

Generalized theta functions for holomorphic triples

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Introduction

Let X be a smooth projective curve over an algebraically closed field k of arbitrary characteristic. A triple on X is a triple (E_2, E_1, φ) where E_1, E_2 are vector bundles on X and $\varphi : E_2 \rightarrow E_1$ is a morphism. For triples, we can introduce the notion of (semi)stability, depending on a real parameter α , and then study the moduli space of α -(semi)stable triples. The properties of triples and their moduli spaces have been investigated in [2] by Bradlow and García-Prada, in [3] by the same authors with Gothen, in [17] by Schmitt in characteristic 0 or in [1] by Álvarez-Cónsul in case of arbitrary characteristic, etc. Particularly, in [2], Bradlow and García-Prada showed that there is a one-to-one correspondence between triples $T = (E_2, E_1, \varphi)$ and the extensions $E_T \in \text{Ext}^1(p^*E_2, p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2))$ on $X \times \mathbb{P}^1$ defined by T . By showing that T is α -(semi)stable if and only if E_T is $H(\alpha)$ -(semi)stable where $H(\alpha)$ is an ample divisor on $X \times \mathbb{P}^1$, Bradlow and García-Prada constructed the moduli space of α -(semi)stable triples as a closed subscheme of the moduli space of semistable vector bundles of fixed type on $X \times \mathbb{P}^1$. Let $C \in |mH(\alpha)|$ be a smooth projective curve on $X \times \mathbb{P}^1$. It is well known that for m large enough, the restriction of E_T to C is still semistable. We then obtain a morphism from moduli space of α -semistable triples to the moduli space of semistable vector bundles on C which allows us to study triples and their moduli space from moduli space of vector bundles on C .

Let $U_C(r, d)$ be the moduli space of semistable vector bundles of rank r and degree d on C . For a given vector bundle F on C such that $\chi(E \otimes F) = 0$ for some $E \in U_C(r, d)$ (hence for all $E \in U_C(r, d)$), we set $\Theta_F = \{E \in U_C(r, d) | H^0(C, E \otimes F) \neq 0\}$. It was proven by Drezet and Narasimhan in [5] that Θ_F is the underlying space of an effective Cartier divisor, called a generalized theta divisor, if it is a proper subset of $U_C(r, d)$. Moreover, for any vector bundle F' such that $[F'] = n[F]$ in the Grothendieck's group of coherent sheaves $K(C)$, we have $\Theta_{F'} \in |n\Theta_F|$. On smooth projective curve C , it was first proven by Faltings in [8] and then by Seshadri in [18] that a vector bundle E is semistable if and only if there exists a vector bundle F such that $E \otimes F$ is cohomologically trivial, i.e. $H^*(X, E \otimes F) = 0$. We say that F is orthogonal to E . It follows that Θ_F is a generalized theta divisor if F is orthogonal to some $E \in U_C(r, d)$. In this case, the line bundle $\theta_F = \mathcal{O}_{U_C(r, d)}(\Theta_F)$ has a section ζ_F , called a generalized theta function, such that its zero divisor is Θ_F . Studying properties of linear

system $|\Theta_F|$: base point freeness, separation properties, ampleness, global generatedness, etc, allows us to construct and study the moduli space of vector bundles on curves by a GIT-free way (see [8], [18], [11], [16], [7], [6], etc). As mentioned above, there is a morphism from moduli space of α -semistable triples on X to the moduli space of semistable vector bundles on C . So there should be a natural way to define and study theta line bundles and theta functions on the moduli space of α -semistable triples.

The aims of this thesis are to extend some properties of α -semistable triples on smooth projective curves to arbitrary characteristic and to add the missing pieces of the study of moduli space of triples: generalized theta line bundles and generalized theta functions. Studying properties of these objects also allows us to define the orthogonal triples and furthermore prepare the ingredients for a GIT- free construction of moduli space of triples.

More in details, we firstly recall in **Chapter 1** the definitions, basic properties of slope semistability and global generatedness of vector bundles on curves. In particular, we recall the existence of the Harder-Narasimhan filtration for vector bundles and also a modified Harder-Narasimhan filtration which will be used later.

In **Chapter 2**, we recall the numerical equivalence of divisors on ruled surfaces, typically on $X \times \mathbb{P}^1$. We then prove the following Bogomolov's inequality for vector bundles on $X \times \mathbb{P}^1$ in case of arbitrary characteristic,

Theorem 1 (Thm. 2.2.4). *Let E be a vector bundle on $X \times \mathbb{P}^1$ and H be an ample divisor. If E is H -semistable then*

$$\Delta(E) = (\text{rank}(E) - 1)c_1^2(E) - 2\text{rank}(E)c_2(E) \leq 0.$$

As a consequence of Bogomolov's inequality, we obtain then a restriction theorem for semistable vector bundles on $X \times \mathbb{P}^1$ (Cor. 2.3.4).

In **Chapter 3** we collect the definitions and basic properties of triples, α -semistability of triples and also the existence of the moduli space of triples. For each triple T , we construct a vector bundle E_T on $X \times \mathbb{P}^1$ (see Section 3.3). Let p, q be the projections from $X \times \mathbb{P}^1$ onto X and \mathbb{P}^1 , respectively. Let $\alpha = \frac{a}{b}$ be a positive rational number. Let us fix $H(\alpha) = aF_p + bF_q$ an ample divisor on $X \times \mathbb{P}^1$ where F_p, F_q are any p -fiber and q -fiber, respectively. We give then a proof of the following result of Bradlow and García-Prada in [2] which works in arbitrary characteristic.

Theorem 2 (Thm. 3.3.4). *The triple $T = (E_2, E_1, \varphi)$ is α -(semi)stable if and only if the extension E_T defined by T is $H(\alpha)$ -(semi)stable.*

Chapter 4 is the main part of the thesis where we define and study properties of generalized theta line bundles and generalized theta functions on moduli space of triples. Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ be a family of α -semistable triples on X parametrized by a scheme S . Using the construction of the determinant line bundle associated to a flat family of coherent

sheaves (see Section 4.1.1) and the triple product (see Section 4.2.1), we can define the determinant line bundle for triples (Def 4.2.5). For any triple $F = (F_2 \xrightarrow{\psi} F_1)$ on X , we then obtain a line bundle $\theta_{F_2 \rightarrow F_1}$ on S . In particular, if $\psi : F_2 \rightarrow F_1$ is surjective and the Euler characteristic of triple product $\mathcal{T}_s \times F$ is zero for all $s \in S$ then $\theta_{F_2 \rightarrow F_1}$ has a section $\zeta_{F_2 \rightarrow F_1}$ (see Section 4.2.4). Let $C \in |mH(\alpha)|$ be a smooth projective curve. As an application of Bogomolov's inequality, $E_T|_C$ is still semistable for any $m \geq m_0 = \lceil \frac{1-r}{r} \Delta(E_T) + 1 \rceil$ (Cor. 2.3.4). As $E_T|_C$ is semistable, the existence of orthogonal bundles F_C allows us to construct a α -orthogonal triple $F = (F_2 \xrightarrow{\psi} F_1)$ (Def 4.2.2) which characterizes the α -semistability of T :

Theorem 3 (Thm. 4.2.3). *Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple on X and $\alpha = \frac{a}{b} \in \mathbb{Q}$. Then T is α -semistable if and only if it has a α -orthogonal triple.*

Let us fix $C \in |m_0H(\alpha)|$. Let \mathcal{M} be the moduli space of α -semistable triples of type (r_1, r_2, d_1, d_2) on X and $U_C(r, d)$ be the moduli space of semistable vector bundles on C of rank $r = r_1 + r_2$ and degree $d = m_0 \deg_{H(\alpha)}(E_T)$. We have a morphism

$$\mathfrak{t} : \mathcal{M} \longrightarrow U_C(r, d)$$

sending T to $E_T|_C$ (Lemma 4.3.1). We define the generalized theta line bundle and generalized theta functions to be the pullback by \mathfrak{t} of the corresponding ones on $U_C(r, d)$ (Def. 4.3.2). Let θ_{F_0} be a basic theta divisor on $U_C(r, d)$ and $|p\Theta_{F_0}|$ be a pluritheta linear series. Let $\mathcal{L} = \mathfrak{t}^*\theta_{F_0}$ be a basic theta line bundle on \mathcal{M} . Then for each p , we have a generalized theta line bundle \mathcal{L}^p and a linear system $V(\mathcal{L}^p) \subset H^0(\mathcal{M}, \mathcal{L}^p)$ spanned by generalized theta functions. The first and most important applications which we have from the existence of α -orthogonal triples are the base point freeness, ampleness and the universal property of the generalized theta line bundle for triples. Let F_C be a vector bundle on C such that $[F_C] = p[F_0] \in K(C)$. Using F_C we construct a triple $F = (F_2 \xrightarrow{\psi} F_1)$ (see proof of Lemma 4.2.4 or (4.27)), such that the line bundle $\theta_{F_2 \rightarrow F_1}$ has a geometric section $\zeta_{F_2 \rightarrow F_1}$ whose zero divisor is defined by

$$\Theta_{F_2 \rightarrow F_1} = \{s \in S | F \text{ is not a } \alpha\text{-orthogonal triple of } \mathcal{T}_s\}$$

(see (4.21) and (4.25)). Let $\text{Comp}(p, F)$ be the set of all triples $F' = (F'_2 \xrightarrow{\psi'} F'_1)$ such that $[F'_i] = [F_i] \in K(X)$. Then all the line bundles $\theta_{F'_2 \rightarrow F'_1}$ are isomorphic to $\theta_{F_2 \rightarrow F_1}$ (see Lemma 4.2.6). Therefore we obtain on S a linear system $V(p, F) \subset H^0(S, \theta_{F_2 \rightarrow F_1})$ spanned by sections $\zeta_{F'_2 \rightarrow F'_1}$. By the result of Esteves and Popa on the base point freeness of pluritheta linear series in [7], we obtain the following base point freeness of linear systems $V(\mathcal{L}^p)$ and $V(p, F)$.

Corollary 4 (Cor. 4.3.3). *For any $p \geq r^2 + r$, then*

- (i) *The linear systems $V(\mathcal{L}^p)$ and $V(p, F)$ are base point free.*

(ii) There exists a triple $F' = (F'_2 \xrightarrow{\psi'} F'_1) \in \text{Comp}(p, F)$ such that

$$\overline{\Theta}_{F'_2 \rightarrow F'_1} = \{T \in \mathcal{M} \mid F' \text{ is not a } \alpha\text{-orthogonal triple of } T\}$$

is a Cartier divisor on \mathcal{M} and

$$\mathcal{L}^p \cong \mathcal{O}_{\mathcal{M}}(\overline{\Theta}_{F'_2 \rightarrow F'_1}).$$

Similar to the construction of extensions on $X \times \mathbb{P}^1$ defined by triples on X , for each family \mathcal{T} of α -semistable triples on X parametrized by S , we obtain an induced family $E_{\mathcal{T}}$ of $H(\alpha)$ -semistable vector bundles on $X \times \mathbb{P}^1$ and then a family $\mathcal{E} = E_{\mathcal{T}}|_C$ of semistable vector bundles on C parametrized by S . We then have a commutative diagram of induced morphisms

$$\begin{array}{ccc} S & \xrightarrow{f_{\mathcal{E}}} & U_C(r, d) \\ & \searrow f_{\mathcal{T}} & \nearrow \iota \\ & \mathcal{M} & \end{array}$$

By the universal property and the ampleness of the generalized theta line bundle θ_{F_C} on $U_C(r, d)$ (Thm. 4.1.2), we obtain the similar properties for \mathcal{L}^p .

Corollary 5 (Cor. 4.3.4). *Let $p \geq 1$ and $F'' = (F''_2 \xrightarrow{\psi''} F''_1)$ be any triple in $\text{Comp}(p, F)$. Then for any family $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ of α -semistable triples on X parametrized by S , the line bundle $\theta_{F''_2 \rightarrow F''_1}$ descends to \mathcal{L}^p by the induced morphism $f_{\mathcal{T}} : S \rightarrow \mathcal{M}$.*

Corollary 6 (Cor. 4.3.5). *The line bundle \mathcal{L} on \mathcal{M} is ample.*

Another important consequence of Theorem 3 is Langton's valuative criterion for triples. Let R be a discrete valuation ring and $X_{\eta} = \eta \times X$ be the generic fiber of the first projection

$$\text{Spec}(R) \times X \rightarrow \text{Spec}(R) = \{0, \eta\}.$$

Theorem 7 (Thm. 4.4.4). *Let T_{η} be a α -semistable triple of type (r_1, r_2, d_1, d_2) on the generic fiber X_{η} . Then there exists a triple \mathcal{T} on X_R such that $\mathcal{T}|_{X_{\eta}} \cong T_{\eta}$ and $T_0 := \mathcal{T}|_{X_0}$ is α -semistable.*

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Chapter 1

Vector bundles on curves

1.1 Semistability of vector bundles

Let X be a smooth projective curve over an algebraically closed field k of arbitrary characteristic. By a vector bundle on X , we mean a locally free coherent sheaf on X . Let E be a nonzero vector bundle on X , $d = \deg(E)$ and $r = \text{rank}(E)$. Then we define the slope of E to be $\mu(E) = \frac{d}{r}$.

Definition 1.1.1. *Let E be a nonzero vector bundle on X . Then E is called (semi)stable if for any nonzero subsheaf $E' \subsetneq E$ we have $\mu(E')(\leq)\mu(E)$.*

Here we use the notation (\leq) introduced by Huybrechts and Lehn in [12, p.11]: If the word “(semi)stable” and the relation sign “ (\leq) ” appear together in a statement, then this statement stands for two assertions, one with “stable” and strict inequality “ $<$ ” and the other with “semistable” and the mild inequality “ \leq ”.

Remark 1.1.2. Assume that

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

is an exact sequence of vector bundles. Then we have

$$\begin{aligned}\deg(E) &= \deg(F) + \deg(G), \\ \text{rank}(E) &= \text{rank}(F) + \text{rank}(G).\end{aligned}$$

It implies that

$$\mu(F)(\leq)\mu(E) \iff \mu(E)(\leq)\mu(G).$$

Hence we can also formulate semistability in terms of quotient bundles: E is semistable if and only if for any nonzero quotient G of E , we have $\mu(E) \leq \mu(G)$.

The following properties of semistable vector bundles can be seen easily by the definition.

Lemma 1.1.3. *Let E be a vector bundle on X and L be a line bundle. Then the following holds:*

(i) L is stable.

(ii) E is (semi)stable if and only if $E \otimes L$ is (semi)stable.

(iii) E is (semi)stable if and only if E^\vee , its dual bundle, is (semi)stable.

Proposition 1.1.4. *Let E and F be vector bundles on X .*

(i) If E and F are both semistable and $\mu(E) > \mu(F)$ then $\text{Hom}_X(E, F) = 0$.

(ii) If E is stable then it is simple, i.e. $\text{End}(E) \cong k$.

Proof. Let $\phi \in \text{Hom}(E, F)$ be a morphism. If $\phi \neq 0$ then $I = \text{Im } \phi \subset F$ is not only a nonzero subsheaf of F but also a quotient of E . Since E and F are semistable, we have

$$\mu(E) \leq \mu(I) \leq \mu(F).$$

This proves (i). For (ii), we show that any endomorphism of E is in form λId_E for some $\lambda \in k$. Let $\phi \in \text{End}(E)$ be a nontrivial morphism and I be the image of ϕ . As we have shown above, $\mu(E) < \mu(I) < \mu(E)$ if $I \subsetneq E$. This is impossible. Hence ϕ must be surjective and hence an isomorphism. If $\phi \neq \lambda \text{Id}_E$ then there exists $x \in X(k)$ such that $\phi_x \in \text{End}_k(E \otimes k(x))$ is not in form λId . Let λ be a eigentvalue of ϕ_x . Then $\phi - \lambda \text{Id}_E \in \text{End}(E)$ is nontrivial but not surjective. Hence it is not an isomorphism. This gives us a contradiction. \square

For any coherent sheaf E on X , $\chi(E)$ is the Euler characteristic of E , defined by

$$\chi(E) = \dim H^0(X, E) - \dim H^1(X, E).$$

Theorem 1.1.5 (Riemann-Roch formula - [9, Thm. 4.1, p.432]). *Let E be a coherent sheaf on smooth projective curve X of genus g_X . Then*

$$\chi(E) = \deg(E) + \text{rank}(E)(1 - g_X).$$

It follows from Riemann-Roch formula that

$$\mu(E) = \frac{\chi(E)}{\text{rank}(E)} + g_X - 1$$

for any nontrivial vector bundle E on X .

Proposition 1.1.6. *Let E and F be vector bundles on X . If $F \neq 0$ and $H^*(E \otimes F) = 0$, i.e. $H^0(X, E) = H^1(X, E) = 0$, then E is semistable.*

Proof. Assume that $F \neq 0$ and $H^*(X, E \otimes F) = 0$. Since $H^*(X, E \otimes F) = 0$, $\chi(E \otimes F) = 0$. It implies that $g_X - 1 = \mu(E \otimes F)$. If E is not semistable then there exists a nontrivial subbundle $G \subsetneq E$ such that $\mu(G) > \mu(E)$. It follows that

$$\mu(G \otimes F) > \mu(E \otimes F) = g_X - 1$$

and hence $\chi(G \otimes F) > 0$. But we also have

$$\chi(G \otimes F) = \dim H^0(X, G \otimes F) - \dim H^1(X, G \otimes F) \leq 0$$

since $H^0(X, G \otimes F) \subseteq H^0(X, E \otimes F) = 0$. This is a contradiction. So E is semistable. \square

Definition 1.1.7. *A vector bundle E on X is called globally generated if the evaluation map*

$$e : H^0(X, E) \otimes \mathcal{O}_X \longrightarrow E$$

is surjective.

Theorem 1.1.8 ([9, Thm. 5.17, p.121; Thm. 5.2, p.228]). *Let E be a vector bundle on X . Then there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

(i) $E(n)$ is globally generated.

(ii) $H^q(X, E(n)) = 0$ for all $q > 0$.

The following properties of globally generated vector bundles are useful later.

Proposition 1.1.9 ([11, Prop. 2.6]). *Let E be a globally generated vector bundle of rank r on X . Then we have the following exact sequences*

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow E \longrightarrow \det(E) \longrightarrow 0$$

and

$$0 \longrightarrow \det(E)^{-1} \longrightarrow \mathcal{O}_X^{r+1} \longrightarrow E \longrightarrow 0.$$

1.2 The Harder-Narasimhan filtration for vector bundles

In this section we will explain the existence and some elementary properties of the Harder-Narasimhan filtration for vector bundles on a smooth projective curve. Moreover, we can make a slight modification on the index set of the Harder-Narasimhan filtration so that it can be indexed by all of rational numbers.

Definition 1.2.1. *Let E be a vector bundle over a smooth projective curve X . A Harder-Narasimhan filtration for E is a flag*

$$0 = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_k = E$$

of subbundles of E having the following two properties:

- (i) E_i/E_{i-1} is semistable for $i = 1, \dots, k$,
- (ii) $\mu(E_{i+1}/E_i) < \mu(E_i/E_{i-1})$ for $i = 1, \dots, k-1$.

By Riemann-Roch formula,

$$\mu(E') \leq g_X - 1 + \dim H^0(X, E')$$

for any nonzero subsheaf E' of E . Hence there exists a subsheaf of E which is maximal among the subsheaves of E of maximal slope, i.e. there exists a subsheaf $F \subseteq E$ such that for all $G \subseteq E$, $\mu(G) \leq \mu(F)$ and in case of equality, $G \subseteq F$. In fact, F is semistable and unique. We call it the maximal destabilizing subsheaf. The existence of the maximal destabilizing subsheaf induces the existence of the Harder-Narasimhan filtration for a vector bundle on X .

Theorem 1.2.2 ([12, Thm. 1.3.4]). *Let E be a vector bundle over a smooth projective curve X , then there exists a unique Harder-Narasimhan filtration for E .*

The following properties come directly from the definition of the Harder-Narasimhan filtration.

Proposition 1.2.3. *Let E be a vector bundle with the Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_k = E.$$

Then we have

- (i) $\mu(E_j/E_i) > \mu(E_t/E_i)$ for $i < j < t$.
- (ii) $\mu(E_t/E_i) > \mu(E_t/E_j)$ for $i < j < t$.
- (iii) $\mu(E_1) > \mu(E_2) > \dots > \mu(E_k) = \mu(E)$.

Let us set $\mu_i = \mu(E_i/E_{i-1})$ for $i = 2, \dots, k-1$ and $\mu_{\max} = \mu(E_1)$, $\mu_{\min} = \mu(E/E_{k-1})$. By a slight modification, we can index the Harder-Narasimhan filtration by all of rational numbers as follows: For each $\rho \in \mathbb{Q}$, we set $E_{(\rho)} = E_i$ where i is the index such that

$$\mu_i \geq \rho > \mu_{i+1}.$$

It follows that

$$E_{(\rho)} = 0 \iff \rho > \mu_{\max} \text{ and } E_{(\rho)} = E \iff \rho \leq \mu_{\min}. \quad (1.1)$$

Proposition 1.2.4. *The following properties are true:*

- (i) *If $\rho > \tau$ then $E_{(\rho)} \subseteq E_{(\tau)}$. If $E_{(\rho)} \not\subseteq E_{(\tau)}$ then $\rho > \tau$.*
- (ii) *$\mu_{\min}(E_{(\tau)}) \geq \tau, \mu_{\max}(E/E_{(\tau)}) < \tau$.*
- (iii) *$(E_{(\rho)})_{(\tau)} = (E_{(\tau)})_{(\rho)} = E_{(\max\{\rho, \tau\})}$.*

Proof. Assume that E has the following Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_k = E.$$

Assume $\rho > \tau$ and $E_{(\tau)} = E_i$. Then $\mu_i \geq \tau > \mu_{i+1}$. It implies that

$$\rho > \mu_i \text{ or } \mu_i \geq \rho > \mu_{i+1}.$$

By definition, $E_{(\rho)} \subseteq E_i = E_{(\tau)}$. If $E_{(\rho)} \not\subseteq E_i$ then $E_{(\rho)} \subseteq E_{i-1}$. Therefore

$$\rho \geq \mu_{i-1} > \tau$$

and we have (i).

