# Oriented cohomology theory on Deligne-Mumford stacks

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# Introduction

**Motivations**

In [**14**], Quillen introduced the notion of an *oriented cohomology theory* on the category of differentiable manifolds. The crucial observation of Quillen is that the complex cobordism theory $MU^*$ is the universal cohomology theory among all *oriented cohomology theories*. In particular, the universality of $MU^*$ comes from the universal formal group law, where the formal group law $F_{fgl}(u,v) \in A^*(pt)[[u,v]]$ associated to an *oriented cohomology theory* $A^*$ is defined as the expression of the first Chern class $c_1(L\otimes M)$ of a tensor product of line bundles as a power series in $c_1(L)$ and $c_1(M)$ by the following formula

$$c_1(L \otimes M) = F_{fgl}(c_1(L), c_1(M)).$$

Quillen's theory of complex cobordism was extended to the algebraic setting by Levine and Morel in [**11**]. They consider the category $Sm_k$ of smooth quasi-projective $k$-schemes with a fixed ground field $k$. They first consider cobordism cycles on a $k$-scheme of finite type, then they impose axioms $(Dim)$, $(Sect)$ and $(FGL)$ on the cobordism cycles functor, giving rise to the theory of algebraic cobordism. $(Dim)$ is just the usual dimension condition on cohomology theory, $(Sect)$ is the classical cobordism and $(FGL)$ gives the formal group law on algebraic cobordism.

We observe that the fact that algebraic cobordism is an *oriented cohomology theory* is highly non-trivial. In particular, a lot of technical results are needed to prove localization sequence, the projective bundle formula and above all, Gysin pullbacks for l.c.i. morphisms.

The main result of Levine and Morel is the following:

Theorem 0.1 (Theorem 7.1.3 in [**11**]). *Assume $k$ admits resolution of singularities. Then there exists a the universal oriented cohomology theory on $Sm_k$, the theory of algebraic cobordism, denoted $\Omega^*$.*

Together with the following two theorems, we get interesting results on $K_0$ and $Ch^*$.

Theorem 0.2 (Theorem 7.1.4 in [**11**]). *Assume $k$ admits resolution of singularities. The canonical morphism $\Omega^* \to K_0[\beta,\beta^{-1}]$ of oriented cohomology theories on $Sm_k$ induces an isomorphism*

$$\Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta,\beta^{-1}] \to K_0[\beta,\beta^{-1}].$$

Theorem 0.3. *Suppose that $k$ has characteristic zero. Then the canonical morphism*

$$\Omega^*_{ad} := \Omega^*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \to Ch^*$$

*induced from $\Omega^* \to Ch^*$ is an isomorphism.*

In particular, these theorems tell us that $Ch^*$ is universal among *oriented cohomology theories* with additive formal group laws, while $K_0[\beta, \beta^{-1}]$ is universal among *oriented cohomology theories* with multiplicative and periodic formal group laws.

Essentially due to Quillen's observation, there is another theorem that tells us information about the algebraic cobordism with rational coefficients, i.e., by applying the twisting construction we can get algebraic cobordism out of $Ch^*$.

THEOREM 0.4. *Let k be a field that admits resolution of singularities. Then the canonical morphism*

$$\Omega^* \to \Omega^*_{ad}[\boldsymbol{t}]^{(\boldsymbol{t})}$$

*is an isomorphism after* $\otimes\mathbb{Q}$*.*

**Overview of the thesis**

The main theme of the thesis is to prove similar results in the setting of Deligne-Mumford stacks.

In Chapter 1 we give the basic definitions and main examples we are interested in. The notion of an oriented Borel-Moore homology theory is introduced since it's defined on the category $DM_k$ of Deligne-Mumford stacks, while an oriented cohomology theory is mainly defined on $SmDM_k$, the category of smooth Deligne-Mumford stacks. Similar as the case of schemes, Chow groups functor is the main example of our theory. We recall the basic definition of rational Chow groups and the proper push-forward, smooth pullback, together with Gysin pullback, and various compatibilities from [**17**]. With slight modifications, we are able to show certain l.c.i. pullback for an oriented theory.

Since we don't find a nice reference for the proof of projective bundle formula, we give a detailed proof of projective bundle formula in case of Deligne-Mumford stacks by using a result of Kresch [**9**, Proposition 4.5.2-Proposition 4.5.5] on stratifications by global quotient Deligne-Mumford stacks. The extended homotopy property, just like the case of schemes, follows from the localization sequence and projective bundle formula.

In chapter 2, we first establish an equivalence of categories between oriented Borel-Moore homology theories and oriented cohomology theories on $SmDM_k$. We introduce the first Chern class operator and show the elementary properties of $\widetilde{c_1}$, thus we can talk about the formal group law on any theory $A^*$.

In section 2, we talk about Chern classes and Whitney product formula.

In section 3, we have to introduce an axiom ($Complete$), which turns out to be a major difference from the case of schemes. Namely, the theory on schemes is automatically complete but not on Deligne-Mumford stacks. In particular, $Ch^*$ always satisfies the axiom ($Complete$).

In section 4, we introduce Todd classes and show its elementary properties. Todd classes lie in the heart of Grothendieck-Riemann-Roch and also the idea of twisting, which is exactly the topic of the last section of this chapter.

In chapter 3, we generalize two Riemann-Roch type theorems of Panin in [**13**]. Let $\varphi : A^* \to B^*$ be a morphism of two oriented cohomology theories on $SmDM_k$, i.e., a natural transformation of the underlying functor, we would like to know when $\varphi$ commutes with push-forward. The answer is positive for strongly projective morphisms when $\varphi$ sends first Chern class of $A^*$ to first Chern class of $B^*$, and even for projective morphisms if both $A^*$ and $B^*$ satisfy the axioms (*Complete*) and (*Extended FGL*).

The proof of this theorem, just like the case of the schemes, is consisted of two steps.

The case of a closed immersion employs deformation to the normal cone first to reduce to the case of an imbedding into the projective bundle of the normal bundle, then splitting principle and induction allows us to reduce to the case of a line bundle. Then it is just a direct computation.

The case of projections are consisted of two parts : projections to a point and projective space bundle to the underlying space. The crucial lemma is the computation of the push-forward of the diagonal imbedding. Then we do the explicit computation and use induction to see that $\varphi$ sends basis of $A$ to basis of $B$. For the case of projective space bundle, we have to make use of formal group law to make a change of basis.

For the second theorem, under some assumptions, we can find a Todd class operator $\widetilde{Td}_\tau$ such that

$$f_B(\widetilde{Td}_\tau(T_Y) \circ \varphi(x)) = \widetilde{Td}_\tau(T_X) \circ \varphi(f_A(x))$$

for any $x \in A^*(Y)$.

In chapter 4, we first use a Chern character map and the first Riemann-Roch theorem to deduce the universality of $K_0[\beta, \beta^{-1}]$ among oriented cohomology theories with multiplicative and periodic formal group law, and $\widehat{K_0}[\beta, \beta^{-1}]$ among oriented cohomology theories with multiplicative and periodic formal group law, which are also complete.

On the other hand, Edidin and Graham proves the following isomorphism with rational coefficients :

$$\widehat{K_0(X)} \to Ch^*(X)_{\mathbb{Q}}.$$

This result, together with Adams operation and the twisting construction, allows us to prove the universality of $Ch^*$ among additive and complete theories.

Finally, we can apply the twisting construction to $Ch^*$ to get rational algebraic cobordism.

**Notations and Conventions**

Unless otherwise mentioned, we work over a fixed ground field $k$.

An *algebraic stack* is an algebraic stack over $k$ in the sense of [**10**], while a *Deligne-Mumford stack* is an algebraic stack in the sense of [**17**], in other words an algebraic stack that admits an etale surjective morphism from a scheme. We always assume the stack is of fine type over $k$.

We will denote by $DM_k$ the category of separated stacks of finite type over Spec $k$. $SmDM_k$ will then represent the full subcategory of $DM_k$ consisting of stacks smooth over Spec $k$.

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## Contents

CHAPTER 1

# Definitions and examples of oriented cohomology theories

## 1. Oriented cohomology theories and oriented Borel-Moore homology theories

In this section we introduce the notions of oriented cohomology theory and Borel-Moore oriented homology theory for Deligne-Mumford stacks following [**11**] with only minor modifications.

Definition 1.1. Let $\mathcal{V}$ be a full subcategory of $DM_k$. $\mathcal{V}$ is said *admissible* if it satisfies the following conditions

(1) $\operatorname{Spec} k$ and the empty scheme $\emptyset$ are in $\mathcal{V}$.
(2) If $Y \to X$ is a smooth morphism in $DM_k$ with $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.
(3) If $X$ and $Y$ are in $\mathcal{V}$, then so is the product $X \times_{\operatorname{Spec} k} Y$.
(4) If $X$ and $Y$ are in $\mathcal{V}$, so is $X \coprod Y$.

In the following definition $\mathbf{R}^*$ will denote the category of commutative, graded rings with unit. Let us also recall that a functor $A^* : \mathcal{V}^{op} \to \mathbf{R}^*$ is said to be additive if $A^*(\emptyset) = 0$ and for any pair $(X, Y) \in \mathcal{V}^2$ the canonical ring map $A^*(X \coprod Y) \to A^*(X) \times A^*(Y)$ is an isomorphism. We also let $\mathcal{V}'$ denote the subcategory of $\mathcal{V}$ whose morphisms are proper.

Definition 1.2. Let $f : X \to Z$, $g : y \to Z$ be morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

We say that $f$ and $g$ are transverse in $\mathcal{V}$ if

(1) The fiber product $W$ is in $\mathcal{V}$.
(2) $Tor_q^{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{O}_X) = 0$ for all $q > 0$.

Specifically, if $\mathcal{V} = SmDM_k$, we require that $W$ is smooth with the condition that $dim_k W = dim_k X + dim_k Y - dim_k Z$.

Definition 1.3. Let $\mathcal{V}$ be an admissible subcategory of $DM_k$. An oriented cohomology theory on $\mathcal{V}$ is given by

(D1). An additive functor $A^* : \mathcal{V}^{\mathrm{op}} \to \mathbf{R}^*$.
(D2). For each proper morphism $f : Y \to X$ in $\mathcal{V}$ of relative codimension $d$, a homomorphism of graded $A^*(X)$-modules:

$$f_* : A^*(Y) \to A^{*+d}(X) \ .$$

Observe that the ring homomorphism $f^* : A^*(X) \to A^*(Y)$ gives $A^*(Y)$ the structure of an $A^*(X)$-module.

These satisfy

(A1). One has $(\mathrm{Id}_X)_* = \mathrm{Id}_{A^*(X)}$ for any $X \in \mathcal{V}$. Moreover, given proper morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathcal{V}$, with $f$ of relative codimension $d$ and $g$ of relative codimension $e$, one has

$$(f \circ g)_* = f_* \circ g_* : A^*(Z) \to A^{*+d+e}(X) \ .$$

(A2). Let $f : X \to Z$, $g : y \to Z$ be transverse morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

Suppose that $f$ is proper of relative dimension $d$ (thus so is $f'$). Then $g^* f_* = f'_* g'^*$.

(PB). Let $E \to X$ be a rank $n$ vector bundle over some $X$ in $\mathcal{V}$, $O(1) \to \mathbb{P}(E)$ the canonical quotient line bundle with zero section $s : \mathbb{P}(E) \to O(1)$. Let $1 \in A^0(\mathbb{P}(E))$ denote the multiplicative unit element. Define $\xi \in A^1(\mathbb{P}(E))$ by

$$\xi := s^*(s_*(1)) \ .$$

Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module, with basis $(1, \xi, ...., \xi^{n-1})$.

(EH). Let $E \to X$ be a vector bundle over some $X$ in $\mathcal{V}$, and let $p : V \to X$ be an $E$-torsor. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

(Weak Localization). For any D-M stack $X$ in $\mathcal{V}$ with closed sub-stack $Y$ and complement $U$, the excision sequence $A_*Y \xrightarrow{i_*} A_*X \xrightarrow{j^*} A_*U$ is exact. Here $i, j$ are just the inclusions.

A morphism of oriented cohomology theories on $\mathcal{V}$ is a natural transformation of functors $\mathcal{V}^{\mathrm{op}} \to \mathbf{R}^*$, it is called an oriented morphism of oriented cohomology theories if furthermore it commutes with the maps $f_*$.

DEFINITION 1.4. Let $\mathcal{V}$ be an admissible subcategory of $DM_k$. An *oriented Borel-Moore homology theory* on $\mathcal{V}$ is given by

(D1). An additive functor $A_* : \mathcal{V}' \to \mathbf{Ab}_*$.

(D2). Pull-back maps as homomorphism of graded groups exist for any regular local immersion or smooth morphism.

(D3). An element $1 \in A_0(Speck)$ and, for each pair $(X, Y)$ of objects in $\mathcal{V}$, a bilinear graded pairing

$$\begin{array}{ccc} A_*(X) \otimes A_*(Y) & \to & A_*(X \times_{Speck} Y) \\ u \otimes v & \mapsto & u \times v \end{array}$$

called the external product, which is associative, commutative and admits 1 as unit element.

These satisfy

(BM1). One has $\mathrm{Id}_X^* = \mathrm{Id}_{A_*(X)}$ for any $X \in \mathcal{V}$ if we view $\mathrm{Id}_X$ as either a smooth morphism or a regular local immersion. Moreover, given composable regular local immersions $i_1$,$i_2$, we have $(i_1 i_2)^* = i_2^* i_1^*$, the same holds for composable smooth morphisms $p_1, p_2$.

We also require that: given a smooth morphism $p : X \to Z$ and a regular local immersion $i : Z \to X$ such that $pi = \mathrm{Id}_Z^*$, then $i^* p^* = \mathrm{Id}$.

(BM2). Let $f : X \to Z$, $g : Y \to Z$ be transverse morphisms in $\mathcal{V}$. Suppose that $f$ is proper and that $g$ is an regular local immersion or smooth morphism, giving the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

Note that $f'$ is proper and $g'$ is a regular immersion or smooth morphism. Then $g^* f_* = f'_* g'^*$.

Given the cartesian square

$$\begin{array}{ccc} W' & \xrightarrow{p'} & X' \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{p} & Z' \end{array}$$

where $i, i'$ regular local immersions, $p, p'$ smooth, then $p'^* i^* = i'^* p^*$.

(BM3). Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms in $\mathcal{V}$. If $f$ and $g$ are proper, then for $u' \in A_*(X')$ and $v' \in A(Y')$ one has

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v') \ .$$

If $f$ and $g$ are regular local immersions or smooth morphisms, then for $u \in A_*(X)$ and $v \in A_*(Y)$ one has

$$(f \times g)^*(u \times v) = f^*(u) \times g^*(v) \ .$$

(PB). For $L \to Y$ a line bundle on $Y \in \mathcal{V}$ with zero-section $s : Y \to L$, define the operator

$$\widetilde{c_1}(L) : A_*(Y) \to A_{*-1}(Y)$$

by $\widetilde{c_1}(\eta) = s^*(s_*(\eta))$. Let $E$ be a rank $n+1$ vector bundle on $X \in \mathcal{V}$, with projective bundle $q : \mathbb{P}(E) \to X$ and canonical quotient line bundle $O(1) \to \mathbb{P}(E)$. For $i \in \{0, ...., n\}$, let

$$\xi^{(i)} : A_{*+i-n}(X) \to A_*(\mathbb{P}(E))$$

be the composition of $q^* : A_{*+i-n}(X) \to A_{*+i}(\mathbb{P}(E))$ with $\widetilde{c_1}(O(1))^i : A_{*+i}(\mathbb{P}(E) \to A_*(\mathbb{P}(E)))$. Then the homomorphism

$$\sum_{i=0}^{n} \xi^{(i)} : \bigoplus_{i=0}^{n} A_{*+i-n}(X) \to A_*(\mathbb{P}(E))$$

is an isomorphism.

(EH). Let $E \to X$ be a vector bundle of rank $r$ over $X \in \mathcal{V}$, and let $p : V \to X$ be an $E$-torsor. Then $p^* : A_*(X) \to A_{*+r}(V)$ is an isomorphism.

(Weak Localization). For any D-M stack $X$ in $\mathcal{V}$ with closed sub-stack $Y$ and complement $U$, the excision sequence $A_*Y \xrightarrow{i_*} A_*X \xrightarrow{j^*} A_*U$ is exact. Here $i, j$ are just the inclusions.

Remark 1.5. By [**11**, Section 5.2.4], a consequence of (*Weak Localization*) and other axioms of oriented Borel-Moore homology theory is the so called cellular decomposition axiom.

$(CD)$. For integers $r, N > 0$, let $W = \mathbb{P}^N \times_{Speck} ... \times_{Speck} \mathbb{P}^N$ ($r$ factors), and let $p_i : W \to \mathbb{P}^N$ be the $i$-th projection. Let $X_0, ...., X_N$ be the standard homogeneous coordinates on $\mathbb{P}^N$, let $n_1, ...., n_r$ be non negative integers, and let $i : Z \to W$ be the subscheme defined by $\prod_{i=1}^{r} p_i^*(X_N)^{n_i} = 0$. Suppose that $Z$ is in $\mathcal{V}$. Then $i_* : A_*(Z) \to A_*(W)$ is injective.

