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The most important thing to remember is this: To be ready at any moment to give up what you are for what you might become.

W.E.B. Du Bois

To my family.

Contents

Introduction

The present thesis is very much influenced by my late teacher Alexandru Lupaş (Arad, România, 5 January 1942 - Sibiu, România, 14 August 2007) and to subsequent work done by my present thesis advisor. To be more specific, the starting point for our introduction is a very short question at the end of Lupaş' paper [63]. He drew the reader's attention to the fact that at least some well-known "Bernstein-type" operators are compositions of the classical Bernstein operators B_n, and other operators. According to our knowledge, Lupaş was the first one to observe such a phenomenon.

It is his merit to have shown that it can be much easier to look at things from an algebraic point of view and to draw conclusions on the behaviour of a composite operator from properties of the building blocks. The present thesis follows this approach and makes an attempt to give explanations which go far beyond Lupaş' original ambition.

We note that when we talk about Bernstein-type operators we mean certain positive linear operators defined on a space $C[a, b]$, where $[a, b]$ is a compact interval of the real axis. When doing so we explicitly exclude mappings such as the Mirakyan-Favard-Szàsz operators taking care of continuous function on the real semi-axis.

An early attempt to give a survey on operators being defined on such compact sets X, where X is a subset of $\mathbb{R}^d, d \geq 1$, was given in three bibliographies [96], [41] and [42]. There an informal attempt was made to define such operators of Bernstein-type. What had always been clear is that the classical Bernstein operators introduced in 1912 (see [12]) play a crucial role in this context and that many of the subsequent papers dealing with the subject are treating modifications of such operators.

There is a good reason to consider the classical B_n operator as a fundamental building block. Several attempts have been made to decompose the Bernstein operator into simpler but non-trivial components. A technical report considering this problem was given by Gonska, Heilmann and Raşa (see [37]). There it is shown that all seemingly natural decompositions fail. In this sense B_n can be considered as a kind of "prime" operator.

Following the tracks laid by both my mentors I also became interested in these so-called Bernstein-type operators, although this notion is so broadly defined that none of us knows exactly what it means.

By introducing two classes of Bernstein-Euler-Jacobi (BEJ) operators we will first rediscover and explain many positive linear operators which have been considered in the literature over the years, but were never explained to be compositions. Thus we bring clarity and avoid the original definitions and formulae which are lengthy, confusing and mostly difficult to be handled.

We start our thesis with several auxiliary results which are indispensable for our considerations in later sections. Many of these are in regard to quantitative approximation theory in connection with shape preservation. Besides of the Bernstein operator an important role will be played by Beta-type operators with Jacobi weights.

Some of their properties are presented in Section 1.3 following in part the fundamental work of Lupaş and Mühlbach.

In the second chapter we discuss the two kinds of BEJ operators and give explicit representations of their moments up to order two. This is motivated by the fact that the second moments of a positive linear operator govern the degree of approximation. The expressions obtained are rather complicated, but for special values of the parameters involved they all reduce to results known from the literature. In Section 2.3 we give a survey of special cases which we were able to detect in the literature and which all can be explained as particular instances of BEJ-type operators. In the next two sections we show how the properties of the building blocks can be used to derive results for the general case. First we show that the operators from both BEJ classes have the (strong) variation-diminishing property (SVDP), and then we use a method developed by Finta to give direct and converse results for some special cases.

Our central Chapter 3 deals with the particular class of operators U_n^ϱ which were introduced by Păltănea in [75]. With the classical Bernstein operator and so-called "genuine" Bernstein-Durrmeyer operators they share the property to reproduce linear functions and this makes them very different from other BEJ-type operators which do not share this feature.

We begin the chapter by recalling the definition provided by Păltănea and in the first section we collect some properties of the operators. They are either new or come from one of the papers of Gonska and Păltănea (see [45], [46]) but for which we have chosen a different path of proof. We remark that although U_n^ϱ coincides with the genuine Bernstein-Durrmeyer operator for the case $\varrho = 1$, when it comes to studying it, its properties are mostly influenced by the Bernstein operator. This will become more and more obvious as we walk through this chapter.

In Section 3.2 we give a representation of the images of monomials that shows exactly how strong this relationship is. Section 3.3 is taken in its entirety from [45] and the results therein will be used repeatedly throughout this work.

After having noted that the images of the monomials for both U_n^ϱ and B_n are quite similar we came across the article on the eigenstructure of B_n written by Cooper and Waldron (see [18]) and this motivated us to find a way to develop similar results for U_n^ϱ. Thus each result in Section 3.4 has a correspondent in this paper, and also enriches the results known for U_n.

In Section 3.5 we give a complete and detailed proof of the SVDP using a slightly different approach than in the general case. At the end we include an observation on the preservation of convexity.

In Section 3.6 we address the topic of global smoothness preservation, having its roots in the preservation of some Lipschitz classes.

The next section is dedicated to some kind of "strong" Voronovskaya-type inequality. The reason why we call this inequality "strong" is that in addition to the convergence of $n(U_n^\varrho f - f)$ towards $\frac{(\varrho+1)}{2(n\varrho+1)}\varphi^2 f''$ it also expresses the degree of approximation depending on the smoothness properties of the function. We came across this result in our attempt to prove a strong converse inequality of type B, as defined by Ditzian and Ivanov in [22].

In Section 3.8 we look at approximation by powers of U_n^ϱ. First we provide quantitative results using different types of moduli and then consider the eigenstructure as means to develop further results.

In Section 3.9 we consider the difference of two classes of operators and provide a direct and a constructive approach to estimate it and then in the next section we tackle the problem of the commutators. Other type of such commutators were already studied in [44].

In Section 3.11 we study the behaviour of U_n^ϱ with respect to Lipschitz classes of order m.

In Sections 3.12 to 3.14 the focus is on the relation they have to certain Lagrange-type interpolators associated to them, a well known feature in the theory of Bernstein operators. Considerations concerning iterated Boolean sums based on a single mapping U_n^ϱ, ϱ and n fixed and a relationship between certain divided differences used in Section 3.12 and the representation of the derivatives $(U_n^\varrho)^{(j)}$ are also included.

In Section 3.15 we give asymptotic formulae for higher order moments using the same approach as in Subsection 1.3.5 where we studied the problem for the Beta operator. These add to our series of Voronovskaya-type results, and help put together an overview on what can be done on the subject.

In the last section we study power series of U_n^ϱ. Using the eigenstructure of the operators we give a non-quantitative convergence result towards the inverse Voronovskaya operators. We include a quantitative statement via a smoothing approach.

Much of the material presented in this thesis was submitted for publication and is presently under consideration or it has already been published.

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Notations and symbols

In this work we shall often make use of the following symbols:

:=	is the sign indicating equal by definition". a:=b" indicates that a is the quantity to be defined or explained, and *b* provides the definition or explanation. b=:a" has the same meaning.
$\mathbb{N}$	the set of natural numbers,
$\mathbb{N}_0$	the set of natural numbers including zero,
$\mathbb{R}$	the set of real numbers,
$[a,b]$	a closed interval,
(a,b) or $]a,b[$	an open interval. Let X be an interval of the real axis.
$B(X)$	the set of all real-valued and *bounded* functions defined on X.
$L^p(X)$	the class of the *Lebesgue p-integrable* functions on $X,\ p \geq 1$.
$\|f\|_p$	is the norm on $L^p(X)$ defined by $\|f\|_p := \left(\int_X |f(x)|^p dx\right)^{1/p},\ p \geq 1$.
$C(X)$	the set of all real-valued and *continuous* functions defined on X.
$C[a,b]$	the set of all real-valued and *continuous* functions defined on the compact interval $[a,b]$. For $f \in B(X)$ or $f \in C(X)$
$\|f\|_\infty$	is the *Chebyshev norm* or *sup-norm*, namely $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$.
$W_{2,\infty}[0,1]$	the set of all real-valued and *continuous* functions that verify f' absolutely continuous and $\|\|f''\|\|_{L_\infty} < \infty$, where $\|\|f''\|\|_{L_\infty} = vrai \sup_{x\in[0,1]} |f''(x)|$.
$C^r[a,b]$	the set of all real-valued, *r-times continuously differentiable* function, $(r \in \mathbb{N})$.
$\mathrm{Lip}_\tau M$	the set of all $C[a,b]-$ functions that verify the *Lipschitz condition*: $|f(x_2) - f(x_1)| \leq M|x_2 - x_1|^\tau,\ \forall x_1, x_2 \in [a,b], 0 < \tau \leq 1,\ M > 0$.
$\prod_n$	$(\prod_n[a,b],\ n \in \mathbb{N}_0)$ the linear space of all real polynomials with the degree at most n.
e_n	denotes the $n-$th *monomial* with $e_n : [a,b] \ni x \mapsto x^n \in \mathbb{R}, n \in \mathbb{N}_0$. For a function $f : X \to \mathbb{R},\ X$ an interval of the real axis we have:
$\Delta_h^k f(x)$	is the finite difference of order $k \in \mathbb{N}$, step $h \in \mathbb{R}\backslash\{0\}$ and starting point $x \in X$. A computing formula: $\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} f(x+ih),\ x+ih \in X,\ i = 0,\ldots,k,\ h \in \mathbb{R},\ h \neq 0$.
D^r or $f^{(r)}$	$r-$th *derivative* of the function $f \in C^r[a,b]$.
$[x_0,\ldots,x_m; f]$	$m-$th divided difference of $f \in \mathcal{F}(X)$ on the not necessarily distinct knots $x_0,\ldots,x_m \in X$.
$a^{\overline{b}}$	are the *rising factorials*

$a^{\overline{b}} := \prod_{i=0}^{b-1}(a+i),\ a \in \mathbb{R},\ b \in \mathbb{N}_0$, where $\prod_{i=0}^{-1} := 1$.

$a^{\underline{b}}$ — are the *falling factorials*

$a^{\underline{b}} := \prod_{i=0}^{b-1}(a-i),\ a \in \mathbb{R},\ b \in \mathbb{N}_0$, where $\prod_{i=0}^{-1} := 1$.

$y^{[m,h]}$ — the *factorial power* of step $h \in \mathbb{R}$ defined by: $y^{[m,h]} := \prod_{i=0}^{m-1}(y-ih)$, $m \in \mathbb{N}_0$. As above $\prod_{i=0}^{-1} := 1$.

Chapter 1

Preliminaries and auxiliary results

1.1 Main tools

1.1.1 Positive linear operators

In this section we will introduce some basic definitions and some basic properties concerning *positive linear operators*. For more information on this topic see [95].

Definition 1.1.1. Let X, Y be two linear spaces of real functions. The mapping $L : X \to Y$ is called a *linear operator* if $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$, for all $f, g \in X$ and for all $\alpha, \beta \in \mathbb{R}$.
If for all $f \geq 0, f \in X$ we have that $Lf \geq 0$, then L is a *positive linear operator*.

Remark 1.1.2. a) The set $\mathcal{L}(X, Y) := \{L : X \to Y \mid L \text{ is a linear operator }\}$ is a real vector space.

b) In order to highlight the argument of the function $Lf \in Y$ we use the notation $L(f; x)$ but also in some rare cases $(Lf)(x)$.

Some elementary inequalities are recalled in the following:

Property 1.1.3. *Let* $L : X \to Y$ *be a positive and linear operator.*

(i) If $f,\ g \in X$ *with* $f \leq g$ *then* $Lf \leq Lg$. *(monotonicity)*

(ii) $\forall f \in X$ *we have* $|Lf| \leq L|f|$.

Definition 1.1.4. Let $L : X \to Y$, where $X \subseteq Y$ are two linear *normed spaces* of real functions. To each operator L we can assign a non-negative number $\|L\|$ defined by

$$\|L\| := \sup_{\substack{f \in X \\ \|f\|=1}} \|Lf\| = \sup_{\substack{f \in X \\ 0<\|f\|\leq 1}} \|Lf\|.$$

By convention, if X is the zero linear space, any operator L which maps X to Y must be the *zero operator* and is assigned the *zero norm*.

It can be easily verified that $\|\cdot\|$ satisfies all the properties of a norm and hence is called *the operator norm*.

Choosing $X = Y = C[a, b]$ the following can be stated regarding the continuity and the operator norm:

Corollary 1.1.5. *If* $L : C[a, b] \to C[a, b]$ *is linear and positive then* L *is also continuous and* $\|L\| = \|Le_0\|$.

The next result provides a necessary and sufficient condition for the convergence of a sequence of positive linear operators towards the identity operator. It was independently discovered and proved by three mathematicians in three consecutive years: T. Popoviciu [80] in 1951, H. Bohman [13] in 1952 and P. P. Korovkin [61] in 1953.

This classical result of approximation theory is mostly known under the name of *Bohman-Korovkin theorem*, because T. Popoviciu's contribution in [80] remained unknown for a long time.

Theorem 1.1.6. *Let $L_n : C[a,b] \to C[a,b]$ be a sequence of positive linear operators. If $\lim_{n\to\infty} L_n e_i = e_i$, $i = 0,1,2$, uniformly on $[a,b]$, then $\lim_{n\to\infty} L_n f = f$ uniformly on $[a,b]$ for every $f \in C[a,b]$.*

Remark 1.1.7. *Due to the above result the monomials e_j, $j = 0,1,2$, play an important role in the approximation theory of linear and positive operators on spaces of continuous functions. They are often called Korovkin test-functions.*

This elegant and simple result has inspired many mathematicians to extend the last theorem in different directions, generalizing the notion of sequence and considering different spaces. In this way a special branch of approximation theory arose, called Korovkin-type approximation theory. A complete and comprehensive exposure on this topic can be found in [6].

1.1.2 Different types of moduli of smoothness

The first modulus of smoothness (continuity) has a long history. It appeared already in 1911 in the Ph. D. thesis of D. Jackson [53], the work that laid the basis for what is known today as *Quantitative Approximation Theory*.

Ditzian and Totik introduced in 1987 what they call a "natural modulus of smoothness" which is considered to be a "better tool to deal with the rate of best approximation, inverse theorems and embedding theorems" (see [23, p.1-4]).

The Ditzian-Totik modulus of smoothness is given by

$$\omega_\varphi^r(f,t)_p \equiv \sup_{0<h\leq t} ||\Delta_{h\varphi}^r f||_{L_p} \tag{1.1}$$

where the function $\varphi(x)$ and the interval in question are related to the problem at hand.

Remark 1.1.8. *A vital feature of (1.1) is that the increment $h\varphi(x)$ varies with x. For $\varphi(x) \equiv 1$, (1.1) is reduced to the classical modulus.*

The main tools to measure the degree of convergence of positive linear operators towards the identity operator are the *moduli of smoothness* of first and second order. For $f \in C[a,b]$ and $\delta \geq 0$ we have

$$\begin{aligned} \omega_1(f;\delta) &:= \sup\{|f(x+h) - f(x)| : x, x+h \in [a,b],\ 0 \leq h \leq \delta\}; \\ \omega_2(f;\delta) &:= \sup\{|f(x+h) - 2f(x) + f(x-h)| : x, x \pm h \in [a,b],\ 0 \leq h \leq \delta\}. \end{aligned}$$

Most of the error estimates in this work are given in terms of the two moduli of smoothness, the Ditzian-Totik second order modulus denoted by $\omega_\varphi^2(f,\cdot)$ and sometimes by $\omega_2^\varphi(f,\cdot)$ or in term of $\tilde{\omega}_1$.

ω_1 inherits its name from the first part of the following property:

Proposition 1.1.9. *Let $f \in C[a,b]$ and $\delta > 0$.*

a) If $\lim\limits_{\delta \to 0^+} \omega_1(f;\delta) = 0$, then f is continuous on $[a,b]$.

b) The following equivalence holds: $f \in Lip_\tau M$ iff $\omega_1(f;\delta) \leq M \cdot \delta^\tau$, where $0 < \tau \leq 1$ and $M > 0$.

A useful modification is represented by the *least concave majorant* of $\omega_1(f;\cdot)$ given by

$$\tilde{\omega}(f;\varepsilon) = \begin{cases} \sup\limits_{\substack{0\leq x\leq \varepsilon\leq y\leq b-a \\ x\neq y}} \frac{(\varepsilon-x)\omega(f,y)+(y-\varepsilon)\omega(f,x)}{y-x} & \text{for} \quad 0 \leq \varepsilon \leq b-a, \\ \tilde{\omega}(f,b-a) = \omega(f,b-a) & \text{if} \qquad \varepsilon > b-a. \end{cases} \tag{1.2}$$

The definition of $\tilde{\omega}(f,\cdot)$ shows that

$$\omega_1(f;\cdot) \leq \tilde{\omega}_1(f;\cdot) \leq 2 \cdot \omega_1(f;\cdot). \tag{1.3}$$

For some further properties of $\tilde{\omega}(f;\cdot)$ see, e.g., V.K. Dzjadyk [24, p. 153ff] or [35].

It was shown by N.P. Korneičuk [60, p. 670] that for any $\varepsilon \geq 0$ and $\xi > 0$ the function $\omega(f;\cdot)$ and its least concave majorant $\tilde{\omega}(f;\cdot)$ are related by the inequality

$$\tilde{\omega}(f;\xi \cdot \varepsilon) \leq (1+\xi) \cdot \omega(f;\varepsilon), \tag{1.4}$$

and that this inequality cannot be improved for each $\varepsilon > 0$ and $\xi = 1, 2 \ldots$.

However we also give estimates, where moduli of higher order are involved. Therefore we give the definition of ω_k, $k \in \mathbb{N}$, as given in 1981 by L. L. Schumaker in his book [90]:

Definition 1.1.10. For $k \in \mathbb{N}$, $\delta \in \mathbb{R}_+$ and $f \in C[a,b]$ the *modulus of smoothness* of order k is defined by

$$\omega_k(f;\delta) := \sup\{|\Delta_h^k f(x)| \mid 0 \leq h \leq \delta,\ x, x+kh \in [a,b]\}. \tag{1.5}$$

Remark 1.1.11. *For clarity sometimes we will write $\omega_k(f;\delta;[a,b])$.*
It is obvious that for $\delta \geq \frac{b-a}{k}$ one has $\omega_k(f;\delta) = \omega_k(f;\frac{b-a}{k})$.

We collect in the following proposition some useful properties of ω_k:

Property 1.1.12. *(see [95])*

1) $\omega_k(f;0) = 0$.

2) $\omega_k(f;\cdot)$ is a positive, continuous and non-decreasing function on $\mathbb{R}_+$.

3) $\omega_k(f;\cdot)$ is sub-addititive, i.e., $\omega_1(f;\delta_1+\delta_2) \leq \omega_1(f;\delta_1) + \omega_1(f;\delta_2)$, $\delta_i \geq 0$, $i = 1,2$.

4) $\forall \delta \geq 0$, $\omega_{k+1}(f;\delta) \leq 2\omega_k(f;\delta)$.

5) If $f \in C^1[a,b]$ then $\omega_{k+1}(f;\delta) \leq \delta \cdot \omega_k(f';\delta)$, $\delta \geq 0$.

6) *If* $f \in C^r[a,b]$ *then* $\omega_r(f;\delta) \le \delta^r \sup_{\delta\in[a,b]} |f^{(r)}(\delta)|$.

7) $\forall \delta > 0$ *and* $n \in \mathbb{N}$, $\omega_k(f;n\delta) \le n^k \omega_k(f;\delta)$.

8) $\forall \delta > 0$ *and* $r > 0$, $\omega_k(f;r\delta) \le (1+[r])^k \omega_k(f;\delta)$, *where* $[a]$ *is the integer part of* a.

9) *If* $\delta \ge 0$ *is fixed, then* $\omega_k(f;\cdot)$ *is a seminorm on* $C[a,b]$.

Corollary 1.1.13. *(see [95])*

1) $\forall \delta > 0, \omega_{k+r}(f;\delta) \le 2^r \omega_k(f;\delta)$, $k,r \in \mathbb{N}$.

2) $\forall 0 < \delta \le 1$, $\omega_{k+1}(f;\delta^k) \le \omega_k(f;\delta)$.

1.1.3 Žuk's function and its applications

Some of the estimates in terms of different moduli of smoothness can be elegantly proven by using as an intermediate a special smoothing function that was constructed by V. Žuk in [103]. Therefore we find it instructive to present here its definition and its relevant properties, see also [38].

Žuk's approach was the following: For $f \in C[a,b]$ he first defined the extension $f_h : [a-h, b+h] \to \mathbb{R}$, with $h > 0$, by

$$f_h(x) := \begin{cases} P_-(x), & a-h \le x \le a, \\ f(x), & a \le x \le b, \\ P_+(x), & b < x \le b+h, \end{cases}$$

where $P_-, P_+ \in \prod_1$ are *the best approximants* to f on the indicated intervals.

Then Žuk defined its function $Z_h f(\cdot)$ (sometimes also denoted by $f_{2,h}(\cdot)$) using the *second order Steklov means*

$$Z_h f(x) := \frac{1}{h} \cdot \int_{-h}^{h} \left(1 - \frac{|t|}{h}\right) f_h(x+t)dt, \; x \in [a,b].$$

It can be shown that $Z_h f \in W_{2,\infty}[a,b]$.

The following estimates were proven in [103, Lemma 1] (or [38, Lemma 2.1])

Lemma 1.1.14. *Let* $f \in C[a,b]$, $0 < h \le \frac{1}{2}(b-a)$. *Then*

$$\begin{aligned} \|f - Z_h f\|_\infty &\le \frac{3}{4} \cdot \omega_2(f;h), \\ \|(Z_h f)''\|_{L_\infty} &\le \frac{3}{2} \cdot h^{-2} \cdot \omega_2(f;h). \end{aligned}$$

Supplementary estimates for lower order derivatives of $Z_h f$ are given in

Lemma 1.1.15. (see [38, Lemma 2.4]) *Let* f, h *and* $Z_h f$ *be given as in Lemma 1.1.14. Then*

$$\begin{aligned} \|(Z_h f)'\|_\infty &\le \frac{1}{h} \cdot \left[2 \cdot \omega_1(f;h) + \frac{3}{2} \cdot \omega_2(f;h)\right], \\ \|Z_h f\|_\infty &\le \|f\|_\infty + \frac{3}{4} \cdot \omega_2(f;h). \end{aligned}$$

Corollary 1.1.16. *As an immediate consequence of the latter lemma, one has the simpler inequalities*

$$\|(Z_h f)'\|_\infty \le \frac{5}{h} \cdot \omega_1(f;h), \text{ and } \|Z_h f\|_\infty \le 4 \cdot \|f\|_\infty.$$

As an application of the upper inequalities the authors proved in [38] the following:

Lemma 1.1.17. (see [38, Lemma 4.1]) *Let $g \in W_{2,\infty}$ and the polynomial $B_n g$, where B_n is the Bernstein operator defined on $[a,b]$ (see Section 1.2 for details). Then for any $\varepsilon > 0$ and a sufficiently large n the following inequalities hold:*

$$\|g - B_n g\|_\infty < \varepsilon, \ \|B_n g\|_\infty \le \|g\|_\infty, \ \|(B_n g)'\|_\infty \le \|g'\|_\infty,$$

and

$$\|(B_n g)''\|_\infty \le \|g''\|_{L_\infty}.$$

In other words, the latter lemma affirms that functions in $W_{2,\infty}[a,b]$ can be approximated well by functions in $C^2[a,b]$, while "retaining important differential characteristics", see [38].

Supplementary results on "smoothing of functions by smoother ones" can be found in [34, Lemma 3.1]. Having further applications in mind, we shall present this assertion below:

Lemma 1.1.18. *Let $I = [0,1]$ and $f \in C^r(I), r \in \mathbb{N}_0$. For any $h \in (0,1]$ and $s \in \mathbb{N}$ there exists a function $f_{h,r+s} \in C^{2r+s}(I)$ with*

(i) $\|f^{(j)} - f^{(j)}_{h,r+s}\|_\infty \le c \cdot \omega_{r+s}(f^{(j)};h)$ *for* $0 \le j \le r$,

(ii) $\|f^{(j)}_{h,r+s}\|_\infty \le c \cdot h^{-j} \cdot \omega_j(f;h)$, *for* $0 \le j \le r+s$,

(iii) $\|f^{(j)}_{h,r+s}\|_\infty \le c \cdot h^{-(r+s)} \cdot \omega_{r+s}(f^{(j-r-s)};h)$, *for* $r+s \le j \le 2r+s$.

Here the constant c depends only on r and s.

Next we present a partial generalization of a theorem of Brudnyĭ which will be used as means to prove some further results.

Theorem 1.1.19. (see [38, Theorem 4.2]) *Let $(B, ||\cdot||_B)$ be a Banach space, and let $H : C[a,b] \to (B, ||\cdot||_B)$ be an operator, where*

(i) $||H(f+g)||_B \le \gamma\{||Hf||_B + ||Hg||_B\}$ *for all* $f, g \in C[a,b]$,

(ii) $||Hf||_B \le \alpha ||f||_C$ *for all* $f \in C[a,b]$,

(iii) $||Hg||_B \le \beta_0 ||g||_C + \beta_1 ||g'||_C + \beta_2 ||g''||_C$ *for all* $g \in C^2[a,b]$.

Then for all $f \in C[a,b], 0 < h \le (b-a)/2$ the following inequality holds:

$$||Hf||_B \le \gamma \left\{ \beta_0 ||f|| + \frac{2\beta_1}{h}\omega_1(f;h) + \frac{3}{4}\left(\alpha + \beta_0 + \frac{2\beta_1}{h} + \frac{2\beta_2}{h}\right)\omega_2(f;h) \right\}.$$

Corollary 1.1.20. (see [38, Corollary 4.3]) *In many cases one has $\gamma = 1$ and $\beta_0 = \beta_1 = 0$, so that the inequality from Theorem 1.1.19 simplifies to*

$$||Hf||_B \le \left(\frac{3\alpha}{4} + \frac{3\beta_2}{2h^2}\right)\omega_2(f;h).$$

1.1.4 K-functionals and their relationship to the moduli

In 1968 J. Peetre introduced in [77] a functional, nowadays called *Peetre's K-functional*, for investigation of interpolation spaces between two Banach spaces. As predicted by Peetre this became another important instrument to measure the smoothness of a function in terms of how well it can be approximated by smoother functions.

It is possible to define the K-functional in a very general context as is presented in [23]. This can be used in applications and in particular for polynomials of best approximation.

Definition 1.1.21. For a positive integer r the $K-$functional of the pair of spaces $L_p(a,b), 1 \leq p \leq \infty$, and a corresponding weighted Sobolev space with the weight function φ^r is given by

$$K_{r,\varphi}(f,t^r)_p = \inf_{g^{(r-1)}\in A.C.} \{||f-g||_p + t^r||\varphi^r g^{(r)}||_p\}.$$

where $g^{(r-1)} \in A.C.$ means that g is $r-1$ differentiable and $g^{(r-1)}$ is absolutely continuous in every closed finite interval $[c,d]$ such that $[c,d] \subset (a,b)$.

Sometimes the notation K_r^φ is used.

For various $K-$functionals probably the most important problem is that of characterizing their behavior using structural properties of the functions. Here the characterization will be done using the modulus of smoothness $\omega_\varphi^r(f,t)_p$. To that end the following equivalence theorem is given

Theorem 1.1.22. *(see [23, p.11]) Suppose r is a positive integer, $f \in L_p(0,1), 1 \leq p \leq \infty$ and $\varphi(x) = \sqrt{x(1-x)}$. Then*

$$M^{-1}\omega_\varphi^r(f,t)_p \leq K_{r,\varphi}(f,t^r)_p \leq M\omega_\varphi^r(f,t)_p,\ 0 < t \leq t_0$$

for some constants $M > 0$ and t_0.

Remark 1.1.23. *This result is also valid if $C[0,1]$ replaces $L_\infty(0,1)$.*

For the applications we have in mind in this general context, it suffices to consider the case $r = 2$.

The classical definition of the K-functional is given below.

Definition 1.1.24. For any $f \in C[a,b]$, $\delta \geq 0$ and integer $s \geq 1$ we call

$$\begin{aligned} K_s(f;\delta)_{[a,b]} &:= K(f;\delta;C[a,b],C^s[a,b]) \qquad (1.6)\\ &:= \inf\{\|f-g\|_\infty + \delta\cdot\|g^{(s)}\|_\infty : g \in C^s[a,b]\}, \end{aligned}$$

Peetre's K-functional of order s.

Whenever there is no doubt about the interval of definition of f we shall use for $K_s(f;\delta)_{[a,b]}$ the abbreviation $K_s(f;\delta)$.

It is clear that the quantity in (1.6) reflects some approximation properties of f: the inequality $K_s(f;\delta) < \varepsilon,\ \delta > 0$ implies that f can be approximated with error $\|f-g\|_\infty < \varepsilon$ in $C[a,b]$ by an element $g \in C^s[a,b]$, whose norm is not to large, $\|g^{(s)}\|_\infty < \frac{\varepsilon}{\delta}$.

The following lemma collects some of the properties of $K_s(f;\cdot)$. They were proven by P.L. Butzer & H. Berens [14], but they can also be found in more recent work on approximation theory as in: [90], [21] and [35].

Lemma 1.1.25. (see Proposition 3.2.3 in [14]) *Let $K_s(f;\cdot)$ be defined as in (1.6).*

1) *The mapping $K_s(f;\delta):\mathbb{R}_+\to\mathbb{R}_+$ is continuous especially at $\delta=0$, i.e.,*

$$\lim_{\delta\to 0^+} K_s(f;\delta)=0=K_s(f;0).$$

2) *For each fixed $f\in C[a,b]$ the application $K_s(f;\cdot):\mathbb{R}_+\to\mathbb{R}_+$ is monotonically increasing and concave function.*

3) *For arbitrary $\lambda,\delta\geq 0$, and fixed $f\in C[a,b]$, one has the inequality*

$$K_s(f;\lambda\cdot\delta)\leq\max\{1,\lambda\}\cdot K_s(f;\delta).$$

4) *For arbitrary $f_1,\ f_2\in C[a,b]$ we have $K_s(f_1+f_2;\delta)\leq K_s(f_1;\delta)+K_s(f_2;\delta),\ \delta\geq 0$.*

5) *For each $\delta\geq 0$ fixed, $K_s(\cdot;\delta)$ is a seminorm on $C[a,b]$, such that*

$$K_s(f;\delta)\leq\|f\|_\infty,$$

for all $f\in C[a,b]$.

The following theorem establishes the close relationship between the K-functional and the moduli of smoothness. K_s and ω_s are related by the following *equivalence relation*, see H. Johnen [54]:

Theorem 1.1.26. *There exist constants C_1 and C_2, depending only on s and $[a,b]$ such that*

$$C_1\cdot\omega_s(f;\delta)\leq K_s(f;\delta^s)\leq C_2\cdot\omega_s(f;\delta), \tag{1.7}$$

for all $f\in C[a,b]$ and $\delta>0$.

In general there are no sharp constants known in the above (double) inequality. However, there are two exceptional cases for $s=1,2$. We present them below.

The following lemma known as *Brudnyĭ's representation theorem* establishes the connection between $K_1(f;\delta)_{[a,b]}$ and the least concave majorant defined at (1.2).

Lemma 1.1.27. *Every function $f\in C[a,b]$ satisfies the equality*

$$K_1\left(f,\delta;C[a,b],C^1[a,b]\right)=\frac{1}{2}\cdot\tilde{\omega}_1(f;2\delta),\ \delta\geq 0. \tag{1.8}$$

More details and also proofs of the above lemma can be found in many different sources, as for example: in the article of B. S. Mitjagin & E. M. Semenov [70], or in the book by R. T. Rockafellar [87], or in the monograph of R. A. DeVore & G. G. Lorentz [21, p. 175] and more recently in a paper of R. Păltănea [76].

Also for the case $s=2$ there is something known about the constants in front of the moduli of smoothness. Thus, H. Gonska proved in [32, p. 31] the following

Lemma 1.1.28. *Let $f\in C[a,b]$ and $0\leq\delta$. Then we have*

$$\frac{1}{4}\cdot\omega_2(f;\delta)\leq K_2\left(f,\frac{\delta^2}{2};C[a,b],C^2[a,b]\right)\ \text{and}$$

$$K_2(f,\delta^2;C[a,b],C^2[a,b])\leq\left(\frac{3}{2}+2\cdot\max\left\{1,\frac{\delta^2}{(b-a)^2}\right\}\right)\cdot\omega_2(f;\delta).$$

In another context, but also very useful for our next applications is the following:

Lemma 1.1.29. *For any $f \in C[a,b]$ and $\delta \geq 0$ the following identity holds,*

$$K(f;\delta;C[a,b],C^2[a,b]) = K(f;\delta;C[a,b],W_{2,\infty}[a,b]), \tag{1.9}$$

where the K-functional on the right hand side can be defined in an analogous way to the other one.

Proof. It is trivial to see that $C^2[a,b] \subset W_{2,\infty}[a,b]$ implies $K(f;\delta;C[a,b],W_{2,\infty}[a,b]) \leq K(f;\delta;C[a,b],C^2[a,b])$. In order to prove the inverse inequality let $\varepsilon > 0$ be fixed and $g \in W_{2,\infty}[a,b]$. Obviously we have $B_n g \in C^2[a,b]$ and furthermore $\|(B_n g)''\|_\infty \leq \|g''\|_{L_\infty}$, see Lemma 1.1.17. Having this in mind, for a sufficiently large $n \in \mathbb{N}$ and $0 \leq \delta$ the following inequality holds:

$$\begin{aligned} K(f;\delta;C[a,b],C^2[a,b]) &\leq \|f - B_n g\|_\infty + \delta \cdot \|(B_n g)''\|_\infty \\ &\leq \|f-g\|_\infty + \|g - B_n g\|_\infty + \delta \cdot \|(B_n g)''\|_\infty \\ &\leq \|f-g\|_\infty + \varepsilon + \delta \cdot \|g''\|_{L_\infty}. \end{aligned}$$

This implies, by passing on the right hand side to the *infimum* for all functions in $W_{2,\infty}[a,b]$ that

$$K(f,\delta;C[a,b],C^2[a,b]) \leq K(f,\delta;C[a,b],W_{2,\infty}[a,b]) + \varepsilon,\ \varepsilon > 0.$$

But ε was arbitrarily chosen, so letting $\varepsilon \to 0$ we arrive at the desired inequality. □

1.1.5 The integral remainder of the Taylor expansion

The following lemma proved to be very useful in the proof of some converse inequalities and in establishing a strong Voronovskaya-type inequality for U_n^ϱ (to be defined later).

Lemma 1.1.30. *Let $R_2(f,u,x) = \int_x^u (u-v) f''(v)dv$ be the integral remainder of f in Taylor expansion. Then for $x,y \in [0,1]$ we have:*

$$|R_2(f,u,x)| \leq \frac{|u-x|}{\varphi^2(x)} \left| \int_u^x \varphi^2(v) f''(v) dv \right|. \tag{1.10}$$

Proof. Lemma 1.1.30 results from Lemma 9.6.1 ([23], p.140). □

1.1.6 Variation diminution

Shape preservation properties of an approximation method are considered to be of great importance in both Approximation Theory and Computer Aided Geometric Design. Among them, we discuss the variation diminution.
We refer to [29], which contains historical remarks clarifying the various meanings of "variation–diminishing" employed in the past.

Thus, the following scheme is valid:

$$GVDP \Rightarrow SVDP \Rightarrow WVDP$$

where

- Geometric Variation Diminishing Property (GVDP): is the verification criteria introduced by Schoenberg in [89, p. 267] - for an operator to have this property it must have the SVDP and also preserve linear functions;
- Strong Variation Diminishing Property (SVDP): see description below;
- Weak Variation Diminishing Property (WVDP): is a method which diminishes the total variation.

Let K be any interval on the real line, and let $f : K \to \mathbb{R}$ be an arbitrary function. For an ordered sequence $x_0 < x_1 < ... < x_n$ of points in K, let $S[f_k]$ denote the number of sign changes in the finite sequence of ordinates $f(x_k)$, where zeros are disregarded. The number of sign changes of f in the interval K is defined by

$$S_K[f] = \sup S[f(x_k)],$$

where the supremum is taken over all ordered finite sets $\{x_k\}$.
Let I and J be two intervals, let U be a subspace of $C(I)$, and suppose that $L : U \to C(J)$ is a linear operator reproducing constant functions.
The operator L is said to be (strongly) variation-diminishing (as an operator from U into $C(J)$) if

$$S_J[Lf] \leq S_I[f], \text{ for all } f \in U.$$

The main result presented in [29] (see Theorem 1 there), which represents a new approach in proving the SVDP, reads as follows.

Theorem 1.1.31. *Let $I = (a, b)$ or $I = (a, \infty)$ with $a \geq 0$, let $w : I \to \mathbb{R}_+$ be a strictly positive continuous weight function, and $[\alpha, \beta] \subset [0, \infty)$. Consider a linear and positive definite functional $A : C(I) \to \mathbb{R}$ having the following properties: there exists a subspace $C_w^{[\alpha,\beta]}(I) \subset C(I)$ such that for $f \in C_w^{[\alpha,\beta]}(I) \subset C(I)$ the function $Lf : (\alpha, \beta) \to \mathbb{R}$ given by $(Lf)(x) := A_t[t^x \cdot w(t) \cdot f(t)]$ is well–defined. If the function Lf has one-sided limits at the endpoints, then*

$$S_{[\alpha,\beta]}[Lf] \;\leq\; S_I[f], \qquad \forall f \in C_w^{[\alpha,\beta]}(I),$$

where, for $x \in \{\alpha, \beta\}$, one understands by $sgn(Lf)(x)$ the sign of the corresponding one-sided limit.

1.1.7 Remarks on (de)compositions of positive linear operators

In [36] a representation of the second moments of compositions of positive linear operators was given. In particular the following general situations were considered:

(i) products of more than two operators;

(ii) the assumption $Qe_i = e_i, i = 0, 1$ is dropped.

In the first case we have:

Theorem 1.1.32. *Suppose that k operators $P_i : C[a, b] \to C[a, b], 1 \leq i \leq k$, are given, satisfying $P_i e_1 = e_1, P_i e_0 = e_0$ for $2 \leq i \leq k$. Then*

$$\left(\prod_{i=1}^{k} P_i\right)((e_1 - x_1)^2; x_1) = \sum_{j=1}^{k} P_1(...P_j((e_1 - x_j)^2; x_j; ...); x_1).$$

In the above the variable x_j is related to the operator P_j in the sense that P_j yields functions of the variable x_j (such as $P_j((e_1 - x_j)^2; x_j)$).
The less convenient case in which some of the operators do not reproduce e_1 and e_0 was also considered. Thus it was shown

Lemma 1.1.33. *Let P, Q, R be positive, linear operators with $Qe_1 \neq e_1 \neq Re_1$. Then the second moment of the composition is given by:*

(i) for two operators

$$\begin{aligned}(P \circ Q))((e_1 - xe_0)^2; x) = P^u(Q((e_1 - ue_0)^2; u); x) - \\ -P((e_1 - xe_0)^2; x) + 2P((e_1 - xe_0)' \cdot (Qe_1 - x); x)\end{aligned} \tag{1.11}$$

(ii) for three operators

$$\begin{aligned}(P \circ (Q \circ R))((e_1 - xe_0)^2; x) = P(Q(R((e_1 - se_0)^2; s); u); x) - \\ -P(Q((e_1 - ue_0)^2; u); x) - P((e_1 - xe_0)^2; x) + \\ +2P(Q((e_1 - ue_0) \cdot (Re_1 - ue_0); u); x) + 2P((e_1 - xe_0) \cdot (QRe_1 - x); x)\end{aligned} \tag{1.12}$$

where for clarity a superscript such as in P^u indicates that the operator P is applied to functions in the variable u.

