in C \cap Y$.

Let $f_i \in k(C_i)^\times$ be the restriction of f to the component C_i having η_i as generic point. Then the divisor of f is, by definition, the cycle on C given by $\sum_{i=i}^{r} n_i \mathrm{div}(f_i)$, where n_i denotes the length of the local ring $\mathcal{O}_{C,\eta_i}$. The group of rational equivalence $\mathcal{R}_0(X,Y)$ is the subgroup of $z_0(X \setminus Y)$ of zero cycles on the open complement of Y that is generated by cycles of the form $\nu_{C,*}(\mathrm{div}(f))$, where $\nu\colon C \to X$ is a Cartier curve relative to Y and f is in $k(C, Y_C)^\times$.

Definition 3.1.2 (See [48], Definition 1.2)**.** *The Levine-Weibel (or cohomological) relative Chow group* $\mathrm{CH}_0(X,Y)$ *is the quotient* $z_0(X \setminus Y)/\mathcal{R}_0(X,Y)$.

Since we are assuming X to be integral, we can actually simplify the definition of rational equivalence, using [42, Lemma 1.4]. In particular, we can assume that the Cartier curves that we consider are integral.

Remark 3.1.3. Let X be an integral variety and let Y be its singular locus. The Levine-Weibel Chow group of X, usually denoted $\mathrm{CH}_0^{LW}(X)$, is the relative Chow group $\mathrm{CH}_0(X,Y)$ of Definition 3.1.2. We remark here that this is not the only possible definition of Chow group of zero cycles on a singular variety (apart, of course, from the usual definition of [16]). In [4, Section 3], we propose an alternative definition, based on the notion of *l.c.i. curves* instead of Cartier curves. The two groups coincide in many interesting cases (see [4, Theorem 3.16]), but the lci version seems to have better functorial properties, as established in [4, Section 3.5].

Recall from [48] the following Definition.

Definition 3.1.4. *Let C be a 1-dimensional reduced excellent scheme over k, Y a closed subscheme of C containing no components of C. We define the Y-normalization of C to be the datum of a reduced 1-dimensional scheme C^Y equipped with a finite birational map*

$$\varphi_{C,Y}\colon C^Y \to C$$

such that

(1) *The normalization of $C \setminus Y$ is $\varphi_{C,Y}^{-1}(C \setminus Y)$,*
(2) *The morphism $\varphi_{C,Y}$ is an isomorphism on some neighborhood of Y.*

It is easy to check that the Y-normalization exists and that it is uniquely determined by the properties i) and ii).

Definition 3.1.5. *Let D be an effective Cartier divisor on X and let Y be a closed subset of X. Assume that Y does not intersect D. For an integral Cartier curve C of X relative to Y such that C intersects properly D, we denote by C^Y its Y_C-normalization. Let D_{C^Y} denote the pull-back of D to C^Y. We define $G(C^Y, Y_C, D_{C^Y})$ as*

$$G(C^Y, Y_C, D_{C^Y}) = k(C^Y, Y_C) \cap G(C^Y, D_{C^Y}),$$

where $G(C^Y, D_{C^Y})$ is defined as in 1.3.1 as the group of rational functions on C^Y that are congruent to 1 along D_{C^Y}. We define the relative Chow group $\mathrm{CH}_0(X, Y|D)$ of 0-cycles with modulus for (X,Y,D) as the cokernel of the homomorphism

$$\tau^{LW}\colon \bigoplus_C G(C^Y, Y_C, D_{C^Y}) \to Z_0(X \setminus Y).$$

where the sum runs over the set of integral Cartier curves C of X relative to Y such that C intersects D properly and the map τ^{LW} is the divisor map.

Remark 3.1.6. Let $\overline{C}$ be a curve without embedded components, and let $Y_{\overline{C}}$ be a closed subset of $\overline{C}$ containing the singular points of $\overline{C}$. Let D be an effective divisor on C not intersecting Y_C and write $C = \overline{C} \setminus D$. The localization sequence of (2.2.1.3) takes in this case the form

$$K_1(\overline{C} - V; D) \xrightarrow{\partial} \coprod_{x \in V} K_0^{|x|}(\overline{C}; D) \simeq Z_0(C \setminus Y_{\overline{C}}) \to K_0(\overline{C}; D) \to K_0(\overline{C} - V; D)$$

where V denotes the set of closed points of $\overline{C}$ not in $Y_{\overline{C}} \cup D$. The divisor map

$$G(\overline{C}, Y_{\overline{C}}, D) \to z_0(C \setminus Y_{\overline{C}})$$

can be identified with the composite

$$G(\overline{C}, Y_{\overline{C}}, D) \to K_1(\overline{C} - V; D) \xrightarrow{\partial} \coprod_{x \in V} K_0^{|x|}(\overline{C}; D) \simeq z_0(C \setminus Y_{\overline{C}}).$$

Noting that $K_0(\overline{C}; D) = Pic(\overline{C}, D) \oplus \mathrm{H}^0(\overline{C}', \mathbb{Z})$, where $\overline{C}'$ denotes the union of the irreducible components of $\overline{C}$ that are disjoint from D, we can prove the following Lemma (see [48, 1.4])

Lemma 3.1.7. *Let $\overline{C}$ be a (separated) purely 1-dimensional scheme of finite type over k, without embedded components. Let Y be a closed subscheme of $\overline{C}$ containing the singular locus of $\overline{C}$ and not containing any component of $\overline{C}$. Let D be an effective Cartier divisor on $\overline{C}$. Then $\mathrm{CH}_0(\overline{C}, Y|D)$ is canonically isomorphic to the relative Picard group $\mathrm{Pic}(\overline{C}, D)$.*

3.2. Rigidity for the Chow group of 0-cycles with modulus. Let $\overline{X}$ be a quasi-projective variety over an algebraically closed field k of exponential characteristic $p \geq 1$. Let D be an effective Cartier divisor on $\overline{X}$ and suppose that the singular locus of $\overline{X}$ (if not empty) is contained in D. Write X for the open complement $\overline{X} \setminus D$. Let C be a smooth curve over k and let Z be a finite correspondence from C to $\overline{X}$ such that $|Z| \subset C \times X$. For a closed point x in C, we denote by $Z(x)$ the cycle

$$Z(x) = p_{2,*}(Z \cdot (p_1^*(x) \cap \{x\} \times X)).$$

It is a 0-cycle on $\overline{X}$, supported outside D. Recall from [3] the following result.

Theorem 3.2.1 ([3], Theorem 2.13)**.** *In the above notations, let n be an integer prime to p. Assume that there exists a dense open subset C_0 of C such that for every $x \in C_0(k)$ one has*

$$n \cdot [Z(x)] = 0 \quad \textit{in } \mathrm{CH}_0(\overline{X}|D)$$

Then the function $x \in C(k) \mapsto [Z(x)]$ is constant.

Remark 3.2.2. Theorem 2.13 in [3] is stated for $\overline{X}$ projective over k. However, the proof (based on the proof of [48, Proposition 4.1]) goes through without this assumption. Since we are going to apply Theorem 3.2.1 for X affine, we stated it in general for quasi-projective varieties.

3.3. An easy moving. In this Subsection, we discuss a result that concerns the image of torsion cycles from curves to affine varieties with modulus. As the reader will soon notice, we owe a great intellectual debt to [42]. We need some preliminaries Lemmas. The first one is taken directly from [42], while the second one is pretty standard, and we refer the reader to [3] for a simple proof.

Lemma 3.3.1 (see [42], Corollary 1.2)**.** *Let X be a quasi-projective variety, smooth over k. Let D be an effective Cartier divisor on X and let $\nu\colon C \hookrightarrow X$ be a reduced curve in X, properly intersecting D. Then there exists a projective closure $\overline{X}$ of X such that, if $\overline{C}$ (resp. $\overline{D}$) denotes the closure of C (resp. D) in $\overline{X}$, then the following hold:*

(1) $\overline{D} \cap \overline{C} = C \cap D$,

(2) $\overline{X} \setminus \overline{D}$ is normal. In particular, the singular locus $\overline{X}_{sing}$ ($\subset \overline{X} - X$) has codimension at least 2 on $\overline{X}$ at each point of $\overline{X}_{sing} \cap (\overline{C} \setminus C)$.

Lemma 3.3.2 ([3], Lemma 2.36)**.** *Let S be the spectrum of a discrete valuation ring, s the closed point of S, η the generic point of S. Let $\pi\colon X \to S$ be a semistable projective curve over S, i.e. π is a projective and flat morphism of relative dimension 1 such that the special fibers $X_{\overline{s}}$ over geometric points of S are reduced, connected curves having only ordinary singularities. Suppose that the family is generically smooth. Let D be an effective Cartier divisor on X such that the composition $\pi_D\colon D \to S$*

of the inclusion $D \hookrightarrow X$ with the morphism π is flat. Then the relative Picard functor $\mathbf{Pic}^0_{(X|D)/S}$ is representable by a scheme (locally) of finite type over S.

The following Proposition is an application of a classical moving argument for 0-cycles on smooth varieties. Its proof is representative of the arguments used repeatedly in [4], Sections 5 and 6 (see, in particular, Lemma 5.4 in *loc.cit.*).

Proposition 3.3.3. *Let X be an affine smooth k-variety of dimension at least 2 and let D be an effective Cartier divisor on it. Let $\nu\colon C \hookrightarrow X$ be an integral curve, properly intersecting D and smooth in a neighborhood of D. Write ν_* for the push-forward map*

$$\nu_*\colon \mathrm{CH}_0(C,|\nu^*(D)) = \mathrm{Pic}(C,\nu^*(D)) \to \mathrm{CH}_0(X|D).$$

Let n be an integer prime to the characteristic of k and let α in $\mathrm{CH}_0(C|\nu^*(D))$ be an n-torsion 0-cycle. Then $\nu_*(\alpha) = 0$ in $\mathrm{CH}_0(X|D)$.

Proof. Let $\overline{X}$ be a compactification of X satisfying conditions (1) and (2) of Lemma 3.3.1. We can find (by [35, Theorem 7]) an integral projective surface $\overline{Y}$ satisfying

i) $\overline{Y} \supset \overline{C}$;
ii) $\overline{Y}$ intersects $\overline{D}$ properly;
iii) $Y = \overline{Y} \cap X$ is an affine. The divisor $D_Y = Y \cap D$ is an effective Cartier divisor on Y and C is a Cartier divisor in a neighborhood of $C \cap D$ in Y (see [42, Lemma 1.3]).
iv) $\overline{Y}$ is normal.

By condition (2) of Lemma 3.3.1, we can moreover assume that $\overline{Y}$ is such that every point of $\overline{C} \setminus C$ is either a smooth point of $\overline{Y}$ or an isolated singularity. By resolving the singularity of $\overline{Y}$ that are on $\overline{C} \setminus C$ and the singularities of $\overline{C} \setminus C$, we can actually assume that every point $x \in \overline{C} \setminus C$ is a regular point of $\overline{Y}$ and that $\overline{C}$ is also regular at x.

Replacing X with Y, we are reduced to the following case

a) X is an integral normal affine surface, and D is an effective Cartier divisor on it.
b) $\overline{X}$ is a projective compactification of X.
c) $\nu\colon C \hookrightarrow X$ admits a compactification $\overline{\nu}\colon \overline{C} \to \overline{X}$ such that $\overline{C}$ is regular at $\overline{C} \setminus C$ and at every point of $\overline{C} \cap \overline{D} = C \cap D$. The surface $\overline{X}$ is regular at every point of $\overline{C} \setminus C$.

Let F denote the closed complement $\overline{X} \setminus X$. Since X is affine, F is the support of a very ample line bundle $\mathcal{L}$ on $\overline{X}$. For d sufficiently large, we can find global sections

$$s_0 \in \mathrm{H}^0(\overline{X}, \mathcal{L}^{\otimes d} \otimes \mathcal{I}_{\overline{C}}) \subset \mathrm{H}^0(\overline{X}, \mathcal{L}^{\otimes d}) \quad \text{and } s_\infty \in \mathrm{H}^0(\overline{X}, \mathcal{L}^{\otimes d})$$

such that $(s_0) = W_0 = \overline{C} + E$, for E integral, intersecting $\overline{D}$ properly and away from $C \cap D$, is a reduced connected curve, and such that $(s_\infty) = W_\infty$ is contained in $\overline{X} \setminus X = F$.

Using s_0 and s_∞, we can define a pencil $P = \{W_t \,|\, t \in \mathbb{P}^1\}$ of hyperplane sections of $\overline{X}$, interpolating between W_0 and W_∞. More precisely, let W be the flat cycle

$$W \subset \overline{X} \times \mathbb{P}^1$$

given by the equation $s_0 + ts_\infty$ for t a rational coordinate on $\mathbb{P}^1$. Then for general t, W_t is integral, intersects $\overline{D}$ properly and misses the singular locus of $\overline{D}_{red}$.

Let S be the spectrum of the local ring of $\mathbb{P}^1$ at 0, s its closed point, η its generic point. We denote by $\pi_S\colon W_S \to S$ the base change of $W \to \mathbb{P}^1$ to S. By construction, the special fibre $(W_S)_s$ coincides with W_0, while the generic fibre $W_\eta \to k(t)$ represents the generic member of the pencil. The family $W_S \to S$ is flat, projective, so $\chi(\mathcal{O}_{W,t}) = \chi(\mathcal{O}_{W,s}) = \chi(\mathcal{O}_{W_0})$. Since W_t is integral and W_0 is reduced and connected, we conclude that the curves in the family have constant arithmetic genus $g = p_a(W_t)$. Hence, the morphism π_S is cohomologically flat in dimension 0 (see [9, page 206]). By Artin's representability theorem ([9, Theorem 8.3.1]), the relative Picard $\mathbf{Pic}^0_{(W_S|D_S)/S} \to S$ is representable by a (locally finitely presented) algebraic space over S, where D_S is the base change of $\overline{D}$ to W_S (and is an horizontal divisor on W_S). Actually, Artin's theorem shows the representability of $\mathbf{Pic}_{W_S/S} \to S$, but $\mathbf{Pic}^0_{(W_S|D_S)/S} \to S$ is a torsor over $\mathbf{Pic}_{W_S/S} \to S$ for the group $G = \pi_{D,*}\mathbb{G}_{m,D}$. Thus $\mathbf{Pic}^0_{(W_S|D_S)/S} \to S$ is representable by an algebraic space as well (see also [55, Theorem 5.2]). Since $h^0(\mathcal{O}_{W_\eta}) = h^0(\mathcal{O}_{W_0})$, by [55, Theorem 4.1.1, Proposition 8.0.1], $\mathbf{Pic}^0_{W_S/S}$ coincides with its maximal separated quotient, and thus it is representable by a scheme, separated and locally of finite type over S. By [19, Lemma 3.6], the same holds for $\mathbf{Pic}^0_{(W_S|D_S)/S}$.

Since the canonical restriction map

$$\mathrm{Pic}(\overline{C}, \nu^*D = \overline{\nu}^*\overline{D}) \xrightarrow{j^*} \mathrm{Pic}(C, \nu^*D)$$

is surjective, as $\overline{C} \cap \overline{D} = C \cap D$, we can lift α to a cycle $\tilde{\alpha}$ in $\mathrm{Pic}(\overline{C}, \nu^*D)$. By assumption, we have that $n\alpha = 0$ in $\mathrm{Pic}(C, \nu^*D)$, so that $n\tilde{\alpha} \in \mathrm{Ker}(j^*)$, that is the subgroup of $\mathrm{Pic}(\overline{C}, \nu^*D)$ that is generated by classes of (regular) points in $\overline{C} \setminus C$. Since $W_0 = \overline{C} + E$, we have a canonical map

$$\mathrm{Pic}(W_0, W_0 \cap \overline{D}) = \mathrm{Pic}(W_S \otimes k(s), D_S \otimes k(s)) \to \mathrm{Pic}(\overline{C}, \nu^*D) \times \mathrm{Pic}(E, E \cap D)$$

that is surjective. The kernel is n-divisible (see, e.g., [49], Lemma 7.5.18).

We choose now $S' \to S$ a DVR dominating S so that there is a section

$$\gamma' \colon S' \to \mathbf{Pic}^0_{(W_{S'}|D_{S'})/S'}$$

satisfying $\gamma'(s') = n\beta_0$, where β_0 lifts to $\mathrm{Pic}(W_{0,S'}, (W_0 \cap \overline{D}) \times_S S')$ the element $(\tilde{\alpha}, 0)$ of

$$\mathrm{Pic}(\overline{C}, \nu^*D) \times \mathrm{Pic}(E, E \cap D).$$

Here s' denotes the closed point of S', above $0 \in \mathbb{P}^1$. Note that, since the class of $n\tilde{\alpha}$ in $\mathrm{Pic}(\overline{C}, \nu^*D)$ is represented by points in $\overline{C} \setminus C$, we can further assume that the divisor Z' on $W_S \times_S S'$ representing γ' is supported on $F_{S'}$.

Let now T be the normalization of an irreducible component of $n^{-1}(\gamma'(S')) \subset \mathbf{Pic}^0_{(W_{S'}|D_{S'})}$ passing through β_0 and let Z be the divisor on $W_{S'} \times_{S'} T$ representing $\gamma \colon T \to \mathbf{Pic}^0_{(W_{S'}|D_{S'})}$. Write $p \colon T \to S$ for the composite map. For $t \in T$, let $Z(t)$ denote the divisor on $W_{p(t)}$ given by

$$Z(t) = p_{1,*}(Z \cdot (p_2^*(t) \cap \{t\} \times W_{S'}))$$

where p_1 and p_2 are the two projections from $W_{S'} \times_{S'} T$. Note that, since both γ' and γ are defining subschemes of the relative Picard scheme $\mathbf{Pic}^0_{(W_{S'}|D_{S'})}$ of line bundles with a trivialization along $D_{S'}$, we have that

$$Z \subset W_{S'} \times_{S'} T \hookrightarrow W \times_{\mathbb{P}^1} T \hookrightarrow \overline{X} \times T$$

is supported away from $\overline{D} \times T$. Taking T smaller if necessary, we can also assume that Z is actually closed in $X \times T$. Moreover, the choice of T gives that

$$nZ(t) = Z'(t), \quad \text{in } \mathrm{Pic}(W_{p(t)}, D_{p(t)})$$

(note that for t in a dense subset of T, $W_{p(t)}$ is actually smooth over k and $D_{p(t)}$ is an effective Cartier divisor on it). Pushing forward to X gives then a map

$$Z \colon T(k) \to \mathrm{CH}_0(X|D)$$

and since $Z'(t)$ is supported on $\overline{X} \setminus X$ for every t, $nZ(t) = 0$ in $\mathrm{CH}_0(X|D)$. In other words, we have a family of n-torsion 0-cycles on X with modulus D, parametrized by T and represented by Z. By the Rigidity Theorem 3.2.1, the family is constant. In particular, whenever T is realized as dense open subset of any curve T^* such that Z extends to a correspondence

$$Z^*\colon T^*(k) \to \mathrm{CH}_0(X|D) \tag{3.3.3.1}$$

then for every $t \in T^*(k)$, the class of $Z^*(t)$ will be the same, since Theorem 3.2.1 only requires to check what happens on a dense set. Let now $\overline{T}$ be a smooth projective model of T, containing T as open dense subset. Let β_∞ a point of $\overline{T} \setminus T$ above $\infty \in \mathbb{P}^1$ and take T^* to be $T \cup \{\beta_\infty\}$. Then since $Z \cap (X \times \beta_\infty) \subset W_\infty \subset \overline{X} \setminus X$, Z is still closed in $X \times T^*$, it defines a correspondence as in (3.3.3.1). In particular, since $Supp(Z^*(\beta_\infty)) = Z \cap X \times \beta_\infty = \emptyset$, $Z^*(\beta_\infty) = 0$ in $\mathrm{CH}_0(X|D)$. Thus

$$\nu_*(\alpha) = Z(\beta_0) = Z^*(\beta_\infty) = 0 \quad \text{in } \mathrm{CH}_0(X|D)$$

completing the proof. □

The following lemma is easy to check.

Lemma 3.3.4. *Let X be an affine smooth k-variety, of dimension at least 2 and let D be an effective Cartier divisor on it. Let $u\colon X' \to X$ be a sequence of blow-ups with center in points lying over $X \setminus D$. Then one has an isomorphism*

$$u_*\colon \mathrm{CH}_0(X'|D) \xrightarrow{\simeq} \mathrm{CH}_0(X|D)$$

We then have the following Corollary to Proposition 3.3.3.

Corollary 3.3.5. *Let X, X', D and u be as in Lemma 3.3.4. Let $\nu\colon C \hookrightarrow X'$ be a smooth integral curve in X', properly intersecting D. Write ν_* for the push-forward map*

$$\nu_*\colon \mathrm{CH}_0(C|\nu^*(D)) \to \mathrm{CH}_0(X'|D) \xrightarrow{\simeq} \mathrm{CH}_0(X|D).$$

Let n be an integer prime to the characteristic of k and let α in $\mathrm{CH}_0(C|\nu^*(D))$ be an n-torsion 0-cycle. Then $\nu_*(\alpha) = 0$ in $\mathrm{CH}_0(X'|D)$ (and a fortiori in $\mathrm{CH}_0(X|D)$).

Proof. Let E be the exceptional divisor of the blow-up $u\colon X' \to X$. We have to consider two cases. If $\nu(C) \subset E$, the C is a projective curve and any cycle $\alpha \in \mathrm{CH}_0(C|\nu^*(D))$ of degree zero satisfies $u_*(\alpha) = 0$. In particular, this applies to torsion cycles. If $\nu(C)$ is not completely contained in the exceptional divisor, write Z for the image $u(C) \subset X$ and write $j\colon Z \to X$ for the inclusion. Since the center of the blow-up is disjoint from D, u induces an isomorphism on a neighborhood of $\nu^*(D)$ between C and Z. In particular, $\nu^*(D) \simeq j^*(D)$ and Z is a complete intersection in a neighborhood of D. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}_0(C|\nu^*(D)) & \xrightarrow{\nu_*} & \mathrm{CH}_0(X'|D) \\
{\scriptstyle u^*}\uparrow & & \downarrow{\scriptstyle u_*} \\
\mathrm{CH}_0(Z|j^*(D)) & \xrightarrow{j_*} & \mathrm{CH}_0(X|D)
\end{array}$$

where we keep writing u for the induced morphism $C \to Z$. The morphism u^* is surjective and, by [49, Lemma 7.5.18], the kernel is n-divisible. In particular, for every torsion cycle $\alpha \in \mathrm{CH}_0(C|\nu^*(D))[n]$, we can find $\beta \in \mathrm{CH}_0(Z|j^*(D))[n]$ such that $u^*(\beta) = \alpha$. But then we have $0 = j_*(\beta) = u_*(\nu_*(\alpha))$, from which we conclude $\nu_*(\alpha) = 0$ since u_* is an isomorphism. □

3.4. Vanishing results. We now come to the main results of this Section. We keep the notations of Lemma 3.3.4.

3.4.1. Let $\nu\colon C \hookrightarrow X'$ be an integral curve in X' intersecting D in the regular locus of D_{red} transversally (we will say that such a curve is in good position with respect to D). Let α be a 0-cycle with modulus (of degree 0) on C, supported on $C_{reg} \setminus \nu^*D$, giving a class $\alpha \in \mathrm{CH}_0(C|\nu^*D)$. Let n be an integer prime to the characteristic of k. Suppose that α is n-divisible in $\mathrm{CH}_0(C|\nu^*D)$.

Then there is a 0-cycle β on C such that $n\beta = \alpha$ in $\mathrm{CH}_0(C|\nu^*D)$. This is the case if, for example, C is smooth or if $C_{sing} \subset D$.

For C a smooth integral complete intersection curve, we define the element

$$n_C^{-1}(\alpha) \in \mathrm{CH}_0(X'|D)$$

to be the class of $\nu_*(\beta)$. By Corollary 3.3.5, since any n-torsion cycle on C with modulus ν^*D vanishes on $\mathrm{CH}_0(X'|D)$, the class is well defined, i.e., it depends (a priori) only on the curve C and not on the choice of the lifting β.

3.4.2. Let $(\overline{X}, D)$ be a pair consisting of a smooth and projective k-variety $\overline{X}$ and an effective Cartier divisor D on it. Let $T_{\overline{X}|D}$ denote the subgroup of the group of degree zero 0-cycle $\mathrm{CH}_0(\overline{X}|D)^0$ that is generated by the images of torsion 0-cycles on proper smooth curves mapping to $\overline{X}$ (and having image not contained in D). In [3], Section 2, we proved that the operation n^{-1} as defined in 3.4 satisfies two important properties (the second implied by the first)

i) Given two smooth curves C_1 and C_2 in $\overline{X}$ such that $\alpha \in C_1 \cap C_2$, one has the equality

$$n_{C_1}^{-1}(\alpha) = n_{C_2}^{-1}(\alpha)$$

in $\mathrm{CH}_0(\overline{X}|D)^0/T_{\overline{X}|D}$.

ii) There is a well defined map

$$n_X^{-1}\colon \mathrm{CH}_0(\overline{X}|D)^0/T_{\overline{X}|D} \to \mathrm{CH}_0(\overline{X}|D)^0/T_{\overline{X}|D}$$

such that, for every $\alpha \in \mathrm{CH}_0(\overline{X}|D)^0/T_{\overline{X}|D}$, we have $n_X^{-1}(n\alpha) = \alpha$ in $\mathrm{CH}_0(\overline{X}|D)^0/T_{\overline{X}|D}$.

The projectivity of $\overline{X}$ was used in an essential way to prove the two statements. Replacing $\overline{X}$ with X affine, we have to modify slightly the arguments using the same strategy as in the proof of 3.3.3, making use of the nice compactification provided by Lemma 3.3.1. Similarly, we can no longer assume that the singularities of $\overline{X}$ are contained in D. Again, we closely follow [42] (or [41], as we did in [3]) to prove the following Lemma.

Lemma 3.4.3. *Let $\nu_1\colon C_1 \hookrightarrow X'$ and $\nu_2\colon C_2 \hookrightarrow X'$ be two smooth integral curves in X' such that $C_i \setminus D$ is regular and intersecting D in the regular locus of D_{red} transversally. Let α be a 0-cycle with modulus on X', supported on $(C_1 \setminus D) \cap (C_2 \setminus D)$ and let n be as above an integer prime to the characteristic of k such that α is n-divisible in the relative Picard groups of C_1 and C_2. Then we have an equality in $\mathrm{CH}_0(X'|D)$*

$$n_{C_1}^{-1}(\alpha) = n_{C_2}^{-1}(\alpha).$$

Proof. First, we note that since X is smooth and since X' is obtained as blow-up along smooth centres, X' is a smooth k-variety as well. We will use a pencil argument, realizing C_1 and C_2 as components of the two special fiber of a $\mathbb{P}^1$-family of curves in X'. For this reason, we can replace C_2 with a general complete intersection of hypersurfaces containing the support of α. In particular, we can assume that C_1 intersects C_2 transversally at each point of $C_1 \cap C_2$. Arguing as in the proof of Proposition 3.3.3, we can assume that X' is a surface admitting a projective model $\overline{X'}$ such that $\overline{X'} \setminus \overline{D}$ is normal and such that, for $i = 1, 2$, $\nu_{C_i}\colon C_i \to X'$ admits a compactification $\overline{\nu_{C_i}}\colon \overline{C_i} \to \overline{X'}$. Using Lemma 3.3.1, (that does not require the curve C to be integral), we can assume that $\overline{C_i} \cap \overline{D} = C_i \cap D$ for $i = 1, 2$, so that the choice of the compactification does not effect the modulus condition of the curves. Again as in the proof of Proposition 3.3.3, by solving the singularities of $\overline{X'}$ that are on $\overline{C_i} \setminus C_i$ and the singularities of $\overline{C_i} \setminus C_i$, we can assume that every point $x \in \overline{C_i} \setminus C_i$ is a regular point of $\overline{X'}$ and that $\overline{C_i}$ is also regular there.

We keep using the notations of Proposition 3.3.3. Let P be the pencil $P = \{W_t | t \in \mathbb{P}^1\}$ of hyperplane sections of $\overline{X'}$ interpolating between C_1 and C_2. More precisely, we require that

i) The generic member W_t is integral and misses the singular locus of $\overline{D}_{red}$. The restriction $W_t \cap X'$ is regular.
ii) The base locus of P contains the support of α and misses $\overline{D}$. The rational map $\overline{X'} \dashrightarrow \mathbb{P}^1$ determined by P becomes a morphism after a single blow-up of each point in the base locus (this is achieved by the condition that C_1 and C_2 intersect transversally).
iii) The special fibers are $W_0 = C_1 + E_1$ and $W_\infty = C_2 + E_2$, where E_i are smooth integral curves, intersect $\overline{D}$ properly and away from $\overline{X'} \setminus X$ and are disjoint from the base locus of P. In addition, W_0 and W_∞ have only ordinary double points as singularities.

The proof of Proposition 2.34 in [3] now goes through: we give a sketch of the argument for completeness. Write W_P for the blow-up of $\overline{X'}$ at every point of the base locus of P and write Z for the divisor on W_P determined by the cycle α. Write $u\colon W_P \to \overline{X'}$ for the blow-down map. The pencil $W_S = W_P \times_{\mathbb{P}^1} S \to S$ for S the spectrum of the local ring of $\mathbb{P}^1$ at the origin is a semistable family, making the relative Picard functor $\mathbf{Pic}^0_{W_S|D_S} \to S$ representable by Lemma 3.3.2. Using the n-divisibility of generalized Jacobians over algebraically closed fields, we can find a DVR $S' \to S$ dominating S so that there is a section

$$\gamma'\colon S' \to \mathbf{Pic}^0_{W_{S'}|D_{S'}}$$

such that $n\gamma'(s') = Z_s$ and such that $n\gamma'(\overline{\eta'}) = Z_{\overline{\eta}}$, where s' dominates $0 \in \mathbb{P}^1$, $\overline{\eta}$ (resp. $\overline{\eta'}$) is a geometric generic point of S (resp. S'). Let Z' be the horizontal Cartier divisor representing γ'. Note that we can assume that Z' is supported on $X' \setminus D$ (in particular, away from $\overline{X'} \setminus X$). This defines a map

$$Z'\colon S' \to \mathrm{CH}_0(X'|D)$$

satisfying $nZ'(s') = u_*(Z_s) = \alpha$ and $nZ'(\overline{\eta'}) = u_*(Z_{\overline{\eta'}}) = \alpha$ in $\mathrm{CH}_0(X'_{\overline{\eta'}}|D_{\overline{\eta'}})$. In particular, the family of cycles $Z' - Z'(s')$ parametrized by S' defines a family of n-torsion cycles in $\mathrm{CH}_0(X'_{\overline{\eta'}}|D_{\overline{\eta'}})$, which is constant by the rigidity Theorem 3.2.1. Hence we have an equality

$$n^{-1}_{C_1}(\alpha) = n^{-1}_{W_{\overline{\eta'}}}(\alpha).$$

Replacing W_0 with W_∞ gives $n^{-1}_{C_2}(\alpha) = n^{-1}_{W_{\overline{\eta'}}}(\alpha)$, completing the proof. □

3.4.4. We resume the notations of Lemma 3.3.4. By Lemma 3.4.3, we have well defined maps

$$n^{-1}_{X'}\colon Z^0_0(X' \setminus D) \to \mathrm{CH}_0(X'|D)$$
$$n^{-1}_{X}\colon Z^0_0(X \setminus D) \to \mathrm{CH}_0(X|D)$$

(note that the proof works perfectly fine if we replace X' with X, and that every 0-cycle on X is supported on a smooth complete intersection integral curve on which it is n-divisible), satisfying $n^{-1}_X(n\alpha) = \alpha$ and $n^{-1}_{X'}(n\alpha) = \alpha$. Since given any two 0-cycles α_1 and α_2 with modulus D we can always find a smooth complete intersection curve in X (or in X') containing the union of their supports and in good position with respect to D, the maps n^{-1}_X and $n^{-1}_{X'}$ are group homomorphisms.

The proof of the following Lemma is identical to the corresponding statement in [3], Lemma 2.38 (taken from [41], Lemma 3.2).

Lemma 3.4.5. *Let $u\colon X' \to X$ be a blow-up at a point $x \in X \setminus D$. Then the following diagram commutes:*

$$\begin{array}{ccc} Z_0^0(X' \setminus D) & \xrightarrow{n_{X'}^{-1}} & \mathrm{CH}_0(X'|D) \\ \downarrow{u_*} & & \downarrow{u_*} \\ Z_0^0(X \setminus D) & \xrightarrow{n_X^{-1}} & \mathrm{CH}_0(X|D) \end{array}$$

We need one last ingredient.

Proposition 3.4.6. *The map n_X^{-1} factors through $\mathrm{CH}_0(X|D)$.*

Proof. Let $\nu\colon C \to X$ be a finite map from a normal curve C to X such that $\nu(C)$ is birational to C and is in good position with respect to D. Let $f \in k(C)^\times$ be a rational function on C that is congruent to 1 modulo ν^*D. We have to show that $n_X^{-1}(\nu_*(\mathrm{div} f)) = 0$ in $\mathrm{CH}_0(X|D)$. Let $u\colon X' \to X$ be the blow-up of X at the singular points of $\nu(C)$ away from D. By the previous Lemma, we have

$$n_X^{-1}(\nu_*(\mathrm{div} f)) = u_*(n_{X'}^{-1}(\nu'_*(\mathrm{div} f)))$$

where we have a factorization $C \to C' \hookrightarrow X'$, for $\nu'\colon C' \to X'$ is the strict transform of $\nu(C)$ in X'. Note that C' is regular away from D. It is then enough to show that $n_{X'}^{-1}(\nu'_*(\mathrm{div} f)) = 0$. But we have $n_{X'}^{-1}(\nu'_*(\mathrm{div} f)) = n_{C'}^{-1}(\mathrm{div}(f))$, since $n_{X'}^{-1}$ of any cycle can be computed by choosing a good curve containing it, thanks to Lemma 3.4.3, and the latter is clearly zero. □

We are finally ready to state and prove the main result of this Section.

Theorem 3.4.7. *Let X be a smooth affine k-variety of dimension at least 2, D an effective Cartier divisor on it. Then the Chow group of zero 0-cycles on X with modulus D, $\mathrm{CH}_0(X|D)$, is torsion free, except possibly for p-torsion if the characteristic of k is $p > 0$.*

Proof. Thanks to Lemma 3.4.6, we have for every n prime to the characteristic of k, a well defined group homomorphism

$$n_X^{-1}\colon\ \mathrm{CH}_0(X|D) \to \mathrm{CH}_0(X|D)$$

and this is, by the remarks in 3.4.4, inverse to the multiplication by n, proving the claim. □

Remark 3.4.8. One should note that Theorem 3.4.7 can not be directly pushed forward to encompass the p-torsion part of the Chow groups. For this, we refer the reader to [4].

4. Cycles with modulus and relative K-theory

Although the sum-modulus condition is by far the most successful in the current development of the theory, there are indeed situations where the strong-sup conditions seems to be better behaved. To give an example - and to justify our choice of including the two conditions in the systematic treatment of Section 1 - we present in this Section a construction of a cycle class map from "higher" 0-cycles with modulus to relative K-groups.

4.1. Generalities on relative and multirelative K-theory. We postpone large part of the discussion on K-theory spaces and spectra to Chapter II, Section 5 (where the choice of the model actually plays a relevant role), and we limit ourselves to introduce the minimal necessary notation. Let Y be a Noetherian separated scheme. Write $\mathbf{K}^{TT}(Y) = \mathbf{K}(Y)$ for the K-theory (Ω)-spectrum of Thomason-Trobaugh on the Waldhausen category of strict perfect complexes on Y. For a closed subscheme $j_Z\colon Z \hookrightarrow Y$, the spectrum of algebraic K-theory of Y relative to Z is the homotopy fiber of the morphism of spectra $\mathbf{K}(Y) \xrightarrow{j_Z^*} \mathbf{K}(Z)$.

$$\mathbf{K}^{TT}(Y;Z) = \mathrm{hofib}(\mathbf{K}(Y) \xrightarrow{j_Z^*} \mathbf{K}(Z))$$

Its homotopy groups, $\pi_*(\mathbf{K}^{TT}(Y;Z)) = K_*(Y;Z)$, are called the *$K$-theory groups of Y relative to Z* or simply groups of *relative K-theory*. By construction, there is an exact sequence of homotopy groups

(4.1.0.1) $$\ldots \to K^{TT}_{*+1}(Z) \to K^{TT}_*(Y;Z) \to K^{TT}_*(Y) \to K^{TT}_*(Z) \to K^{TT}_{*-1}(Y;Z) \to \ldots$$

where $K^{TT}_*(Y)$ and $K^{TT}_*(Z)$ denote, respectively, the Thomason-Trobaugh K-theory groups of Y and Z. When Y is equipped with an ample family of line bundles as in [63, Definition 2.1.1], the choice of the Waldhausen category is not critical, as remarked by [63, 3.4]. To simplify the notation, we will drop the superscript TT from the K-theory spectra.

4.1.1. Let T be another closed subscheme of Y. We denote by $\mathbf{K}^{|T|}(Y)$ or by $\mathbf{K}(Y \text{ on } T)$ the K-theory spectrum of the cosimplicial biWaldhausen category of perfect complexes on Y that are acyclic on the open complement $Y \setminus T$. As customary, we call it the *K-theory spectrum of Y with support on T*. We have already secretly introduced this notation in the localization sequence (2.2.1.3), and we make this fact explicit: when U is itself quasi compact, Thomason's (proto)-localization theorem [63, 5.1] gives a homotopy fiber sequence

$$\mathbf{K}^{|T|}(Y) \to \mathbf{K}(Y) \xrightarrow{\iota_U^*} \mathbf{K}(U)$$

apart possibly from the failure of surjectivity of the map $K_0(Y) \to K_0(U)$. Similarly, for $\mathcal{F}$ a family of supports on Y we denote by $\mathbf{K}^{\mathcal{F}}(Y)$ the corresponding K-theory spectrum. In the relative setting, we get the analogue fibration sequence

$$\mathbf{K}^{|T|}(Y;Z) \to \mathbf{K}(Y;Z) \to \mathbf{K}(U;Z \cap U)$$

where the term $\mathbf{K}^{|T|}(Y;Z)$ is defined as the homotopy fiber of the induced map $\mathbf{K}^{|T|}(Y) \to \mathbf{K}^{|T \cap Z|}(Z)$.

Definition 4.1.2. *Let I be a finite set and let $\mathcal{P}(I)$ be the set of subsets of I, seen as category with morphisms given by inclusions. We call I-cube a functor $\mathcal{X}\colon \mathcal{P}(I) \to \mathcal{C}$ from $\mathcal{P}(I)$ to some category $\mathcal{C}$. An n-cube is an I-cube with respect to a set I that has cardinality n. Given a subset J of I, the inclusion $\mathcal{P}(J) \to \mathcal{P}(I)$ defines a J-subcube of an I-cube $\mathcal{X}$.*

4.1.3. Let Y be a Noetherian separated scheme. Let $Y_1, \ldots, Y_n$ be a set closed subschemes. We define an n-cube of schemes as follows. For every $I \subset \{1, \ldots, n\}$, let Y_I be the subscheme $Y_I = \bigcap_{i \notin I} Y_i$. For $I = \{1, \ldots, n\}$, we conventionally set $Y_\emptyset = Y$, and we write $Y_i = Y_{\{1,\ldots,i-1,i+1,\ldots,n\}}$ for short. If $\varphi_{I,J}\colon I \subset J$, there is a corresponding closed embedding of subschemes of Y

$$\varphi_{I,J}\colon Y_I = \bigcap_{i \notin I} Y_i \to \bigcap_{j \notin J} Y_j = Y_J.$$

Pulling back along $\varphi_{I,J}$ defines a (contravariant) n-cube of spectra $I \mapsto \mathbf{K}(Y_I)$.

