A local proof of the Breuil-Mézard conjecture in the scalar semi-simplification case

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von
Fabian Sander

vorgelegt an der Fakultät für
Mathematik der Universität
Duisburg-Essen

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Gutachter:
Prof. Dr. Vytautas Paškūnas
Prof. Dr. Jan Kohlhaase
Prof. Dr. Ariane Mézard

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Part 1. Introduction

The most celebrated result in mathematics in the 20th century is certainly Andrew Wiles' proof of Fermat's Last Theorem, in which he showed that Galois representations attached to semi-stable elliptic curves come from cuspidal modular forms. More generally, there is the following deep conjecture by Jean-Marc Fontaine and Barry Mazur.

Conjecture 0.1. *(Fontaine-Mazur (1995),* [16]*) Let $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous irreducible representation that is unramified at all but finitely many primes and is not a twist of an even representation with finite image. Then ρ is associated to a cuspidal modular form if and only if it is potentially semi-stable at p.*

Under the assumptions that ρ is odd, i.e. $\det\rho(c) = -1$ for any complex conjugation c in $G_{\mathbb{Q}_p}$, and the Hodge-Tate weights of ρ are distinct, this conjecture was proved by the work Kisin [27] and Matthew Emerton [13]. Frank Calegari has shown that the oddness assumption can be removed under mild conditions on the mod p reduction of ρ if $p > 7$ [8],[23]. A key ingredient in Kisin's proof is the Breuil-Mézard conjecture, which predicts the Hilbert-Samuel multiplicity of potentially semi-stable deformation rings of 2-dimensional $G_{\mathbb{Q}_p}$-representations over $\overline{\mathbb{F}}_p$.

In this thesis we want to give a purely local proof of the conjecture in the so-called scalar semi-simplification case. In the following, we want to explain the Breuil-Mézard conjecture in more detail.

1. Deformations of Galois representations

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer ϖ and residue field k. Let $E/\mathbb{Q}_p$ be a finite field extension and let $\rho\colon G_E \to \mathrm{GL}_n(k)$ be a continuous representation. We define $\mathcal{C}_{\mathcal{O}}$ to be the category of artinian local $\mathcal{O}$-algebras with residue field k. We say that a *lift of ρ to $A \in \mathrm{Obj}(\mathcal{C}_{\mathcal{O}})$* is a continuous representation $\rho_A\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_n(A)$ such that the following diagram commutes

$$\begin{tikzcd} G_E \arrow[r, "\rho_A"] \arrow[dr, "\rho"'] & \mathrm{GL}_n(A) \arrow[d, "\mathrm{pr}"] \\ & \mathrm{GL}_n(k). \end{tikzcd}$$

This leads to the *framed deformation functor*

$$\mathrm{Def}^{\square}\colon \mathcal{C}_{\mathcal{O}} \to \mathrm{Sets}$$
$$A \mapsto \{\text{lifts of } \rho \text{ to } A\}.$$

This functor is pro-representable by a noetherian local $\mathcal{O}$-algebra $R_\rho^{\square}$, the *universal framed deformation ring*, with the associated *universal framed deformation* $\rho^{\square}$.

For any $\mathfrak{p} \in \text{m-Spec}(R_\rho^{\square}[1/p])$, the set of maximal ideals, the residue field $\kappa(\mathfrak{p})$ is a finite extension of $\mathbb{Q}_p$. If we denote its ring of integers by $\mathcal{O}_{\mathfrak{p}}$, we get an associated

p-adic representation $\rho_{\mathfrak{p}}\colon G_E \to \mathrm{GL}_n(\mathcal{O}_{\mathfrak{p}})$ that lifts ρ:

$$\begin{tikzcd} G_E \arrow[r, "\rho^\square"] \arrow[dr, "\rho"'] \arrow[rr, bend left, "\rho_{\mathfrak{p}}"] & \mathrm{GL}_n(R_\rho^\square) \arrow[r, "\bmod\ \mathfrak{p}"] \arrow[d, "\mathrm{pr}"] & \mathrm{GL}_n(\mathcal{O}_{\mathfrak{p}}) \arrow[dl, "\mathrm{pr}"] \\ & \mathrm{GL}_n(k) & \end{tikzcd}$$

Now that we know that we can parameterize p-adic representations by the framed deformation ring $R_\rho^\square$, one can ask whether it is possible to find quotients of it that parameterize representations with certain properties like fixed determinants or being unramified. We want to explain the properties and invariants of of p-adic representations that we need to state the Breuil-Mézard conjecture.

2. p-adic Galois representations

We want to remind the reader of Fontaine's classification of p-adic representations via the so-called *rings of periods* that are equipped with a $G_{\mathbb{Q}_p}$-action and various additional structures. The following definition is taken from [2].

Definition 2.1. *A* ring of periods *is a topological $\mathbb{Q}_p$-algebra B which is an integral domain and is equipped with a G_E-action and the following properties*

(i) $\mathrm{Frac}(B)^{G_E} = B^{G_E}$,

(ii) *if there is a G_E-invariant line $\mathbb{Q}_p \cdot b$ for some $b \in B$, then $b \in B^\times$.*

For a p-adic representation V of G_E we define

$$D_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_E}. \tag{1}$$

Then we obtain a natural map

$$B \otimes_{B^{G_E}} D_B(V) \to B \otimes_{\mathbb{Q}_p} V. \tag{2}$$

This map is injective by property (i), and by property (ii) it is surjective if and only if $\dim_{B^{G_E}} D_B(V) = \dim_{\mathbb{Q}_p} V$. If this holds, then we say that V is B-admissible.

So via a ring of periods B we can define the subcategory of B-admissible p-adic G_E-representations. If B has special structures like a filtration, a Frobenius morphism or a monodromy operator, then for a B-admissible representation V we get these structures induced on $D_B(V)$. So we can try to understand such representations by analyzing the associated module $D_B(V)$. The most important examples of rings of periods are $B_{\mathrm{dR}}, B_{\mathrm{st}}, B_{\mathrm{cr}}$, which have, for example, the following structures (in addition to the G_E-action):

- B_{dR}: a filtration $\{\mathrm{Fil}^i B_{\mathrm{dR}}\}_{i\in\mathbb{Z}}$ that is decreasing, separated and exhaustive, i.e. $\mathrm{Fil}^i B_{\mathrm{dR}} \supseteq \mathrm{Fil}^{i+1} B_{\mathrm{dR}}$ for all $i \in \mathbb{Z}$, $\mathrm{Fil}^i B_{\mathrm{dR}} = B_{\mathrm{dR}}$ for $i \ll 0$ and $\mathrm{Fil}^i B_{\mathrm{dR}} = 0$ for $i \gg 0$,
- B_{st}: a Frobenius endomorphism φ, a monodromy operator N,
- B_{cr}: a filtration (analogous to B_{dR}), a Frobenius endomorphism φ.

Definition 2.2. *Let $* \in \{\mathrm{dR}, \mathrm{st}, \mathrm{cr}\}$. If a p-adic representation V is B_*-admissible, then we say that V is* de Rham *(resp.* semi-stable, crystalline*). Furthermore, we say that V is* potentially semi-stable *(resp.* potentially crystalline*) if there is a finite extension E'/E such that $V|_{G_{E'}}$ is semi-stable (resp. crystalline).*

Remark 2.3. *We have the following inclusion of subcategories*

$$\{\textit{crystalline rep.}\} \subsetneq \{\textit{semi-stable rep.}\} \subsetneq \{\textit{de Rham rep.}\} \subsetneq \{\textit{all adic rep.}\}.$$

Definition 2.4. *Let V be a de Rham representation. The* Hodge-Tate weights *of V are $\{h \in \mathbb{Z} | \operatorname{Fil}^{-h} D_{\mathrm{dR}}(V) \neq \operatorname{Fil}^{-h+1} D_{\mathrm{dR}}(V)\}$.*

There is the following deep theorem of Berger.

Theorem 2.5. *(Berger,[1]) A p-adic representation is de Rham if and only if it is potentially semi-stable.*

Now we want to have closer look at semi-stable representations. We need the following definition.

Definition 2.6. *Let E_0 be the maximal unramified extension of $\mathbb{Q}_p$ inside E. A* (φ, N)-module over E_0 *is a finite dimensional E_0-vector space D with endomorphisms φ, N such that*

- *N is E_0-linear,*
- *φ acts E_0-semilinear via the absolute Frobenius of $G_{E_0/\mathbb{Q}_p}$,*
- *$N\varphi = p\varphi N$, in particular, N is nilpotent.*

A filtered (φ, N)-module over E *is a (φ, N)-module D over E_0 equipped with a decreasing, separated and exhaustive filtration on $E \otimes_{E_0} D$. We call a filtered (φ, N)-module D over E* admissible *if $\dim_E D < \infty$, φ is an isomorphism and for any sub-(φ, N)-module D' the Hodge polygon of D' lies above the Newton polygon of D' and the polygons of D end up at the same point. If $N = 0$, then we say that D is a* φ-module.

Theorem 2.7. *(Colmez-Fontaine,[10]) We have the following equivalences of categories.*

$$\{\textit{semi-stable rep. of } G_E\} \xrightarrow{D_{\mathrm{st}}} \{\textit{admissible filtered } (\varphi, N)\textit{-modules over } E\},$$

$$\{\textit{crystalline rep. of } G_E\} \xrightarrow{D_{\mathrm{cr}}} \{\textit{admissible filtered } \varphi\textit{-modules over } E\}.$$

Remark 2.8. *If there is a finite extension E'/E such that $\rho|_{G_{E'}}$ is semi-stable, then $D_{\mathrm{st},E'}(\rho) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \rho|_{G_{E'}})^{G_{E'}}$ is an admissible filtered $(\varphi, N, G_{E'/E})$-module, i.e. there is a E-linear $G_{E'/E}$-action that commutes with φ and N.*

Now that we know that we can pass from a potentially semi-stable G_E-representation V to an admissible filtered $(\varphi, N, G_{E'/E})$-module, we want to explain how one can use this module to attach a *Weil-Deligne representation* to V.

We let k_E denote the residue field of the ring of integers $\mathcal{O}_E$ and set $q := \# k_E$. The *inertia subgroup of* G_E is defined by the following short exact sequence.

$$1 \longrightarrow I_E \longrightarrow G_E \xrightarrow{v} G_{k_E} \longrightarrow 0. \tag{3}$$

We let $\mathrm{Fr} \in G_{k_E}$ denote the geometric Frobenius and obtain an isomorphism $G_{k_E} \cong \hat{\mathbb{Z}}$ via sending $\mathrm{Fr} \mapsto 1$. Now we can define the *Weil group* of E to be

$$W_E := \{g \in G_E \mid v(g) \in \mathbb{Z}\}. \tag{4}$$

We define a topology on W_E by demanding that its natural subgroup I_E, equipped with its usual topology, is open in W_E.

Definition 2.9. *Let L be a field of characteristic 0. A* Weil-Deligne representation *over L is a pair (r, N), where $r\colon W_E \to \mathrm{GL}(V)$ is a continuous representation of W_E on a finite-dimensional L-vector space V with an open kernel and $N \in \mathrm{End}(V)$ is nilpotent such that for all $g \in W_E$*

$$r(g)Nr(g)^{-1} = q^{-v(g)}N. \tag{5}$$

If (r, N) is a Weil-Deligne representation, then we call $(r|_{I_E}, N)$ an inertial type *and we say that (r, N) is* Frobenius semi-simple *if r is semi-simple.*

Example 2.10. *Let (r, N) be a Weil-Deligne representation over a field L of characteristic 0 as above.*

(i) *If $\dim_L r = 1$, then $N\colon L \to L$ is supposed to be a nilpotent endomorphism, which forces it to be the zero map. Hence 1-dimensional Weil-Deligne representations over L are simply continuous characters $r\colon W_E \to L^\times$ with an open kernel.*

(ii) *We want to construct a Weil-Deligne representation with non-trivial operator N. Let $r\colon W_E \to \mathrm{GL}(V)$ be a continuous finite-dimensional representation with an open kernel. Let $\mathrm{Art}\colon E^\times \xrightarrow{\cong} W_E^{\mathrm{ab}}$ be the Artin map given by local class field theory. Then we can define a W_E-representation*

$$r' := r \oplus r\left|\mathrm{Art}^{-1}\right|_E,$$

and a nilpotent endomorphism N that induces an isomorphism of vector spaces $r \xrightarrow{\cong} r\left|\mathrm{Art}^{-1}\right|_E$ and is zero on $r\left|\mathrm{Art}^{-1}\right|_E$. One easily calculates that in fact

$$r'N(g) = q^{-v(g)}Nr'(g), \text{ for all } g \in W_E.$$

Hence (r', N) is a Weil-Deligne representation.

Now we want to construct a Weil-Deligne representation (r, N) associated to a potentially semi-stable representation $\tilde{\rho}\colon G_E \to \mathrm{GL}(V)$. First we can attach to $\tilde{\rho}$ a $(\varphi, N, \mathrm{Gal}(E'/E))$-module $D_{\mathrm{st}}(\tilde{\rho}|_{G_{E'}})$, where E' is chosen such that $\tilde{\rho}|_{G_{E'}}$ is semi-stable. Then we can define a W_E-action on $D_{\mathrm{st}}(\tilde{\rho}|_{G_{E'}})$ by letting $g \in W_E$ act as

$$r(g) := g\varphi^{-nv(g)},$$

where $q = \#k_E = p^n$ and the action of W_E factors through $\mathrm{Gal}(E'/E)$. Since we have $N\varphi = p\varphi N$ on $D_{\mathrm{st}}(\tilde{\rho}|_{G_{E'}})$, we obtain

$$Nr(g) = q^{-v(g)}r(g)N.$$

We define

$$\mathrm{WD}(\tilde{\rho}) := (r, N).$$

This construction is independent of the choice of E'. Moreover, $\tilde{\rho}$ is semi-stable if and only if r is unramified, and it is crystalline if and only if r is unramified and $N = 0$. Thus $\tilde{\rho}$ is potentially crystalline if and only if $N = 0$.

Now we have finally defined all the invariants we are interested in and can make the following definition.

Definition 2.11. *Let $\tau\colon I_E \to \mathrm{GL}_n(L)$ be an inertial type, let $\mathbf{w}$ be a n-tuple of integers and let $\psi\colon G_E \to \mathcal{O}^\times$ be a continuous character. We say that a continuous potentially semi-stable representation $\tilde{\rho}\colon G_E \to \mathrm{GL}_n(L)$ is* of p-adic Hodge type

$(\mathbf{w},\tau,\psi)$ *if its Hodge-Tate weights are* $\mathbf{w}$*,* $\det\tilde\rho\cong\psi\epsilon$*, where* ϵ *is the cyclotomic character and* $WD(\tilde\rho)|_{I_E}\cong\tau$*.*

The question whether there exists a quotient of the universal framed deformation ring $R_\rho^\square$ that parametrizes p-adic representations of a certain Hodge type was answered by a deep theorem of Mark Kisin.

Theorem 2.12. *(Kisin, [26]) There exists a unique reduced and* $\mathcal{O}$*-torsion free quotient* $R_\rho^\square(\mathbf{w},\tau,\psi)$ *(resp.* $R_\rho^{\square,\mathrm{cr}}(\mathbf{w},\tau,\psi)$*) of* $R_\rho^\square$ *such that for all* $\mathfrak{p}\in\text{m-Spec}(R_\rho^\square[1/p])$ *the associated map* $R_\rho^\square\to R_\rho^\square/\mathfrak{p}$ *factors through* $R_\rho^\square(\mathbf{w},\tau,\psi)$ *(resp.* $R_\rho^{\square,\mathrm{cr}}(\mathbf{w},\tau,\psi)$*) if and only if* $\rho_\mathfrak{p}$ *is potentially semi-stable (resp. potentially crystalline) of p-adic Hodge type* $(\mathbf{w},\tau,\psi)$*.*

The objects on the Galois side of the Breuil-Mézard conjecture will be the special fibers of the rings $R_\rho^\square(\mathbf{w},\tau,\psi)$ (resp. $R_\rho^{\square,\mathrm{cr}}(\mathbf{w},\tau,\psi)$).

3. Automorphic multiplicities for GL_2

We want to start by reminding the reader of the *classical Local Langlands correspondence* that was proved by Harris-Taylor [20] and Henniart [22]. It is given by a family of bijections rec_E:

$$\{\text{smooth irreducible complex representations of } \mathrm{GL}_n(E)\}$$
$$\xrightarrow{\mathrm{rec}_E}$$
$$\{n\text{-dimensional Frobenius-semisimple Weil-Deligne representations of } W_E \text{ over } \mathbb{C}\}.$$

These bijections are compatible with twisting by characters and preserve L-factors and ϵ-factors. By fixing an isomorphism $\iota\colon\mathbb{C}\xrightarrow{\cong}\overline{\mathbb{Q}}_p$ we also get the correspondence for smooth irreducible representations on p-adic vector spaces.

We now specialize to the case $n=2$. To establish a connection between Kisin's semi-stable deformation rings and invariants that arise in the representation theory of $\mathrm{GL}_2(\mathbb{Z}_p)$, we need the *inertial local Langlands correspondence* due to Henniart (where the representation $\sigma^{\mathrm{cr}}(\tau)$ was defined by Kisin in [27]).

