On an inverse problem of financial mathematics with error in the operator

Der Fakultät für Mathematik der Universität Duisburg-Essen
(Standort Essen)

zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigte Dissertation

von

Anna Stephan

aus Berlin

Referent: Prof. Dr. Denis Belomestny
Korreferent: PD Dr. John G. M. Schoenmakers
Korreferent: PD Dr. Volker Krätschmer

Datum der mündlichen Prüfung: 02. April 2014

Acknowledgements

I would like to thank my supervisor Prof. Dr. Denis Belomestny for accompanying me throughout the project with his useful advices. But above all, I am indebted to my family and close friends, especially to Peter Peyk, for their great support and patience.

Symbols

For some real-valued functions f and g

$f(x) = O(g(x)),\ x \to x_0$	$\exists\, a$, some positiv constant, such that $\lvert f(x)\rvert \le a\,\lvert g(x)\rvert$ in some neighbourhood of the point $x = x_0$
$f(x) \lesssim g(x),\ x \to x_0$	$\exists\, b$, some positiv constant, such that $\frac{\lvert f(x)\rvert}{\lvert g(x)\rvert} \le b$ in some neighbourhood of the point $x = x_0$
$f(x) = o(g(x)),\ x \to x_0$	$\frac{f(x)}{g(x)} \to 0$ as $x \to x_0$
$f(x) \asymp g(x),\ x \to x_0$ $\Leftrightarrow$ $f(x) - g(x) = o(g(x)),\ x \to x_0$	$\frac{f(x)}{g(x)} \to 1$ as $x \to x_0$

Abbreviations

We highlight our assumptions by the use of the following abbreviations

AM	**A**ssumption for the noisy **M**odel with fixed design
AI	**A**ssumption for the **I**dealised calibration algorithm
AC	**A**ssumption for the **C**onvergence of the forward operator to the linear smoother
AT	**A**ssumption for the **T**hresholds used by the cutoff regularisation method
AR	**A**ssumption for the thresholds used by the cutoff **R**egularisation method
AV	**A**ssumption for the forward operator $\mathbf{V}_i$
AF	**A**ssumption for the growth of the function $\mathbf{f}_n$
AX	**A**ssumption for the underlying Markov process $(\mathbf{X}_t)$

Contents

Chapter 1

Introduction

This thesis handles a calibration problem for the so-called Markov-functional models (MFMs). The purpose of this work is to improve the theoretical background when dealing with this type of calibration problems and to develop a practical method for solving this problem in realistic situations.

A couple of natural questions arise in this endeavor:

(i) Why do we chose MFMs?
(ii) What do we mean by calibration problem for MFMs?
(iii) Why are calibration problems fundamental in financial mathematics?
(iv) What are we going to do exactly?

In order to give an accurate answer to the questions raised above, we shall first give some background information on the topic of liquid calibration instruments. Basically, this is a summary of Hunt, Kennedy (2000) who have put this very much to the point.

When dealing with liquid calibration instruments in finance mathematics, one usually considers a pricing model for a derivative with observable market data. Nearly all theoretical mathematical approaches to derivative pricing are concerned with a model for an economy, and supply tools to allow one to move from a model of an economy to the pricing model of a derivative within that economy (for more details on the theory of derivative pricing see Part I in Hunt, Kennedy (2000)). Thus, there is always some connection between a model of an economy and the derivative pricing model (cf. Hunt, Kennedy (2000): Chapter 9.1, p. 215). Within an arbitrage-free economy the common approach to the construction of a derivative pricing model follows more or less the same pattern: One starts with a model of the asset price processes in this economy (e.g. specified via some stochastic differential equation (SDE)) in a real-world probability measure $\mathbb{P}$ (cf. Hunt, Kennedy (2000): Chapter 9.2, p. 215 et sqq.). ”Then one chooses a numeraire $\mathbb{N}$ and changes the probability measure from the original measure $\mathbb{P}$ to an equivalent martingale measure [(EMM)] $\mathbb{N}$, under which all $\mathbb{N}$-rebased assets are martingales. Having done this, the value of any derivative can be calculated by taking expectations in this measure $\mathbb{N}$. For most products encountered in practice the value is determined by the joint distribution of a finite number of asset prices on a finite number of dates in a martingale measure. On the other hand, a model for the asset price processes can only be formulated based on information and intuition available in the real-world [...]” (Hunt, Kennedy (2000): Chapter 9.2, p. 215-216). So the link between a real-world probability measure and an EMM should respond to the question how to ”[...] use real-world information to formulate a model which, when we have changed to a martingale measure, will give an asset price distribution which in turn yields a reasonable price for the derivative under consideration” (Hunt, Kennedy (2000): Chapter 9.2, p. 216).

However, for practical use, being familiar with the mathematical theory of pricing derivatives is only the beginning. We must decide on the model of the economy and the assumptions that this model implies. This decision depends on many things, including the particular derivative in question and the need to get numbers out of this model in a reasonable time. Furthermore, in the case that we price derivatives based on real market data, one would not aim to derive the price from model parameters. One would rather select a pricing model first, and then use the market price to deduce the parameters of the pricing model which fit best the observed market prices (cf. Hunt, Kennedy (2000): Chapter 9.1, p. 215 and Chapter 11.1, p. 237). This is what we mean by calibrating the model to the market data. Thus, calibration can be seen as an instrument to ensure that a pricing model matches the observed market price by specifying model parameters. Obviously, only models which **can** be calibrated are of practical interest. That is why robust calibration methods have a key position not only in financial mathematics, but also in all disciplines engaged in modelling using real-world data.

As we already mentioned, dealing with liquid calibration instruments in finance mathematics means dealing with pricing models for a financial derivative with observable market data. Therefore, taking a model from the class of market models – formulated in terms of market rates and as such directly related to tradable assets – seems an appropriate choice. However, since market models capture the joint distribution of market rates, they are high-dimensional (cf. Hunt, Kennedy (2000): Chapter 18). Thus, they are hard to implement and one would try to avoid to do this in practice.

In addition, the choice of short-rate models or instantaneous forward rate models (cf. Hunt, Kennedy (2000): Chapter 17) which model the prices of derivatives as some functions of the underlying process related to instantaneous forward rates is not quite appropriate either. This is because instantaneous forwards cannot be traded in the market and therefore these models have poor calibration properties.

The class of MFMs was introduced by Hunt, Kennedy, Pelsser (2000). The prime reason which lead to their development was the popular request to have models that are capable of exactly replicating prices of liquid calibration instruments in almost the same manner as market models while perpetuating the efficiency of short-rate models in calculating derivative prices (cf. e.g. Hunt, Kennedy (2000): Chapter 19.1, p. 351). And indeed, MFMs comply with this request: they have good calibration properties and allow for efficient implementations.

Summarising, MFMs can be characterised as follows (cf. Hunt, Kennedy (2000): p. 352):

(i) they are arbitrage-free,
(ii) they can readily be calibrated and they price relevant liquid instruments correctly,
(iii) they have realistic and transparent properties,
(iv) they allow for efficient implementation,
(v) they can be used for pricing of multi-temporal derivatives.

The defining characteristic of MFMs is that the prices of the underlying assets, namely, pure discount bond prices, are at any time functions of some low-dimensional Markovian process in some martingale measure (cf. Hunt, Kennedy (2000): Chapter 19.1, p. 351). "This ensures that its implementation is efficient, since it only requires to track the driving Markov process. [...] The freedom of choosing its functional form is what permits accurate calibration of MFMs to relevant market prices, a property not possessed by short-rate models" (Hunt, Kennedy (2000):

Chapter 19.1, p. 351). The freedom to specify the law of the driving Markov process allows us to formulate the model in such a way as to capture well the features of the real market relevant to a given product (cf. Hunt, Kennedy (2000): Chapter 19.1, p. 351).

Put precisely, the MFMs framework is built on modelling the numeraire $\mathbb{N}$ and the terminal discount bond as functionals of a (low-dimensional) Markov process whose dynamics can easily be followed. The functional forms, on the other hand, are obtained by calibration to prices of appropriate liquid derivatives at a finite number of dates. Furthermore, since the discount bonds at earlier times are received by using the martingale property of numeraire-rebased assets, the resulting model is arbitrage-free by construction and the calibration of MFMs can be defined as a backwards induction procedure. At each step of this procedure we use market prices for the relevant liquid financial derivatives with different strikes to get the current numeraire as a function of the state variable (cf. e.g. Hunt, Kennedy (2000): Chapter 19.2 et sqq.). And the aim of the present thesis is to analyse this calibration problem mathematically. It turns out that the underlying mathematical problem is challenging because we are dealing with a sequence of successive nonlinear ill-posed inverse problems with errors in the operators. Such inverse problems have not been yet thorougly studied in the literature and here we are going to fill this gap.

As liquid financial derivative, we will select one of the sufficiently common derivatives seen in the financial market, namely, a digital caplet. A liquid digital derivative is a derivative for which we have some real-world data and which has a payoff of either one or zero at some point in the future, depending on the level of some index rate (cf. Hunt, Kennedy (2000): Chapter 11.5, p. 244). Thus, digital liquid derivatives are easy to price and provide an insight into the calibration procedure. Generally speaking, the value of a financial derivative depends on the underlying asset. In our particular case, the payoff-function of digital caplets depends on forward LIBORs and therefore we choose a LIBOR-MFM which will be introduced in more detail in Chapter 2. In Chapter 3, we will formulate the calibration algorithm based on a full and uncorrupted caplets data set as well as the calibration algorithm as design-based and corrupted with noise, which is a much more realistic approach. In Chapter 4, first, we will rigorously analyse the underlying inverse problem from the point of view of ill-posedeness, regularisation and convergence. From a theoretical point of view, we will study a nonlinear inverse problem with errors in the operator. Second, we will propose a regularisation approach which significantly enhances the stability of the calibration method compared to the extrapolation of digital option prices, usually employed in the field so far (see e.g. Chapter 9.4 in Hunt, Kennedy (2000)), as the extrapolation of digital prices is prone to misspecification errors given the bid ask spread and other frictions on the market. Moreover, we will demonstrate that we can achieve the same convergence rates in a special case of one of the best-known and widely accepted asset price models, namely, the generalised Black-Scholes model (with constant instantaneous volatility). Chapter 5 contains comments about the convergence result. Finally, the mathematical details which we used to verify the convergence rate are collected in Appendix A.

It should be mentioned that the topic of inverse problems with error in the operator has recently drawn much attention in the econometric and statistical literature (see, e.g. Hoffmann, Reiss (2008) and Chen, Reiss (2011)). However, studying a **non**linear inverse problem with errors in the operator – according to our knowledge – has not been reported before (for a recent review of calibration problems in financial mathematics let us refer to Cont, Tankov (2004) and Cont, Tankov (2006)).

Chapter 2

LIBOR-Markov-functional models

We can understand the class of LIBOR-Markov-functional models (LIBOR-MFMs) as a class of example models for MFMs. The representatives of the class of LIBOR-MFMs can be used to price LIBOR-based interest rate derivatives (cf. e.g. Hunt, Kennedy (2000): Chapter 19.4.1). Thus, LIBOR-MFMs are suitable for pricing standard market derivatives like digital caplets.

To distinguish LIBOR-MFMs, we will have to characterise theirs components: Markov-functional interest rate models and LIBOR market models (see Hunt, Kennedy (2000): Chapters 17, 18.2, 19). As we already mentioned in the introduction, Markov-functional interest rate models focus on specifying a low-dimensional process, which is Markovian in some martingale measure, and on formulating the underlying asset prices, namely, pure discount bond prices as function of this process – whereas LIBOR market models focus on modelling the dynamics of LIBOR forward rates. The main idea behind LIBOR-MFMs is to model the pure discount bond as a function of some low-dimensional process (which is Markovian under some martingale measure) at first, and to define LIBORs using some closed form relation between LIBOR forward rate and pure discount bond afterwards.

Let us start with sketching the main properties of the LIBOR-MFMs and then continue with a more detailed consideration of these models.

Assume $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ as a filtered probability space satisfying the usual conditions (see e.g. Appendix 1 in Hunt, Kennedy (2000)) and let X_t be a one-dimensional Markov process adapted to the (augmented) filtration $(\mathcal{F}_t)_{t\geq 0}$. The aim of modelling is to specify the functions

ψ_{ij}, such that the time T_i value of a zero-coupon discount bond with maturity T_j is of the form $\mathcal{D}_{T_iT_j} = \psi_{ij}(X_{T_i}), i = 1,...,n; i \leq j \leq n+1$. Next, using the closed form relation between LIBOR forward rate and pure discount bond, we specify the functions f_i, such that the ith LIBOR forward rate $L^i_{T_i}$ of time T_i for the period $[T_i, T_{i+1}]$ is of the form $L^i_{T_i} = f_i(X_{T_i})$. An essential feature of these types of models is the freedom to choose the functional forms of ψ_{ij} in such a way that market prices of calibration instruments are replicated. Moreover, because the functional form of f_i is implicated by the functional form of ψ_{ij}, we can guarantee that our particular derivative in question will be priced accurately relative to existing products. However, there is one important assumption to be made on the class of functions f_i and ψ_{ij} respectively, if we want the driving Markov process to give us all the information on the correlation structure of discount bonds and LIBOR forward rates. Namely, the functions f_i and ψ_{ij} should be monotone functions of the current state of the Markov chain $X_{T_i}, i = 1,...,n; i \leq j \leq n+1$ (cf. Hunt, Kennedy (2000): p. 356). It should be also noted that the remaining freedom to choose the dynamics of the driving Markov process allows to capture the characteristic calibration product features using the influence of the joint distribution of the market rates (cf. Hunt, Kennedy (2000): Chapter 19.1, p. 351).

Obviously, LIBOR-MFMs are above all MFMs and thus they inherite the representative properties of MFMs. Hence, as we already mentioned in the introduction, LIBOR-MFMs combine the strong points of two types of models. They are capable of exactly replicating prices of liquid calibration instruments, in the same manner as market models, while perpetuating the efficiency of short-rate models in calculating derivative prices. LIBOR-MFMs are tractable since calibration involves the integration of the known probability distribution of the one-dimensional Markov process, and they are efficient since we can avoid to model the joint distribution of market rates (and zero-bonds) explicitly (cf. Hunt, Kennedy (2000): Chapter 19).

Let us now continue by going through the main properties of LIBOR-MFMs step by step. In a more general context of continuous time models, $\mathcal{D}_{tT}$ stands for the time $t \leq T$ value of a zero-coupon discount bond with maturity T, i.e. the price of security making a single payment of 1 at time $T \geq t$. According to the general setting of interest rate models we choose an economy of pure discount bonds. Thus, we denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the asset prices of the economy, $\mathcal{F}_t = \sigma(\{D_{uT} : 0 \leq u \leq t, u \leq T < \infty\})$, where $\{D_{tT} : 0 < t < T < \infty\}$ denotes the complete continuous time term structure of pure discount bonds (cf. Hunt, Kennedy (2000): Chapter 8.2, p. 183 et sqq.). There are several well-known ways to model the dynamics of pure discount bonds when dealing with short-rate models (cf. Hunt, Kennedy (2000): Chapter 17). "The most direct way to specify a term structure model of underlying assets is by explicitly stating the law of the pure discount bonds or an SDE that they satisfy" (Hunt, Kennedy (2000): p. 187). It is also common to give an SDE for the numeraire-rebased discount bonds which is possible within an arbitrage-free economy. "In the latter situation, the numeraire [a price process that is almost surely positive] is often a given bond, and the model specification is usually incomplete in that the law of the numeraire bond is not given and the economy is only modelled until some fixed time T, because this is all that is needed for the application at hand" (Hunt, Kennedy (2000): p. 188). Since we aim to value digital caplets, the problem of finding the value of a derivative within our economy is of great interest. We suppose that there is some time T_{n+1}, at which the value of the derivative will have been determined by the evolution of the economy up to the time T_{n+1}. Therefore, it will be enough to consider an economy comprising only a finite number of these bonds. Let $T_1 < T_2 < \ldots < T_{n+1}$ be a sequence of maturity dates and redefine $\mathcal{F}_t := \sigma\left(\{D_{uT_i} : u \leq t, i \in [1;n+1]\}\right)$. Notice that for the theoretical content of derivative pricing it is still important to be able to extend such models to a continuum. So, at some important points, we will outline the continuous time structure which can easily be reduced to the consideration at some finite number of points. We assume that the derivative in question can be replicated and in keeping the economy arbitrage-free we can use the martingale properties for the derivative pricing. We will allow trading in this economy, buying and selling of the assets throughout time, but we will preclude the injection of external funds into the economy – all trading strategies must be self-financing. The value of derivatives generated in this way by trading in the assets of the economy will be called a price process (cf. Hunt, Kennedy (2000): Chapter 7.1.2, p. 144 et sqq.). Economically, a digital caplet is an option which pays a unit amount at time T_{i+1} if at T_i the ith LIBOR forward rate $L^i_{T_i}$ for the period $[T_i, T_{i+1}]$ is above some strike level K (cf. Hunt, Kennedy (2000): Chapter 11.5.1). At first, we assume that the market price $V_i(K)$ of the digital caplet with payoff $1_{\left\{L^i_{T_i} > K\right\}}$ is available for any $K > 0$ and any $i = 1, \ldots, n$. In terms of continuous time structure models we can introduce the ith LIBOR forward rate of time $0 < t \leq T_i$ for the period $[T_i, T_{i+1}]$ as follows:

$$L^i_t = \frac{\mathcal{D}_{tT_i} - \mathcal{D}_{tT_{i+1}}}{\varsigma_i \mathcal{D}_{tT_{i+1}}}, \tag{2.0.1}$$

where $\varsigma_i = T_{i+1} - T_i$ are the so-called accrual factors (see e.g. Chapter 18.2 in Hunt, Kennedy (2000)). In particular, by (2.0.1) we have a closed form relation between LIBOR forward rate and prices of pure zero bonds. Hence, one (usually) specifies the driving process of LIBOR forward rates only. The dynamics of a zero bond price process follows by (2.0.1)-(2.0.4). There are several approaches for modelling the rates L_t^i. In a LIBOR market model, each of the forward LIBORs L_t^i solves the SDE of the form:

$$dL_t^i = \mu_i(t)L_t^i dt + \sigma_i(t)L_t^i dW_t \tag{2.0.2}$$

for some deterministic, locally bounded, volatility functions $\sigma_i(t)$, where W_t is the standard Brownian motion under $\mathbb{P}$ (see e.g. Chapter 18.2.1 in Hunt, Kennedy (2000)). Thus, given the model is arbitrage-free, the time-zero value of the derivative to be evaluated is given by taking expected values in some martingale measure $\mathbb{N}$ which is equivalent to $\mathbb{P}$. There are some reasons for $\mathbb{N} := \mathbb{Q}^{n+1}$, a martingale measure associated with the numeraire $\mathcal{D}_{\cdot,T_{n+1}}$, being reasonable when dealing with LIBOR-MFMs. For one, the final LIBOR forward rate L_t^n is a martingale under $\mathbb{Q}^{n+1}$. This implies that $dL_t^n = \sigma_n(t)L_t^n dW_t^{n+1}$, where W_t^{n+1} is standard Brownian motion under $\mathbb{Q}^{n+1}$ and we can calculate the law of L_t^n using the Girsanov's theorem. In particular, by the solution of this SDE and $L_{T_n}^n =: f_n(W_{T_n}^{n+1})$ we know the explicit function f_n of the current state of the Brownian motion $W_{T_n}^{n+1}$. Two further details should be noted: first, a Brownian motion is Markovian which follows from its definition, and second, provided that we know the initial value L_0^n we can now recover the dynamics of $L_t^i, i = n-1, ..., 1$ by specifying the functional form of f_i using a backwards iterative calibration procedure (if we assume the functions f_i are monotone increasing). Before proceeding, we would like to emphasise that this model is arbitrage-free. In other words, each of the numeraire-rebased discount bonds

$$\overline{\mathcal{D}}_{tT} = \frac{\mathcal{D}_{tT}}{\mathcal{D}_{tT_{n+1}}} \tag{2.0.3}$$

must be a martingale under $\mathbb{Q}^{n+1}$. This imposes a relationship between the diffusion term and the drift term in the SDE (2.0.2). Namely,

$$\mu_i(t) = -\sum_{j=i+1}^{n} \frac{\varsigma_j L_t^j}{1+\varsigma_j L_t^j}\sigma_i(t)\sigma_j(t), \quad 1 \le i \le n.$$

