Affineness of Deligne-Lusztig Varieties

Dissertation
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)

von
Felix Grelak, M. Sc.,
aus Potsdam

vorgelegt an dem Institut für
Mathematik der Universität
Duisburg-Essen

Essen 2014

Betreuer

Prof. Dr. Ulrich Görtz

Gutachter

Prof. Dr. Ulrich Görtz
Prof. Dr. Sascha Orlik

Datum der Disputation

23. Juni 2014

Vorsitz

Prof. Dr. Georg Hein

Contents

Introduction

In the present work we are concerned with the affineness of Deligne-Lusztig varieties $X(w)$, in particular with the computational decision of the affineness. Deligne-Lusztig varieties are certain schemes over algebraically closed fields k of positive characteristic which are attached to an element of the Weyl group W of a reductive algebraic group G which is defined over a finite field $k_0 \subset k$. We have a Frobenius map F on G and a F-stable Borel subgroup $B = T \ltimes U$ that is also defined over k_0, where T is a maximal torus and U is the unipotent radical.

In the first chapters we recall some basic definitions and the important starting point of research in this area, namely the Deligne-Lusztig criterion for affineness ([11]). This criterion is a combinatorial, sufficient criterion, which we have implemented with the help of *GAP 3* ([33]) and *CVXOPT* ([1]). Next, we comment on further combinatorial, sufficient criteria by Bonnafé, He, Orlik, Rapoport, Rouquier et. al.([5], [22], [31]), which rely on the original criterion. Harashita ([19]) managed to generalize the Deligne-Lusztig criterion along the original proof, see chapter 3. Harashita's criterion has the disadvantage that it is (efficiently) implementable only in a limited way, because one has to check quasi-finiteness of a morphism. Instead, we only check the injectivity. Based on Harashita's work we developed an easy to compute, combinatorial criterion, see Theorem 4.1, resp. Corollary 4.3. We have implemented many of the criteria mentioned above (except the quasi-finiteness criterion) and the result is that these are insufficient to prove the affineness for all Deligne-Lusztig varieties.

Even the quasi-finiteness criterion cannot handle all Deligne-Lusztig varieties: For $G = \mathbb{SL}_7$, $k_0 = \mathbb{F}_2$ and $w = s_4 s_5 s_6 s_5 s_2 s_3 s_4 s_1 s_2 = w^{-1}$, Harashita's quasi-finite criterion does not apply. Furthermore, the cyclic-shift F-conjugacy class consists just of w alone. Therefore the cyclic-shift criterion (Proposition 2.4) does not apply, so this may be a good candidate for $X(w)$ being not affine. One can use the implementation to produce full lists of examples that cannot be handled by the existing criteria. We recapitulate the well-known criteria (and our new one) in the diagram below. Therefore we define the following subsets of W consisting of elements satisfying different (combinations of) affineness criteria:

$(CoxE) := \{w \in W;\ w \text{ is a Coxeter element}\}$ (Remark 2.7)

$(ORH) := \{w \in W;\ w \text{ is of minimal length in } \mathcal{K}(w)\}$ (Theorem 2.5)

$(BR) := \{w \in W;\ w \text{ satisfies Bonnafé-Rouquier's criterion}\}$ (Theorem 2.9)

$(DL) := \{w \in W;\ w \text{ satisfies Deligne-Lusztig's criterion}\}$ (Theorem 1.5)

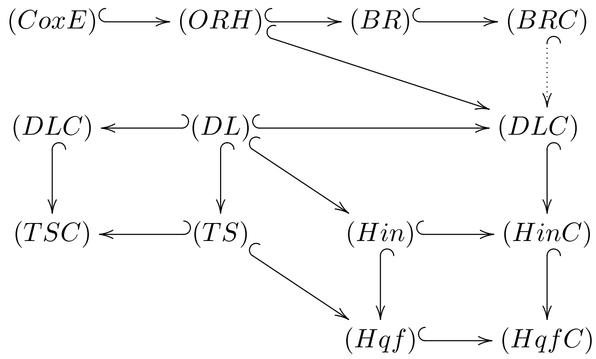
$(TS) := \{w \in W;\ w \text{ satisfies the tangent space criterion}\}$ (Theorem 4.1)

$(Hin) := \{w \in W;\ w \text{ satisfies the injectivity criterion}\}$ (Corollary 3.9)

$(Hqf) := \{w \in W;\ w \text{ satisfies Harashita's quasi-finite criterion}\}$ (Theorem 3.6)

$(XC) := \{w \in W;\ w \in Cyc(v) \text{ for some element } v \text{ of } (X)\}$ (Proposition 2.4)

We have the following diagram of inclusions, that are all strict for general choice of G, except possibly the arrows to (Hqf) and $(HqfC)$:



Here, the arrow from (BRC) to (DLC) is only conjectured. We want to remark that, $(TS), (Hin)$, and (Hqf) strongly depend on (DL) and that the cyclic-shift criterion is very powerful, see the table at the end of chapter 4.

In the second part, we develop a further criterion for affineness. Compared to the well-known criteria mentioned above, this is a sufficient and necessary criterion. Denote by π the T-torsor $G/U \to G/B$. We obtain the following

Theorem 0.1. *Let $X(w)$ be a Deligne-Luztig variety. Then*

$$X(w) \text{ affine } \Leftrightarrow \; H^1\left(\pi^{-1}(X(w)), \mathcal{O}_{\pi^{-1}(X(w))}\right) = 0.$$

For the proof see Theorem 5.11. Moreover, we show that it is sufficient to compute for a finite set $\{\tau_i; 1 \leq i \leq l(w_0)\}$, $w_0 \in W$ the longest element, of elements of certain cohomology groups, whether they are zero-cohomological (Theorem 5.1). We obtain an explicit description of this set: Namely, we attach to a fixed reduced expression for $w_0 \in W$ a filtration $\{U_{l(w_0)} \subset \cdots \subset U_0 = U\}$ of U and get $\mathbb{G}_a$-torsors $\tau_i : U_i \to U_{i-1}$. Then, we consider the restrictions of τ_i to the inverse images of $\pi^{-1}(X(w))$ and regard them as elements of the first cohomology group of $(\pi\tau_1 \ldots \tau_{i-1})^{-1}(X(w))$. Moreover, we state and discuss in detail an implementable algorithm, which checks with methods from linear algebra whether these elements are all zero-cohomological. We give some examples for G of type A, compare them to the algorithm, and comment on other approaches for proving the affineness. We frequently focus on reducing strategies for the implementation. Thereby, we found a new, constructive proof of the well-known fact that all Deligne-Lusztig varieties attached to Coxeter elements are affine (Proposition 5.16).

We have implemented the algorithm for G_0 split of type A with the help of Singular ([10]) and Magma ([7]) and are able to calculate small, well-known examples. Unfortunately, we did not manage to obtain a (reasonably small) bound, which ensures the termination of the algorithm in case there exists an element τ_i that is not zero-cohomological. For practical use of the implementation it would be good to have preferably small bounds if one wants to proof the non-affineness for some Deligne-Lusztig varieties. The main bottleneck of the algorithm is the fact that we have to compute Gröbner bases, which are memory- and time-intensive computations, even with a (not yet existing) kernel implementation of the F_5-algorithm. Right now, we are only able to handle the case $G = \mathbb{SL}_3/\mathbb{F}_q$, but for $G = \mathbb{SL}_4$ we stopped computations after one week. In the last chapter, we therefore focus on some further reducing strategies to speed up computations. However, we have not implemented all the suggested strategies. To state two of them, we suggest to keep attention on the monomial orders for the occurring rings and to concentrate on different reduced expressions for $w_0 \in W$.

I want to thank my advisor Ulrich Görtz for providing the interesting subject. I am deeply grateful for his constant attendance on discussion and explanation of mathematical issues and for his confidence and patience that he puts in me. I would also like to thank my girlfriend Philine, my colleagues at work Christian, Philipp and Ulrich, my family, and my friends for supporting me.

1. Basic Definitions

Let q be a power of a prime and let $k := \overline{\mathbb{F}}_q$ be a fixed algebraic closure. Let G_0 be a connected, reductive algebraic group over $k_0 := \mathbb{F}_q$ and denote by $G := G_0 \times_{\mathbb{F}_q} \operatorname{Spec} k$ the base change from $\mathbb{F}_q$ to k. We have an endomorphism $F : G \to G$ coming from the Frobenius endomorphism on G_0, which is induced by the usual Frobenius $x \mapsto x^q$ on k. The fixed points of G under F are then $G^F = G_0(\mathbb{F}_q)$.

The group G acts on the set X_G of all Borel subgroups of G by conjugation. This set X_G is a projective variety. For us, a variety (over k) is a reduced, separated scheme of finite type over k. We fix once and for all an F-stable, maximal torus T in G and an F-stable Borel group B such that $T \subset B$. We denote by U the unipotent radical of $B = U \rtimes T$. We identify the Weyl group of G with $W = N_G(T)/T$, where $N_G(T)$ denotes the normalizer of T in G. We have an isomorphism $G/B \xrightarrow{\sim} X_G$, $[g] \mapsto gBg^{-1}$. For $w \in W$ denote by $\dot{w} \in N_G(T)$ a representative and by $\mathcal{O}(w) := G.(B, \dot{w}B\dot{w}^{-1})$ the orbit of $(B, \dot{w}B\dot{w}^{-1})$ in $X_G \times X_G$ under the diagonal action. We denote again by F the Frobenius on G/B coming from F. We denote for $w \in W$ by $l(w)$ the length of w, i.e. the length of any reduced expression for w.

Definition 1.1. *Two Borel subgroups B_1, B_2 are in relative position $w \in W$, written $\operatorname{inv}(B_1, B_2) = w$, if and only if $(B_1, B_2) \in \mathcal{O}(w)$.*

Lemma 1.2. *The morphism $W \to G\backslash X_G \times X_G$, $w \mapsto \mathcal{O}(w)$ is an isomorphism. For every $w, w_1, w_2 \in W$ with $l(w) = l(w_1) + l(w_2)$ and $w = w_1 w_2$ we have an isomorphism of varieties $\mathcal{O}(w_1) \times_{X_G} \mathcal{O}(w_2) \xrightarrow{\sim} \mathcal{O}(w)$.*

Proof. By the Bruhat decomposition we have an isomorphism $W \cong B\backslash G/B$. It is straight-forward to check that the mapping $BgB \mapsto (B, gB)$ is well defined and induces an isomorphism $B\backslash G/B \xrightarrow{\sim} G\backslash(G/B \times G/B)$ with inverse mapping $(gB, hB) \mapsto Bg^{-1}hB$. Here G acts diagonally and on each factor simply by multiplication. But as we have an isomorphism of G/B with X_G, we get an induced isomorphism $G\backslash(G/B \times G/B) \xrightarrow{\sim} G\backslash(X_G \times X_G)$, where G acts diagonally and on each factor by conjugation. Under the composition of the isomorphisms above, w is mapped to the G-orbit of $(B, \dot{w}B\dot{w}^{-1})$, which is independent of the chosen representative for w.

For the second statement, we remark that one can rephrase the Bruhat decomposition in terms of the relative position. We have for $w, w_1, w_2 \in W$ as above and $B_1, B_2, B_3 \in X_G$ any Borel subgroups the following:

$$(B_1, B_2) \in \mathcal{O}(w_1), (B_2, B_3) \in \mathcal{O}(w_2) \Rightarrow (B_1, B_3) \in \mathcal{O}(w_1 w_2) = \mathcal{O}(w)$$

$$(B_1, B_3) \in \mathcal{O}(w_1 w_2) \Rightarrow \exists! B_2 \in X_G : (B_1, B_2) \in \mathcal{O}(w_1) \text{ and } (B_2, B_3) \in \mathcal{O}(w_2)$$

So the map $\mathcal{O}(w_1) \times \mathcal{O}(w_2) \to \mathcal{O}(w)$ is bijective on k-valued points. It is easy to check that this induces a birational morphism. As $\mathcal{O}(w)$ is smooth, hence normal, we can apply Zariski's main theorem ([16], Cor. 12.88) to the above varieties to see that this morphism is indeed an isomorphism. □

For G_0 split of type A and rank $n-1$ one can regard G/B as the flag variety of full flags of k^n. That is, given any matrix $A = (a_{ij}) \in G(k)$ we map it to the flag $\operatorname{Flag}(A) := (\langle (a_{i1})_i \rangle \subset \langle (a_{i,1})_i, (a_{i,2})_i \rangle \subset \ldots \subset k^n)$. We denote by $\mathcal{G}(i, n)$ the Grassmannian over k, which is (as a set) defined as (the set of) all i-dimensional subspaces of k^n. We can therefore regard G/B as (closed) subset of $\mathcal{G}(k^n) := \prod_{i=1}^{n-1} \mathcal{G}(i, n)$. We have a closed embedding of $\mathcal{G}(k^n)$ into a certain product of projective spaces, called Plücker embedding, given as follows. Let $1 \leq i \leq n-1$, let

$A^{(i)} \in \mathrm{Mat}_k(n,i)$ be a matrix of rank i, so that $A^{(i)}$ represents a subspace of k^n of dimension i. Fix a basis $e_1, \ldots, e_n$ for k^n consisting of row vectors. We define the Plücker map:

$$pl := (pl_1, \ldots, pl_{n-1}) : \mathcal{G}(k^n) \hookrightarrow \prod_{i=1}^{n-1} \mathbb{P}(\bigwedge^i k^n) \cong \prod_{i=1}^{n-1} \mathbb{P}^{\binom{n}{i}-1}$$

$$pl_i(A^{(i)}) = (a^{(i)}_{1,1}e_1 + \ldots + a^{(i)}_{n,1}e_n) \wedge \ldots \wedge (a^{(i)}_{1,i}e_1 + \ldots + a^{(i)}_{n,i}e_n) = \sum_{|J|=i} p_J(A^{(i)})e_J$$

Here we define $e_J = e_{j_1} \wedge \ldots \wedge e_{j_i}$ for a subset $J = \{j_1, \ldots, j_i\} \subset \{1, \ldots, n\}$ and call p_J the corresponding Plücker coordinate, i.e. $p_J(A^{(i)}) \in k$ is the minor of $A^{(i)}$ obtained by taking the rows with indices in J and all columns. This is well defined, as for any $A^{(i)}, A'^{(i)} \in \mathrm{Mat}_k(n,i)$, representing the same i-dimensional subspace of k^n, there exists an invertible matrix $M \in \mathbb{GL}_i(k)$ such that $A^{(i)}M = A'^{(i)}$. But then $pl_i(A^{(i)})$ and $pl_i(A'^{(i)}) = pl_i(A^{(i)})\det(M)$ differ only by a non-zero constant. For a proof of the fact that this is a (closed) embedding, we refer to [16], Prop. 8.23.

For calculations, we might also fix a basis of $\bigwedge^i k^n$ by fixing some order on the set $\{e_J; J = \{j_1, \ldots, j_i\}, 1 \leq j_1 < \ldots < j_i \leq n\}$ and identify $\mathbb{P}(\bigwedge^i k^n)$ with $\mathbb{P}^{\binom{n}{i}-1}$. In this case, we will also write pl for the composition of the Plücker map with this fixed isomorphism and again write p_J for the Plücker coordinates.

Definition 1.3. (Deligne-Lusztig) *For $w \in W$ let $X(w) \subset X_G$ denote the reduced, locally closed subscheme consisting of all Borel subgroups B' of G such that B' and $F(B')$ are in relative position w. It is called the Deligne-Lusztig variety attached to w.*

We can regard $X(w)$ as the subscheme of $G/B =: X$ given by

$$\left\{gB \in X; g^{-1}F(g) \in BwB\right\}.$$

One can also view $X(w)$ as the scheme theoretic intersection in $X_G \times X_G$ of $\mathcal{O}(w)$ with the graph of F. The scheme $X(w)$ is smooth, locally closed, purely of dimension $l(w)$, G^F-stable and of finite type over k. We have the following

Theorem 1.4. (Lusztig, [29]**)** *For $w \in W$ the following are equivalent:*

(i) $X(w)$ is irreducible.
(ii) The closure $\overline{X(w)}$ is connected.
(iii) The element w is not contained in any F-stable proper standard parabolic subgroup of W.

Proof. See [15], Cor. 1.2. □

The following discussion shows that it is enough to consider only irreducible $X(w)$, if we want to check the affineness of $X(w)$ for all $w \in W$. Let $w \in W$, let $P \subset G$ be the smallest parabolic subgroup of G, such that P contains B, P is F-stable, and P contains all simple reflections s with $s \leq w$ in the Bruhat order. Consider the following commutative diagram:

$$\begin{array}{ccc} X(w) & \longrightarrow & (G/P)(k_0) \\ \big\downarrow & & \big\downarrow \\ G/B & \xrightarrow{\ \tau\ } & G/P\ . \end{array}$$

We have that $X(w)$ is the finite, disjoint union of $\tau^{-1}(\dot{x}P) \cap X(w)$, for $\dot{x}P \in (G/P)(k_0)$. The fibers can be described as $\tau^{-1}(\dot{x}P) \cap X(w) = \{gB; g^{-1}F(g) \in$

$BwB, gP = \dot{x}P\}$. We may assume $F(\dot{x}) = \dot{x}$, so one can easily see that the morphism

$$\tau^{-1}(\dot{x}P) \cap X(w) \to \{gB; g^{-1}F(g) \in BwB, gP = P\}, \quad g \mapsto \dot{x}^{-1}g,$$

is an isomorphism of varieties. Denote by L_P the standard Levi factor, that is L_P contains T and normalizes the unipotent radical U_P of P. Then L_P is again reductive with corresponding Weyl group $W_P \subseteq W$, where W_P is generated by all simple reflections contained in P, so $w \in W_P$. We have $P/B = L_P/B_{L_P}$, where $B_{L_P} = L_P \cap B$, and we get

$$\begin{aligned}\{gB; g^{-1}F(g) \in BwB, gP = P\} &\cong \{gB \in P/B; g^{-1}F(g) \in BwB\} \\ &= \{gB_{L_P} \in L_P/B_{L_P}; g^{-1}F(g) \in B_{L_P}wB_{L_P}\} \\ &=: X_P(w) \subset L_P/B_{L_P}.\end{aligned}$$

Now, $X_P(w)$ is irreducible by Theorem 1.4. Moreover, $X_P(w)$ is affine if and only if $X(w)$ is affine (see also [4]). Obviously, the same discussion applies to many other geometric properties.

Deligne and Lusztig gave a sufficient combinatorial criterion for $X(w)$ being affine in [11], Theorem 9.7. To discuss this result we define $\mathcal{X}(T)$ to be the character group of T, i.e. $\mathcal{X}(T) = \mathrm{Hom}(T, \mathbb{G}_m) = \mathcal{X}(B)$. Let Φ be the root system associated to T, denote by Φ^+ the set of positive roots and by $\Delta \subset \Phi^+$ the set of simple roots determined by B. We have an induced map on Φ by the Frobenius F, denoted again by F. As B and T are F-stable, Φ^+ and Δ are F-stable as well. The elements of Δ are in one to one correspondence to the simple reflections of $W = N_G(T)/T$. The Weyl group W operates on $\mathcal{X}(T)$ and leaves Φ stable. Denote by $C \subset \mathcal{X}(T) \otimes \mathbb{R}$ the fundamental Weyl chamber corresponding to B (or Δ). For $\alpha \in \Phi$ we denote by $\check{\alpha}$ the corresponding coroot. Then

$$C = \{\lambda \in \mathcal{X}(T) \otimes \mathbb{R}; \langle \lambda, \check{\alpha} \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Denote by C^0 the interior of C (w.r.t. to the standard topology of $\mathbb{R}$), which is described by replacing all $\geq$ in the above description by $>$. Denote for two Weyl chambers C_1, C_2 by $D(C_1, C_2)$ the intersection of the closed radicial half spaces containing both C_1 and C_2 and by $D^0(C_1, C_2)$ its interior. For $w \in W$ fix a reduced expression $w = s_{i_1} \ldots s_{i_l}$ and write $v_j = s_{i_1} \ldots s_{i_j}$ for $j = 0, \ldots, l(w)$. Write α_j for the simple root corresponding to s_{i_j}. In this case one can describe $D(w) := D(C, -wC)$ as

$$D(w) = \{\lambda \in \mathcal{X}(T) \otimes \mathbb{R}; \langle \lambda, v_{j-1}(\check{\alpha}_j) \rangle \geq 0 \text{ for } 1 \leq j \leq l(w)\},$$

see [11], proof of Prop. 9.6.1.

Theorem 1.5. (Deligne-Lusztig) *Let $w \in W$. If there exists $\mu \in \mathcal{X}(T) \otimes \mathbb{R}$ such that*

$$\mu \in D^0(C, -w^{-1}C) \text{ and} \tag{1.6}$$

$$F\mu - w\mu \in C^0, \tag{1.7}$$

then $X(w)$ is affine.

Proof. See Remark 3.7. □

We want to remark here that we get exactly the same explicit descriptions for $C^0 \subset \mathcal{X} \otimes \mathbb{Q}$ and $D^0(w) \subset \mathcal{X} \otimes \mathbb{Q}$, where we simply use the same names for these sets as above. Furthermore, if G_0 is split then F operates on the Euclidian space $\mathcal{X}(T) \otimes \mathbb{R}$ simply as multiplication by q.

Corollary 1.8. *Let h be the Coxeter number of W. If $q \geq h$, then $X(w)$ is affine for every $w \in W$.*

Proof. One can find $\mu \in C^0$ such that $\langle \mu, \check{\alpha} \rangle = 1$ for every simple root $\alpha \in \Delta \subset \Phi$. But for $q \geq h$ condition (1.7) is then automatically satisfied, see [11], Thm. 9.7. □

Remark 1.9. *To computationally find such μ, satisfying (1.6) and (1.7), for G of classical type, we identify $\mathcal{X}(T) \otimes \mathbb{R}$ with the space $E = \mathbb{R}^{\mathrm{rk}(G)}$, resp. with a subspace $E \subset \mathbb{R}^{\mathrm{rk}(G)+1}$ of codimension 1 for G of type A. We use the standard identifications for this, see for example* [8]*, Planches I-IX. With the descriptions above, to fulfill the Deligne-Lusztig conditions we have to find a $\lambda = (\lambda_1, \ldots, \lambda_{\mathrm{rk}(G)}) \in E$ such that*

$$\langle -\lambda, v_{j-1}(\check{\alpha_j}) \rangle \leq 0, \quad 1 \leq j \leq l(w),$$

and

$$\langle q\lambda - w\lambda, \check{\alpha} \rangle < 0, \quad \alpha \in \Delta.$$

This can be expressed as the linear optimization problem (LOP) "minimize λ_1" under the side conditions above. For G of type A one has to add the extra conditions $\lambda_i \leq 0$, $1 \leq i \leq r$, to ensure the termination of the algorithm for solving the LOP. We have implemented this with the help of GAP 3 *(*[33]*) and* CVXOPT *(*[1]*).*

Definition 1.10. *Let $\lambda \in \mathcal{X}(T) = \mathcal{X}(B)$, denote by π the T-torsor $G/U \to G/B$ and let $V \subset G/B$ be any open subset. Define the invertible $\mathcal{O}_{G/B}$-module $\mathcal{L}(\lambda)$ by*

$$\mathcal{L}(\lambda)(V) = \left\{ f \in \mathcal{O}_{\pi^{-1}(V)}(\pi^{-1}(V)); f(gt) = \lambda^{-1}(t)f(g) \text{ for all } t \in T, g \in G \right\}.$$

We set $E := G \times^B \mathbb{A}^1 = G \times \mathbb{A}^1 / \sim$, where $\sim$ is defined by $(g, a) \sim (gb, \lambda^{-1}(b)a)$. Denote by σ the map $E \to G/B$, $\sigma(g, a) = gB$. Then one can identify $H^0(V, \mathcal{L}(\lambda))$ with $\{s : V \to E; \sigma \circ s = id_V\}$. We want to remark here that, for G semisimple, the map $\mathcal{X}(T) \to \mathrm{Pic}(G/B)$, $\lambda \mapsto \mathcal{L}(\lambda)$, is an isomorphism, see [26], §8.3.

For general (reductive) G we have $G \cong R(G)G_s$, where $R(G) = \left(\bigcap_{B' \in X_G} B'\right)^0$ is the radical of G and $G_s = (G, G)$ is the commutator subgroup of G, which is semisimple by [34], Corollary 8.1.6. But then the surjective morphism $\alpha : G \to G_s$ induces an isomorphism between W and the Weyl group attached to G_s, T_s, where $T_s = \alpha(T)$ is a maxmial torus, and an isomorphism between the set of Borel subgroups of G containing T and the set of Borel subgroups of G_s containing T_s (see [6], Proposition 11.20). We write $B_s = \alpha(B)$. For $\lambda \in \mathcal{X}(T)$ we denote by $\lambda_s \in \mathcal{X}(T_s)$ the image of λ under the induced map $\mathcal{X}(T) \to \mathcal{X}(\alpha(T))$. One has $G/B \cong G_s/B_s$ and under this isomorphism $\mathcal{L}_{G/B}(\lambda) \cong \mathcal{L}_{G_s/B_s}(\lambda_s)$.