Theorem 3.1. *(Henniart, [21]) Let* $\tau\colon I_E\to\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ *be an inertial type. Then there exists a finite-dimensional* $\overline{\mathbb{Q}}_p$*-representation* $\sigma(\tau)$ *(resp.* $\sigma^{\mathrm{cr}}(\tau)$*) of* $\mathrm{GL}_2(\mathcal{O}_E)$ *such that if* $\tilde\tau$ *is any Frobenius-semisimple Weil-Deligne representation of* W_E *over* $\overline{\mathbb{Q}}_p$*, then* $\mathrm{rec}_E^{-1}(\tilde\tau)\big|_{\mathrm{GL}_2(\mathcal{O}_E)}$ *contains* $\sigma(\tau)$ *if and only if* $\tilde\tau|_{I_E}\cong\tau$ *(resp.* $\tilde\tau|_{I_E}\cong\tau$ *and the monodromy operator* N *is trivial).*

Example 3.2. *(see [21]) Let* $\chi_1,\chi_2\colon\mathcal{O}_E^\times\to\overline{\mathbb{Q}}_p^\times$ *be distinct characters that we consider as characters of* I_E *via local class field theory. Let* n *be minimal such that* $\chi_1\chi_2^{-1}\big|_{1+\mathfrak{m}_E^n}=\mathbb{1}$ *and let* $J_n:=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathcal{O}_E)\mid c\in\mathfrak{m}_E^n\}$*. Then for the inertial type* $\tau=\chi_1\oplus\chi_2$ *the inertial local Langlands correspondence associates the* $\mathrm{GL}_2(\mathcal{O}_E)$*-representation* $\sigma(\tau)=\mathrm{Ind}_{J_n}^{\mathrm{GL}_2(\mathcal{O}_E)}\chi_1\otimes\chi_2$ *given by right translations on the vector space*

$$\{f\colon\mathrm{GL}_2(\mathcal{O}_E)\to\overline{\mathbb{Q}}_p\mid f(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)g)=\chi_1(a)\chi_2(d)f(g)\ \forall g\in\mathrm{GL}_2(\mathcal{O}_E)\}.$$

We have $\sigma(\tau)=\sigma^{\mathrm{cr}}(\tau)$ except the case when $\tau=\chi\oplus\chi$. Then $\sigma(\tau)=\tilde{\mathrm{st}}\otimes\chi\circ\det$, where $\tilde{\mathrm{st}}$ is the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_p)$ inflated to $\mathrm{GL}_2(\mathbb{Z}_p)$, and $\sigma^{\mathrm{cr}}(\tau)=\chi\circ\det$.

Now we are able to define the numbers that appear on the GL_2-side of Breuil-Mézard conjecture. We again fix a residual representation $\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ and a Hodge type $(\mathbf{w}, \tau, \psi)$, where τ is defined over a p-adic field L, as in Definition 2.11. We let $\sigma(\tau)$ (resp. $\sigma^{\mathrm{cr}}(\tau)$) be the smooth irreducible representation of $K := \mathrm{GL}_2(\mathbb{Z}_p)$ associated to τ by Theorem 3.1. By enlarging L if necessary, we can assume that $\sigma(\tau)$ (resp. $\sigma^{\mathrm{cr}}(\tau)$) is defined over L. We define

$$\sigma(\mathbf{w}, \tau) := \sigma(\tau) \otimes \mathrm{Sym}^{b-a-1} L^2 \otimes \det{}^a$$

and let $\overline{\sigma(\mathbf{w}, \tau)}$ be the semi-simplification of the reduction of a K-invariant $\mathcal{O}$-lattice modulo ϖ. One can show that $\overline{\sigma(\mathbf{w}, \tau)}$ is independent of the choice of the lattice. Now any irreducible finite-dimensional K-representation over k is isomorphic to $\sigma_{n,m} := \mathrm{Sym}^n k^2 \otimes \det^m$, where $n, m \in \mathbb{N}$, $0 \le n \le p-1$, $0 \le m \le p-2$. For $\sigma_{n,m}$ we let $a_{n,m}$ denote the multiplicity with which $\sigma_{n,m}$ occurs in $\overline{\sigma(\mathbf{w}, \tau)}$. Analogously we define

$$\sigma^{\mathrm{cr}}(\mathbf{w}, \tau) := \sigma^{\mathrm{cr}}(\tau) \otimes \mathrm{Sym}^{b-a-1} L^2 \otimes \det{}^a$$

and let $a^{\mathrm{cr}}_{n,m}$ denote the multiplicity with which $\sigma_{n,m}$ occurs in $\sigma^{\mathrm{cr}}(\mathbf{w}, \tau)$.

4. Hilbert-Samuel multiplicities

Definition/Lemma 4.1. *Let A be a d-dimensional noetherian local ring with maximal ideal $\mathfrak{m}$, let I be an $\mathfrak{m}$-primary ideal and let M be a finite A-module. Then there exists the* Hilbert-Samuel polynomial $\chi^I_M \in \mathbb{Q}[x]$, *which is characterized by*

$$\chi^I_M(n) = l(M/I^n M) \text{ for all } n \in \mathbb{N}, n \gg 0.$$

It satisfies the following properties, see [29, §13],

- $\deg(\chi^I_M) = \dim M$,
- $\chi^I_M(n) = e(M, I) n^d/d! + \dots$, *for $n \gg 0$ and some $e(M, I) \in \mathbb{Z}$.*

In the special case $M = A$ and $I = \mathfrak{m}$ the number $e(A) := e(A, \mathfrak{m})$ is called the Hilbert-Samuel multiplicity of A.

Example 4.2. *Let A be a local ring of dimension d with maximal ideal $\mathfrak{m}$.*

(i) *If A is regular, one easily computes $\chi^{\mathfrak{m}}_A(n) = \binom{n+d}{d}$ for all $n \in \mathbb{N}$. Thus we have $e(A) = 1$.*

(ii) *One possibility for A to have multiplicity greater than one is the existence of more than one irreducible component of dimension d. For example, for $A = \mathbb{F}_p[\![x, y]\!]/(xy)$ one computes $\chi^{(x,y)}_A(n) = 2n$. Since $\dim A = 1$, we have $e(A) = 2$. More generally, $e(\mathbb{F}_p[\![x, y]\!]/(x^a y^b)) = a + b$.*

5. The Breuil-Mézard conjecture

Now we have defined everything we need to finally state the Breuil-Mézard conjecture.

Conjecture 5.1. *(Breuil-Mézard,* [4]*) For all $n \in \{0, \dots, p-1\}, m \in \{0, \dots, p-2\}$ there exist $\mu_{n,m} \in \mathbb{N}$, only depending on ρ, such that for any Hodge type $(\mathbf{w}, \tau, \psi)$*

$$e(R^{\square}_{\rho}(\mathbf{w}, \tau, \psi)/(\varpi)) = \sum_{n,m} a_{n,m}(\mathbf{w}, \tau) \mu_{n,m},$$

$$e(R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w}, \tau, \psi)/(\varpi)) = \sum_{n,m} a^{\mathrm{cr}}_{n,m}(\mathbf{w}, \tau) \mu_{n,m}.$$

We remark that the conjecture implies that

$$\mu_{n,m} = e(R_\rho^{\square,\mathrm{cr}}((\tilde{m}, \tilde{m}+n+1), \mathbb{1} \oplus \mathbb{1}, \psi)/\varpi),$$

where $\tilde{m}$ is chosen such that $\psi|_{I_{\mathbb{Q}_p}} = \epsilon^{2\tilde{m}+n+1}$. Hence the conjecture predicts that, for every Hodge type $(\mathbf{w}, \tau, \psi)$, the Hilbert-Samuel multiplicities of $R_\rho^{\square}(\mathbf{w}, \tau, \psi)/(\varpi)$ and $R_\rho^{\square,\mathrm{cr}}(\mathbf{w}, \tau, \psi)/(\varpi)$ are determined by the multiplicities occurring in the crystalline deformation rings for small weights. In the case of Hodge-Tate weights $\mathbf{w} = (a, b)$ with $b - a \leq p - 2$, it follows from Fontaine-Laffaille theory that $R_\rho^{\square}(\mathbf{w}, \tau, \psi)$ and $R_\rho^{\square,\mathrm{cr}}(\mathbf{w}, \tau, \psi)$ are power series rings over $\mathcal{O}$ so that we have $e(R_\rho^{\square}(\mathbf{w}, \tau, \psi)/\varpi) = 1$ and $e(R_\rho^{\square,\mathrm{cr}}(\mathbf{w}, \tau, \psi)/\varpi) = 1$.

In the following, we will state a variant of the conjecture due to Matthew Emerton and Toby Gee [14].

Definition 5.2. *Let R be a noetherian ring. Then the* group of d-dimensional cycles *$Z_d(R)$ is the free abelian group generated by all $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\dim R/\mathfrak{p} = d$.*

Conjecture 5.3. *([14]) Let $d := \dim R_\rho^{\square}(\mathbf{w}, \tau, \psi)$. For every smooth irreducible representation σ of K over k there exists a $(d-1)$-dimensional cycle $z(\sigma, \rho)$ of $R_\rho^{\square}$, independent of the considered Hodge type such that there are equalities of $(d-1)$-dimensional cycles*

$$z_{d-1}(R_\rho^{\square}(\mathbf{w}, \tau, \psi)/(\varpi)) = \sum_\sigma m_\sigma(\mathbf{w}, \tau) z(\sigma, \rho),$$

$$z_{d-1}(R_\rho^{\square,\mathrm{cr}}(\mathbf{w}, \tau, \psi)/(\varpi)) = \sum_\sigma m_\sigma^{\mathrm{cr}}(\mathbf{w}, \tau) z(\sigma, \rho),$$

where the sums are taken over all smooth irreducible K-representations over k and $m_\sigma(\mathbf{w}, \tau) = a_{n,m}(\mathbf{w}, \tau)$, $m_\sigma^{\mathrm{cr}}(\mathbf{w}, \tau) = a_{n,m}^{\mathrm{cr}}(\mathbf{w}, \tau)$ for $\sigma = \mathrm{Sym}^n k^2 \otimes \det^m$.

Remark 5.4. (i) *The cycle version of the Breuil-Mézard conjecture implies the multiplicity version. If*

$$z_{d-1}(R_\rho^{\square}(\mathbf{w}, \tau, \psi)/(\varpi)) = \sum_\sigma m_\sigma(\mathbf{w}, \tau) z(\sigma, \rho),$$

then

$$\begin{aligned} e(z_{d-1}(R_\rho^{\square}(\mathbf{w}, \tau, \psi)/(\varpi))) &= e\big(\sum_\sigma m_\sigma(\mathbf{w}, \tau) z(\sigma, \rho)\big) \\ &= \sum_\sigma m_\sigma(\mathbf{w}, \tau) e(z(\sigma, \rho)), \end{aligned}$$

and for $\sigma = \mathrm{Sym}^n k^2 \otimes \det^m$ we set $\mu_{n,m} := e(z(\sigma, \rho))$. The equality in the crystalline case follows analogous.

(ii) *Let $\sigma \cong \mathrm{Sym}^n k^2 \otimes \det^m$. The conjecture implies that*

$$z(\sigma, \rho) = Z(R_\rho^{\square,\mathrm{cr}}((\tilde{m}, \tilde{m}+n+1), \mathbb{1} \oplus \mathbb{1}, \psi)/\varpi),$$

where $\tilde{m}$ is chosen such that $\psi|_{I_{\mathbb{Q}_p}} = \epsilon^{2\tilde{m}+n+1}$.

6. Results

The conjecture has been proved first by Mark Kisin [27] for $p > 2$ in all cases except from $\rho \cong \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$, where ω is the mod p cyclotomic character and χ is some character. Paškūnas gave a new proof of the conjecture for all ρ with scalar endomorphisms when $p \geq 5$, by using only local methods [34]. Furthermore, he developed a general formalism to prove statements like the Breuil-Mézard conjecture. The last missing case (for $p \geq 5$), when $\rho \cong \omega\chi \oplus \chi$, was done by Hu-Tan [24]. We also refer the reader to the various generalizations and modifications of the conjecture in [14], [18], [19].

One surprising consequence of Remark 5.4 is that the Hilbert-Samuel multiplicities of the rings $R_\rho^\square(\mathbf{w}, \tau, \psi)/(\varpi)$ and $R_\rho^{\square,\mathrm{cr}}(\mathbf{w}, \tau, \psi)/(\varpi)$ for any Hodge type are determined and can be calculated by knowing only the multiplicities of the rings $R_\rho^{\square,\mathrm{cr}}(\mathbf{w}, \mathbb{1} \oplus \mathbb{1}, \psi)/(\varpi)$ for small Hodge-Tate weights. Kisin predicted that these numbers are always 1 or 2. We found that these predictions were inaccurate if ρ has scalar semi-simplification. In particular, we show that the multiplicity is 4 if $\rho = \chi \oplus \chi$.

In this thesis we will always assume that $\rho \cong \begin{pmatrix} \chi & \phi \\ 0 & \chi \end{pmatrix}$ for some continuous character $\chi \colon G_{\mathbb{Q}_p} \to k^\times$. A special feature of this situation is that we can assume by twisting that χ is trivial so that the image of ρ is a p-group. Hence it factors through the maximal pro-p quotient of $G_{\mathbb{Q}_p}$ which is a free group in 2 letters, see [31, Thm. 7.5.11]. In the first part of this thesis we give explicit presentations of the crystalline deformation rings in this case when the Hodge-Tate weights are small.

Theorem 6.1. *([35]) Let a be an integer such that $\chi|_{I_{\mathbb{Q}_p}} \equiv \epsilon^a \pmod{\pi}$. The universal deformation ring $R_\rho^{\square,\mathrm{cr}}((a, a+p-1), \mathbb{1} \oplus \mathbb{1}, \psi)$ exists and has a presentation*

$$R_\rho^{\square,\mathrm{cr}}((a, a+p-1), \mathbb{1} \oplus \mathbb{1}, \psi) \cong \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!]/(I_1, I_2, I_3, I_4),$$

where

$$\begin{aligned}
I_1 =& (v + x_{11})(v - x_{11}) - x_{12}x_{21}, \\
I_2 =& (v + x_{11})^2 y_{12} - 2(v + x_{11})x_{12}y_{11} - x_{12}^2 y_{21}, \\
I_3 =& x_{21}^2 y_{12} - 2x_{21}(v - x_{11})y_{11} - (v - x_{11})^2 y_{21}, \\
I_4 =& (v + x_{11})x_{21}y_{12} - 2x_{12}x_{21}y_{11} - x_{12}(v - x_{11})y_{21},
\end{aligned}$$

and $v = \frac{\epsilon(\gamma)^{p-1}-1}{2} \in \mathcal{O}$ and $x_{12} := \hat{x}_{12} + [\phi(\gamma)]$, $y_{12} := \hat{y}_{12} + [\phi(\delta)]$ where $[\phi(\gamma)], [\phi(\delta)]$ denote the Teichmüller lifts of $\phi(\gamma)$ and $\phi(\delta)$ to $\mathcal{O}$.

This enables us to compute the Hilbert-Samuel multiplicities of the special fibers of these deformation rings. Moreover, it turns out that some of the crystalline deformation rings are more complicated than expected.

Theorem 6.2. *([35]) Let a be an integer such that $\chi|_{I_{\mathbb{Q}_p}} \equiv \epsilon^a \pmod{\pi}$. Then $R_\rho^{\square,\mathrm{cr}}((a,a+p-1), \mathbb{1}\oplus\mathbb{1}, \psi)/\pi$ is geometrically irreducible, generically reduced and*

$$e(R_\rho^{\square,\mathrm{cr}}((a,a+p-1), \mathbb{1}\oplus\mathbb{1}, \psi)/\pi) = \begin{cases} 1, & \text{if } \rho\otimes\chi^{-1} \text{ is ramified,} \\ 2, & \text{if } \rho\otimes\chi^{-1} \text{ is unramified, indecomposable,} \\ 4, & \text{if } \rho\otimes\chi^{-1} \text{ is split.} \end{cases}$$

In the last two cases, $R_\rho^{\square,\mathrm{cr}}((a,a+p-1), \mathbb{1}\oplus\mathbb{1}, \psi)$ is not Cohen-Macaulay.

In the second part of this thesis we give a new local proof of the Breuil-Mézard conjecture via a formalism of Paškūnas in our case.

Theorem 6.3. *([36]) Let $p > 2$ and let $(\mathbf{w}, \tau, \psi)$ be a Hodge type. There exists a reduced $\mathcal{O}$-torsion free quotient $R_\rho^\square(\mathbf{w},\tau,\psi)$ (resp. $R_\rho^{\square,\mathrm{cr}}(\mathbf{w},\tau,\psi)$) of $R_\rho^\square$ such that for all $\mathfrak{p} \in \text{m-Spec}(R_\rho^\square[1/p])$, $\mathfrak{p}$ is an element of $\text{m-Spec}\big(R_\rho^\square(\mathbf{w},\tau,\psi)[1/p]\big)$ (resp. $\text{m-Spec}\big(R_\rho^{\square,\mathrm{cr}}(\mathbf{w},\tau,\psi)[1/p]\big)$) if and only if $\rho_\mathfrak{p}^\square$ is potentially semi-stable (resp. potentially crystalline) of p-adic Hodge type $(\mathbf{w},\tau,\psi)$. If $R_\rho^\square(\mathbf{w},\tau,\psi)$ (resp. $R_\rho^{\square,\mathrm{cr}}(\mathbf{w},\tau,\psi)$) is non-zero, then it has Krull dimension 5.*

Furthermore, there exists a four-dimensional cycle $z(\rho)$ of $R_\rho^\square$ such that there are equalities of four-dimensional cycles

$$z_4\big(R_\rho^\square(\mathbf{w},\tau,\psi)/(\varpi)\big) = m_\lambda(\mathbf{w},\tau)z(\rho), \tag{6}$$

$$z_4\big(R_\rho^{\square,\mathrm{cr}}(\mathbf{w},\tau,\psi)/(\varpi)\big) = m_\lambda^{\mathrm{cr}}(\mathbf{w},\tau)z(\rho), \tag{7}$$

where $\lambda := \mathrm{Sym}^{p-2} k^2 \otimes \chi\circ\det$.

We remark that by this result, together with works of Paškūnas [34], Yongquan Hu and Fucheng Tan [24], the whole conjecture is now proved in the 2-dimensional case only by local methods, when $p \geq 5$.