Definition 4.1.4. *The total (homotopy) fiber of $\mathbf{K}(Y_\bullet)$ is, by definition*

$$Fib(\mathbf{K}(Y_\bullet)) = \mathrm{hofib}(\mathbf{K}(Y_\emptyset) \to \underleftarrow{\mathrm{holim}}_{I \neq \emptyset} \mathbf{K}(Y_I)).$$

The iterated homotopy fiber of the K-theory spectra of $Y_\bullet$ is the Ω-spectrum inductively defined by

$$\mathbf{K}(Y;Y_1,\ldots,Y_n) = \mathrm{hofib}(\mathbf{K}(Y;Y_1,\ldots,Y_{n-1}) \to \mathbf{K}(Y_n;Y_1 \cap Y_n,\ldots,Y_{n-1} \cap Y_n))$$

Assume that each intersection scheme Y_I is provided with a family of supports $\mathcal{F}(Y_I)$ such that, for every $i \in I$ and every $Z \in \mathcal{F}(Y_I)$, the intersection of Z with Y_i is contained in

$\mathcal{F}(Y_{I\setminus\{i\}})$. We can repeat the above construction replacing everywhere the K-theory spectra with the corresponding spectra with support, obtaining in this way a corresponding spectrum $\mathbf{K}^{\mathcal{F}}(Y; Y_1, \ldots, Y_n)$.

4.1.5. We present now a dual construction. Let X be a Noetherian separated scheme (admitting an ample family of line bundles, as above) and let $i_Y : Y \hookrightarrow X$ be a closed subscheme of X. Assume that the morphism i_Y is a regular closed immersion. In particular, i_Y is a *perfect* projective morphism in the sense of [63, Definition 2.5.2], and by [63, 3.16.5] there is a well defined push-forward map $(i_Y)_* : \mathbf{K}^{TT}(Y) \to \mathbf{K}^{TT}(X)$. We denote by $\mathbf{K}(X/Y)$ the homotopy cofiber of $(i_Y)_*$,

$$\mathbf{K}(X/Y) = \mathrm{hocof}(\mathbf{K}(Y) \to \mathbf{K}(X)).$$

4.1.6. More generally, suppose that we are given a family of closed subschemes $Y_1, \ldots, Y_n$ of X. As in 4.1.1, consider the n-cocube of schemes (see Definition 4.1.2) $Y_\bullet$, setting $Y_{\{1,\ldots,n\}} = X$. Assume now that, for every $I \subset J \subset \{1, \ldots, n\}$, the morphism $\varphi_{I,J}$ is a regular closed immersion. This gives, again by [63, 3.16.5], a well-defined push-forward map between the K-theory spectra (so, it is a covariant construction)

$$(\varphi_{I,J})_* : \mathbf{K}(Y_I) \to \mathbf{K}(Y_J).$$

We define in this way an n-cube of spectra, $\mathbf{K}^W(Y_\bullet)$. The *total (homotopy) cofiber* of $\mathbf{K}(Y_\bullet)$ is by definition

$$Cof(\mathbf{K}(Y_\bullet)) = \mathrm{hocof}(\underset{I\neq\{1,\ldots,n\}}{\underrightarrow{\mathrm{hocolim}}}\, \mathbf{K}(Y_I) \to \mathbf{K}(Y_{\{1,\ldots,n\}})).$$

Definition 4.1.7. *We keep the above assumptions on X and $Y_1, \ldots, Y_n$. The iterated homotopy cofiber of the K-theory spectra of X and Y_* is the Ω-spectrum defined inductively*

$$\mathbf{K}(X/Y_1, \ldots, Y_n) = \mathrm{hocof}(\mathbf{K}(Y_n/Y_1 \cap Y_n, \ldots, Y_{n-1} \cap Y_n) \to \mathbf{K}(X/Y_1, \ldots, Y_{n-1})).$$

Remark 4.1.8. The following remark holds for the total homotopy fiber as well, but we will see in Chapter II that it plays a more important role for our goals in the cofiber construction. There is unique natural map $\mathbf{K}(X/Y_1, \ldots, Y_n) \to Cof(\mathbf{K}(Y_\bullet))$, that is a homotopy equivalence. The existence of the map and the fact that it is an equivalence is dual to [15, C.6] (this is also dual to [52, Proposition 5.5.4], that the reader can consult for a detailed proof). In particular, for every permutation σ of the set $\{Y_1, \ldots, Y_n\}$, we have maps

$$\mathbf{K}(X/Y_1, \ldots, Y_n) \to Cof(\mathbf{K}(Y_\bullet)) \leftarrow \mathbf{K}(X/Y_{\sigma(1)}, \ldots, Y_{\sigma(n)})$$

that are homotopy equivalences. In particular, there is a canonical "zig-zag" datum joining $\mathbf{K}(X/Y_1, \ldots, Y_n)$ and $\mathbf{K}(X/Y_{\sigma(1)}, \ldots, Y_{\sigma(n)})$, and thus the space of homotopies between different iterated homotopy cofibers is contractible. We will then forget the difference between the choices of order of the set of subschemes Y_*.

4.2. Loopings and relative K-theory. We put ourselves back into the geometric situation. Let Y be a regular k-variety and consider the n-cube of schemes defined by $Y \times \square^n = Y \times (\mathbb{P}^1 \setminus \{1\})^n$. Let $\partial\square^n$ denote the strict normal crossing divisor given by the union of the faces $F^n_{\varepsilon,i}$, for $\varepsilon \in \{0, \infty\}$ and $i \leq n$. Using the homotopy property of K-theory of regular schemes, there is a natural homotopy equivalence (see [43, Theorem 3.1])

$$\mathbf{K}(Y \times \square^n; Y \times \partial\square^n) \to \Omega^n \mathbf{K}(Y)$$

giving the isomorphisms

$$K_0(Y \times \square^n; Y \times \partial\square^n) \xrightarrow{\simeq} K_n(Y).$$

This construction gives a nice delooping of K-theory, and allow us to construct classes in higher K-groups by constructing classes in (multi)-relative K_0. For Y not regular, the canonical morphism $\mathbf{K}(\mathbb{A}^1_Y) \xrightarrow{\iota_0^*} \mathbf{K}(Y)$ fails to be a homotopy equivalence, and the construction has to be a bit modified. We take inspiration from [43] in doing so.

The new ingredient is the following: instead of homotopy invariance, we use the projective bundle formula, available by [63] for any quasi-compact and quasi-separated scheme. We recall the statement.

Theorem 4.2.1 (see [63], Theorem 4.1)**.** *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{E}$ be an algebraic vector bundle of rank r over X and let $\pi\colon \mathbb{P}\mathcal{E}_X \to X$ be the associated projective space bundle. Then there is a natural homotopy equivalence*

$$\prod^r \mathbf{K}(X) \xrightarrow{\sim} \mathbf{K}(\mathbb{P}\mathcal{E}_X) \tag{4.2.1.1}$$

given by the formula $(x_0, x_1, \dots, x_{r-1}) \mapsto \sum_{i=0}^{r-1} \pi^*(x_i) \otimes [\mathcal{O}_{\mathbb{P}\mathcal{E}}(-i)]$.

4.2.2. Let X be a scheme of finite type over k (though the reader is free to keep working with X quasi-compact and quasi-separated in this subsection). By (4.2.1.1), there is an isomorphism

$$K_*(\mathbb{P}^1_X) \simeq K_*(X)[\mathcal{O}] \oplus K_*(X)[\mathcal{O}(-1)], \tag{4.2.2.1}$$

where $K_*(X)[\mathcal{O}]$ and $K_*(X)[\mathcal{O}(-1)]$ are written with respect to the external product

$$K(X) \wedge K(\mathbb{P}^1_{\mathbb{Z}}) \to K(\mathbb{P}^1_X)$$

and $[\mathcal{O}]$ and $[\mathcal{O}(-1)]$ are elements in $K_0(\mathbb{P}^1_{\mathbb{Z}})$. It is convenient for us to change basis for the direct sum decomposition to

$$\{[\mathcal{O}], [\mathcal{O}] - [\mathcal{O}(-1)]\},$$

so to get

$$K_*(\mathbb{P}^1_X) \simeq K_*(X)[\mathcal{O}] \oplus K_*(X)([\mathcal{O}] - [\mathcal{O}(-1)]). \tag{4.2.2.2}$$

For $i \in \{0, 1, \infty\}$, let ι_i be the regular embedding

$$\iota_i\colon X \times \{i\} \to \mathbb{P}^1_X$$

and let $\pi\colon \mathbb{P}^1_X \to X$ be the projection. We have the associated pull-back morphisms

$$\iota_i^*\colon \mathbf{K}(\mathbb{P}^1_X) \to \mathbf{K}(X) \text{ for } i \in \{0, 1, \infty\},$$
$$\pi^*\colon \mathbf{K}(X) \to \mathbf{K}(\mathbb{P}^1_X)$$

and the push-forward morphisms

$$\iota_{i,*}\colon \mathbf{K}(X) \to \mathbf{K}(\mathbb{P}^1_X) \text{ for } i \in \{0, 1, \infty\}.$$

Note that since the projection π has a section, π^* is a split monomorphism, corresponding to the canonical inclusion of the first direct summand of (4.2.2.2). For $\iota = \iota_0$, ι_∞ or ι_1 and for every $j \in \mathbb{Z}$ we have $\iota^*(\mathcal{O}_{\mathbb{P}^1}(j)) = \mathcal{O}$. Thus

$$\iota^*[\mathcal{O}_{\mathbb{P}^1}] = [\mathcal{O}] = 1, \text{ and } \iota^*([\mathcal{O}] - [\mathcal{O}(-1)]) = 0$$

in $K_0(X)$. Hence we see that on the direct sum decomposition of $K_*(\mathbb{P}^1_X)$, the pull back morphisms along the three rational sections all agree and they correspond to the canonical projection on the first component, splitting the pull back along the projection π^*. In particular, one has that the map

$$\mathbf{K}(\mathbb{P}^1_X) \xrightarrow{\iota_0^* - \iota_\infty^*} \mathbf{K}(X) \tag{4.2.2.3}$$

is homotopy equivalent to the zero map.

By the Projection Formula [63, 3.17], the diagram

$$\begin{array}{ccccc} \mathbf{K}(X)\wedge\mathbf{K}(X) & \xrightarrow{\pi^*\wedge\iota_*} & \mathbf{K}(\mathbb{P}^1_X)\wedge\mathbf{K}(\mathbb{P}^1_X) & \xrightarrow{\otimes} & \mathbf{K}(\mathbb{P}^1_X) \\ & \searrow^{\otimes} & & \nearrow_{\iota_*} & \\ & & \mathbf{K}(X) & & \end{array}$$

commutes, up to canonically chosen homotopy. Thus the diagram

$$\begin{array}{ccc} \mathbf{K}(X)\simeq\mathbf{K}[\mathcal{O}_X] & \xrightarrow{\iota_*} & \mathbf{K}(\mathbb{P}^1_X) \\ \downarrow{\scriptstyle \pi^*} & & \\ \mathbf{K}(\mathbb{P}^1_X)[\iota_*\mathcal{O}_X] & & \end{array}$$

commutes as well and we see that the push-forward along the inclusion ι is a monomorphism on the K-groups, split by π_*, that corresponds to the inclusion on the second direct summand of (4.2.2.2), since $[\iota_*\mathcal{O}_X] = [\mathcal{O}] - [\mathcal{O}(-1)]$ in $K_0(\mathbb{P}^1_X)$. In particular, the homotopy cofiber $\mathbf{K}(\mathbb{P}^1_X/X\times\{1\}) = \mathbf{K}(\mathbb{P}^1_X/X\times F^1_1)$ is homotopy equivalent to $\mathbf{K}(X)$ via the projection map π^*. For $\varepsilon\in\{0,\infty\}$, consider the homotopy fiber

$$\mathbf{K}(\mathbb{P}^1_X;X\times F^1_\varepsilon/X\times F^1_1) = \mathrm{hofib}(\mathbf{K}(\mathbb{P}^1_X/X\times F^1_1)\xrightarrow{\iota^*_\varepsilon}\mathbf{K}(X))$$

Since ι^*_ε is a homotopy equivalence, $\mathbf{K}(\mathbb{P}^1_X;X\times F^1_\varepsilon/X\times F^1_1)$ is contractible and thus the iterated homotopy fiber/cofiber

$$\mathbf{K}(\mathbb{P}^1_X;X\times F^1_0,X\times F^1_\infty/X\times F^1_1)\simeq\mathrm{hocof}(\mathbf{K}(X)\xrightarrow{\iota_{1,*}}\mathbf{K}(X\times\mathbb{P}^1;X\times F^1_0,X\times F^1_\infty))$$

is homotopy equivalent to $\Omega\mathbf{K}(X)$.

4.2.3. More generally, consider $X\times(\mathbb{P}^1)^n = X\times\overline{\square}^n$. An iterated application of the projective bundle theorem shows that $\mathbf{K}(X\times(\mathbb{P}^1)^n = X\times\overline{\square}^n)$ decomposes as $2n$-copies of the K-theory spectrum of X, two copies for each copy of $\mathbb{P}^1$ in the closed box $\overline{\square}^n$.

Let $Cof(\mathbf{K}(X\times(\overline{\square}^n)/X\times F^n_{1,\bullet})$ be total homotopy cofiber of the n-cube of schemes $(X\times F^n_{1,i}\hookrightarrow X\times(\mathbb{P}^1)^n)^n_{i=1}$, where $F^n_{1,i}$ denotes the face $y_i=1$ on the i-th copy of $\mathbb{P}^1$, with respect to the push-forward along the inclusion of faces with value 1. It is clear by construction that $Cof(\mathbf{K}(X\times(\overline{\square}^n)/X\times F^n_{1,\bullet})$ is homotopy equivalent to $\mathbf{K}(X)$.

For $\emptyset\neq I\subset\{1,\dots,n\}\times\{0,\infty\}$, consider the subscheme $X\times\partial\overline{\square}^n_I$ of $X\times\overline{\square}^n$ given by

$$X\times\partial\overline{\square}^n_I = \bigcap_{(k,\varepsilon)\notin I} X\times F^n_{k,\varepsilon}\hookrightarrow X\times\overline{\square}^n$$

For fixed I, consider for every $(k,\varepsilon)\in I$, the inclusion of the face $\iota^n_{1,k}\colon X\times F^n_{1,k}\to X\times\partial\overline{\square}^n_I$. This defines another (co)cube of schemes, and a corresponding (co)cube of spectra with maps induced by push-forward

$$\iota^n_{1,k,*}\colon\mathbf{K}(X\times F^n_{1,k})\to\mathbf{K}(X\times\partial\overline{\square}^n_I = X\times F_{I'}\times\overset{\overset{k}{\vee}}{\mathbb{P}^1}\times F_{I''})$$

for a partition $I = I'\cup I''$ with the obvious convention. We denote by $Cof(\mathbf{K}(X\times\partial\overline{\square}^n_I)/X\times F^n_{1,\bullet})$ its total homotopy cofiber. The following Lemma is now proved by descending induction on n.

Lemma 4.2.4. *The total homotopy fiber/cofiber*

$$\begin{aligned}&\mathbf{K}(X\times\overline{\square}^n; X\times\partial\overline{\square}^n/X\times F^n)\\&\quad=\mathrm{hofib}(Cof(\mathbf{K}(X\times\overline{\square}^n)/X\times F^n_{1,\bullet})\to\underset{I\neq\emptyset}{\underleftarrow{\mathrm{holim}}}\,Cof(\mathbf{K}(X\times\partial\overline{\square}^n_I)/X\times F^n_{1,\bullet}))\end{aligned}$$

is homotopy equivalent to the n-th loop $\Omega^n\mathbf{K}(X)$.

4.2.5. Let X be as above and let Y be a closed subset of X (if the reader is still considering X quasi-compact and quasi-separated, she might want to assume that the open complement $U = X\setminus Y$ is quasi compact as well). The closed immersion ι_Y gives a pullback morphism on the K-theory spectra, and gives induced pullback morphisms between the cubical objects $X\times\overline{\square}^n$ and $Y\times\overline{\square}^n$. We denote by $\mathbf{K}((X;Y)\times\overline{\square}^n;(X;Y)\times\partial\overline{\square}^n/(X;Y)\times F^n)$ the homotopy fiber

$$\mathrm{hofib}(\mathbf{K}(X\times\overline{\square}^n; X\times\partial\overline{\square}^n/X\times F^n)\xrightarrow{\iota_Y^*}\mathbf{K}(Y\times\overline{\square}^n; Y\times\partial\overline{\square}^n/Y\times F^n)).$$

By Lemma 4.2.4, we get a natural homotopy equivalence

$$\mathbf{K}((X;Y)\times\overline{\square}^n;(X;Y)\times\partial\overline{\square}^n/(X;Y)\times F^n)\xrightarrow{\sim}\Omega^n\mathbf{K}(X;Y) \tag{4.2.5.1}$$

for the relative K theory spectrum $\mathbf{K}(X;Y)$.

4.3. A cycle class map for 0-cycles with modulus. Let k be a field. Assume that $\overline{X}$ is an integral and regular quasi-projective k-variety, and let D be an effective Cartier divisor on it. Assume that the support $|D|$ of D is a strict normal crossing divisor on $\overline{X}$. We will make systematic use of Adams operations on relative K-theory with support. For their construction, we refer the reader to [44, Section 5 and 7].

4.3.1. Write $d = \dim\overline{X}$. Recall from Definition 1.2.9 that for every $n\geq 0$, the group $z^{d+n}(\overline{X}|D,n)$ is the free abelian group generated by the set $C^{d+n}(\overline{X}|D,n)$ of closed points P in $\overline{X}\times\square^n$ such that $P\notin D\times\square^n$ and $P\notin\overline{X}\times F^n_{i,\varepsilon}$ for $i=1,\dots,n$ and $\varepsilon\notin\{0,\infty\}$. Clearly, this set coincides with the set of closed points in $\overline{X}\times\overline{\square}^n$ that are disjoint from $D\times\overline{\square}^n$ and that do not meet any face $\overline{X}\times F^n_{i,\eta}$, for $i=1,\dots,n$ and $\eta\in\{0,1,\infty\}$.

4.3.2. Take a point P in $C^{d+n}(\overline{X}|D,n)$. Since $\overline{X}$ is regular and quasi-projective, the module $\mathcal{O}_P$ is quasi-isomorphic in the derived category of $\mathcal{O}_{\overline{X}\times\square^n}$-modules to a bounded complex of vector bundles. In particular, we have an isomorphism

$$\mathbb{Q}=K_0(k(P))_{\mathbb{Q}}=K_0(k(P))^{(0)}\xrightarrow{\simeq}K_0^{|P|}(\overline{X}\times\square^n)^{(d+n)}=K_0^{|P|}(\overline{X}\times\overline{\square}^n)^{(d+n)}.$$

The image of the class of 1 along the natural morphism $K_0^{|P|}(\overline{X}\times\overline{\square}^n)^{(d+n)}\to K_0(\overline{X}\times\overline{\square}^n)^{(d+n)}$ defines a class $[\mathcal{O}_P]$, that we call the *fundamental class* of the point P.

Extending this assignment by linearity, we have a group homomorphism

$$z^{d+n}(\overline{X}|D,n)_{\mathbb{Q}}\xrightarrow{\mathrm{cyc}^{d+n}}K_0(\overline{X}\times\overline{\square}^n)^{(d+n)},\quad\sum_{j=1}^r a_j[P_j]\mapsto\sum_{j=1}^r a_j[\mathcal{O}_{P_j}]. \tag{4.3.2.1}$$

Since any P in $C^{d+n}(\overline{X}|D,n)$ is disjoint from $D\times\overline{\square}^n$ and from the boundary divisor $\overline{X}\times\partial\overline{\square}^n$, we have a natural homotopy equivalence between $\mathbf{K}^{|P|}(\overline{X}\times\overline{\square}^n)$ and the multi-relative K-spectrum with support $\mathbf{K}^{|P|}((\overline{X},D)\times\overline{\square}^n;(\overline{X},D)\times\partial\overline{\square}^n)$. The group homomorphism (4.3.2.1) lifts first to a group homomorphism to the relative K_0 group, $K_0(\overline{X}\times\overline{\square}^n;D\times\overline{\square}^n)^{(d+n)}$, and then to the multi-relative K_0 group

$$z^{d+n}(\overline{X}|D,n)_{\mathbb{Q}}\xrightarrow{\mathrm{cyc}^{d+n}}K_0((\overline{X},D)\times\overline{\square}^n;(\overline{X},D)\times\partial\overline{\square}^n)^{(d+n)}.$$

Composing now with the map induced on π_0 by the natural morphism of spectra

$$\mathbf{K}((\overline{X};D)\times\overline{\square}^n;(\overline{X};D)\times\partial\overline{\square}^n)\to\mathbf{K}((\overline{X};D)\times\overline{\square}^n;(\overline{X};D)\times\partial\overline{\square}^n/(\overline{X};D)\times F^n),$$

we finally obtain a group homomorphism

$$z^{d+n}(\overline{X}|D,n)_{\mathbb{Q}}\xrightarrow{\mathrm{cyc}^{d+n}}K_0((\overline{X};D)\times\overline{\square}^n;(\overline{X};D)\times\partial\overline{\square}^n/(\overline{X};D)\times F^n)^{(d+n)}\simeq K_n(\overline{X};D)^{(d+n)},$$

where the last isomorphism follows from (4.2.5.1). We will show that this map factors through the higher Chow group $\mathrm{CH}^{d+n}(\overline{X}|D,n)_{\mathbb{Q},M_{\mathrm{ssup}}}$ defined using the strong sup-condition.

4.4. Exploiting the modulus condition: classes of curves. We want to study now how to relate a 1-cycle with modulus with a suitably defined class in the relative K-groups. We first discuss how the good-position conditions allow us to construct classes in the relative K_0-groups $K_0(\overline{X}\times\overline{\square}^n;\overline{X}\times\partial\overline{\square}^n)$. Essentially, we use the argument of [43, Lemma 2.2].

Given any integral curve $C\subset X\times\square^{n+1}$ that is in good position with respect to every face $X\times F^{n+1}_{i,\varepsilon}$, write $\overline{C}$ for its closure in $\overline{X}\times\overline{\square}^{n+1}$. Let $\mathcal{O}_{\overline{C}}$ be its structure sheaf and write $\iota_{\overline{C}}\colon\overline{C}\to\overline{X}\times\overline{\square}^{n+1}$ for the closed immersion. Since $\overline{X}$ is regular, the coherent $\mathcal{O}_{\overline{X}\times\overline{\square}^{n+1}}$ module $\mathcal{O}_{\overline{C}}$ is quasi-isomorphic to a bounded complex of vector bundles. Suppose moreover that $\overline{C}$ is itself regular. As for the case of points, sending 1 to the class of $\mathcal{O}_{\overline{C}}$ gives an isomorphism

$$K_0(k(\overline{C}))_{\mathbb{Q}}=K_0(k(\overline{C}))^{(0)}\xrightarrow{\simeq}K_0^{|\overline{C}|}(\overline{X}\times\overline{\square}^{n+1})^{(d+n)}.$$

Suppose now that W is an arbitrary purely 1-dimensional cycle in $\overline{X}\times\overline{\square}^n$. Write $z^0(W)_{\mathbb{Q}}$ for the $\mathbb{Q}$-vector space on the components of W. Assume that W is reduced. Removing the 0-dimensional subset W' of singular points of W does not change the group $z^0(W)_{\mathbb{Q}}=z^0(W\setminus W')_{\mathbb{Q}}$. The regularity of $\overline{X}$ gives then an isomorphism (see the argument at page 263 of [43])

$$z^0(W)_{\mathbb{Q}}\xrightarrow{\simeq}K_0^{|W\setminus W'|}(\overline{X}\times\overline{\square}^{n+1}\setminus W')^{(d+n)}\simeq K_0^{|W|}(\overline{X}\times\overline{\square}^{n+1})^{(d+n)}. \tag{4.4.0.1}$$

Write $z^{d+n}(\overline{X}\times\overline{\square}^{n+1})^W$ for the subgroup of $z^{d+n}(\overline{X}\times\overline{\square}^{n+1})$ supported on W. The isomorphism (4.4.0.1) gives then the map

$$z^{d+n}(\overline{X}\times\overline{\square}^{n+1})^W\xrightarrow{\mathrm{cyc}_W}K_0^{|W|}(\overline{X}\times\overline{\square}^{n+1})^{(d+n)}$$

4.4.1. Let now F be a component of $\partial\overline{\square}^n$ and assume that F intersects each component of W properly. We have a commutative diagram

$$\begin{array}{ccc}
z^{d+n}(\overline{X}\times\overline{\square}^{n+1})^W_{\mathbb{Q}} & \xrightarrow{\mathrm{cyc}_W} & K_0^{|W|}(\overline{X}\times\overline{\square}^{n+1})^{(d+n)}\\
\Big\downarrow{\scriptstyle\cdot\overline{X}\times F} & & \Big\downarrow{\scriptstyle\iota^*_{\overline{X}\times F}}\\
z^{d+n}(\overline{X}\times F)^{W\cap\overline{X}\times F}_{\mathbb{Q}} & \xrightarrow{\mathrm{cyc}_{W\cap\overline{X}\times F}} & K_0^{|W\cap\overline{X}\times F|}(\overline{X}\times F)^{(d+n)}.
\end{array}$$

If $[T]$ in $z^{d+n}(\overline{X}\times\overline{\square}^{n+1})^W_{\mathbb{Q}}$ is such that $T\cdot(\overline{X}\times F)=0$, the commutativity of the above diagram implies that the class $\iota^*_{\overline{X}\times F}(\mathrm{cyc}_W[T])$ is trivial in $K_0^{|W\cap\overline{X}\times F|}(\overline{X}\times F)^{(d+n)}$. In particular, the class $\mathrm{cyc}_W[T]$ lifts to the relative K_0-group $K_0^{|W|}(\overline{X}\times\overline{\square}^{n+1};\overline{X}\times F)^{(d+n)}$. Since the K_1 group with support $K_1^{|W\cap\overline{X}\times F|}(\overline{X}\times F)^{(d+n)}$ is equal to zero for weight reasons (see [43, (2.1), p. 261]), this class is well defined.

In particular, suppose that $[T]$ satisfies $T\cdot(\overline{X}\times F^{n+1}_{i,\varepsilon})=0$ for all $i=2,\dots,n+1$, $\varepsilon\in\{0,\infty\}$ and $T\cdot(\overline{X}\times F^{n+1}_{1,\infty})=0$.

Notation 4.4.2. *We write* $\partial'\overline{\square}^{n+1}$ *for the divisor* $(\sum_{i=2}^{n+1} F_{i,0}^{n+1} + F_{i,\infty}^{n+1}) + F_{1,\infty}^{n+1}$ *and* F^{n+1} *(resp.* F^n*) for the divisor* $\sum_{i=1}^{n+1} F_{i,1}^{n+1}$ *(resp. for the divisor* $\sum_{i=1}^{n} F_{i,1}^{n}$*) of* $\overline{\square}^{n+1}$ *(resp. of* $\overline{\square}^{n}$*).*

4.4.3. We can iterate the argument to get inductively a well defined class

$$\mathrm{cyc}_W[T] \in K_0^{|W|}(\overline{X}\times\overline{\square}^{n+1};\overline{X}\times\partial'\overline{\square}^{n+1})^{(d+n)}.$$

Projecting to the iterated cofiber along the faces $F_{i,1}^{n+1}$ for $i=1,\dots,n+1$, gives then a class (that we still denote in the same way)

$$\mathrm{cyc}_W[T] \in K_0^{|W|}(\overline{X}\times\overline{\square}^{n+1};\overline{X}\times\partial'\overline{\square}^{n+1}/\overline{X}\times F^{n+1})^{(d+n)}$$

and forgetting the support we end up with a class

$$\mathrm{cyc}_{d+n}[T] \in K_0(\overline{X}\times\overline{\square}^{n+1};\overline{X}\times\partial'\overline{\square}^{n+1}/\overline{X}\times F^{n+1})^{(d+n)}.$$

4.4.4. Let $N(z^{d+n}(\overline{X}|D,n+1)_{M_{\mathrm{ssup}}})$ be the group of admissible cycles in the normalized complex of $\underline{z}^{d+n}(\overline{X}|D,\bullet)_{M_{\mathrm{ssup}}}$. To simplify this already heavy notation, we suppress the subscript M_{ssup} in what follows.

A cycle in $N(z^{d+n}(\overline{X}|D,n+1))$ is a 1-dimensional cycle Z in $X\times\square^n$ such that, for every face $F_{i,\varepsilon}^{n+1}$, for $i=1,\dots,n+1$, $\varepsilon\in\{0,\infty\}$ but with $(i,\varepsilon)\neq((1,0))$, it satisfies $Z\cdot F_{i,\varepsilon}^{n+1}=0$. Moreover, Z is in good position with respect to the remaining face $X\times F_{1,0}^{n+1}$ and it satisfies the M_{ssup} modulus condition. We can furthermore assume that no component of Z is a vertical coordinate line, i.e., the pullback along a projection $p_j\colon X\times\square^{n+1}\to X\times\square^n$ of a point $P\in X\times\square^n$.

The group of 0-cycles with modulus $\mathrm{CH}^{d+n}(\overline{X}|D,n)_{M_{\mathrm{ssup}}}$ is then the cokernel

$$N(z^{d+n}(\overline{X}|D,n+1))_{\mathbb{Q}} \xrightarrow{\cdot X\times F_{1,0}^{n+1}} z^{d+n}(\overline{X}|D,n)_{\mathbb{Q}} \to \mathrm{CH}^{d+n}(\overline{X}|D,n)_{\mathbb{Q},M_{\mathrm{ssup}}} \to 0.$$

4.4.5. Let Z be a normalized admissible cycle $Z\in N(z^{d+n}(\overline{X}|D,n+1))$. The closure Z in $\overline{X}\times\overline{\square}^{n+1}$ is the closure of its components $Z_1,\dots,Z_r$. We will need to show the following

Claim 4.4.6. *The image of the class* $\mathrm{cyc}_{\overline{Z}}[\overline{Z}]$ *in the cofiber group*

$$K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times F_{j,1}^{n+1})^{(d+n)}$$

vanishes along the restriction to $K_0^{|\overline{Z}|}(D\times\overline{\square}^{n+1}/D\times F_{j,1}^{n+1})^{(d+n)}$.

This would in fact allow us to lift $\mathrm{cyc}_{\overline{Z}}[\overline{Z}]$ to a class in the relative group

$$K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}/\overline{X}\times F_{j,1}^{n+1})^{(d+n)} \tag{4.4.6.1}$$

modulo the image of $K_1^{|\overline{Z}\cap D\times\overline{\square}^{n+1}|}(D\times\overline{\square}^{n+1}/D\times F_{j,1}^{n+1})^{(d+n)}$. In the proof of this claim we will see how the modulus condition on the cycle plays a substantial role.

Remark 4.4.7. Since D is not regular, we cannot conclude as before that $K_1^{|\overline{Z}|}(D\times\overline{\square}^{n+1}/D\times F_{j,1}^{n+1})^{(d+n)}=0$. In fact, the vanishing of $K_1^{|W\cap\overline{X}\times F|}(\overline{X}\times F)^{(d+n)}$ in 4.4.1 is a special case of [43, Claim (2.1)], that uses the regularity of $\overline{X}$ and of the face F in an essential way. Since the class we are after is necessary only to produce relations in the relative K_0, we will not worry about the problem of the choice of the lifting. Of course, this would be the first problem to solve in order to construct a cycle class map for 1-cycles with modulus.

We postpone the proof of Claim 4.4.6 to the next Section 4.5. Write $p_{\overline{X}}^j$ for the natural map

$$K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1})^{(d+n)} \to K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times F_{j,1}^{n+1})^{(d+n)}$$

and assume we can choose a lifting $\beta_j(\overline{Z})$ for $p^j_{\overline{X}}(\mathrm{cyc}_{\overline{Z}}[\overline{Z}])$ in (4.4.6.1). We can trace this class in the iterated relative cofiber

$$K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}/\overline{X}\times F^{n+1})^{(d+n)}$$

as follows. For $i\neq j$, we look at the following commutative diagram

$$\begin{array}{ccc}
K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}/\overline{X}\times F^{n+1}_{j,1})^{(d+n)} & \xrightarrow{\mathrm{rel}_j} & K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times F^{n+1}_{j,1})^{(d+n)}\\
\downarrow p^{j,i}_{(\overline{X},D)} & & \downarrow p^{j,i}_{\overline{X}}\\
K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}/\overline{X}\times (F^{n+1}_{j,1},F^{n+1}_{i,1}))^{(d+n)} & \xrightarrow[\mathrm{rel}_{i,j}]{} & K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times (F^{n+1}_{j,1},F^{n+1}_{i,1}))^{(d+n)}\\
\uparrow p^{i,j}_{(\overline{X},D)} & & \uparrow p^{i,j}_{\overline{X}}\\
K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}/\overline{X}\times F^{n+1}_{i,1})^{(d+n)} & \xrightarrow{\mathrm{rel}_i} & K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times F^{n+1}_{i,1})^{(d+n)}
\end{array}$$

and we define $\beta_{i,j}(\overline{Z})$ to be $p^{j,i}_{(\overline{X},D)}(\beta_j(\overline{Z}))$. We have by construction

$$\mathrm{rel}_{i,j}(p^{j,i}_{(\overline{X},D)}(\beta_j(\overline{Z}))) = p^{j,i}_{\overline{X}}(\mathrm{rel}_j(\beta_j(\overline{Z})) = p^{j,i}_{\overline{X}}(p^j_{\overline{X}}(\mathrm{cyc}_{\overline{Z}}[\overline{Z}])) = p^{i,i}_{\overline{X}}(p^i_{\overline{X}}(\mathrm{cyc}_{\overline{Z}}[\overline{Z}]))$$

(commuting i and j), so that $\beta_{i,j}(\overline{Z})$ is in fact a lift of the image of $\mathrm{cyc}_{\overline{Z}}[\overline{Z}]$ along the natural map to the iterated cofiber

$$K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1})^{(d+n)}\to K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times(F^{n+1}_{j,1},F^{n+1}_{i,1}))^{(d+n)}$$

that therefore dies in the group $K_0^{|\overline{Z}\cap D|}(D\times\overline{\square}^{n+1}/D\times(F^{n+1}_{j,1},F^{n+1}_{i,1}))^{(d+n)}$.

Repeating this process for every face $F^{n+1}_{k,i}$, $k=1,\dots,n+1$, we obtain a class $\beta(\overline{Z})$ in the iterated relative cofiber

$$\beta(\overline{Z})\in K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)},$$

well defined up to a class in the image of the natural map

$$K_1^{|\overline{Z}\cap D\times\overline{\square}^{n+1}|}(D\times\overline{\square}^{n+1}/D\times F^{n+1})^{(d+n)}\to K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)}.$$

4.4.8. By construction, $\beta(\overline{Z})$ maps via the relativization map

$$K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}/\overline{X}\times F^{n+1})^{(d+n)}\to K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times F^{n+1})^{(d+n)}$$

to the image of the fundamental class $\mathrm{cyc}[\overline{Z}]\in K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1})^{(d+n)}$ constructed in (4.4.0.1). By assumption, the cycle Z satisfies $Z\cdot(X\times F^{n+1}_{i,\varepsilon})=0$ for all $i=2,\dots,n+1$, $\varepsilon\in\{0,\infty\}$ and $Z\cdot(X\times F^{n+1}_{1,\infty})=0$. By Remark 1.2.7, Z is already closed in $\overline{X}\times\square^{n+1}$, so that any extra point of intersection of $\overline{Z}$ with a face $\overline{X}\times\overline{F}^{n+1}_{i,\varepsilon}\subset\overline{X}\times\overline{\square}^{n+1}$ is supported on some intersection $\overline{X}\times\overline{F}^{n+1}_{i,\varepsilon}\cap\overline{X}\times F^{n+1}_{k,1}$ for some $k\in\{1,\dots,n+1\}$, $k\neq i$ (we introduce the overline notation $\overline{F}^{n+1}_{i,\varepsilon}$ for sake of clarity).

In particular, the class $\iota^*_{\overline{X}\times F^{n+1}_{i,\varepsilon}}(\mathrm{cyc}[\overline{Z}])$ is trivial in the quotient

$$K_0^{|\overline{Z}\cap\overline{X}\times F^{n+1}_{i,\varepsilon}|}(\overline{X}\times F^{n+1}_{i,\varepsilon}/\overline{X}\times F^{n+1}_{k,1}\cap\overline{X}\times F^{n+1}_{i,\varepsilon})^{(d+n)}. \tag{4.4.8.1}$$

Remark 4.4.9. The scheme $\overline{X}\times F^{n+1}_{i,\varepsilon}$ is regular, and the K_0-group with support on $\overline{Z}\cap\overline{X}\times F^{n+1}_{i,\varepsilon}$ depends only on the set of points $|\overline{Z}\cap\overline{X}\times F^{n+1}_{i,\varepsilon}|$. In particular, we have isomorphisms

$$K_0^{|\overline{Z}\cap\overline{X}\times F^{n+1}_{i,\varepsilon}|}(\overline{X}\times F^{n+1}_{i,\varepsilon}))^{(d+n)}\simeq K_0(k(\overline{Z}\cap\overline{X}\times F^{n+1}_{i,\varepsilon}))^{(0)}\simeq K_0(k(\overline{Z}\cap\overline{X}\times F^{n+1}_{i,\varepsilon}\cap\overline{X}\times F^{n+1}_{k,1}))^{(0)}$$

showing immediately that $\iota^*_{\overline{X}\times F^{n+1}_{i,\varepsilon}}(\mathrm{cyc}[\overline{Z}])$ dies in the quotient (4.4.8.1). This argument fails without the regularity assumption on $\overline{X}$, since we can't identify the K_0 with support with its support. This is precisely the problem of lifting classes to the group (4.4.6.1) that we discussed before.

The vanishing of $\iota^*_{\overline{X}\times F^{n+1}_{i,\varepsilon}}(\mathrm{cyc}[\overline{Z}])$ in the quotient group (4.4.8.1) allow us to lift it to class in the relative group

$$\mathrm{cyc}[\overline{Z}] \in K_0^{\overline{Z}}(\overline{X}\times\overline{\square}^{n+1}; \overline{X}\times F^{n+1}_{i,\varepsilon}/\overline{X}\times F^{n+1}_{k,1})^{(d+n)}$$

well-defined, since the group $K_1^{|\overline{Z}\cap\overline{X}\times F^{n+1}_{i,\varepsilon}|}(\overline{X}\times F^{n+1}_{i,\varepsilon}/\overline{X}\times F^{n+1}_{k,1}\cap\overline{X}\times F^{n+1}_{i,\varepsilon})^{(d+n)}$ is trivial (in fact, the group $K_1^{|\overline{Z}\cap\overline{X}\times F^{n+1}_{i,\varepsilon}|}(\overline{X}\times F^{n+1}_{i,\varepsilon})$ is trivial again by weight reasons, and so is, *a fortiori*, the cofiber group). We repeat the argument for every $i = 2,\ldots,n+1$ and $\varepsilon\in\{0,\infty\}$ and one more time for $i=1$, $\varepsilon=\infty$ to get a class in the iterated cofiber-fiber

$$\mathrm{cyc}[\overline{Z}] \in K_0^{\overline{Z}}(\overline{X}\times\overline{\square}^{n+1}; \overline{X}\times\partial'\overline{\square}^{n+1}/\overline{X}\times F^{n+1})^{(d+n)}$$

where $\partial'\overline{\square}^{n+1}$ and $F^{n+1}_{k,1}$ are defined as in Notation 4.4.2.

The lifting property that we just showed for $\mathrm{cyc}[\overline{Z}]$ implies in fact the same property for the chosen "relative" lift $\beta(\overline{Z})$: this is in fact obvious once one accepts our claim about the vanishing of $\mathrm{cyc}[\overline{Z}]$ in the cofiber groups $K_0^{|\overline{Z}|}(D\times\overline{\square}^{n+1}/D\times F^{n+1}_{j,1})^{(d+n)}$.