1.2 The Bernstein operators

Maybe the best-known and celebrated positive operators are the *Bernstein operators*, introduced by S. N. Bernstein [12] in 1912 in order to prove Weierstrass' fundamental theorem, see [101]. For any $f \in C[0,1]$, $n \in \mathbb{N}$ and $x \in [0,1]$, they are given by

$$B_n(f; x) := \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.13}$$

where the polynomials

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ 0 \leq k \leq n, \tag{1.14}$$

form the *Bernstein basis*. To be formally correct we set for $k < 0$ or $k > n$ that $p_{n,k} := 0$. It is not difficult to define the Bernstein operators on an arbitrary compact interval $[a, b]$, $a < b$. We shall come back many times to the properties of these operators and their generalizations. As references we will mainly use [21] and [95].

1.2.1 Basic properties

The operators $B_n, n \in \mathbb{N}$ defined by (1.13) have the following properties:

1. they are linear and positive;
2. $\sum_{k=0}^{n} p_{n,k}(x) = 1$;

3. $B_n(e_0;x) = 1, B_n(e_1;x) = x, B_n(e_2;x) = x^2 + \dfrac{x(1-x)}{n}$;
4. $||B_n|| = 1$;
5. $\lim\limits_{n\to\infty} B_n(f;x) = f(x)$ uniformly on $[0,1], (\forall) f \in C[0,1]$;
6. $|B_n(f;x) - f(x)| \le k\omega_f(\frac{1}{\sqrt{n}}), (\forall) x \in [0,1]$;
7. $|B_n(f;x) - f(x)| \le \dfrac{3}{4\sqrt{n}}\omega_{f'}(\frac{1}{\sqrt{n}}), (\forall) f \in C^1[0,1], (\forall) x \in [0,1]$.

Remark 1.2.1. *(i) The Bernstein operator interpolates at the end points.*

$$B_n(f,0) = p_{n,0}(0)f(0) = f(0),\ B_n(f,1) = p_{n,n}(1)f(1) = f(1);$$

(ii) The exact value of the constant in property 6. above was found by Sikkema in 1961, namely $k = (4306 + 837\sqrt{6})/5832 \approx 1,089$. Previous estimations were given in 1935 by T. Popoviciu who found $k \le \frac{3}{2}$, and in 1953 by G.G. Lorentz who found $k \le \frac{5}{4}$.

(iii) A finite set of real non-negative functions defined on the interval I, which have the sum equal to 1 is called a partition of unity on I. The set $\{p_{n,k} : k = 0, ..., n\}$ is a blending system of $[0,1]$ for all $n \in \mathbb{N}$.

(iv) In 1966 G. Călugăreanu (see [15]) showed that B_n has n eigenvalues all in the interval $(0,1)$, and they have the following representation:

$$\lambda_k = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot ... \cdot \left(1 - \frac{k-1}{n}\right), k = 1, ..., n.$$

The k-th eigenvalue has an infinity of corresponding eigenvectors given by polynomials of degree k.

1.2.2 Derivatives of Bernstein polynomials

With the usual notations the following relationships hold:

(i) $p'_{n,k}(x) = n(p_{n-1,k-1}(x) - p_{n-1,k}(x)) = \dfrac{k-nx}{x(1-x)}p_{n,k}(x), k = 0, ..., n$ for all $x \in (0,1)$. ($p_{0,0} := 1$ and $p_{s,-1} = p_{s,s+1} := 0, s \in \mathbb{N}_0$).

(ii) $B'_n(f;x) = n\sum\limits_{k=0}^{n-1} \Delta_{1/n} f(\frac{k}{n}) p_{n-1,k}(x) = \sum\limits_{k=0}^{n-1} p_{n-1,k}(x)\left[\dfrac{k}{n}, \dfrac{k+1}{n}; f\right]$.

(iii) $B_n^{(j)}(f;x) = n(n-1) \cdot ... \cdot (n-j+1) \sum\limits_{k=0}^{n-j} p_{n-j,k}(x) \Delta^j_{1/n} f(\frac{k}{n}), j \le n$.

With $B_n^{(j)} f$ we denoted the j-th derivative of the polynomial $B_n f$. In particular $B_n^{(j)}(f;0) = n(n-1) \cdot ... \cdot (n-j+1)\Delta^j_{1/n} f(0), j = 0, ..., n$. This gives the Taylor expansion:

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k_{1/n} f(0) x^k.$$

If f is a polynomial of degree n, then $\Delta^k_{1/n} f(0) = 0$ for $k > n$ and $\Delta^n_{1/n} f(0) \neq 0$. Therefore, the Bernstein polynomial of degree n is itself a polynomial of degree n. For further information on this topic see [21, p. 305-307] and [95, p. 300-303].

1.2.3 Approximation and shape preserving properties

We begin with the following simple but important fact:

Theorem 1.2.2. *(Voronovskaya [100]) If f is bounded on $[0,1]$, differentiable in some neighborhood of x, and has second derivative $f''(x)$ for some $x \in [0,1]$, then*

$$\lim_{n\to\infty} n[B_n(f,x) - f(x)] = \frac{x(1-x)}{2} f''(x). \tag{1.15}$$

If $f \in C^2[0,1]$ the convergence is uniform.

Remark 1.2.3. *(see [21, p.307]) Historically, this has been the first example of saturation: for certain operators, convergence cannot be too fast, even for very smooth functions. The theorem shows that $f(x) - B_n(f;x)$ is of order not better than $1/n$ if $f''(x) \neq 0$.*

Theorem 1.2.4. *For some constant $C > 0$, and for all $f \in C[0,1]$,*

$$|f(x) - B_n(f;x)| \leq C\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right), x \in [0,1]. \tag{1.16}$$

In particular, if $f \in C^1[0,1]$, then

$$|f(x) - B_n(f;x)| \leq C\sqrt{\frac{x(1-x)}{n}}\,\omega_2\left(f', \sqrt{\frac{x(1-x)}{n}}\right), x \in [0,1]. \tag{1.17}$$

Inequalities (1.16) and (1.17) show that the Bernstein polynomials have a slow rate of convergence. This is compensated for by their shape preserving properties.

Theorem 1.2.5. *(i) The polynomial $B_n f$ increases on $[0,1]$ if f is increasing on this interval;*

(ii) For $k = 1, 2, ...$, $B_n f$ is monotone of order k on $[0,1]$ if f has this property;

(iii) $Var B_n f \leq Var f$;

(iv) one has $Z_{(0,1)} B_n f \leq S_{(0,1)} f$ where the first term is the number of zeros of $B_n f$ on $(0,1)$ and the second term is the number of sign changes of f on $(0,1)$.

Proof. See ([21], p.309). □

Remark 1.2.6. *Taking into consideration the theorem above it is clear that the Bernstein operators have the strong variation diminution property as operators from C[0,1] into itself (see also [29, p. 97]).*

1.2.4 Bernstein polynomials for convex functions

We begin with a result established by Stancu in 1967 (see [93]), which is sometimes falsely attributed to B. Averbach (see e.g. [58, p.306]).

Theorem 1.2.7. *For $n = 1, 2, ...,$ we have*

$$B_n(f;x) - B_{n+1}(f;x) = \frac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f \right] p_{n-1,k}(x). \tag{1.18}$$

Proof. We have

$$\begin{aligned} B_{n+1}(f,x) &= f(0)(1-x)^{n+1} + f(1)x^{n+1} + \sum_{v=1}^{n} f\left(\tfrac{v}{n+1}\right) \tbinom{n+1}{v} x^v (1-x)^{n+1-v}, \\ B_n(f,x) &= B_n(f,x)[x + (1-x)] \\ &= f(0)(1-x)^{n+1} + f(1)x^{n+1} + \sum_{v=1}^{n} f\left(\tfrac{v-1}{n}\right) \tbinom{n}{v-1} x^v (1-x)^{n+1-v} \\ &\quad + \sum_{v=1}^{n} f\left(\tfrac{v}{n}\right) \tbinom{n}{v} x^v (1-x)^{n+1-v}. \end{aligned}$$

Subtracting and replacing $v-1$ by k in all sums, we obtain

$$\begin{aligned} B_n(f,x) - B_{n+1}(f,x) &= \sum_{k=0}^{n-1} x^{k+1}(1-x)^{n-k} \times \\ &\quad \left[\tbinom{n}{k} f\left(\tfrac{k}{n}\right) - \tbinom{n+1}{k+1} f\left(\tfrac{k+1}{n+1}\right) + \tbinom{n}{k+1} f\left(\tfrac{k+1}{n}\right) \right] \\ &= \tfrac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} \tbinom{n-1}{k} x^k (1-x)^{n-1-k} \times \\ &\quad \left[\tfrac{n^2(n+1)}{n-k} f\left(\tfrac{k}{n}\right) - \tfrac{n^2(n+1)^2}{(k+1)(n-k)} f\left(\tfrac{k+1}{n+1}\right) + \tfrac{n^2(n+1)}{k+1} f\left(\tfrac{k+1}{n}\right) \right] \\ &= \tfrac{x(1-x)}{n(n+1)} \sum_{k=0}^{n-1} p_{n-1,k}(x) \left\{ \left[\tfrac{k}{n}, \tfrac{k+1}{n+1}, \tfrac{k+1}{n}; f \right] \right\} \end{aligned}$$

□

If f is convex, then all the terms in the sum (1.18) are non-negative and we obtain:

Corollary 1.2.8. *(W.B. Temple 1954, O. Aramă 1957, I.J. Schoenberg 1959, see [99], [9], [88]) If f is convex on $[0,1]$, then for all $n \in \mathbb{N}$ and $x \in (0,1)$*

$$B_n(f;x) \geq B_{n+1}(f;x) \geq f(x). \tag{1.19}$$

The inequalities are strict if f is strictly convex on $[0,1]$.

1.2.5 Converse result

In 1994, H.B. Knoop and X.L. Zhou (see [59]) gave a lower estimate for the Bernstein operators. They proved that:

Theorem 1.2.9. *There exists an absolute constant $C > 0$ such that*

$$C^{-1}\omega_{\varphi}^2(f, n^{-\frac{1}{2}}) \leq ||f - B_n f|| \leq C\omega_{\varphi}^2(f, n^{-\frac{1}{2}})$$

holds, for all $f \in C[0,1]$.

A main tool in the proof is the following lemma which is of crucial importance in the proof of some results in Section 2.5:

Lemma 1.2.10. *Let $f \in C[0,1]$. Then*

$$\frac{1}{n}||\varphi^2 B_n'' f|| \leq C_0||B_n f - f|| \tag{1.20}$$

where $C_0 > 0$ is an absolute constant.

Proof. See the inequality (2.1) in ([59], p.317) □

1.3 The (Euler-Jacobi) Beta type operators

Along with the Bernstein operator presented in the previous section we shall also use as factor operators, in the next chapter, the (Euler-Jacobi) Beta-type operators $\mathcal{B}_r^{a,b}$ of various kinds which will be further discussed below.
The reason for the name is the fact that the operators contain both Euler's Beta function and Jacobi weights.

1.3.1 Definition of operators $\mathcal{B}_r^{a,b}$

Definition 1.3.1. For $f \in C[0,1]$, $r > 0$ and $x \in [0,1]$ we define

(i) in case $a = b = -1$:

$$\mathcal{B}_r^{-1,-1}(f;x) = \begin{cases} f(0), x = 0; \\ \dfrac{\int_0^1 t^{rx-1}(1-t)^{r-rx-1}f(t)dt}{B(rx, r-rx)}, 0 < x < 1; \\ f(1), x = 1. \end{cases}$$

(ii) in case $a = -1, b > -1$:

$$\mathcal{B}_r^{-1,b}(f;x) = \begin{cases} f(0), x = 0; \\ \dfrac{\int_0^1 t^{rx-1}(1-t)^{r-rx+b}f(t)dt}{B(rx, r-rx+b+1)}, 0 < x \leq 1. \end{cases}$$

(iii) in case $a > -1, b = -1$:

$$\mathcal{B}_r^{a,-1}(f;x) = \begin{cases} \dfrac{\int_0^1 t^{rx+a}(1-t)^{r-rx-1}f(t)dt}{B(rx+a+1, r-rx)}, 0 \leq x < 1; \\ f(1), x = 1. \end{cases}$$

(iv) in case $a, b > -1$:

$$\mathcal{B}_r^{a,b}(f;x) = \frac{\int_0^1 t^{rx+a}(1-t)^{r-rx+b} f(t)dt}{B(rx+a+1, r-rx+b+1)}, 0 \le x \le 1.$$

Remark 1.3.2. *We prefer to use the Jacobi notation $\alpha, \beta \ge -1$ for natural r and $a, b \ge -1$ for the real r.*

Remark 1.3.3. *When discussing this class of operators one must refer to the papers of Mühlbach [71] and Lupaş in [62] where the first special cases were considered.*
Case $a = b = -1$: This case can be traced back to a paper by Mühlbach [71] who used a real number $\frac{1}{\lambda} > 0$. The same case with a natural n instead of the r was investigated by Lupaş in [62], where the operator was denoted by $\overline{\mathbb{B}}_n$ (see [62, p.63]).
Case $a = b = 0$: These were called Beta operators by Lupaş (see [62, p.37]) and denoted by $\mathbb{B}_n$.

1.3.2 Basic properties

The operators $\mathcal{B}_r^{a,b}, r > 0$ as given in Definition 1.3.1 have the following properties:

1. they are linear and positive;

2. $\mathcal{B}_r^{a,b}(e_0;x) = 1, \mathcal{B}_r^{a,b}(e_1;x) = \dfrac{rx+a+1}{r+a+b+2}$,

   $\mathcal{B}_r^{a,b}(e_2;x) = \dfrac{(rx+a+1)(rx+a+2)}{(r+a+b+2)(r+a+b+3)}$;

3. $\lim\limits_{r\to\infty} \mathcal{B}_r^{a,b}(f;x) = f(x)$ uniformly on $[0,1], (\forall) f \in C[0,1]$;

For the case $a = b = -1$ we have the following additional properties:

4. the operator $\mathcal{B}_r^{-1,-1}$ reproduces linear functions;

5. it interpolates at the end points

$$\mathcal{B}_r^{-1,-1}(f,0) = f(0), \ \mathcal{B}_r^{-1,-1}(f,1) = f(1);$$

6. $||\mathcal{B}_r^{-1,-1}|| = 1$.

Lemma 1.3.4. *Let $f \in C[0,1]$ convex. If $s > r > 0$, then*

$$\mathcal{B}_r^{-1,-1}(f;x) \ge \mathcal{B}_s^{-1,-1}(f;x). \tag{1.21}$$

Remark 1.3.5. *Lemma 1.3.4 is a consequence of [5, Theorem 1] and it was proved using methods specific to probability theory.*

1.3.3 Moments and their recursion

We shall focus on the building blocks $\mathcal{B}_n^{\alpha,\beta}$ for n a natural number, $\alpha, \beta \geq -1$ and on their moments of all orders. As is well known, knowledge of their behavior is essential for asymptotic statements as, for example, Voronovskaya-type results.

Definition 1.3.6. Let $\alpha, \beta \geq -1, n > 1, m \in \mathbb{N}_0$ and $x \in [0,1]$, then the moment of order m is defined by

$$T_{n,m}^{\alpha,\beta}(x) = \mathcal{B}_n^{\alpha,\beta}((e_1 - xe_0)^m; x).$$

Theorem 1.3.7.

$$T_{n,0}^{\alpha,\beta}(x) = 1,\ T_{n,1}^{\alpha,\beta}(x) = \frac{\alpha + 1 - (\alpha+\beta+2)x}{n+\alpha+\beta+2} \tag{1.22}$$

and for $m \geq 1$ we have the following recursion formula

$$\begin{aligned}(n+m+\alpha+\beta+2)T_{n,m+1}^{\alpha,\beta}(x) &= mXT_{n,m-1}^{\alpha,\beta}(x) + \\ &+[m+\alpha+1-(2m+\alpha+\beta+2)x]T_{n,m}^{\alpha,\beta}(x)\end{aligned} \tag{1.23}$$

where $X = x(1-x)$.

Proof. Below we will repeatedly use the function $\psi(t) = t(1-t), t \in [0,1]$. Let $f \in C^1[0,1], \alpha, \beta \geq -1, 0 < x < 1$. Then

$$\mathcal{B}_n^{\alpha,\beta}(\psi f'; x) = \frac{\int\limits_0^1 t^{nx+\alpha}(1-t)^{n-nx+\beta} t(1-t) f'(t)dt}{B(nx+\alpha+1, n-nx+\beta+1)}.$$

Using integration by parts we obtain

$$\begin{aligned}\mathcal{B}_n^{\alpha,\beta}(\psi f'; x) &= \frac{1}{B(nx+\alpha+1, n-nx+\beta+1)}\Big[t^{nx+\alpha+1}(1-t)^{n-nx+\beta+1}f(t)\Big|_0^1 \\ &\quad - \int_0^1 f(t)[(nx+\alpha+1)t^{nx+\alpha}(1-t)^{n-nx+\beta+1} - \\ &\quad -(n-nx+\beta+1)t^{nx+\alpha+1}(1-t)^{n-nx+\beta}]dt\Big] \\ &= \frac{\int\limits_0^1 f(t)t^{nx+\alpha}(1-t)^{n-nx+\beta}[t(n-nx+\beta+1)-(1-t)(nx+\alpha+1)]dt}{B(nx+\alpha+1, n-nx+\beta+1)} \\ &= \frac{\int\limits_0^1 f(t)t^{nx+\alpha}(1-t)^{n-nx+\beta}[n(t-x)-(\alpha+1)+t(\alpha+\beta+2)]dt}{B(nx+\alpha+1, n-nx+\beta+1)}\end{aligned}$$

and taking into consideration the identity

$$\begin{aligned}&n(t-x)-(\alpha+1)+t(\alpha+\beta+2) = \\ &= ((e_1 - xe_0)(n+\alpha+\beta+2) + [x(\alpha+\beta+2)-(\alpha+1)]e_0)(t)\end{aligned}$$

we can now write

$$\mathcal{B}_n^{\alpha,\beta}(\psi f';x) = \mathcal{B}_n^{\alpha,\beta}([(e_1 - xe_0)(n+\alpha+\beta+2) + (x(\alpha+\beta+2) - (\alpha+1))e_0]f;x). \tag{1.24}$$

In (1.24) we choose $f = (e_1 - xe_0)^m$ and use the fact that $t(1-t) = (X + X'(e_1 - xe_0) - (e_1 - xe_0)^2)(t)$:

$$m\mathcal{B}_n^{\alpha,\beta}([X(e_1-xe_0)^{m-1} + X'(e_1-xe_0)^m - (e_1-xe_0)^{m+1}];x) =$$
$$\mathcal{B}_n^{\alpha,\beta}([(n+\alpha+\beta+2)(e_1-xe_0)^{m+1} - (\alpha+1-(\alpha+\beta+2)x)(e_1-xe_0)^m];x).$$

The equality above becomes successively:

$$mXT_{n,m-1}^{\alpha,\beta}(x) + mX'T_{n,m}^{\alpha,\beta}(x) - mT_{n,m+1}^{\alpha,\beta}(x) = (n+\alpha+\beta+2)T_{n,m+1}^{\alpha,\beta}(x) -$$
$$-[\alpha+1-(\alpha+\beta+2)x]T_{n,m}^{\alpha,\beta}(x);$$

$$(m+n+\alpha+\beta+2)T_{n,m+1}^{\alpha,\beta}(x) = mXT_{n,m-1}^{\alpha,\beta}(x) +$$
$$+[m+\alpha+1-(\alpha+\beta+2+2m)x]T_{n,m}^{\alpha,\beta}(x).$$

So (1.23) is established for $0 < x < 1$. Due to the continuity, it is valid also for $x \in \{0,1\}$. □

In particular we have:

Corollary 1.3.8. *For $\alpha = \beta = 0$ we have $\mathcal{B}_n^{0,0} = \mathbb{B}_n$ (Lupaş notation) with the corresponding recurrence formula for the moments:*

$$(n+m+2)T_{n,m+1}^{0,0}(x) = mXT_{n,m-1}^{0,0}(x) + (m+1)X'T_{n,m}^{0,0}(x)$$

where $T_{n,0}^{0,0}(x) = 1$, $T_{n,1}^{0,0}(x) = \dfrac{X'}{n+2}$.
For $\alpha = \beta = -1$ we have $\mathcal{B}_n^{-1,-1} = \overline{\mathbb{B}}_n$ (Lupaş notation). Then the recurrence formula becomes

$$(n+m)T_{n,m+1}^{-1,-1}(x) = mXT_{n,m-1}^{-1,-1}(x) + mX'T_{n,m}^{-1,-1}(x)$$

where $T_{n,0}^{-1,-1}(x) = 1$, $T_{n,1}^{-1,-1}(x) = 0$.

The next proposition contains another kind of recurrence formula for the moments.

Proposition 1.3.9. *Let $i \geq 0$ and $j \geq 0$ be integers. Then*

$$T_{n,m}^{\alpha+i,\beta+j}(x) = \frac{(n+\alpha+\beta+2)^{\overline{i+j}}}{(nx+\alpha+1)^{\overline{i}}(nx+\beta+1)^{\overline{j}}} \sum_{k=0}^{i+j} \frac{[x^i(1-x)^j]^{(k)}}{k!} T_{n,m+k}^{\alpha,\beta}(x). \tag{1.25}$$

Proof. Using the definition of the Beta operator it is easy to show that

$$\mathcal{B}_n^{\alpha,\beta}(t^i(1-t)^j f(t);x) = \frac{(nx+\alpha+1)^{\overline{i}}(nx+\beta+1)^{\overline{j}}}{(n+\alpha+\beta+2)^{\overline{i+j}}} \mathcal{B}_n^{\alpha+i,\beta+j}(f(t);x). \tag{1.26}$$

The following equation

$$t^i(1-t)^j = \sum_{k=0}^{i+j} \frac{[x^i(1-x)^j]^{(k)}}{k!}(t-x)^k \tag{1.27}$$

is a consequence of Taylor's formula. Next using (1.27) and the fact that the Beta operator is linear we get

$$\mathcal{B}_n^{\alpha,\beta}(t^i(1-t)^j f(t);x) = \sum_{k=0}^{i+j} \frac{[x^i(1-x)^j]^{(k)}}{k!}\mathcal{B}_n^{\alpha,\beta}((t-x)^k f(t);x). \tag{1.28}$$

Combining (1.26) and (1.28) we arrive at

$$\begin{aligned}\mathcal{B}_n^{\alpha+i,\beta+j}(f(t);x) &= \frac{(n+\alpha+\beta+2)^{\overline{i+j}}}{(nx+\alpha+1)^{\overline{i}}(nx+\beta+1)^{\overline{j}}}\times \\ &\times \sum_{k=0}^{i+j} \frac{[x^i(1-x)^j]^{(k)}}{k!}\mathcal{B}_n^{\alpha,\beta}((t-x)^k f(t);x).\end{aligned}$$

For $f(t) = (t-x)^m$ we obtain (1.25). □

Remark 1.3.10. *Another recurrence formula for the moments of* $\mathcal{B}_n^{-1,-1}$ *can be found in [71, Satz 3].*

1.3.4 The moments of order two

Since the second moment controls to a certain extent the approximation properties of $\mathcal{B}_n^{\alpha,\beta}$, it is useful to have a closer look at it. From Theorem 1.3.7 we obtain

$$\begin{aligned}T_{n,2}^{\alpha,\beta}(x) = \frac{(\alpha+1)(\alpha+2)+(n-2(\alpha+1)(\alpha+\beta+3))x}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} + \\ + \frac{(-n+6+(\alpha+\beta)(\alpha+\beta+5))x^2}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)}.\end{aligned} \tag{1.29}$$

(I). First, let us remark that

$$\lim_{\alpha\to\infty} T_{n,2}^{\alpha,\beta}(x) = (1-x)^2, \text{ uniformly on } [0,1],$$

and

$$\lim_{\beta\to\infty} T_{n,2}^{\alpha,\beta}(x) = x^2, \text{ uniformly on } [0,1]. \tag{1.30}$$

Roughly speaking, a large value of α (with a fixed β) suggests a better approximation near 1, and we draw a similar conclusion from (1.30).

(II). Now let $\beta = \alpha \geq -1$. Consider the sequence $s_n := \frac{\sqrt{4n+1}-5}{4}, n \geq 1$. In this case,

$$T_{n,2}^{\alpha,\alpha}(x) = \frac{(\alpha+1)(\alpha+2)-(-n+6+2\alpha(2\alpha+5))x(1-x)}{(n+2\alpha+2)(n+2\alpha+3)}.$$

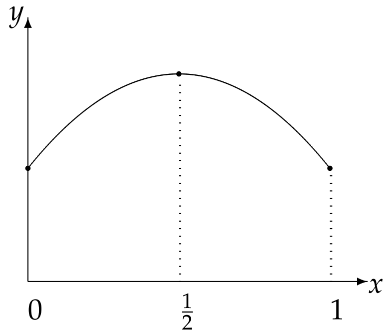
Therefore,

$$T_{n,2}^{\alpha,\alpha}(0) = T_{n,2}^{\alpha,\alpha}(1) = \frac{(\alpha+1)(\alpha+2)}{(n+2\alpha+2)(n+2\alpha+3)},$$

and

$$T_{n,2}^{\alpha,\alpha}\left(\frac{1}{2}\right) = \frac{1}{4(n+2\alpha+3)}.$$

(i) If $-1 \leq \alpha < s_n$, the graph of $T_{n,2}^{\alpha,\alpha}$ has the following form:

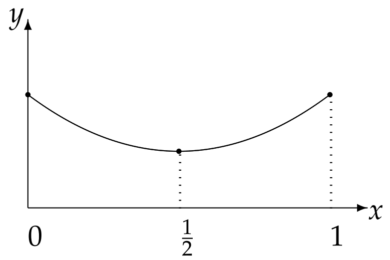


This suggests a better approximation near the end points.

(ii) If $\alpha = s_n$, $T_{n,2}^{\alpha,\alpha}$ is a constant function, namely

$$T_{n,2}^{s_n,s_n}(x) = \left(\frac{\sqrt{4n+1}-1}{4n}\right)^2, x \in [0,1].$$

(iii) For $\alpha > s_n$, the graph looks like



and indicates a better approximation near $\frac{1}{2}$.

(iv) In the extreme cases, when $\alpha = -1$, respectively $\alpha \to \infty$, we have $T_{n,2}^{-1,-1}(x) = \dfrac{x(1-x)}{n+1}$, respectively $\lim\limits_{\alpha\to\infty} T_{n,2}^{\alpha,\alpha}(x) = \left(\dfrac{1-2x}{2}\right)^2$.

1.3.5 Asymptotic formulae

Here we present first two asymptotic formulae for higher order moments of $\mathcal{B}_n^{\alpha,\beta}$ in order to arrive at Voronovskaya-type results.

Theorem 1.3.11. *For $\alpha, \beta \geq -1$ and all $l \geq 1$ one has*

$$(P_l): \begin{cases} \lim\limits_{n\to\infty} n^l T_{n,2l}^{\alpha,\beta}(x) = (2l-1)!!X^l, \\ \lim\limits_{n\to\infty} n^l T_{n,2l-1}^{\alpha,\beta}(x) = X^{l-1}\left[(l-1)!2^{l-1}X' \sum\limits_{k=1}^{l-1} \dfrac{(2k-1)!!}{(2k-2)!!} + \right. \\ \left. +(2l-1)!!(\alpha+1-(\alpha+\beta+2)x)\right]. \end{cases} \tag{1.31}$$

The convergence is uniform on $[0,1]$.

Proof. We shall prove the proposition by induction on $l \geq 1$. $T_{n,1}^{\alpha,\beta}$ and $T_{n,2}^{\alpha,\beta}$ are given by (1.22), respectively (1.29), and it is easy to prove that (P_1) is true. Suppose that (P_l) is true. According to (1.23) and (1.31),

$$\begin{aligned}
&\lim_{n\to\infty} n^{l+1} T_{n,2l+1}^{\alpha,\beta}(x) = \lim_{n\to\infty} n^{l+1} \frac{2lX}{n+2l+\alpha+\beta+2} T_{n,2l-1}^{\alpha,\beta}(x) + \\
&+ \lim_{n\to\infty} n^{l+1} \frac{2l+\alpha+1-(4l+\alpha+\beta+2)x}{n+2l+\alpha+\beta+2} T_{n,2l}^{\alpha,\beta}(x) \\
&= 2lX^l \left[(l-1)! 2^{l-1} X' \sum_{k=1}^{l-1} \frac{(2k-1)!!}{(2k-2)!!} + (2l-1)!!(\alpha+1-(\alpha+\beta+2)x) \right] + \\
&+ [2l+\alpha+1-(4l+\alpha+\beta+2)x](2l-1)!!X^l \\
&= X^l [2^l l! X' \sum_{k=1}^{l-1} \frac{(2k-1)!!}{(2k-2)!!} \\
&+ (2l-1)!!(2l(\alpha+1) - 2l(\alpha+\beta+2)x + 2l + \alpha + 1 - (4l+\alpha+\beta+2)x)] \\
&= X^l [2^l l! X' \sum_{k=1}^{l} \frac{(2k-1)!!}{(2k-2)!!} - (2l)!!X' \frac{(2l-1)!!}{(2l-2)!!} + \\
&+ (2l-1)!!((2l+1)(\alpha+1-(\alpha+\beta+2)x) + 2l - 4lx] \\
&= X^l \left[2^l l! X' \sum_{k=1}^{l} \frac{(2k-1)!!}{(2k-2)!!} + (2l+1)!!(\alpha+1-(\alpha+\beta+2)x \right]
\end{aligned}$$

and this proves the first formula in (1.31) for $l+1$ instead of l. Similarly,

$$\begin{aligned}
&\lim_{n\to\infty} n^{l+1} T_{n,2l+2}^{\alpha,\beta}(x) = \lim_{n\to\infty} n^{l+1} \frac{(2l+1)X}{n+2l+1+\alpha+\beta+2} T_{n,2l}^{\alpha,\beta}(x) + \\
&+ \lim_{n\to\infty} n^{l+1} \frac{2l+1+\alpha+1-(4l+2+\alpha+\beta+2)x}{n+2l+1+\alpha+\beta+2} T_{n,2l+1}^{\alpha,\beta}(x) \\
&= (2l+1)X(2l-1)!!X^l = (2l+1)!!X^{l+1},
\end{aligned}$$

which is the second formula in (1.31) for $l+1$ instead of l. This concludes the proof by induction. □

The following result of Sikkema (see [92, p. 241]) will be used below. Note also the 1962 result of Mamedov [66] dealing with a similar problem.

Theorem 1.3.12. *Let* $L_n : B[a,b] \to C[c,d], [c,d] \subseteq [a,b]$, *be a sequence of positive linear operators. Let the function* $f \in B[a,b]$ *be* $q-$*times differentiable at* $x \in [c,d]$, *where* $q \geq 2$ *is a natural number. Let* $\varphi : \mathbb{N} \to \mathbb{R}$ *be a function such that*

(i) $\lim_{n\to\infty} \varphi(n) = \infty$,

(ii) $L_n((e_1-x)^q;x) = \dfrac{c_q(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right)$, $n \to \infty$, *where* $c_q(x)$ *does not depend on* n,

(iii) there exists an even number $m > q$ *such that* $L_n((e_1-x)^m;x) = o\left(\frac{1}{\varphi(n)}\right)$, $n \to \infty$.

Then

$$\lim_{n\to\infty} \varphi(n) \left\{ L_n(f;x) - \sum_{r=0}^{q} \frac{L_n((e_1-x)^r;x)}{r!} f^{(r)}(x) \right\} = 0.$$

Corollary 1.3.13. *(i) Theorem 1.3.12 can be rewritten in the form*

$$\lim_{n\to\infty} \varphi(n) \left\{ L_n(f;x) - \sum_{r=0}^{q-1} \frac{L_n((e_1-x)^r;x)}{r!} f^{(r)}(x) \right\} = c_q(x) \frac{f^{(q)}(x)}{q!}.$$

(ii) If in addition to the assumption of Theorem 1.3.12, one assumes that

$$L_n((e_1-x)^r;x) = \frac{c_r(x)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right),\ n\to\infty, r=1,2,...,q,$$

where the functions c_r are independent of n, then one also has

$$\lim_{n\to\infty} \varphi(n) \{L_n(f;x) - f(x)L_n(e_0;x)\} = \sum_{r=1}^{q} c_r(x) \frac{f^{(r)}(x)}{q!}.$$

That is, all derivatives now appear on the right hand side which is independent of n.

As a consequence of Corollary 1.3.13 (ii) we have the following Voronovskaya-type relation.

Corollary 1.3.14. *Let $f \in C^2[0,1]$. Then*

$$\lim_{n\to\infty} n \left\{ \mathcal{B}_n^{\alpha,\beta}(f;x) - f(x) \right\} = \frac{x(1-x)}{2} f''(x) + [\alpha+1-(\alpha+\beta+2)x]f'(x),$$

uniformly on $[0,1]$.

Proof. For $\varphi(n) = n$ and $q = 2$ as given in Corollary 1.3.13 (ii),

$$\lim_{n\to\infty} n \left\{ \mathcal{B}_n^{\alpha,\beta}(f;x) - f(x) \right\} = \sum_{r=1}^{2} c_r(x) \frac{f^{(r)}(x)}{r!} = c_1(x)\frac{f'(x)}{1!} + c_2(x)\frac{f''(x)}{2!}$$

where $c_r(x) = \lim_{n\to\infty} nT_{n,r}^{\alpha,\beta}(x)$. By using Lemma 1.3.11 with $l = 1$ we get

$$\begin{aligned} c_1(x) &= \alpha+1-(\alpha+\beta+2)x \\ c_2(x) &= X, \end{aligned}$$

and this concludes the proof. □

Remark 1.3.15. *As a consequence of Lemma 1.3.11 and Corollary 1.3.13 (i) we deduce similarly that for $f \in C^{2l}[0,1]$,*

$$\lim_{n\to\infty} n^l \left\{ \mathcal{B}_n^{\alpha,\beta}(f(t);x) - \sum_{k=0}^{2l-1} \frac{f^{(k)}(x)}{k!} T_{n,k}^{\alpha,\beta}(x) \right\} = \frac{(2l-1)!!}{(2l)!} X^l f^{(2l)}(x), l \geq 1. \quad (1.32)$$

From this we get also

$$\begin{aligned} \lim_{n\to\infty} n^l \left\{ \mathcal{B}_n^{\alpha,\beta}(f(t);x) - \sum_{k=0}^{2l-2} \frac{f^{(k)}(x)}{k!} T_{n,k}^{\alpha,\beta}(x) \right\} &= \frac{(2l-1)!!}{(2l)!} X^l f^{(2l)}(x) + \\ + \frac{X^{l-1}}{(2l-1)!} \left[(l-1)!2^{l-1}X' \sum_{k=1}^{l-1} \frac{(2k-1)!!}{(2k-2)!!} + \right. & \qquad (1.33) \\ \left. +(2l-1)!!(\alpha+1-(\alpha+\beta+2)x) \right] f^{(2l-1)}(x). \end{aligned}$$

Remark 1.3.16. *In order to compare the result above with a special previous result for the case $\alpha = \beta = -1$ we manipulate the left hand side of (1.33) for $l = 2$ by writing*

$$\lim_{n\to\infty} n[n(\mathcal{B}_n^{\alpha,\beta}(f(t);x) - f(x)) - (\alpha+1-(\alpha+\beta+2)x)f'(x) - \frac{X}{2}f''(x)]$$
$$= \lim_{n\to\infty} n^2\left[(\mathcal{B}_n^{\alpha,\beta}(f(t);x) - f(x) - T_{n,1}^{\alpha,\beta}(x)f'(x) - T_{n,2}^{\alpha,\beta}(x)\frac{f''(x)}{2}\right] +$$
$$+ \lim_{n\to\infty} n[nT_{n,1}^{\alpha,\beta}(x) - (\alpha+1-(\alpha+\beta+2)x)]f'(x) +$$
$$+\frac{1}{2}\lim_{n\to\infty} n[nT_{n,2}^{\alpha,\beta}(x) - X]f''(x).$$

By using (1.22), (1.31) and (1.33) with $l = 2$, we get

$$\lim_{n\to\infty} n[n(\mathcal{B}_n^{\alpha,\beta}(f(t);x) - f(x)) - \frac{X}{2}f''(x) - (\alpha+1-(\alpha+\beta+2)x)f'(x)]$$
$$= \frac{1}{8}X^2 f^{IV}(x) + \frac{1}{6}X(3\alpha+5-(3\alpha+3\beta+10)x)f'''(x) -$$
$$-(\alpha+\beta+2)(\alpha+1-(\alpha+\beta+2)x)f'(x) + \frac{1}{2}f''(x)[(\alpha+1)(\alpha+2) -$$
$$-(2\alpha^2+2\alpha\beta+10\alpha+4\beta+11)x + x^2((\alpha+\beta)(\alpha+\beta+7)+11)].$$

For $\alpha = \beta = -1$, this reduces to

$$\lim_{n\to\infty} n[n(\mathcal{B}_n^{-1,-1}(f;x) - f(x)) - \frac{X}{2}f''(x)] =$$
$$= \frac{1}{24}\left(3X^2 f^{IV}(x) + 8X(1-2x)f'''(x) - 12Xf''(x)\right).$$

This result can be also deduced from [4, Remark 3].

1.3.6 Iterates of $\mathcal{B}_n^{\alpha,\beta}$

1. $\alpha = \beta = -1$. In this case $\mathcal{B}_n^{-1,-1}$ are positive linear operators preserving linear functions, and $\mathcal{B}_n^{-1,-1}e_2(x) = \dfrac{nx(nx+1)}{n(n+1)} > e_2(x)$, for $0 < x < 1$. Consequently

$$\lim_{m\to\infty}\left(\mathcal{B}_n^{-1,-1}\right)^m f(x) = (1-x)f(0) + xf(1), f \in C[0,1],$$

uniformly on $[0,1]$ (see [83]).