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Part 2. Hilbert-Samuel multiplicities of certain deformation rings

1. Introduction

Let $p > 2$ be a prime. Let k be a finite field of characteristic p, E be a finite totally ramified extension of $W(k)[1/p]$ with ring of integers $\mathcal{O}$ and uniformizer π. For a given continuous representation $\bar{\rho}\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ we consider the universal framed deformation ring $R^{\square}_{\bar{\rho}}$ and the universal framed deformation $\rho^{univ}\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R^{\square}_{\bar{\rho}})$. For all $\mathfrak{p} \in \text{m-Spec}(R^{\square}_{\bar{\rho}}[1/p])$, the set of maximal ideals of $R^{\square}_{\bar{\rho}}[1/p]$, we can specialize the universal representation at $\mathfrak{p}$ to obtain the representation

$$\rho_{\mathfrak{p}}\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R^{\square}_{\bar{\rho}}[1/p]/\mathfrak{p}),$$

where $R^{\square}_{\bar{\rho}}[1/p]/\mathfrak{p}$ is a finite extension of $\mathbb{Q}_p$. Let $\tau\colon I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ be a representation with an open kernel, where $I_{\mathbb{Q}_p}$ is the inertia subgroup of $G_{\mathbb{Q}_p}$. We also fix integers a, b with $b \geq 0$ and a continuous character $\psi\colon G_{\mathbb{Q}_p} \to \mathcal{O}^{\times}$ such that $\overline{\psi\epsilon} = \det(\bar{\rho})$, where ϵ is the cyclotomic character. Kisin showed in [26] that there exist unique reduced $\mathcal{O}$-torsion free quotients $R^{\square,\psi}_{\bar{\rho}}(a,b,\tau)$ and $R^{\square,\psi}_{\bar{\rho},cris}(a,b,\tau)$ of $R^{\square}_{\bar{\rho}}$ with the property that $\rho_{\mathfrak{p}}$ factors through $R^{\square,\psi}_{\bar{\rho}}(a,b,\tau)$ resp. $R^{\square,\psi}_{\bar{\rho},cris}(a,b,\tau)$ if and only if $\rho_{\mathfrak{p}}$ is potentially semi-stable resp. potentially crystalline with Hodge-Tate weights $(a, a+b+1)$ and has determinant $\psi\epsilon$ and inertial type τ. If τ is trivial then $R^{\square,\psi}_{\bar{\rho},cris}(a,b) := R^{\square,\psi}_{\bar{\rho},cris}(a,b,\mathbb{1}\oplus\mathbb{1})$ parametrizes all the crystalline lifts of $\bar{\rho}$ with Hodge-Tate weights $(a, a+b+1)$ and determinant $\psi\epsilon$. The Breuil-Mézard conjecture, proved by Kisin for almost all $\bar{\rho}$, see also [4], [5], [14], [24], [34], says that the Hilbert-Samuel multiplicity of the ring $R^{\square,\psi}_{\bar{\rho}}(a,b,\tau)/\pi$ can be determined by computing certain automorphic multiplicities, which do not depend on $\bar{\rho}$, and the Hilbert-Samuel multiplicities of $R^{\square,\psi}_{\bar{\rho},cris}(a,b)$ in low weights for $0 \leq a \leq p-2, 0 \leq b \leq p-1$. For most $\bar{\rho}$, the Hilbert-Samuel multiplicities of $R^{\square,\psi}_{\bar{\rho},cris}(a,b)$ have already been determined. Our goal is to compute the Hilbert-Samuel multiplicity of the ring $R^{\square,\psi}_{\bar{\rho},cris}(a,b)$ with $0 \leq a \leq p-2, 0 \leq b \leq p-1$, when

$$\bar{\rho}\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k), \quad g \mapsto \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}.$$

One may show that $R^{\square,\psi}_{\bar{\rho},cris}(a,b)$ is zero if either $b \neq p-2$ or the restriction of χ to $I_{\mathbb{Q}_p}$ is not equal to ϵ^a modulo π.

Theorem 1.1. *Let a be an integer with $0 \leq a \leq p-2$ such that $\chi|_{I_{\mathbb{Q}_p}} \equiv \epsilon^a \pmod{\pi}$. Then $R^{\square,\psi}_{\bar{\rho},cris}(a,p-2)/\pi$ is geometrically irreducible, generically reduced and*

$$e(R^{\square,\psi}_{\bar{\rho},cris}(a,p-2)/\pi) = \begin{cases} 1, & \text{if } \bar{\rho}\otimes\chi^{-1} \text{ is ramified,} \\ 2, & \text{if } \bar{\rho}\otimes\chi^{-1} \text{ is unramified, indecomposable,} \\ 4, & \text{if } \bar{\rho}\otimes\chi^{-1} \text{ is split.} \end{cases}$$

In the last two cases, $R^{\square,\psi}_{\bar{\rho},cris}(a,p-2)$ is not Cohen-Macaulay.

The multiplicity 4 does not seem to have been anticipated in the literature, see for example [27, 1.1.6]. Our method is elementary in the sense that we do not use any integral p-adic Hodge theory. The only p-adic Hodge theoretic input is that if

ρ is a crystalline lift of $\bar{\rho}$ with Hodge-Tate weights $(0, p-1)$, then we have an exact sequence

$$0 \longrightarrow \epsilon^{p-1}\chi_1 \longrightarrow \rho \longrightarrow \chi_2 \longrightarrow 0,$$

where $\chi_1, \chi_2 \colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ are unramified characters. This allows us to convert the problem into a linear algebra problem, which we solve in Lemma 2.4. This gives us an explicit presentation of the ring $R^{\square,\psi}_{\bar{\rho},cris}(a, p-2)$, using which we compute the multiplicities in §4. Our argument gives a proof of the existence of $R^{\square.\psi}_{\bar{\rho},cris}(a, p-2)$ independent of [26]. After writing this note we discovered that the idea to convert the problem into linear algebra already appears in [37].

2. The universal deformation ring

After twisting we may assume that $\chi = 1$ and $a = 0$ so that

$$\bar{\rho}(g) = \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}.$$

Since the image of $\bar{\rho}$ in $\mathrm{GL}_2(k)$ is a p-group, the universal representation factors through the maximal pro-p quotient of $G_{\mathbb{Q}_p}$, which we denote by G. We have the following commuting diagram

$$\begin{tikzcd} G_{\mathbb{Q}_p} \arrow[r] \arrow[d] & G \arrow[d] \\ G^{ab}_{\mathbb{Q}_p} \arrow[r] & G^{ab}_{\mathbb{Q}_p}(p) \cong G^{ab} \end{tikzcd}$$

where $G^{ab}_{\mathbb{Q}_p} := \mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ is the maximal abelian quotient of $G_{\mathbb{Q}_p}$ and can be described by the exact sequence

$$1 \longrightarrow \mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p^{ur}) \longrightarrow G^{ab}_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{F}_p} \longrightarrow 1,$$

where $\mathbb{Q}_p^{ur}$ is the maximal unramified extension of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$. Local class field theory implies that the natural map

$$G^{ab}_{\mathbb{Q}_p} \to \mathrm{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$$

is an isomorphism, where μ_{p^∞} is the group of p-power order roots of unity in $\bar{\mathbb{Q}}_p$. The cyclotomic character ϵ induces an isomorphism

$$\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \xrightarrow[\epsilon]{\cong} \mathbb{Z}_p^\times$$

and $\mathrm{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \cong \hat{\mathbb{Z}}$, hence

$$G^{ab} \cong (1 + p\mathbb{Z}_p) \times \mathbb{Z}_p,$$

where the map onto the first factor is given by ϵ^{p-1}. We choose a pair of generators $\bar{\gamma}, \bar{\delta}$ of G^{ab} such that $\bar{\gamma} \mapsto (1+p, 0)$ and $\bar{\delta} \mapsto (1, 1)$. From [31, Thm 7.5.11] G is a free pro-p group of rank 2. We can choose generators γ, δ that lift $\bar{\gamma}, \bar{\delta}$. The way we choose these generators will be of importance in the following.

Lemma 2.1. *Let $\eta \colon G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ be a continuous character such that $\eta \equiv 1(p)$. Then $\eta = \epsilon^k \chi$ for an unramified character χ if and only if $\eta(\gamma) = \epsilon(\gamma)^k$ and $p - 1 | k$.*

Proof. " $\Rightarrow$ " : Since γ maps to identity in $\mathrm{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$, we clearly have $\chi(\gamma) = 1$ for every unramified character χ. Hence $\epsilon(\gamma)^k \equiv 1(p)$, which implies $p-1|k$.

" $\Leftarrow$ " : From $\eta\epsilon^{-k}(\gamma) = 1$ and the fact that δ maps to the image of identity in the maximal pro-p quotient of $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, we see that $\eta\epsilon^{-k} = \chi$ for an unramified character χ. $\square$

Since G is a free pro-p group generated by γ and δ, to give a framed deformation of $\bar{\rho}$ to $(A, \mathfrak{m}_A)$ is equivalent to give two matrices in $\mathrm{GL}_2(A)$ which reduce to $\bar{\rho}(\gamma)$ and $\bar{\rho}(\delta)$ modulo $\mathfrak{m}_A$. Since $p \neq 2$ we can write

$$R_{\bar{\rho}}^{\square} = \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, t_\gamma, y_{11}, \hat{y}_{12}, y_{21}, t_\delta]\!]$$

and the universal framed deformation is given by

$$\begin{aligned} \rho^{univ} \colon G &\to \mathrm{GL}_2(R_{\bar{\rho}}^{\square}), \\ \gamma &\mapsto \begin{pmatrix} 1 + t_\gamma + x_{11} & x_{12} \\ x_{21} & 1 + t_\gamma - x_{11} \end{pmatrix}, \\ \delta &\mapsto \begin{pmatrix} 1 + t_\delta + y_{11} & y_{12} \\ y_{21} & 1 + t_\delta - y_{11} \end{pmatrix}, \end{aligned}$$

where $x_{12} := \hat{x}_{12} + [\phi(\gamma)]$, $y_{12} := \hat{y}_{12} + [\phi(\delta)]$ where $[\phi(\gamma)], [\phi(\delta)]$ denote the Teichmüller lifts of $\phi(\gamma)$ and $\phi(\delta)$ to $\mathcal{O}$.

Remark 2.2. *We note that there are essentially* 3 *different cases:*

(1) $\bar{\rho}$ *is ramified* $\Leftrightarrow \phi(\gamma) \neq 0 \Leftrightarrow x_{12} \in (R_{\bar{\rho}}^{\square})^\times$,

(2) $\bar{\rho}$ *is unramified, non-split* $\Leftrightarrow \phi(\gamma) = 0, \phi(\delta) \neq 0 \Leftrightarrow x_{12} \in \mathfrak{m}_{R_{\bar{\rho}}^{\square}}, y_{12} \in (R_{\bar{\rho}}^{\square})^\times$,

(3) $\bar{\rho}$ *is split* $\Leftrightarrow \phi(\gamma) = 0, \phi(\delta) = 0 \Leftrightarrow x_{12}, y_{12} \in \mathfrak{m}_{R_{\bar{\rho}}^{\square}}$.

Let $\psi \colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a continuous character such that $\overline{\psi\epsilon} = \mathbb{1}$ and let $R_{\bar{\rho}}^{\square,\psi}$ be the quotient of $R_{\bar{\rho}}^{\square}$ which parametrizes lifts of $\bar{\rho}$ with determinant $\psi\epsilon$. Since γ, δ generate G as a group, we obtain

$$\begin{aligned} R_{\bar{\rho}}^{\square,\psi} &\cong R_{\bar{\rho}}^{\square}/(\det(\rho^{univ}(\gamma)) - \psi\epsilon(\gamma), \det(\rho^{univ}(\delta)) - \psi\epsilon(\delta)) \\ &\cong \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!], \end{aligned}$$

because we can eliminate the parameters t_γ, t_δ via applying Hensel's Lemma to the relations $(1+t_\gamma)^2 = \psi\epsilon(\gamma) + x_{11}^2 + x_{12}x_{21}, (1+t_\delta)^2 = \psi\epsilon(\delta) + y_{11}^2 + y_{12}y_{21}$. We let $v := \frac{1-\epsilon^{p-1}(\gamma)}{2}$ and define four polynomials

$$I_1 := (v + x_{11})(v - x_{11}) - x_{12}x_{21}, \tag{8}$$

$$I_2 := (v + x_{11})^2 y_{12} - 2(v + x_{11})x_{12}y_{11} - x_{12}^2 y_{21}, \tag{9}$$

$$I_3 := x_{21}^2 y_{12} - 2x_{21}(v - x_{11})y_{11} - (v - x_{11})^2 y_{21}, \tag{10}$$

$$I_4 := (v + x_{11})x_{21}y_{12} - 2x_{12}x_{21}y_{11} - x_{12}(v - x_{11})y_{21}. \tag{11}$$

Since for every representation with Hodge-Tate weights $(0, p-1)$ the determinant is a character of Hodge-Tate weight $p-1$ and $R_{\bar{\rho},cris}^{\square,\psi}(0, p-2)$ parametrizes all lifts $\rho_{\mathfrak{p}}$ with determinant $\psi\epsilon$, we let from now on ψ have Hodge-Tate weight $p-2$, as otherwise $R_{\bar{\rho},cris}^{\square,\psi}(0, p-2)$ would be trivial.

Definition 2.3. *We set*

$$R := R_{\bar{\rho}}^{\square,\psi}/(I_1, I_2, I_3, I_4).$$

Our goal is to show that $R^{\square,\psi}_{\bar\rho,cris}(0,p-2)$ is isomorphic to R.

Lemma 2.4. *If* $\mathfrak{p} \in$ m-Spec$(R^{\square,\psi}_{\bar\rho}[1/p])$, *then* $\mathfrak{p} \in$ m-Spec$(R[1/p])$ *if and only if* $\rho_{\mathfrak{p}}$ *is reducible and* $\rho_{\mathfrak{p}}(\gamma)$ *acts on a one-dimensional* G*-invariant subspace with eigenvalue* $\epsilon^{p-1}(\gamma)$.

Proof. Let $\mathfrak{p} \in$ m-Spec$(R^{\square,\psi}_{\bar\rho}[1/p])$ such that $\rho_{\mathfrak{p}}$ is reducible and $\rho_{\mathfrak{p}}(\gamma)$ acts on a one-dimensional G-invariant subspace with eigenvalue $\epsilon^{p-1}(\gamma)$. Since by assumption $\det(\rho_{\mathfrak{p}}(\gamma)) = \psi\epsilon(\gamma) = \epsilon(\gamma)^{p-1}$ and since $\epsilon(\gamma)^{p-1}$ is an eigenvalue of $\rho_{\mathfrak{p}}(\gamma)$, the other eigenvalue must be 1. Therefore we can write $1+t_\gamma = \frac{\epsilon(\gamma)^{p-1}+1}{2}$ and obtain

$$\begin{aligned} 0 &= \det\begin{pmatrix} 1+t_\gamma+x_{11}-\epsilon(\gamma)^{p-1} & x_{12} \\ x_{21} & 1+t_\gamma-x_{11}-\epsilon(\gamma)^{p-1}\end{pmatrix} \\ &= (v+x_{11})(v-x_{11}) - x_{12}x_{21}. \end{aligned}$$

If we now take $\mathfrak{p}$ as above but with $I_1 := (v+x_{11})(v-x_{11})-x_{12}x_{21} \in \mathfrak{p}$, it is easy to see that the vectors $v_1 = \begin{pmatrix} -x_{12} \\ v+x_{11}\end{pmatrix}$ and $v_2 = \begin{pmatrix} v-x_{11} \\ -x_{21}\end{pmatrix}$ are eigenvectors for $\rho_{\mathfrak{p}}(\gamma)$ with eigenvalue $\epsilon(\gamma)^{p-1}$ if they are non-zero. But at least one of them is non-zero because otherwise we obtain $v=0$ and thus $\epsilon(\gamma)^{p-1}=1$, which is a contradiction to the definition of γ. So $\rho_{\mathfrak{p}}$ is reducible with an invariant subspace on which $\rho_{\mathfrak{p}}(\gamma)$ acts by $\epsilon(\gamma)^{p-1}$ if and only if the vectors $v_1, v_2, \rho^{univ}(\delta)v_1, \rho^{univ}(\delta)v_2$ are pairwise linear dependent. It is easy to check that this is equivalent to the satisfaction of the equations $I_1 = I_2 = I_3 = I_4 = 0$. $\square$

Lemma 2.5.

$$\text{m-Spec}(R[1/p]) = \text{m-Spec}(R^{\square,\psi}_{\bar\rho}(0,p-2)[1/p]).$$

Proof. From [25, Prop.3.5(i)] we know that every crystalline lift $\rho_{\mathfrak{p}}$ of a reducible 2-dimensional representation $\bar\rho$, such that $\rho_{\mathfrak{p}}$ has Hodge-Tate-weights $(0,p-1)$, is reducible itself. Moreover, [6, Thm. 8.3.5] says that if ρ is a reducible 2-dimensional crystalline representation, then there are unramified characters χ_1,χ_2 and an exact sequence

$$0 \longrightarrow \epsilon^{p-1}\chi_1 \longrightarrow \rho \longrightarrow \chi_2 \longrightarrow 0.$$

Thus $\rho_{\mathfrak{p}}(\gamma)$ acts on the invariant subspace as $\epsilon(\gamma)^{p-1}$ and hence from Lemma 2.4 it is clear that

$$\text{m-Spec}(R[1/p]) \supset \text{m-Spec}(R^{\square,\psi}_{\bar\rho}(0,p-2)[1/p]).$$

For the other inclusion we note that it is also clear from Lemma 2.4 that any maximal ideal $\mathfrak{p} \in$ m-Spec$(R[1/p])$ gives rise to a reducible representation $\rho_{\mathfrak{p}}$ such that $\rho_{\mathfrak{p}}(\gamma)$ acts on the invariant subspace as $\epsilon(\gamma)^{p-1}$ and that the other eigenvalue of $\rho_{\mathfrak{p}}(\gamma)$ is 1. So we obtain with Lemma 2.1 that $\rho_{\mathfrak{p}}$ is an extension of two crystalline characters

$$0 \to \eta_1 \to * \to \eta_2 \to 0$$

where the Hodge-Tate-weight of η_1 is equal to $p-1$ and the weight of η_2 is equal to 0. Then we can conclude from [30, Prop. 1.28] that it is semi-stable and from [6, Thm. 8.3.5, Prop. 8.3.8] that it is crystalline and hence $\mathfrak{p} \in$ m-Spec$(R^{\square,\psi}_{\bar\rho}(0,p-2)[1/p])$. $\square$

Remark 2.6. *We have the following identities mod I_1:*

$$x_{21}I_2 = (v + x_{11})I_4, \tag{12}$$

$$(v - x_{11})I_2 = x_{12}I_4, \tag{13}$$

$$x_{21}I_4 = (v + x_{11})I_3, \tag{14}$$

$$(v - x_{11})I_4 = x_{12}I_3. \tag{15}$$

3. Reducedness

In order to show that $R^{\square,\psi}_{\bar{\rho}}(0, p-2)$ is equal to R, it is enough to show that R is reduced and $\mathcal{O}$-torsion free, since then the assertion follows from Lemma 2.5, as $R[1/p]$ is Jacobson because R is a quotient of a formal power series ring over a complete discrete valuation ring.