(see p. 341 in Hunt, Kennedy (2000)). Notice that by combining (2.0.1) with (2.0.3) it follows:

$$\overline{\mathcal{D}}_{tT_i} = (1+\varsigma_i L_t^i)\overline{\mathcal{D}}_{tT_{i+1}}, \quad i = 1, \ldots, n. \tag{2.0.4}$$

As mentioned before (due to the non-arbitrage), in order to be able to evaluate the price of interest rate derivatives (like a digital caplet) it is sufficient to determine the functional form associated with the terminal numeraire bonds $\mathcal{D}_{T_iT_{n+1}}, i = 1,\ldots,n$. Then the functional form of former times bonds can be found using the martingale property for numeraire-rebased assets under $\mathbb{Q}^{n+1}$ and the fact that the underlying process X_t is Markovian under $\mathbb{Q}^{n+1}$. In particular, having specified X_t, its (conditional) probability distribution is known. It is a normal distribution in case we make use of (2.0.2). Thus, at the very least we need to assume the existence of the (transition) density function of X_t. Using the martingale property for numeraire-rebased assets under $\mathbb{Q}^{n+1}$ and the fact that X_t is Markovian under $\mathbb{Q}^{n+1}$, for $i \le j \le n+1$, we get

$$\mathcal{D}_{T_iT_j} = \mathcal{D}_{T_iT_{n+1}} \mathrm{E}_{\mathbb{Q}^{n+1}}\left[\frac{\mathcal{D}_{T_jT_j}}{\mathcal{D}_{T_jT_{n+1}}}\middle|\mathcal{F}_{T_i}\right] \tag{2.0.5}$$

$$= \psi_{i(n+1)}(X_{T_i})\int_{-\infty}^{\infty}\frac{1}{\psi_{j(n+1)}(u)}\phi_{j|i}(u|X_{T_i})du,$$

where $\phi_{j|i}(u|x)$ denotes the transition density of X_{T_j} given X_{T_i}, and X_t is a one-dimensional Markov process under $\mathbb{Q}^{n+1}$. Hence, if the Markov process X_t and thus his transition density is specified, we can extract the functional form of the numeraire discount bond $\mathcal{D}_{T_iT_{n+1}}$ at times T_i, $i = 1,\ldots,n$ from market observed derivative prices and thus recover the dynamics of L_t^i. Actually, the main advantage of the LIBOR-MFMs is that this calibrating procedure can be done efficiently (in contrast to pure market models). This is possible due to the assumption that the ith forward LIBOR rate $L_{T_i}^i$ is a monotone increasing function f_i of the current state of the Markov chain $X_{T_i}, i = 1,\ldots,n$ and that f_n is explicitly known. Then the functional form of $\mathcal{D}_{T_n,T_{n+1}}$ can be determined using (2.0.1) via

$$\mathcal{D}_{T_n,T_{n+1}} = \frac{1}{1+\varsigma_n L_{T_n}^n}.$$

In the following, we will demonstrate how market prices of the calibrating one-period digital caplets can be used to deduce (numerically) the functional forms of the numeraire $\mathcal{D}_{T_iT_{n+1}}, i < n$.

A one-period digital caplet expiring at time $T_i, i = 1,\ldots,n$, corresponding to the ith LIBOR forward rate $L_{T_i}^i$ having strike K has a payoff at time T_i of

$$V_{T_i}^i(K) = \mathcal{D}_{T_i,T_{i+1}} 1_{\left\{L_{T_i}^i > K\right\}} \tag{2.0.6}$$

(see e.g. Chapter 11.5 in Hunt, Kennedy (2000)). Applying the fundamental theorem of asset pricing (see e.g. Corollary 7.34 in Hunt, Kennedy (2000)) its value at time $t = 0 = T_0$ is given by

$$V_i(K) := V^i_{T_0}(K) = \mathcal{D}_{0T_{n+1}} \mathrm{E}_{\mathbb{Q}^{n+1}} \left[\frac{\mathcal{D}_{T_iT_{i+1}}}{\mathcal{D}_{T_iT_{n+1}}} 1_{\left\{L^i_{T_i} > K\right\}} \right] \tag{2.0.7}$$

$$= \mathcal{D}_{0T_{n+1}} \mathrm{E}_{\mathbb{Q}^{n+1}} \left[\overline{\mathcal{D}}_{T_iT_{i+1}} 1_{\left\{L^i_{T_i} > K\right\}} \right],$$

where we make use of (2.0.3) in the last equation. To determine the functional form of the numeraire $\mathcal{D}_{T_iT_{n+1}}, i < n$, we proceed as in Chapter 19.3.1 in Hunt, Kennedy (2000). More specifically, we work backwards iteratively from the terminal time T_n. Consider the ith step in this procedure and suppose that the function $\psi_{j(n+1)}$ has already been determined for $j = i+1, \dots, n$. At time T_i consider the problem of finding a function g_i satisfying

$$V_i(K) = \mathcal{D}_{0T_{n+1}} \mathrm{E}_{\mathbb{Q}^{n+1}} \left[\frac{\mathcal{D}_{T_iT_{i+1}}}{\mathcal{D}_{T_iT_{n+1}}} 1_{\left\{X_{T_i} > g_i(K)\right\}} \right]$$

$$= \psi_{0(n+1)}(X_0) \int_{g_i(K)}^{\infty} Q_i(v)\phi_i(v)dv \tag{2.0.8}$$

with

$$Q_i(v) := \int_{-\infty}^{\infty} \frac{1}{\psi_{(i+1)(n+1)}(u)} \phi_{(i+1)|i}(u|v)du,$$

where ϕ_i is the marginal density of X_{T_i}. One more time, in (2.0.8) we used the martingale property of (the numeraire-rebased asset) $\left(\frac{\mathcal{D}_{tT_{i+1}}}{\mathcal{D}_{tT_{n+1}}}\right)_{t=T_i}$ under $\mathbb{Q}^{n+1}$ and the Markov property of X_t under $\mathbb{Q}^{n+1}$ to get

$$\frac{\mathcal{D}_{T_iT_{i+1}}}{\mathcal{D}_{T_iT_{n+1}}} = \mathrm{E}_{\mathbb{Q}^{n+1}} \left[\frac{\mathcal{D}_{T_{i+1}T_{i+1}}}{\mathcal{D}_{T_{i+1}T_{n+1}}} \middle| X_{T_i} = v \right]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\psi_{(i+1)(n+1)}(u)} \phi_{(i+1)|i}(u|v)du = Q_i(v) \tag{2.0.9}$$

for all $v \in \mathbb{R}$ satisfying $X_{T_i}(\omega) = v, \omega \in \Omega$.

Since the initial value of the zero-coupon bond $\mathcal{D}_{0T_{n+1}}$ is quoted on the market, $\psi_{0(n+1)}$ is known. By comparing (2.0.7) with (2.0.8) it follows that g_i is an inverse function for f_i, which is well defined due to the monotony of f_i. In particular, if g_i is found we can compute f_i.

Next, the value of the function $\psi_{i(n+1)}$ at the point $u \in \mathbb{R}$ is obtained by combining (2.0.1) for $t = T_i$ with (2.0.9):

$$\psi_{i(n+1)}(u) = \frac{1}{(1+\varsigma_i f_i(u))Q_i(u)}.$$

Finally, (2.0.9) allows for backwards iterative estimation methods, as for all $v \in \mathbb{R}$ satisfying $X_{T_{i-1}}(\omega) = v, \omega \in \Omega$ we have

$$Q_{i-1}(v) = \frac{\mathcal{D}_{T_{i-1}T_i}}{\mathcal{D}_{T_{i-1}T_{n+1}}} = \mathrm{E}_{\mathbb{Q}^{n+1}}\left[\frac{1}{\mathcal{D}_{T_iT_{n+1}}}\middle| X_{T_{i-1}} = v\right]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\psi_{i(n+1)}(u)} \phi_{i|(i-1)}(u|v)du$$

$$= \mathrm{E}_{\mathbb{Q}^{n+1}}\left[\frac{Q_i}{\mathcal{D}_{T_iT_{i+1}}}\middle| X_{T_{i-1}} = v\right]$$

$$= \int_{-\infty}^{\infty} (1+\varsigma_i f_i(u))Q_i(u)\phi_{i|(i-1)}(u|v)du,$$

where in the last equation we used (2.0.1) with $t = T_i$ to get $\mathcal{D}_{T_iT_{i+1}} = \frac{1}{1+\varsigma_i L^i_{T_i}}$.

In practice there are only a finite number of caplets available on the market. So one faces the problem of solving the equation (2.0.8) based on the data

$$V_i(K_1), \ldots, V_i(K_L).$$

One common way to address this problem is based on the interpolation of the price function V_i (see e.g. Chapter 9.4 in Hunt, Kennedy (2000)). However, this approach can suffer from instability and lead to a misspecification, given the bid-ask spread and other frictions (i.e. costs associated with rebalancing a portfolio) on the market. In the next chapter we will focus on a more robust calibration algorithm.

Chapter 3

Calibration

3.1 Idealised calibration algorithm

(AI) At first, let us formulate the "idealised" calibration algorithm for finding f_i based on full and uncorrupted caplets data. Suppose that a monotone increasing non-negative function $f_n(x)$ and a non-negative function $\psi_{0(n+1)}(x)$ are given. Moreover, assume that the Markov chain $\left(X_{T_j}\right)_{j=0,\ldots,n}$ starts at some point x_0, has the one-step transition densities $\phi_{(j+1)|j}$ and the marginal densities ϕ_j. Without loss of generality we may assume $\psi_{0(n+1)}(x_0) = 1$.

1. Initialization:

$$Q_n(v) \equiv 1, \forall v \in \mathbb{R}.$$

2. ith step: suppose that the functions $f_j(\cdot)$ and $Q_j(\cdot)$ have already been determined for $j = i+1, \ldots, n$. Then, based on the market caplet prices $V_i(K), K > 0$, define $f_i(x) := g_i^{-1}(x)$ with g_i solving the equation:

$$\int_{g_i(K)}^{\infty} Q_i(v)\phi_i(v)dv = V_i(K), \quad K > 0, \tag{3.1.1}$$

where

$$Q_i(v) = \int_{-\infty}^{\infty} (1 + \varsigma_{i+1} f_{i+1}(u)) Q_{i+1}(u) \phi_{(i+1)|i}(u|v) du.$$

As can easily be seen, the most difficult part of the algorithm is the computation of the function f_i . In fact, this problem is equivalent to the problem of finding the inverse function $V_i^{\leftarrow}$ for V_i, since

$$f_i(x) = V_i^{\leftarrow}(J_i(x)) \tag{3.1.2}$$

with

$$J_i(x) = \int_x^\infty Q_i(v)\phi_i(v)dv.$$

This ”idealised” algorithm is of little practical relevance, because usually we do not have the values of V_i for all $K > 0$ at our disposal. Moreover, the caplet prices can be contaminated with noise. In the next section we formulate the regularised calibration algorithm using only a finite number of caplets.

3.2 Regularised calibration algorithm in realistic situations

The aim of the calibration algorithm is to approximate the ”link” function f_i by numerically inverting the function V_i and then applying (3.1.2). Unfortunately, we do not know V_i exactly, but rather its values on a strike grid $\mathbf{K} = \{0 < K_1 < K_2 < \ldots K_L\}$ corrupted with noise:

(AM)

$$\widetilde{V}_i(K_l) := V_i(K_l) + \sigma_{il}\varepsilon_{il}, \quad l = 1, \ldots, L, \tag{3.2.1}$$

where

- $\sigma_{il} > 0$
- ε_{il} are independent, centered r. v. with $\mathrm{E}\varepsilon_{il}^2 = 1$
- $\sigma_{il}\varepsilon_{il}$ are bounded.

As we already discussed in Section 3.1, the inverse function of V_i is of great interest. Regrettably, V_i as a function on $\mathbb{R}^+$ is not invertible. To illustrate this, consider (2.0.6): once there is a strike, say $\tilde{K}$, satisfying $V_i(\tilde{K}) = 0$, it follows that $V_i(\hat{K}) = 0$ for all strikes $\hat{K} > \tilde{K}$. Moreover, such a $\tilde{K}$ exists due to the boundedness of the forward LIBORs, thus V_i is not a one-to-one function. Hence, in order to compute the inverse of V_i, we have to regularise the problem. One way to do this is to use the observation that at the regions where the function $V_i(K)$ is decreasing and one-to-one both V_i and its inverse have the same measurement properties. Hence, with

$$V_i^{\leftarrow}(x) := \inf\left\{K \in \mathbb{R}^+ : V_i(K) \geq x\right\}$$

we have

$$\int_0^\infty 1_{\{V_i(K)>x\}} dK = \int_0^\infty 1_{\{K<V_i^{\leftarrow}(x)\}} dK = V_i^{\leftarrow}(x). \tag{3.2.2}$$

First, we smooth the indicator function in (3.2.2) via a strictly monotone, uniformly Lipschitz distribution function Φ of a distribution supported on $[-1,1]$ to get

$$V_{i,h}^{\leftarrow}(x) := \int_0^\infty \Phi\left(\frac{V_i(K)-x}{h}\right) dK,$$

where h is a bandwidth. Suppose that Φ is uniformly Lipschitz on $\mathbb{R}$ with the Lipschitz constant $\mathcal{L}_\Phi$. The function $V_{i,h}^{\leftarrow}(x)$ can be approximated in turn using

$$\widehat{V}_{i,h}^{\leftarrow}(x) := \sum_{l=1}^{L-1} (K_{l+1}-K_l)\Phi\left(\frac{\widehat{V}_{i,L}(K_l)-x}{h}\right),$$

where $\widehat{V}_{i,L}$ is a linear smoother, i.e.

$$\widehat{V}_{i,L}(K_l) := \sum_{j=1}^{L} w_{j,L}(K_l;\lambda_L)\widetilde{V}_i(K_j), \tag{3.2.3}$$

where $w_{j,L}$ are some weights depending on a regularisation parameter λ_L.

For example, the well-known Nadaraya-Watson estimator can be obtained using the weights

$$w_{j,L}(x;\lambda_L) = \frac{\mathcal{K}\left(\frac{K_j-x}{\lambda_L}\right)}{\sum_{i=1}^L \mathcal{K}\left(\frac{K_i-x}{\lambda_L}\right)} 1_{\left\{\sum_{i=1}^L \mathcal{K}\left(\frac{K_i-x}{\lambda_L}\right)\neq 0\right\}},$$

where $\mathcal{K}$ is a symmetric kernel supported on $[-1,1]$ and λ_L is a bandwidth.
Typically, the weights $w_{j,L}(x;\lambda_L)$ of linear regression estimators satisfy the equality

$$\sum_{j=1}^{L} w_{j,L}(x;\lambda_L) = 1, \quad \forall x$$

(see e.g. Section 1.5, p. 33 in Tsybakov (2009)). Moreover, since the kernel is defined as a non-negative real-valued integrable function $\mathcal{K}$, we have

$$w_{j,L}(x;\lambda_L) \geq 0, \quad \forall j.$$

If smoothing splines are used, then, we will have

$$W = B(B^\top B + \lambda_L \Omega)^{-1} B^\top,$$

where $B_{ji} = B_j(K_i)$, $B_1, \ldots, B_N$, are a basis for the natural splines (such as the B-splines with $N = L+4$) and the matrix Ω has the entries

$$\Omega_{jk} = \int B''_j(x) B''_k(x) dx, \quad j,k = 1,\ldots,N$$

for the matrix $W = (w_{j,L}(K_l;\lambda_L))_{j,l=1}^{L}$ (see e.g. Theorem 5.81 in Wasserman (2006)).

We assume that $\widehat{V}_{i,L}$ converges to V_i as $L \to \infty$ and $\lambda_L \to 0$ in the following sense

(AC)

$$\mathbb{P}\left(\max_{l \leq L} \sup_{x \in [K_1, K_l]} \left\{ \gamma_l |\widehat{V}_{i,l}(x) - V_i(x)| \right\} > c_1 \right) \leq c_2 L^{-r} \tag{3.2.4}$$

for some sequence $\gamma_L \to \infty$, $L \to \infty$; $r > 0$ and some positive constants c_1 and c_2.

Discussion **(AC)** holds for many well-known linear smoothers, such as local polynomial smoothers and smoothing splines. The main idea of the proof is described below. First, one can write

$$\widehat{V}_{i,l}(x) - V_i(x) = \left[\sum_{j=1}^{l} w_{j,l}(x;\lambda_l) V_i(K_j) - V_i(x) \right] + \sum_{j=1}^{l} \sigma_{ij} w_{j,l}(x;\lambda_l) \varepsilon_{ij}. \tag{3.2.5}$$

As the term in parentheses, mentioned in (3.2.5), fixes the bias of $\widehat{V}_{i,l}$, defined as $b(x) = \mathbb{E}\left[\widehat{V}_{i,l}(x)\right] - V_i(x)$, and is deterministic, the other one is stochastic and can be estimated using the Talagrand exponential inequality (see e.g. Giné, Guillou (2001)) and the Montgomery-Smith's maximal inequality (see e.g. Montgomery-Smith (1993)). In this way, we obtain

$$\mathbb{P}\left(\max_{l\le L}\left\{\gamma_l \sup_x \left|\sum_{j=1}^{l} \sigma_{ij} w_{j,l}(x;\lambda_l)\varepsilon_{ij}\right|\right\} > a\right)$$

$$\le 9\mathbb{P}\left(\gamma_L \sup_x \left|\sum_{j=1}^{L} \sigma_{ij} w_{j,L}(x;\lambda_L)\varepsilon_{ij}\right| > a/30\right) \le cL^{-r} \tag{3.2.6}$$

for some positive constants a, c and $r>0$ under some assumptions on the sequence of regularisation parameters $\lambda_L \to 0$, $L\to\infty$. With some more details: from the very beginning of the proof of Corollary 4 in Montgomery-Smith (1993) we get

$$\mathbb{P}\left(\sup_{1\le l\le L} \|S_l\| > a\right) \le 3 \sup_{1\le l\le L} \mathbb{P}\left(\|S_l\| > a/3\right), \tag{3.2.7}$$

where $S_l := \sum_{j=1}^{l} \varepsilon_{ij}$ and $\|S_l\| := \gamma_l \sup_x \left|\sum_{j=1}^{l} \sigma_{ij} w_{j,l}(x;\lambda_l)\varepsilon_{ij}\right|$. Moreover, by Theorem 1 in Montgomery-Smith (1993) we know that

$$\mathbb{P}\left(\|S_l\| > a/3\right) \le 3\mathbb{P}\left(\|S_L\| > a/30\right). \tag{3.2.8}$$

Combining (3.2.7) with (3.2.8) provides the first inequality of (3.2.6). The second inequality follows with (2.12) in Giné, Guillou (2001).