As $E_{(\tau)} = E_i$, the Harder-Narasimhan filtration of $E_{(\tau)}$ is induced from the one of E ,

$$0 = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_i.$$

By definition, we have

$$\mu_{i+1} < \tau \leq \mu_i = \mu(E_i/E_{i-1}) = \mu_{\min}(E_{(\tau)}).$$

Similarly,

$$0 = E_i/E_i \subset E_{i+1}/E_i \subset E_{i+2}/E_i \subset \dots \subset E_k/E_i = E/E_i$$

is the Harder-Narasimhan filtration of $E/E_{(\tau)}$. Then

$$\mu_{\max}(E/E_{(\tau)}) = \mu(E_{i+1}/E_i) = \mu_{i+1} < \tau$$

and we get (ii).

If $\rho \leq \tau$ then $(E_{(\tau)})_{(\rho)} = E_{(\tau)}$ since $\rho \leq \mu_{\min}(E_{(\tau)})$ by (1.1) and (ii). If $\rho > \tau$ then

$$E_{(\rho)} = E_{(\max\{\rho, \tau\})} \subseteq E_{(\tau)} \subseteq E$$

by (i). Therefore $E_{(\rho)} = (E_{(\rho)})_{(\rho)} \subseteq (E_{(\tau)})_{(\rho)} \subseteq E_{(\rho)}$. It proves (iii). \square

Corollary 1.2.5. *Let $f : E \rightarrow F$ be a morphism of vector bundles on X . Then for every $\rho \in \mathbb{Q}$, there exists a unique morphism*

$$f_\rho : E_{(\rho)} \rightarrow F_{(\rho)}$$

making the following diagram commutative

$$\begin{array}{ccc} E_{(\rho)} & \xrightarrow{f_\rho} & F_{(\rho)} \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & F \end{array}$$

Proof. Assume that E and F have the Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_k = E$$

and

$$0 = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_{k'} = F,$$

respectively. Let $\rho \in \mathbb{Q}$ and $E_{(\rho)} = E_i \subseteq E$. The claim is trivial if $f(E_{(\rho)}) = 0$. Assume that $f(E_{(\rho)}) \neq 0$. Let j be the minimal index such that $f(E_{(\rho)}) \subseteq F_j$. Let $\tau = \mu(F_j/F_{j-1})$, then $F_{(\tau)} = F_j$ and the composition

$$E_i \xrightarrow{f} F_j \xrightarrow{\pi} F_j/F_{j-1}$$

is not zero. Suppose that $F_{(\tau)} \not\subseteq F_{(\rho)}$, then $\rho > \tau$. By definition, F_j/F_{j-1} is semistable with slope τ and for $1 \leq \nu \leq i$, $E_\nu/E_{\nu-1}$ is semistable with slope $\mu_\nu \geq \rho > \tau$. It is implied by Proposition 1.1.4 that $\text{Hom}(E_\nu/E_{\nu-1}, F_j/F_{j-1}) = 0$ for all $1 \leq \nu \leq i$. It follows that $\text{Hom}(E_i, F_j/F_{j-1}) = 0$. This is a contradiction since $\pi \circ f$ is not zero. \square

For each $\tau \in \mathbb{Q}$, we define the quotient

$$\text{gr}_\tau(E) := E_{(\tau)} / (\cup_{\rho > \tau} E_{(\rho)}).$$

It is clear that $\text{gr}_\tau(E) \neq 0$ if and only if $\tau = \mu(E_i/E_{i-1})$ for some $i \neq 0$. In that case,

$$\text{gr}_{\mu(E_i/E_{i-1})}(E) = E_i/E_{i-1}.$$

The following lemma will be used later.

Lemma 1.2.6. *Let E and F be two vector bundles on X with $\mu(E) > \mu(F)$. Then there exists a rational number τ such that*

$$\frac{\text{rank}(E_{(\tau)})}{\text{rank}(E)} > \frac{\text{rank}(F_{(\tau)})}{\text{rank}(F)}.$$

Proof. We set $M = \{\tau \in \mathbb{Q} \mid \text{rank}(\text{gr}_\tau(E)) + \text{rank}(\text{gr}_\tau(F)) > 0\}$. Then M is a finite set and we could write $M = \{\tau_1, \tau_2, \dots, \tau_n\}$ with $\tau_i > \tau_{i+1}$. We define two functions $\text{rank}_E, \text{rank}_F$ from M to \mathbb{N} by setting $\text{rank}_E(\tau_i) = \text{rank}(\text{gr}_{\tau_i}(E))$ and $\text{rank}_F(\tau_i) = \text{rank}(\text{gr}_{\tau_i}(F))$, respectively. Now we have

$$\begin{aligned} \mu(E) &= \frac{\deg(E)}{\text{rank}(E)} = \frac{\sum_{i=1}^n \deg(\text{gr}_{\tau_i}(E))}{\text{rank}(E)} \\ &= \sum_{i=1}^n \tau_i \frac{\text{rank}_E(\tau_i)}{\text{rank}(E)} = \sum_{i=1}^n (\tau_i - \tau_{i+1}) \sum_{j=1}^i \frac{\text{rank}_E(\tau_j)}{\text{rank}(E)}, \end{aligned}$$

where we set $\tau_{n+1} := 0$. Therefore we obtain

$$\mu(E) = \sum_{i=1}^{n-1} (\tau_i - \tau_{i+1}) \sum_{j=1}^i \frac{\text{rank}_E(\tau_j)}{\text{rank}(E)} + \tau_n.$$

The same formula holds for the slope of F . Since $\mu(E) > \mu(F)$,

$$\sum_{i=1}^{n-1} (\tau_i - \tau_{i+1}) \left(\sum_{j=1}^i \frac{\text{rank}_E(\tau_j)}{\text{rank}(E)} - \sum_{j=1}^i \frac{\text{rank}_F(\tau_j)}{\text{rank}(F)} \right) > 0.$$

We have $\tau_i > \tau_{i+1}$ for all i . It implies that there exists i such that

$$\sum_{j=1}^i \frac{\text{rank}_E(\tau_j)}{\text{rank}(E)} - \sum_{j=1}^i \frac{\text{rank}_F(\tau_j)}{\text{rank}(F)} > 0 \Leftrightarrow \frac{\text{rank}(E_{(\tau_i)})}{\text{rank}(E)} > \frac{\text{rank}(F_{(\tau_i)})}{\text{rank}(F)}.$$

□

Chapter 2

Bogomolov's inequality for vector bundles on $X \times \mathbb{P}^1$ in case of arbitrary characteristic

2.1 Numerical equivalence of divisors on ruled surfaces

Definition 2.1.1. *A geometrically ruled surface is a surface Y , together with a surjective morphism $\pi : Y \rightarrow X$ to a smooth curve X such that the fiber Y_x is isomorphic to \mathbb{P}^1 for every point $x \in X$ and π admits a section.*

One of the simplest examples of ruled surfaces which is our main subject is the ruled surface $X \times \mathbb{P}^1$ together with the first projection $p : X \times \mathbb{P}^1 \rightarrow X$, where X is a smooth projective curve.

Definition 2.1.2. *A divisor D on a surface Y is numerically equivalent to zero, written $D \equiv 0$, if the intersection number $D.E$ is zero for any divisor E . We say that D and E are numerically equivalent, written $D \equiv E$, if $D - E \equiv 0$.*

Let $\text{Num } Y$ be the quotient of $\text{Pic } Y$ by the subgroup of divisor classes numerically equivalent to zero. Then $\text{Num } Y$ is a finitely generated abelian group. The intersection pairing induces a nondegenerate bilinear pairing $\text{Num } Y \times \text{Num } Y \rightarrow \mathbb{Z}$.

Proposition 2.1.3 ([9, Prop. 2.3, p. 370]). *Let $\pi : Y \rightarrow X$ be a ruled surface, X_0 be a section and f be any fiber. Then*

$$\text{Pic } Y \cong \mathbb{Z} \oplus \pi^* \text{Pic } X$$

where \mathbb{Z} is generated by X_0 . Also

$$\text{Num } Y \cong \mathbb{Z} \oplus \mathbb{Z},$$

generated by X_0, f and satisfying $X_0.f = 1, f^2 = 0$.

On our ruled surface $X \times \mathbb{P}^1$, let $p : X \times \mathbb{P}^1 \rightarrow X$ and $q : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projections. We can choose X_0 to be any q -fiber F_q and f be any p -fiber F_p . It implies that for any divisor class $D \in \text{Num}(X \times \mathbb{P}^1)$, we have $D \equiv aF_p + bF_q$ for some $a, b \in \mathbb{Z}$ where $F_p^2 = F_q^2 = 0, F_p.F_q = 1$. In particular, we have

Proposition 2.1.4 ([9, Prop. 2.20, p. 382]). *A divisor $H \equiv aF_p + bF_q$ on $X \times \mathbb{P}^1$ is ample if and only if $a, b \in \mathbb{Z}_{>0}$.*

Let $H \equiv aF_p + bF_q$ be an ample divisor on $X \times \mathbb{P}^1$, $D \equiv cF_p + dF_q$. Then $H^2 = 2ab > 0$. If $D.H = 0$ then $ad + bc = 0$. It implies that

$$D^2 = 2cd \leq 0. \tag{2.1}$$

We have $D^2H^2 = 4abcd$, $(D.H)^2 = (ad + bc)^2$. Hence

$$(D.H)^2 - D^2H^2 = (ad - bc)^2 \geq 0 \tag{2.2}$$

for any divisor D .

2.2 Bogomolov's inequality

In this section, we will prove Bogomolov's inequality for torsion free coherent sheaves on the ruled surface $X \times \mathbb{P}^1$ in case of arbitrary characteristic. Firstly, we recall the Grothendieck's splitting theorem for vector bundles on \mathbb{P}^1 .

Theorem 2.2.1 (Grothendieck's splitting theorem - [12, Thm. 1.3.1]). *Let E be a vector bundle of rank r on \mathbb{P}^1 . Then there exists a unique r -tuple $(a_1, a_2, \dots, a_r) \in \mathbb{Z}^r$ with $a_1 \leq a_2 \leq \dots \leq a_r$, such that*

$$E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Recall that $p : X \times \mathbb{P}^1 \rightarrow X$ and $q : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ are the projections.

Lemma 2.2.2. *Let E be a vector bundle of rank r on $X \times \mathbb{P}^1$. Suppose that for any p -fiber F_p we have $E|_{F_p} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. Then there exists a vector bundle F on X such that $E \cong p^*F$.*

Proof. See [18, Lm. 4.1]. □

Proposition 2.2.3. *Let E be a vector bundle on $X \times \mathbb{P}^1$. Suppose that*

$$R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

Then there exist vector bundles E_0 and E_1 on X such that the following sequence of vector bundles on $X \times \mathbb{P}^1$

$$0 \longrightarrow p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^*E_0 \longrightarrow E \longrightarrow 0 \quad (2.3)$$

is exact.

Proof. Let $x \in X$ be any closed point and E be a vector bundle on $X \times \mathbb{P}^1$ such that $R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. Then E_x , the restriction of E on the fiber of p over x , is a vector bundle on $F_p = x \times \mathbb{P}^1 \cong \mathbb{P}^1$. We have

$$R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes k(x) \cong H^1(\mathbb{P}^1, E_x(-1)).$$

So the condition $R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ implies that

$$H^1(\mathbb{P}^1, E_x(-1)) = 0$$

for all $x \in X$. On $\mathbb{P}^1 \times \mathbb{P}^1$ we have the following resolution of diagonal

$$0 \longrightarrow p_1^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_{\Delta_{\mathbb{P}^1}} \longrightarrow 0, \quad (2.4)$$

where p_1, p_2 are the first and second projection from $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the following commutative diagram

$$\begin{array}{ccccc} & & X \times \mathbb{P}^1 & \xrightarrow{q} & \mathbb{P}^1 \\ & \nearrow p & \uparrow p_{13} & & \uparrow p_2 \\ X & & X \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{p_{23}} & \mathbb{P}^1 \times \mathbb{P}^1 \\ & \searrow p & \downarrow p_{12} & & \downarrow p_1 \\ & & X \times \mathbb{P}^1 & \xrightarrow{q} & \mathbb{P}^1, \end{array}$$

where p_{ij} are the projections. Pulling back the exact sequence (2.4) to $X \times \mathbb{P}^1 \times \mathbb{P}^1$ by p_{23} and using the commutativity of the above diagram, we get a short exact sequence on $X \times \mathbb{P}^1 \times \mathbb{P}^1$,

$$0 \longrightarrow (p_{12})^*q^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes (p_{13})^*q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{X \times \mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_{X \times \Delta_{\mathbb{P}^1}} \longrightarrow 0.$$

Tensoring this exact sequence with $(p_{12})^*E$ and then applying the functor $(p_{13})_*$, we have a long exact sequence

$$\begin{aligned} 0 &\longrightarrow (p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow (p_{13})_*(p_{12})^*E \\ &\longrightarrow (p_{13})_*((p_{12})^*E|_{X \times \Delta_{\mathbb{P}^1}}) \longrightarrow R^1(p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1). \end{aligned} \quad (2.5)$$

Consider the following commutative diagram,

$$\begin{array}{ccc} X \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{p_{13}} & X \times \mathbb{P}^1 \\ p_{12} \downarrow & & \downarrow p \\ X \times \mathbb{P}^1 & \xrightarrow{p} & X. \end{array}$$

Since all the maps in this diagram are flat, we have

$$R^1(p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \cong p^*R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

It implies that the sequence

$$\begin{aligned} 0 &\longrightarrow (p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \\ &\longrightarrow (p_{13})_*(p_{12})^*E \longrightarrow (p_{13})_*((p_{12})^*E|_{X \times \Delta_{\mathbb{P}^1}}) \longrightarrow 0 \end{aligned}$$

is exact.

We have $(p_{13})_*((p_{12})^*E|_{X \times \Delta_{\mathbb{P}^1}}) \cong E$. To complete the proof, we need to show that

$$(p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \cong p^*E_0$$

and

$$(p_{13})_*(p_{12})^*E \cong p^*E_1$$

for some vector bundles E_0, E_1 on X . By Lemma 2.2.2, it is clear because

$$(p_{13})_*(p_{12}^*E)|_{F_p} \cong \mathcal{O}_{\mathbb{P}^1} \otimes H^0(\mathbb{P}^1, E|_{F_p})$$

and

$$(p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))]|_{F_p} \cong \mathcal{O}_{\mathbb{P}^1} \otimes H^0(\mathbb{P}^1, E|_{F_p} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$$

for any F_p be any p -fiber. □

Let H be an ample divisor on $X \times \mathbb{P}^1$ and E be a vector bundle of rank r . We define the H -degree and H -slope of E by setting

$$\deg_H(E) = c_1(E).H \text{ and } \mu_H(E) = \frac{\deg_H(E)}{r}. \quad (2.6)$$

Then E is called H -(semi)stable if for any nonzero subsheaf $F \subseteq E$ we have $\mu_H(F) (\leq) \mu_H(E)$.

Theorem 2.2.4 (Bogomolov's inequality). *Let E be a vector bundle of rank r and H be an ample divisor on $X \times \mathbb{P}^1$. If E is H -semistable then*

$$\Delta(E) = (r - 1)c_1^2(E) - 2rc_2(E) \leq 0,$$

where $c_1(E), c_2(E)$ are the first and second Chern classes of E .

Proof. Suppose that E is H -semistable but $\Delta(E) > 0$. Then we will construct a subsheaf $E' \subset E$ which destabilizes E .

I - **Construction of E' .** By Serre's vanishing theorem we can show that for any vector bundle E on $X \times \mathbb{P}^1$, there exists a line bundle L such that $R^1 p_*(E \otimes L) = 0$. Since the semistability of E is unchanged when we tensor E with L and $\Delta(E) = \Delta(E \otimes L)$, by replacing E with $E \otimes L$ if necessary, we can assume that $R^1 p_*(E \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. It follows from Proposition 2.2.3 that E has a resolution

$$0 \longrightarrow p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^* E_0 \longrightarrow E \longrightarrow 0, \quad (2.7)$$

where E_1, E_0 are vector bundles on X . Assume that E_i has rank r_i and degree d_i for $i = 1, 2$. By computing Chern characters up to numerical equivalence, we have

$$\text{ch}(p^* E_0) \equiv r_0 + d_0 F_p, \text{ch}(p^* E_1) \equiv r_1 + d_1 F_p, \text{ch}(q^* \mathcal{O}_{\mathbb{P}^1}(-1)) \equiv 1 - F_q,$$

$$\text{ch}(p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1)) \equiv \text{ch}(p^* E_1) \text{ch}(q^* \mathcal{O}_{\mathbb{P}^1}(-1)) \equiv r_1 + d_1 F_p - r_1 F_q - d_1 [pt],$$

$$\text{ch}(E) \equiv \text{ch}(p^* E_0) - \text{ch}(p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1)) \equiv r_0 - r_1 + (d_0 - d_1) F_p + r_1 F_q + d_1 [pt].$$

Therefore

$$c_1(E) = \text{ch}_1(E) \equiv (d_0 - d_1) F_p + r_1 F_q,$$

$$c_2(E) = \frac{c_1^2(E) - 2 \text{ch}_2(E)}{2} \equiv (d_0 r_1 - d_1 r_1 - d_1) [pt].$$

It follows that

$$\Delta(E) = 2(r_0 d_1 - r_1 d_0) = 2r_0 r_1 (\mu(E_1) - \mu(E_0)).$$

Assume that $\Delta(E) > 0$, then $\mu(E_1) > \mu(E_0)$. Lemma 1.2.6 allows us to choose $\tau \in \mathbb{Q}$ such that

$$\frac{\text{rank}(E_{1(\tau)})}{r_1} > \frac{\text{rank}(E_{0(\tau)})}{r_0}.$$

We have $p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{F_q} \subset p^* E_0|_{F_q}$ for any q -fiber $F_q \cong X$. Then $E_1 \subset E_0$ and hence $E_{1(\tau)} \subset E_{0(\tau)}$ by Corollary 1.2.5. Particularly, $E_{0(\tau)}$ and $E_{1(\tau)}$ are nontrivial. So

$$\frac{\text{rank}(E_{1(\tau)})}{r_1} > \frac{\text{rank}(E_{0(\tau)})}{r_0} \iff \frac{\text{rank}(E_{1(\tau)})}{\text{rank}(E_{0(\tau)})} > \frac{r_1}{r_0}.$$

We choose such τ that $\frac{\text{rank}(E_{1(\tau)})}{\text{rank}(E_{0(\tau)})}$ becomes maximal. Assume that $\text{rank}(E_{1(\tau)}) = r_{1\tau}$ and $\text{rank}(E_{0(\tau)}) = r_{0\tau}$. If $\mu(E_{1(\tau)}) > \mu(E_{0(\tau)})$, it follows from Lemma 1.2.6 that there exists $\rho \in \mathbb{Q}$ such that

$$\frac{\text{rank}((E_{1(\tau)})_{(\rho)})}{r_{1\tau}} > \frac{\text{rank}((E_{0(\tau)})_{(\rho)})}{r_{0\tau}}$$

where $(E_{1(\tau)})_{(\rho)}$ and $(E_{0(\tau)})_{(\rho)}$ are both nontrivial. It follows that

$$\frac{\text{rank}((E_{1(\tau)})_{(\rho)})}{\text{rank}((E_{0(\tau)})_{(\rho)})} > \frac{r_{1\tau}}{r_{0\tau}} > \frac{r_1}{r_0}.$$

This contradicts the choice of τ . Hence

$$\mu(E_{1(\tau)}) \leq \mu(E_{0(\tau)}). \quad (2.8)$$

Since $E_{1(\tau)}$ is subbundle of $E_{0(\tau)}$, the image of the composition

$$p^*E_{1(\tau)} \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow p^*E_0$$

is contained in $p^*E_{0(\tau)}$. We obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*E_{1(\tau)} \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & p^*E_{0(\tau)} & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi & & \\ 0 & \longrightarrow & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & p^*E_0 & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

Let $E' = \text{Im}(\varphi)$ and $K = \text{Ker}(\varphi)$. We have $\mathcal{O}_{\mathbb{P}^1}^{r_0 r} \cong p^*E_{0(\tau)}|_{F_p} \twoheadrightarrow E''|_{F_p}$, equivalently, $E''|_{F_p}$ is globally generated. Since $K \subset p^*(E_1/E_{1(\tau)}) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)$, $H^0(\mathbb{P}^1, K|_{F_p}) = 0$. Therefore $K \neq E''$ and then $E' \neq 0$.