2. $\alpha > -1, \beta = -1$. Then $\mathcal{B}_n^{\alpha,-1}$ are positive linear operators preserving constant functions, $\mathcal{B}_n^{\alpha,-1}f(1) = f(1)$ for all $f \in C[0,1]$, and

$$\mathcal{B}_n^{\alpha,-1}e_2(x) = \frac{(nx+\alpha+1)(nx+\alpha+2)}{(n+\alpha+1)(n+\alpha+2)} > e_2(x), 0 \le x < 1.$$

Therefore

$$\lim_{m\to\infty}\left(\mathcal{B}_n^{\alpha,-1}\right)^m f(x) = f(1), f \in C[0,1],$$

uniformly on $[0,1]$ (see [83]).

3. $\alpha = -1, \beta > -1$. As in the previous case, one proves that

$$\lim_{m\to\infty}\left(\mathcal{B}_n^{-1,\beta}\right)^m f(x) = f(0), f \in C[0,1].$$

4. $\alpha > -1, \beta > -1$. In this case we have for all $k \geq 0$,

$$\mathcal{B}_n^{\alpha,\beta} e_k(x) = \frac{(nx+\alpha+1)^{\overline{k}}}{(n+\alpha+\beta+2)^{\overline{k}}}, x \in [0,1].$$

From this we get

$$\mathcal{B}_n^{\alpha,\beta} e_k(x) = \frac{1}{(n+\alpha+\beta+2)^{\overline{k}}} \sum_{j=0}^{k} s_{k-j}(k,\alpha) n^j x^j, \tag{1.34}$$

where $s_{k-j}(k,\alpha)$ are elementary symmetric sums of the numbers $\alpha+1, \alpha+2, ..., \alpha+k$; in particular $s_0(k,\alpha) = 1$ and

$$s_1(k,\alpha) = (\alpha+1) + ... + (\alpha+k) = k\alpha + \frac{k(k+1)}{2}. \tag{1.35}$$

It follows that the numbers

$$\lambda_{n,k} := \frac{n^k}{(n+\alpha+\beta+2)^{\overline{k}}}, k \geq 0,$$

are eigenvalues of $\mathcal{B}_n^{\alpha,\beta}$, and to each of them there corresponds a monic eigenpolynomial $q_{n,k}$ with $deg\ q_{n,k} = k$. Let $q \in \Pi$ and $d = deg\ q$. Then q has a decomposition

$$q = a_{n,0}(q)q_{n,0} + a_{n,1}(q)q_{n,1} + ... + a_{n,d}(q)q_{n,d}$$

with some coefficients $a_{n,k}(q) \in \mathbb{R}$. Since $\lambda_{n,0} = 1$ and $q_{n,0} = e_0$ we get

$$(\mathcal{B}_n^{\alpha,\beta})^m p = a_{n,0}(q)e_0 + \sum_{k=1}^{d} a_{n,k}(q)\lambda_{n,k}^m q_{n,k},\ m \geq 1$$

and so

$$\lim_{m\to\infty} (\mathcal{B}_n^{\alpha,\beta})^m q = a_{n,0}(q)e_0,\ p \in \Pi. \tag{1.36}$$

Consider the linear functional $\mu_n : \Pi \to \mathbb{R}, \mu_n(q) = a_{n,0}(q)$, and the linear operator $P_n : \Pi \to \Pi$,

$$P_n q = \mu_n(q)e_0,\ q \in \Pi.$$

Then (1.36) becomes

$$\lim_{m\to\infty} (\mathcal{B}_n^{\alpha,\beta})^m q = P_n q,\ q \in \Pi. \tag{1.37}$$

Obviously P_n is positive, and so μ_n is positive; moreover, $||\mu_n|| = 1$ because $\mu_n(e_0) = 1$. By the Hahn-Banach theorem, μ_n can be extended to a norm-one linear functional on $C[0,1]$. Since Π is dense in $C[0,1]$, the extension is unique and the extended functional $\mu_n : C[0,1] \to \mathbb{R}$ is also positive. Now P_n can be extended from Π to $C[0,1]$ by setting $P_n : C[0,1] \to \Pi, P_n f = \mu_n(f)e_0, f \in C[0,1]$. Remark that

$$||(\mathcal{B}_n^{\alpha,\beta})^m|| = ||P_n|| = 1,\ m \geq 1. \tag{1.38}$$

Using again the fact that Π is dense in $C[0,1]$, we get from (1.37) and (1.38)

$$\lim_{m\to\infty} (\mathcal{B}_n^{\alpha,\beta})^m f = P_n f,\ f \in C[0,1]. \tag{1.39}$$

On the other hand, from (1.34) we deduce the following recurrence formula for the computation of $P_n e_k, k \geq 1$:

$$\left((n+\alpha+\beta+2)^{\overline{k}} - n^k\right) P_n e_k = \sum_{j=0}^{k-1} s_{k-j}(k,\alpha) n^j P_n e_j.$$

Since $P_n e_k = \mu_n(e_k)e_0$, we get for $n \geq 1$ and $k \geq 1$

$$\mu_n(e_k) = \sum_{j=0}^{k-1} s_{k-j}(k,\alpha) \frac{n^j}{(n+\alpha+\beta+2)^{\overline{k}} - n^k} \mu_n(e_j). \tag{1.40}$$

Using (1.40) it is easy to prove by induction on k that there exists

$$\mu(e_k) := \lim_{n\to\infty} \mu_n(e_k), k \geq 0, \tag{1.41}$$

and, moreover,

$$\mu(e_k) = \frac{s_1(k,\alpha)}{(\alpha+\beta+2)+ ... +(\alpha+\beta+k+1)} \mu(e_{k-1}),$$

i.e., taking (1.35) into account,

$$\mu(e_k) = \frac{2\alpha+k+1}{2\alpha+2\beta+k+3} \mu(e_{k-1}),\ k \geq 1.$$

Since $\mu(e_0) = 1$, it follows that

$$\mu(e_k) = \frac{(2\alpha+2)^{\overline{k}}}{(2\alpha+2\beta+4)^{\overline{k}}},\ k \geq 0.$$

This can be rewritten as

$$\mu(e_k) = \frac{B(2\alpha+k+2, 2\beta+2)}{B(2\alpha+2, 2\beta+2)} = \frac{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} e_k(t)dt}{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1}dt},$$

so that

$$\mu(q) = \frac{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} q(t)dt}{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1}dt}, q \in \Pi.$$

Consider the extension of μ to $C[0,1]$, i.e.,

$$\mu(f) = \frac{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} f(t)dt}{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1}dt}, f \in C[0,1],$$

and the positive linear operator $P : C[0,1] \to \Pi, Pf = \mu(f)e_0, f \in C[0,1]$. According to (1.41), $\lim_{n\to\infty} \mu_n(q) = \mu(q), q \in \Pi$, i.e.,

$$\lim_{n\to\infty} P_n q = Pq, q \in \Pi. \tag{1.42}$$

Since $||P_n|| = ||P|| = 1, n \geq 1$, we conclude from (1.42) that $\lim_{n\to\infty} P_n f = Pf, f \in C[0,1]$. Thus, for the operators P_n described in (1.39) we have proved:

Theorem 1.3.17. *Let $\alpha > -1, \beta > -1$. Then for each $f \in C[0,1]$ and $n \geq 1$,*

$$\lim_{n\to\infty} P_n f = \frac{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} f(t)dt}{\int_0^1 t^{2\alpha+1}(1-t)^{2\beta+1} dt} e_0.$$

For $\alpha = \beta = 0$, this result was obtained, with different methods, in [11].

1.3.7 Variation diminution

Theorem 1.3.18. *The operators $\mathcal{B}_r^{a,b}$ as given in Definition 1.3.1 have the (strong) variation-diminishing property, that is,*

$$S_{[0,1]}[\mathcal{B}_r^{a,b} f] \leq S_{[0,1]}[f] \text{ for all } f \in C[0,1].$$

Proof. (i) in case $a = b = -1$:

$$S_{[0,1]}[\mathcal{B}_r^{-1,-1} f] = S_{[0,1]}\left[\int_0^1 t^{rx-1}(1-t)^{r-rx-1} f(t)\, dt\right].$$

Substituting $\left(\frac{t}{1-t}\right)^r = u$ the above integral becomes

$$\frac{1}{r}\int_0^\infty u^x \cdot \frac{u^{-1}}{(u^{\frac{1}{r}}+1)^r} \cdot f\left(\frac{u^{\frac{1}{r}}}{u^{\frac{1}{r}}+1}\right) du.$$

Obviously, the number of sign changes of $f(t), t \in [0,1]$ equals the number of sign changes of the function $g(u) = f\left(\frac{u^{\frac{1}{r}}}{u^{\frac{1}{r}}+1}\right)$, $u \in [0,\infty)$. Applying Theorem 1.1.31 for the functional $A(g) = \int_0^\infty g(u)\, du$ with $w(u) = \frac{u^{-1}}{(u^{\frac{1}{r}}+1)^r}$ we get that the operators $\mathcal{B}_r^{-1,-1}$ have the (strong) variation–diminishing property on $C[0,1]$. That means

$$S_{[0,1]}[\mathcal{B}_r^{-1,-1} f] \leq S_{[0,1]}[f].$$

(ii) in case $a = -1, b > -1$:

$$\begin{aligned} S_{[0,1]}[\mathcal{B}_r^{a,-1} f] &= S_{[0,1]}\left[\int_0^1 t^{rx-1}(1-t)^{r(1-x)-1} t^{a+1} f(t)dt\right] \\ &\leq S_{[0,1]}\left[t^{a+1} f(t)\right] = S_{[0,1]}[f]. \end{aligned}$$

(iii) in case $a > -1, b = -1$:

$$\begin{aligned} S_{[0,1]}[\mathcal{B}_r^{-1,b} f] &= S_{[0,1]}\left[\int_0^1 t^{rx-1}(1-t)^{r(1-x)-1}(1-t)^{b+1} f(t)dt\right] \\ &\leq S_{[0,1]}\left[(1-t)^{b+1} f(t)\right] = S_{[0,1]}\left[f\right]. \end{aligned}$$

(iv) in case $a, b > -1$:
In [29] it was shown that

$$\begin{aligned} S_{[0,1]}[\mathcal{B}_r^{a,b} f] &= S_{[0,1]}\left[\int_0^1 t^{rx}(1-t)^{r(1-x)} t^a (1-t)^b f(t)dt\right] \\ &\leq S_{[0,1]}\left[t^a(1-t)^b f(t)\right] = S_{[0,1]}\left[f\right]. \end{aligned}$$

□

Chapter 2

The Bernstein-Euler-Jacobi (BEJ) class of composition operators

Many operators arising in the theory of positive linear operators are compositions of other mappings of this type. Many times the classical Bernstein operator B_n is one of the building blocks. Other frequently used factor operators are Beta-type operators $\mathcal{B}_r^{a,b}$ of various kinds. Our intention is to use these building blocks to provide an overview of the various operators that fit into this pattern. This comes to emphasize the importance of understanding the singular pieces that form the composition. As one will notice we will refer many times to the properties of the building blocks presented in the previous chapter.

2.1 BEJ of first kind

2.1.1 Definition

We introduce and study a class of positive linear operators that are given by

Definition 2.1.1. For $r > 0, a, b \geq -1, n, m > 1$ we define $R_{m,n}^{(r,a,b)} : C[0,1] \to C[0,1]$ by

$$R_{m,n}^{(r,a,b)} = B_m \circ \mathcal{B}_r^{a,b} \circ B_n. \tag{2.1}$$

Here $\mathcal{B}_r^{a,b}$ is the Euler-Jacobi Beta operator defined in the previous chapter and B_n, B_m are the n-th and m-th Bernstein operators.

The purpose of introducing such an operator is to explain known operators using decomposition.

Lemma 2.1.2. *(Images of the monomials up to degree* 2*)*

$$R_{m,n}^{(r,a,b)}(e_0, x) = e_0$$

$$R_{m,n}^{(r,a,b)}(e_1, x) = \frac{re_1 + a + 1}{r + a + b + 2}$$

$$R_{m,n}^{(r,a,b)}(e_2, x) = \frac{(n-1)\left(r^2\left(\frac{m-1}{m}e_2 + \frac{e_1}{m}\right) + 2are_1 + 3re_1 + a^2 + 3a + 2\right)}{n(r+a+b+2)(r+a+b+3)} + \\ + \frac{re_1 + a + 1}{n(r+a+b+2)}$$

Remark 2.1.3. *The BEJ operator of first kind reproduces constants. For special choices of a and b, namely when $a = b = -1$ it also reproduces linear functions.*

2.1.2 Moments up to order 2

Definition 2.1.4. Let $r > 0, a, b \geq -1, n, m > 1, s \in \mathbb{N}_0$ and $x \in [0,1]$, then the moment of order s is given by

$$M_s(x) := R_{m,n}^{(r,a,b)}((e_1 - xe_0)^s; x).$$

Lemma 2.1.5. *(Explicit representation of the moments until order 2)*

$$M_0(x) := R_{m,n}^{(r,a,b)}((e_1 - xe_0)^0; x) = e_0,$$

$$M_1(x) := R_{m,n}^{(r,a,b)}((e_1 - xe_0)^1; x) = \frac{a+1-e_1(a+b+2)}{r+a+b+2}$$

$$\begin{aligned} M_2(x) = & \frac{e_2[mn(a^2+b^2+5a+5b+2ab+6-r)+r^2(1-m-n)]}{mn(r+a+b+2)(r+a+b+3)} - \\ & -\frac{e_1[mn(2a^2+2ab+8a+2b+6-r)+r^2(1-m-n)+mr(a-b)]}{mn(r+a+b+2)(r+a+b+3)} + \\ & +\frac{e_0[mn(a+1)(a+2)+m(r(a+1)+ab+a+b+1)]}{mn(r+a+b+2)(r+a+b+3)} \end{aligned}$$

Proof. The proof of the formula for the first moments is straightforward and we shall omit it.
In order to express the second moments of $R_{m,n}^{(r,a,b)}$ we shall employ (1.12). Thus we obtain

$$\begin{aligned} M_2(x) &= (B_m \circ \mathcal{B}_r^{a,b} \circ B_n)((e_1 - xe_0)^2; x) \\ &= B_m(\mathcal{B}_r^{a,b}(B_n((e_1 - ze_0)^2; z); y); x) \\ &\quad -B_m(\mathcal{B}_r^{a,b}((e_1 - ye_0)^2; y); x) - \\ &\quad -B_m((e_1 - xe_0)^2; x) + \\ &\quad +2B_m(\mathcal{B}_r^{a,b}((e_1 - ye_0)(B_n e_1 - ye_0); y); x) + \\ &\quad +2B_m((e_1 - xe_0)(\mathcal{B}_r^{a,b} B_n e_1 - xe_0); x) \end{aligned} \tag{2.2}$$

$$\begin{aligned} & B_m(\mathcal{B}_r^{a,b}(B_n((e_1 - ze_0)^2; z); y); x) = B_m\left(\mathcal{B}_r^{a,b}\left(\frac{e_1 - e_2}{n}; y\right); x\right) \\ & = B_n\left(\frac{re_1 + a + 1}{n(r+a+b+2)} - \frac{(re_1+a+1)(re_1+a+2)}{n(r+a+b+2)(r+a+b+3)}; x\right) \\ & = \frac{rx+a+1}{n(r+a+b+2)} - \frac{r^2\left(\frac{m-1}{m}x^2 + \frac{x}{m}\right) + 2arx + 3rx + a^2 + 3a + 2}{n(r+a+b+2)(r+a+b+3)} \end{aligned}$$

$$\begin{aligned} & B_m(\mathcal{B}_r^{a,b}((e_1 - ye_0)^2; y); x) = B_m\left(\frac{a^2+b^2+2ab+5a+5b+6-r}{(r+a+b+2)(r+a+b+3)}e_2 - \right. \\ & \left. - \frac{2ab+2a^2+8a+2b+6-r}{(r+a+b+2)(r+a+b+3)}e_1 + \frac{a^2+3a+2}{(r+a+b+2)(r+a+b+3)}e_0; x\right) \\ & = \frac{a^2+b^2+2ab+5a+5b+6-r}{(r+a+b+2)(r+a+b+3)}\left(\frac{m-1}{m}x^2 + \frac{x}{m}\right) - \\ & - \frac{2ab+2a^2+8a+2b+6-r}{(r+a+b+2)(r+a+b+3)}x + \frac{a^2+3a+2}{(r+a+b+2)(r+a+b+3)} \end{aligned}$$

$$B_m((e_1 - xe_0)^2; x) = \frac{x(1-x)}{m}$$

$$2B_m(\mathcal{B}_r^{a,b}((e_1 - ye_0)(B_n e_1 - ye_0); y); x) = 2B_m(\mathcal{B}_r^{a,b}((e_1 - ye_0)^2; y); x)$$

$$\begin{aligned} 2B_m((e_1 - xe_0)(\mathcal{B}_r^{a,b} B_n e_1 - xe_0); x) &= 2B_m((e_1 - xe_0)(\frac{re_1 + a + 1}{r+a+b+2} - xe_0); x) \\ &= \frac{2r}{r+a+b+2}\left(\frac{m-1}{m}x^2 + \frac{x}{m}\right) - \frac{2rx^2}{r+a+b+2}. \end{aligned}$$

If we substitute all these equations in (2.2) we obtain the expression for the second moments given above. □

2.2 BEJ of second kind

2.2.1 Definition

The second class of positive linear operators that we consider are given by

Definition 2.2.1. For $r, s > 0, a, b, c, d \geq -1, n, m > 1$ we define $R_n^{s,c,d;r,a,b} : C[0,1] \to C[0,1]$ by

$$R_n^{s,c,d;r,a,b} = \mathcal{B}_s^{c,d} \circ B_n \circ \mathcal{B}_r^{a,b}. \tag{2.3}$$

Here $\mathcal{B}_r^{a,b}$ and $\mathcal{B}_s^{c,d}$ are Euler-Jacobi Beta operators and B_n the n-th Bernstein operator.

Lemma 2.2.2. *(Images of the monomials up to degree 2)*

$$R_n^{s,c,d;r,a,b}(e_0, x) = e_0$$

$$R_n^{s,c,d;r,a,b}(e_1, x) = \frac{r(se_1 + c + 1)}{(r+a+b+2)(s+c+d+2)} + \frac{a+1}{(r+a+b+2)}$$

$$\begin{aligned} R_n^{s,c,d;r,a,b}(e_2, x) = &\frac{r^2}{n(r+a+b+2)(r+a+b+3)}\left[\frac{(n-1)(se_1+c+1)(se_1+c+2)}{(s+c+d+2)(s+c+d+3)} + \right. \\ &\left. + \frac{se_1+c+1}{s+c+d+2}\right] + \frac{(se_1+c+1)(2ar+3r)}{(s+c+d+2)(r+a+b+2)(r+a+b+3)} \\ &+ \frac{a^2+3a+2}{(r+a+b+2)(r+a+b+3)} \end{aligned}$$

Remark 2.2.3. *The BEJ operator of second kind reproduces constants. For special choices of a, b and c, d, namely when $a = b = c = d = -1$ it also reproduces linear functions.*

2.2.2 Moments up to order 2

Definition 2.2.4. Let $r > 0, a, b, c, d \geq -1, n > 1, t \in \mathbb{N}_0$ and $x \in [0,1]$, then the moment of order t is given by

$$M_t(x) := R_n^{s,c,d;r,a,b}((e_1 - xe_0)^t; x).$$

Lemma 2.2.5. *(Explicit representation of the moments until order 2)*

$$M_0(x) := R_n^{s,c,d;r,a,b}((e_1 - xe_0)^0; x) = e_0,$$

$$M_1(x) := R_n^{s,c,d;r,a,b}((e_1 - xe_0)^1; x) = \frac{-e_1[r(c+d+2)+(a+b+2)(s+c+d+2)]}{(r+a+b+2)(s+c+d+2)} + \\ + \frac{r(c+1)+(s+c+d+2)(a+1)}{(r+a+b+2)(s+c+d+2)}$$

$$M_2(x) = \frac{(n-1)(a^2+b^2+2ab+5a+5b+6-r)(se_1+c+1)(se_1+c+2)}{n(r+a+b+2)(r+a+b+3)(s+c+d+2)(s+c+d+3)} + \\ + \frac{(a^2+b^2+2ab+5a+5b+6-r-n(2ab+2a^2+8a+2b+6-r))(se_1+c+1)}{n(r+a+b+2)(r+a+b+3)(s+c+d+2)} + \\ + \frac{a^2+3a+2}{(r+a+b+2)(r+a+b+3)} + \frac{(se_1+c+1)(se_1+c+2)}{n(s+c+d+2)(s+c+d+3)} - \\ - \frac{se_1+c+1}{n(s+c+d+2)} - \frac{c^2+d^2+2cd+5c+5d+6-s}{(s+c+d+2)(s+c+d+3)} e_2 + \\ + \frac{2cd+2c^2+8c+2d+6-s}{(s+c+d+2)(s+c+d+3)} e_1 - \frac{c^2+3c+2}{(s+c+d+2)(s+c+d+3)} + \\ + \frac{2r(se_1+c+1)}{n(r+a+b+2)(s+c+d+2)} - \frac{2r(se_1+c+1)(se_1+c+2)}{n(r+a+b+2)(s+c+d+2)(s+c+d+3)} + \\ + \frac{2r(se_1+c+1)(se_1+c+2)}{(r+a+b+2)(s+c+d+2)(s+c+d+3)} + \frac{2(a+1-e_1(2r+a+b+2))(se_1+c+1)}{(r+a+b+2)(s+c+d+2)} - \\ - \frac{2(a+1)e_1}{r+a+b+2} + 2e_2.$$

Proof. The proof of the formula for the first moments is straightforward and we shall omit it.
In order to express the second moments of $R_n^{s,c,d;r,a,b}$ we shall employ once again (1.12). Thus we obtain

$$\begin{aligned} M_2(x) &= (\mathcal{B}_s^{c,d} \circ B_n \circ \mathcal{B}_r^{a,b})((e_1 - xe_0)^2; x) \\ &= \mathcal{B}_s^{c,d}(B_n(\mathcal{B}_r^{a,b}((e_1 - ze_0)^2; z); y); x) \\ &\quad - \mathcal{B}_s^{c,d}(B_n((e_1 - ye_0)^2; y); x) - \\ &\quad - \mathcal{B}_s^{c,d}((e_1 - xe_0)^2; x) + \\ &\quad + 2\mathcal{B}_s^{c,d}(B_n((e_1 - ye_0)(\mathcal{B}_r^{a,b} e_1 - ye_0); y); x) + \\ &\quad + 2\mathcal{B}_s^{c,d}((e_1 - xe_0)(B_n \mathcal{B}_r^{a,b} e_1 - xe_0); x) \end{aligned} \tag{2.4}$$

$$\mathcal{B}_s^{c,d}(B_n(\mathcal{B}_r^{a,b}((e_1 - ze_0)^2; z); y); x) = \frac{a^2+3a+2}{(r+a+b+2)(r+a+b+3)} \\ + \frac{(a^2+b^2+2ab+5a+5b+6-r-n(2ab+2a^2+8a+2b+6-r))(sx+c+1)}{n(r+a+b+2)(r+a+b+3)(s+c+d+2)} \\ + \frac{(n-1)(a^2+b^2+2ab+5a+5b+6-r)(sx+c+1)(sx+c+2)}{n(r+a+b+2)(r+a+b+3)(s+c+d+2)(s+c+d+3)}$$

$$\mathcal{B}_s^{c,d}(B_n((e_1 - ye_0)^2; y); x) = \mathcal{B}_s^{c,d}\left(\frac{e_1(1-e_1)}{n}; x\right) \\ = \frac{sx+c+1}{n(s+c+d+2)} - \frac{(sx+c+1)(sx+c+2)}{n(s+c+d+2)(s+c+d+3)}$$

$$\mathcal{B}_s^{c,d}((e_1 - xe_0)^2; x) = \frac{c^2 + d^2 + 2cd + 5c + 5d + 6 - s}{(s + c + d + 2)(s + c + d + 3)} x^2 + \\ + \frac{2cd + 2c^2 + 8c + 2d + 6 - s}{(s + c + d + 2)(s + c + d + 3)} x + \frac{c^2 + 3c + 2}{(s + c + d + 2)(s + c + d + 3)}$$

$$(e_1 - y)(\mathcal{B}_r^{a,b} e_1 - y) = (e_1 - y) \left(\frac{re_1 + a + 1}{r + a + b + 2} - y \right)$$

$$B_n((e_1 - y)(\mathcal{B}_r^{a,b} e_1 - y); y) = \frac{r}{r + a + b + 2} B_n(e_2; y) + \\ + \left[\frac{a + 1}{r + a + b + 2} - y - \frac{ry}{r + a + b + 2} \right] B_n(e_1; y) + \left(y^2 - y \frac{a + 1}{r + a + b + 2} \right) B_n(e_0; y) = \\ = \frac{r}{n(r + a + b + 2)} (y - y^2)$$

$$2\mathcal{B}_s^{c,d}(B_n((e_1 - ye_0)(\mathcal{B}_r^{a,b} e_1 - ye_0); y); x) = 2\mathcal{B}_s^{c,d} \left(\frac{r}{n(r + a + b + 2)} (e_1 - e_2); x \right) = \\ = \frac{2r(sx + c + 1)}{n(r + a + b + 2)(s + c + d + 2)} - \frac{2r(sx + c + 1)(sx + c + 2)}{n(r + a + b + 2)(s + c + d + 2)(s + c + d + 3)}$$

$$(e_1 - x)(B_n \mathcal{B}_r^{a,b} e_1 - x) = (e_1 - x) \left(\frac{re_1 + a + 1}{r + a + b + 2} - x \right)$$

$$2\mathcal{B}_s^{c,d}((e_1 - x)(B_n \mathcal{B}_r^{a,b} e_1 - x); x) = \frac{2r}{r + a + b + 2} \mathcal{B}_s^{c,d}(e_2; x) + \\ +2 \left(\frac{a + 1}{r + a + b + 2} - x - \frac{xr}{r + a + b + 2} \right) \mathcal{B}_s^{c,d}(e_1; x) + 2 \left(x^2 - \frac{x(a + 1)}{r + a + b + 2} \right) \mathcal{B}_s^{c,d}(e_0; x) = \\ = \frac{2r(sx + c + 1)(sx + c + 2)}{(r + a + b + 2)(s + c + d + 2)(s + c + d + 3)} + 2 \frac{[a + 1 - x(2r + a + b + 2)](sx + c + 1)}{(r + a + b + 2)(s + c + d + 2)} - \\ - \frac{2x(a + 1)}{r + a + b + 2} + 2x^2$$

If we substitute all these equations in (2.4) we obtain the expression for the second moments given above. □

2.3 Particular cases

The general setting can be adapted in such a way that for different values of the indices we find many known operators. We use the convention $B_\infty = \mathcal{B}_\infty^{a,b} = Id$.
Thus, the first two tables contain all the particular cases of both classes that we were able to locate in the literature. In the next three we give the general form of the specific operators, if it exists, and cite the articles where they were first mentioned.
In our opinion it is impossible to get a complete overview because of the great amount of articles being published every day, but we are confident that most of the cases are included.
We also include the description of the second moments of these cases and differentiate between operators that reproduce linear functions and those that do not.

BEJ first kind						BEJ second kind						
m	**r**	**a**	**b**	**n**	Other notation	**s**	**c**	**d**	**n**	**r**	**a**	**b**
n	n	-1	-1	∞	U_n	∞	-	-	n	n	-1	-1
n	n	0	0	∞	M_n	∞	-	-	n	n	0	0
n	n	α>-1	α>-1	∞	$D^{<\alpha>}$	∞	-	-	n	n	α>-1	α>-1
n	n	a>-1	b>-1	∞	M_n^{ab}	∞	-	-	n	n	a>-1	b>-1
n	$n \cdot \varrho$	-1	-1	∞	U_n^{ϱ}	∞	-	-	n	$n \cdot \varrho$	-1	-1
n	n·c	>-1	>-1	∞	P_n	∞	-	-	n	$n \cdot c$	>-1	>-1
∞	n	-1	-1	n	L_n^{∇}	n	-1	-1	n	∞	-	-
∞	n	0	0	n	$V_n^{0,0}$	n	0	0	n	∞	-	-
∞	n	α >-1	β>-1	n	$V_n^{\alpha,\beta}$	n	α>-1	β>-1	n	∞	-	-
∞	$1/\alpha$	-1	-1	n	S_n^{α}, Q_n	$1/\alpha$	-1	-1	n	∞	-	-
∞	$n \cdot \varrho$	c>-1	d>-1	n	$Q_n^{\varrho,c,d}$	$n \cdot \varrho$	c>-1	d>-1	n	∞	-	-
-	-	-	-	-	$\mathbb{B}_\infty^{(\alpha,\lambda)}$	$1/\alpha$	-1	-1	∞	$1/\lambda$	-1	-1
-	-	-	-	-	F_n^{α}	$1/\alpha$	-1	-1	n	n	-1	-1
-	-	-	-	-	$\mathbb{B}_n^{(\alpha,\lambda)}$	$1/\alpha$	-1	-1	n	$1/\lambda$	-1	-1
n	∞	-	-	n+1	D_n	-	-	-	-	-	-	-
m	∞	-	-	n	$R_{m,n}^{\infty}$	-	-	-	-	-	-	-
m	$n \cdot \varrho$	-1	-1	n	$R_{m,n}^{\varrho}$	-	-	-	-	-	-	-
∞	n	-1	-1	∞	$\overline{\mathbb{B}}_n, \mathcal{B}_n^{-1,-1}$	n	-1	-1	∞	∞	-	-
						∞	-	-	∞	n	-1	-1
∞	$1/\lambda$	-1	-1	∞	$\tilde{\mathbb{B}}_\lambda,\ T_\lambda$	$1/\lambda$	-1	-1	∞	∞	-	-
						∞	-	-	∞	$1/\lambda$	-1	-1

Table 2.3.1: Particular cases overview - part 1

BEJ first kind						BEJ second kind						
m	**r**	**a**	**b**	**n**	Other notation	**s**	**c**	**d**	**n**	**r**	**a**	**b**
∞	n	0	0	∞	$\mathbb{B}_n$, $\mathcal{B}_n^{0,0}$	n	0	0	∞	∞	-	-
						∞	-	-	∞	n	0	0
∞	n	-1	β> -1	∞	$\mathcal{B}_n^{-1,\beta}$	n	-1	β> -1	∞	∞	-	-
						∞	-	-	∞	n	-1	β> -1
∞	n	α> -1	-1	∞	$\mathcal{B}_n^{\alpha,-1}$	n	α> -1	-1	∞	∞	-	-
						∞	-	-	∞	n	α> -1	-1
∞	n	α> -1	β>-1	∞	$\mathcal{B}_n^{\alpha,\beta}$	n	α> -1	β>-1	∞	∞	-	-
						∞	-	-	∞	n	α> -1	β> -1
n	∞	-	-	∞	B_n	∞	-	-	n	∞	-	-
∞	∞	-	-	n								

Table 2.3.2: Particular cases overview - part 2

Name	General form	References
Bernstein operator	$B_n(f;x) := \sum\limits_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)$	[12]
Mühlbach Beta type operator	$T_\lambda[f;x] = \int\limits_0^1 f(t) K_\lambda(t,x) dt, T_\lambda[f;0] = f(0), T_\lambda[f;1] = f(1)$ $K_\lambda(t,x) = \psi(t; \frac{x}{\lambda}, \frac{1-x}{\lambda}), \psi(t;p,q) = \frac{t^{p-1}(1-t)^{q-1}}{B(p,q)}$ sometimes the following notation is used: $\tilde{\mathbb{B}}_\lambda(f;x)$ (see [78])	[71]
Lupaş Beta operator of the first kind	$\mathbb{B}_n(f;x) = \dfrac{1}{B(nx+1, n+1-nx)} \int\limits_0^1 t^{nx}(1-t)^{n(1-x)} f(t) dt$	[62, p.37]
Lupaş Beta operator of the second kind	$\overline{\mathbb{B}}_n(f;x) = \dfrac{1}{B(nx, n-nx)} \int\limits_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t) dt,$ $(x \in (0,1), \overline{\mathbb{B}}_n(f;0) = f(0), \overline{\mathbb{B}}_n(f;1) = f(1))$	[62, p.63]
genuine Bernstein-Durrmeyer operator	$U_n(f;x) = \sum\limits_{k=1}^{n-1} p_{n,k}(x) \int\limits_0^1 p_{n-2,k-1}(t) f(t) dt + f(0) p_{n,0}(x) + f(1) p_{n,n}(x)$	[16] [31]
classical Durrmeyer	$M_n(f;x) = (n+1) \sum\limits_{k=0}^{n} p_{n,k}(x) \int\limits_0^1 p_{n,k}(t) f(t) dt$	[25]
Durrmeyer with Jacobi weights	$M_n^{ab}(f;x) := \sum\limits_{k=0}^{n} p_{n,k}(x) \dfrac{\int\limits_0^1 f(t) p_{n,k}(t) t^a (1-t)^b dt}{\int\limits_0^1 p_{n,k}(t) t^a (1-t)^b dt}$	[73]
Păltănea operator	$U_n^\varrho(f;x) = \sum\limits_{k=1}^{n-1} \left(\int\limits_0^1 \dfrac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho, (n-k)\varrho)} f(t) dt \right) p_{n,k}(x) + f(0)(1-x)^n + f(1) x^n$	[75]
a Stancu type operator	$L_n^\nabla(f;x) = C_n(f;x) = \dfrac{1}{n^{\overline{n}}} \sum\limits_{k=0}^{n} \binom{n}{k} (nx)^{\overline{k}} (n-nx)^{\overline{k}} f\left(\frac{k}{n}\right)$	[64]

Table 2.3.3: General form of known operators - part 1

Name	General form	References
Bernstein-Durrmeyer with symmetric weight	$D^{<\alpha>}(f;x) = \sum\limits_{k=0}^{n} p_{n,k}(x) \frac{(2\alpha+2)^{\overline{n}}}{(\alpha+1)^{\overline{k}}(\alpha+1)^{\overline{n-k}}} \int\limits_0^1 \frac{t^{k+\alpha}(1-t)^{n+k+\alpha}}{B(\alpha+1,\alpha+1)} f(t)dt$	[63]
Mache-Zhou operator	$P_n(f;x) = \sum\limits_{k=0}^{n} p_{n,k}(x) \frac{\int\limits_0^1 f(t)t^{ck+a}(1-t)^{c(n-k)+b}dt}{B(ck+a+1,c(n-k)+b+1)}, a,b > -1$	[65]
Stancu-type operator with parameters a, b	$Q_n^{\varrho,a,b}(f;x) = \sum\limits_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} \frac{B(n\varrho x+a+k+1, n\varrho(1-x)+n-k+b+1)}{B(n\varrho x+a+1, n\varrho(1-x)+b+1)}$	Gonska handwritten notes, 18 March 2009.
Lupaş operator with Jacobi weights	$V_n^{\alpha,\beta}(f;x) = \sum\limits_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) \frac{(nx+\alpha+1)^{\overline{k}}(n-nx+\beta+1)^{\overline{n-k}}}{(n+\alpha+\beta+2)^{\overline{n}}}$	Raşa handwritten notes, 19 August 2008.
Lupaş operator	$(\mathbb{B}_n \circ B_n)(f;x) = V_n^{0,0}(f;x) = \sum\limits_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) \frac{(nx+1)^{\overline{k}}(n-nx+1)^{\overline{n-k}}}{(n+2)^{\overline{n}}}$	[63]
Stancu operator	$S_n^{\alpha}(f;x) = \sum\limits_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} \frac{\prod\limits_{i=0}^{k-1}(x+i\alpha)\prod\limits_{j=0}^{n-k-1}(1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\cdot\ldots\cdot(1+(n-1)\alpha)}$	[94]
Stancu operator (Mühlbach notation)	$Q_n[f](x) = Q_n[f;x,a] = \sum\limits_{k=0}^{n} f(x_{nk})q_{nk}(x,a),$ $f(x_{nk}) = f(\frac{k}{n}), q_{nk}(x,a) = \binom{n}{k}\frac{\varphi_k(x,a)\varphi_{n-k}(1-x,a)}{\varphi_n(1,a)}, \varphi_k(x,a) = \prod\limits_{l=0}^{k-1}(x+la), \varphi_0(x,a) := 1$	[71]

Table 2.3.4: General form of known operators - part 2

Name	General form	References
Finta operator	$F_n^{\alpha}(f;x) = f(0)w_{n,0}^{(\alpha)}(x) + f(1)w_{n,n}^{(\alpha)}(x) + \sum\limits_{k=1}^{n-1} w_{n,k}^{(\alpha)}(x)\int\limits_0^1 (n-1)p_{n-2,k-1}(t)f(t)dt,$ $w_{n,k}^{(\alpha)}(x) := \binom{n}{k}\frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}$	[27]
general Beta-type operators	$\mathbb{B}_n^{(\alpha,\lambda)}(f;x) = (\tilde{\mathbb{B}}_{\alpha} \circ B_n \circ \tilde{\mathbb{B}}_{\lambda})(f;x)$	[78]
Piţul Beta operator	$\mathbb{B}_{\infty}^{(\alpha,\lambda)}(f;x) = (\tilde{\mathbb{B}}_{\alpha} \circ \tilde{\mathbb{B}}_{\lambda})(f;x)$	[78]
D_n	$D_n(f;x) = (B_n \circ B_{n+1})(f;x)$	[43]

Table 2.3.5: General form of known operators - part 3

Notation	Second moments
* U_n	$U_n((e_1 - xe_0)^2;x) := \dfrac{2X}{n+1}$
M_n	$M_n((e_1 - xe_0)^2;x) := \dfrac{2X(n-3)+2}{(n+2)(n+3)}$
$D^{<\alpha>}$	$D^{<\alpha>}((e_1 - xe_0)^2;x) := \dfrac{X(2n-4a^2-10a-6)+(a+1)(a+2)}{(n+2a+2)(n+2a+3)}$
M_n^{ab}	$M_n^{ab}((e_1 - xe_0)^2;x) :=$ $\dfrac{a^2+b^2+2ab+5a+5b+6-2n}{(n+a+b+2)(n+a+b+3)} \cdot x^2 - \dfrac{2a^2+2ab+8a+2b+6-2n}{(n+a+b+2)(n+a+b+3)} \cdot x + \dfrac{(a+1)(a+2)}{(n+a+b+2)(n+a+b+3)}$
* U_n^{ϱ}	$U_n^{\varrho}((e_1 - xe_0)^2;x) := \dfrac{X(1+\varrho)}{n\varrho+1}$
P_n	$P_n((e_1 - xe_0)^2;x) :=$ $\dfrac{a^2+b^2+2ab+5a+5b+6-nc-nc^2}{(nc+a+b+2)(nc+a+b+3)} \cdot x^2 - \dfrac{(2a^2+2ab+8a+2b+6-nc-nc^2)\cdot x+(a+1)(a+2)}{(nc+a+b+2)(nc+a+b+3)}$