Lemma 3.1. *If $\mathcal{O} = W(k)$, then R is an $W(k)$-torsion-free integral domain.*

Proof. We distinguish two cases.

If $\bar{\rho}$ is ramified, i.e. x_{12} is invertible, we consider that for every complete local ring A with $a \in \mathfrak{m}_A, u \in A^\times$, there is a canonical isomorphism $A[\![z]\!]/(uz - a) \cong A$. Using this we see from (8),(9),(13) and (15) that

$$\begin{aligned} R =& \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!]/(I_1, I_2) \\ \cong& \mathcal{O}[\![x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]\!], \end{aligned}$$

which shows the claim.

In the second case, where $\bar{\rho}$ is unramified, i.e. $x_{12} \notin R^\times$, we consider the ideal $I := (\pi, x_{11}, x_{12}, x_{21})$ and have

$$\mathrm{gr}_I R^{\square,\psi}_{\bar{\rho}} \cong k[\![y_{11}, \hat{y}_{12}, y_{21}]\!][\pi, x_{11}, x_{12}, x_{21}].$$

Since $\mathcal{O} = W(k)$ we have $v \in I \setminus I^2$. Hence the elements I_1, I_2, I_3, I_4, regarded as polynomials in the variables $\pi, x_{11}, x_{12}, x_{21}$, are homogeneous of degree 2, so that

$$\mathrm{gr}_I R \cong k[\![y_{11}, \hat{y}_{12}, y_{21}]\!][\pi, x_{11}, x_{12}, x_{21}]/(I_1, I_2, I_3, I_4),$$

see [12, Ex. 5.3]. Because R is noetherian it follows from [12, Cor. 5.5.] that it is enough to show that $\mathrm{gr}_I R$ is an integral domain.

We define

$$A := k[\![y_{11}, \hat{y}_{12}, y_{21}]\!][x_{11}, x_{12}, x_{21}, \pi]/(I_1)$$

and look at the map

$$\phi\colon A \to A[x_{12}^{-1}]/(I_2).$$

The latter ring is isomorphic to $k[\![y_{11}, \hat{y}_{12}, y_{21}]\!][x_{11}, x_{12}, x_{12}^{-1}, \pi]/(I_2)$ and since I_2 is irreducible it is an integral domain. So we would be done by showing that $\ker(\phi) = (I_2, I_3, I_4)$. The inclusion $(I_2, I_3, I_4) \subset \ker(\phi)$ is clear from (13) and (15). For the other one we consider the fact that

$$\ker(\phi) = \{a \in A : \ \exists n \in \mathbb{N} \cup \{0\}, b, c, d \in A : x_{12}^n a = bI_2 + cI_3 + dI_4\}.$$

To show that $\ker(\phi) \subset (I_2, I_3, I_4)$, we let $a \in A$ and n be minimal with the property that there exist $b, c, d \in A$ such that

$$x_{12}^n a = bI_2 + cI_3 + dI_4. \tag{16}$$

If $n = 0$ there is nothing to show. Now we assume that $n > 0$ and consider the prime ideal $\mathfrak{p} := (x_{12}, v - x_{11}) \subset A$ and see that

$$A/\mathfrak{p} \cong k[\![y_{11}, y_{12}, y_{21}]\!][x_{11}, x_{21}]$$

is a unique factorization domain. We also observe that

$$I_2 \equiv y_{12}(v + x_{11})^2 \bmod \mathfrak{p}, \tag{17}$$

$$I_3 \equiv y_{12}x_{21}^2 \bmod \mathfrak{p}, \tag{18}$$

$$I_4 \equiv y_{12}(v + x_{11})x_{21} \bmod \mathfrak{p}. \tag{19}$$

Modulo $\mathfrak{p}$ (16) becomes

$$0 \equiv y_{12}b(v + x_{11})^2 + y_{12}cx_{21}^2 + y_{12}d(v + x_{11})x_{21}. \tag{20}$$

Since $A/\mathfrak{p}$ is a UFD there are $b_1, c_1 \in A$ such that

$$b \equiv b_1x_{21} \bmod \mathfrak{p}, \tag{21}$$

$$c \equiv c_1(v + x_{11}) \bmod \mathfrak{p}, \tag{22}$$

and we see that

$$d \equiv -\frac{b_1x_{21} + c_1(v + x_{11})}{2} \bmod \mathfrak{p}. \tag{23}$$

Hence we can find $b_2, b_3, c_2, c_3, d_1, d_2 \in A$ such that

$$\begin{aligned} b &= b_1x_{21} + b_2x_{12} + b_3(v - x_{11}), \\ c &= c_1(v + x_{11}) + c_2x_{12} + c_3(v - x_{11}), \\ d &= -\frac{b_1x_{21} + c_1(v + x_{11})}{2} + d_1x_{12} + d_2(v - x_{11}). \end{aligned}$$

Substituting this in (16) we use the relations (12)-(15) to get

$$x_{12}^n a = bI_2 + cI_3 + dI_4 \tag{24}$$

$$\begin{aligned} &= x_{12}(b_2I_2 + b_3I_4 + c_2I_3 + d_1I_4 + d_2I_3) \\ &\quad + \frac{1}{2}(b_1(v + x_{11}) + c_1x_{21})I_4 + (v - x_{11})c_3I_3. \end{aligned} \tag{25}$$

Modulo $\mathfrak{p}$ we get $b_1(v + x_{11}) + c_1x_{21} \equiv 0$ and hence there are $b_4, b_5, b_6, c_4, c_5, c_6$ with

$$b_1 = x_{21}b_4 + x_{12}b_5 + (v - x_{11})b_6, \tag{26}$$

$$c_1 = (v + x_{11})c_4 + x_{12}c_5 + (v - x_{11})c_6. \tag{27}$$

Hence we can rewrite (25) to

$$x_{12}^n a = x_{12}z + \frac{1}{2}(b_4 + c_4)(v + x_{11})^2I_3 + (v - x_{11})c_3I_3, \tag{28}$$

for a certain $z \in (I_2, I_3, I_4)$. So with (28) we see that $b_4 + c_4 \equiv 0$ modulo $\mathfrak{p}$ and $c_3 \equiv 0$ modulo the prime ideal $\mathfrak{p}' := (x_{12}, v + x_{11})$. Therefore we can find some $c_7, c_8, e_1, e_2 \in A$ with

$$\begin{aligned} c_3 &= c_7x_{12} + c_8(v + x_{11}), \\ b_4 + c_4 &= e_1x_{12} + e_2(v - x_{11}). \end{aligned}$$

But since we have $(v + x_{11})(v - x_{11}) = x_{12}x_{21}$ in A we can finally transform (28) to

$$x_{12}^n a = x_{12}z',$$

for some $z' \in (I_2, I_3, I_4)$, which shows that $x_{12}^{n-1} a \in (I_2, I_3, I_4)$, since A is an integral domain. But this is a contradiction to the minimality of n. $\square$

Proposition 3.2. *R is reduced and $\mathcal{O}$-torsion free for any choice of $\mathcal{O}$.*

Proof. Since $\mathcal{O}$ is flat over $W(k)$ and we have seen in Lemma 3.1 that

$$S := W(k)[\![x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!]/(I_1, I_2, I_3, I_4)$$

is an integral domain, we get an injection

$$\mathcal{O} \otimes_{W(k)} S \to \mathcal{O} \otimes_{W(k)} \mathrm{Quot}(S).$$

As S is $W(k)$-torsion-free by Lemma 2.5, we obtain an isomorphism

$$\mathcal{O} \otimes_{W(k)} \mathrm{Quot}(S) \xrightarrow{\cong} \mathcal{O}[1/p] \otimes_{W(k)[1/p]} \mathrm{Quot}(S).$$

Since $\mathcal{O}[1/p]$ is a separable field extension of $W(k)[1/p]$, we deduce that $\mathcal{O}[1/p] \otimes_{W(k)[1/p]} \mathrm{Quot}(S)$ is reduced and $\mathcal{O}$-torsion free. $\square$

4. The Multiplicity

We want to compute the Hilbert-Samuel-Multiplicity of the ring R/π for the given representation

$$\bar{\rho} \colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k), \quad g \mapsto \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}.$$

We denote the maximal ideal of R/π by $\mathfrak{m}$.

Theorem 4.1.

$$e(R/\pi) = \begin{cases} 1, & \text{if } \bar{\rho} \text{ is ramified,} \\ 2, & \text{if } \bar{\rho} \text{ is unramified, indecomposable,} \\ 4, & \text{if } \bar{\rho} \text{ is split.} \end{cases}$$

Proof. If we set $J := y_{12}x_{21} + 2x_{11}y_{11} + x_{12}y_{21} \in R$ we obtain modulo (π, J) the relations

$$I_2 \equiv - x_{12} J, \tag{29}$$

$$I_3 \equiv x_{21} J, \tag{30}$$

$$I_4 \equiv x_{11} J. \tag{31}$$

We split the proof into 3 cases as in Remark 2.2. If $\bar{\rho}$ is ramified, i.e. x_{12} is invertible, we see as in the proof of Lemma 3.1 that

$$R/\pi \cong k[\![x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!]/(x_{11}^2 + x_{12}x_{21}, J) \tag{32}$$

$$\cong k[\![x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]\!]. \tag{33}$$

Hence it is a regular local ring and therefore $e(R/\pi) = 1$.

Let us assume in the following that $\bar{\rho}$ is unramified, i.e. $x_{12} = \hat{x}_{12} \in \mathfrak{m}_R$, and we can consider the exact sequence

$$0 \to (R/\pi)/\mathrm{Ann}_{R/\pi}(J) \to R/\pi \to R/(\pi, J) \to 0. \tag{34}$$

From (29)-(31) we see that $x_{11}, x_{12}, x_{21} \in \mathrm{Ann}_{R/\pi}(J)$. But since $I_2, I_3, I_4 \in R/\pi$ are homogeneous polynomials of degree 2 in the variables x_{11}, x_{12}, x_{21}, no element in $k[\![y_{11}, \hat{y}_{12}, y_{21}]\!]$ annihilates J. Thus $\mathrm{Ann}_{R/\pi}(J) = (x_{11}, x_{12}, x_{21})$, which is a prime ideal. We obtain $\dim((R/\pi)/\mathrm{Ann}_{R/\pi}(J)) = 3$. From (32) we see that $\dim R/\pi \geq 4$

and since $\{x_{12}-x_{21}, x_{12}-\hat{y}_{12}, x_{12}-y_{21}, y_{11}\}$ can be checked to be a system of parameters for R/π, we get $\dim R/\pi = 4$. Therefore (34) gives us $e(R/\pi) = e(R/(\pi, J))$, see [29, Thm. 14.6]. We obtain that

$$\begin{aligned} R/(\pi, J) \cong & k[\![x_{11}, x_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!]/(x_{11}^2 + x_{12}x_{21}, J) \\ \cong & (k[\![x_{11}, x_{12}, x_{21}]\!]/(x_{11}^2 + x_{12}x_{21}))[\![y_{11}, \hat{y}_{12}, y_{21}]\!]/(J) \end{aligned}$$

is a complete intersection of dimension 4. So if $\mathfrak{q} \subset R/(\pi, J)$ is an ideal generated by 4 elements, such that $R/(\pi, J, \mathfrak{q})$ has finite length as a $R/(\pi, J)$-module, then these elements form a regular sequence in $R/(\pi, J)$ and $e_{\mathfrak{q}}(R/(\pi, J)) = l(R/(\pi, J, \mathfrak{q}))$, see [29, Thm. 17.11]. Besides, if there exists an integer n such that $\mathfrak{q}\mathfrak{m}^n = \mathfrak{m}^{n+1}$, then $e(R/(\pi, J)) = e_{\mathfrak{q}}(R/(\pi, J))$, see [29, Thm. 14.13]. So to finish the proof it would suffice to find such an ideal $\mathfrak{q}$.

If $\bar{\rho}$ is indecomposable, i.e. $\phi(\delta)$ is non-zero and therefore y_{12} is a unit in R, we can write the equation $J = 0$ as

$$x_{21} = -y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})$$

and $I_1 = 0$ as

$$x_{11}^2 = x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})$$

so that

$$R/(\pi, J) \cong k[\![x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]\!]/(x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})).$$

Hence it is clear that $x_{12}, x_{21}, y_{11}, \hat{y}_{12}$ form a system of parameters for $R/(\pi, J)$ that generates an ideal $\mathfrak{q}$ with $\mathfrak{q}\mathfrak{m} = \mathfrak{m}^2$. So we obtain

$$e_{\mathfrak{q}}(R/(\pi, J)) = l(R/(\pi, J, \mathfrak{q})) = l(k[\![x_{11}]\!]/(x_{11}^2)) = 2,$$

and hence $e(R/\pi) = 2$.

If $\bar{\rho}$ is split, which is equivalent to $x_{12}, y_{12} \notin R^\times$, we define

$$\mathfrak{q} := (x_{12} - x_{21}, x_{12} - y_{12}, x_{12} - y_{21}, y_{11})$$

and claim that $\mathfrak{q}\mathfrak{m}^2 = \mathfrak{m}^3$. If we write

$$\mathfrak{m} = (x_{12} - x_{21}, x_{12} - y_{12}, x_{12} - y_{21}, y_{11}, x_{11}, x_{12})$$

we just have to check that $x_{11}^3, x_{11}^2x_{12}, x_{11}x_{12}^2, x_{12}^3 \in \mathfrak{q}\mathfrak{m}^2$. Therefore it is enough to see that

$$\begin{aligned} x_{11}^2 =& x_{11}y_{11} - \frac{1}{2}(x_{12} - y_{12})x_{21} - \frac{1}{2}(x_{21} - y_{21})x_{12} \;\in \mathfrak{m}\mathfrak{q}, \\ x_{12}^2 =& -x_{11}^2 + x_{12}(x_{12} - x_{21}) \;\in \mathfrak{m}\mathfrak{q}. \end{aligned}$$

Hence

$$e(R/\pi) = l(R/(\pi, J, \mathfrak{q})) = l(k[\![x_{11}, x_{12}]\!]/(x_{11}^2, x_{12}^2)) = 4.$$

□

Corollary 4.2. *If $\bar{\rho}$ is unramified, then the ring R is not Cohen-Macaulay.*

Proof. Since R is $\mathcal{O}$-torsion free, π is R-regular and hence R is CM if and only if R/π is CM. In (34) we have constructed a non-zero submodule of R/π of dimension strictly less than the dimension of R/π. In particular, we showed that $\mathfrak{p} := (x_{11}, x_{12}, x_{21})$ is an associated prime of R/π with $\dim((R/\pi)/\mathfrak{p}) < \dim R/\pi$. It follows from [7, Thm. 2.1.2(a)] that R/π cannot be CM. □

Proposition 4.3. $\mathrm{Spec}(R/\pi)$ *is geometrically irreducible and generically reduced.*

To prove the Proposition we need the following Lemma. As in the proof of Theorem 4.1 we define $J := y_{12}x_{21} + 2x_{11}y_{11} + x_{12}y_{21}$.

Lemma 4.4. *$R/(\pi, J)$ is an integral domain.*

Proof. We again distinguish between 3 cases as in Remark 2.2. If $\bar{\rho}$ is ramified, i.e. x_{12} is invertible, we have already seen in the proof of Theorem 4.1 that

$$\begin{aligned} R/(\pi, J) &\cong k[\![x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!]/(x_{11}^2 + x_{12}x_{21}, J) \\ &\cong k[\![x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]\!]. \end{aligned}$$

If $\bar{\rho}$ is unramified and indecomposable, i.e. $x_{12} = \hat{x}_{12} \in \mathfrak{m}_R, y_{12} \in R^\times$ we saw that

$$R/(\pi, J) \cong k[\![x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]\!]/(x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})).$$

Since the element $x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})$ is irreducible in the unique factorization domain $k[\![x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]\!]$, the quotient is an integral domain. If $\bar{\rho}$ is unramified and split, i.e. $x_{12}, y_{12} \in \mathfrak{m}_R$, let $\mathfrak{n}$ denote the maximal ideal of $R/(\pi, J)$. It is enough to show that the graded ring $\mathrm{gr}_\mathfrak{n} R/(\pi, J)$ is a domain. Since J is homogeneous we have

$$\mathrm{gr}_\mathfrak{n} R/(\pi, J) \cong k[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}]/(x_{11}^2 + x_{12}x_{21}, J).$$

We set $A := k[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}]/(x_{11}^2 + x_{12}x_{21})$ and have to prove that $(J) \subset A$ is a prime ideal. We look at the localization map $A \xrightarrow{\iota} A[y_{21}^{-1}]$, which is an inclusion because y_{21} is regular in A. This gives us a map $A \xrightarrow{\bar{\iota}} A[y_{21}^{-1}]/(J)$. Since

$$A[y_{21}^{-1}]/(J) \cong k[x_{11}, x_{21}, y_{11}, y_{12}, y_{21}, y_{21}^{-1}]/(x_{11}^2 - x_{21}y_{21}^{-1}(2x_{11}y_{11} + x_{21}y_{12}))$$

is a domain, we would be done by showing that $\ker(\bar{\iota}) = (J)$. We have

$$\ker(\bar{\iota}) = \{a \in A : y_{21}^i a = bJ \text{ for some } i \in \mathbb{Z}_{\geq 0}, b \in A : y_{21} \nmid b\}.$$

But since $(y_{21}) \subset A$ is a prime ideal and y_{21} does not divide J, we see that $i = 0$ in all these equations and hence $\ker(\bar{\iota}) = (J)$. □

Proof of the Proposition. Let $\mathfrak{p}$ be a minimal prime ideal of $S := R/\pi$. It follows from (29)-(31) that $J^2 = 0$ and thus $J \in \mathrm{rad}(S) = \bigcap_{\mathfrak{p} \text{ minimal}} \mathfrak{p}$. So Lemma 4.4 gives us that JS is the only minimal prime ideal of S, hence $\mathrm{Spec}(S)$ is irreducible. If we replace the field k by an extension k', we obtain the irreducibility of $\mathrm{Spec}(S \otimes_k k')$ analogously, thus $\mathrm{Spec}(S)$ is geometrically irreducible.