Concerning the rates γ_l, they depend on the properties of the functions V_i. For example, if $V_i \in W^{m,2}(\mathbb{R}^+), m \ge 1$ with

$$W^{m,p}(\mathbb{R}^+) := \left\{f \in C^{m-1}(\mathbb{R}^+) : f^{(m)} \in L^p(\mathbb{R}^+),\, f^{(m-1)} \text{ abs. continuous }\right\}$$

and smoothing splines of m-th order are used, then $\frac{1}{\gamma_l} = (l^{-1}\log l)^{m/(2m+1)}$ with a proper choice of the sequence λ_l (see Eggermont, LaRiccia (2009): p. 100, equation (1.7) and use the fact that almost surely convergence implies convergence in probability).

Suppose that the estimates $\widetilde{f}_{i+1},\dots,\widetilde{f}_{n-1}$ and $\widetilde{Q}_{i+1},\dots,\widetilde{Q}_{n-1}$ are already constructed, where $\|\widetilde{f}_j\|_\infty < K_L$ for $j = i+1,\dots,n$. Define now the estimate for Q_i via

$$\widetilde{Q}_i(v) = \int_{-\infty}^{\infty} (1+\varsigma_{i+1}\widetilde{f}_{i+1}(u))\widetilde{Q}_{i+1}(u)\phi_{(i+1)|i}(u|v)du$$

and set

$$\widetilde{J}_i(y) = \int_y^{\infty} \widetilde{Q}_i(v)\phi_i(v)dv.$$

Since $\widetilde{Q}_i$ and as a result, $\widetilde{J}_i(y)$ may not any longer have the nice properties of the original functions Q_i and J_i respectively, we use cutoff regularisation method to regularise $\widetilde{J}_i(y)$ by setting

$$\hat{J}_i(y) := \begin{cases} J_i^+(y), & \widetilde{J}_i(y) > J_i^+(y), \\ \widetilde{J}_i(y), & \widetilde{J}_i(y) \in [J_i^-(y), J_i^+(y)], \\ J_i^-(y), & \widetilde{J}_i(y) < J_i^-(y), \end{cases}$$

where J_i^- and J_i^+ are two threshold functions on $\mathbb{R}$ satisfying

(AT)

$$0 < J_i^-(y) \le J_i^+(y) \le A_i \le 1.$$

Define

$$\widetilde{f}_i(y) := \widehat{V}^{\leftarrow}_{i,h}(\hat{J}_i(y)).$$

It turns out that with a proper choice of the thresholds J_i^- and J_i^+, the estimate $\widetilde{f}_i$ converges to f_i, as the mesh size of the strike grid $\mathbf{K}$ or the noise levels σ_{il} in (3.2.1) tend to zero.

Chapter 4

Convergence of the regularised calibration algorithm

In order to prove the convergence of the regularised calibration algorithm, we need several assumptions regarding the class of forward operators V_i and the threshold functions J_i^-, J_i^+.

4.1 Assumptions on the class of forward operators and on the thresholds

The following main assumptions concern the functions V_i, $i = 1, \ldots, n$ and their asymptotic behaviour.

(AV) Each function V_i is a non-negative, monotone decreasing and two times continuously differentiable function on $\mathbb{R}^+$ satisfying

$$V_i(0) = A_i = \psi_{0(i+1)}(x_0).$$

Moreover, we assume the representation

$$V_i(z) \quad = \quad \begin{cases} \exp\left(-z^{2/p_-} l_i^-(z)\right), & z > z_0, \\ A_i - \exp\left(-z^{-2/p_+} l_i^+(z)\right), & 0 \le z \le z_0, \end{cases}$$

for some $2 \leq p_- \leq 4$ and some $p_+ > 1$ and $z_0 > 0$, where l_i^+ and l_i^- are monotone decreasing and monotone increasing slowly varying functions, at zero and infinity respectively. That is, l_i^+ and l_i^- fulfill

$$\frac{l_i^-(\lambda z)}{l_i^-(z)} \to 1, \quad z \to +\infty$$

and

$$\frac{l_i^+(\lambda z)}{l_i^+(z)} \to 1, \quad z \to +0$$

for any $\lambda > 0$.

Remark. **(AV)** is motivated by the definition of V_i as the price of the digital caplet

$$V_i(K) = \mathcal{D}_{0T_{n+1}} \mathrm{E}_{\mathbb{Q}^{n+1}} \left[\overline{\mathcal{D}}_{T_i T_{i+1}} 1_{\left\{L^i_{T_i} > K\right\}} \right].$$

The typical forms of the function $V_i(K)$ and its inverse can be seen in Figure 4.1.1.

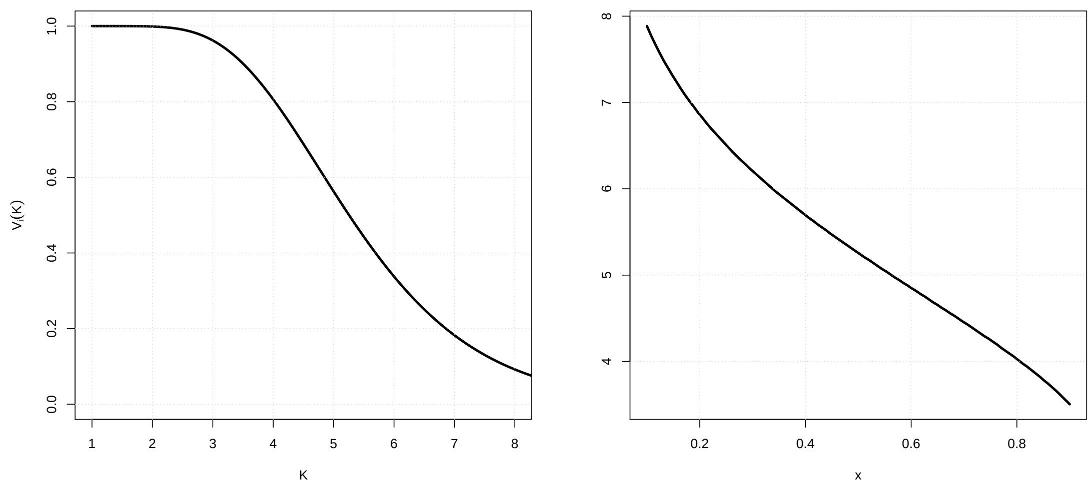

FIGURE 4.1.1: Function $V_i(K)$ (left) and its inverse $V_i^{\leftarrow}(x)$ (right).

The next step is to impose some restrictions on the thresholds.

(AR) The threshold functions J_i^- and J_i^+ satisfy

$$\overline{\mathcal{H}}_i(J_i(y) - h) \leq \frac{J_i^-(y) - h}{J_i(y) - h} \leq \frac{1}{1+\theta} \left[\theta + \left(\frac{p_-}{2} - 1 \right) \right]$$

for $J_i(y) \in [h, x_0]$, $h > 0$ and

$$\overline{\mathcal{H}}_i(A_i - J_i(y) - h) \le \frac{A_i - J_i^+(y) - h}{A_i - J_i(y) - h} \le \frac{1}{1+\theta}\left[\theta - \left(\frac{p_+}{2} + 1\right)\right]$$

for $J_i(y) \in (x_0, A_i - h]$, $h > 0$. With $2 \le p_- \le 4$, $\theta > \frac{p_+}{2} + 1$ each function $\overline{\mathcal{H}}_i(x)$ is non-negative, monotone increasing, bounded above by A_i and slowly varying at $x = 0$ satisfying $\overline{\mathcal{H}}_i(0) = 0$, as well as two times continuously differentiable for $x \le x_0$.

4.2 Main result / Theorem 1

Let $\mathbf{K}$ be the uniform grid with $K_{l+1} - K_l = \Delta$, $l = 1, \dots, L-1$. The following theorem shows that the regularised algorithm indeed converges as $\Delta \to 0$, $K_L \to \infty$ and $K_1 \to 0$. The convergence rates of the estimates $\widetilde{f}_i$ in terms of the weighted L_1 distance:

$$\rho_i(L) := \int_{-\infty}^{\infty} Q_i(y) \left| f_i(y) - \widetilde{f}_i(y) \right| \phi_i(y) dy$$

are of order $1/\sqrt{L_{\gamma,\Delta}}$ up to a slowly varying function of $L_{\gamma,\Delta}$, where $L_{\gamma,\Delta}$ is a generalised "sample size" defined as $L_{\gamma,\Delta} = \min\{\gamma_L, 1/\Delta\}$ and $L \to \infty$.

Theorem 1. *Let $\mathbf{K}$ be the uniform grid with $K_{l+1} - K_l = \Delta$, $l = 1, \dots, L-1$.*
Set $L_{\gamma,\Delta} := \min\{\gamma_L, 1/\Delta\}$, where γ_L is defined as in (3.2.4). If $\Delta \cdot L \asymp \log^{p_-/2+1}(L_{\gamma,\Delta})$, $K_1 \asymp L_{\gamma,\Delta}^{-1/2}$ and $h \asymp L_{\gamma,\Delta}^{-1/2}$, $L_{\gamma,\Delta} \to \infty$, then it holds with probability 1 that

$$\rho_i(L) = O\left(\frac{\mathcal{S}_i(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}\right), \quad L_{\gamma,\Delta} \to \infty,$$

where $\mathcal{S}_i$ is a slowly varying function at infinity.

4.3 Relationship with Black-Scholes Model / Theorem 2

In a special case of the Black-Scholes (BS) model with constant instantaneous volatility $\sigma_i(t) = \sigma_i$ the price at time zero of a digital caplet is given by

$$V_i(z) = \mathcal{D}_{0,T_{i+1}} \Phi(d_i(z)),$$

where Φ is the cumulative normal distribution function,

$$d_i(z) = \frac{\log\left(L_0^i/z\right)}{\sigma_i\sqrt{T_i}} - \frac{1}{2}\sigma_i\sqrt{T_i} \tag{4.3.1}$$

and $\mathcal{D}_{0,T_{i+1}}$ is quoted on the market (see e.g. equation (12) in Hunt, Kennedy, Pelsser (2000)). As a result,

$$V_i(z) \quad = \quad \begin{cases} \exp\left(-\frac{\log^2(z)}{2\sigma_i^2 T_i} - \log\left(\frac{\log(z)}{\sigma_i\sqrt{T_i}}\right)\right), & z \to \infty, \\ A_i - \exp\left(-\frac{1}{2\sigma_i^2 T_i}\log^2\left(\frac{1}{z}\right) - \log\left(\frac{\log\left(\frac{1}{z}\right)}{\sigma_i\sqrt{T_i}}\right)\right), & z \to +0 \end{cases}$$

and the BS model does not formally fulfil the assumption **(AV)**. However, the statement of Theorem 1 continues to hold with a proper choice of the asymptotic behaviour of $\Delta \cdot L$ and functions J_i^-, J_i^+. To illustrate this, let us formulate a theorem for the generalised BS model.

Theorem 2. *Suppose that instead of **(AV)** and **(AR)** the following conditions hold*

$$V_i(z) \quad = \quad \begin{cases} \exp\left(-\frac{\log^2(z)}{2\sigma_i^2 T_i} - l\left(\log(z)\right)\right), & z > z_0, \\ A_i - \exp\left(-\frac{1}{2\sigma_i^2 T_i}\log^2\left(\frac{1}{z}\right) - l\left(\log\left(\frac{1}{z}\right)\right)\right), & 0 \le z \le z_0, \end{cases}$$

where l is a positive, monotone increasing, slowly varying function at infinity. Furthermore, the threshold functions J_i^- and J_i^+ satisfy

$$\overline{\mathcal{H}}_i(J_i(y) - h) \le \frac{J_i^-(y) - h}{J_i(y) - h} \le A_i \tag{4.3.2}$$

for $J_i(y) \in [h, x_0], h > 0$ and

$$\overline{\mathcal{H}}_i(A_i - J_i(y) - h) \le \frac{A_i - J_i^+(y) - h}{A_i - J_i(y) - h} \le A_i \tag{4.3.3}$$

for $J_i(y) \in (x_0, A_i - h], h > 0$; each function $\overline{\mathcal{H}}_i(x)$ is non-negative, monotone increasing, bounded above by A_i and slowly varying at $x = 0$ with $\overline{\mathcal{H}}_i(0) = 0$ as well as two times continuously differentiable for $x \le x_0$. Then, if $\Delta \cdot L \asymp \exp\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}, L_{\gamma,\Delta} \to \infty$ and under the assumptions of Theorem 1, it holds with probability 1 that

$$\rho_i(L) = O\left(\frac{\mathcal{S}_i(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}\right), \quad L_{\gamma,\Delta} \to \infty,$$

where $\mathcal{S}_i$ is a slowly varying function at infinity.

4.4 The choice of regularisation parameters

In order to choose the threshold functions $J_i^-(y)$ and $J_i^+(y)$ appropriately, one can use some prior information on the asymptotic behaviour of J_i. At first, let us make two following assumptions:

(AX) The underlying Markov process (X_t) is given by (B_t), where B_t is the standard Brownian motion. In this case we have

$$\phi_{(i+1)|i}(u|v) = \frac{1}{\sqrt{2\pi\alpha_i}} e^{-(v-u)^2/2\alpha_i} =: k_i(u-v),$$

where α_i is some positive real number.

(AF) The function $f_n(x)$ is bounded on any set of the form $[-\infty, A)$ for each fixed $A \in \mathbb{R}$, and has at most polynomial growth as $x \to +\infty$, i.e.

$$\check{c}_1 x^{q_-} \leq f_n(x) \leq \check{c}_2 x^{q_+}, \quad x \to +\infty$$

for some positive constants $\check{c}_1, \check{c}_2$ and some $q_-, q_+ \geq 0$.

Under assumptions **(AX)**, **(AV)** and **(AF)** we have

$$\tilde{c}_1 |v|^{q_-} \leq Q_{n-1}(v) \leq \tilde{c}_2 |v|^{q_+},$$

where $Q_{n-1}(v) = \int_{-\infty}^{\infty} (1 + \varsigma_n f_n(u)) k_{n-1}(u-v) du$ and $\tilde{c}_1$, $\tilde{c}_2$ are some positive constants.
As a result,

$$\tilde{c}_1 \int_x^{\infty} |v|^{q_-} e^{-v^2/(2\alpha_{n-1})} dv \leq J_{n-1}(x) \leq \tilde{c}_2 \int_x^{\infty} |v|^{q_+} e^{-v^2/(2\alpha_{n-1})} dv, \tag{4.4.1}$$

where

$$\int_x^{\infty} v^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv \asymp \alpha_{n-1} \frac{x^{q_\pm - 1}}{e^{x^2/(2\alpha_{n-1})}}, \quad x \to +\infty. \tag{4.4.2}$$

Since

$$\int_x^\infty v^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv$$

$$= -\alpha_{n-1} \int_x^\infty v^{q_\pm - 1} \left[\frac{-2v}{2\alpha_{n-1}} \right] e^{-v^2/(2\alpha_{n-1})} dv$$

$$= -\alpha_{n-1} \left\{ e^{-v^2/(2\alpha_{n-1})} v^{q_\pm - 1} |_x^\infty \right\}$$

$$+ \alpha_{n-1} \int_x^\infty (q_\pm - 1) \, v^{q_\pm - 2} e^{-v^2/(2\alpha_{n-1})} dv$$

$$= \alpha_{n-1} \frac{x^{q_\pm - 1}}{e^{x^2/(2\alpha_{n-1})}}$$

$$+ \alpha_{n-1} (q_\pm - 1) \int_x^\infty \frac{v^{q_\pm}}{v^2} e^{-v^2/(2\alpha_{n-1})} dv, \quad x \to \infty$$

and so

$$\left| \int_x^\infty v^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv - \alpha_{n-1} \frac{x^{q_\pm - 1}}{e^{x^2/(2\alpha_{n-1})}} \right|$$

$$= \alpha_{n-1} (q_\pm - 1) \int_x^\infty \frac{v^{q_\pm}}{v^2} e^{-v^2/(2\alpha_{n-1})} dv$$

$$\leq \frac{\alpha_{n-1} (q_\pm - 1)}{x^2} \int_x^\infty v^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv$$

$$= o\left(\int_x^\infty v^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv \right), \quad x \to \infty,$$

which is (4.4.2).

Proceeding, we use the fact that $\frac{\breve{l}_i^-(x^2)}{(x^2)^r} \to 0, x \to \infty$ for all $r > 0$. By choosing $r = 1/2$, we have $x^{p_-} \breve{l}_i^-(x^2) = o\left(x^{p_- + 1}\right)$. Combining this with the representation of $V_i^\leftarrow, 1 \leq i \leq n$, specified in Appendix (A.1.1), we get

$$f_{n-1}(x) = V_{n-1}^\leftarrow (J_{n-1}(x)) \asymp x^{p_- + 1}, \quad x \to +\infty.$$

Continuing backwards in this way, we derive

$$J_i(x) \asymp \alpha_{i-1} x^{p_-} e^{-x^2/(2\alpha_i)}, \quad x \to +\infty \tag{4.4.3}$$

as well as

$$f_i(x) \asymp x^{p_-+1}, \quad x \to +\infty$$

for all $i = 1, \dots, n-2$.

Let us go to the case $x \to -\infty$ which is similar up to and including (4.4.1).
Proceeding, we determinate the asymptotic equivalent for $\int_x^\infty |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv,\ x \to -\infty$.

Denote

$$\int_x^\infty |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv = \int_{-\infty}^\infty |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv$$

$$- \int_{-\infty}^x |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv, \quad x \to -\infty.$$

Next,

$$\int_{-\infty}^\infty |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv = \int_{-\infty}^A |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv$$

$$+ \int_A^\infty |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv$$

$$\leq A_{n-1} + \int_A^\infty v^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv, \quad A \to \infty.$$

As we already know

$$\int_A^\infty v^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv \asymp \frac{\alpha_{n-1} A^{q_\pm - 1}}{e^{A^2/(2\alpha_{n-1})}}, \quad A \to +\infty$$

and

$$\int_{-\infty}^x |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv \asymp \frac{\alpha_{n-1} |x|^{q_\pm - 1}}{e^{x^2/(2\alpha_{n-1})}}, \quad x \to -\infty.$$

Then it follows

$$\int_x^\infty |v|^{q_\pm} e^{-v^2/(2\alpha_{n-1})} dv \leq A_{n-1} - \frac{\alpha_{n-1}\,|x|^{q_\pm - 1}}{e^{x^2/(2\alpha_{n-1})}} + \frac{\alpha_{n-1}\,|x|^{q_\pm - 1}}{e^{x^2/(2\alpha_{n-1})}}$$

$$\asymp A_{n-1} - \frac{\alpha_{n-1}\,|x|^{q_\pm - 1}}{e^{x^2/(2\alpha_{n-1})}}, \quad x \to -\infty.$$

Again, it follows with the representation of $V_i^{\leftarrow}, 1 \leq i \leq n$, specified in Appendix (A.1.1)

$$f_{n-1}(x) = V_{n-1}^{\leftarrow}(J_{n-1}(x)) \asymp |x|^{-p_+ + 1}, \quad x \to -\infty.$$

Proceeding backwards in a similar way, we derive

$$J_i(x) \asymp A_i - \alpha_{i-1}|x|^{-p_+} e^{-x^2/(2\alpha_i)}, \quad x \to -\infty \tag{4.4.4}$$

as well as

$$f_i(x) \asymp |x|^{-p_+ + 1}, \quad x \to -\infty$$

for all $i = 1, \ldots, n-2$. The asymptotic relations (4.4.3) and (4.4.4) indicate that the threshold functions

$$J_i^-(x) = \hat{c}_i x^{p_-} e^{-x^2/(2\alpha_i)}$$

and

$$J_i^+(x) = A_i - \check{c}_i |x|^{-p_+} e^{-x^2/(2\alpha_i)}$$

would satisfy **(AR)** and comply with the requirements of Lemma 7 if $\hat{c}_i > 0$ small enough and $\check{c}_i > 0$ big enough.