A morphism of oriented Borel-Moore homology theories on $\mathcal{V}$ is a natural transformation of functors $\mathcal{V}' \to \mathbf{Ab}_*$ which respects the element 1 and commutes with the external product. It is called an oriented morphism if it commute with maps $f^*$.

Lemma 1.6. *If a regular immersion or a smooth morphism $f : X \to Y$ can be factored as a regular local immersion $i$ followed by a smooth morphism $p$, then $f^* = i^*p^*$.*

Proof. Note that if $pi$ and $p$ are smooth, we have the conclusion that $i$ is also smooth.

Recall that a local embedding just means a representable unramified morphism of finite type. Then if $pi$ and $p$ are both local embeddings, we know that $i$ is a local embedding. But we know by assumption that $p$ is smooth, $pi$ is regular, so $i$ is also regular.

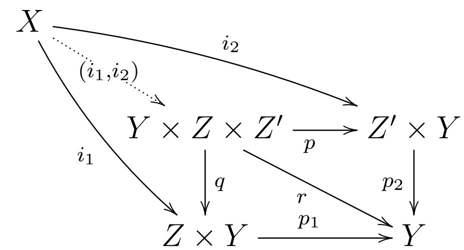
Then the lemma is true because of (BM1) for composable maps.

□

Lemma 1.7. *If a morphism $f : X \to Y$ can be factored as a regular local immersion $i : X \to Z \times_{Speck} Y$ followed by the projection $p : Z \times_{Speck} Y \to Y$, for some smooth DM stack $Z$ over $k$, then $i^*p^*$ is independent of the factorization for an Oriented Borel-Moore homology theory . We set $f^* = i^*p^*$.*

Proof. Suppose $Z, Z', i_1, p_1, i_2, p_2$ gives different factorizations, we just need to show $i_1^*p_1^* = i_2^*p_2^*$.

We just consider the following commutative diagram and use the previous lemma together with (BM1) for composable smooth morphisms.



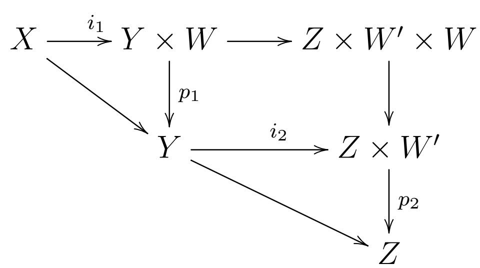

∎

Definition 1.8. We call $f$ in the previous lemma a strongly local complete intersection morphism. We usually denote it by strong l.c.i. morphism for short.

Lemma 1.9. *Let $f : X \to Y$ and $g : Y \to Z$ be two strong l.c.i morphisms. Then $gf$ is still a strong l.c.i. morphism and $(gf)^* = f^*g^*$.*

Proof. Suppose $W, W', i_1, p_1, i_2, p_2$ gives factorizations for $f$ and $g$.

We consider the following commutative diagram.




This gives $(gf)^*$ the desired factorization of a regular local immersion followed by a smooth projection. Note that the commutative square in the diagram is a cartesian diagram, then $(gf)^* = f^*g^*$ follows from definition of $f^*$ and (BM2).

∎

Remark 1.10. From above lemmas, we can get a simpler but weaker version of an oriented Borel-Moore homology theory. Namely, we have only strong l.c.i. morphisms and proper morphisms, satisfying the usual compatibilities between functorial pull-back and push-forward. Since in particular, we would like to apply the theory to arbitrary morphisms between smooth DM stacks.

Lemma 1.11. *Leg $f : X \to Y$ be an arbitrary morphism between smooth DM stacks . Then $f$ is a strong l.c.i. morphism.*

Proof. We have the obvious graph inclusion of $X$ into $X \times Y$ followed by the projection into $Y$.

We are left to show that the graph inclusion $\gamma_f = (1, f)$ is a regular local immersion. Consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\gamma_f} & X \times Y \\ & \searrow^{\mathrm{id}_X} & \downarrow p_1 \\ & & X \end{array}$$

By [**17**, Lemma (1.3)], $\gamma_f$ is representable since $\mathrm{id}_X$ is representable. We are again in the same situation where $p_1$ is smooth, $\mathrm{id}_X$ is a regular embedding. This implies that $\gamma_f$ is a regular local immersion. □

REMARK 1.12. Specifically, we have $(gf)^* = f^*g^*$ in an oriented Borel-Moore homology theory for $f : X \to Y$ and $g : Y \to Z$ where $X, Y, Z$ are all smooth.

In the previous definition the abbreviations $(PB)$ and $(EH)$ stands respectively for *projective bundle formula* and *extended homotopy property*. The morphisms $f^*$ are called *pull-backs*, while the morphisms $f_*$ are called *push-forwards*.

## 2. Recollection on Chow groups of Deligne-Mumford stacks

By a *Deligne-Mumford stack*, we mean an algebraic stack in the sense of [**17**] and [**2**]. In this section we recall some constructions of Chow groups for a Deligne-Mumford stack and prove some elementary properties in the sense of [**8**]. These give examples of Oriented Borel-Moore homology theories.

### 2.1. Chow groups for Deligne-Mumford stacks.

DEFINITION 2.1. Let $X$ be a D-M stack. A cycle of dimension $n$ on $X$ is an element of the free abelian group $Z_n(X)$ generated by all integral closed sub-stacks of dimension $n$. Set

$$Z_*(X) = \bigoplus_n Z_n(X).$$

The group of rational equivalences on $X$ is

$$W_*(X) = \bigoplus_n W_n(X)$$

where

$$W_n(X) = \bigoplus_G k(G)^*.$$

The direct sum is taken over all integral sub-stacks $G$ of $X$ of dimension $k + 1$.

By restricting $Z_*$ and $W_*$ to the etale site of a D-M stack $X$, we get two sheaves on $X$, denoted by $\mathcal{Z}_*$ and $\mathcal{W}_*$. Here the groups of global sections of $\mathcal{W}_*$ and $\mathcal{Z}_*$ coincide with the $W_*(X)$ and $Z_*(X)$ defined above. There is a morphism of sheaves

$$\partial \colon \mathcal{W}_* \to \mathcal{Z}_*.$$

Hence we get a homomorphism

$$\partial_X \colon W_*(X) \to Z_*(X).$$

DEFINITION 2.2. The Chow group $Ch_*(X)$ of $X$ is the cokernel of $\partial_X$. We also get the rational Chow group $Ch_*(X)_{\mathbb{Q}}$ of $X$ by $\otimes\mathbb{Q}$.

Let us also recall the definition of a quasi-coherent sheaf on a Deligne-Mumford stack.

DEFINITION 2.3. A quasi-coherent sheaf $\mathcal{F}$ on $X$ consists of the following data.
(i) For each atlas(an etale surjective morphism $U \to X$), a quasi-coherent sheaf $\mathcal{F}_U$ on $U$.
(ii) For each morphism $\varphi : U \to V$ between atlases, it induces an isomorphism of of sheaves $\mathcal{F}_U \to \varphi^*\mathcal{F}_V$ which satisfies the cocycle condition.
A coherent sheaf resp. vector bundle is a quasi-coherent sheaf $\mathcal{F}$ where $\mathcal{F}_U$ is coherent resp. coherent and locally free.

**2.2. Flat pullback and proper push-forward.** In this section, we review the construction of Gillet and Vistoli [**17**] on flat pullback and proper push-forward.

DEFINITION 2.4. Let $f : X \to Y$ be a separated dominant morphism of integral DM stacks.
Define
$$deg(X/Y) = deg(X \times_Y V/V)$$
if $f$ is representable and $V \to Y$ is an atlas of $Y$.
Define
$$deg(X/Y) = deg(U/Y)/deg(U/X)$$
in general and $U \to X$ is an atlas of $X$.

The following lemma and proposition on properties of degree are from [**17**] , Section 1.

LEMMA 2.5. *The degree is well defined. If $f : X \to Y$ and $g : Y \to Z$ are separated dominant morphisms of integral stacks, then*
$$deg(X/Z) = deg(X/Y)deg(Y/Z).$$

For a DM stack $X$, let $I_X$ denote the inertia group stack of $X$ see [Definition1.12] [**17**], By [Lemma 1.13][**17**], $I_X \to X$ is finite. Let $\delta(X) = deg(I_X/X)$, i.e. $\delta(X)$ is the order of the automorphism group of a general geometric point of $X$.

PROPOSITION 2.6. *Let $f : X \to Y$ be a separated dominant morphism of integral stacks, then*
$$deg(X/Y) = \delta(Y)/\delta(X)[k(X) : k(Y)].$$

Now we can use degree to define flat pullback and proper push-forward.

DEFINITION 2.7. Let $f : X \to Y$ be a morphism of DM stacks.
(i) If $f$ is flat, we define the flat pullback
$$f^* : Z_*(Y) \to Z_*(X)$$
by $f^*[Y'] = [Y' \times_Y X]$ for any closed integral sub-stack $Y'$ of $Y$.

(ii)If $f$ is proper(not necessarily representable), the proper push-forward

$$Z_*(X)_{\mathbb{Q}} \to Z_*(Y)_{\mathbb{Q}}$$

is defined by $f^*[X'] = deg(X'/Y')[Y']$ where $X'$ is an integral sub-stack of $X$ and $Y'$ is its image in $Y$.

Proposition 2.8. *The flat pullback and proper push-forward pass to rational equivalence. Thus we get flat pullback*

$$f^* : Ch_*(Y) \to Ch_*(X)$$

*and proper push-forward*

$$f_* : Ch_*(X)_{\mathbb{Q}} \to Ch_*(Y)_{\mathbb{Q}}.$$

*Totether with the lemma and proposition before, we see that $Ch_*$ is a contravairant functor for flat morphisms, and a covariant functor for proper morphisms.*

Lemma 2.9. *Given a cartesian square as follows.*

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

*Here $f$ is a proper morphism of stacks and $g$ is flat. Then*

$$g^* f_* = f'_* g'^* : Ch_*(X)_{\mathbb{Q}} \to Ch_*(Y)_{\mathbb{Q}}$$

Proof. These are [3.6-3.9] in [**17**]. □

Definition 2.10. For $i : D \hookrightarrow X$ an inclusion of effective Cartier divisor, suppose $Y$ is a closed integral sub-stack of dimension $n$. We define $D \cdot Y$ as follows: if $Y \nsubseteq D$, $D$ restricts to a Cartier divisor $i^*D$ on $Y$, we set the associated Weil divisor of $j^*D$ as $D \cdot Y$ in $Ch_{n-1}(D)$; if $Y \subseteq D$, $D \cdot Y$ is the class in $Ch_{n-1}(D)$ represented by $[C]$, for any effective Cartier divisor $C$ on $Y$ such that $\mathcal{O}_Y(C)$ is isomorphic to $i^*\mathcal{O}_X(D)$. Then we can define a pullback $i^* : Ch_*(X) \to Ch_{*-1}(D)$ by

$$i^*(Y) = D \cdot Y$$

and extend linearly.

## 3. Localization sequence, Projective bundle formula and the extended homotopy property

By mimicking Fulton's proofs in [**8**] chapter 1-3, we would like to prove the exactness of the localization sequence, the projective bundle formula and the extended homotopy property hold for $Ch_*$ in this section.

Since the push-forward maps are only defined for rational Chow groups (in case the morphism is not representable), from now on we only use Chow groups with rational coefficients in order to make the whole theory work. We omit the $\mathbb{Q}$ from the notation for convenience.

In case of Deligne-Mumford stacks, most of the properties are exactly the same as in the case of schemes. Here we just list the main ingredients below while going through the proof of localization sequence.

The localization property we will prove is:

$(Loc)$For any D-M stack $X$ with closed sub-stack $Y$ and complement $U$, the excision sequence $Ch_kY \xrightarrow{i_*} Ch_kX \xrightarrow{j^*} Ch_kU \to 0$ is exact. Here $i, j$ are just the inclusions.

The proof is the same as for schemes, using:

(i)The sequence $Z_kY \xrightarrow{i_*} Z_kX \xrightarrow{j^*} Z_kU \to 0$ is exact. This follows from the very definitions of $X, Y, U$, and $i_*, j^*$.

(ii)$R(\bar{W}_i) = R(W_i)$ for $W$ sub-stacks of $U$. Here $R(W_i)$ denotes groups of rational functions on $W_i$ and $\bar{W}_i$ is the closure in $X$.

See the proof of Proposition 1.8 in [**8**].

**3.1. Projective bundle formula.** The projective bundle formula is:

$(PB)$ For $L \to Y$ a line bundle on $Y$ with zero-section $s : Y \to L$, define the operator

$$\widetilde{c_1}(L) : Ch_*(Y) \to Ch_{*-1}(Y)$$

by $\widetilde{c_1}(\eta) = s^*(s_*(\eta))$. Let $E$ be a rank $n+1$ vector bundle on $X$ where $X$ is a D-M stack, with projective bundle $q : \mathbb{P}(E) \to X$ and canonical quotient line bundle $O(1) \to \mathbb{P}(E)$. For $i \in \{0, ...., n\}$, let

$$\xi^{(i)} : Ch_{*+i-n}(X) \to Ch_*(\mathbb{P}(E))$$

be the composition of $q^* : Ch_{*+i-n}(X) \to Ch_{*+i}(\mathbb{P}(E))$ with $\widetilde{c_1}(O(1))^i : Ch_{*+i}(\mathbb{P}(E) \to Ch_*(\mathbb{P}(E)))$. Then the homomorphism

$$\sum_{i=0}^{n} \xi^{(i)} : \bigoplus_{i=0}^{n} Ch_{*+i-n}(X) \to Ch_*(\mathbb{P}(E))$$

is an isomorphism.

We first give a proof in case of schemes. Let $X$ be a scheme of finite type over a ground field.

Proposition 3.1. *Let $p : E \to X$ be a trivial vector bundle of rank $n$, i.e. $E = X \times \mathbb{A}^n$. Then the flat pull-back*

$$p^* : Ch_*(X) \to Ch_{*+n}(E)$$

*is surjective for all $k$.*

Proof. We may assume $n = 1$ and continue inductively. See the proof of [**8**, Proposition 1.9]

□

Proposition 3.2. *$(PB)$ hold for trivial vector bundles.*

Proof. In case of a trivial bundle $E = \mathcal{O}_X^{n+1}$, we have $\mathbb{P}(E) = \mathbb{P}^n_X$, and $\mathcal{O}(1)$ is the invertible sheaf with $q^*\mathcal{O}(1)$ the $\mathcal{O}_X$-module generated by $X_0, ..., X_n$.

Now everything is essentially a consequence of elementary properties and computations of Chern classes of explicit vector bundles.

We show injectivity by constructing an inverse of

$$\sum_{i=0}^{n} \xi^{(i)} : \bigoplus_{i=0}^{n} Ch_{*+i-n}(X) \to Ch_*(\mathbb{P}(E))$$

which we denote by

$$\prod_{i=0}^{n} \sigma^{(i)} : Ch_*(\mathbb{P}(E)) \to \bigoplus_{i=0}^{n} Ch_{*+i-n}(X)$$

where $\sigma^0 := q_*(\widetilde{c_1}(O(1))^n$ . We then define inductively

$$\sigma^m := q_*(\widetilde{c_1}(O(1))^{n-m}(Id - \sum_{j=0}^{m-1} \widetilde{c_1}(\mathcal{O}(1))^j q^* \sigma_j)).$$

We can verify that

$$\sum_{i=0}^{n} \xi^{(i)} \prod_{i=0}^{n} \sigma^{(i)} = Id$$

. See [Lemma 3.5.3] in [**11**] for example.

For surjectivity, we use induction. $n = 0$ is trivial.

Denote $i_{n-1} : \mathbb{P}^{n-1}_X \to \mathbb{P}^n_X$; let $j : \mathbb{A}^n_X \to \mathbb{P}^n_X$ be the open complement.

We consider the following commutative diagram, first arrow the evident inclusion,

$$\begin{array}{ccc} \bigoplus_{i=0}^{n-1} Ch_{*+i-n+1}(X) & \longrightarrow & \bigoplus_{i=0}^{n} Ch_{*+i-n}(X) \\ \downarrow{\scriptstyle \sum_{i=0}^{n-1}\xi^{(i)}} & & \downarrow{\scriptstyle \sum_{i=0}^{n}\xi^{(i)}} \\ Ch_*(\mathbb{P}^{n-1}_X) & \xrightarrow{i_{n-1*}} & Ch_*(\mathbb{P}^n_X) \end{array}$$

Since $j^*(\mathcal{O}(1))$ is the trivial bundle, it follows that

$$j^*\xi^{(i)} = 0$$

for $i > 0$.

Using the localization sequence

$$Ch_k(\mathbb{P}^{n-1}_X) \to Ch_k(\mathbb{P}^n_X) \xrightarrow{j^*} Ch_k(\mathbb{A}^n_X) \to 0,$$

we see that $\sum_{i=0}^{n} \xi^{(i)}$ is surjective if

$$p^* = j^*\xi^o : Ch_{*-n}(X) \to Ch_*(\mathbb{A}^n_X)$$

is. That was shown in the previous proposition. □

Remark 3.3. The proof of injectivity actually works for arbitrary vector bundles.