* these operators reproduce linear functions

Table 2.3.6: Second moments - part 1

Notation	Second moments
* L_n^{∇}	$L_n^{\nabla}((e_1 - xe_0)^2; x) := \frac{2X}{2+1}$
$V_n^{0,0}$	$V_n^{0,0}((e_1 - xe_0)^2; x) := \frac{X(2n^2 - 6n) + 3n + 1}{n(n+2)(n+3)}$
$V_n^{\alpha,\beta}$	$V_n^{\alpha,\beta}((e_1 - xe_0)^2; x) := \frac{\alpha^2 + \beta^2 + 2\alpha\beta + 5\alpha + 5\beta + 6 - 2n}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} \cdot x^2 - \frac{2\alpha^2 + 2\alpha\beta + 9\alpha + \beta + 6 - 2n}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} \cdot x + \frac{n\alpha^2 + 4n\alpha + \alpha\beta + 3n + \alpha + \beta + 1}{n(n+\alpha+\beta+2)(n+\alpha+\beta+3)}$
* S_n^{α}	$S_n^{\alpha}((e_1 - xe_0)^2; x) := \frac{2X}{n+1}$
$Q_n^{\varrho,c,d}$	$Q_n^{\varrho,c,d}((e_1 - xe_0)^2; x) := \frac{c^2 + d^2 + 2cd + 5c + 5d + 6 - n\varrho - n\varrho^2}{(n\varrho+c+d+2)(n\varrho+c+d+3)} \cdot x^2 - \frac{2c^2 + 2cd + 8c + 2d + 6 - n\varrho - n\varrho^2}{(n\varrho+c+d+2)(n\varrho+c+d+3)} \cdot x + \frac{nc^2 + 3nc + nc\varrho + n\varrho + 2n + c + d + cd + 1}{n(n\varrho+c+d+2)(n\varrho+c+d+3)}$
* $\mathbb{B}_{\infty}^{(\alpha,\lambda)}$	$\mathbb{B}_{\infty}^{\alpha,\lambda}((e_1 - xe_0)^2; x) := \frac{X(\alpha + \lambda + \alpha\lambda)}{(1+\alpha)(1+\lambda)}$
* F_n^{α}	$F_n^{\alpha}((e_1 - xe_0)^2; x) := \frac{X(n\alpha + \alpha + 2)}{(\alpha+1)(n+1)}$
* $\mathbb{B}_n^{(\alpha,\lambda)}$	$\mathbb{B}_n^{(\alpha,\lambda)}((e_1 - xe_0)^2; x) := \frac{X(n\alpha + n\lambda + n\alpha\lambda + 1)}{n(1+\alpha)(1+\lambda)}$
* D_n	$D_n((e_1 - xe_0)^2; x) := \frac{2X}{n+1}$
* $R_{m,n}^{\infty}$	$R_{m,n}^{\infty}((e_1 - xe_0)^2; x) := \frac{X(n+m-1)}{nm}$

* these operators reproduce linear functions

Table 2.3.7: Second moments - part 2

Notation	Second moments
* $R^{\varrho}_{m,n}$	$R^{\varrho}_{m,n}((e_1 - xe_0)^2; x) := \dfrac{X(n\varrho + m\varrho - \varrho + m)}{m(n\varrho + 1)}$
* $\overline{\mathbb{B}}_n$, $\mathcal{B}_n^{-1,-1}$	$\overline{\mathbb{B}}_n((e_1 - xe_0)^2; x) := \dfrac{X}{n+1}$
* $\widetilde{\mathbb{B}}_\lambda$, T_λ	$\widetilde{\mathbb{B}}_\lambda((e_1 - xe_0)^2; x) := \dfrac{\lambda X}{1+\lambda}$
$\mathbb{B}_n$, $\mathcal{B}_n^{0,0}$	$\mathbb{B}_n((e_1 - xe_0)^2; x) := \dfrac{X(n-6)+2}{(n+2)(n+3)}$
$\mathcal{B}_n^{-1,\beta}$	$\mathcal{B}_n^{-1,\beta}((e_1 - xe_0)^2; x) := \dfrac{nX + (\beta+1)(\beta+2)x^2}{(n+\beta+1)(n+\beta+2)}$
$\mathcal{B}_n^{\alpha,-1}$	$\mathcal{B}_n^{\alpha,-1}((e_1 - xe_0)^2; x) := \dfrac{nX + (\alpha+1)(\alpha+2)(x-1)^2}{(n+\alpha+1)(n+\alpha+2)}$
$\mathcal{B}_n^{\alpha,\beta}$	$\mathcal{B}_n^{\alpha,\beta}((e_1 - xe_0)^2; x) := \dfrac{\alpha^2 + \beta^2 + 2\alpha\beta + 5\alpha + 5\beta + 6 - n}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} \cdot x^2 - \dfrac{2\alpha^2 + 2\alpha\beta + 8\alpha + 2\beta + 6 - n}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} \cdot x + \dfrac{(\alpha+1)(\alpha+2)}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)}$
* B_n	$B_n((e_1 - xe_0)^2; x) := \dfrac{X}{n}$

* these operators reproduce linear functions

Table 2.3.8: Second moments - part 3

2.4 Variation diminution

Theorem 2.4.1. *The BEJ operators of first, respectively second kind as given in Definitions 2.1.1 and 2.2.1 have the (strong) variation-diminishing property, that is,*

$$S_{[0,1]}[R_{m,n}^{(r,a,b)}f] \leq S_{[0,1]}[f] \text{ for all } f \in C[0,1] \tag{2.5}$$

$$S_{[0,1]}[R_n^{s,c,d;r,a,b}f] \leq S_{[0,1]}[f] \text{ for all } f \in C[0,1]. \tag{2.6}$$

Proof. It is known that the Bernstein operators satisfy the SVDP. Thus we have

$$S_{[0,1]}[R_{m,n}^{(r,a,b)}f] \leq S_{[0,1]}[(\mathcal{B}_r^{a,b} \circ B_n)f].$$

In Section 1.3.7 we have shown that $\mathcal{B}_r^{a,b}$ also satisfy the SVDP. Thus we have

$$S_{[0,1]}[R_{m,n}^{(r,a,b)}f] \leq S_{[0,1]}[(\mathcal{B}_r^{a,b} \circ B_n)f] \leq S_{[0,1]}[B_n f] \leq S_{[0,1]}[f]$$

for all $f \in C[0,1]$. Similarly it can be shown that (2.6) holds, but we'll skip that proof. □

2.5 Direct and converse results

Direct and strong converse inequality of type A, in the terminology of [22], exist for the Bernstein operators in [59], for "genuine" Bernstein-Durrmeyer operators in [72], for a special selection of Stancu operators in [26] and for the Finta operator in [28].

Using the method presented by Finta in [26] and [28] we can give such results for two more cases, that is, for the composition of two different Bernstein operators and for a particular case of the general composition that reproduces linear functions.

The results presented in this section have been published in [97].

2.5.1 Case I - a composition of two Bernstein operators

We define

$$R_{m,n}^{\infty}f = (B_m \circ B_n)(f,x) = \sum_{k=0}^{m} p_{m,k}(x) B_n f\left(\frac{k}{m}\right),$$

a positive linear operator that reproduces linear functions, where $p_{m,k}$ are the Bernstein basis polynomials with $p_{m,k}(x) = \binom{m}{k}x^k(1-x)^{m-k}$, $x \in [0,1]$.
For this first case we need the following result:

Lemma 2.5.1. *Let $f \in C[0,1]$. Then*

$$||R_{m,n}^{\infty}f - B_n f|| \leq \frac{1}{m}||\varphi^2 B_n'' f||. \tag{2.7}$$

Proof. $R_{m,n}^{\infty}f = B_m(B_n f) = \sum\limits_{k=0}^{m} p_{m,k}(x) B_n f(\frac{k}{m})$ where $p_{m,k} = \binom{m}{k}x^k(1-x)^{m-k}$. Hence

$$(R_{m,n}^{\infty}f - B_n f)(x) = \sum_{k=0}^{m} p_{m,k}(x)\left(B_n f\left(\frac{k}{m}\right) - B_n f(x)\right)$$

and by Taylor expansion with integral remainder, namely

$$B_nf\left(\frac{k}{m}\right)=B_nf(x)+\left(\frac{k}{m}-x\right)B_n'f\,(x)+\int\limits_x^{\frac{k}{m}}\left(\frac{k}{m}-u\right)B_n''f(u)du,$$

we can write

$$R_{m,n}^{\infty}f-B_nf=\sum_{k=0}^{m}p_{m,k}(x)\left[\left(\frac{k}{m}-x\right)B_n'f\,(x)+\int\limits_x^{\frac{k}{m}}\left(\frac{k}{m}-u\right)B_n''f(u)du\right].$$

By simple computation we obtain:

$$\sum_{k=0}^{m}p_{m,k}(x)\left(\frac{k}{m}-x\right)=0, \tag{2.8}$$

and

$$\sum_{k=0}^{m}p_{m,k}(x)\left(\frac{k}{m}-x\right)^2=\frac{\varphi^2(x)}{m}. \tag{2.9}$$

Then in view of (2.8), (2.9) and Lemma 1.1.30, for $x\neq 0,1$, we obtain

$$\begin{aligned}||R_{m,n}^{\infty}f-B_nf|| &\leq \sum_{k=0}^{m}p_{m,k}(x)\left|\int\limits_x^{\frac{k}{m}}\left(\frac{k}{m}-u\right)B_n''f(u)du\right|\\ &\leq \frac{||\varphi^2B_n''f||}{\varphi(x)^2}\sum_{k=0}^{m}p_{m,k}(x)\left(\frac{k}{m}-x\right)^2\\ &= \frac{1}{m}||\varphi^2B_n''f||,\end{aligned} \tag{2.10}$$

which completes the proof. □

Theorem 2.5.2. *Let $f\in C[0,1]$. Then there exists a constant $C>0$ such that*

$$||R_{m,n}^{\infty}f-f||\leq C\omega_{\varphi}^2(f,n^{-1/2})_{C[0,1]}. \tag{2.11}$$

Proof. We have

$$||R_{m,n}^{\infty}f-f||\leq||R_{m,n}^{\infty}f-B_nf||+||B_nf-f||. \tag{2.12}$$

Let $g\in W_{\infty}^2(\varphi)$. In view of Lemma 2.5.1 and [21, Lemma 7.4, p.324] we obtain

$$\begin{aligned}||R_{m,n}^{\infty}f-B_nf|| &\leq \frac{1}{m}||\varphi^2B_n''f||\\ &\leq \frac{1}{m}\{||\varphi^2B_n''(f-g)||+||\varphi^2B_n''g||\}\\ &\leq \frac{1}{m}\{2n||f-g||+12||\varphi^2g''||\}\\ &\leq 12\frac{n}{m}\{||f-g||+\frac{1}{n}||\varphi^2g''||\}.\end{aligned}$$

So

$$\begin{aligned} ||R^{\infty}_{m,n}f - B_nf|| &\leq 12\frac{n}{m}\inf\{||f-g|| + \frac{1}{n}||\varphi^2 g''|| : g \in W^2_{\infty}(\varphi)\} \\ &= 12\frac{n}{m}K_{2,\varphi}(f,n^{-1})_{C[0,1]}. \end{aligned}$$

Because $K_{2,\varphi}(f,n^{-1})_{C[0,1]}$ is equivalent with $\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]}$ in view of Theorem 1.1.22 we obtain the existence of a constant $C_1 \neq C_1(f,n,m) > 0$ such that

$$||R^{\infty}_{m,n}f - B_nf|| \leq 12\frac{n}{m}C_1\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]}. \tag{2.13}$$

On the other hand it has been shown in [23] that for some constant $C_2 \neq C_2(f,n,m) > 0$

$$||B_nf - f|| \leq C_2\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]}, \tag{2.14}$$

for every $f \in C[0,1]$. Thus, by (2.12), (2.13) and (2.14) we obtain for a constant $C = 12\frac{n}{m}C_1 + C_2$ the estimate (2.11). □

Corollary 2.5.3. *Under the assumption of Theorem 2.5.2 we have*

$$||R^{\infty}_{m,n}f - f|| \leq C||B_nf - f|| \tag{2.15}$$

where $C > 0$ is constant.

Proof. In view of [59] we have for some absolute constant $M > 0$

$$M\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]} \leq ||B_nf - f||. \tag{2.16}$$

Thus by Theorem 2.5.2 we get (2.15). □

Theorem 2.5.4. *Let $\alpha_1 = C_0\frac{n}{m} < 1$, where C_0 denotes the absolute constant in Lemma 1.2.10 and the pair (n,m) is chosen accordingly. Then there exists a constant $C > 0$ such that for all $f \in C[0,1]$ we have*

$$C^{-1}||B_nf - f|| \leq ||R^{\infty}_{m,n}f - f|| \leq C||B_nf - f|| \tag{2.17}$$

and

$$C^{-1}\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]} \leq ||R^{\infty}_{m,n}f - f|| \leq C\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]}. \tag{2.18}$$

Proof. We have

$$\begin{aligned} ||B_nf - f|| &\leq ||R^{\infty}_{m,n}f - f|| + ||R^{\infty}_{m,n}f - B_nf|| \\ &\leq ||R^{\infty}_{m,n}f - f|| + C_0\frac{n}{m}||B_nf - f|| \end{aligned}$$

in view of Lemma 1.2.10 and (2.7). But $\alpha_1 = C_0\frac{n}{m} < 1$, by assumption, and therefore

$$||B_nf - f|| \leq ||R^{\infty}_{m,n}f - f|| + \alpha_1||B_nf - f||.$$

So

$$(1-\alpha_1)||B_nf - f|| \leq ||R^{\infty}_{m,n}f - f||.$$

Hence by Corollary 2.5.3 we obtain (2.17) for some $C > 0$. The inequalities in (2.18) are direct consequences of (2.16) and (2.17). Thus the theorem is proved. □

2.5.2 Case II - BEJ of first kind general composition

We define

$$\begin{aligned} R^{\varrho}_{m,n}f(x) &= (B_m \circ \mathcal{B}^{-1,-1}_{n\varrho} \circ B_n)(f,x) \\ &= \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{\frac{n\varrho k}{m}-1}(1-t)^{n\varrho-\frac{n\varrho k}{m}-1} B_n f(t)dt}{B(\frac{n\varrho k}{m}, n\varrho - \frac{n\varrho k}{m})}, \end{aligned}$$

a positive linear operator that reproduces linear functions with $\varrho > 0$.

Lemma 2.5.5. *Let $f \in C[0,1]$ and $\varrho > 0$. Then*

$$||R^{\varrho}_{m,n}f - B_n f|| \le \frac{n\varrho + m}{m(n\varrho+1)} ||\varphi^2 B''_n f||. \tag{2.19}$$

Proof.

$$R^{\varrho}_{m,n}f(x) = \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{\frac{n\varrho k}{m}-1}(1-t)^{n\varrho-\frac{n\varrho k}{m}-1} B_n f(t)dt}{B(\frac{n\varrho k}{m}, n\varrho - \frac{n\varrho k}{m})}, \quad x \in (0,1).$$

Hence

$$R^{\varrho}_{m,n}f(x) - B_n f(x) = \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{\frac{n\varrho k}{m}-1}(1-t)^{n\varrho-\frac{n\varrho k}{m}-1} [B_n f(t) - B_n f(x)]dt}{B(\frac{n\varrho k}{m}, n\varrho - \frac{n\varrho k}{m})}$$

and by Taylor expansion with integral remainder, that is,

$$B_n f(t) = B_n f(x) + (t-x)B'f(x) + \int_x^t (t-u)B''_n f(u)du,$$

we can write

$$R^{\varrho}_{m,n}f(x) - B_n f(x) =$$

$$= \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{\frac{n\varrho k}{m}-1}(1-t)^{n\varrho-\frac{n\varrho k}{m}-1} \left[(t-x)B'f(x) + \int_x^t (t-u)B''_n f(u)du\right] dt}{B(\frac{n\varrho k}{m}, n\varrho - \frac{n\varrho k}{m})}.$$

By simple computations we obtain

$$\sum_{k=0}^{m} p_{m,k}(x) \frac{1}{B(\frac{n\varrho k}{m}, n\varrho - \frac{n\varrho k}{m})} \int_0^1 t^{\frac{n\varrho k}{m}-1}(1-t)^{n\varrho-\frac{n\varrho k}{m}-1}(t-x)dt = 0 \tag{2.20}$$

and

$$\sum_{k=0}^{m} p_{m,k}(x) \frac{1}{B(\frac{n\varrho k}{m}, n\varrho - \frac{n\varrho k}{m})} \int_0^1 t^{\frac{n\varrho k}{m}-1}(1-t)^{n\varrho-\frac{n\varrho k}{m}-1}(t-x)^2dt = \frac{n\varrho+m}{m(n\varrho+1)}\varphi^2(x), \tag{2.21}$$

respectively. Then in view of (2.20), Lemma 1.1.30 and (2.21), we get

$$\begin{aligned} |R^{\varrho}_{m,n}f(x) - B_n f(x)| &\leq \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{\frac{n\varrho k}{m}-1}(1-t)^{n\varrho-\frac{n\varrho k}{m}-1}\left|\int_x^t (t-u)B''_n f(u)du\right| dt}{B(\frac{n\varrho k}{m}, n\varrho - \frac{n\varrho k}{m})} \\ &\leq \frac{||\varphi^2 B''_n f||}{\varphi^2(x)} \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{\frac{n\varrho k}{m}-1}(1-t)^{n\varrho-\frac{n\varrho k}{m}-1}(t-x)^2 dt}{B(\frac{n\varrho k}{m}, n\varrho - \frac{n\varrho k}{m})} \\ &= \frac{n\varrho + m}{m(n\varrho+1)}||\varphi^2 B''_n f|| \end{aligned}$$

which completes the proof. □

Theorem 2.5.6. *Let $f \in C[0,1]$. Then there exists a constant $C > 0$ such that*

$$||R^{\varrho}_{m,n}f - f|| \leq C\omega^2_{\varphi}(f, n^{-1/2})_{C[0,1]}. \tag{2.22}$$

Proof. We have

$$||R^{\varrho}_{m,n}f - f|| \leq ||R^{\varrho}_{m,n}f - B_n f|| + ||B_n f - f||. \tag{2.23}$$

Let $g \in W^2_{\infty}(\varphi)$. In view of Lemma 2.5.5 and [21, Lemma 7.4, p.324] we obtain

$$\begin{aligned} ||R^{\varrho}_{m,n}f - B_n f|| &\leq \frac{n\varrho + m}{m(n\varrho+1)}||\varphi^2 B''_n f|| \\ &\leq \frac{n\varrho + m}{m(n\varrho+1)}\{||\varphi^2 B''_n(f-g)|| + ||\varphi^2 B''_n g||\} \\ &\leq \frac{n\varrho + m}{m(n\varrho+1)}\{2n||f-g|| + 12||\varphi^2 g''||\} \\ &\leq 12\frac{n(n\varrho + m)}{m(n\varrho+1)}\{||f-g|| + \frac{1}{n}||\varphi^2 g''||\}. \end{aligned}$$

So

$$\begin{aligned} ||R^{\varrho}_{m,n}f - B_n f|| &\leq 12\frac{n(n\varrho + m)}{m(n\varrho+1)} \inf\{||f-g|| + \frac{1}{n}||\varphi^2 g''|| : g \in W^2_{\infty}(\varphi)\} \\ &= 12\frac{n(n\varrho + m)}{m(n\varrho+1)} K_{2,\varphi}(f, n^{-1})_{C[0,1]}. \end{aligned}$$

Because $K_{2,\varphi}(f, n^{-1})_{C[0,1]}$ is equivalent to $\omega^2_{\varphi}(f, n^{-1/2})_{C[0,1]}$ in view of Theorem 1.1.22, we obtain the existence of a constant $C_1 \neq C_1(f, n, m, \varrho) > 0$ such that

$$||R^{\varrho}_{m,n}f - B_n f|| \leq 12\frac{n(n\varrho + m)}{m(n\varrho+1)} C_1 \omega^2_{\varphi}(f, n^{-1/2})_{C[0,1]}. \tag{2.24}$$

On the other hand it has been shown in [23] that for some constant $C_2 \neq C_2(f, n, m, \varrho) > 0$

$$||B_n f - f|| \leq C_2 \omega^2_{\varphi}(f, n^{-1/2})_{C[0,1]}. \tag{2.25}$$

for every $f \in C[0,1]$. Thus, by (2.23), (2.24) and (2.25) for a constant $C = 12\frac{n(n\varrho + m)}{m(n\varrho+1)}C_1 + C_2$ we obtain (2.22). □

Corollary 2.5.7. *Under the assumption of* Theorem 2.5.6 *we have*

$$||R^{\varrho}_{m,n}f - f|| \leq C||B_n f - f|| \tag{2.26}$$

where $C > 0$ is constant.

Proof. In view of [59] we have for some absolute constant $M > 0$

$$M\omega^2_{\varphi}(f, n^{-1/2})_{C[0,1]} \leq ||B_n f - f|| \tag{2.27}$$

Thus by Theorem 2.5.6 we get (2.26). □

Theorem 2.5.8. *Let $\alpha_2 = C_0 \dfrac{n(n\varrho + m)}{m(n\varrho + 1)} < 1$, where C_0 denotes the absolute constant in Lemma 1.2.10, $\varrho \geq 1$ and the triplet (n, m, ϱ) is chosen accordingly. Then there exists a constant $C > 0$ such that for all $f \in C[0, 1]$ we have*

$$C^{-1}||B_n f - f|| \leq ||R^{\varrho}_{m,n}f - f|| \leq C||B_n f - f|| \tag{2.28}$$

and

$$C^{-1}\omega^2_{\varphi}(f, n^{-1/2})_{C[0,1]} \leq ||R^{\varrho}_{m,n}f - f|| \leq C\omega^2_{\varphi}(f, n^{-1/2})_{C[0,1]}. \tag{2.29}$$

Proof. We have

$$\begin{aligned} ||B_n f - f|| &\leq ||R^{\varrho}_{m,n}f - f|| + ||R^{\varrho}_{m,n}f - B_n f|| \\ &\leq ||R^{\varrho}_{m,n}f - f|| + C_0 \frac{n(n\varrho + m)}{m(n\varrho + 1)} ||B_n f - f|| \end{aligned}$$

in view of Lemma 1.2.10 and (2.19). But $\alpha_2 = C_0 \dfrac{n(n\varrho + m)}{m(n\varrho + 1)} < 1$ by assumption, and therefore

$$||B_n f - f|| \leq ||R^{\varrho}_{m,n}f - f|| + \alpha_2||B_n f - f||.$$

So

$$(1 - \alpha_2)||B_n f - f|| \leq ||R^{\varrho}_{m,n}f - f||.$$

Hence by Corollary 2.5.7 we obtain (2.28) for some $C > 0$. The inequalities in (2.29) are direct consequences of (2.27) and (2.28). Thus the theorem is proved. □

Remark 2.5.9. *This method can be applied for compositions of linear positive operators that reproduce linear functions under the assumption that a strong converse inequality as the one given in Lemma 1.2.10 exists for the operator on the right hand side.*
This method does not cover all the cases of the composition as a result of the restrictions applied to the constants.

Chapter 3

The class of operators U_n^ϱ linking the Bernstein and the genuine Bernstein-Durrmeyer operators

Denote by $C[0,1]$ the space of continuous, real-valued functions on $[0,1]$ and by Π_n the space of polynomials of degree at most $n \in \mathbb{N}_0 := \{0, 1, 2...\}$.

Definition 3.0.10. Let $\varrho > 0$ and $n \in \mathbb{N}_0, n \geq 1$. Define the operator $U_n^\varrho : C[0,1] \to \Pi_n$ by

$$\begin{aligned} U_n^\varrho(f;x) &:= \sum_{k=0}^{n} F_{n,k}^\varrho(f) p_{n,k}(x) \\ &:= \sum_{k=1}^{n-1} \left(\int_0^1 \frac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho,(n-k)\varrho)} f(t)dt \right) p_{n,k}(x) + \\ &\quad + f(0)(1-x)^n + f(1)x^n, \end{aligned} \tag{3.1}$$

$f \in C[0,1], x \in [0,1]$ and $B(\cdot,\cdot)$ is Euler's Beta function. The fundamental functions $p_{n,k}$ are defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq k \leq n, \quad k, n \in \mathbb{N}_0, \quad x \in [0,1].$$

For $\varrho = 1$ and $f \in C[0,1]$, we obtain

$$\begin{aligned} U_n^1(f;x) = U_n(f;x) &= (n-1) \sum_{k=1}^{n-1} \left(\int_0^1 f(t) p_{n-2,k-1}(t) dt \right) p_{n,k}(x) \\ &\quad + (1-x)^n f(0) + x^n f(1), \end{aligned} \tag{3.2}$$

where U_n are the "genuine" Bernstein-Durrmeyer operators (see [16], [31]), while for $\varrho \to \infty$, for each $f \in C[0,1]$ the sequence $U_n^\varrho(f;x)$ converges uniformly to the Bernstein polynomial

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x). \tag{3.3}$$

The U_n^ϱ were introduced in [75] by R. Păltănea and further investigated in [45] and [46].

Remark 3.0.11. *The "genuine" Bernstein-Durrmeyer operators were studied by many authors. See for example the Habilitationschrift from D. Kacsó which has a whole chapter dedicated to the operators [55, Chapter 3] and the citations therein for further information.*

As a consequence of the extensive investigations that we've conducted on the class of operators U_n^{ϱ}, several articles have been published or find themselves currently under review. We feel that in this sense an overview is in order. Thus, we find in [48] Sections 3.2, 3.4 and Subsection 3.8.2, in [57] some properties given in Section 3.1, but also Sections 3.5, 3.6 and Subsection 3.8.1, in [85] part of Section 3.1 and Sections 3.9, 3.10 and 3.11, in [51] Sections 3.12, 3.13 and 3.14 and in [49] Section 3.16.

3.1 Properties

U_n^{ϱ} share many properties common for the well–known operators $B_n, U_n, \mathcal{B}_n^{-1,-1}$, such as being positive linear operators preserving linear functions.

Basic properties of the functionals $F_{n,k}^{\varrho} : C[0,1] \to \mathbb{R}$ are the following

$$F_{n,k}^{\varrho}(e_m) = \frac{(k\varrho)^{\overline{m}}}{(n\varrho)^{\overline{m}}}, 0 \le k \le n, \text{ and } e_m(x) = x^m, x \in [0,1], \text{ for } m \ge 0. \tag{3.4}$$

This implies

$$U_n^{\varrho}(e_0) = e_0, \quad U_n^{\varrho}(e_1) = e_1.$$

Lemma 3.1.1. *If f is convex on $C[0,1]$, then*

$$U_n^{\varrho}(f;x) \ge U_{n+1}^{\varrho}(f;x) \ge f(x), 0 < x < 1. \tag{3.5}$$

The inequalities are strict when f is strictly convex on $[0,1]$.

Proof. This result is a consequence of Corollary 1.2.8 and Lemma 1.3.4.

We choose $s = (n+1)\varrho$ and $r = n\varrho$ in (1.21) and we compose to the left with the $(n+1)$-st Bernstein operator. We get then

$$(B_{n+1} \circ \mathcal{B}_{n\varrho}^{-1,-1})(f;x) \ge (B_{n+1} \circ \mathcal{B}_{(n+1)\varrho}^{-1,-1})(f;x) = U_{n+1}^{\varrho}(f;x). \tag{3.6}$$

Next in the inequality below we compose to the right with $\mathcal{B}_{n\varrho}^{-1,-1}(f;x)$

$$B_n(f;x) \ge B_{n+1}(f;x)$$

and get

$$U_n^{\varrho}(f;x) = (B_n \circ \mathcal{B}_{n\varrho}^{-1,-1})(f;x) \ge (B_{n+1} \circ \mathcal{B}_{n\varrho}^{-1,-1})(f;x). \tag{3.7}$$

Combining (3.6) and (3.7) we get (3.5). □

Lemma 3.1.2. *If f is convex on $C[0,1]$, then*

$$U_n^{\varrho}(f;x) \ge B_n(f;x), 0 < x < 1. \tag{3.8}$$

The inequality is strict if f is strictly convex on $[0,1]$.

Proof. In [85] it is shown that for $f \in C[0,1]$ convex and $0 < \varrho < \sigma$,

$$U_n^{\varrho}(f;x) \ge U_n^{\sigma}(f;x).$$

Letting $\sigma \to \infty$ in the inequality above we get (3.8). □

In [45] the authors proved that for each $n \geq 1$ and $f \in C[0,1]$,

$$\lim_{\varrho \to \infty} U_n^\varrho f = B_n f, \text{ uniformly on } [0,1].$$

Thus, for n fixed and $\varrho \in [1, \infty)$, the operators U_n^ϱ constitute a link between the genuine Bernstein-Durrmeyer operators U_n and the Bernstein operators B_n. The authors of [46] proved that for $n \geq 1$ and $f \in C[0,1]$,

$$\lim_{\varrho \to 0^+} U_n^\varrho f = B_1 f, \text{ uniformly on } [0,1]. \tag{3.9}$$

Moreover, they proved

Theorem 3.1.3. *For $U_n^\varrho, 0 < \varrho < \infty, n \geq 1$, we have*

$$|U_n^\varrho f(x) - B_1 f(x)| \leq \frac{9}{4}\omega_2\left(f; \sqrt{\frac{n\varrho - \varrho}{n\varrho + 1}x(1-x)}\right).$$

In what follows, we give a different proof of (3.9). First of all, we have

$$F_{n,k}^\varrho(e_j) = \frac{k\varrho(k\varrho+1)\cdot \ldots \cdot (k\varrho + j - 1)}{n\varrho(n\varrho+1)\cdot \ldots \cdot (n\varrho + j - 1)}, \; j \geq 0, \; 0 \leq k \leq n,$$

and consequently,

$$\lim_{\varrho \to 0^+} F_{n,k}^\varrho(e_0) = 1 \tag{3.10}$$

and

$$\lim_{\varrho \to 0^+} F_{n,k}^\varrho(e_j) = \frac{k}{n}, \;\; j = 1, 2, \ldots. \tag{3.11}$$

Now let $p \in \Pi, p = a_0 e_0 + a_1 e_1 + \ldots + a_m e_m$ for some $a_0, a_1, \ldots, a_m \in \mathbb{R}$. Then, according to (3.10) and (3.11),

$$\lim_{\varrho \to 0^+} F_{n,k}^\varrho(p) = a_0 + (a_1 + \ldots + a_m)\frac{k}{n} = p(0) + (p(1) - p(0))\frac{k}{n}.$$

This leads to

$$\lim_{\varrho \to 0^+} U_n^\varrho p = \sum_{k=0}^{n} \left(p(0) + (p(1) - p(0))\frac{k}{n}\right) p_{n,k} = p(0)e_0 + (p(1) - p(0))e_1,$$

and so

$$\lim_{\varrho \to 0^+} U_n^\varrho p = B_1 p, \;\; p \in \Pi. \tag{3.12}$$

Since Π is dense in $C[0,1]$, and $||U_n^\varrho|| = ||B_1|| = 1$, (3.9) is a consequence on (3.12).

In the sequel we shall be concerned with shape preserving properties of the operators U_n^ϱ. In [45, Theorem 4.1], the authors proved that for $n \geq 1$ and $\varrho > 0$, the operators U_n^ϱ transform k-convex functions into k-convex functions. Basically this means that if $f^{(k)} \geq 0$, then $(U_n^\varrho)^{(k)} f \geq 0, k \geq 0$; see [45] for the complete terminology. Here we shall present briefly another proof of this theorem.
First, let $\alpha \geq -1, \beta \geq -1$ be real numbers. For $r > 0$ consider the kernel

$$(x,y) \in [0,1] \times]0,1[\to K_r^{a,b}(x,y) := \frac{y^{rx+a}(1-y)^{r(1-x)+b}}{B(rx+a+1, r(1-x)+b+1)},$$

and the operator

$$\mathcal{B}_r^{a,b} f(x) := \int_0^1 K_r^{a,b}(x,y) f(y) dy, \; f \in C[0,1], \; x \in [0,1].$$

Let us remark that the kernel $K_r^{a,b}$ can be represented also as

$$K_r^{a,b}(x,y) = \frac{e^{a \log y + (r+b)\log(1-y)} \cdot e^{rx(\log y - \log(1-y))}}{B(rx+a+1, r(1-x)+b+1)}.$$

According to [58, Theorem 1.1, part (a), p. 99], and [58, (1.5), p. 100], $K_r^{a,b}$ is a totally positive kernel. Moreover, a direct computation yields

$$\mathcal{B}_r^{a,b} e_k(x) = \frac{(rx+a+1)(rx+a+2)\cdot \ldots \cdot (rx+a+k)}{(r+a+b+2)\cdot \ldots \cdot (r+a+b+k+1)}.$$

Thus, for any $k \geq 0, \mathcal{B}_r^{a,b} e_k$ is a polynomial of degree k with leading coefficient

$$a_{r,k}^{a,b} := \frac{r^k}{(r+a+b+2)\cdot \ldots \cdot (r+a+b+k+1)}.$$

By [10, Theorem 2.3 and Remark 2.5], $\mathcal{B}_r^{a,b}$ transforms k-convex functions into k-convex functions, $k \geq 0$. Since the Bernstein operator B_n does the same, we conclude that $B_n \circ \mathcal{B}_r^{a,b}$ preserves k-convexity. In particular, $U_n = B_n \circ \mathcal{B}_r^{-1,-1}$ preserves k-convexity, and this is the content of [45, Theorem 4.1].

3.2 Images of the monomials

More generally we have

Theorem 3.2.1. *The images of the monomials under U_n^{ϱ} can be written as*

$$U_n^{\varrho}(e_m) = \frac{1}{(n\varrho)^{\overline{m}}} \sum_{l=0}^{m} c_{m-l}^{(m)} (n\varrho)^l B_n(e_l) \tag{3.13}$$

where the coefficients $c_j^{(m)}, \; j = 0,1,\ldots,m$ are given by the elementary symmetric sums:

$$\begin{aligned}
&c_0^{(m)} := 1, \quad c_m^{(m)} := 0, \\
&c_1^{(m)} = 1+2+\ldots+(m-1) = \frac{m(m-1)}{2}, \\
&c_2^{(m)} = 1\cdot 2 + 1\cdot 3 + \ldots + 1\cdot(m-1) + 2\cdot 3 + \ldots + (m-2)\cdot(m-1), \\
&\ldots \\
&c_{m-1}^{(m)} = 1\cdot 2\cdot 3\cdot \ldots \cdot (m-1) = (m-1)!.
\end{aligned} \tag{3.14}$$

Proof.

$$
\begin{aligned}
U_n^\varrho(e_m;x) &= \sum_{k=0}^{n} F_{n,k}^\varrho(e_m)p_{n,k}(x)\\
&= \sum_{k=0}^{n} \frac{k\varrho(k\varrho+1)\cdot\ldots\cdot(k\varrho+m-1)}{n\varrho(n\varrho+1)\cdot\ldots\cdot(n\varrho+m-1)}p_{n,k}(x)\\
&= \frac{1}{(n\varrho)^{\overline{m}}}\sum_{k=0}^{n} k\varrho(k\varrho+1)\cdot\ldots\cdot(k\varrho+m-1)p_{n,k}(x)\\
&= \frac{1}{(n\varrho)^{\overline{m}}}\sum_{k=0}^{n}[c_0^{(m)}(k\varrho)^m + c_1^{(m)}(k\varrho)^{m-1}+\ldots+c_{m-1}^{(m)}k\varrho]p_{n,k}(x)\\
&= \frac{1}{(n\varrho)^{\overline{m}}}\left\{c_0^{(m)}\varrho^m\sum_{k=0}^{n}k^m p_{n,k}(x)+c_1^{(m)}\varrho^{m-1}\sum_{k=0}^{n}k^{m-1}p_{n,k}(x)+\ldots\right.\\
&\qquad\left.\ldots+c_{m-1}^{(m)}\varrho\sum_{k=0}^{n}kp_{n,k}(x)\right\}\\
&= \frac{1}{(n\varrho)^{\overline{m}}}\left\{c_0^{(m)}\varrho^m n^m\sum_{k=0}^{n}\frac{k^m}{n^m}p_{n,k}(x)+c_1^{(m)}\varrho^{m-1}n^{m-1}\sum_{k=0}^{n}\frac{k^{m-1}}{n^{m-1}}p_{n,k}(x)+\ldots\right.\\
&\qquad\left.\ldots+c_{m-1}^{(m)}n\varrho\sum_{k=0}^{n}\frac{k}{n}p_{n,k}(x)\right\}\\
&= \frac{1}{(n\varrho)^{\overline{m}}}\left\{c_0^{(m)}\varrho^m n^m B_n(e_m;x)+c_1^{(m)}\varrho^{m-1}n^{m-1}B_n(e_{m-1};x)+\ldots\right.\\
&\qquad\left.\ldots+c_{m-1}^{(m)}n\varrho B_n(e_1;x)\right\}\\
&= \frac{1}{(n\varrho)^{\overline{m}}}\sum_{l=0}^{m}c_{m-l}^{(m)}(n\varrho)^l B_n(e_l;x).
\end{aligned}
$$

□

Remark 3.2.2. *This representation of the images of the monomials highlights the close relationship between B_n and U_n^ϱ.*

3.3 The moments of U_n^ϱ

In [45] the following formulas for the moments of U_n^ϱ are proved.