II - E' **destabilizes** E . It follows from Proposition 2.1.4 that any ample divisor H on $X \times \mathbb{P}^1$ is in form $aF_p + bF_q$ for some $a, b \in \mathbb{Z}_{>0}$. Suppose that

$$c_1(E') \equiv m'F_p + n'F_q, c_1(E) = mF_p + nF_q,$$

then $\mu_H(E') = \frac{c_1(E') \cdot H}{r'} = \frac{m'b + n'a}{r'}$ and $\mu_H(E) = \frac{mb + na}{r}$. We have

$$\deg(E'|_{F_p}) = n', \deg(E'|_{F_q}) = m'.$$

If $\mu(E'|_{F_p}) > \mu(E|_{F_p})$ and $\mu(E'|_{F_q}) > \mu(E|_{F_q})$, then $m'/r' > m/r$ and $n'/r' > n/r$. It follows that

$$\mu_H(E') = \frac{an' + bm'}{r'} > \frac{am + bn}{r} = \mu_H(E).$$

Therefore, to show that E' destabilizes E , it is suffice to show that $E'|_{F_p}$ destabilizes $E|_{F_p}$ and $E'|_{F_q}$ destabilizes $E|_{F_q}$ for any F_p and F_q .

+ $\mu(E'|_{F_p}) > \mu(E|_{F_p})$: Let F_p be any p -fiber. Restricting the resolution (2.7) of E to F_p , we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{r_1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{r_0} \longrightarrow E|_{F_p} \longrightarrow 0.$$

It implies that

$$\text{rank}(E|_{F_p}) = r_0 - r_1, \deg(E|_{F_p}) = r_1.$$

Similarly,

$$\text{rank}(E''|_{F_p}) = r_{0\tau} - r_{1\tau}, \deg(E''|_{F_p}) = r_{1\tau}.$$

As we have seen, $\frac{r_{1\tau}}{r_{0\tau}} > \frac{r_1}{r_0}$, so

$$\mu(E''|_{F_p}) = \frac{r_{1\tau}}{r_{0\tau} - r_{1\tau}} > \mu(E|_{F_p}) = \frac{r_1}{r_0 - r_1}.$$

Restricting the following exact sequence

$$0 \longrightarrow K \longrightarrow E'' \longrightarrow E' \longrightarrow 0 \quad (2.9)$$

to F_p , we get an exact sequence

$$0 \longrightarrow K|_{F_p} \longrightarrow E''|_{F_p} \longrightarrow E'|_{F_p} \longrightarrow 0. \quad (2.10)$$

We have $\mu(E''|_{F_p}) = \frac{r_{1\tau}}{r_{0\tau} - r_{1\tau}} > 0$ by the choice of τ . Since

$$K \subset p^*(E_1/E_{1(\tau)}) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)$$

then $K|_{F_p} \subset \mathcal{O}_{\mathbb{P}^1}^{r_1 - r_{1\tau}}$. It follows from Theorem 2.2.1 that

$$\mu(K|_{F_p}) \leq -1 < \mu(E''|_{F_p}).$$

From the exactness of (2.10),

$$\mu(K|_{F_p}) < \mu(E''|_{F_p}) \implies \mu(E'|_{F_p}) > \mu(E''|_{F_p}) > \mu(E|_{F_p}).$$

+ $\mu(E'|_{F_q}) > \mu(E|_{F_q})$: Let F_q be any q -fiber. Restricting (2.9) to F_q , we have an exact sequence

$$0 \longrightarrow K|_{F_q} \longrightarrow E''|_{F_q} \longrightarrow E'|_{F_q} \longrightarrow 0 \quad (2.11)$$

where $K|_{F_q} \subseteq E_1/E_{1(\tau)}$. By (iv) of Proposition 1.2.4, $\mu_{\max}(E_1/E_{1(\tau)}) < \tau$, hence

$$\mu(K|_{F_q}) \leq \mu_{\max}(E_1/E_{1(\tau)}) < \tau.$$

Consider the following exact sequence

$$0 \longrightarrow E_{1(\tau)} \longrightarrow E_{0(\tau)} \longrightarrow E''|_{F_q} \longrightarrow 0.$$

Since $\mu(E_{0(\tau)}) \geq \mu(E_{1(\tau)})$ (see (2.8)),

$$\mu(E''|_{F_q}) \geq \mu(E_{0(\tau)}) \geq \tau.$$

Hence $\mu(K|_{F_q}) < \mu(E''|_{F_q})$. It follows from the sequence (2.11) that

$$\mu(E'|_{F_q}) > \mu(E''|_{F_q}).$$

Similarly, since $\mu(E_1) > \mu(E_0)$ and the sequence

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E|_{F_q} \longrightarrow 0$$

is exact, we have

$$\mu(E_0) > \mu(E|_{F_q}).$$

It is clear by definition that $\mu(E_{0(\tau)}) \geq \mu(E_0)$. So we have

$$\mu(E'|_{F_q}) > \mu(E''|_{F_q}) \geq \mu(E_{0(\tau)}) \geq \mu(E_0) > \mu(E|_{F_q}).$$

□

Remark 2.2.5. Let E be a torsion free coherent sheaf on $X \times \mathbb{P}^1$ and $E^\vee = \text{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X)$ be the dual sheaf of E . Then $E^{\vee\vee} = (E^\vee)^\vee$ is a vector bundle of the same rank and $T := E^{\vee\vee}/E$ is a sheaf of rank zero. Consider the following exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow T \longrightarrow 0$$

It is easy to see that $\Delta(E) = \Delta(E^{\vee\vee}) - 2 \text{rank}(E) \text{length}(T) \leq \Delta(E^{\vee\vee})$. Moreover, we have known that E is H -semistable if and only if $E^{\vee\vee}$ is H -semistable. Therefore, Bogomolov's inequality remains true for torsion free coherent sheaves.

2.3 Restriction theorems

Using Bogomolov's inequality, we can consider the semistability of vector bundles on $X \times \mathbb{P}^1$ restricted to a smooth curve.

Lemma 2.3.1 (cf. [13, Thm. 5.1]). *Let $0 = E_0 \subset E_1 \subset \dots \subset E_m = E$ be the Harder-Narasimhan filtration of E . Set $F_i = E_i/E_{i-1}$, $r_i = \text{rank}(F_i)$, $\mu_i = \mu(F_i)$. Then we have*

$$\frac{\Delta(E)}{r} = \sum_{i=1}^m \frac{\Delta(F_i)}{r_i} + \frac{1}{r} \sum_{i < j} r_i r_j \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right)^2. \quad (2.12)$$

Proof. We have

$$\begin{aligned} \Delta(E) &= (r-1)c_1^2(E) - 2rc_2(E) \\ &= r(c_1^2(E) - 2c_2(E)) - c_1^2(E) \\ &= 2 \text{ch}_0(E) \text{ch}_2(E) - \text{ch}_1^2(E), \end{aligned}$$

where $\text{ch}_i(E)$ is the i^{th} term in the expression of the Chern character of E :

$$\text{ch}(E) = r + c_1(E) + \frac{(c_1^2(E) - 2c_2(E))}{2}.$$

Since Chern character is additive with respect to an exact sequence, we have

$$\frac{\Delta(E)}{r} = 2 \text{ch}_2(E) - \frac{c_1^2(E)}{r} = 2 \sum_{i=1}^m \text{ch}_2(F_i) - \frac{(\sum_{i=1}^m c_1(F_i))^2}{r},$$

$$\sum_{i=1}^m \frac{\Delta(F_i)}{r_i} = 2 \sum_{i=1}^m \text{ch}_2(F_i) - \sum_{i=1}^m \frac{c_1^2(F_i)}{r_i}.$$

Therefore the quality (2.12) is equivalent to

$$\sum_{i=1}^m \frac{c_1^2(F_i)}{r_i} = \frac{1}{r} \left[\left(\sum_{i=1}^m c_1(F_i) \right)^2 + \sum_{i < j} r_i r_j \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right)^2 \right]. \quad (2.13)$$

But it is clear because

$$\begin{aligned} \left(\sum_{i=1}^m c_1(F_i) \right)^2 + \sum_{i < j} r_i r_j \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right)^2 &= \sum_{i=1}^m c_1^2(F_i) + \sum_{i < j} \left(\frac{r_j c_1^2(F_i)}{r_i} + \frac{r_i c_1^2(F_j)}{r_j} \right) \\ &= \sum_{j=1}^m c_1^2(F_j) + \sum_{j=1}^m \frac{(\sum_{i \neq j} r_i) c_1^2(F_j)}{r_j} \\ &= r \sum_{j=1}^m \frac{c_1^2(F_j)}{r_j}. \end{aligned}$$

□

Proposition 2.3.2 (cf. [13, Thm. 5.1]). *Let E be any rank r torsion free sheaf on $X \times \mathbb{P}^1$ and H be an ample divisor. Then we have*

$$H^2 \cdot \Delta(E) \leq r^2 (\mu_{\max} - \mu) (\mu - \mu_{\min}).$$

Proof. Let $0 = E_0 \subset E_1 \subset \dots \subset E_m = E$ be the Harder-Narasimhan filtration of E . Set $F_i = E_i/E_{i-1}$, $r_i = \text{rank}(F_i)$, $\mu_i = \mu(F_i)$. We have

$$\frac{\Delta(E)}{r} = \sum \frac{\Delta(F_i)}{r_i} + \frac{1}{r} \sum_{i < j} r_i r_j \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right)^2 \leq \frac{1}{r} \sum_{i < j} r_i r_j \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right)^2$$

since $\Delta(F_i) \leq 0$ for all i . It follows from (2.2) that

$$H^2 \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right)^2 \leq \left(\left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right) H \right)^2.$$

Since H is ample, $H^2 > 0$. We have

$$\begin{aligned} \frac{\Delta(E)}{r} &\leq \frac{1}{r H^2} \sum_{i < j} r_i r_j \left(\left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right) H \right)^2 \\ &= \frac{1}{r H^2} \sum_{i < j} r_i r_j (\mu_i - \mu_j)^2. \end{aligned}$$

By the definition of the Harder-Narasimhan filtration, we have

$$\mu_{\max} = \mu_1 > \mu_2 > \dots > \mu_m = \mu_{\min},$$

$$r = \sum_{i=1}^m r_i,$$

$$r\mu(E) = \sum_{i=1}^m r_i\mu_i.$$

Moreover

$$\begin{aligned} \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 &= r \left(\sum_{i=1}^{m-1} \left(\sum_{j \leq i} r_j (\mu_j - \mu) \right) (\mu_i - \mu_{i+1}) \right) \\ &\leq r \left(\sum_{i=1}^{m-1} \left(\sum_{j \leq i} r_j (\mu_1 - \mu) \right) (\mu_i - \mu_{i+1}) \right) \\ &= r(\mu_1 - \mu) \left(\sum_{i=1}^{m-1} \left(\sum_{j \leq i} r_j \right) (\mu_i - \mu_{i+1}) \right) \\ &= r(\mu_{\max} - \mu) \left(\sum_{i=1}^m r_i \mu_i - r\mu_m \right) \\ &= r^2(\mu_{\max} - \mu)(\mu - \mu_{\min}). \end{aligned}$$

So we have

$$\frac{\Delta(E)}{r} \leq \frac{1}{rH^2} \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq \frac{r^2(\mu_{\max} - \mu)(\mu - \mu_{\min})}{rH^2}.$$

It implies that

$$H^2\Delta(E) \leq r^2(\mu_{\max} - \mu)(\mu - \mu_{\min}).$$

□

Theorem 2.3.3 ([13, Thm. 5.2]). *Let E be a vector bundle on $X \times \mathbb{P}^1$ of rank $r \geq 2$. Assume that E is H -stable. Let $C \in |mH|$ be a smooth curve. If $m \geq \frac{1-r}{r}\Delta(E) + 1$ then $E|_C$ is stable.*

Proof. Assume that $E|_C$ is not stable. Then there exists a quotient Q of $E|_C$ which is a vector bundle of rank R and degree d on C such that $\mu(E|_C) \geq \mu(Q)$. Let E' be the kernel of the composition $E \rightarrow E|_C \rightarrow Q$ and consider the following exact sequence

$$0 \rightarrow E' \rightarrow E \rightarrow Q \rightarrow 0,$$

where Q is considered as a sheaf of rank zero on $X \times \mathbb{P}^1$. By computing with Chern character, we have

$$\text{ch}(Q) \equiv RC + \left(d - \frac{RC^2}{2}\right)[pt].$$

Therefore,

$$\text{ch}(E') = \text{ch}(E) - \text{ch}(Q) \equiv r + (c_1(E) - RmH) + \left(\frac{\Delta(E) + c_1^2(E)}{2r} - d + \frac{Rm^2H^2}{2}\right)[pt].$$

It follows that

$$\begin{aligned}\Delta(E') &= 2 \operatorname{ch}_0(E') \operatorname{ch}_2(E') - \operatorname{ch}_1^2(E') \\ &= \Delta(E) + 2rR(\mu(E|_C) - \mu(Q)) + m^2H^2R(r - R) \\ &\geq \Delta(E) + m^2H^2R(r - R).\end{aligned}$$

If E' is semistable, $\Delta(E') \leq 0$ by Bogomolov's inequality. So we have

$$-\Delta(E) \geq m^2H^2R(r - R) \geq (r - 1)m^2H^2$$

since $1 \leq R \leq r - 1$. It implies then

$$\frac{1 - r}{r} \Delta(E) + 1 \geq \frac{(r - 1)^2}{r} m^2H^2 + 1.$$

But it is a contradiction since $r \geq 2$, $H^2 \geq 1$ and $m \geq \frac{1-r}{r} \Delta(E) + 1$. So E' can not be semistable. In that case, we have

$$\mu_{\max}(E') - \mu(E') = \mu_{\max}(E') - \mu(E) + \frac{R}{r}mH^2 = \frac{R}{r}mH^2 - \frac{dr' - d'r}{rr'},$$

where r', d' are the rank and degree of the maximal destabilizing subbundle of E' . Since E is stable, $dr' - d'r \geq 1$. Moreover $1 \leq r' \leq r - 1$, then

$$\mu_{\max}(E') - \mu(E') \leq \frac{R}{r}mH^2 - \frac{1}{r(r - 1)}.$$

Consider the following diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & E' & \longrightarrow & S & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E(-C) & \longrightarrow & E & \longrightarrow & E|_C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & Q & \xrightarrow{\operatorname{Id}} & Q & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where S is the kernel of $E|_C \rightarrow Q$ on C and the rows are exact. By the kernel-cokernel exact sequence,

$$0 \longrightarrow E(-C) \longrightarrow E' \longrightarrow S \longrightarrow 0$$

is an exact sequence where S is a sheaf considered on $X \times \mathbb{P}^1$. In particular $E'^{\vee} \subset E(-C)^{\vee}$ and $E(-C)^{\vee}$ are still stable. Therefore

$$\mu(E') - \mu_{\min}(E') = \mu(E(-C)) + \frac{r - R}{r}mH^2 - \mu_{\min}(E')$$

$$\begin{aligned}
 &= \mu_{\max}(E'^{\vee}) - \mu(E(-C)^{\vee}) + \frac{r-R}{r}mH^2 \\
 &\leq \frac{r-R}{r}mH^2 - \frac{1}{r(r-1)}.
 \end{aligned}$$

Now we apply Proposition 2.3.2 for E' and obtain

$$\begin{aligned}
 0 &\geq H^2\Delta(E') - r^2(\mu_{\max}(E') - \mu(E'))(\mu(E') - \mu_{\min}(E')) \\
 &\geq H^2\Delta(E) + m^2H^4R(r-R) - r^2\left(\frac{R}{r}mH^2 - \frac{1}{r(r-1)}\right)\left(\frac{r-R}{r}mH^2 - \frac{1}{r(r-1)}\right) \\
 &= H^2\Delta(E) + \frac{rH^2}{r-1}m + \frac{1}{(r-1)^2}.
 \end{aligned}$$

Therefore

$$\frac{rH^2}{1-r}m \geq H^2\Delta(E) + \frac{1}{(r-1)^2} \iff m \leq \frac{1-r}{r}\Delta(E) - \frac{1}{r(r-1)H^2}.$$

This contradicts the assumption that $m \geq \frac{1-r}{r}\Delta(E) + 1$. \square

Corollary 2.3.4 (cf. [13, Cor. 5.4]). *Let E be a vector bundle of rank $r \geq 2$ on $X \times \mathbb{P}^1$ and $C \in |mH|$ be a general smooth projective curve. Assume that E is H -semistable. If*

$$m \geq \frac{1-r}{r}\Delta(E) + 1$$

then $E|_C$ is semistable.

Proof. Assume that we have a short exact sequence of vector bundles on $X \times \mathbb{P}^1$

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0,$$

where $\mu_H(E_i) = \mu_H(E)$. Let $C \in |mH|$ be a general smooth projective curve. By restricting to C , the sequence

$$0 \longrightarrow E_1|_C \longrightarrow E|_C \longrightarrow E_2|_C \longrightarrow 0$$

is still exact and $\mu(E_1|_C) = \mu(E_2|_C) = \mu(E|_C)$. It is easy to see that $E|_C$ is semistable if $E_i|_C$ is semistable for $i = 1, 2$. By induction on the length of a Jordan-Hölder filtration which depends only on E itself, it is enough to assume that E has a Jordan-Hölder filtration of length 2,

$$0 = E_0 \subset E_1 \subset E.$$

Set $E_2 = E/E_1$ and consider the exact sequence

$$0 \longrightarrow E_1|_C \longrightarrow E|_C \longrightarrow E_2|_C \longrightarrow 0$$

where $\mu(E_1|_C) = \mu(E_2|_C) = \mu(E|_C)$. By Theorem 2.3.3, $E_i|_C$ are semistable if

$$m \geq \frac{1-r_i}{r_i}\Delta(E_i) + 1.$$

As in (2.12), we have

$$\frac{\Delta(E)}{r} = \frac{\Delta(E_1)}{r_1} + \frac{\Delta(E_2)}{r_2} + \frac{r_1 r_2}{r} \left(\frac{c_1(E_1)}{r_1} - \frac{c_1(E_2)}{r_2} \right)^2.$$

It follows that

$$\begin{aligned} \frac{1-r}{r} \Delta(E) + 1 &= \frac{1-r_1}{r_1} \Delta(E_1) + 1 + \frac{1-r_2}{r_2} \Delta(E_2) \\ &\quad + \left[(1-r) \frac{r_1 r_2}{r} \left(\frac{c_1(E_1)}{r_1} - \frac{c_1(E_2)}{r_2} \right)^2 - \frac{r_2}{r_1} \Delta(E_1) - \frac{r_1}{r_2} \Delta(E_2) \right] \end{aligned}$$

We have $\Delta(E_i) \leq 0$ since E_i are semistable. Moreover $\mu(E_2) = \mu(E_1)$. It follows from (2.1) that

$$\left(\frac{c_1(E_1)}{r_1} - \frac{c_1(E_2)}{r_2} \right)^2 \leq 0.$$

So we have

$$\begin{aligned} \frac{1-r}{r} \Delta(E) + 1 &\geq \frac{1-r_1}{r_1} \Delta(E_1) + 1 + \frac{1-r_2}{r_2} \Delta(E_2) \\ &\geq \max \left\{ \frac{1-r_1}{r_1} \Delta(E_1) + 1, \frac{1-r_2}{r_2} \Delta(E_2) + 1 \right\}. \end{aligned}$$

Hence for any

$$m \geq \frac{1-r}{r} \Delta(E) + 1,$$

$E_i|_C$ is semistable. Therefore, $E|_C$ is semistable.

□

Chapter 3

Triples and their moduli spaces

3.1 Triples and their stability

Let X be a smooth projective curve over an algebraically closed field k . We consider the first notions of triples and their stability.

Definition 3.1.1. *A triple T on X consists of two vector bundles E_1, E_2 on X , together with a morphism $\varphi : E_2 \rightarrow E_1$ and it is denoted by $T = (E_2 \xrightarrow{\varphi} E_1)$ or simply $T = (E_2, E_1, \varphi)$.*

Let $T = (E_2, E_1, \varphi)$ and $T' = (E'_2, E'_1, \varphi')$ be triples on X . Then a morphism of triples, $f : T \rightarrow T'$, is a pair (f_2, f_1) of morphisms such that the following diagram commutes

$$\begin{array}{ccc} E_2 & \xrightarrow{\varphi} & E_1 \\ \downarrow f_2 & & \downarrow f_1 \\ E'_2 & \xrightarrow{\varphi'} & E'_1. \end{array} \quad (3.1)$$

In particular, f is an isomorphism if f_1 and f_2 are both isomorphisms.

If $\text{rank}(E_i) = r_i$ and $\text{deg}(E_i) = d_i$ for $i = 1, 2$, then we say that T is of type (r_1, r_2, d_1, d_2) .

To introduce the notion of (semi)stability for triples, we first precise the subtriples of a given one and then define the slope which depends on a real parameter.

Definition 3.1.2. *A triple $T' = (E'_2, E'_1, \varphi')$ is called a subtriple of T if E'_i is a subbundle of E_i for $i = 1, 2$ and the commutative diagram (3.1) holds for $f = (j_2, j_1)$ where $j_i : E'_i \hookrightarrow E_i$ are the inclusions. If $E'_1 = E'_2 = 0$ then T' is called trivial subtriple. T' is called proper subtriple if it is nontrivial and $T' \neq T$.*

Example 3.1.3. Let $T = (L, E, \varphi)$ be a triple where L is a line bundle on X . Then any subtriple T' of T is either $T' = (0, E', 0)$ where $E' \subset E$ is any subbundle or $T' = (L, E', \varphi)$ where E' is a subbundle of E such that $\varphi(L) \subset E'$.