PROPOSITION 3.4.

$$\sum_{i=0}^{n} \xi^{(i)} : \bigoplus_{i=0}^{n} Ch_{*+i-n}(X) \to Ch_*(\mathbb{P}(E))$$

*is surjective, thus an isomorphism.*

PROOF. For any vector bundle $E$ we can find a closed subscheme $Y$ of $X$ such that the complement $U$ is an open set over which $E$ is trivial. Using Noetherian induction and $(Loc)$, it suffices to prove it for $X = U$, which is shown in the previous proposition. □

In the next step, we'd like to show projective bundle formula for global quotient Deligne-Mumford stacks of the form $X = [V/G]$ where $V$ is a scheme and $G$ is a group scheme over the ground field acting on $V$.

LEMMA 3.5. *All the vector bundles on $X = [V/G]$ is of the form $[E_V/G]$ where $E_V$ is a vector bundle on $V$.*

PROOF. $V \times_X V \rightrightarrows V \to X$ gives a presentation of $X$ where $G$ acts on $V$. Suppose $E$ is a vector bundle on $X$. We consider the following $G$-equivariant cartesian diagram

$$\begin{array}{ccc} E_V & \longrightarrow & E \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

Then $E_V \times_E E_V \rightrightarrows E_V \to E$ gives the presentation of $E$ with $G$ acting on $E_V$, this presentation gives exactly the stack $[E_V/G]$. □

PROPOSITION 3.6. *Let $X = [V/G]$, $E$ be a vector bundle on $X$. We have an isomorphism*

$$\sum_{i=0}^{n} \xi^{(i)} : \bigoplus_{i=0}^{n} Ch_{*+i-n}(X) \to Ch_*(\mathbb{P}(E))$$

PROOF. From the previous lemma, we can identify $\mathbb{P}(E)$ with $[\mathbb{P}(E_V)/G]$.

By [**4**, Proof of Theorem 3] we can interpret $Ch_*(\mathbb{P}(E))$ as invariant Chow groups of schemes, which we denote by $Ch_*(\mathbb{P}(E_V))^G$ and $\bigoplus_{i=0}^{n} Ch_{*+i-n}(X)$ as $\bigoplus_{i=0}^{n} Ch_{*+i-n}(V)^G$.

Then, the proposition follows from the projective bundle formula of schemes and the fact that $G$-action of $E_V$ is induced from $G$-action of $V$. See the following diagram.

$$\begin{array}{ccc} G \times E_V & \longrightarrow & E_V \\ \downarrow & & \downarrow \\ G \times V & \longrightarrow & V \end{array}$$

□

The following proposition is taken from [**9**, Proposition 4.5.2-Proposition 4.5.5]. We use it for the proof of surjectivity of $(PB)$.

Proposition 3.7. *Every Deligne-Mumford stack $X$ admits a stratification by global quotient Deligne-Mumford stacks, i.e.,there exists a stratification of $X^{red}$ by locally closed sub-stacks $U_i$, here for each $i$,$U_i$ is isomorphic to a stack of the form $[V_i/G_i]$ where $V_i$ is a quasi-projective scheme and $G_i$ is an algebraic group acting on $V_i$.*

Remark 3.8. By our definition of Chow groups, $X^{red}$ and $X$ have the same Chow groups which vanish for $i < 0$ and $i > \dim X$.

We establish a lemma first which will be used for injectivity of $(PB)$.

Lemma 3.9. *For $\alpha$ in $Ch_{*+i-n}(X)$, we always have*

$$q_*\xi^{(i)}(\alpha) = 0, i < n$$

*and*

$$q_*\xi^{(i)}(\alpha) = \alpha, i = n.$$

Proof. We can see that $q_*\xi^{(i)}$ sends $Ch_{*+i-n}(X)$ to $Ch_*(X)$. We may assume $S$ to be the integral sub-stack of dimension $s$ representing $\alpha$. Let $f$ be the embedding $f : S \to X$.

We have $q_*\xi^{(i)}(\alpha) = f_*(q_*\xi_S^{(i)}(\alpha))$, here $\xi_S^{(i)}$ we mean pulling back $E$ to $S$. But $\xi_S^{(i)}(\alpha))$ lies in $Ch_{s+i}S = 0$.

For $i = n$, we first notice

$$q_*\xi^{(n)}(\alpha) = m \cdot \alpha$$

where $m$ is a rational number. To compute $m$, we may restrict to an open sub-stack of $X$ which is a global quotient $[V/G]$. Then all the computations are exactly the same as the case of schemes as long as we remember that there is always a group action on $E$ and $\mathbb{P}(E)$ induced by the action on base scheme. There we have $m = 1$.

□

Theorem 3.10. *$(PB)$ holds for any Deligne-Mumford stack $X$.*

Proof. We prove surjectivity first using localization sequence. By Proposition 3.7, we may choose a sub-stack which contains the generic point in the stratification. Take $Y = X/U$ to be the closed complement.

Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\bigoplus_{i=0}^n Ch_{*+i-n}(Y) & \longrightarrow & \bigoplus_{i=0}^n Ch_{*+i-n}(X) & \longrightarrow & \bigoplus_{i=0}^n Ch_{*+i-n}(U) & \longrightarrow & 0 \\
\downarrow i & & \downarrow \sum_{i=0}^n \xi^{(i)} & & \downarrow j & & \downarrow \\
Ch_*(\mathbb{P}(E_Y)) & \longrightarrow & Ch_*(\mathbb{P}(E)) & \longrightarrow & Ch_*(\mathbb{P}(E_U)) & \longrightarrow & 0
\end{array}$$

We know $j$ is surjective by the previous case and $i$ is surjective by noetherian induction. This gives surjectivity.

For injectivity, suppose we have $(\alpha_0, ......, \alpha_n)$ goes to 0. We apply the lemma to deduce $\alpha_n = 0$. Then we apply the lemma inductively to get $\alpha_i = 0$, for $0 \leq i \leq n$.

□

**3.2. The extended homotopy property.** For the extended homotopy property, we have the following proposition.

Proposition 3.11. *(EH)Let $E \to X$ be a vector bundle of rank n over some $X$ , and let $p : V \to X$ be an $E$-torsor. Then $p^* : Ch_*(X) \to Ch_{*+n}(V)$ is an isomorphism.*

Proof. Let $\mathcal{F}$ be the sheaf of sections of $F$, and $\mathcal{F}^\vee$ be the dual. An affine bundle is described as follow:

$$0 \to \mathcal{F} \to \mathcal{E} \xrightarrow{\pi} \mathcal{O}_X \to 0$$

and $\mathcal{V} := \pi^{-1}(1)$ is the sheaf of sections of $V$.

Consider the dual sequence:

$$0 \to \mathcal{O}_X \to \mathcal{E}^\vee \to \mathcal{F}^\vee \to 0.$$

Denote by $p : \mathbb{P}(\mathcal{E}^\vee) \to X$ and $q : \mathbb{P}(\mathcal{F}^\vee) \to X$ , and the inclusion $i : \mathbb{P}(\mathcal{F}^\vee) \to \mathbb{P}(\mathcal{E}^\vee)$. The complement of $\mathbb{P}(\mathcal{F}^\vee)$ in $\mathbb{P}(\mathcal{E}^\vee)$ is isomorphic to $V$.

We have a localization sequence

$$Ch_{*+n}(\mathbb{P}(\mathcal{F}^\vee) \xrightarrow{i_*} Ch_{*+n}(\mathbb{P}(\mathcal{E}^\vee) \xrightarrow{j^*} Ch_{*+n}V \to 0.$$

There is also an obvious short exact sequence

$$\bigoplus_{i=0}^{n-1} Ch_{*+i+1}(X) \xrightarrow{\kappa} \bigoplus_{i=0}^{n} Ch_{*+i}(X) \xrightarrow{\mu} Ch_*X \to 0.$$

By projective bundle formula for $\mathbb{P}(\mathcal{F}^\vee)$ and $\mathbb{P}(\mathcal{E}^\vee)$, we have isomorphisms between $Ch_{*+n}(\mathbb{P}(\mathcal{F}^\vee)$ and $\bigoplus_{i=0}^{n-1} Ch_{*+i+1}(X)$, $Ch_{*+n}(\mathbb{P}(\mathcal{E}^\vee)$ and $\bigoplus_{i=0}^{n} Ch_{*+i}(X)$. By 5-lemma, it's enough to show the commutativity of the two short exact sequences below:

$$\begin{array}{ccccccc}
Ch_{*+n}(\mathbb{P}(\mathcal{F}^\vee) & \xrightarrow{i_*} & Ch_{*+n}(\mathbb{P}(\mathcal{E}^\vee) & \xrightarrow{j^*} & Ch_{*+n}V & \longrightarrow & 0 \\
\downarrow{\scriptstyle \sum_{i=0}^{n-1}\xi^{(i)}} & & \downarrow{\scriptstyle \sum_{i=0}^{n}\xi^{(i)}} & & \downarrow{\scriptstyle p^*} & & \downarrow \\
\bigoplus_{i=0}^{n-1} Ch_{*+i+1}(X) & \xrightarrow{\kappa} & \bigoplus_{i=0}^{n} Ch_{*+i}(X) & \xrightarrow{\mu} & Ch_*X & \longrightarrow & 0
\end{array}$$

Note that $j^*\mathcal{O}(1) = \mathcal{O}$ and we also have $\widetilde{c_1}(\mathcal{O}) = 0$ and commutativity of $j^*$ and $\widetilde{c_1}$. This establishes the commutative square on the right.

For the commutativity of the left square: We need to show $\widetilde{c_1}(\mathcal{O}_E(1))^j \circ p^* = i_*\widetilde{c_1}(\mathcal{O}_F(1))^{j-1} \circ q^*$ for $j \geq 1$, this could be achieved inductively from the case $j = 1$. This follows from the fact that $\mathcal{O}_E(1)$has a section vanishing precisely on $\mathbb{P}(\mathcal{E}^\vee)$.

## 4. Pull-back for regular local immersions

Let $f : X \to Y$ be a regular local immersion of codimension $d$, and $g$ be a morphism $T \to Y$, where $T$ is a pure dimensional DM stack. We form the cartesian diagram

$$\begin{array}{ccc}
W & \xrightarrow{f'} & T \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{f} & Y
\end{array}$$

Here $g'$ is still a local immersion. Let $C_{W/T}$ be the normal cone to $W$ in $T$, $N = g'^* N_{X/Y}$ the pullback of the normal bundle to $X$ in $Y$. There is a natural closed embedding of $C_{W/T}$ into $N$. Let $s : W \to N$ be the 0-section of $N$. Let $s^* : Ch_*(N) \to Ch_*(W)$ be the inverse of the pull-back isomorphism for a vector bundle on a D-M stack.

Define the intersection product

$$(X.T) \in Ch_{k-d}(W)$$

by

$$(X.T) = s^*[C_{W/T}].$$

If $Z$ is a D-M stack and $Z \to Y$ a morphism. Let $y$ be a cycle on $Y$, $y = \sum_i m_i[T_i]$ for sub-stacks $T_i$ of $Y$. Set $V = X \times_Y Z$ and $W_i = X \times_Y T_i$, we denote $h_i : T_i \to V$ the natural embedding.

Definition 4.1. The Gysin homomorphism

$$f^* : Z_*(Z) \to Ch_*(V)$$

is defined by

$$f^*(y) = \sum_i m_i h_{i*}(F.T_i).$$

The following theorems are due to Vistoli in Section 3 and Section 4, [**17**].

Theorem 4.2. *Let $f : X \to Y$ be a regular local immersion of D-M stacks. If $Z \to Y$ is a morphism of D-M stacks. Then the Gysin homomorphism*

$$f^* : Z_*(Z) \to Ch_*(V)$$

*passes to rational equivalence.*

We also call the resulting homomorphism

$$f^* : Ch_*(Z) \to Ch_*(V)$$

the Gysin homomorphism.

Theorem 4.3. *Consider a fiber diagram of D-M stacks*

$$\begin{array}{ccccc} X' & \xrightarrow{p} & X & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow f \\ Y' & \xrightarrow{q} & Y & \longrightarrow & G \end{array}$$

*where $f$ is a regular local embedding of stacks.*
*(i) If $q$ is proper, then*

$$f^* q_* = p_* f^* : Ch_*(Y') \to Ch_*(X).$$

*(ii) If $q$ is flat, then*

$$f^* q^* = p^* f^* : Ch_*(Y) \to Ch_*(X').$$

THEOREM 4.4. *Let be two $f_1 : F_1 \to G_1$ $f_2 : F_2 \to G_2$ be two regular local embeddings. Consider the fiber diagram of D-M stacks below*

$$\begin{array}{ccccc} Z & \longrightarrow & X_2 & \longrightarrow & F_2 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f_2} \\ X_1 & \longrightarrow & Y & \longrightarrow & G_2 \\ \downarrow & & \downarrow & & \\ F_1 & \xrightarrow{f_1} & G_1 & & \end{array}$$

*Then*

$$f_1^* f_2^* = f_2^* f_1^* : Ch_*(Y) \to Ch_*(Z).$$

□

REMARK 4.5. ($Ch_*$ as an oriented Borel-Moore homology theory on $DM_k$.) To get an oriented Borel-Moore homology theory on $DM_k$, we first need to see that external product exist, which is exactly the same as in the case of schemes, see [Section 1.10] in [**8**].

From the very constructions and all the theorems of our pull-back, push-forward, Gysin pull-back for regular local immersions, we see that $(BM1)(BM2)(BM3)(PB)(EH)$ are all satisfied. $(Weak\ Localization)$, as a consequence of $(Loc)$, is also satisfied.

REMARK 4.6. From the very definitions and constructions, we see that they all agree with the usual case of schemes when we require our Deligne-Mumford stack to be a scheme, see [**8**] chapter 1-3 to compare with the definions there.

CHAPTER 2

# Properties of oriented cohomology theories

## 1. Formal group laws on $DM_k$

By a formal group law we mean a commutative 1-dimensional formal group law. In this section we will produce formal group laws out of an oriented cohomology theory or an oriented Borel-Moore homology theory. Before doing that, we first show how to get an oriented cohomology theory out of an oriented Borel-Moore homology theory.

Let $A_*$ be an oriented Borel-Moore homology theory on $SmDM_k$. For $Y$ in $SmDM_k$ of pure dimension $d$ over $k$, let $A^n(Y) = A_{d-n}(Y)$. Then we extend to any $Y \in SmDM_k$ linearly over the connected components of $Y$. $Y$ being smooth, we know that

$$\delta_Y : Y \to Y \times_k Y$$

is a regular embedding, so we can define a product on $A^*(Y)$ by

$$a \cup_Y b : \delta_Y^*(a \times b).$$

This makes $A^*(Y)$ into a commutative graded ring with unit $1_Y$, natural with respect to pullbacks.

Similarly, for any morphism $f : Z \to Y$ in $SmDM_k$, the graph embedding

$$(Id_Z, f) : Z \to Z \times Y$$

makes $A^*(Z)$ into a graded $A^*(Y)$-module by

$$a \cup_f b : (Id_Z, f)^*(a \times b).$$

Conversely, if $A^*$ ia a cohomology theory on $SmDM_k$, let $A_*$ be the corresponding homological grading, $A_n(X) = A^{\dim X - n}(X)$. Define the external product by

$$a \times b := p_1^*(a) \cup p_2^*(b).$$

Definition 1.1. Let $p : X \to \mathrm{Spec}\ k$ be a strongly l.c.i. morphism, $X \in \mathcal{V}$. Define the fundamental class of $X$, $1_X \in A_*(X)$ by $1_X = p_X^*(1)$, where $1 \in A_0(k)$ is the unit element.

Proposition 1.2. *The operations $A_* \to A^*, A^* \to A_*$ gives equivalences of the category of oriented Borel-Moore homology theories on $SmDM_k$ with the category of oriented cohomology theories on $SmDM_k$.*

Proof. Suppose we are given an oriented Borel-Moore homology theory first. Most of the properties of oriented cohomology theory are routinely checked using properties of external product and the observation that any morphism $X \to Y$ in $SmDM_k$ is strongly l.c.i. . To show that $A^*$ is an oriented cohomology theory on $SmDM_k$, we need to show that

1. If $f : Y \to X$ is a proper morphism in $SmDM_k$ of relative dimension $d$, then the push-forward $f_* : A^*(Y) \to A^{*-d}(X)$ is $A^*(X)$-linear.

2. For a line bundle $p : L \to X$ on $X \in SmDM_k$, the Chern class endomorphism $\widetilde{c_1} : A^*(X) \to A^{*+1}(X)$ is given by cup product with $s^*(s_*(1_X))$ where $s : X \to L$ is the zero section.

These two facts follow just from the definition and properties of oriented Borel-Moore homology theory. We omit the details here. See the proof of [**11**] [Proposition 5.2.1].

On the other hand, given an oriented cohomology theory $A^*$ on $SmDM_k$. Since all the morphisms in $SmDM_k$ are strongly l.c.i., we get axioms of OBM homology theory from OCT immediately except $(BM3)$. For $(BM3)$, we have the factorization $(f \times g)_* = (f \times id_{Y'})_* \circ (id_X \times g)_*$ and then we use $(A1)$ and $(A2)$.

□

REMARK 1.3. For $SmDM_k$, by [**11**] [Proposition 5.2.1], $(CD)$ is a consequence of all the axioms without $(Weak\ Localization)$ in the definition of an oriented Borel-Moore homology theory.