$\mathrm{Spec}(S)$ is called generically reduced if $S_\mathfrak{p}$ is reduced for any minimal prime ideal $\mathfrak{p}$. We have already seen that the only minimal prime ideal is $\mathfrak{p} = JS$. By localizing (34) we obtain $S_\mathfrak{p} \cong R/(\pi, J)$. Lemma 4.4 implies that $S_\mathfrak{p}$ is reduced.

□

Part 3. A local proof of the Breuil-Mézard conjecture in the scalar semi-simplification case

1. Introduction

Let $p>2$ be a prime number, k be a finite field of characteristic p and L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer ϖ. Let $\rho\colon G_{\mathbb{Q}_p}\to \mathrm{GL}_2(k)$ be a continuous representation of the form

$$\rho(g)=\begin{pmatrix}\chi(g) & \phi(g)\\ 0 & \chi(g)\end{pmatrix}, \forall g\in G_{\mathbb{Q}_p}, \tag{35}$$

so that the semi-simplification of ρ is isomorphic to $\chi\oplus\chi$. Let $R^{\square}_{\rho}$ denote the associated universal framed deformation ring of ρ and let $\rho^{\square}$ be the universal framed deformation. For any $\mathfrak{p}\in \text{m-Spec}(R^{\square}_{\rho}[1/p])$, the set of maximal ideals, the residue field $\kappa(\mathfrak{p})$ is a finite extension of $\mathbb{Q}_p$. We denote its ring of integers by $\mathcal{O}_{\mathfrak{p}}$ and get an associated representation $\rho^{\square}_{\mathfrak{p}}\colon G_{\mathbb{Q}_p}\to \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ that lifts ρ. Let $\tau\colon I_{\mathbb{Q}_p}\to\mathrm{GL}_2(L)$ be a representation of the inertia group of $\mathbb{Q}_p$ with an open kernel, $\psi\colon G_{\mathbb{Q}_p}\to\mathcal{O}^{\times}$ a continuous character and let $\mathbf{w}=(a,b)$ be a pair of integers with $b>a$. We say that $\rho^{\square}_{\mathfrak{p}}$ is of p-adic Hodge type $(\mathbf{w},\tau,\psi)$ if it is potentially semi-stable with Hodge-Tate weights $\mathbf{w}$, $\det\rho_{\mathfrak{p}}\cong\psi\epsilon$, $\psi|_{I_{\mathbb{Q}_p}}=\epsilon^{a+b}\det\tau$ and $\mathrm{WD}(\rho^{\square}_{\mathfrak{p}})\big|_{I_{\mathbb{Q}_p}}\cong\tau$, where ϵ is the cyclotomic character and $\mathrm{WD}(\rho^{\square}_{\mathfrak{p}})$ is the Weil-Deligne representation associated to $\rho^{\square}_{\mathfrak{p}}$ by Fontaine [15].

By a result of Henniart [21] there exists a unique smooth irreducible $K:=\mathrm{GL}_2(\mathbb{Z}_p)$-representation $\sigma(\tau)$ and a modification $\sigma^{\mathrm{cr}}(\tau)$ defined by Kisin [27, 1.1.4] such that for any smooth absolutely irreducible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation π with associated Weil-Deligne representation $\mathrm{LL}(\pi)$ via the classical local Langlands correspondence, we have $\mathrm{Hom}_K(\sigma(\tau),\pi)\neq 0$ (resp. $\mathrm{Hom}_K(\sigma^{\mathrm{cr}}(\tau),\pi)\neq 0$) if and only if $\mathrm{LL}(\pi)|_{I_{\mathbb{Q}_p}}\cong\tau$ (resp. $\mathrm{LL}(\pi)|_{I_{\mathbb{Q}_p}}\cong\tau$ and the monodromy operator N on $\mathrm{LL}(\pi)$ is trivial). We have $\sigma(\tau)\not\cong\sigma^{\mathrm{cr}}(\tau)$ only if $\tau=\chi\oplus\chi$, in which case $\sigma(\tau)=\tilde{\mathrm{st}}\otimes\chi\circ\det$ and $\sigma^{\mathrm{cr}}(\tau)=\chi\circ\det$, where $\tilde{\mathrm{st}}$ is the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_p)$, inflated to $\mathrm{GL}_2(\mathbb{Z}_p)$, and χ is considered as a character of $\mathbb{Z}_p^{\times}$ via local class field theory. By enlarging L if necessary, we can assume that $\sigma(\tau)$ (resp. $\sigma^{\mathrm{cr}}(\tau)$) is defined over L. We define $\sigma(\mathbf{w},\tau):=\sigma(\tau)\otimes\mathrm{Sym}^{b-a-1}L^2\otimes\det^a$ and let $\overline{\sigma(\mathbf{w},\tau)}$ be the semi-simplification of the reduction of a K-invariant $\mathcal{O}$-lattice modulo ϖ. One can show that $\overline{\sigma(\mathbf{w},\tau)}$ is independent of the choice of the lattice. For every irreducible smooth finite-dimensional K-representation σ over k we let $m_{\sigma}(\mathbf{w},\tau)$ denote the multiplicity with which σ occurs in $\overline{\sigma(\mathbf{w},\tau)}$. Analogously we define $\sigma^{\mathrm{cr}}(\mathbf{w},\tau):=\sigma^{\mathrm{cr}}(\tau)\otimes\mathrm{Sym}^{b-a-1}L^2\otimes\det^a$ and let $m^{\mathrm{cr}}_{\sigma}(\mathbf{w},\tau)$ denote the multiplicity with which σ occurs in $\overline{\sigma^{\mathrm{cr}}(\mathbf{w},\tau)}$.

We prove the following theorem.

Theorem 1.1. *Let $p>2$ and let $(\mathbf{w},\tau,\psi)$ be a Hodge type. There exists a reduced $\mathcal{O}$-torsion free quotient $R^{\square}_{\rho}(\mathbf{w},\tau,\psi)$ (resp. $R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w},\tau,\psi)$) of $R^{\square}_{\rho}$ such that for all $\mathfrak{p}\in\text{m-Spec}(R^{\square}_{\rho}[1/p])$, $\mathfrak{p}$ is an element of $\text{m-Spec}\left(R^{\square}_{\rho}(\mathbf{w},\tau,\psi)[1/p]\right)$ (resp. $\text{m-Spec}\left(R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w},\tau,\psi)[1/p]\right)$) if and only if $\rho^{\square}_{\mathfrak{p}}$ is potentially semi-stable (resp. potentially crystalline) of p-adic Hodge type $(\mathbf{w},\tau,\psi)$. If $R^{\square}_{\rho}(\mathbf{w},\tau,\psi)$ (resp. $R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w},\tau,\psi)$) is non-zero, then it has Krull dimension 5.*

Furthermore, there exists a four-dimensional cycle $z(\rho)$ *of* $R_\rho^\square$ *such that there are equalities of four-dimensional cycles*

$$z_4\big(R_\rho^\square(\mathbf{w},\tau,\psi)/(\varpi)\big) = m_\lambda(\mathbf{w},\tau)z(\rho), \tag{36}$$

$$z_4\big(R_\rho^{\square,\mathrm{cr}}(\mathbf{w},\tau,\psi)/(\varpi)\big) = m_\lambda^{\mathrm{cr}}(\mathbf{w},\tau)z(\rho), \tag{37}$$

where $\lambda := \mathrm{Sym}^{p-2} k^2 \otimes \chi \circ \det$.

The equality of cycles also implies the analogous equality of Hilbert-Samuel multiplicities. Hence the above theorem proves the Breuil-Mézard conjecture [4], as stated in [27], in our case. This case has also been handled by Kisin in [27] using global methods, see also the errata in [19]. However, our proof is purely local and by our results, together with works of Paškūnas [34], Yongquan Hu and Fucheng Tan [24], the whole conjecture is now proved in the 2-dimensional case only by local methods, when $p \geq 5$.

2. Formalism

We quickly recall a formalism due to Paškūnas used by him to prove the Breuil-Mézard conjecture for residual representations with scalar endomorphisms in [34]. Let R be a complete local notherian commutative $\mathcal{O}$-algebra with residue field k. Let G be a p-adic analytic group, K a compact open subgroup and P its pro-p Sylow subgroup. Let N be a finitely generated $R[\![K]\!]$-module, let V be a continuous finite dimensional L-representation of K, and Θ be an $\mathcal{O}$-lattice in V which is invariant under the action of K. Let

$$M(\Theta) := \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \mathrm{Hom}_{\mathcal{O}}(\Theta, \mathcal{O})), \mathcal{O}). \tag{38}$$

This is a finitely generated R-module [34, Lemma 2.15]. Let d denote the Krull dimension of $M(\Theta)$. Recall that Pontryagin duality $\lambda \mapsto \lambda^\vee$ induces an anti-equivalence of categories between discrete $\mathcal{O}$-modules and compact $\mathcal{O}$-modules [31, (5.2.2)-(5.2.3)]. For any λ in $\mathrm{Mod}^{\mathrm{sm}}_K(\mathcal{O})$, the category of smooth K-representations on $\mathcal{O}$-torsion modules, we define

$$M(\lambda) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \lambda^\vee)^\vee. \tag{39}$$

Then $M(\lambda)$ is also a finitely generated R-module [34, Cor. 2.5]. We define $\mathrm{Mod}^{\mathrm{pro}}_G(\mathcal{O})$ to be the category of compact $\mathcal{O}[\![K]\!]$-modules with an action of $\mathcal{O}[G]$, such that the restriction to $\mathcal{O}[K]$ of both actions coincide. Pontryagin duality induces an anti-equivalence of categories between $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{pro}}_G(\mathcal{O})$. For any $R[1/p]$-module m of finite length, we choose a finitely generated R-submodule m^0 with $\mathrm{m} \cong \mathrm{m}^0 \otimes_{\mathcal{O}} L$ and define

$$\Pi(\mathrm{m}) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\mathrm{m}^0 \otimes_R N, L). \tag{40}$$

By [34, Lemma 2.21], $\Pi(\mathrm{m})$ is an admissible unitary L-Banach space representation of G.

Theorem 2.1 (Paškūnas,[34])**.** *Let* $\mathfrak{a}$ *be the* R*-annihilator of* $M(\Theta)$*. If the following hold*

(a) N *is projective in* $\mathrm{Mod}^{\mathrm{pro}}_K(\mathcal{O})$*,*
(b) $R/\mathfrak{a}$ *is equidimensional and all the associated primes are minimal,*
(c) *there exists a dense subset* Σ *of* $\mathrm{Supp}\, M(\Theta)$*, contained in* $\mathrm{m\text{-}Spec}\, R[1/p]$*, such that for all* $\mathfrak{n} \in \Sigma$ *the following hold:*

(i) $\dim_{\kappa(\mathfrak{n})} \operatorname{Hom}_K(V, \Pi(\kappa(\mathfrak{n}))) = 1$,
(ii) $\dim_{\kappa(\mathfrak{n})} \operatorname{Hom}_K(V, \Pi(R_{\mathfrak{n}}/\mathfrak{n}^2)) \leq d$,

then $R/\mathfrak{a}$ is reduced, of dimension d and we have an equality of $(d-1)$-dimensional cycles

$$z_{d-1}(R/(\varpi, \mathfrak{a})) = \sum_{\sigma} m_\sigma z_{d-1}(M(\sigma)),$$

where the sum is taken over the set of isomorphism classes of smooth irreducible k-representations of K and m_σ is the multiplicity with which σ occurs as a subquotient of Θ/ϖ.

We want to specify the following criterion in our situation, which allows us to check the first two conditions of Theorem 2.1.

Theorem 2.2 (Paškūnas,[34]). *Suppose that R is Cohen-Macaulay and N is flat over R. If*

$$\operatorname{projdim}_{\mathcal{O}[\![P]\!]} k\hat{\otimes}_R N + \max_{\sigma}\{\dim_R M(\sigma)\} \leq \dim R, \tag{41}$$

where the maximum is taken over all the irreducible smooth k-representations of K, then the following holds:

(o) *(41) is an equality,*
(i) *N is projective in $\operatorname{Mod}_K^{\mathrm{pro}}(\mathcal{O})$,*
(ii) *$M(\Theta)$ is a Cohen-Macaulay module,*
(iii) *$R/\operatorname{ann}_R M(\Theta)$ is equidimensional, and all the associated prime ideals are minimal.*

We start with the following setup. Let $\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ be a continuous representation of the form $\rho(g) = \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}$, as in (35). After twisting we may assume that χ is trivial so that for all $g \in G_{\mathbb{Q}_p}$

$$\rho(g) = \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}. \tag{42}$$

Let $\psi\colon \mathbb{Q}_p^\times \to \mathcal{O}^\times$ be a continuous character with $\psi\epsilon \equiv \mathbb{1} \mod \varpi$. Let R be a complete local noetherian $\mathcal{O}$-algebra and let

$$\rho_R\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R) \tag{43}$$

be a continuous representation with determinant $\psi\epsilon\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ such that $\rho_R \equiv \rho \mod \mathfrak{m}_R$. Let $R^{\mathrm{ps},\psi}$ denote the universal deformation ring that parametrizes 2-dimensional pseudo-characters of $G_{\mathbb{Q}_p}$ lifting the trace of the trivial representation and having determinant $\psi\epsilon$. Let $T\colon G_{\mathbb{Q}_p} \to \mathcal{O}$ be the associated universal pseudo-character. Since $\operatorname{tr}\rho_R$ is a pseudo-character lifting $\operatorname{tr}\rho$, the universal property of $R^{\mathrm{ps},\psi}$ induces a morphism of $\mathcal{O}$-algebras

$$R^{\mathrm{ps},\psi} \to R. \tag{44}$$

Let from now on $G := \mathrm{GL}_2(\mathbb{Q}_p)$, P the subgroup of upper triangular matrices and $K := \mathrm{GL}_2(\mathbb{Z}_p)$. Let I_1 be the subgroup of K which consists of the matrices that are upper unipotent modulo p. In particular, I_1 is a maximal pro-p Sylow subgroup of K. We let ω be the mod p cyclotomic character, via local class field theory considered as $\omega\colon \mathbb{Q}_p^\times \to k^\times, x \mapsto x\,|x| \mod p$, and define

$$\pi := (\operatorname{Ind}_P^G \mathbb{1} \otimes \omega^{-1})_{\mathrm{sm}}. \tag{45}$$

We let $\mathrm{Mod}^{\mathrm{sm}}_{G,\psi}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_{G}(\mathcal{O})$ that consists of smooth G-representations with central character ψ and denote by $\mathrm{Mod}^{\mathrm{lfin}}_{G,\psi}(\mathcal{O})$ its full subcategory of representations that are locally of finite length. We denote by $\mathrm{Mod}^{\mathrm{pro}}_{G,\psi}(\mathcal{O})$ resp. $\mathfrak{C}(\mathcal{O})$ the full subcategories of $\mathrm{Mod}^{\mathrm{pro}}_{G}(\mathcal{O})$ that are anti-equivalent to $\mathrm{Mod}^{\mathrm{sm}}_{G,\psi}(\mathcal{O})$ resp. $\mathrm{Mod}^{\mathrm{lfin}}_{G,\psi}(\mathcal{O})$ via Pontryagin duality. We see that π is an object of $\mathrm{Mod}^{\mathrm{lfin}}_{G,\psi}(\mathcal{O})$. Let $\tilde{P}$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$. We define $\tilde{E} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P})$. Paškūnas has shown in [33, Cor. 9.24] that the center of $\tilde{E}$ is isomorphic to $R^{\mathrm{ps},\psi}$ and

$$\tilde{E} \cong (R^{\mathrm{ps},\psi}\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![G_{\mathbb{Q}_p}]\!])/J, \tag{46}$$

where J denotes the closure of the ideal generated by $g^2 - T(g)g + \psi\epsilon(g)$ for all $g \in G_{\mathbb{Q}_p}$ [33, Cor. 9.27]. The representation ρ_R induces a morphism of $\mathcal{O}$-algebras $\mathcal{O}[\![G_{\mathbb{Q}_p}]\!] \to M_2(R)$. Together with the morphism (44) we obtain a morphism of $R^{\mathrm{ps},\psi}$-algebras

$$R^{\mathrm{ps},\psi}\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![G_{\mathbb{Q}_p}]\!] \to M_2(R). \tag{47}$$

The Cayley-Hamilton theorem tells us that this morphism is trivial on J, so that we get a morphism of $R^{\mathrm{ps},\psi}$-algebras

$$\eta\colon \tilde{E} \to M_2(R). \tag{48}$$

We define

$$M^{\square}(\sigma) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}\left((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P}, \sigma^\vee\right)^\vee. \tag{49}$$

Our goal is to prove the following theorem that enables us to check the condition of Paškūnas' theorem 2.2 for $N = (R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P}$ in the last section. We let $\mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}$ denote the length of a minimal projective resolution in $\mathrm{Mod}^{\mathrm{pro}}_{I_1,\psi}(\mathcal{O})$.

Theorem 2.3. *Let ρ and ρ_R be as before. We consider R as an $R^{\mathrm{ps},\psi}$-module via (44). Assume that $\dim R = \dim R^{\mathrm{ps},\psi} + \dim R/\mathfrak{m}_{R^{\mathrm{ps},\psi}}R$. Then*

$$\mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}\left(k\hat{\otimes}_R((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P})\right) + \max_{\sigma}\{\dim_R M^{\square}(\sigma)\} \le \dim R.$$

In particular, the inequality holds if R is flat over $R^{\mathrm{ps},\psi}$.

We start with computing the first summand.