Furthermore, we will see in section A.2 that under assumptions **(AX)** and **(AF)** in case of the BS model the choice

$$J_i^-(x) = \frac{\hat{c}_i e^{\sigma_{i+1}\sqrt{T_{i+1}}x/\sqrt{\alpha_{i+1}} - x^2/(2\alpha_i)}}{x\sqrt{\alpha_{i+1}} - \sigma_{i+1}\alpha_i\sqrt{T_{i+1}}}$$

and

$$J_i^+(x) = A_i - \frac{\check{c}_i e^{-\sigma_{i+1}\sqrt{T_{i+1}}|x|/\sqrt{\alpha_{i+1}} - x^2/(2\alpha_i)}}{\sqrt{\alpha_{i+1}}\,|x| - \sigma_{i+1}\alpha_i\sqrt{T_{i+1}}}$$

for $\hat{c}_i > 0$ small enough and $\check{c}_i > 0$ big enough, would satisfy J_i^- and J_i^+ requirements of Theorem 2.

Chapter 5

Conclusion

By studying a nonlinear inverse problem with errors in the operator and its ill-posedness phenomena, we try to fill a gap in the literature of inverse problems in option pricing. Our assumptions about the noisy model with fixed design enable a nonparametric regression approach for the estimation of the (errorless) forward operator. The nonparametric approach only assumes that the (errorless) forward operator is of a given nonparametric class. Our assumptions about this nonparametric class include the asymptotic behaviour of the (errorless) forward operator which allows us to analyse in detail the observed ill-posedness phenomena and ways of regularisation. In order to overcome the local ill-posedness, we propose a novel regularisation method based on the generalised inverse, smoothing techniques, the Riemann integral, the linear nonparametric regression estimator and the cutoff regularisation method. We use the cutoff regularisation method to insure that the estimation error at each step of the calibration procedure does not cause too much trouble at the next step. Our model is tolerable for diverse linear nonparametric regression estimators, provided that the linear smoother of the forward operator corrupted with noise is close to the (errorless) forward operator for rather big amounts of points of the strike grid. It turns out that with a proper choice of the thresholds, the regularised calibration algorithm converges, when the mesh size of the strike grid or the noise levels tend to zero. By verifying the convergence rates in terms of the weighted L_1 distance, we can see that the special case of the generalised Black-Scholes model does not formally fulfil the assumption about the asymptotic behaviour of the (errorless) forward operator. However, the main result concerning the convergence rates holds, and the analysis can be done in a similar manner as in the previously studied case. Nevertheless, the optimality of the convergence rate remains an open question.

Appendix A

Appendix: Mathematical details

A.1 Technical Lemmas and Proofs relating to Theorem 1

Lemma 3. *The assumption **(AV)** implies that the inverse function $V_i^{\leftarrow}$ is bounded on any subset of $[0,A_i]$ not containing 0 and its first two derivatives are bounded on any subset of $(0,A_i)$. Moreover, the following representations hold*

$$V_i^{\leftarrow}(x) = \begin{cases} \log^{p_-/2}\left(\frac{1}{x}\right)\breve{l}_i^-\left(\log\frac{1}{x}\right), & 0 \leq x \leq x_0, \\ \log^{-p_+/2}\left(\frac{1}{A_i-x}\right)\breve{l}_i^+\left(\log\frac{1}{A_i-x}\right), & x_0 < x \leq A_i \end{cases} \tag{A.1.1}$$

and

$$\left(V_i^{\leftarrow}\right)^{(j)}(x) = \begin{cases} \frac{(-1)^j p_-}{2}\left(\frac{1}{x}\right)^j \log^{p_-/2-1}\left(\frac{1}{x}\right)\breve{l}_i^-\left(\log\frac{1}{x}\right), & 0 \leq x \leq x_0, \\ -\frac{p_+}{2}\left(\frac{1}{A_i-x}\right)^j \log^{-p_+/2-1}\left(\frac{1}{A_i-x}\right)\breve{l}_i^+\left(\log\frac{1}{A_i-x}\right), & x_0 < x \leq A_i \end{cases} \tag{A.1.2}$$

for $j=1,2$ and some $x_0 \in (0,A_i)$, where $\breve{l}_i^+$ and $\breve{l}_i^-$ are slowly varying functions. Furthermore,

$$\frac{\left(V_i^{\leftarrow}\right)'(x)}{x\left(V_i^{\leftarrow}\right)''(x)} \asymp -1 + \left(\frac{p_-}{2} - 1\right)\log^{-1}\frac{1}{x}, \quad x \to 0 \tag{A.1.3}$$

and

$$\frac{\left(V_i^{\leftarrow}\right)'(x)}{(A_i - x)\left(V_i^{\leftarrow}\right)''(x)} \asymp 1 + \left(\frac{p_+}{2} + 1\right)\log^{-1}\frac{1}{A_i - x}, \quad x \to A_i. \tag{A.1.4}$$

Lemma 4. *The representations (A.1.1) and (A.1.2) and the continuity of* $(V_i^{\leftarrow})^{(j)}$, $j = 1,2$ *(see Proof of Lemma 3) imply that there are constants* $C_{i,1} > 0$ *and* $C_{i,2} > 0$, *such that*

$$\sup_{x\in[a,b]} \left|(V_i^{\leftarrow})'(x)\right| \le C_{i,1} \max\left\{\left|(V_i^{\leftarrow})'(a)\right|, \left|(V_i^{\leftarrow})'(b)\right|\right\} \tag{A.1.5}$$

and

$$\sup_{x\in[a,b]} \left|(V_i^{\leftarrow})''(x)\right| \le C_{i,2} \max\left\{\left|(V_i^{\leftarrow})''(a)\right|, \left|(V_i^{\leftarrow})''(b)\right|\right\} \tag{A.1.6}$$

for $0 \le a < b \le A_i$.

Lemma 5. *For any* $h > 0$ *and any fixed* $x \ge V_i(K_L) + h$ *it holds that*

$$\left|V_{i,h}^{\leftarrow}(x) - \widehat{V}_{i,h}^{\leftarrow}(x)\right| \le K_1 + \frac{K_L \cdot \mathcal{L}_\Phi}{h}\left[\mathfrak{R}_i(L) + \frac{\Delta\|V_i'\|_\infty}{2}\right],$$

where

$$\mathfrak{R}_i(L) := \sup_{x\in[K_1,K_L]} |V_i(x) - \widehat{V}_{i,L}(x)|.$$

Lemma 6. *It holds for any* $x \in (h, A_i - h)$ *that*

$$\left|V_{i,h}^{\leftarrow}(x) - V_i^{\leftarrow}(x)\right| \le 2hC_{i,1}\max\left\{\left|(V_i^{\leftarrow})'(x-h)\right|, \left|(V_i^{\leftarrow})'(x+h)\right|\right\}.$$

The next lemma shows that with a proper choice of the thresholds $J_i^-(y)$ and $J_i^+(y)$, $V_i^{\leftarrow}(J_i(y))$ and $V_i^{\leftarrow}(\widehat{J}_i(y))$ are close, provided $J_i(y)$ and $\widetilde{J}_i(y)$ are close.

Lemma 7. *Set*

$$\mathcal{H}_i^-(x) := \frac{1}{1+\theta}\left[\theta + \left(\frac{p_-}{2} - 1\right)\log^{-1}\frac{1}{x}\right],$$

$$\mathcal{H}_i^+(x) := \frac{1}{1+\theta}\left[\theta - \left(\frac{p_+}{2} + 1\right)\log^{-1}\frac{1}{x}\right]$$

for some $\theta > \frac{p_+}{2} + 1$ *and* $x \in [0; A_i]$, *then*

$$\left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right| \le \xi_i(y)\left(J_i(y) - \widetilde{J}_i(y)\right)^2,$$

where

$$\xi_i(y) := \left\{ \frac{1}{2} + c(1+\theta) \right\} q_i(y)$$

for some $c > 0$ *with*

$$q_i(y) := \sup_{x \in [J_i^-(y), J_i^+(y)]} \left| (V_i^{\leftarrow})''(x) \right|,$$

provided that

$$J_i^-(y) \leq J_i(y) \mathcal{H}_i^-(J_i(y)), \quad J_i(y) \in [0, x_0] \tag{A.1.7}$$

and

$$A_i - J_i^+(y) \leq (A_i - J_i(y)) \mathcal{H}_i^+(A_i - J_i(y)), \quad J_i(y) \in (A_i - x_0, A_i]. \tag{A.1.8}$$

Lemma 8. *It holds for any fixed* $0 < x_0 < A_i$ *and* $j = 1, 2$ *that*

$$\int_\varepsilon^{x_0} \left| (V_i^{\leftarrow})^{(j)} (x \overline{\mathcal{H}}_i(x)) \right| dx \lesssim \left(\frac{1}{\varepsilon} \right)^{j-1} S_{i,j}^- \left(\frac{1}{\varepsilon} \right),$$

$$\int_{x_0}^{A_i - \varepsilon} \left| (V_i^{\leftarrow})^{(j)} (A_i - (A_i - x) \overline{\mathcal{H}}_i(A_i - x)) \right| dx \lesssim \left(\frac{1}{\varepsilon} \right)^{j-1} S_{i,j}^+ \left(\frac{1}{\varepsilon} \right)$$

as $\varepsilon \to 0$, *where* $S_{i,1}^\pm$ *and* $S_{i,2}^\pm$ *are slowly varying functions.*

A.1.1 Proof of Lemma 3

As a preliminary, we shall assume the existence of an inverse function $V_i^{\leftarrow}$.

Next, notice that $V_i^{\leftarrow}(A_i) = 0$ due to $V_i(0) = A_i$. Using Proposition 1.3.6 (v) in Bingham, Goldie, Teugels (1987) and the continuity of V_i, we have $\lim_{z\to\infty} V_i(z) = 0$, and the set $\{x : V_i(x) = 0\}$ is not empty. Define $b = \inf\{x : V_i(x) = 0\} < \infty$. Using **(AV)**, $V_i : [0,b] \to [0,A_i]$ is monotone decreasing and two times continuously differentiable. Thus, we have the same properties for $V_i^{\leftarrow} : [0,A_i] \to [0,b]$ (see for instance Section 12, Theorem 1 in Forster (1983)).

In particular, $(V_i^{\leftarrow})^{(j)}[0,A_i], j = 0,1,2$ are bounded (see for instance Section 11, Theorem 2 in Forster (1983)). It follows that non-negative constants $M_i^{(j)}, j = 0,1,2$ exist, satisfying $\left|(V_i^{\leftarrow})^{(j)}(x)\right| \le M_i^{(j)}$ for all $x \in [0,A_i]$. Hence, it holds that $M_i^{(j)} : \left|(V_i^{\leftarrow})^{(j)}(x)\right| \le M_i^{(j)}$ for all $x \in A \subseteq [0,A_i]$. As a result, $(V_i^{\leftarrow})^{(j)}, j = 0,1,2$ is bounded on any subset of $[0,A_i]$.

Consider now V_i on $\mathbb{R}^+$. $V_i[0,b]$ safisfies the above discussed relations as well as $V_i(\mathbb{R}^+ \setminus [0,b]) = 0$ due to the continuity and the decreasing monotony of V_i. This is the reason why $V_i^{\leftarrow}(0) = \mathbb{R}^+ \setminus [0,b]$ is an unbounded set which yields the function $V_i^{\leftarrow}(x)$ being unbounded in $x = 0$.

Next, let's suppose $(V_i^{\leftarrow})^{(j)}, j = 1,2$ are of the sort mentioned in (A.1.2), and let's use Proposition 1.3.6 ((i)-(v)) in Bingham, Goldie, Teugels (1987) to verify $\lim_{x\to 0}\left|(V_i^{\leftarrow})^{(j)}(x)\right| = \infty$, $j = 1,2$. To be more precisely: verify that $\log x$ is slowly varying at infinity, thus $\log\frac{1}{x}$ is slowly varying at 0. Then use (iii) to get $\check{l}^{-}\left(\log\frac{1}{x}\right)$ is slowly varying at infinity as $x \to 0$. Finally, use (i), (v) to get $\left|\log^{p_-/2-1}\left(\frac{1}{x}\right)\check{l}^{-}\left(\log\frac{1}{x}\right)\right| \to \infty$, $p_- \ge 2$ as $x \to 0$, and combine this with the fact that $\left|\left(\frac{1}{x}\right)^j\right| \to \infty$, as $x \to 0$. Similarly, $\lim_{x\to A_i}\left|(V_i^{\leftarrow})^{(j)}(x)\right| = \infty, j = 1,2$.

As a result, $V_i^{\leftarrow}$ is bounded on any subset of $[0,A_i]$ not containing 0, and $(V_i^{\leftarrow})', (V_i^{\leftarrow})''$ are bounded on any subset of $[0,A_i]$ not containing 0 and A_i, provided that the inverse function $V_i^{\leftarrow}$ exists and the representations $(V_i^{\leftarrow})^{(j)}, j = 0,1,2$ hold. To complete the proof, let us close this gap.

Let us prove the representations (A.1.1) and (A.1.2) for $j = 0,1,2$ and $0 \le x \le x_0$.

Denote $f_i(z) := z^{2/p_-} l_i^-(z)$, then the function $h_i(z) := \log(f_i(e^z))$ satisfies

$$\lim_{z\to+\infty} h_i'(z) = \lim_{z\to+\infty;\ u\to 0} \frac{h_i(z+u)-h_i(z)}{u}$$

$$= \lim_{z\to+\infty;\ u\to 0} \frac{\log\left(\frac{f_i(e^{z+u})}{f_i(e^z)}\right)}{u}$$

$$= \lim_{z\to+\infty;\ u\to 0} \frac{\log\left(\frac{(e^{z+u})^{2/p_-} l_i^-(e^{z+u})}{(e^z)^{2/p_-} l_i^-(e^z)}\right)}{u}$$

$$= \lim_{u\to 0} \frac{u\cdot 2/p_-}{u} = 2/p_-,$$

where we used the fact that the logarithmic function is continuous, and with $\lambda = e^u$ it holds that

$$\lim_{z\to\infty} \frac{l_i^-(e^{z+u})}{l_i^-(e^z)} = 1$$

(see Theorem 1.3.3 in Bingham, Goldie, Teugels (1987)). Since $h_i(z) \to \infty$ as $z \to \infty$, h_i has an inverse $h_i^{\leftarrow}(y)$ for large enough y, and $(h_i^{\leftarrow})'(y) = \frac{1}{h_i'(h_i^{\leftarrow}(y))} = \frac{1}{h_i'(z)} \to p_-/2$ as $z \to \infty$. Hence, for $z > z_0$ we have

$$(h_i^{\leftarrow})(z) = z\frac{p_-}{2} + c(z) + \int_{z_0}^{z} \eta(t)dt$$

$$=: z\frac{p_-}{2} + \breve{h}_i^-(z),$$

where c and η are measurable and bounded functions on any subset of $[z_0, z]$ and

$c(z) \to c > 0, \eta(z) \to 0$ as $z \to \infty$. It follows that f_i has an inverse $f_i^{\leftarrow}(y)$ for large enough y, and so V_i has an inverse $V_i^{\leftarrow}(x)$ satisfying

$$V_i^{\leftarrow}(x) = f_i^{\leftarrow}\left(\log\frac{1}{x}\right) = \exp\left(h_i^{\leftarrow}\left(\log\log\frac{1}{x}\right)\right)$$

$$= \log^{p_-/2}\left(\frac{1}{x}\right)\exp\left(\breve{h}_i^-\left(\log\log\frac{1}{x}\right)\right)$$

for all $x < x_0$, where in the second equation we used $h_i(z) = \log(f_i(e^z))$ as $z \to \infty$ to justify $f_i(q) = \exp(h_i(\log(q)))$ with $q = e^z$ as $z \to \infty$ and so $f_i^{\leftarrow}(y) = \exp(h_i^{\leftarrow}(\log(y)))$ for y large enough. According to the representation theorem for slowly varying functions (see Theorem 1.3.1 in Bingham, Goldie, Teugels (1987)) there is a slowly varying function, say $\check{l}_i^{-}$, such that

$$V_i^{\leftarrow}(x) = \log^{p_-/2}\left(\frac{1}{x}\right)\check{l}_i^{-}\left(\exp\left(\log\log\frac{1}{x}\right)\right) = \log^{p_-/2}\left(\frac{1}{x}\right)\check{l}_i^{-}\left(\log\frac{1}{x}\right)$$

for all $x < x_0$.

Let us now focus on the first and second derivative of $V_i^{\leftarrow}(x), x < x_0$.
Set $f(y) := \log^{p_-/2}(y)L(\log y)$ for some slowly varying function L, then

$$f'(y) = \frac{p_-}{2y}\log^{p_-/2-1}(y)L(\log y) + \frac{1}{y}\log^{p_-/2}(y)L'(\log y)$$

$$= \frac{1}{y}\log^{p_-/2-1}(y)L(\log y)\left(\frac{p_-}{2} + \frac{L'(\log y)}{L(\log y)}\log y\right)$$

and

$$f''(y) = \frac{p_-}{2y^2}\left(\frac{p_-}{2}-1\right)\log^{p_-/2-2}(y)L(\log y) - \frac{p_-}{2y^2}\log^{p_-/2-1}(y)L(\log y)$$

$$+\frac{p_-}{2y^2}\log^{p_-/2-1}(y)L'(\log y) - \frac{1}{y^2}\log^{p_-/2}(y)L'(\log y)$$

$$+\frac{p_-}{2y^2}\log^{p_-/2-1}(y)L'(\log y) + \frac{1}{y^2}\log^{p_-/2}(y)L''(\log y)$$

$$= \frac{\log^{p_-/2-2}(y)}{y^2}L(\log y)\times$$

$$\times\left[\frac{p_-}{2}\left(\frac{p_-}{2}-1\right) - \frac{p_-}{2}\log(y) - \log^2(y)\frac{L'(\log y)}{L(\log y)} + \log(y)\left(p_-\nu(y) + \log(y)\tau(y)\right)\right]$$

$$= -\frac{\log^{p_-/2-1}(y)}{y^2}L(\log y)\left(\frac{p_-}{2} + \log(y)\frac{L'(\log y)}{L(\log y)} - \frac{p_-}{2}\left(\frac{p_-}{2}-1\right)\log^{-1}y + o(\log^{-1}y)\right),$$

where we used the fact that

$$\nu(y) := \frac{L'(\log y)}{L(\log y)} = o(\log^{-1} y), \quad \tau(y) := \frac{L''(\log y)}{L(\log y)} = o(\log^{-2} y), \quad y \to \infty.$$

Since

$$\frac{d}{dx} f(1/x) = -x^{-2} f'(1/x)$$

and

$$\frac{d^2}{dx^2} f(1/x) = 2x^{-3} f'(1/x) + x^{-4} f''(1/x)$$

we have for $x < x_0$

$$(V_i^{\leftarrow})'(x) = -\frac{1}{x} \log^{p_-/2-1}\left(\frac{1}{x}\right) L\left(\log\frac{1}{x}\right)\left[\frac{p_-}{2} + \frac{L'(-\log x)}{L(-\log x)} \log\frac{1}{x}\right]$$

and

$$(V_i^{\leftarrow})''(x) = \frac{1}{x^2} \log^{p_-/2-1}\left(\frac{1}{x}\right) L\left(\log\frac{1}{x}\right) \times$$

$$\times \left[\frac{p_-}{2} + \frac{L'(-\log x)}{L(-\log x)} \log\frac{1}{x} + \frac{p_-}{2}\left(\frac{p_-}{2} - 1\right)\log^{-1}\frac{1}{x} + o\left(\log^{-1}\frac{1}{x}\right)\right]$$

with $L = \breve{l}_i^-$. Obviously, (A.1.3) follows. To conclude this proof, we shall verify the existence of $V_i^{\leftarrow}(x)$, $x_0 < x \le A_i$ and that $(V_i^{\leftarrow}(x))^{(j)}$ can be represented as claimed for $j = 0,1,2$, $x_0 < x \le A_i$ and some $x_0 \in (0, A_i)$.