Proposition 1.11. *With the notations above, we have*

$$\mathcal{L}(\lambda) \textit{ ample } \Leftrightarrow \lambda \in -C^0.$$

Proof. See [24], II.4.4 Proposition. □

Remark 1.12. *In loc. cit., the B-operation on $G \times \mathbb{A}^1 / \sim$ is given by $(g, a) \sim (gb, \lambda(b)a)$, we thus have here a change of sign compared to loc. cit., i.e. $\lambda \in -C^0$ instead of $\lambda \in C^0$.*

Remark 1.13. *In the proof of loc. cit. slightly more is shown, namely if $\lambda \in -C^0$, then $\mathcal{L}(\lambda)$ is actually very ample. Moreover one can show that in this situation ampleness and very ampleness are equivalent.*

Theorem 1.14. (Haastert, [17]**)** *For every $w \in W$, $X(w)$ is quasi-affine.*

Proof. Let $w \in W$ and $\dot{w} \in N(T)$ be a representative for w. We fix some isomorphism $T \cong \mathbb{G}_m^r$, let $t = (t_i)_i \in T$ and let $\varphi : (t_i)_i \mapsto \varphi((t_i)_i) = \prod \varphi_i(t_i)$ be a character. Then the induced dual morphism $L_w^* : \mathcal{X}(T) \to \mathcal{X}(T)$ of the Lang map $L^w : T \to T$, $t \mapsto t^{-1}\dot{w}F(t)\dot{w}^{-1}$, is given by

$$L_w^*(\varphi) : (t_i)_i \mapsto \prod \varphi_i(L^w(t)_i) = \prod \varphi_i(t_i^{-1}F(t_{w^{-1}(i)})).$$

The surjectivity of L^w (see [6], §16.4 Cor.) yields that L_w^* is injective. As $\mathcal{X}(T)$ is free of finite rank, the image of L_w^* has finite index in $\mathcal{X}(T)$. Thus for every $\mu \in -C^0$, there exist $n \in \mathbb{Z}_{>0}, \lambda \in \mathcal{X}(T)$ such that $n\mu = L_w^*(\lambda) = -\lambda + F(w^{-1}\lambda)$.

The line bundle $\mathcal{L}(-\lambda + F(w^{-1}\lambda))$ is very ample. Furthermore the restriction of $\mathcal{L}(-\lambda + F(w^{-1}\lambda))$ to $X(w)$ is trivial (see [11], 9.6, page 149), so $\mathcal{O}_{X(w)}$ is also ample. But this is equivalent to saying that $X(w)$ is quasi-affine (see [16], Prop. 13.80). □

Remark 1.15. *One can generalize this argument to generalized Deligne-Lusztig varieties (see Definition 3.1), to show that all distinguished Deligne-Lusztig varieties are quasi-affine. See* [19]*, Thm. 3.1.1, for a complete proof.*

2. Other affineness Criteria

In this chapter we will state further affineness criteria deduced from Deligne-Lusztig's criterion. We keep the notation of the last chapter. In particular B, U, T and $W = N_G(T)/T$ are fixed as above. Denote by S the set $\{s_i\}_{i\in\mathcal{I}}$ of simple reflections in W. Observe that the Frobenius $F : G \to G$ also induces an automorphism of W, since T is F-stable. We denote this automorphism again by F. Under this, S is fixed as B is F-stable. We recall some definitions of [5].

Definition 2.1. *Let $w, w' \in W$.*

- *(i) We say that w' is F-conjugate to w if and only if there exists a $v \in W$ such that $w' = v^{-1}wF(v)$. We denote this relation by $w' \sim_F w$ and write $\mathcal{K}(w) = \{w' \in W; w' \sim_F w\}$.*
- *(ii) We say that w' is F-conjugate by cyclic shift to w if and only if there exist $n \in \mathbb{N}$, $(x_i)_{i\leq n}$, $(y_i)_{i\leq n}$ and $(w_i)_{i\leq n+1}$ in W such that*

$$w_1 = w, w_{n+1} = w', \forall i : w_i = x_i y_i, w_{i+1} = y_i F(x_i)$$
$$\text{and } l(w_i) = l(w_{i+1}) = l(x_i) + l(y_i).$$

We denote this relation by $w' \overset{F}{\leftrightarrow} w$ and let $Cyc(w) = \{w' \in W; w' \overset{F}{\leftrightarrow} w\}$.

Remark 2.2. *It is easy to see that these are both equivalence relations. Moreover, let $w, w' \in W, w' \overset{F}{\leftrightarrow} w$ and $(x_i)_i, (y_i)_i$ be two sequences as in the definition. By setting $v := x_1 \dots x_n$, we see that $w' \sim_F w$, so $Cyc(w) \subseteq \mathcal{K}(w)$. On the other hand, it is in general not true that for every $w' \in \mathcal{K}(w)$ with $l(w') = l(w)$ we have $w' \in Cyc(w)$. But for minimal length elements this is proved by Orlik and Rapoport in* [31]*, see below.*

Also, we have the following

Lemma 2.3. *Let $s, t \in S$ be simple reflections and let $w \in W$ be such that $l(w) = l(swt)$. Then either $w = swt$ or $l(sw) = l(w) - 1$ or $l(wt) = l(w) - 1$.*

Proof. See [11], Lemma 1.6.4. □

Proposition 2.4. *Let $w, w' \in W$ with $w' \overset{F}{\leftrightarrow} w$. Then $X(w')$ and $X(w)$ are simultaneously affine, resp. non-affine.*

Proof. We recall the proof from [5], Prop. 2, resp. [11], pages 107-108. Observe that we are in the situation that [11] called "Case 1": We assume by induction that $w = x_1y_1$ and $w' = y_1F(x_1)$ with $l(w) = l(x_1) + l(y_1) = l(y_1) + l(F(x_1)) = l(w')$. For $\dot{x} \in X(w)$ there is a unique Borel subgroup $\dot{x}' =: \sigma(\dot{x})$ such that $(\dot{x}, \dot{x}') \in \mathcal{O}(x_1)$ and $(\dot{x}', F(\dot{x})) \in \mathcal{O}(y_1)$, so $\dot{x}' \in X(w')$. As above, $\mathcal{O}(v)$ denotes the G-orbit in $X_G \times X_G$ corresponding to $v \in W$ under the isomorphism $W \xrightarrow{\sim} G\backslash X_G \times X_G$ of Lemma 1.2. With the same argument applied to $y_1F(x_1) = F(x_1)F(y_1) = F(w)$ we also get a morphism $X(w') \to X(F(w))$. The commutative diagram

$$\begin{array}{ccc} X(w) & \xrightarrow{\sigma} & X(w') \\ \downarrow F & \swarrow \tau & \downarrow F \\ X(F(w)) & \longrightarrow & X(F(w')) \end{array}$$

induces that τ is universally bijective, since F is a universal homeomorphism. Thus σ is universally bijective, too. As F is finite and τ is separated, σ is finite. Thus the result follows. □

Now Orlik and Rapoport have conjectured that for every element w of minimal length in its F-conjugacy class $\mathcal{K}(w)$, one has that $X(w)$ is affine. In [31] they proved this, for G_0 split of classical type, by finding for every such w a certain minimal length element that is F-cyclic conjugated to w and satisfies the Deligne-Lusztig criterion. They conclude by using that for minimal length elements v of $\mathcal{K}(w)$ we have an identity $Cyc(v) = \{v' \in \mathcal{K}(w) = \mathcal{K}(v); l(v') = l(v)\}$. Xuhua He (see [22]) proved the conjecture of Orlik and Rapoport in full generality (by proving the missing parts case by case) by a similar strategy. We have

Theorem 2.5. (Orlik-Rapoport, He, Bonnafé-Rouquier) *Let $w \in W$ be an element of minimal length in the F-conjugacy class $\mathcal{K}(w)$. Then $X(w)$ is affine.*

Proof. See [31], § 5, and [22], Theorem 1.3. □

Remark 2.6. *As this is a corollary of the Deligne-Lusztig criterion, from the computational point of view only with Proposition 2.4 one can prove the affineness of further Deligne-Lusztig varieties.*

Remark 2.7. *As a corollary of the theorem, one can conclude that $X(w)$ is affine if w is a (twisted) Coxeter element, i.e. every reduced expression of w contains for every F-orbit of $s_i \in S$ exactly one element. There are several more proofs of this result, e.g. by Deligne and Lusztig for G_0 split of type A (see* [11]*, §2.2), by Lusztig (see* [28]*, Cor. 2.8) and by Hansen (see* [21]*, §2, Remark 5). We give another proof for G_0 split of type A in Proposition 5.16.*

Bonnafé and Rouquier gave a slightly more general criterion generalizing Theorem 2.5 (see [5], §4). We recall the main steps of this proof (see [5], Theorem B), as it admits a different approach compared to the proof of the Deligne-Lusztig criterion. It is rather a generalization of [11], §2.2. We need some notation.

Definition 2.8. *The braid monoid associated to (W, S) is the monoid B^+ with presentation*

$$B^+ = \langle (\underline{x})_{x\in W}; \forall x, x' \in W, l(x'x) = l(x') + l(x) : \underline{x'x} = \underline{x'}\,\underline{x} \rangle .$$

We again denote by F the automorphism of B^+, coming from the extension of F to W. We set, for any sequence $(x_1, \ldots, x_r) \subset W$,

$$\mathcal{O}(x_1, \ldots, x_r) := \mathcal{O}(x_1) \times_{X_G} \cdots \times_{X_G} \mathcal{O}(x_r).$$

We have that $\mathcal{O}(x_1, \ldots, x_r) \cong \mathcal{O}(y_1, \ldots, y_s)$, whenever $\underline{x_1} \cdots \underline{x_r} = \underline{y_1} \cdots \underline{y_s}$ in B^+, see [12], App. 2. Now assume, we have for $(x_1, \ldots, x_r)$ a $b \in B^+$ such that $\underline{x_1} \cdots \underline{x_r} = \underline{w_0} b = \underline{w_0}\ \underline{b_1} \cdots \underline{b_s}$ in B^+, where $b_i \in W$. For $v \in W$ we denote by $\dot{v}$ a fixed representative for v as above. We define $b_0 := w_0$. In [5], page 1204, it is shown that the map

$$G \times \prod_{i=1}^{s} \left(U\dot{b}_i \cap \dot{b}_i U^- \right) \longrightarrow \underbrace{\left\{ (g_{-1}U, \ldots, g_s U) \in (G/U)^{s+2}; \forall 0 \le i \le s, g_{i-1}^{-1} g_i \in U\dot{b}_i \right\}}_{=:\widetilde{\mathcal{O}}(b_0,\ldots,b_s)},$$

$$(g, h_0, \ldots, h_s) \mapsto (gU, g\dot{b}_0 U, \ldots, g\dot{b}_0 h_1 \cdots h_s U),$$

is an isomorphism of varieties.

One can define an action of T on the right on $\widetilde{\mathcal{O}}(b_0, \cdots, b_s)$ such that $\mathcal{O}(b_0, \cdots, b_s)$ can be identified with the quotient of $\widetilde{\mathcal{O}}(b_0, \cdots, b_s)$ by T. But $\widetilde{\mathcal{O}}(b_0, \cdots, b_s)$ is affine and T is reductive, whence the quotient is affine, too (see [6], Cor 8.21).

Now assume for $w \in W$ and $x_i = F^i(w)$ we have such a $b \in B^+$. Then we can identify $X(w)$ with the closed subvariety $\left\{ (\dot{x}', F(\dot{x}'), \cdots, F^{r-1}(\dot{x}')); \dot{x}' \in X \right\} \cap \mathcal{O}(w, F(w), \cdots, F^{r-1}(w))$ of X_G^r. As $\mathcal{O}(w, F(w), \cdots, F^{r-1}(w))$ is affine, we have the following result:

Theorem 2.9. (Bonnafé-Rouquier) *Let I be an F-stable subset of S, w_I the longest element in the subgroup $W_I \subseteq W$ generated by I and let $w \in W_I$ be such that there exists*

$$r \geq 1, b \in B^+ : \underline{w}F(\underline{w}) \cdots F^{r-1}(\underline{w}) = \underline{w_I}b.$$

Then $X(w)$ is affine.

Remark 2.10. *Computing the cyclic-shift F-conjugacy class yields the problem that by definition there is no bound on the length $n \in \mathbb{N}$ of the sequences $(x_i)_i$ and $(y_i)_i$ which yield a cyclic-shift from w' to w, for $w, w' \in W$. But one can take the obvious bound $n_{\max} = l(w_0)$, where w_0 is the longest element in W. This bound seems to be far away from being optimal. For the implementation observe that one has covered the whole of $Cyc(w)$, as soon as for some $n \in \mathbb{N}$ one cannot find sequences $(x_i)_i$ and $(y_i)_i$ of length $n+1$ that yield a new $w' \in W$ which is cyclic-shift F-conjugated to w.*

Remark 2.11. *One can introduce the notation "good element" for elements of W, see* [14]*. Then one has that all good elements satisfy the conditions of the last theorem. By definition, all good elements are of minimal length in their conjugacy classes. Bonnafé and Rouquier showed in* [5] *that every F-conjugacy class contains a good element. Furthermore, $X(w)$ is affine if w is a good element, see loc. cit. Therefore the criteria "F-conjugated by cyclic-shift to a good element" and "F-conjugated by cyclic-shift to an element of minimal length in its conjugacy class" coincide.*

Remark 2.12. *In* [5] *it is shown that there exist groups G and elements $w \in W_G$ such that the Deligne-Lusztig criterion does not apply, but Theorem 2.9 does. To compute this, Bonnafé and Rouquier used their own (resp. Jean Michel's) implementation of this criterion. However, they were not able to find an element $w \in W$ that is not F-conjugated by cyclic-shift to an element satisfying the Deligne-Lusztig criterion.*

3. A criterion of Harashita

In this chapter, following Harashita ([19]), we generalize the notion of Deligne-Lusztig varieties to obtain a generalization of Theorem 1.5. We keep the notations of chapter 1.

We denote by W_I the subgroup of W generated by the simple reflections s_α for $\alpha \in I \subseteq \Delta$ and by $P_I = BW_IB$ the standard parabolic subgroup attached to I. Let X_I be the set of parabolic subgroups of G conjugate to P_I. This is a smooth projective scheme over k. For two subsets $I, J \subseteq \Delta$ and $w \in W$ we denote by $\mathcal{O}_{IJ}(w)$ the orbit of $(P_I, \dot{w}P_J\dot{w}^{-1})$ under the diagonal action of G on $X_I \times X_J$. We have

$$X_I \times X_J = \coprod_{w \in W_I \backslash W / W_J} \mathcal{O}_{IJ}(w).$$

Definition 3.1. (Harashita) *The generalized Deligne-Lusztig variety $X_I(w)$ associated to $w \in W_I\backslash W/W_{F(I)}$ is the locally closed subscheme of X_I consisting of the parabolic subgroups P such that $(P, F(P)) \in \mathcal{O}_{I,F(I)}(w)$. We call $X_I(w)$ distinguished if $I = \tilde{w}F(I) \subset \Phi$, where $\tilde{w}$ is the representative of minimal length in W for $w \in W_I\backslash W/W_{F(I)}$.*

The idea of Harashita to prove the affineness for $X(w)$ is as follows. We want to find a line bundle on G/B such that the restriction to $X(w)$ is ample and trivial. Whereas Deligne-Lusztig consider as candidates only line bundles on G/B which are ample, Harashita also considers line bundles coming via pull-back from an ample line bundle on some G/P_J. While these are not ample on G/B, the restriction to $X(w)$ is ample under certain conditions. This yields a generalization of the Deligne-Lusztig criterion.

Let U_I be the unipotent radical of P_I and let L_I be the standard Levi subgroup of $P_I = U_I \rtimes L_I$. That is, L_I contains T and normalizes U_I. For the character groups we identify $\mathcal{X}(P_I) = \mathcal{X}(L_I) = \{\lambda \in \mathcal{X}(T); \langle\lambda, \check{\alpha}\rangle = 0 \text{ for } \alpha \in I\}$. We denote by π_I the L_I-torsor $G/U_I \to X_I$ and define for $\lambda \in \mathcal{X}(P_I)$ the following line bundle $\mathcal{L}_I(\lambda)$ on X_I:

$$\mathcal{L}_I(\lambda)(V) = \left\{ f \in \mathcal{O}_{\pi_I^{-1}(V)}(\pi_I^{-1}(V)); f(gx) = \lambda(x)^{-1}f(g) \text{ for all } x \in L_I, g \in G/U_I \right\},$$

$V \subset X_I$ any open subscheme. We have

Lemma 3.2. *Let $w \in W$, let $I \subseteq \Delta$ such that $I = \tilde{w}F(I) \subseteq \Delta$, let $\lambda \in \mathcal{X}(P_I)$. On $\mathcal{O}_{I,w^{-1}I}(w)$ we have an isomorphism*

$$\Psi(\dot{w}) : \mathrm{pr}_1^*\mathcal{L}_I(\lambda) \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{L}_{w^{-1}I}(w^{-1}\lambda)$$

Proof. See [19], Prop. 2.1.2. □

Remark 3.3. *Let $I \subset \Delta$, $w \in W$ such that $X_I(w)$ is distinguished. One may ask if $X_I(w)$ is quasi-affine. One knows for general $\mu \in -C_I^0$ that $\mathcal{L}_I(\mu)$ is (very) ample (see* [26]*, 8.3). Moreover, if $\mu = L_w^*(\lambda)$ for some $\lambda \in \mathcal{X}(P_I)$, then the restriction of $\Psi(\dot{w})$ to $X_I(w)$, considered as a section of $\mathrm{pr}_1^*\mathcal{L}_I(\lambda)^{-1} \otimes \mathrm{pr}_2^*\mathcal{L}_{w^{-1}I}(w^{-1}\lambda)$, yields a nowhere vanishing section of $\mathcal{L}_I(\mu)$ on $X_I(w)$. Hence the ample line bundle $\mathcal{L}_I(\mu)$ is isomorphic to $\mathcal{O}_{X_I(w)}$. Thus $\mathcal{O}_{X_I(w)}$ is ample, so $X_I(w)$ is quasi-affine. This generalizes Theorem 1.14, as we saw in the proof of this theorem that there always exist $\lambda \in \mathcal{X}(B)$ such that $L_w^*(\lambda) \in -C_\emptyset^0 = -C^0$. Harashita has shown that in the generalized case for distinguished $X_I(w)$ these special $\lambda \in \mathcal{X}(P_I)$ always exist also, see* [19]*, proof of Thm. 3.1.1.*

Remark 3.4. *If we omit the condition $I = \tilde{w}F(I)$ from the definition above, there exist examples such that $X_I(w)$ is not quasi-affine. Let $G = \mathbb{SL}_3$, $w = s_1$, $I = \{\alpha_2\}$,* *where α_2 is the root attached to s_2. We have $wI \not\subset \Delta$, so particularly $I \neq wF(I)$. Then one can check by a straight-forward computation that $X_I(w) \subset G/P_I$ is a closed subvariety of the projective scheme G/P_I of dimension bigger than zero, thus it is not quasi-affine. There are more well-known examples of non-affine $X_I(w)$, see* [2], *Introduction.*

Next we want to determine when it is possible to extend the isomorphism $\Psi(\dot{w})$ to the closure of $\mathcal{O}_{I,w^{-1}I}(w)$ and its behaviour at the boundary. We introduce some notation. Let $\langle I \rangle \subset \mathcal{X}(T)$ be the submodule generated by the elements of I. Let $\Sigma_I = \Phi \backslash \langle I \rangle$, resp. $\Sigma_I^{\pm} = \Phi^{\pm} \backslash \langle I \rangle$. Let $w \in W_I \backslash W / W_{F(I)}$, such that $X_I(w)$ is distinguished. Let $v = \tilde{w} \in W$ be the minimal length representative of w which satisfies that any other representative of w can be written in the form xvy, $x \in W_I, y \in W_{F(I)}$, with $l(xvy) = l(x) + l(v) + l(y)$, see [8], Chapter IV, Exercises §1, 3). We define a map

$$\kappa : \Phi^+ \cap v\Phi^- \longrightarrow W_I \backslash W / W_{F(I)}$$

as follows. Let $l = l(v)$ and $v = s_1 \cdots s_l$ be a reduced expression. For $1 \leq i \leq l$ we write $v = v_i s_i v_i'$, where $v_i = s_1 \cdots s_{i-1}$ and $v_i' = s_{i+1} \cdots s_l$. Let $\alpha_i \in \Delta$ be the root associated to s_i. Then $\Phi^+(v) := \Phi^+ \cap v\Phi^-$ is equal to $\{v_i\alpha_i; 1 \leq i \leq l\}$. Now we define $\kappa(v_i\alpha_i)$ to be the class of $v_i v_i'$.

Denote by

$$D_I^0(w) = \left\{ \lambda \in \mathcal{X}(P_I) \otimes \mathbb{Q}; \langle \lambda, \check{\alpha} \rangle > 0 \text{ for } \alpha \in \Sigma_I^+ \cap v\Sigma_{F(I)}^- \right\}$$

and by $D_I(w)$ the set, where $>$ is replaced by $\geq$. For $\lambda \in D_I(w)$ we define

$$\partial_I^\lambda(w) = \kappa\left(\{ \alpha \in \Sigma_I^+ \cap v\Sigma_{F(I)}^-; \langle \lambda, \check{\alpha} \rangle > 0 \} \right).$$

By assumption, we have $W_I v W_{F(I)} = W_I v$. It is easy to check that $\partial_I^\lambda(w)$ is independent of the chosen reduced expression of v.

Denote by Π_I the set $\Delta \backslash I$ and define the chamber C_I^0 in $\mathcal{X}(P_I) \otimes \mathbb{Q}$ by

$$C_I^0 = \{ \mu \in \mathcal{X}(P_I) \otimes \mathbb{Q}; \langle \mu, \check{\alpha} \rangle > 0 \text{ for } \alpha \in \Pi_I \}.$$

Proposition 3.5. *Let $\lambda \in \mathcal{X}(P_I)$, $w \in W$. The isomorphism $\Psi(\dot{w})$ extends to the closure of $\mathcal{O}_{I,w^{-1}I}(w)$ in $X_I \times X_{w^{-1}I}$ if and only if $\lambda \in D_I(w)$. If this is the case, then it vanishes precisely on the closures of $\mathcal{O}_{I,w^{-1}I}(w')$ for $w' \in \partial_I^\lambda(w)$.*

Proof. See [19], Prop. 2.2.1. □

We set

$$X_I^\lambda(w) := \overline{X_I(w)} \backslash \bigcup_{w' \in \partial_I^\lambda(w)} \overline{X_I(w')}.$$

Theorem 3.6. (Harashita) *Let $w \in W$, let $I \subset \Delta$ with $I = \tilde{w}F(I)$ and let $\lambda \in D_I(w)$. Now let $I \subseteq J \subseteq \Delta$ such that $Fw^{-1}\lambda - \lambda \in -C_J^0$. If the restriction of $\tau_{IJ} : X_I \to X_J$ to $X_I^\lambda(w)$ is quasi-finite, then $X_I^\lambda(w)$ is affine.*

Proof. See below.

Remark 3.7. *If $X_I^\lambda(w)$ is affine, then $X_I(w)$ is affine, see* [19], *Remark 3.2.2 (3).*

Remark 3.8. *For $I = J = \emptyset$ this yields the original Deligne-Lusztig criterion 1.5.*

Corollary 3.9. (Injectivity criterion) *Let $w \in W$, let $I \subset \Delta$ with $I = \tilde{w}F(I)$ and let $\lambda \in D_I(w)$. Now let $I \subseteq J \subseteq \Delta$ such that $Fw^{-1}\lambda - \lambda \in -C_J^0$ and let $\mathcal{C}_I(w) := P_I w P_{F(I)}$. If for any $w' \in W_I \backslash W / W_{F(I)}$ such that $X_I(w') \subset X^\lambda(w)$, and for any $u \in W_J \backslash W_I$ we have*

$$u^{-1}\mathcal{C}_I(w')F(u) \cap \mathcal{C}_I(w') = \emptyset,$$

then $\tau_{IJ}|_{X_I^\lambda(w)}$ is injective. Then in particular $X_I^\lambda(w)$ is affine.