LEMMA 1.4. *Let $A$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$. Take $X \in \mathcal{V}$, $L \to X$ a line bundle with sheaf of sections $\mathcal{L}$. Let $s : X \to L$ a section such that the induced map $\times s : \mathcal{O}_X \to \mathcal{L}$ is injective, and let $i : D \to X$ be the divisor defined by $s = 0$. Suppose that $D$ is in $\mathcal{V}$. Then $\widetilde{c_1} = i_* i^*$*

PROOF. Note first that $i^*$ is well defined since it is a regular embedding. Let $s_0 : X \to L$ be the zero section. Both $s$ and $s_0$ are regular embeddings. We show first

$$s_0^* = s^* : A_*(L) \to A_{*-1}(X).$$

We define the map $s(t) : X \times \mathbb{A}^1 \to L$ defined by $s(t) = ts + (1-t)s_0$ where $\mathbb{A}^1$ has parameter $t$. Note that

$$(s(t), Id_{\mathbb{A}^1}) : X \times \mathbb{A}^1 \to L \times \mathbb{A}^1$$

is a regular embedding, hence $s(t)$ is strongly l.c.i. morphism.

Let $i_0, i_1 : X \to X \times \mathbb{A}^1$ be the sections with value $0, 1$, it follows from $(EH)$ that $i_0^* = i_1^*$. Then

$$s_0^* = i_0^* s(t)^* = i_1^* s(t)^* = s^*.$$

Next we consider the cartesian square

$$\begin{array}{ccc} D & \xrightarrow{i} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle s_0} \\ X & \xrightarrow{s} & L \end{array}$$

By $(BM2)$,

$$s^* s_0^* = i_* i^*.$$

But we already have $s_0^* = s^*$. This finishes the proof.

□

PROPOSITION 1.5. *Let $A$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$. We have the following properties:*

*1.(Sect) For any smooth D-M stack $Y$, any line bundle on $Y$ and any section $s$ of $L$ which is transverse to the zero section of $L$, one has*

$$\widetilde{c_1}(1_Y) = i_*(1_Z)$$

*where $Z$ is defined by $s = 0$ and $i : Z \to Y$ is the resulting closed immersion.*

*2. Let $L$ be a line bundle on some $X \in \mathcal{V}$. If $f : Y \to X$ is a smooth morphism in $\mathcal{V}$, then*

$$\widetilde{c_1}(f^*L) \circ f^* = f^* \circ \widetilde{c_1}(L).$$

*If $f : Y \to X$ is a proper morphism in $\mathcal{V}$, then*

$$\widetilde{c_1}(L) \circ f_* = f_* \circ \widetilde{c_1}(f^*L).$$

*3. If $L, M$ are line bundles on $X \in \mathcal{V}$, then*

$$\widetilde{c_1}(L) \circ \widetilde{c_1}(M) = \widetilde{c_1}(M) \circ \widetilde{c_1}(L).$$

*4. Let $X, Y \in \mathcal{V}$, and $L \to X$ be a line bundle on $X$. For $\alpha \in A_*(X), \beta \in A_*(Y)$, we have*

$$\widetilde{c_1}(L)(\alpha) \times \beta = \widetilde{c_1}(p_1^*L)(\alpha \times \beta)$$

*where $p_1$ is just the first projection from $X \times Y \to X$.*

*5. Let $A$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$. Take $X \in \mathcal{V}$, $L \to X$ a line bundle that admits a section $s$ of $L$ which is transverse to the zero section of $L$, $i : Z \to Y$ is the closed immersion and $Z$ is defined by $s = 0$. Then the image of $\widetilde{c_1} : A_*(X) \to A_{*-1}(X)$ is contained in the image of $i_* : A_{*-1}(Z) \to A_{*-1}(X)$*

PROOF. The property (1) follows from the previous lemma and functoriality of pullback.

(2) follows from $(BM1)$ and $(BM2)$ applied to the transverse cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & f^*L \\ {\scriptstyle f^*s}\downarrow & & \downarrow{\scriptstyle f_L} \\ X & \xrightarrow{s} & L \end{array}$$

(3) follows from $(BM2)$ applied to the transverse cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{s_M} & L \\ {\scriptstyle s_L}\downarrow & & \downarrow{\scriptstyle \widetilde{s}_M} \\ X & \xrightarrow{\widetilde{s}_L} & L \bigoplus M \end{array}$$

(4) follows from $(BM3)$.

(5) follows from the previous lemma since

$$\widetilde{c_1}(x) = i_*(i^*(x))$$

for any $x \in A_*(X)$.

□

The following theorems are taken from [**11**] Section [4.1] and [5.2] once we only consider Deligne-Mumford stacks that are already schemes.

THEOREM 1.6. *There is a unique power series*

$$F_A(u,v) = \sum_{i,j} a_{i,j} u^i v^j \in A_*(k)[[u,v]]$$

*with* $a_{i,j} \in A_{i+j-1}(k)$*, such that, for any integers* $n > 0, m > 0$ *we have in the endomorphism ring of* $A_*(\mathbb{P}^n \times \mathbb{P}^m)$*:*

$$F_A(\widetilde{c_1}(pr_1^*(\gamma_n)), \widetilde{c_1}(pr_x^*(\gamma_m))) = \widetilde{c_1}(pr_1^*(\gamma_n) \otimes pr_2^*(\gamma_m)).$$

*Moreover,* $(A_*(k), F_A(u,v))$ *is a commutative formal group law. Here* $\gamma_i$ *denotes* $\mathcal{O}(1)$ *on* $\mathbb{P}^i$*.*

THEOREM 1.7. *Let* $A$ *be an oriented Borel-Moore homology theory on some admissible subcategory* $\mathcal{V}$ *of* $Sch_k$*. Then for any smooth quasi-projective scheme* $Y$ *and any* $L, M$ *line bundles on* $Y$*, one has*

$$F_A(\widetilde{c_1}(L), \widetilde{c_1}(M))(1_Y) = \widetilde{c_1}(L \otimes M)(1_Y) \in A_*(Y).$$

*Here* $F_A$ *is the power series in theorem 1.6.*

REMARK 1.8. If we can apply Jouanolou's trick for smooth Deligne-Mumford stacks, i.e., there exist a vector bundle $E \to Y$ of rank $r$, and a torsor $\pi : T \to Y$ under $E \to Y$ such that $T$ is an affine and smooth scheme with finite group action, we can show the same result for $DM_k$ directly by reducing everything to projective spaces as in the case of 1.6 and then the result follows from the theorem, namely: For $A$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$. $Y$ is a smooth Deligne-Mumford stack and $L, M$ any line bundle on $Y$, one has

$$F_A(\widetilde{c_1}(L), \widetilde{c_1}(M))(1_Y) = \widetilde{c_1}(L \otimes M)(1_Y) \in A_*(Y).$$

Where $F_A$ is the power series in 1.6.

We conlcude this section with the following definitions concerning formal group laws.

DEFINITION 1.9. Let $A^*$ be an oriented cohomology theory on $SmDM_k$.

(1) We say that $A^*$ is ordinary if there is a formal group law $F_A(u,v) = u + v$ such that
$$F_A(\widetilde{c_1}(L), \widetilde{c_1}(M))(1_Y) = \widetilde{c_1}(L \otimes M)(1_Y) \in A_*(Y)$$
for $L, M$ line bundles on $Y \in SmDM_k$.

(2) We say that $A^*$ is multiplicative if there is a formal group law $F_A(u,v) = u+v-buv$ for some $b \in A^{-1}(k)$; moreover, if $b$ is a unit, we say $A^*$ is periodic.

## 2. Chern classes in an OBM homology theory

In this section, $A_*$ will be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$.

Let us recall $(PB)$ first. For $L \to Y$ a line bundle on $Y \in \mathcal{V}$ with zero-section $s : Y \to L$, define the operator

$$\widetilde{c_1}(L) : A_*(Y) \to A_{*-1}(Y)$$

by $\widetilde{c_1}(\eta) = s^*(s_*(\eta))$. Let $E$ be a rank $n+1$ vector bundle on $X \in \mathcal{V}$, with projective bundle $q : \mathbb{P}(E) \to X$ and canonical quotient line bundle $O(1) \to \mathbb{P}(E)$. For $i \in \{0, ...., n\}$, let

$$\xi^{(i)} : A_{*+i-n}(X) \to A_*(\mathbb{P}(E))$$

be the composition of $q^* : A_{*+i-n}(X) \to A_{*+i}(\mathbb{P}(E))$ with $\widetilde{c_1}(O(1))^i : A_{*+1}(\mathbb{P}(E) \to A_*(\text{Proj}(E)))$. Then the homomorphism

$$\sum_{i=0}^{n} \xi^{(i)} : \bigoplus_{i=0}^{n} A_{*+i-n}(X) \to A_*(\mathbb{P}(E))$$

is an isomorphism.

We consider

$$\widetilde{c_i}(E) : A_*(X) \to A_{*-i}(X)$$

for $i \in \{0, ...., n, n+1\}$, with $\widetilde{c_o}(E) = 1$, and satisfying the equation below as homomorphisms $A_*(X) \to A_{*-1}(\mathbb{P}(E))$:

$$\sum_{i=0}^{n+1} (-1)^i \widetilde{c_1}(O(1))^{n+1-i} \circ q^* \circ \widetilde{c_i}(E) = 0.$$

By the isomorphism in $(PB)$, any element in $A_{*-1}(\mathbb{P}(E))$ corresponds to a unique element in $\sum_{i=0}^{n} \xi^{(i)} : \bigoplus_{i=0}^{n} A_{*+i-n-1}(X)$ via the map $\sum_{i=0}^{n+1} (-1)^i \widetilde{c_1}(O(1))^{n+1-i} \circ q^*$, so $\widetilde{c_i}(E)$ are uniquely determined. The homomorphisms $\widetilde{c_i}(E)$ are called the $i$-th Chern class operator of $E$.

THEOREM 2.1. *The Chern class operators satisfy the following properties:*

*(1) Given vector bundles $E \to X$ and $F \to X$ on $X \in \mathcal{V}$ one has*

$$\widetilde{c_i}(E) \circ \widetilde{c_j}(F) = \widetilde{c_j}(F) \circ \widetilde{c_i}(E)$$

*for all $i, j$.*

*(2) For any line bundle $L$, $\widetilde{c_1}(L)$ agrees with the one given in $(PB)$ which occurs in an oriented B-M homology theory.*

*(3) For any smooth equi-dimensional morphism $Y \to X$ in $\mathcal{V}$ and any vector bundle $E \to X$ over $X$, one has*

$$\widetilde{c_i}(f^*(E)) \circ f^* = f^* \circ \widetilde{c_i}(E).$$

*(4) For a short exact sequence of vector bundles over $X$,*

$$0 \to E_1 \to E_2 \to E_3 \to 0.$$

*Then for any positive integer $n$ one has the following equation in $End(A_*(X))$*

$$\widetilde{c_n}(E_2) = \sum_{i=0}^{n} \widetilde{c_i}(E_1) \widetilde{c_{n-i}}(E_3).$$

*(5) For any proper morphism $Y \to X$ in $\mathcal{V}$, and any vector bundle $E \to X$ over $X$ one has*

$$\widetilde{c_i}(E) \circ f_* = f_* \circ \widetilde{c_i}(f^*E).$$

*Moreover, the Chern class operators are characterized by* $(1-4)$.

To prove the theorem, we first establish the following lemmas that are essentially from [**11**, **Lemma** 4.1.18-4.1.19]. Only the proof of Lemma 2.2 differs from there.

With all these lemmas, the proof of the theorem is exactly the same as the proof of [**11**, **Proposition** 4.1.15] where one uses splitting principle and naturality of $\widetilde{c_i}$.

LEMMA 2.2. *Let $A_*$ be an oriented B-M homology theory. Let $X \in \mathcal{V}$, let $D_1, D_2, ....., D_n$ be effective Cartier divisors on $X$ such that $\cap_{i=1}^n D_i = \emptyset$. Let $L_1, ...., L_n$ be line bundles on $X$, and let $M_i = L_i \otimes \mathcal{O}_X(D)$, $i = 1, ...., n$. Then*

$$\prod_{i=1}^n (\widetilde{c_1}(M_i) - \widetilde{c_1}(L_i) = 0$$

*as an operator on $A_*(X)$.*

PROOF. Consider any $\alpha \in A_*(X)$. We would like to show

$$\prod_{i=1}^n (\widetilde{c_1}(M_i) - \widetilde{c_1}(L_i)(\alpha) = 0.$$

For each $i$, take $U_i$ to be the complement of $D_i$ in $X$.

Since $M_i$ and $L_i$ are isomorphic on $U_i$, it follows that $(\widetilde{c_1}(M_i) - \widetilde{c_1}(L_i)(\alpha)$ goes to 0 in $A_*(U_i)$.

Denote $p_i : D_i \to X$. We can use ($Weak$ *Localization*) for $U_i, D_i, X$ here to get

$$(\widetilde{c_1}(M_i) - \widetilde{c_1}(L_i)(\alpha) \subset p_i(A_*(D_i))$$

by the exactness in the middle.

Inductively, we have

$$\prod_{i=1}^n (\widetilde{c_1}(M_i) - \widetilde{c_1}(L_i)(\alpha) \subset A_*(\emptyset) = 0.$$

□

LEMMA 2.3. *Let $A_*$ be an oriented B-M homology theory and $X \in \mathcal{V}$. Let $L_1, ...., L_n$ be line bundles on $X$ and let $E = \oplus_{i=1}^n L_i$. Then $\widetilde{c_p}(E)$ is the $p$-th elementary symmetric polynomial in the first Chern class operators $\widetilde{c_1}(L_1), ...., \widetilde{c_1}(L_n)$.*

REMARK 2.4.
(1) Comparing to [**11**],we can't use the same proof as there since we don't have a general formal group law on $\mathcal{V}$. That's why we have to add the axiom ($Weak$ *Localization*) which doesn't show up in the case of schemes.
(2) In the proof of [**11**, **Lemma** 4.1.18], one needs to show $\prod_{i=1}^n (\widetilde{c_1}(M_i) - \widetilde{c_1}(L_i) = 0$ as an operator on $A_*(X)$, but axiom ($FGL$) only allows one to apply the equation of Chern class operators for fundamental class $1_Y$. In order to proceed in the proof

of Whitney product formula, one can use (*Extended Nilp*) and (*Extended FGL*) as mentioned in [**11**, Remark 5.2.9].

(*Extended Nilp*). For each smooth Deligne-Mumford stack $Y$ there exists an integer $N_Y$ such that, for each family $(E_1, E_2, ...., E_n)$ of vector bundles on $Y$ with $n > N_Y$, one has

$$\widetilde{c_1}(E_1), ...., \widetilde{c_1}(E_n) = 0$$

as operators on $A_*(Y)$.

Note that our definition of (*ExtendNilp*) is slightly different from [**11**]. In the original definition of Levine-Morel, they use line bundles instead of vector bundles. We denote it by (*Extended Nilp*)′

(*Extended Nilp*)′. For each smooth Deligne-Mumford stack $Y$ there exists an integer $N_Y$ such that, for each family $(L_1, L_2, ...., L_n)$ of line bundles on $Y$ with $n > N_Y$, one has

$$\widetilde{c_1}(L_1), ...., \widetilde{c_1}(L_n) = 0$$

as operators on $A_*(Y)$.

(*Extended FGL*). Let $\varphi_A : \mathbb{L} \to A_*(k)$ be the homomorphism giving the $\mathbb{L}$-structure and let $F_A \in A_*(k)[[u, v]]$ be the image of the universal formal group law $F_{\mathbb{L}} \in \mathbb{L}_*[[u, v]]$ by $\varphi_A$. Then for any smooth Deligne-Mumford stack $Y$ and any $L, M$ line bundles on $Y$, one has

$$F_A(\widetilde{c_1}(L), \widetilde{c_1}(M)) = \widetilde{c_1}(L \otimes M)$$

as operators on $A_*(Y)$.

(3) (*Extended Nilp*)′ and (*Extended FGL*) are satisfied when we restrict our attention to the category of smooth quasi-projective schemes $Sm_k$. Indeed, we can make use of Jouanolou's trick to find $X'$ affine scheme and $p : X' \to X$ smooth and quasi-projective such that $P^* : A_*(X) \to A_{*+r}(X')$ is an isomorphism.

For any $L$ on $X$, we find a morphism $f : X' \to \mathbb{P}^n$ such that $p^*L \cong f^*(\mathcal{O}(1))$. $f$ gives $A_*(X')$ the $A_*(\mathbb{P}^n)$-module structure and we have

$$p^*(\widetilde{c_1}(L)(x)) = \widetilde{c_1}(p^*(L))(p^*(x)) = c_1(\mathcal{O}(1)) \cap_f p^*(x)$$

for all $x \in A_*(X)$, where $c_1(\mathcal{O}(1)) := \widetilde{c_1}(\mathcal{O}(1))(1_{\mathbb{P}^n})$. Thus the operator being 0 is reduced to the case applying the fundamental classes of projective spaces.

A similar proof gives us (*Extended FGL*) in this case.

In case of schemes, (*Extended Nilp*)′ and (*Extended Nilp*) agree because of Whitney product formula. But Whitney product formula is a consequence of (*Extended Nilp*)′ and (*Extended FGL*) in the case of schemes. See [**11**] [Section 4.1.7] for a complete proof.