Redenote $f_i(z) := z^{-2/p_+} l_i^+(z)$, then the function $h_i(z) = \log(f_i(e^z))$ satisfies $h_i'(z) \to -2/p_+$ as $z \to +0$. In a similar manner as above we get the existence of the inverse of h_i, say $h_i^{\leftarrow}(y)$, for small enough y satisfying $(h_i^{\leftarrow})'(y) \to -p_+/2$ as $z \to +0$. Hence,

$$V_i^{\leftarrow}(x) = \exp\left(h_i^{\leftarrow}\left(\log\log\frac{1}{A_i - x}\right)\right) = \log^{-p_+/2}\left(\frac{1}{A_i - x}\right) \breve{l}_i^+\left(\log\frac{1}{A_i - x}\right)$$

for all $x_0 < x \le A_i$. Reset $f(y) := \log^{-p_+/2}(y) L(\log y)$ for some slowly varying function L, then

$$f'(y) = \frac{1}{y} \log^{-p_+/2-1}(y) L(\log y)\left(\frac{-p_+}{2} + \frac{L'(\log y)}{L(\log y)} \log y\right)$$

and

$$f''(y) = \frac{\log^{p_-/2-2}(y)}{y^2} L(\log y) \times$$

$$\times \left[\frac{p_+}{2} \left(\frac{p_+}{2} + 1 \right) + \frac{p_+}{2} \log(y) - \log^2(y) \frac{L'(\log y)}{L(\log y)} + \log(y) \left(\log(y) \tau(y) - p_- \nu(y) \right) \right]$$

$$= -\frac{\log^{-p_+/2-1}(y)}{y^2} L(\log y) \times$$

$$\times \left(\frac{-p_+}{2} + \log(y) \frac{L'(\log y)}{L(\log y)} - \frac{p_+}{2} \left(\frac{p_+}{2} + 1 \right) \log^{-1} y + o(\log^{-1} y) \right),$$

where, one more time, we used the fact that

$$\nu(y) = \frac{L'(\log y)}{L(\log y)} = o(\log^{-1} y), \quad \tau(y) = \frac{L''(\log y)}{L(\log y)} = o(\log^{-2} y), \quad y \to \infty.$$

Since

$$\frac{d}{dx} f\left(\frac{1}{A_i - x} \right) = \frac{f'\left(\frac{1}{A_i - x} \right)}{(A_i - x)^2}$$

and

$$\frac{d^2}{dx^2} f\left(\frac{1}{A_i - x} \right) = 2(A_i - x)^{-3} f'\left(\frac{1}{A_i - x} \right) + (A_i - x)^{-4} f''\left(\frac{1}{A_i - x} \right),$$

we have for $x_0 < x \le A_i$ and some $x_0 \in (0, A_i)$

$$(V_i^{\leftarrow})'(x) = \frac{1}{A_i - x} \log^{-p_+/2-1} \left(\frac{1}{A_i - x} \right) L\left(\log \frac{1}{A_i - x} \right) \times$$

$$\times \left[-\frac{p_+}{2} + \frac{L'(-\log(A_i - x))}{L(-\log(A_i - x))} \log \frac{1}{A_i - x} \right],$$

as well as

$$(V_i^{\leftarrow})''(x) = \frac{1}{(A_i-x)^2}\log^{-p_+/2-1}\left(\frac{1}{A_i-x}\right) L\left(\log\frac{1}{A_i-x}\right) \times$$

$$\times \left[\frac{-p_+}{2} + \frac{L'(-\log(A_i-x))}{L(-\log(A_i-x))}\log\frac{1}{A_i-x} + \frac{p_+}{2}\left(\frac{p_+}{2}+1\right)\log^{-1}\frac{1}{A_i-x} + o\left(\log^{-1}\frac{1}{A_i-x}\right)\right]$$

with $L = \check{l}_i^+$. Obviously, (A.1.4) follows. □

A.1.2 Proof of Lemma 4

We will first prove the assertion for $0 < a < b < A_i$. In agreement with Lemma 3, $(V_i^{\leftarrow})'$ is bounded on any subset of $[a,b]$. Thus, due to the consistency of $(V_i^{\leftarrow})'$ on $[a,b]$ and due to the fact that $[a,b]$ is a closed interval, $(V_i^{\leftarrow})'$ attains its maximum and minimum. Furthermore, due to monotony of $(V_i^{\leftarrow})'$ its extreme values are on the boundary of $[a,b]$.
Since the image of $(V_i^{\leftarrow})'([a,b])$ is a subset of $\mathbb{R}^-$, it follows that $\left|(V_i^{\leftarrow})'\right| : [a,b] \to \mathbb{R}^+$ is continuous, bounded and monotone as well. Hence, $\left|(V_i^{\leftarrow})'\right|$ attains its extreme values on the boundary of its domain. Suppose c is an intermediate value between $\min\left\{\left|(V_i^{\leftarrow})'(a)\right|, \left|(V_i^{\leftarrow})'(b)\right|\right\}$ and $\max\left\{\left|(V_i^{\leftarrow})'(a)\right|, \left|(V_i^{\leftarrow})'(b)\right|\right\}$.
Define a function $\pi : [a,b] \to \mathbb{R}$ by $\pi(p) := \left|(V_i^{\leftarrow})'(p)\right| - c$. Then π is a continuous function satisfying $\min\{\pi(a),\pi(b)\} < 0 < \max\{\pi(a),\pi(b)\}$. Using the intermediate value theorem, some $x \in [a,b]$ exists, such that $\pi(x) = 0$, and so $c = \left|(V_i^{\leftarrow})'(x)\right|$. Since c is arbitrarily picked, the function $\left|(V_i^{\leftarrow})'\right|$ takes any values $\left|(V_i^{\leftarrow})'(x)\right|$, $x \in [a,b]$. Thus, for $0 < a < b < A_i$ we get $\sup_{x\in[a,b]}\left|(V_i^{\leftarrow})'(x)\right| = \max\left\{\left|(V_i^{\leftarrow})'(a)\right|, \left|(V_i^{\leftarrow})'(b)\right|\right\}$.

To complete the proof, we will verify the assertion for the two remaining cases: $a = 0$ and $b = A_i$. Using (A.1.2) we have $\left|(V_i^{\leftarrow}(x))'\right| \asymp \left|\frac{p_-/2}{x\log x}\log^{p_-/2}\left(\frac{1}{x}\right)\check{l}_i^-\left(\log\frac{1}{x}\right)\right|, x \searrow a$. The right side of the last equation is maximal in $x = a$. Hence, $\left|(V_i^{\leftarrow}(x))'\right| \le \hat{C}_{i,1}\left|(V_i^{\leftarrow}(a))'\right|, x \searrow a$ for some positive constant $\hat{C}_{i,1}$.
Furthermore, again with (A.1.2) we get $\left|(V_i^{\leftarrow}(x))'\right| \le \check{C}_{i,1}\left|(V_i^{\leftarrow}(b))'\right|, x \nearrow b$ for some positive constant $\check{C}_{i,1}$ and thus $\sup_{\{x=a,\ x=b\}}\left|(V_i^{\leftarrow})'(x)\right| \le C_{i,1}\max\left\{\left|(V_i^{\leftarrow})'(a)\right|, \left|(V_i^{\leftarrow})'(b)\right|\right\}$ with $C_{i,1} = \max\left\{\hat{C}_{i,1}, \check{C}_{i,1}\right\}, a = 0, b = A_i$ and (A.1.5) as required.
The assertion (A.1.6) follows readily by choosing $(V_i^{\leftarrow})''$ in place of $(V_i^{\leftarrow})'$. □

A.1.3 Proof of Lemma 5

Since Φ is defined as a function with support $[-1;1]$ and for $K \in [K_L;\infty), h \geq 0$, we have $\frac{V_i(K)-x}{h} < -1 \Leftrightarrow x > h + V_i(K) \Rightarrow x > h + V_i(K_L)$ due to the decreasing monotony of V_i. Furthermore, because Φ is per definition right-continuous and due to the asymptotic property on the boundary of its domain, we have $\lim_{y \to -1} \Phi(y) = \Phi(-1) = 0$. It follows that $\int_{K_L}^{\infty} \Phi\left(\frac{V_i(K)-x}{h}\right) dK = 0$ for all $x \geq V_i(K_L) + h$. Hence,

$$\left|V_{i,h}^{\leftarrow}(x) - \widehat{V}_{i,h}^{\leftarrow}(x)\right| = \left|\int_0^{\infty} \Phi\left(\frac{V_i(K)-x}{h}\right) dK - \sum_{l=1}^{L-1} (K_{l+1} - K_l)\, \Phi\left(\frac{\widehat{V}_{i,L}(K_l)-x}{h}\right)\right|$$

$$= \left|\int_0^{K_L} \Phi\left(\frac{V_i(K)-x}{h}\right) dK - \sum_{l=1}^{L-1} \int_{K_l}^{K_{l+1}} \Phi\left(\frac{\widehat{V}_{i,L}(K_l)-x}{h}\right) dK\right|$$

$$= \left|\int_0^{K_1} \Phi\left(\frac{V_i(K)-x}{h}\right) dK\right.$$

$$\left. + \sum_{l=1}^{L-1} \int_{K_l}^{K_{l+1}} \left(\Phi\left(\frac{V_i(K)-x}{h}\right) - \Phi\left(\frac{\widehat{V}_{i,L}(K_l)-x}{h}\right)\right) dK\right|.$$

Combining the mean value theorem for integration with the fact that Φ is a distribution function and thus $\Phi\left(\frac{V_i(K)-x}{h}\right) \leq 1, \forall K \geq 0$ holds, we make use of the triangle inequality and of the Lipschitz continuity of Φ to get

$$\left|V_{i,h}^{\leftarrow}(x) - \widehat{V}_{i,h}^{\leftarrow}(x)\right| \leq K_1 + \frac{\mathcal{L}_{\Phi}}{h} \sum_{l=1}^{L-1} \int_{K_l}^{K_{l+1}} \left(\left|V_i(K_l) - \widehat{V}_{i,L}(K_l)\right| + |V_i(K_l) - V_i(K)|\right) dK.$$

Once again, using the mean value theorem we have

$$\int_{K_l}^{K_{l+1}} |V_i(K_l) - V_i(K)|\,dK$$

$$\leq \left\|V_i'\right\|_\infty \int_{K_l}^{K_{l+1}} |K_l - K|\,dK$$

$$= \left\|V_i'\right\|_\infty \int_{K_l}^{K_{l+1}} (K - K_l)\,dK$$

$$= \left\|V_i'\right\|_\infty \left[\frac{1}{2}K_{l+1}^2 - \frac{1}{2}K_l^2 - (K_{l+1} - K_l)\,K_l\right]$$

$$= \left\|V_i'\right\|_\infty \left[\frac{1}{2}(K_{l+1} - K_l)^2 - K_l^2 + K_l^2\right]$$

$$= (K_{l+1} - K_l)\,\frac{\Delta \left\|V_i'\right\|_\infty}{2}$$

and thus by $\sum_{l=1}^{L-1} (K_{l+1} - K_l) = K_L - K_1$ we get

$$\left|V_{i,\overleftarrow{h}}(x) - \widehat{V}_{i,\overleftarrow{h}}(x)\right| \leq K_1 + \frac{(K_L - K_1)\mathcal{L}_\Phi}{h}\left[\mathfrak{R}_i(L) + \frac{\Delta \left\|V_i'\right\|_\infty}{2}\right]$$

$$\leq K_1 + \frac{K_L \mathcal{L}_\Phi}{h}\left[\mathfrak{R}_i(L) + \frac{\Delta \left\|V_i'\right\|_\infty}{2}\right],$$

where the derivative V_i' is uniformly bounded on $\mathbb{R}^+$ due to **(AV)**. Indeed, we have, for instance, for $z > z_0$

$$\left|V_i'(z)\right| = \left|z^{2/p_- - 1}\left(\frac{2}{p_-} l_i^-(z) + z\left(l_i^-\right)'(z)\right)\exp\left(-z^{2/p_-} l_i^-(z)\right)\right|$$

$$\lesssim z^{2/p_- - 1} l_i^-(z)\exp\left(-z^{2/p_-} l_i^-(z)\right)$$

on (z_0, ∞) due to $z\left(l_i^-\right)'(z) = o(l_i^-(z))$ (see, e.g. Bingham, Goldie, Teugels (1987)). □

A.1.4 Proof of Lemma 6

Due to the support of Φ we have with $\varphi(K) := \Phi\left(\frac{V_i(K)-x}{h}\right) - 1_{\{V_i(K)>x\}}$

$$\left|V_{i,h}^{\leftarrow}(x) - V_i^{\leftarrow}(x)\right| = \left|\int_0^{V_i^{\leftarrow}(x+h)} \varphi(K)dK + \int_{V_i^{\leftarrow}(x+h)}^{V_i^{\leftarrow}(x-h)} \varphi(K)dK + \int_{V_i^{\leftarrow}(x-h)}^{\infty} \varphi(K)dK\right|.$$

Combining (3.2.2) with the fact that $V_i^{\leftarrow}$ is monotone decreasing and thus

$$V_i^{\leftarrow}(x+h) \leq V_i^{\leftarrow}(x) \leq V_i^{\leftarrow}(x-h), \quad \forall h \geq 0$$

holds, we get

$$\int_0^{V_i^{\leftarrow}(x+h)} 1_{\{V_i(K)>x\}} dK = V_i^{\leftarrow}(x+h),$$

as well as

$$\int_{V_i^{\leftarrow}(x-h)}^{\infty} 1_{\{V_i(K)>x\}} dK = 0.$$

Furthermore, for $K \in [V_i^{\leftarrow}(x-h), \infty)$ and $h > 0$ (due to the fact that $V_i^{\leftarrow}$ is monotone decreasing) we have

$$V_i(K) \leq x-h \Leftrightarrow V_i(K) - x \leq -h \Rightarrow \frac{V_i(K)-x}{h} \leq -1.$$

Moreover, using the monotony of Φ and the fact that the image of a distribution function is a subset of $[0,1]$, we get for $K \in [V_i^{\leftarrow}(x-h), \infty)$ and $h > 0$

$$\Phi\left(\frac{V_i(K)-x}{h}\right) \leq \Phi(-1) = 0$$

and thus

$$\int_{V_i^{\leftarrow}(x-h)}^{\infty} \Phi\left(\frac{V_i(K)-x}{h}\right) dK = 0.$$

Next, with a similar argumentation we have for $K \in [0; V_i^{\leftarrow}(x+h)]$ and $h > 0$

$$V_i(K) \geq x+h \Leftrightarrow V_i(K) - x \geq h \Rightarrow \frac{V_i(K)-x}{h} \geq 1$$

$$\Rightarrow \Phi\left(\frac{V_i(K)-x}{h}\right) \geq \Phi(1) = 1.$$

Hence,

$$\int_0^{V_i^{\leftarrow}(x+h)} \Phi\left(\frac{V_i(K)-x}{h}\right) dK = V_i^{\leftarrow}(x+h).$$

Finally, again using the fact that Φ is a distribution function and thus $\Phi\left(\frac{V_i(K)-x}{h}\right) \leq 1$ holds for any $K > 0$, we have $\left|\Phi\left(\frac{V_i(K)-x}{h}\right) - 1_{\{V_i(K)>x\}}\right| \leq 1$ for any $K > 0$. Therefore,

$$\begin{aligned} \left|V_{i,h}^{\leftarrow}(x) - V_i^{\leftarrow}(x)\right| &\leq |V_i^{\leftarrow}(x-h) - V_i^{\leftarrow}(x+h)| \\ &\leq 2h \sup_{\upsilon \in [x-h;\, x+h]} \left|(V_i^{\leftarrow})'(\upsilon)\right|, \end{aligned}$$

where in the last inequality we make use of the mean value theorem. We complete the proof using (A.1.5). □

A.1.5 Proof of Lemma 7

Notice that the definition of $\mathcal{H}_i^{\pm}(J_i(y))$ implies that $J_i^-(y) \le J_i(y) \le J_i^+(y)$ holds and consider 3 cases.

CASE $J_i^-(y) \le \widetilde{J}_i(y) \le J_i^+(y)$:

$$\left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widetilde{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \frac{1}{2}\left|(V_i^{\leftarrow})''(x)\right|\left|J_i(y) - \widetilde{J}_i(y)\right|^2, \quad x \in (J_i(y), \widetilde{J}_i(y))$$

$$\le \frac{1}{2}\left|(V_i^{\leftarrow})''(x)\right|\left|J_i(y) - \widetilde{J}_i(y)\right|^2, \quad x \in (J_i^-(y), J_i^+(y))$$

$$\le \frac{q_i(y)}{2}\left|J_i(y) - \widetilde{J}_i(y)\right|^2.$$

CASE $\widetilde{J}_i(y) < J_i^-(y)$: Due to the definition of $\mathcal{H}_i^-$ and (A.1.3) we have

$$J_i(y) - J_i^-(y) \ge J_i(y)\left[1 - \mathcal{H}_i^-(J_i(y))\right]$$

$$= \frac{J_i(y)}{1+\theta}\left[1 - \left(\frac{p_-}{2} - 1\right)\log^{-1}\frac{1}{J_i(y)}\right]$$

$$\asymp \frac{1}{1+\theta}\left|\frac{(V_i^{\leftarrow})'(J_i(y))}{(V_i^{\leftarrow})''(J_i(y))}\right|, \quad J_i(y) \to 0.$$

Since the function $\frac{(V_i^{\leftarrow})'(x)}{(V_i^{\leftarrow})''(x)}$ is continuous, there is a constant $\check{c} > 0$, such that

$$\left|(V_i^{\leftarrow})'(J_i(y))\right| = \left|(V_i^{\leftarrow})''(J_i(y))\right| \left|\frac{(V_i^{\leftarrow})'(J_i(y))}{(V_i^{\leftarrow})''(J_i(y))}\right|$$

$$\leq \check{c}(1+\theta)\left|(V_i^{\leftarrow})''(J_i(y))\right|\left(J_i(y) - J_i^{-}(y)\right).$$

Henceforth, we get

$$\left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^{-}(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^{-}(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i^{-}(y) - \widetilde{J}_i(y) + J_i(y) - J_i^{-}(y)\right)\right|$$

$$\leq \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^{-}(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - J_i^{-}(y)\right)\right|$$

$$+ \left|(V_i^{\leftarrow})'(J_i(y))\right|\left(J_i^{-}(y) - \widetilde{J}_i(y)\right)$$

$$\leq q_i(y)\left(\frac{1}{2} + \check{c}(1+\theta)\right)\left|J_i(y) - \widetilde{J}_i(y)\right|^2,$$

using $(J_i^{-}(y), J_i(y)) \subseteq (J_i^{-}(y), J_i^{+}(y))$ and $(J_i^{-}(y), J_i(y)) \subseteq (\widetilde{J}_i(y), J_i(y))$ in the last inequality.