In particular, we finally obtain a class — that we keep denoting $\beta(\overline{Z})$ — in the following multi-relative K-group:

$$\beta(\overline{Z}) \in K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}; (\overline{X},D)\times\partial'\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)}, \tag{4.4.9.1}$$

which is well defined up to a relative K_1-class, supported on $|\overline{Z}\cap D\times\overline{\square}^{n+1}|$. Forgetting the support gives a class

$$\beta_{d+n}[\overline{Z}] \in K_0((\overline{X},D)\times\overline{\square}^{n+1}; (\overline{X},D)\times\partial'\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)}$$

well defined up to elements in the image of $K_1(D\times\overline{\square}^{n+1}; D\times\partial'\overline{\square}^{n+1}/D\times F^{n+1})^{(d+n)}$.

We can actually give a more precise statement. If we write Φ_Z for the composite map

$$\begin{array}{c} K_1^{|\overline{Z}\cap D\times\overline{\square}^{n+1}|}(D\times\overline{\square}^{n+1}; D\times\partial'\overline{\square}^{n+1}/D\times F^{n+1})^{(d+n)} \\ \big\downarrow \\ K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1}; (\overline{X},D)\times\partial'\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)} \\ \big\downarrow \\ K_0((\overline{X},D)\times\overline{\square}^{n+1}; (\overline{X},D)\times\partial'\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)} \end{array}$$

we see that the class $\beta_{d+n}[\overline{Z}]$ is well defined up to the image of Φ_Z. Write $G_1^{d+n}(\overline{X}|D,n+1)$ for the limit

$$G_1^{d+n}(\overline{X}|D,n+1) = \varinjlim_{Z\in N(z^{d+n}(\overline{X}|D,n+1))_{\mathbb{Q}}} K_1^{|\overline{Z}\cap D\times\overline{\square}^{n+1}|}(D\times\overline{\square}^{n+1}; D\times\partial'\overline{\square}^{n+1}/D\times F^{n+1})^{(d+n)}$$

and let $\Phi\colon G_1^{d+n}(\overline{X}|D,n+1)\to K_0((\overline{X},D)\times\overline{\square}^{n+1}; (\overline{X},D)\times\partial'\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)}$ be the induced map. We can at this point define a map

$$N(z^{d+n}(\overline{X}|D,n+1))_{\mathbb{Q}} \xrightarrow{\beta_{d+n}} \widetilde{K_0}((\overline{X},D)\times\overline{\square}^{n+1}; (\overline{X},D)\times\partial'\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)}$$

where $\widetilde{K_0}$ denotes the quotient of the multi-relative K_0 group by the image of Φ. Using this construction, we can finally prove the following

Theorem 4.4.10. *The cycle class map*

$$z^{d+n}(\overline{X}|D,n)_{\mathbb{Q}} \xrightarrow{\mathrm{cyc}^{d+n}} K_0((\overline{X};D)\times\overline{\square}^n;(\overline{X};D)\times\partial\overline{\square}^n/(\overline{X};D)\times F^n)^{(d+n)} \simeq K_n(\overline{X};D)^{(d+n)}$$

factors through $\mathrm{CH}^{d+n}(\overline{X}|D,n)_{\mathbb{Q}}$.

Proof. We have to show that the composition $\mathrm{cyc}^{d+n}\circ(-\cdot X\times F^{n+1}_{1,0})$ is trivial. Let $Z\in N(z^{d+n}(\overline{X}|D,n+1))_{\mathbb{Q}}$ and choose a class $\beta[\overline{Z}]$ as in (4.4.9.1), lifting the canonical class $\mathrm{cyc}[\overline{Z}]$. Let $W=|\overline{Z}\cap\overline{X}\times\overline{F}^{n+1}_{1,0}|$ and $W'=|Z\cap\overline{X}\times F^{n+1}_{1,0}|$. Clearly, the finite set of points $W\setminus W'$ is contained in the union of the faces $F^{n+1}_{k,1}$ of $\overline{\square}^{n+1}$. We have

$$\begin{array}{ccccc}
K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1})^{(d+n)} & \longrightarrow & K_0^{|\overline{Z}|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times F^{n+1})^{(d+n)} & & \\
\downarrow{\scriptstyle \iota^*_{\overline{X}\times\overline{F}^{n+1}_{1,0}}} & & \downarrow{\scriptstyle \iota^*_{\overline{X}\times\overline{F}^{n+1}_{1,0}}} & & \\
K_0^{W}(\overline{X}\times F^{n+1}_{1,0})^{(d+n)} & \longrightarrow & K_0^{|W|}(\overline{X}\times F^{n+1}_{1,0}/\overline{X}\times F^{n})^{(d+n)} & \xrightarrow{\simeq} & K_0^{|W'|}(\overline{X}\times F^{n+1}_{1,0}/\overline{X}\times F^{n})^{(d+n)}
\end{array}$$

Where the last isomorphism follows from the same argument used in 4.4.8. In particular, the class of $\iota^*_{\overline{X}\times\overline{F}^{n+1}_{1,0}}(\mathrm{cyc}[\overline{Z}])$ in the cofiber group $K_0^W(\overline{X}\times\overline{F}^{n+1}_{1,0}/\overline{X}\times F^n)^{(d+n)}$ agrees with the class $\iota^*(\overline{X}\times F^{n+1})(\mathrm{cyc}[Z])\in K_0^{|W'|}(\overline{X}\times\overline{\square}^{n+1}/\overline{X}\times F^n)^{(d+n)}$, so that we don't see the "extra intersection points" given by the closure of Z in $\overline{X}\times\overline{\square}^{n+1}$. Thus

$$\mathrm{cyc}^{d+n}\circ(-\cdot X\times F^{n+1}_{1,0})(Z)=\iota^*_{\overline{X}\times F^{n+1}}(\mathrm{cyc}[Z])=\iota^*_{\overline{X}\times\overline{F}^{n+1}_{1,0}}(\mathrm{cyc}[\overline{Z}])=\iota^*_{\overline{X}\times\overline{F}^{n+1}_{1,0}}(\beta(\overline{Z}))$$

in

$$K_0^{|W'|}(\overline{X}\times F^{n+1}_{1,0}/\overline{X}\times F^n)^{(d+n)}=K_0^{|W'|}((\overline{X};D)\times\overline{\square}^n;(\overline{X};D)\times\partial\overline{\square}^n/(\overline{X};D)\times F^n)^{(d+n)}.$$

Let $K_0^{d+n}(\overline{X}|D,n)$ be the limit

$$K_0^{d+n}(\overline{X}|D,n)=\varinjlim_{P\in(z^{d+n}(\overline{X}|D,n))_{\mathbb{Q}}} K_0^{|P|}((\overline{X},D)\times\overline{\square}^n;(\overline{X},D)\times\partial\overline{\square}^n/(\overline{X};D)\times F^n)^{(d+n)}$$

and $K_0^{d+n}(\overline{X}|D,n+1)$ be the limit

$$K_0^{d+n}(\overline{X}|D,n+1)=\varinjlim_{Z\in N(z^{d+n}(\overline{X}|D,n+1))_{\mathbb{Q}}} K_0^{|\overline{Z}|}((\overline{X},D)\times\overline{\square}^{n+1};(\overline{X},D)\times\partial'\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)}.$$

By construction, the cycle class map cyc^{d+n} factors through $K_0^{d+n}(\overline{X}|D,n)$, and we have a commutative diagram

$$\begin{array}{ccccc}
N(z^{d+n}(\overline{X}|D,n+1))_{\mathbb{Q}} & \xrightarrow{\beta_{d+n}} & K_0^{d+n}(\overline{X}|D,n+1) & \xrightarrow{\Psi} & \\
\downarrow{\scriptstyle \cdot X\times F^{n+1}_{1,0}} & & \downarrow{\scriptstyle \iota^*_{\overline{X}\times F^{n+1}_{1,0}}} & & \searrow \\
z^{d+n}(\overline{X}|D,n)_{\mathbb{Q}} & \longrightarrow & K_0^{d+n}(\overline{X}|D,n) & \longrightarrow & K_0((\overline{X};D)\times\overline{\square}^n;(\overline{X};D)\times\partial\overline{\square}^n/(\overline{X};D)\times F^n)^{(d+n)}
\end{array}$$

where the bottom composite $z^{d+n}(\overline{X}|D,n)_{\mathbb{Q}}\to K_0((\overline{X};D)\times\overline{\square}^n;(\overline{X};D)\times\partial\overline{\square}^n/(\overline{X};D)\times F^n)^{(d+n)}$ is cyc^{d+n}.

Now note that the map Ψ has to factors through

$$K_0((\overline{X},D)\times\overline{\square}^{n+1};(\overline{X},D)\times\partial'\overline{\square}^{n+1}/(\overline{X},D)\times F^{n+1})^{(d+n)}$$

But the latter group is zero. In fact, let Y be either $\overline{X} \times \overline{\square}^n$ or $D \times \overline{\square}^n$ and identify $Y \times \mathbb{P}^1$ with $\overline{X} \times \overline{\square}^{n+1}$ or with $D \times \overline{\square}^{n+1}$ accordingly. We have then the exact sequence of relative K-groups

$$\ldots \to K_p(Y \times \mathbb{P}^1; Y \times \{\infty\}/Y \times \{1\}) \to K_p(Y \times \mathbb{P}^1/Y \times \{1\}) \to K_p(Y \times \{0\}) \to \ldots$$

However, the projective bundle formula tells us that the maps $K_p(Y \times \mathbb{P}^1/Y \times \{1\}) \to K_p(Y \times \{0\})$ are all isomorphisms, so that the groups $K_p(Y \times \mathbb{P}^1; Y \times \{\infty\}/Y \times \{1\})$ are trivial. Thus the composition $\mathrm{cyc}^{d+n} \circ (- \cdot X \times F^{n+1}_{1,0})$ is trivial, proving the Theorem. □

4.5. Lifting classes. In this Section we finally explain how to construct the desired lifting. We resume the notations of 4.4.4, so suppose that Z is a cycle in $N(z^{d+n}(\overline{X}|D, n+1))$. Note that the class $\mathrm{cyc}_{\overline{Z}}[\overline{Z}_k] \in K_0^{|\overline{Z}|}(\overline{X} \times \overline{\square}^{n+1})^{(d+n)}$ is the image of $\mathrm{cyc}_{\overline{Z}_k}[\overline{Z}_k] \in K_0^{|\overline{Z}_k|}(\overline{X} \times \overline{\square}^{n+1})^{(d+n)}$ via the natural map

$$\rho_{\overline{Z}_k, \overline{Z}} \colon K_0^{|\overline{Z}_k|}(\overline{X} \times \overline{\square}^{n+1})^{(d+n)} \to K_0^{|\overline{Z}|}(\overline{X} \times \overline{\square}^{n+1})^{(d+n)}$$

In particular, the class $\mathrm{cyc}_{\overline{Z}}[\overline{Z}]$ is the sum $\sum_{k=1}^{r} \rho_{\overline{Z}_k, \overline{Z}} \mathrm{cyc}_{\overline{Z}_k}[\overline{Z}_k]$. It is then enough to show that each $\mathrm{cyc}_{\overline{Z}_k}[\overline{Z}_k]$ projects to a class in $K_0^{|\overline{Z}_k|}(\overline{X} \times \overline{\square}^{n+1}/\overline{X} \times F^{n+1}_{j,1})^{(d+n)}$ that vanishes when restricted to $K_0^{|\overline{Z}_k|}(D \times \overline{\square}^{n+1}/D \times F^{n+1}_{j,1})^{(d+n)}$. In order to show it, we exploit the modulus condition.

By definition, the modulus condition on a cycle W is tested on its irreducible components. Consider now the case of our cycle Z. The strong sup-modulus conditions, satisfied by each Z_k, takes the following form. Write $\overline{Z}_k$ for the closure of Z_k in $\overline{X} \times \overline{\square}^{n+1}$. Let $\varphi_{\overline{Z}_k} \colon \overline{Z}_k^N \to \overline{X} \times \overline{\square}^{n+1}$ be the normalization morphism followed by the natural inclusion. Then, there exists $j = j(k) \in \{1, \ldots, n+1\}$ such that

$$\varphi^*_{\overline{Z}_k}(D \times \overline{\square}^{n+1}) \leq \varphi^*_{\overline{Z}_k}(\overline{X} \times F^{n+1}_{j,1}).$$

In particular, we have an inclusion of sets $\overline{Z}_k \cap D \times \overline{\square}^{n+1} \subseteq \overline{X} \times F^{n+1}_{j,1}$. Without loss of generality, we can assume that the intersection of $\overline{Z}_k$ with the divisor $D \times \overline{\square}^{n+1}$ is given by a single closed point P, that coincides with the intersection of $\overline{Z}_k$ with $\overline{X} \times F^{n+1}_{j,1}$. We have then the following commutative diagram

(4.5.0.1)
$$\begin{array}{ccc}
K_0^{|\overline{Z}_k|}(\overline{X} \times \overline{\square}^{n+1}/\overline{X} \times F^{n+1}_{j,1})^{(d+n)} & \longrightarrow & K_0^{|P|}(D \times \overline{\square}^{n+1}/D \times F^{n+1}_{j,1})^{(d+n)} \\
\uparrow{\scriptstyle p^j_{\overline{X}}} & & \uparrow \\
K_0^{|\overline{Z}_k|}(\overline{X} \times \overline{\square}^{n+1})^{(d+n)} & \xrightarrow{\iota^*_{D \times \overline{\square}^{n+1}}} & K_0^{|P|}(D \times \overline{\square}^{n+1})^{(d+n)} \\
\uparrow{\scriptstyle i^{n+1}_{j,1,*}} & & \uparrow{\scriptstyle i_{D \times F^{n+1}_{j,1},*}} \\
K_0^{|P|}(\overline{X} \times F^{n+1}_{j,1})^{(d+n-1)} & \longrightarrow & K_0^{|P|}(D \times F^{n+1}_{j,1})^{(d+n-1)}
\end{array}$$

We have to show that the restriction $\iota^*_{D \times \overline{\square}^{n+1}}([\mathcal{O}_{\overline{Z}_k}]) = \iota^*_{D \times \overline{\square}^{n+1}}(\mathrm{cyc}_{\overline{Z}_k}[\overline{Z}_k])$ of the fundamental class of $\overline{Z}_k$ along the divisor $D \times \overline{\square}^{n+1}$ is the image of a class $\alpha_{Z_k, j}$ in $K_0^{|P|}(D \times F^{n+1}_{j,1})^{(d+n)}$ along the push-forward $i_{D \times F^{n+1}_{j,1},*}$.

4.5.1. In order to treat uniformly the case where $\overline{Z}_k$ is not regular in a neighborhood of D, we change the notation a bit and consider the following slightly more general situation (see also [5, Section 5], where we use the same convention). Let Y be a smooth (connected) k-variety of dimension $d+1$, equipped with a smooth divisor F and an effective Cartier divisor D. Asume that F and D satisfy together the following condition:

(★) There is no common component of D and F, and $D_{red} + F$ is a (reduced) simple normal crossing divisor on Y.

Definition 4.5.2. *Let C be an integral curve contained in $X = Y - (F + D)$. Write $\overline{C}$ for the closure of C in Y and $\overline{C}^N$ for the normalization of $\overline{C}$. Let $\varphi_{\overline{C}}\colon \overline{C}^N \to Y$ be the natural map. We say that C satisfies the modulus condition with respect to the divisor D and the face F if the following inequality of Cartier divisors on $\overline{C}^N$ holds:*

$$\varphi^*_{\overline{C}}(D) \leq \varphi^*_{\overline{C}}(F)$$

Write $\iota_D\colon D \to Y$ (resp. $\iota_F\colon F \to Y$) for the inclusion of D (resp. of F) in Y and write $j_{D,F}\colon D \cap F \to D$ for the inclusion of the intersection of D and F inside D. Let C be an integral curve satisfying the modulus condition. Assume for simplicity that $\overline{C} \cap D$ is given by a single point P. In the current setting, the diagram (4.5.0.1) takes the following form:

$$\begin{array}{ccc}
K_0^{|\overline{C}|}(Y/F)^{(d)} & \longrightarrow & K_0^{|P|}(D/D \cap F)^{(d)} \\
\uparrow & & \uparrow \\
K_0^{\overline{C}}(Y)^{(d)} & \xrightarrow{\iota_D^*} & K_0^{|P|}(D)^{(d)} \\
\uparrow{\scriptstyle \iota_{F,*}} & & \uparrow{\scriptstyle j_{D,F,*}} \\
K_0^{|P|}(F)^{(d-1)} & \longrightarrow & K_0^{|P|}(D \cap F)^{(d-1)}
\end{array}$$

We write $[\overline{C}]$ for the fundamental class of $\overline{C}$ in $K_0^{\overline{C}}(Y)^{(d)}$ given by the class of the structure sheaf $\mathcal{O}_{\overline{C}}$.

Proposition 4.5.3. *The restriction $\iota_D^*([\overline{C}]) = \iota_D^*([\mathcal{O}_{\overline{C}}])$ of the fundamental class of $\overline{C}$ along the divisor D is the image of a class $\alpha_{\overline{C}}$ in $K_0^{|P|}(D \cap F)$ along the push-forward $j_{D,F,*}$.*

Proof. We start by assuming that $\overline{C}$ is regular in a neighborhood of P. Since $\overline{C}$ is not contained in D, the module $\mathcal{O}_{\overline{C}}$ is $\mathcal{O}_D$-torsion free and we have an equality

$$\iota_D^*([\mathcal{O}_{\overline{C}}]) = [\mathcal{O}_{\overline{C}} \otimes_{\mathcal{O}_Y} \mathcal{O}_D]$$

in $K_0^{|P|}(D)$, and $\mathcal{O}_{\overline{C}} \otimes_{\mathcal{O}_Y} \mathcal{O}_D$ is a module of finite homological dimension over $\mathcal{O}_D$. The class $\iota_D^*([\mathcal{O}_{\overline{C}}])$ is supported on P by assumption, and we can work locally around P in the following sense. Let $\mathcal{O}_P$ be the local ring of $\mathcal{O}_Y$ at P. Since Y is regular at P, $\mathcal{O}_P$ is a Noetherian regular equicharacteristic local ring. By Cohen's structure theorem ([11]), the completion $\widehat{\mathcal{O}}_P$ of $\mathcal{O}_P$ is isomorphic to a power series ring $K[\![x_1, \ldots, x_d, t_1]\!]$, where t_1 is the image in $\mathcal{O}_P$ of a local parameter for the smooth divisor F. Up to reordering, the ideal I_D of the divisor D in $\widehat{\mathcal{O}}_P$ will be then generated by an element $\prod_{i=1}^s x_i^{m_i}$. By [63, Proposition 3.19], we can replace the group $K_0^{|P|}(D)$ with $K_0^{|P|}(\mathrm{Spec}(\widehat{\mathcal{O}}_P/I_D))$ and the group $K_0^{|P|}(D \cap F)$ with $K_0^{|P|}(\mathrm{Spec}(\widehat{\mathcal{O}}_P/(I_D, t_1)))$. Let $\widehat{\mathcal{O}}_{\overline{C},P}$ be the completion of the local ring of $\overline{C}$ at P. After this reduction, the class $\iota_D^*([\mathcal{O}_{\overline{C}}])$ we are after is (the class of) the module $\widehat{\mathcal{O}}_{\overline{C},P} \otimes \widehat{\mathcal{O}}_P/I_D$. Since $\overline{C}$ is regular at P, we can assume that the image of one of the paramenters x_i or t_1 is a local parameter for $\overline{C}$. Without loss of generality (the proof is substantially identical in other cases), we assume that this role is played by x_d. Thus, we can write

$$x_i = x_d^{a_i} v_i, \quad t_1 = x_d^b u_1, \quad \text{for } i = 1, \ldots, d-1. \tag{4.5.3.1}$$

and elements $v_i, u_1 \in \widehat{\mathcal{O}}^{\times}_{\overline{Z}_k,P}$, that we can write as power series in x_d. Note that since the curve is actually passing through the point P, the exponents a_i and b have to be positive. The modulus condition gives then the following inequality

$$\sum_{i=1}^{s} m_i a_i \leq b. \tag{4.5.3.2}$$

Write J for the ideal of $K[\![x_1, \ldots, x_d, t_1]\!]$ defined by the equations (4.5.3.1). By (4.5.3.2), the ideal $(J, \prod_{i=1}^{s} x_i^{m_i})$ and the ideal $(\prod_{i=1}^{s} x_i^{m_i}, t_1, x_i - x_d^{a_i} v_i)_i$ coincide. Now, the module

$$\begin{aligned} M_{\overline{C},P} &= K[\![x_1, \ldots, x_d, t_1]\!]/(\prod_{i=1}^{s} x_i^{m_i}, t_1, x_i - x_d^{a_i} v_i)_i \\ &\simeq (K[\![x_1, \ldots, x_d]\!]/(\prod_{i=1}^{s} x_i^{m_i}))[\![t_1]\!]/(t_1, x_i - x_d^{a_i} v_i)_i \end{aligned}$$

has finite homological dimension as module over $(K[\![x_1, \ldots, x_d]\!]/(\prod_{i=1}^{s} x_i^{m_i}))[\![t_1]\!]/(t_1)$, and is supported on P.

It gives the a well defined class $[M_{\overline{C},P}]$ in the K_0 group with support $K_0^{|P|}(\mathrm{Spec}(\widehat{\mathcal{O}}_P/(I_D, t_1)))$ that satisfies

$$j_{D,F,*}[M_{\overline{C},P}] = [\widehat{\mathcal{O}}_{\overline{C},P} \otimes \widehat{\mathcal{O}}_P/I_D] = \iota_D^*([\mathcal{O}_{\overline{C}}])$$

as required.

We now deal with the case where $\overline{C}$ is not necessarily regular in a neighborhood of $\overline{C} \cap D$. Let $\varphi\colon \overline{C}^N \to \overline{C}$ be the normalization morphism. It fits in a commutative diagram

$$\begin{array}{ccc} \overline{C}^N & \stackrel{j}{\hookrightarrow} & Y \times \mathbb{P}^M \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle p} \\ \overline{C} & \hookrightarrow & Y \end{array}$$

where p is the natural projection. The curve $\overline{C}^N$ is now regular and embedded in the smooth variety $Y \times \mathbb{P}^M = \mathbb{P}^M_Y$, and satisfies the modulus condition with respect to $\mathbb{P}^M_D$ and the face $\mathbb{P}^M_F$. In particular, we can apply the normal case to conclude that the class of $\overline{C}^N$ in the cofiber group $K_0^{|\overline{C}^N|}(Y \times \mathbb{P}^M / F \times \mathbb{P}^M)$ dies when restricted to $D \times \mathbb{P}^M$. The projection formula of [63, Proposition 3.18] gives an equality of K_0-classes $p_*[\mathcal{O}_{\overline{C}^N}] = [p_*\mathcal{O}_{\overline{C}^N}] = [\mathcal{O}_{\overline{C}}] + [S]$, where S is a coherent sheaf on $\overline{C}$ supported on finitely many points $y_1, \ldots, y_r$. The class $[p_*\mathcal{O}_{\overline{C}^N}]$ in $K_0^{|\overline{C}|}(Y)^{(d)}$ maps then to a class in the cofiber group $K_0^{|\overline{C}|}(Y/F)^{(d)}$ that is mapped to zero by construction when restricted to $K_0^{|P|}(D/F \cap D)^{(d)}$. Let now be $T \subset \overline{C}$ be the (closed) of the points y_i. Since Y is regular, we have an isomorphism $K_0^{|\overline{C}|}(Y)^{(d)} \xrightarrow{\simeq} K_0^{|\overline{C} \setminus T|}(Y \setminus T)^{(d)}$. In particular, as the sheaf S is supported on T, we have $p_*[\mathcal{O}_{\overline{C}^N}] = [p_*\mathcal{O}_{\overline{C}^N}] = [\mathcal{O}_{\overline{C}}]$ in $K_0^{|\overline{C}|}(Y)^{(d)}$. We can thus replace $[\mathcal{O}_{\overline{C}}]$ with the push-forward class $[p_*\mathcal{O}_{\overline{C}^N}]$ and we are done. □

Applying Proposition 4.5.3 to the setting of 4.5, we can deduce the following Proposition, proving Claim 4.4.6.

Proposition 4.5.4. *The restriction $\iota^*_{D \times \overline{\square}^{n+1}}([\mathcal{O}_{\overline{Z}_k}]) = \iota^*_{D \times \overline{\square}^{n+1}}(\mathrm{cyc}_{\overline{Z}_k}[\overline{Z}_k])$ of the fundamental class of $\overline{Z}_k$ along the divisor $D \times \overline{\square}^{n+1}$ is the image of a class $\alpha_{Z_k,j}$ in $K_0^{|P|}(D \times F_{j,1}^{n+1})^{(d+n)}$ along the push-forward $i_{D \times F_{j,1}^{n+1},*}$.*

We conclude this section stating some properties of our cycle class map for $n = 0$. For the proofs, we refer the reader to [4]. The setting is the usual one.

Theorem 4.5.5 ([4], see Theorem 11.6)**.** *Let $\overline{X}$ be a smooth quasi-projective scheme of dimension $d \geq 1$ over a perfect field k and let $D \subset \overline{X}$ be an effective Cartier divisor. Then, the cycle class map*

$$\mathrm{cyc}^d \colon \mathrm{CH}^d(\overline{X}|D, 0) \simeq \mathrm{CH}_0(\overline{X}|D) \to K_0(\overline{X}; D)$$

is injective if k is algebraically closed and $\overline{X}$ is affine.

In fact, when $\overline{X}$ has dimension 2, this theorem can be improved in the following sense.

Theorem 4.5.6 ([4], see Theorem 12.4 and 12.5)**.** *Let $\overline{X}$ be a smooth quasi-projective surface over a perfect field k and $D \subset \overline{X}$ an effective Cartier divisor. Then there is a short exact sequence*

$$0 \to \mathrm{CH}_0(\overline{X}|D) \xrightarrow{\mathrm{cyc}^2} K_0(\overline{X}; D) \to \mathrm{Pic}(\overline{X}, D) \to 0.$$

In view of Proposition 2.1.5, when $\overline{X}$ is affine we can identify $\mathrm{Pic}(\overline{X}, D)$ with $\mathrm{CH}^1(\overline{X}|D)$, thus completely describing the relative K_0-group of the pair $(\overline{X}, D)$ in terms of Chow groups with modulus. We discussed in [4] further applications of this decomposition result.

CHAPTER II

Additive homotopy theory of Schemes

Notations and conventions. Throughout this Chapter we fix a base field k for which we assume to have resolution of singularities (see Section 1.3). Unless specified otherwise, all schemes will be assumed to be separated and of finite type over k. We write $\mathbf{Sm}(k)$ for the category of smooth quasi-projective k-schemes.

1. Categories of schemes with moduli conditions

1.1. Schemes with compactifications. Let X be a smooth k-scheme and let ∂X be a reduced codimension 1 closed subscheme of X with irreducible components $\partial X_1, \dots, \partial X_N$. We say that ∂X is a strict normal crossing divisor if for every subset I of $\{1, \dots, N\}$, the subscheme $\partial X_I = \bigcap_{i \in I} \partial X_i$ is smooth over k and of pure codimension $|I|$ in X. We will denote by ∂X_* the set of irreducible components of a normal crossing divisor ∂X and we write $|\partial X| = \cup_{i=1}^N \partial X_i$ for the support of ∂X. If $T_1, \dots, T_r$ are smooth integral codimension one subschemes of X such that their union is a strict normal crossing divisor, we say that the set $T_1, \dots, T_r$ form a normal crossing divisor on X.

Definition 1.1.1. *The category $\mathbf{Sm}_{\log}(k)$ is the category of pairs $(X, \partial X)$, where X is a smooth k-scheme and ∂X is a strict normal crossing divisor on X (possibly empty). A morphism*

$$f\colon (X, \partial X) \to (Y, \partial Y)$$

of pairs in $\mathbf{Sm}_{\log}(k)$ is a k-morphism $f\colon X \to Y$ such that for every irreducible component ∂Y_i of ∂Y, the reduced inverse image $f^{-1}(\partial Y_i)_{\mathrm{red}}$ is a strict normal crossing divisor on X and satisfies $f(X \setminus |\partial X|) \subseteq Y \setminus |\partial Y|$.

Suppose we are given two pairs $(X, \partial X)$ and $(Y, \partial Y)$ in $\mathbf{Sm}_{\log}(k)$. Write $\partial X_1, \dots, \partial X_M$ for the components of ∂X $\partial Y_1, \dots, \partial Y_N$ for the components of ∂Y. Their product $(X, \partial X) \times (Y, \partial Y)$ is by definition the pair $(X \times Y, X \times \partial Y + \partial X \times Y)$, where $(X \times \partial Y + \partial X \times Y)$ is by definition the normal crossing divisor formed by $X \times \partial Y_1, \dots, X \times \partial Y_N, \partial X_1 \times Y, \dots \partial X_M \times Y$. It's easy to see that $\times$ is the categorical product in $\mathbf{Sm}_{\log}(k)$. The terminal object in $\mathbf{Sm}_{\log}(k)$ is the pair $(\mathrm{Spec}(k), \emptyset)$.

Definition 1.1.2. *Let $f\colon (X, \partial X) \to (Y, \partial Y)$ be a morphism in $\mathbf{Sm}_{\log}(k)$. We say that f is minimal if $\partial X = f^{-1}(\partial Y)_{\mathrm{red}}$.*

1.1.3. Let $\omega\colon \mathbf{Sm}_{\log}(k) \to \mathbf{Sm}(k)$ be the functor $(X, \partial X) \mapsto X \setminus |\partial X|$. This functor has an obvious left adjoint

$$\lambda\colon \mathbf{Sm}(k) \leftrightarrows \mathbf{Sm}_{\log}(k)\colon \omega$$

that sends a smooth k-scheme X to the pair $(X, \emptyset)$. Indeed, any morphism $f\colon (X, \emptyset) \to (Y, \partial Y)$ in $\mathbf{Sm}_{\log}(k)$ is given by a morphism of k-schemes $f\colon X \to Y$ that has to factor through the open embedding $Y \setminus \partial Y = \omega((Y, \partial Y)) \to Y$. Note that the functors λ and ω both commute with products. There is also another functor

$$F\colon \mathbf{Sm}_{\log}(k) \to \mathbf{Sm}(k), \quad P = (X, \partial X) \mapsto F(P) = X,$$

that does not have any obvious left adjoint. If confusion does not arise, given a smooth k-scheme X, we will write just X in $\mathbf{Sm}_{\log}(k)$ for the pair $(X, \emptyset) = \lambda(X)$.

1.1.4. Let $\mathbb{P}^1$ be the projective line over k and let y be the standard rational coordinate on it. For every $n \geq 1$ we have a distinguished object $\overline{\square}^n$ in $\mathbf{Sm}_{\log}(k)$, defined as $\overline{\square}^n = ((\mathbb{P}^1)^n, F^n_\infty)$, where F^n_∞ denotes the normal crossing divisor $\sum_{i=1}^n (y_i = \infty)$. There are also maps $\iota^n_{\varepsilon,i} \colon \overline{\square}^n \hookrightarrow \overline{\square}^{n+1}$ for $\varepsilon \in \{0,1\}, i = 1, \ldots n+1$ given by the inclusion with

$$(\iota^n_{\varepsilon,i})^*(y_j) = y_j \text{ for } 1 \leq j < i\, , (\iota^n_{\varepsilon,i})^*(y_j) = y_{j-1} \text{ for } i < j \leq n+1 \text{ and } (\iota^n_{\varepsilon,i})^*(y_i) = \varepsilon,$$

as well as projections $p^n_i \colon \overline{\square}^n \to \overline{\square}^{n-1}$ for $i = 1, \ldots, n$ induced by $p^n_i \colon (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$ that forgets the i-th coordinate. We will denote by $\delta_n \colon \overline{\square}^n \to \overline{\square}^n \times \overline{\square}^n$ the diagonal map.

Remark 1.1.5. To get a feeling of the importance of the additional datum of a "divisor at infinity" ∂X on a smooth scheme X, consider the pairs $\mathbb{A}^1 = (\mathbb{A}^1, \emptyset)$ and $\overline{\square}^1 = (\mathbb{P}^1, \infty)$ and $\mathbb{P}^1 = (\mathbb{P}^1, \emptyset)$. The are canonical maps

$$\mathbb{A}^1 \to \overline{\square}^1 \to \mathbb{P}^1$$

and there are no-noncostant maps in the opposite directions, so that the three objects are all distinct in $\mathbf{Sm}_{\log}(k)$.

1.2. Modulus pairs à la Kahn-Saito-Yamazaki. We recall from [29] the following constructions.

Definition 1.2.1. *A modulus pair $M = (\overline{M}, M^\infty)$ consists of a scheme $\overline{M} \in \mathbf{Sch}(k)$ and an effective Cartier divisor $M^\infty \subset \overline{M}$, possibly empty, such that $\overline{M}$ is locally integral and the open (dense) subset $M^o = \overline{M} \setminus M^\infty$ is smooth and separated over k. A pair M is called proper if $\overline{M} \to \mathrm{Spec}(k)$ is proper.*

Definition 1.2.2. *Let M_1, M_2 be two modulus pairs. Consider a scheme-theoretic morphism*

$$f \colon M^o_1 \to M^o_2$$

over k. Let $\overline{\Gamma}_f \subset \overline{M}_1 \times_k \overline{M}_2$ be the closure of the graph of f and let p_1, p_2 be the two projections

$$p_1 \colon \overline{M}_1 \times \overline{M}_2 \to \overline{M}_1, \quad p_2 \colon \overline{M}_1 \times \overline{M}_2 \to \overline{M}_2.$$

Let $\varphi \colon \overline{\Gamma}^N_f \to \overline{M}_1 \times \overline{M}_2$ be the composition of the normalization morphism of the closure of the graph with the inclusion. We say that f is admissible for the pair M_1, M_2 if

i) the composition morphism $p_1 \circ \varphi \colon \overline{\Gamma}^N_f \to \overline{M}_1$ is proper,

ii) there is an inequality $\varphi^* p_1^*(M_1^\infty) \geq \varphi^* p_2^*(M_2^\infty)$ as Weil divisors on $\overline{\Gamma}^N_f$.

We denote by $\underline{\mathbf{M}}\mathbf{Sm}$ the category having objects modulus pairs and morphism admissible morphisms between them. With $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, we denote the subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ whose morphisms satisfy the additional condition that $p_1 \circ \varphi \colon \overline{\Gamma}^N_f \to \overline{M}_1$ is finite. Finally, we denote by $\mathbf{MSm}$ the full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ whose objects are proper modulus pairs.

Remark 1.2.3. Note the difference between the admissibility condition of a morphism in $\underline{\mathbf{M}}\mathbf{Sm}$ and a morphism in $\mathbf{Sm}_{\log}$, even in the case a morphism $f \colon M_1 \to M_2$ in $\underline{\mathbf{M}}\mathbf{Sm}$ is induced by a morphism of smooth schemes $f \colon \overline{M_1} \to \overline{M_2}$ (e.g. if f is a map in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$). In this situation, condition ii) in 1.2.2 reads

$$\nu^* M_1^\infty \geq \nu^* f^* M_2^\infty \tag{1.2.3.1}$$

where $\nu \colon \overline{M_1}^N \to \overline{M_1}$ is the normalization morphism. The inequality in (1.2.3.1) is an inequality of effective Weil divisors on a normal variety. Suppose now that both $\overline{M_1}$ and $\overline{M_2}$ are smooth and that the divisors M_1^∞ and M_2^∞ have SNC support. We immediately see that $M_1^\infty \geq f^* M_2^\infty$ implies that f gives rise to a map in $\mathbf{Sm}_{\log}(k)$ according to Definition 1.1.1. On the other hand, since the admissibility condition in $\mathbf{Sm}_{\log}(k)$ is checked on the reduced pull-back of the divisor, it is not true that a map in $\mathbf{Sm}_{\log}(k)$ gives rise to a map of pairs in $\underline{\mathbf{M}}\mathbf{Sm}$ (see below for a useful example).

1.2.4. For $M, N \in \underline{\mathbf{M}}\mathbf{Sm}$, we define their tensor product $L = M \otimes N$ by $\overline{L} = \overline{M} \times \overline{N}$ and $L^\infty = M^\infty \times \overline{N} + \overline{M} \times N^\infty$. This gives the categories $\underline{\mathbf{M}}\mathbf{Sm}$ and $\mathbf{MSm}$ a symmetric monoidal structure, with unit the modulus pair $(\mathrm{Spec}(k), \emptyset)$.

As noticed in [29], Warning 1.12, the tensor product $M_1 \otimes M_2$ does not have the universal property of products, since, for example, the diagonal morphism $M \xrightarrow{\Delta} M \otimes M$ is not admissible as soon as M^∞ is not empty. Indeed, let $M = (\mathbb{P}^1, \infty)$. Then $M \otimes M = (\mathbb{P}^1 \times_k \mathbb{P}^1, \infty \times \mathbb{P}^1 + \mathbb{P}^1 \times \infty)$. The diagonal map $\delta\colon \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is not admissible in $\underline{\mathbf{M}}\mathbf{Sm}$, since

$$\delta^*(\infty \times \mathbb{P}^1 + \mathbb{P}^1 \times \infty) = 2 \cdot \infty \not\leq \infty.$$

On the other hand, the map $\delta = \delta_1$ is an admissible morphism in $\mathbf{Sm}_{\log}(k)$ between $\overline{\square}^1$ and $\overline{\square}^1 \times \overline{\square}^1 = \overline{\square}^2$.

1.3. Inverting birational maps. We need to embed the category $\mathbf{Sm}_{\log}(k)$ in a larger category with the same objects but where we allow some morphims to be defined only after a proper birational transformation. Recall that we are assuming that k *admits resolution of singularities*, i.e. that the following two conditions hold:

(1) for any reduced scheme of finite type X over k, there exists a proper birational morphism $f\colon \tilde{X} \to X$ such that $\tilde{X}$ is smooth,
(2) for any smooth scheme X over k and a proper surjective morphism $Y \to X$ which has a section over a dense open subset U of X, there exists a sequence of blow-ups $X_n \to X_{n-1} \to \dots X_0 = X$, with smooth centers lying over $X \setminus U$, and a morphism $X_n \to Y$ such that the composition $X_n \to Y \to X$ is the structure morphism $X_n \to X$.

Definition 1.3.1. *Let $P = (X, \partial X) \in \mathbf{Sm}_{\log}(k)$. We denote by $\mathfrak{B}_P$ the category of admissible blow-ups of P. An object of $\mathfrak{B}_P$ is a morphism $\pi\colon P' = (X', \partial X') \to P$ with $P' \in \mathbf{Sm}_{\log}(k)$ induced by a projective birational map $\pi\colon X' \to X$ that restricts to an isomorphism on the open complements $\omega(X') \simeq \omega(X)$ and such that $|\partial X'| = |\pi^{-1}(\partial X)|$. Morphisms in $\mathfrak{B}_P$ are the minimal morphisms in $\mathbf{Sm}_{\log}(k)$ over P. If $\mathcal{S}_b$ denotes the class of admissible blow-ups of pairs, $\mathfrak{B}_P$ is the full subcategory of the comma category $\mathbf{Sm}_{\log}(k)/P$ given by the objects $P' \xrightarrow{s} P$ with $s \in \mathcal{S}_b$.*

In other words, an object in $\mathfrak{B}_P$ is a blow-up with center in a closed subscheme $Z \subset \partial X \subset X$.