## 3. Completion

Let $A^*$ be an oriented Borel-Moore homology theory on an admissible theory $\mathcal{V}$ of $DM_k$ . For $n \geq 1$, let $I_A^{(n)}(X)$ be the $A_*(k)$-submodule of $A_*(X)$ generated by

$$\{\widetilde{c_1}(E_1) \circ \widetilde{c_1}(E_2) \circ ...\widetilde{c_1}(E_n)(\alpha) \colon E_i \to X \text{ are vector bundles on } X, \alpha \in A_*(X)\}.$$

REMARK 3.1. Let $A^*$ be the oriented cohomology theory associated to $A_*$ on $\mathcal{V}$. Then $I_A^{(1)}(X)$ is just the $A^*(k)$-ideal generated over $A^*(k)$ by ${c_1}^A(E)$ for all vector bundles $E \to X$. Denote$I_A^{(1)}(X) = I_A(X)$, then we have

$$I_A^{(n)}(X) = I_A(X)^n$$

for all $n \geq 1$.

We can consider the completion, setting

$$\hat{A}_*(X) = \varprojlim_n \frac{A_*(X)}{I_A^{(n)}(X)}.$$

DEFINITION 3.2. (1) We say the theory $A_*$ is complete if the natural map $A_*(X) \to \hat{A}_*(X)$ is always an isomorphism for all $X \in \mathcal{V}$.
(2) In general, an oriented Borel-Moore homology theory is not complete. We say the theory $A_*$ satisfies the axiom (*Complete*) if $A_*$ is complete.

REMARK 3.3. (1) Let $A_*$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$ satisfying the axiom (*Complete*), it makes sense to consider the axiom (*Extended FGL*), as the power series $F_A(\widetilde{c_1}(L), \widetilde{c_1}(M))$ makes sense as an operator on $A_*(Y)$.
(2) As a ring homomorphism, any morphism of oriented cohomology theories is automatically continuous with respect to the $I_A^{(*)}$-adic topology.
(3) An oriented cohomology theory $A^*$ on an admissible subcategory $\mathcal{V}$ of $Sch_k$ in the sense of [**11**, Definition 1.1.2], it is always complete. The idea goes as follows.
First we use $(EH)$ and Jouanolou's trick to replace $X$ by an affine torsor $\pi : T \to X$. We may assume that all vector bundles are sums of line bundles. On affine schemes all line bundles are very ample, by [**11**] [Lemma 2.3.9], we have

$$c_1(L_1) \circ ... \circ c_1(L_n) = 0$$

for $n$ bigger than the dimension of the scheme.
So $I_A(X)^n = 0$ for $n$ big enough.
(4) In general, (*Extended Nilp*) implies that $I_A^{(n)}(X) = 0$ for $n$ big enough. So the theory $A_*$ satisfying (*Extended Nilp*) is always complete.

## 4. Todd classes

DEFINITION 4.1. Let $A_*$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$ satisfying the axiom (*Complete*). Take $\tau = (\tau_i) \in \prod_0^\infty A_i(k)$, with $\tau_0 = 1$. Define the inverse Todd class operator of a line bundle $L \to X$ to be the operator on $A_*(X)$ given by the infinite sum

$$\widetilde{Td}_\tau^{-1}(L) = \sum_{i=0}^{\infty} \widetilde{c_1}(L)^i \tau_i.$$

Completeness of $A_*$ is needed here to make sure that $\widetilde{Td}_\tau^{-1}(L)$ is a well-defined degree 0 endomorphism of $A_*(X)$.

Proposition 4.2. *Let $A_*$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$ satisfying (Complete). Take $\tau = (\tau_i) \in \prod_0^\infty A_i(k)$, with $\tau_0 = 1$. Then we can define an endomorphism*

$$\widetilde{Td}_\tau^{-1}(E) : A_*(X) \to A_*(X)$$

*of degree 0 for each $X \in \mathcal{V}$ and every vector bundle $E$ on $X$ such that $\widetilde{Td}_\tau^{-1}(E)$ satisfies and is uniquely characterized by the following properties:*

*(1) Given vector bundles $E \to X$ and $F \to X$ on $X \in \mathcal{V}$ one has*

$$\widetilde{Td}_\tau^{-1}(E) \circ \widetilde{Td}_\tau^{-1}(F) = \widetilde{Td}_\tau^{-1}(F) \circ \widetilde{Td}_\tau^{-1}(E).$$

*(2) For any line bundle $L$, we have*

$$\widetilde{Td}_\tau^{-1}(L) = \sum_{i=0}^{\infty} \widetilde{c}_1(L)^i \tau_i.$$

*(3) For any smooth equi-dimensional morphism $Y \to X$ in $\mathcal{V}$ and any vector bundle $E \to X$ over $X$, one has*

$$\widetilde{Td}_\tau^{-1}(f^*(E)) \circ f^* = f^* \circ \widetilde{Td}_\tau^{-1}(E).$$

*(4) For a short exact sequence of vector bundles over $X$,*

$$0 \to E_1 \to E_2 \to E_3 \to 0.$$

*one has the following equation in $End(A_*(X))$*

$$\widetilde{Td}_\tau^{-1}(E_2) = \widetilde{Td}_\tau^{-1}(E_1)\widetilde{Td}_\tau^{-1}(E_3).$$

*(5) For any proper morphism $Y \to X$ in $\mathcal{V}$, and any vector bundle $E \to X$ over $X$ one has*

$$\widetilde{Td}_\tau^{-1}(E) \circ f_* = f_* \circ \widetilde{Td}_\tau^{-1}(f^*E).$$

*$\widetilde{Td}_\tau^{-1}(E)$ is called the inverse Todd class operator of $E$.*

Proof. Apply the splitting principle, we may assume that $E_1, E_3$ are both direct sum of line bundles, and that $E_2 = E_1 \oplus E_3$. Consider the power series $f(t) := \sum_{i=0}^\infty \tau_i t^i$, where we give $t_i$ degree $-1$ and $\tau_i$ degree 0, then the product $f(t_1)....f(t_n)$ is a sum

$$f(t_1)....f(t_n) = \sum_{i=0}^{\infty} P_i(t_1, ...., t_n)$$

with $P_i$ symmetric polynomial of degree$-i$ . By construction, we would like our $\widetilde{Td}_\tau^{-1}(E)$ to satisfy property (1) and (3). First we write

$$P_i(t_1, ...., t_n) = Q_{i,n}(\sigma_1(t_*), ...., \sigma_i(t_*))$$

where $\sigma_1(t_*), ...., \sigma_i(t_*)$ are elementary symmetric polynomials. We observe that $Q_{i,n}$ is independent of $n$ for $n \geq i$. Set $Q_i = Q_{i,i}$. For any vector bundle $E \to X$, we set

$$\widetilde{Td}_\tau^{-1}(E) = \sum_{i=0}^{\infty} Q_i(\widetilde{c_1}(E), ...., \widetilde{c_i}(E))$$

where $\widetilde{c_i}(E)$ is set to be the zero endomorphism for $i > rank(E)$. By construction and properties of Chern class operators, the properties $(1) - (5)$ are satisfied.

□

COROLLARY 4.3. *The assignment* $E \to \widetilde{Td}_\tau^{-1}(E) \in Aut(A_*(X))$ *extends to a group homomorphism:*

$$\widetilde{Td}_\tau^{-1} : K_0(X) \to Aut(A_*(X)).$$

PROOF. This follows from property (4) of the proposition above. □

## 5. Twisting a theory

The idea of this section goes back to Quillen in [**14**].

**5.1. The twisting on $DM_k$.** Let $A_*$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$ that satisfies the axiom $(Complete)$ and $\tau = (\tau_i) \in \prod_0^\infty A_i(k)$, with $\tau_0 = 1$. We construct a new oriented Borel-Moore homology theory on $\mathcal{V}$, denoted by $A_*^{(\tau)}$, as follows.

$A_*^{(\tau)}(X) = A_*(X)$ and the push-forward maps stay the same: $f_*^{(\tau)} = f_*$.

For any smooth morphism $f : Y \to X$ we have the bundle of *vertical tangent vectors* $T_f$, defined as the dual of the bundle with sheaf of sections the relative Kaḧler differentials $\Omega^1_{Y/X}$. The *virtual normal bundle* of $f$, $N_f$, is the element of $K_0(Y)$ defined by $N_f := -T_f$.

For a regular immersion $i : Y \to X$, $N_i$ is just the usual normal bundle. For a strong l.c.i. morphism $f : Y \to X$, one factors $f$ as $f = p \circ i$, with $i : Y \to P$ a closed immersion and $p : P \to X$ a smooth morphism. Then define $N_f \in K_0(X)$ by

$$N_f := [N_i] + i^*[N_p] = [N_i] - i^*[T_p].$$

Note that the independence of the choice of factorization of $f$ follows from [Appendix B.7] in [**8**], especially B.7.4, B.7.4 and B.7.6

We define

$$f^*_{(\tau)} := \widetilde{Td}_\tau^{-1}(N_f) \circ f^*.$$

For any line bundle $L$ over $X$ we define

$$\widetilde{c_1}^{(\tau)}(L) := \widetilde{Td}_\tau^{-1}(L) \circ \widetilde{c_1}(L).$$

Noting that $\widetilde{Td}_\tau^{-1}$ has all the compatible properties with pullback and push-forward, $(BM1) - (BM3)$ are satisfied. Following with the fact that $\widetilde{Td}_\tau^{-1}(L)$ is an automorphism of $A_*(X)$, we wee that $(PB), (EH), (Weak\ Localization)$ are satisfied. So we do get a new oriented Borel-Moore homology theory on $\mathcal{V}$.

Suppose that $A_*$ satisfies (*Extended FGL*), let us have a look what happens to the formal group law in the new theory. By the definition of $\widetilde{c_1}^{(\tau)}(L)$, we have

$$\widetilde{c_1}^{(\tau)}(L) = \lambda_{(\tau)}(\widetilde{c_1})$$

where $\lambda_{(\tau)}(u) = \sum_{i\geq 0} \tau_i \cdot u^{i+1} \in A_*(k)[[u]]$. We observe that there is a unique power series $\lambda^{-1}_{(\tau)}(u)$ such that $\lambda^{-1}_{(\tau)}(\lambda_{(\tau)}(u)) = u$. Recall that the formal group law is given by the equation

$$F^{(\tau)}_A(\widetilde{c_1}^{(\tau)}(L), \widetilde{c_1}^{(\tau)}(M)) = \widetilde{c_1}^{(\tau)}(L \otimes M).$$

Inserting the identity

$$\widetilde{c_1}^{(\tau)}(L) = \lambda_{(\tau)}(\widetilde{c_1}),$$

we have

$$F^{(\tau)}_A(\lambda_{(\tau)}(u), \lambda_{(\tau)}(v)) = \lambda_{(\tau)}(F_A(u, v)).$$

Finally we get the formal group law on the new theory as follows:

$$F^{(\tau)}_A(u, v) = \lambda_{(\tau)}(F_A(\lambda^{-1}_{(\tau)}(u), \lambda^{-1}_{(\tau)}(v))).$$

**5.2. The twisting on $SmDM_k$.** If we restrict our attention to $SmDM_k$ we can do a similar twisting. Let $f_\tau(t) = \sum_{i\geq 0} \tau_i t^i$ and we let $\tau^{-1} \in \prod_0^\infty A_i(k)$ be the sequence such that

$$f_{\tau^{(-1)}}(t) \cdot f_\tau(t) = 1.$$

We define the Todd class of $E$ for a vector bundle $E \to X$, by

$$\widetilde{Td}_\tau(E) = \widetilde{Td}^{-1}_{\tau^{(-1)}}(E).$$

For $X \in SmDM_k$, let $T_X$ be the tangent bundle of $X$. For an arbitrary morphism $f : Y \to X$ in $SmDM_k$, we have the *virtual tangent bundle* $T_f \in K_0(Y)$:

$$T_f = [T_Y] - [f^*T_X] \in K_0(Y).$$

If we factor $f$ as $f = p \circ i$ where $i : Y \to Y \times X$ the closed immersion and $p : Y \times X \to X$ the projection, we know that the two definitions of $T_f$ agree in $K_0$ by [Appendix B.7] in [**8**]. We still set $N_f := -T_f$.

Define the new theory $A^\tau_*$ on $SmDM_k$ by $A^\tau_*(X) = A_*(X)$, pullbacks stay the same. For a proper morphism $f : Y \to X$, we set

$$f^\tau_* = f_* \circ \widetilde{Td}_\tau(T_f)$$

while

$$\widetilde{c_1}^\tau(L) := \widetilde{Td}^{-1}_\tau(L) \circ \widetilde{c_1}(L) = \widetilde{Td}_\tau(-L) \circ \widetilde{c_1}(L).$$

The same as in the previous subsection, we still get an oriented Borel-Moore homology theory.

LEMMA 5.1. *Let $X \in SmDM_k$, with tangent bundle $T_X$. Then the automorphism*

$$\widetilde{Td}_\tau^{-1}(T_X) : A_*^{(\tau)}(X) \xrightarrow{\sim} A_*^\tau(X)$$

*gives an isomorphism of complete Borel-Moore homology theories on $SmDM_k$. Thus they have the same formal group law if it exist.*

PROOF. It follows from the easy computation that

$$\widetilde{Td}(T_X) = (\widetilde{Td}_\tau^{-1}(T_X))^{-1}$$

and the corresponding equality for $\widetilde{c}_1$. □

EXAMPLE 5.2. Let us consider the Borel-Moore homology theory on $DM_k$

$$X \rightarrowtail Ch_*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]$$

obtained from $Ch_*$ by the extension of scalars $\mathbb{Q} \subset \mathbb{Q}[\beta, \beta^{-1}]$, here $\beta$ has degree 1.

We apply our twisting for family $\tau$ given by

$$\lambda_\tau(u) = \frac{1 - e^{-\beta u}}{\beta}.$$

We denote the new theory by $Ch_*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]^{td}$.

As we know the formal group law on $Ch_*$ is given by $F(u, v) = u + v$, we can compute the formal group law of $Ch_*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]^{td}$ as the multiplicative one:

$$F_m(u, v) = u + v - \beta uv.$$

EXAMPLE 5.3. Let $A_*$ be an oriented Borel-Moore homology theory on some admissible subcategory $\mathcal{V}$ of $DM_k$, satisfying the axiom (*Complete*). Let $t_1, t_2, \ldots$ be variables, and set $t_0 = 1$. We consider the twisting

$$X \rightarrowtail A_*(X)[\mathbf{t}]^{(\mathbf{t})},$$

where $\tau = \mathbf{t} = (t_0, t_1, \ldots)$. We know that $A_*[\mathbf{t}]^{(\mathbf{t})}$ is still an oriented Borel-Moore homology theory.

**5.3. The case of schemes.** Suppose we are working with the category of quasi-projective schemes $Sch_k$, as introduced in [**11**]. We note that in $Sch_k$, proper morphisms are projective morphisms, smooth schemes are additionally quasi-projective. Denote $\Omega_*$ to be the algebraic cobordism on $Sch_k$. We have the following theorems from [**11**] [Theorem 4.1.28] [Theorem 7.1.4].

THEOREM 5.4. *Let $k$ be a field that admits resolution of singularities. We denote $\Omega_*^{ad}$ to be the theory*

$$X \rightarrowtail \Omega_*^{ad}(X) := \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}.$$

*Then the canonical morphism*

$$\Omega_* \to \Omega_*^{ad}[\boldsymbol{t}]^{(\boldsymbol{t})}$$

*is an isomorphism after $\otimes\mathbb{Q}$.*

THEOREM 5.5. *Suppose furthermore that* $k$ *has characteristic zero. Then the canonical morphism*

$$\Omega_*^{ad} \to Ch_*$$

*induced from* $\Omega_* \to Ch_*$ *is an isomorphism.*

CHAPTER 3

# Theorems of Riemann-Roch type for oriented cohomology theories

This chapter is inspired by the work of Panin in [**13**].

In [**13**, Section 1], Panin defines a ring cohomology theory, an integration on a ring cohomology theory, Euler class, perfect integration, an oriented cohomology pre-theory on the category of smooth schemes, morphisms and oriented morphisms between oriented cohomology pre-theories. They correspond to an additive functor from $SmDM_k$ to $\mathbf{R}^*$ satisfying $(EH)$, push-forward and various compatibilities including $(Weak\ Localization)$, Chern class operator applied to the fundamental class, projective bundle formula, an oriented cohomology theory on $SmDM_k$, morphisms and oriented morphisms between oriented cohomology theories.

## 1. The general Riemann-Roch theorem

We give the following definitions first.

Definition 1.1. (1) A strongly projective morphism in $SmDM_k$ is a morphism $f : X \to Y$ that factors as a closed immersion $i : X \to \mathbb{P}^n_Y$ followed by a projection $p : \mathbb{P}^n_Y \to Y$.

(2) A projective morphism in $SmDM_k$ is a morphism $f : X \to Y$ that factors as a closed immersion $i : X \to \mathbb{P}_Y(\mathcal{E})$ followed by a projection $p : \mathbb{P}_Y(\mathcal{E}) \to Y$.