CASE $\widetilde{J}_i(y) > J_i^+(y)$: Due to the definition of $\mathcal{H}_i^+$ and (A.1.4) we have

$$J_i^+(y) - J_i(y) = J_i^+(y) - A_i - (J_i(y) - A_i)$$

$$\geq (J_i(y) - A_i)\left[\mathcal{H}_i^+(A_i - J_i(y)) - 1\right]$$

$$= \frac{A_i - J_i(y)}{1+\theta}\left[1 + \left(\frac{p_+}{2} + 1\right)\log^{-1}\frac{1}{A_i - J_i(y)}\right]$$

$$\asymp \frac{1}{1+\theta}\left|\frac{(V_i^{\leftarrow})'(J_i(y))}{(V_i^{\leftarrow})''(J_i(y))}\right|, \quad J_i(y) \to A_i.$$

Once again, we use the continuity of the function $\frac{(V_i^{\leftarrow})'(x)}{(V_i^{\leftarrow})''(x)}$ to assure the existence of a constant $\hat{c} > 0$, such that

$$\left|(V_i^{\leftarrow})'(J_i(y))\right| = \left|(V_i^{\leftarrow})''(J_i(y))\right|\left|\frac{(V_i^{\leftarrow})'(J_i(y))}{(V_i^{\leftarrow})''(J_i(y))}\right|$$

$$\leq \hat{c}(1+\theta)\left|(V_i^{\leftarrow})''(J_i(y))\right|\left(J_i^+(y) - J_i(y)\right).$$

Henceforth, we get

$$\left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^+(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^+(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i^+(y) - \widetilde{J}_i(y) + J_i(y) - J_i^+(y)\right)\right|$$

$$\leq \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^+(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - J_i^+(y)\right)\right|$$

$$+ \left|(V_i^{\leftarrow})'(J_i(y))\right|\left|J_i^+(y) - \widetilde{J}_i(y)\right|$$

$$\leq q_i(y)\left(\frac{1}{2} + \hat{c}(1+\theta)\right)\left|J_i(y) - \widetilde{J}_i(y)\right|^2,$$

using $(J_i(y), J_i^+(y)) \subseteq (J_i^-(y), J_i^+(y))$; $(J_i(y), J_i^+(y)) \subseteq (J_i(y), \widetilde{J}_i(y))$ and $(J_i^+(y), \widetilde{J}_i(y)) \subseteq (J_i(y), \widetilde{J}_i(y))$ in the last inequality. Finally, we complete the proof by choosing $c := \max\{\hat{c}, \check{c}\}$. □

A.1.6 Proof of Lemma 8

We will prove only the first asymptotic relation, the second one can be proven in a similar manner. Due to the representations (A.1.2) and the properties of slowly varying functions, there are slowly varying functions $\check{L}_{i,j}^-$ such that

$$(V_i^{\leftarrow})^{(j)}(x\psi(x)) = \left(\frac{1}{x}\right)^j \check{L}_{i,j}^-\left(\frac{1}{x}\right), \quad x \in [0, x_0], \quad j = 1, 2 \tag{A.1.9}$$

with $\psi(x) := \overline{\mathcal{H}}_i(x)$. Indeed,

$$(V_i^{\leftarrow})^{(j)}(x\psi(x)) = \frac{(-1)^j p_-}{2}\left(\frac{1}{x\psi(x)}\right)^j \log^{p_-/2-1}\left(\frac{1}{\psi(x)x}\right)\check{l}_i^-\left(\log\frac{1}{x\psi(x)}\right),$$

where

$$\check{L}_{i,j}^-(y) = \frac{(-1)^j p_-}{2}\left(\frac{1}{\psi(1/y)}\right)^j \log^{p_-/2-1}\left(\frac{y}{\psi(1/y)}\right)\check{l}_i^-\left(\log\frac{y}{\psi(1/y)}\right)$$

is slowly varying as $y \to \infty$. Finally, the Karamata's theorem (Proposition 1.5.8 and Proposition 1.5.9 in Bingham, Goldie, Teugels (1987)) implies that

$$\int_\varepsilon^{x_0}\left|\left(\frac{1}{x}\right)\check{L}_{i,1}^-\left(\frac{1}{x}\right)\right|dx = S_{i,1}^-\left(\frac{1}{\varepsilon}\right) \tag{A.1.10}$$

and

$$\int_\varepsilon^{x_0}\left|\left(\frac{1}{x}\right)^2\check{L}_{i,1}^-\left(\frac{1}{x}\right)\right|dx = \left(\frac{1}{\varepsilon}\right)S_{i,2}^-\left(\frac{1}{\varepsilon}\right) \tag{A.1.11}$$

for some slowly varying $S_{i,1}^-$ and $S_{i,2}^-$. □

A.1.7 Proof of Theorem 1

We begin by specifying three lemmas which will help us to prove this theorem.

Lemma 9. *Set* $\mathcal{A}_{i,h,\delta} := \{\delta + h \le J_i(y) \le A_i - \delta - h\}$, $\delta, h \ge 0$, *then*

$$\int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| f_i(y) - \widetilde{f}_i(y) \right| \phi_i(y) dy \le \mathcal{R}_{i,0} + \mathcal{R}_{i,1} \left\| J_i - \widetilde{J}_i \right\|_\infty + \mathcal{R}_{i,2} \left\| J_i - \widetilde{J}_i \right\|_\infty^2$$

holds with

$$\mathcal{R}_{i,0} := A_i \left(K_1 + \frac{K_L \mathcal{L}_\Phi}{h} \left[\mathfrak{R}_i(L) + \frac{\Delta \|V_i'\|_\infty}{2} \right] \right) + 2hC_{i,1} \left(S_{i,1}^- \left(\frac{1}{\delta} \right) + S_{i,1}^+ \left(\frac{1}{\delta} \right) \right),$$

$$\mathcal{R}_{i,1} := V_i^{\leftarrow}(h + \delta),$$

$$\mathcal{R}_{i,2} := C_{i,2} \frac{1}{\delta} \left(S_{i,2}^- \left(\frac{1}{\delta} \right) + S_{i,2}^+ \left(\frac{1}{\delta} \right) \right),$$

where

$$K_L > V_i^{\leftarrow}(\delta \overline{\mathcal{H}}_i(\delta)).$$

Proof. We have

$$\begin{aligned}
\left| f_i(y) - \widetilde{f}_i(y) \right| &= \left| V_i^{\leftarrow}(J_i(y)) - \widehat{V}_{i,h}^{\leftarrow}(\widehat{J}_i(y)) \right| \\
&\le \left| V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) \right| + \left| V_i^{\leftarrow}(\widehat{J}_i(y)) - \widehat{V}_{i,h}^{\leftarrow}(\widehat{J}_i(y)) \right| \\
&\le \left| V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) \right| + \left| V_i^{\leftarrow}(\widehat{J}_i(y)) - V_{i,h}^{\leftarrow}(\widehat{J}_i(y)) \right| \\
&\quad + \left| V_{i,h}^{\leftarrow}(\widehat{J}_i(y)) - \widehat{V}_{i,h}^{\leftarrow}(\widehat{J}_i(y)) \right|.
\end{aligned}$$

Next, we use Lemma 5 to estimate $\left| V_{i,h}^{\leftarrow}(\widehat{J}_i(y)) - \widehat{V}_{i,h}^{\leftarrow}(\widehat{J}_i(y)) \right|$. Combining the chosen regularisation method with the decreasing monotony of V_i and **(AR)**, we have for $K_L > V_i^{\leftarrow}(\delta \overline{\mathcal{H}}_i(\delta))$

$$\widehat{J}_i(y) \in [J_i^-(y), J_i^+(y)] \subseteq [J_i^-(y), A_i] \subseteq [\delta \overline{\mathcal{H}}_i(\delta) + h, A_i] \subseteq [V_i(K_L) + h, A_i].$$

Hence, $\int_{K_L}^{\infty} \Phi\left(\frac{V_i(K)-\widehat{J}_i(y)}{h}\right) dK = 0$ for all $\widehat{J}_i(y)$, and thus

$$\left|V_{i,h}^{\leftarrow}(\widehat{J}_i(y)) - \widehat{V}_{i,h}^{\leftarrow}(\widehat{J}_i(y))\right| \leq K_1 + \frac{K_L \mathcal{L}_\Phi}{h}\left[\mathfrak{R}_i(L) + \frac{\Delta \|V_i'\|_\infty}{2}\right].$$

Continuing, we use Lemma 6 to estimate $\left|V_i^{\leftarrow}(\widehat{J}_i(y)) - V_{i,h}^{\leftarrow}(\widehat{J}_i(y))\right|$.
For any $\widehat{J}_i(y) \in (h, A_i - h), h \geq 0$ and some constant $C_{i,1} > 0$ we have

$$\begin{aligned}\left|V_i^{\leftarrow}(\widehat{J}_i(y)) - V_{i,h}^{\leftarrow}(\widehat{J}_i(y))\right| &\leq 2h \sup_{\xi \in \left[\widehat{J}_i(y)-h, \widehat{J}_i(y)+h\right]} \left|(V_i^{\leftarrow})'(\xi)\right| \\ &\leq 2h \sup_{\xi \in \left[(J_i(y)-h)\overline{\mathcal{H}}_i(J_i(y)-h), A_i-(A_i-J_i(y)-h)\overline{\mathcal{H}}_i(A_i-J_i(y)-h)\right]} \left|(V_i^{\leftarrow})'(\xi)\right| \\ &\leq 2hC_{i,1} r(J_i(y))\end{aligned} \tag{A.1.12}$$

with

$$r(x) = \max\left\{\left|(V_i^{\leftarrow})'((x-h)\overline{\mathcal{H}}_i(x-h))\right|, \left|(V_i^{\leftarrow})'(A_i-(A_i-x-h)\overline{\mathcal{H}}_i(A_i-x-h))\right|\right\}.$$

Let us study (A.1.12) closely to assure that
$(J_i(y)-h)\overline{\mathcal{H}}_i(J_i(y)-h) \leq \widehat{J}_i(y) - h$ and $A_i - (A_i - J_i(y) - h)\overline{\mathcal{H}}_i(A_i - J_i(y) - h) \geq \widehat{J}_i(y) + h$ hold. First, $\widehat{J}_i(y) \in [J_i^-(y), J_i^+(y)]$ by construction and so $\widehat{J}_i(y) \geq J_i^-(y)$. Using **(AR)** we get for some $x_0 \in (h; A_i - h)$ and $h \leq J_i(y) \leq x_0$

$$(J_i(y)-h)\overline{\mathcal{H}}_i(J_i(y)-h) \leq J_i^-(y) - h \leq \widehat{J}_i(y) - h.$$

Next, with a similar argumentation, we have $A_i \geq J_i^+(y);\ \widehat{J}_i(y) \leq J_i^+(y);\ h \geq 0$ by construction. Again, with **(AR)** for some $x_0 \in (h; A_i - h)$ and $J_i(y) \in (x_0, A_i - h)$ we get

$$\begin{aligned}&(A_i - J_i(y) - h)\overline{\mathcal{H}}_i(A_i - J_i(y) - h) \leq A_i - J_i^+(y) - h \leq A_i - \widehat{J}_i(y) - h \\ &\Rightarrow A_i - (A_i - J_i(y) - h)\overline{\mathcal{H}}_i(A_i - J_i(y) - h) \geq \widehat{J}_i(y) + h.\end{aligned}$$

Continuing, we use integration by substitution and the mean value theorem for integration as well as the fact that $\int_{\mathcal{A}_{i,h,\delta}} Q_i(y)\phi_i(y)dy \leq A_i$ holds to get

$$\int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| V_i^{\leftarrow}(\widehat{J}_i(y)) - \widehat{V}_{i,h}^{\leftarrow}(\widehat{J}_i(y)) \right| \phi_i(y)dy$$

$$\leq \left\{ K_1 + \frac{K_L \mathcal{L}_\Phi}{h} \left[\mathfrak{R}_i(L) + \frac{\Delta \|V_i'\|_\infty}{2} \right] \right\} \int_{\mathcal{A}_{i,h,\delta}} Q_i(y)\phi_i(y)dy$$

$$+ 2hC_{i,1} \int_{\mathcal{A}_{i,h,\delta}} -J_i'(y) r(J_i(y))dy$$

$$\leq A_i \left(K_1 + \frac{K_L \mathcal{L}_\Phi}{h} \left[\mathfrak{R}_i(L) + \frac{\Delta \|V_i'\|_\infty}{2} \right] \right) + 2hC_{i,1} \int_{A_i-\delta-h}^{\delta+h} r(x)dx.$$

Next, consider

$$0 < \int_{\delta+h}^{A_i-\delta-h} r(x)dx = \int_{\delta+h}^{\bar{x}_0} \left| (V_i^{\leftarrow})' \left((x-h)\overline{\mathcal{H}}_i(x-h) \right) \right| dx \tag{A.1.13}$$

$$+ \int_{\bar{x}_0}^{A_i-\delta-h} \left| (V_i^{\leftarrow})' \left(A_i - (A_i - x - h)\overline{\mathcal{H}}_i(A_i - x - h) \right) \right| dx$$

$$= \int_{\delta}^{\bar{x}_0} \left| (V_i^{\leftarrow})' \left(x\overline{\mathcal{H}}_i(x) \right) \right| dx - \int_{\bar{x}_0-h}^{\bar{x}_0} \left| (V_i^{\leftarrow})' \left(x\overline{\mathcal{H}}_i(x) \right) \right| dx$$

$$+ \int_{\bar{x}_0}^{A_i-\delta} \left| (V_i^{\leftarrow})' \left(A_i - (A_i - x)\overline{\mathcal{H}}_i(A_i - x) \right) \right| dx$$

$$- \int_{\bar{x}_0}^{\bar{x}_0+h} \left| (V_i^{\leftarrow})' \left(A_i - (A_i - x)\overline{\mathcal{H}}_i(A_i - x) \right) \right| dx$$

$$\leq \int_{\delta}^{\bar{x}_0} \left| (V_i^{\leftarrow})' \left(x\overline{\mathcal{H}}_i(x) \right) \right| dx + \int_{\bar{x}_0}^{A_i-\delta} \left| (V_i^{\leftarrow})' \left(A_i - (A_i - x)\overline{\mathcal{H}}_i(A_i - x) \right) \right| dx,$$

where

$$\bar{x}_0 := \min \left\{ \tilde{x}_0 \in (h+\delta, A_i - h - \delta) : \left| (V_i^{\leftarrow})' \left(\hat{x}_0 \overline{\mathcal{H}}_i(\hat{x}_0) \right) \right| \leq \left| (V_i^{\leftarrow})' \left(A_i - \check{x}_0 \overline{\mathcal{H}}_i(\check{x}_0) \right) \right| \right\}$$

with $\hat{x}_0 := \tilde{x}_0 - h$ and $\check{x}_0 := A_i - \tilde{x}_0 - h$.

Before proceeding, we shall make sure that $\bar{x}_0$ exists and (A.1.13) holds.

Using **(AR)** we have for all x satisfying $A_i - h \geq x > x_0$

$$(A_i - x - h)\overline{\mathcal{H}}_i(A_i - x - h) \leq A_i - x - h.$$

Moreover,

$$(x-h)\overline{\mathcal{H}}_i(x-h) + A_i - x - h \leq A_i \Leftrightarrow \overline{\mathcal{H}}_i(x-h) \leq \frac{x+h}{x-h}.$$

Since $\overline{\mathcal{H}}_i$ is bounded by $A_i \leq 1$ and $\frac{x+h}{x-h} \geq 1$, the last inequation is always true, and thus

$$(x-h)\overline{\mathcal{H}}_i(x-h) \leq A_i - (A_i - x - h)\overline{\mathcal{H}}_i(A_i - x - h) \tag{A.1.14}$$

holds for all x satisfying $A_i - h \geq x > x_0$.

Continuing, we use the monotony of $(V_i^{\leftarrow})'$ to get

$$\left|(V_i^{\leftarrow})'\left((x-h)\overline{\mathcal{H}}_i(x-h)\right)\right| \leq \left|(V_i^{\leftarrow})'\left(A_i - (A_i - x - h)\overline{\mathcal{H}}_i(A_i - x - h)\right)\right|$$

for all x satisfying $A_i - h \geq x > x_0$.

Furthermore, as we already know $(V_i^{\leftarrow})'(0)$ is unbounded and the relation above is not valid for $x \to \delta + h$ as $\delta \to 0$. In this case we have

$$\left|(V_i^{\leftarrow})'\left((x-h)\overline{\mathcal{H}}_i(x-h)\right)\right| \geq \left|(V_i^{\leftarrow})'\left(A_i - (A_i - x - h)\overline{\mathcal{H}}_i(A_i - x - h)\right)\right|$$

which is (A.1.13).

Now, combining Lemma 8 with the fact that due to the absolute value in the definition of $r(x)$ and $(V_i^{\leftarrow})'(x) < 0, \forall x$ the order of the constraints by integral is negligible, we get

$$\int_{A_i-\delta-h}^{\delta+h} r(x)dx \leq S_{i,1}^{-}\left(\frac{1}{\delta}\right) + S_{i,1}^{+}\left(\frac{1}{\delta}\right), \quad \delta \leq \bar{x}_0 - h.$$

Continuing, since $\bar{x}_0 \in (h+\delta, A_i - h - \delta)$ and the condition $\delta \leq \bar{x}_0 - h$ is always satisfied, we come up to

$$\int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| V_i^{\leftarrow}(\widehat{J}_i(y)) - \widehat{V}_{i,h}^{\leftarrow}(\widehat{J}_i(y)) \right| \phi_i(y) dy \leq \mathcal{R}_{i,0}.$$

Let us now verify that

$$\int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) \right| \phi_i(y) dy \leq \mathcal{R}_{i,1} \left\| J_i - \widetilde{J}_i \right\|_{\infty} + \mathcal{R}_{i,2} \left\| J_i - \widetilde{J}_i \right\|_{\infty}^2$$

holds.