Lemma 2.4.

$$\mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}\left(k\hat{\otimes}_R((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P})\right) = 3.$$

Proof. We have

$$k\hat{\otimes}_R((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P}) \cong (k\oplus k)\hat{\otimes}_{\tilde{E}}\tilde{P}.$$

Because of $k\hat{\otimes}_{\tilde{E}}\tilde{P} \cong \pi^\vee$, see [33, Lemma 9.1], and since $\tilde{P}$ is flat over the local ring $\tilde{E}$, $(k\oplus k)\hat{\otimes}_{\tilde{E}}\tilde{P}$ is an extension of $\pi^\vee$ by itself. Thus

$$\mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}\left(k\hat{\otimes}_R((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P})\right) = \mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}\,\pi^\vee.$$

The rest of the proof works analogous to the proof of [34, Prop. 6.21], the respective cohomology groups are calculated in [33, Cor. 10.4]. $\square$

Lemma 2.5. *Let R, N, σ be as before,* m *a compact R-module. Then*

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}\hat{\otimes}_R N, \sigma^\vee)^\vee \cong \mathrm{m}\hat{\otimes}_R \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \sigma^\vee)^\vee.$$

Proof. Since m is compact, we can write it as an inverse limit $\mathrm{m} = \varprojlim \mathrm{m}_i$ of finitely generated R-modules. Also the completed tensor product is defined as an inverse limit, so that we obtain

$$\begin{aligned}\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}\hat{\otimes}_R N, \sigma^\vee) &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\varprojlim(\mathrm{m}_i\hat{\otimes}_R N), \sigma^\vee)\\ &\cong \mathrm{Hom}_K(\sigma, \varinjlim(\mathrm{m}_i\hat{\otimes}_R N)^\vee).\end{aligned}$$

The universal property of the inductive limit yields a morphism

$$\varinjlim \mathrm{Hom}_K(\sigma, (\mathrm{m}_i\hat{\otimes}_R N)^\vee) \to \mathrm{Hom}_K(\sigma, \varinjlim(\mathrm{m}_i\hat{\otimes}_R N)^\vee),$$

which is easily seen to be injective. For the surjectivity we have to show that every K-morphism from σ to $\varinjlim(\mathrm{m}_i\hat{\otimes}_R N)^\vee$ factors through some finite level. But this follows from the fact that σ is a finitely generated K-representation. This implies

$$\begin{aligned}\mathrm{Hom}_K(\sigma, \varinjlim(\mathrm{m}_i\hat{\otimes}_R N)^\vee) &\cong \varinjlim \mathrm{Hom}_K(\sigma, (\mathrm{m}_i\hat{\otimes}_R N)^\vee)\\ &\cong \varinjlim \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}_i\hat{\otimes}_R N, \sigma^\vee).\end{aligned}$$

Since the statement holds for finitely generated m by [34, Prop. 2.4], taking the Pontryagin duals yields

$$\begin{aligned}\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}\hat{\otimes}_R N, \sigma^\vee)^\vee &\cong \varprojlim \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}_i\hat{\otimes}_R N, \sigma^\vee)^\vee\\ &\cong \varprojlim \mathrm{m}_i\hat{\otimes}_R \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \sigma^\vee)^\vee\\ &\cong \mathrm{m}\hat{\otimes}_R \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \sigma^\vee)^\vee.\end{aligned}$$

□

For the rest of the section we set $N = \tilde{P}$ so that $M(\sigma) = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\tilde{P}, \sigma^\vee)^\vee$.

Lemma 2.6. *Let σ be a smooth irreducible K-representation over k. Then $M(\sigma) \neq 0$ if and only if $\mathrm{Hom}_K(\sigma, \pi) \neq 0$. Moreover, $\dim_{R^{\mathrm{ps},\psi}} M(\sigma) \leq 1$.*

Proof. By [33, Cor. 9.25], we know that $\tilde{E}$ is a free $R^{\mathrm{ps},\psi}$-module of rank 4. Hu-Tan have shown in [34, Prop. 2.9] that $M(\sigma)$ is a cyclic $\tilde{E}$-module, thus $M(\sigma)$ is a finitely generated $R^{\mathrm{ps},\psi}$-module. Furthermore, $M(\sigma)$ is a compact $\tilde{E}$-module, see for example [17, §IV.4, Cor.1]. The same way as in Lemma 2.5 one can show that

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(k\hat{\otimes}_{\tilde{E}}\tilde{P}, \sigma^\vee)^\vee \cong k\hat{\otimes}_{\tilde{E}} M(\sigma). \tag{50}$$

By [33, Prop. 1.12], we have $k\hat{\otimes}_{\tilde{E}}\tilde{P} \cong \pi^\vee$ so that (50) implies

$$k\hat{\otimes}_{\tilde{E}} M(\sigma) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\pi^\vee, \sigma^\vee)^\vee \cong \mathrm{Hom}_K(\sigma, \pi). \tag{51}$$

Hence Nakayama lemma gives us that $M(\sigma) \neq 0$ if and only if $\mathrm{Hom}_K(\sigma, \pi) \neq 0$. If this holds, it follows again from [34, Prop. 2.4] that, if we let J denote the annihilator of $M(\sigma)$ as $\tilde{E}$-module, there is an isomorphism of rings $\tilde{E}/J \cong k[\![S]\!]$. Again by [33, Cor. 9.24], $R^{\mathrm{ps},\psi}$ is isomorphic to the center of $\tilde{E}$. If we let J^{ps} denote the annihilator of $M(\sigma)$ as $R^{\mathrm{ps},\psi}$-module, we get an inclusion

$$R^{\mathrm{ps},\psi}/J^{\mathrm{ps}} \hookrightarrow \tilde{E}/J \cong k[\![S]\!]. \tag{52}$$

Hence it suffices to show that $\dim_{R^{\mathrm{ps},\psi}} k[\![S]\!] \leq 1$, which is equivalent to the existence of an element $x \in \mathfrak{m}_{R^{\mathrm{ps},\psi}}$ that does not lie in J^{ps}. We assume that $\mathfrak{m}_{R^{\mathrm{ps},\psi}} \subset J^{\mathrm{ps}}$. Then we have a finite dimensional k-vector space $M(\sigma)/\mathfrak{m}_{R^{\mathrm{ps},\psi}} M(\sigma) \cong M(\sigma)$, on which $\tilde{E}/J \cong k[\![S]\!]$ acts faithfully, which is impossible. □

The proof of the theorem is now just a combination of the above Lemmas.

Proof of Theorem 2.3. Let σ be such that $M^{\square}(\sigma) \neq 0$. Then we see from Lemma 2.5 that

$$M^{\square}(\sigma) \cong (R \oplus R)\hat{\otimes}_{\tilde{E},\eta} M(\sigma).$$

Since $\tilde{E}$ is a finite $R^{\mathrm{ps},\psi}$-module by [33, Cor. 9.17], we have

$$\begin{aligned}\dim_R M^{\square}(\sigma) &= \dim_R (R \oplus R)\hat{\otimes}_{\tilde{E},\eta} M(\sigma)\\ &\leq \dim_R (R \oplus R) \otimes_{R^{\mathrm{ps},\psi}} M(\sigma).\end{aligned}$$

By [7, A.11] we know that for a morphism of local rings $A \to B$ and non-zero finitely generated modules M, N over A resp. B, we have

$$\dim_B M \otimes_A N \leq \dim_A M + \dim_B N/\mathfrak{m}_A N. \tag{53}$$

Since we already know from Lemma 2.6 that $\dim_{R^{\mathrm{ps},\psi}} M(\sigma) = 1$, we obtain from (53) that

$$\dim_R \left((R \oplus R) \otimes_{R^{\mathrm{ps},\psi}} M(\sigma)\right) \leq 1 + \dim R/\mathfrak{m}_{R^{\mathrm{ps},\psi}} R.$$

This expression depends only on the structure of R as an $R^{\mathrm{ps},\psi}$-module and the assumption of the theorem implies

$$\dim_R \left((R \oplus R) \otimes_{R^{\mathrm{ps},\psi}} M(\sigma)\right) \leq 1 + \dim R - \dim R^{\mathrm{ps},\psi}.$$

From the explicit description of $R^{\mathrm{ps},\psi}$ in [33, Cor. 9.13] we know in particular that $R^{\mathrm{ps},\psi} \cong \mathcal{O}[\![t_1, t_2, t_3]\!]$ and thus $\dim R^{\mathrm{ps},\psi} = 4$. The statement is now an immediate consequence of Lemma 2.4. $\square$

3. Flatness

Let again $\rho \cong \begin{pmatrix} \mathbb{1} & \phi \\ 0 & \mathbb{1} \end{pmatrix}$. Our goal in this section is to show that the universal framed deformation of ρ with fixed determinant satisfies the conditions of Theorem 2.3. Let $G_{\mathbb{Q}_p}(p)$ be the maximal pro-p quotient of $G_{\mathbb{Q}_p}$. Since $p > 2$, it is a free pro-p group on 2 generators γ, δ [31, Thm. 7.5.11]. Since the image of ρ is a p-group, it factors through $G_{\mathbb{Q}_p}(p)$. We have shown in [35] that the universal framed deformation ring $R^{\square}_{\rho}$ of ρ is isomorphic to $\mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, t_\gamma, y_{11}, \hat{y}_{12}, y_{21}, t_\delta]\!]$ and the universal framed deformation is given by

$$\rho^{\square} \colon G_{\mathbb{Q}_p}(p) \to \mathrm{GL}_2(R^{\square}_{\rho}), \tag{54}$$

$$\gamma \mapsto \begin{pmatrix} 1 + t_\gamma + x_{11} & x_{12} \\ x_{21} & 1 + t_\gamma - x_{11} \end{pmatrix}, \tag{55}$$

$$\delta \mapsto \begin{pmatrix} 1 + t_\delta + y_{11} & y_{12} \\ y_{21} & 1 + t_\delta - y_{11} \end{pmatrix}, \tag{56}$$

where $x_{12} := \hat{x}_{12} + [\phi(\gamma)]$, $y_{12} := \hat{y}_{12} + [\phi(\delta)]$ and $[\phi(\gamma)], [\phi(\delta)]$ denote the Teichmüller lifts of $\phi(\gamma)$ and $\phi(\delta)$ to $\mathcal{O}$. Let $\psi \colon G_{\mathbb{Q}_p} \to \mathcal{O}^{\times}$ be a continuous character with $\psi\epsilon \equiv 1 \bmod \varpi$. To find the quotient $R^{\square,\psi}_{\rho}$ of $R^{\square}_{\rho}$ that parametrizes lifts of ρ with determinant $\psi\epsilon$, we have to impose the conditions $\det(\rho^{\square}(\gamma)) = \psi\epsilon(\gamma)$ and $\det(\rho^{\square}(\delta)) = \psi\epsilon(\delta)$. Therefore, analogous to [35], we define the ideal

$$I := \left((1 + t_\gamma)^2 - x_{11}^2 - x_{12}x_{21} - \psi\epsilon(\gamma), (1 + t_\delta)^2 - y_{11}^2 - y_{12}y_{21} - \psi\epsilon(\delta)\right) \subset R^{\square,\psi}_{\rho}$$

and obtain

$$R^{\square,\psi}_{\rho} := \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, t_\gamma, y_{11}, \hat{y}_{12}, y_{21}, t_\delta]\!]/I. \tag{57}$$

Let again $R^{\mathrm{ps},\psi}$ denote the universal deformation ring that parametrizes 2-dimensional pseudo-characters of $G_{\mathbb{Q}_p}$ with determinant $\psi\epsilon$ that lift the trace of the trivial 2-dimensional representation. Paškūnas has shown in [33, 9.12,9.13] that $R^{\mathrm{ps},\psi}$ is isomorphic to $\mathcal{O}[\![t_1,t_2,t_3]\!]$ and the universal pseudo-character is uniquely determined by

$$\begin{aligned} T\colon G_{\mathbb{Q}_p}(p) &\to \mathcal{O}[\![t_1,t_2,t_3]\!] \\ \gamma &\mapsto 2(1+t_1) \\ \delta &\mapsto 2(1+t_2) \\ \gamma\delta &\mapsto 2(1+t_3) \\ \delta\gamma &\mapsto 2(1+t_3). \end{aligned}$$

Since the trace $T^{\square}$ of $\rho^{\square}$ is a pseudo-deformation of $2\cdot\mathbb{1}$ to $R^{\square}_{\rho}$, we get an induced morphism

$$\phi\colon \mathcal{O}[\![t_1,t_2,t_3]\!] \to R^{\square,\psi}_{\rho} \tag{58}$$
$$t_1 \mapsto T^{\square}(\gamma) = t_\gamma \tag{59}$$
$$t_2 \mapsto T^{\square}(\delta) = t_\delta \tag{60}$$
$$t_3 \mapsto T^{\square}(\gamma\delta) = T^{\square}(\delta\gamma) = (1+t_\gamma)(1+t_\delta) + \frac{1}{2}z - 1, \tag{61}$$

where $z = x_{12}y_{21} + 2x_{11}y_{11} + x_{21}y_{12}$.

Proposition 3.1. *The map* (58) *makes $R^{\square,\psi}_{\rho}$ into a flat $\mathcal{O}[\![t_1,t_2,t_3]\!]$-module.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}[\![t_1,t_2,t_3]\!]$. Since $R^{\square,\psi}_{\rho}$ is a regular local ring modulo a regular sequence, it is Cohen-Macaulay. Since $\mathcal{O}[\![t_1,t_2,t_3]\!]$ is regular, the statement is equivalent to

$$\dim \mathcal{O}[\![t_1,t_2,t_3]\!] + \dim R^{\square,\psi}_{\rho}/\mathfrak{m}R^{\square,\psi}_{\rho} = \dim R^{\square,\psi}_{\rho},$$

see for example [12, Thm. 18.16]. But since $\dim \mathcal{O}[\![t_1,t_2,t_3]\!] = 4$, $\dim R^{\square,\psi}_{\rho} = 7$ by (57) and

$$R^{\square,\psi}_{\rho}/\mathfrak{m}R^{\square,\psi}_{\rho} \cong k[\![x_{11},\hat{x}_{12},x_{21},y_{11},\hat{y}_{12},y_{21}]\!]/(x_{11}^2 + x_{12}x_{21}, y_{11}^2 + y_{12}y_{21}, z)$$

by (58)-(61), it just remains to prove that

$$\dim k[\![x_{11},\hat{x}_{12},x_{21},y_{11},\hat{y}_{12},y_{21}]\!]/(x_{11}^2 + x_{12}x_{21}, y_{11}^2 + y_{12}y_{21}, z) = 3.$$

We distinguish 3 cases: If $x_{12} \in (R^{\square,\psi}_{\rho})^{\times}$, we obtain

$$R^{\square,\psi}_{\rho}/\mathfrak{m}R^{\square,\psi}_{\rho} \cong k[\![x_{11},\hat{x}_{12},y_{11},\hat{y}_{12}]\!]/(y_{11}^2 - y_{12}x_{12}^{-1}(2x_{11}y_{11} - y_{12}x_{11}^2x_{12}^{-1})),$$

so that $\{x_{11},\hat{x}_{12},\hat{y}_{12}\}$ is a system of parameters for $R^{\square,\psi}_{\rho}/\mathfrak{m}R^{\square,\psi}_{\rho}$. Analogously, if $y_{12} \in (R^{\square,\psi}_{\rho})^{\times}$, then $\{y_{11},\hat{y}_{12},\hat{x}_{12}\}$ is a system of parameters. So the only case left is when $x_{12}, y_{12} \notin (R^{\square,\psi}_{\rho})^{\times}$. But it is easy to see that in this case $\{x_{12}, y_{21}, x_{21}-y_{12}\}$ is a system of parameters for $R^{\square,\psi}_{\rho}/\mathfrak{m}R^{\square,\psi}_{\rho}$, which finishes the proof. □

4. Locally algebraic vectors

In this section we want to adapt the strategy of [34, §4] to show that part c) of Paškūnas' Theorem 2.1 holds in the following setting. Let from now on $R := R_\rho^{\square,\psi}$, $\pi \cong (\mathrm{Ind}_P^G \mathbb{1} \otimes \omega^{-1})_{\mathrm{sm}}$, $\tilde{P}$ a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$. Let $N := (R \oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P}$, where the $\tilde{E}$-module structure on $R \oplus R$ is induced by $\rho^\square$, as in (48).

In [33, §5.6] Paškūnas defines a covariant exact functor

$$\check{\mathbf{V}} \colon \mathfrak{C}(\mathcal{O}) \to \mathrm{Mod}^{\mathrm{pro}}_{G_{\mathbb{Q}_p}}(\mathcal{O}), \tag{62}$$

which is a modification of Colmez' Montreal functor, see [9]. It satisfies

$$\check{\mathbf{V}}((\mathrm{Ind}_P^G \chi_1 \otimes \chi_2\omega^{-1})^\vee) = \chi_1, \tag{63}$$

so that in our case

$$\check{\mathbf{V}}((\mathrm{Ind}_P^G \mathbb{1} \otimes \omega^{-1})^\vee) = \mathbb{1}. \tag{64}$$

For an admissible unitary L-Banach space representation Π of G with central character ψ and an open bounded G-invariant lattice Θ in Π, we define

$$\Theta^d := \mathrm{Hom}_{\mathcal{O}}(\Theta, \mathcal{O}), \tag{65}$$

which lies in $\mathfrak{C}(\mathcal{O})$. We also define

$$\check{\mathbf{V}}(\Pi) := \check{\mathbf{V}}(\Theta^d) \otimes_{\mathcal{O}} L, \tag{66}$$

which is independent of the choice of Θ.