The goal is to prove the following theorem:

Theorem 1.2. *(1) Let $\varphi : A^* \to B^*$ be a morphism of two oriented cohomology theories on $SmDM_k$, i.e., a natural transformation of the underlying functor. Suppose $\varphi(c_1^A(L)) = c_1^B(L)$ for any line bundle $L$. Then $ch_A : \varphi : A^* \to B^*$ commutes with strongly projective morphisms, i.e., for $f : X \to Y$ a strongly projective morphism, we have*

$$f_B \circ \varphi^X = \varphi^Y \circ f_A$$

*where $f_A, f_B$ denote the push-forward in theory $A^*$ and $B^*$.*

*(2) Furthermore, if both $A^*$ and $B^*$ satisfy the axioms $(Complete)$ and $(Extended\ FGL)$. Then $ch_A : \varphi : A^* \to B^*$ commutes with projective morphisms, i.e., for $f : X \to Y$ a projective morphism, we have*

$$f_B \circ \varphi^X = \varphi^Y \circ f_A.$$

By definition, push-forwards are compatible with composition, the theorem are reduced to the following two lemmas.

LEMMA 1.3. *Let $\varphi : A^* \to B^*$ be a morphism of two oriented cohomology theories on $SmDM_k$. Suppose $\varphi(c_1^A(L)) = c_1^B(L)$ for any line bundle $L$. Then, for any $i : X \to Y$ a closed immersion in $SmDM_k$, we have*

$$i_B \circ \varphi^X = \varphi^Y \circ i_A.$$

The proof for the case of a closed immersion is the use of classical idea on deformation to the normal cone. It doesn't differ too much from the discussions in [**11**] [Section 4.2] [Proposition 4.2.9] and [**13**] [Section 1.8] [Theorem 1.8.3]. The proof will be sketched in the next section.

LEMMA 1.4. *(1) Let $\varphi : A^* \to B^*$ be a morphism of two oriented cohomology theories on $SmDM_k$. Suppose $\varphi(c_1^A(L)) = c_1^B(L)$ for any line bundle $L$. For $X \in SmDM_k$, $p : X \times \mathbb{P}^n \to X$ we have*

$$p_B \circ \varphi^{X\times\mathbb{P}^n} = \varphi^X \circ p_A.$$

*(2) Furthermore, if both $A^*$ and $B^*$ satisfy the axioms $(Complete)$ and $(Extended\ FGL)$. Then for any vector bundle $E$ with sheaf of sections $\mathcal{E}$, $q : \mathbb{P}_X(\mathcal{E}) \to X$ a projection in $SmDM_k$, we have*

$$q_B \circ \varphi^{P_X(\mathcal{E})} = \varphi^X \circ q_A.$$

The proof of this lemma is the main part of this chapter. It will be discussed later.

## 2. The case of a closed immersion

We prove Lemma 1.3 in this section. We recall the statement first.

Let $\varphi : A^* \to B^*$ be a morphism of two oriented cohomology theories on $SmDM_k$. Suppose $\varphi(c_1^A(L)) = c_1^B(L)$ for any line bundle $L$. Then, for any $i : X \to Y$ a closed immersion in $SmDM_k$, we have

$$i_B \circ \varphi^X = \varphi^Y \circ i_A.$$

Let $i : Y \to X$ be a closed imbedding of smooth Deligne-Mumford stacks with normal bundle $N = N_{X/Y}$. First we may assume that $Y$ is connected since both $A$ and $B$ are additive. Then the normal bundle has a constant rank. We prove the above statement for the closed imbedding $s : Y \to \mathbb{P}(1 \oplus N)$ first, then we use the deformation to the normal cone to deduce Lemma 1.3. See the next two lemmas for the precise statement.

LEMMA 2.1. *Let $Y$ be smooth and $E$ be a vector bundle of rank $n$ over $Y$. Let $p : \mathbb{P}(1 \oplus N) \to Y$ be the associated projective bundle. Let $s : Y \to \mathbb{P}(1 \oplus N)$ be a section identifying $Y$ with $\mathbb{P}(1)$ in $\mathbb{P}(1 \oplus N)$. Then Lemma 1.3 holds for $s$, i.e.,*

$$s_B \circ \varphi^X = \varphi^Y \circ s_A.$$

PROOF. By splitting principle and extended homotopy property for both $A$ and $B$, we may assume that $E$ is a sum of line bundles. Let $E = L_1 \oplus L_2 \oplus ... \oplus L_n$.

Let $F_i = 1 \oplus L_1 \oplus L_2 \oplus ... \oplus L_i$ and $s_i : \mathbb{P}(F_{i-1}) \to \mathbb{P}(F_i)$ be the closed imbedding. Then we have $s = s_n \circ ... \circ s_1$. Thus the statement of the lemma is reduced to $s_1$.

For any $\alpha \in A(Y)$, we have $\alpha = s_1^A(\beta)$ for an element $\beta \in A(\mathbb{P}(1 \oplus L_1))$ because of projective bundle formula for $A$.Then we have

$$s_B \circ \varphi^Y(\alpha) = s_B(\varphi^Y(s_1^A(\beta)) = s_B(s^B(\varphi^{\mathbb{P}(1\oplus L_1)}(\beta))) = c_1^B(L_1) \cup \varphi^{\mathbb{P}(1\oplus L_1)}(\beta)$$

and

$$\varphi(s_A(\alpha)) = \varphi(s_A s^A(\beta)) = \varphi(c_1(L_1) \cup \beta) = \varphi(c_1^A(L_1)) \cup \varphi^{\mathbb{P}(1\oplus L_1)}(\beta).$$

Then the lemma follows from the assumption saying that $\varphi(c_1^A(L)) = c_1^B(L)$ for any line bundle $L$. □

The following lemma finishes the proof of Lemma 1.3.

LEMMA 2.2. *Let $i : Y \to X$ be a closed imbedding with normal bundle $N = N_{X/Y}$. Let $p : \mathbb{P}(1 \oplus N) \to Y$ be the associated projective bundle. Let $s : Y \to \mathbb{P}(1 \oplus N)$ be a section identifying $Y$ with $\mathbb{P}(1)$ in $\mathbb{P}(1 \oplus N)$. If the statement of Lemma 1.3 holds for $s$, then it holds for $i$ as well.*

PROOF. We recall briefly the construction of deformation to the normal cone.

Consider the blow up $X_t$ of $X \times \mathbb{A}^1$ with center $Y \times 0$. The pre-image of $Y \times 0$ is the projective bundle $\mathbb{P}(1 \oplus N)$ and morphism is exactly $p : \mathbb{P}(1 \oplus N) \to Y$. We have a commutative diagram where both squares are transversal cartesian.

$$\begin{array}{ccccc}
Y & \xrightarrow{p_0} & Y \times \mathbb{A}^1 & \xleftarrow{p_1} & Y \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle i_t} & & \downarrow{\scriptstyle i} \\
\mathbb{P}(1 \oplus N) & \xrightarrow{j_0} & X_t & \xleftarrow{j_1} & X
\end{array}$$

.Here 0 and 1 just means the inclusion at points 0 and 1. This gives commutative diagrams for both theory $A$ and theory $B$.

$$\begin{array}{ccccc}
A(Y) & \xleftarrow{p_0^A} & A(Y \times \mathbb{A}^1) & \xrightarrow{p_1^A} & A(Y) \\
\downarrow{\scriptstyle s_A} & & \downarrow{\scriptstyle (i_t)_A} & & \downarrow{\scriptstyle i_A} \\
A(\mathbb{P}(1 \oplus N)) & \xleftarrow{j_0^A} & A(X_t) & \xrightarrow{j_1^A} & A(X)
\end{array}$$

We have furthermore the following commutative diagram:

$$\begin{array}{ccccc}
A(Y) & \xleftarrow{p_0^A} & A(Y \times \mathbb{A}^1) & \xrightarrow{p_1^A} & A(Y) \\
\downarrow{\scriptstyle a_0} & & \downarrow{\scriptstyle a_t} & & \downarrow{\scriptstyle a_1} \\
B(\mathbb{P}(1 \oplus N)) & \xleftarrow{j_0^B} & B(X_t) & \xrightarrow{j_1^B} & B(X)
\end{array}$$

Here $a_0 = s_B \circ \varphi - \varphi \circ s_A, a_t = (i_t)_B \circ \varphi - \varphi \circ (i_t)_A, a_1 = i_B \circ \varphi - \varphi \circ i_A$.

In order to finish the proof, we need to show $a_1 = 0$ under the assumption that $a_0 = 0$. Since $p_1^A$ is an isomorphism, it's enough to show that $a_t = 0$.

Let $V_t = X_t - Y \times \mathbb{A}^1$, $j_t : V_t \to X_t$ be the open inclusion. Then $j_t^B \circ (i_t)_B = j_t^A \circ (i_t)_A = 0$ because of the following transversal diagram and $A(\emptyset) = 0$.

$$\begin{array}{ccc} \emptyset & \longrightarrow & X - Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{\;i\;} & X \end{array}$$

So we have $j_t^B \circ a_t = 0$. We also have $j_0^B \circ a_t = 0$ since $a_0 = 0$.

Using (*Weak Localization*) on $B$ and the geometry of the deformation to the normal cone, we know that $ker(j_0^B) \cap ker(j_t^B) = 0$(see [**13**] Lemma 1.4.2, for example ). So $a_t$ must be 0. □

## 3. The case of projections

Now let's focus on the case of projections. To prove Lemma 1.4, we establish some lemmas first.

Let $A$ be an oriented cohomology theory. We set $\{\mathbb{P}^r\} = (p_r)_A(1) \in A(k)$, here $p_r : \mathbb{P}^r \to pt$ is just the projection. Observe that $\{\mathbb{P}^0\} = 1$ by definition. Set $\xi_m$ to be $c_1(\mathcal{O}(1)) \in A(\mathbb{P}^m)$. We know that $(1, \xi, ..., \xi^m)$ form a free basis of $A(\mathbb{P}^m)$ as $A(k)$-module.

LEMMA 3.1. *Let $i_{r,n} : \mathbb{P}^r \to \mathbb{P}^n$ be the linear inclusion. Then $(i_{r,n})_A(\xi_r^j) = \xi_n^{j+n-r}$ in $A(\mathbb{P}^n)$. In particular, $(i_{0,n})_A(1) = \xi_n^n$. Note that for any morphism $f$, we denote $f_A$ to be the push-forward and $f^A$ to be the pullback in the theory $A$.*

PROOF. Let $i_m : \mathbb{P}^{m-1} \to \mathbb{P}^m$ be the linear inclusion. Then we have a chain of relations in $A(\mathbb{P}^n)$:

$$(i_m)_A(\xi_{m-1}^j) = (i_m)_A(i_m^A)(\xi_m^j)) = c_1((O)(1)) \cdot \xi_m^j = \xi_m^{j+1}.$$

Then we use

$$i_{r,n} = i_n \circ ... \circ i_{r+1}$$

to deduce the formula. □

LEMMA 3.2. *We have*

$$(p_n)_A(\xi_n^r) = \{\mathbb{P}^{n-r}\}.$$

*In particular, we have*

$$(p_n)_A(\xi_n^n) = 1.$$

PROOF. By the previous lemma we have

$$(i_{n-r,n})_A(1) = (\xi_n^r)$$

by setting $j = 0$ and $r$ to be $n - r$. Then

$$(p_n)_A(\xi_n^r) = (p_n \circ i_{n-r,n})_A(1) = (p_{n-r})_A(1) = \{\mathbb{P}^{n-r}\}.$$

□

LEMMA 3.3. *Let $\triangle : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal imbedding. We identify $A(\mathbb{P}^n) \otimes_{A(k)} A(\mathbb{P}^n)$ with $A(\mathbb{P}^n \times \mathbb{P}^n)$ by cup-product. The relation in $A(\mathbb{P}^n \times \mathbb{P}^n)$ is given by*

$$\triangle_A(1) = \xi_n^n \otimes 1 + 1 \otimes \xi_n^n + \sum_{i>0,j>0} a_{i,j}\xi_n^i \otimes \xi_n^j$$

*for elements $a_{i,j}$ in $A(k)$.*

PROOF. We consider the transversal diagram.

$$\begin{array}{ccc} pt & \xrightarrow{(id,j)} & pt \times \mathbb{P}^n \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle (j\times id)} \\ \mathbb{P}^n & \xrightarrow{\triangle} & \mathbb{P}^n \times \mathbb{P}^n \end{array}$$

Here $j$ is just the inclusion of a point.

By the compatibility of transversal diagrams and the particular case of Lemma 3.1 we have

$$(j \times id)^A(\triangle_A(1)) = (id, j)_A(j^A(1)) = \xi_n^n.$$

By symmetry we have

$$(id \times j)^A(\triangle_A(1)) = \xi_n^n.$$

We consider the commutative diagram of rings.

$$\begin{array}{ccc} A(\mathbb{P}^n) \otimes_{A(k)} A(\mathbb{P}^n) & \xrightarrow{\beta} & A(\mathbb{P}^n) \oplus A(\mathbb{P}^n) \\ {\scriptstyle \cup}\downarrow & & \downarrow{\scriptstyle id} \\ A(\mathbb{P}^n \times \mathbb{P}^n) & \xrightarrow{\alpha} & A(\mathbb{P}^n) \oplus A(\mathbb{P}^n) \end{array}$$

Here

$$\alpha = ((j \times id)^A, (id \times j)^A), \beta(a \otimes b) = j^A(a)b + aj^A(b).$$

The kernel of $j^A$ is the $A(\mathbb{P}^n)$-module generated by $\xi_n$. So the kernel of $\beta$ is generated by $\xi_n^i \otimes \xi_n^j$ with all $i > 0$ and $j > 0$. But

$$\alpha(\triangle_A(1)) = \alpha(\xi_n^n \otimes 1 + 1 \otimes \xi_n^n)$$

, we know that $\alpha(\triangle_A(1))$ is exactly given by the desired formula.

□

**3.1. The case for strongly projective morphisms.** To prove part 1 of Lemma 1.4, By (BM2) and the following cartesian square

$$\begin{array}{ccc} \mathbb{P}^n & \longrightarrow & pt \\ \downarrow & & \downarrow \\ X \times \mathbb{P}^n & \xrightarrow{p} & X \end{array}$$

, it suffices to prove the statement for $p_n : \mathbb{P}^n \to pt$. Note that we also know that $(1, \xi, ..., \xi^{n-1})$ form a free basis of $A(\mathbb{P}^n)$ as $A(k)$-module for any oriented cohomology theory $A$ or $B$. That is, we need to show $\varphi((p_n)_A(\xi_A^i))$ and $(p_n)_B(\varphi(\xi_B^i))$ coincide in $A^*(k)$ for all $i$ , where this $A^*$ is the theory mentioned in our lemma.

By assumption, we have $\varphi(\xi_A) = \xi_B$. Then we have $\varphi(\xi_A^i) = \xi_B^i$ for any non-negative integer $i$. By Lemma 3.2 we have

$$(p_n)_A(\xi_A^i) = \{\mathbb{P}^{n-i}\}_A$$

and

$$(p_n)_B(\xi_B^i) = \{\mathbb{P}^{n-i}\}_B.$$

So it's enough to check the relation

$$\varphi(\{\mathbb{P}^{n-i}\}_A) = \{\mathbb{P}^{n-i}\}_B$$

for all integers $0 \leq i \leq n$.

This is the lemma below.

Lemma 3.4. *Let $\varphi : A^* \to B^*$ be a morphism of two oriented cohomology theories on $SmDM_k$. Suppose $\varphi(c_1^A(L)) = c_1^B(L)$ for any line bundle $L$. Then in $B^*(k)$ we always have*

$$\varphi(\{\mathbb{P}^n\}_A) = \{\mathbb{P}^n\}_B$$

*for any $n$ non-negative integers.*

Proof. We do induction on $n$. The case of 0 is nothing but the easy computation of the basic definitions.

Now we assume that the relation $\varphi(\{\mathbb{P}^m\}_A) = \{\mathbb{P}^m\}_B$ holds for all $m < n$ and we prove the relation $\varphi(\{\mathbb{P}^n\}_A) = \{\mathbb{P}^n\}_B$.

By Lemma 3.3 we have

$$\triangle_A(1) = \xi_A^n \otimes 1 + 1 \otimes \xi_A^n + \sum_{i>0,j>0} b_{i,j}^n \xi_A^i \otimes \xi_A^j$$

in $A(\mathbb{P}^n \times \mathbb{P}^n)$ and

$$\triangle_B(1) = \xi_B^n \otimes 1 + 1 \otimes \xi_B^n + \sum_{i>0,j>0} a_{i,j}^n \xi_B^i \otimes \xi_B^j$$

in $B(\mathbb{P}^n \times \mathbb{P}^n)$.

for some elements $a_{i,j}^n$ in $A(k)$ and $b_{i,j}^n$ in $B(k)$.