Proof. Let w' be as above. If $\tau_{IJ}|_{X_I(w')}$ is not injective, then there exists $x = \dot{g}^{-1}F(\dot{g}) \in \mathcal{C}_I(w')$ and $h \in P_J \backslash P_I$ such that $h^{-1}xF(h) \in \mathcal{C}_I(w')$. In particular, $h \neq id$ and thus there exists an $u \in W_J \backslash W_I$ with $u^{-1}xF(u) \in \mathcal{C}_I(w')$, so $x \in u\mathcal{C}_I(w')F(u^{-1}) \cap \mathcal{C}_I(w')$.

On the other hand, if $\tau_{IJ}|_{X_I(w')}$ is injective, then we have for every $u \in W_J \backslash W_I$ that $u^{-1}\mathcal{C}_I(w')F(u) \cap \mathcal{C}_I(w') = \emptyset$. □

Proof (of Theorem 3.6). Let $X := X_I^\lambda(w)$ and let $\bar{X}$ be the normal closure of X in X_I. As for $\lambda \in D_I(w)$ the set $\partial_I^\lambda(w)$ equals the set $\partial_I^{n\lambda}(w)$ for every $n \in \mathbb{N}$, we may assume that $\lambda \in \mathcal{X}(P_I) \subset \mathcal{X}(P_I) \otimes \mathbb{Q}$. Denote by $\mathcal{L}$ the line bundle $\mathcal{L}_I(Fw^{-1}\lambda - \lambda)$. From Proposition 3.5 we get a map $\Psi(\dot{w})$, which we again consider as a section of $\mathrm{pr}_1^*\mathcal{L}_I(\lambda)^{-1} \otimes \mathrm{pr}_2^*\mathcal{L}_{w^{-1}I}(w^{-1}\lambda)$, as well as the following Cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & V \\ \big\downarrow & \square & \big\downarrow \\ \bar{X} & \xrightarrow{\varphi} & \mathbb{P}(H^0(\bar{X}, \mathcal{L})^\vee) \end{array}$$

Here V denotes the open subscheme of $\mathbb{P}(H^0(\bar{X}, \mathcal{L})^\vee)$ where $\Psi(\dot{w})$ does not vanish. As φ is proper, ψ is also proper. Moreover V is affine, so for proving the affineness of X it is sufficient to show that ψ is quasi-finite, see [16] Cor. 12.89. So let $v \in \mathrm{im}\ \psi$ and let $x \in X$ be such that $\psi(x) = v$. Let $\lambda' \in \mathcal{X}(T)$ be such that $\langle \lambda', \check{\alpha} \rangle = 0$, for all $\alpha \in J$, and $\langle \lambda', \check{\alpha} \rangle = \langle \lambda, \check{\alpha} \rangle$, for all $\alpha \in \Delta \backslash J$. Let $\mathcal{L}' = \mathcal{L}_J(Fw^{-1}\lambda' - \lambda')$ so that $\tau_{IJ}^*\mathcal{L}' = \mathcal{L}$. Since $\mathcal{L}$ is the pullback of $\mathcal{L}'$, we obtain the following commutative diagram

$$\begin{array}{ccc} X_I & \longrightarrow & \mathbb{P}(H^0(X_I, \mathcal{L})^\vee) \\ {\scriptstyle \tau_{IJ}}\big\downarrow & & \big\downarrow \\ X_J & \longrightarrow & \mathbb{P}(H^0(X_J, \mathcal{L}')^\vee) \end{array}$$

We denote by φ' the composition $X \hookrightarrow X_I \to \mathbb{P}(H^0(X_I, \mathcal{L})^\vee)$. As the restriction of every element of $H^0(X_I, \mathcal{L})$ is contained in $H^0(\bar{X}, \mathcal{L})$, the point $v' := \varphi'(x)$ is determined by v, i.e. $\psi^{-1}(v) \subset \varphi'^{-1}(v')$. As by assumption $Fw^{-1}\lambda - \lambda \in -C_J^0$, the line bundle $\mathcal{L}'$ is very ample on X_J, see [24], II.4.4 Remarks 1). Thus $X_J \hookrightarrow \mathbb{P}(H^0(\tau_{IJ}(\bar{X}), \mathcal{L}')^\vee)$ is an embedding. Lemma 3.10 yields that the natural map $\mathbb{P}(H^0(\tau_{IJ}(X_I), \mathcal{L}')^\vee) \to \mathbb{P}(H^0(X_I, \mathcal{L})^\vee)$ is an isomorphism. Thus φ' factors through

$$\bar{X} \hookrightarrow X_I \xrightarrow{\tau_{IJ}} X_J \hookrightarrow \mathbb{P}(H^0(X_I, \mathcal{L})^\vee).$$

But now, since $\tau_{IJ|X_I^\lambda(w)}$ is quasi-finite, $\varphi'^{-1}(v')$ is finite and the claim follows. □

Lemma 3.10. *With the notation of the proof of the last theorem, we have an isomorphism* $H^0(X_I, \mathcal{L}) \cong H^0(X_J, \mathcal{L}')$.

Proof. As $\mathcal{L}', \mathcal{L}$ are locally free rank-1 sheaves, one can find affine open covers $\{U_i'\}_i$ of X_J and $\{\tau_{IJ}^{-1}(U_i')\}_i = \{U_i\}_i$ of X_I such that $\mathcal{L}'_{|U_i'} \cong \mathcal{O}_{U_i'}$ and $(\tau_{IJ}^*\mathcal{L}')_{|U_i} \cong \mathcal{O}_{U_i}$, where $U_i \cong U_i' \times L_J/(L_J \cap P_I)$. But $L_J/(L_J \cap P_I) \cong P_J/P_I$ is an irreducible, projective subscheme of G/P_I. Therefore we have $H^0(U_i, \mathcal{L}_{|U_i}) \cong H^0(U_i', \mathcal{L}') \otimes_k H^0(P_J/P_I, \mathcal{O}_{P_J/P_I}) \cong H^0(U_i', \mathcal{O}_{U_i'})$ for all i and similarly $H^0(U_i \cap U_j, \mathcal{L}_{|U_i \cap U_j}) \cong H^0(U_i' \cap U_j', \mathcal{L}'_{|U_i' \cap U_j'})$. But this forces also that the global sections $H^0(X_I, \mathcal{L})$ and $H^0(X_J, \mathcal{L}')$ coincide. □

Remark 3.11. *Computationally we are only interested in the case* $I = \emptyset$ *and* G_0 *split of classical type. As in the original criterion one can attach a linear optimization problem (LOP) to the conditions* $\lambda \in D(w)$ *and* $Fw^{-1}\lambda - \lambda \in -C_J^0$

above. We are only able to (efficiently) implement the injectivity criterion 3.9. The intersection of Schubert varieties $u^{-1}\mathcal{C}_I(w')F(u) \cap \mathcal{C}_I(w')$ *for some* $u \in W_J \backslash \{id\}$, $w' \in W$*, can be computed by the fact that for every simple reflection* s *we have* $\mathcal{C}_I(w')s = \mathcal{C}_I(w') \cup \mathcal{C}_I(w's)$ *if* $l(w's) < l(w')$*, and* $\mathcal{C}_I(w')s = \mathcal{C}_I(w's)$ *if* $l(w's) > l(w')$.

Remark 3.12. *By Theorem 3.6 even in the case* $I = J = \emptyset$ *this criterion is a little more general compared to the one of Deligne and Lusztig, as we only have to find a* $\lambda \in D(w)$*, instead of* $\lambda \in D^0(w)$*. We have not implemented this generalization in the implementation of the Deligne-Lusztig criterion. There exist examples where one can see this difference, see Remark 4.10.*

4. A criterion by Tangent Spaces

We use the notation and conditions of the last chapter. In this chapter, we first comment on the criterion of Harashita, as in general it is not easy to compute whether a given map is quasi-finite. We suggest to check the restrictive, but more efficiently computable condition that a map is quasi-finite if certain tangent spaces are zero-dimensional. We will obtain a finite set $\{T_e(v); v \in \Delta^\lambda(w) \subset W\}$ of tangent spaces such that it is enough to check the dimension of these to decide whether $X(w)$ is affine. From this we deduce an easy-to-compute criterion for G_0 split of type A, as in this case computing dimensions of certain tangent spaces comes down to calculate inversion of permutations.

Second, we give examples and discuss the criteria so far from a computational point of view.

We have as a corollary of Theorem 3.6 the following combinatorial criterion:

Theorem 4.1. *Let $w' \in W_I \backslash W / W_{FI}$, $I = \widetilde{w'}FI$, $\lambda \in D_I(w')$ and $I \subset J \subseteq \Delta$ such that $Fw'^{-1}\lambda - \lambda \in -C_J^0$. If for all $w \in \Delta^\lambda(w') := \left\{w \leq w' | \forall v' \in \partial_I^\lambda(w') : w \not\leq v'\right\}$ we have $T_e\left(\mathcal{C}_I(w)w^{-1} \cap P_J\right) = T_e(P_I)$, then $X_I(w')$ is affine.*

Proof. If $\tau_{IJ}|_{X_I^\lambda(w')}$ is quasi-finite, then $X_I(w')$ is affine by Theorem 3.6 and Remark 3.7. But $\tau_{IJ}|_{X_I^\lambda(w')}$ is quasi-finite if and only if for all $w \in \Delta^\lambda(w')$ we have that $\tau_{IJ}|_{X_I(w)}$ is quasi-finite. Fix some $w \in \Delta^\lambda(w')$. We have that $\tau_{IJ}|_{X_I(w)}$ is quasi-finite if and only if for every $x = \dot{g}^{-1}F(\dot{g}) \in P_I w P_{F(I)}$ the inverse image $\tau_{IJ}^{-1}(\dot{g}P_J) = \{\dot{g}hP_I; h \in P_J/P_I \cap h^{-1}xF(h) \in \mathcal{C}_I(w)\}$ is finite. But this is finite if and only if for all $x \in \mathcal{C}_I(w) := P_I w P_{FI}$ the set

$$Q_x := \left\{h \in P_J/P_I; h^{-1}xF(h) \in P_I w P_{F(I)}\right\}$$

is finite or, equivalently, zero-dimensional. We have $Q_x = g^{-1}X_I(w) \cap P_J/P_I$, where $g \in G$ such that $g^{-1}Fg = x$. If for all $z \in Q_x$ the tangent space $T_z(Q_x)$ is zero, then Q_x is finite.

For locally closed $z \in Z \subseteq G/P_I$ we get from the Cartesian diagram

$$\begin{array}{ccc} P_I \subseteq \tilde{Z} = \pi_I^{-1}(Z) & \hookrightarrow & G \\ \downarrow & & \downarrow \pi_I \\ Z & \hookrightarrow & G/P_I \end{array}$$

a corresponding Cartesian diagram for the tangent spaces ($\dot{z} \in \tilde{Z}, \pi_I(\dot{z}) = z$)

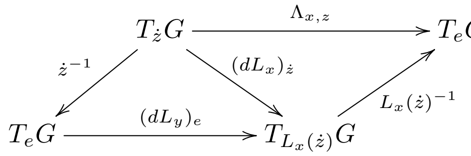
$$\begin{array}{ccc} T_{\dot{z}}\tilde{Z} & \hookrightarrow & T_{\dot{z}}G \\ \downarrow & & \downarrow d\pi_{I\dot{z}} \\ T_z Z & \hookrightarrow & T_z(G/P_I), \end{array}$$

where π_I denotes the projection and where the kernel is just $\ker(d\pi_I)_{\dot{z}} = T_{\dot{z}}(\pi_I^{-1}(z)) = T_{\dot{z}}(\dot{z}P_I)$, so $T_z(G/P_I) = T_{\dot{z}}(G)/T_{\dot{z}}(\dot{z}P_I)$ and $T_z Z = T_{\dot{z}}\tilde{Z}/T_{\dot{z}}(\dot{z}P_I)$.

We define $\tilde{X}_I(w) := \{\dot{h} \in G | \dot{h}^{-1}F\dot{h} \in \mathcal{C}_I(w)\} = \pi_I^{-1}\left(X_I(w)\right)$ and thus we get for fixed $z \in Q_x$:

$$\begin{aligned} T_z(Q_x) &= T_z\left(g^{-1}X_I(w) \cap P_J/P_I\right) = T_z\left(g^{-1}X_I(w)\right) \cap T_z(P_J/P_I) \\ &= T_{\dot{z}}(g^{-1}\tilde{X}_I(w))/T_{\dot{z}}(\dot{z}P_I) \ \cap \ T_{\dot{z}}(P_J)/T_{\dot{z}}(\dot{z}P_I). \end{aligned}$$

Now, we write $L_x : G \to G, \gamma \mapsto \gamma^{-1}xF\gamma$, for the Lang map and define $y := L_x(\dot{z})$. For $s \in G$ we denote by s the map $T_xG \to T_{sx}G$ induced by the left multiplication $G \to G$, $\gamma \mapsto s\gamma$. Consider the map $\Lambda_{x,z}$, defined as follows



.

One can calculate the differential of the Lang map as $(dL_y)_e(M) = -My = -ML_x(\dot{z})$, $M \in \dot{z}T_eG = T_{\dot{z}}G$, see [6], V.16. So $\Lambda_{x,z}(M) = -(L_x(\dot{z}))^{-1}\dot{z}^{-1}ML_x(\dot{z})$ and one can see that $(dL_y)_e$ and $\Lambda_{x,z}$ are isomorphisms. But for $\dot{h} \in \tilde{X}_I(w)$ we have $L_x(g^{-1}\dot{h}) = \dot{h}^{-1}gxF(g^{-1})F(\dot{h}) = \dot{h}^{-1}F(\dot{h}) \in \mathcal{C}_I(w)$. By the surjectivity of L_e we thus get $L_x(g^{-1}\tilde{X}_I(w)) = \mathcal{C}_I(w)$, so altogether we have that $\Lambda_{x,z}$ induces an isomorphism $T_{\dot{z}}\left(g^{-1}\tilde{X}_I(w)\right) \cong T_{L_x(\dot{z})}\left(\mathcal{C}_I(w)\right) \cong T_e\left(L_x(\dot{z})^{-1}\mathcal{C}_I(w)\right)$.

That $T_z(Q_x)$ is zero-dimensional is now equivalent with

$$\Lambda_{x,z}^{-1}\left(T_e\left(L_x(\dot{z})^{-1}\mathcal{C}_I(w)\right)\right) \cap T_{\dot{z}}(P_J) = T_{\dot{z}}(g^{-1}\tilde{X}_I(w)) \cap T_{\dot{z}}(P_J) \overset{!}{=} T_{\dot{z}}(\dot{z}P_I)$$

$$\Leftrightarrow \dot{z}^{-1}\Lambda_{x,z}^{-1}\left(T_e\left(L_x(\dot{z})^{-1}\mathcal{C}_I(w)\right)\right) \cap T_e(P_J) \overset{!}{=} T_e(P_I) \text{ for } \dot{z} \in P_J. \tag{4.2}$$

Recall that we have fixed $x \in \mathcal{C}_I(w), z \in Q_x$ and $y = L_x(\dot{z}) = \dot{z}^{-1}xF\dot{z}$. Since $z \in Q_x$, we have $y \in \mathcal{C}_I(w)$. But for $N \in T_e(y^{-1}\mathcal{C}_I(w)) \subset T_e(G)$ we have $\dot{z}^{-1}\Lambda_{x,z}^{-1}(N) = \dot{z}^{-1}(-\dot{z}yNy^{-1}) = -yNy^{-1} = \Lambda_{y,e}^{-1}(N)$, so we get $\dot{z}^{-1}\Lambda_{x,z}^{-1}\left(T_e\left(L_x(\dot{z})^{-1}\mathcal{C}_I(w)\right)\right) = \Lambda_{y,e}^{-1}\left(T_e\left(L_y(\dot{e})^{-1}\mathcal{C}_I(w)\right)\right)$. Thus it is enough to check (4.2) for $\dot{z} = e$, $y = x$.

Now fix $y \in \mathcal{C}_I(w) = P_IwP_{FI}$ and write $y = c^{-1}wb$ with $c \in P_I$ and $b \in P_{FI}$. Then we get

$$\begin{aligned} \Lambda_{y,e}^{-1}\left(T_e\left(L_y(e)^{-1}\mathcal{C}_I(w)\right)\right) \cap T_e(P_J) &= T_e(\mathcal{C}_I(w)y^{-1}) \cap T_e(P_J) \\ &= T_e\left(\mathcal{C}_I(w)w^{-1}c \cap P_J\right) \\ &\cong T_e\left(\mathcal{C}_I(w)w^{-1} \cap P_J\right), \end{aligned}$$

where the last isomorphism is induced by $G \to G$, $\gamma \mapsto c\gamma c^{-1}$. Since $T_e\left(\mathcal{C}_I(w)y^{-1}\right)$ and $T_e\left(\mathcal{C}_I(w)w^{-1}\right)$ both contain $T_e(P_I)$, we have that $T_e(Q_y)$ is zero if and only if $T_e(\mathcal{C}_I(w)w^{-1} \cap P_J) = T_e(P_I)$. □

Corollary 4.3. *Let $I = \emptyset$, λ, J as in Theorem 4.1 above, $w = s_{i_1} \dots s_{i_l}$ a reduced expression. Then $X(w)$ is affine if for every $v \in \Delta^\lambda(w) := \left\{v \leq w | \forall v' \in \partial_I^\lambda(w) : v \not\leq v'\right\}$ the set*

$$M_v := \left\{\alpha \in \Phi_J | \alpha < 0, v^{-1}(\alpha) > 0\right\}$$

is empty.

Proof. Let v be in $\Delta^\lambda(w)$ and write $U'_v := U \cap vU^-v^{-1}$, where U^- denotes the opposite group of U in G. Under the locally closed embedding $\rho_v : U'_v \times B \hookrightarrow G$, $(u, b) \mapsto uvbv^{-1}$ one has $\mathcal{C}_I(v)v^{-1} = \text{im } \rho_v$.

The groups P_J, U'_v and vBv^{-1} are direct spanned by the root groups (and T)

$$\begin{aligned} P_J &= \prod_{\alpha>0} U_\alpha \times T \times \prod_{0>\alpha\in\Phi_J} U_\alpha, \\ U'_v &= \prod_{\substack{\alpha<0 \\ v(\alpha)>0}} U_{v(\alpha)} \quad \text{and} \\ vBv^{-1} &= T \times \prod_{\alpha>0} U_{v(\alpha)}. \end{aligned}$$

We write $\mathfrak{t} := T_e(T) \subseteq \mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{g}_\alpha := T_e(U_\alpha) \subseteq \mathfrak{g}$ for α a root. Now we can calculate the tangent space as

$$\begin{aligned} T_e(\mathcal{C}(v)v^{-1}) &= \mathfrak{t} \oplus \bigoplus_{\substack{\alpha<0 \\ v(\alpha)>0}} \mathfrak{g}_{v(\alpha)} \oplus \bigoplus_{\alpha>0} \mathfrak{g}_{v(\alpha)} \\ &= \mathfrak{t} \oplus \bigoplus_{\substack{\alpha>0 \\ v^{-1}(\alpha)<0}} \mathfrak{g}_\alpha \oplus \bigoplus_{v^{-1}(\alpha)>0} \mathfrak{g}_\alpha \\ &= \mathfrak{t} \oplus \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \oplus \bigoplus_{\substack{\alpha<0 \\ v^{-1}(\alpha)>0}} \mathfrak{g}_\alpha. \end{aligned}$$

By Theorem 4.1 for $I = \emptyset$, we have to check whether $T_e\left(\mathcal{C}(v)v^{-1} \cap P_J\right) = T_e(B)$. But now this is equivalent to say that the set $M_v = \{\alpha \in \Phi_J | \alpha < 0, v^{-1}(\alpha) > 0\}$ is empty. □

Consider the special case G_0 split of type A and rank $n-1$, so that $W = \mathbb{S}_n$. Calculating the dimension of the tangent space comes down to calculating inversions.

Corollary 4.4. *For G_0 split of type A and rank $n-1$, $w = s_{i_1} \dots s_{i_l}$ a reduced expression, let $\Delta^\lambda(w)$ be as in Corollary 4.3. Then $X(w)$ is affine if for every $v \in \Delta^\lambda(w)$ the set*

$$N_v := \left\{k | s_k \in W_J, v^{-1}(k+1) < v^{-1}(k)\right\}$$

is empty.

Proof. By Corollary 4.3 we have to show, for every v as above, that $M_v = \emptyset$ if and only if $N_v = \emptyset$. We can identify Φ with $\{\alpha^{ij} \in \mathbb{R}^n | i \neq j, \alpha^{ij}_i = 1, \alpha^{ij}_j = -1 \text{ and } \forall k \neq i,j : \alpha^{ij}_k = 0\}$, where $\Phi^+ = \{\alpha^{ij} \in \Phi | i < j\}$ and the condition $w(\alpha^{ij}) > 0$ just means $w(i) < w(j)$. Furthermore we identify Φ_J with

$$\left\{\alpha^{ij} \in \Phi | \exists u \in W_J : u(i) = j\right\}.$$

But then we can identify M_v with

$$\left\{\alpha^{ij} \in \Phi | i > j, v^{-1}(i) < v^{-1}(j), \exists u \in W_J : u(i) = j\right\}.$$

Since N_v is in bijection with $\left\{\alpha^{k+1,k} \in \Phi | s_k \in W_J, v^{-1}(k+1) < v^{-1}(k)\right\} \subseteq M_v$, we clearly have that N_v is empty whenever M_v is empty.

For the other direction let us assume we have an inversion (j,i) of v^{-1} for which $\alpha^{ij} \in \Phi_J$. That is, $i > j$, $v^{-1}(i) < v^{-1}(j)$ and there is an $u \in W_J$ such that $u(i) = j$. We then have $s_k \in J$ for all $k = j, \dots, i-1$ and the assumption $N_v = \emptyset$ implies $v^{-1}(j) < v^{-1}(j+1) < \ldots < v^{-1}(i-1) < v^{-1}(i)$ and therefore yields a contradiction to $v^{-1}(i) < v^{-1}(j)$. □

Remark 4.5. *With the notations of Theorem 4.1, for finding a (minimal) set $J \subset \Delta$ such that $Fw^{-1}\lambda - \lambda \in -C^0_J$, one can first set $J = I$ and then repeatedly enlarge J by some $s \in \Delta$ and check whether one can solve the linear optimization problem (LOP) attached to J. But we suggest another, mostly faster approach for the computational checking of the affineness: First find one maximal (according to its cardinality) subset $J' \subseteq \Delta$ such that the conditions of Cor. 4.4 are satisfied. Afterwards, check if one can solve the LOP attached to J'. If not, then one cannot use the criterion 4.4. In the implementation of Harashita's injectivity criterion 3.9 we use the same strategy to first detect maximal sets J' for which this criterion would apply and then check if the attached LOP is solvable.*

Example 4.6. *By Corollary 3.9 one can check injectivity of the map* $\tau_{IJ}|_{X_I^\lambda(w)}$ *by showing that for all* w' *with* $X_I(w') \subseteq X_I^\lambda(w)$ *and for any* $u \in W_J \backslash W_I$ *one has* $u^{-1}\mathcal{C}_I(w')Fu \cap \mathcal{C}_I(w') = \emptyset$. *Let* $q = 2$ *and* $k_0 = \mathbb{F}_q$. *Consider* $G = \mathbb{SL}_4(k)$, $W = \mathbb{S}_4$, $w = s_1$, $I = \emptyset$, $J = \{s_2, s_3\}$. *Take for* B *the upper triangular matrices and for* T *the diagonal matrices. The character group of* T *is isomorphic to* $\mathbb{Z}^4/(1,1,1,1)\mathbb{Z}$.