Lemma 4.1. *N satisfies the following three properties (see* [34, §4]*):*

- (N0) *$k\hat{\otimes}_R N$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is finitely generated over $\mathcal{O}[\![K]\!]$,*
- (N1) *$\mathrm{Hom}_{\mathrm{SL}_2(\mathbb{Q}_p)}(1, N^\vee) = 0$,*
- (N2) *$\check{\mathbf{V}}(N) \cong \rho^\square$ as $R[\![G_{\mathbb{Q}_p}]\!]$-modules.*

Proof. As we have already seen in the proof of Lemma 2.4, $k\hat{\otimes}_R N$ is an extension of $\pi^\vee$ by itself. Since π is absolutely irreducible and admissible we get $(N0)$. From [33, Lemma 5.53] we obtain that

$$\check{\mathbf{V}}(\rho^\square \hat{\otimes}_{\tilde{E},\eta}\tilde{P}) \cong \rho^\square \hat{\otimes}_{\tilde{E},\eta}\check{\mathbf{V}}(\tilde{P}), \tag{67}$$

and since $\check{\mathbf{V}}(\tilde{P})$ is a free $\tilde{E}$-module of rank 1 by [33, Cor. 5.55], also $(N2)$ holds. For $(N1)$ we notice that $\pi^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$. Since $\tilde{P}$ is a projective envelope of $\pi^\vee$, $\tilde{P}^\vee$ is an injective envelope of π. Since G acts on $(\tilde{P}^\vee)^{\mathrm{SL}_2(\mathbb{Q}_p)}$ via the determinant, we must have $(\tilde{P}^\vee)^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$. □

Remark 4.2. *Let m be a $R[1/p]$-module of finite length. Then Lemma 4.1 implies that*

$$\check{\mathbf{V}}(\Pi(\mathrm{m})) \cong \mathrm{m} \otimes_R \check{\mathbf{V}}(N),$$

see [34, Rmk. 4.2, Lemma 4.3]*.*

The following Proposition is analogous to [34, 4.14] and shows that condition (i) of part c) of Paškūnas' Theorem 2.1 is satisfied in our setting.

Proposition 4.3. *Let V be either $\sigma(\mathbf{w},\tau)$ or $\sigma^{\mathrm{cr}}(\mathbf{w},\tau)$, let $\mathfrak{p}\in \text{m-Spec}(R[1/p])$ and $\kappa(\mathfrak{p}):=R[1/p]/\mathfrak{p}$. Then*

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V,\Pi(\kappa(\mathfrak{p})))\leq 1.$$

If $V=\sigma(\mathbf{w},\tau)$, then $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V,\Pi(\kappa(\mathfrak{p})))=1$ if and only if $\rho^{\square}_{\mathfrak{p}}$ is potentially semi-stable of type $(\mathbf{w},\tau,\psi)$.
If $V=\sigma^{\mathrm{cr}}(\mathbf{w},\tau)$, then $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V,\Pi(\kappa(\mathfrak{p})))=1$ if and only if $\rho^{\square}_{\mathfrak{p}}$ is potentially crystalline of type $(\mathbf{w},\tau,\psi)$.

Proof. Let $F/\kappa(\mathfrak{p})$ be a finite extension. We have

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V,\Pi(\kappa(\mathfrak{p})))=\dim_F \mathrm{Hom}_K(V\otimes_{\kappa(\mathfrak{p})}F,\Pi(\kappa(\mathfrak{p}))\otimes_{\kappa(\mathfrak{p})}F),$$

see for example [33, Lemma 5.1]. Thus by replacing $\kappa(\mathfrak{p})$ by a finite extension, we can assume without loss of generality that $\rho^{\square}_{\mathfrak{p}}$ is either absolutely irreducible or reducible. Since $\rho^{\square}_{\mathfrak{p}}$ is a lift of $\rho\cong\begin{pmatrix}\mathbb{1} & * \\ 0 & \mathbb{1}\end{pmatrix}$ and N satisfies (N0), (N1) and (N2) by Lemma 4.1, the only case that is not handled in [34, 4.14] is when $\rho^{\square}_{\mathfrak{p}}$ is an extension

$$0\longrightarrow \chi_1\longrightarrow \rho^{\square}_{\mathfrak{p}}\longrightarrow \chi_2\longrightarrow 0\ ,$$

where χ_1,χ_2 are two characters that have the same Hodge-Tate weight. Such a representation is clearly never of any Hodge-type with distinct Hodge-Tate weights, so it is enough to show that $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V,\Pi(\kappa(\mathfrak{p}))=0$. It follows, for example from [13, Prop. 3.4.2], that $\Pi(\kappa(\mathfrak{p}))$ is an extension of $\Pi_2:=(\mathrm{Ind}_P^G\chi_2\otimes\chi_1\epsilon^{-1})_{\mathrm{cont}}$ by $\Pi_1:=(\mathrm{Ind}_P^G\chi_1\otimes\chi_2\epsilon^{-1})_{\mathrm{cont}}$. If we denote the locally algebraic vectors of Π_i by Π_i^{alg}, then [33, Prop. 12.5] tells us that $\Pi_1^{\mathrm{alg}}=\Pi_2^{\mathrm{alg}}=0$. But this implies that also $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}=0$, and since V is a locally algebraic representation, we have

$$\mathrm{Hom}_K(V,\Pi(\kappa(\mathfrak{p})))\cong \mathrm{Hom}_K(V,\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}})=0.$$

□

To apply Paškūnas Theorem 2.1, we have to find a set of 'good' primes of $R[1/p]$ that is dense in $\mathrm{Supp}\, M(\Theta)$.

Definition 4.4. *Let $\Sigma\subset \mathrm{Supp}\, M(\Theta)\cap \text{m-Spec}(R[1/p])$ consist of all primes $\mathfrak{p}$ such that either $\Pi(\kappa(\mathfrak{p}))$ is reducible but non-split or $\Pi(\kappa(\mathfrak{p}))$ is absolutely irreducible and $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is irreducible.*

Proposition 4.5. *Σ is dense in $\mathrm{Supp}\, M(\Theta)$.*

Proof. We already know that $M(\Theta)$ is Cohen-Macaulay by applying Theorem 2.3 to Paškūnas' Theorem 2.2. Since R is $\mathcal{O}$-torsion free and $R[1/p]$ is Jacobson, it suffices to show that the dimension of the complement of Σ in $\mathrm{Supp}\, M(\Theta)\cap\text{m-Spec}(R[1/p])$ is strictly smaller than the dimension of $R[1/p]$, which is equal to 4.

Let first $\mathfrak{p}\in \text{m-Spec}\, R[1/p]$ be such that $\Pi(\kappa(\mathfrak{p}))$ is absolutely irreducible and $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is reducible. By a result of Colmez [9, Thm. VI.6.50] we know that in this case we have $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}\cong\pi\otimes W$, where W is an irreducible algebraic G-representation and $\pi\cong(\mathrm{Ind}_P^G\chi\,|.|\otimes\chi\,|.|^{-1})_{\mathrm{sm}}$ for some smooth character χ. In particular, if the Hodge-Tate weights are $\mathbf{w}=(a,b)$, we have $W\cong \mathrm{Sym}^{b-a-1}L^2\otimes{\det}^a$. But since $\det\rho^{\square}=\psi\epsilon$, the product of the central characters of π and W must be ψ, so that we obtain $\chi^2\epsilon^{a+b}=\psi$, which can only be satisfied by a finite number of characters χ. By a result of Berger-Breuil [3, Cor. 5.3.2], the universal unitary completion of Π^{alg} is topologically irreducible in this case and therefore isomorphic to Π. Hence there are only finitely many absolutely irreducible Banach space representations $\Pi(\kappa(\mathfrak{p}))$ such that $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is reducible. Moreover, all of them give rise to a point $x_{\mathfrak{p}} \in$ m-Spec $R^{\mathrm{ps},\psi}[1/p]$ by taking the trace of the associated $G_{\mathbb{Q}_p}$-representation $\rho_{\mathfrak{p}}^{\square} = \check{\mathbf{V}}(\Pi(\kappa(\mathfrak{p})))$. We already know from Proposition 3.1 that R is flat over $R^{\mathrm{ps},\psi}$ and $\dim R/\mathfrak{m}_{R^{\mathrm{ps},\psi}} R = 3$. Thus, above every prime $x_{\mathfrak{p}}$ there lies only an at most 3-dimensional family of primes $\mathfrak{p} \in$ m-Spec $R[1/p]$ such that $\Pi(\kappa(\mathfrak{p}))$ is absolutely irreducible and $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is reducible.

Let now $\mathfrak{p} \in$ Supp $M(\Theta)$ be such that, after extending scalars if necessary, $\rho_{\mathfrak{p}}^{\square}$ is split. Hence from Proposition 4.3 we know that $\rho_{\mathfrak{p}}^{\square}$ is potentially semi-stable of a Hodge type $(\mathbf{w}, \tau, \psi)$ determined by Θ, where $\mathbf{w} = (a, b)$, $\tau = \chi_1 \oplus \chi_2$ and $\chi_i \colon I_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ have finite image. We claim that the closed subset of m-Spec $R_{\rho}^{\square}[1/p]$ consisting of points of the Hodge type above, is of dimension at most 3. As before, $\rho^{\square}$ factors through the maximal pro-p quotient $G_{\mathbb{Q}_p}(p)$ of $G_{\mathbb{Q}_p}$, which is a free pro-p group of rank 2, generated by a 'cyclotomic' generator γ and an 'unramified' generator δ. From our assumptions we see that for every representation $\rho_{\mathfrak{p}}^{\square}$ of the type above there are unramified characters μ_1, μ_2 such that up to conjugation

$$\rho_{\mathfrak{p}}^{\square} \sim \begin{pmatrix} \epsilon^b \chi_1 \mu_1 & 0 \\ 0 & \epsilon^a \chi_2 \mu_2 \end{pmatrix}. \tag{68}$$

As in (54), we have $R_{\rho}^{\square} \cong \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, t_{\gamma}, y_{11}, \hat{y}_{12}, y_{21}, t_{\delta}]\!]$ with the universal framed deformation determined by

$$\rho^{\square}(\gamma) = \begin{pmatrix} 1 + t_{\gamma} + x_{11} & x_{12} \\ x_{21} & 1 + t_{\gamma} - x_{11} \end{pmatrix}, \tag{69}$$

$$\rho^{\square}(\delta) = \begin{pmatrix} 1 + t_{\delta} + y_{11} & y_{12} \\ y_{21} & 1 + t_{\delta} - y_{11} \end{pmatrix}. \tag{70}$$

Since the trace is invariant under conjugation, we get the following identities from (68)-(70):

$$I_1 \colon \epsilon^b \chi_1(\gamma) + \epsilon^a \chi_2(\gamma) = 2(1 + t_{\gamma}), \tag{71}$$

$$I_2 \colon \mu_1(\delta) + \mu_2(\delta) = 2(1 + t_{\delta}). \tag{72}$$

We get

$$R_{\rho}^{\square}/(I_1, I_2) \cong \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!].$$

Moreover, using (71),(72), we get the following relations for the determinants:

$$I_3 \colon x_{11}^2 + x_{12} x_{21} = \frac{1}{4}(\epsilon^a \chi_1(\gamma) - \epsilon^b \chi_2(\gamma))^2, \tag{73}$$

$$I_4 \colon y_{11}^2 + y_{12} y_{21} = \frac{1}{4}(\mu_1(\delta) - \mu_2(\delta))^2. \tag{74}$$

Since we assume the representation $\rho_{\mathfrak{p}}^{\square}$ to be split, it is, in particular, abelian. This can be summed up in the following relations:

$$I_5 : 0 = x_{12} y_{21} - x_{21} y_{12}, \tag{75}$$

$$I_6 : 0 = x_{12} y_{11} - x_{11} y_{12}, \tag{76}$$

$$I_7 : 0 = x_{21} y_{11} - x_{11} y_{21}. \tag{77}$$

We want to find a system of parameters $\mathcal{S}$ for $R_\rho^\square/(I_1,\dots,I_7)$ of length at most 4. If $x_{12}\in(R_\rho^\square)^\times$, it is easy to check that $\mathcal{S}=\{\varpi,\hat{x}_{12},\hat{y}_{12},x_{11}\}$ is such a system. Analogously, if $y_{12}\in(R_\rho^\square)^\times$, we can take $\mathcal{S}=\{\varpi,\hat{x}_{12},\hat{y}_{12},y_{11}\}$. In the last case, when $x_{12},y_{12}\in\mathfrak{m}_{R_\rho^\square}$, which means that $\hat{x}_{12}=x_{12}$, $\hat{y}_{12}=y_{12}$, we can take $\mathcal{S}=\{\varpi,x_{12},y_{21},x_{21}-y_{12}\}$. Thus $\dim R_\rho^\square/(I_1,\dots,I_7)\le 4$ and since R is $\mathcal{O}$-torsion free, we obtain

$$\dim R_\rho^\square[1/p]/(I_1,\dots,I_7)\le 3, \tag{78}$$

which proves the claim. $\square$

The next step is to prove that part c)ii) of Paškūnas' Theorem 2.1 is satisfied for all $\mathfrak{p}\in\Sigma$. The following definition is analogous to [34, 4.17].

Definition 4.6. *Let* $\mathrm{Ban}^{\mathrm{adm}}_{G,\psi}(L)$ *be the category of admissible* L*-Banach space representations of* G *with central character* ψ *and let* Π *in* $\mathrm{Ban}^{\mathrm{adm}}_{G,\psi}(L)$ *be absolutely irreducible. Let* $\mathcal{E}$ *be the subspace of* $\mathrm{Ext}^1_{G,\psi}(\Pi,\Pi)$ *that is generated by extensions* $0\to\Pi\to E\to\Pi\to 0$ *such that the resulting sequence of locally algebraic vectors* $0\to\Pi^{\mathrm{alg}}\to E^{\mathrm{alg}}\to\Pi^{\mathrm{alg}}\to 0$ *is exact. We say that* Π *satisfies (RED), if* $\Pi^{\mathrm{alg}}\neq 0$ *and* $\dim\mathcal{E}\le 1$.

The following lemma is a generalization of [34, Lemma 4.18] which avoids the assumption $\dim_L \mathrm{Hom}_G(\Pi,E)=1$.

Lemma 4.7. *Let* $\Pi\in\mathrm{Ban}^{\mathrm{adm}}_{G,\psi}(L)$ *be absolutely irreducible. Let* $n\ge 1$ *and let*

$$0\to\Pi\to E\to\Pi^{\oplus n}\to 0 \tag{79}$$

be an exact sequence in $\mathrm{Ban}^{\mathrm{adm}}_{G,\psi}(L)$*. Let* V *be either* $\sigma(\mathbf{w},\tau)$ *or* $\sigma^{\mathrm{cr}}(\mathbf{w},\tau)$*. If* Π^{alg} *is irreducible and* Π *satisfies (RED), then*

$$\dim_L \mathrm{Hom}_K(V,E)\le\dim_L\mathrm{Hom}_G(\Pi,E)+1.$$

Proof. Since Π^{alg} is irreducible, we obtain by [34, Lemma 4.10] and [21] that $\dim_L\mathrm{Hom}_K(V,\Pi)=1$. We apply the functors $\mathrm{Hom}_G(\Pi,\_)$ and $\mathrm{Hom}_K(V,\_)$ to the sequence (79) to obtain the following diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_G(\Pi,\Pi) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_G(\Pi,E) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_G(\Pi,\Pi^{\oplus n}) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^1_{G,\psi}(\Pi,\Pi) \arrow[d, "\alpha"] \\
0 \arrow[r] & \mathrm{Hom}_K(V,\Pi) \arrow[r] & \mathrm{Hom}_K(V,E) \arrow[r] & \mathrm{Hom}_K(V,\Pi^{\oplus n}) \arrow[r] & \mathrm{Ext}^1_{K,\psi}(V,\Pi),
\end{tikzcd}$$

where Ext^1 means the Yoneda extensions in $\mathrm{Ban}^{\mathrm{adm}}_{G,\psi}(L)$ resp. $\mathrm{Ban}^{\mathrm{adm}}_{K,\psi}(L)$. The diagram yields an exact sequence

$$0\longrightarrow\mathrm{Hom}_G(\Pi,E)\longrightarrow\mathrm{Hom}_K(V,E)\longrightarrow\ker(\alpha),$$

and therefore

$$\dim_L\mathrm{Hom}_K(V,\Pi)\le\dim_L\mathrm{Hom}_G(\Pi,E)+\dim_L\ker(\alpha). \tag{80}$$

The irreducibility of Π^{alg} implies that $\ker(\alpha)$ is equal to the space $\mathcal{E}$ of Definition 4.6. Since we assume that Π satisfies (RED), we are done. $\square$

Lemma 4.8. *Let $\mathfrak{p} \in \Sigma$. If $\mathrm{End}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}) = \kappa(\mathfrak{p})$, then*

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}} \left(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \rho_{\mathfrak{p}}^{\square}\right) = 4.$$

If $\rho_{\mathfrak{p}}^{\square}$ is reducible such that there is a non-split exact sequence

$$0 \longrightarrow \delta_2 \longrightarrow \rho_{\mathfrak{p}}^{\square} \longrightarrow \delta_1 \longrightarrow 0,$$

with $\delta_1\delta_2^{-1} \neq \mathbb{1}, \epsilon^{\pm 1}$, then

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}} \left(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \delta_1\right) = 4.$$

Proof. We start with the exact sequence

$$0 \longrightarrow \mathfrak{p}/\mathfrak{p}^2 \longrightarrow R[1/p]/\mathfrak{p}^2 \longrightarrow \kappa(\mathfrak{p}) \longrightarrow 0. \tag{81}$$

Tensoring (81) with $\rho^{\square,\psi}[1/p]$ over $R[1/p]$ and applying the functor $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\_, \rho_{\mathfrak{p}}^{\square})$ yields the exact sequence

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}} \left(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \rho_{\mathfrak{p}}^{\square}\right) \longrightarrow \mathrm{Hom}_{G_{\mathbb{Q}_p}} \left(\mathfrak{p}/\mathfrak{p}^2 \otimes_{R[1/p]} \rho^{\square,\psi}[1/p], \rho_{\mathfrak{p}}^{\square}\right) \xrightarrow{\partial} \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \rho_{\mathfrak{p}}^{\square}).$$

Since we assume $\mathrm{End}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}) = \kappa(\mathfrak{p})$, we have

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}} \left(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \rho_{\mathfrak{p}}^{\square}\right) = 1 + \dim_{\kappa(\mathfrak{p})} \ker(\partial).$$

We see that

$$\ker(\partial) = \{\phi\colon R \to \kappa(\mathfrak{p})[\epsilon] \mid \rho^{\square,\psi}[1/p] \otimes_{R[1/p],\phi} \kappa(\mathfrak{p})[\epsilon] \cong \rho_{\mathfrak{p}}^{\square} \oplus \rho_{\mathfrak{p}}^{\square} \text{ as } G_{\mathbb{Q}_p}\text{-reps.}\}.$$