Proposition 1.3.2. *The class $\mathcal{S}_b$ enjoys a calculus of right fractions. In particular, for every $P, Q \in \mathbf{Sm}_{\log}(k)$, the natural map*

$$\varinjlim_{P' \in \mathfrak{B}_P} \mathrm{Hom}_{\mathbf{Sm}_{\log}(k)}(P', Q) \to \mathrm{Hom}_{\mathbf{Sm}_{\log}(k)[\mathcal{S}_b^{-1}]}(P, Q)$$

is an isomorphism. Moreover, since for any $P \in \mathbf{Sm}_{\log}(k)$ the category $\mathfrak{B}_P$ contains a small cofinal subcategory, then the Hom *sets of $\mathbf{Sm}_{\log}(k)[\mathcal{S}_b^{-1}]$ are small.*

Proof. We recall from [17, I.2.2] the conditions that a class of morphisms Σ has to satisfy in order to enjoy calculus of right fractions (see also [29, Appendix A.5]). These conditions are: a) the identities of $\mathbf{Sm}_{\log}(k)$ are in Σ; b) if $u\colon X \to Y$ and $v\colon Y \to Z$ are in Σ, then their composition $v \circ u$ is also in Σ; c) for each diagram $X' \xrightarrow{s} X \xleftarrow{u} Y$ where $s \in \Sigma$, there exists a commutative square

(1.3.2.1)
$$\begin{array}{ccc} Y' & \xrightarrow{u'} & X' \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{u} & X \end{array}$$

with $s' \in \Sigma$; d) if $f, g\colon X \rightrightarrows Y$ are morphisms of $\mathbf{Sm}_{\log}(k)$ and if $f\colon Y \to Y'$ is a morphism of Σ such that $s \circ f = s \circ g$, then there exists a morphism $t\colon X' \to X$ such that $f \circ t = g \circ t$. For $P \in \mathbf{Sm}_{\log}(k)$, write $\Sigma \downarrow P$ for the full subcategory of the comma category $\mathbf{Sm}_{\log}(k)/P$ given by the objects $P' \xrightarrow{s} P$ with $s \in \Sigma$. If a class of arrows Σ enjoys calculus of right fractions, there is an isomorphism $\mathrm{colim}_{P' \in \Sigma \downarrow P} \mathrm{Hom}_{\mathbf{Sm}_{\log}(k)}(P', Q) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{Sm}_{\log}(k)[\Sigma^{-1}]}(P, Q)$, natural in P and Q, by [17, Proposition I.2.4] (see also [29, A.9]). We limit ourselves to prove that $\mathcal{S}_b$ enjoys calculus or right fractions to prove the Proposition. All the stated conditions are obvious except possibly for c). Given pairs $(X, \partial X)$, $(X', \partial X')$ and $(Y, \partial Y)$ with maps $f\colon (Y, \partial Y) \to (X, \partial X)$ and $\pi\colon (X', \partial X') \to (X, \partial X)$ with $\pi \in \mathfrak{B}_{(X, \partial X)}$, we define the pair $(Y', \partial Y')$ as follows. Set $\tilde{Y} = Y \times_X X'$ and write p_Y for the projection $\tilde{Y} \to Y$. By assumption, $\tilde{Y} \setminus |p_Y^{-1}(f^{-1}(\partial X))|$ is isomorphic to $Y \setminus |f^{-1}\partial X|$. By resolution of singularities, we can find a projective birational map
$$\pi'\colon Y' \to \tilde{Y} \to Y$$
that is obtained as sequence of blow-ups with smooth centers lying over ∂Y and such that $\partial Y' := (\pi')^{-1}(\partial Y)_{\mathrm{red}}$ is a normal crossing divisor on Y'. Then $(Y', \partial Y') \to (Y, \partial Y)$ is a morphism in $\mathfrak{B}_{(Y, \partial Y)}$. The induced morphism $(Y', \partial Y) \to (X', \partial X')$ is clearly admissible and gives a commutative square like (1.3.2.1) as required. □

Write $\mathbf{BSm}_{\log}(k)$ for the localized category $\mathbf{Sm}_{\log}(k)[\mathcal{S}_b^{-1}]$. We denote by

(1.3.2.2)
$$v\colon \mathbf{Sm}_{\log}(k) \to \mathbf{BSm}_{\log}(k)$$

the localization functor: it is clearly faithful, and by [17, I.3.6] commutes with finite direct and inverse limits that exist in $\mathbf{Sm}_{\log}(k)$. Note here that finite products exist in $\mathbf{Sm}_{\log}(k)$ and there is a terminal object $(\mathrm{Spec}(k), \emptyset)$, but it seems that arbitrary fiber products are not representable in $\mathbf{Sm}_{\log}(k)$ (we construct in 1.5.2 fiber products in $\mathbf{Sm}_{\log}(k)$ where one of the maps is a smooth morphism that is minimal in the sense of Definition 1.1.2). If this is the case, by [61, Tag 04AS], $\mathbf{Sm}_{\log}(k)$ does not have all small limits.

1.3.3. If an object in $\mathbf{Sm}_{\log}(k)$ has empty boundary divisor, our definition does not allow more morphisms to appear in $\mathbf{BSm}_{\log}(k)$. More precisely, we have
$$\mathrm{Hom}_{\mathbf{Sm}_{\log}(k)}((X, \emptyset), (Y, \partial Y)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{BSm}_{\log}(k)}((X, \emptyset), (Y, \partial Y))$$
for every $(Y, \partial Y)$ in $\mathbf{Sm}_{\log}(k)$. As a consequence of this fact, we can extend the adjunction of 1.1.3 to the localized category $\mathbf{BSm}_{\log}(k)$. Indeed, we first notice that since every admissible blow-up $\mathfrak{B}_P$ for a given pair $P = (X, \partial X)$ does not change the open complement $X \setminus \partial X$. The universal property of the localization allows then to define a functor
$$\omega\colon \mathbf{BSm}_{\log}(k) \to \mathbf{Sm}(k), \quad (X, \partial X) \mapsto X \setminus \partial X,$$
that restricts to the functor ω of 1.1.3 when composed with the localization functor v. The above observation shows then that we have a bijection
$$\mathrm{Hom}_{\mathbf{Sm}(k)}(X, (Y \setminus \partial Y)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{BSm}_{\log}(k)}((X, \emptyset), (Y, \partial Y))$$
so that the composite functor $\lambda\colon \mathbf{Sm}(k) \to \mathbf{Sm}_{\log}(k) \xrightarrow{v} \mathbf{BSm}_{\log}(k)$ is left adjoint to ω.

A similar situation shows up in the case $\dim X = 1$. In this case, morphisms from $(X, \partial X)$ to any other object of $\mathbf{Sm}_{\log}(k)$ do not change if we pass to $\mathbf{BSm}_{\log}(k)$, since every morphism in $\mathfrak{B}_{(X,\partial X)}$ is an isomorphism already in $\mathbf{Sm}_{\log}(k)$.

Remark 1.3.4. Let $\mathbf{Psh}(\mathbf{Sm}_{\log}(k))$ (resp. $\mathbf{Psh}(\mathbf{BSm}_{\log}(k))$) be the category of presheaves of sets on $\mathbf{Sm}_{\log}(k)$ (resp. on $\mathbf{BSm}_{\log}(k)$). Then the functor v induces a string of adjoint functors $(v_!, v^*, v_*)$ (where each functor is the left adjoint to the the following one) between the categories of presheaves: v^* is induced by composition with v, while $v_!$ (resp. v_*) is the left (resp. right) Kan extension of v. Since v is a localization, then $v_!$ is also a localization or, equivalently (by [17, Proposition I.1.3]), v^* is fully faithful. The functor v^* identifies $\mathbf{Psh}(\mathbf{BSm}_{\log}(k))$ with the subcategory of $\mathbf{Psh}(\mathbf{Sm}_{\log}(k))$ of presheaves that invert the morphisms in $\mathcal{S}_b$.

1.4. Modulus data. In the spirit of [29], we introduce a category of *modulus data* over k, that will be the basic object for our constructions.

Definition 1.4.1. *A modulus datum M consists of a triple $M = (\overline{M}; \partial M, D_M)$, where $\overline{M} \in$ $\mathbf{Sm}(k)$ is a smooth k-scheme, ∂M is a strict normal crossing divisor on M (possibly empty), D_M is an effective Cartier divisor on M (again, the case $D_M = \emptyset$ is allowed), and the total divisor $|D_M|_{\mathrm{red}} + \partial M$ is a strict normal crossing divisor on $\overline{M}$.*

Let M_1, M_2 be two modulus data. A morphism $f\colon M_1 \to M_2$ is called admissible if it is a morphism of k-schemes $f\colon \overline{M_1} \to \overline{M_2}$ that satisfies the following conditions.

- *i) The map f is a morphism in $\mathbf{Sm}_{\log}(k)$ between $(\overline{M_1}, \partial M_1)$ and $(\overline{M_2}, \partial M_2)$, i.e. for every irreducible component $\partial M_{2,k}$ of ∂M_2 we have $|f^*(\partial M_2)| \subseteq |\partial M_1|$.*
- *ii) The divisor $f^*(D_{M_2})$ is defined, and satisfies $f^*(D_{M_2}) \geq D_{M_1}$ as Weil divisors on $\overline{M_1}$.*

We denote by $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ the category having objects modulus data and morphisms admissible morphisms. If one inverts the inequality in condition ii) above, we obtain a "dual" category $\underline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. We will refer to condition ii) as the modulus condition on morphism. If equality holds in ii), we will say that the morphism f is minimal with respect to the modulus condition. If f is minimal also with respect to the boundary divisors ∂M_1 and ∂M_2 in the sense of Definition 1.1.2, we will simply say that f is a minimal morphism of modulus data. Finally, given a modulus datum $M = (\overline{M}; \partial M, D_M)$, we will say that ∂M is the boundary divisor of the datum M and that D_M is its modulus divisor.

Remark 1.4.2. The condition that $|D_M|_{\mathrm{red}} + \partial M$ forms a strict normal crossing divisor on $\overline{M}$ is *not* necessary for the construction of our motivic homotopy category $\overline{\mathbf{M}}\mathcal{H}(k)$. It will show up only in Section 5, where we define the K-theory space associated to a modulus datum M.

Definition 1.4.3. *The category of modulus pairs $\overline{\mathbf{M}}\mathbf{Sm}(k)$ is the category having objects pairs (M, D_M) where M is a smooth k-scheme and D_M is an effective Cartier divisor on it such that the open complement $M^o = M \setminus |D_M|$ is dense. A morphism of pairs $M_1 \xrightarrow{f} M_2$ is a morphism of k-schemes such that $f^*(D_{M_2}) \geq D_{M_1}$ as Weil (or Cartier) divisors on M_1.*

Remark 1.4.4. We are here using the opposite inequality of the definition given in [29]. A possible way for unifying the two notions would be to allow *non effective* modulus pairs $M = (\overline{M}, D_M)$ in $\underline{\mathbf{M}}\mathbf{Sm}$ where D_M is a Cartier divisor on $\overline{M}$, not necessarily effective. Then, one can fix the direction of the inequality of the definition of modulus pairs as done in [29], Definition 1.1, and embed our category $\overline{\mathbf{M}}\mathbf{Sm}(k)$ in $\underline{\mathbf{M}}\mathbf{Cor}$ by sending $(\overline{M}, D_M)$ to $(\overline{M}, -(D_M))$.

1.4.5. There is an obvious fully faithful functor $u\colon \mathbf{Sm}_{\log}(k) \to \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, that sends a pair $P = (X, \partial X)$ to the modulus datum $u(P) = (X; \partial X, \emptyset)$, as well as a "forgetful" functor $F\colon \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k) \to \overline{\mathbf{M}}\mathbf{Sm}$ that sends a modulus datum $(\overline{M}; \partial M, D_M)$ to the modulus pair

$(\overline{M}, D_M)$. They fit together in a commutative square of categories

$$\begin{array}{ccccc} \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k) & \xrightarrow{F} & \overline{\mathbf{M}}\mathbf{Sm}(k), & \quad (\overline{M}; \partial M, D_M) \longmapsto (\overline{M}, D_M) \\ {\scriptstyle u}\uparrow & & {\scriptstyle u}\uparrow & \\ \mathbf{Sm}_{\log}(k) & \xrightarrow{F} & \mathbf{Sm}(k) & \quad (X, \partial X) \longmapsto X \end{array}$$

1.5. Fiber products. As for products, fiber products do not exist in general in the categories $\mathbf{Sm}_{\log}(k)$ and $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. We have, however, the following useful proposition.

Proposition 1.5.1. *Let $f\colon M = (\overline{M}; \partial M, D_M) \to N = (\overline{N}; \partial N, D_N)$ be a minimal morphism in $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ such that the underlying morphism of schemes $f\colon \overline{M} \to \overline{N}$ is smooth. Then, for every $g\colon L = (\overline{L}; \partial L, D_L) \to N$, the fiber product $L \times_N M$ exists in $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$.*

Proof. Since f is smooth, the fiber product $\overline{M}' = \overline{L} \times_{\overline{N}} \overline{M}$ is also smooth over k. Write $f'\colon \overline{M}' \to L$ for the base-change map, and define $\partial M'$ to be the divisor $(f')^{-1}(\partial L)_{\mathrm{red}}$. Each component of $\partial M'$ is the inverse image of a (smooth) component of ∂L along a smooth map, so it is smooth over k. Similarly, each face $\partial M'_I = \cap_{i \in I} \partial M'_i$ is smooth over k and of pure codimension $|I|$ on $\overline{M}'$. Thus, $\partial M'$ defined in this way is a strict normal crossing divisor on $\overline{M}'$. By construction, it's clear that the morphisms f' and $g'\colon \overline{M}' \to \overline{M}$ are admissible morphisms in $\mathbf{Sm}_{\log}(k)$, and that f' is minimal. As for the modulus condition, set $D_{M'}$ to be the divisor $(f')^*(D_L)$. Then we have

$$(g')^*(D_M) = (g')^* f^* D_N = (f')^* g^* D_N \geq (f')^* D_L = D_{M'},$$

where the first equality follows from the minimality requirement on f, so that the maps g' and f' are both satisfying the modulus condition, and therefore are admissible morphisms in $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. We are left to show that the universal property of the fiber product is satisfied by the modulus datum M', but this is straightforward. □

We deduce from the case $D_M = D_L = \emptyset$ the analogous statement for $\mathbf{Sm}_{\log}(k)$.

Corollary 1.5.2. *Let $f\colon (X, \partial X) \to (Y, \partial Y)$ be a minimal morphism in $\mathbf{Sm}_{\log}(k)$ such that $f\colon X \to Y$ is smooth. Then, for every map $g\colon (Z, \partial Z) \to (Y, \partial Y)$, the fiber product $(X, \partial X) \times_{(Y, \partial Y)} (Z, \partial Z)$ is representable in $\mathbf{Sm}_{\log}(k)$.*

Of course, there is nothing to say in case ∂X and ∂Y are also empty.

1.6. Monoidal structure on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. We extend the product in $\mathbf{Sm}_{\log}(k)$ to a symmetric monoidal structure on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$.

Definition 1.6.1. *Let $M, N \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ be modulus data. We define the modulus datum $L = M \otimes N$ by*

$$L = (\overline{M} \times \overline{N}; \partial L = \partial M \times \overline{N} + \overline{M} \times \partial M, D_L = D_M \times \overline{N} + \overline{M} \times D_N).$$

The category $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ equipped with the tensor product $\otimes$ is a symmetric monoidal category, with unit object $\mathbb{1} = (\mathrm{Spec}(k), \emptyset)$. In a similar fashion, we define a symmetric monoidal product $\otimes$ on the category of modulus pairs $\overline{\mathbf{M}}\mathbf{Sm}(k)$.

1.6.2. When $M = (\overline{M}; \partial M, \emptyset)$ and $N = (\overline{N}; \partial N, \emptyset)$, then we can check that $M \otimes N = M \times N = (\overline{M} \times \overline{N}; \partial L, \emptyset)$ is the categorical product of M and N. In particular, the functor $u\colon \mathbf{Sm}_{\log}(k) \to \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ is strict monoidal (when one considers on $\mathbf{Sm}_{\log}(k)$ the monoidal structure given by the cartesian product). The forgetful functor $F\colon \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k) \to \overline{\mathbf{M}}\mathbf{Sm}(k)$ is also strict monoidal.

Remark 1.6.3. For arbitrary objects M, N in $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, our choice of admissibility condition for morphisms prevents the existence of projection maps $M \otimes N \to M$ or $M \otimes N \to N$. However, if $M = (\overline{M}; \partial M, \emptyset)$ and $N = (\overline{N}; \partial N, D_M)$, then the map $M \otimes N \to N$ induced by the projection $\overline{M} \times \overline{N} \to \overline{N}$ is clearly admissible.

1.7. A digression on interval objects in monoidal categories. Let $(\mathcal{M}, \otimes, \mathbb{1})$ be a symmetric monoidal category with unit object $\mathbb{1}$.

Definition 1.7.1. *An object I in $\mathcal{M}$ is called a weak interval in $\mathcal{M}$ if there exist a map $p_I \colon I \to \mathbb{1}$ (the "projection") and monomorphisms $\iota^I_\varepsilon \colon \mathbb{1} \to I$ for $\varepsilon = 0, 1$ (the "inclusions at 0 and 1") that satisfy*

$$p_I \circ \iota^I_0 = p_I \circ \iota^I_1 = \mathrm{id}_{\mathbb{1}} .$$

Let $\mathcal{M}$ be a symmetric monoidal category equipped with a weak interval I. An I-$\otimes$ $-$homotopy between two maps $f, g \colon X \rightrightarrows Y$ is the datum of a morphism $H \colon X \otimes I \to Y$ in $\mathcal{M}$ such that $f = H \circ (\mathrm{id}_X \otimes \iota^I_0)$ and $g = H \circ (\mathrm{id}_X \otimes \iota^I_1)$.

Definition 1.7.2. *An object I in $\mathcal{M}$ is called an interval in $\mathcal{M}$ if it is a weak interval $(I, \iota^I_0, \iota^I_1, p_I)$ that is additionally equipped with a multiplication map*

$$\mu \colon I \otimes I \to I,$$

verifying the identities $\mu \circ (\mathrm{id}_I \otimes \iota^I_0) = \iota^I_0 \circ p_I$ and $\mu \circ (\mathrm{id}_I \otimes \iota^I_1) = \mathrm{id}_I$.

The notion of interval object presented here is more general then the definition of Voevodsky in [64] and agrees with the definition proposed in [29, 5].

1.7.3. Any weak interval object I in $\mathcal{M}$ determines a co-cubical object $I^\bullet \colon \mathbf{Cube} \to \mathcal{M}$ by setting

$$I^n = I^{\otimes n}, \quad p^n_{i,I} = \mathrm{id}_I^{\otimes (i-1)} \otimes p_I \otimes \mathrm{id}_I^{\otimes (n-1)}, \quad \delta^n_{i,\varepsilon} = \mathrm{id}_I^{\otimes (i-1)} \otimes \iota^I_\varepsilon \otimes \mathrm{id}_I^{\otimes (n-1)}$$

for $\varepsilon \in \{0, 1\}$, $i = 1, \ldots, n$. If I is moreover an interval object (so that is equipped with a multiplication map), the same formulas work to give an extended co-cubical object, $I^\bullet \colon \mathbf{ECube} \to \mathcal{M}$, where $\mathbf{ECube}$ is the extended cubical category introduced in 1.1.4.

Conversely, given a strict monoidal extended cocubical object C in $\mathcal{M}$, where the monoidal structure on $\mathbf{ECube}$ is given by cartesian product, one can easily check that $I = C([1])$ is an interval object in $\mathcal{M}$.

1.7.4. In arbitrary monoidal categories there are no diagonal morphisms, as remarked in [29], Remark 5.9, so that given a weak interval object one can — a priori — only develop a cubical theory and not a simplicial theory. Fortunately, we will consider for our applications an interval object $(I, \iota^I_1, \iota^I_0, p_I, \mu)$ that is equipped with an extra map $\delta_I \colon I \to I^{\otimes 2}$ such that the compositions $(\mathrm{id}_I \otimes p_I) \circ \delta$ and $(p_I \otimes \mathrm{id}_I) \circ \delta$ are the identity on I.

Following [64, 2.2], we can then construct a universal cosimplicial object in $\mathcal{M}$ as follows.

Definition 1.7.5. *Set $\Delta^n_I = I^{\otimes n}$ for every n. For $i = 0, \ldots, n$, let*

$$d^i \colon [n-1] = \{0, \ldots, n-1\} \to [n] = \{0, \ldots, n\} \quad (\textit{resp. } s^i \colon [n+1] \to [n])$$

be the standard $i-$th face (resp. $i-$th degeneracy) in the simplicial category Δ. Define

$$\Delta_I(d^i) = \begin{cases} \iota^I_0 \otimes \mathrm{id}_I^{\otimes (n-1)} & \text{if } i = 0, \\ \mathrm{id}_I^{\otimes (n-1)} \otimes \iota^I_1 & \text{if } i = n, \\ \mathrm{id}_I^{\otimes (i-1)} \otimes \delta_I \otimes \mathrm{id}_I^{\otimes (n-i-1)} & \text{if } 1 \leq i \leq n-1 \end{cases} \tag{1.7.5.1}$$

and

$$\Delta_I(s^i) = \mathrm{id}_I^{\otimes i} \otimes p_I \otimes \mathrm{id}_I^{\otimes (n-i)} . \tag{1.7.5.2}$$

It is easy to check that this data define a cosimplicial object in $\mathcal{M}$, that we will denote by $\Delta_I^\bullet$.

Remark 1.7.6. The formulas in (1.7.5.1) and (1.7.5.2) are not explicit in [64]. Voevodsky's construction holds in a $\otimes$-category that is a site with products, the tensor structure being given by cartesian products of objects, and this fact is used in the formulation of *loc.cit.*. It is not hard (but a bit tedious) to deduce from Voevodsky's formulas our definitions.

1.8. A distinguished interval in $\mathbf{Sm}_{\log}(k)$. We specialize the result of the previous subsection to our case of interest.

Consider the object $\overline{\square} = \overline{\square}^1 = (\mathbb{P}^1, \infty)$ in $\mathbf{Sm}_{\log}(k)$. We have two distinguished admissible morphisms in $\mathbf{Sm}_{\log}(k)$

$$\iota_0^{\overline{\square}}, \iota_1^{\overline{\square}} \colon \mathrm{Spec}(k) = (\mathrm{Spec}(k), \emptyset) \rightrightarrows \overline{\square}$$

induced by the inclusions of the points 0 and 1 in $\mathbb{P}^1$. There is also a projection

$$p_{\overline{\square}} \colon \overline{\square} \to (\mathrm{Spec}(k), \emptyset) =: \mathbb{1},$$

indued by the structure map $\mathbb{P}^1 \to \mathrm{Spec}(k)$ and satisfying the obvious property that

$$p_{\overline{\square}} \circ \iota_0^{\overline{\square}} = p_{\overline{\square}} \circ \iota_1^{\overline{\square}} = \mathrm{id}_{\mathbb{1}},$$

making $(\overline{\square}, \iota_0^{\overline{\square}}, \iota_1^{\overline{\square}}, p_{\overline{\square}})$ a weak interval object in $\mathbf{Sm}_{\log}(k)$.

Let $\mathbf{BSm}_{\log}(k)$ as in 1.3 be the localization of the category $\mathbf{Sm}_{\log}(k)$ to admissible blow-ups, and consider $\overline{\square}$ as object there. There is an extra multiplication map

$$\mu \colon (\mathbb{P}^1 \times \mathbb{P}^1, F_\infty^2) \to \overline{\square}$$

in $\mathbf{BSm}_{\log}(k)$ induced by the following diagram:

(1.8.0.1)
$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^1. \\ \uparrow{\scriptstyle \pi} & \nearrow{\scriptstyle \tilde{\mu}} & \\ \mathbf{Bl}_{0\times\infty,\infty\times 0}(\mathbb{P}^1 \times \mathbb{P}^1) & & \end{array}$$

Here, $\pi \colon B = \mathbf{Bl}_{0\times\infty,\infty\times 0}(\mathbb{P}^1 \times \mathbb{P}^1) \to (\mathbb{P}^1 \times \mathbb{P}^1)$ is the blow-up along the closed subscheme $(0 \times \infty \cup \infty \times 0) \subset F_\infty^2$. It is easy to see that B is smooth over k (since it is the blow-up along a regularly embedded subscheme), and that it agrees with the closure in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of the graph of the rational map $\mu \colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ given by the multiplication map $\mu \colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, $(x, y) \mapsto xy$ (and we will constantly use this identification in what follows). The map $\tilde{\mu}$ in (1.8.0.1) is then identified with the composition $B \hookrightarrow (\mathbb{P}^1)^3 \xrightarrow{p_3} \mathbb{P}^1$, where p_3 is the projection to the third factor.

On B we have two distinguished divisors, that we denote by $\tilde{F}_\infty^2$ and E_2 respectively. The divisor $\tilde{F}_\infty^2 = \tilde{F}_{\infty,1}^2 + \tilde{F}_{\infty,2}^2 = \mathbb{P}^1 \times \infty \times \infty + \infty \times \mathbb{P}^1 \times \infty$ is the strict transform of the boundary divisor F_∞^2 on $\mathbb{P}^1 \times \mathbb{P}^1$ along the map π. The divisor $E^2 = E_1^2 + E_2^2 = 0 \times \infty \times \mathbb{P}^1 + \infty \times 0 \times \mathbb{P}^1$ is the exceptional divisor of the blow-up. Together, $(\tilde{F}_{\infty,1}^2, \tilde{F}_{\infty,2}^2, E_1^2, E_2^2)$ form a strict normal crossing divisor on B, that we simply denote by ∂B. Then, the pair $(B, \partial B)$ is a well-defined object in $\mathbf{Sm}_{\log}(k)$ and the map $\tilde{\mu}$ is an admissible morphism in $\mathbf{Sm}_{\log}(k)$. Note that we clearly have $\pi^{-1}(F_\infty^2) = \partial B$, so that π is a minimal morphism in $\mathbf{Sm}_{\log}(k)$, that is therefore an admissible blow-up for $(\mathbb{P}^1 \times \mathbb{P}^1, F_\infty^2)$ (and hence becomes invertible in $\mathbf{BSm}_{\log}(k)$). To conclude, we have constructed a well-defined morphism

$$\mu \colon (\mathbb{P}^1 \times \mathbb{P}^1, F_\infty^2) \to \overline{\square}, \quad \text{in } \mathbf{BSm}_{\log}(k)$$

as required.

To show that $\overline{\square}$ with this multiplication morphism defines an interval object in $\mathbf{BSm}_{\log}(k)$, we still need to check that the axioms of Definition 1.7.2 hold. First, note that there is a natural monoidal structure on $\mathbf{BSm}_{\log}(k)$ that makes the localization functor v strict monoidal. Namely, given pairs $(X, \partial X)$ and $(Y, \partial Y)$, define

$$(X, \partial X) \otimes (Y, \partial Y) = (X \times Y, \partial X \times Y, X \times \partial Y). \tag{1.8.0.2}$$

This assignments coincides with the cartesian product in $\mathbf{Sm}_{\log}(k)$. The following Lemma is an easy exercise.

Lemma 1.8.1. *The product $(\mathbb{P}^1, \infty) \otimes (\mathbb{P}^1, \infty) = (\mathbb{P}^1 \times \mathbb{P}^1, F^2_\infty)$ in $\mathbf{BSm}_{\log}(k)$ is the categorical product of $(\mathbb{P}^1, \infty)$ with itself in $\mathbf{BSm}_{\log}(k)$. Thus we have equalities $\overline{\square} \times \overline{\square} = \overline{\square} \otimes \overline{\square} = (\mathbb{P}^1 \times \mathbb{P}^1, F^2_\infty)$ in $\mathbf{BSm}_{\log}(k)$.*

Consider now the inclusions at 0 and 1 of $\overline{\square}^1$ in $\mathbf{Sm}_{\log}(k)$. First, we note that the morphisms

$$\mathrm{id}_{\overline{\square}} \times \iota_1^{\overline{\square}} \colon \overline{\square} \times \mathbb{1} \simeq \overline{\square} \to \mathbb{P}^1 \times \mathbb{P}^1, \quad \iota_1^{\overline{\square}} \times \mathrm{id}_{\overline{\square}} \colon \mathbb{1} \times \overline{\square} \simeq \overline{\square} \to \mathbb{P}^1 \times \mathbb{P}^1,$$

automatically factors through π, since their image is disjoint from the center of the blow-up. Explicitly, we have the morphism $\iota_1^{\overline{\square}} \times \mathrm{id}_{\overline{\square}} \colon \overline{\square} \to B$ given by the diagonal embedding $1 \times \Delta_{\mathbb{P}^1} \hookrightarrow B$ induced by $x \mapsto (1, x, \mu(1, x) = x)$, and $\mathrm{id}_{\overline{\square}} \times \iota_1^{\overline{\square}}$ given by the "twisted" diagonal embedding $x \mapsto (x, 1, \mu(x, 1) = x)$. These maps are clearly admissible in $\mathbf{Sm}_{\log}(k)$. Since the morphism $\tilde{\mu}$ is induced by the third projection, we immediately see that we have identities

$$\tilde{\mu} \circ (\mathrm{id}_{\overline{\square}} \times \iota_1^{\overline{\square}}) = \tilde{\mu} \circ (\iota_1^{\overline{\square}} \times \mathrm{id}_{\overline{\square}}) = \mathrm{id}_{\overline{\square}} \quad \text{in } \mathbf{Sm}_{\log}(k),$$

that descend to the corresponding identities in $\mathbf{BSm}_{\log}(k)$ once we replace $\tilde{\mu}$ with μ.

The inclusions at 0 given by $\mathrm{id}_{\overline{\square}} \times \iota_0^{\overline{\square}}$ and $\iota_0^{\overline{\square}} \times \mathrm{id}_{\overline{\square}}$ have image in $\mathbb{P}^1 \times \mathbb{P}^1$ that is clearly not disjoint from the center of the blow-up. We explicitly lift them to B by taking the strict transform of their image. Explicitly, for $\iota_0^{\overline{\square}} \times \mathrm{id}_{\overline{\square}}$ (the other case is identical) we have

$$(\mathbb{P}^1, \infty) \xrightarrow{\iota_0^{\overline{\square}} \times \mathrm{id}_{\overline{\square}}} (0 \times \mathbb{P}^1 \times 0, 0 \times \infty \times 0) \hookrightarrow B.$$

Then $\partial B \cap (0 \times \infty \times 0) = (0 \times \infty \times 0)$, so that the map is admissible. The composition $(0 \times \mathbb{P}^1 \times 0) \hookrightarrow B \xrightarrow{p_3}$ is the constant morphism to $0 \in \mathbb{P}^1$, so that we have identities

$$\tilde{\mu} \circ (\mathrm{id}_{\overline{\square}} \times \iota_0^{\overline{\square}}) = \tilde{\mu} \circ (\iota_0^{\overline{\square}} \times \mathrm{id}_{\overline{\square}}) = \iota_0^{\overline{\square}} \circ p_{\overline{\square}} \quad \text{in } \mathbf{Sm}_{\log}(k),$$

that descend to the corresponding identities in $\mathbf{BSm}_{\log}(k)$ once we replace $\tilde{\mu}$ with μ. To summarize, we have proved the following

Proposition 1.8.2. *The quintuple $(\overline{\square}, \iota_0^{\overline{\square}}, \iota_1^{\overline{\square}}, p_{\overline{\square}}, \mu)$ makes $\overline{\square}$ into an interval object for the category $\mathbf{BSm}_{\log}(k)$.*

2. Topologies on modulus data

Before moving to the definition of motivic spaces, we review here the properties of the Grothendieck topologies that we use on the categories $\mathbf{Sm}_{\log}(k)$ and $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. This section is the analogue in our context of [29, 3.2].

2.1. A recollection on cd-structures. Recall the following definition from [65].

Definition 2.1.1 ([65], 2.1)**.** *Let $\mathcal{C}$ be a small category with an initial object 0 and P be a set of commutative squares in $\mathcal{C}$. We say that P forms a cd-structure on $\mathcal{C}$ if whenever $Q \in P$ and Q' is isomorphic to Q, then Q' is also in P. The squares of the collection P are called distinguished squares of P.*

Definition 2.1.2. *The cd-topology t_P associated with a cd-structure P is the Grothendieck topology on $\mathcal{C}$ generated by coverings sieves of the following form:*

- *i) The empty sieve is a covering sieve of the initial object 0,*
- *ii) The sieve generated by morphisms of the form $\{A \to X, Y \to X\}$ where $A \to X$ and $Y \to X$ are two sides of a square in P of the form*

(2.1.2.1)
$$\begin{array}{ccc} B & \xrightarrow{e_B} & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{e} & X \end{array}$$

is a covering sieve.

The class S_P of simple coverings is the smallest class of families of morphisms of the form $\{U_i \to X\}_{i \in I}$ satisfying the following two conditions:

- *(1) any isomorphism is in S_P,*
- *(2) for a distinguished square Q of the form (2.1.2.1) of P and families $\{p_i \colon Y_i \to Y\}_{i \in I}$ and $\{q_j \colon A_j \to A\}_{j \in J}$ that are in S_P, the family $\{p \circ p_i, e \circ q_j\}_{i,j}$ is also in S_P.*

Definition 2.1.3 ([65], 2.3)**.** *A cd-structure is called complete if any covering sieve of an object X which is not isomorphic to the initial object 0 contains a sieve that is generated by a simple covering.*

Lemma 2.1.4 ([65], 2.5)**.** *If P is a cd-structure such that any morphism with values in 0 is an isomorphism and for any distinguished square Q and any morphism $f \colon X' \to X$, the pull back square f^*Q is also distinguished, then P is complete.*

As remarked by Voevodsky in *loc.cit.*, the topology associated with any complete cd-structure is necessarily Noetherian, i.e. any t_P-covering has a finite refinement. In particular, if $\rho(X)$ denotes the t_P sheaf associated with the presheaf represented by X, one has the following

Lemma 2.1.5 ([65], 2.8)**.** *Let P be a complete cd-structure. Then for any X in $\mathcal{C}$, the sheaf (of sets) $\rho(X)$ is a compact object in $Shv(\mathcal{C}, t_P)$, i.e. $\mathrm{Hom}(\rho(X), -)$ commutes with filtered colimits of sheaves.*

Lemma 2.1.6 ([65], 2.9)**.** *Let P be a complete cd-structure and let F be a presheaf of sets on $\mathcal{C}$. Then F is a sheaf in the t_P-topology if $F(0) = *$ and $F(Q)$ is a pull-back square for every distinguished square Q of P.*

The two notions of being *bounded* and *regular* for a *cd*-structure will play an important role in characterizing the fibrations in the local projective model structure for simplicial presheaves given by [6].

Definition 2.1.7. *Let P be a cd-structure on $\mathcal{C}$. Then P is called regular if for any distinguished square Q in P of the form (2.1.2.1), one has*

- *(1) Q is a pull-back square,*
- *(2) the morphism e is a monomorphism in $\mathcal{C}$,*
- *(3) The morphism of sheaves*

$$\Delta \amalg \rho(e_B) \times \rho(e_B) \colon \rho(Y) \amalg (\rho(B) \times_{\rho(A)} \rho(B)) \to \rho(Y) \times_{\rho(X)} \rho(Y)$$

is surjective.

Condition (3) of Definition 2.1.7 can be hard to check. Voevodsky provides in [65, 2.11] a sufficient condition for P to be regular.

Lemma 2.1.8. *A cd-structure P is regular provided that conditions (1) and (2) of Definition 2.1.7 are satisfied together with the following condition:*

(3′) *For every distinguished square Q of the form (2.1.2.1), the objects $Y \times_X Y$ and $B \times_A B$ exist and the square*

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B \times_A B & \longrightarrow & Y \times_X Y \end{array}$$

where the vertical arrows are the diagonals, is distinguished.

As a consequence of the definition, we obtain the following characterization of sheaves in the topology associated with a complete regular cd-structure

Proposition 2.1.9. *Let P be a complete regular cd-structure on a category $\mathcal{C}$. Then a presheaf of sets F is a sheaf in t_P if and only if $F(0) = *$ and for any distinguished square Q, $F(Q)$ is a pull-back square.*

The next notion is used to define dimension for objects in a category with a cd-structure, and to introduce a class of cd-structures of finite dimension (in an appropriate sense).

Definition 2.1.10. *A density structure on a category $\mathcal{C}$ with an initial object 0 is a function that assigns to each object X in $\mathcal{C}$ a sequence $D_i(X)_{i\geq 0}$ of families of morphisms satisfying the following conditions:*

(1) *X is the codomain of elements of $D_i(X)$ for all i,*
(2) *$(0 \to X)$ is an element of $D_0(X)$ for all X,*
(3) *isomorphisms belong to D_i for all i,*
(4) *$D_{i+1}(X) \subset D_i(X)$ for all X,*
(5) *If $j \colon U \to V$ is in $D_i(V)$ and $f \colon V \to X$ is in $D_i(X)$, then $f \circ j \in D_i(X)$.*

We say that a density structure is locally of finite dimension if for any X there exists n such that any element in $D_{n+1}(X)$ is an isomorphism. The smallest such n is called the dimension of X with respect to D and we write $\dim_D(X)$.

Remark 2.1.11. The main example of density structure is coming from the dimension of a Noetherian topological space T. Let $\mathcal{C}$ be the category of open subsets of T, with morphisms given by inclusions. For $V \in \mathcal{C}$, we set $D_i(V)$ to be the set of open embeddings $U \to V$ such that the codimension of the complement $V \setminus U$ is at least i. It's easy to see that if T has finite dimension as Noetherian space, then the dimension of T with respect to this density structure is the dimension in the usual sense.

Definition 2.1.12. *A density structure D is said to be reducing for a cd-structure P if any distinguished square in P has a refinement which is reducing with respect to D, i.e. if Q is of the form (2.1.2.1), then for every $B_0 \in D_i(B), Y_0 \in D_{i+1}(Y), A_0 \in D_{i+1}(A)$, there exist a $X' \in D_{i+1}(X)$ and a primed distinguished square Q'*

$$Q' = \begin{array}{ccc} B' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & X' \end{array}$$

and a morphism of squares $Q' \to Q$ such that the induced map $X' \to X$ coincide with the morphism $X' \to X$ in $D_{i+1}(X)$ and such that every over component factor through B_0, Y_0, A_0 respectively. We say that a cd-structure P is bounded if there exists a reducing density structure of locally finite dimension for it.

2.2. Topologies on $\mathbf{Sm}_{\log}(k)$ and $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. The definitions of this section are adapted from [29] to our setting. Let σ be either the Zariski or the Nisnevich topology on $\mathbf{Sm}(k)$.

Definition 2.2.1. *A morphism $p\colon (U, \partial U) \to (X, \partial X)$ in $\mathbf{Sm}_{\log}(k)$ is called a σ-covering for $(X, \partial X)$ if $p\colon U \to X$ is a σ-cover of $\mathbf{Sm}(k)$ and if p is a minimal morphism in $\mathbf{Sm}_{\log}(k)$.*

By Corollary 1.5.2, the pull-back of a σ-covering along any morphism $f\colon (Y, \partial Y) \to (X, \partial X)$ is still a σ-covering, so that the above definition gives rise to a Grothendieck topology t_σ on $\mathbf{Sm}_{\log}(k)$.

The topology t_σ is the Grothendieck topology associated with a cd-structure P_σ. The distinguished squares P_σ are defined as follows. Let $(X; \partial X)$ be a pair in $\mathbf{Sm}_{\log}(k)$. For $\sigma = \mathrm{Zar}$, let $i\colon U \hookrightarrow X$ and $j\colon V \hookrightarrow X$ be two open embeddings (for the Zariski topology on X). Then we have the pairs $(U, \partial U)$ and $(V, \partial V)$ in $\mathbf{Sm}_{\log}(k)$, where $\partial U = U \cap \partial X$ and $\partial V = V \cap \partial X$ are strict normal crossing divisors on U and V respectively. The distinguished squares P_{Zar} on $\mathbf{Sm}_{\log}(k)$ over $(X, \partial X)$ are then given by the pull-back squares

$$\begin{array}{ccc} (U \cap V, \partial X \cap (U \cap V)) & \longrightarrow & (U, \partial U) \\ \downarrow & & \downarrow \\ (V, \partial V) & \longrightarrow & (X, \partial X) \end{array}$$

for U and V running on the set of open subschemes of X.

In a similar fashion, an elementary Nisnevich square (i.e. a distinguished square in P_{Nis}) in $\mathbf{Sm}_{\log}(k)$ is a pull-back square of the form

$$\begin{array}{ccc} (U \times_X Y, \partial Y \cap p^{-1}(U)) & \longrightarrow & (Y, \partial Y) \\ \downarrow & & \downarrow p \\ (U, \partial U) & \xrightarrow{\ j\ } & (X, \partial X) \end{array}$$

where $j\colon U \hookrightarrow X$ is an open embedding, $p\colon Y \to X$ is an étale morphism such that

$$(p^{-1}(X \setminus U))_{\mathrm{red}} \to (X \setminus U)_{\mathrm{red}}$$

is an isomorphism, ∂Y is the strict normal crossing divisor $p^{-1}(\partial X)$ on Y and $\partial U = U \cap \partial X$.