An explicit calculation shows that $\lambda := (2,1,3,4) \in D_I^0(w)$ *fulfills* $Fw^{-1}\lambda - \lambda \in -C_J^0$. *Since* $\lambda \in D_I^0(w)$, *the only* w' *such that* $X_I(w') \subseteq X_I^\lambda(w)$ *is* w *itself. Since* $Bs_3Bs_1Bs_3B = Bs_3s_1B \cup Bs_1B$ *the injectivity fails, but on the other hand we have* $w^{-1}(2) < w^{-1}(3) < w^{-1}(4)$, *so by Corollary 4.4 all tangent spaces are zero. In fact we have more:* $\lambda_2 := (2,1,4,5) \in D_I^0(w)$ *fulfills* $Fw^{-1}\lambda_2 - \lambda_2 \in -C_I^0$, *so* $X(w)$ *is affine by the original criterion of Deligne and Lusztig (Theorem 1.5).*

We make a direct check for the quasi-finiteness of $\tau_{IJ}|_{X(s_1)}$, *where* $\tau_{IJ} : G/B \to G/P_J$. *We cover* G/B *by the open charts* $vU^-B/B, v \in W$ *and identify it with the flag variety of full flags in* k^4. *Furthermore,* $G/P_J \cong \mathbb{P}^3$, *and when we restrict to the standard open chart* U^-B/B, *then the flag* $\mathcal{F}_0$, *identified with the matrix* $A_0 = \left(a_{ij}^0\right) \in \mathbb{SL}_4(k)$, *where* $a_{ii}^0 = 1$ *and* $a_{ij}^0 = 0$ *for* $1 \leq i < j \leq 4$, *gets mapped under* τ_{IJ} *to* $\tau_{IJ}(\mathcal{F}_0) = (1 : a_{21}^0 : a_{31}^0 : a_{41}^0) \in \mathbb{P}^3$. *The fiber over* $\tau_{IJ}(\mathcal{F}_0)$ *under* $\tau_{IJ}|_{vU^-B/B}$ *is therefore*

$$\left\{ vA := v \begin{pmatrix} 1 & & & \\ a_{21} & 1 & & \\ a_{31} & a_{32} & 1 & \\ a_{41} & a_{42} & a_{43} & 1 \end{pmatrix} \middle| \begin{array}{c} A \in \mathbb{SL}_4(k) \text{ and } \tau_{IJ}(\mathcal{F}_0) = \\ (a_{v^{-1}(1),1} : a_{v^{-1}(2),1} : a_{v^{-1}(3),1} : a_{v^{-1}(4),1}) \end{array} \right\}.$$

We have

$$(vA)^{-1}F(vA) = A^{-1}F(A) = \begin{pmatrix} 1 & & & \\ a_{21}^q - a_{21} & 1 & & \\ x & a_{32}^q - a_{32} & 1 & \\ y & z & a_{43}^q - a_{43} & 1 \end{pmatrix},$$

where we write

$$\begin{aligned} x &:= a_{31}^q - a_{31} - a_{21}a_{32}^q + a_{21}a_{32} \\ y &:= a_{41}^q - a_{21}a_{42}^q + a_{43}^q(a_{21}a_{32} - a_{31}) + a_{43}a_{31} - a_{41} - a_{21}(a_{43}a_{32} - a_{42}) \\ z &:= a_{42}^q - a_{42} + a_{32}(a_{43} - a_{43}^q). \end{aligned}$$

Then $(vA)^{-1}F(vA) \in X(s_1)$ *if and only if*

$$\begin{aligned} y &= 0 \\ x &= 0 \\ a_{21}^q - a_{21} &\neq 0 \\ z &= 0 \\ a_{32}^q - a_{32} &= 0 \\ a_{43}^q - a_{43} &= 0. \end{aligned}$$

But this implies $a_{42}^q - a_{42} = 0$, *so* $a_{32}, a_{42}, a_{43} \in \mathbb{F}_q$. *Furthermore, since* $a_{11} = 1$, a_{21}, a_{31} *and* a_{41} *are uniquely determined by the condition*

$$(1 : a_{21}^0 : a_{31}^0 : a_{41}^0) = (a_{v^{-1}(1),1} : a_{v^{-1}(2),1} : a_{v^{-1}(3),1} : a_{v^{-1}(4),1}) \in \mathbb{P}^3.$$

This equation is solvable if and only if $a_{v(1),1}^0 \neq 0$. *Thus the fiber of* $\tau_{IJ}(\mathcal{F}_0)$ *on the chart* vU^-B/B *is empty for* $a_{v(1),1}^0 = 0$ *and*

$$\left(\tau_{IJ}^{-1}|_{X(s_1)}(\tau_{IJ}(\mathcal{F}_0))\right) \cap vU^{-1}B/B$$

$$= \left\{ v \begin{pmatrix} 1 & & & \\ a^0_{v(2),1}/a^0_{v(1),1} & 1 & & \\ a^0_{v(3),1}/a^0_{v(1),1} & a_{32} & 1 & \\ a^0_{v(4),1}/a^0_{v(1),1} & a_{42} & a_{43} & 1 \end{pmatrix} | a_{32}, a_{42}, a_{43} \in \mathbb{F}_q \right\}$$

otherwise. Thus $\tau_{IJ}|_{X(s_1)}$ is quasi-finite.

We define the following subsets of W consisting of elements satisfying different (combinations of) affineness criteria:

$(CoxE) := \{w \in W;\ w$ is a Coxeter element$\}$ (Remark 2.7)

$(ORH) := \{w \in W;\ w$ is of minimal length in $\mathcal{K}(w)\}$ (Theorem 2.5)

$(BR) := \{w \in W;\ w$ satisfies Bonnafé-Rouquier's criterion$\}$ (Theorem 2.9)

$(DL) := \{w \in W;\ w$ satisfies Deligne-Lusztig's criterion$\}$ (Theorem 1.5)

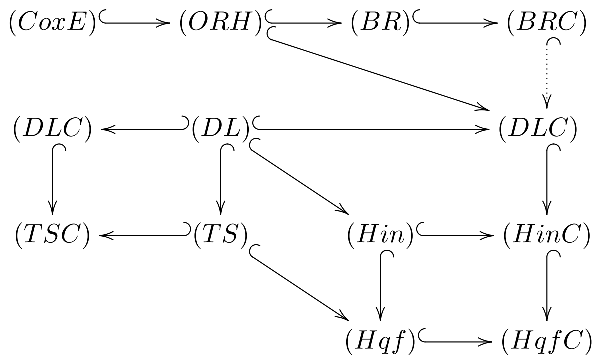
$(TS) := \{w \in W;\ w$ satisfies the tangent space criterion$\}$ (Theorem 4.1)

$(Hin) := \{w \in W;\ w$ satisfies the Injectivity criterion$\}$ (Corollary 3.9)

$(Hqf) := \{w \in W;\ w$ satisfies Harashita's quasi-finite criterion$\}$ (Theorem 3.6)

$(XC) := \{w \in W;\ w \in Cyc(v)$ for some element v of $(X)\}$ (Proposition 2.4)

For G_0 split of classical type, i.e. G_0 of type A, B, C, or D, these criteria are implemented in *GAP 3* via the functions `IsAffineMinimalLength,` resp. `GoodCoxeterWord`, for (ORH), `IsAffineDL` for (DL), `IsAffineTS` for (TS), `IsAffineHin` for (Hin) and `FCyclicShiftClass` for (XC). We have the following diagram of inclusions:



Remark 4.7. *We want to remark here that the arrow $(BRC) \hookrightarrow (DLC)$ is only conjectured (cf Remark 2.12). We want to remark that $(ORHC) \hookrightarrow (DLC)$ is strict and in general far away from being an equality. Observe that the inclusions $(X) \hookrightarrow (XC)$ are all strict, at least for a general choice of G. By the table below, for G_0 of classical type, one could conjecture that there might be an inclusion of (TSC) into $(HinC)$, but one should check this for bigger examples, as for example for G_0 of type A and rank 6 there are only 17, resp. 18, elements $w \in W$ such that $w \in (TSC)$, resp. $w \in (HinC)$, but $w \notin (DLC)$.*

Group	#tested w's	(DL)	(DLC)	(TSC)	$(HinC)$	$\neg(Hqf)$	$\#\{Cyc(w)\}$
$D_4/\mathbb{F}_2$	126	66	112	113	113	14	11
$B_4/\mathbb{F}_2$	300	170	264	275	275	19	20
$A_5/\mathbb{F}_2$	444	442	442	444	444	0	0
$D_5/\mathbb{F}_2$	1520	777	1386	1407	1407	134	85
$B_5/\mathbb{F}_2$	3270	1596	2725			545	261
$A_6/\mathbb{F}_2$	3414	3362	3382	3399	3400	8	10

Remark 4.8. *Here, in the second column we list the number of elements for which we checked the different criteria. That is, all elements $v \in W$ for which every reduced expression for v contains every simple reflection, except for v a Coxeter element or $v = w_0$. The numbers in the other columns indicate how many of these fulfill the criterion of this column.*

In the column before the last column we list the number of elements which are definitively not contained in (Hqf), that is, for which the minimal set J equals Δ. This might be smaller than the number of all elements which are not contained in (Hqf), while the last column indicates the number of different F-cyclic-shift conjugacy classes left. In all cases listed above, one can read off the number of elements that cannot be handled by some of the implemented criteria from the table simply by subtracting the number in column $(HinC)$ from the number of tested elements.

To produce such tables, one can use the functions `PerformanceTest`, `TestAffine-Combination` *and* `RemainingMostInteresting` *from our implementation.*

Example 4.9. *Consider $G = \mathbb{GL}_7/\mathbb{F}_2$, $W = \mathbb{S}_7$, $I = \emptyset$ and*

$$w = s_4s_5s_6s_5s_2s_3s_4s_1s_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 1 & 7 & 2 & 6 & 4 \end{pmatrix} = w^{-1}.$$

With [1] *and* [33] *we then calculate with our implementation that the minimal J for which there exists λ with $Fw^{-1}\lambda - \lambda \in -C_J^0$ is $J = \Delta$. So in this case $\tau_{IJ}|_{X(w)}$ cannot be quasi-finite. Furthermore we calculate the cyclic-shift F-conjugacy class to consist just of w alone, while for example $v = s_1s_3s_5 \in \mathcal{K}(w)$ is of lower length. So this may be a good candidate for $X(w)$ being not affine.*

Remark 4.10. *We make some further remarks on the results of the computations. For $G = \mathbb{GL}_7/\mathbb{F}_3$ we already have that all $X(w)$ are affine by (DLC), but in this case the Coxeter number $h = 6$ is not smaller than $q = 3$.*

We get that $X(w)$ and $X(w^{-1})$ have not to be simultaneously affine, e.g. for $G = \mathbb{GL}_7/\mathbb{F}_2$, $W = \mathbb{S}_7$, we have that $w = s_2s_1s_3s_4s_3s_2s_5s_6s_5s_4$ is not treatable by any criterion above, but $X(w^{-1})$ is affine by (DLC). For $w = s_4s_5s_6s_5s_1s_2s_3s_4s_1 \neq w^{-1}$ we calculated that for $J = \{2, 3, 5\}$ we can find some $\lambda \in D(w)$ such that $Fw^{-1}\lambda - \lambda \in -C_J^0$ and where we have $\langle \lambda, w_i\check{\alpha}_i \rangle = 0$ for $i \in \{2, 6, 9\} \subset \{1, \ldots, l(w) = 9\}$ (with the notation of Theorem 3.6). For this w we have $w \notin (TS)$, but $w \in (Hin)$. This is the only example we have found so far that is contained in $(HinC)\backslash(TSC)$.

5. A cohomological Criterion

Keep the notation of the last chapters. In particular, we have fixed T, U, B and the root system Φ. We denote by U^- the opposite group of $U = \prod_{\alpha>0} U_\alpha$ in G. Fix $w \in W$ with $s_i \leq w$ for all i. Then we know that the associated Deligne-Lusztig variety

$$X(w) = \{g \in G | g^{-1}F(g) \in BwB\}/B$$

is irreducible (see Theorem 1.4). Denote by $w_0 \in W$ the longest element and set $N := l(w_0)$. From now on, we assume G_0 to be split.

The idea is, to check whether $X(w)$ is affine by computing if the Čech cohomology groups all vanish. By [30] it is enough to check that $H^i(X(w), \mathcal{O}_{X(w)}) = 0$ for all $i > 0$. We will consider the inverse image $X(w)^{(0)}$ of $X(w)$ under the T-torsor $G/B \to G/U$ and will later see, that $X(w)$ is affine if and only if $H^1(X(w)^{(0)}, \mathcal{O}_{X(w)^{(0)}}) = 0$. Moreover, we give a construction, independent of w, of a finite set $\{\tau_i; 1 \leq i \leq N\}$ of elements in certain H^1 groups such that

$$X(w) \text{ is affine } \Leftrightarrow \forall i : \tau_i = 0.$$

So in some sense we obtain a computable criterion to check whether $X(w)$ is affine. However, the problem is that for quasi-affine, non-affine X, some of the cohomology groups $H^i(X, \mathcal{O}_X)$ are infinite-dimensional, e.g. $H^{n-1}(\mathbb{A}^n \backslash \{0\}, \mathcal{O}_{\mathbb{A}^n \backslash \{0\}})$. In any case, the spaces arising in the Čech complex are infinite-dimensional as k-vector spaces.

Theorem 5.1. *There exists an explicit algorithm that checks whether $X(w)$ is affine. More precisely, one can define, for $1 \leq i \leq N$, varieties $X(w)^{(i)}$, affine coverings $\{\mathcal{U}_v^{(i)}\}_{v \in W}$ for $X(w)^{(i)}$, and elements $\tau_i \in \check{H}^1(X(w)^{(i)}, \mathcal{O}_{X(w)^{(i)}})$ such that*

$$X(w) \textit{ affine } \Leftrightarrow \forall i : \tau_i = 0.$$

We will prove this in several steps and give explicit and computable descriptions of $X(w)^{(i)}, \{\mathcal{U}_v^{(i)}\}$ and τ_i. Moreover, for G of type A, we have implemented such an algorithm, see chapter 6.

Lemma 5.2. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be quasi-coherent sheaves on separated schemes X_1, X_2 of finite type over k. Then for every $i \in \mathbb{N}$ we have*

$$H^i(X_1 \times X_2, pr_1^*\mathcal{F}_1 \otimes pr_2^*\mathcal{F}_2) = \bigoplus_{\mu+\nu=i} H^\mu(X_1, \mathcal{F}_1) \otimes H^\nu(X_2, \mathcal{F}_2).$$

Proof. See [25], Prop. 9.2.4. □

Remark 5.3. *Lemma 5.2 in particular implies that for every $m \in \mathbb{N}$*

$$H^1(X \times \underbrace{\mathbb{G}_a \times \cdots \times \mathbb{G}_a}_{m \text{ times}}, \mathcal{O}_{X \times \mathbb{G}_a \times \cdots \times \mathbb{G}_a}) = H^1(X, \mathcal{O}_X) \otimes_k k[D]^{\otimes m}.$$

Lemma 5.4. *Let $Z_m \xrightarrow{\tau_m} \ldots \xrightarrow{\tau_1} Z_0$ be a chain of $\mathbb{G}_a$-torsors and let Z_m be affine. Then*

$$Z_0 \textit{ is affine } \Leftrightarrow \tau_1, \ldots, \tau_m \textit{ are trivial.}$$

Proof. If all τ_i are trivial, then $Z_m \cong Z_0 \times \mathbb{G}_a^m$. We claim that in this case Z_0 is affine also. In fact, let $\mathcal{F}_1$, $\mathcal{F}_2$ be quasi-coherent sheaves on Z_0, resp. on $\mathbb{G}_a^m$, and define $\mathcal{F} := pr_1^*\mathcal{F}_1 \otimes pr_2^*\mathcal{F}_2$. We will also denote by $\mathcal{F}$ the sheaf on Z_m isomorphic to $\mathcal{F}$. By Lemma 5.2 we have for every $i \geq 1$

$$H^i(Z_m, \mathcal{F}) \cong H^i(Z_0 \times \mathbb{G}_a^m, \mathcal{F}) = \bigoplus_{\mu+\nu=i} H^\mu(Z_0, \mathcal{F}_1) \otimes H^\nu(\mathbb{G}_a^m, \mathcal{F}_2).$$

As $\mathbb{G}_a^m$ is affine, all factors of the sum but the first vanish, so we get

$$H^i(Z_m, \mathcal{F}) \cong H^i(Z_0, \mathcal{F}_1) \otimes H^0(\mathbb{G}_a^m, \mathcal{F}_2) = H^i(Z_0, \mathcal{F}_1) \otimes R,$$

where $R = H^0(\mathbb{G}_a^m, \mathcal{F}_2)$ is a non-trivial k-algebra. As Z_m is affine, we have $H^i(Z_m, \mathcal{F}) = 0$, so by Serre's criterion of affineness (see [27], §5, Thm. 2.23), Z_0 is affine.

On the other hand, assume $\tau_1, \dots, \tau_{i-1}$ are trivial, but τ_i is not. Then, by Remark 5.3, τ_i corresponds to a non-zero element $g^{(i)} \in H^1(Z_0, \mathcal{O}_{Z_0}) \otimes k[D_1, \dots, D_i]$, so $H^1(Z_0, \mathcal{O}_{Z_0})$ cannot be zero and Z_0 is not affine. □

We have an affine open cover of G/B, given by

$$G/B = \bigcup_{v \in W} \mathcal{U}_v,$$

$$\mathcal{U}_v = \{vuB | u \in U^-\} \cong vU^- \cong U^-.$$

For the projections $G \xrightarrow{\psi} G/U \xrightarrow{\pi} G/B$ we define

$$X(w)^{(0)} = \pi^{-1}(X(w)) \quad \text{and} \quad \mathcal{U}_v^{(0)} = \pi^{-1}(\mathcal{U}_v).$$

Proposition 5.5. *Let G be a as above, let $H \triangleleft B \subset G$ be a normal, closed subgroup. Let $G/H \xrightarrow{\tau} G/B$ be the projection. Then we have that τ is a B/H-torsor, trivialized by the open cover $\{\mathcal{U}_v\}_v$ of G/B.*

Proof. Let $v \in W$. We identify $\mathcal{U}_v$ with vU^-. Let B/H operate on G/H by multiplication from the right. We have to show that there exists a B/H-equivariant isomorphism ψ_v such that the right triangle of the following diagram commutes:

$$\begin{array}{ccccc} G/H & \hookleftarrow & \pi^{-1}(\mathcal{U}_v) & \xrightarrow{\psi_v} & \mathcal{U}_v \times B/H \\ \downarrow & & \downarrow & \swarrow_{pr_1} & \\ G/B & \hookleftarrow & \mathcal{U}_v & & \end{array}$$

The multiplication from the right by B yields a commutative diagram

$$\begin{array}{ccccccc} & & vU^- \times B & \xrightarrow{\sim} & vU^-B & & \\ & & \downarrow & & \downarrow & & \\ (vU^- \times B)/H & = & vU^- \times B/H & \xrightarrow{\sim} & vU^-B/H & = & \pi^{-1}(\mathcal{U}_v). \end{array}$$

Now for $g = [\dot{g}] \in \pi^{-1}(\mathcal{U}_v)$, we define $\psi_v(g)$ as follows. With $v \in W$, let $u^- \in U^-$, write $\dot{g} = vu^-b$ uniquely by the Bruhat decomposition and define $\psi_v(g) := (vu^-, bH)$. This is well defined, as for $h \in H \triangleleft B$ we have $\dot{g}h = vu^-bh$, so $\psi_v([\dot{g}h]) = (vu^-, bH)$. Furthermore ψ_v is B/H-equivariant, since for $b'H \in B/H$ we have:

$$\psi_v([\dot{g}].b'H) = \psi_v([\dot{g}b']) = (vu^-, bb'H) = (vu^-, bH).b'H$$

Thus τ is a B/H-torsor. □

Lemma 5.6. *Let V be a connected, nilpotent algebraic group of positive dimension. Then the following holds: For every proper closed subgroup $V' \subset V$ we have* $\dim V' < \dim N_V(V')$.

Proof. See [23], § 17.4 Prop. □

Remark 5.7. *With Lemma 5.6 we have in particular for every $V' \subset V$ of codimension 1 that V' is normal in V.*

Proposition 5.8. *There exists a filtration of $U = U_0 \supset \cdots \supset U_N = \{1\}$ by closed, normal subgroups such that $U_{j-1}/U_j \cong \mathbb{G}_a$. More concretely, for every reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$, define $v_{N-j} = s_{i_1} \cdots s_{i_j}$, $j = 1, \ldots, N$. Then*

$$U_j := \prod_{\substack{\alpha>0 \\ v_{N-j}(\alpha)<0}} U_\alpha$$

fulfills the assumption.

Proof. First, we want to make a remark. The statement of the proposition is a well-known fact, for example in [6], Cor. 15.5, the following is shown: For k perfect, U unipotent, connected and solvable, U splits over k. Here we give an explicit construction.

As above, let Φ^+ denote the set of positive roots associated to B. Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced expression. Then we know that $U = \prod_{\substack{\alpha>0 \\ w_0(\alpha)<0}} U_\alpha$. Moreover, for every subset $I \subset \Phi^+$ of the form $I = \{\alpha > 0; v(\alpha) < 0\}$ for some $v \in W$, the product $\prod_{\alpha \in I} U_\alpha$ equals the group generated by all the closed subgroups $U_\alpha, \alpha \in I$, and is again closed and connected (see [34], Lemma 8.3.5). As these groups are all unipotent, they are nilpotent (see [23], Cor. 17.5).

We let $v_{N-j} = s_{i_1} \cdots s_{i_j} \in W$ for $j = 1, \ldots, N$. Then for all j we have $l(v_{N-j}) = j$ and $v_j \leq v_{j-1}$. We set $U_j := \prod_{\substack{\alpha>0 \\ v_{N-j}(\alpha)<0}} U_\alpha$. As $\dim(U_j) = l(v_{N-j})$ (see [34], Lemma 8.3.6) and $U_j \subset U_{j-1}$, we can apply Remark 5.7 repeatedly to U_j and U_{j-1}. Thus we see that U_j is normal in U_{j-1}. Denote by $\alpha_j \in \Phi^+$ the unique positive root such that $v_{N-j}(\alpha_j) < 0$ and $v_{N-(j-1)}(\alpha_j) > 0$; then $U_{j-1}/U_j \cong U_{\alpha_j} \cong \mathbb{G}_a$. Thus the claim follows. $\square$

Corollary 5.9. *Given $G \xrightarrow{\psi} G/U \xrightarrow{\pi} G/B$, we have that π is a T-torsor and the U-torsor ψ can be written as a sequence of $\mathbb{G}_a$-torsors.*

Proof. From Proposition 5.5 we get that π, resp. $\pi \circ \psi$, is a $B/U = T$-torsor, resp. a B-torsor. From Proposition 5.8 we know that U has a filtration by normal, closed subgroups. We fix such a filtration and denote by $U_2 \subset U_1 \subset U$ two of its steps. Thus there exist two subsets $I_2 \subset I_1 \subset \Phi^+$ such that $U_i = \prod_{\alpha \in I_i} U_\alpha$. Consider the commutative diagram

$$\begin{array}{ccc} G/U_2 & \xrightarrow{\pi_2} & G/B \\ \downarrow{\scriptstyle\tau} & \nearrow{\scriptstyle\pi_1} & \\ G/U_1 & & \end{array}.$$

We can apply Proposition 5.5 for U_1 and U_2 and get a commutative diagram

$$\begin{array}{ccc} \mathcal{U}_v \times B/U_2 & \xrightarrow{\pi_2} & \mathcal{U}_v \\ \downarrow{\scriptstyle\tau} & \nearrow{\scriptstyle\pi_1} & \\ \mathcal{U}_v \times B/U_1 & & \end{array}.$$

As the multiplication $\prod U_\alpha \to U$ is an isomorphism of varieties, we have $\mathcal{U}_v \times B/U_2 \cong (\mathcal{U}_v \times B/U_1) \times U_1/U_2$. Whence τ is a U_1/U_2-torsor. $\square$

Now we are ready to prove the theorem. Note that the restrictions of π and ψ to (the inverse image of) $X(w) \subset G/B$ are also T-, resp. U-torsors, trivialized by (the inverse images of) $\mathcal{U}_v \cap X(w)$.

Denote by $L : G \to G, g \mapsto g^{-1}F(g)$ the Lang map L_{id}, which is surjective. We have the following Cartesian diagram

(5.10)
$$\begin{array}{ccc} G & \xrightarrow{\quad L \quad} & G \\ \uparrow & \square & \uparrow \\ \psi^{-1} \circ \pi^{-1}(X(w)) = \{g \in G | L(g) \in BwB\} & \longrightarrow & BwB. \end{array}$$

But the Lang map is clearly affine, so the second line is also an affine morphism, as affineness for morphisms is stable under base change (see [16], Prop. 12.3). As $BwB \cong \mathbb{A}^{l(w)} \times B \cong \mathbb{A}^{l(w)+N} \times T$ is affine, the left hand side of (5.10) is also affine. Furthermore, T is reductive, so $X(w)^{(0)}$ and $X(w) = X(w)^{(0)}/T$ are simultaneously affine ([32], Theorem A). As the restriction $X(w) \cap \mathcal{U}_v$ to any open chart is affine, the restriction $X(w)^{(0)} \cap \mathcal{U}_v^{(0)}$ is also affine. We fix a reduced expression for w_0 and get by Proposition 5.8 a filtration $\{U_i\}_{1 \leq i \leq N}$ of U. We denote for $1 \leq i \leq N$ by

$$\tau_i : G/U_i \longrightarrow G/U_{i-1}$$

the projections and define

$$X(w)^{(i)} = \tau_i^{-1}(X(w)^{(i-1)}) \text{ and } \mathcal{U}_v^{(i)} = \tau_i^{-1}(\mathcal{U}_v^{(i-1)}).$$

By Proposition 5.5, all τ_i are $\mathbb{G}_a$-torsors, trivialized by $\{\mathcal{U}_v^{(i)}\}_v$ and so are the restrictions of τ_i to $X(w)^{(i)}$ are. Applying Lemma 5.4 yields $X(w)^{(0)}$ is affine if and only if all τ_i are successively trivial. Moreover, we see that $X(w)^{(i)} \cap \mathcal{U}_v^{(i)}$ is affine for all $v \in W, 1 \leq i \leq N$. Thus we have proved most of the following theorem:

Theorem 5.11. *We have $X(w)$ affine if and only if the set $\{\tau_i; 1 \leq i \leq N\}$ of $\mathbb{G}_a$-torsors consists of trivial torsors.*
In particular, $X(w)$ is affine if and only if $H^1(X(w)^{(0)}, \mathcal{O}_{X(w)^{(0)}}) = 0$.