Let $\phi \in \ker(\partial)$ and let $\hat{R}$ be the $\mathfrak{p}$-adic completion of $R[1/p]$. Then we can identify $\hat{R}$ with the universal framed deformation ring that parametrizes lifts of $\rho_{\mathfrak{p}}^{\square}$ with determinant $\psi\epsilon$ [28, (2.3.5)] and ϕ induces a morphism $\hat{R} \to \kappa(\mathfrak{p})[\epsilon]$. If we denote the adjoint representation of $\rho_{\mathfrak{p}}^{\square}$ by $\mathrm{ad}\,\rho_{\mathfrak{p}}^{\square}$, there is a natural isomorphism

$$\mathrm{Hom}_{\kappa(\mathfrak{p})-\mathrm{Alg}}(\hat{R}, \kappa(\mathfrak{p})[\epsilon]) \cong Z^{1,\psi}(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square}), \tag{82}$$

where $Z^{1,\psi}(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square})$ denotes the space of cocyles that correspond to deformations with determinant $\psi\epsilon$. Here the morphism $\phi \in \mathrm{Hom}_{\kappa(\mathfrak{p})-\mathrm{Alg}}(\hat{R}, \kappa(\mathfrak{p})[\epsilon])$ that corresponds to a lift $\tilde{\rho}$ of $\rho_{\mathfrak{p}}^{\square}$ is mapped to the cocycle Φ that appears in the equality

$$\tilde{\rho}(g) = \rho_{\mathfrak{p}}^{\square}(g)(1 + \Phi(g)\epsilon).$$

Since $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \rho_{\mathfrak{p}}^{\square}) \cong H^1(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square})$, we obtain that

$$\ker(\partial) = \{\phi \in Z^{1,\psi}(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square}) \mid \phi = 0 \text{ in } H^1(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square})\}.$$

Hence $\ker(\partial) \cong B^{1,\psi}(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square})$, the corresponding coboundaries. There is an exact sequence

(83)

$$0 \longrightarrow (\mathrm{ad}\,\rho_{\mathfrak{p}}^{\square})^{G_{\mathbb{Q}_p}} \longrightarrow \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square} \longrightarrow Z^1(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square}) \longrightarrow H^1(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square}) \longrightarrow 0,$$

where the middle map is given by $x \mapsto (g \mapsto gx - x)$. Since by assumption $\mathrm{End}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}) = \kappa(\mathfrak{p})$, we see from (83) that

$$\dim_{\kappa(\mathfrak{p})} B^{1,\psi}(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square}) = 3.$$

Let now $\rho_{\mathfrak{p}}^{\square}$ be reducible such that there is a non-split exact sequence

$$0 \longrightarrow \delta_2 \longrightarrow \rho_{\mathfrak{p}}^{\square} \longrightarrow \delta_1 \longrightarrow 0 \ ,$$

with $\delta_1 \neq \delta_2$. Tensoring (81) with $\rho^{\square,\psi}[1/p]$ and applying the functor $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\_, \delta_1)$ gives us an exact sequence

(84)

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\rho^{\square}[1/p]/\mathfrak{p}^2, \delta_1\right) \longrightarrow \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\mathfrak{p}/\mathfrak{p}^2 \otimes_{R[1/p]} \rho^{\square,\psi}[1/p], \delta_1\right) \xrightarrow{\partial'} \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \delta_1).$$

Since $\delta_1 \neq \delta_2$ we have $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}(\rho_{\mathfrak{p}}^{\square}, \delta_1) = 1$ and therefore

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \delta_1) = 1 + \dim_{\kappa(\mathfrak{p})} \ker(\partial'). \tag{85}$$

Moreover, we obtain isomorphisms

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\mathfrak{p}/\mathfrak{p}^2 \otimes_{R[1/p]} \rho^{\square,\psi}[1/p], \delta_1\right) \cong (\mathfrak{p}/\mathfrak{p}^2)^* \cong \mathrm{Hom}_{\kappa(\mathfrak{p})-\mathrm{Alg}}(\hat{R}^{\square}, \kappa(\mathfrak{p})[\epsilon]) \tag{86}$$

$$\cong Z^{1,\psi}(G_{\mathbb{Q}_p}, \mathrm{ad}\, \rho_{\mathfrak{p}}^{\square}). \tag{87}$$

From (83) we obtain again that the kernel of the natural surjection

$$Z^1(G_{\mathbb{Q}_p}, \mathrm{ad}\, \rho_{\mathfrak{p}}^{\square}) \longrightarrow\!\!\!\!\!\rightarrow H^1(G_{\mathbb{Q}_p}, \mathrm{ad}\, \rho_{\mathfrak{p}}^{\square}) \cong \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \rho_{\mathfrak{p}}^{\square}) \tag{88}$$

is 3-dimensional. Hence (84), and (86)-(88) give us an induced map

$$\bar{\partial}' \colon \mathrm{Ext}^{1,\psi}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \rho_{\mathfrak{p}}^{\square}) \to \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \delta_1)$$

with

$$\dim_{\kappa(\mathfrak{p})} \ker(\partial') = 3 + \dim_{\kappa(\mathfrak{p})} \ker(\bar{\partial}'). \tag{89}$$

Since $\mathrm{End}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}) = \kappa(\mathfrak{p})$, also the universal (non-framed) deformation ring $\hat{R}^{\mathrm{un}}$ of $\rho_{\mathfrak{p}}^{\square}$ exists, that parametrizes deformations of $\rho_{\mathfrak{p}}^{\square}$ with determinant $\psi\epsilon$. Therefore we can use the same argument as in the proof of [34, Lemma 4.20.], with $\rho_{\mathfrak{p}}^{\square}$ instead of $\rho_{\mathfrak{p}}^{\mathrm{un}}$, to obtain that $\ker(\bar{\partial}') = \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\delta_1, \delta_2)/\mathcal{L}$, where $\mathcal{L}$ is the subspace corresponding to $\rho_{\mathfrak{p}}^{\square}$. Since we assume $\delta_1\delta_2^{-1} \neq \mathbb{1}, \epsilon^{\pm 1}$, we have $\dim_{\kappa(\mathfrak{p})} \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\delta_1, \delta_2) = 1$ and obtain from (85) and (89) that

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \delta_1) = 4.$$

□

Corollary 4.9. *Let V be either $\sigma(\mathbf{w}, \tau)$ or $\sigma^{\mathrm{cr}}(\mathbf{w}, \tau)$ and let Θ be a K-invariant $\mathcal{O}$-lattice in V. Then for all $\mathfrak{p} \in \Sigma$,*

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K\left(V, \Pi(R[1/p]/\mathfrak{p}^2)\right) \leq 5.$$

Proof. Let $\mathfrak{p} \in \Sigma$. If $\Pi(\kappa(\mathfrak{p}))$ is absolutely irreducible, then also $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is irreducible. By the same argument as in [34, Thm. 4.19] that uses a result of Dospinescu [11, Thm. 1.4, Prop. 1.3], we obtain that $\Pi(\kappa(\mathfrak{p}))$ satisfies (RED). From the exact sequence

$$0 \longrightarrow \mathfrak{p}/\mathfrak{p}^2 \longrightarrow R[1/p]/\mathfrak{p}^2 \longrightarrow \kappa(\mathfrak{p}) \longrightarrow 0 \tag{90}$$

we obtain an exact sequence of unitary Banach space representations

$$0 \longrightarrow \Pi(\kappa(\mathfrak{p})) \longrightarrow \Pi(R[1/p]/\mathfrak{p}^2) \longrightarrow \Pi(\kappa(\mathfrak{p}))^{\oplus n} \longrightarrow 0 \ . \tag{91}$$

Thus we can apply Lemma 4.7 and obtain

$$\dim_{\kappa(\mathfrak{p})} \operatorname{Hom}_K \left(V, \Pi(R[1/p]/\mathfrak{p}^2)\right) \leq \dim_{\kappa(\mathfrak{p})} \operatorname{Hom}_G \left(\Pi(\kappa(\mathfrak{p})), \Pi(R[1/p]/\mathfrak{p}^2)\right) + 1.$$

The contravariant functor $\check{\mathbf{V}}$ induces an injection

(92)

$$\operatorname{Hom}_G \left(\Pi(\kappa(\mathfrak{p})), \Pi(R[1/p]/\mathfrak{p}^2)\right) \hookrightarrow \operatorname{Hom}_{G_{\mathbb{Q}_p}} \left(\check{\mathbf{V}}(\Pi(R[1/p]/\mathfrak{p}^2)), \check{\mathbf{V}}(\Pi(\kappa(\mathfrak{p})))\right).$$

Since the target is isomorphic to $\operatorname{Hom}_{G_{\mathbb{Q}_p}} \left(\rho^{\square}[1/p]/\mathfrak{p}^2, \rho^{\square}_{\mathfrak{p}}\right)$ by Remark 4.2, the claim follows from Lemma 4.8.

Let now $\Pi(\kappa(\mathfrak{p}))$ be reducible. Then, as in the proof of Proposition 4.3, it comes from an exact sequence

$$0 \longrightarrow \delta_2 \longrightarrow \rho^{\square}_{\mathfrak{p}} \longrightarrow \delta_1 \longrightarrow 0 \,, \tag{93}$$

with $\delta_1\delta_2^{-1} \neq \mathbb{1}, \epsilon^{\pm 1}$. We obtain an associated exact sequence

$$0 \longrightarrow \Pi_1 \longrightarrow \Pi(\kappa(\mathfrak{p})) \longrightarrow \Pi_2 \longrightarrow 0 \,, \tag{94}$$

where $\check{\mathbf{V}}(\Pi_i) = \delta_i$, $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}} = \Pi_1^{\mathrm{alg}}$ and (94) splits if and only if (93) splits, see [13, Prop. 3.4.2]. Furthermore, Π_1 is irreducible and, again as in [34, Thm. 4.19], Π_1 satisfies (RED). If we let E be the closure of the locally algebraic vectors in $\Pi(R[1/p]/\mathfrak{p}^2)$, we obtain an isomorphism

$$\operatorname{Hom}_K(V, \Pi(R[1/p]/\mathfrak{p}^2)) \cong \operatorname{Hom}_K(V, E).$$

Now (91) gives rise to another exact sequences of unitary Banach space representations

$$0 \longrightarrow \Pi_1 \longrightarrow E \longrightarrow \Pi_1^{\oplus m} \longrightarrow 0. \tag{95}$$

Since Π_1 satisfies (RED), we can apply Lemma 4.7 to obtain

$$\dim_{\kappa(\mathfrak{p})} \operatorname{Hom}_K(V, E) \leq \dim_{\kappa(\mathfrak{p})} \operatorname{Hom}_G(\Pi_1, E) + 1.$$

Because of the inclusions

$$\operatorname{Hom}_G(\Pi_1, E) \hookrightarrow \operatorname{Hom}_G(\Pi_1, \Pi(R[1/p]/\mathfrak{p}^2)) \hookrightarrow \operatorname{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \delta_1)$$

we obtain

$$\dim_{\kappa(\mathfrak{p})} \operatorname{Hom}_K(V, E) \leq \dim_{\kappa(\mathfrak{p})} \operatorname{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p], \delta_1) + 1.$$

But by Lemma 4.8 $\dim_{\kappa(\mathfrak{p})} \operatorname{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p], \delta_1) = 4$, and we are done. $\square$

Now we are finally able to prove the main theorem. We let again $\chi\colon G_{\mathbb{Q}_p} \to k^\times$ be a continuous character and let

$$\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$$
$$g \mapsto \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}.$$

Theorem 4.10. *Let $p > 2$ and let $(\mathbf{w}, \tau, \psi)$ be a Hodge type. There exists a reduced $\mathcal{O}$-torsion free quotient $R^{\square}_{\rho}(\mathbf{w}, \tau, \psi)$ (resp. $R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w}, \tau, \psi)$) of $R^{\square}_{\rho}$ such that for all $\mathfrak{p} \in$ m-Spec$(R^{\square}_{\rho}[1/p])$, $\mathfrak{p}$ is an element of m-Spec$\left(R^{\square}_{\rho}(\mathbf{w}, \tau, \psi)[1/p]\right)$ (resp. m-Spec$\left(R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w}, \tau, \psi)[1/p]\right)$) if and only if $\rho^{\square}_{\mathfrak{p}}$ is potentially semi-stable (resp. potentially crystalline) of p-adic Hodge type $(\mathbf{w}, \tau, \psi)$. If $R^{\square}_{\rho}(\mathbf{w}, \tau, \psi)$ (resp. $R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w}, \tau, \psi)$) is non-zero, then it has Krull dimension 5.*

Furthermore, there exists a four-dimensional cycle $z(\rho) := z_4(M(\lambda))$ *of* $R^{\square}_{\rho}$*, where* $\lambda := \mathrm{Sym}^{p-2} k^2 \otimes \chi \circ \det$*, such that there are equalities of four-dimensional cycles*

$$z_4\big(R^{\square}_{\rho}(\mathbf{w}, \tau, \psi)/(\varpi)\big) = m_{\lambda}(\mathbf{w}, \tau) z(\rho), \tag{96}$$

$$z_4\big(R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w}, \tau, \psi)/(\varpi)\big) = m^{\mathrm{cr}}_{\lambda}(\mathbf{w}, \tau) z(\rho). \tag{97}$$

Proof. We set $N := (R \oplus R)\hat{\otimes}_{\tilde{E},\eta} \tilde{P}$, as in Theorem 2.3. Hence, if we let $\mathfrak{a}$ be the R-annihilator of $M(\Theta)$, we obtain from Proposition 4.3 and [34, Prop. 2.22], analogous to [34, Thm. 4.15], that for any K-invariant $\mathcal{O}$-lattice Θ in $\sigma(\mathbf{w}, \tau)$ (resp. $\sigma^{\mathrm{cr}}(\mathbf{w}, \tau)$) $R/\sqrt{\mathfrak{a}} \cong R^{\square}_{\rho}(\mathbf{w}, \tau, \psi)$ (resp. $R/\sqrt{\mathfrak{a}} \cong R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w}, \tau, \psi)$). Since R is Cohen-Macaulay, Proposition 3.1 shows that we can apply Theorem 2.3 in our situation. Let Z be the center of G and let $Z_1 := I_1 \cap Z$. Since $p > 2$, there exists a continuous character $\sqrt{\psi}\colon Z_1 \to \mathcal{O}^{\times}$ with $\sqrt{\psi}^2 = \psi$. Twisting by $\sqrt{\psi} \circ \det$ induces an equivalence of categories between $\mathrm{Mod}^{\mathrm{pro}}_{I_1,\psi}(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{pro}}_{I_1/Z_1}(\mathcal{O})$. In this way we can use Theorem 2.3 to show the inequality of Theorem 2.2 for the setup $G = \mathrm{GL}_2(\mathbb{Q}_p)/Z_1$, $K = \mathrm{GL}_2(\mathbb{Z}_p)/Z_1$ and $P = I_1/Z_1$. Hence we obtain from Paškūnas' Theorem 2.2 that the conditions a) and b) of the criterion 2.1 for the Breuil-Mézard conjecture are satisfied. We let Σ be as in Definition 4.4. Since we know from Corollary 4.5 that Σ is dense in $\mathrm{Supp}\, M(\Theta)$, condition (i) of part c) follows from Proposition 4.3. As already remarked in the proof of Proposition 4.5, we have $\dim M(\Theta) = 5$. Thus condition (ii) of part c) is the statement of Corollary 4.9. Hence Theorem 2.1 says that there are equations of the form

$$z_4\big(R^{\square}_{\rho}(\mathbf{w}, \tau, \psi)/(\varpi)\big) = \sum_{\sigma} m_{\sigma}(\mathbf{w}, \tau) z_4(M(\sigma)), \tag{98}$$

$$z_4\big(R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w}, \tau, \psi)/(\varpi)\big) = \sum_{\sigma} m^{\mathrm{cr}}_{\sigma}(\mathbf{w}, \tau) z_4(M(\sigma)). \tag{99}$$

where the sum runs over all isomorphism classes of smooth irreducible K-representations over k. By Lemma 2.6 we have that $M(\sigma) \neq 0$ if and only if σ lies in the K-socle of π. We let K_1 denote the kernel of the projection $K \to \mathrm{GL}_2(\mathbb{F}_p)$ and let B denote the subgroup of upper triangular matrices of $\mathrm{GL}_2(\mathbb{F}_p)$. Let now σ be a smooth irreducible K-representation in the K-socle of π. Since K_1 is a normal pro-p subgroup of K, we must have $\sigma^{K_1} \neq 0$ and thus $\sigma = \sigma^{K_1}$. There are isomorphisms of K-representations

$$\pi^{K_1} \cong \big((\mathrm{Ind}^{K}_{P \cap K} \mathbb{1} \otimes \omega^{-1})_{\mathrm{sm}}\big)^{K_1} \cong \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{F}_p)}_{B} \mathbb{1} \otimes \omega^{-1}, \tag{100}$$

and it follows from [32, Lemma 4.1.3] that the K-socle of π^{K_1} is isomorphic to $\mathrm{Sym}^{p-2} k^2 \otimes \chi \circ \det$, in particular, it is irreducible. Therefore there is only a single cycle $z(\rho) = z_4(M(\mathrm{Sym}^{p-2} k^2 \otimes \chi \circ \det))$ on the right hand side of (98) and (99). $\square$

Remark 4.11. *If* $\tau = \mathbb{1} \oplus \mathbb{1}$ *and* $\mathbf{w} = (a, b)$ *with* $b - a \leq p - 1$*, then the right hand side of* (97) *is non-trivial if and only if* $b - a = p - 1$*, in which case the Hilbert-Samuel multiplicity of* $z(\rho)$ *is equal to the multiplicity of* $R^{\square,\mathrm{cr}}_{\rho}(\mathbf{w}, \mathbb{1} \oplus \mathbb{1}, \psi)/(\varpi)$*. In* [35]*, we computed that this multiplicity is* 1 *if* $\rho \otimes \chi^{-1}$ *is ramified,* 2 *if* $\rho \otimes \chi^{-1}$ *is unramified and indecomposable, and* 4 *if* $\rho \otimes \chi^{-1}$ *is split.*

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