The following Proposition is an immediate application of Lemmas 2.1.4 and 2.1.8, using the known results for the Nisnevich and Zariski topology on $\mathbf{Sm}(k)$.

Proposition 2.2.2. *The set of elementary Nisnevich (resp. Zariski) squares P_{Nis} (resp. P_{Zar}) on the category $\mathbf{Sm}_{\log}(k)$ defines a complete and regular cd-structure. In particular, a presheaf of sets F on $\mathbf{Sm}_{\log}(k)$ is a sheaf in the Nisnevich (resp. Zariski) topology if and only if for any elementary square Q, the square of sets $F(Q)$ is cartesian.*

We can add the modulus divisor to the picture, obtaining two complete and regular cd-structures on the category of modulus data.

Definition 2.2.3. *Let $\sigma \in \{\mathrm{Zar}, \mathrm{Nis}\}$. A morphism $p\colon U \to M$ of modulus data is called a σ-cover if*

i) The underlying morphism of schemes $\overline{p}\colon \overline{U} \to \overline{M}$ is a σ-cover of $\mathbf{Sm}(k)$,
ii) p is a minimal morphism of modulus data (so that $D_U = p^*(D_M)$ and $\partial U = (p^{-1})(\partial M)$).

The class of σ-covers defines a Grothendieck topology on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, using Proposition 1.5.1 instead of Corollary 1.5.2. The topology t_σ on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ is the Grothendieck topology associated with a complete and regular cd-structure P_σ. For $\sigma = \mathrm{Nis}$, a distinguished square in

$\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ is a pull-back square of the form

$$\begin{array}{ccc} (\overline{U}\times_{\overline{M}}\overline{Y},\partial Y\cap p^{-1}(\overline{U}),D_{\overline{U}\times_{\overline{M}}\overline{Y}}) & \longrightarrow & (\overline{Y};\partial Y,D_Y) \\ \downarrow & & \downarrow p \\ (\overline{U};\partial U,D_U) & \xrightarrow{j} & (\overline{M};\partial M,D_M) \end{array} \tag{2.2.3.1}$$

where $j\colon \overline{U}\hookrightarrow \overline{M}$ is an open embedding, $p\colon \overline{Y}\to\overline{M}$ is an étale morphism such that

$$(p^{-1}(\overline{M}\setminus\overline{U}))_{\mathrm{red}}\to(\overline{M}\setminus\overline{U})_{\mathrm{red}}$$

is an isomorphism, and

$$\partial Y = p^{-}1(\partial M),\quad D_Y = p^*D_M,\quad \partial U = U\cap\partial M,\quad D_U = D_M\cap U.$$

The cd-structures P_{Nis} and P_{Zar} on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ are also bounded in the sense of Definition 2.1.12. A density structure $D_i(-)$ that works for both cd-structures was introduced by Voevodsky in [66, 2] and used in [29, Definition 3.11]. In our context, it takes the following form. Recall that a sequence of points $x_0,\dots,x_d$ of a topological space X is called *an increasing sequence* of *length d* if $x_i\neq x_{i+1}$ and $x_i\in\overline{\{x_{i+1}\}}$.

Definition 2.2.4. *Let $M=(\overline{M};\partial M,D_M)$ be a modulus datum. Define $D_i(M)$ as the class of morphisms of modulus data $j\colon U\to M$ where $U=(\overline{U};\partial U,D_U)$ is the minimal datum associated to a dense open embedding $j\colon\overline{U}\hookrightarrow\overline{M}$ such that for any $z\in\overline{M}\setminus\overline{U}$ there exists an increasing sequence $z=x_0,x_1,\dots,x_d$ in $\overline{M}$ of length d.*

Lemma 2.2.5. *The assignment $M\mapsto D_i(M)_{i\geq 0}$ defines a density structure on the category of modulus data, that is compatible with the standard density structure on $\mathbf{Sm}(k)$ defined in [66, 2]. The cd-structures P_{Nis} and P_{Zar} on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ are bounded with respect to this density structure.*

Proof. Let σ be either the Zariski or the Nisnevich topology. We need to show that the density structure $D_i(M)_{i\geq 0}$ is reducing for the cd-structures P_{Nis} and P_{Zar} on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. By definition, this property depends only on the small site M_σ attached to any fixed modulus datum M. Let $(\overline{M})_\sigma$ be the usual small σ-site on the underlying scheme $\overline{M}$. Then, the forgetful functor $F\colon\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)\to\mathbf{Sm}(k)$ that sends a modulus datum N to the underlying scheme $\overline{N}$ defines an isomorphism of sites $M_\sigma\xrightarrow{F_M}(\overline{M})_\sigma$ (this was already observed by [29, Lemma 3.9] in the context of modulus pairs). The claim now follows from [66, Proposition 2.10]. □

Recall that a point of a site T is a functor $x^*\colon\mathbf{Shv}(T)\to\mathbf{Set}$ which commutes with finite limits and all colimits. A site T has *enough points* if isomorphisms can be tested stalkwise, i.e. if there is a set x_i^* of points such that the induced functor $(x_i^*)\colon\mathbf{Shv}(T)\to\prod_i\mathbf{Set}$ is faithful.

Question 2.2.6. *Does the site $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ with the Nisnevich or the Zariski topology defined above have enough points?*

Lemma 2.2.7. *The Nisnevich and the Zariski topologies on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ are sub-canonical.*

Proof. It's straightforward to check that every representable presheaf on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ sends a distinguished square to a Cartesian square. □

3. Motivic spaces with modulus

3.1. Generalities and first definitions. Let $\mathcal{S}$ be the category of simplicial sets, $\mathcal{S}=\Delta^{\mathrm{op}}\mathbf{Set}$, with simplicial function objects $\mathcal{S}(-,-)$. Recall that the standard n-simplex Δ^n denotes the representable presheaf $\mathrm{Hom}_\Delta(-,[n])$.

Let T be a site. We write $\mathbf{Psh}(T)$ for the category of presheaves (of sets) on T and $s\mathbf{Psh}(T) = \Delta^{\mathrm{op}}\mathbf{Psh}(T)$ for the category of simplicial objects in $\mathbf{Psh}(T)$. Equivalently, an object $\mathscr{X}$ of $s\mathbf{Psh}(T)$ can be seen as a set-valued presheaf on $T \times \Delta$, or as a presheaf on T with values in simplicial sets.

To give an object $\mathscr{X}$ of $s\mathbf{Psh}(T)$, it is equivalent to give a collection of presheaves of sets $\mathscr{X}_n$ for $n \geq 0$ together with *faces* and *degeneracies*

$$d_i^n \colon \mathscr{X}_n \to \mathscr{X}_{n-1}, \quad n \geq 1, \quad i = 0, \dots, n$$
$$s_i^n \colon \mathscr{X}_n \to \mathscr{X}_{n+1} \quad n \geq 0, \quad i = 0, \dots, n$$

that are subject to the usual simplicial identities (see, for example, [18, 1]). Given any simplicial set K, we will denote by K the constant presheaf on T with value K. With this convention, we have a standard cosimplicial object $\Delta^\bullet$ in $s\mathbf{Psh}(T)$

$$\Delta \to s\mathbf{Psh}(T), \quad n \mapsto \Delta^n$$

where the latter is seen as constant presheaf. From this, we derive the usual simplicial structure on $s\mathbf{Psh}(T)$ (i.e. the enrichment of $s\mathbf{Psh}(T)$ in $\mathcal{S}$). The *simplicial function complex* $\mathcal{S}(\mathscr{X}, \mathscr{Y})$ is the simplicial set having n-simplices given by $\mathrm{Hom}_{s\mathbf{Psh}(T)}(\mathscr{X} \times \Delta^n, \mathscr{Y})$ and simplicial structure given using the structure of $\Delta^\bullet$.

For every object U in T, we denote by h_U (or simply by U if no confusion arises) the Yoneda functor $h_U(X) = \mathrm{Hom}_T(X, U)$ considered as discrete simplicial set or, equivalently, as simplicial presheaf of simplicial dimension 0.

3.2. Monoidal structures on presheaves categories. In this section we present some general material on symmetric monoidal structures for simplicial presheaves. We will then specialize these general results for the construction of a closed symmetric monoidal model structure on the category of motivic spaces with modulus.

Let $\mathcal{C}$ be a small category. Write $\mathbf{Psh}(\mathcal{C})$ for the category of presheaves of sets on $\mathcal{C}$ and $s\mathbf{Psh}(\mathcal{C})$ for the category of simplicial presheaves on $\mathcal{C}$. To simplify the notation, given an object X of $\mathcal{C}$ we will denote by X the representable (simplicial) presheaf h_X if no confusion occurs. As recalled above, $s\mathbf{Psh}(\mathcal{C})$ is a $\mathcal{S}$-category in a standard way.

3.2.1. Assume now that $\mathcal{C}$ is a small symmetric monoidal category, with tensor product $\otimes$ and unit $\mathbb{1}$. There is a natural extension of the monoidal structure on $\mathcal{C}$ to a symmetric monoidal structure on $\mathbf{Psh}(\mathcal{C})$ via *Day convolution*, introduced by Day in [12], that makes the Yoneda functor $h_{\mathcal{C}}$ strong monoidal. The existence of the monoidal structure follows formally from the general theory of left Kan extensions. Given two presheaves F and G, their convolution product is the coend

$$F \otimes^{\mathrm{Day}} G = \int^{X,Y \in \mathcal{C}} F(X) \times G(Y) \times \mathrm{Hom}_{\mathcal{C}}(-, X \otimes Y).$$

If the reader wishes to ignore the coend symbol, we can write the same thing as follows. Given that any presheaf is colimit of representable presheaves, write $F = \mathrm{colim}_{X \downarrow F}\, h_X$ and $G = \mathrm{colim}_{Y \downarrow G}\, h_Y$. Then, their convolution $F \otimes^{\mathrm{Day}} G$ is the colimit $\mathrm{colim}_{X,Y}\, h_{X \otimes Y}$. It follows immediately from the definition that the Yoneda functor is strong (symmetric) monoidal. It is also clear that the unit for the convolution product is the representable presheaf $h_{\mathbb{1}}$. It is also formal to see that the monoidal structure on presheaves given by Day convolution is closed, i.e. there exists an internal hom $[-,-]$ that is right adjoint to $\otimes^{\mathrm{Day}}$. This is characterized by

$$[F, G](X) = \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(h_X \otimes^{\mathrm{Day}} F, G), \quad \text{for all } X \in \mathcal{C}.$$

Unless required for clarity, we will drop the superscript and write simply $\otimes$ for the tensor product of presheaves. Recall (see e.g. [25]) that a monoidal category $(\mathcal{D}, \star, \mathbb{1}_{\mathcal{D}})$ is called monoidally co-complete if $\mathcal{D}$ is co-complete and all the endofunctors $X \star (-)$, $(-) \star Y$ for X, Y in $\mathcal{D}$ are co-continuous. By [25, Proposition 4.1], the category $\mathbf{Psh}(\mathcal{C})$ on a small symmetric monoidal category is monoidally co-complete. This construction is universal in the following sense.

Proposition 3.2.2 (Theorem 5.1 [25]). *Let $\mathcal{D}$ be a monoidally co-complete category. Then, the functor $[\mathbf{Psh}(\mathcal{C}), \mathcal{D}]_{\otimes} \to [\mathcal{C}, \mathcal{D}]_{\otimes}$ between the categories of strong monoidal functors from $\mathbf{Psh}(\mathcal{C})$ to $\mathcal{D}$ and the category of strong monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ induced by the Yoneda functor is an equivalence.*

Remark 3.2.3. Here's a situation where the previous Proposition turns out to be useful. Let $u\colon \mathcal{C} \to \mathcal{D}$ be a strict symmetric monoidal functor. Then u gives rise to a string of adjoint functors between the categories of presheaves

$$(u_!, u^*, u_*), \quad \mathbf{Psh}(\mathcal{D}) \xrightarrow{u^*} \mathbf{Psh}(\mathcal{C})$$

where each functor is the left adjoint to the the following one. The left adjoint $u_!$ to the restriction u^* is defined via left Kan extension, so that Proposition 3.2.2 implies that it is strong monoidal.

3.2.4. There is a natural way of extending Day convolution from the category of presheaves on $\mathcal{C}$ to the category of simplicial presheaves, so that the sequence of embeddings

$$\mathcal{C} \hookrightarrow \mathbf{Psh}(\mathcal{C}) \hookrightarrow s\mathbf{Psh}(\mathcal{C})$$

is a sequence of strong monoidal functors (recall here that we identify presheaves of sets with discrete simplicial presheaves, i.e. simplicial presheaves of simplicial dimension zero). Given two simplicial presheaves F, G we define $F \otimes G$ by

$$(F \otimes G)_n = F_n \otimes G_n, \quad n \geq 0 \tag{3.2.4.1}$$

and this gives $s\mathbf{Psh}(\mathcal{C})$ the structure of a closed symmetric monoidal category. We keep writing $[-,-]$ for the internal hom for Day convolution.

Remark 3.2.5. The category of simplicial presheaves on $\mathcal{C}$ is enriched over $\mathcal{S}$ in the following way. The product $F \times K$ of a simplicial presheaf F with a simplicial set K is defined on sections by $(F \times K)(U) = F(U) \times K$. Alternatively, we can simply think to $X \times K$ as the product of X with the constant simplicial presheaf K.

The functor $\mathcal{S} \to s\mathbf{Psh}(\mathcal{C})$ given by $K \mapsto \mathbb{1} \otimes^{\mathrm{Day}} K$ is easily seen to be endowed with the structure of symmetric monoidal functor.

Remark 3.2.6. There is a pointed variant of Day convolution. Let $s\mathbf{Psh}(\mathcal{C})_{\bullet}$ be the category of pointed simplicial presheaves, i.e. the category of presheaves of pointed simplicial sets on $\mathcal{C}$, and let $(-)_+\colon s\mathbf{Psh}(\mathcal{C}) \to s\mathbf{Psh}(\mathcal{C})_{\bullet}$ be the canonical "add base point" functor (left adjoint to the forgetful functor). By mimicking the definition of smash product $\wedge$ of pointed simplicial preshaves starting from the cartesian product, we can define a symmetric monoidal structure $\otimes_{\bullet}^{\mathrm{Day}}$ on $s\mathbf{Psh}(\mathcal{C})_{\bullet}$ (see Section 4.8 for details). This is the unique symmetric monoidal structure on $s\mathbf{Psh}(\mathcal{C})_{\bullet}$ that has $\mathbb{1}_+$ as unit and that makes $(-)_+$ strong monoidal.

We will come back later on the behaviour of Day convolution with respect to different model structures on simplicial presheaves.

3.3. Motivic spaces with modulus and interval objects. We begin with the following definition.

Definition 3.3.1. *Let $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ be the category of modulus data over k. A motivic space with modulus is a contravariant functor $\mathscr{X}: \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k) \to \mathcal{S}$ i.e. a simplicial presheaf on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. We let $\overline{\mathbf{M}}\mathcal{M}(k)$ denote the category of motivic spaces with modulus.*

Since $\overline{\mathbf{M}}\mathcal{M}(k)$ is a category of simplicial presheaves on a small category, it is a locally finitely presentable bicomplete $\mathcal{S}$-category, with simplicial function complex defined as above. In particular, finite limits commute with filtered colimits. The following fact is standard.

Lemma 3.3.2. *Every motivic space with modulus is filtered colimit of finite limits of spaces of the form $h_M \times \Delta^n$, for $M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ a modulus datum and $n \geq 0$.*

Apart from the category of motivic spaces $\overline{\mathbf{M}}\mathcal{M}(k)$, there are two other categories of simplicial presheaves that will play an important rôle in what follows.

Definition 3.3.3. *The category of motivic spaces with compactifications, $\mathcal{M}_{\log}(k)$ is the category of simplicial presheaves on $\mathbf{Sm}_{\log}(k)$, i.e. $\mathcal{M}_{\log}(k) = s\mathbf{Psh}(\mathbf{Sm}_{\log}(k))$. The category of birational motivic spaces with compactifications, $B\mathcal{M}_{\log}(k)$ is the category of simplicial presheaves on the localized category $\mathbf{BSm}_{\log}(k)$, i.e. $B\mathcal{M}_{\log}(k) = s\mathbf{Psh}(\mathbf{BSm}_{\log}(k))$. Finally, we let $\mathcal{M}_k$ denote the category of motivic spaces over k in the sense of Morel-Voevodsky, i.e. $\mathcal{M}_k = s\mathbf{Psh}(\mathbf{Sm}(k))$.*

The categories $\overline{\mathbf{M}}\mathcal{M}(k)$, $\mathcal{M}_{\log}(k)$ and $B\mathcal{M}_{\log}(k)$ are closed symmetric monoidal categories, where we consider on $\overline{\mathbf{M}}\mathcal{M}(k)$ Day convolution induced by the monoidal structure 1.6 on $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ (and the usual Cartesian product on the other categories).

3.3.4. Recall from the discussion in Section 1.3 (with the notations of (1.3.2.2)) that there is a canonical faithful functor

$$v\colon \mathbf{Sm}_{\log}(k) \to \mathbf{BSm}_{\log}(k)$$

and from 1.4.5 that there is a fully faithful embedding

$$u\colon \mathbf{Sm}_{\log}(k) \hookrightarrow \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k).$$

They are both strict monoidal functors.

These functors extend to the presheaves categories, giving a pletora of adjunctions

$$(u_!, u^*, u_*), \quad u^*\colon \overline{\mathbf{M}}\mathcal{M}(k) \leftrightarrows \mathcal{M}_{\log}(k)\colon u_*$$

$$(v_!, v^*, v_*), \quad v^*\colon B\mathcal{M}_{\log}(k) \leftrightarrows \mathcal{M}_{\log}(k)\colon v_*.$$

Note that from general principle the restriction functors u^* and v^* are exact and the functors $u_!$ and $v_!$ are right exact and strong monoidal by Proposition 3.2.2.

Definition 3.3.5. *We denote by I the object of $\overline{\mathbf{M}}\mathcal{M}(k)$ given by*

$$I = u_! v^*(\overline{\square}) = u_! v^*(h_{(\mathbb{P}^1,\infty)}).$$

We will use the interval structure of $\overline{\square} = \overline{\square}^1$ on $\mathbf{BSm}_{\log}(k)$ to show that I is an interval object in the symmetric monoidal category $\overline{\mathbf{M}}\mathcal{M}(k)$. We start from the following simple observation.

Lemma 3.3.6. *The representable simplicial presheaf $\overline{\square}$ is an interval object in $B\mathcal{M}_{\log}(k)$ and a weak interval object in $\overline{\mathbf{M}}\mathcal{M}(k)$.*

Proof. The maps $(\iota_0^{\overline{\square}}, \iota_1^{\overline{\square}}, p_{\overline{\square}})$ extend obviously to maps in $B\mathcal{M}_{\log}(k)$. Since the Yoneda embedding preserves (small) limits, we have $h_{\overline{\square}} \times h_{\overline{\square}} = h_{\overline{\square}\times\overline{\square}} = h_{\overline{\square}\otimes\overline{\square}}$, so that the multiplication μ also extends, with the required compatibilities, to $B\mathcal{M}_{\log}(k)$. The statement for $\overline{\mathbf{M}}\mathcal{M}(k)$ is also clear. □

3.3.7. Let $(X, \partial X)$ be an object of $\mathbf{Sm}_{\log}(k)$. From the adjunction $(v_!, v^*)$ we get a natural map $\eta_X\colon h_{(X,\partial X)} \to v^*(v_! h_{(X,\partial X)}) = v^*(h_{(X,\partial X)})$ (since $v_!$ commutes with Yoneda). Evaluated on an object $(Y, \partial Y)$ of $\mathbf{Sm}_{\log}(k)$, the map η_X corresponds to the inclusion

$$\mathrm{Hom}_{\mathbf{Sm}_{\log}(k)}((Y,\partial Y),(X,\partial X)) \hookrightarrow \mathrm{Hom}_{\mathbf{BSm}_{\log}(k)}((Y,\partial Y),(X,\partial X)).$$

We will still denote by η_X the morphism in $\overline{\mathbf{M}}\mathcal{M}(k)$ given by $u_!(\eta_X)$. This is the map of motivic spaces

$$h_{(X;\partial X,\emptyset)} = u_!(h_{(X,\partial X)}) \xrightarrow{u_!\eta_X} u_!(v^*(h_{(X,\partial X)})).$$

For $X = \overline{\square}$, the above construction gives a canonical *comparison* morphism of motivic spaces $\eta\colon \overline{\square} \to I$.

Proposition 3.3.8. *The motivic space I is an interval object in $\overline{\mathbf{M}}\mathcal{M}(k)$ for the Day convolution product.*

Proof. We start by proving that the simplicial presheaf $v^*(\overline{\square})$ in $\mathcal{M}_{\log}(k)$ is an interval object for the usual product of presheaves. Since the terminal object of $\mathbf{Sm}_{\log}(k)$ is $(\mathrm{Spec}(k), \emptyset)$ and since Yoneda preserves small limits, the simplicial presheaf represented by $(\mathrm{Spec}(k), \emptyset)$ is just the constant simplicial having one element in simplicial degree 0 (the "point", denoted pt). Since v^* is exact, we have that $v^*(\mathrm{pt}) = \mathrm{pt}$, and therefore we automatically obtain maps

$$\iota_\varepsilon^{v^*(\overline{\square})}\colon\ \mathrm{pt} = v^*(\mathrm{pt}) \rightrightarrows v^*(\overline{\square}), \quad \varepsilon \in \{0,1\}, \quad p_{v^*(\overline{\square})}\colon v^*(\overline{\square}) \to \mathrm{pt}$$

satisfying the identities $p_{v^*(\overline{\square})} \circ \iota_\varepsilon^{v^*(\overline{\square})} = \mathrm{id}_{\mathrm{pt}}$. Let now $\mu\colon \overline{\square} \times \overline{\square} \to \overline{\square}$ be the multiplication map in $B\mathcal{M}_{\log}(k)$. Since v^* is exact, we have $v^*(\overline{\square} \times \overline{\square}) = v^*(\overline{\square}) \times v^*(\overline{\square})$ so that we get a map

$$v^*\mu\colon v^*(\overline{\square}) \times v^*(\overline{\square}) \to v^*(\overline{\square})$$

and we have

$$\begin{aligned} v^*(\mu) \circ (\mathrm{id}_{v^*\overline{\square}} \times v^*(\iota_0^{\overline{\square}})) &= v^*(\mu) \circ v^*(\mathrm{id}_{\overline{\square}} \times (\iota_0^{\overline{\square}})) = v^*(\mu \circ (\mathrm{id}_{\overline{\square}} \times (\iota_0^{\overline{\square}})) \\ &= v^*(\iota_0^{\overline{\square}} \circ p_{\overline{\square}}) = \iota_0^{v^*\overline{\square}} \circ p_{v^*\overline{\square}}, \end{aligned}$$

$$v^*(\mu) \circ (\mathrm{id}_{v^*\overline{\square}} \times v^*(\iota_1^{\overline{\square}})) = v^*(\mu) \circ v^*(\mathrm{id}_{\overline{\square}} \times (\iota_1^{\overline{\square}})) = v^*(\mu \circ (\mathrm{id}_{\overline{\square}} \times (\iota_1^{\overline{\square}})) = \mathrm{id}_{v^*\overline{\square}},$$

completing the proof that $v^*(\overline{\square})$ is an interval in $\mathcal{M}_{\log}(k)$. As for I, we first notice that

$$u_!(h_{(\mathrm{Spec}(k),\emptyset)}) = \mathbb{1}$$

is the unit for Day convolution on $\overline{\mathbf{M}}\mathcal{M}(k)$. Applying the functor $u_!$ we then obtain the morphisms $\iota_0^I, \iota_1^I, p_I$ that make I into a weak interval on $\overline{\mathbf{M}}\mathcal{M}(k)$. As for the multiplication, it's enough to show that $u_!(v^*(\overline{\square}) \times v^*(\overline{\square})) = u_!(v^*(\overline{\square} \times \overline{\square})) \simeq I \otimes I$, since the identities involving $u_!(\mu)$ will be then automatically satisfied by functoriality (or using the same chain of equalities as above). Slightly more generally, let $F \in \mathbf{Psh}(\mathbf{Sm}_{\log}(k))$ be a presheaf of sets and consider it as simplicial presheaf of simplicial dimension zero. Write $F = \mathrm{colim}_{U \downarrow F} h_U$, for $U \in \mathbf{Sm}_{\log}(k)$. Then

$$\begin{aligned} u_!(F \times F) &= u_!(\operatorname*{colim}_{U\downarrow F} h_U \times \operatorname*{colim}_{U'\downarrow F} h_{U'}) =^{\dagger} u_!(\operatorname*{colim}_{U,U'} h_{U\times U'}) \\ {}^{\dagger\dagger} &= \operatorname*{colim}_{U,U'} u_!(h_{U\times U'}) = \operatorname*{colim}_{U,U'}(h_{u_!(U\times U')}) = \operatorname*{colim}_{U,U'}(h_{u(U)\otimes u(U')}) \\ &= \operatorname*{colim}_{U,U'}(h_{u(U)} \otimes h_{u(U')}) =^{\ddagger} \operatorname*{colim}_{U\downarrow F} h_{u(U)} \otimes \operatorname*{colim}_{U'\downarrow F} h_{u(U')} \\ &= u_!(F) \otimes u_!(F). \end{aligned}$$

The equality † follows from the fact that colimits commute with finite fiber products in a category of presheaves (Giraud's axiom), while †† follows from the fact that $u_!$ commutes with colimits. For the equality ‡ we have used the fact that $\mathbf{Psh}(\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k))$ is monoidally co-complete for Day convolution product. The other equalities are trivial, using the fact that $u_!$ commutes with Yoneda and that $u(U) \otimes u(U') = u(U) \times u(U')$ in $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ for every U, U' in $\mathbf{Sm}_{\log}(k)$ by 1.6.2. Specializing these equalities to the case $F = v^*(\overline{\square})$ gives the required statement. □

Remark 3.3.9. Our method of transporting the interval structure from $\mathbf{BSm}_{\log}(k)$ to $\overline{\mathbf{M}}\mathcal{M}(k)$ looks quite general. It seems plausible that one can repeat a similar argument by replacing the category of simplicial presheaves on $\mathbf{BSm}_{\log}(k)$ with the category of extended co-cubical objects in $\mathcal{S}$, i.e. the category of (strong) monoidal functors $[\mathbf{ECube}, \mathcal{S}]_{\otimes}$, or even with the category of extended cubical object in a category of presheaves with values in monoidal model category $\mathcal{M}$.

Remark 3.3.10. In the references [51], [64] and [6] (for $\mathbb{A}^1$-theory) and [29] (for the modulus-theory), the interval objects considered are always representable (either by $\mathbb{A}^1$ or by the modulus pair $(\mathbb{P}^1, 1)$). Here, we are pushing the ideas of [29] further to get a theory that works more generally for interval objects in categories of presheaves that are not necessarily representable.

3.3.11. Let $\mathbb{A}^1$ denote the representable simplicial presheaf $h_{(\mathbb{A}^1;\emptyset,\emptyset)} = h_{u(\mathbb{A}^1,\emptyset)}$. It is clearly an interval object in $\overline{\mathbf{M}}\mathcal{M}(k)$ for the cartesian product as well as for Day convolution product, since by 1.6.2 we have $\mathbb{A}^1 \otimes \mathbb{A}^1 = \mathbb{A}^1 \times \mathbb{A}^1$. From the admissible morphism $j\colon (\mathbb{A}^1, \emptyset) \hookrightarrow (\mathbb{P}^1, \infty)$ in $\mathbf{Sm}_{\log}(k)$ we get maps of motivic spaces

$$\mathbb{A}^1 \to \overline{\square} \to I \text{ in } \overline{\mathbf{M}}\mathcal{M}(k), \quad \mathbb{A}^1 \to \overline{\square} \to v^*(\overline{\square}) \text{ in } \mathcal{M}_{\log}(k)$$

that we will use to compare the different interval structures on $\overline{\mathbf{M}}\mathcal{M}(k)$, on $\mathcal{M}_{\log}(k)$ and on $\mathcal{M}(k)$.

3.3.12. We start by comparing $\mathbb{A}^1$ and $v^*(\overline{\square})$ with the standard interval $\mathbb{A}^1$ in the category of motivic spaces $\mathcal{M}(k)$. Recall that thes adjoint pair $\lambda\colon \mathbf{Sm}(k) \leftrightarrows \mathbf{Sm}_{\log}(k)\colon \omega$ give rise to a string of four adjoint functors

$$(\lambda_!, \lambda^* = \omega_!, \lambda_* = \omega^*, \omega_*), \quad \omega_! = \lambda^*\colon \mathcal{M}_{\log}(k) \leftrightarrows \mathcal{M}(k)\colon \lambda_*$$

(i.e. the functor $\omega_!$ as in turn a left adjoint). The functors λ^* and ω^* clearly commute with products. Since λ commutes with products and $\lambda_!$ commutes with Yoneda, the same argument used in the proof of Proposition 3.3.8 (or even Proposition 3.2.2 directly) shows that $\lambda_!$ is monoidal with respect to the cartesian product.

Lemma 3.3.13. *There are canonical isomorphisms $\lambda^*(\mathbb{A}^1) \simeq \lambda^* v^*(\overline{\square}) \simeq \mathbb{A}^1$ as interval objects of $\mathcal{M}(k)$.*

Proof. Since $\lambda^* = \omega_!$, it's clear that $\lambda^*(\mathbb{A}^1) = \omega_!(h_{(\mathbb{A}^1,\emptyset)}) = h_{\omega(\mathbb{A}^1,\emptyset)} = h_{\mathbb{A}^1} = \mathbb{A}^1$. For the second statement, it's enough to check that $\lambda^* v^*(\overline{\square}) \simeq \mathbb{A}^1$ as presheaves on $\mathbf{Sm}(k)$, and we just have to play with adjunctions. For $X \in \mathbf{Sm}(k)$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Psh}(\mathbf{Sm}(k))}(h_X, \lambda^* v^*(\overline{\square})) &= \mathrm{Hom}_{\mathbf{Psh}(\mathbf{Sm}_{\log}(k))}(\lambda_!(h_X), v^*(\overline{\square})) \\
&= \mathrm{Hom}_{\mathbf{Psh}(\mathbf{BSm}_{\log}(k))}(v_!(h_{(X,\emptyset)}), \overline{\square}) \\
&= \mathrm{Hom}_{\mathbf{BSm}_{\log}(k)}((X, \emptyset), (\mathbb{P}^1, \infty)) \\
&= \mathrm{Hom}_{\mathbf{Sm}(k)}(X, \mathbb{A}^1).
\end{aligned}$$

The fact that the isomorphisms are compatible with the interval structure is a tautology from the definitions. □

Remark 3.3.14. Note that the interval I of $\overline{\mathbf{M}}\mathcal{M}(k)$ is obtained from $v^*(\overline{\square})$ by applying the left adjoint $u_!$ of the restriction functor u^*. The exact same construction, using $\omega_! = \lambda^*$ instead of $u_!$ produces, in view of Lemma 3.3.13, nothing but the usual interval $\mathbb{A}^1$ in the Morel-Voevodsky category of motivic spaces.

3.3.15. Before moving forward to give more refined comparisons, we ask the following question. Let $F\colon \mathbf{Sm}_{\log}(k) \to \mathbf{Sm}(k)$ be the forgetful functor $(X, \partial X) \to X$. Then, F defines the usual set of adjoint functors between the categories of spaces $\mathcal{M}_{\log}(k)$ and $\mathcal{M}(k)$, namely $(F_!, F^*, F_*)$. The general principle illustrated above allows us to construct the object $J \in \mathcal{M}$ as $J = F_!(v^*(\overline{\square}))$. Since F is strict monoidal for the cartesian product, the argument of Proposition 3.3.8 goes through to show that J is in fact an interval object in $\mathcal{M}(k)$ for the usual product of simplicial presheaves.

The canonical adjunction map $\eta\colon \overline{\square} \to v^*(\overline{\square})$ gives then a map $\mathbb{P}^1 \to J$ in $\mathcal{M}$. By construction, this map cannot be an isomorphism (since $\mathbb{P}^1$ does not have any interval structure on $\mathcal{M}(k)$), and is also easy to see that J is not isomorphic to $\mathbb{A}^1$. One could therefore try to develop a machinery for the localization of $\mathcal{M}(k)$ to J, in the same spirit of what we will do for $\overline{\mathbf{M}}\mathcal{M}(k)$ with I. We don't know, at the moment, what would be the outcome of such construction.

4. Motivic homotopy categories with modulus

4.1. Model structures on simplicial presheaves. The category of simplicial sets $\mathcal{S}$ carries a well-known cofibrantly generated model structure (the Quillen model structure). The generating cofibrations are the inclusion of boundaries

$$I = \{\iota_n\colon \partial\Delta^n \hookrightarrow \Delta^n\}, \qquad \text{for } n \geq 0$$

and the generating trivial cofibrations are the horn inclusions

$$J = \{j^n_k\colon \Lambda^n_k \hookrightarrow \Delta^n\}, \qquad \text{for } n \geq 1, 0 \leq k \leq n.$$

With this definition, the fibrations are exactly the Kan fibrations and the cofibrations are all injective maps (monomorphisms). The category of simplicial presheaves $s\mathbf{Psh}(T)$ on a small Grothendieck site T carries two natural model structures, the *injective* and the *projective* model structure presenting the same homotopy category, i.e. having the same class of weak equivalences:

Definition 4.1.1. *A map $f\colon A \to B$ in $s\mathbf{Psh}(T)$ in $s\mathbf{Psh}(T)$ is called an objectwise (or levelwise or sectionwise) simplicial weak equivalence if $f(X)\colon A(X) \to B(X)$ is a weak equivalence of simplicial sets for each $X \in T$. A map $f\colon A \to B$ is called an objectwise Kan fibration if $f(X)\colon A(X) \to B(X)$ is a Kan fibration of simplicial sets for each $X \in T$.*

The injective model structure on $s\mathbf{Psh}(T)$ was introduced by Heller in [21].

Theorem 4.1.2 (Heller)**.** *The category of simplicial presheaves on a small site forms a proper simplicial cellular model category having for weak equivalence the objectwise simplicial weak equivalences, for cofibrations all monomorphisms and for fibrations the maps having the right lifting property with respect to trivial cofibrations.*

We will refer to fibrations for the injective model structure as *(simplicial) injective fibrations*. The category of simplicial presheaves with the injective model stucture will be denoted, following the usual convention, $s\mathbf{Psh}(T)_{\mathrm{inj}}$. Note that every object is cofibrant for the injective structure. The second model structure on $s\mathbf{Psh}(T)$, the projective one, goes back to Quillen.

Theorem 4.1.3 (see [22], Theorem 11.6.1)**.** *The category of simplicial presheaves on a small site forms a proper simplicial cellular model category having for weak equivalences the objectwise simplicial* *weak equivalences, for fibrations the objectwise Kan fibrations and for cofibrations the maps having the left lifting property with respect to trivial fibrations.*

We denote by $s\mathbf{Psh}(T)_{\mathrm{proj}}$ the category of simplicial presheaves with the projective model structure.

Remark 4.1.4. We refer the reader to [22, 12] for the definition of cellular model category. Both the injective and the projective model structure on simplicial presheaves are cellular (see [28], around 7.19 for a comment on the cellularity of the injective model structure). By definition, a cellular model category is cofibrantly generated. For the projective model structure, a set of generators is given by

(4.1.4.1)
$$I = \{\mathrm{id}_X \times \iota_n \colon h_X \times \partial\Delta^n \to h_X \times \Delta^n\}, \quad \text{for } n \geq 0, X \in T$$
$$J = \{\mathrm{id}_X \times j^n_k \colon h_X \times \Lambda^n_k \to h_X \times \Delta^n\}, \quad \text{for } n \geq 1, 0 \leq k \leq n, X \in T,$$

from which we get immediately that every representable (simplicial) presheaf is cofibrant for the projective structure. We choose a functorial cofibrant replacement $(-)^{\mathrm{c}} \to \mathrm{id}_{s\mathbf{Psh}T}$, so that for every object $\mathscr{X} \in s\mathbf{Psh}(T)$, there is an objectwise trivial fibration $\mathscr{X}^{\mathrm{c}} \to \mathscr{X}$ with $\mathscr{X}^{\mathrm{c}}$ cofibrant. It's possible to write down generating cofibrations for the injective model structure as well, but since we are not going to use the explicit form, we omit it.

Scholium 4.1.5. *There are different choices that one can make for the cofibrant replacement functor. We recall here a very convenient one introduced by Dugger* [13, Section 2]*. Let F be an object in* $\mathbf{Psh}(T)$*. Define $\tilde{Q}F$ to be the simplicial presheaf whose $n-$th level is*

$$\tilde{Q}F_n = \coprod_{h_{U_n} \to h_{U_{n-1}} \to \dots \to h_{U_0} \to F} (h_{U_n})$$

where U_n runs on the set of representable presheaves mapping to F, and whose faces and degeneracies are the obvious ones, coming from compositions of maps and inserting identities. One can prove that $\tilde{Q}F$ is cofibrant and that the natural map $\tilde{Q}F \to F$ is a weak equivalence (see [13, Lemma 2.7]*). For an arbitrary object $\mathscr{X}$ of* $s\mathbf{Psh}(T)$*, applying $\tilde{Q}$ in every simplicial dimension gives a bi-simplicial presheaf. We write $Q\mathscr{X}$ for the diagonal. By* [13, Proposition 2.8]*, $Q\mathscr{X}$ is cofibrant for every simplicial presheaf $\mathscr{X}$, and the map $Q\mathscr{X} \to \mathscr{X}$ is a weak equivalence.*

Remark 4.1.6. The injective model structure $s\mathbf{Psh}(T)_{\mathrm{inj}}$ is also cofibrantly generated, but the description of the set of generating trivial cofibration is not very explicit.

4.1.7. Let $u \colon T \to T'$ be a functor between small categories (for example, a functor between small sites). The usual string of adjoint functors between the categories of simplicial presheaves on $\mathcal{C}$ and on $\mathcal{D}$ behave nicely with respect to the model structures. More precisely, we have the following

Proposition 4.1.8. *Let $u^* \colon s\mathbf{Psh}(T) \to s\mathbf{Psh}(T')$ has both a left adjoint $u_!$ and a right adjoint u_*. The pair $(u_!, u^*)$ is a Quillen adjunction for the projective model structure and the pair (u^*, u_*) is a Quillen adjunction for the injective model structure on the two functor categories.*

4.1.9. Recall (see [23, Definition 4.2.6]) that a model category $\mathcal{M}$ that is also a monoidal category with product $\otimes$ and unit $\mathbb{1}$ is called a *monoidal model category* if the following conditions are satisfied

(1) Let $Q\mathbb{1} \xrightarrow{q} \mathbb{1}$ be the cofibrant replacement for the unit $\mathbb{1}$. Then the natural map $Q\mathbb{1} \otimes X \to \mathbb{1} \otimes X \simeq X$ is a weak equivalence for all cofibrant X.
(2) Given two cofibrations $f \colon U \to V$ and $g \colon W \to X$ in $\mathcal{M}$, their push-out product
$$f \square g \colon (V \otimes W) \amalg_{U \otimes W} (U \otimes X) \to V \otimes X$$
is a cofibration, which is trivial if either f or g is.

We refer to the first condition as the *unit axiom* and to the second condition as the *pushout product axiom*.

Suppose that a small site T carries a symmetric monoidal structure $\otimes$, that extends via Day convolution to $s\mathbf{Psh}(T)$ (see Section 3.2).

Proposition 4.1.10. *The projective model structure on simplicial presheaves is a (symmetric) monoidal model category with respect to Day convolution.*

Proof. The unit axiom is automatically satisfied, since $\mathbb{1} = h_{\mathbb{1}}$ for the unit object of T for the product $\otimes$, and since every representable presheaf is automatically cofibrant for the projective structure (given the explicit set of generating cofibrations). By [23, Corollary 4.2.5] it's enough to show that the pushout product of two generating cofibrations is still a cofibrations and that the pushout product of a generating trivial cofibration with a generating cofibration is a trivial cofibration.