Theorem 3.3.1. *For $x, y \in [0,1]$, we have*

$$U_n^\varrho(e_0;x) = 1,\ \ U_n^\varrho(e_1 - ye_0;x) = x - y$$

and for $r \geq 1$ and $\Psi(x) = x(1-x)$

$$
\begin{aligned}
U_n^\varrho((e_1-ye_0)^{r+1};x) &= \frac{\varrho\Psi(x)}{n\varrho+r}(U_n^\varrho((e_1-ye_0)^r;x))'_x+\\
&+\frac{(1-2y)r+n\varrho(x-y)}{n\varrho+r}(U_n^\varrho((e_1-ye_0)^r;x))+\frac{r\Psi(y)}{n\varrho+r}(U_n^\varrho((e_1-ye_0)^{r-1};x)).
\end{aligned}
$$

For brevity we set $M_{n,r}^\varrho(x) := U_n^\varrho((e_1-xe_0)^r;x), n \geq 1, r \geq 0, x \in [0,1]$. It is immediate that

$$(M_{n,r}^\varrho(x))' = (U_n^\varrho((e_1-ye_0)^r;x))'_x|_{y=x} - rM_{n,r-1}^\varrho(x). \tag{3.15}$$

Using (3.15) and setting $y = x$ in Theorem 3.3.1, we obtain the following recursion for the central moments:

Corollary 3.3.2. *The following relations are true*

$$M_{n,0}^\varrho(x) = 1, M_{n,1}^\varrho(x) = 0,$$

and for $r \geq 1$

$$M_{n,r+1}^\varrho(x) = \frac{r(\varrho+1)\Psi(x)}{n\varrho+r} M_{n,r-1}^\varrho(x) + \frac{(1-2x)r}{n\varrho+r} M_{n,r}^\varrho(x) + \frac{\varrho\Psi(x)}{n\varrho+r}(M_{n,r}^\varrho(x))'. \tag{3.16}$$

In particular

$$M_{n,2}^\varrho(x) = \frac{(\varrho+1)\Psi(x)}{n\varrho+1}, \tag{3.17}$$

$$M_{n,3}^\varrho(x) = \frac{(\varrho+1)(\varrho+2)\Psi(x)\Psi'(x)}{(n\varrho+1)(n\varrho+2)},$$

$$M_{n,4}^\varrho(x) = \frac{3\varrho(\varrho+1)^2\Psi^2(x)n}{(n\varrho+1)(n\varrho+2)(n\varrho+3)} + \frac{-6(\varrho+1)(\varrho^2+3\varrho+3)\Psi^2(x)+(\varrho+1)(\varrho+2)(\varrho+3)\Psi(x)}{(n\varrho+1)(n\varrho+2)(n\varrho+3)}.$$

3.4 The eigenstructure of U_n^ϱ

There are many applications that arise from having a complete description of the eigenstructure. Applications of the results presented in this section can be found in Sections 3.12, 3.13, 3.16 and Subsection 3.8.2.

3.4.1 Diagonalisation and description of the eigenfunctions

We shall use the *Stirling numbers of second kind* $S(k,j)$ defined by

$$x^k = \sum_{j=0}^{k} S(k,j)x(x-1)...(x-j+1).$$

The following identity holds (see [17], Theorem A [1b], p.204):

$$S(k,j) = \frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}i^k,\ 0 \leq j \leq k. \tag{3.18}$$

Consider the eigenfunction equation

$$U_n^\varrho p_{\varrho,k}^{(n)} = \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)} \tag{3.19}$$

with respect to the basis of monomials $\{e_0, e_1, ..., e_n\}$. Since U_n^ϱ is degree reducing, we have to solve an upper triangular system. This will be done in the proof of the next theorem.

Remark 3.4.1. *The operator U_n^ϱ reproduces linear polynomials, which are therefore eigenfunctions corresponding to the eigenvalue* 1.

Theorem 3.4.2. *The operator U_n^ϱ can be represented in diagonal form*

$$U_n^\varrho f = \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \text{ for all } f \in C[0,1], \tag{3.20}$$

with $\lambda_{\varrho,k}^{(n)}$ and $p_{\varrho,k}^{(n)}$ its eigenvalues and eigenfunctions and $\mu_{\varrho,k}^{(n)}$ the dual functionals of $p_{\varrho,k}^{(n)}$. The eigenvalues are given by

$$\lambda_{\varrho,k}^{(n)} = \varrho^{k-1} \frac{(n-1)(n-2)...(n-k+1)}{(n\varrho+1)(n\varrho+2)...(n\varrho+k-1)} = \frac{\varrho^k n!}{(n\varrho)^{\overline{k}}(n-k)!} \tag{3.21}$$

and they satisfy

$$1 = \lambda_{\varrho,0}^{(n)} = \lambda_{\varrho,1}^{(n)} > \lambda_{\varrho,2}^{(n)} > \lambda_{\varrho,3}^{(n)} > ... > \lambda_{\varrho,n}^{(n)} > 0.$$

The eigenfunction for $\lambda_{\varrho,k}^{(n)}$ is a polynomial of degree k given by

$$p_{\varrho,k}^{(n)}(x) = \sum_{j=0}^{k} c^\varrho(j,k,n)x^j = x^k - \frac{k}{2}x^{k-1} + \text{ lower order terms,} \tag{3.22}$$

where the coefficients can be computed using the recurrence formula

$$\begin{aligned} c^\varrho(k,k,n) &:= 1, \\ c^\varrho(k-1,k,n) &:= -\frac{k}{2}, \\ c^\varrho(k-j,k,n) &:= \frac{(n\varrho)^{\overline{k}}}{\varrho^{k-j}[\varrho^j(n-k+1)^{\overline{j}} - (n\varrho+k-j)^{\overline{j}}]} \times \\ & \sum_{i=0}^{j-1} \sum_{l=k-j}^{k-i} \frac{c^\varrho(k-i,k,n)}{(n\varrho)^{\overline{k-i}}} c_{k-i-l}^{(k-i)} \varrho^l S(l,k-j), \; j = 2,...,k. \end{aligned} \tag{3.23}$$

Proof. The eigenvalues of U_n^ϱ are determined from the upper triangular system of equations (3.19). They can be found on the diagonal and are equal to the coefficients of the terms with the highest degree of $U_n^\varrho(e_m)$. As we have seen before, we can write

$$\begin{aligned} U_n^\varrho(e_m;x) = \frac{1}{(n\varrho)^{\overline{m}}} \Big\{ & c_0^{(m)} \varrho^m \sum_{k=0}^{n} k^m p_{n,k}(x) + c_1^{(m)} \varrho^{m-1} \sum_{k=0}^{n} k^{m-1} p_{n,k}(x) + ... \\ & ... + c_{m-1}^{(m)} \varrho \sum_{k=0}^{n} k p_{n,k}(x) \Big\} \end{aligned}$$

and because

$$\sum_{k=0}^{n} k^m p_{n,k}(x) = n(n-1)(n-2) \cdot ... \cdot (n-m+1)x^m + \text{ terms of lower degree}$$

the eigenvalues are given by

$$\begin{aligned} \lambda_{\varrho,m}^{(n)} &= \frac{1}{(n\varrho)^{\overline{m}}} \varrho^m n(n-1)(n-2) \cdot ... \cdot (n-m+1) \\ &= \frac{\varrho^m \cdot n!}{(n\varrho)^{\overline{m}}(n-m)!}. \end{aligned}$$

The linear polynomials are eigenfunctions for the eigenvalues $\lambda_{\varrho,0}^{(n)} = \lambda_{\varrho,1}^{(n)} = 1$, for which $p_{\varrho,0}^{(n)}(x) = 1, p_{\varrho,1}^{(n)}(x) = x - \frac{1}{2}$ are clearly a basis which satisfies (3.22) and (3.23).
It remains to consider the 1-dimensional $\lambda_{\varrho,k}^{(n)}$- eigenspace of polynomials of exact degree $k = 2, 3, ..., n$.
We shall plug into (3.13)

$$B_n(e_m; x) = \sum_{j=0}^{m} a(j, m, n)x^j,$$

where

$$a(j, m, n) = \frac{S(m, j)n!}{n^m(n-j)!}, 0 \le j \le m \le n,$$

as it was considered in [18] and we obtain

$$U_n^\varrho(e_m; x) = \frac{1}{(n\varrho)^{\overline{m}}} \sum_{l=0}^{m} c_{m-l}^{(m)}(n\varrho)^l \sum_{r=0}^{l} a(r, l, n)x^r.$$

Express the eigenfunctions in the form

$$p_{\varrho,k}^{(n)}(x) = \sum_{s=0}^{k} c^\varrho(s, k, n)x^s, \quad c^\varrho(k, k, n) := 1. \tag{3.24}$$

Then the eigenfunction equation (3.19) gives:

$$\begin{aligned}
\lambda_{\varrho,k}^{(n)} \sum_{r=0}^{k} c^\varrho(r, k, n)x^r &= \sum_{s=0}^{k} \frac{c^\varrho(s, k, n)}{(n\varrho)^{\overline{s}}} \sum_{l=0}^{s} c_{s-l}^{(s)}(n\varrho)^l \sum_{r=0}^{l} a(r, l, n)x^r \\
&= \sum_{s=0}^{k} \frac{c^\varrho(s, k, n)}{(n\varrho)^{\overline{s}}} \sum_{r=0}^{s} \sum_{l=r}^{s} c_{s-l}^{(s)}(n\varrho)^l a(r, l, n)x^r \\
&= \sum_{r=0}^{k} \sum_{s=r}^{k} \frac{c^\varrho(s, k, n)}{(n\varrho)^{\overline{s}}} \sum_{l=r}^{s} c_{s-l}^{(s)}(n\varrho)^l a(r, l, n)x^r.
\end{aligned}$$

Equating the coefficients of x^r above gives for $0 \le r \le k$:

$$\lambda_{\varrho,k}^{(n)} c^\varrho(r, k, n) = \sum_{s=r}^{k} \frac{c^\varrho(s, k, n)}{(n\varrho)^{\overline{s}}} \sum_{l=r}^{s} c_{s-l}^{(s)}(n\varrho)^l a(r, l, n).$$

Into this we make first the substitution $s = k - i$ and subsequently $r = k - j$ to obtain

$$\lambda_{\varrho,k}^{(n)} c^\varrho(k-j, k, n) = \sum_{i=0}^{j} \frac{c^\varrho(k-i, k, n)}{(n\varrho)^{\overline{k-i}}} \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)}(n\varrho)^l a(k-j, l, n).$$

which, for $k>1$, can be solved for $c^\varrho(k-j,k,n)$ to give

$$\begin{aligned}
c^\varrho(k-j,k,n) &= \left(\lambda_{\varrho,k}^{(n)} - \frac{(n\varrho)^{k-j}}{(n\varrho)^{\overline{k-j}}}a(k-j,k-j,n)\right)^{-1} \sum_{i=0}^{j-1} \frac{c^\varrho(k-i,k,n)}{(n\varrho)^{\overline{k-i}}} \times \\
&\quad \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)}(n\varrho)^l a(k-j,l,n) \\
&= \left(\frac{\varrho^k\cdot n!}{(n\varrho)^{\overline{k}}(n-k)!} - \frac{(n\varrho)^{k-j}}{(n\varrho)^{\overline{k-j}}}\frac{S(k-j,k-j)\cdot n!}{n^{k-j}(n-k+j)!}\right)^{-1} \times \\
&\quad \sum_{i=0}^{j-1} \frac{c^\varrho(k-i,k,n)}{(n\varrho)^{\overline{k-i}}} \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)}(n\varrho)^l \frac{S(l,k-j)n!}{n^l(n-k+j)!} \\
&= \left(\varrho^{k-j}\left[\frac{\varrho^j}{(n\varrho)^{\overline{k}}(n-k)!} - \frac{1}{(n\varrho)^{\overline{k-j}}(n-k+j)!}\right]\right)^{-1} \times \\
&\quad \sum_{i=0}^{j-1} \frac{c^\varrho(k-i,k,n)}{(n\varrho)^{\overline{k-i}}} \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)}\varrho^l \frac{S(l,k-j)}{(n-k+j)!} \\
&= \frac{(n-k+j)!(n\varrho)^{\overline{k}}}{\varrho^{k-j}[\varrho^j(n-k+1)^{\overline{j}} - (n\varrho+k-j)^{\overline{j}}]} \times \\
&\quad \sum_{i=0}^{j-1} \frac{c^\varrho(k-i,k,n)}{(n\varrho)^{\overline{k-i}}} \sum_{l=k-j}^{k-i} c_{k-i-l}^{(k-i)}\varrho^l \frac{S(l,k-j)}{(n-k+j)!}.
\end{aligned}$$

From here we get easily the equations (3.23). In particular, for $j=1$ we get

$$c^\varrho(k-1,k,n) = \frac{c_1^{(k)} + c_0^{(k)}\varrho\frac{k(k-1)}{2}}{(n-k+1)\varrho - (n\varrho+k-1)} = -\frac{k}{2} \tag{3.25}$$

because $c_0^{(k)} = 1$ and $c_1^{(k)} = 1+2+...+(k-1) = \frac{k(k-1)}{2}$ and $S(k-1,k-1) = 1, S(k,k-1) = \frac{k(k-1)}{2}$. □

Theorem 3.4.3. *The dual functional $\mu_{\varrho,k}^{(n)} \in span\{f \to F_{n,j}^\varrho(f); j=0,1,...,n\}$ defined on $C[0,1]$ satisfies*

$$\mu_{\varrho,k}^{(n)}(p_{\varrho,i}^{(n)}) = \delta_{i,k};\ i,k=0,1,...,n,$$

and is given by

$$\mu_{\varrho,k}^{(n)}(f) = \sum_{j=0}^n v^\varrho(j,k,n)F_{n,j}^\varrho(f);\ k=0,1,...,n, \tag{3.26}$$

where the $(n+1)\times(n+1)$ matrix of coefficients $V := [v^\varrho(j,k,n)]_{j,k=0}^n$ is the inverse of $P := [F_{n,j}^\varrho(p_{\varrho,i}^{(n)})]_{i,j=0}^n$.

Proof. The biorthogonality condition $\mu_{\varrho,k}^{(n)}(p_{\varrho,i}^{(n)}) = \delta_{i,k}$ follows easily from (3.19) and (3.20). Using (3.26) it can be written as

$$\sum_{j=0}^n F_{n,j}^\varrho(p_{\varrho,i}^{(n)})v^\varrho(j,k,n) = \delta_{i,k},$$

i.e., $PV = I$, and so $V = P^{-1}$. □

Theorem 3.4.4. *The eigenfunctions and the dual functionals satisfy the equations*

$$p_{\varrho,k}^{(n)}(x) = (-1)^k p_{\varrho,k}^{(n)}(1-x), \quad \mu_{\varrho,k}^{(n)}(f) = (-1)^k (f \circ R), \tag{3.27}$$

where $R(x) = 1 - x$ is reflection about the point $\frac{1}{2}$. The eigenfunctions of degree ≥ 2 can be factored as follows:

$$\begin{aligned} p_{\varrho,2j}^{(n)}(x) &= x(x-1)q(x-1/2), \\ p_{\varrho,2j+1}^{(n)}(x) &= x(x-1/2)(x-1)q(x-1/2), \quad j = 1,2,\ldots. \end{aligned} \tag{3.28}$$

In each case q is an even monic polynomial.

Proof. From (3.1) it follows that

$$U_n^\varrho(f \circ R) = (U_n^\varrho f) \circ R, \tag{3.29}$$

so that

$$U_n^\varrho(p_{\varrho,k}^{(n)} \circ R) = (U_n^\varrho p_{\varrho,k}^{(n)}) \circ R = \lambda_{\varrho,k}^{(n)}(p_{\varrho,k}^{(n)} \circ R),$$

and $p_{\varrho,k}^{(n)} \circ R$ is a $\lambda_{\varrho,k}^{(n)}$- eigenfunction. For $k = 0, 1$ the property (3.27) of $p_{\varrho,k}^{(n)}$ is obvious, and for $k \geq 2$ the eigenfunction $p_{\varrho,k}^{(n)} \circ R$ must be a scalar multiple of $p_{\varrho,k}^{(n)}$ (the eigenspace is 1- dimensional). By equating the coefficients of x^k yields

$$p_{\varrho,k}^{(n)} = (-1)^k p_{\varrho,k}^{(n)} \circ R.$$

So $p_{\varrho,k}^{(n)}$ is even (odd) about the point $1/2$ when k is even (odd). In particular, the zeros of $p_{\varrho,k}^{(n)}$ are symmetric about $1/2$. Moreover, (3.29) implies that

$$\begin{aligned} \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f \circ R) &= \lambda_{\varrho,k}^{(n)}(p_{\varrho,k}^{(n)} \circ R)\mu_{\varrho,k}^{(n)}(f) \\ &= \lambda_{\varrho,k}^{(n)}(-1)^k p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \end{aligned}$$

and equating the coefficients of $p_{\varrho,k}^{(n)}$ in the preceding relation we get

$$\mu_{\varrho,k}^{(n)}(f) = (-1)^k \mu_{\varrho,k}^{(n)}(f \circ R).$$

Taking $j = k$ in (3.23) and using $S(m, 0) = 0, m \geq 1$, we obtain $c^\varrho(0, k, n) = 0, k \geq 2$. Thus, for $k \geq 2, x = 0$ is a zero of $p_{\varrho,k}^{(n)}$, and by the symmetry property so is $x = 1$. Further, when k is odd the symmetry property of the zeros implies that $x = 1/2$ must be a zero of $p_{\varrho,k}^{(n)}$, which proves (3.28). This completes the proof. □

3.4.2 Asymptotics of the eigenfunctions

We show that for each $\varrho > 0$ and $k \geq 0$ the sequence $(p_{\varrho,k}^{(n)})_{n\geq 1}$ is convergent.

Theorem 3.4.5. *For $0 \leq j \leq k$,*

$$\lim_{n\to\infty} c^\varrho(j,k,n) = c^*(j,k),$$

where

$$c^*(0,1) = -\frac{1}{2}, c^*(j,k) := \prod_{i=1}^{k-j} \frac{(k+1-i)(k-i)}{i(i-2k+1)}, \quad (j,k) \neq (0,1). \tag{3.30}$$

This means that, $p_{\varrho,k}^{(n)}$ converges uniformly on $[0,1]$ to $p_k^* \in \Pi_k$ as $n \to \infty$, where

$$p_k^*(x) := \sum_{j=0}^{k} c^*(j,k)x^j = x^k - \frac{k}{2}x^{k-1} + \frac{k(k-1)(k-2)}{4(2k-3)}x^{k-2} - ....$$

Proof. Noticing that $p_{\varrho,0}^{(n)}(x) = 1 = p_0^*(x), p_{\varrho,1}^{(n)}(x) = x - 1/2 = p_1^*(x)$, it is sufficient to prove the result for $k \geq 2$. This will be done using induction on j in order to prove that $\lim_{n\to\infty} c^\varrho(k-j,k,n)$ exists and is given by (3.30). Since $c^\varrho(k,k,n) = 1$, this result holds for $j = 0$. Suppose it is true for $\lim_{n\to\infty} c^\varrho(k-i,k,n), i = 0,...,j-1$, where $0 < j \leq k$. Since for all $j > 0$,

$$\begin{aligned}\varrho^{k-j}[\varrho^j(n-k+1)^{\bar{j}} - (n\varrho+k-j)^{\bar{j}}] = \quad & -\varrho^{k-1}(\varrho+1)\frac{j(2k-j-1)}{2}n^{j-1} + \\ & + \text{lower order powers of } n,\end{aligned}$$

taking the limit as $n \to \infty$ on both sides of

$$\begin{aligned}c^\varrho(k-j,k,n) \quad := \quad & \frac{(n\varrho)_k}{\varrho^{k-j}[\varrho^j(n-k+1)^{\bar{j}} - (n\varrho+k-j)^{\bar{j}}]} \times \\ & \sum_{i=0}^{j-1}\sum_{l=k-j}^{k-i} \frac{c^\varrho(k-i,k,n)}{(n\varrho)^{\overline{k-i}}} c_{k-i-l}^{(k-i)} \varrho^l S(l,k-j)\end{aligned}$$

and using the induction hypothesis gives

$$\begin{aligned}\lim_{n\to\infty} c^\varrho(k-j,k,n) \quad = \quad & -\frac{\varrho^k}{\varrho^{k-1}(\varrho+1)\frac{j(2k-j-1)}{2}\varrho^{k-j+1}} \times \\ & [c^*(k-j+1,k)c_1^{(k-j+1)}\varrho^{k-j}S(k-j,k-j) \\ & + c^*(k-j+1,k)c_0^{(k-j+1)}\varrho^{k-j+1}S(k-j+1,k-j)].\end{aligned}$$

But $c_0^{(k-j+1)} = 1, c_1^{(k-j+1)} = 1 + 2 + ... + (k-j) = \frac{1}{2}(k-j)(k-j+1), S(k-j,k-j) =$

1 and $S(k-j+1,k-j)=\binom{k-j+1}{2}=\frac{1}{2}(k-j)(k-j+1)$; so we get

$$\begin{aligned}\lim_{n\to\infty} c^\varrho(k-j,k,n) &= \frac{2\varrho}{(\varrho+1)\varrho^{k-j+1}j(j-2k+1)}\times \\ &\quad \left[\frac{(k-j)(k-j+1)}{2}c^*(k-j+1,k)\varrho^{k-j}\right. \\ &\quad \left. + \frac{(k-j)(k-j+1)}{2}c^*(k-j+1,k)\varrho^{k-j+1}\right] \\ &= \frac{2\varrho^{k-j}(\varrho+1)(k-j)(k-j+1)}{2\varrho^{k-j}(\varrho+1)j(j-2k+1)}c^*(k-j+1,k) \\ &= \frac{(k-j)(k-j+1)}{j(j-2k+1)}c^*(k-j+1,k) \\ &= \frac{(k-j)(k-j+1)}{j(j-2k+1)}\prod_{i=1}^{j-1}\frac{(k-i)(k-i+1)}{i(i-2k+1)} \\ &= \prod_{i=1}^{j}\frac{(k-i)(k-i+1)}{i(i-2k+1)},\end{aligned}$$

which completes the induction. □

3.4.3 The structure of the dual functionals

In the first part of this subsection we provide a recurrence relation for calculating the coefficients $v^\varrho(j,k,n)$ of the dual functional $\mu_{\varrho,k}^{(n)}$, i.e.,

$$\mu_{\varrho,k}^{(n)}(f)=\sum_{j=0}^{n}v^\varrho(j,k,n)F_{n,j}^\varrho(f), k=0,1,...,n.$$

Let $n\geq 1$ be fixed. For each $j\in\{0,1,...,n\}$ there exists a unique polynomial $l_{\varrho,j}^{(n)}$ of degree $\leq n$ satisfying

$$F_{n,i}^\varrho(l_{\varrho,j}^{(n)})=\delta_{i,j}. \tag{3.31}$$

Its coefficients can be determined from a system of linear equations with non-zero determinant. Indeed, consider the positive linear functionals $F_{n,i}^\varrho : C[0,1]\to\mathbb{R}$, $\varrho>0$, and search the polynomials $l_{\varrho,j}^{(n)}\in\Pi_n$ of the form $l_{\varrho,j}^{(n)}=c_{j0}e_0+c_{j1}e_1+...+c_{jn}e_n$ so that $F_{n,i}^\varrho(l_{\varrho,j}^{(n)})=\delta_{i,j}$. For a fixed j we have $F_{n,i}^\varrho(l_{\varrho,j}^{(n)})=c_{j0}F_{n,i}^\varrho(e_0)+c_{j1}F_{n,i}^\varrho(e_1)+...+c_{jn}F_{n,i}^\varrho(e_n)=\delta_{i,j}$ which can be written as a system of linear equations:

$$\begin{cases} c_{j0}F_{n,0}^\varrho(e_0)+c_{j1}F_{n,0}^\varrho(e_1)+...+c_{jn}F_{n,0}^\varrho(e_n)=\delta_{0,j} \\ c_{j0}F_{n,1}^\varrho(e_0)+c_{j1}F_{n,1}^\varrho(e_1)+...+c_{jn}F_{n,1}^\varrho(e_n)=\delta_{1,j} \\ ... \\ c_{j0}F_{n,n}^\varrho(e_0)+c_{j1}F_{n,n}^\varrho(e_1)+...+c_{jn}F_{n,n}^\varrho(e_n)=\delta_{n,j}.\end{cases}$$

We claim that

$$A:=\begin{vmatrix} F_{n,0}^\varrho(e_0) & F_{n,0}^\varrho(e_1) & ... & F_{n,0}^\varrho(e_n) \\ F_{n,1}^\varrho(e_0) & F_{n,1}^\varrho(e_1) & ... & F_{n,1}^\varrho(e_n) \\ ... & ... & ... & ... \\ F_{n,n}^\varrho(e_0) & F_{n,n}^\varrho(e_1) & ... & F_{n,n}^\varrho(e_n)\end{vmatrix}\neq 0.$$

We have seen that $F_{n,i}^\varrho(e_m) = \dfrac{(i\varrho)^{\overline{m}}}{(n\varrho)^{\overline{m}}}$, so the determinant becomes

$$A = \begin{vmatrix} 1 & \frac{(0\varrho)^{\overline{1}}}{(n\varrho)^{\overline{1}}} & \frac{(0\varrho)^{\overline{2}}}{(n\varrho)^{\overline{2}}} & \dots & \frac{(0\varrho)^{\overline{n}}}{(n\varrho)^{\overline{n}}} \\ 1 & \frac{(1\varrho)^{\overline{1}}}{(n\varrho)^{\overline{1}}} & \frac{(1\varrho)^{\overline{2}}}{(n\varrho)^{\overline{2}}} & \dots & \frac{(1\varrho)^{\overline{n}}}{(n\varrho)^{\overline{n}}} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{(n\varrho)^{\overline{1}}}{(n\varrho)^{\overline{1}}} & \frac{(n\varrho)^{\overline{2}}}{(n\varrho)^{\overline{2}}} & \dots & \frac{(n\varrho)^{\overline{n}}}{(n\varrho)^{\overline{n}}} \end{vmatrix}$$

Elementary manipulations of the determinant yield that

$$A = \frac{1}{(n\varrho)^{\overline{1}}(n\varrho)^{\overline{2}}\dots(n\varrho)^{\overline{n}}} \begin{vmatrix} 1 & 0\cdot\varrho & (0\cdot\varrho)^2 & \dots & (0\cdot\varrho)^n \\ 1 & \varrho & (\varrho)^2 & \dots & (\varrho)^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n\varrho & (n\varrho)^2 & \dots & (n\varrho)^n \end{vmatrix} =$$

$$= \frac{1}{(n\varrho)^{\overline{1}}(n\varrho)^{\overline{2}}\dots(n\varrho)^{\overline{n}}} \prod_{0\le i<j\le n} (j\cdot\varrho - i\cdot\varrho) \neq 0,$$

which means that $l_{\varrho,j}^{(n)}$ is uniquely determined. We have by (3.1),

$$U_n^\varrho(l_{\varrho,j}^{(n)}) = \sum_{k=0}^{n} F_{n,k}^\varrho(l_{\varrho,j}^{(n)})p_{n,k}$$

and by (3.20) and (3.26),

$$U_n^\varrho(l_{\varrho,j}^{(n)}) = \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)} \sum_{i=0}^{n} v^\varrho(i,k,n) F_{n,i}^\varrho(l_{\varrho,j}^{(n)}).$$

By using (3.31) and (3.24) we get successively

$$\begin{aligned} p_{n,j}(x) &= \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)}(x) v^\varrho(j,k,n), \quad j=0,1,\dots,n, \\ \tbinom{n}{j} x^j (1-x)^{n-j} &= \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} \sum_{s=0}^{k} c^\varrho(s,k,n) x^s v^\varrho(j,k,n), \\ \tbinom{n}{j} x^j \sum_{i=0}^{n-j} (-1)^i \tbinom{n-j}{i} x^i &= \sum_{s=0}^{n}\sum_{l=s}^{n} \lambda_{\varrho,l}^{(n)} c^\varrho(s,l,n) v^\varrho(j,l,n) x^s. \end{aligned}$$

For $i = n-j-k$, equating the coefficients of x^{n-k} we get

$$(-1)^{n-j-k}\binom{n}{j}\binom{n-j}{k} = \sum_{l=n-k}^{n} \lambda_{\varrho,l}^{(n)} c^\varrho(n-k,l,n) v^\varrho(j,l,n).$$

Setting now $s = n-l$, we get

$$\begin{aligned} (-1)^{n-j-k}\tbinom{n}{j}\tbinom{n-j}{k} = & \sum_{s=0}^{k-1} \lambda_{\varrho,n-s}^{(n)} v^\varrho(j,n-s,n) c^\varrho(n-k,n-s,n) + \\ & \lambda_{\varrho,n-k}^{(n)} v^\varrho(j,n-k,n). \end{aligned}$$

For $k = 0$ this reduces to

$$v^\varrho(j,n,n) = (-1)^{n-j}\frac{(n\varrho)^{\overline{n}}}{\varrho^n j!(n-j)!}, \tag{3.32}$$

while for $k = 1, ..., n$ we get

$$\begin{aligned} v^\varrho(j,n-k,n) &= \frac{(-1)^{n-j-k}(n\varrho)^{\overline{n-k}}}{\varrho^{n-k}j!(n-j-k)!} - \\ &- \sum_{s=0}^{k-1}\tfrac{k!}{s!}\varrho^{k-s}\frac{(n\varrho)^{\overline{n-k}}}{(n\varrho)^{\overline{n-s}}}v^\varrho(j,n-s,n)c^\varrho(n-k,n-s,n). \end{aligned} \tag{3.33}$$

Now (3.32) and (3.33) constitute the required recurrence.

In the sequel we shall study the limits of the dual functionals, acting on polynomials, as $n \to \infty$. Consider the linear functionals $\mu_k^* : C[0,1] \to \mathbb{R}$ defined by

$$\begin{aligned} &\mu_0^*(f) := \frac{f(0)+f(1)}{2}, \quad \mu_1^*(f) := f(1) - f(0), \\ &\mu_k^*(f) := \frac{1}{2}\tbinom{2k}{k}\left((-1)^k f(0) + f(1) - k\int_0^1 f(x)P_{k-2}^{(1,1)}(2x-1)dx\right), k \geq 2, \end{aligned}$$

where $(P_j^{(1,1)}(x))_{j\geq 0}$ are the Jacobi polynomials, orthogonal with respect to the weight $(1-t)(1+t)$ on the interval $[-1,1]$.
These functionals were introduced in [18], where it was proved that they are limits of the dual functionals in the setting of Bernstein operators. We shall obtain a similar result for the operators U_n^ϱ.

Theorem 3.4.6. *Let $k \geq 0$ and $\varrho > 0$ be fixed. For every $f \in \Pi$,*

$$\lim_{n\to\infty} \mu_{\varrho,k}^{(n)}(f) = \mu_k^*(f). \tag{3.34}$$

Proof. First we prove that for each $j \geq 0$,

$$\lim_{n\to\infty} \mu_{\varrho,j}^{(n)}(f) = \mu_j^*(f), \;\; f \in \Pi_j. \tag{3.35}$$

So, let $f \in \Pi_j$. Because U_n^ϱ is degree reducing and $\lim_{n\to\infty} U_n^\varrho f = f$ (see [45], [46]), we have

$$U_n^\varrho f = \sum_{i=0}^{j} \lambda_{\varrho,i}^{(n)} p_{\varrho,i}^{(n)} \mu_{\varrho,i}^{(n)}(f) \to f = \sum_{i=0}^{j} p_i^* \mu_i^*(f), \; n \to \infty.$$

The last equality is a consequence of [18, (4.18)]. Since the above convergence takes place in the finite dimensional space Π_j, we may consider the coefficients of x^j in order to obtain

$$\lambda_{\varrho,j}^{(n)} \mu_{\varrho,j}^{(n)}(f) \to \mu_j^*(f).$$

Together with $\lambda_{\varrho,j}^{(n)} \to 1$, this leads to (3.35).
We shall prove by induction on $r \geq 0$ that

$$\lim_{n\to\infty} \mu_{\varrho,k}^{(n)}(f) = \mu_k^*(f), \text{ for all } k \geq 0, \;\; f \in \Pi_{k+r}, \tag{3.36}$$

and this will complete the proof of (3.34).
For $r = 0$, (3.36) is a consequence of (3.35). Suppose that (3.36) is also true for $1, ..., r-1$, and let $f \in \Pi_{k+r}$. As before, we have

$$U_n^\varrho f = \sum_{i=0}^{k+r} \lambda_{\varrho,i}^{(n)} p_{\varrho,i}^{(n)} \mu_{\varrho,i}^{(n)}(f) \to f = \sum_{i=0}^{k+r} p_i^* \mu_i^*(f).$$

By considering the coefficients of x^k as $n \to \infty$ we get

$$\lambda_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f) + \sum_{i=1}^{r} \lambda_{\varrho,k+i}^{(n)} c^\varrho(k,k+i,n) \mu_{\varrho,k+i}^{(n)}(f) \to \mu_k^*(f) + \sum_{i=1}^{r} c^*(k,k+i) \mu_{k+i}^*(f). \tag{3.37}$$

We know that for all $i = 1, ..., r$,

$$\lambda_{\varrho,k+i}^{(n)} \to 1, c^\varrho(k,k+i,n) \to c^*(k,k+i).$$

By the induction hypothesis, $\mu_{\varrho,k+i}^{(n)}(f) \to \mu_{k+i}^*(f),\ i = 1, ..., r$. Now (3.37) implies

$$\lambda_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f) \to \mu_k^*(f),$$

and so $\mu_{\varrho,k}^{(n)}(f) \to \mu_k^*(f)$. This concludes the induction. □

Remark 3.4.7. *For $\varrho \to \infty$, each result of this section has a corresponding one in [18], concerning the Bernstein operators B_n.*
For $\varrho = 1$ we cover some results concerning the eigenstructure of the genuine Bernstein-Durrmeyer operators, scattered in the literature; see, e.g., [31], [39], [40] and the references therein.

3.5 Variation diminution

In Chapter 2 we have proved that all the operators that belong to the BEJ classes have this property. In particular, for B_n and U_n the proof can be found in [88] and [30], respectively.
In what follows we present the detailed proof for U_n^ϱ.

Theorem 3.5.1. *The operators U_n^ϱ have the (strong) variation-diminishing property, that is,*

$$S_{[0,1]}[U_n^\varrho f] \le S_{[0,1]}[f] \text{ for all } f \in C[0,1].$$

Proof. We use the fact that $U_n^\varrho = B_n(\mathcal{B}_{n\varrho}^{-1,-1})$ and that the Bernstein operators B_n are (strongly) variation–diminishing. Thus we have

$$S_{[0,1]}[U_n^\varrho f] \le S_{[0,1]}[\mathcal{B}_{n\varrho}^{-1,-1} f] = S_{[0,1]}[\left[\int_0^1 t^{n\varrho x-1}(1-t)^{n\varrho-n\varrho x-1} f(t)\,dt\right].$$

Substituting $\left(\frac{t}{1-t}\right)^{n\varrho} = u$ the above integral becomes

$$\frac{1}{n\varrho} \int_0^\infty u^x \cdot \frac{u^{\frac{1}{n\varrho}-2}}{(u^{\frac{1}{n\varrho}}+1)^{n\varrho}} \cdot f\left(\frac{u^{\frac{1}{n\varrho}}}{u^{\frac{1}{n\varrho}}+1}\right) du.$$

Obviously, the number of sign changes of $f(t), t \in [0,1]$ equals the number of sign changes of the function $g(u) = f\left(\frac{u^{\frac{1}{n\varrho}}}{u^{\frac{1}{n\varrho}}+1}\right), u \in [0,\infty)$. Applying Theorem 1.1.31 for the functional $A(g) = \int_0^\infty g(u)\,du$ with $w(u) = \frac{u^{\frac{1}{n\varrho}-2}}{(u^{\frac{1}{n\varrho}}+1)^{n\varrho}}$ we get that the operators U_n^ϱ have the (strong) variation–diminishing property on $C[0,1]$. □

Remark 3.5.2. *As degree $U_n^\varrho e_i = i,\ i = 0,1,\ldots,n$ (with $e_i(x) = x^i$, see [45, Lemma 3.5]) and U_n^ϱ have the (strong) variation–diminishing property, it follows from [29, Theorem 7] that $U_n^\varrho, n \in \mathbb{N}$, preserve the convexity of order i, for $i = 0,1,\ldots,n$ (i.e., $U_n^\varrho f$ is convex of order i, provided that f is convex of order i). This preservation of convexity by U_n^ϱ was proved first by H. Gonska and R. Păltănea (see [45, Theorem 4.1], where also more details about the terminology and historical references can be found) and also at the end of Section 3.1, both using different methods.*

3.6 Global smoothness preservation

Over the last decades there has been considerable interest in the preservation of global smoothness in various contexts. This intensive research culminated in the book by G. Anastassiou and S. Gal [8].
The results in this section generalize the corresponding statements available in the literature for both Bernstein (see [19]) and genuine Bernstein–Durrmeyer operators (see [55, S.3.3.2]) and they supplement results on the behavior of the operators U_n^ϱ with respect to Lipschitz classes very recently given in [85].
To that end, we use first the following result given earlier by C. Cottin and H. Gonska [19, Theorem 2.2].

Lemma 3.6.1. *Let $k \geq 0$ and $s \geq 1$ be integers, and let $I = [a,b]$ and $I' = [c,d] \subset [a,b]$ be compact intervals with non-empty interior. Furthermore, let $L : C^k(I) \to C^k(I')$ be a linear operator having the following properties:*

(i) *L is convex of orders $k-1$ and $k+s-1$,*

(ii) *L maps $C^{k+s}(I)$ into $C^{k+s}(I')$,*

(iii) $L(\Pi_{k-1}) \subseteq \Pi_{k-1}$ *and* $L(\Pi_{k+s-1}) \subseteq \Pi_{k+s-1}$

(iv) $L(C^k(I)) \not\subset \Pi_{k-1}$

Then for all $f \in C^k(I)$ and all $\delta \geq 0$ we have

$$K_s(D^k Lf;\delta)_{I'} \leq \frac{1}{k!}||D^k Le_k|| \cdot K_s\left(f^{(k)}; \frac{1}{(k+s)^{\underline{s}}} \frac{||D^{k+s} Le_{k+s}||}{||D^k Le_k||}\delta\right). \tag{3.38}$$

First we provide the corresponding quantitative statement regarding the smoothing effect of the operators U_n^ϱ.

Theorem 3.6.2. *Let $k \geq 0$ and $s \geq 1$ be fixed integers. Then for all $n \geq k+s$, all $f \in C^k[0,1]$ and all $\delta \geq 0$ the following inequality holds*

$$K_s(D^k U_n^\varrho f;\delta)_{[0,1]} \leq \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} K_s\left(f^{(k)};\varrho^s \frac{(n-k)^{\underline{s}}(n\varrho)^{\overline{k}}}{(n\varrho)^{\overline{k+s}}}\delta\right)_{[0,1]}. \tag{3.39}$$

Proof. We verify each statement of Lemma 3.6.1.
i) According to [45, Theorem 4.1]: The operators $U_n^\varrho, \varrho > 0, n \geq 1$, are convex of order $r-1$ for all $0 \leq r \leq n$.
ii) U_n^ϱ is a polynomial operator so the general assumption and condition (ii) is satisfied.
iii) According to [45, Corollary 4.2]: $U_n^\varrho(\Pi_{k-1}) \subseteq \Pi_{k-1}$ for $\varrho > 0, 0 \leq k \leq n$.
iv) Consider the k-th monomial $e_k \in C^k[0,1]$. From the assumption that $n \geq k+s$ it follows that $U_n^\varrho e_k \in \Pi_k \setminus \Pi_{k-1}$, so that condition (iv) is also verified.
The images of the monomials under U_n^ϱ can be written in the form given in (3.13). Since (see [33, p.429])

$$D^l B_n e_l = \frac{n^{\underline{l}}}{n^l} l!$$

then

$$D^m U_n^\varrho e_m = m!\varrho^m \frac{n^{\underline{m}}}{(n\varrho)^{\overline{m}}}, \quad m \in \{k, k+s\}.$$

Plugging these expressions into inequality (3.38) we get (3.39). □

We now consider two special cases of $s \geq 1$ which are of particular interest. The first is the case $s = 1$ leading to

Proposition 3.6.3. *Let $k \geq 0$ be a fixed integer. Then for all $n \geq k+1, f \in C^k[0,1]$ and $\delta \geq 0$ we have*

$$\begin{aligned}\omega_1(D^k U_n^\varrho f;\delta) &\leq \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} \widetilde{\omega}_1\left(f^{(k)};\frac{\varrho(n-k)}{n\varrho+k}\delta\right)\\ &\leq 1 \cdot \widetilde{\omega}_1(f^{(k)};\delta) \leq 2 \cdot \omega_1(f^{(k)};\delta).\end{aligned}$$

where $\widetilde{\omega}_1(f,\cdot)$ denotes the least concave majorant of $\omega_1(f,\cdot)$ and is given by

$$\widetilde{\omega}_1(f,t) := \begin{cases} \sup\limits_{\substack{0 \leq x \leq t \leq y \leq 1\\ x \neq y}} \dfrac{(t-x)\omega_1(f,y)+(y-t)\omega_1(f;x)}{y-x}, & \text{for } 0 \leq t \leq 1,\\ \omega_1(f,t), & \text{for } t > 1. \end{cases}$$

The leftmost inequality is best possible in the sense that for e_{k+1} both sides are equal and do not vanish.