Definition 3.1.4. Let $T = (E_2, E_1, \varphi)$ be a triple of type (r_1, r_2, d_1, d_2) . For any real number $\alpha \in \mathbb{R}$, we define

$$\deg_\alpha(T) = d_1 + d_2 - 2\alpha r_1$$

and

$$\mu_\alpha(T) = \frac{\deg_\alpha(T)}{r_1 + r_2} = \frac{d_1 + d_2 - 2\alpha r_1}{r_1 + r_2}. \quad (3.2)$$

The triple T is called α -(semi)stable if for all nontrivial subtriples T' of T , $\mu_\alpha(T')(\leq)\mu_\alpha(T)$.

Example 3.1.5. We consider the simplest triples $T = (0, E, 0)$ or $T = (E, 0, 0)$. Then

$$\mu_\alpha(T')(\leq)\mu_\alpha(T) \iff \mu(E')(\leq)\mu(E)$$

where $E' \subsetneq E$ and $T' = (0, E', 0)$ or $T' = (E', 0, 0)$. Hence the α -(semi)stability of T is in fact equivalent to the (semi)stability of E for any α . From now on, we just consider triples T with E_1 and E_2 are both nontrivial.

Remark 3.1.6. The above definition of α -degree and α -slope slightly differ from the definitions given in [2],[4], etc, where the authors defined $\deg_\alpha(T) = d_1 + d_2 + \alpha r_2$. This modification does not change the (semi)stability of triples and is useful when we connect the semistability of extension bundles on $X \times \mathbb{P}^1$ defined by T with the semistability of T itself.

Lemma 3.1.7. For a short exact sequence of triples on X ,

$$\begin{array}{ccccccc} 0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0 \\ & & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' & & \\ 0 & \longrightarrow & E'_1 & \longrightarrow & E_1 & \longrightarrow & E''_1 & \longrightarrow & 0, \end{array} \quad (3.3)$$

we have

$$\mu_\alpha((E'_2, E'_1, \varphi'))(\leq)\mu_\alpha((E_2, E_1, \varphi)) \iff \mu_\alpha((E_2, E_1, \varphi))(\leq)\mu_\alpha((E''_2, E''_1, \varphi'')).$$

Proof. Since the rows are exact, we have

$$d_i = d'_i + d''_i, r_i = r'_i + r''_i$$

for $i = 1, 2$. It follows that

$$\mu_\alpha((E_2, E_1, \varphi)) = \frac{\deg_\alpha(E_2, E_1, \varphi)}{r_1 + r_2} = \frac{\deg_\alpha(E'_2, E'_1, \varphi') + \deg_\alpha(E''_2, E''_1, \varphi'')}{(r'_1 + r'_2) + (r''_1 + r''_2)}.$$

Hence

$$\frac{\deg_\alpha(E_2, E_1, \varphi)}{r_1 + r_2}(\geq)\frac{\deg_\alpha(E'_2, E'_1, \varphi')}{r'_1 + r'_2} \iff \frac{\deg_\alpha(E_2, E_1, \varphi)}{r_1 + r_2}(\leq)\frac{\deg_\alpha(E''_2, E''_1, \varphi'')}{r''_1 + r''_2}.$$

□

Let $T = (E_2, E_1, \varphi)$ be a triple and L be a line bundle on X . We define the triple $T \otimes L$ to be the triple $(E_2 \otimes L, E_1 \otimes L, \varphi \otimes \text{Id}_L)$.

Lemma 3.1.8. *Let $T = (E_2, E_1, \varphi)$ be a triple on X and L a line bundle. Then T is α -semistable if and only if $T \otimes L$ is α -semistable.*

Proof. We have

$$\mu_\alpha(T \otimes L) = \mu_\alpha(T) + \deg(L).$$

Moreover, any subtriple of $T \otimes L = (E_2 \otimes L, E_1 \otimes L, \varphi \otimes \text{Id}_L)$ is induced from a subtriple of T by tensoring with L . Hence T is α -semistable if and only if $T \otimes L$ is. \square

It is well known that any stable vector bundle E is simple, i.e. $\text{End } E \cong k$. For stable triples, an analogous result holds true.

Definition 3.1.9. *Let $T = (E_2, E_1, \varphi)$ be a triple. Let us set*

$$\text{End}(T) = \{(u, v) \in \text{End}(E_2) \oplus \text{End}(E_1) \mid \varphi u = v \varphi\}.$$

We say that T is simple if $\text{End}(T) \cong k$, i.e. the only elements in $\text{End}(T)$ are of the form $(\lambda \text{Id}_{E_2}, \lambda \text{Id}_{E_1})$ for some $\lambda \in k$.

Proposition 3.1.10 ([2, Prop. 3.10]). *Let $T = (E_2, E_1, \varphi)$ be a α -stable triple and $(u, v) \in \text{End}(T)$. Then either (u, v) is trivial or both u, v are isomorphisms.*

Proof. Suppose that u and v are both neither nontrivial nor isomorphism. Then

$$T' = (\text{Ker } u, \text{Ker } v, \varphi) \text{ and } T'' = (\text{Im } u, \text{Im } v, \varphi)$$

are nontrivial subtriples of T . T is α -stable, then

$$\mu_\alpha(T') < \mu_\alpha(T) \text{ and } \mu_\alpha(T'') < \mu_\alpha(T). \quad (3.4)$$

We have an exact sequence of triples

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0.$$

It follows from lemma 3.1.7 that $\mu_\alpha(T'') > \mu_\alpha(T)$ if $\mu_\alpha(T) < \mu_\alpha(T')$. This contradicts (3.4). \square

Corollary 3.1.11 ([2, Cor. 3.12]). *If T is α -stable, then it is simple.*

For any triple $T = (E_2, E_1, \varphi)$ there is always a dual triple $T^\vee = (E_1^\vee, E_2^\vee, \varphi^\vee)$, where φ^\vee is the transpose of φ , i.e. the image of φ via the canonical isomorphism

$$\text{Hom}(E_2, E_1) \cong \text{Hom}(E_1^\vee, E_2^\vee).$$

From the one-to-one correspondence between subbundles of a given bundle E and subbundles of E^\vee , it follows that there is also a one-to-one correspondence between subtriples of T and subtriples of T^\vee . Therefore the α -(semi)stability of T is related to that of T^\vee .

Proposition 3.1.12 ([2, Prop. 3.16]). *The triple T is α -(semi)stable if and only if its dual triple T^\vee is α -(semi)stable*

Proof. By symmetric property, it is enough to show that T is α -(semi)stable if T^\vee is. Let $T' = (E'_2, E'_1, \varphi')$ be nontrivial subtriple of T and T'' be the quotient of T by T' . Then we have a short exact sequence of triples,

$$0 \longrightarrow (T'')^\vee \longrightarrow T^\vee \longrightarrow (T')^\vee \longrightarrow 0.$$

As T^\vee is α -(semi)stable, we have $\mu_\alpha(T^\vee)(\geq)\mu_\alpha((T'')^\vee)$. It follows from Lemma 3.1.7 that $\mu_\alpha(T^\vee)(\leq)\mu_\alpha((T')^\vee)$. But we have

$$\begin{aligned} \mu_\alpha(T^\vee)(\leq)\mu_\alpha((T')^\vee) &\iff \frac{d_1 + d_2 + 2\alpha r_2}{r_1 + r_2} (\geq) \frac{d'_1 + d'_2 + 2\alpha r'_2}{r'_1 + r'_2} \\ &\iff \frac{d_1 + d_2 - 2\alpha r_1 + 2\alpha(r_1 + r_2)}{r_1 + r_2} (\geq) \frac{d'_1 + d'_2 - 2\alpha r'_1 + 2\alpha(r'_1 + r'_2)}{r'_1 + r'_2} \\ &\iff \mu_\alpha(T)(\geq)\mu_\alpha(T'). \end{aligned}$$

Hence T is α -(semi)stable. □

3.2 Constraints on the parameter α

The constraints on α for α -(semi)stability of triples were first given by Bradlow and García-Prada in [2] where the authors defined the α -degree of $T = (E_2, E_1, \varphi)$ to be $d_1 + d_2 + \alpha r_2$. Using our definition of the α -degree

$$\deg_\alpha(T) := d_1 + d_2 - 2\alpha r_1,$$

we have,

$$\mu_\alpha(T) = \frac{\deg_\alpha(T)}{r_1 + r_2} = \frac{d_1 + d_2 - 2\alpha r_1}{r_1 + r_2} = \frac{d_1 + d_2 + 2\alpha r_2}{r_1 + r_2} - 2\alpha.$$

It follows that T is α -(semi)stable if and only if T is 2α -(semi)stable by definition of Bradlow and García-Prada. Hence the constraints on α with respect to our definition can be induced by [2, Sec 3.3].

Proposition 3.2.1 (cf. [2, Lm. 3.5]). *Let $T = (E_2, E_1, 0)$ be a triple. Then T is α -semistable if and only if $\alpha = \frac{\mu(E_1) - \mu(E_2)}{2}$ and both bundles E_1, E_2 are semistable. In this case T can not be α -stable.*

Proposition 3.2.2 (cf. [2, Prop. 3.17]). *Let $T = (E_2, E_1, \varphi)$ be a α -(semi)stable triple where $\varphi \neq 0$. Then $\alpha(\geq)\frac{\mu(E_1) - \mu(E_2)}{2}$ and $\alpha(\geq)0$.*

Proposition 3.2.3 (cf. [2, Prop. 3.18]). *Let $T = (E_2, E_1, \varphi)$ be a triple with $r_1 \neq r_2$ and $\varphi \neq 0$. If T is α -(semi)stable then*

$$\alpha(\leq) \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right) \frac{\mu(E_1) - \mu(E_2)}{2}. \quad (3.5)$$

Proof. Let $I = \text{Im } \varphi$ and $K = \text{Ker } \varphi$. Assume that $r_1 > r_2$ then φ can not be surjective. Then $T'' = (E_2, I, \varphi)$ is proper subtriple of T .

If φ is not injective, $T' = (K, 0, \varphi)$ is also nontrivial. As T is α -(semi)stable, we have

$$\mu_\alpha(T')(\leq) \mu_\alpha(T) \iff r_K(d_1 + d_2) - d_K(r_1 + r_2)(\geq) 2\alpha r_1 r_K \quad (3.6)$$

and

$$\begin{aligned} \mu_\alpha(T'')(\leq) \mu_\alpha(T) &\iff \frac{2d_2 - d_K - 2\alpha(r_2 - r_K)}{2r_2 - r_K}(\leq) \frac{d_1 + d_2 - 2\alpha r_1}{r_1 + r_2} \\ &\iff 2(d_1 r_2 - d_2 r_1) + d_K(r_1 + r_2) - r_K(r_1 + r_2)(\geq) 2\alpha(r_2(r_1 - r_2) + r_K(r_1 + r_2)) - 2\alpha_1 r_K. \end{aligned} \quad (3.7)$$

Let (3.6) add to (3.7), we get

$$d_1 r_2 - d_2 r_1(\geq) \alpha(r_2(r_1 - r_2) + r_K(r_1 + r_2)).$$

As $\varphi \neq 0$ and T is α -(semi)stable, we have $\alpha(\geq)0$. Therefore

$$d_1 r_2 - d_2 r_1(\geq) \alpha r_2(r_1 - r_2) \quad (3.8)$$

since $r_K > 0$.

If φ is injective, i.e. $K = 0$ then

$$\mu_\alpha(T'')(\leq) \mu_\alpha(T) \iff d_1 r_2 - d_2 r_1(\geq) \alpha r_2(r_1 - r_2).$$

So we always have

$$d_1 r_2 - d_2 r_1(\geq) \alpha(r_2(r_1 - r_2)).$$

It implies that

$$\alpha(\leq) \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right) \frac{(\mu(E_1) - \mu(E_2))}{2}.$$

If $r_1 < r_2$ then the dual triple $T^\vee = (E_1^\vee, E_2^\vee, \varphi^\vee)$ is also α -(semi)stable by Propostion 3.1.12. Applying the same computation for T^\vee , we get

$$\begin{aligned} \deg(E_2^\vee) \text{rank}(E_1^\vee) - \deg(E_1^\vee) \text{rank}(E_2^\vee)(\geq) \alpha \text{rank}(E_1^\vee)(\text{rank}(E_2^\vee) - \text{rank}(E_1^\vee)) \\ \iff d_1 r_2 - d_2 r_1(\geq) \alpha r_1(r_2 - r_1) \\ \iff \alpha(\leq) \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right) \frac{\mu(E_1) - \mu(E_2)}{2}. \end{aligned}$$

□

If $r_1 = r_2$ then T could be α -(semi)stable for any $\alpha \geq \frac{\mu(E_1) - \mu(E_2)}{2}$. For example, the triple $T = (\mathcal{O}_X, \mathcal{O}_X, \text{Id}_{\mathcal{O}_X})$, then for any $\alpha(\geq)0$, T is α -(semi)stable.

3.3 The corresponding extensions on $X \times \mathbb{P}^1$ of triples

3.3.1 The construction of extensions

Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple on X and $\alpha = \frac{a}{b} \in \mathbb{Q}$ such that $\gcd(a, b) = 1$. We will construct a vector bundle E_T on $X \times \mathbb{P}^1$ such that its semistability with respect to an ample divisor $H(\alpha)$ is equivalent to the α -semistability of T .

On \mathbb{P}^1 we have the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{e} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0. \quad (3.9)$$

Let $p : X \times \mathbb{P}^1 \longrightarrow X$ and $q : X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the projections. By pulling back the exact sequence (3.9) by q and then taking tensor product with p^*E_1 , we obtain a short exact sequence

$$0 \longrightarrow p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\psi} p^*E_1 \longrightarrow 0 \quad (3.10)$$

where $V := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ and $\psi := \text{Id}_{p^*E_1} \otimes q^*e$.

Let us set E_T as the pullback of two morphisms

$$\psi : p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^*E_1$$

and

$$p^*\varphi : p^*E_2 \longrightarrow p^*E_1.$$

It is locally given by

$$E_T(U) = \{(\eta, \nu) \in p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)(U) \oplus p^*E_2(U) \mid (p^*\varphi)(\nu) = \psi(\eta)\}$$

for any open subset $U \subseteq X \times \mathbb{P}^1$. Since ψ is surjective, we get also a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) & \longrightarrow & E_T & \longrightarrow & p^*E_2 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) & \longrightarrow & p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & p^*E_1 \longrightarrow 0 \end{array} \quad (3.11)$$

with exact rows. Furthermore, we have an exact sequence

$$0 \longrightarrow E_T \longrightarrow p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E_2 \longrightarrow p^*E_1 \longrightarrow 0. \quad (3.12)$$

Lemma 3.3.1. *Let $T = (E_2 \xrightarrow{\varphi} E_1)$, $T' = (E'_2 \xrightarrow{\varphi'} E'_1)$ be triples on X . Then any morphism $f : T \longrightarrow T'$ induces a morphism $\bar{f} : E_T \longrightarrow E_{T'}$ of vector bundles on $X \times \mathbb{P}^1$ defined by T and T' , respectively. In particular, if f is injective then \bar{f} is.*

Proof. Assume that $f = (f_2, f_1) : T \longrightarrow T'$, i.e. we have a commutative diagram

$$\begin{array}{ccc} E_2 & \xrightarrow{\varphi} & E_1 \\ \downarrow f_2 & & \downarrow f_1 \\ E'_2 & \xrightarrow{\varphi'} & E'_1. \end{array}$$

By the construction of E_T and $E_{T'}$, we have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{T'} & \longrightarrow & p^*E'_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E'_2 & \longrightarrow & p^*E'_2 \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_T & \longrightarrow & p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E_2 & \longrightarrow & p^*E_2 \longrightarrow 0 \end{array}$$

where the square on the right hand side commutes. From the exactness of the rows, there exists a morphism $\bar{f} : E_{T'} \longrightarrow E_T$ making the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{T'} & \longrightarrow & p^*E'_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E'_2 & \longrightarrow & p^*E'_2 \longrightarrow 0 \\ & & \downarrow \bar{f} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_T & \longrightarrow & p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E_2 & \longrightarrow & p^*E_2 \longrightarrow 0 \end{array}$$

commutative. It is clear that, \bar{f} is injective if f is injective. \square

3.3.2 $\mathrm{SL}(2)$ -invariant subbundles

It follows from Lemma 3.3.1 that any subtriple of T defines a subbundle of E_T . But it is obviously not true that every subbundle of E_T is obtained by this way. Hence we would like to look for the subbundles of E_T which are defined by subtriples of T .

The graded ring $S = k[x, y]$ is generated by S_1 , the space of homogeneous polynomials of degree 1, as k -algebra. On $\mathbb{P}^1 = \mathrm{Proj} S$, any $s \in S_d$, a homogeneous polynomial of degree d , determines in a natural way a global section of $\mathcal{O}_{\mathbb{P}^1}(d)$. Then x, y determine the standard basis of k -vector space $V = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. On V , $\mathrm{SL}(2) := \mathrm{SL}_2(V)$ acts naturally by

$$g(Ax + By) = \begin{pmatrix} A \\ B \end{pmatrix}^t g \begin{pmatrix} x \\ y \end{pmatrix}$$

for any $g \in \mathrm{SL}(2)$ and $Ax + By \in V$. We then have a natural action of $\mathrm{SL}(2)$ on S_d given by

$$g(s(x, y)) = s(g(x), g(y))$$

for any $s = s(x, y) \in S_d$ and $g \in \mathrm{SL}(2)$. This action is compatible with the action of $\mathrm{SL}(2)$ on \mathbb{P}^1 given by

$$g \circ P = (du - bv : -cu + av)$$

for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$ and $P = (u : v) \in \mathbb{P}^1$. Assume that $P = Z(\mathfrak{p}) \in \mathbb{P}^1$, we have

$$\mathcal{O}_{\mathbb{P}^1}(n)_P \cong S(n)_{(\mathfrak{p})}$$

where $S(n)_i = S_{n+i}$ for any $i \geq 0, n \in \mathbb{Z}$ and an isomorphism

$$S(n)_{(\mathfrak{p})} \cong S(n)_{(g(\mathfrak{p}))} \quad (3.13)$$

for any $g \in \mathrm{SL}(2)$. In other words, we get a natural $\mathrm{SL}(2)$ -equivariant structure on $\mathcal{O}_{\mathbb{P}^1}(n)$ for any n ,

$$\Delta^n : q_{\mathrm{SL}(2)}^* \mathcal{O}_{\mathbb{P}^1}(n) \cong \delta^* \mathcal{O}_{\mathbb{P}^1}(n)$$

locally given by (3.13) where $\delta : \mathrm{SL}(2) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the action and $q_{\mathrm{SL}(2)} : \mathrm{SL}(2) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the second projection.

Let $\mathrm{SL}(2)$ act trivially on X , then $\mathrm{SL}(2)$ acts on $X \times \mathbb{P}^1$ by

$$g \circ (x, P) = (x, g \circ P)$$

for any $(x, P) \in X \times \mathbb{P}^1$. Under this action, p^*E_2 and p^*E_1 are $\mathrm{SL}(2)$ -invariant vector bundles and the morphism

$$p^*\varphi : p^*E_2 \rightarrow p^*E_1$$

is $\mathrm{SL}(2)$ -equivariant.

On \mathbb{P}^1 we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{e_0} \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow 0 \quad (3.14)$$

where e_0 is the evaluation map defined by

$$(e_0)_P(s \otimes \rho) = s_P \cdot \rho.$$

Here s_P is the image of global section $s \in V$ in the stalk $\mathcal{O}_{\mathbb{P}^1}(1)_P, P \in \mathbb{P}^1$ and $\rho \in \mathcal{O}_{\mathbb{P}^1, P}$. Moreover, for any $g \in \mathrm{SL}(2)$, $\Delta_{(g, P)}^1(s_P)$ is the image of the global section $g(s)$ in the stalk $\mathcal{O}_{\mathbb{P}^1}(1)_{g \circ P}$. So we have the following commutative diagram

$$\begin{array}{ccc} V \otimes \mathcal{O}_{\mathbb{P}^1, P} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(1)_P \\ \downarrow g \otimes \Delta_{(g, P)} & & \downarrow \Delta_{(g, P)}^1 \\ V \otimes \mathcal{O}_{\mathbb{P}^1, g \circ P} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(1)_{g \circ P} \end{array}$$

for any $g \in \mathrm{SL}(2)$ and $P \in \mathbb{P}^1$. It implies that e_0 is a $\mathrm{SL}(2)$ -equivariant morphism and hence $\mathcal{O}_{\mathbb{P}^1}(-1)$ is $\mathrm{SL}(2)$ -invariant. As $e = e_0 \otimes \mathrm{Id}_{\mathcal{O}_{\mathbb{P}^1}(-1)}$, the Euler sequence (3.9) is $\mathrm{SL}(2)$ -equivariant. It follows that the morphism

$$\psi = \mathrm{Id}_{p^*E_1} \otimes q^*e : p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow p^*E_1$$

is $\mathrm{SL}(2)$ -equivariant. Being the fullback of two $\mathrm{SL}(2)$ -equivariant morphisms, $p^*\varphi$ and ψ , E is also $\mathrm{SL}(2)$ -invariant.