Since $J_i(y) \in [J_i^-(y), J_i^+(y)] \subseteq (0, A_i]$, Lemma 7 and the first-order Taylor series approximation give

$$V_i^{\leftarrow}(J_i(y)) - \left\{ V_i^{\leftarrow}(\widehat{J}_i(y)) + (V_i^{\leftarrow})'(J_i(y)) \left(J_i(y) - \widetilde{J}_i(y) \right) \right\} =: R_1(J_i(y)),$$

where $|R_1(J_i(y))| \leq \xi_i(y) \left(J_i(y) - \widetilde{J}_i(y) \right)^2$ and $\xi_i(y)$ is such that $\left| (V_i^{\leftarrow})''(x) \right| \leq \xi_i(y)$ for all x in $[J_i^-(y), J_i^+(y)]$. From this it follows that

$$\left| V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) \right| \leq \left| (V_i^{\leftarrow})'(J_i(y)) \left(J_i(y) - \widetilde{J}_i(y) \right) + \xi_i(y) \left(J_i(y) - \widetilde{J}_i(y) \right)^2 \right|.$$

Combining the mean value theorem for integration and the triangle inequality we get

$$\begin{aligned}
&\int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) \right| \phi_i(y) dy \\
&\leq \int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| (V_i^{\leftarrow})'(J_i(y)) \left(J_i(y) - \widetilde{J}_i(y) \right) \right| \phi_i(y) dy \\
&+ \int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| \xi_i(y) \left(J_i(y) - \widetilde{J}_i(y) \right)^2 \right| \phi_i(y) dy \\
&\leq \left\| J_i - \widetilde{J}_i \right\|_{\infty} \int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| (V_i^{\leftarrow})'(J_i(y)) \right| \phi_i(y) dy \\
&+ \left\| J_i - \widetilde{J}_i \right\|_{\infty}^2 \int_{\mathcal{A}_{i,h,\delta}} Q_i(y) \left| \xi_i(y) \right| \phi_i(y) dy.
\end{aligned}$$

Next,

$$\int_{\mathcal{A}_{i,h,\delta}} Q_i(y)\left|(V_i^{\leftarrow})'(J_i(y))\right|\phi_i(y)dy = -\int_{\mathcal{A}_{i,h,\delta}} (V_i^{\leftarrow})'(J_i(y))dJ_i(y)$$

$$= V_i^{\leftarrow}(h+\delta) - V_i^{\leftarrow}(A_i - h - \delta)$$

$$\leq V_i^{\leftarrow}(h+\delta)$$

due to the monotony of $V_i^{\leftarrow}$. By (A.1.6) it holds that

$$\int_{\mathcal{A}_{i,h,\delta}} Q_i(y)\left|\xi_i(y)\right|\phi_i(y)dy \leq C_{i,2}\int_{h+\delta}^{A_i-h-\delta} q(x)dx$$

with

$$q(x) = \max\left\{\left|(V_i^{\leftarrow})''((x-h)\overline{\mathcal{H}}_i(x-h))\right|, \left|(V_i^{\leftarrow})''(A_i-(A_i-x-h)\overline{\mathcal{H}}_i(A_i-x-h))\right|\right\}.$$

Combining (A.1.14) with (A.1.2) we get

$$\left|(V_i^{\leftarrow})''\left((x-h)\overline{\mathcal{H}}_i(x-h)\right)\right| \leq \left|(V_i^{\leftarrow})''\left(A_i-(A_i-x-h)\overline{\mathcal{H}}_i(A_i-x-h)\right)\right|$$

for all x satisfying $A_i - h \geq x > x_0$ and

$$\left|(V_i^{\leftarrow})''\left((x-h)\overline{\mathcal{H}}_i(x-h)\right)\right| > \left|(V_i^{\leftarrow})''\left(A_i-(A_i-x-h)\overline{\mathcal{H}}_i(A_i-x-h)\right)\right|$$

for $x \to \delta + h$ as $\delta \to 0$. Thus, with monotony of $(V_i^{\leftarrow})''$, a $\check{x}_0 \in [\delta+h, A_i-\delta-h]$ exists, such that

$$\check{x}_0 = \min\left\{\tilde{x}_0 \in (h+\delta, A_i-h-\delta) : \left|(V_i^{\leftarrow})''(\hat{x}_0\overline{\mathcal{H}}_i(\hat{x}_0))\right| \leq \left|(V_i^{\leftarrow})''(A_i-\check{x}_0\overline{\mathcal{H}}_i(\check{x}_0))\right|\right\}$$

and

$$\left|(V_i^{\leftarrow})''((x-h)\overline{\mathcal{H}}_i(x-h))\right| > \left|(V_i^{\leftarrow})''(A_i-(A_i-x-h)\overline{\mathcal{H}}_i(A_i-x-h))\right|$$

holds for all x satisfying $h \leq x < \check{x}_0$, and

$$\left|(V_i^{\leftarrow})''\left((x-h)\overline{\mathcal{H}}_i(x-h)\right)\right| \leq \left|(V_i^{\leftarrow})''\left(A_i-(A_i-x-h)\overline{\mathcal{H}}_i(A_i-x-h)\right)\right|$$

holds for all x satisfying $A_i - h \geq x \geq \check{x}_0$.

With the same argumentation as in the case of the integration of $r(x)$ (discussed above) we get

$$\int_{h+\delta}^{A_i-h-\delta} q(x)dx \leq \frac{1}{\delta}\left\{S_{i,2}^{-}\left(\frac{1}{\delta}\right)+S_{i,2}^{+}\left(\frac{1}{\delta}\right)\right\}.$$

Finally, joining by triangle inequality completes the proof. □

Lemma 10. *Denote*

$$\overline{\mathcal{A}}_{i,h,\delta} := \left\{\{-\infty \leq J_i(y) \leq \delta+h\} \bigcup \{A_i-\delta-h \leq J_i(y) \leq \infty\}\right\}$$

$$= \left\{\{0 \leq J_i(y) \leq \delta+h\} \bigcup \{A_i-\delta-h \leq J_i(y) \leq A_i\}\right\}$$

with $\delta, h \geq 0$*, then it holds that*

$$\int_{\overline{\mathcal{A}}_{i,h,\delta}} Q_i(y)\left|f_i(y)-\widetilde{f}_i(y)\right|\phi_i(y)dy \leq 2(\delta+h)(A_i+K_L).$$

Proof. Due to (3.1.2)

$$f_i(y) \leq A_i, \quad y \in \mathbb{R}.$$

Moreover, because the image of a distribution function is a subset of $[0;1]$ and due to the fact that $K_{l+1} - K_l > 0$ for all $l \in [1, L-1]$, we have

$$\widetilde{f}_i(y) = \widehat{V}_{i,h}^{\leftarrow}(\hat{J}_i(y))$$

$$= \sum_{l=1}^{L-1}(K_{l+1}-K_l)\Phi\left(\frac{\widetilde{V}_i(K_l)-\hat{J}_i(y)}{h}\right)$$

$$\leq K_L - K_1 < K_L.$$

Thus,

$$\sup_y \left| f_i(y) - \widetilde{f}_i(y) \right| \leq K_L + A_i$$

and

$$\int_{\overline{\mathcal{A}}_{i,h,\delta}} Q_i(y) \left| f_i(y) - \widetilde{f}_i(y) \right| \phi_i(y) dy$$

$$\leq (A_i + K_L) \left\{ \int_{\{0 \leq J_i(y) \leq \delta + h\}} dJ_i(y) + \int_{\{A_i - \delta - h \leq J_i(y) \leq A_i\}} dJ_i(y) \right\}$$

$$\leq 2(A_i + K_L)(h + \delta).$$

□

Lemma 11. *Denote*

$$\rho_i := \int_{-\infty}^{\infty} Q_i(y) \left| f_i(y) - \widetilde{f}_i(y) \right| \phi_i(y) dy$$

and

$$\vartheta_i := \int_{-\infty}^{\infty} \left| Q_i(v) - \widetilde{Q}_i(v) \right| \phi_i(v) dv.$$

Furthermore, introduce the notation $K_{j,i}$ for the jth strike of the ith digital caplet. Then it holds that

$$\rho_i \leq \widetilde{\mathcal{R}}_{i,0} + \mathcal{R}_{i,1} \varsigma_{i+1} \rho_{i+1} + \mathcal{R}_{i,1} \left(1 + \varsigma_{i+1} K_{L,(i+1)}\right) \vartheta_{i+1}$$

$$+ 3\mathcal{R}_{i,2} \left[\vartheta_{i+1}^2 + \varsigma_{i+1}^2 \left(\rho_{i+1}^2 + K_{L,(i+1)}^2 \vartheta_{i+1}^2 \right) \right]$$

with

$$\widetilde{\mathcal{R}}_{i,0} := \mathcal{R}_{i,0} + 2(\delta + h)(A_i + K_{L,i})$$

and

$$\vartheta_i \leq (1 + \varsigma_{i+1} K_{L,(i+1)}) \vartheta_{i+1} + \varsigma_{i+1} \rho_{i+1},$$

provided that $K_{L,i} > V_i^{\leftarrow}(\delta \overline{\mathcal{H}}_i(\delta))$.

Proof. For any fixed $v \in \mathbb{R}$ we have

$$
\begin{aligned}
\left|Q_i(v) - \widetilde{Q}_i(v)\right| = & \left|\int_{-\infty}^{\infty} \left\{Q_{i+1}(u)\phi_{(i+1)|i}(u|v) + \varsigma_{i+1} f_{i+1}(u) Q_{i+1}(u)\phi_{(i+1)|i}(u|v)\right\} du \right. \\
& \left. - \int_{-\infty}^{\infty} \left\{\widetilde{Q}_{i+1}(u)\phi_{(i+1)|i}(u|v) + \varsigma_{i+1} \widetilde{f}_{i+1}(u) \widetilde{Q}_{i+1}(u)\phi_{(i+1)|i}(u|v)\right\} du \right| \\
& \leq \int_{-\infty}^{\infty} \left|Q_i(u) - \widetilde{Q}_i(u)\right| \phi_{(i+1)|i}(u|v) du \\
& + \int_{-\infty}^{\infty} \varsigma_{i+1}\phi_{(i+1)|i}(u|v) \left| f_{i+1}(u) Q_{i+1}(u) - \widetilde{f}_{i+1}(u) Q_{i+1}(u) \right. \\
& \left. + \widetilde{f}_{i+1}(u) Q_{i+1}(u) - \widetilde{f}_{i+1}(u) \widetilde{Q}_{i+1}(u) \right| du \\
& \leq \int_{-\infty}^{\infty} \left|Q_i(u) - \widetilde{Q}_i(u)\right| \phi_{(i+1)|i}(u|v) du \\
& + \int_{-\infty}^{\infty} \varsigma_{i+1}\phi_{(i+1)|i}(u|v) Q_{i+1}(u) \left| f_{i+1}(u) - \widetilde{f}_{i+1}(u) \right| du \\
& + \int_{-\infty}^{\infty} \varsigma_{i+1}\phi_{(i+1)|i}(u|v) \widetilde{f}_{i+1}(u) \left| Q_{i+1}(u) - \widetilde{Q}_{i+1}(u) \right| du \\
& \leq \int_{-\infty}^{\infty} \left|Q_{i+1}(u) - \widetilde{Q}_{i+1}(u)\right| \phi_{(i+1)|i}(u|v) \left\{1 + \varsigma_{i+1}\widetilde{f}_{i+1}(u)\right\} du \\
& + \int_{-\infty}^{\infty} \varsigma_{i+1}\phi_{(i+1)|i}(u|v) Q_{i+1}(u) \left| f_{i+1}(u) - \widetilde{f}_{i+1}(u) \right| du \\
& \leq \left\{1 + \varsigma_{i+1} K_{L,(i+1)}\right\} \int_{-\infty}^{\infty} \left|Q_{i+1}(u) - \widetilde{Q}_{i+1}(u)\right| \phi_{(i+1)|i}(u|v) du \\
& + \varsigma_{i+1} \int_{-\infty}^{\infty} \left| f_{i+1}(u) - \widetilde{f}_{i+1}(u) \right| Q_{i+1}(u) \phi_{(i+1)|i}(u|v) du.
\end{aligned}
$$

Since $\phi_{(i+1)|i}(u|v)$ is a non-negative, continuous density function, we can use Fubini's theorem for integration and other characteristics, such that

(i) $\phi_{i+1,i}(u,v) = \phi_i(v)\phi_{(i+1)|i}(u|v)$,

(ii) $\phi_{i+1}(u) = \int_{-\infty}^{\infty} \phi_{i+1,i}(u,v)dv$,

(iii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{i+1,i}(u,v)dvdu = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{i+1,i}(u,v)dudv$.

Thus,

$$\int_{-\infty}^{\infty} \left|Q_i(v) - \widetilde{Q}_i(v)\right| \phi_i(v)dv \leq (1+\varsigma_{i+1}K_{L,(i+1)}) \int_{-\infty}^{\infty} \left|Q_{i+1}(u) - \widetilde{Q}_{i+1}(u)\right| \int_{-\infty}^{\infty} \phi_{i+1,i}(u,v)dvdu$$

$$+ \varsigma_{i+1} \int_{-\infty}^{\infty} Q_{i+1}(u) \left|f_{i+1}(u) - \widetilde{f}_{i+1}(u)\right| \int_{-\infty}^{\infty} \phi_{i+1,i}(u,v)dvdu$$

$$= (1+\varsigma_{i+1}K_{L,(i+1)}) \int_{-\infty}^{\infty} \left|Q_{i+1}(u) - \widetilde{Q}_{i+1}(u)\right| \phi_{i+1}(u)du$$

$$+ \varsigma_{i+1} \int_{-\infty}^{\infty} Q_{i+1}(u) \left|f_{i+1}(u) - \widetilde{f}_{i+1}(u)\right| \phi_{i+1}(u)du$$

which is the same as

$$\vartheta_i \leq (1+\varsigma_{i+1}K_{L,(i+1)})\vartheta_{i+1} + \varsigma_{i+1}\rho_{i+1}. \tag{A.1.15}$$

Furthermore,

$$\sup_y \left|J_i(y) - \widetilde{J}_i(y)\right| \leq \sup_y \left| \int_y^{\infty} \left[Q_i(v) - \widetilde{Q}_i(v)\right] \phi_i(v)dv \right|$$

$$\leq \int_{-\infty}^{\infty} \left|Q_i(v) - \widetilde{Q}_i(v)\right| \phi_i(v)dv$$

and thus

$$\left\|J_i - \widetilde{J}_i\right\|_{\infty} \leq \vartheta_i \tag{A.1.16}$$

as well as

$$\left\|J_i-\widetilde{J}_i\right\|_\infty^2 \le \vartheta_i^2.$$

Combining (A.1.15) with the fact that $(\sum_{i=1}^n a_i)^2 \le n\sum_{i=1}^n a_i^2, \forall a_i \in \mathbb{R}^+$ holds, we get

$$\left\|J_i-\widetilde{J}_i\right\|_\infty^2 \le 3\vartheta_{i+1}^2 + 3\varsigma_{i+1}^2\left(\rho_{i+1}^2 + K_{L,(i+1)}^2\vartheta_{i+1}^2\right). \tag{A.1.17}$$

We complete the proof by using Lemmas 9-10 as well as (A.1.15), (A.1.17), (A.1.16). □

Let us continue with the proof of Theorem 1. Denote

$$L_{\gamma,\Delta} := \min\{\gamma_L, 1/\Delta\}.$$

Let δ satisfy $\delta \asymp L_{\gamma,\Delta}^{-1/2}, L_{\gamma,\Delta} \to \infty$. Using (A.1.1) we get

$$V_i^{\leftarrow}(\delta\overline{\mathcal{H}}_i(\delta)) = \log^{p_-/2}\left(\frac{1}{\delta\overline{\mathcal{H}}_i(\delta)}\right)\breve{l}_i^-\left(\log\left(\frac{1}{\delta\overline{\mathcal{H}}_i(\delta)}\right)\right)$$

$$= o\left(\log^{p_-/2+1}\left(\frac{1}{\delta\overline{\mathcal{H}}_i(\delta)}\right)\right), \quad \delta \to 0.$$

Next, $K_{L,i} = K_{1,i} + (L-1)\Delta \asymp \log^{p_-/2+1}(L_{\gamma,\Delta}), L_{\gamma,\Delta} \to \infty$.
Let us verify $\frac{V_i^{\leftarrow}(\delta\overline{\mathcal{H}}_i(\delta))}{K_{L,i}} \to 0, L_{\gamma,\Delta} \to \infty$ to assure that $K_{L,i} > V_i^{\leftarrow}(\delta\overline{\mathcal{H}}_i(\delta)), L_{\gamma,\Delta} \to \infty$ holds.
As we already know, we have

$$\frac{V_i^{\leftarrow}(\delta\overline{\mathcal{H}}_i(\delta))}{\log^{p_-/2+1}\left(\frac{\sqrt{L_{\gamma,\Delta}}}{\overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right)}\right)} \to 0, \quad L_{\gamma,\Delta} \to \infty$$

using $\delta \asymp L_{\gamma,\Delta}^{-1/2}, L_{\gamma,\Delta} \to \infty$. Since

$$K_{L,i} \ge \log^{p_-/2+1}\left(\frac{\sqrt{L_{\gamma,\Delta}}}{\overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right)}\right), \quad L_{\gamma,\Delta} \to \infty \tag{A.1.18}$$

and

$$\log^{p_-/2+1}\left(\frac{\sqrt{L_{\gamma,\Delta}}}{\overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right)}\right) \to \infty, \quad L_{\gamma,\Delta} \to \infty,$$

we get

$$\frac{V_i^{\leftarrow}(\delta\overline{\mathcal{H}}_i(\delta))}{K_{L,i}} \to 0, \quad L_{\gamma,\Delta} \to \infty,$$

and so we can make use of Lemmas 9-11.

With some more details: due to the monotony of the logarithm function (A.1.18) holds in the case that

$$L_{\gamma,\Delta} \geq \frac{\sqrt{L_{\gamma,\Delta}}}{\overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right)}, \quad L_{\gamma,\Delta} \to \infty \Leftrightarrow \overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right) \geq \frac{1}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty. \tag{A.1.19}$$

Indeed, the last inequality in (A.1.19) is always true due to $\frac{\frac{1}{\sqrt{L_{\gamma,\Delta}}}}{\overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right)} \to 0, L_{\gamma,\Delta} \to \infty.$

Next, introduce the events

$$\mathcal{D}_{i,L} := \left\{ \mathfrak{R}_i(L) \leq c_1 \gamma_L^{-1} \right\}, \quad i = 1, \ldots, n.$$

Due to Lemma 11 $\rho_n \leq \widetilde{\mathcal{R}}_{n,0}$, and we have

$$\widetilde{\mathcal{R}}_{n,0} = A_n \left(K_{1,n} + \frac{K_{L,n} \mathcal{L}_\Phi}{h} \left[\mathfrak{R}_n(L) + \frac{\Delta \|V_n'\|_\infty}{2} \right] \right)$$

$$+ 2hC_{n,1} \left(S_{n,1}^- \left(\frac{1}{\delta} \right) + S_{n,1}^+ \left(\frac{1}{\delta} \right) \right) + 2(\delta + h)(A_n + K_{L,n})$$

$$\leq \hat{C}_n \left[K_{1,n} + \frac{K_{L,n}}{h} \max\{\gamma_L^{-1}, \Delta\} + h \left(S_{n,1}^- \left(\frac{1}{\delta} \right) + S_{n,1}^+ \left(\frac{1}{\delta} \right) \right) + (\delta + h)(A_n + K_{L,n}) \right]$$

$$\leq \check{C}_n \left[K_{1,n} + \frac{K_{1,n} + \Delta L}{\sqrt{L_{\gamma,\Delta}}} + \frac{1}{\sqrt{L_{\gamma,\Delta}}} \left(S_{n,1}^- \left(\sqrt{L_{\gamma,\Delta}} \right) + S_{n,1}^+ \left(\sqrt{L_{\gamma,\Delta}} \right) \right) + \frac{1}{\sqrt{L_{\gamma,\Delta}}} \right]$$

$$=: \frac{\mathcal{S}_{\rho,n}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty$$

on $\mathcal{D}_{n,L}$ for some slowly varying function at infinity $\mathcal{S}_{\rho,n}$ and some constants $\hat{C}_n, \check{C}_n$. Since $\vartheta_n = 0$, it holds on $\mathcal{D}_{n,L} \cap \mathcal{D}_{n-1,L}$ with some constants $\hat{C}_{n-1}, \check{C}_{n-1}$

$$\rho_{n-1} \leq \widetilde{\mathcal{R}}_{n-1,0} + \mathcal{R}_{n-1,1} \varsigma_n \rho_n + 3\mathcal{R}_{n-1,2} \varsigma_n^2 \rho_n^2$$

$$\leq \frac{\check{C}_{n-1}}{\sqrt{L_{\gamma,\Delta}}} \left[2 + \log^{p_-/2+1}(L_{\gamma,\Delta}) + S_{n-1,1}^- \left(\sqrt{L_{\gamma,\Delta}} \right) + S_{n-1,1}^+ \left(\sqrt{L_{\gamma,\Delta}} \right) \right]$$

$$+ \frac{\varsigma_n}{\sqrt{L_{\gamma,\Delta}}} \mathcal{S}_{\rho,n}(L_{\gamma,\Delta}) V_{n-1}^{\leftarrow}(h + \delta)$$

$$+ \hat{C}_{n-1} \varsigma_n^2 \frac{\mathcal{S}_{\rho,n}^2(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}} \left(S_{n-1,2}^- \left(\sqrt{L_{\gamma,\Delta}} \right) + S_{n-1,2}^+ \left(\sqrt{L_{\gamma,\Delta}} \right) \right)$$

$$=: \frac{\mathcal{S}_{\rho,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty,$$

due to $V^{\leftarrow}_{n-1}(h+\delta) \asymp \log^{p_-/2}\left(\frac{\sqrt{L_{\gamma,\Delta}}}{2}\right)\check{l}^{-}_{n-1}\left(\log\frac{\sqrt{L_{\gamma,\Delta}}}{2}\right), L_{\gamma,\Delta} \to \infty$ by (A.1.1), which is slowly varying at infinity as a product of two slowly varying functions (at infinity).