Now we take the diagonal embedding $\triangle : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$. Being a closed imbedding we know that $\triangle$ commute with $\varphi$. On the other hand we have $\varphi(\xi_A) = \xi_B$. But in $B(\mathbb{P}^n \times \mathbb{P}^n)$ we have $\xi_B^i \otimes \xi_B^j$ for $i > 0, j > 0$ are independent. This proves that

$$\varphi(a_{i,j}^n) = b_{i,j}^n.$$

Consider $pr_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ be the projection to the 2-nd factor. Note that $pr_1 \circ \triangle = id_{\mathbb{P}^n}$. Applying the push-forward to $\triangle_A(1)$ on both sides of the equation. We get

$$1 = 1 + \{\mathbb{P}^n\}_A \xi_A^n + \sum_{i>0,j>0} a_{i,j}^n \{\mathbb{P}^{n-i}\}_A \xi_A^j.$$

But the family$\{1, \xi_A, ..., \xi_A^n\}$ form a free basis of the $A(k)$-module $A(\mathbb{P}^n)$. So the sum of coefficients at $\xi_B^n$ is 0. This concludes

$$\{\mathbb{P}^n\}_A = -a_{1,n}^n\{\mathbb{P}^{n-1}\}_A - a_{2,n}^n\{\mathbb{P}^{n-2}\}_A - ... - a_{n,n}^n\{\mathbb{P}^0\}_A.$$

The same procedure gives us

$$\{\mathbb{P}^n\}_A = -a_{1,n}^n\{\mathbb{P}^{n-1}\}_B - a_{2,n}^n\{\mathbb{P}^{n-2}\}_B - ... - a_{n,n}^n\{\mathbb{P}^0\}_B.$$

By the inductive hypothesis we have

$$\varphi(\{\mathbb{P}^m\}_A) = \{\mathbb{P}^m\}_B$$

holds for all $m < n$ and we prove earlier that

$$\varphi(a_{i,j}^n) = b_{i,j}^n$$

We get the required relation

$$\varphi(\{\mathbb{P}^n\}_A) = \{\mathbb{P}^n\}_B.$$

□

**3.2. The case for projective morphisms.** To prove part 2 of Lemma 1.4, we are interested in the projection $p : \mathbb{P}_X(\mathcal{E}) \to X$. Thanks to splitting principle, we may assume that $\mathcal{E}$ is the sum of line bundles $\mathcal{E} = \oplus_{i=0}^n L_i$ for line bundles $L_i$ on $X$ since by $(EH)$ we can check all the identities after pulling back.

Then we can replace $i_{r,n} : \mathbb{P}^r \to \mathbb{P}^n$ by $j_{r,n} : \mathbb{P}_X(\oplus_{i=0}^{n-r} L_i) \to \mathbb{P}_X(\oplus_{i=0}^n L_i)$. We observe that all the previous discussions still go through with minor modifications.

Specifically, we use the family $\{1, \xi_1, \xi_1\xi_2, ..., \xi_1...\xi_n\}$ and $\{1, \xi, ..., \xi^n\}$ at the same time. We denote $\xi^{(i)} = \xi_1...\xi_i$ where $\xi_i = c_1(\mathcal{O}(1)\otimes p^*L_i^{-1})$, $p : \mathbb{P}_X(\oplus_{i=0}^n L_i) \to X$ is the projection, and $\xi = c_1(\mathcal{O}(1))$ as before.

First we note that Lemma 3.2 and the special case of Lemma 3.1 still hold for $\xi^{(i)}$. That is, we still have

$$(j_{0,n})_A(1) = \xi_n^{(n)}$$

and

$$(p_n)_A(\xi_n^{(r)}) = \{\mathbb{P}_X(\oplus_{i=0}^{n-r} L_i)\}.$$

This gives us the same formula in Lemma 3.3 with the family $\{1, \xi^{(1)}, \xi^{(2)}, ..., \xi^{(n)}\}$. For Lemma 3.4, the same proof still goes through with the family $\{1, \xi^{(1)}, \xi^{(2)}, ..., \xi^{(n)}\}$ except that we have to show that all the objects in this family are independent of each other additionally. We notice that both theories $A$ and $B$ satisfy the axioms (*Complete*) and (*Extended FGL*). With these ingredients, we have the following claim which finishes the proof.

CLAIM. *The family $\{1, \xi^{(1)}, \xi^{(2)}, ..., \xi^{(n)}\}$ is a change of basis for the free basis $\{1, \xi, ..., \xi^n\}$ of the $A(X)$-module $A(\mathbb{P}_X(\oplus_{i=0}^n L_i))$(respectively for theory B).*

PROOF. Suppose the formal group law of $A$ is given by $F_A(u, v) = u + v + \sum_{i,j\geq 1} u^i v^j$. Denote $c_1(p^*L_i^{-1}) = \zeta$.

Then we have

$$\xi_i = c_1(\mathcal{O}(1) \otimes p^* L_i^{-1}) = (1 + \sum_{i \geq 0, j \geq 1} \xi^i \zeta^j)\xi + c_1(p^* L_i^{-1}).$$

By the naturality of $c_1$, we have $c_1(p^* L_i^{-1}) = p^*(c_1(L_i^{-1}))$. So,

$$\xi_i = (1 + \sum_{i \geq 0, j \geq 1} \xi^i \zeta^j)\xi + c_1(p^* L_i^{-1}) = p^*(1 + \sum_{i \geq 0, j \geq 1} \xi^i c_1(L_i^{-1})^j)\xi + p^* c_1(L_i^{-1}).$$

Since we assume the theory to be complete, $1 + \sum_{i \geq 0, j \geq 1} \xi^i c_1(L_i^{-1})^j$ is a unit in $A(X)$ since $j$ begins from 1. Taking the products, we have the following equation for each $i$.

$$\xi^{(i)} = a_i \xi^i + a_{i-1} \xi^{i-1} + \ldots + a_0$$

where each $a_i$ is a unit in $A(X)$. Then the transformation matrix is upper-triangular with all diagonal elements being units. Thus the claim is proved.

□

## 4. The Riemann-Roch theorem with Todd classes

Applying the twisting construction, we can get a Riemann-Roch theorem which looks like the classical Grothendieck-Riemann-Roch.

For any oriented cohomology theory $B^*$. We set $B^*(\mathbb{P}^\infty)$ to be

$$B^*(\mathbb{P}^\infty) = \varprojlim_n B^*(\mathbb{P}^n).$$

By projective bundle formula, $B^*(\mathbb{P}^\infty) = B^*(k)[[t]]$ where $t = c_1^B(\mathcal{O}(1))$. Let $\varphi : A^* \to B^*$ be a natural transformation of two oriented cohomology theories on $SmDM_k$, then $\varphi^{\mathbb{P}^\infty}(c_1^A(\mathcal{O}(1))) \in B^*(\mathbb{P}^\infty)$, we set $\lambda(t) \in B^*(k)[[t]]$ as

$$\lambda(t) = \varphi^{\mathbb{P}^\infty}(c_1^A(\mathcal{O}(1)) = \sum_{i=0}^{\infty} a_{i-1} t^i.$$

Note that $a_i \in B^i(k)$ since $\varphi$ is homomorphism of graded rings. We are interested in the natural transformation $\varphi$ such that $a_{-1} = 0, a_0 = 1$.

Definition 4.1. We say that $\varphi$ is standard if $a_{-1} = 0, a_0 = 1$.

Theorem 4.2. *Let $\varphi : A^* \to B^*$ be a standard morphism of two oriented cohomology theories on $SmDM_k$. Furthermore, $B^*$ satisfies the axioms (Complete) and (Extended FGL), $A^*$ satisfies (Extended FGL).*

*In addition, suppose that $\varphi^X(c_1^A(L)) = \lambda(c_1^B(L))$ for any line bundle $L \to X$ on $X$. Then for any projective morphism $f : Y \to X$ in $SmDM_k$. There exists a Todd class operator $\widetilde{Td}_\tau$ such that*

$$f_B(\widetilde{Td}_\tau(T_Y) \circ \varphi(x)) = \widetilde{Td}_\tau(T_X) \circ \varphi(f_A(x))$$

*for any $x \in A*(Y)$.*

PROOF. Let's assume that $A$ satisfies the axiom ($Complete$) first. In order to apply R-R Theorem in the previous section, we can use the twisting construction to change the 1st Chern class on $B^*$.

Let $\tau = (a_i) \in \prod_0^\infty B^i(k)$. We consider the new oriented cohomology theory $B^\tau$ on $SmDM_k$. Recall that in the new theory, we have:

$$f_*^\tau = f_* \circ \widetilde{Td}_\tau(T_f),$$

and

$$c_1^\tau(L) = \lambda_\tau(c_1(L)).$$

where $T_f = [T_Y] - [f^*T_X] \in K_0(Y)$.

In particular, we have

$$c_1^{\tau,B}(L) = \lambda(c_1^B(L)) = \varphi^X(c_1^A(L))$$

for any line bundle $L \to X$. Then, by the Riemann-Roch theorem in the previous section, we have

$$f_B^\tau(\varphi(x)) = \varphi(f_A(x))$$

for any $x \in A*(Y)$. Expressing this out, we get:

$$f_B(\widetilde{Td}_\tau(T_f) \circ \varphi(x)) = \varphi(f_A(x))$$

$$f_B(\widetilde{Td}_\tau(T_Y) \circ \widetilde{Td}_\tau^{-1}(f^*T_X) \circ \varphi(x)) = \varphi(f_A(x)).$$

By projection formula,

$$f_B(\widetilde{Td}_\tau(T_Y) \circ \varphi(x)) = \widetilde{Td}_\tau(T_X) \circ \varphi(f_A(x))$$

for any $x \in A*(Y)$.

Now we can treat the case when $A$ does not necessarily satisfy the axiom ($Complete$). Via the completion process, we can see that all the properties of $\varphi : A \to B$ passes to $\hat{\varphi} : \hat{A} \to B$ except possibly that $\hat{A}$ might not satisfy ($Weak\ Localization$). Note that:

(1) It's enough to prove the theorem for $\hat{\varphi} : \hat{A} \to B$ under the same hypothesis except that $\hat{A}$ might not satisfy ($Weak\ Localization$).
(2) We only use ($Weak\ Localization$) to prove Whitney product formula for Chern classes and the case of a closed immersion.
(3) We are doing the twisting construction only on theory $B$ which is already complete.
(4) In the process of proving Riemann-Roch in the previous section, we don't make use of Whitney product formula except the case of a closed immersion.
(5) As we have seen, we only use ($Weak\ Localization$) on theory $B$ when we try to show the lemma on deformation to the normal cone.

Thus the theorem still holds without assuming $A$ to satisfy the axiom ($Complete$).

□

REMARK 4.3. (1) The assumption saying that $\varphi^X(c_1^A(L)) = \lambda(c_1^B(L))$ for any line bundle $L \to X$ on $X$ is automatic in case of projective schemes. The idea is easy as follows:

We first embed $X$ into a projective space via $i : X \to \mathbb{P}^n$ such that $i^*(\mathcal{O}(1)) = L$. We deduce that $a_{i,L} = a_{i,n}$ in the series $\lambda(t)$. Then for $n < m$, we consider the linear embedding $\mathbb{P}^n \to \mathbb{P}^m$ to deduce that $a_{i,n} = a_{i,m}$. Pass to limit, we get the desired result. Note that we are using [**11**] [Lemma 2.3.9] saying that for oriented cohomology theory, we always have

$$c_1(L_1) \circ ... \circ c_1(L_j) = 0$$

for $j > dim_k(X)$ and $(L_1, ..., L_j)$ a family of line bundles over $X$ generated by its global sections.

(2) The assumption is also satisfied for smooth quasi-projective scheme by using Jouanolou's trick. See for example the argument in part (3) of Remark 2.4.

CHAPTER 4

# Universal theory in oriented cohomology theories

## 1. Completion and Riemann-Roch theorem for quotient Deligne-Mumford stacks

In this section we introduce a Riemann-Roch theorem for quotient Deligne-Mumford stacks due to Edidin and Graham, which suggests us to consider certain completion of an oriented cohomology theory $A^*$.

**1.1. A Riemann-Roch theorem.** Let $X = [V/G]$ be a quotient Deligne-Mumford stack, denote $G_0(X)$ to be the Grothendieck ring of coherent sheaves on $X$, $K_0(X)$ to be the Grothendieck ring of locally free sheaves on $X$. The ring $K_0(X)$ has a distinguished augmentation ideal, and we denote by $\widehat{G_0(X)}$ the completion of $G_0(X)_{\mathbb{Q}}$ with respect to the topology generated by this ideal. The following theorem is from [**6**, Theorem 3.5].

THEOREM 1.1. *There is a homomorphism $\tau_X : G_0(X) \to Ch^*(X)_{\mathbb{Q}}$ which factors through an isomorphism $\widehat{G_0(X)} \to Ch^*(X)_{\mathbb{Q}}$. The map $\tau_X$ is covariant for proper representable morphisms and when $X$ is smooth and $V$ a vector space then*

$$\tau_X(V) = ch(V)Td(T_X)$$

*where $ch$ is the Chern character and $Td$ is the usual Todd class with exponential and $T_X$ is the tangent bundle of $X$.*

REMARK 1.2. (1) $Ch^*(X)_{\mathbb{Q}}$ is complete with respect to the topology generated by the augmentation ideal $J$ , see [**5**, Section 6]. Here $J$ is the augmentation ideal of $A^*(X)$, note that here we mean $A^*(X)$ by operational Chow group and $Ch^i(X) = A^i(X)$ is isomorphic when $X$ is smooth, this is proved in [**4**, Theorem 4].

(2) $K_0(X)$ is naturally identified with the equivariant Grothendieck ring $K_0(G, V)$, and $G_0(X)$ is naturally identified with the equivariant Grothendieck ring $G_0(G, V)$. Furthermore, $Ch^*(X)$ is defined to be the equivariant Chow groups $Ch^*_G(V)$ which agrees with Kresch's integral Chow groups. Tensoring with $\mathbb{Q}$, they all agree with Vistoli's rational Chow groups. See [**6**, Section 3.2].

(3) By a general theorem of Thomason, we have the following result:

(See [**15**] [Thm 2.18]) For $X = [V/G]$ a Deligne-Mumford quotient stack over a field, suppose $V$ is a smooth quasi-projective scheme. Then $X$ satisfies the resolution property, namely, every equivariant coherent sheaf is a quotient of an equivariant locally free sheaf.

(4) If $X$ is smooth and has the resolution property- that is, every coherent sheaf is the quotient of a locally free sheaf, then the natural map $K_0(X) \to G_0(X)$ is an isomorphism since we can always find a finite resolution of locally free sheaves for any coherent sheaf on $X$. The finiteness comes from the regularity of $X$.

From now on we restrict our attention to the subcategory of $SmDM_k$ where all the objects are quotient stacks of the form $X = [V/G]$ where $V$ is smooth and quasi-projective, i.e., those Deligne-Mumford stacks that are already quotient stacks with smooth and quasi-projective moduli spaces. Denote it by $GQ_k$.

From now on,we use the convention that all the theories we are interested in, either Chow groups or Grothendieck groups, or even a general oriented cohomology theory, are equipped with rational coefficients.

The condition of a Deligne-Mumford stack being a quotient stack is not very restrictive. Indeed, we haven't seen any example of a separated Deligne-Mumford stack that is not a quotient stack. Moreover, there are lots of interesting results showing that certain kind of Deligne-Mumford stacks are quotient stacks. For example, if $X$ satisfies the resolution property, then $X$ is a global quotient, see [**3**] Theorem 2.14. Another interesting result of Totaro in [**16**, Theorem 1.2] says:

Theorem 1.3. *Let $X$ be a smooth Deligne-Mumford stack over a field $k$. Suppose that $X$ has finite stabilizer group and that the stabilizer group is generically trivial. Let $B$ be the Keel-Mori coarse moduli space of $X$. If the algebraic space $B$ is a scheme with affine diagonal, then the stack $X$ has the resolution property.*

**1.2. An aside on $K_0[\beta, \beta^{-1}]$.** For $X$ smooth in $GQ_k$, we consider a graded ring $K_0(X)[\beta, \beta^{-1}] := K_0(X) \otimes_{\mathbb{Z}} \mathbb{Q}[\beta, \beta^{-1}]$, here $\mathbb{Q}[\beta, \beta^{-1}]$ is the ring of Laurent polynomial in a variable $\beta$ of degree $-1$. View $K_0$ as a functor on $GQ_k$, we would like to show that $K_0[\beta, \beta^{-1}]$ is an oriented cohomology theory on $GQ_k$.

We may define the pull-backs for any morphism $f : Y \to X$ by the formula

$$f^*([\mathcal{E}] \cdot \beta^n) := [f^*(\mathcal{E})] \cdot \beta^n$$

for $\mathcal{E}$ a locally free coherent sheaf on $X$ and $n \in \mathbb{Z}$.

For each proper morphism $f : Y \to X$ of relative codimension $d$, we define

$$f_*([\mathcal{E}] \cdot \beta^n) := \sum_{i=0} (-1)^i [R^i f_*(\mathcal{E})] \cdot \beta^{n-d} \in K_0(X)[\beta, \beta^{-1}]$$

for $\mathcal{E}$ a locally free coherent sheaf on $Y$ and $n \in \mathbb{Z}$. By [**7**] or [**12**] we know the coherence of $[R^i f_*(\mathcal{E})]$, but for smooth $X$ we can identify $G_0(X)$ with $K_0(X)$ by taking a finite locally free resolution of a coherent sheaf.

Using standard results of K-theory, we can see that this defines an oriented cohomology theory on $GQ_k$. See [**5**] for example, where one identifies $\widehat{K_0(X)}$ with $Ch^*(X)$.

Furthermore, for a line bundle $L \to X$ with zero section $s : X \to L$, projection $\pi : L \to X$ and sheaf of sections $\mathcal{L}$, one has

$$s^*(s_*(1_X)) = s^*([\mathcal{O}_{s(X)}]\beta^{-1}) = s^*(1 - [\pi^*(\mathcal{L}^\vee])\beta^{-1} = (1 - [\mathcal{L}^\vee]\beta^{-1}.$$

So we have $c_1^K(L) = (1 - [\mathcal{L}^\vee])\beta^{-1}$.