Proposition 3.3.2. *Let E be a vector bundle on X and $F \subseteq p^*E$ be a subbundle of p^*E on $X \times \mathbb{P}^1$ which is $\mathrm{SL}(2)$ -invariant. Then $F \cong p^*E'$ for some subbundle $E' \subseteq E$.*

Proof. To prove, we will show that $F|_{F_p} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ for any p -fiber F_p . Then Lemma 2.2.2 implies that $F \cong p^*E'$ for some $E' \subseteq E$. As a consequence of Proposition 1.1.9, there exists a filtration

$$0 = E_0 \subseteq E_1 \subseteq \dots \subseteq E_r = E$$

of subbundles such that E_i/E_{i-1} is a line bundle for each $i = 1, \dots, r$. Pulling back this sequence by p gives us a filtration for p^*E ,

$$0 = p^*E_0 \subseteq p^*E_1 \subseteq \dots \subseteq p^*E_r = p^*E,$$

with the same property. We intersect this sequence with F to get a filtration for F ,

$$0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_{r'} = F,$$

where $F_i = p^*E_i \cap F$, $r' = \mathrm{rank}(F)$ such that F_i/F_{i-1} is a line bundle. For each $i = 1, \dots, r'$, we consider the following $\mathrm{SL}(2)$ -equivariant diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_i & \longrightarrow & F_{i+1} & \longrightarrow & F_{i+1}/F_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p^*E_i & \longrightarrow & p^*E_{i+1} & \longrightarrow & p^*E_{i+1}/p^*E_i \longrightarrow 0 \end{array}$$

Restricting this diagram to $F_p \cong \mathbb{P}^1$, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_i|_{F_p} & \longrightarrow & F_{i+1}|_{F_p} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(a_i) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p^*E_i|_{F_p} & \longrightarrow & p^*E_{i+1}|_{F_p} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0 \end{array}$$

which is still a $\mathrm{SL}(2)$ -equivariant diagram. Consider the $\mathrm{SL}(2)$ -equivariant short exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a_i) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow T_i \longrightarrow 0 \quad (3.15)$$

where T_i is torsion sheaf for $i = 1, \dots, r'$. Since $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(a_i)$ are $\mathrm{SL}(2)$ -invariant, T is also $\mathrm{SL}(2)$ -invariant. Moreover, $\mathrm{SL}(2)$ acts transitively on \mathbb{P}^1 . It follows that $\mathrm{Supp} T_i$ is whole \mathbb{P}^1 if it is non-empty. This contradicts the fact that T_i is torsion. Hence $T_i = 0$ and $\mathcal{O}_{\mathbb{P}^1}(a_i) \cong \mathcal{O}_{\mathbb{P}^1}$. But $F|_{F_p} \cong \bigoplus_{i=1}^{r'} \mathcal{O}_{\mathbb{P}^1}(a_i)$, therefore

$$F|_{F_p} \cong \bigoplus_{i=1}^{r'} \mathcal{O}_{\mathbb{P}^1}.$$

□

We get immediately the following corollary.

Corollary 3.3.3. *Let E_T be the vector bundle on $X \times \mathbb{P}^1$ defined by triple $T = (E_2 \xrightarrow{\varphi} E_1)$ on X and $E' \subset E_T$ be a $\mathrm{SL}(2)$ -invariant subbundle. Then E' is defined by a subtriple T' of T*

Proof. Let K and I be the kernel and image of morphism $E' \rightarrow E_T \rightarrow p^*E_2$. Then K and I are $\mathrm{SL}(2)$ -invariant bundles making the following diagram commutative

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & I \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) & \longrightarrow & E_T & \longrightarrow & p^*E_2 \longrightarrow 0. \end{array} \quad (3.16)$$

By Proposition 3.3.2, $I = p^*E'_2$ and $K = p^*E'_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)$ for some $E'_1 \subset E_1$ and $E'_2 \subset E_2$. Taking the pushforward of the diagram (3.16) by p , we get a commutative square

$$\begin{array}{ccc} E'_2 & \longrightarrow & E'_1 \otimes R^1p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & E_1 \otimes R^1p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2)). \end{array}$$

By Serre's duality, we can take two isomorphisms

$$E'_1 \otimes R^1p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \cong E'_1,$$

$$E_1 \otimes R^1p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \cong E_1$$

such that the diagram

$$\begin{array}{ccc} E'_2 & \xrightarrow{\varphi'} & E'_1 \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{\varphi} & E_1 \end{array}$$

commutes, i.e. $T' = (E'_2 \xrightarrow{\varphi'} E'_1)$ is a subtriple of T . □

For a given triple $T = (E_2, E_1, \varphi)$ of type (r_1, r_2, d_1, d_2) , we set

$$\alpha_m = \frac{\mu(E_1) - \mu(E_2)}{2} \text{ and } \alpha_M = \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right) \frac{\mu(E_1) - \mu(E_2)}{2} \quad (3.17)$$

if $r_1 \neq r_2$ and $\alpha_M = +\infty$ if $r_1 = r_2$. We have seen from Propositions 3.2.2 and 3.2.3 that T can not be α -semistable unless

$$\alpha_m \leq \alpha \leq \alpha_M \text{ and } \alpha \geq 0.$$

Let $\alpha = \frac{a}{b} \in \mathbb{Q}$ such that $a, b > 0$, $\gcd(a, b) = 1$ and $\alpha_m \leq \alpha \leq \alpha_M$. Let E_T be the vector bundle on $X \times \mathbb{P}^1$ defined by T as in (3.11) and (3.12). It is obvious that

$$E_T \in \text{Ext}^1(p^*E_2, p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)).$$

On $X \times \mathbb{P}^1$, $H(\alpha) = aF_p + bF_q$ is an ample divisor. We have

$$\deg_{H(\alpha)}(E_T) = b(d_1 + d_2) - 2ar_1$$

and then the $H(\alpha)$ -slope of E_T

$$\mu_{H(\alpha)}(E_T) = \frac{b(d_1 + d_2) - 2ar_1}{r_1 + r_2} = b\mu_\alpha(T).$$

Theorem 3.3.4. *The triple $T = (E_2, E_1, \varphi)$ is α -(semi)stable if and only if the extension E_T defined by T is $H(\alpha)$ -(semi)stable.*

Proof. Assume that the bundle E_T defined by T is $H(\alpha)$ -(semi)stable. Let $T' = (E'_2 \xrightarrow{\varphi'} E'_1)$ be a nontrivial subtriple of T . It follows from Lemma 3.3.1 that the bundle $E'_{T'}$ defined by T' is a subbundle of E_T . Hence

$$\mu_\alpha(T') = \frac{\mu_{H(\alpha)}(E'_{T'})}{b} (\leq) \frac{\mu_{H(\alpha)}(E_T)}{b} = \mu_\alpha(T),$$

i.e. T is α -(semi)stable.

Assume that T is α -(semi)stable and E_T is not $H(\alpha)$ -(semi)stable. Then there exists a maximal destabilizing subsheaf $E' \subseteq E_T$. As E_T is $\text{SL}(2)$ -invariant, E' is also $\text{SL}(2)$ -invariant because $g^*E' \subseteq E_T$ for every $g \in \text{SL}(2)$ and the action of g does not change the rank E' and the $\deg_{H(\alpha)}(E')$ as $\text{SL}(2)$ is connected. By Corollary 3.3.2, $E' = E'_{T'}$ for some subtriple $T' = (E'_2 \xrightarrow{\varphi'} E'_1)$ of T . We have

$$\mu_\alpha(T') = \frac{\mu_{H(\alpha)}(E'_{T'})}{b} (\geq) \frac{\mu_{H(\alpha)}(E_T)}{b} = \mu_\alpha(T).$$

This contradicts the α -(semi)stability of T . □

3.4 Moduli space of triples

Let S be a scheme of finite type over k and $p : S \times X \rightarrow S$ be the first projection. A family of triples on X parametrized by S (or a S -family of triples on X) is a triple $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ on $S \times X$ where \mathcal{E}_i are families of vector bundles on X parametrized by S . Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ and $\mathcal{T}' = (\mathcal{E}'_2 \xrightarrow{\varphi'_S} \mathcal{E}'_1)$ be S -families of triples. We say that they are equivalent if there exists a line bundle L on S and isomorphisms $\psi_i : \mathcal{E}_i \rightarrow \mathcal{E}'_i \otimes p^*L$ such that

$$\varphi_S = \psi_1^{-1} \circ (\varphi'_S \otimes p^* \text{Id}_L) \circ \psi_2.$$

For a given $\alpha \in \mathbb{Q}_{>0}$ we say that $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi^S} \mathcal{E}_1)$ is a S -family of α -(semi)stable triples of type $(\underline{r}, \underline{d}) = (r_1, r_2, d_1, d_2)$ if $\mathcal{T}_s = (\mathcal{E}_{2s} \xrightarrow{\varphi_s} \mathcal{E}_{1s})$ is α -(semi)stable and \mathcal{E}_{is} have rank r_i and degree $d_i, i = 1, 2; s \in S$. Let Sch/k and $Sets$ be the category of schemes of finite type over k and the category of sets, respectively. We consider the following contravariant functor:

$$\begin{aligned} \underline{\mathcal{M}}_\alpha^{(s)s}(\underline{r}, \underline{d}) : Sch/k &\longrightarrow Sets \\ S &\longmapsto \{\text{Equivalence classes of } S - \text{families of } \alpha - \text{(semi)stable} \\ &\quad \text{triples of type } (\underline{r}, \underline{d})\}. \end{aligned}$$

There are several approaches to construct the moduli space of α -(semi)stable triples on curves, i.e. the scheme corepresents $\underline{\mathcal{M}}_\alpha^{(s)s}(\underline{r}, \underline{d})$. As we have seen, a triple $T = (E_2 \xrightarrow{\varphi} E_1)$ is α -(semi)stable if and only if the vector bundle E_T on the ruled surface $X \times \mathbb{P}^1$ defined by T is $H(\alpha)$ -(semi)stable. It implies that we can embed the functor $\underline{\mathcal{M}}_\alpha^{(s)s}(\underline{r}, \underline{d})$ into the moduli functor of $H(\alpha)$ -(semi)stable vector bundles on $X \times \mathbb{P}^1$ which is corepresented by a projective scheme of finite type over k (cf. [13], [18]). By this way, Brallow and García-Prada (cf. [2]) constructed the moduli space of α -stable triples on compact Riemann surfaces.

From another point of view, any triple $T = (E_2 \xrightarrow{\varphi} E_1)$ on X can be seen as a representation of the quiver Q ,

$$\bullet \longrightarrow \bullet,$$

consisting of 2 vertices and one arrow, in the category of vector bundles on X . As the semistability of representations is given, we also have the concepts of the Harder-Narasimhan filtration, Jordan-Hölder filtration, the associated graded object for a representation and then the S -equivalence as usual. The moduli space of α -semistable triples is then just a special case of the moduli space of twisted representation of quivers given by A.Schmitt (cf. [17, Thm. 3.7.2]) as the base field k is the field of complex numbers or by Álvarez-Cónsul (cf. [1]) for characteristic p . From there, we obtain the moduli space of α -semistable triples on smooth projective curve X .

Theorem 3.4.1. *There exist a projective scheme $\mathcal{M}_\alpha^{ss}(\underline{r}, \underline{d})$ which corepresents the functor $\underline{\mathcal{M}}_\alpha^{ss}(\underline{r}, \underline{d})$. The closed points of $\mathcal{M}_\alpha^{ss}(\underline{r}, \underline{d})$ are in bijection to the set of S -equivalence classes of α -semistable triples of type (r_1, r_2, d_1, d_2) . There is an open subscheme $\mathcal{M}_\alpha^s(\underline{r}, \underline{d})$ which corepresents the subfunctor $\underline{\mathcal{M}}_\alpha^s(\underline{r}, \underline{d})$ and whose closed points correspond to the isomorphism classes of α -stable triples.*

Proof. See [17, Thm. 3.7.2] or [1, Thm. 8] □

Chapter 4

Generalized theta line bundles

4.1 Theta divisors for vector bundles on curves

4.1.1 Determinant line bundles

Let X be a smooth projective variety of dimension n . We define the Grothendieck's groups $K(X)$ and $K^0(X)$ to be the quotients of free abelian group generated by coherent and locally free sheaves, respectively, by the subgroup generated by all expressions $E - E' - E''$, wherever there is an exact sequence $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$. Moreover, the tensor product turns $K^0(X)$ into a commutative ring with $1 = [\mathcal{O}_X]$ and gives $K(X)$ a module structure over $K^0(X)$.

Let S be a connected scheme of finite type over k . Then the projection $p : S \times X \rightarrow S$ is a smooth morphism of relative dimension n . Let \mathcal{E} be a flat family of coherent sheaves on X parametrized by S . We have a homomorphism $p_! : K^0(S \times X) \rightarrow K^0(S)$, given by

$$p_!([\mathcal{E}]) = \sum_{i=0}^n (-1)^i [R^i p_* \mathcal{E}]$$

(cf. [12, Cor. 2.1.11]). Let $q : S \times X \rightarrow X$ be the second projection. Then we have a well defined morphism

$$\begin{aligned} \lambda_{\mathcal{E}} : K(X) &\longrightarrow \text{Pic}(S) \\ [F] &\mapsto \det(p_!(\mathcal{E} \otimes q^*[F])) \end{aligned} \tag{4.1}$$

which is the composition of morphisms

$$K(X) \xrightarrow{q^*} K^0(S \times X) \xrightarrow{[\mathcal{E}]} K^0(S \times X) \xrightarrow{p_!} K^0(S) \xrightarrow{\det} \text{Pic}(S).$$

For each class $[F] \in K(X)$, the line bundle $\lambda_{\mathcal{E}}([F])$ is called a determinant line bundle associated to \mathcal{E} at the class $[F]$. The following properties of $\lambda_{\mathcal{E}}$ can be seen easily by using

the projection formula for direct images of sheaves and the basic properties of the determinant of locally free sheaves.

Lemma 4.1.1 (cf. [12, Lm. 8.1.2]). *Let \mathcal{E} be a S -flat family of coherent sheaves and $[F]$ be any class in $K(X)$.*

(i) *(Additive property) If*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is a short exact sequence of S -flat families of coherent sheaves then

$$\lambda_{\mathcal{E}}([F]) = \lambda_{\mathcal{E}'}([F]) \otimes \lambda_{\mathcal{E}''}([F]).$$

Similarly, if we have a short exact sequence of coherent sheaves on X

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0,$$

then $\lambda_{\mathcal{E}}([F]) \cong \lambda_{\mathcal{E}}([F']) \otimes \lambda_{\mathcal{E}}([F''])$.

(ii) *(Base-change property) If $f : S' \longrightarrow S$ is any morphism, then*

$$f^*(\lambda_{\mathcal{E}}([F])) = \lambda_{f^*\mathcal{E}}([F])$$

where $f_X := f \times \text{Id}_X : S' \times X \longrightarrow S \times X$.

(iii) *(Projection property) If L is a line bundle on S then*

$$\lambda_{\mathcal{E} \otimes p^*L}([F]) = \lambda_{\mathcal{E}}([F]) \otimes L^{\chi(\mathcal{E}_s \otimes F)}$$

for any $s \in S$.

4.1.2 Generalized theta functions and generalized theta divisors

Let X be a smooth projective curve and r, d be integers, $r \geq 2$. Let $U_X(r, d)$ be the moduli space of S -equivalence classes of vector bundles on X of rank r and degree d . Then $U_X(r, d)$ is a normal projective variety whose smooth locus is the fine moduli space of stable vector bundles $U_X^s(r, d)$, unless $g = 2, r = 2$ and d is even, when $U_X(r, d)$ is smooth. Drezet and Narasimhan (cf. [5]) even showed that, they are locally factorial schemes. Let F be a nonzero vector bundle on X . We call F a complementary vector bundle of $U_X(r, d)$ if $\chi(E \otimes F) = 0$ for all $E \in U_X(r, d)$. Let $h = \gcd(r, d)$ and $r_0 = \frac{r}{h}, d_0 = \frac{d}{h}$. By Riemann-Roch formula, it is easy to see that F has rank pr_0 and degree $p(r_0(g_X - 1) - d_0)$ for some $p \in \mathbb{Z}_{>0}$. In particular, to a generic vector bundle F_0 of rank r_0 and degree $r_0(g_X - 1) - d_0$ we can associate an effective Cartier divisor Θ_{F_0} on $U_X(r, d)$ (cf. [5, Section 0.2]) supported on the set

$$\Theta_{F_0} = \{E \in U_X(r, d) \mid H^0(E \otimes F) \neq 0\},$$

called a basic theta divisor.

Let S be a connected scheme and \mathcal{E} be a family of vector bundles of rank r and degree d parametrized by S . We will show by the following construction that for each vector bundle F on X such that $\chi(\mathcal{E}_s \otimes F) = 0$ for some $s \in S$ (hence for all $s \in S$), the line bundle $\lambda_{\mathcal{E}}([F])^{-1}$ has a geometric section $\zeta_{\mathcal{E}}(F)$ whose zero divisor is given by

$$\{s \in S \mid H^0(X, \mathcal{E}_s \otimes F) \neq 0\}. \quad (4.2)$$

Indeed, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$H^0(X, \mathcal{E}_s \otimes \mathcal{O}_X(-n)) = 0$$

for all $s \in S$. Let F be any vector bundle of rank r' on X . Proposition 1.1.9 implies that for n large enough, there exists a short exact sequence of vector bundles on X

$$0 \longrightarrow \det(F)^{-1}(-(1+r')n) \xrightarrow{\varphi_F(n)} \mathcal{O}_X^{r'+1}(-n) \longrightarrow F \longrightarrow 0. \quad (4.3)$$

We consider such large n that it is also bigger than n_0 and denote by $n \gg n_0$. Pulling back this exact sequence by q and then tensoring with \mathcal{E} , we get an exact sequence on $S \times X$

$$0 \longrightarrow \mathcal{E} \otimes q^* \det(F)^{-1}(-(1+r')n) \xrightarrow{\text{Id}_{\mathcal{E}} \otimes q^* \varphi_F(n)} \mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n) \longrightarrow \mathcal{E} \otimes q^* F \longrightarrow 0.$$

Pushing forward to S by p , we have

$$\begin{aligned} 0 \longrightarrow p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)) &\longrightarrow p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)) \longrightarrow p_*(\mathcal{E} \otimes q^* F) \\ &\longrightarrow R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)) \xrightarrow{R^1(\varphi_F(n))} R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)) \\ &\longrightarrow R^1 p_*(\mathcal{E} \otimes q^* F) \longrightarrow 0 \end{aligned} \quad (4.4)$$

where $R^1(\varphi_F(n)) = R^1 p_*(\text{Id}_{\mathcal{E}} \otimes q^* \varphi_F(n))$. For any $s \in S$, we have

$$p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)) \otimes k(s) \cong H^0(X, \mathcal{E}_s^{r'+1}(-n)) = 0.$$

Hence the exact sequence (4.4) becomes

$$\begin{aligned} 0 \longrightarrow p_*(\mathcal{E} \otimes q^* F) &\longrightarrow R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)) \xrightarrow{R^1(\varphi_F(n))} R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)) \\ &\longrightarrow R^1 p_*(\mathcal{E} \otimes q^* F) \longrightarrow 0. \end{aligned} \quad (4.5)$$

Then we have

$$\lambda_{\mathcal{E}}([F])^{-1} \cong \det(R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)))^{-1} \otimes \det(R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n))). \quad (4.6)$$

Assume that for some $m \gg n_0$, we have another resolution of F of form

$$0 \longrightarrow \det(F)^{-1}(-(1+r')m) \xrightarrow{\varphi_F(m)} \mathcal{O}_X^{r'+1}(-m) \longrightarrow F \longrightarrow 0. \quad (4.7)$$

Then we have the same long exact sequence as (4.5) for m . It implies that

$$\begin{aligned} & \det(R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)))^{-1} \otimes \det(R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n))) \\ & \cong \det(R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)m)))^{-1} \otimes \det(R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-m))), \end{aligned} \quad (4.8)$$

equivalently, $\lambda_{\mathcal{E}}([F])^{-1}$ is independent of the choice of n provided $n \gg n_0$. It is then clear that

$$\lambda_{\mathcal{E}}([F]) \cong \lambda_{\mathcal{E}}([F']) \quad (4.9)$$

for any F, F' such that $\text{rank}(F) = \text{rank}(F')$ and $\det(F) = \det(F')$.

Assume that $\chi(\mathcal{E}_s \otimes F) = 0$ for some $s \in S$ (hence for all $s \in S$). It follows from Riemann - Roch formula that

$$\text{rank}(p_*(\mathcal{E} \otimes q^* F)) = \text{rank}(R^1 p_*(\mathcal{E} \otimes q^* F)).$$

From the exactness of (4.5) we have

$$\text{rank}(R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n))) = \text{rank}(R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n))).$$

It implies that

$$R^1(\varphi_F(n)) : R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)) \longrightarrow R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n))$$

is a morphism of vector bundles of the same rank. Therefore the line bundle

$$\lambda_{\mathcal{E}}([F])^{-1} \cong \det(R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)))^{-1} \otimes \det(R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)))$$

has a geometric section $\zeta_{\mathcal{E}}(F) = \det(R^1(\varphi_F(n)))$. Since $\chi(\mathcal{E}_s \otimes F) = 0$ for all $s \in S$, by the construction, we see that the zero divisor of $\zeta_{\mathcal{E}}(F)$ parametrizes all points $s \in S$ such that $H^0(S, \mathcal{E}_s \otimes F) \neq 0$ (cf. [11, Prop. 4.1]).

Assume further that \mathcal{E} is a family of semistable vector bundles of rank r and degree d on X parametrized by S . Then there exists a morphism

$$f_{\mathcal{E}} : S \longrightarrow U_X(r, d), s \mapsto \mathcal{E}_s$$

induced by the family \mathcal{E} . Let $[F] \in K(X)$ be any class where F is a complementary vector bundle of $U_X(r, d)$. Then $\chi(\mathcal{E}_s \otimes F) = 0$ for all $s \in S$. We obtain on S a line bundle $\lambda_{\mathcal{E}}([F])^{-1}$ and a generalized theta function $\zeta_{\mathcal{E}}(F)$.