First, let X, X' be objects of T and let K be a simplicial set. We have for every $n \geq 0$

$$((h_X \times K) \otimes^{\mathrm{Day}} h_{X'})_n = (\coprod_{k \in K_n} h_X) \otimes^{\mathrm{Day}} h_{X'} = \coprod_{k \in K_n} (h_X \otimes^{\mathrm{Day}} h_{X'}) = ((h_X \otimes h_{X'}) \times K)_n,$$

where the first and the third equalities follow from the definition of the simplicial structure on $s\mathbf{Psh}(T)$ and the second equality follows from the fact that $\otimes^{\mathrm{Day}}$ commutes with colimits. The simplicial identities are clear, and thus we get $(h_X \times K) \otimes h_{X'} \simeq (h_{X'} \otimes h_X) \times K$. Similarly, for X, X' in T and K, L in $\mathcal{S}$ we have the canonical isomorphism

$$(h_X \times K) \otimes (h_{X'} \times K) \simeq (h_X \otimes h_{X'}) \times (K \times L). \tag{4.1.10.1}$$

Let now $f_i \colon K_i \to L_i$ for $i = 1, 2$ be morphisms of simplicial sets and let X, X' be again objects of T. The source of the Day pushout product of $\mathrm{id}_X \times f_1$ and $\mathrm{id}_{X'} \times f_2$ takes the form

$$((h_X \times K_1) \otimes (h_{X'} \times L_2)) \amalg_{(h_X \times K_1) \otimes (h_{X'} \times K_2)} ((h_X \times L_1) \otimes (h_{X'} \times K_2)).$$

Using (4.1.10.1) and the fact that the product of simplicial presheaves commutes with coproducts (since it is a left adjoint), we can write $(\mathrm{id}_X \times f_1) \square^{\mathrm{Day}} (\mathrm{id}_{X'} \times f_2)$ as

$$(h_X \otimes h_{X'}) \times (K_1 \times L_2 \amalg_{K_1 \times K_2} L_1 \times K_2) \xrightarrow{\mathrm{id}_{h_X \otimes h_{X'}} \times (f_1 \square f_2)} (h_X \otimes h_{X'}) \times (L_1 \times L_2).$$

Replacing f_1 and f_2 with the explicit generating (trivial) cofibrations of $\mathcal{S}$ we get that the pushout product axiom for $\mathcal{S}$ implies the axiom for $s\mathbf{Psh}(T)$. $\square$

Remark 4.1.11. The pointed version of Day convolution presented in 3.2.6 gives rise to a symmetric monoidal model structure on the category of pointed simplicial presheaves $s\mathbf{Psh}(T)_\bullet$ with the projective model structure. This follows from Proposition 4.1.10 together with [23, Proposition 4.2.9]. Alternatively, one can repeat the proof of Proposition 4.1.10 replacing the product of simplicial presheaves with the smash product of the pointed counterparts.

4.2. Local model structures. Following Jardine (see [28] or [26]), we can put the topology in the picture as follows. Let T be again a small Grothendieck site and let $s\mathbf{Psh}(T)$ be the category of simplicial presheaves on T. We denote by $\pi_n(\mathscr{X})$ the n-homotopy presheaf of $\mathscr{X}$. For a presheaf F on T, we write $\widetilde{F}$ for the associated sheaf.

Definition 4.2.1. *A map $\mathscr{X} \to \mathscr{Y}$ of simplicial presheaves is called a local weak equivalence (for the topology σ on T) if and only if*

(1) the map $\widetilde{\pi_0}(X) \to \widetilde{\pi_0}(Y)$ is an isomorphism of sheaves

(2) the diagram of presheaves maps

$$\begin{array}{ccc} \pi_n(\mathscr{X}) & \longrightarrow & \pi_n(\mathscr{Y}) \\ \downarrow & & \downarrow \\ \mathscr{X}_0 & \longrightarrow & \mathscr{Y}_0 \end{array}$$

induce pull-back diagrams of associated sheaves for $n \geq 1$.

Equivalently, (see for example [65]*, around Lemma 3.5) one can reformulate condition (2) as follows. For any object X of T, any $x \in \mathscr{X}(X)$ and any $n \geq 1$ the morphism of associated sheaves $\widetilde{\pi_n}(\mathscr{X}, x) \to \widetilde{\pi_n}(\mathscr{Y}, f(x))$ on the overcategory T/X defined by f is an isomorphism.*

Remark 4.2.2. Since sheafification is an exact functor, one can check easily that every objectwise weak equivalence is a local weak equivalence.

Remark 4.2.3. If the site T (or, rather, the topos $\mathbf{Psh}(T)$) has enough points, there is yet another description of local weak equivalences. A *stalkwise weak equivalence* is a map of simplicial presheaves which induces a weak equivalence of simplicial sets in all stalks. The equivalence between the two approaches is proved by Jardine in [26].

Definition 4.2.4. *We say that a map $p\colon \mathscr{X} \to \mathscr{Y}$ of simplicial presheaves on T is an injective fibration or a global fibration if it has the right lifting property with respect to every map $f\colon \mathscr{A} \to \mathscr{B}$ that is a monomorphism and a local weak equivalence.*

Recall that an *injective cofibration* is simply a monomorphism of simplicial presheaves. We use the word injective fibrations as this is now common in literature, though some authors still use the term global fibration. If necessary, we will distinguish injective fibrations for the injective-local model structure from fibrations for the injective model structure on simplicial presheaves by adding the word *simplicial* to the latter class.

Theorem 4.2.5 (See [28], Theorem 5.8)**.** *The category of simplicial presheaves on a small site T with the class of local weak equivalences, injective cofibrations and global fibrations is a proper simplicial cellular closed model category. We write $s\mathbf{Psh}(T)^{loc}_{\mathrm{inj}}$ to denote this structure and we call it the local injective model structure.*

Remark 4.2.6. In [28], *loc.cit.*, the model structure is only specified to be cofibrantly generated. The cellularity is however remarked later in Section 7 when dealing with localization problems.

4.2.7. *Localizations.* Given any model category $\mathcal{M}$ and a set of morphisms S, we say that an object X of $\mathcal{M}$ is S-local (see [22, 3.1.4]) if it is fibrant and for every map $f\colon A \to B$ the induced map f^* between the homotopy function complexes is a weak equivalence. A map $g\colon X \to Y$ is called an $S-$local equivalence if for every S-local object Z the induced map g^* between the homotopy function complexes is a weak equivalence. Recall the following general Theorem, due to Hirschhorn in the presented form:

Theorem 4.2.8 ([22], Theorem 4.1.1)**.** *Let $\mathcal{M}$ be a left proper cellular model category and let S be a set of maps in $\mathcal{M}$. Then, the left Bousfield localization of $\mathcal{M}$ with respect to S exists. That is, there is a model structure $L_S\mathcal{M}$ on the underlying category of $\mathcal{M}$ in which the weak equivalences are the S-local equivalences of $\mathcal{M}$, the cofibrations are the same cofibrations of $\mathcal{M}$ and the class of fibrations is the class of maps having the right lifting property with respect to those maps that are both cofibrations and S-local equivalences. The fibrant objects of $L_S\mathcal{M}$ are precisely the S-local objects of $\mathcal{M}$. Moreover, $L_S\mathcal{M}$ is a left proper cellular model category, that has a natural structure of simplicial model category if $\mathcal{M}$ has one.*

4.2.9. Let $\mathcal{M}$ be a (symmetric) monoidal model category. Write $\otimes$ for its monoidal product and take a set of morphisms S. In general, the Bousfield localization $L_S\mathcal{M}$ (whenever exists) will not inherit the structure of monoidal model category. The following Proposition gives a convenient criterion for checking if the localization to S behaves well with respect to the monoidal structure. The proof is standard, and we refer to [10].

Proposition 4.2.10 ([10], Proposition 5.6)**.** *Let $\mathcal{M}$ be a left proper cellular symmetric monoidal model category. Let S be a set of morphisms in $\mathcal{M}$. Assume that the following conditions hold:*

i) $\mathcal{M}$ admits generating sets of (trivial) cofibrations consisting of maps between cofibrant objects;

ii) For every cofibrant object X, the functor $X \otimes (-)$ sends the elements of S to S-local weak equivalences;

iii) The unit object for the monoidal structure is cofibrant.

Then the left Bousfield localization $L_S\mathcal{M}$ with respect to S (that exists by Theorem 4.2.8) is a symmetric monoidal model category.

The following Corollary is an immediate application of Proposition 4.2.10, using the fact that Day convolution makes the projective model structure on simplicial presheaves symmetric monoidal by Proposition 4.1.10.

Corollary 4.2.11 ([10], Theorem 5.7)**.** *Suppose that $\mathcal{M} = s\mathbf{Psh}(T)_{\mathrm{proj}}$ is the category of simplicial presheaves on a site T, equipped with the Day convolution product. Let S be a set of maps between cofibrant objects in $\mathcal{M}$. Assume that for every object X of T and every map $s\colon \mathscr{X} \to \mathscr{Y}$ in S, the map $\mathrm{id} \otimes s\colon h_X \otimes \mathscr{X} \to h_X \otimes \mathscr{Y}$ is an S-local equivalence. Then the left Bousfield localization $L_S\mathcal{M}$ with respect to S is a symmetric monoidal model category for Day convolution.*

4.2.12. When the topology on T is defined by a regular, complete and bounded cd-structure P, a result of Voevodsky ([65], Proposition 3.8) presents the local injective structure of Jardine as left Bousfield localization of the injective structure of 4.1.2 to the class of maps given by distinguished squares. More precisely, let T be a site as above and let $X \in T$. For Q a distinguished square of the form (2.1.2.1) for the cd-structure P on T, write $P(Q)$ for the simplicial homotopy push-out of the diagram $(A \leftarrow B \to Y)$. There is a natural map $P(Q) \to X$. Write Σ_P for the class of maps

$$\Sigma_P = \{P(Q) \to X\}_Q \cup \{\emptyset \to h_\emptyset\}$$

where Q runs on the set of distinguished squares for the cd-structure P on T, and $\emptyset$ is the initial object of $\mathbf{Psh}(T)$ and $h_\emptyset$ is presheaf represented by the initial object of T.

Theorem 4.2.13 ([65])**.** *Let T be a small site whose topology is defined by a complete bounded and regular cd-structure. Then the local injective model structure on $s\mathbf{Psh}(T)$ is the left Bousfield localization of the (global) injective model structure 4.1.2 to the class Σ_P.*

4.2.14. Together with the injective local model structure, there is a *projective local model structure* on simplicial presheaves due to Blander [6]. We recall here the following useful facts about it.

Proposition 4.2.15 ([6], Lemma 4.1)**.** *Let T be a site with an initial object $\emptyset$ whose topology is defined by a complete bounded regular cd-structure P. Then a simplicial presheaf F on T is local projective fibrant if and only if $F(U)$ is a Kan simplicial set for all U in T and if for every distinguished square Q of P, the square $F(Q)$ is a homotopy pull-back square of simplicial sets. Such presheaves are called flasque.*

Theorem 4.2.16 (Blander [6], Lemma 4.3, Voevodsky [65])**.** *The local projective model structure on the category of simplicial presheaves on a site T equipped with a complete regular bounded cd-structure P is the left Bousfield localization of the projective model structure of 4.1.3 to the class Σ_P.*

We write $s\mathbf{Psh}(T)^{loc}_{\mathrm{proj}}$ to denote the local projective model structure on $s\mathbf{Psh}(T)$. By the general theory of Bousfield localization, the local projective and the local injective model structures on $s\mathbf{Psh}(T)$ are both cellular, proper and simplicial. We recall the following Lemma.

Lemma 4.2.17. *The identity functor from the local projective model structure to the local injective model structure is a left Quillen equivalence.*

The weak equivalences in both model structure agree and are precisely the local weak equivalences. Note that left proper is automatic in both cases by Bousfield localization, while right properness follows from [6, Lemma 3.4] (for the projective case) and [28, Lemma 4.37] (for the injective case).

Proposition 4.2.18. *Let T be a small site whose topology is defined by a complete bounded and regular cd-structure. Assume that for every distinguished square Q in T and every object Z in T, the product $Q \otimes Z$ is still a distinguished square in T. Then Day convolution makes $s\mathbf{Psh}(T)^{loc}_{\mathrm{proj}}$ a monoidal model category.*

Proof. Thanks to Blander's Theorem 4.2.16, the local projective model structure is a (left) Bousfield localization of the projective structure, and we have an explicit description of the set of maps that we are inverting. In the notations of 4.2.12, write $\varphi_X \colon P(Q) \to X$ for the natural map from the simplicial homotopy push-out of a distinguished square having $X \in T$ as bottom right corner. According to Corollary 4.2.11, we have to check that for every representable presheaf h_Z, the induced map $\varphi_X \otimes \mathrm{id}_Z \colon P(Q) \otimes h_Z \to h_X \otimes h_Z = h_{X\otimes Z}$ is an S-local equivalence for $S = \Sigma_P$. Write Q_Z for the tensor product square $Q \otimes Z$ in T. By assumption, Q_Z is a distinguished square, and the bottom right corner of it is the product $X \otimes Z$. The natural map $P(Q_Z) \to h_{X\otimes Z}$ is an element of Σ_P by assumption, and factors through the product $P(Q) \otimes h_Z$. Since the projective model structure is monoidal for Day convolution, the tensor product $(-) \otimes h_Z$ commutes with homotopy colimits. In particular, the map $P(Q_Z) \to P(Q) \otimes h_Z$ is a weak equivalence. By the 2 out of 3 property of S-local equivalences, we conclude that the required map $\varphi_Z \otimes \mathrm{id}_Z$ is an S-local equivalence, completing the proof. □

4.3. Interval-local objects and I-homotopies. We start from the following general definition. Let T be a (small) site and let $s\mathbf{Psh}(T)$ be the category of simplicial presheaves on T. Suppose that T carries a symmetric monoidal structure $\otimes$, that extends via Day convolution to $s\mathbf{Psh}(T)$. Finally, suppose that there exists an interval object I for the $\otimes$-structure on $s\mathbf{Psh}(T)$. We consider on $s\mathbf{Psh}(T)$ both the injective and the projective local model structures.

Definition 4.3.1. *A simplicial presheaf $\mathscr{X}$ is called projective I-local (resp. injective I-local) if:*

i) $\mathscr{X}$ is fibrant for the projective local model structure on $s\mathbf{Psh}(T)$ (resp. $\mathscr{X}$ is fibrant for the injective local model structure on $s\mathbf{Psh}(T)$)

ii) For every $\mathscr{Y}$ in $s\mathbf{Psh}T$, the map between the homotopy function complexes

$$\mathrm{Map}(\mathscr{Y} \otimes I, \mathscr{X}) \to \mathrm{Map}(\mathscr{Y}, \mathscr{X})$$

induced by $\mathrm{id} \otimes \iota^I_0$ is a weak equivalence.

Remark 4.3.2. If $\mathcal{M}$ is a simplicial model category, a homotopy function complex between a cofibrant object X and a fibrant object Y is weakly equivalent to the simplicial mapping space (or simplicial function complex, in the terminology of 3.1) $\mathcal{S}(X, Y)$. Since every object is cofibrant in $s\mathbf{Psh}(T)_{\mathrm{inj}}$, an injective fibrant simplicial presheaf $\mathscr{X}$ is injective I-local if for every $\mathscr{Y}$, the natural map between the simplicial function complexes

$$\mathcal{S}(\mathscr{Y} \otimes I, \mathscr{X}) \to \mathcal{S}(\mathscr{Y}, \mathscr{X})$$

is a weak equivalence.

On the other hand, not every object is cofibrant for the projective structure. We can then reformulate condition ii) of Definition 4.3.1 using the simplicial function complex and the functorial cofibrant replacement $(-)^{c}$ as follows: a projective fibrant simplicial presheaf $\mathscr{X}$ is projective I-local if for every object $\mathscr{Y}$, the natural map between the simplicial function complexes

$$\mathcal{S}((\mathscr{Y})^{c} \otimes I^{c}, \mathscr{X}) \to \mathcal{S}((\mathscr{Y})^{c}, \mathscr{X})$$

is a weak equivalence. Note that we are using here the fact that $\otimes$ is left Quillen bi-functor, so that it preserves cofibrant objects.

Remark 4.3.3. Note that the cofibrant replacement $(I)^{c} \to I$ was chosen to be functorial. This gives automatically the object $(I)^{c}$ the structure of interval object in $s\mathbf{Psh}(T)_{\mathrm{proj}}$. This applies, in particular, in the case $T = \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ and $s\mathbf{Psh}(T) = \overline{\mathbf{M}}\mathcal{M}(k)$.

Definition 4.3.4. *A morphism $g\colon \mathscr{X} \to \mathscr{Y}$ is called a projective $I-$weak equivalence (resp. an injective I-weak equivalence) if for any projective (resp. injective) I-local object $\mathscr{Z}$ the induced map between the homotopy function complexes*

$$g^{*}\colon \mathrm{Map}(\mathscr{Y}, \mathscr{Z}) \to \mathrm{Map}(\mathscr{X}, \mathscr{Z})$$

is a weak equivalence. We write $\mathbf{W}_{I}^{\mathrm{proj}}$ (resp. $\mathbf{W}_{I}^{\mathrm{inj}}$) for the class of projective (resp. injective) I-weak equivalences.

Definition 4.3.5. *A morphism $p\colon \mathscr{X} \to \mathscr{Y}$ is called a projective I-fibration (resp. an injective I-fibration) if it has the right lifting property with respect to all maps that are both projective cofibrations (resp. monomorphisms) and projective (resp. injective) I-weak equivalences.*

Specializing Hirschhorn's Theorem 4.2.8 to the case $\mathcal{M} = s\mathbf{Psh}(T)_{\mathrm{inj}}$ or $s\mathbf{Psh}(T)_{\mathrm{proj}}$ with S given by $\mathbf{W}_{I}^{\mathrm{inj}}$ or $\mathbf{W}_{I}^{\mathrm{proj}}$ respectively, produces the *I-localized model structure* (injective or projective), denoted $L_{I}(s\mathbf{Psh}(T))$ with the relevant subscript. We will denote the homotopy category of $L_{I}(s\mathbf{Psh}(T))^{loc}_{\mathrm{proj}}$ (resp. $L_{I}(s\mathbf{Psh}(T))^{loc}_{\mathrm{inj}}$) by $\mathcal{H}(T, I)^{loc}_{\mathrm{proj}}$ (resp. by $\mathcal{H}(T, I)^{loc}_{\mathrm{inj}}$) or simply by $\mathcal{H}(T, I)$ if it is clear wich model structure is considered.

Remark 4.3.6. It is not obvious (at least, not a priori) that the categories $L_{I}(s\mathbf{Psh}(T))^{loc}_{\mathrm{proj}}$ and $L_{I}(s\mathbf{Psh}(T))^{loc}_{\mathrm{inj}}$ are Quillen equivalent. For the category $\mathcal{M}(k)$ of motivic spaces over k, a comparison between the two localized model structures is given, for example, in [14, Theorem 2.17]: the identity functor id is the left adjoint of a Quillen equivalence between the motivic model structure (in the sense of [14, 2.12], built as localization of the projective model structure on simplicial presheaves) and the Goerss-Jardine model structure (built as localization of the injective model structure on simplicial presheaves). In our case, we can argue as follows.

Proposition 4.3.7. *The identity functor $\mathrm{id}_{s\mathbf{Psh}(T)}\colon L_{I}(s\mathbf{Psh}(T))^{loc}_{\mathrm{proj}} \to L_{I}(s\mathbf{Psh}(T))^{loc}_{\mathrm{inj}}$ is the left adjoint of a Quillen equivalence.*

Proof. Recall first that the identity functor is a Quillen equivalence from the local projective to the local injective model structure before I-localization. Let $I^{c} \to I$ be the (functorially chosen) cofibrant replacement of I in the projective local model structure on $s\mathbf{Psh}(T)$. Note in particular that $I^{c} \to I$ is a local weak equivalence between two injective-cofibrant objects. Write $\mathbf{W}_{I^{c}}^{\mathrm{inj}}$ for the class of I^{c}-injective weak equivalences, defined replacing I with I^{c} in Definitions 4.3.1 and 4.3.4. We claim that $\mathbf{W}_{I^{c}}^{\mathrm{inj}} = \mathbf{W}_{I}^{\mathrm{inj}}$. Start with $f\colon A \to B \in \mathbf{W}_{I^{c}}^{\mathrm{inj}}$ and let $\mathscr{Z}$ be an injective I-local object. Then $\mathscr{Z}$ is globally fibrant (i.e. fibrant for the injective local model structure) and thus (using that by Proposition 4.2.18 $-\otimes \mathscr{X}$ preserves local weak equivalences) the map $\mathcal{S}(I \otimes \mathscr{X}, \mathscr{Z}) \to \mathcal{S}(I^{c} \otimes \mathscr{X}, \mathscr{Z})$ is a simplicial homotopy equivalence for every $\mathscr{X}$ (note that since $\mathscr{Z}$ is fibrant and $I^{c} \otimes \mathscr{X}$ and $I \otimes \mathscr{X}$ are both cofibrant, the simplicial mapping spaces are fibrant simplicial sets). We conclude that the map

$$\mathcal{S}(I^c \otimes \mathscr{X}, \mathscr{Z}) \to \mathcal{S}(\mathscr{X}, \mathscr{Z})$$

is a weak equivalence. In particular, the object $\mathscr{Z}$ is I^c-local and thus the map f is an injective I-weak equivalence. We can reverse the argument, and start from $\mathscr{Z}$ injective I^c-local to get that every injective I-weak equivalence is an injective I^c-weak equivalence.

Note now that the classes of maps $\mathbf{W}^{\text{proj}}_{I^c}$ and $\mathbf{W}^{\text{proj}}_{I}$ do clearly coincide by definition. We are then reduced to show that $\mathbf{W}^{\text{proj}}_{I^c} = \mathbf{W}^{\text{inj}}_{I^c}$. This can be done using the same argument of [14, Theorem 2.17]. We recall the argument for completeness. Suppose that $f\colon A \to B$ is a projective I^c-weak equivalence and take $\mathscr{Z}$ injective I^c-fibrant. Then $\mathscr{Z}$ is globally fibrant, hence fibrant for the projective local model structure. The map of simplicial mapping spaces

$$(f^c)^*\colon \mathcal{S}(B^c, \mathscr{Z}) \to \mathcal{S}(A^c, \mathscr{Z})$$

is then a weak equivalence. Since the maps $A^c \to A$ and $B^c \to B$ are local weak equivalences, we conclude that $f \in \mathbf{W}^{\text{inj}}_{I^c}$. Conversely, start from $\mathscr{Z}$ projective I^c-local and choose a local weak equivalence $\mathscr{Z} \to \mathscr{Z}'$ with $\mathscr{Z}'$ globally fibrant (this exists by general principle, see [28] or [26]). Then $\mathscr{Z}'$ is easily seen to be injective I^c-fibrant. For every $f\colon A \to B$ injective I^c-weak equivalence, we then have a diagram of simplicial mapping spaces

$$\begin{array}{ccc} \mathcal{S}(B^c, \mathscr{Z}) & \longrightarrow & \mathcal{S}(A^c, \mathscr{Z}) \\ \downarrow & & \downarrow \\ \mathcal{S}(B^c, \mathscr{Z}') & \longrightarrow & \mathcal{S}(A^c, \mathscr{Z}') \end{array}$$

where the vertical arrows are weak equivalences and the bottom horizontal arrow is a weak equivalence since f^c is an injective I^c-weak equivalence. Thus the top horizontal arrow is a weak equivalence as well, showing that f is a projective I^c-weak equivalence. The fact that the identity functor gives a Quillen equivalence is now obvious. □

4.3.8. Let $f, g\colon \mathscr{X} \to \mathscr{Y}$ be two morphisms of simplicial presheaves. As in 1.7, an *elementary I-$\otimes$-homotopy from f to g* is a morphism $H\colon \mathscr{X} \otimes I \to \mathscr{Y}$ satisfying $H \circ \iota^I_0 = f$ and $H \circ \iota^I_1 = g$. Two morphisms are called *I-$\otimes$-homotopic* if they can be connected by a sequence of elementary I-$\otimes$-homotopies. A morphism $f\colon \mathscr{X} \to \mathscr{Y}$ is called a *strict I-$\otimes$-homotopy equivalence* if there is a morphism $g\colon \mathscr{Y} \to \mathscr{X}$ such that $f \circ g$ and $g \circ f$ are I-$\otimes$-homotopic to the identity (of $\mathscr{Y}$ and of $\mathscr{X}$ respectively).

Lemma 4.3.9 (cfr. [51], Lemma 3.6)**.** *Any strict I-$\otimes$-homotopy equivalence $f\colon \mathscr{X} \to \mathscr{Y}$ is an I-weak equivalence for both the injective and the projective I-localized structure on simplicial presheaves.*

Proof. We have to show that the compositions of f with an I-$\otimes$-homotopy inverse are equal to the corresponding identities in the homotopy categories $\mathcal{H}(T, I)_{\text{proj}}$ and $\mathcal{H}(T, I)_{\text{inj}}$. But it's clear from the definition that two elementary I-$\otimes$-homotopic maps are equal in the I-homotopy category (and this does not really depend on the choice of the injective/projective model structure). □

Remark 4.3.10. The results of the previous paragraphs hold if the object I is only assumed to be a weak interval object in $s\mathbf{Psh}(T)$. In fact, the multiplication map μ does not play any role in the definition of I-local object nor in the general localization Theorem 4.2.8. In particular, since the notion of I-$\otimes$-homotopy makes sense for any weak interval I (see 1.7), Lemma 4.3.9 continues to hold in this generality.

4.4. Comparison of intervals. Let $\overline{\mathbf{M}}\mathcal{M}(k)$ be the category of motivic spaces with modulus introduced in 3.3.1. Let $I = u_! v^*(\overline{\square})$ be the distinguished interval object of 3.3.5. Let $\eta\colon \overline{\square} \to I$ be the natural map constructed in 3.3.7.

Since $\overline{\square}$ is a weak interval in $\overline{\mathbf{M}}\mathcal{M}(k)$, we can talk about $\overline{\square}$-$\otimes$-homotopies between morphisms of motivic spaces with modulus.

Proposition 4.4.1. *Let $\mathscr{X}$ be any motivic space with modulus. Then the map $\mathrm{id} \otimes \iota_0^I\colon \mathscr{X} = \mathscr{X} \otimes \mathbb{1} \to \mathscr{X} \otimes I$ is a strict $\overline{\square}$-$\otimes$-homotopy equivalence.*

Proof. Let $p = \mathrm{id} \otimes p_I\colon \mathscr{X} \otimes I \to \mathscr{X} \otimes \mathbb{1} = \mathscr{X}$ be the projection morphism. Since the composition $p \circ (\mathrm{id} \otimes \iota_0^I)$ is the identity on $\mathscr{X}$, it's enough to show that there exists a $\overline{\square}$-$\otimes$-homotopy between $(\mathrm{id} \otimes \iota_0^I) \circ p$ and the identity on $\mathscr{X} \otimes I$. Write H for the map

$$H = H_{\overline{\square}}\colon (\mathscr{X} \otimes I) \otimes \overline{\square} \xrightarrow{\mathrm{id}_{\mathscr{X}} \otimes \mathrm{id}_I \otimes \eta} \mathscr{X} \otimes (I \otimes I) \xrightarrow{\mathrm{id}_{\mathscr{X}} \otimes \mu} \mathscr{X} \otimes I.$$

We need to check that $H \circ (\mathrm{id} \otimes \iota_0^{\overline{\square}}) = (\mathrm{id} \otimes \iota_0^I) \circ p$ and that $H \circ (\mathrm{id} \otimes \iota_1^{\overline{\square}}) = \mathrm{id}$. By adjunction, the compositions $\eta \circ \iota_\varepsilon^{\overline{\square}}$ for $\varepsilon = 0, 1$ agree with the map ι_ε^I, so that $\mathrm{id}_{\mathscr{X} \otimes I} \otimes \iota_\varepsilon^{\overline{\square}} = \mathrm{id}_{\mathscr{X} \otimes I} \otimes \iota_\varepsilon^I$. The required identities then follow from the interval structure on I. ☐

Lemma 4.3.9 and Proposition 4.4.1 together give the following

Corollary 4.4.2. *For every motivic space $\mathscr{X}$, the map $\mathscr{X} \to \mathscr{X} \otimes I$ is a $\overline{\square}$-weak equivalence.*

Corollary 4.4.3. *Let $\mathscr{X}$ be a $\overline{\square}$-local object (for either the projective or the injective model structure on $\overline{\mathbf{M}}\mathcal{M}(k)$). Then $\mathscr{X}$ is I-local.*

Proof. By definition, a $\overline{\square}$-local object $\mathscr{X}$ satisfies the following condition: for every $\overline{\square}$-weak equivalence $f\colon \mathscr{Y} \to \mathscr{Z}$, the induced map f^* on homotopy function complexes is a weak equivalence. In particular, the map

$$\mathrm{Map}(\mathscr{Y} \otimes I, \mathscr{X}) \to \mathrm{Map}(\mathscr{Y}, \mathscr{X})$$

is a weak equivalence for every $\mathscr{Y}$, since $\mathscr{Y} \to \mathscr{Y} \otimes I$ is a $\overline{\square}$-weak equivalence by Corollary 4.4.2. But this is precisely the condition that a fibrant object $\mathscr{X}$ has to satisfy for being I-local. ☐

Proposition 4.4.4. *Let $\mathscr{X}$ be an $\mathbb{A}^1$-local motivic space with modulus. Then $\mathscr{X}$ is I-local.*

Proof. Let $\theta\colon \mathbb{A}^1 \to I$ be the canonical map of 3.3.11, induced by adjunction by the identity morphism on $\mathbb{A}^1$ in $\mathbf{Sm}(k)$. It's enough to show that the map $\mathrm{id}_{\mathscr{X}} \otimes \iota_0^I\colon \mathscr{X} \to \mathscr{X} \otimes I$ is an $\mathbb{A}^1$ strict homotopy equivalence. An $\mathbb{A}^1$-$\otimes$-homotopy inverse is given by the projection $p\colon \mathscr{X} \otimes I \to \mathscr{X}$, and the homotopy between $(\mathrm{id}_{\mathscr{X}} \otimes \iota_0^I) \circ p$ is the map

$$H_{\mathbb{A}^1}\colon (\mathscr{X} \otimes I) \otimes \mathbb{A}^1 \xrightarrow{\mathrm{id}_{\mathscr{X} \otimes I} \otimes \theta} \mathscr{X} \otimes (I \otimes I) \xrightarrow{\mathrm{id}_{\mathscr{X}} \otimes \mu} \mathscr{X} \otimes I.$$

The argument is then formally identical to the one given in Proposition 4.4.1 and Corollaries 4.4.2 and 4.4.3. ☐

We can also ask for the relation between the interval objects $v^*(\overline{\square})$ and $\mathbb{A}^1 = (\mathbb{A}^1, \varnothing)$ in $\mathcal{M}_{\log}(k)$. The functor $\omega_!$ is particularly well behaved, since it sends $\mathbb{A}^1$-local and $\overline{\square}$-local objects in $\mathcal{M}_{\log}(k)$ to $\mathbb{A}^1$-local objects in $\mathcal{M}(k)$. In the other direction, we have the following

Proposition 4.4.5. *Let $\mathscr{X}$ be an $\mathbb{A}^1$-local motivic space in $\mathcal{M}(k)$. Then, $\lambda_*(\mathscr{X})$ is $v^*(\overline{\square})$-local in $\mathcal{M}_{\log}(k)$.*

4.5. A singular functor. We specialize the results of the previous sections to $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, equipped with the Nisnevich topology introduced in Section 2.2. This is the topology associated to a complete bounded regular cd-structure by Proposition 2.2.2 and Proposition 2.2.5. In particular, we can apply Theorem 4.2.13.

4.5.1. Let $\overline{\mathbf{M}}\mathcal{M}(k)^{loc}_{\mathrm{inj}}$ be the category of motivic spaces with modulus (over k) equipped with the local injective (for the Nisnevich topology) model structure and let I be again the distinguished interval object of 3.3.5. To simplify the notation, we write $\overline{\mathbf{M}}\mathcal{M}(k)^{I-loc}_{\mathrm{inj}}$ for the I-localization of the local-injective model structure on $\overline{\mathbf{M}}\mathcal{M}(k)$. By Theorem 4.2.8, $\overline{\mathbf{M}}\mathcal{M}(k)^{I-loc}_{\mathrm{inj}}$ is a left proper cellular simplicial model category. Right properness does not follow formally.

For the category of motivic spaces $\mathcal{M}(k)$ (without modulus), properness of the $\mathbb{A}^1$-local (injective) model structure is proved in [51], Theorem 3.2 and, using a different technique, in [27], Theorem A.5. The proof of Morel and Voevodsky makes use of the endofunctor Sing_*, that plays also an important role in the construction of a fibrant replacement functor.

Since we will work constantly in the category $\overline{\mathbf{M}}\mathcal{M}(k)$, we will omit the locution "with modulus" for a motivic space $\mathscr{X}$ (i.e. for an object of $\overline{\mathbf{M}}\mathcal{M}(k)$).

4.5.2. We introduce in this section an endofunctor $\mathrm{Sing}^{\otimes}_I(-)$ on $\overline{\mathbf{M}}\mathcal{M}(k)$ that plays in our theory the role of Sing_*. Our results look formally like the corresponding statements in [51, Section 3], but the proofs are different.

We start by noticing that the interval I comes equipped with an extra diagonal map

$$\delta\colon I \to I \otimes I$$

induced by the diagonal $\delta = (\mathbb{P}^1, \infty) \to (\mathbb{P}^1 \times \mathbb{P}^1, F^2_\infty)$ in $\mathbf{Sm}_{\log}(k)$. Thus, we can use the formulas of 1.7.5 for constructing a cosimplicial object $\Delta^\bullet_I$ in $\overline{\mathbf{M}}\mathcal{M}(k)$ whose n-th term is $I^{\otimes n}$. Similarly, we write $\Delta^\bullet_{\mathbb{A}^1}$ for the cosimplicial object deduced from $\mathbb{A}^1$ with the standard interval structure.

Definition 4.5.3. *Let $\mathscr{X}$ be a motivic space with modulus. We write $\mathrm{Sing}^{\otimes}_I(\mathscr{X})$ for the diagonal simplicial presheaf of the bi-simplicial presheaf*

$$\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Psh}(\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)), \quad ([n],[m]) \mapsto [\Delta^m_I = I^{\otimes m}, \mathscr{X}_n].$$

For every $n \geq 0$ there is canonical isomorphism $[\mathbb{1}, \mathscr{X}_n] = \mathscr{X}_n$, giving by composition a natural transformation $s\colon \mathrm{id} \to \mathrm{Sing}^{\otimes}_I$. Since the right adjoint $[-,-]$ to Day convolution is left exact, for any $\mathscr{X}$ the morphism $s_{\mathscr{X}}\colon \mathscr{X} \to \mathrm{Sing}^{\otimes}_I(\mathscr{X})$ is a monomorphism and therefore a cofibration for the local-injective model structure.

Proposition 4.5.4. *Let $f, g\colon \mathscr{X} \rightrightarrows \mathscr{Y}$ be two morphisms of simplicial presheaves and let H be an elementary I-$\otimes$-homotopy between them. Then there exists an elementary simplicial homotopy between $\mathrm{Sing}^{\otimes}_I(f)$ and $\mathrm{Sing}^{\otimes}_I(g)$.*

Proof. It is enough to show that there exists a simplicial homotopy

$$\mathrm{Sing}^{\otimes}_I(\mathscr{X}) \times \Delta^1 \to \mathrm{Sing}^{\otimes}_I(\mathscr{X} \otimes I)$$

between the natural maps $\mathrm{Sing}^{\otimes}_I(\mathrm{id}_{\mathscr{X}} \otimes \iota^I_0)$ and $\mathrm{Sing}^{\otimes}_I(\mathrm{id}_{\mathscr{X}} \otimes \iota^I_1)$. Thus, we have to construct a map of presheaves

$$H_n\colon \mathrm{Sing}^{\otimes}_I(\mathscr{X})_n \times \Delta^1_n (= \mathrm{Hom}_\Delta([n],[1])) \to \mathrm{Sing}^{\otimes}_I(\mathscr{X} \otimes I)_n = [I^{\otimes n}, \mathscr{X}_n \otimes I]$$

for every n, compatible with faces and degeneracies. Note here that I is a discrete simplicial presheaf, so that $\mathscr{X}_n \otimes I = (\mathscr{X} \otimes I)_n$ according to the definition of the extension of Day convolution to simplicial presheaves (see (3.2.4.1)). For every $M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, we have

$$\mathrm{Sing}^{\otimes}_I(\mathscr{X})_n(M) = \mathrm{Hom}_{\mathbf{Psh}(\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k))}(M \otimes I^{\otimes n}, \mathscr{X}_n)$$

so that a section over M of $\mathrm{Sing}_I^{\otimes}(\mathscr{X})_n$ is a map of presheaves $\alpha_M\colon M\otimes I^{\otimes n}\to\mathscr{X}_n$. Limits of presheaves are computed objectwise, and Δ_n^1 is a constant presheaf, therefore a section over M of $\mathrm{Sing}_I^{\otimes}(\mathscr{X})_n\times\Delta_n^1$ is a pair (α_M, f) for $f\in\mathrm{Hom}_\Delta([n],[1])$. Let $\Delta(f)\colon I^{\otimes n}\to I$ be the induced morphism given by the cosimplicial structure of $\Delta_I^\bullet$. Then we can consider the composition

$$M\otimes I^{\otimes n}\xrightarrow{\mathrm{id}_M\otimes\delta_n} M\otimes I^{\otimes n}\otimes I^{\otimes n}\xrightarrow{\mathrm{id}_M\otimes I^{\otimes n}\otimes\Delta(f)} M\otimes I^{\otimes n}\otimes I\xrightarrow{\alpha_M\otimes\mathrm{id}_I}\mathscr{X}_n\otimes I \tag{4.5.4.1}$$

that defines a section over M of $\mathrm{Sing}_I^{\otimes}(\mathscr{X}\otimes I)_n=[I^{\otimes n},\mathscr{X}_n\otimes I]$. For $\varphi\colon M'\to M$ there is a restriction map $\mathrm{Sing}_I^{\otimes}(\mathscr{X})_n(M)\xrightarrow{\varphi^*}\mathrm{Sing}_I^{\otimes}(\mathscr{X})_n(M')$ that sends a section α_M to the composite $\alpha_M\circ(\varphi\otimes\mathrm{id})$. The assignment (4.5.4.1) is clearly compatible with φ^* and thus defines a morphism of presheaves of sets. It is easy to check that it is also compatible with the simplicial structure and that it gives indeed an homotopy between $\mathrm{Sing}_I^{\otimes}(\mathrm{id}_{\mathscr{X}}\otimes\iota_0^I)$ and $\mathrm{Sing}_I^{\otimes}(\mathrm{id}_{\mathscr{X}}\otimes\iota_1^I)$. □

Corollary 4.5.5. *For any motivic space with modulus $\mathscr{X}$, the morphism*

$$\mathrm{Sing}_I^{\otimes}(\mathscr{X})\xrightarrow{\mathrm{Sing}_I^{\otimes}(\mathrm{id}_{\mathscr{X}}\otimes\iota_0^I)}\mathrm{Sing}_I^{\otimes}(\mathscr{X}\otimes I)$$

is a simplicial homotopy equivalence.