Proof. It remains to prove the last statement. Assume there is a $\mathbb{G}_a$-torsor τ_i (of smallest index) such that τ_i is not trivial. But then, by Remark 5.3, τ_i can be regarded as a non-zero element of $H^1(X(w)^{(0)}, \mathcal{O}_{X(w)^{(0)}}) \otimes k[D]^{\otimes i-1}$ and therefore as an element of $H^1(X(w)^{(0)}, \mathcal{O}_{X(w)^{(0)}})$. □

Remark 5.12. *Neeman has proven a slightly weaker, but more general version of the last statement of the theorem, namely that for any quasi-affine scheme X one has X is affine if and only if $H^i(X, \mathcal{O}_X) = 0$ for every $i \geq 1$ (see* [30]*, Thm. 4.1). In his Diploma thesis (*[17]*), Haastert has shown a similar result: For any quasi-affine scheme X one has X is affine if $H^i(X, \mathcal{O}_X)$ is finite-dimensional for every $i \geq 1$.*

We first comment on strategies to reduce the computational complexity, before we calculate the cohomology.

Lemma 5.13. *Let X/k be an integral scheme, $\mathfrak{U} = \{\mathcal{U}_i\}_{1 \leq i \leq m}$ an affine open cover of X, $(\dot{g}_{i,j})_{i,j} \in \prod_{i<j} \Gamma(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}_X)$ a representative for a cocycle $g \in \check{H}^1(\mathfrak{U}, \mathcal{O}_X)$. Then we have:*

$$g = 0$$

(5.14)
$$\Leftrightarrow \bigcap_{1 \leq i \leq m} \left(\Gamma(\mathcal{U}_i, \mathcal{O}_X) + \dot{g}_{1,i|\mathcal{U}_1 \cap \mathcal{U}_i}\right) \neq \emptyset,$$

where we set $\dot{g}_{1,1} := 0$ and where we consider the intersection in $\Gamma(\bigcap_i \mathcal{U}_i, \mathcal{O}_X)$.

Proof. We have the following equality on $\mathcal{U}_1 \cap \mathcal{U}_i \cap \mathcal{U}_j$:

$$\dot{g}_{1,i} + \dot{g}_{i,j} = \dot{g}_{1,j}$$

Now assume

$$\forall j \exists f_j \in \Gamma(\mathcal{U}_j, \mathcal{O}_X) : f_1 - f_j = \dot{g}_{1,j} \text{ on } \mathcal{U}_1 \cap \mathcal{U}_j. \tag{5.15}$$

Then (5.15) implies

$$f_1 - f_i + \dot{g}_{i,j} = f_1 - f_j \Leftrightarrow \dot{g}_{i,j} = f_i - f_j \text{ on } \mathcal{U}_1 \cap \mathcal{U}_i \cap \mathcal{U}_j.$$

But the last equality holds on all of $\mathcal{U}_i \cap \mathcal{U}_j$, since X is integral.

Consider the first coboundary operator

$$\partial^1 : \prod_i \Gamma(\mathcal{U}_i, \mathcal{O}_X) \to \prod_{i<j} \Gamma(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}_X)$$
$$(f_i)_i \mapsto (f_{i|\mathcal{U}_i \cap \mathcal{U}_j} - f_{j|\mathcal{U}_i \cap \mathcal{U}_j})_{i,j}$$

and denote the restricted coboundary operator by

$$\tilde{\partial}^1 : \prod_i \Gamma(\mathcal{U}_i, \mathcal{O}_X) \to \prod_{1<j} \Gamma(\mathcal{U}_1 \cap \mathcal{U}_j, \mathcal{O}_X)$$
$$(f_i)_i \mapsto (f_{1|\mathcal{U}_1 \cap \mathcal{U}_j} - f_{j|\mathcal{U}_1 \cap \mathcal{U}_j})_j.$$

So altogether we can say

$$\begin{aligned}
& g = 0 \\
\Leftrightarrow\ & (\dot{g}_{i,j})_{i,j} \in \operatorname{im}(\partial^1) \\
\Leftrightarrow\ & (\dot{g}_{1,j})_j \in \operatorname{im}(\tilde{\partial}^1) \\
\Leftrightarrow\ & \exists f_1 \in \Gamma(\mathcal{U}_1, \mathcal{O}_X) \forall j > 1 \exists f_j \in \Gamma(\mathcal{U}_j, \mathcal{O}_X) : f_{1|\mathcal{U}_1 \cap \mathcal{U}_j} - f_{j|\mathcal{U}_1 \cap \mathcal{U}_j} = \dot{g}_{1,j|\mathcal{U}_1 \cap \mathcal{U}_j} \\
\Leftrightarrow\ & \bigcap_{1 \le j \le m} \left(\Gamma(\mathcal{U}_j, \mathcal{O}_X) + \dot{g}_{1,j|\mathcal{U}_1 \cap \mathcal{U}_j}\right) \neq \emptyset.
\end{aligned}$$

□

Sometimes it is possible to reduce the calculations above even more. Observe that for many $w \in W$, one does not need all charts $\mathcal{U}_v$ to cover the whole of $X(w)$. In fact, all extremes can occur: For $w = id$ one needs all charts to cover $X(id)$, which consists exactly of the $\mathbb{F}_q$-rational points, while the next proposition shows that there exist $w \in W$ such that only a single chart is needed. In general if there exists some $v \in W$ such that $X(w) \cap \mathcal{U}_{id} \cap \mathcal{U}_v = X(w) \cap \mathcal{U}_{id}$, then it also holds that $X(w) \cap \mathcal{U}_v \cap \mathcal{U}_{v'} = X(w) \cap \mathcal{U}_v$ for every v' in the subgroup $H := \langle v \rangle \subseteq G$. More generally, define H to be the subgroup generated by all $v' \in W$ such that $X(w) \cap \mathcal{U}_{id} \cap \mathcal{U}_{v'} = X(w) \cap \mathcal{U}_{id}$. Therefore to cover the whole of $X(w)$ it is enough to choose a representative for every orbit of $H \circlearrowleft G$ and take the charts attached to them.

Proposition 5.16. *Let $G = \mathbb{GL}_n$, $k_0 = \mathbb{F}_q$, $w = s_1 \cdots s_{n-1}$. Then $X(w)$ is contained in every open chart $\mathcal{U}_v$, i.e.*

$$X(w) \cap \bigcap \mathcal{U}_v = X(w).$$

In particular, $X(w)$ is affine, where we have an explicit description of $\Gamma(X(w), \mathcal{O}_{X(w)})$.

Proof. From [11], page 117 first line, we know that we have

$$X(w) = \left\{(\mathcal{F}_i) \in G/B; \mathcal{F}_i = \mathcal{F}_1 \oplus F(\mathcal{F}_1) \oplus \cdots \oplus F^{i-1}(\mathcal{F}_1)\right\}.$$

Here we regard G/B as the space of full flags $\mathcal{F}$ over $k = \overline{\mathbb{F}}_q$. In particular, we have $\det(A) \neq 0$, where

$$A := \begin{pmatrix} x_1 & x_1^q & \cdots & x_1^{q^{n-1}} \\ \vdots & \vdots & & \vdots \\ x_n & x_n^q & \cdots & x_n^{q^{n-1}} \end{pmatrix}$$

represents a flag $(\mathcal{F}_i) \in X(w)$. It is enough to show that $X(w) \subset \mathcal{U}_{w_0}$, where w_0 is the longest element in W. By Lemma 6.1 below we have to show that for all $i = 1, \ldots, n-1$, we have

$$\det(A_{\{n,\ldots,n-i+1\}\times\{1,\ldots,i\}}) \neq 0.$$

Let $i = 1, 1 \leq l \leq n$, and assume there exists an j, $1 \leq j \leq n$, such that $x_j = 0$. It follows that $x_j^{q^l} = 0$ and that $\det(A) = 0$.

We proceed by induction on i. For $i \geq 2$, we denote by $A_j^{(i-1)}$ the left $(i-1 \times i-1)$-submatrix of A consisting of the rows labelled by $\{n, \ldots, \widehat{j}, \ldots, n-i+1\}$ and the columns labelled by $\{1, \ldots, i-1\}$. By induction we can assume that

$$\det(A_j^{(i-1)}) \neq 0 \text{ for all } n-i+1 \leq j \leq n.$$

With the notation above, we have to show that $\det(A_{n-i}^{(i)}) \neq 0$. But by developing this determinant by the last column, we get

$$\det(A_{n-i}^{(i)}) = \sum_{r=n}^{n-i+1} (-1)^r x_r^{q^{i-1}} \det(A_r^{(i-1)}).$$

If we assume this to be zero, we find $\alpha_r, \beta_r \in k$ such that $\beta_r^{q^{i-1}} = \alpha_r$ and

$$x_{n-i+1}^{q^{i-1}} = \sum_{r=n}^{n-i+2} \alpha_r x_r^{q^{i-1}} = (\sum_{r=n}^{n-i+2} \beta_r x_r)^{q^{i-1}}.$$

But then $\det(A) = 0$, because

$$A_{\{n-i+1\}\times\{1,\ldots,n\}} \in \left\langle A_{\{n-i+2\}\times\{1,\ldots,n\}}, \cdots, A_{\{n\}\times\{1,\ldots,n\}} \right\rangle. \qquad \square$$

Remark 5.17. *Essentially the same proof works for all $v \in W$ at once, i.e one can directly show that $X(w) \subset \bigcap \mathcal{U}_v$.*

Under the assumptions of Lemma 5.13 assume that X is locally of finite type over k. We then can exhaust every $\Gamma(\mathcal{U}_i, \mathcal{O}_X)$ by finitely generated k-algebras $A_i^{(n)}, n \in \mathbb{N}$, which are subrings of $\Gamma(\mathcal{U}_i, \mathcal{O}_X) \subseteq \Gamma(\bigcap_i \mathcal{U}_i, \mathcal{O}_X)$, such that $A_i^{(n)} \subseteq A_i^{(n+1)}$ and $\Gamma(\mathcal{U}_i, \mathcal{O}_X) = \bigcup_n A_i^{(n)}$.

Assume, we have given $(\dot{g}_{i,j})_{i,j} \in \prod_{i<j} \Gamma(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}_X)$, a representative for the zero cocycle $0 = g \in \check{H}^1(\mathfrak{U}, \mathcal{O}_X)$. Then there exists a tuple (n_i) of positive integers such that the intersection $\bigcap_i (A_i^{(n_i)} + \dot{g}_{1,i}) \subset \bigcup_i A_i^{(n_i)} \subset \Gamma(\bigcap_i \mathcal{U}_i, \mathcal{O}_X)$ is already non-empty.

We now apply this discussion to the variety $X = X(w)$, which is integral by Theorem 1.4. We can use the trivializing affine covers $\{\mathcal{U}_v^{(i)}\}$ to calculate the Čech cohomology, which is in fact isomorphic to the cohomology. We denote by $\mathrm{im}(\partial^{1,(i)})$ the image of the first coboundary operator for $X(w)^{(i)}$ and by $\mathrm{im}(\tilde{\partial}^{1,(i)})$ the restricted one. This leads to the following algorithm from which Theorem 5.1 follows:

Input	$w \in W, G, B, T, U, q$, an integer l, denoting the bound, a Čech cover $\mathfrak{U}$.
Output	*"$X(w)$ is affine." or "Cannot decide whether $X(w)$ is affine or not."*.
Step 1	Compute all occurring rings and maps.
Step 2	Calculate a filtration, calculate for every τ_i a representative $\dot{g}^{(i)}$ for the corresponding cocycle.
Step 3	For $i = 1, \ldots, N$ check whether $\dot{g}^{(i)} \in \mathrm{im}(\partial^{1,(i)})$. If not, return *"Cannot decide whether $X(w)$ is affine or not."*.
Step 4	Return *"$X(w)$ is affine."*.

6. The case G_0 split of type A

In this chapter, we consider the case G_0 split of type A. We will give explicit descriptions for the equations defining $X(w)$. From this we deduce descriptions of the affine coordinate rings of $X(w)^{(i)}$ on the affine open charts to calculate the Čech cohomology. Thereby we obtain an explicit algorithm for the affineness criterion Theorem 5.1, which we have implemented.

So, let $n \in \mathbb{N}, G = \mathbb{GL}_n$, $k_0 = \mathbb{F}_q$, $W = \mathbb{S}_n$, and B the subgroup of upper triangular matrices, T the diagonal matrices, U the unipotent upper triangular matrices. We allow ourself to write G, B, U, T etc. for the set of closed points $G(k), B(k), U(k), T(k)$ etc., when this causes no confusion. We frequently write for every $v \in W = N_G(T)/T$ again v for the attached permutation matrix $P_v = (\delta_{v(i),i})_i = (\delta_{i,v^{-1}(i)})_i$, where $\delta_{i,j}$ is the Kronecker delta. This is a representative for v in $N_G(T)$. We identify the variety $X(w)$ with its closed points:

$$X(w) = \left\{\dot{x} = (x_{ij})_{ij} \in G; \dot{x}^{-1}F(\dot{x}) \in BwB\right\}/B$$

We have $N = l(w_0) = n(n-1)/2$ and let $\{\mathcal{U}_v\}_{v\in W} = \{\{vuB | u \in U^-\}/B\}$ be a suitable affine cover. Here U^- denotes the unipotent lower triangular matrices.

Now we want to work out the equations describing $X(w)$. For any matrix A and any index sets $I, J \subseteq \{1, \cdots, n\}$ with $|I| = |J|$, we denote by $A_{\{I\}\times\{J\}}$ the submatrix of A consisting of the rows labelled by I and columns specified in J. As we are mostly interested in the (non-)vanishing of the determinant function of such submatrices, there is mostly no need to specify an order on these sets. Whenever this is however necessary, we take the obvious order.

Lemma 6.1. *For every $A \in G(k)$ and every $v \in W$, we have:*

$$\begin{aligned}
&AB \in \mathcal{U}_v \\
\Leftrightarrow\ &\exists b = b^{(v)}(A) \in B(k) : Ab \in vU^-(k) \\
\Leftrightarrow\ &\forall i = 1, \dots, n-1 : \det(A_{\{v(1),\dots,v(i)\}\times\{1,\dots,i\}}) \neq 0
\end{aligned}$$

Moreover, if $AB \in \mathcal{U}_v$ then $b^{(v)}(A) \in B(k)$ is unique.

Proof. First we prove this for $v = id$. For the proof of "$\Rightarrow$" in the last line, we observe that the inequalities hold, as they hold for all elements of $U^-(k)$ and that multiplication from the right by some $b \in B$ does not change the considered minors.

For the other direction, we have to find some $b = b^{(id)}(A) \in B$ such that $Ab \in U^-(k)$. If we denote by $A^{(i)}$ the upper left submatrix of A of size $(i \times i)$ and write $y^{(i)} = (0, \dots, 0, 1)^t \in k^i$, we need to find an element $b = (b_{ij}) \in B$ whose entries satisfy the following system of linear equations:

$$A^{(i)} \left(b_l^{(i)}\right)_l = y^{(i)}, 1 \leq i \leq n$$

By assumption, $A^{(i)}$ has full rank, so we can solve all these equations uniquely. Thus, $(b_l^{(i)})$ is the ith column of the inverse matrix of $A^{(i)}$. Especially we have

$$b_i^{(i)} = (-1)^{i+i} \det(A^{(i)})^{-1} \det(A^{(i-1)}) \neq 0,$$

as $(-1)^{i+i}\det(A^{(i-1)})$ is the (i,i)-th entry of the adjoint matrix to $A^{(i)}$. Thus we have $b := (b_l^{(i)})_{1\leq l\leq i\leq n} \in B$ and $Ab \in U^-(k)$.

To handle the case of general v, note that $v^{-1}A = (a_{v(i),j})_{i,j}$. Thus we have

$$\begin{aligned}
AB \in \mathcal{U}_v \Leftrightarrow\ & v^{-1}AB \in \mathcal{U}_{id} \\
\Leftrightarrow\ & \forall i = 1, \dots, n-1 : \det(v^{-1}A_{\{1,\dots,i\}\times\{1,\dots,i\}}) \neq 0 \\
\Leftrightarrow\ & \forall i = 1, \dots, n-1 : \det(A_{\{v(1),\dots,v(i)\}\times\{1,\dots,i\}}) \neq 0, \text{ as desired.} \quad \square
\end{aligned}$$

Proposition 6.2. *For every $M \in G(k)$ we have:*

$$M \in BvB$$

$$\Leftrightarrow$$

$$\forall i = 1, \dots, n-1, v(i) < j \leq n : \det(M_{\{v(1),\dots,v(i-1),j\}\times\{1,\dots,i\}}) = 0 \tag{6.3}$$

$$\forall i = 1, \dots, n-1 : \det(M_{\{v(1),\dots,v(i)\}\times\{1,\dots,i\}}) \neq 0 \tag{6.4}$$

Proof. As in the proof of the preceding lemma we observe that the considered equalities and inequalities hold for $P_v \in BvB$. Let $b' \in B$, then we have $b'P_v = (b'_{i,v(i)})_i$ and we easily see that (6.3) holds. Moreover

$$\det(b'P_v)_{\{v(1),\dots,v(i)\}\times\{1,\dots,i\}} = \prod_{j=1}^{i} b'_{v(j),v(j)} \neq 0,$$

so (6.4) also holds. Let $I = \{v(1), \dots, v(i-1), j\}$ for some $j \geq v(i)$, let $J = \{1, \dots, i\}$ and let $H = \{1, \dots, n\}$. For any $M \in G$ and any $b \in B$ we have

$$(Mb)_{I\times J} = M_{I\times H} b_{H\times J} = M_{I\times J} b_{J\times J}.$$

But now $\det((Mb)_{I\times J}) = \det(M_{I\times J})\det(b_{J\times J})$, so (6.3) and (6.4) hold for M and Mb simultaneously, in particular for $M = b'P_v$.

For the other direction we use induction. We assume that the claimed (in)equations (6.3) and (6.4) hold. We further assume by induction that there exists i_0 such that the ith column of M is the standard basis column vector $e_{v(i)}$ for all $i < i_0$ and that the $v(i)$th row of M is the standard basis row vector e_i. Write $j_0 := v(i_0)$, then we have

$$0 \neq \det(M_{\{v(1),\dots,v(i_0-1),j_0\}\times\{1,\dots,i_0\}}) =: d.$$

Developing the determinant by the last column, we get

$$\begin{aligned} d = &(-1)^{i_0+j_0} M_{\{j_0\}\times\{i_0\}} \det(M_{\{v(1),\dots,v(i_0-1)\}\times\{1,\dots,i_0-1\}}) \\ &+ \sum_{s=1}^{i_0-1} (-1)^{i_0+s} M_{\{v(s)\}\times\{i_0\}} \det(M_{\{v(1),\dots,\widehat{v(s)},\dots v(i_0)\}\times\{1,\dots,i_0-1\}}). \end{aligned}$$

But for $s \in \{1, \dots, i_0 - 1\}$ we have $M_{\{v(s)\}\times\{l\}} \neq 0$ if and only if $l = s$ by the assumption on the rows of M. So $M_{\{v(s)\}\times\{i_0\}} = 0$ and all but the first term cancel. This shows that $M_{\{j_0\}\times\{i_0\}} \neq 0$. By induction we have $M_{\{j_0\}\times\{i\}} = 0$ for all $i < i_0$. We claim that we also have $M_{\{r\}\times\{i_0\}} = 0$ for all $r > j_0$. In fact, let $r > j_0$, $r \notin \{v(1), \dots, v(i_0 - 1)\}$ and assume $M_{\{j_0\}\times\{i\}} \neq 0$. Then we see by developing by the rth row that

$$\det(M_{\{v(1),\dots,v(i_0-1),r\}\times\{1,\dots,i_0\}}) = \pm M_{\{r\}\times\{i_0\}} \neq 0.$$

This is a contradiction to (6.3). But if $r > j_0$ and $r \in \{v(1), \dots, v(i_0 - 1)\}$ we have $M_{\{r\}\times\{i_0\}} = 0$ by induction.
Therefore we can modify M (by multiplying with suitable elements $b, b' \in B$ from the left and right) such that the j_0th row of M is equal to the i_0th standard base row vector e_{i_0} and the i_0th column of M is equal to the j_0th standard base column vector e_{j_0}. Thus the claim follows by induction. □

For $v \in W$ we denote by U'_v the subgroup $vU^-v^{-1} \cap U$ of G as in chapter 4. By [6], Thm. 14.12, we know that the map $U'_v \times B \to BvB$, given by $(u, b) \mapsto uvb$, is an isomorphism of varieties. Next we want to show that the subscheme of G given by (6.3) and (6.4) is reduced. To do so, we prove the following

Lemma 6.5. *Let $v \in W$, let $Z \subset G$ be the subscheme defined by (6.3) and (6.4). Let R be any k-algebra. Then the map*

$$U'_v(R) \times B(R) \longrightarrow Z(R), (u, b) \mapsto uvb,$$

is bijective.

Proof. For the injectivity assume we have $u, u' \in U'_v(R)$ and $b, b' \in B(R)$ such that $u'vb' = uvb$. We have $(u')^{-1}u = v\tilde{u}v^{-1}$ for some $\tilde{u} \in U^-(R)$, as $(u')^{-1}u \in U'_v(R)$. Thus we see from

$$u'vb' = uvb \Leftrightarrow b'b^{-1} = v^{-1}(u')^{-1}uv = \tilde{u} \in U^-(R)$$

that $b' = b$ and then also that $u' = u$.