Theorem 4.1.2 (cf. [5, Thm. D, part (a)], [15, Cor. 1.6, Cor. 1.7] or [12, Thm. 8.1.5]). *Let $[F]$ be a class in $K(X)$ such that F is a complementary vector bundle of $U_X(r, d)$. There exists a unique ample line bundle $\mathbb{L}([F])$ on $U_X(r, d)$ such that for any family \mathcal{E} of semistable vector bundles of rank r and degree d on X parametrized by S , we have*

$$\lambda_{\mathcal{E}}([F])^{-1} \cong f_{\mathcal{E}}^* \mathbb{L}([F]). \quad (4.10)$$

Moreover, the geometric section $\zeta_{\mathcal{E}}(F)$ descends by $f_{\mathcal{E}}$ to a section of $\mathbb{L}([F])$ on $U_X(r, d)$ whose zero divisor is supported on the set

$$\Theta_F = \{E \in U_X(r, d) \mid H^0(X, E \otimes F) \neq 0\}.$$

We denote this section by ζ_F and call it a generalized theta function. As $\mathbb{L}([F])$ is independent of the choice of representations of class $[F] \in K(X)$, we denote it again by θ_F and call it a generalized theta line bundle. If ζ_F is a nonzero section, i.e. there exists $E \in U_X(r, d)$ such that $H^0(E \otimes F) = 0$, then Θ_F is a divisor, called a generalized theta divisor, and $\theta_F = \mathcal{O}_{U_X(r, d)}(\Theta_F)$. In particular, if we take $F = F_0$ then $\lambda_{\mathcal{E}}([F_0])^{-1}$ descends to basic theta line bundle $\theta_{F_0} = \mathcal{O}_{U_X(r, d)}(\Theta_{F_0})$. Let F'_0 be another vector bundle such that $\Theta_{F'_0}$ is a basic theta divisor. We have a morphism of Picard groups $\det^* : \text{Pic}(J^d(X)) \rightarrow \text{Pic}(U_X(r, d))$ induced from the canonical morphism $\det : U_X(r, d) \rightarrow J^d(X)$. As $\text{Pic}^0(X) \subset \text{Pic}(J^d(X))$, we have

$$\theta_{F'_0} \cong \theta_{F_0} \otimes \det^*(\det(F'_0) \otimes \det(F_0)^{-1}) \quad (4.11)$$

(cf. [5, Thm. D]).

4.1.3 Pluritheta linear series on $U_X(r, d)$

Proposition 1.1.6 gives us a sufficient condition under which a vector bundle is semistable: E is semistable if there exists a nonzero vector bundle F such that $E \otimes F$ is cohomologically trivial, i.e. $H^*(X, E \otimes F) = 0$. In fact, it was proven by Faltings in [8], Seshadri in [18, Thm. 6.2], etc, that this is also a necessary condition for semistability of vector bundles on curves. In [6, Thm. 2], Esteves even showed that we can find such vector bundle F with fixed determinant. We recall this result for $E \in U_X(r, d)$.

Theorem 4.1.3 ([6, Thm. 2]). *Let $E \in U_X(r, d)$ be a semistable vector bundle on X and $\{L_p \mid p > 0\}$ be a sequence of line bundles such that $\deg(L_p) = p(r_0(g_X - 1) - d_0)$ for every integer $p > 0$. Then there is a vector bundle F on X of rank pr_0 such that $\det(F) \cong L_p$ and $E \otimes F$ is cohomologically trivial for a certain $p > 0$.*

Let F_0 be a complementary vector bundle of $U_X(r, d)$ such that Θ_{F_0} is a basic theta divisor. For each integer p , let us set $L_p = (\det(F_0))^p$ and consider the complementary vector bundles F of rank pr_0 and determinant L_p . It follows from Theorem 4.1.2 and (4.11) that for any such F , the generalized theta line bundle θ_F is ample and isomorphic to $\theta_{F_0}^p$. In particular, if ζ_F is a nonzero section, then $\Theta_F \in |p\Theta_{F_0}|$. We call $|p\Theta_{F_0}|$ a pluritheta linear series. What we are interested in are the properties of this linear series as base point freeness, separated properties, very ampleness, etc. These properties should be related to a bound on the rank of F .

Theorem 4.1.4 ([7, Thm. B]). *Let Θ_{F_0} be a basic theta divisor on $U_X(r, d)$. For each $p \geq r^2 + r$, the linear series $|p\Theta_{F_0}|$ on $U_X(r, d)$ separates points and is very ample on the smooth locus $U_X^s(r, d)$.*

4.2 Determinant line bundles and invariant sections for triples

4.2.1 Triple product

For any triples $T = (E_2 \xrightarrow{\varphi} E_1)$ and $F = (F_2 \xrightarrow{\psi} F_1)$ on X , we define the triple product of T and F to be the triple

$$T \times F := (E_2 \otimes F_1 \oplus E_1 \otimes F_2 \xrightarrow{(\varphi \otimes \text{Id}_{F_1})\pi_1 - (\text{Id}_{E_1} \otimes \psi)\pi_2} E_1 \otimes F_1) \quad (4.12)$$

where $\pi_1 : E_2 \otimes F_1 \oplus E_1 \otimes F_2 \rightarrow E_2 \otimes F_1$ and $\pi_2 : E_2 \otimes F_1 \oplus E_1 \otimes F_2 \rightarrow E_1 \otimes F_2$ are the projections. It is easy to see that the triple product is exact, i.e. if

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0 \quad (4.13)$$

is an exact sequence of triples on X then for any triple F , the sequence

$$0 \rightarrow T' \times F \rightarrow T \times F \rightarrow T'' \times F \rightarrow 0$$

is exact.

Definition 4.2.1. *For a given triple $T = (E_2 \xrightarrow{\varphi} E_1)$ on X , we call $\chi(T)$ the Euler Characteristic of T , defined by*

$$\chi(T) = \chi(E_2) - \chi(E_1),$$

where $\chi(E_i)$ is the Euler Characteristic of $E_i, i = 1, 2$.

For an exact sequence of triples as (4.13), we have

$$\chi(T) = \chi(T') + \chi(T'').$$

4.2.2 α -orthogonal triples

Let $T = (E_2 \xrightarrow{\varphi} E_1)$ and $F = (F_2 \xrightarrow{\psi} F_1)$ be triples on X . Let K be the kernel of

$$\varphi \times \psi := (\varphi \otimes \text{Id}_{F_1})\pi_1 - (\text{Id}_{E_1} \otimes \psi)\pi_2$$

as in (4.12). We then have a short exact sequence

$$0 \longrightarrow K \longrightarrow E_2 \otimes F_1 \oplus E_1 \otimes F_2 \xrightarrow{\varphi \times \psi} E_1 \otimes F_1.$$

Assume that $T = (E_2 \xrightarrow{\varphi} E_1)$ is of type (r_1, r_2, d_1, d_2) . As in (3.17), let us set

$$\alpha_m = \frac{\mu(E_1) - \mu(E_2)}{2} \quad \text{and} \quad \alpha_M = \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right) \frac{\mu(E_1) - \mu(E_2)}{2}$$

if $r_1 \neq r_2$ and $\alpha_M = +\infty$ if $r_1 = r_2$. From now on, by a rational number $\alpha = \frac{a}{b} \in \mathbb{Q}$, we mean a rational number $\alpha = \frac{a}{b} \in \mathbb{Q}$ such that $a, b > 0$, $(a, b) = 1$ and $\alpha_m \leq \alpha \leq \alpha_M$.

Definition 4.2.2. Let $T = (E_2 \xrightarrow{\varphi} E_1)$, $F = (F_2 \xrightarrow{\psi} F_1)$ be triples on X and $\alpha = \frac{a}{b} \in \mathbb{Q}$. Then F is called a α -orthogonal triple of T if it satisfies the following properties:

- (i) ψ is surjective,
- (ii) $\text{rank}(F_2) = 2\text{rank}(F_1)$, $\mu(F_2) = \mu(F_1) - \alpha$,
- (iii) $H^*(X, K) = 0$ where K is the kernel of $\varphi \times \psi$.

Similar to the semistability of vector bundles, we can characterize the α -semistability of triples by a perpendicularity condition.

Theorem 4.2.3. Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple of type (r_1, r_2, d_1, d_2) on X and $\alpha = \frac{a}{b} \in \mathbb{Q}$. Then T is α -semistable if and only if it has a α -orthogonal triple.

Proof. Assume that $F = (F_2 \xrightarrow{\psi} F_1)$ is a α -orthogonal triple of $T = (E_2 \xrightarrow{\varphi} E_1)$. If T is not α -semistable, there exist a subtriple $T' = (E'_2 \xrightarrow{\varphi'} E'_1)$ which destabilizes T , i.e. $\mu_\alpha(T') > \mu_\alpha(T)$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & E'_2 \otimes F_1 \oplus E'_1 \otimes F_2 & \longrightarrow & E'_1 \otimes F_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & E_2 \otimes F_1 \oplus E_1 \otimes F_2 & \longrightarrow & E_1 \otimes F_1 & \longrightarrow & 0 \end{array}$$

where the rows are exact.

Assume that F is of type (R_1, R_2, D_1, D_2) . Then

$$R_2 = 2R_1, D_2 = 2D_1 - 2\alpha R_1$$

by condition (ii) in Definition 4.2.2. We have

$$\text{rank}(K) = (r_1 + r_2)R_1 \quad \text{and} \quad \deg(K) = (r_1 + r_2)D_1 + R_1(d_1 + d_2) - 2\alpha r_1 R_1.$$

It follows that

$$\mu(K) = \mu_\alpha(T) + \mu(F_1). \tag{4.14}$$

Similarly, $\mu(K') = \mu_\alpha(T') + \mu(F_1)$. Since $H^*(X, K) = 0$ then $\chi(K) = 0$. Moreover, $K' \subseteq K$ implies that $\chi(K') \leq 0$. Therefore

$$\mu(K) = g_X - 1 \geq \mu(K') \implies \mu_\alpha(T) \geq \mu_\alpha(T').$$

This is a contradiction. Hence T is α -semistable.

Assume that T is α -semistable. We will construct a α -orthogonal triple $F = (F_2 \xrightarrow{\psi} F_1)$ of T . This is done by the following lemma.

Lemma 4.2.4. *Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple on X of type (r_1, r_2, d_1, d_2) and $\alpha = \frac{a}{b} \in \mathbb{Q}$. If T is α -semistable then it has a α -orthogonal triple.*

Proof. Let E_T be the extension on $X \times \mathbb{P}^1$ defined by T . It follows from Theorem 3.3.4 that E_T is $H(\alpha)$ -semistable of rank $r = r_1 + r_2$ and degree $b(d_1 + d_2) - 2ar_1$. By computing the Chern character up to numerical equivalence, we have

$$\text{ch}(E_T) \equiv r_1 + r_2 + (d_1 + d_2)F_p - 2r_1F_q - 2d_1[pt].$$

Therefore

$$\Delta(E_T) = 2 \text{ch}_0(E_T) \text{ch}_2(E_T) - \text{ch}_1^2(E_T) = 4(r_1d_2 - r_2d_1). \quad (4.15)$$

Let us fix $C \in |m_0H(\alpha)|$ a smooth projective curve where

$$m_0 = \lceil \frac{1-r}{r} \Delta(E_T) + 1 \rceil = \lceil \frac{4(1-(r_1+r_2))(d_2r_1-d_1r_2)}{r_1+r_2} + 1 \rceil. \quad (4.16)$$

Here we denote by $\lceil x \rceil$ the smallest integral number which is not less than x for any $x \in \mathbb{Q}$. Applying Corollary 2.3.4 for E_T , we see that $E_T|_C$ is again a semistable vector bundle of rank r and degree

$$d = m_0 \deg_{H(\alpha)}(E_T) = bm_0\mu_\alpha(T).$$

By Theorem 4.1.3, there exists a vector bundle F_C on C such that $H^*(C, E_T|_C \otimes F_C) = 0$. Let f be the composition $C \hookrightarrow X \times \mathbb{P}^1 \rightarrow X$ which is a finite morphism. Consider the following diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) & \longrightarrow & E_T & \longrightarrow & p^*E_2 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) & \longrightarrow & p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & p^*E_1 \longrightarrow 0 \end{array}$$

By restricting the above diagram to C and then twisting with F_C , the square on the left hand side becomes

$$\begin{array}{ccc} E_T|_C \otimes F_C & \longrightarrow & p^*E_2|_C \otimes F_C \\ \downarrow & & \downarrow \\ [p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)]|_C \otimes F_C & \longrightarrow & p^*E_1|_C \otimes F_C \end{array}$$

We take the pushforward of this diagram to X by f and set

$$K = f_*(E_T|_C \otimes F_C), F_1 = f_*F_C, F_2 = f_*((V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))|_C \otimes F_C).$$

As f is finite,

$$R^1 f_*([p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)]|_C \otimes F_C) = 0.$$

So we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & f_*([p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)]|_C \otimes F_C) & \longrightarrow & K & \longrightarrow & E_2 \otimes F_1 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \varphi \otimes \text{Id}_{F_1} \\ 0 & \longrightarrow & f_*([p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)]|_C \otimes F_C) & \longrightarrow & E_1 \otimes F_2 & \xrightarrow{\text{Id}_{F_1} \otimes \psi} & E_1 \otimes F_1 \longrightarrow 0, \end{array}$$

where $\psi := f_*(q^*e|_C \otimes \text{Id}_{F_C}) : F_2 \rightarrow F_1$ is surjective. It implies that K is the pullback of two morphisms

$$\varphi \otimes \text{Id}_{F_1} : E_2 \otimes F_1 \rightarrow E_1 \otimes F_1 \text{ and } \text{Id}_{E_1} \otimes \psi : E_1 \otimes F_2 \rightarrow E_1 \otimes F_1.$$

Equivalently, K is the kernel of $\varphi \times \psi$, the morphism of triple product

$$(E_2 \xrightarrow{\varphi} E_1) \times (F_2 \xrightarrow{\psi} F_1).$$

We then have a short exact sequence of vector bundles,

$$0 \longrightarrow K \longrightarrow E_2 \otimes F_1 \oplus E_1 \otimes F_2 \xrightarrow{\varphi \times \psi} E_1 \otimes F_1 \longrightarrow 0.$$

Since f is finite,

$$H^*(X, K) = H^*(X, f_*(E_T|_C \otimes F_C)) = H^*(C, E_T|_C \otimes F_C) = 0.$$

So $F = (F_2 \xrightarrow{\psi} F_1)$ will be the triple which we are looking for if the numerical invariants of F_1 and F_2 satisfy condition (ii) in Definition 4.2.2.

Now we compute the rank and degree of $F_1 = f_*(F_C)$. Let us set

$$h = \gcd(r, d), r_0 = \frac{r}{h}, d_0 = \frac{d}{h}.$$

As we have seen,

$$\text{rank } F_C = pr_0, \text{deg}(F_C) = p(r_0(gc - 1) - d_0) \quad (4.17)$$

for some $p \in \mathbb{Z}_{>0}$. Let G be a vector bundle on C of rank R and degree D . Since $C \in |m_0H(\alpha)|$ where $H(\alpha) = aF_p + bF_q$, $f : C \rightarrow X$ is finite of degree m_0b . For any $x \in X$,

$$\dim(f_*G \otimes k(x)) = m_0bR.$$

Hence f_*G is a vector bundle of rank m_0bR . As f is finite, we have

$$\dim H^*(C, G) = \dim H^*(X, f_*G).$$

So Riemann-Roch formula implies that $\chi(f_*G) = \chi(G)$. But

$$\chi(f_*G) = \chi(G) \iff \deg(f_*G) + \text{rank}(f_*G)(1 - g_X) = D + R(1 - g_C).$$

it implies that $\deg(f_*G) = RB + D$ where $B = m_0b(g_X - 1) + 1 - g_C$. Now we have,

$$\begin{aligned} \text{rank}(F_1) &= m_0bpr_0, \\ \text{rank}(F_2) &= 2m_0bpr_0 = 2\text{rank}(F_1), \\ \deg(F_1) &= pr_0(m_0b(g_X - 1) + 1 - g_C) + p(r_0(g_C - 1) - d_0) \\ &= m_0bpr_0(g_X - 1) - pd_0, \\ \deg(F_2) &= 2pr_0(m_0b(g_X - 1) + 1 - g_C) + \deg(V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)|_C \otimes F_C) \\ &= 2pr_0(m_0b(g_X - 1) + 1 - g_C) - 2kr_0m_0a + 2p(r_0(g_C - 1) - d_0) \\ &= 2(m_0bpr_0(g_X - 1) - pd_0) - 2pr_0m_0a \\ &= 2\deg(F_1) - 2\alpha\text{rank}(F_1). \end{aligned}$$

It follows that

$$\text{rank}(F_2) = 2\text{rank}(F_1), \mu(F_2) = \mu(F_1) - \alpha.$$

□

The theorem is then completely proven. □

4.2.3 Determinant line bundles for triples

Let S be a connected scheme of finite type over k , $p : S \times X \rightarrow S$ and $q : S \times X \rightarrow X$ be the projections. Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ be a family of triples on X parametrized by S . For any triple $F = (F_2 \xrightarrow{\psi} F_1)$ on X , the triple product of \mathcal{T} and q^*F ,

$$\mathcal{T} \times q^*F = (\mathcal{E}_2 \otimes q^*F_1 \oplus \mathcal{E}_1 \otimes q^*F_2 \xrightarrow{\varphi_S \times q^*\psi} \mathcal{E}_1 \otimes q^*F_1),$$

is again a S -family of triples.

Definition 4.2.5. Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ be a family of triples on X parametrized by S and $F = (F_2 \xrightarrow{\psi} F_1)$ a triple on X . Then the line bundle

$$\mathcal{L}_{\mathcal{T} \times q^*F} := \lambda_{\mathcal{E}_1}([F_2]) \otimes \lambda_{\mathcal{E}_2}([F_1]) \otimes (\lambda_{\mathcal{E}_1}([F_1]))^{-1},$$

where $\lambda_{\mathcal{E}_i}$ are morphisms defined in (4.1), is called the determinant line bundle associated to \mathcal{T} at F . As \mathcal{T} is fixed, we denote $\mathcal{L}_{\mathcal{T} \times q^*F}$ just by $\mathcal{L}_{F_2 \rightarrow F_1}$.

We will show that the determinant line bundle associated to \mathcal{T} at F defined above has similar properties of the determinant line bundle associated to a family of vector bundles as in Lemma 4.1.1: let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$, $\mathcal{T}' = (\mathcal{E}'_2 \xrightarrow{\varphi'_S} \mathcal{E}'_1)$ and $\mathcal{T}'' = (\mathcal{E}''_2 \xrightarrow{\varphi''_S} \mathcal{E}''_1)$ be S -families of triples on X such that the following sequence

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}'' \longrightarrow 0$$

is exact. Since the triple product is exact, the sequence

$$0 \longrightarrow \mathcal{T}' \times q^*F \longrightarrow \mathcal{T} \times q^*F \longrightarrow \mathcal{T}'' \times q^*F \longrightarrow 0$$

is exact. Precisely, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}'_2 \otimes q^*F_1 \oplus \mathcal{E}'_1 \otimes q^*F_2 & \longrightarrow & \mathcal{E}_2 \otimes q^*F_1 \oplus \mathcal{E}_1 \otimes q^*F_2 & \longrightarrow & \mathcal{E}''_2 \otimes q^*F_1 \oplus \mathcal{E}''_1 \otimes q^*F_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}'_1 \otimes q^*F_1 & \longrightarrow & \mathcal{E}_1 \otimes q^*F_1 & \longrightarrow & \mathcal{E}''_1 \otimes q^*F_1 \longrightarrow 0, \end{array} \quad (4.18)$$

where the rows are exact. From the exactness of the rows in (4.18) we have

$$\lambda_{\mathcal{E}_1}([F_2]) \otimes \lambda_{\mathcal{E}_2}([F_1]) \cong \lambda_{\mathcal{E}'_1}([F_2]) \otimes \lambda_{\mathcal{E}'_2}([F_1]) \otimes \lambda_{\mathcal{E}''_1}([F_2]) \otimes \lambda_{\mathcal{E}''_2}([F_1]),$$

$$\lambda_{\mathcal{E}_1}([F_1]) \cong \lambda_{\mathcal{E}'_1}([F_1]) \otimes \lambda_{\mathcal{E}''_1}([F_1]).$$

It is implied by the definition that

$$\begin{aligned} \mathcal{L}_{\mathcal{T} \times q^*F} &:= \lambda_{\mathcal{E}_1}([F_2]) \otimes \lambda_{\mathcal{E}_2}([F_1]) \otimes (\lambda_{\mathcal{E}_1}([F_1]))^{-1} \\ &\cong \lambda_{\mathcal{E}'_1}([F_2]) \otimes \lambda_{\mathcal{E}'_2}([F_1]) \otimes \lambda_{\mathcal{E}''_1}([F_2]) \otimes \lambda_{\mathcal{E}''_2}([F_1]) \otimes \lambda_{\mathcal{E}'_1}([F_1])^{-1} \otimes \lambda_{\mathcal{E}''_1}([F_1])^{-1} \\ &\cong [\lambda_{\mathcal{E}'_1}([F_2]) \otimes \lambda_{\mathcal{E}'_2}([F_1]) \otimes \lambda_{\mathcal{E}'_1}([F_1])^{-1}] \otimes [\lambda_{\mathcal{E}''_1}([F_2]) \otimes \lambda_{\mathcal{E}''_2}([F_1]) \otimes \lambda_{\mathcal{E}''_1}([F_1])^{-1}] \\ &\cong \mathcal{L}_{\mathcal{T}' \times q^*F} \otimes \mathcal{L}_{\mathcal{T}'' \times q^*F}. \end{aligned}$$

Similarly, if $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ is an exact sequence of triples, we have

$$\mathcal{L}_{\mathcal{T} \times q^*F} \cong \mathcal{L}_{\mathcal{T} \times q^*F'} \otimes \mathcal{L}_{\mathcal{T} \times q^*F''}.$$

Using the distributive property of pullback over tensor product, the base-change property is also fulfilled, i.e. if $f : S' \longrightarrow S$ be a morphism of scheme then

$$f^* \mathcal{L}_{\mathcal{T} \times q^*F} = \mathcal{L}_{f_X^* \mathcal{T} \times q^*F},$$

where $f_X = f \times \text{Id}_X : S' \times X \longrightarrow S \times X$ and $q' : S' \times X \longrightarrow X$ is the second projection. For any line bundle L on S , we have

$$\mathcal{T} \otimes p^*L = (\mathcal{E}_2 \otimes p^*L \xrightarrow{\varphi_S \times p^* \text{Id}_L} \mathcal{E}_1 \otimes p^*L)$$

and then

$$\begin{aligned}
\mathcal{L}_{\mathcal{T} \otimes p^* L \times q^* F} &= \lambda_{\mathcal{E}_1 \otimes p^* L}([F_2]) \otimes \lambda_{\mathcal{E}_2 \otimes p^* L}([F_1]) \otimes (\lambda_{\mathcal{E}_1 \otimes p^* L}([F_1]))^{-1} \\
&= \mathcal{L}_{\mathcal{T} \times q^* F} \otimes L^{\chi(\mathcal{E}_{1s} \otimes F_2) + \chi(\mathcal{E}_{2s} \otimes F_1) - \chi(\mathcal{E}_{1s} \otimes F_1)} \\
&= \mathcal{L}_{\mathcal{T} \times q^* F} \otimes L^{\chi(\mathcal{E}_s \times F)}
\end{aligned}$$

for some $s \in S$.