Proof. By Proposition 4.5.4, it's enough to show that the map $\mathrm{id}_{\mathscr{X}}\otimes\iota_0^I$ is an I-$\otimes$-homotopy equivalence. A I-$\otimes$-homotopy inverse is given by the projection map $p\colon\mathscr{X}\otimes I\xrightarrow{\mathrm{id}\otimes p_I}\mathscr{X}$. The composition $p\circ(\mathrm{id}_{\mathscr{X}}\otimes\iota_0^I)$ is clearly the identity, and an homotopy between $(\mathrm{id}_{\mathscr{X}}\otimes\iota_0^I)\circ p$ and the identity of $\mathscr{X}\otimes I$ is given by the multiplication map $\mathscr{X}\otimes I\otimes I\xrightarrow{\mathrm{id}\otimes\mu}\mathscr{X}\otimes I$. □

Lemma 4.5.6. *The map $p^*\colon\mathscr{X}=[\mathbb{1},\mathscr{X}]\to[I,\mathscr{X}]$ induced by the projection $p_I\colon I\to\mathbb{1}$ is a strict I-$\otimes$-homotopy equivalence.*

Proof. A homotopy inverse is given by $\iota_0^*\colon[I,\mathscr{X}]\to[\mathbb{1},\mathscr{X}]$ induced by ι_0^I. Since the composition $\iota_0^*\circ p^*$ is the identity on $\mathscr{X}$ we just need to show (as in the previous corollary) that there is a I-$\otimes$-homotopy between the composition in the other direction $p^*\circ\iota_0^*$ and the identity morphism. By adjunction, we have a canonical isomorphism $[I\otimes I,\mathscr{X}]\simeq[I,[I,\mathscr{X}]]$ that gives

$$\mathrm{Hom}_{\mathbf{\overline{M}}\mathcal{M}(k)}([I,\mathscr{X}],[I\otimes I,\mathscr{X}])\simeq\mathrm{Hom}_{\mathbf{\overline{M}}\mathcal{M}(k)}([I,\mathscr{X}]\otimes I,[I,\mathscr{X}]),$$

so that getting a homotopy $[I,\mathscr{X}]\otimes I\to[I,\mathscr{X}]$ it's equivalent to specifying a map $[I,\mathscr{X}]\to[I\otimes I,\mathscr{X}]$ satisfying the required identities. The multiplication map $I\otimes I\to I$ does the job. □

Proposition 4.5.7. *For any motivic space with modulus $\mathscr{X}$, the morphism $s_{\mathscr{X}}\colon\mathscr{X}\to\mathrm{Sing}_I^{\otimes}(\mathscr{X})$ is an I-weak equivalence.*

Proof. This is formally identical to [51, Corollary 3.8], using Lemma 4.5.6. □

The functor $\mathrm{Sing}_I^{\otimes}$ has formally a left adjoint that is constructed, as usual, by left Kan extension. More precisely, we recall the following definition that is valid in every category of simplicial presheaves on a small site.

Definition 4.5.8. *Let $D^\bullet$ be a cosimplicial object in $s\mathbf{Psh}(T)$. We denote by $|-|_{D^\bullet}$ the left Kan extension $\mathrm{Lan}_{\Delta^\bullet}(D^\bullet)(-)$ of $D^\bullet$ along the functor $\Delta\to s\mathbf{Psh}(T)$ that sends $[n]$ to the constant simplicial presheaf Δ^n.*

There is a canonical isomorphism $|\Delta^n|_{D^\bullet}=D^n=D([n])$. We can write an explicit description as follows. Let $\mathscr{X}$ be a simplicial presheaf and identify every $\mathscr{X}_n$ with a simplicial presheaf of dimension zero. Then $|\mathscr{X}|_{D^\bullet}$ is the co-equalizer (in $s\mathbf{Psh}(T)$) of the following diagram

$$\coprod_{\varphi\in\mathrm{Hom}_\Delta([m],[n])}\mathscr{X}_n\times D^m\underset{g}{\overset{f}{\rightrightarrows}}\coprod_n\mathscr{X}_n\times D^n \tag{4.5.8.1}$$

where the n-th term of the map $f\colon \mathscr{X}_n \times D^m \to \mathscr{X}_n \times D^n$ is induced by the cosimplicial structure of $D^\bullet$ and the m-th term of the map $g\colon \mathscr{X}_n \times D^m \to \mathscr{X}_m \times D^m$ is induced by the simplicial structure of $\mathscr{X}$.

Remark 4.5.9. We specialize to the case $D = \Delta^\bullet \otimes \Delta_I^\bullet$ in $\overline{\mathbf{M}}\mathcal{M}(k)$. By definition, the functor $\mathrm{Sing}_I^{\otimes}(-)$ satisfies, for any $\mathscr{Y} \in \overline{\mathbf{M}}\mathcal{M}(k)$,

$$\begin{aligned}\mathrm{Sing}_I^{\otimes}(\mathscr{Y})_n &= \mathrm{Hom}_{\overline{\mathbf{M}}\mathcal{M}(k)}(\Delta^n, \mathrm{Sing}_I^{\otimes}(\mathscr{Y})) = \mathrm{Hom}_{\overline{\mathbf{M}}\mathcal{M}(k)}(\Delta^n, [I^{\otimes n}, \mathscr{Y}]) \\ &= \mathrm{Hom}_{\overline{\mathbf{M}}\mathcal{M}(k)}(\Delta^n \otimes I^{\otimes n}, \mathscr{Y}) = \mathrm{Hom}_{\overline{\mathbf{M}}\mathcal{M}(k)}(|\Delta^n|_{\Delta^\bullet \otimes \Delta_I^\bullet}, \mathscr{Y})\end{aligned}$$

so that it's clear by construction that $|-|_{\Delta^\bullet \otimes \Delta_I^\bullet}$ is its left adjoint.

We summarize the properties of the functor $\mathrm{Sing}_I^{\otimes}(-)$ proved so far in the following Theorem.

Theorem 4.5.10. *The endofunctor $\mathrm{Sing}_I^{\otimes}(-)$ of $\overline{\mathbf{M}}\mathcal{M}(k)$ commutes with limits and takes the morphism $\mathrm{id} \otimes \iota_0^I\colon \mathscr{X} \to \mathscr{X} \otimes I$ to a simplicial weak equivalence for every motivic space $\mathscr{X}$. Moreover, the canonical natural transformation $\mathrm{id} \to \mathrm{Sing}_I^{\otimes}(-)$ is both a monomorphism and an injective I-weak equivalence.*

Proof. $\mathrm{Sing}_I^{\otimes}(-)$ commutes with limits since it is by construction a right adjoint. The other statements are precisely the content of Corollary 4.5.5 and Proposition 4.5.7. □

To continue or construction, we need to further study the properties of the adjoint $|-|_{\Delta^\bullet \otimes \Delta_I^\bullet}$ to $\mathrm{Sing}_I^{\otimes}(-)$.

Remark 4.5.11. For $D^\bullet = \Delta^\bullet$, we have $\mathrm{Lan}_{\Delta^\bullet}(D^\bullet)(-) = \mathrm{Lan}_{\Delta^\bullet}(\Delta^\bullet)(-) \simeq \mathrm{id}$, that is, $|-|_{\Delta^\bullet}$ is the (pointwise) left Kan extension of the functor $\Delta^\bullet$ along itself and is isomorphic to the identity functor.

Lemma 4.5.12 (see [51], Lemmas 3.9 and 3.10)**.** *Let $D^\bullet$ be a cosimplicial object in $\overline{\mathbf{M}}\mathcal{M}(k)$. Then the following conditions are equivalent*

i) the morphism $D^0 \amalg D^0 \to D^1$ induced by the cofaces is a monomorphism,
ii) the functor $|-|_{D^\bullet}$ preserves monomorphism.

Lemma 4.5.12 clearly applies to $D^\bullet = \Delta^\bullet \otimes I^\bullet$. In this case we have $D^0 = \Delta^0 \otimes I^{\otimes 0} = \Delta^0 \otimes \mathbb{1} = \mathrm{pt}$ and the maps $D^0 \rightrightarrows D^1 = \Delta^1 \otimes I$ induced by the cofaces are two distinct rational points $\mathrm{pt} \rightrightarrows \Delta^1 \otimes I$.

Lemma 4.5.13. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two motivic spaces. Then the natural map $\varphi\colon (\mathscr{X} \times \mathscr{Y}) \otimes I \to \mathscr{X} \times (\mathscr{Y} \otimes I)$ is an I-homotopy equivalence, and hence an I-weak equivalence.*

Proof. The existence of the map is guaranteed by universal property given the existence of morphisms $(\mathscr{X} \times \mathscr{Y}) \otimes I \xrightarrow{p_1 \otimes \mathrm{id}} \mathscr{X} \otimes I \xrightarrow{\mathrm{id} \otimes p_I} \mathscr{X}$ and $(\mathscr{X} \times \mathscr{Y}) \otimes I \xrightarrow{p_2 \otimes \mathrm{id}} \mathscr{Y} \otimes I$ for p_1, p_2 the projections $\mathscr{X} \times \mathscr{Y} \to \mathscr{X}$ and $\mathscr{X} \times \mathscr{Y} \to \mathscr{Y}$. Let ν be the morphism

$$\mathscr{X} \times (\mathscr{Y} \otimes I) \xrightarrow{\mathrm{id} \times (\mathrm{id} \otimes p_I)} \mathscr{X} \times \mathscr{Y} \xrightarrow{\mathrm{id} \otimes \iota_0^I} (\mathscr{X} \times kY) \otimes I.$$

We claim that $\nu \circ \varphi$ is I-$\otimes$-homotopic to the identity of $(\mathscr{X} \times \mathscr{Y}) \otimes I$ and that $\varphi \circ \nu$ is I-$\otimes$-homotopic to the identity of $\mathscr{X} \times (\mathscr{Y} \otimes I)$. For the first composition, a homotopy is given by the multiplication map

$$(\mathscr{X} \times \mathscr{Y}) \otimes I \otimes I \xrightarrow{\mathrm{id} \otimes \mu} (\mathscr{X} \times \mathscr{Y}) \otimes I$$

and the required identities follow from the fact that the morphism $(\mathscr{X} \times \mathscr{Y}) \otimes I \to \mathscr{X} \times \mathscr{Y}$ factors through φ. For the second composition, we consider the morphism H defined as

$$(\mathscr{X} \times (\mathscr{Y} \otimes I)) \otimes I \to \mathscr{X} \times ((\mathscr{Y} \otimes I) \otimes I) = \mathscr{X} \times (\mathscr{Y} \otimes (I \otimes I)) \xrightarrow{\mathrm{id} \times (\mathrm{id} \otimes \mu)} \mathscr{X} \times (\mathscr{Y} \otimes I)$$

where the first map is given again by universal property. It is easy to check that this is indeed the required homotopy. $\square$

Remark 4.5.14. Let A be the class of morphisms $\{\mathscr{X} \xrightarrow{\mathrm{id}\otimes\iota_0^I} \mathscr{X}\otimes I\}_{\mathscr{X}\in\overline{\mathbf{M}}\mathcal{M}(k)}$. Following [51, Definition 2.1], we say that a local injective fibrant motivic space $\mathscr{Y}$ is A-local if for any $\mathscr{Z}$ in $\overline{\mathbf{M}}\mathcal{M}(k)$ and any $\mathscr{X} \xrightarrow{\mathrm{id}\otimes\iota_0^I} \mathscr{X}\otimes I$ in A, the map

$$\mathrm{Map}(\mathscr{Z}\times(\mathscr{X}\otimes I),\mathscr{Y}) \to \mathrm{Map}(\mathscr{Z}\times\mathscr{X},\mathscr{Y})$$

is a weak equivalence of simplicial sets. By Lemma 4.5.13, an object $\mathscr{Z}$ is A-local if and only if it is (injective) I-local according to Definition 4.3.1.

Lemma 4.5.15. *For any motivic space with modulus $\mathscr{X}$, the morphism $|\mathscr{X}|_{\Delta^\bullet\otimes\Delta_I^\bullet} \to |\mathscr{X}|_{\Delta^\bullet} \simeq \mathscr{X}$ induced by the projection $p_I\colon \Delta^\bullet\otimes\Delta_I^\bullet \to \Delta^\bullet$ is an I-weak equivalence.*

Proof. The functor $|-|_{\Delta^\bullet\otimes\Delta_I^\bullet}$ commutes with colimits and we can do induction on the skeleton to the reduce to the case $\mathscr{X} = \mathscr{Y}\times\Delta^r$ for some simplicial presheaf $\mathscr{Y}$ of simplicial dimension 0 and some $r \geq 0$. The co-equalizer (4.5.8.1) then takes the form

$$\coprod_{\varphi\in\mathrm{Hom}_\Delta([m],[n])} \mathscr{Y}\times\Delta^r[n]\times(\Delta^m\otimes I^{\otimes m}) \overset{f}{\underset{g}{\rightrightarrows}} \coprod_n \mathscr{Y}\times\Delta^r[n]\times(\Delta^n\otimes I^{\otimes n}).$$

The term $\mathscr{Y}$ is constant and the functor $\mathscr{Y}\times(-)$ commutes with arbitrary colimits in $\overline{\mathbf{M}}\mathcal{M}(k)$. So we can identify $|\mathscr{Y}\times\Delta^r|_{\Delta^\bullet\otimes I^\bullet}$ with $\mathscr{Y}\times(\Delta^r\otimes I^{\otimes r})$. The morphism $|\mathscr{Y}\times\Delta^r|_{\Delta^\bullet\otimes\Delta_I^\bullet} \to \mathscr{Y}\times\Delta^r$ is then identified with the projection $\mathscr{Y}\times(\Delta^r\otimes I^{\otimes r}) \to \mathscr{Y}\times\Delta^r$. By Lemma 4.5.13 (applied several times), the product $\mathscr{Y}\times(\Delta^r\otimes I^{\otimes r})$ is I-weakly equivalent to the product $(\mathscr{Y}\times\Delta^r)\otimes I^{\otimes r}$, and the projection $p_I^r\colon(\mathscr{Y}\times\Delta^r)\otimes I^{\otimes r} \to \mathscr{Y}\times\Delta^r$ factors through the canonical map $(\mathscr{Y}\times\Delta^r)\otimes I^{\otimes r} \to \mathscr{Y}\times(\Delta^r\otimes I^{\otimes r})$. An easy induction shows that p_I^r is an I-weak equivalence, and the statement follows. $\square$

Proposition 4.5.16. *The functor $\mathrm{Sing}_I^{\otimes}(-)$ preserves injective I-fibrations.*

Proof. Equivalently (using the same strategy of [51], Corollary 3.13), we show that its left adjoint $|-|_{\Delta^\bullet\otimes\Delta_I^\bullet}$ preserves injective cofibrations (i.e. monomorphisms of presheaves) and injective I-weak equivalences. The first property is provided by Lemma 4.5.12, while the second property is provided by Lemma 4.5.15 together with the 2 out of 3 property of injective I-weak equivalences. $\square$

4.6. Properness. We can now prove right properness of the injective I-local model structure $\overline{\mathbf{M}}\mathcal{M}(k)^{loc}_{\mathrm{inj}}$. The proof is a combination of arguments due to Jardine in [27, Appendix A] and Morel-Voevodksy [51, pp. 77–82].

Lemma 4.6.1. *Let $\mathscr{X},\mathscr{Y},\mathscr{E}$ be three motivic spaces with modulus. Let $p\colon\mathscr{E}\to\mathscr{X}\times(\mathscr{Y}\otimes I)$ be an injective I-fibration. Then, in the cartesian square*

(4.6.1.1)
$$\begin{array}{ccc}
\mathscr{E}\times_{\mathscr{X}\times(\mathscr{Y}\otimes I)}(\mathscr{X}\times\mathscr{Y}) & \xrightarrow{f} & \mathscr{E} \\
\downarrow & & \downarrow p \\
\mathscr{X}\times\mathscr{Y} & \xrightarrow{\mathrm{id}_{\mathscr{X}}\times(\mathrm{id}_{\mathscr{Y}}\otimes\iota_0^I)} & \mathscr{X}\times(\mathscr{Y}\otimes I)
\end{array}$$

the morphism f is an injective I-weak equivalence.

Proof. This statement is proved by using a combination of the established properties of the functor $\mathrm{Sing}_I^{\otimes}(-)$. Applying $\mathrm{Sing}_I^{\otimes}(-)$ to the top arrow of (4.6.1.1) gives the following diagram

$$\begin{CD} \mathrm{Sing}_I^{\otimes}(\mathscr{E} \times_{\mathscr{X} \times (\mathscr{Y} \otimes I)} (\mathscr{X} \times \mathscr{Y})) @>{\mathrm{Sing}_I^{\otimes}(f)}>> \mathrm{Sing}_I^{\otimes}(\mathscr{E}) \\ @AAA @AAA \\ \mathscr{E} \times_{\mathscr{X} \times (\mathscr{Y} \otimes I)} (\mathscr{X} \times \mathscr{Y}) @>{f}>> \mathscr{E} \end{CD}$$

whose vertical maps are injective I-weak equivalences by Theorem 4.5.10. We will show that $\mathrm{Sing}_I^{\otimes}(f)$ is a simplicial weak equivalence. Note that again by Theorem 4.5.10, $\mathrm{Sing}_I^{\otimes}(-)$ commutes with limits, so that applying it to (4.6.1.1) gives another cartesian square. Using Lemma 4.5.13 and Theorem 4.5.10, we see that $\mathrm{Sing}_I^{\otimes}(\mathscr{X} \times \mathscr{Y}) \to \mathrm{Sing}_I^{\otimes}(\mathscr{X} \times (\mathscr{X} \otimes I))$ is a simplicial weak equivalence. By Proposition 4.5.16, the morphism $\mathrm{Sing}^I(\mathscr{E}) \to \mathrm{Sing}_I^{\otimes}(\mathscr{X} \times (\mathscr{Y} \otimes I))$ is a simplicial fibration. Since the injective model structure on simplicial presheaves is proper, we conclude that $\mathrm{Sing}_I^{\otimes}(f)$ is a weak equivalence as required. □

Lemma 4.6.2. *Suppose we are given morphisms of motivic spaces $\mathscr{X} \xrightarrow{\mathrm{id} \otimes \iota_0^I} \mathscr{X} \otimes I \xrightarrow{g} \mathscr{Y}$ and an injective I-fibration $p\colon \mathscr{E} \to \mathscr{Y}$. Then the induced map*

$$\mathscr{E} \times_{\mathscr{Y}} \mathscr{X} \to \mathscr{E} \times_{\mathscr{X}} (\mathscr{X} \otimes I)$$

is an injective I-weak equivalence.

Proof. The class of fibrations in $\overline{\mathbf{M}}\mathcal{M}(k)^{loc}_{\mathrm{inj}}$ is closed under pull-backs (as in any closed model category), so that the morphism $\mathscr{E} \times_{\mathscr{X}} (\mathscr{X} \otimes I) \to \mathscr{X} \otimes I$ is an injective I-fibration. The statement then follows from Lemma 4.6.1. □

Lemma 4.6.3. *Suppose we are given a morphism of motivic spaces $\mathscr{X} \xrightarrow{g} \mathscr{Y}$ and an injective I-fibration $p\colon \mathscr{E} \to \mathscr{Y}$ with $\mathscr{Y}$ fibrant for the injective I-local model structure. Suppose moreover that g is an injective I-weak equivalence. Then the top horizontal arrow in the pull-back square*

$$\begin{CD} \mathscr{E} \times_{\mathscr{Y}} \mathscr{X} @>>> \mathscr{E} \\ @VVV @VV{p}V \\ \mathscr{X} @>{g}>> \mathscr{Y} \end{CD}$$

is an injective I-weak equivalence.

Proof. Choose a factorization of g as $\mathscr{X} \xrightarrow{j} \mathscr{W} \xrightarrow{q}$ with q an injective I-fibration and j an elementary I-cofibration (see [51, p. 75] where the class B_1 of elementary A-cofibrations is introduced or [27, p. 537]). Since $\mathscr{Y}$ is fibrant, $\mathscr{W}$ is I-fibrant as well. Since g and j are injective I-weak equivalences, q is an injective I-weak equivalence between I-fibrant objects, and it is therefore a local weak equivalence. Since p is a global fibration and the injective local model structure is proper, q pulls back along p to a local weak equivalence (and thus to an I-weak equivalence) $\mathscr{W} \times_{\mathscr{Y}} \mathscr{E} \to \mathscr{E}$. We are then left to show that the natural map $\mathscr{E} \times_{\mathscr{Y}} \mathscr{X} \to \mathscr{W} \times_{\mathscr{Y}} \mathscr{E}$ is an I-weak equivalence. This follows from [27, Lemma A.3], using Lemma 4.6.2 instead of [27, Lemma A.1]. □

Write $\varphi_{\mathscr{X}}\colon \mathscr{X} \to \mathcal{L}\mathscr{X}$ for a functorial I-injective fibrant model of a motivic space $\mathscr{X}$. The morphism $\varphi_{\mathscr{X}}$ is by construction a cofibration and an injective I-weak equivalence, while the map $\mathcal{L}\mathscr{X} \to \mathrm{pt}$ is an injective I-fibration. Note that the existence of $\mathcal{L}\mathscr{X}$ is guaranteed by the fact that $\overline{\mathbf{M}}\mathcal{M}(k)^{loc}_{\mathrm{inj}}$ is cofibrantly generated.

Theorem 4.6.4. *Let $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{E}$ be motivic spaces and suppose we are given a diagram*

$$\begin{array}{ccc} \mathscr{E} \times_{\mathscr{Y}} \mathscr{X} & \xrightarrow{\gamma} & \mathscr{E} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathscr{X} & \xrightarrow{g} & \mathscr{Y} \end{array}$$

where g is an injective I-weak equivalence and p is an injective I-fibration. Then the induced morphism $\gamma\colon \mathscr{E} \times_{\mathscr{Y}} \mathscr{X} \to \mathscr{E}$ is also an injective I-weak equivalence.

Proof. We can construct a commutative square of the form

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{j} & \mathscr{E}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ \mathscr{Y} & \xrightarrow{\varphi_{\mathscr{Y}}} & \mathcal{L}\mathscr{Y} \end{array}$$

such that the upper horizontal arrow j is a cofibration and an I-weak equivalence and p' is an injective I-fibration. By Lemma 4.6.3, the induced map $\varphi'\colon \mathscr{E}' \times_{\mathcal{L}\mathscr{Y}} \mathscr{Y} \to \mathscr{E}'$ is an injective I-weak equivalence between I-fibrant objects over $\mathscr{Y}$, and therefore the morphism $\theta\colon \mathscr{E} \to \mathscr{E}' \times_{\mathcal{L}\mathscr{Y}} \mathscr{Y}$ is also an injective I-weak equivalence. Since both objects are fibrant over $\mathscr{Y}$, θ is also a simplicial homotopy equivalence, and so also a local weak equivalence (cfr. [51, Lemma 2.30]). In the cartesian square

$$\begin{array}{ccc} \mathscr{X} \times_{\mathscr{Y}} \mathscr{E} & \xrightarrow{\gamma} & \mathscr{E} \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} \\ \mathscr{X} \times_{\mathcal{L}\mathscr{Y}} \mathscr{E}' & \xrightarrow{\gamma'} & \mathscr{Y} \times_{\mathcal{L}\mathscr{Y}} \mathscr{E}' \end{array}$$

the morphism γ' is an injective I-weak equivalence, since $\varphi' \circ \gamma'$ is an injective I-weak equivalence by Lemma 4.6.3 (and we have already noticed that φ' is an injective I-weak equivalence). We are then left to show that θ' is an injective I-weak equivalence to conclude. But this map is in fact a local weak equivalence, since the morphism θ is one and the injective local model structure on simplicial presheaves is proper. □

Remark 4.6.5. Theorem 4.6.4 is merely a reformulation in our context of [27, Theorem A.5]. This is a consequence of a more general statement, as explained in [28, 7.3]. Jardine proves in *loc. cit.* Lemma 7.25, right properness for the f-local theory where $f\colon A \to B$ is a cofibration of simplicial presheaves, satisfying certain conditions, analogue to the property established by Lemmas 4.6.2 and 4.6.3. On the other hand, the category $\overline{\mathbf{M}}\mathcal{M}(k)^{I-loc}_{\mathrm{inj}}$ is not obtained as f-local theory (in the sense of Jardine), since we are inverting $\mathscr{X} \to \mathscr{X} \otimes I$ and not $\mathscr{X} \to \mathscr{X} \times I$. This explains why, for example, the proof of [27, Lemma A.1] does not go through in our context and we need to follow more closely [51], using the singular functor $\mathrm{Sing}^{\otimes}_{I}(-)$.

Remark 4.6.6. Having Theorem 4.6.4 at hand, it should be possible to use the strategy of [6, Lemma 3.4] to show that the I-local projective model structure on $\overline{\mathbf{M}}\mathcal{M}(k)$ is also right proper. The proof in *loc.cit.* uses in an essential way the right properness of the injective structure.

4.7. Nisnevich B.G. property. Let $\overline{\mathbf{M}}\mathcal{M}(k)^{I-loc}_{\mathrm{inj}}$ be again the category of motivic spaces with modulus equipped with the I-local injective model structure. To shorten the notation, we will refer to it as the *(injective) motivic model structure*. One can define in a similar way a projective variant. We set the following notation.

Definition 4.7.1. *The class of I-local-weak equivalences will be called the class of motivic weak equivalences. We say that a map $f\colon \mathscr{X} \to \mathscr{Y}$ in $\overline{\mathbf{M}}\mathcal{M}(k)$ is motivic fibration if it is an injective I-fibrations. An object $\mathscr{X}$ is motivic fibrant if the structure morphism is a motivic fibration.*

Note that by Proposition 4.3.7 we don't need to specify if we consider the injective or the projective model structure in the definition of motivic weak equivalences.

The *unstable unpointed motivic homotopy category with modulus* $\overline{\mathbf{M}}\mathcal{H}(k)$ over k is the homotopy category associated to the model structure $\overline{\mathbf{M}}\mathcal{M}(k)^{I-loc}_{\mathrm{inj}}$.

For $M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ and $\mathscr{X} \in \overline{\mathbf{M}}\mathcal{M}(k)$, we write $\mathscr{X}(M\otimes I)$ for the simplicial set $\mathrm{Map}(M\otimes I, \mathscr{X})$. The morphism $\mathrm{id}\otimes \iota_0^I$ induces then a morphism a simplicial sets $\mathscr{X}(M\otimes I) \to \mathscr{X}(M)$.

Lemma 4.7.2. *A motivic space $\mathscr{X} \in \overline{\mathbf{M}}\mathcal{M}(k)$ is motivic fibrant if and only if*

(1) $\mathscr{X}$ is globally fibrant (see Definition 4.2.4), and

(2) for every $M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, the map $\mathscr{X}(M\otimes I) \to \mathscr{X}(M)$ is a weak equivalence.

Definition 4.7.3. *A motivic space with modulus is called I-invariant if for every $M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, the map $\mathscr{X}(M\otimes I) \to \mathscr{X}(M)$ is a weak equivalence.*

Definition 4.7.4. *A motivic space with modulus $\mathscr{X}$ is said to be Nisnevich excisive (or to have the B.G. property with respect to Nisnevich squares) if $\mathscr{X}(\emptyset)$ is contractible and the square*

$$\begin{array}{ccc} \mathscr{X}(\overline{M};\partial M, D_M) & \xrightarrow{j^*} & \mathscr{X}(\overline{U};\partial U, D_U) \\ \downarrow{\scriptstyle p^*} & & \downarrow \\ \mathscr{X}(\overline{Y};\partial Y, D_Y) & \longrightarrow & \mathscr{X}(\overline{U}\times_{\overline{M}}\overline{Y}, \partial Y\cap p^{-1}(\overline{U}), D_{\overline{U}\times_{\overline{M}}\overline{Y}}) \end{array}$$

is homotopy cartesian for every elementary Nisnevich square of the form (2.2.3.1).

The following Proposition is the analogue of [51, Proposition 1.16] in our setting.

Proposition 4.7.5 (B.G. property for motivic spaces)**.** *Let $\mathscr{X}$ be a motivic space with modulus. Then the following are equivalent*

(1) $\mathscr{X}$ is Nisnevich excisive and I-invariant

(2) Any motivic fibrant replacement $\mathscr{X} \to \mathcal{L}\mathscr{X}$ (i.e. a fibrant replacement for the I-injective model structure) is a sectionwise weak equivalence.

Proof. We follow the proof of [51, Proposition 1.16]. The fact that the second condition implies the first one follows from the explicit description of fibrant objects in a Bousfield localization. Indeed, it's clear that if $\mathscr{X} \to \mathcal{L}\mathscr{X}$ is a sectionwise weak-equivalence, then $\mathscr{X}$ is I-invariant, because if the map $\mathscr{X} \to \mathcal{L}\mathscr{X}$ is a sectionwise weak equivalence, for every motivic space $\mathscr{Z}$ the induced map $\mathrm{Map}(\mathscr{Z}, \mathscr{X}) \to \mathrm{Map}(\mathscr{Z}, \mathcal{L}\mathscr{X})$ is weak equivalence of simplicial sets, and this applies in particular to $\mathscr{Z} = M\otimes I$. To show the B.G. property, we combine Lemma 4.7.2 with Theorem 4.2.13, noting that the B.G. property is invariant with respect to weak equivalences of presheaves i.e., sectionwise weak equivalences (see [51, Remark 3.1.14]).

Conversely, suppose that $\mathscr{X}$ satisfies condition (1). Let $\varphi_{\mathscr{X}}\colon \mathscr{X} \to \mathcal{L}\mathscr{X}$ be a motivic fibrant replacement. Using [65, Proposition 3.8] we are left to show that $\varphi_{\mathscr{X}}$ is a local injective fibrant replacement to deduce that it is a sectionwise weak equivalence. Write a factorization of $\varphi_{\mathscr{X}}$ as

$$\mathscr{X} \xrightarrow{\psi} \mathscr{Y} \xrightarrow{\varphi'} \mathcal{L}\mathscr{X}$$

in the local injective model structure with ψ a trivial local cofibration (so a monomorphism that is also a local weak equivalence) and φ' a local injective fibration. Note that ψ is a motivic weak equivalence, so that also φ' is a motivic weak equivalence. Since φ' is a local injective fibration and $\mathcal{L}\mathscr{X}$ is local injective fibrant, $\mathscr{Y}$ is also local injective fibrant. Hence ψ is a trivial local cofibration with target a local injective fibrant object, and therefore it is a fibrant replacement for $\mathscr{X}$ in the local injective model structure. In particular, $\mathscr{Y}$ is local projective fibrant, hence by Proposition 4.2.15 it has the B.G. property with respect to Nisnevich squares (it is flasque in the sense of [65, Definition 3.3]). By [65, Lemma 3.5] (see also [51, Lemma 1.18]), the map ψ is then a sectionwise weak equivalence and so by Lemma 4.7.2 we conclude that $\mathscr{Y}$ is motivic fibrant. Since φ' is now an injective I-weak equivalence between injective I-fibrant objects, we conclude that φ' is an objectwise simplicial weak equivalence by [22, Theorem 3.2.13.(1)]. □

Corollary 4.7.6. *Any morphism $f\colon \mathscr{X} \to \mathscr{Y}$ between motivic spaces with modulus satisfying the equivalent conditions of Proposition 4.7.5 is a motivic weak equivalence if and only if it is a sectionwise weak equivalence.*

Proof. Taking motivic fibrant replacements of $\mathscr{X}$ and of $\mathscr{Y}$ we get a diagram

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{f} & \mathscr{Y} \\ {\scriptstyle \varphi_{\mathscr{X}}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathscr{Y}}} \\ \mathcal{L}\mathscr{X} & \xrightarrow{f'} & \mathcal{L}\mathscr{Y} \end{array}$$

with vertical arrows sectionwise weak equivalences. The induced morphism f' is a motivic weak equivalence if (and only if) f is a motivic weak equivalence. The local Whitehead Lemma [22, Theorem 3.2.13] implies that f' is an objectwise weak equivalence in this case, and thus so is f. □

4.8. Pointed variants. We write $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ for the category of *pointed motivic spaces with modulus*. Objects are pairs, $(\mathscr{X}, x)$, where $\mathscr{X}$ is a motivic space with modulus and $x\colon \mathrm{pt} \to \mathscr{X}$ is a fixed basepoints. Equivalently, a pointed motivic space with modulus is a contravariant functor

$$\mathscr{X} : \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k) \to \mathcal{S}_\bullet$$

from $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ to the category $\mathcal{S}_\bullet$ of pointed simplicial sets. We have a canonical adjunction

$$(-)_+ \colon \overline{\mathbf{M}}\mathcal{M}(k) \leftrightarrows \overline{\mathbf{M}}\mathcal{M}_\bullet(k)$$

where $(-)_+$ is the "add base point functor", left adjoint to the forgetful functor. The category $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ inherits by [22, 7.6.5] two natural model structures, induced respectively by the I-local injective and projective model structure on $\overline{\mathbf{M}}\mathcal{M}(k)$.

Theorem 4.8.1. *The injective I-local structure on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ is proper, cellular and simplicial, while the projective I-local structure on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ is left proper, cellular and simplicial (but see Remark 4.6.6 on right properness). A morphism $f\colon (\mathscr{X}, x) \to (\mathscr{Y}, y)$ is an injective (resp. projective) I-weak equivalence if and only if the underlying morphism of unpointed motivic spaces $\mathscr{X} \to \mathscr{Y}$ is a weak equivalence in the I-localized injective (resp. projective) motivic model structure. The statement remains true if one replaces the projective I-localized model structure on $\overline{\mathbf{M}}\mathcal{M}(k)$ with the I^c-localized projective structure on $\overline{\mathbf{M}}\mathcal{M}(k)$.*

By Proposition 4.3.7, the identity functor is a (left) Quillen equivalence between the projective and the injective motivic model structure on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$. We denote by $\overline{\mathbf{M}}\mathcal{H}_\bullet(k)$ the *unstable pointed motivic homotopy category with modulus* over k.

Remark 4.8.2. We note that the objectwise projective model structure on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ (i.e., the model structure on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ given before any Bousfield localization) has an explicit set of generating cofibrations and trivial cofibrations that are immediately deduced from (4.1.4.1), namely

(4.8.2.1)

$$I = \{\mathrm{id}_M \wedge \iota_{n,+} \colon h_M \wedge (\partial\Delta^n)_+ \to (h_X)_+ \wedge (\Delta^n)_+\}, \quad \text{for } n \geq 0, M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$$
$$J = \{\mathrm{id}_M \wedge j^n_{k_+} \colon (h_M)_+ \wedge (\Lambda^n_k)_+ \to (h_X)_+ \wedge (\Delta^n)_+\}, \quad \text{for } n \geq 1, 0 \leq k \leq n, M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k),$$

where, clearly, $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ can be replaced by any site T.

4.8.3. There are two natural closed monoidal structures on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$, induced respectively by the cartesian product and by Day convolution on $\overline{\mathbf{M}}\mathcal{M}(k)$ (see Remark 3.2.6).

i) *Smash product*. We define the smash product objectwise, $\mathscr{X} \wedge \mathscr{Y}(M) = \mathscr{X}(M) \wedge \mathscr{Y}(M)$ for $\mathscr{X}$ and $\mathscr{Y}$ pointed motivic spaces. With this definition $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ is naturally enriched over $\mathcal{S}_\bullet$. The simplicial function space is given degreewise by the pointed simplicial set $\mathcal{S}_\bullet(\mathscr{X}, \mathscr{Y})$

$$\mathcal{S}_\bullet(\mathscr{X}, \mathscr{Y})_n = \mathrm{Hom}_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(\mathscr{X} \wedge (\Delta[n])_+, \mathscr{Y}),$$

and internal hom given by

$$\mathscr{H}om_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(\mathscr{X}, \mathscr{Y})(M) = \mathcal{S}_\bullet(\mathscr{X}, \mathscr{Y})(\mathscr{X} \wedge (h_M)_+, \mathscr{Y}).$$

ii) *Pointed Day convolution*. For two pointed motivic spaces $\mathscr{X}$ and $\mathscr{Y}$, we define their pointed convolution $\mathscr{X} \otimes^{\mathrm{Day}}_\bullet \mathscr{Y}$ by means of the following push-out diagram of unpointed simplicial presheaves

$$\begin{array}{ccc} (\mathscr{X} \otimes^{\mathrm{Day}} \mathrm{pt}) \coprod (\mathrm{pt} \otimes^{\mathrm{Day}} \mathscr{Y}) & \longrightarrow & \mathscr{X} \otimes^{\mathrm{Day}} \mathscr{Y} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathscr{X} \otimes^{\mathrm{Day}}_\bullet \mathscr{Y}. \end{array}$$

We have, in particular, $(\mathscr{X} \otimes \mathscr{Y})_+ = (\mathscr{X})_+ \otimes^{\mathrm{Day}}_\bullet (\mathscr{Y})_+$ for unpointed motivic spaces $\mathscr{X}$ and $\mathscr{Y}$. The unit for $\otimes^{\mathrm{Day}}_\bullet$ is $\mathbb{1}_+$. This definition makes the add base point functor $(-)_+$ strict monoidal. We denote by $[-,-]_\bullet$ the pointed version of the internal hom for Day convolution.

Lemma 4.8.4. *Let $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)^{I^c-loc}_{proj}$ denote the category of pointed motivic spaces equipped with the I^c-local projective model structure. Then taking the smash product with any motivic space $\mathscr{X}$ preserves projective I^c-weak equivalences.*

Proof. The argument given in [14, Lemma 2.18 - Lemma 2.20] works almost *verbatim* in our setting, so we just sketch the proof. First, we prove that smashing with any cofibrant space $\mathscr{X}$ preserves projective I^c-weak equivalences. Given a pointed motivic space $\mathscr{Z}$ that is I^c-fibrant and a pointed motivic space $\mathscr{X}$ that is cofibrant, the internal hom $\mathscr{H}om_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(\mathscr{X}, \mathscr{Z})$ is clearly objectwise fibrant since the category of pointed simplicial presheaves on any small site T equipped with the projective structure is a monoidal model category for the smash product (for a self-contained proof of this fact one can mimic the proof of Proposition 4.1.10 replacing Day convolution with the smash product). In particular the internal hom $\mathscr{H}om_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(\mathscr{X}, -)$ is a right Quillen functor. Let now Λ' be the set of maps

$$\Lambda' = \Sigma_P \cup \{(h_M)_+ \otimes^{\mathrm{Day}}_\bullet I^c_+ \to (h_M)_+\}_{M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)}$$

where Σ_P is defined as in 4.2.12. Let Λ be the set of pushout product maps $f \square g$ where $f \in \Lambda'$ and $g \in \{(\partial\Delta^n)_+ \hookrightarrow (\Delta^n)_+\}$. To show that $\mathscr{H}om_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(\mathscr{X}, \mathscr{Z})$ is I^c-projective fibrant, it's enough to show that for every generating cofibration $i = \mathrm{id}_M \wedge \iota_{n,+} \colon h_M \wedge (\partial\Delta^n)_+ \to (h_X)_+ \wedge$

$(\Delta^n)_+$, the push-out product of i and any $f \in \Lambda$ is still a composition of pushouts of maps in Λ. For this is enough to use the pointed version of (4.1.10.1), that replaces the fourth listed point in the proof of [14, Lemma 2.18]. To conclude, we have to show that for every I^c-weak equivalence $f\colon \mathscr{Y} \to \mathscr{Y}'$, every I^c-projective fibrant $\mathscr{Z}$, and every cofibrant $\mathscr{X}$, the map $\mathrm{Map}(f \wedge \mathscr{X}, \mathscr{Z}) = \mathcal{S}_\bullet((f \wedge \mathscr{X})^c, \mathscr{Z})$ is a weak equivalence. By the above argument, $\mathscr{H}om_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(\mathscr{X}, \mathscr{Z})$ is I^c-projective fibrant, so that the natural map between the simplicial function spaces

$$\mathcal{S}_\bullet(f^c \wedge \mathscr{X}, \mathscr{Z}) \to \mathcal{S}_\bullet(f^c, \mathscr{H}om_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(\mathscr{X}, \mathscr{Z})) = \mathrm{Map}(f, \mathscr{H}om_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(\mathscr{X}, \mathscr{Z})),$$

induced by the closed monoidal structure on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$, is a weak equivalence. In particular, this shows that $f^c \wedge \mathscr{X}$ is an I^c-projective weak equivalence. Since the I^c-local structure is obtained by left Bousfield localization from the projective (objectwise) model structure on simplicial presheaves, and the latter is monoidal with respect to the smash product, we conclude as in [14] that $f^c \wedge \mathscr{X}$ is an I^c-projective weak equivalence if and only if $(f \wedge \mathscr{X})^c$ is an I^c-projective weak equivalence.