We now prove the surjectivity. Let $M = (m_{ij}) \in GL_n(R)$. We have $v(m_{ij})v^{-1} = (m_{v^{-1}(i),v^{-1}(j)})_{i,j}$ and therefore we see that we can describe $U'_v(R)$ as

$$vU^-(R)v^{-1} \cap U = \{u \in U(R); u_{ij} = 0 \text{ for all } i < j \text{ with } v^{-1}(i) < v^{-1}(j)\},$$

see also Corollary 4.3. Fix some $1 \leq i_0 \leq n-1$ and assume that the ith column of M is the standard basis column vector $e_{v(i)}$ for all $i < i_0$ and that the $v(i)$th row of M is the standard basis row vector e_i. By imitating the proof of Proposition 6.2 and by multiplying with suitable $b \in B(R)$ from the right, we can assume that $m_{v(i_0),i_0} = 1$ and $m_{r,i_0} = 0$ for all $r > v(i_0)$. It remains to show that we can find $u^{(i_0)} \in U'_v(R)$ such that

$$[u^{(i_0)}M]_{r,i_0} = 0 \text{ for all } r < v(i_0), \tag{6.6}$$

so that the i_0th column of $u^{(i_0)}M$ is equal to the standard basis column vector $e_{v(i_0)}$. Observe that, for $r \in \{v(1), \dots, v(i_0 - 1)\}$, we know by assumption on the rows that $m_{r,i_o} = 0$. For all other $r < v(i_0)$ we then can find some $j_r > i_0$ such that $v(j_r) = r$. But then we have

$$v^{-1}(r) = v^{-1}(v(j_r)) = j_r > i_0 = v^{-1}(v(i_0)) \text{ and } r < i_0.$$

Thus first setting $u^{(i_0)} = (\delta_{ij})$ and then setting

$$u^{(i_0)}_{r,v(i_0)} = -m_{r,i_0}, \text{ for all } r < i_0, r \notin \{v(1), \dots v(i_0 - 1)\},$$

yields an $u^{(i_0)} \in U'_v(R)$ such that (6.6) holds. □

We now want to describe the affine coordinate ring of $X(w)$ on the open affine charts. Therefore we use the following notation: Let $x = (x_{ij})_{ij} \in U^-(k[X_{ij}]_{j<i})$ with $x_{ij} = X_{ij}$ for $j < i$, let $M = (x_{ij})^{-1}(x^q_{ij})$. We set

$$\begin{aligned}
\dot{R}_w :=& k[X_{ij}]_{j<i}[L_w], \\
I_w :=& \left(\{\det M_{\{w(1),\dots,w(i-1),j\}\times\{1,\dots,i\}}; 1 \leq i \leq n-1, w(i) < j \leq n\}\right), \\
l_w :=& \prod_{i=1}^{n-1} \det M_{\{w(1),\dots,w(i)\}\times\{1,\dots,i\}}, \\
l^{(v)} :=& \prod_{l=1}^{n-1} \det (x_{ij})_{\{v(1),\dots,v(l)\}\times\{1,\dots,l\}} \text{ and} \\
R_w :=& \dot{R}_w / \left(I_w + (L_w l_w - 1)\right).
\end{aligned}$$

Let $\mathcal{D} = \left\{\det(x_{ij})_{\{v(1),\dots,v(i)\}\times\{1,\dots,i\}}; v \in W, i = 1, \dots, n-1\right\} \subset k[X_{ij}]_{j<i}$ and let $R^{loc}_w = R_w[L_f]_{f\in\mathcal{D}}/(\{L_f f - 1; f \in \mathcal{D}\})$ be the localisation by all occurring minors. For every $v \in W$ there exist unique $(b^{(v)}_{ij}) \in B(R^{loc}_w)$ and $x^{(v)} = (x^{(v)}_{ij}) \in U^-(R^{loc}_w)$

such that $(x_{ij}) = vx^{(v)}(b_{ij}^{(v)})$. If we write $R_w^{(v)} = \left\langle \{x_{ij}^{(v)}; j < i\} \right\rangle_{k-alg} \subseteq R_w^{loc}$, we can view $R_w^{(v)}$ as the image of R_w under the morphism

$$\varphi_v : R_w \hookrightarrow R_w^{loc}, X_{ij} \mapsto x_{ij}^{(v)}.$$

Under this point of view we have $\varphi_{id} = id$ and, by construction, $(\varphi_v)^{-1} = \varphi_{v^{-1}}$ on the image.

Proposition 6.7. *Let $v, v' \in W$. Then*

$$\Gamma\left(X(w) \cap \mathcal{U}_v, \mathcal{O}_{X(w)\cap\mathcal{U}_v}\right) = \varphi_{v^{-1}}(R_w) \cong R_w,$$
$$\Gamma\left(X(w) \cap \mathcal{U}_v \cap \mathcal{U}_{v'}, \mathcal{O}_{X(w)\cap\mathcal{U}_v\cap\mathcal{U}_{v'}}\right) = \left\langle R_w^{(v)}, R_w^{(v')} \right\rangle_{k-alg} =: R_w^{(v,v')} \subseteq R_w^{loc} \text{ and}$$
$$\Gamma\left(X(w) \cap \bigcap_v \mathcal{U}_v, \mathcal{O}_{X(w)\cap\bigcap_v\mathcal{U}_v}\right) = R_w^{loc}.$$

Proof. It remains to show that these equations define a reduced scheme. To this end we consider the following diagram:

$$\begin{array}{ccccc} U^- & \supset & \{g \in U^-; g^{-1}F(g) \in BwB\}_{\text{red}} & \xrightarrow{\sim} & X(w) \cap U^-B/B \subset X(w) \\ \downarrow L & & \downarrow & & \\ U^- & \supset & (BwB \cap U^-)_{\text{red}} & & \end{array}$$

As this diagram is set-theoretically a cartesian one and since the Lang map L is étale, the above diagram is cartesian on the scheme level. As $BwB \cap U^-$ is an open subscheme of $BwB/B \subset G/B$, it is enough to show that BwB/B with the canonical reduced structure is defined by the equations and inequations in Proposition 6.2. But this follows from Lemma 6.5. □

Actually, there is another way to prove the reducedness of BwB/B by stating well-known equations for $\mathcal{C}(w) = \overline{BwB/B}$ and by showing that they define the same scheme as (6.3) and (6.4).
The following will also yield the equations defining $\overline{X(w)} \cap U^-B/B$. We denote by $W^{(i)}$ the set of all increasing sequences of length i with entries in $\{1, \ldots, n\}$: $W^{(i)} = \{(\tau_j)_j; \tau_j < \tau_{j+1}, \tau_j \in \{1, \ldots, n\}\}$, which is in bijection to $\mathbb{S}_n/(\mathbb{S}_i \times \mathbb{S}_{n-i})$. This bijection is defined as follows: We map $\tau \in W^{(i)}$ to the coset containing $\dot{\tau} = (\tau_1 \cdots \tau_i \cdots \tau_n)$. Here, for $j \geq i+1$, we define τ_j to be the smallest number in $\{1, \ldots, n\} \backslash \{\tau_1, \ldots, \tau_{j-1}\}$. We then have that $\dot{\tau}$ is the unique representative of minimal length of τ in $W = \mathbb{S}_n$. In the same way, let w_i denote the minimal length representative of $[w] \in \mathbb{S}_n/(\mathbb{S}_i \times \mathbb{S}_{n-i})$. Hence w_i corresponds to the ordered set $\{w(1), \ldots, w(i)\}_>$ in $W^{(i)}$. Furthermore, let p_τ denote the Plücker coordinate, that is for $\tau \in W^{(i)}, M \in G(k)$:

$$p_\tau(M) = \det M_{\{\tau_1,\ldots,\tau_i\}\times\{1,\ldots,i\}}.$$

Theorem 6.8. (Lakshmibai-Gonciulea) *For a Schubert variety $\mathcal{C}(w)$, $G = \mathbb{SL}_n$, the ideal sheaf of $\mathcal{C}(w)$ in G/B is generated by*

$$\bigcup_{1\leq i\leq n-1} \left\{p_\tau; \tau \in W^{(i)}, \dot{\tau} \not\leq w_i\right\}.$$

For a proof see [26], Thm. 5.1.3. As we have $\mathbb{SL}_n/B_{\mathbb{SL}_n} = \mathbb{GL}_n/B_{\mathbb{GL}_n}$, we will work with $G = \mathbb{GL}_n$ as above.

We fix i, denote by $(r_1, \ldots, r_i) \in W^{(i)}$ the increasing ordered set $\{w(1), \ldots, w(i)\}_>$ and denote for $j \geq i+1$ by r_j the smallest number in $\{1, \ldots, n\} \backslash \{r_1, \ldots, r_{j-1}\}$. We have $w_i = (r_1 \cdots r_i \cdots r_n)$. It is not difficult to see that

$$w_i = s_{r_1 - 1} \cdots s_1 \ldots s_{r_i - 1} \cdots s_i$$

is a reduced expression for w_i. Also, $\dot{\tau} = s_{\tau_1 - 1} \cdots s_1 \ldots s_{\tau_i - 1} \cdots s_i$ is a reduced expression for the unique representative of $\tau \in W^{(i)}$. This yields the following description for the Bruhat order:

Lemma 6.9. *With the above notation, we have:*

$$\dot{\tau} \leq w_i \Leftrightarrow \tau_j \leq r_j, \textit{ for all } 1 \leq j \leq i.$$

Proof. Only the "if" part is left to prove. Assume there exists a $j \leq i$ such that $r_j < \tau_j < \tau_i$. By Theorem 2.1.5 of [3] we have

$$\dot{\tau} \leq w_i \Leftrightarrow \dot{\tau}[i_1, i_2] \leq w_i[i_1, i_2], \text{ for all } i_1, i_2 \in \{1, \ldots, n\},$$

where we define $x[i_1, i_2] := \# \{a \in \{1, \ldots, i_1\}; x(a) \geq i_2\}$ for $x \in W$. But choosing $i_1 = j$ and $i_2 = r_j + 1 \leq \tau_j$ we see that $\dot{\tau}[j, r_j + 1] = 1$ and $w_i[j, r_j + 1] = 0$. $\square$

But now, we have $\dot{\tau} \not\leq w_i$ if and only if there exists a j, $1 \leq j \leq i$, such that $\tau_j > r_j$.
Let R be any k-algebra and let $M \in G(R)$ be any matrix satisfying (6.3) and (6.4). We have to show that

$$\det(M_{\{\tau_1, \ldots, \tau_i\} \times \{1, \ldots, i\}}) = 0. \tag{6.10}$$

Denote by j, $1 \leq j \leq i$, the biggest index such that $\tau_j > r_j \in \{w(1), \ldots, w(i)\}$. Let $m_1 < \ldots < m_j, 1 \leq m_l \leq i$, be such that, for $1 \leq l \leq j$, we have $w(m_l) \in \{r_1, \ldots, r_j\}$. We know that $r_1 < \ldots < r_j < \tau_j < \ldots < \tau_i$. Therefore by (6.3) we know that

$$\det(M_{\{w(1), \ldots, w(m_l - 1), \tau_r\} \times \{1, \ldots, m_l\}}) = 0, \text{ for all } j \leq r \leq i \text{ and } 1 \leq l \leq j. \tag{6.11}$$

Let

$$d_s := \det(M_{\{w(1), \ldots, w(m_l - 1), \tau_r\} \times \{1, \ldots, \widehat{s}, \ldots, m_l\}}), \text{ for } 1 \leq s \leq m_l, 1 \leq l \leq j.$$

We develop the determinants of (6.11) by the τ_r-th row and get, for all $j \leq r \leq i$ and $1 \leq l \leq j$:

$$0 = \sum_{s=1}^{m_l} (-1)^s m_{\tau_r, s} d_s$$

By (6.4) we have $d_{m_l} \neq 0$, for all l, thus these equations are equivalent to

$$m_{\tau_r, m_l} = \sum_{s=1}^{m_l - 1} (-1)^{m_l + s} d_s d_{m_l}^{-1} m_{\tau_r, s}, \text{ for all } j \leq r \leq i, 1 \leq l \leq j.$$

But using all these equations repeatedly, we obtain for all $1 \leq t \leq i$ and all $1 \leq s \leq i$, $s \notin \{m_1, \ldots, m_j\}$, some $\alpha_s^{(t)} \in R$ such that

$$m_{\tau_r, t} = \sum_{\substack{s=1 \\ s \notin \{m_1, \ldots, m_j\}}}^{i} \alpha_s^{(t)} m_{\tau_r, s} \text{ for all } j \leq r \leq i. \tag{6.12}$$

We want to remark here that in fact for $j = i$ this means that the τ_j-th row of M consists of zeros in the first i entries. But in this case (6.10) follows immediately. In the case $j = 1$ we get (6.10) from the multi-linearity of the determinant.
In the other cases we develop the determinant of (6.10) repeatedly by the first $j - 1$ rows. For $s = \{s_1, \ldots, s_{j-1}\} \subset \{1, \ldots, i\}$,$\# s = j - 1$, we let $s^c = \{1, \ldots, i\} \backslash s$ be

the complement and $M_s = M_{\{\tau_j,\ldots,\tau_i\}\times s^c}$ be the associated submatrix of M, so we get:

$$(6.10) \Leftrightarrow 0 = \sum_{s\in\mathcal{S}} \left(\prod_{r=1}^{j-1} m_{\tau_r,s_r}(-1)^{s_r}\right) \det M_s,$$

where we sum over $\mathcal{S} = \big\{(s_1,\ldots,s_{j-1}) \subset \{1,\ldots,i\}^{j-1}; \#\{s_1,\ldots,s_{j-1}\} = j-1\big\}$. But by (6.12) and the multi-linearity of the determinant, we find that all the above $(i-j+1)\times(i-j+1)$-matrices vanish:

$$\det(M_s) = 0 \text{ for all } s \subset \{1,\ldots,i\}, \#s = j-1.$$

This shows (6.10). Altogether this would also prove Proposition 6.7. $\square$

Next, we want to restrict to $X(w)^{(0)}$, so we want to describe the affine coordinate rings for the affine cover $\{\mathcal{U}_v^{(0)}\}$. Therefore we define

$$\begin{aligned}
\dot{R}_w^{(0)} &:= k[X_{ij}]_{j<i}[L_w][T_i, S_1],\\
I_w^{(0)} &:= \left(I_w + (L_w l_w - 1) + (S_1 \prod_i T_i - 1)\right) \subset \dot{R}_w^{(0)},\\
R_w^{(0)} &:= \dot{R}_w^{(0)}/I_w^{(0)} \text{ and}\\
R_w^{loc,(0)} &:= R_w^{loc}[T_i, S_1]/(S_1 \prod_i T_i - 1).
\end{aligned}$$

Furthermore, we define

$$\begin{aligned}
\dot{R}_w^{(i)} &:= \dot{R}_w^{(0)}[D_1,\ldots,D_i],\\
I_w^{(i)} &:= \left(I_w^{(0)}\right) \subset \dot{R}_w^{(i)},\\
R_w^{(i)} &:= \dot{R}_w^{(i)}/I_w^{(i)} \text{ and}\\
R_w^{loc,(i)} &:= R_w^{loc,(0)}[D_1,\ldots,D_i].
\end{aligned}$$

We see from

$$\Gamma(\mathcal{U}_v^{(0)} \cap X(w)^{(0)}, \mathcal{O}_{X(w)^{(0)}}) = \Gamma(\mathcal{U}_v \cap X(w), \mathcal{O}_{X(w)}) \otimes k[T_1,\ldots,T_n]_{T_1\cdots T_n}$$

that $\Gamma(\mathcal{U}_{id}^{(0)} \cap X(w)^{(0)}, \mathcal{O}_{X(w)^{(0)}}) = R_w^{(0)}$.

Next, we want to extend the maps φ_v to endomorphisms $\varphi_v^{(i)}$ on $R_w^{loc,(i)}$. By that, we will also find representatives $\left(g_{id,v}^{(i)}\right)_{v^{-1}} \in \prod_{id\neq v^{-1}} \varphi_{v^{-1}}^{(i)}(R_w^{(i)})$ of the cocycles corresponding to the torsors τ_i. For this, let us fix a reduced expression $w_0 = s_{i_1}\ldots s_{i_N}$, set $v_j = s_{i_1}\ldots s_{i_{N-j}}$, define $U_j = \prod_{\alpha>0, v_j(\alpha)<0} U_\alpha$ and let $\tau_j : G/U_j \to G/U_{j-1}$ be the attached torsors. Denote by $\alpha_j \in \Phi^+$ the unique positive root such that $v_{j-1}(\alpha_j) < 0$ and $v_j(\alpha_j) > 0$. We will consider $\mathbb{G}_a \cong U_{j-1}/U_j \cong U_{\alpha_j}$ as a subgroup of $U = \prod_{1\leq j\leq N} U_{\alpha_j}$, where we fix an order of the product by the reduced expression of w_0 above.

The general strategy to determine the cocycle attached to τ_i is as follows. We only determine the restricted cocycle in the sense of Lemma 5.13. We can calculate it without restricting to $X(w)$. As τ_i is a $\mathbb{G}_a$-torsor, we have the following diagram:

$$\begin{array}{ccccc}
G/U_i & \hookleftarrow & \mathcal{U}_v^{(i)} & \xrightarrow[\psi_v^{(i)}]{\sim} & \mathcal{U}_v^{(i-1)} \times U_{\alpha_i}\\
\downarrow{\scriptstyle \tau_i} & \square & \downarrow & \swarrow{\scriptstyle pr_1} & \\
G/U_{i-1} & \hookleftarrow & \mathcal{U}_v^{(i-1)} & &
\end{array}$$

Let $V := \mathcal{U}_{id}^{(i-1)} \cap \mathcal{U}_v^{(i-1)}$ and let $R^{(v)(i-1)} = \Gamma(V, \mathcal{O}_V)$. Then the isomorphism $(\psi_{v|V}^{(i)}) \circ (\psi_{id|V}^{(i)})^{-1} \in \mathrm{Isom}(\mathbb{G}_a, \mathbb{G}_a)(V)$ corresponds to an isomorphism of $R^{(v)(i-1)}$-algebras

$$R^{(v)(i-1)}[D_i] \to R^{(v)(i-1)}[D_i], \quad D_i \mapsto h_{id,v}^{(i)} D_i + \dot{g}_{id,v}^{(i)},$$

where $\dot{g}_{id,v}^{(i)} \in R^{(v)(i-1)}$ and it is easy to check that $h_{id,v}^{(i)} = 1$. But then $\left(\dot{g}_{id,v}^{(i)}\right)_{v^{-1}}$ is the (restricted) cocycle attached to τ_i.

For simplicity we define $U_{\alpha_0} := T$. For all $v \in W$ and all $0 \leq j \leq N$, we identify

$$\mathcal{U}_v^{(j)} \cong vU^- \times U_{\alpha_0} \times \ldots \times U_{\alpha_j}$$

as usual. That is, given gU_j, $g \in G$, there exist unique $x^{(v)}(g) \in U^-$ and $d_r^{(v)}(g) \in U_{\alpha_r}$, $0 \leq r \leq j$, such that

$$gU_j = vx^{(v)}(g)d_0^{(v)}(g) \cdots d_j^{(v)}(g)U_j.$$

We now define

$$\psi_v^{(i)}(gU_i) = \left(vx^{(v)}(g)d_0^{(v)}(g) \cdots d_{i-1}^{(v)}(g)U_{i-1}, d_i^{(v)}(g)\right).$$

Recall that we have defined matrices $x, x^{(v)} \in U^-$, $b^{(v)} \in B$ such that $x = vx^{(v)}b^{(v)}$. We can write $u_0^{(v)} := (b^{(v)})^{-1} = \tilde{d}_0^{(0)} \cdots \tilde{d}_N^{(0)} \in \prod_{j \geq 0} U_{\alpha_j}$ uniquely. For $1 \leq j \leq N$, denote by $d_j \in U_{\alpha_j}$ the matrix with non-trivial entry D_j, denote by $d_0 \in T$ the matrix $\mathrm{diag}(T_i)$, and let $d_0^{(v)} = \tilde{d}_0^{(0)} d_0$.

For $1 \leq i \leq N$, we define

$$u_i^{(v)} := (d_{i-1}^{(v)})^{-1} u_{i-1}^{(v)} d_{i-1} = \tilde{d}_i^{(i)} \cdots \tilde{d}_N^{(i)} \in \prod_{j \geq i} U_{\alpha_j} \text{ and } d_i^{(v)} := \tilde{d}_i^{(i)} d_i \tag{6.13}$$

and get $d_i^{(v)} \in U_{\alpha_i}$, $u_i^{(v)} \in U_i$ and

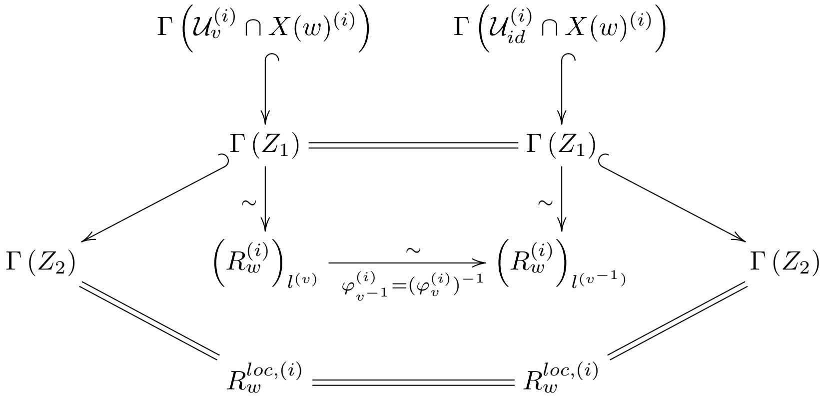
$$xd_0 \cdots d_i = vx^{(v)} d_0^{(v)} \cdots d_i^{(v)} u_i^{(v)}.$$

As we stated above, the non-trivial entry of the matrix $d_i^{(v)} \in U_{\alpha_i}$ is of the form $D_i + \dot{g}_{id,v}^{(i)}$ for some $\dot{g}_{id,v}^{(i)} \in R^{(v),(i-1)}$ and $\left(\dot{g}_{id,v}^{(i)}\right)_{v^{-1}}$ is a representative for the (restricted) cocycle attached to τ_i. We write $d_0^{(v)} = \mathrm{diag}(t_j^{(v)})$.

Now, we restrict to $X(w)$. By extending φ_v by $T_j \mapsto t_j^{(v)}$ we get an endomorphism $\varphi_v^{(0)}$ on $R_w^{loc,(0)}$. We are now able to extend $\varphi_v^{(i-1)}$ to a map $\varphi_v^{(i)}$ on $R_w^{loc,(i)}$, where the image of D_i is given by $D_i + g_{id,v}^{(i)}$. Here $g_{id,v}^{(i)} \in R_w^{loc,(i)}$ denotes the canonical image of $\dot{g}_{id,v}^{(i)}$.

This induces isomorphisms of the coordinate rings in the following commutative diagram, where we denote for short by $\Gamma(Z)$ the coordinate ring $\Gamma\left(Z, \mathcal{O}_{X(w)^{(i)}}\right)$ for any subscheme Z of $X(w)^{(i)}$. For the moment we define $Z_1 = \mathcal{U}_{id}^{(i)} \cap \mathcal{U}_v^{(i)} \cap X(w)^{(i)}$

and $Z_2 = \bigcap_v \mathcal{U}_v^{(i)} \cap X(w)^{(i)}$. We have:



Now, we would like to exhaust all occurring rings to get a finite linear problem. Set $\dot{R}_w^{loc} = (\dot{R}_w)[L_f]_{f\in\mathcal{D}}$. We will define a lift $\dot{\varphi}_v : \dot{R}_w \to \dot{R}_w^{loc}$ of φ_v as follows. Recall that in the proof of Lemma 6.1 we saw, how to compute for $x = (X_{ij}) \in U^-(\dot{R}_w^{loc})$, $v \in W$ and $A = v^{-1}x$ some (unique) $b^{(v)} \in B(\dot{R}_w^{loc})$ such that $Ab^{(v)} \in U^-(\dot{R}_w^{loc})$. The ith column of $b^{(v)}$ is given by $(b_{li}^{(v)})_l = (A^{(i)})^{-1}y^{(i)}$, where $y^{(i)} = (0,\ldots,0,1)^t \in k^i$ and $A^{(i)}$ denotes the submatrix $x_{\{v(1),\ldots,v(i)\}\times\{1,\ldots,i\}}$ of A. Denote for any matrix M by $\mathrm{Adj}(M)$ the adjoint matrix in the sense of Cramer's rule and let $l_i^{(v)} = \det(x_{\{v(1),\ldots,v(i)\}\times\{1,\ldots,i\}}) \in \mathcal{D}$. Then we have $(A^{(i)})^{-1} = (l_i^{(v)})^{-1}\mathrm{Adj}(A^{(i)})$. If we replace $(l_i^{(v)})^{-1}$ by the variable $L_{l_i^{(v)}}$, we have

$$(b_{li}^{(v)})_l := L_{l_i^{(v)}}\mathrm{Adj}(A^{(i)})y^{(i)} \in \dot{R}_w^{loc}, \text{ for every } 1 \le l \le i.$$

But then it follows that $x^{(v)} = v^{-1}x(b^{(v)})^{-1} \in U^-(\dot{R}_w^{loc})$ and we define the image of X_{ij} under $\dot{\varphi}_v$ to be $x_{ij}^{(v)}$.

Defining the image of L_w under some lift is more subtle (and less natural). For the moment, let $\dot{\varphi}_v$ denote the lift described above for the variables X_{ij} and $L_{l_i^{(v^{-1})}}$. As $\dot{\varphi}_v(l_w^{-1}) \in k[X_{ij}]_{l_w l^{(v)}}/(I_w)$, it is always possible to find some $h \in k[X_{ij}][L_{l_i^{(v)}}]$ such that $l_w \equiv h\dot{\varphi}_v(l_w) \mod (I_w)$. We choose such an h and set $\dot{\varphi}_v(L_w) = L_w h$.

Now, define a lift of $\varphi_v^{(i)}$. For this, observe that every $t_i^{(v)}$ can be written as $t_i^{(v)} = T_i^{-1}h$ for some $h \in \dot{R}_w^{(v)}[T_j, S_1]_{j\neq i}$. Therefore we set $\dot{\varphi}_v^{(i)}(T_l) = hS_1\prod_{j\neq l} T_j$.

Actually, we cannot control how the degrees of $\varphi_v^{(i)}(f)$ raise for any $f \in \Omega^{m_2^i} = \{f \in \dot{R}_w^{loc}; \deg(f) \le m_2^i\}$, $m_2^i \in \mathbb{N}$. But if we fix some lift $\dot{\varphi}_v{}^{(i)}$ for fixed i we have the following:

(i) For every $m_2^i \in \mathbb{N}$ there exists $m_1^i \in \mathbb{N}$ such that for every $f \in \Omega^{m_2^i}$ we have $\dot{\varphi}_v^{(i)}(f)$ has total degree at most m_1^i.