Proof. Theorem 3.6.2 gives in this particular case

$$K_1(D^k U_n^\varrho f;\delta)_{[0,1]} \leq \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} K_1\left(f^{(k)};\frac{\varrho(n-k)}{(n\varrho+k)}\delta\right)_{[0,1]}.$$

For the K-functional K_1 it is known from Brudnyĭ's representation theorem (see, e.g. [70, p.1258]) that $K_1(f,\delta) = \dfrac{1}{2}\widetilde{\omega}_1(f,2\delta)$. Using this representation on both sides of

the inequality involving K_1 and (1.3) leads to our first assertion.
Furthermore, for the function $e_{k+1}(x) = x^{k+1}$ it can be easily verified by using the property $\omega(c \cdot e_1 + d \cdot e_0 : \delta) = |c| \cdot \delta, c, d \in \mathbb{R}$ (the same for $\widetilde{\omega}_1$). Thus, for $n \geq k+1$ and $\delta > 0$, both sides in the leftmost inequality above equal

$$(k+1)! \cdot \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} \cdot \frac{\varrho(n-k)}{n\varrho + k} \cdot \delta > 0.$$

□

Thus it follows

Corollary 3.6.4. *For a fixed integer $k \geq 0$ the following assertion holds for all $n \in \mathbb{N}$. If $f^{(k)} \in Lip_M(\tau; [0,1])$ for some $M \geq 0$ and some $0 < \tau \leq 1$, then $D^k U_n^{\varrho} f$ is in the same Lipschitz class.*

The second case we discuss in more detail is $s = 2$. Here we get

Proposition 3.6.5. *Let $k \geq 0$ be a fixed integer. Then for all $n \geq k+2, f \in C^k[0,1]$ and $\delta \geq 0$ we have*

$$\begin{aligned} \omega_2(D^k U_n^{\varrho} f; \delta) &\leq 3 \cdot \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} \left[1 + \varrho^2 \frac{(n-k)(n-k-1)}{2(n\varrho+k)(n\varrho+k+1)} \right] \omega_2\left(f^{(k)}; \delta\right) \\ &\leq \frac{9}{2} \omega_2(f^{(k)}; \delta). \end{aligned}$$

Proof. From Theorem 3.6.2 with $s = 2$ we arrive at

$$\begin{aligned} K_2(D^k U_n^{\varrho} f; \delta)_{[0,1]} &\leq \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} K_2\left(f^{(k)}; \varrho^2 \frac{(n-k)(n-k-1)}{(n\varrho+k)(n\varrho+k+1)} \delta\right)_{[0,1]} \\ &\leq K_2(f^{(k)}; \delta)_{[0,1]}. \end{aligned}$$

In our further argumentation we shall employ Žuk's function $Z_h f$ defined in Section 1.1.3. Thus we avoid using the statement of Theorem 3.6.2 and the equivalence between the K-functional and the modulus ω_2, which would deteriorate the constants. First recall the identity

$$K_2(f; \delta) = K(f; \delta; C[0,1], C^2[0,1]) = K(f; \delta; C[0,1], W_{2,\infty}[0,1]).$$

Let now $f \in C^k[0,1], 0 < \delta < \frac{1}{2}$ be arbitrary given, and let $|h| \leq \delta$. Then for a typical difference figuring in the definition of $\omega_2(D^k U_n^{\varrho} f; \delta)$ we have

$$\begin{aligned} &|D^k U_n^{\varrho} f(x-h) - 2D^k U_n^{\varrho} f(x) + D^k U_n^{\varrho} f(x+h)| = \\ &\quad |\{D^k U_n^{\varrho}(f-g; x-h) - 2D^k U_n^{\varrho}(f-g; x) + D^k U_n^{\varrho}(f-g; x+h)\} + \\ &\qquad \{D^k U_n^{\varrho}(g; x-h) - 2D^k U_n^{\varrho}(g; x) + D^k U_n^{\varrho}(g; x+h)\}| \end{aligned}$$

where $g \in C^k[0,1]$ with $g^{(k)} \in W_{2,\infty}[0,1]$ arbitrarily chosen.
The absolute value of the first term in braces can be estimated from above by

$$4||D^k U_n^{\varrho}(f-g)||_\infty \leq 4\varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} ||(f-g)^{(k)}||_\infty.$$

For the modulus of the second expression in braces we have

$$\begin{aligned}
&|D^kU_n^\varrho(g;x-h) - 2D^kU_n^\varrho(g;x) + D^kU_n^\varrho(g;x+h)| \\
&= |D^{k+2}U_n^\varrho(g;\xi)| \cdot h^2 \text{ (for some } \xi \text{ between } x-h \text{ and } x+h) \\
&\leq |D^{k+2}U_n^\varrho g| \cdot h^2 \leq \varrho^{k+2} \frac{n^{\underline{k+2}}}{(n\varrho)^{\overline{k+2}}} \cdot h^2 \cdot ||g^{(k+2)}||_{L_\infty}.
\end{aligned}$$

We substitute now the function $g^{(k)} \in W_{2,\infty}[0,1]$ by Žuk's function $Z_h(f^{(k)})$, yielding

$$||(f-g)^{(k)}|| = ||f^{(k)} - Z_h(f)|| \leq \frac{3}{4} \cdot \omega_2(f^{(k)};h)$$

and

$$||g^{(k+2)}||_{L_\infty} = ||Z_h''(f)||_{L_\infty} \leq \frac{3}{2} \cdot \frac{1}{h^2} \cdot \omega_2(f^{(k)};h).$$

Combining these estimates and taking into account the preceding steps we obtain

$$\begin{aligned}
\omega_2(D^kU_n^\varrho f;\delta) &\leq 4 \cdot \frac{3}{4}\varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}}\omega_2(f^{(k)};\delta) + \frac{3}{2} \cdot \varrho^{k+2} \frac{n^{\underline{k+2}}}{(n\varrho)^{\overline{k+2}}} \cdot \omega_2(f^{(k)};h) \\
&= 3 \cdot \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} \left[1 + \varrho^2 \frac{(n-k)(n-k-1)}{2(n\varrho+k)(n\varrho+k+1)}\right] \omega_2\left(f^{(k)};\delta\right) \\
&\leq \frac{9}{2}\omega_2(f^{(k)};\delta).
\end{aligned}$$

□

Defining Lipschitz classes with respect to the second order modulus by

$$\text{Lip}_M^*(\tau,[0,1]) := \left\{ f \in C[0,1] : \omega_2(f;\delta) \leq M \cdot \delta^\tau,\ 0 \leq \delta \leq \frac{1}{2} \right\},\ 0 < \tau \leq 2,$$

we get

Corollary 3.6.6. *For a fixed integer $k \geq 0$ the following assertion holds for all $n \in \mathbb{N}$. If $f^{(k)} \in Lip_M^*(\tau;[0,1])$ for some $M \geq 0$ and some $0 < \tau \leq 2$, then*

$$D^kU_n^\varrho f \in Lip_{4.5M}^*(\tau;[0,1]).$$

3.7 Strong Voronovskaya-type inequality

In the attempt to prove a strong converse inequality of type B, as defined by Z. Ditzian and K.G. Ivanov in [22] we came across the following strong Voronovskaya-type inequality which is of interest by itself. The reason why we call this inequality "strong" is that in addition to the convergence of $U_n^\varrho f - f$ towards $\frac{(\varrho+1)}{2(n\varrho+1)}\varphi^2 f''$ it also expresses the degree of approximation depending on the smoothness properties of the function.

Theorem 3.7.1. *For $f \in C^3[0,1], 1 < p \leq \infty$ and U_n^ϱ given by (3.1), we have*

$$\left|\left|U_n^\varrho f - f - \frac{(\varrho+1)}{2(n\varrho+1)}\varphi^2 f''\right|\right|_p \leq C(p)C(\varrho)\left\{\frac{||\varphi^3 f'''||_p}{(n\varrho+1)\sqrt{\varrho[(n-1)\varrho+1]}} + \right. \\ \left. + \frac{\varrho[(n-1)\varrho+1]}{(n\varrho+1)^4}||f'''||_p\right\} \tag{3.40}$$

where $\varphi(x) = \sqrt{x(1-x)}$ *and* $C(\varrho) = \varrho^2 d_\varrho + \varrho(\varrho+1)\sqrt{c_\varrho}$ *with* $c_\varrho = \dfrac{5\varrho^2 + 13\varrho + 12}{\varrho^2}$, $d_\varrho = 6 + \dfrac{(\varrho+1)(\varrho+2)}{\varrho^2}$.

Proof. We expand $f(t)$ by the Taylor formula

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \frac{1}{2}\int_x^t (t-v)^2 f'''(v)dv. \tag{3.41}$$

Taking into account that

$$U_n^\varrho(t-x;x) = 0 \text{ and } U_n^\varrho((t-x)^2;x) = \frac{\varrho+1}{n\varrho+1}\varphi^2(x)$$

we get from (3.41)

$$U_n^\varrho f - f - \frac{(\varrho+1)}{2(n\varrho+1)}\varphi^2 f'' = \frac{1}{2}U_n^\varrho\left(\int_x^t (t-v)^2 f'''(v)dv; x\right) := I_n(f;x).$$

We define $\varphi_n(x) = \max\left\{\varphi(x), E := \sqrt{\dfrac{\varrho[(n-1)\varrho+1]}{(n\varrho+1)^2}}\right\}$ and observe that, for v between t and x,

$$\frac{|t-v|}{\varphi_n^2(v)} \leq \frac{|t-v|}{\varphi^2(v)} \leq \frac{|t-x|}{\varphi^2(x)} \text{ and } \frac{|t-v|}{\varphi_n^2(v)} \leq \frac{|t-x|}{E^2}$$

and hence

$$\frac{|t-v|}{\varphi_n^2(v)} \leq \frac{|t-x|}{\varphi_n^2(x)}.$$

Therefore,

$$\begin{aligned} |I_n(f;x)| &= \left|\frac{1}{2}U_n^\varrho\left(\int_x^t (t-v)^2 f'''(v)dv; x\right)\right| \\ &\leq \frac{1}{2}U_n^\varrho\left(\left|\int_x^t (t-v)^2 f'''(v)dv\right|; x\right) \end{aligned}$$

Using inequality (9.6.1) in [23],

$$|I_n(f;x)| \leq U_n^\varrho\left(\frac{|t-x|^3}{\varphi_n^3(x)}\left|\frac{1}{t-x}\int_x^t |\varphi_n^3(v)f'''(v)|dv\right|; x\right).$$

We write $g(v) := |\varphi_n^3(v)f'''(v)|$ and $M(g,x) := \sup\limits_{0\le t\le 1}\left|\frac{1}{t-x}\int\limits_x^t g(v)dv\right|$ (see e.g. [98, Chapter 2]).

$$|I_n(f;x)| \le U_n^{\varrho}\left(\frac{|t-x|^3}{\varphi_n^3(x)}M(g,x);x\right) = \frac{M(g,x)}{\varphi_n^3(x)}U_n^{\varrho}(|t-x|^3;x). \tag{3.42}$$

To evaluate the expression $\frac{U_n^{\varrho}(|t-x|^3;x)}{\varphi_n^3(x)}$ we split the interval $[0,1]$ as suggested in the proof of Theorem 4.2 in [46]. Thus we have

(i) for $x \in \left[0, \frac{\varrho}{n\varrho+1}\right] \cup \left[1-\frac{\varrho}{n\varrho+1}, 1\right] \Leftrightarrow \varphi_n(x) = E \ge \varphi(x)$,

$$\frac{U_n^{\varrho}(|t-x|^3;x)}{\varphi_n^3(x)} \le \frac{\varrho^2 d_{\varrho}}{(n\varrho+1)\sqrt{\varrho[(n-1)\varrho+1]}}.$$

(ii) for $x \in \left[\frac{\varrho}{n\varrho+1}, 1-\frac{\varrho}{n\varrho+1}\right] \Leftrightarrow \varphi_n(x) = \varphi(x) \ge E$,

$$\frac{U_n^{\varrho}(|t-x|^3;x)}{\varphi_n^3(x)} \le \frac{(\varrho+1)\sqrt{\varrho c_{\varrho}}}{(n\varrho+1)^{3/2}}.$$

In general

$$\begin{aligned}\frac{U_n^{\varrho}(|t-x|^3;x)}{\varphi_n^3(x)} &\le \frac{\varrho^2 d_{\varrho}}{(n\varrho+1)\sqrt{\varrho[(n-1)\varrho+1]}} + \frac{\varrho(\varrho+1)\sqrt{c_{\varrho}}}{(n\varrho+1)\sqrt{\varrho(n\varrho+1)}}\\ &\le \frac{\varrho^2 d_{\varrho}+\varrho(\varrho+1)\sqrt{c_{\varrho}}}{(n\varrho+1)\sqrt{\varrho[(n-1)\varrho+1]}} = \frac{C(\varrho)}{(n\varrho+1)\sqrt{\varrho[(n-1)\varrho+1]}}.\end{aligned}$$

Returning to (3.42) we can write

$$|I_n(f;x)| \le \frac{C(\varrho)}{(n\varrho+1)\sqrt{\varrho[(n-1)\varrho+1]}}M(g,x).$$

We now recall an inequality about maximal functions given in [98, Theorem 1 (c)]

$$||M(g,\cdot)||_p \le C(p)||g||_p, \text{ for } 1 < p \le \infty.$$

Therefore

$$\begin{aligned}||I_n(f;x)||_p &\le C(p)\frac{C(\varrho)}{(n\varrho+1)\sqrt{\varrho[(n-1)\varrho+1]}}||\varphi_n^3 f'''||_p\\ &\le C(p)C(\varrho)\left\{\frac{||\varphi^3 f'''||_p}{(n\varrho+1)\sqrt{\varrho[(n-1)\varrho+1]}} + \frac{\varrho[(n-1)\varrho+1]}{(n\varrho+1)^4}||f'''||_p\right\}.\end{aligned}$$

□

Remark 3.7.2. *For the particular case $\varrho \to \infty$, namely for the Bernstein operator (3.40) becomes:*

$$\left|\left|B_n f - f - \frac{1}{2n}\varphi^2 f''\right|\right|_p \le C(p)(7+\sqrt{5})\left\{\frac{1}{n\sqrt{n-1}}||\varphi^3 f'''||_p + \frac{n-1}{n^4}||f'''||_p\right\}.$$

A similar result was proved by Ditzian and Ivanov in [22, Lemma 8.3], namely:

$$\left|\left|B_n f - f - \frac{1}{2n}\varphi^2 f''\right|\right|_\infty \le n^{-3/2}||\varphi^3 f'''||_\infty.$$

If we compare the two inequalities we notice that what we gained through generality we lost in precision, but still we can see that in both cases the order of approximation is the same: $O(n^{-3/2})$.

Remark 3.7.3. *For the particular case $\varrho = 1$, namely for the genuine Bernstein-Durrmeyer operator (3.40) becomes:*

$$\left|\left|U_n f - f - \frac{1}{n+1}\varphi^2 f''\right|\right|_p \le C(p)(12+2\sqrt{30})\left\{\frac{1}{(n+1)\sqrt{n}}||\varphi^3 f'''||_p + \frac{n}{(n+1)^4}||f'''||_p\right\}.$$

We have not found a similar inequality to compare this with. The closest we got is a result of Parvanov and Popov [72]

$$\left|\left|U_n f - f - \frac{1}{n}\varphi^2 f''\right|\right|_\infty \le \frac{1}{2n^2}||\varphi^2(\varphi^2 f'')''||_\infty.$$

We notice that in this case the function must admit a derivative of order four.

3.8 Approximation by powers of U_n^ϱ

3.8.1 Upper and lower inequalities

The operators U_n^ϱ are of the form given in [56] for certain general positive linear operators preserving linear functions, so that we can apply the general results provided there for iterates of such operators. We have namely

$$U_n^\varrho(e_2; x) = \left(1 - \frac{\varrho+1}{n\varrho+1}\right)x^2 + \frac{\varrho+1}{n\varrho+1}x,$$

Hence an application of Theorem 6 as well as of Corollaries 7, 8 and 10 in [56] (with the coefficient of x^2 in the above $a_n = 1 - \frac{\varrho+1}{n\varrho+1}$) yields the following statements.

Corollary 3.8.1. *Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\Phi : [0,1] \to \mathbb{R}$ be a function such that Φ^2 is concave. Then for $n, k \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1]$ the following pointwise estimate holds for the iterates of U_n^ϱ*

$$|[U_n^\varrho]^k(f;x) - f(x)| \le 2 \cdot K_2^\Phi\left(f; \frac{\varphi^2(x)}{\Phi^2(x)} \cdot \frac{1-(1-\frac{\varrho+1}{n\varrho+1})^k}{2}\right).$$

Corollary 3.8.2. *Let $\Phi : [0,1] \to \mathbb{R}$ be an admissible step–weight function of the Ditzian–Totik modulus and such that Φ^2 is concave. Then for all $n, k \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1]$, we have*

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq c \cdot \omega_2^\Phi \left(f; \frac{\varphi(x)}{\Phi(x)} \cdot \sqrt{\frac{1-(1-\frac{\varrho+1}{n\varrho+1})^k}{2}} \right),$$

where the constant c depends only on the function Φ.
In particular, for $\Phi = \varphi^\lambda$, $\lambda \in [0,1]$, $x \in [0,1]$ we get

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq c \cdot \omega_2^{\varphi^\lambda} \left(f; \varphi^{1-\lambda}(x) \cdot \sqrt{\frac{1-(1-\frac{\varrho+1}{n\varrho+1})^k}{2}} \right).$$

In terms of the classical modulus of smoothness we have

Corollary 3.8.3. *For all $f \in C[0,1]$, $n, k \in \mathbb{N}$, $x \in [0,1]$, and each $h > 0$ we have the following pointwise estimate*

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq \left[1 + \frac{1}{2h^2} \cdot \left(1 - (1 - \frac{\varrho+1}{n\varrho+1})^k \right) \cdot x(1-x) \right] \cdot \omega_2(f;h).$$

Taking, in particular, $h = \sqrt{\left(1-(1-\frac{\varrho+1}{n\varrho+1})^k\right) \cdot x(1-x)}$, and $h = \sqrt{1-(1-\frac{\varrho+1}{n\varrho+1})^k}$, yields

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{\left(1-(1-\frac{\varrho+1}{n\varrho+1})^k\right) \cdot x(1-x)} \right), \text{ and}$$

$$\|[U_n^\varrho]^k f - f\| \leq \frac{9}{8} \cdot \omega_2 \left(f; \sqrt{1-(1-\frac{\varrho+1}{n\varrho+1})^k} \right),$$

respectively.

Furthermore, in terms of the second order Ditzian–Totik modulus we get

Corollary 3.8.4. *For all $f \in C[0,1]$, $n, k \in \mathbb{N}$, and $h \in \left(0, \frac{1}{2}\right]$ there holds the uniform estimate*

$$\|[U_n^\varrho]^k f - f\| \leq \left[1 + \frac{3}{2h^2} \cdot \left(1 - (1-\frac{\varrho+1}{n\varrho+1})^k \right) \right] \cdot \omega_2^\varphi(f;h).$$

For the particular choice $h = \sqrt{1-(1-\frac{\varrho+1}{n\varrho+1})^k}$, this gives

$$\|[U_n^\varrho]^k f - f\| \leq \frac{5}{2} \cdot \omega_2^\varphi \left(f; \sqrt{1-(1-\frac{\varrho+1}{n\varrho+1})^k} \right).$$

Remark 3.8.5. *Note that for $n \in \mathbb{N}$, and $0 < \varrho < \infty$, one has $0 \leq 1 - \dfrac{\varrho+1}{n\varrho+1} < 1$ and $1 - \dfrac{\varrho+1}{n\varrho+1} \to 1$, for $n \to \infty$, so, for k fixed the results in the above imply uniform convergence as $n \to \infty$. For n fixed and $k \to \infty$ one has $[U_n^\varrho]^k \to B_1 f$ (see [46]).*

Applying the general results given above for $k = 1$ (no iterates) we get the following direct estimates, which supplement the corresponding results given by R. Păltănea [75, Theorem 2.3].

Corollary 3.8.6. *Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\Phi : [0,1] \to \mathbb{R}$ be an admissible step–weight function of the Ditzian–Totik modulus such that Φ^2 is concave. Then for $f \in C[0,1]$ and $x \in [0,1]$ the following estimates hold for U_n^ϱ:*

$$|U_n^\varrho(f;x) - f(x)| \le 2 \cdot K_2^\Phi \left(f; \frac{\varphi^2(x)}{\Phi^2(x)} \cdot \frac{\varrho+1}{2(n\varrho+1)} \right)$$

and

$$|U_n^\varrho(f;x) - f(x)| \le c \cdot \omega_2^\Phi \left(f; \frac{\varphi(x)}{\Phi(x)} \cdot \sqrt{\frac{\varrho+1}{2(n\varrho+1)}} \right),$$

where the constant c depends only on the function Φ.
In particular, for $\Phi = \varphi^\lambda,\ \lambda \in [0,1],\ x \in [0,1]$ we get

$$|U_n^\varrho(f;x) - f(x)| \le c \cdot \omega_2^{\varphi^\lambda} \left(f; \varphi^{1-\lambda}(x) \cdot \sqrt{\frac{\varrho+1}{2(n\varrho+1)}} \right).$$

Furthermore, in terms of the second order Ditzian–Totik modulus with $h = \sqrt{\frac{\varrho+1}{n\varrho+1}}$ respectively $h = \sqrt{\frac{1}{n\varrho+1}}$, one has the uniform estimates

$$||U_n^\varrho(f;x) - f(x)|| \le \frac{5}{2} \cdot \omega_2^{\varphi^\lambda} \left(f; \sqrt{\frac{\varrho+1}{n\varrho+1}} \right),$$

$$||U_n^\varrho(f;x) - f(x)|| \le \frac{5+3\varrho}{2} \cdot \omega_2^{\varphi^\lambda} \left(f; \sqrt{\frac{1}{n\varrho+1}} \right).$$

In terms of the classical modulus of smoothness we get for the particular choices $h = \sqrt{\frac{\varrho+1}{n\varrho+1}x(1-x)}$ respectively $h = \sqrt{\frac{x(1-x)}{n\varrho+1}}$ the local estimates

$$|U_n^\varrho(f;x) - f(x)| \le \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{\frac{\varrho+1}{n\varrho+1}x(1-x)} \right),$$

$$|U_n^\varrho(f;x) - f(x)| \le \frac{3+\varrho}{2} \cdot \omega_2 \left(f; \sqrt{\frac{x(1-x)}{n\varrho+1}} \right),$$

and for $h = \sqrt{\frac{\varrho+1}{n\varrho+1}}$ respectively $h = \sqrt{\frac{1}{n\varrho+1}}$ the global estimates

$$|U_n^\varrho(f;x) - f(x)| \le \frac{9}{8} \cdot \omega_2 \left(f; \sqrt{\frac{\varrho+1}{n\varrho+1}} \right), \tag{3.43}$$

$$|U_n^\varrho(f;x) - f(x)| \le \frac{9+\varrho}{8} \cdot \omega_2 \left(f; \sqrt{\frac{1}{n\varrho+1}} \right). \tag{3.44}$$

Remark 3.8.7. *The reason we choose two different representations for h is that one case is better suited for the instance when $\varrho \to \infty$, and the other for small values of ϱ. However these are not the only choices available. One can manipulate h in order to get the best possible estimate.*

Concerning the magnitude of the constants appearing in inequalities like (3.44), we show

Theorem 3.8.8. *The best possible constant c in the uniform estimate*

$$|U_n^{\varrho}(f;x) - f(x)| \le c \cdot \omega_2\left(f; \sqrt{\frac{1}{n\varrho+1}}\right). \tag{3.45}$$

cannot be smaller than 1 *for* $1 \le \varrho < \infty$.

Proof. Recall that for convex functions $f \in C[0,1]$ one has $U_n^{\varrho} \ge f$ and $B_n f \ge f$. Moreover, according to Lemma 3.1.2, it holds $U_n^{\varrho} f \ge B_n f$, thus

$$0 \le B_n f(x) - f(x) \le U_n^{\varrho} f(x) - f(x),\ x \in [0,1],$$

implying

$$||B_n f - f|| \le ||U_n^{\varrho} f - f||.$$

Let now n and ϱ be fixed, $0 < \varepsilon < \frac{1}{n\varrho}$, and consider the convex function

$$f_\varepsilon(x) = \begin{cases} 0, & 0 \le x \le 1-\varepsilon, \\ \frac{1}{\varepsilon}x + 1 - \frac{1}{\varepsilon}, & 1-\varepsilon < x \le \varepsilon. \end{cases}$$

We have

$$B_n f_\varepsilon(x) = \sum_{k=0}^{n-1} p_{n,k}(x) f_\varepsilon\left(\frac{k}{n}\right) + x^n \cdot f_\varepsilon(1) = x^n,$$

thus

$$||B_n f_\varepsilon - f_\varepsilon|| = \max_{x\in[0,1]} \left(B_n f_\varepsilon(x) - f_\varepsilon(x)\right) = B_n f_\varepsilon(1-\varepsilon) - f_\varepsilon(1-\varepsilon) = (1-\varepsilon)^n.$$

Next we compute

$$\omega_2\left(f_\varepsilon; \frac{1}{\sqrt{n\varrho+1}}\right) = \sup_{\substack{|h| \le \frac{1}{\sqrt{n\varrho+1}} \\ x \pm h \in [0,1]}} |f_\varepsilon(x-h) - 2f_\varepsilon(x) + f_\varepsilon(x+h)| = f_\varepsilon(1) = 1,$$

since the largest possible value for the second order difference is obtained for $x = 1-\varepsilon, h = \varepsilon (< \frac{1}{n\varrho} \le \frac{1}{\sqrt{n\varrho+1}})$. Hence, there holds

$$\frac{||B_n f_\varepsilon - f_\varepsilon||}{\omega_2\left(f_\varepsilon; \frac{1}{\sqrt{n\varrho+1}}\right)} \le (1-\varepsilon)^n.$$

Assume that there exists a constant a such that
$$\frac{||B_n f_\varepsilon - f_\varepsilon||}{\omega_2\left(f_\varepsilon; \dfrac{1}{\sqrt{n\varrho+1}}\right)} \leq a < 1.$$
This gives $(1-\varepsilon)^n \leq a < 1$. With n fixed, we let $\varepsilon \to 0$, and obtain a contradiction ($\lim\limits_{\varepsilon\to 0}(1-\varepsilon)^n < 1$). Thus
$$\|B_n f_\varepsilon - f_\varepsilon\| \leq c_1 \cdot \omega_2\left(f_\varepsilon; \sqrt{\frac{1}{n\varrho+1}}\right) \text{ with } c_1 \geq 1.$$
Since
$$||B_n f_\varepsilon - f_\varepsilon|| \leq ||U_n^\varrho f_\varepsilon - f_\varepsilon||,$$
it follows that
$$||U_n^\varrho f_\varepsilon - f_\varepsilon|| \leq c \cdot \omega_2\left(f_\varepsilon; \sqrt{\frac{1}{n\varrho+1}}\right), \text{ with } c \geq 1,$$
so for the best constant in (3.45) it also holds $c \geq 1$. □

We apply now the results of Corollaries 13 – 16 in [56] for iterates of U_n^ϱ. This shows that lower inequalities in terms of the classical moduli, corresponding to the upper ones in the above, are not possible. More precisely, for $k \in \mathbb{N}$ fixed, we have

Corollary 3.8.9. *Lower inequalities of the form*
$$C(f)\omega_2\left(f; \sqrt{1-(1-\frac{\varrho+1}{n\varrho+1})^k}\right) \leq \|[U_n^\varrho]^k(f) - f\| \quad \text{for all } f \in C[0,1]$$
do not hold.

Corollary 3.8.10. *The lower pointwise estimates*
$$C(f)\omega_2\left(f; \sqrt{\left(1-(1-\frac{\varrho+1}{n\varrho+1})^k\right) x(1-x)}\right) \leq |[U_n^\varrho]^k(f;x) - f(x)| \text{ for } f \in C[0,1]$$
do not hold.

Corollary 3.8.11. *Let $0 < \lambda \leq 1$ be fixed. The lower pointwise estimates*
$$C(f)\omega_2\left(f; \varphi^{1-\lambda}(x)\sqrt{\frac{1-(1-\frac{\varrho+1}{n\varrho+1})^k}{2}}\right) \leq |[U_n^\varrho]^k(f;x) - f(x)|, \ f \in C[0,1],$$
do not hold.

Moreover, we have

Corollary 3.8.12. *For $l \geq 3$ it is not possible to have an inequality of the type*
$$C(f) \cdot \omega_l\left(f; \sqrt[l]{1-(1-\frac{\varrho+1}{n\varrho+1})^k}\right) \quad \leq \quad \|[U_n^\varrho]^k(f) - f\|$$
for all $f \in C[0,1]$ and all $n \in \mathbb{N}$.

3.8.2 Applications using the eigenstructure

By Theorem 3.4.2,

$$(U_n^\varrho)^i f = \sum_{k=0}^{n} (\lambda_{\varrho,k}^{(n)})^i p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad f \in C[0,1],\ i = 1,2,.... \tag{3.46}$$

The linear function $B_1(f;x) = f(0)(1-x) + f(1)x$ is the uniform limit of the overiterated operator images $(U_n^\varrho)^i f$, as $i \to \infty$ according to [46, Remark 3.2]. More generally we have

Corollary 3.8.13. *Suppose $(g_j)_{j\geq 1}$ is a sequence of polynomials with $g_j(0) = 0$ and*

$$\lim_{j\to\infty} g_j(\lambda_{\varrho,k}^{(n)}) = G(\varrho,k,n), \quad k = 0,1,...,n.$$

Then

$$\lim_{j\to\infty} (g_j(U_n^\varrho))f = \sum_{k=0}^{n} G(\varrho,k,n) p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \tag{3.47}$$

the convergence being uniform.

Proof. By using (3.46) we get

$$(g_j(U_n^\varrho))f = \sum_{k=0}^{n} g_j(\lambda_{\varrho,k}^{(n)}) p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad f \in C[0,1],\ j = 1,2,....$$

Letting $j \to \infty$ yields (3.47). □

Lemma 3.8.14. *Suppose that j_n is a sequence of positive integers with*

$$\lim_{n\to\infty} \frac{j_n}{n} = t,$$

then

$$\lim_{n\to\infty} (\lambda_{\varrho,k}^{(n)})^{j_n} = e^{-\frac{k(k-1)}{2}(\frac{1}{\varrho}+1)t}, \text{ for all } k,\ 0 \leq t < \infty, \tag{3.48}$$

and

$$\lim_{n\to\infty} (\lambda_{\varrho,k}^{(n)})^{j_n} = 0, \text{ for all } k \geq 2,\ t = \infty. \tag{3.49}$$

Proof. Let

$$y_n = (\lambda_{\varrho,k}^{(n)})^{j_n - nt} = \left[\left(1+\tfrac{1}{n\varrho}\right)^{-1} ... \left(1+\tfrac{k-1}{n\varrho}\right)^{-1} \left(1-\tfrac{1}{n}\right) ... \left(1-\tfrac{k-1}{n}\right)\right]^{j_n - nt}.$$

Then

$$\begin{aligned}
\log y_n &= (j_n - nt)\left[\log\left(1+\frac{1}{n\varrho}\right)^{-1} + ... + \log\left(1+\frac{k-1}{n\varrho}\right)^{-1} + \right. \\
&\qquad \left. + \log\left(1-\frac{1}{n}\right) + ... + \log\left(1-\frac{k-1}{n}\right)\right] \\
&= \left(\frac{j_n}{n} - t\right)\left(-\frac{k(k-1)}{2}\frac{\varrho+1}{\varrho} + O\left(\frac{1}{n}\right)\right) \\
&\to 0, \quad n \to \infty.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} (\lambda_{\varrho,k}^{(n)})^{j_n - nt} = \lim_{n\to\infty} y_n = 1. \tag{3.50}$$

But

$$\begin{aligned} \lim_{n\to\infty} (\lambda_{\varrho,k}^{(n)})^{nt} &= \lim_{n\to\infty} \left(1+\frac{1}{n\varrho}\right)^{-nt} \dots \left(1+\frac{k-1}{n\varrho}\right)^{-nt} \left(1-\frac{1}{n}\right)^{nt} \dots \left(1-\frac{k-1}{n}\right)^{nt} \\ &= e^{-\frac{k(k-1)}{2}(\frac{1}{\varrho}+1)t}. \end{aligned} \tag{3.51}$$

Combining (3.50) and (3.51) gives (3.48). For $t \to \infty$ we obtain (3.49). □

Corollary 3.8.15. *Suppose that*

$$\lim_{n\to\infty} \frac{j_n}{n} = t.$$

Then for $0 \le t < \infty$,

$$\begin{aligned} \lim_{n\to\infty} (U_n^{\varrho})^{j_n} f &= \sum_{k=0}^{s} e^{-\frac{k(k-1)}{2}(\frac{1}{\varrho}+1)t} p_k^* \mu_k^*(f) \\ &= \sum_{k=0}^{\infty} e^{-\frac{k(k-1)}{2}(\frac{1}{\varrho}+1)t} p_k^* \mu_k^*(f), \ \textit{for all } f \in \Pi_s, \end{aligned} \tag{3.52}$$

and for $t = \infty$,

$$\lim_{n\to\infty} (U_n^{\varrho})^{j_n} f = B_1 f = \sum_{k=0}^{1} p_k^* \mu_k^*(f), \ \textit{for all } f \in \Pi. \tag{3.53}$$

The convergence in (3.52) and (3.53) is uniform.

Proof. Suppose that $f \in \Pi_s$. Since U_n^{ϱ} is degree reducing, (3.46) gives

$$(U_n^{\varrho})^{j_n} f = \sum_{k=0}^{s} (\lambda_{\varrho,k}^{(n)})^{j_n} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), \quad n \ge s.$$

Take the limit as $n \to \infty$ in the above and use Lemma 3.8.14, Theorem 3.4.5 and Theorem 3.4.6 to obtain (3.53) and the first equality in (3.52). The second equality in (3.52) follows from (4.19) in [18]. □

3.9 The difference $U_n^{\varrho} - U_n^{\sigma}$

A first approach in order to study this difference is based on a method presented in [43]. We need the following result from that paper :

Theorem 3.9.1. *Let* $A, B : C[0,1] \to C[0,1]$ *be positive linear operators such that*

$$(A - B)((e_1 - x)^i)(x) = 0 \textit{ for } i = 0, 1, \dots, n \textit{ and } x \in [0,1],$$

also satisfying $Ae_0 = Be_0 = e_0$. *Then for all* $f \in C[0,1], x \in [0,1]$ *we have*

$$|(A - B)(f)(x)| \le c_1 \cdot \omega_{n+1}\left(f; \sqrt{\frac{1}{2}(A+B)(|e_1 - x|^{n+1})(x)}\right).$$

Here c_1 *is an absolute constant independent of* f, x *and* A *and* B, *and* $\omega_{n+1}(f, \cdot)$ *denotes the* $(n+1)$*-st order modulus of smoothness.*

We choose $A = U_n^\varrho$ and $B = U_n^\sigma$. Both operators reproduce linear functions so we have

$$(U_n^\varrho - U_n^\sigma)((e_1 - x)^i)(x) = 0 \text{ for } i = 0, 1, x \in [0,1].$$

We recall equation (3.17), which states that the second moments for U_n^ϱ and U_n^σ are given by

$$M_{n,2}^t(x) = \frac{(t+1)x(1-x)}{nt+1}$$

where $t = \varrho, \sigma$. As a consequence of Theorem 3.9.1 the following statement holds:

Proposition 3.9.2.

$$\begin{aligned} |(U_n^\varrho - U_n^\sigma)(f)(x)| &\leq c_1 \cdot \omega_2 \left(f; \sqrt{\frac{1}{2}(U_n^\varrho + U_n^\sigma)(|e_1 - x|^2)(x)} \right) \\ &\leq c_1 \cdot \omega_2 \left(f; \sqrt{\frac{1}{2} \frac{2n\varrho\sigma + (n+1)(\varrho+\sigma) + 2}{(n\varrho+1)(n\sigma+1)} x(1-x)} \right). \end{aligned}$$

Another approach is described in the sequel. Consider the Beta-type operator $\mathcal{B}_r^{-1,-1}$ as given in Definition 1.3.1. It is not difficult to see that

$$U_n^\varrho = B_n \circ \mathcal{B}_{n\varrho}^{-1,-1}, \tag{3.54}$$

where $B_n : C[0,1] \to \Pi_n$ is the classical Bernstein operator. Taking into consideration Lemma 1.3.4 we are in a position to state

Theorem 3.9.3. *Let $f \in C[0,1], n \geq 1, \varrho > 0, \sigma > 0$. Then*

$$|(U_n^\varrho - U_n^\sigma)f(x)| \leq \frac{9}{4}\omega_2 \left(f; \sqrt{\frac{(n-1)|\sigma - \varrho|}{(n\varrho+1)(n\sigma+1)} x(1-x)} \right),$$

where ω_2 is the second order modulus of smoothness.