For the general case, simply replace $\mathscr{X}$ with $\mathscr{X}^c \to \mathscr{X}$, where $(-)^c$ denote a functorially chosen cofibrant replacement. The morphism $\mathscr{X}^c \to \mathscr{X}$ is an objectwise weak equivalence, so it is preserved by smashing with any motivic space. As for $\mathscr{X}^c$ we can apply the previous claim. □

Proposition 4.8.5. *The smash product preserves I-weak equivalences and cofibrations for the injective I-local model structure on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$, and induces a symmetric closed monoidal structure on the unstable motivic homotopy category $\overline{\mathbf{M}}\mathcal{H}_\bullet(k)$.*

Proof. By the description of weak equivalences in the pointed model category given by Theorem 4.8.1 and the equivalence between the classes of I^c-projective weak equivalences and injective I-weak equivalences given by Proposition 4.3.7, Lemma 4.8.4 implies immediately that I-weak equivalences are preserved under smash product. Since cofibrations in the injective structure are monomorphisms, they are clearly preserved by smash product. This gives the homotopy category $\overline{\mathbf{M}}\mathcal{H}_\bullet(k)$ the desired structure of monoidal category. To show that $\overline{\mathbf{M}}\mathcal{H}_\bullet(k)$ is closed with respect to this monoidal structure, it's enough to use the fact that $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)^{I^c-loc}_{\mathrm{proj}}$ is a monoidal model category for the smash product. This is a consequence of Proposition 4.2.10, using Lemma 4.8.4, together with left properness of the I^c-local projective model structure on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$. □

4.9. A formal representability result. We close this Section with a general representability result in the I-homotopy category. We will then specialize this result to obtain a (weak) representability Theorem for relative K-theory on modulus data.

4.9.1. Let $\overline{\mathbf{M}}\mathcal{H}_\bullet(k)$ be again the pointed unstable motivic homotopy category with modulus. For $\mathscr{X}$ and $\mathscr{Y}$ motivic spaces with modulus, we set

$$[\mathscr{X}, \mathscr{Y}]_{\overline{\mathbf{M}}\mathcal{H}_\bullet(k)} = \mathrm{Hom}_{\overline{\mathbf{M}}\mathcal{H}_\bullet(k)}(\mathscr{X}, \mathscr{Y}),$$

(not to be confused with the internal hom for Day convolution, that we denoted $[-,-]$). Let $M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ be any modulus datum. Evaluation at M determines a Quillen pair

$$(M)_+ \wedge (-)\colon \mathcal{S}_\bullet \leftrightarrows \overline{\mathbf{M}}\mathcal{M}_\bullet(k)\colon \mathrm{Ev}_M = \mathcal{S}_\bullet((M)_+, -) \tag{4.9.1.1}$$

(where we write M instead of h_M for short) for the injective model structure $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)^{I-loc}_{\mathrm{inj}}$ on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$, since $(M)_+ \wedge (-)$ preserves monomorphisms and sectionwise weak equivalences.

4.9.2. Write S^1 for the constant simplicial presheaf given by $\Delta^1/\partial\Delta^1$ and S^n for the n-th simplicial sphere $(S^1)^{\wedge n}$. The space S^1 is naturally pointed by the image of $\partial\Delta^1$, so that we can consider it as object in $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$. Smashing with S^1 defines an endofunctor on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$, that is a left Quillen functor by Proposition 4.8.5. As customary, we write $\Sigma(-)$ for $S^1 \wedge (-)$ (the suspension functor) and $\Omega^1(-) = \mathscr{H}om_{\overline{\mathbf{M}}\mathcal{M}_\bullet(k)}(S^1, -)$ for its right adjoint:

$$\Sigma\colon \overline{\mathbf{M}}\mathcal{M}_\bullet(k) \leftrightarrows \overline{\mathbf{M}}\mathcal{M}_\bullet(k)\colon \Omega^1.$$

Theorem 4.9.3. *Let $\mathscr{X}$ be a pointed motivic space with modulus that satisfies the equivalent conditions of Proposition 4.7.5. Then, for any pointed simplicial set K and any modulus datum M, we have a natural isomorphism*

$$[K, \mathscr{X}(M)]_{\mathcal{S}_\bullet} \simeq [(M)_+ \wedge K, \mathscr{X}]_{\overline{\mathbf{M}}\mathcal{H}_\bullet(k)}.$$

Proof. The proof is a formal consequence of the results collected so far. Let $\varphi_{\mathscr{X}}\colon \mathscr{X} \to \mathcal{L}\mathscr{X}$ be a fibrant replacement for $\mathscr{X}$ in the injective I-local model structure. By Proposition 4.7.5, $\varphi_{\mathscr{X}}$ is a sectionwise weak equivalence. Then we have

$$[K, \mathscr{X}(M)]_{\mathcal{S}_\bullet} \simeq [K, \mathcal{L}\mathscr{X}(M)]_{\mathcal{S}_\bullet} \simeq [K, \mathbf{R}\mathrm{Ev}_M(\mathscr{X})]_{\mathcal{S}_\bullet} \simeq [(M)_+ \wedge K, \mathscr{X}]_{\overline{\mathbf{M}}\mathcal{H}_\bullet(k)}$$

where the last isomorphism follows from the Quillen adjunction displayed in (4.9.1.1). □

We state as separate Corollary the following result. It is a direct consequence of Theorem 4.9.3 and Corollary 4.4.3.

Corollary 4.9.4. *Let $\mathscr{X}$ be a pointed motivic space with modulus that is Nisnevich excisive in the sense of Definition 4.7.4 and $\overline{\square}^1$-$\otimes$-invariant. Then, for any $n \geq 0$ and any modulus datum M, we have a natural isomorphism*

$$\pi_n(\mathscr{X}(M)) \simeq [S^n \wedge (M)_+, \mathscr{X}]_{\overline{\mathbf{M}}\mathcal{H}_\bullet(k)}.$$

4.10. Stable homotopy theory. We build from the category $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ of pointed motivic spaces a category of motivic spectra in the standard way. Since the applications that we present later involve only S^1-spectra, there are no difficulties in carry over the theory. The main reference is [24]. We follow the convention of writing the smash product on the left.

Definition 4.10.1. *Let T be any pointed simplicial presheaf. A T-spectrum consists of pointed motivic spaces $(\mathscr{X}^n)_n$ for $n \geq 0$ and pointed maps $\sigma\colon T \wedge \mathscr{X}^n \to \mathscr{X}^{n+1}$. We denote by $\mathbf{Spt}_T(\overline{\mathbf{M}}\mathcal{M}_\bullet(k))$ (or by $\mathbf{Spt}_T(\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k))$) the category of T-spectra of motivic spaces with modulus. For $n \geq 0$, we have an adjunction*

$$F_n\colon \overline{\mathbf{M}}\mathcal{M}_\bullet(k) \leftrightarrows \mathbf{Spt}_T(\overline{\mathbf{M}}\mathcal{M}_\bullet(k))\colon Ev_n$$

where the right adjoint Ev_n (the evaluation functor) is defined as $(\mathscr{X}) \mapsto \mathscr{X}^n$.

Following [24, Definition 1.7], we call a morphism of spectra $f\colon \mathscr{X} \to \mathscr{Y}$ a *level injective fibration* if all the maps $f_n\colon \mathscr{X}^n \to \mathscr{Y}^n$ are injective I-fibrations (i.e., fibrations in the injective I-local structure $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)^{I-loc}_{\mathrm{inj}}$). We call $f\colon \mathscr{X} \to \mathscr{Y}$ a *level equivalence* if all the maps $f_n\colon \mathscr{X}^n \to \mathscr{Y}^n$ are motivic weak equivalences. A *cofibration* is a map having the left lifting property with respect to all trivial level fibrations.

The following result is formally identical to [24, Theorem 1.13], and gives a first model structure on $\mathbf{Spt}_T(\overline{\mathbf{M}}\mathcal{M}_\bullet(k))$.

Proposition 4.10.2. *The category $\mathbf{Spt}_T(\overline{\mathbf{M}}\mathcal{M}_\bullet(k))$ of T spectra of motivic spaces with modulus, together with the classes of cofibrations, level fibrations and level equivalences satisfies the axioms for a proper closed simplicial model category.*

The following definition holds in greater generality, but we limit ourselves to state it in the case $T = S^1$. Note again that by Proposition 4.8.5, smashing with S^1 defines a left Quillen endofunctor on $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$. This is enough to apply the results of [24, Section 3].

Definition 4.10.3 (see [24], Definition 3.3)**.** *Let J be the set of generating cofibrations of $\overline{\mathbf{M}}\mathcal{M}_\bullet(k)$ with respect the injective I-local model structure. We define a set of maps S in $\mathbf{Spt}_{S^1}(\overline{\mathbf{M}}\mathcal{M}_\bullet(k))$ by*

$\{F_{n+1}(S^1 \wedge C) \xrightarrow{\zeta_n^C} F_n C\}$ *where* C *varies in the set of domains and codomains of the maps in* J *and* ζ_n^C *is the adjoint to the identity map of* $T \wedge C$*. We define the stable model structure on* S^1*-spectra* $\mathbf{Spt}_{S^1}(\overline{\mathbf{M}}\mathcal{M}_\bullet(k))$ *to be the (left) Bousfield localization of the level model structure on* $\mathbf{Spt}_{S^1}(\overline{\mathbf{M}}\mathcal{M}_\bullet(k))$ *to the set of maps* S*. We call* S*-local weak equivalences of spectra stable weak equivalences and* S*-fibrations stable fibrations.*

5. K-theory spaces of modulus data

This final section is devoted to the construction of a K-theory functor from the category of modulus data $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ to the homotopy category of presheaves of S^1-spectra over $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. The *K-theory of modulus data* reduces to usual relative K-theory for a modulus pair

$$M = (\overline{M}, \emptyset, D_M) = (\overline{M}, D_M),$$

and, by Proposition 5.2.9 below, is $\overline{\square}$-invariant. Thus, from the results of the preceding sections, is I-invariant.

We do not investigate, in this text, the problem of rectifying our construction to get well-defined functor from $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ to S^1-spectra. With such rectification at hand, the machinery developed before, together with the Nisnevich excision property of [63, Theorem 10.8], would allow us to conclude quite formally using the stable version of Corollary 4.9.4 the representability of relative K-theory in our category. We leave this to a future work, but we hope that the Definitions in this Section will work as a motivating example in justifying the construction of our motivic homotopy category $\overline{\mathbf{M}}\mathcal{M}(k)$.

Part of the notation was introduced in Chapter I, but little emphasis was put there on the homotopy-side of the picture. We present here a more systematic treatment of the subject.

5.1. K-theory spaces and relative K-theory spaces. Let X be a separated Noetherian scheme admitting an ample family of line bundles in the sense of [63, Definition 2.1.1]. We denote by $\mathcal{C}\mathbf{P}(X)$ the category of bounded complexes of *big vector bundles* on X, in the sense of Friedlander-Grayson-Suslin (see [15, C.5] or [69, IV.10.5], where the construction is attributed to Thomason): we recall its definition.

Definition 5.1.1. *Let* $\mathbf{Sch}/X$ *be the category of schemes of finite type over* X*. A big vector bundle* $\mathcal{E}$ *over* X *is the choice, for every* $Y \in \mathbf{Sch}/X$*, of a locally free coherent* $\mathcal{O}_Y$*-module* $\mathcal{E}_Y$ *on* Y *and of an isomorphism* $f^*\mathcal{E}_Y \to \mathcal{E}_Z$ *for every map* $f\colon Z \to Y$ *over* X *such that when* $f = \mathrm{id}_Z$*, the corresponding isomorphism is simply the identity on* $\mathcal{E}_Y$*. We denote by* $\mathbf{P}(X)$ *the exact category of big vector bundles over* X *and by* $\mathcal{C}\mathbf{P}(X)$ *the category of bounded chain complexes on* $\mathbf{P}(X)$*. The assignment* $X \mapsto \mathbf{P}(X)$ *is a contravariant functor to exact categories.*

5.1.2. The category $\mathcal{C}\mathbf{P}(X)$ becomes in the standard way a Waldhausen category having quasi-isomorphisms as weak equivalences and degree-wise split monomorphisms as cofibrations. We can then consider the Waldhausen K-theory Ω-spectrum $\mathbf{K}^W(X)$ associated to $\mathcal{C}\mathbf{P}(X)$. Given $f\colon Y \to X$ a morphism of finite type, the exact functor f^* between the categories of big vector bundles defines a morphism of spectra $f^*\colon \mathbf{K}^W(X) \to \mathbf{K}^W(Y)$. Moreover, if $g\colon Z \to Y$ is another scheme of finite type over Y, there is a strictly commutative diagram of spectra (see again [15, B.5])

$$\begin{array}{ccc} \mathbf{K}^W(X) & \xrightarrow{f^*} & \mathbf{K}^W(Y) \\ \big\downarrow{\scriptstyle (f\circ g)^*} & & \big\downarrow{\scriptstyle g^*} \\ \mathbf{K}^W(Z) & = & \mathbf{K}^W(Z). \end{array}$$

Instead of the category $\mathbf{P}(X)$, we could have considered the category of *standard vector bundles* $\mathbf{V}(X)$ on X in the sense of [34, Definition 3.1]. By [34, Theorem 3.10], $\mathbf{V}(X)$ is a small exact category, equivalent to the category $\mathcal{P}(X)$ of locally free $\mathcal{O}_X$-modules of finite rank. As for $\mathbf{P}(X)$, the assignment $X \mapsto \mathbf{V}(X)$ defines a functor from the category of schemes to the category of (small) exact categories and not just a pseudo-functor. The two approaches are equivalent for our purposes.

Remark 5.1.3. The functoriality in the category $\mathbf{V}(X)$ is particularly well-behaved with respect to the twisted sheaves $\mathcal{O}_{\mathbb{P}^r_X}(n)$ on the projective space $\mathbb{P}^r_X$ over X, in the following sense. Let $f\colon X \to Y$ be a map of schemes and let $\varphi\colon \mathbb{P}^r_X \to \mathbb{P}^r_Y$ the induced morphism. Then (see [34, Proposition 3.12]) we have the equality (not simply a natural isomorphism) $\varphi^*\mathcal{O}_{\mathbb{P}^r_Y}(n) = \mathcal{O}_{\mathbb{P}^r_X}(n)$ for every n. Here $\mathcal{O}(n)$ denotes (a functorially chosen) standard vector bundle corresponding to the locally free sheaf $\mathcal{O}(n)$ on the projective space.

5.1.4. As in [53, Example A.12], we write $K^W(X)$ for the loop space of the first term of $\mathbf{K}^W(X)$, namely

$$K^W(X) = \Omega \mathbf{K}^W_1(X) = \Omega\,\mathrm{Sing}\,|w\mathcal{S}_\bullet(\mathcal{C}\mathbf{P}(X))|.$$

It is a simplicial set having the Waldhausen K-theory groups as homotopy groups. Thanks to the assumption on X, we know that, by [63, Theorem 1.11.7, Proposition 3.10], the space $K^W(X)$ has the same homotopy type of the zeroth space of $K^{naive}(X)$ and of the zeroth space of Thomason-Trobaugh's K-theory spectrum $\mathbf{K}^{TT}(X) = K(X \text{ on } X)$ of [63, 1.5.3]. By [63, Theorem 1.11.2], the homotopy groups $\pi_*(K^W(X))$ coincide with Quillen's higher K-theory groups $K_*(X)$. Suppose now that X is a smooth k-scheme. The assignment $X \mapsto K^W(X)$ defines a pointed motivic space in the sense of [14] (or [53]), i.e., a simplicial presheaf on $\mathbf{Sm}(k)$, that is fibrant in the projective motivic model structure on $\mathcal{M}(k)$ (see again [53, Example A.12]).

5.1.5. Let X be again a separated Noetherian scheme and let Y be a closed subscheme. As in Chapter I, we denote by $\mathbf{K}^{TT}(X;Y)$ the spectrum of algebraic K-theory of X relative to Y, defined as the homotopy fiber of the morphism of spectra $\mathbf{K}^{TT}(X) \to \mathbf{K}^{TT}(Y)$,

$$\mathbf{K}^{TT}(X;Y) = \mathrm{hofib}(\mathbf{K}^{TT}(X) \to \mathbf{K}^{TT}(Y)).$$

When X (and, a fortiori, Y by [63, 2.1.2]) admits an ample family of line bundles, we have already recalled that the non-negative K-theory groups of X and Y can be obtained as homotopy groups of the K-theory space $K^W(X)$ and $K^W(Y)$ respectively. We now briefly recall the construction of a relative K-theory *space* (see [68, Definition 1.5.4] or [69, IV.8.5.3]), whose homotopy groups in degree $n \geq 0$ will coincide with the relative K-groups defined above.

Construction 5.1.6. Let $f\colon Y \to X$ be the inclusion of Y into X, and suppose again that X is a Noetherian scheme equipped with an ample family of line bundles. Pull-back of vector bundles defines an exact functor between the Waldhausen categories $\mathcal{C}\mathbf{P}(X)$ and $\mathcal{C}\mathbf{P}(Y)$:

$$f^*\colon \mathcal{C}\mathbf{P}(X) \to \mathcal{C}\mathbf{P}(Y).$$

For every n, let $S_n(f)$ denote the category $S_n\mathcal{C}\mathbf{P}(X) \times_{S_n(\mathcal{C}\mathbf{P}(Y))} S_{n+1}\mathcal{C}\mathbf{P}(Y)$, whose objects are pairs

$$(\mathcal{E}_*, \mathcal{F}_*) = (\mathcal{E}_1 \rightarrowtail \mathcal{E}_2 \ldots \rightarrowtail \mathcal{E}_n, \mathcal{F}_0 \rightarrowtail \ldots \rightarrowtail \mathcal{F}_n)$$

such that $f^*(\mathcal{E}_*) = (\mathcal{F}_1/\mathcal{F}_0 \rightarrowtail \ldots \rightarrowtail \mathcal{F}_n/\mathcal{F}_0)$. Thus, for $n = 1$, the category $S_1(f)$ is the category of pairs

$$(\mathcal{E}_1, \mathcal{F}_0 \rightarrowtail \mathcal{F}_1 \twoheadrightarrow \mathcal{F}_{01}) \in S_1(\mathcal{C}\mathbf{P}(X)) \times S_2(\mathcal{C}\mathbf{P}(Y)) = \mathcal{C}\mathbf{P}(X) \times S_2(\mathcal{C}\mathbf{P}(Y))$$

where $\mathcal{E}_1$ is a bounded complex of (big) vector bundles on X and $\mathcal{F}_0 \rightarrowtail \mathcal{F}_1 \twoheadrightarrow \mathcal{F}_{01}$ is an extension of complexes of vector bundles on Y such that the restriction $f^*(\mathcal{E}_1)$ satisfies $f^*(\mathcal{E}_1) = \mathcal{F}_{01}$.

Remark 5.1.7. There is a (common, see for example [69, IV.8.5]) abuse of notation in the fiber product of Waldhausen categories $S_n\mathcal{C}\mathbf{P}(X) \times_{S_n(\mathcal{C}\mathbf{P}(Y))} S_{n+1}\mathcal{C}\mathbf{P}(Y)$. In fact, the 1-fiber product of categories does not come equipped naturally with a structure of Waldhausen category (pushouts along cofibrations do not need to exist). The correct construction to consider (that enjoys a natural structure of Waldhausen category) is rather the 2-fiber product of the functors f^* and $d^0\colon PS_\bullet(\mathcal{C}\mathbf{P}(Y)) = S_{\bullet+1}\mathcal{C}\mathbf{P}(Y) \to S_\bullet\mathcal{C}\mathbf{P}(Y)$. However, as remarked by Waldhausen in [68, Definition 1.5.4], in the case of interest there is an equivalence of categories between the 1-fiber product and the 2-fiber product, justifying the abuse of notation.

This construction produces a simplicial Waldhausen category $S_\bullet(f)$. Considering the category $\mathcal{C}\mathbf{P}(Y)$ as constant simplicial Waldhausen category, we have an inclusion functor $\mathcal{C}\mathbf{P}(Y) \to S_\bullet(f)$, that combined with the projection to the first factor gives a sequence of simplicial (Waldhausen) categories

$$\mathcal{C}\mathbf{P}(Y) \to S_\bullet(f) \to S_\bullet\mathcal{C}\mathbf{P}(X)$$

in which the composition of the two maps is trivial. By [68, Proposition 1.5.5], the sequence

$$wS_\bullet\mathcal{C}\mathbf{P}(Y) \to wS_\bullet(S_\bullet(f)) \to wS_\bullet(S_\bullet\mathcal{C}\mathbf{P}(X))$$

is a homotopy fibration sequence. We write $K^W(X;Y)$ for the double loop space

$$K^W(X;Y) = \Omega^2 \operatorname{Sing} |wS_\bullet(S_\bullet(f))|$$

and we call it the *relative K-theory space*. We denote by $K_n(X;Y) = K_n^W(X;Y)$ the homotopy groups $\pi_n(K^W(X;Y))$. They fit in a long exact sequence

$$\ldots \to K_n(X;Y) \to K_n(X) \to K_n(Y) \to \ldots \to K_0(X) \to K_0(Y) \to K_{-1}(X;Y) \to 0$$

and agree with the Thomason-Trobaugh relative K-theory groups displayed in (4.1.0.1) for $n \geq 0$.

Remark 5.1.8. The group $K_{-1}(X;Y) = \pi_1(wS_\bullet(S_\bullet(f)))$ defined above is the cokernel of the map $K_0(X) \to K_0(Y)$, that does not have to be surjective (see [69, IV.8.11]). It does not agree, in general, with the group $K_{-1}(X;Y) = \pi_{-1}(\mathbf{K}^B(X;Y))$, where $\mathbf{K}^B(X;Y)$ denotes Bass-Thomason relative K-theory spectrum, but it will whenever X is a Noetherian regular scheme, since $K^B_{-1}(X) = K_{-1}(X) = 0$ in that case.

5.1.9. Write (X,Y) for a pair of schemes, where X is a (separated) Noetherian scheme and Y is a closed subscheme of X. Let $f\colon (X,Y) \to (X',Y')$ be a morphism of pairs of schemes i.e., $f\colon X \to X'$ is a morphism of schemes that restricts to a morphism $f_Y\colon Y \to Y' \subset X'$. Assume again that X and X' have ample families of line bundles. The morphism f induces a strict commutative square of spaces

$$\begin{array}{ccc} K^W(X') & \xrightarrow{\iota_{Y'}^*} & K^W(Y') \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f_Y^*} \\ K^W(X) & \xrightarrow{\iota_Y^*} & K^W(Y) \end{array}$$

that gives in turn a morphism between the relative K-theory spaces $f^*\colon K^W(X';Y') \to K^W(X;Y)$. Indeed, since $X \mapsto \mathcal{C}\mathbf{P}(X)$ is a contravariant functor from the category of Noetherian schemes to the category of small Waldhausen categories, we have a strict commutative diagram of simplicial Waldhausen categories

$$\begin{array}{ccc} S_\bullet(\mathcal{C}\mathbf{P}(X')) & \xrightarrow{\iota_{Y'}^*} & S_\bullet(\mathcal{C}\mathbf{P}(Y')) \\ \downarrow f^* & & \downarrow f_Y^* \\ S_\bullet(\mathcal{C}\mathbf{P}(X)) & \xrightarrow{\iota_Y^*} & S_\bullet(\mathcal{C}\mathbf{P}(Y)) \end{array}$$

and a corresponding commutative diagram of bisimplicial categories

$$\begin{array}{ccccc} wS_\bullet\mathcal{C}\mathbf{P}(Y') & \longrightarrow & wS_\bullet(S_\bullet(\iota_{Y'})) & \longrightarrow & wS_\bullet(S_\bullet\mathcal{C}\mathbf{P}(X')) \\ \downarrow & & \downarrow & & \downarrow \\ wS_\bullet\mathcal{C}\mathbf{P}(Y) & \longrightarrow & wS_\bullet(S_\bullet(\iota_Y)) & \longrightarrow & wS_\bullet(S_\bullet\mathcal{C}\mathbf{P}(X)) \end{array}$$

where the central vertical arrow is simply induced by the universal property of fiber product of small categories. Taking realization of the two central bisimplicial categories gives then the required morphism of simplicial sets

$$f^*\colon \Omega^2 \operatorname{Sing}|wS_\bullet(S_\bullet(\iota_{Y'}))| \to \Omega^2 \operatorname{Sing}|wS_\bullet(S_\bullet(\iota_Y))|.$$

Compatibility with composition is readily verified (using again the fact that $\mathcal{C}\mathbf{P}(-)$ is strictly functorial on schemes), so that the assignment $(X;Y) \mapsto K^W(X';Y')$ defines a contravariant functor from the category of pairs and morphisms of pairs to the category of pointed simplicial sets.

5.1.10. We specialize our construction to the geometric situation. Let $\overline{\mathbf{M}}\mathbf{Sm}(k)$ be the category of modulus pairs over k introduced in Definition 1.4.3. Objects are pairs $M = (\overline{M}, D_M)$ where $\overline{M}$ is a smooth and separated k-scheme. In particular, $\overline{M}$ admits an ample family of line bundles. The effective Cartier divisor D_M can be considered as closed subscheme of $\overline{M}$, and we can then set $K^{\overline{\mathbf{M}}}(M) = K^W(\overline{M}; D_M)$. This is the *K-theory space* of the modulus pair M. The following Lemma is a consequence of the discussion above.

Lemma 5.1.11. *The assignment $M \mapsto K^{\overline{\mathbf{M}}}(M)$ defines a contravariant functor from the category $\overline{\mathbf{M}}\mathbf{Sm}(k)$ of modulus pairs to the category of pointed simplicial sets.*

Proof. Let $f\colon M \to N$ be a morphism of pairs. By definition, f determines a morphisms of (smooth) k-schemes $\overline{M} \to \overline{N}$ that satisfies $f^*(D_N) \geq D_M$ as Weil (or Cartier) divisors on $\overline{M}$. Translated in terms of closed subschemes of $\overline{M}$, this condition gives the existence of a closed embedding $D_M \xrightarrow{i} f^*(D_N) \hookrightarrow \overline{M}$. We have thus maps in the other directions between the corresponding categories of bounded complexes of big vector bundles

$$\mathcal{C}\mathbf{P}(\overline{M}) \xrightarrow{(\iota_{f^*(D_N)})^*} \mathcal{C}\mathbf{P}(f^*(D_N)) \xrightarrow{i^*} \mathcal{C}\mathbf{P}(D_M)$$

whose composition is exactly the restriction morphism $\mathcal{C}\mathbf{P}(\overline{M}) \xrightarrow{\iota_{D_M}^*} \mathcal{C}\mathbf{P}(D_M)$. It is then clear that there is a map between the relative K-theory spaces $K^W(\overline{M}; f^*(D_N)) \xrightarrow{i^*} K^W(\overline{M}; D_M)$, induced by the identity on $\overline{M}$. The construction of 5.1.9 gives another morphism of pointed spaces $K^W(\overline{N}; D_N) \to K^W(\overline{M}; f^*(D_N))$, that composed with i^* above give the required pullback map $f^*\colon K^{\overline{\mathbf{M}}}(N) \to K^{\overline{\mathbf{M}}}(M)$. □

If we wish to work with the K-theory spectra instead of the K-theory spaces, we have the following

Lemma 5.1.12. *Let $M = (\overline{M}, D_M)$ be a modulus pair. The assignment $M \mapsto \mathbf{K}^W(\overline{M}; D_M)$ defines a contravariant functor from the category $\overline{\mathbf{M}}\mathbf{Sm}(k)$ of modulus pairs to the category of S^1-spectra.*

Remark 5.1.13. The subscript $(-)^W$ on top of the K-theory spectrum $\mathbf{K}^W(\overline{M}; D_M)$ has been so far a reminder of the fact that we are using the category of big vector bundles as Waldhausen category to build the spectra $\mathbf{K}(\overline{M})$ and $\mathbf{K}(D_M)$. As remarked before, the choice of the category is not crucial, and the advantage in using $\mathcal{C}\mathbf{P}(-)$ is in the strict functoriality of the resulting spectrum. We construct in the next section a K-theory spectrum $\mathbf{K}(X/\partial X_*)$ for a scheme with compactification $(X, \partial X) \in \mathbf{Sm}_{\log}(k)$ and then a spectrum $\mathbf{K}(M)$ a modulus datum $M \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$. For those objects we don't have, unfortunately, a good categorical model that gives a strictly functorial construction. Since the very definition of $\mathbf{K}(X/\partial X_*)$ and of $\mathbf{K}(\overline{M})$ involves taking push-forward along certain maps, the best solution is to work with the categories of strict perfect complexes $\mathbf{Perf}(-)$. To avoid introducing heavier notations, we keep writing $\mathbf{K}^W(X)$ to denote the Waldhausen K-theory spectrum constructed from $\mathbf{Perf}(X)$, and we ask the reader to be forgiving.

5.2. Iterated homotopy cofibers and motivic K-spectra with modulus. We specialize the situation of Chapter I, Section 4.1.6, to our case of interest.

Definition 5.2.1. *Let $(Y_1, \dots, Y_n, X)$ be a tuple given by a Noetherian separated equidimensional scheme X admitting an ample family of line bundles and a set $Y_1, \dots, Y_n$ of irreducible closed subschemes of X. We say that the tuple is good (on X) if the following hold.*

i) The subschemes Y_i are effective Cartier divisors on X,
ii) For every $I \subset J \subset \{1, \dots, n\}$, the morphism $\varphi_{I,J}\colon Y_I \to Y_J$ is a regular closed immersion
iii) For every subset $I \subset \{1, \dots, n\}$, the subscheme Y_I has pure codimension $n - |I|$.

5.2.2. Let $(Y_1, \dots, Y_n, X)$ be a good tuple. Let Y be the union $Y_1 \cup \dots \cup Y_n$. For $I = \{1, \dots, n\} \setminus \{i_0, \dots, i_p\}$, write $Y_I = Y_{i_0,\dots,i_p} = Y_{i_0} \cap \dots \cap Y_{i_p} \xrightarrow{\iota_{i_0,\dots,i_p}} Y$ for the inclusion of $Y_{i_0,\dots,i_p}$ in Y and $Y_{i_0,\dots,i_p} \xrightarrow{j_{i_0,\dots,i_p}} X$ for the inclusion of $Y_{i_0,\dots,i_p}$ in X. The proof of the following Lemma is elementary by induction.

Lemma 5.2.3. *There is an exact sequence of $\mathcal{O}_Y$-modules, of finite homological dimension over $\mathcal{O}_X$,*

$$0 \to \mathcal{O}_Y \xrightarrow{d^0} \bigoplus_{i=1}^{n} \iota_{i,*}\mathcal{O}_{Y_i} \xrightarrow{d^1} \dots \bigoplus_{i_0<i_1<\dots<i_p} \iota_{i_0,\dots,i_p,*}\mathcal{O}_{Y_{i_0,\dots,i_p}} \xrightarrow{d^p} \dots \iota_{1,\dots,n,*}\mathcal{O}_{Y_1\cap\dots\cap Y_n} \to 0 \tag{5.2.3.1}$$

where the maps $d^p\colon \bigoplus_{i_0<i_1<\dots<i_p} \iota_{i_0,\dots,i_p,*}\mathcal{O}_{Y_{i_0,\dots,i_p}} \to \bigoplus_{i_0<i_1<\dots<i_{p+1}} \iota_{i_0,\dots,i_{p+1},*}\mathcal{O}_{Y_{i_0,\dots,i_{p+1}}}$ are given by alternating sum $d^p = \sum_{k=0}^{p+1}(-1)^k\delta_k^p$ where δ_k^p is the restriction from $\mathcal{O}_{i_0,\dots,\check{i_k},\dots,i_{p+1}}$ to $\mathcal{O}_{i_0,\dots,i_{p+1}}$.

Construction 5.2.4. Let $f\colon Z \to X$ be a morphism of finite type and assume that $f^{-1}(Y_1) = \bigcup_{i=1}^r Z_i$, for $Z_1, \dots Z_r$ a set of irreducible subschemes of Z that form a good tuple on Z. For any subset $I \subset \{1, \dots, r\}$, we write $f_I\colon Z_I \to Y_1$ for the restriction of f to Z_I. We define an exact functor between the categories of perfect complexes by setting

$$\sigma\colon \mathbf{Perf}(Y_1) \to \prod_{I\subset\{1,\dots,r\}} \mathbf{Perf}(Z_I), \quad \mathcal{F}_\bullet \mapsto (f_I^*(\mathcal{F}_\bullet)[r - |I| - 1]).$$

Let $\nu_I\colon Z_I \to Z$ be the natural inclusion. The push forward $(\nu_I)_*$ is an exact functor from the category $\mathbf{Perf}(Z_I)$ to the category $\mathbf{Perf}(Z)$, and composing the sum of push forwards with the functor σ gives an exact functor (defined up to homotopy - see Remark 5.2.5)

$$\psi = (\sum_I (\nu_I)_*) \circ \sigma\colon \mathbf{Perf}(Y_1) \to \mathbf{Perf}(Z).$$

We denote by ψ_* the induced morphism between the K-theory spectra $\mathbf{K}^W(Y_1) \to \mathbf{K}^W(Z)$.

Remark 5.2.5. The product of Waldhausen categories $\prod_{I\subset\{1,\dots,r\}} \mathbf{Perf}(Z_I)$ is equivalent to the sum $\coprod_{I\subset\{1,\dots,r\}} \mathbf{Perf}(Z_I)$. This is remarked by Barwick, [1, Proposition 4.11], where it is shown that the ∞-category $\mathbf{Wald}_\infty$ of (small) Waldhausen ∞-category, with morphisms given by exact functors, has finite sums. In particular, finite products and coproducts of Waldhausen ∞-categories coincide in the homotopy category of $\mathbf{Wald}_\infty$. Alternatively, we can take directly the K-theory spectra of all the categories involved and, working in **HoSpt**, we can convert products into coproducts.

Applying Lemma 5.2.3, we can show that there is a homotopy equivalence $\psi_* \xrightarrow{\simeq} f^* j_{1,*}$. First, we note that the base change formula of [63] gives an equivalence $j_{W,*} f_W^* \xrightarrow{\simeq} f^* j_{1,*}$, where W denote the scheme-theoretic inverse image $f^{-1}(Y_1)$, j_W is the inclusion of W in Z and f_W is the restriction of f to W. Given any perfect complex $\mathcal{E}_\bullet$ on Y_1, we can tensor the exact sequence (5.2.3.1) for the tuple $Z_1, \dots, Z_n$ and use the projection formula to get

$$0 \to f_W^*(\mathcal{E}_\bullet) \xrightarrow{d^0} \bigoplus_{i=1}^{r} \iota_{i,*}(f_i^* \mathcal{E}_\bullet) \to \dots \to \iota_{1,\dots,r,*} f_{1,\dots,r}^*(\mathcal{E}_\bullet) \to 0.$$

Applying $j_{W,*}$ gives an exact sequence of perfect complexes on Z

$$0 \to j_{W,*} f_W^*(\mathcal{E}_\bullet) \xrightarrow{d^0} \bigoplus_{i=1}^{r} \nu_{i,*}(f_i^* \mathcal{E}_\bullet) \to \dots \to \nu_{1,\dots,r,*} f_{1,\dots,r}^*(\mathcal{E}_\bullet) \to 0. \tag{5.2.5.1}$$

The map sum $\bigvee_{I\subset\{1,\dots,r\}} \mathbf{K}(Z_I) \to \mathbf{K}^W(Z)$ factors by construction through the homotopy colimit of the spectra $\mathbf{K}(Z_I)$:

$$\mathbf{K}^W(Y_1) \xrightarrow{\sigma} \bigvee_{I\subset\{1,\dots,r\}} \mathbf{K}(Z_I) \to \underrightarrow{\operatorname{hocolim}}_{I\neq\{1,\dots,n\}} \mathbf{K}^W(Z_I) \to \mathbf{K}^W(Z).$$

We argue now that this composition agrees up to homotopy with the morphism $j_{W,*} f_W^*(-)$. This is a direct consequence of the displayed equation (5.2.5.1) and of Waldhausen additivity theorem (see [69, V.1.2] and its Corollary).

5.2.6. This construction specializes to the geometric case. Let $(X, \partial X) \in \mathbf{Sm}_{\log}(k)$ be a "scheme with compactification" in the sense of Definition 1.1.1. This is the datum of a smooth k-scheme X and of a strict normal crossing divisor ∂X on X. Write $\partial X_1, \dots, \partial X_n$ for the (smooth) irreducible components of ∂X. For every $i = 1, \dots, n$, we have a regular closed embedding $\iota_i \colon \partial X_i \to X$, and by definition every $\partial X_I = \bigcap_{i\in I} \partial X_i$ is smooth over k, of pure codimension $|I|$. In particular, every inclusion $\partial X_I \to \partial X_J$ is a regular closed embedding and thus the datum $(X, \partial X)$ is a good tuple in the sense of Definition 5.2.1. We denote by $\mathbf{K}^W(X/\partial X_*)$ the iterated homotopy cofiber $\mathbf{K}^W(X/\partial X_1 \dots, \partial X_n)$ of Definition 4.1.7.

Lemma 5.2.7. *Let $f\colon (Y, \partial Y) \to (X, \partial X)$ be a morphism in $\mathbf{Sm}_{\log}(k)$. Then there is a well-defined pull-back of K-theory spectra*

$$f^* \colon \mathbf{K}^W(X/\partial X_*) \to \mathbf{K}^W(Y/\partial Y_*).$$

Proof. Is enough to assume $|\partial X| = 1$, i.e. that ∂X is irreducible. Write $\partial Y_1, \dots, \partial Y_n$ for the components of ∂Y. The admissibility condition of f implies that $(f^{-1}(\partial X))_{\mathrm{red}} = \bigcup_{i=1}^{r} \partial Y_i$ for some r. There is then a commutative diagram of spectra

$$\begin{array}{ccc} \mathbf{K}^W(\partial X) & \xrightarrow{(\iota_{\partial X})_*} & \mathbf{K}^W(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathbf{K}^W(\bigcup_{i=1}^{r} \partial Y_i) & \longrightarrow & \mathbf{K}^W(Y) \end{array}$$

and an induced morphism between the homotopy cofibers $\mathbf{K}^W(X/\partial X) \to \mathbf{K}^W(Y/\bigcup_{i=1}^r \partial Y_i)$. The Construction 5.2.4 gives a factorization of the map $\mathbf{K}^W(\partial X) \xrightarrow{f^*} \mathbf{K}^W(\bigcup_{i=1}^r \partial Y_i) \to \mathbf{K}^W(Y)$ via the homotopy colimit of the K-spectra of the components ∂Y_i, and we obtain by composition the required morphism $\mathbf{K}^W(X/\partial X) \xrightarrow{f^*} \mathbf{K}^W(Y/\partial Y_*)$. □

A combination of Lemma 5.2.7 and Lemma 5.1.12 gives the following result

Proposition 5.2.8. *Let $M = (\overline{M}; \partial M, D_M) \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ be a modulus datum. In particular, the sum $\partial M + |D_M|$ forms a strict normal crossing divisor on $\overline{M}$. We define its K-theory spectrum as the homotopy fiber between the iterated cofibers*

$$\mathbf{K}(M) = \mathrm{hofib}(\mathbf{K}^W(\overline{M}/\partial M_*) \to \mathbf{K}^W(D_M/(D_M \cap \partial M)_*))$$

The assignment $M \mapsto \mathbf{K}(M)$ gives a pseudo functor from the category of modulus data $\overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$ to the category of spectra **HoSpt**.

Proposition 5.2.9. *For any modulus datum $M = (\overline{M}; \partial M, D_M) \in \overline{\mathbf{M}}\mathbf{Sm}_{\log}(k)$, the projection $\pi\colon M \otimes \overline{\square} \to M$ induces a homotopy equivalence*

$$\pi^*\colon \mathbf{K}(M) \xrightarrow{\simeq} \mathbf{K}(M \otimes \overline{\square})$$

Proof. This is a consequence of the definition and of the projective space bundle formula. □

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