(ii) $Q^{m_1^i} := \left\langle \dot{\varphi}_v^{(i)}(f); f \in \Omega^{m_2^i} \right\rangle_{k-vec} \subset Q^{m_1^i+1} \subset \dot{R}_w^{loc,(i)}$

(iii) $\bigcup_{m\in\mathbb{N}} Q^{m^i} = \dot{R}_w^{loc,(i)}$

(iv) Let $J^{m^i} := Q^{m^i} \cap I_w^{(i)}$, then we have $Q^{m^i}/J^{m^i} \subset Q^{m^i+1}/J^{m^i+1} \subset \dot{R}_w^{loc,(i)}$ and $\bigcup_{m\in\mathbb{N}} Q^{m^i}/J^{m^i} = \dot{R}_w^{loc,(i)}$.

Thus we have:

$$(5.14) \Leftrightarrow \forall i \exists m_1^i, m_2^i \in \mathbb{N} : \emptyset \neq \bigcap_{v\in W} \left(\varphi_{v^{-1}}^{(i)}(\Omega^{m_2^i}/(\Omega^{m_2^i} \cap J^{m_2^i})) - g_{id,v}^{(i)}\right) \subset Q^{m_1^i}/J^{m_1^i}$$

Now we can view $Q^{m_1^i}/J^{m_1^i}$ as a k-vector space. We therefore have to solve a finite, linear problem: Deciding whether the intersection of affine subspaces is empty or not. If this intersection is non-empty, the cocycle attached to τ_i is a coboundary. Define $I_w^{loc,(i)}$ through $R_w^{loc,(i)} = \dot{R}_w^{loc,(i)}/I_w^{loc,(i)}$.

Moreover, we see from the above descriptions that everything is defined over $k_0 = \mathbb{F}_q$, and even over $\mathbb{F}_p$, where p is the characteristic of k_0. Therefore, it is enough to consider all rings above as defined over $\mathbb{F}_p$ and then consider the attached $\mathbb{F}_p$-vector spaces. In the implementation, we use this point of view.

For a fixed filtration we give a slightly more precise algorithm:

Step 1 (Rings and Maps)
Set $i = 0$. Calculate the ideal $I_w^{(i)}$.
Calculate the ideal $I_w^{loc,(i)}$.
Calculate the transition functions $\dot{\varphi}_v^{(i)}$.

Step 2 (Cocycles)
Calculate $(g_{id,v}^{(i)})$ by using Step 1.4.

Step 3 (Linearize)
Choose bounds m_1^i and m_2^i and exhaust $\varphi_v^{(i)}(R_w)$ by taking as generators all images of monomials of degree at most m_2^i in $\dot{R}_w$ under the maps $\dot{\varphi}_v^{(i)}$. Then transfer to the quotient ring.
Exhaust $R_w^{loc,(i)}$ by taking as generators all monomials of degree at most m_1^i in $\dot{R}_w^{loc,(i)}$ an transfer to the quotient ring.
Calculate an isomorphism to the k-vector space $Q^{m_1^i}/J^{m_1^i}$ and map all subrings and the cocycles into this space.
Calculate the intersection of the occurring affine spaces.
In case of emptiness, break and return ''Cannot decide whether $X(w)$ is affine or not.''
Else store an element of the intersection and repeat Step 1-3 for $i = i + 1$ by using former calculations in Step 1 and Step 2.

Step 4 (Review)
If you are not sure, whether the chosen bounds fulfill the conditions above, check whether the element of the intersection, transferred back to $R_w^{loc,(i)}$, in fact shows the triviality of τ_i.

7. Examples

Example 7.1. *Let $n = 2$, $G = \mathbb{GL}_2$, $k_0 = \mathbb{F}_q$, and B, T, U as above. So we get $W = \mathbb{S}_2 = \{id, s_1\}$ and since $U \cong \mathbb{G}_a$, in $G \xrightarrow{\psi} G/U \xrightarrow{\pi} G/B$ we have that $\psi = \tau_1$ is already a $\mathbb{G}_a$-torsor. We take $w = s_1$ and calculate for $\dot{x} \in U^-(k)$:*

$$\dot{x}^{-1}F(\dot{x}) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ x^q & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^q - x & 1 \end{pmatrix}.$$

This is in Bs_1B if and only if $x^q - x \neq 0$. For $\dot{x} \in U^-(k)$, we can find a $b \in B(k)$ and some $y \in k$ with $(s_1)^{-1}\dot{x}b = \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}$ if and only if $\begin{pmatrix} y & 1 \\ 1+xy & x \end{pmatrix} \in B(k)$. That is equivalent to saying that $y = -x^{-1} \in k$ is invertible. So we get:

$$\begin{aligned} X(w) \cap \mathcal{U}_{id} &= \left\{ \dot{x} \in U^- | x^q - x \neq 0 \right\} /B \\ X(w) \cap \mathcal{U}_{s_1} &= \left\{ s_1\dot{y} \in U^- | y^q - y \neq 0 \right\} /B \\ X(w) \cap \mathcal{U}_{id} \cap \mathcal{U}_{s_1} &= \left\{ \dot{x} \in U^- | x^q - x \neq 0 \wedge x \neq 0 \right\} /B \end{aligned}$$

Let $\dot{x} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in U^-(k), t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in T(k)$. To describe $X(w)^{(0)} \cap \mathcal{U}_{id}^{(0)} \cap \mathcal{U}_{s_1}^{(0)}$, we have to find some $u' \in U(k)$ such that there exist (unique) $\dot{y} = \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} \in s_1U^-(k)$ and $t' = \begin{pmatrix} t_1' & 0 \\ 0 & t_2' \end{pmatrix} \in T(k)$ with $\dot{x}tu' = \dot{y}t'$. We know from the calculation above that $y = -x^{-1}$, so

$$u' = t^{-1}\dot{x}^{-1}\dot{y}t' = \begin{pmatrix} -t_1^{-1}t_1'x^{-1} & t_1^{-1}t_2' \\ 0 & t_2^{-1}t_2'x \end{pmatrix}.$$

As $u' \in U$, $t_1' = -t_1x$ and $t_2' = t_2x^{-1}$. Thus we get the following descriptions on G/U:

$$\begin{aligned} \dot{R}_{s_1} &= k[X, H] \\ I_{s_1} &= (H(X^q - X) - 1) \\ \dot{R}_{s_1}^{(0)} &= k[X, H, T_1, T_2, S] \\ I_{s_1}^{(0)} &= (H(X^q - X) - 1, ST_1T_2 - 1) \\ \dot{R}_{s_1}^{(id,s_1),(0)} = \dot{R}_{s_1}^{loc,(0)} &= k[X, H, L, T_1, T_2, S] \\ I_{s_1}^{loc,(0)} &= (H(X^q - X) - 1, ST_1T_2 - 1, XL - 1) \\ \varphi_{s_1}^{(0)} &= \{X \mapsto -L, H \mapsto -X^{q+1}H, T_1 \mapsto -T_1X, T_2 \mapsto T_2L, S \mapsto -S\} \\ (g_{id,v}^{(1)})_v &= (0, LT_2^2S) \\ \varphi_{s_1,id}^0(g_{id,s_1}^{(1)}) &= LT_2^2S = g_{id,s_1}^{(1)} \end{aligned}$$

But since $\varphi_{s_1}^{(0)}(-T_2^2SHX) = T_2^2SHL = g_{id,s_1}^{(1)}$ we have

$$\tilde{\partial}^{1,(0)}((-(X^{q-1} - 1)T_2^2SH, 0)) = (g_{id,v}^{(1)})_v.$$

So $(g_{id,v}^{(1)})_v$ is a coboundary. This is the result of the implemented algorithm, but actually we could have seen more easily that this is a coboundary: From the condition $X^q \neq X$ in $R_{s_1}^{(0)}$ we get especially that X is already invertible, since $(X^{q-1} - 1)H \equiv (X^{q-1} - 1)/(X^q - X) \equiv 1/X$. Therefore $(\dot{f}_{id,v})_v = (0, LT_2^2S)$ would work as well. But we see that for computational confirmation of the affineness of $X(s_1)$, we have to exhaust the rings $\dot{R}_{s_1}^{(0)}$ and $\dot{R}_{s_1}^{loc,(0)}$ at least up to degree q+3.

Now we give another example to discuss another approach on proving the affineness:

Example 7.2. *Let* $n = 3$*,* $G = \mathbb{GL}_3$*,* $k_0 = \mathbb{F}_q$*, and* B, T, U *as above. So we get* $W = \mathbb{S}_3 = \{id, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$*, where we fix exactly this order on* W*, whenever we use* W *as an index set. We take the filtration* $U_0 = \{0\} \triangleleft U_1 = \{u \in U; u_{1,2} = u_{1,3} = 0\} \triangleleft U_2 = \{u \in U; u_{1,2} = 0\} \triangleleft U_3 = U$*. We can do most calculations of Step 1 without restricting to the case of a fixed* $w \in W$*:*

Let $x := \begin{pmatrix} 1 & 0 & 0 \\ X_1 & 1 & 0 \\ X_2 & X_3 & 1 \end{pmatrix}$ *and* $t := \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}$*. For every* $v \in W$ *we make an easy calculation to get the unique* $x^{(v)} \in U^-$*,* $t^{(v)} \in T$ *and* $u^{(v)} \in U$ *such that* $xt = vx^{(v)}t^{(v)}u^{(v)}$*. We summarize this in the following table, where we identify the 3-dimensional groups* U^-, T *and* U *with (subsets of)* k^3 *in the following way:* $r \in U^- \mapsto (r_{21}, r_{31}, r_{32})$*,* $r \in T \mapsto (r_{11}, r_{22}, r_{33})$ *and* $r \in U \mapsto (r_{12}, r_{13}, r_{23})$*.*

s_1 :

$$u^{(v)} \mathrel{\widehat{=}} (T_2/(T_1X_1), 0, 0)$$
$$t^{(v)} \mathrel{\widehat{=}} (T_1X_1, T_2/X_1, T_3)$$
$$x^{(v)} \mathrel{\widehat{=}} (1/X_1, X_2/X_1, X_2 - X_1X_3)$$

s_2 :

$$u^{(v)} \mathrel{\widehat{=}} (0, 0, -T_3/X_3)$$
$$t^{(v)} \mathrel{\widehat{=}} (T_1, T_2X_3, -T_3/X_3)$$
$$x^{(v)} \mathrel{\widehat{=}} (X_2, X_1, 1/X_3)$$

s_1s_2 :

$$u^{(v)} \mathrel{\widehat{=}} (T_2/(T_1X_1), 0, T_3X_1/(T_2(X_1X_3 - X_2)))$$
$$t^{(v)} \mathrel{\widehat{=}} (T_1X_1, T_2(X_1X_3 - X_2)/X_1, T_3/(X_1X_3 - X_2))$$
$$x^{(v)} \mathrel{\widehat{=}} (X_2/X_1, 1/X_1, 1/(X_1X_3 - X_2))$$

s_2s_1 :

$$u^{(v)} \mathrel{\widehat{=}} (T_2X_3/(T_1X_3), T_3/(T_1X_2), T_3/(T_2X_3))$$
$$t^{(v)} \mathrel{\widehat{=}} (T_1X_2, -T_2X_3/X_2, -T_3/X_3)$$
$$x^{(v)} \mathrel{\widehat{=}} (1/X_2, X_1/X_2, (X_1X_3 - X_2)/X_3)$$

$s_1s_2s_1$:

$$u^{(v)} \mathrel{\widehat{=}} (T_2X_3/(T_1X_2), T_3/(T_1X_2), T_3X_1/(T_2(X_1X_3 - X_2)))$$
$$t^{(v)} \mathrel{\widehat{=}} (T_1X_2, -T_2(X_1X_3 - X_2)/X_2, T_3/(X_1X_3 - X_2))$$
$$x^{(v)} \mathrel{\widehat{=}} (X_1/X_2, 1/X_2, X_3/(X_1X_3 - X_2))$$

We have $U_3/U_2 \cong \mathbb{G}_a \circlearrowright (G/U_2 \cap \mathcal{U}_v^{(1)})$ *by multiplication from the right:* $xU_2.\alpha = x \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} U_2$*,* $U_2/U_1 \cong \left\{ \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \circlearrowright (G/U_1 \cap \mathcal{U}_v^{(2)})$ *and* $U_1/U_0 \cong \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \right\} \circlearrowright G \cap \mathcal{U}_v^{(3)}$ *likewise.*

We have $\mathcal{U}_{id}^{(0)} = \{xt; x \in U^-, t \in T\}/U_3 \cong U^- \times T$ *and*

$$\forall v \in W, xtU \in \mathcal{U}_{id}^{(0)} \cap \mathcal{U}_v^{(0)} \exists! u^{(v)} \in U_3, y \in U^-, s \in T : xtu^{(v)} = vys$$

Let $x \in U^-, t \in T$. *Now, since* τ_1 *is a* U_3/U_2*-torsor, there exists for every* $\alpha \in U_3/U_2 \cong \mathbb{G}_a$ *a unique* $\alpha' \in U_3/U_2$ *with*

$$xtU_2.\alpha = xt\alpha U_2 = vys\alpha' U_2 = vysU_2.\alpha' \Leftrightarrow \alpha' U_2 = (s^{-1}y^{-1}v^{-1}xt)\alpha U_2,$$

as $U_2 \lhd U_3$. *But* $s^{-1}y^{-1}v^{-1}xt = (u^{(v)})^{-1} \in U_3$, *so*

$$\begin{pmatrix} 1 & \alpha' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} U_2 = (u^{(v)})^{-1} \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} U_2 \Leftrightarrow \alpha' = (u^{(v)})^{-1}_{1,2} + \alpha = \alpha - u^{(v)}_{1,2}.$$

One can easily check that for $xt\alpha U_1 \in \tau_1^{-1}(\mathcal{U}_{id}^{(0)} \cap \mathcal{U}_v^{(0)})$ *and for* $\beta \in U_2/U_1$, *the unique* $\beta' \in U_2/U_1$ *with* $xt\alpha U_1.\beta = vys\alpha' U_1.\beta'$ *is*

$$\beta' = \beta + \alpha u^{(v)}_{2,3} - u^{(v)}_{1,3}.$$

Analogously, we calculate for $\gamma, \gamma' \in U_1/U_0 = U_1$:

$$\gamma' = \gamma - u^{(v)}_{2,3}.$$

For the moment, we set $\dot{R}^{loc} = k[X_1, X_2, X_3, T_1, T_2, T_3]_{X_1X_3(X_1X_3-X_2)T_1T_2T_3}$. *Observe that* $\mathcal{D}$ *does not contain the minor* X_2. *To summarize, we have the following representatives for the cocycles attached to the torsors* τ_i:

$$\left(g^{(1)}_{id,v}\right)_{v^{-1}} = (0, h_1, 0, h_1, h_2, h_2) \in \prod_v \dot{R}^{loc},$$
$$\text{where } h_1 := -T_2/(T_1X_1), h_2 := -T_2X_3/(T_1X_2).$$
$$\left(g^{(2)}_{id,v}\right)_{v^{-1}} = (0, 0, D_1h'_1, D_1h'_3, h'_2 + D_1h'_1, h'_2 + D_1h'_3) \in \prod_v \dot{R}^{loc}[D_1],$$
$$\text{where } h'_1 := T_3/(T_2X_3), h'_2 := -T_3/(T_1X_2), h'_3 := T_3X_1/(T_2(X_1X_3 - X_2)).$$
$$\left(g^{(3)}_{id,v}\right)_{v^{-1}} = (0, 0, h''_1, h''_2, h''_1, h''_2) \in \prod_v \dot{R}^{loc}[D_1, D_2], \text{ where } h''_1 := h'_1, h''_2 := h'_3.$$

The transition functions $\varphi^{(i)}_v$, *given for every* $v \in W$ *by the matrices* $x^{(v)}$ *and* $t^{(v)}$, *just permute (up to a sign) the entries of the* $(g^{(i)})_{id,v}$*'s. Now we can restrict to* $X(w)$ *for some* $w \in W$, *by just modding out some conditions in* $\dot{R}^{loc}$, *resp. in* $k[X_1, X_2, X_3, T_1, T_2, T_3]_{T_1T_2T_3}$.
So, let $w = s_2s_1$ *be a Coxeter element. Then we already know that* $X(s_2s_1)$ *is affine, see Remark 2.7 or Proposition 5.13. The conditions for* $\Gamma(X(w), \mathcal{O}_{X(w)})$ *are* $X_1^q - X_1 \neq 0$, $X_3^q - X_3 \neq 0$ *and* $f_w := X_2^q - X_2 - X_3(X_1^q - X_1) = 0$. *Therefore we see that in*

$$\Gamma(X(w)^{(0)}, \mathcal{O}_{X(w)^{(0)}}) = \left(k[X_1, X_2, X_3, T_1, T_2, T_3]_{T_1T_2T_3(X_1^q - X_1)(X_3^q - X_3)}\right) / (f_w)$$

we actually can invert not just X_1 *and* X_3, *but also* $(X_1X_3 - X_2)^q$, *since otherwise*

$$\begin{aligned} 0 &= (X_1X_3 - X_2)^q - (X_1X_3 - X_2) = (X_1X_3)^q - X_2^q + X_2 - (X_1X_3) \\ \Leftrightarrow 0 &= (X_1X_3)^q - X_3(X_1^q - X_1) - (X_1X_3) = X_1^q(X_3^q - X_3). \end{aligned}$$

So we see that all the cocyles are trivially coboundaries, e.g. the cocycle

$$\left(f^{(i)}_v\right)_v := \left(-\varphi^{(i)}_v(g^{(i)}_{id,v^{-1}})\right)_v$$

maps to $g^{(i)}$.

To look for a more subtle example, we now set $w = s_1 s_2 s_1$, the longest element in W. We get the conditions $X_2^q - X_2 - X_3(X_1^q - X_1) \neq 0$ and $X_2^q - X_2 - X_3^q(X_1^q - X_1) \neq 0$, so one can check (by using the function `subalgebra_containment` from SINGULAR for example) that neither X_1 or X_3, nor $X_1X_3 - X_2$ are invertible in $\Gamma(X(w), \mathcal{O}_{X(w)})$, so we really have to calculate something. To this end, we will embed $X(w)$ into some projective space and see that we can regard $X(w)$ as principal open subset, so that it is affine. Moreover, we will use Serre's affineness criterion to compute cocycles that are mapped to $(g^{(1)}_{id,v^{-1}})_v$ under the first coboundary operator.

Lemma 7.3. *We have an isomorphism of varieties*

$$p : G/B \to X' := \{(x, y); x \cdot y = 0\} \subseteq \mathbb{P}^2 \times \mathbb{P}^2,$$

where the vanishing of $x \cdot y := x_0y_0 + x_1y_1 + x_2y_2$ is independent of the chosen representatives $(x_0, x_1, x_2), (y_0, y_1, y_2)$ for $x, y \in \mathbb{P}^2$.

Proof. To define the isomorphism on closed points, let $A = (a_{ij}) \in G(k)$ and define $p(A) := ([a_{i1}], [(a_{i1}) \times (a_{i2})])$, where $\times$ denotes the usual cross or vector product in k^3. Since A is invertible, this forces $p(A) \in \mathbb{P}^2 \times \mathbb{P}^2$. We have $x(A) \cdot y(A) = 0$, where the latter equality is independent of the chosen representatives. Furthermore, by a straight-forward computation one sees that for every $b \in B(k)$ we have $p(Ab) = p(A)$, so p is well defined.
On the other hand, we give an inverse map. Let $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2$ with $x \cdot y = 0$. It is enough to give the construction for (x, y) contained in the open chart $\{(x, y) \in X'; x_0 \neq 0\}$. We now chose representatives $(1, x_1, x_2)$ and (y_0, y_1, y_2). We have to find a solution $\underline{a}^{(2)}$ for the following system of linear equations

$$\begin{pmatrix} 0 & -x_2 & x_1 \\ x_2 & 0 & -1 \\ -x_1 & 1 & 0 \end{pmatrix} \underline{a}^{(2)} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ x_2 & 0 & -1 \\ -x_1 & 1 & 0 \end{pmatrix} \underline{a}^{(2)} = \begin{pmatrix} y_0 + x_1y_1 + x_2y_2 \\ y_1 \\ y_2 \end{pmatrix}.$$

The equivalence is given by simple Gauss transformation on the rows and we have $0 = x \cdot y = y_0 + x_1y_1 + x_2y_2$. So we see that $\underline{a}^{(2)} = (0, y_2, -y_1)^t$ is a solution. We set

$$p^{-1}((x, y)) := A(x, y) := \left(x_i, \underline{a}^{(2)}_i, ((x_j)_j \times (\underline{a}^{(2)}_j)_j)_i\right)_{0 \leq i \leq 2}.$$

The point $A(x, y)B$ is uniquely determined and independent of the chosen representative for y. □

To describe the image of $X(w)$ under this isomorphism, we regard G/B as the variety of full flags of k^3. Now $X(w)$ contains exactly the flags $\mathcal{F} = (F_1, F_2, F_3)$ such that the relative position of $\mathcal{F}$ and $F(\mathcal{F})$ is w. We have

$$\begin{aligned} & w = \mathrm{inv}(\mathcal{F}, F(\mathcal{F})) \\ \Leftrightarrow & \dim(F_1 \cap F(F_2)) = 0 \wedge \dim(F(F_1) \cap F_2) = 0 \\ \Leftrightarrow & \dim(\langle F_1, F(F_2)\rangle) = 3 \wedge \dim(\langle F(F_1), F_2\rangle) = 3. \end{aligned}$$

The last equalities just mean that for any generators $(a_i)_i$ and $((a_i)_i, (b_i)_i)$ for F_1 and F_2 (independent of the choice) the following determinants do not vanish:

$$\det(((a_i), (a_i^q), (b_i^q))) \neq 0 \wedge \det(((a_i^q), (a_i), (b_i))) \neq 0$$

Back to our projective interpretation, we can express the last equalities as $x \cdot F(y) \neq 0$ and $y \cdot F(x) \neq 0$. Thus we get the following

Lemma 7.4. *Under the isomorphism p above we have*

$$p(X(w)) = \{(x, y) \in X'; x \cdot F(y) \neq 0 \wedge F(x) \cdot y \neq 0\}.$$

Consider the Segre embedding $sg : \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$, $(x,y) \mapsto [x_i y_j]_{3i+j}$. We let $z := \begin{pmatrix} z_0 & z_1 & z_2 \\ z_3 & z_4 & z_5 \\ z_6 & z_7 & z_8 \end{pmatrix}$, denote by D the set of all $(2{\times}2)$-minors of z and let $l'_w := z_0^{q+1} + z_4^{q+1} + z_8^{q+1} + z_2 z_6^q + z_6 z_2^q + z_5 z_7^q + z_7 z_5^q + z_3 z_1^q + z_1 z_3^q$. Fix some $x = [(x_i)_i], y = [(y_i)_i] \in \mathbb{P}^2$. We set $z_l := x_i y_j, l = 3i + j, 0 \leq i, j \leq 2$. Under the morphism sg the equation $0 = x \cdot y$ corresponds to the equation $tr(z) = z_0 + z_4 + z_8 = 0$ and one can easily calculate that the inequation $0 \neq (x \cdot F(y))(F(x) \cdot y)$ corresponds to $0 \neq l'_w$. Thus we have

$$sg \circ p(X(w)) = \{[(z_l)_l]; \mathrm{tr}(z) = 0, \forall d \in D : d = 0, l'_w \neq 0\}.$$

Analogously, we let $Z := \begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_3 & Z_4 & Z_5 \\ Z_6 & Z_7 & Z_8 \end{pmatrix}$, we denote by D' the set of $(2{\times}2)$-minors of Z, let $L'_w := Z_0^{q+1} + Z_4^{q+1} + Z_8^{q+1} + Z_2 Z_6^q + Z_6 Z_2^q + Z_5 Z_7^q + Z_7 Z_5^q + Z_3 Z_1^q + Z_1 Z_3^q$ and $\mathcal{R}' = k[Z_0, \ldots, Z_8]/(\mathrm{tr}(Z) \cup \{D'\})$.

$\mathcal{R}'$ equipped with the usual grading is a graded ring and L'_w is homogeneous of degree $q+1$. Thus, if we attach to $\mathcal{R}'$ the projective spectrum Proj $\mathcal{R}'$, we can regard $X(w)$ as principal open subset $D_+(L'_w) = \mathrm{Spec}\ \mathcal{R}'_{(L'_w)}$, see [16], §(13.2). In particular, we see that $X(w)$ is affine.