Proof. Suppose that $0 < \varrho < \sigma$ and set $r := n\varrho, s := n\sigma$. According to (1.21), we have for each convex function $g \in C[0,1]$,

$$\mathcal{B}_{n\varrho}^{-1,-1} g \geq \mathcal{B}_{n\sigma}^{-1,-1} g.$$

This entails

$$B_n(\mathcal{B}_{n\varrho}^{-1,-1} g) \geq B_n(\mathcal{B}_{n\sigma}^{-1,-1} g). \tag{3.55}$$

Now (3.54) and (3.55) yield

$$U_n^\varrho g \geq U_n^\sigma g, \quad g \in C[0,1] \text{ convex}. \tag{3.56}$$

Let $x \in [0,1]$ be fixed. Consider the functional $\Phi : C[0,1] \to \mathbb{R}$,

$$\Phi(f) := U_n^\varrho f(x) - U_n^\sigma f(x), \ f \in C[0,1].$$

The linear functional Φ is bounded on $C[0,1]$ endowed with the uniform norm; moreover, Φ is different from 0, and according to (3.56),

$$\Phi(g) \geq 0, \quad g \in C[0,1] \text{ convex}.$$

By a result of T. Popoviciu [79] (see also [82]) it follows that for each $f \in C[0,1]$ there exist distinct points t_0, t_1, t_2 in $[0,1]$ such that

$$\Phi(f) = \Phi(e_2)[t_0, t_1, t_2; f], \tag{3.57}$$

where $[t_0, t_1, t_2; f]$ is the divided difference of the function f on the nodes t_0, t_1, t_2. According to [45],

$$U_n^{\varrho} e_2(x) = x^2 + \frac{\varrho+1}{n\varrho+1} x(1-x),$$

so that

$$\Phi(e_2) = U_n^{\varrho} e_2(x) - U_n^{\sigma} e_2(x) = \frac{(n-1)(\sigma-\varrho)}{(n\varrho+1)(n\sigma+1)} x(1-x).$$

On the other hand, if $g \in C^2[0,1]$, then

$$[t_0, t_1, t_2; g] = \frac{1}{2} g''(\xi)$$

for some $\xi \in [0,1]$. Thus (3.57) leads to

$$U_n^{\varrho} g(x) - U_n^{\sigma} g(x) = \frac{(n-1)(\sigma-\varrho)}{(n\varrho+1)(n\sigma+1)} x(1-x) \frac{g''(\xi)}{2}, \; g \in C^2[0,1].$$

This entails

$$|U_n^{\varrho} g(x) - U_n^{\sigma} g(x)| \le \frac{(n-1)(\sigma-\varrho)}{2(n\varrho+1)(n\sigma+1)} x(1-x) ||g''||_{\infty}, \; g \in C^2[0,1].$$

As a consequence of Theorem 1.1.19 and Corollary 1.1.20,
for $h^2 = \dfrac{(n-1)(\sigma-\varrho)}{(n\varrho+1)(n\sigma+1)} x(1-x)$, $\alpha = 2$ and $\beta_2 = \dfrac{h^2}{2}$ we obtain

$$\begin{aligned} |(U_n^{\varrho} - U_n^{\sigma})(f)(x)| &\le \left(2 \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{2}\right) \omega_2 \left(f; \sqrt{\frac{(n-1)|\sigma-\varrho|}{(n\varrho+1)(n\sigma+1)} x(1-x)}\right) \\ &\le \frac{9}{4} \omega_2 \left(f; \sqrt{\frac{(n-1)|\sigma-\varrho|}{(n\varrho+1)(n\sigma+1)} x(1-x)}\right). \end{aligned}$$

□

Remark 3.9.4. *The presence on the difference $|\sigma - \varrho|$ in Theorem 3.9.3 allows us to state that the estimation given there is much better then the one given in Proposition 3.9.2.*

3.10 The commutators $[U_n^{\varrho}; U_n^{\sigma}]$ and $[U_m^{\varrho}; U_n^{\varrho}]$

The problem of studying the commutator $[A; B] := AB - BA$ of two positive linear operators A and B was raised by A. Lupaş in [63]. Some answers to Lupaş's problem can be found in [44]. Here we shall study the commutators $[U_n^{\varrho}; U_n^{\sigma}]$ and $[U_m^{\varrho}; U_n^{\varrho}]$. First of all, we need information about the moments of the investigated operators. Let $M_{n,j}^{\varrho}(x) := U_n^{\varrho}(e_1 - xe_0)^j(x)$, be the j-th moment of U_n^{ϱ}.

By using the recurrence formula for the moments given by (3.16) it is not difficult to prove by induction on j that

$$M_{n,j}^{\varrho}(x) = \mathcal{O}\left(n^{-[\frac{j+1}{2}]}\right) \tag{3.58}$$

uniformly with respect to $x \in [0,1]$. Now let

$$M_{n,r}^{\varrho,\sigma}(x) := U_n^\varrho U_n^\sigma (e_1 - xe_0)^r(x)$$

be the r-th moment of $U_n^\varrho U_n^\sigma$. According to [47, Theorem 4],

$$M_{n,r}^{\varrho,\sigma}(x) = \sum_{\substack{i,k\geq 0\\ i+k=r}} \sum_{j=k}^{r} \binom{r}{k} \frac{1}{(j-k)!} M_{n,j}^{\varrho}(x)(M_{n,i}^{\sigma}(x))^{(j-k)}. \tag{3.59}$$

Combining (3.58) and (3.59) (see also [47, Corollary 1]), we get

$$M_{n,r}^{\varrho,\sigma}(x) = \mathcal{O}\left(n^{-[\frac{r+1}{2}]}\right)$$

uniformly with respect to $x \in [0,1]$. Now by a result of P.C. Sikkema [91] we have

$$U_n^\varrho U_n^\sigma f(x) = \sum_{r=0}^{6} \frac{f^{(r)}(x)}{r!} M_{n,r}^{\varrho,\sigma}(x) + o(n^{-3})$$

uniformly with respect to $x \in [0,1]$, for each $f \in C^6[0,1]$. It follows that for $f \in C^6[0,1]$,

$$(U_n^\varrho U_n^\sigma - U_n^\sigma U_n^\varrho)f(x) = \sum_{r=0}^{6} \frac{f^{(r)}(x)}{r!} (M_{n,r}^{\varrho,\sigma}(x) - M_{n,r}^{\sigma,\varrho}(x)) + o(n^{-3}). \tag{3.60}$$

A combination of hand calculations and MAPLE shows that

$$M_{n,r}^{\varrho,\sigma}(x) - M_{n,r}^{\sigma,\varrho}(x) = 0, r = 0,1,2,3, \tag{3.61}$$

$$\lim_{n\to\infty} n^3(M_{n,4}^{\varrho,\sigma}(x) - M_{n,4}^{\sigma,\varrho}(x)) = \frac{(\sigma-\varrho)(\varrho+1)(\sigma+1)}{\varrho^2\sigma^2} x(1-x), \tag{3.62}$$

$$\lim_{n\to\infty} n^3(M_{n,r}^{\varrho,\sigma}(x) - M_{n,r}^{\sigma,\varrho}(x)) = 0, r = 5,6, \tag{3.63}$$

uniformly with respect to $x \in [0,1]$. From (3.60)-(3.63) we derive

Theorem 3.10.1. *For each $f \in C^6[0,1]$ one has*

$$\lim_{n\to\infty} n^3(U_n^\varrho U_n^\sigma - U_n^\sigma U_n^\varrho)f(x) = \frac{(\sigma-\varrho)(\varrho+1)(\sigma+1)}{\varrho^2\sigma^2} x(1-x)f^{(4)}(x),$$

uniformly with respect to $x \in [0,1]$.

In particular, we see that U_n^ϱ and U_n^σ do not commute if $\varrho \neq \sigma$. On the other hand it is well known (see [31]) that U_n^1 and U_m^1 (i.e., the genuine Bernstein-Durrmeyer operators) do commute. A combination of hand calculations and MAPLE shows that

$$(U_m^\varrho U_n^\varrho - U_n^\varrho U_m^\varrho)e_r(x) = 0, r = 0,1,2,3,$$

and

$$(U_m^\varrho U_n^\varrho - U_n^\varrho U_m^\varrho)e_4(x) = \frac{\varrho^3(\varrho-1)(\varrho+1)^2(m-1)(n-1)(n-m)}{(m\varrho+1)(m\varrho+2)(m\varrho+3)(n\varrho+1)(n\varrho+2)(n\varrho+3)}.$$

We see that U_m^ϱ and U_n^ϱ do not commute if $\varrho \neq 1, m \neq 1, n \neq 1$ and $m \neq n$.

3.11 The behavior of U_n^ϱ with respect to Lipschitz classes of order m

Fix an integer $m \geq 0$ and $M > 0$. We say that a function $f \in C[0,1]$ belongs to the Lipschitz class $Lip_m(M)$ if

$$|\Delta_h^m f(x)| \leq Mh^m$$

for all $x \in [0,1]$ and $h > 0$ such that $x + mh \in [0,1]$; $\Delta_h^m f(x)$ stands for the m-th order difference of f with step h at x. According to [10, Proposition 2.1], $f \in Lip_m(M)$ if and only if $\frac{M}{m!}e_m \pm f$ are m- convex functions.

Theorem 3.11.1. *If $f \in Lip_m(M)$, then for all $n \geq 1, \varrho > 0$,*

$$U_n^\varrho f \in Lip_m\left(\frac{M\varrho^m n(n-1)\cdot\ldots\cdot(n-m+1)}{m!(n\varrho)(n\varrho+1)\cdot\ldots\cdot(n\varrho+m-1)}\right).$$

Proof. Let $f \in Lip_m(M)$. Then $\frac{M}{m!}e_m \pm f$ are m-convex functions, so that

$$\frac{M}{m!}U_n^\varrho e_m \pm U_n^\varrho f$$

are m-convex functions. Since

$$U_n^\varrho e_m(x) = \frac{\varrho^m n(n-1)\cdot\ldots\cdot(n-m+1)}{(n\varrho)(n\varrho+1)\cdot\ldots\cdot(n\varrho+m-1)}x^m + \text{terms of lower degree,}$$

we deduce that

$$\frac{M}{m!}\cdot\frac{\varrho^m n(n-1)\cdot\ldots\cdot(n-m+1)}{(n\varrho)(n\varrho+1)\cdot\ldots\cdot(n\varrho+m-1)}e_m \pm U_n^\varrho f$$

are m-convex functions.
This means that $U_n^\varrho f$ belongs to the class

$$Lip_m\left(\frac{M\varrho^m n(n-1)\cdot\ldots\cdot(n-m+1)}{m!(n\varrho)(n\varrho+1)\cdot\ldots\cdot(n\varrho+m-1)}\right).$$

□

Let now $M > 0$ and $0 < \gamma \leq 1$. Define

$$Lip(\gamma, M) := \{f \in C[0,1] : |f(x) - f(y)| \leq M|x-y|^\gamma, x, y \in [0,1]\},$$

and remark that $Lip(1, M) = Lip_1(M)$. Let ω be the usual modulus of continuity.

Theorem 3.11.2. *For all $n \geq 1$ and $\varrho > 0$,*

a) $\omega(U_n^\varrho f, \delta) \leq 2\omega(f, \delta), f \in C[0,1], \delta > 0$;

b) $U_n^\varrho(Lip(\gamma, M)) \subset Lip(\gamma, M)$.

Proof. According to Theorem 3.11.1, $U_n^\varrho(Lip_1(M)) \subset Lip_1(M)$, hence $U_n^\varrho(Lip(1, M)) \subset Lip(1, M)$. Now the statements a) and b) follow from [7, Corollary 6 and 7]. □

3.12 Lagrange-type operators associated with U_n^ϱ

We will present the relationship between the U_n^ϱ operators and some Lagrange-type operators, using their eigenstructure, thus extending in a natural way results known for Lagrange interpolation (see [95, p.116-126]).

3.12.1 A first description of L_n^ϱ

Let $\varrho > 0$ and $n \geq 1$ be fixed. Consider the functionals $F_{n,k}^\varrho : C[0,1] \to \mathbb{R}, k = 0,1,...,n$, defined by

$$F_{n,0}^\varrho(f) = f(0), F_{n,n-1}^\varrho(f) = f(1),$$
$$F_{n,k}^\varrho(f) = \int_0^1 \frac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho,(n-k)\varrho)} f(t)dt,\ k = 1,...,n-1.$$

Remember that the operator $U_n^\varrho : C[0,1] \to \Pi_n$ is given by

$$U_n^\varrho(f;x) := \sum_{k=0}^n F_{n,k}^\varrho(f) p_{n,k}(x), f \in C[0,1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, x \in [0,1]$. With a slight abuse of notation consider also the operator $U_n^\varrho : \Pi_n \to \Pi_n$. Its eigenvalues $\lambda_{n,k}^{(n)}$ and eigenfunctions $p_{n,k}^{(n)}$, $k = 0,1,...,n$, are described in Section 3.4; in particular,

$$1 = \lambda_{\varrho,0}^{(n)} = \lambda_{\varrho,1}^{(n)} > \lambda_{\varrho,2}^{(n)} > \lambda_{\varrho,3}^{(n)} > ... > \lambda_{\varrho,n}^{(n)} > 0,$$

which means that $U_n^\varrho : \Pi_n \to \Pi_n$ is invertible. Consider the inverse operator $(U_n^\varrho)^{-1} : \Pi_n \to \Pi_n$ (note the domain of definition here!) and define $L_n^\varrho : C[0,1] \to \Pi_n$ by

$$L_n^\varrho = (U_n^\varrho)^{-1} \circ U_n^\varrho. \tag{3.64}$$

Then $U_n^\varrho(L_n^\varrho f) = U_n^\varrho(f), f \in C[0,1]$, which leads to

$$F_{n,k}^\varrho(L_n^\varrho f) = F_{n,k}^\varrho(f), f \in C[0,1], k = 0,1,...,n. \tag{3.65}$$

(3.65) expresses an interpolatory property with respect to the functionals $F_{n,0}^\varrho, ..., F_{n,n}^\varrho$; more precisely, given $f \in C[0,1], L_n^\varrho f$ is the unique polynomial in Π_n satisfying (3.65). In particular, $L_n^\varrho p = p, \forall p \in \Pi_n$. It is known (see [45]) that

$$\lim_{\varrho\to\infty} F_{n,k}^\varrho(f) = f\left(\frac{k}{n}\right), f \in C[0,1], k = 0,1,...,n. \tag{3.66}$$

This entails

$$\lim_{\varrho\to\infty} U_n^\varrho(f) = B_n f,\ \text{uniformly on } [0,1], \tag{3.67}$$

for all $f \in C[0,1]$; here B_n denotes the classical Bernstein operator on $C[0,1]$. Let L_n be the Lagrange operator on $C[0,1]$ based on the nodes $0, \frac{1}{n}, ..., \frac{n-1}{n}, 1$. It is easy to see that

$$L_n = B_n^{-1} \circ B_n, \tag{3.68}$$

where

$$C[0,1] \xrightarrow{B_n} \Pi_n \xrightarrow{(B_n)^{-1}} \Pi_n.$$

We will see that

$$\lim_{\varrho\to\infty} L_n^\varrho(f) = L_n f, \text{ uniformly on } [0,1], \tag{3.69}$$

for all $f \in C[0,1]$. If we interpret (3.67) by saying that $U_n^\infty = B_n$, then (3.69) can be interpreted as $L_n^\infty = L_n$. On the other hand, one has according to (3.20)

$$U_n^\varrho f = \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), f \in C[0,1],$$

where $\mu_{\varrho,k}^{(n)}$ are the dual functionals of $p_{\varrho,k}^{(n)}$. This leads to

$$L_n^\varrho(f) = (U_n^\varrho)^{-1}(U_n^\varrho f) = \sum_{k=0}^{n} \lambda_{\varrho,k}^{(n)} \frac{1}{\lambda_{\varrho,k}^{(n)}} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f),$$

i.e.,

$$L_n^\varrho(f) = \sum_{k=0}^{n} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f), f \in C[0,1]. \tag{3.70}$$

So the relationship between U_n^ϱ and L_n^ϱ, expressed by (3.20) and (3.70), is similar to the relationship between $B_n = U_n^\infty$ and $L_n = L_n^\infty$, described in [18, Section 6].
To conclude this section let us recall that

$$U_n^\varrho = B_n \circ \mathcal{B}_{n\varrho}^{-1,-1}.$$

From (3.64) and (3.68) it follows that

$$L_n^\varrho = (\mathcal{B}_{n\varrho}^{-1,-1})^{-1} \circ L_n \circ \mathcal{B}_{n\varrho}^{-1,-1}, \ \varrho > 0, \tag{3.71}$$

i.e., the operators L_n^ϱ and L_n are similar.

3.12.2 A concrete approach to L_n^ϱ

In order to obtain other representations of the operators L_n^ϱ we shall use a classical method described, for example, in [81, Section 1.2], [20], [67, Section 1.3]. Let $n \geq 1, \varrho > 0$ and $f \in C[0,1]$ be fixed. Then $L_n^\varrho f \in \Pi_n$ has the form $L_n^\varrho f = c_0e_0 + c_1e_1 + ... + c_ne_n$, where $e_j(x) = x^j, x \in [0,1], j \geq 0$, and $c_j \in \mathbb{R}$. According to (3.65), the coefficients $c_0, ..., c_n$ satisfy the system of equations

$$\begin{cases} L_n^\varrho f = c_0e_0 + c_1e_1 + ... + c_ne_n \\ F_{n,0}^\varrho(f) = c_0F_{n,0}^\varrho(e_0) + c_1F_{n,0}^\varrho(e_1) + ... + c_nF_{n,0}^\varrho(e_n) \\ ... \\ F_{n,n}^\varrho(f) = c_0F_{n,n}^\varrho(e_0) + c_1F_{n,n}^\varrho(e_1) + ... + c_nF_{n,n}^\varrho(e_n). \end{cases}$$

By eliminating $c_0, ..., c_n$, we get

$$\begin{vmatrix} L_n^\varrho f & e_0 & e_1 & ... & e_n \\ F_{n,0}^\varrho(f) & F_{n,0}^\varrho(e_0) & F_{n,0}^\varrho(e_1) & ... & F_{n,0}^\varrho(e_n) \\ ... & .... & ... & ... & ... \\ F_{n,n}^\varrho(f) & F_{n,n}^\varrho(e_0) & F_{n,n}^\varrho(e_1) & ... & F_{n,n}^\varrho(e_n) \end{vmatrix} = 0. \tag{3.72}$$

Since $F_{n,i}^{\varrho}(e_m) = \dfrac{(i\varrho)^{\overline{m}}}{(n\varrho)^{\overline{m}}}$, from (3.72) we get after elementary computations:

$$L_n^\varrho f = -V(0, \frac{1}{n}, ..., \frac{n-1}{n}, 1)^{-1} \begin{vmatrix} 0 & e_0 & \frac{(n\varrho)^{\overline{1}}}{(n\varrho)} e_1 & ... & \frac{(n\varrho)^{\overline{n}}}{(n\varrho)^n} e_n \\ F_{n,0}^\varrho(f) & 1 & \frac{(0\varrho)^{\overline{1}}}{(n\varrho)} & ... & \frac{(0\varrho)^{\overline{n}}}{(n\varrho)^n} \\ F_{n,1}^\varrho(f) & 1 & \frac{(1\varrho)^{\overline{1}}}{(n\varrho)} & ... & \frac{(1\varrho)^{\overline{n}}}{(n\varrho)^n} \\ ... & .... & ... & ... & ... \\ F_{n,n}^\varrho(f) & 1 & \frac{(n\varrho)^{\overline{1}}}{(n\varrho)} & ... & \frac{(n\varrho)^{\overline{n}}}{(n\varrho)^n} \end{vmatrix} \tag{3.73}$$

where V is the Vandermonde determinant. Now we are in the position to prove (3.69).

Theorem 3.12.1. *For each $f \in C[0,1]$ we have*

$$\lim_{\varrho\to\infty} L_n^\varrho f = L_n f, \text{ uniformly on } [0,1].$$

Proof. Let us remark that

$$\lim_{\varrho\to\infty} \frac{(j\varrho)^{\overline{k}}}{(n\varrho)^k} = \left(\frac{j}{n}\right)^k. \tag{3.74}$$

From (3.66), (3.73), and (3.74) we deduce

$$\lim_{\varrho\to\infty} L_n^\varrho f = -V\left(0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right)^{-1} \begin{vmatrix} 0 & e_0 & e_1 & ... & e_n \\ f(0) & 1 & 0 & ... & 0 \\ f(\frac{1}{n}) & 1 & \frac{1}{n} & ... & (\frac{1}{n})^n \\ ... & .... & ... & ... & ... \\ f(\frac{n-1}{n}) & 1 & \frac{n-1}{n} & ... & (\frac{n-1}{n})^n \\ f(1) & 1 & 1 & ... & 1 \end{vmatrix}. \tag{3.75}$$

Since the right hand-side of (3.75) is $L_n f$ (see, e.g., [95, Section 3.1], [67, Section 1.3].), the proof is complete. ☐

3.12.3 The associated divided difference

The coefficient of e_n in the expression of $L_n f$ is the divided difference of f at the nodes $0, \frac{1}{n}, ..., \frac{n-1}{n}, 1$, and is given by (see e.g. [95, Section 2.6]):

$$\left[0, \frac{1}{n}, ..., \frac{n-1}{n}, 1; f\right] = V\left(0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right)^{-1} \times \tag{3.76}$$

$$\times \begin{vmatrix} 1 & 0 & 0 & ... & 0 & f(0) \\ 1 & \frac{1}{n} & (\frac{1}{n})^2 & ... & (\frac{1}{n})^{n-1} & f(\frac{1}{n}) \\ ... & .... & ... & ... & ... & ... \\ 1 & \frac{n-1}{n} & (\frac{n-1}{n})^2 & ... & (\frac{n-1}{n})^{n-1} & f(\frac{n-1}{n}) \\ 1 & 1 & 1 & ... & 1 & f(1) \end{vmatrix}.$$

Let us denote by $[F_{n,0}^\varrho, F_{n,1}^\varrho, ..., F_{n,n}^\varrho; f]$ the coefficient of e_n in $L_n^\varrho f$.

Theorem 3.12.2. *For each $f \in C[0,1]$ we have*

$$[F_{n,0}^{\varrho}, F_{n,1}^{\varrho}, ..., F_{n,n}^{\varrho}; f] = \frac{(n\varrho)^{\overline{n}}}{(n\varrho)^n} V\left(0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right)^{-1} \times \tag{3.77}$$

$$\times \begin{vmatrix} 1 & 0 & 0 & ... & 0 & F_{n,0}^{\varrho}(f) \\ 1 & \frac{1}{n} & (\frac{1}{n})^2 & ... & (\frac{1}{n})^{n-1} & F_{n,1}^{\varrho}(f) \\ ... & .... & ... & ... & ... & ... \\ 1 & \frac{n-1}{n} & (\frac{n-1}{n})^2 & ... & (\frac{n-1}{n})^{n-1} & F_{n,n-1}^{\varrho}(f) \\ 1 & 1 & 1 & ... & 1 & F_{n,n}^{\varrho}(f) \end{vmatrix}.$$

Proof. From (3.73) we get immediately

$$[F_{n,0}^{\varrho}, F_{n,1}^{\varrho}, ..., F_{n,n}^{\varrho}; f] = \frac{(n\varrho)^{\overline{n}}}{(n\varrho)^n} V\left(0, \tfrac{1}{n}, ..., \tfrac{n-1}{n}, 1\right)^{-1} \times$$

$$\times \begin{vmatrix} 1 & \frac{(0\varrho)^{\overline{1}}}{n\varrho} & ... & \frac{(0\varrho)^{\overline{n-1}}}{(n\varrho)^{n-1}} & F_{n,0}^{\varrho}(f) \\ 1 & \frac{(1\varrho)^{\overline{1}}}{n\varrho} & ... & \frac{(1\varrho)^{\overline{n-1}}}{(n\varrho)^{n-1}} & F_{n,1}^{\varrho}(f) \\ ... & ... & ... & ... & ... \\ 1 & \frac{(n\varrho)^{\overline{1}}}{n\varrho} & ... & \frac{(n\varrho)^{\overline{n-1}}}{(n\varrho)^{n-1}} & F_{n,n}^{\varrho}(f) \end{vmatrix} =$$

$$= \frac{(n\varrho)^{\overline{n}}/(n\varrho)^n}{(n\varrho)^{\frac{n(n-1)}{2}} V\left(0, \tfrac{1}{n}, ..., \tfrac{n-1}{n}, 1\right)} \begin{vmatrix} 1 & (0\varrho)^{\overline{1}} & ... & (0\varrho)^{\overline{n-1}} & F_{n,0}^{\varrho}(f) \\ 1 & (1\varrho)^{\overline{1}} & ... & (1\varrho)^{\overline{n-1}} & F_{n,1}^{\varrho}(f) \\ ... & ... & ... & ... & ... \\ 1 & (n\varrho)^{\overline{1}} & ... & (n\varrho)^{\overline{n-1}} & F_{n,n}^{\varrho}(f) \end{vmatrix} =$$

$$= \frac{(n\varrho)^{\overline{n}}/(n\varrho)^n}{(n\varrho)^{\frac{n(n-1)}{2}} V\left(0, \tfrac{1}{n}, ..., \tfrac{n-1}{n}, 1\right)} \begin{vmatrix} 1 & 0 & ... & 0 & F_{n,0}^{\varrho}(f) \\ 1 & \varrho & ... & \varrho^{n-1} & F_{n,1}^{\varrho}(f) \\ ... & ... & ... & ... & ... \\ 1 & n\varrho & ... & (n\varrho)^{n-1} & F_{n,n}^{\varrho}(f) \end{vmatrix}$$

and this leads us to (3.77). □

Remark 3.12.3. *From (3.66), (3.76) and (3.77) we derive*

$$\lim_{\varrho \to \infty} [F_{n,0}^{\varrho}, F_{n,1}^{\varrho}, ..., F_{n,n}^{\varrho}; f] = \left[0, \frac{1}{n}, ..., \frac{n-1}{n}, 1; f\right]$$

for all $f \in C[0,1]$. Moreover, let $f \in C[0,1]$ and Φ_n a (Lagrange-type) polynomial with $\Phi_n \in \Pi_n, \Phi_n(\frac{j}{n}) = F_{n,j}^{\varrho}(f), j = 0, ..., n$. From (3.76) and (3.77) it is easy to deduce

$$[F_{n,0}^{\varrho}, F_{n,1}^{\varrho}, ..., F_{n,n}^{\varrho}; f] = \frac{(n\varrho)^{\overline{n}}}{(n\varrho)^n} \left[0, \frac{1}{n}, ..., \frac{n-1}{n}, 1; \Phi_n\right].$$

The last (classical) divided difference can be computed by recurrence; see [95].

Remark 3.12.4. *Using (3.70), we see that*

$$\mu_{\varrho,n}^{(n)} = [F_{n,0}^{\varrho}, F_{n,1}^{\varrho}, ..., F_{n,n}^{\varrho}; \cdot]. \tag{3.78}$$

For the Bernstein operator (i.e., for $\varrho \to \infty$), (3.78) can be found in [18, p.164] .

Remark 3.12.5. *$u_{n+1}^\varrho := e_{n+1} - L_n^\varrho e_{n+1}$ is the unique monic polynomial in Π_{n+1} such that $L_n^\varrho u_{n+1}^\varrho = 0$. For example, $u_{n+1}^\infty = x(x-1)(x-\frac{1}{n})\cdot \ldots \cdot (x - \frac{n-1}{n})$. Moreover, $u_{n+1}^1(x) = x(x-1)J_{n-1}(x)$, where $J_0(x), J_1(x), \ldots$ are the monic Jacobi polynomials, orthogonal on $[0,1]$ with respect to the weight function $x(1-x)$. Indeed, $F_{n,0}^1(u_{n+1}^1) = F_{n,n}^1(u_{n+1}^1) = 0$, and $\int_0^1 t^{k-1}(1-t)^{n-k-1}u_{n+1}^1(t)dt = \int_0^1 t^{k-1}(1-t)^{n-k-1}t(t-1)J_{n-1}(t)dt = 0$ (since for all $k = 1, \ldots, n-1$, $t^{k-1}(1-t)^{n-k-1}$ is a polynomial of degree $n-2$). This implies $F_{n,k}^1(u_{n+1}^1) = 0, k = 1, \ldots, n-1$, and so $L_n^1(u_{n+1}^1) = 0$.*

Now we shall prove a general result.

Theorem 3.12.6. *The polynomial u_{n+1}^ϱ has $n+1$ distinct roots in $[0,1]$.*

Proof. By using Remark 3.12.5 and (3.71) we get $((\mathcal{B}_{n\varrho}^{-1,-1})^{-1} \circ L_n \circ \mathcal{B}_{n\varrho}^{-1,-1})(u_{n+1}^\varrho) = 0$, which entails $L_n(\mathcal{B}_{n\varrho}^{-1,-1}u_{n+1}^\varrho) = 0$. Now the same Remark 3.12.5 yields

$$\mathcal{B}_{n\varrho}^{-1,-1}u_{n+1}^\varrho = \frac{(n\varrho)^{n+1}}{(n\varrho)^{\overline{n+1}}}u_{n+1}^\infty.$$

So $\mathcal{B}_{n\varrho}^{-1,-1}u_{n+1}^\varrho$ has $n+1$ distinct roots in $[0,1]$. According to [57], u_{n+1}^ϱ has at least $n+1$ distinct roots in $[0,1]$; to finish the proof, it suffices to remark that u_{n+1}^ϱ is a polynomial of degree $n+1$. □

Now let us recall the representation of L_n in terms of the fundamental Lagrange polynomials:

$$L_n f(x) = \sum_{k=0}^{n} l_{n,k}(x) f\left(\frac{k}{n}\right), f \in C[0,1], x \in [0,1].$$

Using (3.71) we infer that L_n^ϱ has a similar representation, namely

$$L_n^\varrho f(x) = \sum_{k=0}^{n} l_{n,k}^\varrho(x) F_{n,k}^\varrho(f),$$

where

$$l_{n,k}^\varrho := (\mathcal{B}_{n\varrho}^{-1,-1})^{-1}(l_{n,k}), k = 0, 1, \ldots, n. \tag{3.79}$$

Theorem 3.12.7. *For each $k = 0, 1, \ldots, n$, the polynomial $l_{n,k}^\varrho$ has n distinct roots in [0,1].*

Proof. Since, according to (3.79), $\mathcal{B}_{n\varrho}^{-1,-1}(l_{n,k}^\varrho) = l_{n,k}$, the proof is similar to that of Theorem 3.12.6 and we omit it. □

In what follows we shall establish mean value theorems for the generalized divided difference and for the remainder $R_n^\varrho f := f - L_n^\varrho$.

Theorem 3.12.8. *Let $n \geq 1, \varrho > 0$ and $f \in C[0,1]$ be given. Then there exist $0 = t_0 < t_1 < \ldots . < t_n = 1$ such that*

$$R_n^\varrho f(t_i) = 0,\ i = 0, 1, \ldots, n. \tag{3.80}$$

Proof. According to (3.65), $F_{n,k}^\varrho(R_n^\varrho f) = 0, k = 0, 1, ..., n$, i.e.

$$R_n^\varrho f(0) = R_n^\varrho f(1) = 0, \tag{3.81}$$

$$\int_0^1 t^{k\varrho-1}(1-t)^{(n-k)\varrho-1} R_n^\varrho f(t)dt = 0, k = 1, ..., n-1. \tag{3.82}$$

Set $x := \left(\frac{t}{1-t}\right)^\varrho$, $j := k-1$, and $h(x) := R_n^\varrho f\left(\frac{x^{1/\varrho}}{1+x^{1/\varrho}}\right), x \geq 0$. Then (3.82) becomes

$$\int_0^\infty (1+x^{1/\varrho})^{-n\varrho} x^j h(x)dx = 0,\ j = 0, 1, ..., n-2. \tag{3.83}$$

Suppose that the number of the roots of h in $(0, +\infty)$ is at most $n-2$, i.e. $\{x \in (0, +\infty) : h(x) = 0\} = \{x_1, ..., x_r\}, r \leq n-2$. Then there exists a polynomial $p \in \Pi_{n-2}$ such that $\{x \in (0, +\infty) : p(x) = 0\} \subset \{x_1, ..., x_r\}$ and, moreover,

$$\int_0^\infty (1+x^{1/\varrho})^{-n\varrho} p(x)h(x)dx > 0. \tag{3.84}$$

Obviously (3.84) contradicts (3.83), which means that h has at least $n-1$ roots in $(0, +\infty)$. It follows that $R_n^\varrho f$ has at least $n-1$ roots in $(0, 1)$. Together with (3.81), this proves the theorem. □

Corollary 3.12.9. *Let $n \geq 1, \varrho > 0$ and $f \in C^n[0,1]$ be given. Then there exists $\xi \in (0,1)$ such that*

$$[F_{n,0}^\varrho, F_{n,1}^\varrho, ..., F_{n,n}^\varrho; f] = \frac{f^{(n)}(\xi)}{n!}.$$

Proof. According to Theorem 3.12.8, $R_n^\varrho f$ has at least $n+1$ roots in $[0,1]$. It follows that $(R_n^\varrho f)^{(n)}$ has at least a root $\xi \in (0,1)$. Thus

$$0 = (R_n^\varrho f)^{(n)}(\xi) = f^{(n)}(\xi) - n![F_{n,0}^\varrho, F_{n,1}^\varrho, ..., F_{n,n}^\varrho; f],$$

and the proof is finished. □

Let now $n \geq 1, \varrho > 0$ and $f \in C^{n+1}[0,1]$ be given. Consider the points $t_0, t_1, ..., t_n$ satisfying (3.80), and let $\omega(t) = (t-t_0) \cdot ... \cdot (t-t_n)$.

Corollary 3.12.10. *Let $x \in [0,1] \setminus \{t_0, t_1, ..., t_n\}$. Under the above assumption there exists $\eta_x \in (0,1)$ such that*

$$R_n^\varrho f(x) = \omega(x)\frac{f^{(n+1)}(\eta_x)}{(n+1)!}.$$

Proof. Consider the function $w(t) = \omega(x)R_n^\varrho f(t) - \omega(t)R_n^\varrho f(x), t \in [0,1]$. Then $x, t_0, ..., t_n$ are roots of w, which means that there exists $\eta_x \in (0,1)$ such that $w^{(n+1)}(\eta_x) = 0$. Now it suffices to remark that $w^{(n+1)}(t) = \omega(x)f^{(n+1)}(t) - (n+1)!R_n^\varrho f(x)$. □

Corollaries 3.12.9 and 3.12.10 generalize the mean value theorems for the divided difference and the remainder in classical Lagrange interpolation; see [95, Section 3.1], [67, Section 1.4].

3.13 Iterated Boolean sums of the operators U_n^ϱ

The eigenstructure of U_n^ϱ helps describe the convergence behavior of iterated Boolean sums based on a single mapping U_n^ϱ, with ϱ and n fixed.
For $M \geq 1$, let

$$\oplus^M U_n^\varrho = I - (I - U_n^\varrho)^M.$$

be the iterated Boolean sum of U_n^ϱ; here I stands for the identity operator on $C[0,1]$. Iterated Boolean sums of the classical Bernstein operator and modifications thereof were investigated by numerous authors in the past, among them G. Mastroianni and M.R. Occorsio (see [68],[69]). Some historical information on this method which may be traced to I.P. Natanson can be found in [52]. From a general result of H.J. Wenz [102, Theorem 2] it follows that $\lim_{M\to\infty} \oplus^M U_n^\varrho f = L_n^\varrho f, f \in C[0,1], n \geq 1$. With the notation from the preceding sections, we can say more, namely

Theorem 3.13.1. *Let $n \geq 2$ and $f \in C[0,1]$ be given. Then*

$$\lim_{M\to\infty} (1 - \lambda_{\varrho,n}^{(n)})^{-M} (\oplus^M U_n^\varrho f - L_n^\varrho f) = -[F_{n,0}^\varrho, F_{n,1}^\varrho, ..., F_{n,n}^\varrho; f] p_{\varrho,n}^{(n)}, \tag{3.85}$$

uniformly on $[0,1]$.

Proof. We have, according to (3.20)

$$\begin{aligned} \oplus^M U_n^\varrho f &= (I - (I - U_n^\varrho)^M) f = \sum_{i=1}^{M} (-1)^{i+1} \binom{M}{i} (U_n^\varrho)^i f \\ &= \sum_{i=1}^{M} (-1)^{i+1} \binom{M}{i} \sum_{k=0}^{n} (\lambda_{\varrho,k}^{(n)})^i p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f) \\ &= \sum_{k=0}^{n} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f) \sum_{i=1}^{M} (-1)^{i+1} \binom{M}{i} (\lambda_{\varrho,k}^{(n)})^i \\ &= \sum_{k=0}^{n} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f) (1 - (1 - (\lambda_{\varrho,k}^{(n)})^M). \end{aligned}$$

Combined with (3.70) this yields

$$\oplus^M U_n^\varrho f - L_n^\varrho f = -\sum_{k=0}^{n} p_{\varrho,k}^{(n)} \mu_{\varrho,k}^{(n)}(f) (1 - \lambda_{\varrho,k}^{(n)})^M,$$

i.e.

$$(1 - \lambda_{\varrho,n}^{(n)})^{-M} (\oplus^M U_n^\varrho f - L_n^\varrho f) = -p_{\varrho,n}^{(n)} \mu_{\varrho,n}^{(n)}(f) - \sum_{k=0}^{n-1} \mu_{\varrho,k}^{(n)}(f) \mu_{\varrho,k}^{(n)}(f) \left(\frac{1 - \lambda_{\varrho,k}^{(n)}}{1 - \lambda_{\varrho,n}^{(n)}} \right)^M.$$

Since $0 < \dfrac{1 - \lambda_{\varrho,k}^{(n)}}{1 - \lambda_{\varrho,n}^{(n)}} < 1, k = 0, ..., n-1$, we get

$$\lim_{M\to\infty} (1 - \lambda_{\varrho,n}^{(n)})^{-M} (\oplus^M U_n^\varrho f - L_n^\varrho f) = -\mu_{\varrho,n}^{(n)}(f) p_{\varrho,n}^{(n)}.$$

To conclude the proof it suffices to use (3.78). □

Remark 3.13.2. *For $\varrho \to \infty$, (3.85) was obtained in [86, Theorem 26.7].*

3.14 The derivatives of U_n^ϱ

In this section we show that there is a natural relationship between the derivatives of the operator images and the divided differences $[...;\Phi_n]$ which we introduced in Remark 3.12.3.