By splitting principle, for any vector bundle $E$ of rank $n$, we have $c_1^K(E) = (n - [\mathcal{E}^\vee])\beta^{-1}$.

We notice that $\{c_1^K(E)\}$ for all vector bundles $E$ generate the augmentation ideal of $K_0(X)[\beta, \beta^{-1}]$.

From the relation

$$(1 - [(\mathcal{L} \otimes \mathcal{M})^\vee]) = (1 - [\mathcal{L}^\vee]) + (1 - [\mathcal{M}^\vee]) - (1 - [\mathcal{L}^\vee])(1 - [\mathcal{M}^\vee]).$$

We know that the formal group law is

$$F_m(u, v) = u + v - \beta uv.$$

**1.3. A Chern character map.** Let $A^*$ be an oriented cohomology theory on $GQ_k$, we want to construct a morphism of functors on $GQ_k$ from $K_0[\beta, \beta^{-1}] \to A^*$ if $A^*$ is multiplicative and periodic. We will show that this morphism is compatible with pullback, external products and $\widetilde{c_1}$, i.e., a morphism of oriented cohomology theories compatible with 1st Chern class.

For $X \in GQ_k$, let $E$ be a vector bundle of rank $r$ on $X$ with sheaf of sections $\mathcal{E}$, we define $ch_A : K_0[\beta, \beta^{-1}] \to A^*$ as follows:

$$ch_A(\mathcal{E} \cdot \beta^n) := (r - b \cdot \widetilde{c_1}^A(E^\vee))b^n(1_X).$$

Now let $L$ and $M$ be line bundles on $X \in GQ_k$ with sheaves of sections $\mathcal{L}$ and $\mathcal{M}$. Since we assume $A^*$ to be multiplicative and periodic, we have the formal group laws on $\widetilde{c_1}$.

$$\widetilde{c_1}(L \otimes M)(1_X) = (\widetilde{c_1}(L) + \widetilde{c_1}(M) - b\widetilde{c_1}(L)\widetilde{c_1}(M))(1_X).$$

An easy computation shows that

$$ch_A(\mathcal{L} \otimes \mathcal{M}) = ch_A(\mathcal{L})ch_A(\mathcal{M})$$

and

$$ch_A(c_1^K(L)(\mathcal{M})) = c_1^A(L) \circ ch_A(\mathcal{M}).$$

For example, we have

$ch_A(c_1^K(L)(\mathcal{M})) = ch_A(\beta^{-1}(1 - [\mathcal{L}^\vee])\mathcal{M})$
$= b^{-1}(ch_A(\mathcal{M}) - ch_A(\mathcal{L}^\vee \otimes \mathcal{M}))$
$= b^{-1}(b\widetilde{c_1}(L \otimes M^\vee)(1_X) - b\widetilde{c_1}(M^\vee)(1_X))$
$= c_1^A(L)(1 - bc_1(M^\vee))$
$= c_1^A(L) \circ ch_A(\mathcal{M})$.

Then, by the splitting principle, this tells us

PROPOSITION 1.4. *For all locally free sheaves $\mathcal{E}$ and $\mathcal{F}$, and line bundles $L$, we have*

$$ch_A(\mathcal{E} \otimes \mathcal{F}) = ch_A(\mathcal{E})ch_A(\mathcal{F}).$$
$$ch_A(c_1^K(L)(\mathcal{E})) = c_1^A(L) \circ ch_A(\mathcal{E}).$$

Still, by the splitting principle and properties of Chern class operator, especially the naturality of $\widetilde{c_1}$. We see that $ch_A$ is compatible with pullbacks and external products.

Recall the definition of oriented morphisms between oriented cohomology theories, we notice that in order to show the universality of $K_0[\beta, \beta^{-1}]$, we still need to show that $ch_A$

commutes with push-forward. That is essentially one form of Grothendieck-Riemann-Roch which we have already established in the previous chapter.

**1.4. Completion.** Before we have seen the homomorphism $\tau_X : K_0(X) \to Ch^*(X)$ that factors through an isomorphism $\widehat{K_0(X)} \to Ch^*(X)$. Furthermore, we have the diagram after extension of scalars.

$$\begin{array}{ccc} K_0(X)[\beta, \beta^{-1}] & \xrightarrow{\text{Completion}} & \widehat{K_0(X)}[\beta, \beta^{-1}] \\ & {\scriptstyle \tau_X} \searrow \quad \swarrow {\scriptstyle \sim} & \\ & Ch^*(X)[\beta, \beta^{-1}] & \end{array}$$

In the previous section, we have constructed the morphism of oriented cohomology theories $ch_A : K_0[\beta, \beta^{-1}] \to A^*$ , a natural question is: is there a morphism of oriented cohomology theories $\vartheta_A : Ch^*[\beta, \beta^{-1}] \to A^*$?

The answer is yes: there is a morphism of oriented cohomology theories $\vartheta_A : Ch^*[\beta, \beta^{-1}] \to A^*$ when the theory $A^*$ is complete with additive formal group law.

That is the reason we introduce the notion of the completion of a theory $A^*$ and the axiom (*Complete*).

## 2. Universalities

To motivate, let us mention the theorems from [**11**] first.

THEOREM 2.1 (Theorem 7.1.3 in [**11**]). *Assume $k$ admits resolution of singularities. Then there exists a theory algebraic cobordism, denoted $\Omega^*$, considered as an oriented cohomology theory on $Sm_k$, is the universal oriented cohomology theory on $Sm_k$.*

THEOREM 2.2 (Theorem 7.1.4 in [**11**]). *Assume $k$ admits resolution of singularities. The canonical morphism $\Omega^* \to K_0[\beta, \beta^{-1}]$ of oriented cohomology theories on $Sm_k$ induces an isomorphism*

$$\Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to K_0[\beta, \beta^{-1}].$$

We assume that $k$ admits resolution of singularities because it is needed to show that $\Omega^*$ is an oriented cohomology theory. In general, there is a theorem saying that:

THEOREM 2.3 (Theorem 1.2.3 in [**11**]). *Let $k$ be an arbitrary field. Let $A^*$ be an oriented cohomology theory with multiplicative and periodic formal group law on $Sm_k$, then there exists one and only one morphism of oriented cohomology theories on $Sm_k$*

$$K_0[\beta, \beta^{-1}] \to A^*.$$

By Theorem 1.2 in Chapter 3, if we restrict our attention only to strongly projective and projective morphisms in $GQ_k$ in the definition of an oriented cohomology theory where proper morphisms were originally considered. We have the following theorem.

THEOREM 2.4. *(1) Let $A^*$ be an oriented cohomology theory with multiplicative and periodic formal group law, then there exists one and only one oriented morphism of oriented cohomology theories(with strongly projective push-forwards) on $GQ_k$.*

$$K_0[\beta, \beta^{-1}] \to A^*.$$

*(2) Let $A^*$ be an oriented cohomology theory with multiplicative and periodic formal group law, which is also complete. then there exists one and only one oriented morphism of oriented cohomology theories(with projective push-forwards) on $GQ_k$.*

$$\widehat{K_0}[\beta, \beta^{-1}] \to A^*.$$

PROOF. The uniqueness of (1) follows from the requirement that the morphism is oriented. Recall that oriented morphisms commute with pullbacks and push-forwards, but we know that $\widetilde{c_1} = i_* i^*$ for $i$ a zero section of the line bundle. This implies that oriented morphisms sends the first Chern class to the first Chern class. Recall that $c_1^K(L) = (1 - [\mathcal{L}^\vee])\beta^{-1}$, $ch_A(c_1^K(L)(\mathcal{M})) = c_1^A(L) \circ ch_A(\mathcal{M})$ for any $\mathcal{M}$. Together with splitting principle, we see that any oriented morphism must be give by the Chern character map.

Once we have a unique oriented morphism of oriented cohomology theories $K_0[\beta, \beta^{-1}] \to A^*$, we can extend it uniquely to $\widehat{K_0}[\beta, \beta^{-1}] \to A^*$ since the ring homomorphism is continuous with respect to the $I_K$-topology.

Now the proof of the theorem is just applications of the results on Chern character map and theorems in Chapter 3. We don't need to assume any extra axiom since the theories we begin with already satisfy those axioms.

□

Now let $A^*$ be any oriented cohomology theory with additive formal group law, which is also complete.

Let us consider

$$X \rightarrowtail A^*(X) \otimes \mathbb{Q}[\beta, \beta^{-1}]$$

obtained from $A^*$ by the extension of scalars $\mathbb{Q} \subset \mathbb{Q}[\beta, \beta^{-1}]$, where $\beta$ has degree 1.

We apply our twisting for family $\tau$ given by

$$\lambda_\tau(u) = \frac{1 - e^{-\beta u}}{\beta}.$$

We denote the new theory by $A^*[\beta, \beta^{-1}]^{td}$.

Just like Example 5.2 in Chapter 2, we can compute the formal group law of $A^*[\beta, \beta^{-1}]^{td}$ as the multiplicative one:

$$F_m(u, v) = u + v - \beta uv.$$

By the theorem above, we have an oriented morphism of oriented cohomology theories(with projective push-forwards) on $GQ_k$

$$\widehat{K_0}[\beta, \beta^{-1}] \to A^*[\beta, \beta^{-1}]^{td}.$$

Via the isomorphism $\widehat{K_0(X)} \to Ch^*(X)$ and commutative diagram

$$\begin{array}{ccc} K_0(X)[\beta,\beta^{-1}] & \xrightarrow{\text{Completion}} & \widehat{K_0(X)}[\beta,\beta^{-1}] \\ & \searrow^{\tau_X} \quad \swarrow^{\sim} & \\ & Ch^*(X)[\beta,\beta^{-1}] & \end{array}$$

, where $\tau_X(V) = ch(V)Td(T_X)$, we get an oriented morphism of oriented cohomology theories(with projective push-forwards) on $GQ_k$

$$ch^A : Ch^*(X)[\beta,\beta^{-1}]^{td} \to A^*(X)[\beta,\beta^{-1}]^{td}.$$

Note that $Ch^*[\beta,\beta^{-1}]^{td}$ is an oriented cohomology theory that is already complete since $Ch^n(X) = 0$ for $n > dim_k X$.

DEFINITION 2.5. We define the Adams operation $\psi_k$ on $A^*(X)[\beta,\beta^{-1}]^{td} = A^*(X)[\beta,\beta^{-1}]$ by $\psi_k(\alpha\beta^n) = k^i\alpha\beta^n$ where $\alpha \in A^i(X)$.

For $\widehat{K_0(X)}[\beta,\beta^{-1}]$, we have $\psi_k(L) = L^{\otimes k}$ for any line bundle $L$ on $X$.

PROPOSITION 2.6. *$\psi_k$ commutes with $ch^A$.*

PROOF. It follows from the direct computation. Take $L \in \widehat{K_0(X)}$ for any $L$ a line bundle on $X$.

$$ch^A(\psi_k L) = ch^A(L^{\otimes k}) = 1-(1-e^{-\beta c_1^A(L^\vee)}) = e^{\beta c_1^A(L)} = \sum \frac{\beta^n c_1^A(L^{\otimes k})^n}{n!} = \sum \frac{\beta^n k^n c_1^A(L)^n}{n!}$$

, the last equality comes from additive formal group law on $A^*$. On the other hand, we have

$$\psi_k^{A^*[\beta,\beta^{-1}]}(ch^A(L)) = \sum \frac{\beta^n k^n c_1^A(L)^n}{n!}.$$

□

In particular, the oriented morphism $ch^A : Ch^*(X)[\beta,\beta^{-1}]^{td} \to A^*(X)[\beta,\beta^{-1}]^{td}$ commutes with $\psi_k$.

We then apply the inverse Todd class to $ch^A$. Without confusion, we still denote the oriented morphism by $ch^A : Ch^*(X)[\beta,\beta^{-1}] \to A^*(X)[\beta,\beta^{-1}]$. Note that it still commutes with $\psi_k$.

Note that $ch^A$ is a ring homomorphism which preserves degree, so $\beta$ must go to something of degree $-1$. If $ch^A(\beta) \neq m\beta$ for some $m \in A^0$, then $ch^A$ will not commute with $\psi_k$. For the same reason, we must have $ch^A(Ch^n) \subseteq A^n$.

We know that $ch^A : Ch^*(X)[\beta,\beta^{-1}] \to A^*(X)[\beta,\beta^{-1}]$ is an oriented morphism. Restricting this oriented morphism to $Ch^*(X) \subset Ch^*(X)[\beta,\beta^{-1}]$, we finally get the following theorem, analogous to the case of schemes:

THEOREM 2.7. *Let $k$ be a field of characteristic 0. Let $A^*$ be any oriented cohomology theory with additive formal group law, which is also complete. Then there exists a unique* *oriented morphism of oriented cohomology theories(for projective push-forwards)*

$$\vartheta : Ch^* \to A^*.$$

REMARK 2.8. By Remark 3.3 in Chapter 2, an oriented cohomology theory $A^*$ on the category of smooth quasi-projective schemes $Sm_k$ is always complete.

We may also think of $Sm_k$ as a full subcategory of $SmDM_k$ by requiring the Deligne-Mumford stack to be a scheme. Then an oriented cohomology theory gives rise to an oriented cohomology theory on $Sm_k$ without too much difficulty. For example, Remark 4.6 of chapter 1 tells us $Ch^*$ restricted to $Sm_k$ is an the usual oriented cohomology theory $Ch^*$ on $Sm_k$. Thus Theorem 2.4 is a generalization of Theorem 2.3, while Theorem 2.7 is a generalization of the case of schemes, as we have seen in Theorem 5.5 of Chapter 2.

## 3. Rational algebraic cobordism on Deligne-Mumford stacks

Following Quillen and Levine-Morel, we would like to prove the following theorem:

THEOREM 3.1. *Let $k$ be a field of characteristic 0. Then $Ch^*[\boldsymbol{t}]^{(\boldsymbol{t})}$ is the universal theory among complete theories, i.e., for any oriented cohomology theory $A^*$ on $GQ_k$ such that it satisfies $(Extended\ FGL)$ and $A^*(k)$ contains all rational numbers, there exists a unique oriented morphism of oriented cohomology theories(for projective push-forwards)*

$$\vartheta : Ch^*[\boldsymbol{t}]^{(\boldsymbol{t})} \to A^*.$$

*Note that $\boldsymbol{t} = (t_i)$ where all $t_i$ are formal variables.*

PROOF. We notice first that $Ch^*[\mathbf{t}]$ satisfies the axiom $(Complete)$ since $Ch^*(X) = 0$ for $n > dim_k X$. Then Todd twist doesn't change the completeness, so $Ch^*[\mathbf{t}]^{(\mathbf{t})}$ satisfies the axiom $(Complete)$.

The proof uses the following classical lemma, see Corollary 7.15 in [**1**], for example.

LEMMA 3.2. *Let $R$ be a commutative $\mathbb{Q}$-algebra and let $F(u, v) \in R[[u, v]]$ be a commutative formal group law of rank one over $R$. Then there exists a unique power series $\ell_F(u) = \sum_i \tau_i u^{i+1} \in R[[u, v]]$ such that $\tau_0 = 1$ and satisfying*

$$\ell_F(F(u, v)) = \ell_F(u) + \ell_F(v).$$

Let $A^*$ be an arbitrary theory satisfying the conditions in the theorem. Recall that when we twist a theory by $\tau$, the new formal group law is given by :

$$F_A^{(\tau)}(u, v) = \lambda_{(\tau)}(F_A(\lambda_{(\tau)}^{-1}(u), \lambda_{(\tau)}^{-1}(v))).$$

By the lemma and our assumption on $A^*$, there exists a unique $\tau$ as in the power series $\ell_F(u) = \sum_i \tau_i u^{i+1} \in R[[u, v]]$ such that $\tau_0 = 1$, $\tau_i \in A^i(k)$ and satisfying

$$F_A^{(\tau)}(u, v) = u + v.$$

Thus $A^\tau$ is an additive theory, then there exists an oriented morphism of oriented cohomology theories

$$\vartheta : Ch^* \to A^{\tau *}.$$

Note that $Ch^*(k)[\mathbf{t}]^{(\mathbf{t})} = Ch^*(k)[\mathbf{t}]$ by the definition of twisting. We define the ring homomorphism

$$\varphi_A : Ch^*(k)[\mathbf{t}] \to A^{\tau*}(k)$$

by $\mathbf{t} \to \tau^{-1}$ and $Ch^*(k) \to A^*(k)$ is just the restriction of $\vartheta : Ch^* \to A^{\tau*}$ to a point. Recall that $\tau^{-1}$ is the series given by $f_{\tau^{(-1)}}(t) \cdot f_\tau(t) = 1$. Recall that $A^*(X)$ is always a $A^*(k)$-module for any oriented cohomology theory $A$. We can extend $\varphi_A$ to an oriented morphism

$$\varphi_A : Ch^*[\mathbf{t}] \to A^{\tau*}.$$

Since we have $\mathbf{t} \to \tau^{-1}$, we can apply the twisting $\mathbf{t}$ to $Ch^*[\mathbf{t}]$ and $\tau_{-1}$ to $A^{\tau*}$ to get an oriented morphism

$$\vartheta : Ch^*[\mathbf{t}]^{(\mathbf{t})} \to A^*.$$

□

REMARK 3.3. By Remark 2.8 of this chapter, the above theorem is a generalization of Theorem 5.4 in chapter 2.

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