Let us check explicitly that the cohomological criterion for affineness is in fact satisfied (at least for the first step). Inspired by the above discussion, we have an embedding (coming from the Plücker embedding of G/B)

$$\iota : G/U \hookrightarrow \mathbb{A}^3 \backslash \{0\} \times \mathbb{A}^3 \backslash \{0\} \times \mathbb{A}^1 \backslash \{0\},$$

$$(a_{ij}) \mapsto ((a_{11}, a_{21}, a_{31}), (a_{21}a_{32} - a_{22}a_{31}, a_{31}a_{12} - a_{11}a_{32}, a_{11}a_{22} - a_{12}a_{21}), \det(a_{ij})).$$

We get

$$\iota(G/U) = \{(x, y, t); x \cdot y = 0, x \neq 0, y \neq 0, t \neq 0\}$$

and

$$\iota(X(w)^{(0)}) = \{(x, y, t); x \cdot y = 0, F(x) \cdot y \neq 0, x \cdot F(y) \neq 0, x \neq 0, y \neq 0, t \neq 0\}.$$

Since $x = 0$, resp. $y = 0$, forces $F(x) \cdot y = 0$, the conditions $x \neq 0$ and $y \neq 0$ are automatically satisfied in $\iota(X(w)^{(0)})$. If we define $\widetilde{l}_w = (A_1^q B_1 + A_2^q B_2 + A_3^q B_3)(A_1 B_1^q + A_2 B_2^q + A_3 B_3^q)$, we have

$$\Gamma(\iota(X(w)^{(0)}), \mathcal{O}_{\iota(X(w)^{(0)})}) \cong \mathcal{R},$$

where

$$\mathcal{R} = (k[A_1, A_2, A_3, B_1, B_2, B_3][T]/(A_1 B_1 + A_2 B_2 + A_3 B_3))_{T\widetilde{l}_w}.$$

Locally we get $\Gamma(\iota(X(w)^{(0)} \cap \mathcal{U}_v^{(0)}), \mathcal{O}_{\iota(X(w)^{(0)})}) \cong \mathcal{R}_{l^{(v)}}$, where

$$(l^{(v)})_v = (A_1 B_3, A_2 B_3, A_1 B_2, A_2 B_1, A_3 B_2, A_3 B_1),$$

with $v \in W$ in the order fixed above. To proceed, we claim that $1 \in \left\langle l^{(v)} \right\rangle_{\mathcal{R}}$, e.g. $1 = \sum l^{(v)} h_v$ with

$$\begin{aligned}
h_{id} =& \frac{(A_3B_1)^q + A_3B_1(A_1B_3)^{q-1}}{\widetilde{l}_w} \\
h_{s_1} =& \frac{(A_3B_2)^q + A_3B_2(A_2B_3)^{q-1} + A_3B_2(A_2B_2)^{q-1} + A_3B_2(A_3B_3)^{q-1}}{\widetilde{l}_w} \\
h_{s_2} =& \frac{(A_2B_1)^q + A_2B_1(A_1B_2)^{q-1}}{\widetilde{l}_w} \\
h_{s_1s_2} =& 0 \\
h_{s_2s_1} =& 0 \\
h_{s_1s_2s_1} =& 0.
\end{aligned}$$

By following the lines of the (constructive) proof of Serre's affineness criterion, see [16], Lemma 12.33, we can easily calculate that the cocycle

$$(\iota(g^{(1)})_{id,v})_v = (0, B_3/(A_1A_2), 0, B_3/(A_1A_2), B_3B_2/(A_1A_3B_3), B_2/(A_1A_3)) =$$
$$\Big(0, B_3^3/(l^{(id)}l^{(s_1)}), 0, B_1B_3^2/(l^{(id)}l^{(s_1s_2)}), B_2^2B_3/(l^{(id)}l^{(s_2s_1)}), B_1B_2B_3/(l^{(id)}l^{(s_1s_2s_1)})\Big)$$

is in the image of the first coboundary operator, e.g. if we set $\widetilde{f}_{id}^{(1)} := h_{s_1}/l_{id}$ and the $\widetilde{f}_v^{(1)}$ deduced from this. Back to our main interpretation, with $l_w = (X_2^q - X_2 - X_3(X_1^q - X_1))\,(X_2^q - X_2 - X_3^q(X_1^q - X_1))$, we let

$$\begin{aligned}
\dot{f}_{id}^{(1)} :=& \iota^{-1}(\widetilde{f}_{id}^{(1)}) = T_2X_2X_3(X_1^{q-1} + (X_1X_3)^{q-1} + X_2^{q-1} + (X_2X_3)^{q-1})/(l_wT_1) \\
\equiv& T_2^2T_3SHX_2X_3(X_1^{q-1} + (X_1X_3)^{q-1} + X_2^{q-1} + (X_2X_3)^{q-1})
\end{aligned}$$

and $\dot{f}_v^{(1)} := \varphi_v^{(1)}(\dot{f}_{id}^{(1)} - g_{id,v^{-1}}^{(1)})$. Then we see that $(\dot{f}_v^{(1)})_v$ maps to $(g_{id,v^{-1}}^{(1)})_v$.

Remark 7.5. *In general, one cannot use this strategy for a computational decision of the affineness of $X(w)$, as in general we cannot calculate $\Gamma(X(w), \mathcal{O}_{X(w)})$.*

8. Remarks on the Implementation

We keep the notation of chapter 6. In this chapter, we will focus on the implementation of the affineness criterion Theorem 5.1 for G of type A. First, we will describe the maps φ_v and the cocycle attached to the torsor $G/U_1 \to G/U$ in a little more detail.

Lemma 8.1. *Let $G = \mathbb{GL}_n$ and $k_0 = \mathbb{F}_q$. Let B, T, U be as usual and let $U_1 = \{u \in U; u_{1,2} = 0\} \triangleleft U$.*

(i) Denote by $x = (x_{ij}) \in U^-\left(k[X_{ij}]_{j<i}\right)$ and $t \in T\left(k[X_{ij}]_{j<i}[T_i^{\pm 1}]\right)$ the matrices $x_{i,j} = X_{ij}, j < i$, and $t = \operatorname{diag}(T_i)$. Then denote by $x^{(v)} = (x_{ij}^{(v)}) \in U^-\left((k[X_{ij}]_{j<i})_{l^{(v)}}\right)$, $t^{(v)} = \operatorname{diag}(t_i^{(v)}) \in T\left(k[X_{ij}]_{j<i}[T_i^{\pm 1}]\right)$ and $u^{(v)} \in U\left((k[X_{ij}]_{j<i}[T_i^{\pm 1}]_{l^{(v)}})\right)$ the unique matrices such that $xtu^{(v)} = vx^{(v)}t^{(v)}$. Then the $\mathbb{G}_a$-torsor $G/U_1 \xrightarrow{\tau_1} G/U$ is given by the cocycle

$$(-u_{1,2}^{(v)})_v \in \prod_{v \in W} \left(k[X_{ij}]_{j<i}[T_i^{\pm 1}]\right)_{l^{(v)}},$$

where $l^{(v)} = \prod_{l=1}^{n-1} \det(x_{\{v(1),\dots,v(l)\}\times\{1,\dots,l\}}) \in k[X_{ij}]_{j<i}$.

(ii) Explicitly, $u_{1,2}^{(v)} = T_1^{-1} T_2 m_{1,2}^{(v)} (m_{2,2}^{(v)})^{-1}$, where $m^{(v)} = x^{-1} v x^{(v)} \in B^-\left(k[X_{ij}]_{l^{(v)}}\right)$.

(iii) Denote the isomorphism $\left(k[X_{ij}]_{j<i}[T_i^{\pm 1}]\right)_{l^{(v^{-1})}} \xrightarrow{\sim} \left(k[X_{ij}]_{j<i}[T_i^{\pm 1}]\right)_{l^{(v)}}$, given by $X_{ij} \mapsto x_{ij}^{(v)}$ and $T_i \mapsto t_i^{(v)}$, by $\dot\varphi_v$ and denote by $l(v)$ the Coxeter length of $v \in W$. Then we have

$$\dot\varphi_v(T_i) = T_i \tilde t_i^{(v)}, \text{ where } \tilde t_i^{(v)} \in (k[X_{ij}]_{j<i})_{l^{(v)}},$$

$$\dot\varphi_v(\prod_i T_i) = (-1)^{l(v)} \prod_i T_i,$$

$$\text{and for all } f \in (k[X_{ij}]_{j<i})_{l^{(v^{-1})}} : \dot\varphi_v(f) \in (k[X_{ij}]_{j<i})_{l^{(v)}}$$

.

Proof. For the first part, we refer to the examples for the cases $n = 2$ and $n = 3$. The case of $n \geq 4$ admits the same proof: On every open chart $\mathcal{U}_v^{(1)}$ we have an action of $U/U_1 \cong \{u \in U; u_{ij} = 0, 2 \leq i < j \leq n \wedge u_{1j} = 0, 3 \leq j \leq n\} \cong \mathbb{G}_a$ by multiplication from the right. Given $x, t, x^{(v)}, t^{(v)}, u^{(v)}$ and some $\alpha \in \mathbb{G}_a$ there exists a unique $\alpha^{(v)} \in \mathbb{G}_a$ such that

$$xtU_1.\alpha = vx^{(v)}t^{(v)}U_1.\alpha^{(v)}.$$

As $U_1 \triangleleft U$, this is equivalent to saying that $\alpha^{(v)}U_1 = (vx^{(v)}t^{(v)})^{-1}xt\alpha U_1$. Since $(vx^{(v)}t^{(v)})^{-1}xt = (u^{(v)})^{-1}$, the latter is equivalent to saying that $(u^{(v)})^{-1}\alpha U_1 = \alpha^{(v)}U_1$, so

$$\alpha^{(v)} = -u_{12}^{(v)} + \alpha.$$

For the second part, we observe that

$$t_i^{(v)} = T_i (m_{i,i}^{(v)})^{-1},$$

since $u_{i,i}^{(v)} = 1$. But then

$$u_{1,2}^{(v)} = T_1^{-1} T_2 m_{1,2}^{(v)} (m_{2,2}^{(v)})^{-1}.$$

The first equation of part three follows from (the proof of) part two, as we have an explicit description for $\dot\varphi_v$.
For the second equality, we observe that

$$1 = \det(u^{(v)}) = \underbrace{\det(t)^{-1}}_{=\prod T_i^{-1}} \cdot \underbrace{\det(x^{-1}vx^{(v)})}_{=\prod m_{i,i}^{(v)}} \cdot \underbrace{\det(t^{(v)})}_{=\dot\varphi_v(\prod T_i)}.$$

But $\prod m_{i,i}^{(v)} = (-1)^{l(v)} = \det(v)$, as x and $x^{(v)}$ are elements of U^- and therefore $\det(x^{-1}) = \det(x^{(v)}) = 1$. But then $\dot{\varphi}_v(\prod T_i) = \prod T_i(m_{i,i}^{(v)})^{-1} = (-1)^{l(v)} \prod T_i$. □

Lemma 8.2. *With the same notation as in Lemma 8.1, fix $w \in W$ such that $X(w)$ is integral. If we restrict to $X(w)^{(0)}$, i.e going to $\left(k[X_{ij}][T_i^{\pm 1}]/(I_w)\right)_{l_w}$, we have furthermore: If there exist $(f_v)_v \in \prod_v \left(k[X_{ij}][T_i^{\pm 1}]/I_w\right)_{l_w}$ such that $f_{id} - \varphi_v(f_v) = -u_{1,2}^{(v)}$ for all $v \in W$, then there exist $(\tilde{f}_v)_v \in \prod_v \left(k[X_{ij}]/I_w\right)_{l_w}$ such that $\tilde{f}_{id} - \varphi_v(\tilde{f}_v) = -T_1 T_2^{-1} u_{1,2}^{(v)}$.*

Proof. Let $S := \left(k[X_{ij}]/I_w\right)_{l_w}$ and let $(f_v)_v \in \prod_v S[T_i^{\pm 1}]$ such that for all $v \in W$

$$f_{id} - \varphi_v(f_v) = -u_{1,2}^{(v)} = T_1^{-1} T_2 \tilde{u}_{12}^{(v)}, \tag{8.3}$$

for some $\tilde{u}_{12}^{(v)} \in S_{l(v)}$. We write $f_v = T_1^{-1} T_2 \tilde{f}_v + f_v^{(2)}$ with $\tilde{f}_v \in S_{l(v)}$, so

$$(8.3) \Leftrightarrow T_1^{-1} T_2 \tilde{u}_{12}^{(v)} = T_1^{-1} T_2 \tilde{f}_{id} - \varphi_v(T_1^{-1} T_2 \tilde{f}_v) + f_{id}^{(2)} - \varphi_v(f_v^{(2)}).$$

By assumption, we have that S, and therefore $S[T_i^{\pm 1}]$, is integral, so we have

$$f_{id}^{(2)} - \varphi_v(f_v^{(2)}) \in T_1^{-1} T_2 S_{l(v)}.$$

Since for all $v \in W$ and for all $n_{\underline{i}} \in \mathbb{Z}^n$ we have $\varphi_v(T_{\underline{i}}^{n_{\underline{i}}}) = T_{\underline{i}}^{n_{\underline{i}}} h(n_{\underline{i}})$, for some $h(n_{\underline{i}}) \in S_{l(v)}$, and $\varphi_v(S) = S_{l(v)}$, the last term is equal to zero in $S[T_i^{\pm 1}]_{l(v)}$ for all v. □

Inspired by that, we propose a time saving (i.e. dimension reducing) approach for the computational decision of the triviality of τ_1:
We define $S = \left(k[X_{ij}]_{j<i}/I_w\right)_{l_w}$. Given the cocycle $(g_{id,v})_v := (-u_{1,2}^{(v)})_v$, we set $\tilde{r}_v = \varphi_v(T_1^{-1} T_2) T_1 T_2^{-1} \in (S)_{l(v)}$ and $\tilde{g}_v = -T_1 T_2^{-1} g_{id,v} \in (S)_{l(v)}$. Now assume that the intersection

$$\bigcap_{v \in W} \left(\tilde{r}_v \varphi_v(S) + \tilde{g}_v\right) \subset S^{(loc)} := (S)_{\prod_v l(v)} \tag{8.4}$$

is not empty. Then there exist $\tilde{f}_v \in (S)_{l(v)}$ such that

$$\tilde{f}_{id} = \tilde{r}_{id} \varphi_{id}(\tilde{f}_{id}) + \tilde{g}_{id} = \tilde{r}_v \varphi_v(\tilde{f}_v) + \tilde{g}_v$$

for every v. But if we multiply the last equality by $T_1^{-1} T_2$ and set $f_{id} = T_1^{-1} T_2 \tilde{f}_{id}$ and $f_v = \varphi_v^{-1}(f_{id} - g_{id,v})$, this shows that $(g_{id,v})_v$ is a coboundary. On the other hand, if $(g_{id,v})_v$ is a coboundary, then by Lemma 8.2 the intersection (8.4) is not empty.

Remark 8.5. *Let $N \in \mathbb{N}$, let U, V be subspaces of k^N, generated by the column vectors $u_1, \ldots, u_{m_1}$, resp. $v_1, \ldots, v_{m_2}$. Let $u, v \in k^N$ be any vectors. We want to compute the intersection $Z := (U + u) \cap (V + v) \subset k^N$. First, observe that for all $z \in Z$ we have $Z = (U \cap V) + z$. But Z is non-empty if and only if there exist $\alpha_i, \beta_i \in k$ such that*

$$u + \sum_{i=1}^{m_1} \alpha_i u_i = v + \sum_{i=1}^{m_2} \beta_i v_i \Leftrightarrow u - v = \sum_{i=1}^{m_1} \alpha_i(-u_i) + \sum_{i=1}^{m_2} \beta_i v_i.$$

This means, there exists some $x \in k^{1+m_1+m_2}$, regarded as row vector such that $x_1 \neq 0$ and $x \cdot (u - v, -u_1, \ldots, -u_{m_1}, v_1, \ldots, v_{m_2})^t = 0$. In this case we can define $z_0 := u + \sum_{i=2}^{m_1+1} (x_i/x_1) u_i$ and get $Z = z_0 + (U \cap V)$.

Remark 8.6. *We have implemented the cohomological criterion within* SINGULAR. *With that we checked the example at the end of the last chapter (i.e. $G = \mathbb{GL}_3, w = w_0$), which is also available in the programming code (* `ExSL3-w0-Step1.sing`*). Observe that we have to exhaust S_w^{loc} at least up to degree 11 (in 8 variables) and the subring S at least up to degree 8 (in 4 variables). To get an impression, how big things are, this means that for the proof of the triviality of the first torsor τ_1 we are computing within an $\mathbb{F}_q$-vectorspace of dimension 7044, where we have to intersect subspaces that are generated by 494 column vectors.*

As the occurring subspaces (and affine shifts) are relatively sparse, one should store them as sparse matrices. Unfortunately, SINGULAR *does not offer this, so we use* MAGMA *instead for the linear part of the computations.*

Remark 8.7. *There are two major bottlenecks in the implementation. First, we (frequently) have to compute Gröbner bases. For example, if $G = \mathbb{GL}_3$, we have to find Gröbner bases of ideals with about 6-8 generators in degree up to q^3 in a ring with 14-16 variables. We optionally implemented the use of the* SINGULAR *implementation of the F_5-algorithm of Eder (see* [13]*). This is often faster than the standard options, but it would be even faster if one could use a kernel implementation of the F_5-algorithm.*

Second, when exhausting $R_w^{loc,(i)}$, we are going over to (subspaces of) a quotient vector space of a big vector space. But the mapping of an element of the big space into the quotient corresponds to reduce a monomial modulo an ideal (in our case $I_w^{(i)}$). This can take quite a while, depending on how big the Gröbner base of the ideal is and mainly how far we exhaust the subrings.

Remark 8.8. *Related to the second bottleneck, one would like to know, how far one has to exhaust, to ensure that the intersection of the subspaces is empty if and only if the considered torsor is trivial. In general, we do not have a (natural, reasonably small) bound on the total degree of $f_{id}^{(i)}$. Second, even if we would have such a bound, it is not clear, how the total degree of $f_v^{(i)} := (\varphi_v^{(i)}(f_{id}^{(i)} - g_{id,v}^{(i)}) \bmod I_w^{(i)})$ might grow, even as one can (computationally) find a (big) bound for the degree of $f_v^{(i)}$, depending on the bound of $f_{id}^{(i)}$.*

For example, even in the reduced treatment of the example above one has at first a bound for the degree of 17 for $\tilde{f}_v^{(1)}$. But as we already know an explicit element of the intersection, we found for the special choice of $\tilde{f}_v^{(1)}$ that a bound of 11 after reducing modulo $I_w^{(1)}$ is sufficient. But this has not to be always the case. In fact, it can even occur that the total degree of a polynomial raises after reducing modulo some ideal. This highly depends on the chosen (global) monomial order, but we don't have played around a lot with different orders. Thus we have no suggestion for a preferred order.

Remark 8.9. *For $n \geq 3$, we always have that the inverse order of some reduced expression for w_0 induces a different filtration $\{U'_N \subset \cdots \subset U'_0\}$ of U. But for this filtration the cocycle attached to $\tau'_1 : G/U'_1 \to G/U'_0$ admits the same description as the cocycle attached to $\tau_N : G/U_N \to G/U_{N-1}$. Thus, we can reduce the computations of the last steps a little bit by computing whether the $\mathbb{G}_a$-torsors $\tau_1, \ldots, \tau_{\lceil N/2 \rceil}$ and $\tau'_1, \ldots, \tau'_{\lfloor N/2 \rfloor}$ are trivial. We have not implemented this, but it should be possible to reduce calculations even more if one thinks about all different reduced expressions for w_0 at once.*

References

[1] **M. Andersen, J. Dahl, L. Vandenberghe**: *CVXOPT Python Software for Convex Optimization.* Version 1.1.5 (2012) `http://abel.ee.ucla.edu/cvxopt/.`

[2] **R. Bédard**: *On the Brauer liftings for modular representations.* J. Algebra 93(2) (1985), pp. 332-353.

[3] **A. Björner, F. Brenti**: *Combinatorics of Coxeter Groups.* Springer (2005).

[4] **C. Bonnafé, R. Rouquier**: *On the irreducibility of Deligne–Lusztig varieties.* C. R. Acad. Sci. Paris, Ser. I 343 (2006), pp. 37-39.

[5] **C. Bonnafé, R. Rouquier**: *Affineness of Deligne-Lusztig varieties for minimal length elements.* J. Algebra 320(2008), pp. 1200-1206.

[6] **A. Borel**: *Linear Algebraic Groups, Second Enlarged Edition.* Springer New York (1991).

[7] **W. Bosma, J. Cannon, C. Playoust**: *The Magma algebra system. I. The user language.* J. Symbolic Comput., 24 (1997), pp 235–265.

[8] **N. Bourbaki**: *Groupes et algèbres de Lie. Chapitres 4, 5 et 6.* Masson, Paris (1981).

[9] **M. Brion**: *Lectures on the geometry of flag varieties.* `http://arxiv.org/abs/math/0410240` (2004).

[10] **W. Decker, G. Greuel, G. Pfister, H. Schönemann**: Singular 3-1-6 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2012).

[11] **P. Deligne, G. Lusztig**: *Representations of reductive groups over finite fields.* Ann. of Math.(2)103 (1976), no. 1, pp. 103-161.

[12] **P. Deligne**: *Action du groupe des tresses sur une catégorie.* Invent. math.128(1) (1997), pp. 159-175.

[13] **C. Eder**: *On the criteria of the F_5 algorithm.* `http://arxiv.org/abs/0804.2033`(2008).

[14] **M. Geck, S. Kim, G. Pfeiffer**: *Minimal length elements in twisted conjugacy classes of finite Coxeter groups.* J. Algebra 229 (2000), pp. 570-600.

[15] **U. Görtz**: *On the connectedness of Deligne-Lusztig varieties.* Repr. Th. 13 (2009).

[16] **U. Görtz, T. Wedhorn**: *Algebraic Geometry I. Schemes. With Examples and Exercises* Vieweg+Teubner Verlag, Wiesbaden (2010).

[17] **B. Haastert**: *Über die Deligne-Lusztig Varietäten reduktiver Gruppen.* Diplom-Arbeit Universität Bonn (1984).

[18] **B. Haastert**: *Die Quasiaffinität der Deligne-Lusztig-Varietäten.* J. Alg. 102(1986), pp. 186-193.

[19] **S. Harashita**: *On the affineness of distinguished Deligne-Lusztig varieties.* J. Algebra 390 (2013), pp. 290-297.

[20] **R. Hartshorne**: *Algebraic Geometry. Graduate Texts in Mathematics.* Springer-Verlag, New York (1977).

[21] **S. Have Hansen**: *Picard Groups of Deligne-Lusztig Varieties - with a View toward Higher Codimensions.* Contributions to Algebra and Geometry 43(**1**) (2002), pp. 9-26.

[22] **X. He**: *On the affineness of Deligne-Lusztig varieties.* J. Algebra 320(2008), pp. 1207-1219.

[23] **J. Humphreys**: *Linear Algebraic Groups, 4ed.* Springer, New York (1995).

[24] **J. C. Jantzen**: *Representations of Algebraic Groups. Second Edition.* American Mathematical Society (2003).

[25] **G. Kempf**: *Algebraic Varieties.* Cambridge University Press (1993).

[26] **V. Lakshmibai, N. Gonciulea**: *Flag Varieties.* Hermann Éditeurs des sciences et des arts, Paris (2001).

[27] **Q. Liu**: *Algebraic Geometry and arithmetic Curves.* Oxford University Press, New York (2002).

[28] **G. Lusztig**: *Coxeter Orbits and Eigenspaces of Frobenius.* Invent. Math. 38(1976), pp. 101-159.

[29] **G. Lusztig**: *Representations of finite Chevalley groups.* Regional Conf. Series in Math. 39, AMS (1978).

[30] **A. Neeman**: *Steins, affines and Hilbert's fourteenth problem.* Ann. of Math. (2) 127 (1988), pp. 229-244.

[31] **S. Orlik, M. Rapoport**: *Deligne-Lusztig varieties and period domains over finite fields.* J. Algebra 320(2008), pp. 1220-1234.

[32] **R. Richardson**: *Affine coset spaces of reductive algebraic groups.* Bull. London Math. Soc.9 (1977), no. 1, pp. 38-41.

[33] **M. Schönert et al.** : *GAP – Groups, Algorithms, and Programming – version 3 release 4 patchlevel 4.* Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany (1997).

[34] **T.A. Springer**: *Linear Algebraic Groups. Second Edition.* Birkhäuser, Boston (1998).