Theorem 3.14.1. *With the usual notation the following relationships hold:*

(i) $(U_n^\varrho(f;x))' = n\sum_{k=0}^{n-1} p_{n-1,k}(x)\Delta^1 F_{n,k}^\varrho(f) = \sum_{k=0}^{n-1} p_{n-1,k}(x)\left[\frac{k}{n},\frac{k+1}{n};\Phi_n\right]$;

(ii) $$(U_n^\varrho(f;x))^{(j)} = n(n-1)\cdot\ldots\cdot(n-j+1)\sum_{k=0}^{n-j} p_{n-j,k}(x)\Delta^j F_{n,k}^\varrho(f)$$
$$= n(n-1)\cdot\ldots\cdot(n-j+1)\sum_{k=0}^{n-j} p_{n-j,k}(x)\frac{j!}{n^j}\left[\frac{k}{n},\ldots,\frac{k+j}{n};\Phi_n\right];$$

(iii) $U_n^\varrho(f;x) = \sum_{k=0}^{n}\binom{n}{k}\Delta^k F_{n,0}^\varrho(f)e_k(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{n^k}\left[0,\frac{1}{n},\ldots,\frac{k}{n};\Phi_n\right]e_k(x)$;

where as before $\Phi_n\left(\frac{k}{n}\right) = F_{n,k}^\varrho(f)$.

Proof. (i) The forward difference was defined in [45, p. 792] by:

$$\Delta^j F_{n,k}^\varrho(f) = \sum_{i=0}^{j}\binom{j}{i}(-1)^{i+j}F_{n,k+i}^\varrho(f).$$

Thus we have

$$\left[\frac{k}{n},\frac{k+1}{n};\Phi_n\right] = \frac{\Phi_n(\frac{k+1}{n}) - \Phi_n(\frac{k}{n})}{\frac{k+1}{n}-\frac{k}{n}} = n[F_{n,k+1}^\varrho(f) - F_{n,k}^\varrho(f)] = n\Delta^1 F_{n,k}^\varrho(f);$$

(ii) The first equality can be found in [45, p. 792]. It remains to show that

$$\Delta^j F_{n,k}^\varrho(f) = \frac{j!}{n^j}\left[\frac{k}{n},\ldots,\frac{k+j}{n};\Phi_n\right].$$

We have that $\Delta^{j+1}F_{n,k}^\varrho(f) = \Delta(\Delta^j F_{n,k}^\varrho(f)) = \Delta^j F_{n,k+1}^\varrho(f) - \Delta^j F_{n,k}^\varrho(f)$. By using the recurrence formula for divided differences (see e.g. [95, p.104]) we get:

$$\begin{aligned}\Delta^j F_{n,k+1}^\varrho(f) - \Delta^j F_{n,k}^\varrho(f) &= \frac{j!}{n^j}\cdot\frac{j+1}{n}\cdot\frac{\left[\frac{k+1}{n},\ldots,\frac{k+j+1}{n};\Phi_n\right] - \left[\frac{k}{n},\ldots,\frac{k+j}{n};\Phi_n\right]}{\frac{k+j+1}{n}-\frac{k}{n}}\\ &= \frac{(j+1)!}{n^{j+1}}\left[\frac{k}{n},\ldots,\frac{k+j+1}{n};\Phi_n\right] = \Delta^{j+1}F_{n,k}^\varrho(f).\end{aligned}$$

(iii) We apply Taylor's formula to U_n^ϱ of degree n

$$U_n^\varrho(f;x) = \sum_{j=0}^{n}\frac{(U_n^\varrho f)^{(j)}(0)}{j!}x^k$$

and show that $(U_n^\varrho(f;x))^{(j)} = n(n-1)\cdot\ldots\cdot(n-j+1)\Delta^j F_{n,0}^\varrho(f)$. To this end we take $x=0$ in (ii); because $p_{n-j,0}(0)=1$ and for all $k\geq 1$, $p_{n-j,k}(0)=0$, from $\sum_{k=0}^{n-j}$ only the first term remains, which concludes the proof. □

Remark 3.14.2. *In the case $\varrho \to \infty$ we can find the analogues of the above relationships in [95, p. 300-302].*

3.15 Asymptotic formula

Here we present first two asymptotic formulae for higher order moments of U_n^ϱ in order to arrive at Voronovskaya-type results.

Theorem 3.15.1. *The following proposition*

$$(P_l): \begin{cases} \lim\limits_{n\to\infty} n^l M_{n,2l}^\varrho(x) = (2l-1)!! \left(\dfrac{\varrho+1}{\varrho}\right)^l X^l \\ \lim\limits_{n\to\infty} n^l M_{n,2l-1}^\varrho(x) = X^{l-1}X'(l-1)!2^{l-2}\dfrac{(\varrho+1)^{l-1}}{\varrho^l}(\varrho+2)\sum\limits_{k=1}^{l-1}\dfrac{(2k-1)!!}{(2k-2)!!} \end{cases} \tag{3.86}$$

holds true for all $l \geq 1$. The convergence is uniform on $[0,1]$.

Proof. We shall prove the proposition by induction on $l \geq 1$. $M_{n,1}^\varrho$ and $M_{n,2}^\varrho$ are given Corollary 3.3.2, and it is easy to prove that (P_1) is true. Suppose that (P_l) is true. According to (3.16) and (3.86)

$$\begin{aligned} \lim_{n\to\infty} n^{l+1} M_{n,2l+1}^\varrho(x) &= \lim_{n\to\infty} n^{l+1}\frac{2l(\varrho+1)X}{n\varrho+2l} M_{n,2l-1}^\varrho(x) + \lim_{n\to\infty} n^{l+1}\frac{2lX'}{n\varrho+2l}M_{n,2l}^\varrho(x) \\ &\quad + \lim_{n\to\infty} n^{l+1}\frac{\varrho X}{n\varrho+2l}(M_{n,2l}^\varrho(x))' \\ &= \frac{2l(\varrho+1)X}{\varrho}X^{l-1}X'(l-1)!2^{l-2}\frac{(\varrho+1)^{l-1}}{\varrho^l}(\varrho+2)\sum_{k=1}^{l-1}\frac{(2k-1)!!}{(2k-2)!!} + \\ &\quad + \frac{2lX'}{\varrho}(2l-1)!!\left(\frac{\varrho+1}{\varrho}\right)^l X^l + X(2l-1)!!\left(\frac{\varrho+1}{\varrho}\right)^l X^{l-1}X'l \\ &= X^l X' l! 2^{l-1}\frac{(\varrho+1)^l}{\varrho^{l+1}}(\varrho+2)\sum_{k=1}^{l}\frac{(2k-1)!!}{(2k-2)!!} \end{aligned}$$

and this proves the second formula in (3.86) for $l+1$. Similarly,

$$\begin{aligned} \lim_{n\to\infty} n^{l+1} M_{n,2l+2}^\varrho(x) &= \lim_{n\to\infty} n^{l+1}\frac{(2l+1)(\varrho+1)X}{n\varrho+2l+1}M_{n,2l}^\varrho(x) + \\ &+ \lim_{n\to\infty} n^{l+1}\frac{(2l+1)X'}{n\varrho+2l+1}M_{n,2l+1}^\varrho(x) + \lim_{n\to\infty} n^{l+1}\frac{\varrho X}{n\varrho+2l+1}(M_{n,2l+1}^\varrho(x))' \\ &= \frac{(2l+1)(\varrho+1)X}{\varrho}(2l-1)!!\left(\frac{\varrho+1}{\varrho}\right)^l X^l = \left(\frac{\varrho+1}{\varrho}\right)^{l+1}(2l+1)!!X^{l+1}, \end{aligned}$$

which is the first formula in (3.86) for $l+1$. This concludes the proof by induction on l. ☐

Remark 3.15.2. *In the proof above we have used the fact that $M_{n,2l}^\varrho \in \Pi_{2l}$ for all $n \geq 1$. Together with the first formula in (3.86) this means that*

$$\lim_{n\to\infty} n^l (M_{n,2l}^\varrho(x))' = (2l-1)!!\left(\frac{\varrho+1}{\varrho}\right)^l (X^l)'.$$

As a consequence of Corollary 1.3.13 (ii) we have the following Voronovskaya-type relation which can also be found in [46, Theorem 5.2].

Corollary 3.15.3. *Let* $f \in C^2[0,1]$. *Then*

$$\lim_{n\to\infty} n\left\{U_n^{\varrho}(f;x) - f(x)\right\} = \frac{\varrho+1}{2\varrho}x(1-x)f''(x),$$

uniformly on $[0,1]$.

Proof. For $\varphi(n) = n$ and $q = 2$ as given in Corollary 1.3.13 (ii),

$$\lim_{n\to\infty} n\left\{U_n^{\varrho}(f;x) - f(x)\right\} = \sum_{r=1}^{2} c_r(x)\frac{f^{(r)}(x)}{r!} = c_1(x)\frac{f'(x)}{1!} + c_2(x)\frac{f''(x)}{2!}$$

where $c_r(x) = \lim\limits_{n\to\infty} nM_{n,r}^{\varrho}(x)$. By using Lemma 3.15.1 with $l = 1$ we get

$$\begin{aligned} c_1(x) &= 0 \\ c_2(x) &= \frac{\varrho+1}{\varrho}X, \end{aligned}$$

and this concludes the proof. □

Remark 3.15.4. *As a consequence of Theorem 3.15.1 and Corollary 1.3.13 (ii) we deduce similarly that for* $f \in C^{2l}[0,1]$,

$$\lim_{n\to\infty} n^l\left\{U_n^{\varrho}(f(t);x) - \sum_{k=0}^{2l-1}\frac{f^{(k)}(x)}{k!}M_{n,k}^{\varrho}(x)\right\} = \frac{(2l-1)!!}{(2l)!}\left(\frac{\varrho+1}{\varrho}\right)^l X^l f^{(2l)}(x), l \geq 1.$$

From this we get also

$$\lim_{n\to\infty} n^l\left\{U_n^{\varrho}(f(t);x) - \sum_{k=0}^{2l-2}\frac{f^{(k)}(x)}{k!}M_{n,k}^{\varrho}(x)\right\} = \frac{(2l-1)!!}{(2l)!}\left(\frac{\varrho+1}{\varrho}\right)^l X^l f^{(2l)}(x) +$$

$$+X^{l-1}X'\frac{(l-1)!}{(2l-1)!}2^{l-2}\frac{(\varrho+1)^{l-1}}{\varrho^l}(\varrho+2)\sum_{k=1}^{l-1}\frac{(2k-1)!!}{(2k-2)!!}f^{(2l-1)}(x) \tag{3.87}$$

$$= X^{l-1}\frac{(\varrho+1)^{l-1}}{\varrho^l(2l)!}\times$$

$$\times\left[(\varrho+1)X(2l-1)!!f^{(2l)}(x) + l(\varrho+2)X'(2l-2)!!\sum_{k=1}^{l-1}\frac{(2k-1)!!}{(2k-2)!!}f^{(2l-1)}(x)\right].$$

Remark 3.15.5. *Another Voronovskaya-type result for* U_n^{ϱ} *can be determined from:*

$$\lim_{n\to\infty} n\left[n(U_n^{\varrho}(f(t);x) - f(x)) - \frac{\varrho+1}{2\varrho}Xf''(x)\right] =$$

$$= \lim_{n\to\infty} n^2\left[U_n^{\varrho}(f(t);x)f(x) - M_{n,2}^{\varrho}(x)\frac{f''(x)}{2}\right] + \frac{1}{2}\lim_{n\to\infty} n\left[nM_{n,2}^{\varrho}(x) - \frac{\varrho+1}{\varrho}X\right]f''(x).$$

Thus by using (3.87) with $l = 2$ *we get*

$$\lim_{n\to\infty} n\left[n(U_n^{\varrho}(f(t);x) - f(x)) - \frac{\varrho+1}{2\varrho}Xf''(x)\right] =$$

$$= X\frac{\varrho+1}{8\varrho^2}\left[(\varrho+1)Xf^{IV}(x) + \frac{2}{3}(\varrho+2)X'f'''(x) - 4f''(x)\right].$$

3.16 Power series of the operators U_n^ϱ

In [74] R. Păltănea defined power series of Bernstein operators (with n fixed) and studied their approximation behavior for functions defined on the space $C_0[0,1] := \{f|f(x) = x(1-x)h(x), h \in C[0,1]\}$ to some extent. This article motivated a number of authors to study similar problems or give different proofs of Păltănea's main result. See [1], [2], [3], [84]. In this section we study power (geometric) series of the operators U_n^ϱ, thus bridging the gap between power series of Bernstein operators and such of the genuine operators U_n.

Our main results will concern the convergence of the series as n (the degree of the polynomials inside the series) tends to infinity. The first non-quantitative theorem will essentially use the eigenstructure of the U_n^ϱ presented in detail in Section 3.4. The second result describes the degree of convergence to the "inverse Voronovskaya operators" $-A_\varrho^{-1}$ using a smoothing (K- functional) approach and makes use of exact representations of the moments.

The quantitative statement also holds in the limiting case of Bernstein operators, thus supplementing the original work of R. Păltănea.

As already shown in Section 3.4 the numbers

$$\lambda_{\varrho,j}^{(n)} := \frac{\varrho^j n!}{(n\varrho)^{\bar{j}}(n-j)!}, \quad j = 0,1,...,n, \tag{3.88}$$

are eigenvalues of U_n^ϱ. To each of them there corresponds a monic eigenpolynomial $p_{\varrho,j}^{(n)}$ such that $deg\ p_{\varrho,j}^{(n)} = j,\ j = 0,1,...,n$. In particular,

$$p_{\varrho,0}^{(n)}(x) = 1, p_{\varrho,1}^{(n)}(x) = x - \frac{1}{2}, x \in [0,1]. \tag{3.89}$$

From (3.28) we get

$$p_{\varrho,j}^{(n)}(0) = p_{\varrho,j}^{(n)}(1) = 0, j = 2,...,n. \tag{3.90}$$

Obviously $U_n^\varrho f$ can be decomposed with respect to the basis $\{p_{\varrho,0}^{(n)}, p_{\varrho,1}^{(n)},...,p_{\varrho,n}^{(n)}\}$ of Π_n; this allows us to introduce the dual functionals $\mu_{\varrho,j}^{(n)} : C[0,1] \to \mathbb{R}, j = 0,1,...,n,$ by means of the formula

$$U_n^\varrho f = \sum_{j=0}^{n} \lambda_{\varrho,j}^{(n)} \mu_{\varrho,j}^{(n)}(f) p_{\varrho,j}^{(n)}, f \in C[0,1]. \tag{3.91}$$

In particular, since U_n^ϱ restricted to Π_n is bijective, we have

$$p = \sum_{j=0}^{n} \mu_{\varrho,j}^{(n)}(p) p_{\varrho,j}^{(n)}, p \in \Pi_n. \tag{3.92}$$

Now consider the numbers

$$\lambda_{\varrho,j} := -\frac{\varrho+1}{2\varrho}(j-1)j, j = 0,1,... \tag{3.93}$$

and the monic polynomials

$$p_0^*(x) = 1, p_1^*(x) = x - \frac{1}{2}, p_j^*(x) = x(x-1)P_{j-2}^{(1,1)}(2x-1), j \geq 2, \tag{3.94}$$

where $P_i^{(1,1)}(x)$ are Jacobi polynomials, orthogonal with respect to the weight $(1-x)(1+x)$ on $[-1,1], i \geq 0$. Moreover, consider the linear functionals $\mu_j^* : C[0,1] \to \mathbb{R}$, defined as

$$\mu_0^*(f) = \frac{f(0)+f(1)}{2}, \mu_1^*(f) = f(1) - f(0), \tag{3.95}$$

$$\mu_j^*(f) = \frac{1}{2}\binom{2j}{j}[(-1)^j f(0) + f(1) - j\int_0^1 f(x)P_{j-2}^{(1,1)}(2x-1)dx], j \geq 2. \tag{3.96}$$

It is easy to verify that

$$\lim_{n\to\infty} n(\lambda_{\varrho,j}^{(n)} - 1) = \lambda_{\varrho,j}, j \geq 0. \tag{3.97}$$

The following result can be found in [48].

Theorem 3.16.1. *([48]) For each $j \geq 0$ we have*

$$\lim_{n\to\infty} p_{\varrho,j}^{(n)} = p_j^*, \text{ uniformly on } [0,1], \tag{3.98}$$

$$\lim_{n\to\infty} \mu_{\varrho,j}^{(n)}(p) = \mu_j^*(p), p \in \Pi. \tag{3.99}$$

3.16.1 The power series A_n^ϱ

Consider the space

$$C_0[0,1] := \{f | f(x) = x(1-x)h(x), h \in C[0,1]\}.$$

For $f \in C_0[0,1], f(x) = x(1-x)h(x)$, define the norm

$$||f||_0 := ||h||_\infty.$$

Endowed with the norm $||\cdot||_0, C_0[0,1]$ is a Banach space. Obviously,

$$||f||_\infty \leq \frac{1}{4}||f||_0, f \in C_0[0,1]. \tag{3.100}$$

Lemma 3.16.2. *As a linear operator on $(C_0[0,1], ||\cdot||_0), U_n^\varrho$ has the norm*

$$||U_n^\varrho||_0 = \frac{(n-1)\varrho}{n\varrho+1} < 1. \tag{3.101}$$

Proof. Let $f \in C_0[0,1], f(x) = x(1-x)h(x), h \in C[0,1]$. By straightforward computation we get $U_n^\varrho f(x) = x(1-x)u(x)$, where

$$u(x) = n(n-1)\sum_{k=1}^{n-1} \frac{\int_0^1 t^{k\varrho}(1-t)^{(n-k)\varrho}h(t)dt}{k(n-k)B(k\varrho,(n-k)\varrho)} p_{n-2,k-1}(x).$$

It follows immediately that $U_n^\varrho f \in C_0[0,1]$ and

$$||U_n^\varrho f||_0 = ||u||_\infty \leq \frac{(n-1)\varrho}{n\varrho+1}||h||_\infty = \frac{(n-1)\varrho}{n\varrho+1}||f||_0.$$

Thus

$$||U_n^\varrho||_0 \leq \frac{(n-1)\varrho}{n\varrho+1}. \tag{3.102}$$

On the other hand, let $g(x) = x(1-x), x \in [0,1]$. Then $||g||_0 = 1$ and $U_n^\varrho g(x) = x(1-x)\dfrac{(n-1)\varrho}{n\varrho+1}$, which entails $||U_n^\varrho g||_0 = \dfrac{(n-1)\varrho}{n\varrho+1}$ and so

$$||U_n^\varrho||_0 \geq \frac{(n-1)\varrho}{n\varrho+1}. \tag{3.103}$$

Now (3.101) is a consequence of (3.102) and (3.103). □

According to Lemma 3.16.2, it is possible to consider the operator $A_n^\varrho : C_0[0,1] \to C_0[0,1]$,

$$A_n^\varrho := \frac{\varrho}{n\varrho+1}\sum_{k=0}^{\infty}(U_n^\varrho)^k, n \geq 1. \tag{3.104}$$

For later purposes we also introduce the notation

$$A_n^\infty := \frac{1}{n}\sum_{k=0}^{\infty}(B_n)^k,\ n \geq 1,$$

in order to have Păltănea's power series included.

By using (3.101) we get $||A_n^\varrho||_0 \leq \dfrac{\varrho}{\varrho+1}$, and with the same function $g(x) = x(1-x)$ we find

$$||A_n^\varrho||_0 = \frac{\varrho}{\varrho+1}, n \geq 1. \tag{3.105}$$

Let $p \in \Pi_m \cap C_0[0,1]$, i.e., $p(0) = p(1) = 0$. Then $m \geq 2$. Let $n \geq m$. From (3.89), (3.90) and (3.92) we derive

$$p = \sum_{j=2}^{m} \mu_{\varrho,j}^{(n)}(p) p_{\varrho,j}^{(n)}$$

and, moreover,

$$(U_n^\varrho)^k p = \sum_{j=2}^{m} (\lambda_{\varrho,j}^{(n)})^k \mu_{\varrho,j}^{(n)}(p) p_{\varrho,j}^{(n)}, k \geq 0, \text{ for all } n \geq m.$$

According to (3.104), for all $p \in \Pi_m \cap C_0[0,1]$ and $n \geq m$,

$$A_n^\varrho p = \frac{\varrho}{n\varrho+1}\sum_{j=2}^{m}\frac{1}{1-\lambda_{\varrho,j}^{(n)}}\mu_{\varrho,j}^{(n)}(p) p_{\varrho,j}^{(n)}.$$

By using (3.97), (3.98) and (3.99) we get

$$\lim_{n\to\infty} A_n^\varrho p = \frac{\varrho}{\varrho+1}\sum_{j=2}^{m}\frac{2}{j(j-1)}\mu_j^*(p) p_j^*, \tag{3.106}$$

uniformly on $[0,1]$, for all $p \in \Pi_m \cap C_0[0,1]$.

3.16.2 The Voronovskaya operator A_ϱ

It was proved in [46, p. 918] that

$$\lim_{n\to\infty} n(U_n^\varrho g(x) - g(x)) = \frac{\varrho+1}{2\varrho} x(1-x)g''(x), g \in C^2[0,1],$$

uniformly on $[0,1]$. We need the following result.

Theorem 3.16.3. *The operator* $\{y \in C^2[0,1] |\ y(0) = y(1) = 0\} \to C_0[0,1]$ *defined by*

$$A_\varrho y(x) := \frac{\varrho+1}{2\varrho} x(1-x)y''(x), x \in [0,1],$$

is bijective, and

$$||A_\varrho^{-1} f||_\infty \le \frac{\varrho}{4(\varrho+1)} ||f||_0, f \in C_0[0,1]. \tag{3.107}$$

Proof. Obviously A_ϱ is injective. To prove the surjectivity, let $f \in C_0[0,1], f(x) = x(1-x)h(x), h \in C[0,1]$. It is a matter of calculus to verify that the function

$$-\frac{2\varrho}{\varrho+1} F_\infty(h;x) = y(x) := -\frac{2\varrho}{\varrho+1}\left[(1-x)\int_0^x th(t)dt + x\int_x^1 (1-t)h(t)dt\right], x \in [0,1],$$

is in $C^2[0,1], y(0) = y(1) = 0$, and $A_\varrho y = f$. Therefore A_ϱ is bijective. Moreover, for $x \in [0,1], y = A_\varrho^{-1}(f) \Rightarrow -y(x) = -A_\varrho^{-1}(f;x) = +\frac{2\varrho}{\varrho+1} F_\infty(h;x)$

$$\begin{aligned} |A_\varrho^{-1} f(x)| &\le \frac{2\varrho}{\varrho+1}\left[(1-x)\int_0^x tdt + x\int_x^1 (1-t)dt\right] ||h||_\infty \\ &= \frac{\varrho}{\varrho+1} x(1-x)||h||_\infty \le \frac{\varrho}{4(\varrho+1)} ||f||_0, \end{aligned}$$

and this leads to (3.107). □

Remark 3.16.4. *Further below we will use the notations* $\Psi(x) = x(1-x)$, *and*

$$-A_\infty^{-1}(\Psi h) := 2 \cdot F_\infty(h),\ h \in C[0,1],$$

in order to also cover the Bernstein case.

Another useful result reads as follows.

Lemma 3.16.5. *For all* $p \in \Pi \cap C_0[0,1]$ *we have*

$$\lim_{n\to\infty} A_n^\varrho p = -A_\varrho^{-1} p, \tag{3.108}$$

uniformly on $[0,1]$.

Proof. The polynomials p_j^* from (3.94) satisfy

$$x(1-x)(p_j^*)''(x) = -j(j-1)p_j^*(x), x \in [0,1], j \geq 0$$

(see, e.g., [18, p.155]). This yields $A_\varrho p_j^* = -\frac{\varrho+1}{2\varrho} j(j-1)p_j^*, j \geq 0$, and, moreover,

$$A_\varrho \left(\sum_{j=2}^{m} \frac{2}{j(j-1)} \mu_j^*(p) p_j^* \right) = -\frac{\varrho+1}{\varrho} \sum_{j=2}^{m} \mu_j^*(p) p_j^* \tag{3.109}$$

for all $p \in \Pi_m \cap C_0[0,1]$. According to ([18, (4.18)]), $\sum_{j=2}^{m} \mu_j^*(p) p_j^* = p$, so that (3.109) yields

$$\frac{\varrho}{\varrho+1} \sum_{j=2}^{m} \frac{2}{j(j-1)} \mu_j^*(p) p_j^* = -A_\varrho^{-1} p, \tag{3.110}$$

for all $p \in \Pi_m \cap C_0[0,1]$. Now (3.108) is a consequence of (3.106) and (3.110). □

3.16.3 The convergence of A_n^ϱ on $C_0[0,1]$

The main result of the section is contained in

Theorem 3.16.6. *For all* $f \in C_0[0,1]$,

$$\lim_{n\to\infty} A_n^\varrho f = -A_\varrho^{-1} f,$$

uniformly on $[0,1]$.

Proof. Let $f \in C_0[0,1], f(x) = x(1-x)h(x), h \in C[0,1]$. Consider the polynomials $p_i(x) := x(1-x)B_i h(x)$, where B_i are the classical Bernstein operators, $i \geq 1$. Then $p_i \in C_0[0,1], i \geq 1$, and $\lim_{i\to\infty} ||p_i - f||_0 = \lim_{i\to\infty} ||B_i h - h||_\infty = 0$. Let $\varepsilon > 0$ and fix $i \geq 1$ such that

$$||p_i - f||_0 \leq \frac{2\varrho+2}{3\varrho+2}\varepsilon. \tag{3.111}$$

Then, according to Lemma 3.16.5, there exists n_ε such that

$$||A_n^\varrho p_i + A_\varrho^{-1} p_i||_\infty \leq \frac{2\varrho+2}{3\varrho+2}\varepsilon, n \geq n_\varepsilon. \tag{3.112}$$

Now using (3.100) and (3.105) we infer

$$||A_n^\varrho f - A_n^\varrho p_i||_\infty \leq \frac{1}{4}||A_n^\varrho f - A_n^\varrho p_i||_0 \leq \frac{1}{4}||A_n^\varrho||_0 ||f - p_i||_0 \leq \frac{\varrho}{4(\varrho+1)} \frac{2\varrho+2}{3\varrho+2}\varepsilon,$$

so that

$$||A_n^\varrho f - A_n^\varrho p_i||_\infty \leq \frac{\varrho}{2(3\varrho+2)}\varepsilon. \tag{3.113}$$

On the other hand, (3.107) and (3.111) yield

$$||A_\varrho^{-1} f - A_\varrho^{-1} p_i||_\infty \leq \frac{\varrho}{4(\varrho+1)} ||f - p_i||_0 \leq \frac{\varrho}{2(3\varrho+2)}\varepsilon. \tag{3.114}$$

Finally, using (3.112), (3.113) and (3.114) we obtain, for all $n \geq n_\varepsilon$,

$$||A_n^\varrho f + A_\varrho^{-1} f||_\infty \leq ||A_n^\varrho f - A_n^\varrho p_i||_\infty + ||A_n^\varrho p_i + A_\varrho^{-1} p_i||_\infty + ||A_\varrho^{-1} f - A_\varrho^{-1} p_i||_\infty \leq \varepsilon,$$

and this concludes the proof. □

On $(C[0,1], ||\cdot||_\infty)$ consider the linear operator $H_n^\varrho := A_n^\varrho - (-A_\varrho^{-1})$ given by

$$\begin{aligned} C[0,1] \ni h \mapsto A_n^\varrho(\Psi h; x) &= \frac{\varrho}{n\varrho+1} \sum_{k=0}^{\infty} (U_n^\varrho)^k (\Psi h; x) \in C_0[0,1] \\ C[0,1] \ni h \mapsto -A_\varrho^{-1}(\Psi h; x) &= \frac{2\varrho}{\varrho+1} \left[(1-x) \int_0^x th(t)dt + x \int_x^1 (1-t)h(t)dt \right] \\ &= \frac{2\varrho}{\varrho+1} F_\infty(h; x) \in C_0[0,1] \end{aligned}$$

Theorem 3.16.7. *Let* $h \in C[0,1], \varrho > 0, n \geq \frac{4\varrho+6}{\varrho}, \varepsilon = \sqrt{\frac{\varrho+2}{n\varrho+2}} \leq \frac{1}{2}$ *and* $\Psi(x) = x(1-x)$. *Then*

$$|H_n^\varrho(h;x)| \leq \Psi(x) \left[\frac{2\varrho}{3(\varrho+1)} \sqrt{\frac{\varrho+2}{n\varrho+2}} \omega_1(h;\varepsilon) + \right. \tag{3.115}$$

$$\left. + \frac{3}{4} \left(\frac{2\varrho}{\varrho+1} + \frac{2\varrho}{3(\varrho+1)} \sqrt{\frac{\varrho+2}{n\varrho+2}} + \frac{7(\varrho+3)}{6(\varrho+1)} \right) \omega_2(h;\varepsilon) \right].$$

Proof. Let $h \in C[0,1]$ be fixed, and $g \in C^2[0,1]$ be arbitrary.
Then $|H_n^\varrho(h;x)| \leq |H_n^\varrho(h-g;x)| + |H_n^\varrho(g;x)| = |E_1| + |E_2|$. Here

$$\begin{aligned} |E_1| &= |A_n^\varrho(\Psi(h-g);x) - (-A_\varrho^{-1}(\Psi(h-g);x))| \\ &= |A_n^\varrho(\Psi(h-g);x) - \frac{2\varrho}{\varrho+1} F_\infty(h-g;x)| \\ &\leq ||h-g||_\infty A_n^\varrho(\Psi;x) + \frac{2\varrho}{\varrho+1} |F_\infty(h-g;x)| \\ &= ||h-g||_\infty \frac{\varrho}{\varrho+1} \Psi(x) + \frac{2\varrho}{\varrho+1} ||h-g||_\infty \frac{1}{2} \Psi(x) \\ &= \frac{2\varrho}{\varrho+1} \Psi(x) |h-g||_\infty \end{aligned}$$

and

$$|E_2| = |A_n^\varrho(\Psi g;x) - (-A_\varrho^{-1}(\Psi g;x))|.$$

For $g \in C^2[0,1]$ one has $F_\infty := F_\infty(g) \in C^4[0,1]$, $F_\infty'' = -g, F_\infty''' = -g', F_\infty^{(4)} = -g''$. Moreover, by Taylor's formula we obtain for any points $y, t \in [0,1]$:

$$F_\infty(t) = F_\infty(y) + F_\infty'(y)(t-y) + \frac{1}{2} F_\infty''(y)(t-y)^2 + \frac{1}{6} F_\infty'''(y)(t-y)^3 + \Theta_y(t) \tag{3.116}$$

where

$$\Theta_y(t) := \frac{1}{6} \int_y^t (t-u)^3 F_\infty^{(4)}(u) du.$$

Fix y and consider (3.116) as an equality between two functions in the variable t. Applying to this equality the operator $U_n^\varrho(\cdot, y)$ one arrives at

$$\begin{aligned} U_n^\varrho(F_\infty, y) &= F_\infty(y) + \frac{1}{2}F_\infty''(y)U_n^\varrho((t-y)^2;y) + \frac{1}{6}F_\infty'''(y)U_n^\varrho((t-y)^3;y) + U_n^\varrho(\Theta_y;y) \\ &= F_\infty(y) - \frac{1}{2}g(y)U_n^\varrho((t-y)^2;y) - \frac{1}{6}g'(y)(y)U_n^\varrho((t-y)^3;y) + U_n^\varrho(\Theta_y;y). \end{aligned}$$

This implies

$$\frac{1}{2}g(y)U_n^\varrho((e_1-y)^2;y) - F_\infty(y) + U_n^\varrho(F_\infty, y) = -\frac{1}{6}g'(y)U_n^\varrho((e_1-y)^3;y) + U_n^\varrho(\Theta_y;y).$$

In the above equality we rewrite the left hand side as $\frac{1}{2}g(y)U_n^\varrho((e_1-y)^2;y) - (I - U_n^\varrho)(F_\infty, y)$. Thus we have

$$g(y)U_n^\varrho((e_1-y)^2;y) - 2(I-U_n^\varrho)(F_\infty,y) = -\frac{1}{3}g'(y)(y)U_n^\varrho((e_1-y)^3;y) + 2U_n^\varrho(\Theta_y;y).$$

Application of A_n^ϱ yields

$$\begin{aligned} A_n^\varrho(g(\cdot)U_n^\varrho((e_1-\cdot)^2;\cdot);x) - 2A_n^\varrho \circ (I-U_n^\varrho)(F_\infty,x) = \\ -\frac{1}{3}A_n^\varrho(g'(\cdot)U_n^\varrho((e_1-\cdot)^3;\cdot);x) + 2A_n^\varrho(Q;x) \end{aligned} \tag{3.117}$$

where $Q(y) := U_n^\varrho(\Theta_y;y)$. Note that the first five moments are given by Corollary 3.3.2. In the above expression we have $2A_n^\varrho \circ (I-U_n^\varrho)(F_\infty,x) = \dfrac{2\varrho}{n\varrho+1}F_\infty(x) = \dfrac{2\varrho}{n\varrho+1}F_\infty(g;x)$.

Also $A_n^\varrho(g(\cdot)U_n^\varrho((e_1-\cdot)^2;\cdot);x) = A_n^\varrho(g(\cdot)\dfrac{\varrho+1}{n\varrho+1}\Psi(\cdot);x) = \dfrac{\varrho+1}{n\varrho+1}A_n^\varrho(\Psi g;x)$.

Hence (3.117) can be written as

$$\begin{aligned} &\left|\frac{\varrho+1}{n\varrho+1}A_n^\varrho(\Psi g;x) - \frac{2\varrho}{n\varrho+1}F_\infty(g;x)\right| \\ &\quad = \left|-\frac{1}{3}g'(\cdot)U_n^\varrho(((e_1-\cdot)^3;\cdot);x) - 2A_n^\varrho(Q;x)\right| \\ &\quad \le \frac{1}{3}\left|A_n^\varrho\left(\frac{(\varrho+1)(\varrho+2)}{(n\varrho+1)(n\varrho+2)}\Psi'(\cdot)\Psi(\cdot);x\right)\right| + |2A_n^\varrho(Q;x)| \\ &\quad \le \frac{1}{3}\frac{(\varrho+1)(\varrho+2)}{(n\varrho+1)(n\varrho+2)}||g'||_\infty \frac{\varrho}{\varrho+1}\Psi(x) + |2A_n^\varrho(Q;x)|. \end{aligned}$$

Multiplying the outermost sides of the latter inequality by $\frac{n\varrho+1}{\varrho+1}$ gives

$$\begin{aligned} |E_2| &= \left|A_n^\varrho(\Psi g;x) - \frac{2\varrho}{\varrho+1}F_\infty(g;x)\right| \\ &\le \frac{\varrho(\varrho+2)}{3(n\varrho+2)(\varrho+1)}\Psi(x)||g'||_\infty + 2\frac{n\varrho+1}{\varrho+1}|A_n^\varrho(Q;x)|. \end{aligned}$$

In the last summand we have $Q(y) = U_n^\varrho(\Theta_y; y)$ thus

$$\begin{aligned}|U_n^\varrho(\Theta_y;y)| &\leq \frac{1}{6}U_n^\varrho((e_1-y)^4;y)||g''||_\infty \\ &\leq \frac{1}{6}\cdot\frac{7}{4}\cdot\frac{(\varrho+1)(\varrho+2)(\varrho+3)}{\varrho(n\varrho+1)(n\varrho+2)}\Psi(y)||g''||_\infty.\end{aligned}$$

Hence

$$\begin{aligned}\frac{2(n\varrho+1)}{\varrho+1}|A_n^\varrho(Q;x)| &\leq \frac{2(n\varrho+1)}{\varrho+1}\cdot\frac{7}{24}\cdot\frac{(\varrho+1)(\varrho+2)(\varrho+3)}{\varrho(n\varrho+1)(n\varrho+2)}A_n^\varrho(\Psi;x)||g''||_\infty \\ &= \frac{7}{12}\cdot\frac{(\varrho+2)(\varrho+3)}{(\varrho+1)(n\varrho+2)}\Psi(x)||g''||_\infty.\end{aligned}$$

This leads to

$$\begin{aligned}|E_2| &\leq \frac{\varrho(\varrho+2)}{3(n\varrho+2)(\varrho+1)}\Psi(x)||g'||_\infty + \frac{7}{12}\cdot\frac{(\varrho+2)(\varrho+3)}{(\varrho+1)(n\varrho+2)}\Psi(x)||g''||_\infty \\ &= \frac{(\varrho+2)}{3(n\varrho+2)(\varrho+1)}\Psi(x)\left\{\varrho||g'||_\infty + \frac{7}{4}(\varrho+3)||g''||_\infty\right\}.\end{aligned}$$

Hence for $h \in C[0,1]$ fixed, $g \in C^2[0,1]$ arbitrary we have

$$\begin{aligned}&|H_n^\varrho(h;x)| = |E_1| + |E_2| \\ &\leq \frac{2\varrho}{\varrho+1}\Psi(x)||h-g||_\infty + \frac{(\varrho+2)}{3(n\varrho+2)(\varrho+1)}\Psi(x)\left\{\varrho||g'||_\infty + \frac{7}{4}(\varrho+3)||g''||_\infty\right\}\end{aligned}$$

Next we choose $g = h_\varepsilon, 0 < \varepsilon = \sqrt{\frac{\varrho+2}{n\varrho+2}} \leq \frac{1}{2}$ and by applying Lemmas 1.1.14 and 1.1.15 we obtain

$$\begin{aligned}&||h-g||_\infty \leq \frac{3}{4}\omega_2(h;\varepsilon) \\ &||g'|| \leq \frac{1}{\varepsilon}[2\omega_1(h;\varepsilon) + \frac{3}{2}\omega_2(h;\varepsilon)] \\ &||g''|| \leq \frac{3}{2\varepsilon^2}\omega_2(h;\varepsilon).\end{aligned}$$

Thus

$$\begin{aligned}|H_n^\varrho(h;x)| \leq \Psi(x)\Bigg[&\frac{2\varrho}{3(\varrho+1)}\sqrt{\frac{\varrho+2}{n\varrho+2}}\omega_1(h;\varepsilon) + \\ &+\frac{3}{4}\left(\frac{2\varrho}{\varrho+1} + \frac{2\varrho}{3(\varrho+1)}\sqrt{\frac{\varrho+2}{n\varrho+2}} + \frac{7(\varrho+3)}{6(\varrho+1)}\right)\omega_2(h;\varepsilon)\Bigg].\end{aligned}$$

□

Remark 3.16.8. *If we let $1 \leq \varrho \to \infty$, then for all $n \geq 10$*

$$\begin{aligned}\lim_{\varrho\to\infty}|H_n^\varrho(h;x)| &= \lim_{\varrho\to\infty}|A_n^\varrho(\Psi h;x) - (-A_\varrho^{-1})(\Psi h;x)| \\ &= |A_n^\infty(\Psi h;x) - (-A_\infty^{-1})(\Psi h;x)| \\ &\leq 3\Psi(x)\left[\frac{1}{\sqrt{n}}\omega_1\left(h;\frac{1}{\sqrt{n}}\right) + \omega_2\left(h;\frac{1}{\sqrt{n}}\right)\right].\end{aligned}$$

This is a quantitative form of Păltănea's convergence result in [74, Theorem 3.2].

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