Since $\lambda_{\mathcal{E}_i}([F])$ depends only on the class $[F] \in K(X)$ (see (4.9)), we get the following similar property for determinant line bundles for triples.

Lemma 4.2.6. *Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ be a family of triples on X parametrized by S and $F = (F_2 \xrightarrow{\psi} F_1), F' = (F'_2 \xrightarrow{\psi'} F'_1)$ be triples on X . If $\text{rank}(F'_i) = \text{rank}(F_i)$ and $\det(F'_i) = \det(F_i)$ for $i = 1, 2$ then*

$$\mathcal{L}_{\mathcal{T} \times q^* F'} \cong \mathcal{L}_{\mathcal{T} \times q^* F}.$$

4.2.4 Invariant sections

Let $F = (F_2 \xrightarrow{\psi} F_1)$ be a triple on X where ψ is surjective and $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ be a family of triples on X parametrized by S . By taking the triple product, we have an exact sequence of vector bundles on $S \times X$,

$$0 \longrightarrow \mathcal{K}_F \longrightarrow \mathcal{E}_2 \otimes q^* F_1 \oplus \mathcal{E}_1 \otimes q^* F_2 \longrightarrow \mathcal{E}_1 \otimes q^* F_1 \longrightarrow 0. \quad (4.19)$$

Hence

$$\mathcal{L}_{F_2 \rightarrow F_1} := \lambda_{\mathcal{E}_1}([F_2]) \otimes \lambda_{\mathcal{E}_2}([F_1]) \otimes (\lambda_{\mathcal{E}_1}([F_1]))^{-1} \cong \lambda_{\mathcal{K}_F}([\mathcal{O}_X]), \quad (4.20)$$

where $\lambda_{\mathcal{K}_F}([\mathcal{O}_X])$ is the determinant line bundle associated to S -family of vector bundles \mathcal{K}_F . Let us set $\theta_{F_2 \rightarrow F_1} := \mathcal{L}_{F_2 \rightarrow F_1}^{-1}$. Then $\theta_{F_2 \rightarrow F_1} \cong \lambda_{\mathcal{K}_F}([\mathcal{O}_X])^{-1}$. For any $s \in S$, we have an exact sequence of vector bundles on X

$$0 \longrightarrow (\mathcal{K}_F)_s \longrightarrow \mathcal{E}_{2s} \otimes F_1 \oplus \mathcal{E}_{1s} \otimes F_2 \longrightarrow \mathcal{E}_{1s} \otimes F_1 \longrightarrow 0.$$

It follows that $\chi((\mathcal{K}_F)_s) = \chi(\mathcal{E}_s \times F)$. As we have seen, if $\chi((\mathcal{K}_F)_s) = 0$ for all $s \in S$, the line bundle $\lambda_{\mathcal{K}_F}([\mathcal{O}_X])^{-1}$ has a geometric section, denoted by $\zeta_{\mathcal{K}_F}$. Hence $\theta_{F_2 \rightarrow F_1}$ also has a geometric section, denoted by $\zeta_{F_2 \rightarrow F_1}$, via the isomorphism $\theta_{F_2 \rightarrow F_1} \cong \lambda_{\mathcal{K}_F}([\mathcal{O}_X])^{-1}$. The zero divisor of $\zeta_{F_2 \rightarrow F_1}$, denoted by $\Theta_{F_2 \rightarrow F_1}$, parametrizes the same points as the zero divisor of $\zeta_{\mathcal{K}_F}$,

$$\Theta_{F_2 \rightarrow F_1} = \{s \in S \mid H^0(X, (\mathcal{K}_F)_s) \neq 0\}. \quad (4.21)$$

Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be any triple of type (r_1, r_2, d_1, d_2) and $\alpha = \frac{a}{b}$ be a rational number. Let E_T be the extension bundle of rank $r = r_1 + r_2$ defined by T and $m_0 = \lceil \frac{1-r}{r} \Delta(E_T) + 1 \rceil$.

Let us set $d = m_0 \deg_{H(\alpha)}(E_T)$ and $h = \gcd(r, d)$. For each positive integer p , let $\text{Comp}(p)$ be the set of triples $F = (F_2 \xrightarrow{\psi} F_1)$ which have the following numerical invariants

$$\begin{aligned} \text{rank}(F_1) &= pm_0br_0, \deg(F_1) = p(m_0br_0(g_X - 1) - d_0), \\ \text{rank}(F_2) &= 2 \text{rank}(F_1), \deg(F_2) = 2 \deg(F_1) - \alpha \text{rank}(F_1). \end{aligned} \quad (4.22)$$

where $r_0 = \frac{r}{h}$, $d_0 = \frac{d}{h}$. These triples can be constructed as in the proof of Lemma 4.2.4. Let us set

$$\text{Comp} = \cup_{p \in \mathbb{Z}_{>0}} \text{Comp}(p). \quad (4.23)$$

In particular, for any $F \in \text{Comp}$ and any triple T of type (r_1, r_2, d_1, d_2) , we have

$$\begin{aligned} \chi(T \times F) &= \chi(E_2 \otimes F_1 \oplus E_1 \otimes F_2) - \chi(E_1 \otimes F_1) \\ &= (d_1 + d_2 - 2\alpha r_1) \text{rank}(F_1) + \deg(F_1)(r_1 + r_2) + r_{F_1}(r_1 + r_2)(1 - g_X) \\ &= (d_1 + d_2 - 2\alpha r_1)m_0bpr_0 + r(m_0bpr_0(g_X - 1) - pd_0) + rm_0bpr_0(1 - g_X) \\ &= (d_1 + d_2 - 2\alpha r_1)m_0bpr_0 - rpd_0 = 0 \end{aligned} \quad (4.24)$$

since $m_0b(d_1 + d_2 - 2\alpha r_1) = m_0 \deg_{H(\alpha)}(E_T)$.

Assume that $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi^S} \mathcal{E}_1)$ is a S -family of triples of type (r_1, r_2, d_1, d_2) . By the definition of α -orthogonal triples, for any $F = (F_2 \xrightarrow{\psi} F_1) \in \text{Comp}$, the zero divisor of geometric section $\zeta_{F_2 \rightarrow F_1}$ is described as follows

$$\Theta_{F_2 \rightarrow F_1} = \{s \in S \mid F \text{ is not a } \alpha\text{-orthogonal triple of } \mathcal{T}_s\}. \quad (4.25)$$

4.3 Base point freeness

4.3.1 Generalized theta line bundles for triples

Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a α -semistable triple of type (r_1, r_2, d_1, d_2) on X where $\alpha = \frac{a}{b}$ and E_T be the extension on $X \times \mathbb{P}^1$ defined by T . We know that E_T is $H(\alpha)$ -semistable. Moreover, the restriction of E_T to smooth projective curve $C' \in |mH(\alpha)|$ is still semistable for any $m \geq m_0 = \lceil \frac{1-r}{r} \Delta(E_T) + 1 \rceil$. It is easy to see that T and T' are S -equivalent α -semistable triples if and only if E_T and $E_{T'}$, the extensions defined by T and T' , respectively, are S -equivalent. We will show that for $m \gg m_0$, the S -equivalence is preserved under the restriction to a smooth projective curve $C' \in |mH(\alpha)|$.

Lemma 4.3.1. *Let E_T and $E_{T'}$ be $H(\alpha)$ -semistable vector bundles on $X \times \mathbb{P}^1$ defined by triples T and T' , respectively. Then for m large enough, $E_T|_{C'}$ and $E_{T'}|_{C'}$ are S -equivalent if and only if T and T' are S -equivalent.*

Proof. Assume that

$$0 \subset T^1 \subset T^2 \subset \dots \subset T^n = T$$

is a Jordan-Hölder filtration of T . By the construction of extensions, it follows that

$$0 \subset E_{T^1} \subset E_{T^2} \subset \dots \subset E_{T^n} = E_T$$

is a Jordan-Hölder filtration of E_T . Since $E_{T^i}/E_{T^{i-1}}$ are stable of $H(\alpha)$ -slope $\mu_{H(\alpha)}(E_T)$ for all $i \geq 1$, for any $m \geq m_0$, $E_{T^i}|_{C'}/E_{T^{i-1}}|_{C'}$ are stable of the same slope (see also the proof of Corollary 2.3.4) since

$$E_{T^i}|_{C'}/E_{T^{i-1}}|_{C'} \cong (E_{T^i}/E_{T^{i-1}})|_{C'}.$$

Hence

$$0 \subset E_{T^1}|_{C'} \subset E_{T^2}|_{C'} \subset \dots \subset E_{T^n}|_{C'} = E_T|_{C'}$$

is again a Jordan-Hölder filtration of $E_T|_{C'}$. So we have

$$\text{gr}(E_T|_{C'}) \cong \text{gr}(E_T)|_{C'}.$$

It implies that $\text{gr}(E_T|_C) \cong \text{gr}(E_{T'}|_C)$ if $\text{gr}(T) \cong \text{gr}(T')$.

Conversely, it is enough to show that for m large enough, if E and E' are $H(\alpha)$ -stable vector bundles on $X \times \mathbb{P}^1$ of the same slope such that $E|_{C'}$ and $E'|_{C'}$ are isomorphic and stable then $E \cong E'$. Since E and E' are stable of the same slope, then

$$E \cong E' \iff H^0(X \times \mathbb{P}^1, E^\vee \otimes E') \neq 0.$$

Consider the following exact sequence on $X \times \mathbb{P}^1$,

$$0 \longrightarrow E^\vee \otimes E' \otimes \mathcal{O}_{X \times \mathbb{P}^1}(-mH(\alpha)) \longrightarrow E^\vee \otimes E' \longrightarrow (E^\vee \otimes E')|_{C'} \longrightarrow 0.$$

We then obtain a long exact sequence of cohomology

$$\begin{aligned} 0 &\longrightarrow H^0(X \times \mathbb{P}^1, E^\vee \otimes E' \otimes \mathcal{O}_{X \times \mathbb{P}^1}(-mH(\alpha))) \longrightarrow H^0(X \times \mathbb{P}^1, E^\vee \otimes E') \\ &\longrightarrow H^0(X \times \mathbb{P}^1, (E^\vee \otimes E')|_{C'}) \longrightarrow H^1(X \times \mathbb{P}^1, E^\vee \otimes E' \otimes \mathcal{O}_{X \times \mathbb{P}^1}(-mH(\alpha))) \end{aligned}$$

where $H^0(X \times \mathbb{P}^1, (E^\vee \otimes E')|_{C'}) \cong H^0(C', E^\vee|_{C'} \otimes E'|_{C'}) \neq 0$ since $E|_{C'} \cong E'|_{C'}$. We have $mH(\alpha) = maF_p + mbF_q$ and $a, b > 0$. As

$$E = E_T \in \text{Ext}^1(p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2), p^*E_2), E' = E_{T'} \in \text{Ext}^1(p^*E'_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2), p^*E'_2),$$

by Künneth formula, the vanishing of

$$H^0(X \times \mathbb{P}^1, E^\vee \otimes E' \otimes \mathcal{O}_{X \times \mathbb{P}^1}(-mH(\alpha))) \text{ and } H^1(X \times \mathbb{P}^1, E^\vee \otimes E' \otimes \mathcal{O}_{X \times \mathbb{P}^1}(-mH(\alpha)))$$

is induced from the vanishing of

$$H^0(X, E_j^\vee \otimes E'_i(-ma)) \text{ and } H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-mb+2))$$

which is satisfied for $m \gg m_0$. It follows that

$$H^0(X \times \mathbb{P}^1, E^\vee \otimes E') \cong H^0(X \times \mathbb{P}^1, (E^\vee \otimes E')|_{C'}) \neq 0.$$

Hence $E \cong E'$. □

This lemma implies that for any $m \geq m_0$, we have a well defined morphism between moduli spaces,

$$\begin{aligned} \mathfrak{t}_m : M_\alpha^{ss}(\underline{r}, \underline{d}) &\longrightarrow U_{C'}(r, d') \\ T &\mapsto E_T|_{C'}. \end{aligned} \tag{4.26}$$

and \mathfrak{t}_m is an injective if $m \gg m_0$.

Let F_C be a complementary vector bundle of $U_C(r, d)$ where $C \in |m_0 H(\alpha)|$. Then the generalized theta line bundle θ_{F_C} has a section ζ_{F_C} whose zero divisor is Θ_{F_C} . Let us set

$$\mathfrak{t} = \mathfrak{t}_{m_0} : M_\alpha^{ss}(\underline{r}, \underline{d}) \longrightarrow U_C(r, d).$$

Definition 4.3.2. *We call the pullback $\mathfrak{t}^*\theta_{F_C}$, $\mathfrak{t}^*\zeta_{F_C}$ again a generalized theta line bundle and a generalized theta function, respectively, on $M_\alpha^{ss}(\underline{r}, \underline{d})$.*

4.3.2 Linear systems

Assume that $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ is a family of α -semistable triples of type (r_1, r_2, d_1, d_2) on X parametrized by a connected scheme S . For any $F = (F_2 \xrightarrow{\psi} F_1) \in \text{Comp}$, the line bundle $\theta_{F_2 \rightarrow F_1}$ has a geometric section $\zeta_{F_2 \rightarrow F_1}$ whose zero divisor is

$$\Theta_{F_2 \rightarrow F_1} = \{s \in S \mid F = (F_2 \xrightarrow{\psi} F_1) \text{ is not a } \alpha\text{-orthogonal triple of } \mathcal{T}_s\}.$$

For simplicity, we denote by \mathcal{M} the moduli space $\mathcal{M}_\alpha^{ss}(\underline{r}, \underline{d})$. Consider the following diagram

$$S \xrightarrow{f_{\mathcal{T}}} \mathcal{M} \xrightarrow{\mathfrak{t}} U_C(r, d)$$

where $f_{\mathcal{T}}$ is induced from family \mathcal{T} . Let us recall the linear systems which exist on these schemes:

+ On $U_C(r, d)$: we have pluritheta linear series $|p\Theta_{F_0}|$ where Θ_{F_0} is a basic theta divisor. For any complementary vector bundle F_C such that $\det(F_C) \cong L_p$, the generalized theta line bundle θ_{F_C} has a section ζ_{F_C} whose zero divisor is Θ_{F_C} . Moreover, θ_{F_C} is isomorphic to $\theta_{F_0}^p$ for such any F_C .

+ On \mathcal{M} : we set $\mathcal{L} = \mathfrak{t}^*\theta_{F_0}$ and call it a basic theta line bundle. For each p , we have a generalized theta line bundle \mathcal{L}^p and generalized theta functions $\mathfrak{t}^*(\zeta_{F_C})$ for any such complementary vector bundles F_C . Let $V(\mathcal{L}^p) \subset H^0(\mathcal{M}, \mathcal{L}^p)$ be the linear space spanned by sections $\mathfrak{t}^*\zeta_{F_C}$.

+ On S : Let us fix F_C a complementary vector bundle F_C of $U_C(r, d)$ as above, we consider the following triple constructed as in the proof of Lemma 4.2.4,

$$F = (F_2 \xrightarrow{\psi} F_1) := (f_*(V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)|_C \otimes F_C) \xrightarrow{\psi} f_*F_C), \quad (4.27)$$

where f is the composed morphism $C \hookrightarrow X \times \mathbb{P}^1 \rightarrow X$ and $q : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the projection. It implies that $F \in \text{Comp}(p)$ and then $\chi(\mathcal{E}_s \times F) = 0$ for any $s \in S$. Let $\text{Comp}(p, F) \subset \text{Comp}(p)$ be the subset of triples $F' = (F'_2 \xrightarrow{\psi'} F'_1) \in \text{Comp}(p)$ such that $[F'_i] = [F_i] \in K(X), i = 1, 2$. It follows from Lemma 4.2.6 that all line bundles $\theta_{F'_2 \rightarrow F'_1}$ are isomorphic to $\theta_{F_2 \rightarrow F_1}$. Hence we obtain a unique line bundle $\theta_{F_2 \rightarrow F_1}$ and a linear system $V(p, F) \subset H^0(S, \theta_{F_2 \rightarrow F_1})$ spanned by geometric sections $\zeta_{F'_2 \rightarrow F'_1}$ where $F' = (F'_2 \xrightarrow{\psi'} F'_1) \in \text{Comp}(p, F)$.

4.3.3 Corollaries

The separation property of $|p\Theta_{F_0}|$ on $U_C(r, d)$ implies that for any semistable vector bundle $E_C \in U_C(r, d)$ and any $p \geq p_0 := r^2 + r$, there exists a complementary vector bundle F_C of determinant L_p such that $\Theta_{F_C} \in |p\Theta_{F_0}|$ and $E_C \notin \text{Supp } \Theta_{F_C}$. As a consequence, we obtain similar result for the base point freeness of the linear systems mentioned above on S and \mathcal{M} .

For any triple $F' = (F'_2 \xrightarrow{\psi'} F'_1) \in \text{Comp}(p, F)$, we set

$$\overline{\Theta}_{F'_2 \rightarrow F'_1} := \{T \in \mathcal{M} | F' = (F'_2 \xrightarrow{\psi'} F'_1) \text{ is not a } \alpha\text{-orthogonal triple of } T\}.$$

Corollary 4.3.3. *For any $p \geq p_0$, then*

(i) *The linear systems $V(\mathcal{L}^p)$ and $V(p, F)$ are base point free.*

(ii) *There exists a triple $F' = (F'_2 \xrightarrow{\psi'} F'_1) \in \text{Comp}(p, F)$ such that $\overline{\Theta}_{F'_2 \rightarrow F'_1}$ is a Cartier divisor on \mathcal{M} and*

$$\mathcal{L}^p \cong \mathcal{O}_{\mathcal{M}}(\overline{\Theta}_{F'_2 \rightarrow F'_1}).$$

Proof. Let T be any α -semistable triple on X . For the proof, we may assume that there exists $s \in S$ such that $\mathcal{E}_s = T$. Let E_T be the extension defined by T on $X \times \mathbb{P}^1$. Then $\mathfrak{t}(T) = E_T|_C \in U_C(r, d)$. Since $|p\Theta_{F_0}|$ is base point free, there exists a complementary vector

bundle F'_C such that $\det(F'_C) \cong L_p$, $\Theta_{F'_C} \in |p\Theta_{F_0}|$ and $E_T|_C \notin \text{Supp } \Theta_{F'_C}$. It implies that $\mathfrak{t}^*(\zeta_{F'_C})(T) \neq 0$. Let

$$F' = (F'_2 \xrightarrow{\psi'} F'_1) := (f_*(V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)|_C \otimes F'_C) \xrightarrow{\psi'} f_*F'_C),$$

be the triple constructed from F'_C . Then $F' \in \text{Comp}(p, F)$ and moreover, it is a α -orthogonal triple of $T = \mathcal{E}_s$. Therefore $\zeta_{F'_2 \rightarrow F'_1}(s) \neq 0$. This proves (i) and moreover implies that $\mathfrak{t}^*\Theta_{F'_C}$ is a Cartier divisor. In particular, we have

$$\begin{aligned} \mathfrak{t}^*\Theta_{F'_C} &= \{T \in \mathcal{M} | E_T|_C \in \Theta_{F'_C}\} \\ &= \{T \in \mathcal{M} | H^0(C, E_T|_C \otimes F'_C) \neq 0\} \\ &= \{T \in \mathcal{M} | F' = (F'_2 \xrightarrow{\psi'} F'_1) \text{ is not a } \alpha\text{-orthogonal triple of } T\} \\ &= \overline{\Theta}_{F'_2 \rightarrow F'_1}. \end{aligned} \tag{4.28}$$

It follows that

$$\mathcal{L}^p = \mathfrak{t}^*\theta_{F_0}^p \cong \mathfrak{t}^*\mathcal{O}_{U_C(r,d)}(\Theta_{F'_C}) \cong \mathcal{O}_{\mathcal{M}}(\overline{\Theta}_{F'_2 \rightarrow F'_1}).$$

□

By the construction of the extensions defined by triples, for each S -family of α -semistable triples $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ we obtain a family $E_{\mathcal{T}} = (E_{\mathcal{T}_s})_{s \in S}$ of $H(\alpha)$ -semistable extensions on $X \times \mathbb{P}^1$ parametrized by S . Restricting this family to C , we get a family of semistable vector bundles on C ,

$$\mathcal{E} = (E_{\mathcal{T}_s}|_C)_{s \in S}.$$

We also have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f_{\mathcal{E}}} & U_C(r, d) \\ & \searrow f_{\mathcal{T}} & \nearrow \mathfrak{t} \\ & \mathcal{M} & \end{array} \tag{4.29}$$

where $f_{\mathcal{E}}$ is the morphism induced by \mathcal{E} . For any complementary vector bundle F_C of $U_C(r, d)$, it is implied by Theorem 4.1.2 that the line bundle $\lambda_{\mathcal{E}}([F_C])^{-1}$ descends to θ_{F_C} by $f_{\mathcal{E}}$. We obtain a similar universal property for the generalized theta line bundles on \mathcal{M} .

Corollary 4.3.4. *Let $p \geq 1$ and $F'' = (F''_2 \xrightarrow{\psi''} F''_1)$ be any triple in $\text{Comp}(p, F)$. Then for any family $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ of α -semistable triples on X parametrized by S , the line bundle $\theta_{F''_2 \rightarrow F''_1}$ descends to \mathcal{L}^p .*

Proof. Since $f_{\mathcal{E}} = \mathfrak{t} \circ f_{\mathcal{T}}$, $\lambda_{\mathcal{E}}([F_C])^{-1} \cong f_{\mathcal{T}}^* \mathcal{L}^p$ for any complementary vector bundle F_C such that $\det(F_C) = L_p$. Consider the following diagram

$$\begin{array}{ccccc}
 S \times C & \xrightarrow{q_C} & C & & \\
 \swarrow p_C & \searrow i_S & \downarrow i & & \\
 S & & S \times X \times \mathbb{P}^1 & \xrightarrow{f} & X \times \mathbb{P}^1 \xrightarrow{q} \mathbb{P}^1 \\
 \swarrow p_X & \searrow p_{12} & \downarrow p_{23} & \searrow p & \\
 S \times X & \xrightarrow{q_X} & X & &
 \end{array} \tag{4.30}$$

By the construction of $E_{\mathcal{T}}$, we have an exact sequence

$$0 \longrightarrow E_{\mathcal{T}} \longrightarrow p_{12}^* \mathcal{E}_2 \oplus p_{12}^* \mathcal{E}_1 \otimes V \otimes (q \circ p_{23})^* \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p_{12}^* \mathcal{E}_1 \longrightarrow 0.$$

Restricting to $S \times C$, we have

$$0 \longrightarrow \mathcal{E} \longrightarrow p_{12}^* \mathcal{E}_2|_{S \times C} \oplus p_{12}^* \mathcal{E}_1 \otimes V \otimes (q \circ p_{23})^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{S \times C} \longrightarrow p_{12}^* \mathcal{E}_1|_{S \times C} \longrightarrow 0$$

(see (3.12) and (3.11)). By twisting with $q_C^* F_C$, we pass to the following exact sequence

$$\begin{aligned}
 0 \longrightarrow \mathcal{E} \otimes q_C^* F_C &\longrightarrow p_{12}^* \mathcal{E}_2|_{S \times C} \otimes q_C^* F_C \oplus p_{12}^* \mathcal{E}_1 \otimes V \otimes (q \circ p_{23})^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{S \times C} \otimes q_C^* F_C \\
 &\longrightarrow p_{12}^* \mathcal{E}_1|_{S \times C} \otimes q_C^* F_C \longrightarrow 0.
 \end{aligned} \tag{4.31}$$

Since f is finite morphism, $(f_S)_*$ is an exact functor. We then obtain a short exact sequence on $S \times X$,

$$\begin{aligned}
 0 &\longrightarrow (f_S)_*(\mathcal{E} \otimes q_C^* F_C) \\
 &\longrightarrow (f_S)_*(p_{12}^* \mathcal{E}_2|_{S \times C} \otimes q_C^* F_C) \oplus (f_S)_*(p_{12}^* \mathcal{E}_1 \otimes V \otimes (q \circ p_{23})^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{S \times C} \otimes q_C^* F_C) \\
 &\longrightarrow (f_S)_*(p_{12}^* \mathcal{E}_1|_{S \times C} \otimes q_C^* F_C) \longrightarrow 0.
 \end{aligned} \tag{4.32}$$

By projection formula and flat base change, we obtain the following exact sequence

$$0 \longrightarrow (f_S)_*(\mathcal{E} \otimes q_C^* F_C) \longrightarrow \mathcal{E}_2 \otimes q_X^* F_1 \oplus \mathcal{E}_1 \otimes q_X^* F_2 \longrightarrow \mathcal{E}_1 \otimes q_X^* F_1 \longrightarrow 0.$$

Since $p_C = p_X \circ f_S$ and $(f_S)_*$ is exact,

$$\theta_{F_2 \rightarrow F_1} \cong \lambda_{(f_S)_*(\mathcal{E} \otimes q_C^* F_C)}([\mathcal{O}_X])^{-1} \cong \lambda_{\mathcal{E}}([F_C])^{-1}.$$

Now for any $F'' = (F_2'' \xrightarrow{\psi''} F_1'') \in \text{Comp}(p, F)$,

$$\theta_{F_2'' \rightarrow F_1''} \cong \theta_{F_2 \rightarrow F_1} \cong f_{\mathcal{T}}^* \mathcal{L}^p.$$

□

We consider the numerical equivalence on $K(X \times \mathbb{P}^1)$ which generalizes the numerical equivalence of divisors defined in Chapter 2 (cf. [12, §8, p.178]). Let u and u' be classes in $K(X \times \mathbb{P}^1)$. Then they are said to be numerically equivalent: $u \equiv u'$, if their difference is contained in the radical of the quadratic form $(a, b) \mapsto \chi(a \cdot b)$. Let

$$K(X \times \mathbb{P}^1)_{\text{Num}} = K(X \times \mathbb{P}^1) / \equiv .$$

It is easy to see that for any triple T of fixed type (r_1, r_2, d_1, d_2) , the corresponding extension E_T is contained in a fixed class $\mathbf{e} \in K(X \times \mathbb{P}^1)_{\text{Num}}$. Let $U_{X \times \mathbb{P}^1}(\mathbf{e})$ be the moduli space of $H(\alpha)$ -semistable vector bundles on $X \times \mathbb{P}^1$ of numerical class \mathbf{e} .

Corollary 4.3.5. *The line bundle \mathcal{L} on \mathcal{M} is ample.*

Proof. It is enough to show that, \mathcal{L}^p is ample for some $p \geq p_0$. Let F_C be a complementary vector bundle of $U_C(r, d)$ such that $\mathcal{L}^p = \mathbf{t}^* \theta_{F_C}$. Let $C' \in |mC|$ where m large enough such that the morphism

$$\mathbf{t}_m : \mathcal{M} \longrightarrow U_{C'}(r, md)$$

is an injection (see (4.26)). If we can find a complementary vector bundle $F_{C'}$ of $U_{C'}(r, md)$ such that $\mathbf{t}_m^* \theta_{F_{C'}} \cong \mathcal{L}^{mp}$, we are done since $\mathbf{t}_m^* \theta_{F_{C'}}$ is ample.

Assume that F_C has rank R and degree D . Consider the class $[F_C] \in K(X \times \mathbb{P}^1)$, we have

$$\text{ch}([F_C]) = R \text{ch}([\mathcal{O}_C]) + D = R.C + (D - \frac{RC^2}{2}).$$

Let K be the canonical divisor on $X \times \mathbb{P}^1$, then

$$g_C = \frac{1}{2}C^2 + \frac{1}{2}C.K + 1 \text{ and } g_{C'} = \frac{1}{2}C'^2 + \frac{1}{2}C'.K + 1.$$

As $C' \in |mC|$, we have

$$g_{C'} - 1 = m(g_C - 1) + \frac{m(m-1)}{2}C^2.$$

Let $F_{C'}$ be a vector bundle on C' of rank R and degree $D' = mD + \frac{m(m-1)}{2}RC^2$. We then have

$$rD' + Rmd + rR(1 - g_{C'}) = m(rD + Rd + rR(1 - g_C)) = 0,$$

i.e. $F_{C'}$ is a complementary vector bundle of $U_{C'}(r, md)$. Moreover,

$$\text{ch}([\mathcal{O}_{C'}]) = m \text{ch}([\mathcal{O}_C]) - \frac{m(m-1)}{2}C^2$$

and then

$$\text{ch}([F_{C'}]) = R \text{ch}([\mathcal{O}_{C'}]) + D' = R(m \text{ch}([\mathcal{O}_C]) - \frac{m(m-1)}{2}C^2) + D' = m \text{ch}([F_C]).$$

It implies that $[F_{C'}] = m[F_C] \in K(X \times \mathbb{P}^1)$.

Consider the following factors of \mathfrak{t} and \mathfrak{t}_m

$$\begin{array}{ccc}
 & U_C(r, d) & \\
 & \nearrow \mathfrak{t} & \uparrow i^* \\
 \mathcal{M} & \longrightarrow U_{X \times \mathbb{P}^1}(\mathfrak{e}) & \\
 & \searrow \mathfrak{t}_m & \downarrow i_m^* \\
 & U_{C'}(r, md) &
 \end{array} \tag{4.33}$$

where i^* and i_m^* are restrictions induced from the inclusions

$$i : C \hookrightarrow X \times \mathbb{P}^1 \text{ and } i_m : C' \hookrightarrow X \times \mathbb{P}^1,$$

respectively. Since $[F_{C'}] = m[F_C]$, it implies that $i_m^* \theta_{F_{C'}} \cong (i^* \theta_{F_C})^m$. Therefore, we have

$$\mathfrak{t}_m^* \theta_{F_{C'}} \cong (\mathfrak{t}^* \theta_{F_C})^m \cong \mathcal{L}^{mp}.$$

□

4.4 Langton's valuative criterion for triples

4.4.1 Elementary transformations

Let R be a discrete valuation ring where $\text{Spec } R = \{\eta, 0\}$ and $X_R = \text{Spec } R \times X$. Let us denote by $X_\eta = \eta \times X$ the generic fiber and X_0 the closed fiber of the first projection

$$p_R : \text{Spec } R \times X \longrightarrow \text{Spec } R.$$

Then X_0 is a Cartier divisor on X_R and any vector bundle on X_0 is a sheaf on X_R of projective dimension 1. Let \mathcal{E} be a vector bundle on X_R and E_0 be the restriction of \mathcal{E} on closed fiber X_0 . We recall now the construction of elementary transformations of \mathcal{E} along a quotient bundle of E_0 , from which we can similarly construct the elementary transformations of triples.

Let E be a vector bundle on X_0 which is a quotient of E_0 . Then the kernel \mathcal{E}' of the composition

$$\pi : \mathcal{E} \rightarrow E_0 \xrightarrow{\pi_0} E$$

is a vector bundle on X_R . We call it an elementary transformation of \mathcal{E} along E . If \mathcal{E}' is an elementary transformation of \mathcal{E} along E , we have a short exact sequence of sheaves on X_R

$$0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \xrightarrow{\pi} E \rightarrow 0. \tag{4.34}$$

By restricting this exact sequence to X_η , we see easily that \mathcal{E}' and \mathcal{E} coincide over the generic fiber. On closed fiber, E becomes a subbundle of \mathcal{E}' .

Similarly, let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_R} \mathcal{E}_1)$ be a triple on X_R and $T = (E_2 \xrightarrow{\varphi} E_1)$ be a quotient of T_0 , the restriction of \mathcal{T} on the closed fiber X_0 . Let $\phi : \mathcal{T} \rightarrow T$ be the composed morphism of triples on X_R .

Definition 4.4.1. *The triple $\mathcal{T}' = (\mathcal{E}'_2 \xrightarrow{\varphi'_R} \mathcal{E}'_1)$ is called an elementary transformation of \mathcal{T} along T if it satisfies the following short exact sequence of triples*

$$0 \rightarrow \mathcal{T}' \rightarrow \mathcal{T} \xrightarrow{\phi} T \rightarrow 0. \quad (4.35)$$

Notice that, the sequence (4.35) is a short exact sequence of triples of sheaves on X_R . But \mathcal{T}' is really a triple on X_R , i.e. \mathcal{E}'_1 and \mathcal{E}'_2 are vector bundle on X_R . As above, we can also conclude that the elementary transformations of \mathcal{T} along T do not change the triple over the generic point. But they transfer the quotient T of T_0 to a subtriple of \mathcal{T}' over the closed point.

Lemma 4.4.2. *Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple on X which is not α -semistable and $T' = (E'_2 \xrightarrow{\varphi'} E'_1)$ be the maximal destabilizing subtriple of T . Then the quotient*

$$T'' = T/T' := (E''_2 \xrightarrow{\varphi''} E''_1)$$

is again a triple, i.e. E''_i is vector bundle on X , $i = 1, 2$.

Proof. Let F'_i be the torsion subsheaf of E''_i , $i = 1, 2$. It is clear that $\varphi''(F'_2) \subset F'_1$. Then $F_i = E''_i/F'_i$ are vector bundles and $F = (F_2 \xrightarrow{\varphi''} F_1)$ is a triple. We have a short exact sequence

$$0 \longrightarrow F' \longrightarrow T'' \longrightarrow F \longrightarrow 0$$

where $F' = (F'_2 \xrightarrow{\varphi''} F'_1)$. Let G be the kernel of the composed morphism $T' \rightarrow T'' \rightarrow F$. We obtain a short exact sequence

$$0 \longrightarrow T' \longrightarrow G \longrightarrow F' \longrightarrow 0.$$

It implies that,

$$\mu_\alpha(G) = \mu_\alpha(T') + \frac{\text{length}(F'_1) + \text{length}(F'_2)}{r'_1 + r'_2} \geq \mu_\alpha(T'),$$

where r'_i is the rank of E'_i , $i = 1, 2$. Since T' is maximal destabilizing subtriple of T , $T' = G$. It follows that $F'_i = 0$ for $i = 1, 2$. Hence E''_i are vector bundles. \square

4.4.2 Valuative criterion

Lemma 4.4.3 ([10, Lm. 6.1]). *Let X be a smooth projective curve. Any vector bundle E on X_η can be extended to a vector bundle \mathcal{E} on X_R .*

In [10, Thm. 6.4], Hein gave a proof of Langton's theorem for semistable vector bundles of rank 2 and determinant ω_X using the elementary transformations and the cohomological characterization of semistability of vector bundles. By the same method, we can prove the following Langton's theorem for α -semistable triples (for the original Langton's valuative criterion, see [14]).

Theorem 4.4.4. *Let T_η be a α -semistable triple of type (r_1, r_2, d_1, d_2) on the generic fiber X_η . Then there exists a triple \mathcal{T} on X_R such that $\mathcal{T}|_{X_\eta} \cong T_\eta$ and $T_0 := \mathcal{T}|_{X_0}$ is α -semistable.*

Proof. Assume that $T_\eta = (E_{\eta 2} \xrightarrow{\varphi_\eta} E_{\eta 1})$ on X_η . It is implied from Lemma 4.4.3 that there exist vector bundles \mathcal{E}_2 and \mathcal{E}_1 on X_R which extend $E_{\eta 2}$ and $E_{\eta 1}$, respectively. We consider the morphism $\varphi_\eta \in \text{Hom}_{X_\eta}(E_{\eta 2}, E_{\eta 1})$ as a section of $\mathcal{E}_2^\vee \otimes \mathcal{E}_1$ over the open set X_η . By a standard argument as in [9, Lm. 5.14, p. 118], we can extend this section to obtain a section $\varphi_R \in \text{Hom}_{X_R}(\mathcal{E}_2, \mathcal{E}_1)$. Hence we get an extension $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_R} \mathcal{E}_1)$ of T_η on X_R .

Let $p_R : X_R \rightarrow \text{Spec } R$ and $q_R : X_R \rightarrow X$ be the projections. For any $F = (F_2 \xrightarrow{\psi} F_1) \in \text{Comp}$, we take the triple product $\mathcal{T} \times q_R^* F$ and obtain a short exact sequence of vector bundles on X_R ,

$$0 \longrightarrow \mathcal{K}_F \longrightarrow \mathcal{E}_2 \otimes q_R^* F_1 \oplus \mathcal{E}_1 \otimes q_R^* F_2 \longrightarrow \mathcal{E}_1 \otimes q_R^* F_1 \longrightarrow 0.$$

We now define an integer $\text{bad}(\mathcal{T})$

$$\text{bad}(\mathcal{T}) := \min_{F \in \text{Comp}} \{\text{bad}(\mathcal{K}_F)\}$$

where $\text{bad}(\mathcal{K}_F)$ is the number introduced first by Hein in [10]: it is equal to ∞ if $\text{Supp}(R^1 p_{R*} \mathcal{K}_F)$ is whole $\text{Spec } R$ and to $\text{length}(R^1 p_{R*} \mathcal{K}_F)$ otherwise.

Since T_η is α -semistable, it follows from Lemma 4.2.4 that there exists $F_\eta \in \text{Comp}$ such that $H^1(X_\eta, K_\eta) = 0$ where K_η is the restriction of \mathcal{K}_{F_η} to X_η . Therefore $R^1 p_{R*} \mathcal{K}_{F_\eta}$ is zero at generic point η . So we have

$$\text{bad}(\mathcal{T}) \leq \text{length}(R^1 p_{R*} \mathcal{K}_{F_\eta}),$$

which is finite. It is also implied by Theorem 4.2.3 and base change that

$$T_0 \text{ is } \alpha\text{-semistable} \iff \text{bad}(\mathcal{T}) = 0.$$

Assume that T_0 is not α -semistable. Then $\text{bad}(\mathcal{T})$ is positive. It is implied by Lemma 4.4.2 that there exists a quotient T of T_0 such that

$$\mu_\alpha(T) < \mu_\alpha(T_0).$$

Let \mathcal{T}' be an elementary transformation of \mathcal{T} along T . If we can show that

$$\text{bad}(\mathcal{T}') < \text{bad}(\mathcal{T})$$

then after taking a finite number of elementary transformations, we obtain the right one, i.e. an elementary transformation \mathcal{T}' such that $\text{bad}(\mathcal{T}')$ is zero.

For any $F \in \text{Comp}$, we have (see (4.24))

$$\chi(T_s \times F) = \chi(K_s) = 0$$

where T_s and K_s are the restriction of \mathcal{T} and \mathcal{K}_F , respectively, on the fiber over $s \in \text{Spec } R$. Choose a triple $F = (F_2 \xrightarrow{\psi} F_1) \in \text{Comp}$ such that

$$\text{bad}(\mathcal{T}) = \text{length}(R^1 p_{R*} \mathcal{K}_F).$$

It implies that $R^1 p_{R*} \mathcal{K}_F$ is supported only on the closed point $0 \in \text{Spec } R$. So we have

$$H^0(X_\eta, K_\eta) = H^1(X_\eta, K_\eta) = 0.$$

Hence $p_{R*} \mathcal{K}_F = 0$ since it is torsion free and zero at generic point.

Consider the following exact sequence of triples on X_R ,

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{T} \longrightarrow T \longrightarrow 0.$$

By taking triple product with $q_R^* F$ and applying the kernel-cokernel sequence to the obtained diagram, we have the following exact sequence

$$0 \longrightarrow \mathcal{K}'_F \longrightarrow \mathcal{K}_F \longrightarrow K \longrightarrow 0.$$

Since $p_{R*} \mathcal{K}_F = 0$, by applying functor p_{R*} to this exact sequence, we get an exact sequence

$$0 \longrightarrow p_{R*} K \longrightarrow R^1 p_{R*} \mathcal{K}'_F \longrightarrow R^1 p_{R*} \mathcal{K}_F \longrightarrow R^1 p_{R*} K \longrightarrow 0.$$

We have $p_{R*} K = H^0(X_0, K)$ and $R^1 p_{R*} K = H^1(X_0, K)$ which are modules of finite length on R . Hence $R^1 p_{R*} \mathcal{K}'_F$ has finite length on R . In particular, we have

$$\text{length}(R^1 p_{R*} \mathcal{K}'_F) = \text{length}(R^1 p_{R*} \mathcal{K}_F) + \chi(K).$$

Assume that $T = (E_2 \xrightarrow{\varphi} E_1)$ on $X_0 \cong X$, then

$$0 \longrightarrow K \longrightarrow E_1 \otimes F_2 \oplus E_2 \otimes F_1 \longrightarrow E_1 \otimes F_1 \longrightarrow 0,$$

is an exact sequence. We have

$$\mu(K) = \mu_\alpha(T) + \mu(F_1)$$

(see also (4.14)). Similarly $\mu(K_0) = \mu_\alpha(T_0) + \mu(F_1)$. It implies that $\mu(K) < \mu(K_0)$ since $\mu_\alpha(T_0) > \mu_\alpha(T)$. So we have

$$\chi(K) < \chi(K_0) = 0$$

and then

$$\text{length}(R^1 p_{R*} \mathcal{K}'_F) < \text{length}(R^1 p_{R*} \mathcal{K}_F).$$

□

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