Moreover,

$$\vartheta_{n-1} \leq \varsigma_n \rho_n = \frac{\varsigma_n}{\sqrt{L_{\gamma,\Delta}}}\mathcal{S}_{\rho,n-1}(L_{\gamma,\Delta})$$

$$=: \frac{\mathcal{S}_{\vartheta,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty.$$

On $\mathcal{D}_{n,L} \cap \mathcal{D}_{n-1,L} \cap \mathcal{D}_{n-2,L}$ we have with some constants $\hat{C}_{n-2}, \check{C}_{n-2}$:

$$\rho_{n-2} \leq \widetilde{\mathcal{R}}_{n-2,0} + \mathcal{R}_{n-2,1}\varsigma_{n-1}\rho_{n-1}$$

$$+\mathcal{R}_{n-2,1}\left(1+\varsigma_{n-1}K_{L,n-1}\right)\vartheta_{n-1} + 3\mathcal{R}_{n-2,2}\left[\vartheta^2_{n-1} + \varsigma^2_{n-1}\left(\rho^2_{n-1} + K^2_{L,n-1}\vartheta^2_{n-1}\right)\right]$$

$$\leq \frac{\check{C}_{n-2}}{\sqrt{L_{\gamma,\Delta}}}\left[2+\log^{p_-/2+1}(L_{\gamma,\Delta}) + S^{-}_{n-2,1}\left(\sqrt{L_{\gamma,\Delta}}\right) + S^{+}_{n-2,1}\left(\sqrt{L_{\gamma,\Delta}}\right)\right]$$

$$+\frac{\varsigma_{n-1}}{\sqrt{L_{\gamma,\Delta}}}\mathcal{S}_{\rho,n-1}(L_{\gamma,\Delta})V^{\leftarrow}_{n-2}(h+\delta) + V^{\leftarrow}_{n-2}(h+\delta)\left(1+\varsigma_{n-1}\log^{p_-/2+1}(L_{\gamma,\Delta})\right)\frac{\mathcal{S}_{\vartheta,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}$$

$$+\frac{\hat{C}_{n-2}}{\sqrt{L_{\gamma,\Delta}}}\left[\mathcal{S}^2_{\vartheta,n-1}(L_{\gamma,\Delta}) + \varsigma^2_{n-1}\left(\mathcal{S}^2_{\rho,n-1}(L_{\gamma,\Delta}) + \mathcal{S}^2_{\vartheta,n-1}(L_{\gamma,\Delta})\log^{p_-/2+1}(L_{\gamma,\Delta})\right)\right]\times$$

$$\times\left(S^{-}_{n-2,2}\left(\sqrt{L_{\gamma,\Delta}}\right) + S^{+}_{n-2,2}\left(\sqrt{L_{\gamma,\Delta}}\right)\right)$$

$$=: \frac{\mathcal{S}_{\rho,n-2}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty$$

and

$$\begin{aligned}\vartheta_{n-2} &\leq (1+\varsigma_{n-1}K_{L,n-1})\,\vartheta_{n-1}+\varsigma_{n-1}\rho_{n-1}\\ &\leq \left(1+\varsigma_{n-1}\log^{p_-/2+1}(L_{\gamma,\Delta})\right)\frac{\mathcal{S}_{\vartheta,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}+\frac{\varsigma_{n-1}\mathcal{S}_{\rho,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}\\ &=: \frac{\mathcal{S}_{\vartheta,n-2}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}},\quad L_{\gamma,\Delta}\to\infty.\end{aligned}$$

Continuing in the same way, we derive after the $n-(i-1)$th step on $\bigcap_{j=i}^{n}\mathcal{D}_{j,L}$ for some slowly varying functions $\mathcal{S}_{\rho,i}$ and $\mathcal{S}_{\vartheta,i}$:

$$\rho_i \leq \frac{\mathcal{S}_{\rho,i}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}},\quad \vartheta_i \leq \frac{\mathcal{S}_{\vartheta,i}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}},\quad L_{\gamma,\Delta}\to\infty$$

with $\mathcal{S}_{\vartheta,i}(L_{\gamma,\Delta}) = \left(1+\varsigma_{i+1}\log^{p_-/2+1}(L_{\gamma,\Delta})\right)\mathcal{S}_{\vartheta,i+1}(L_{\gamma,\Delta})+\varsigma_{i+1}\mathcal{S}_{\rho,i+1}(L_{\gamma,\Delta}),\quad L_{\gamma,\Delta}\to\infty.$

Set $L_k := 2^k$ and introduce the events

$$\mathcal{A}_{i,k} := \left\{\max_{L_{k-1}<L\leq L_k}\gamma_L\mathfrak{R}_i(L) > c_1\right\},\quad k\in\mathbb{N}.$$

For any $L>0$ there is a natural number k with

$$\overline{\mathcal{D}}_{i,L}\subseteq\mathcal{A}_{i,k}$$

yielding

$$\overline{\bigcap_{j=i}^{n}\mathcal{D}_{j,L}}\subseteq\bigcup_{j=i}^{n}\mathcal{A}_{j,k}.$$

Furthermore, using (3.2.4)-(3.2.6) and the σ-Subadditivity we get

$$\mathbb{P}\left(\bigcup_{j=i}^{n}\mathcal{A}_{j,k}\right)\leq\sum_{j=1}^{n}\mathbb{P}\left(\mathcal{A}_{j,k}\right)\leq nc_2L_k^{-r}.$$

Thus, with $\frac{1}{2^r} < 1$ for all $r > 0$ we have

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\bigcup_{j=i}^{n} \mathcal{A}_{j,k}\right) \leq nc_2 \sum_{k=1}^{\infty} \left(\frac{1}{2^r}\right)^k$$

$$= nc_2 \left[\sum_{k=0}^{\infty} \left(\frac{1}{2^r}\right)^k - 1\right]$$

$$= nc_2 \left(\frac{1}{1-\frac{1}{2^r}} - 1\right) < \infty.$$

As a result, using the Lemma of Borel-Cantelli and the limit comparison test we get

$$\mathbb{P}\left(\limsup_{L} \overline{\bigcap_{j=i}^{n} \mathcal{D}_{j,L}}\right) = \mathbb{P}\left(\limsup_{k} \bigcup_{j=i}^{n} \mathcal{A}_{j,k}\right) = 0.$$

□

A.2 Technical Lemmas and Proofs relating to Theorem 2

Lemma 12. *If we assume that the market values are explicitly given by the Black-Scholes-type formula, then the asymptotic behaviour of the functions V_i, $i = 1,\dots,n$ can be characterised as follows: Each function V_i is a non-negative, monotone decreasing and two times continuously differentiable function on $\mathbb{R}^+$ with $V_i(0) = A_i = \mathcal{D}_{0,T_{i+1}}$. Moreover, each function V_i has the representation*

$$V_i(z) = \begin{cases} \exp\left(-\frac{1}{2\sigma_i^2 T_i}\log^2(z) - \log\left(\frac{\log(z)}{\sigma_i\sqrt{T_i}}\right)\right), & z > z_0, \\ A_i - \exp\left(-\frac{1}{2\sigma_i^2 T_i}\log^2\left(\frac{1}{z}\right) - \log\left(\frac{\log\left(\frac{1}{z}\right)}{\sigma_i\sqrt{T_i}}\right)\right), & 0 \le z \le z_0 \end{cases}$$

for some $z_0 > 0$.

Lemma 13. *The asymptotic representation for V_i in Theorem 2 implies that the inverse function $V_i^{\leftarrow}$ is bounded on any subset of $[0,A_i]$ not containing 0 and its first two derivatives are bounded on any subset of $(0,A_i)$. Moreover, the following representations hold*

$$V_i^{\leftarrow}(x) = \begin{cases} \exp\sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(m_i(k_i(x))\right)}, & 0 \le x < x_0, \\ \exp\left(-\sqrt{k_i(A_i - x) - 2\sigma_i^2 T_i l\left(m_i(k_i(A_i - x))\right)}\right), & x_0 \le x \le A_i, \end{cases} \tag{A.2.1}$$

where $m_i(k_i(x)) := \sqrt{k_i(x) + O\left(l\left(\sqrt{k_i(x)}\right)\right)}$ and $k_i(x) := 2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)$.
Furthermore,

$$\left(V_i^{\leftarrow}\right)^{(j)}(x) = \begin{cases} \dfrac{\sigma_i^2 T_i \exp(n_i(k_i(x)))\left(1 + \frac{\sigma_i^2 T_i}{n_i(k_i(x))}\right)^{j-1}}{(-x)^j n_i(k_i(x))}, & 0 \le x < x_0, \\ \dfrac{-\sigma_i^2 T_i \exp(-n_i(k_i(A_i - x)))\left(1 - \frac{\sigma_i^2 T_i}{n_i(k_i(A_i - x))}\right)^{j-1}}{(A_i - x)^j n_i(k_i(A_i - x))}, & x_0 \le x \le A_i \end{cases} \tag{A.2.2}$$

for $j = 1,2$ and some $x_0 \in (0,A_i)$, where $n_i(k_i(x)) := \sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}$ and g, f are some asymptotic real-valued functions defined as $g(y) := \sqrt{y(f(y)+1)}$ with $f(y) := O\left(\frac{l(\sqrt{y})}{y}\right), y \to \infty$, yielding

$$\frac{\left(V_i^{\leftarrow}\right)'(x)}{x\left(V_i^{\leftarrow}\right)''(x)} \asymp -1 + \frac{\sigma_i^2 T_i}{n_i(k_i(x))}, \quad x \to 0 \tag{A.2.3}$$

as well as

$$\frac{(V_i^{\leftarrow})'(x)}{(A_i - x)(V_i^{\leftarrow})''(x)} \asymp 1 + \frac{\sigma_i^2 T_i}{n_i(k_i(A_i - x))}, \quad x \to A_i. \tag{A.2.4}$$

The representations (A.2.1) and (A.2.2) and the continuity of $(V_i^{\leftarrow}(x))^{(j)}$, $j = 1,2$ *imply that there are constants* $C_{i,3} > 0$ *and* $C_{i,4} > 0$, *such that*

$$\sup_{x \in [a,b]} \left|(V_i^{\leftarrow})'(x)\right| \le C_{i,3} \max\left\{\left|(V_i^{\leftarrow})'(a)\right|, \left|(V_i^{\leftarrow})'(b)\right|\right\} \tag{A.2.5}$$

and

$$\sup_{x \in [a,b]} \left|(V_i^{\leftarrow})''(x)\right| \le C_{i,4} \max\left\{\left|(V_i^{\leftarrow})''(a)\right|, \left|(V_i^{\leftarrow})''(b)\right|\right\} \tag{A.2.6}$$

for $0 \le a < b \le A_i$.

Let us continue with an analysis quite related to the case of the forward operator satisfying **(AV)**. We begin by observing that some assumptions still hold, particularly,

(AX) in force

(AF) in force, i.e we have $J_n(x) \to 0, x \to \infty$ and due to (A.2.1) we get

$$f_n(x) = \exp\sqrt{k_n(J_n(x)) - 2\sigma_n^2 T_n l\left(\sqrt{k_n(J_n(x)) + O\left(l\left(\sqrt{k_n(J_n(x))}\right)\right)}\right)}$$

$$\le \exp\left(\sqrt{2\sigma_n^2 T_n \log\frac{1}{J_n(x)}}\right), \quad J_n(x) \to 0.$$

Continuing, we use the fact that $\sqrt{2\sigma_n^2 T_n \log\frac{1}{J_n(x)}} \le q_+ \log\frac{1}{J_n(x)}, J_n(x) \to 0$ for all $q_+ > 0$ and so

$$f_n(x) \le x^{q_+}, \quad x \to \infty.$$

Moreover, combining the monotony of the exponential function with the fact that $\log(x) \leq \sqrt{x}, x \to \infty$ holds, we get for $x \to \infty$

$$2\sigma_n^2 T_n \log(x) \asymp k_n(J_n(x)) - 2\sigma_n^2 T_n l\left(\sqrt{k_n(J_n(x)) + O\left(l\left(\sqrt{k_n(J_n(x))}\right)\right)}\right) \leq f_n(x).$$

Let us choose $q_+ = 1$, resulting in

$$2\sigma_n^2 T_n \log(x) \leq f_n(x) \leq x, \quad x \to \infty.$$

Next, we can do exactly the same procedure as in the Chapter 4.4, yielding

$$f_{n-1}(x) \asymp \exp\left(\frac{\sigma_{n-1}\sqrt{T_{n-1}}}{\sqrt{\alpha_{n-1}}} x\right), \quad x \to \infty. \tag{A.2.7}$$

With some more details, we have

$$k_2 \left|\log(v)\right| \leq Q_{n-1}(v) \leq k_1 \left|v\right|$$

with $Q_{n-1}(v) = \int_{-\infty}^{\infty}(1 + \varsigma_n f_n(u))k_{n-1}(u-v)du$ and some positive constants k_1, k_2. Next,

$$k_2 \int_x^{\infty} \left|\log(v)\right| e^{-v^2/(2\alpha_{n-1})} dv \leq J_{n-1}(x) \leq k_1 \int_x^{\infty} \left|v\right| e^{-v^2/(2\alpha_{n-1})} dv$$

which is asymptotically equivalent to

$$\frac{k_2 \alpha_{n-1}\left(x^2 \log(x) + \alpha_{n-1}\right)}{x^3 e^{x^2/(2\alpha_{n-1})}} \leq J_{n-1}(x) \leq \frac{k_1 \alpha_{n-1}}{e^{x^2/(2\alpha_{n-1})}}, \quad x \to \infty \tag{A.2.8}$$

due to

$$\int_x^{\infty} v e^{-v^2/(2\alpha_{n-1})} dv \asymp \frac{\alpha_{n-1}}{e^{x^2/(2\alpha_{n-1})}}, \quad x \to \infty$$

(see (4.4.2)) and

$$\int_x^{\infty} \log(v) e^{-v^2/(2\alpha_{n-1})} dv \asymp \frac{\alpha_{n-1}\left(x^2 \log(x) + \alpha_{n-1}\right)}{x^3 e^{x^2/(2\alpha_{n-1})}}, \quad x \to \infty. \tag{A.2.9}$$

Indeed, as $x \longrightarrow \infty$ we obtain

$$\int_x^\infty \log(v) e^{-v^2/(2\alpha_{n-1})} dv = \int_x^\infty \frac{\log(v)}{[-v/\alpha_{n-1}]} [-v/\alpha_{n-1}] e^{-v^2/(2\alpha_{n-1})} dv$$

$$= -\alpha_{n-1} \left\{ \frac{\log(v)}{v} e^{-v^2/(2\alpha_{n-1})} \Big|_x^\infty - \int_x^\infty \frac{1-\log(v)}{v^2} e^{-v^2/(2\alpha_{n-1})} dv \right\}$$

$$= \frac{\alpha_{n-1} \log(x)}{x e^{x^2/(2\alpha_{n-1})}} - \alpha_{n-1} \int_x^\infty \frac{\log(v)}{v^2} e^{-v^2/(2\alpha_{n-1})} dv$$

$$+ \alpha_{n-1} \int_x^\infty v^{-2} e^{-v^2/(2\alpha_{n-1})} dv.$$

Hence,

$$\left| \int_x^\infty \log(v) e^{-v^2/(2\alpha_{n-1})} dv - \alpha_{n-1} \left\{ \frac{\log(x)}{x e^{x^2/(2\alpha_{n-1})}} + \int_x^\infty v^{-2} e^{-v^2/(2\alpha_{n-1})} dv \right\} \right|$$

$$= \alpha_{n-1} \int_x^\infty \frac{\log(v)}{v^2} e^{-v^2/(2\alpha_{n-1})} dv$$

$$\leq \alpha_{n-1} \frac{1}{x^2} \int_x^\infty \log(v) e^{-v^2/(2\alpha_{n-1})} dv$$

$$= o\left(\int_x^\infty \log(v) e^{-v^2/(2\alpha_{n-1})} dv \right), \quad x \longrightarrow \infty$$

and thus

$$\int_x^\infty \log(v) e^{-v^2/(2\alpha_{n-1})} dv \asymp \alpha_{n-1} \left\{ \frac{\log(x)}{x e^{x^2/(2\alpha_{n-1})}} + \int_x^\infty v^{-2} e^{-v^2/(2\alpha_{n-1})} dv \right\}, \quad x \longrightarrow \infty.$$

Finally, use (4.4.2) with $q_\pm = -2$ which is (A.2.9).

Now, notice that $V_{n-1}^{\leftarrow}$ is a monotone decreasing function and use (A.2.8) to obtain

$$V_{n-1}^{\leftarrow}\left(\frac{k_1 \alpha_{n-1}}{e^{x^2/(2\alpha_{n-1})}} \right) \leq V_{n-1}^{\leftarrow}(J_{n-1}(x)) \leq V_{n-1}^{\leftarrow}\left(\frac{k_2 \alpha_{n-1}\left(x^2 \log(x) + \alpha_{n-1}\right)}{x^3 e^{x^2/(2\alpha_{n-1})}} \right) \tag{A.2.10}$$

for $x \longrightarrow \infty$.

Next, on account of Lemma 13 we have $V_{n-1}^{\leftarrow}(x) \asymp \exp\left(\sqrt{2\sigma_{n-1}^2 T_{n-1}\log(\frac{1}{x})}\right), x \to 0$. So the left hand side and the right hand side of (A.2.10) are both $\asymp \exp\left(\frac{\sigma_{n-1}\sqrt{T_{n-1}}}{\sqrt{\alpha_{n-1}}}x\right), x \to \infty$, which is (A.2.7).

Let us do the same procedure to get an asymptotic equivalent of $f_{n-2}(x), x \to \infty$.

Using (A.2.7) we get

$$\tilde{k}_2 \exp\left(\frac{\sigma_{n-1}\sqrt{T_{n-1}}}{\sqrt{\alpha_{n-1}}}v\right) \le Q_{n-2}(v) \le \tilde{k}_1 \exp\left(\frac{\sigma_{n-1}\sqrt{T_{n-1}}}{\sqrt{\alpha_{n-1}}}v\right)$$

with some positive constants $\tilde{k}_1, \tilde{k}_2$. As a result,

$$\frac{\tilde{k}_2 e^{\sigma_{n-1}\sqrt{T_{n-1}}x/\sqrt{\alpha_{n-1}} - x^2/(2\alpha_{n-2})}}{x/\alpha_{n-2} - \sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}}} \le J_{n-2}(x) \le \frac{\tilde{k}_1 e^{\sigma_{n-1}\sqrt{T_{n-1}}x/\sqrt{\alpha_{n-1}} - x^2/(2\alpha_{n-2})}}{x/\alpha_{n-2} - \sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}}}, \quad x \to \infty$$

due to

$$\int_x^\infty e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}} - v^2/(2\alpha_{n-2})} dv$$

$$= \int_x^\infty \frac{e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}} - v^2/(2\alpha_{n-2})}}{\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}} - v/\alpha_{n-2}} \left(\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}} - v/\alpha_{n-2}\right) dv$$

$$= \left.\frac{e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}} - v^2/(2\alpha_{n-2})}}{\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}} - v/\alpha_{n-2}}\right|_x^\infty$$

$$- \int_x^\infty \frac{1/\alpha_{n-2}}{\left(\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}} - v/\alpha_{n-2}\right)^2} e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}} - v^2/(2\alpha_{n-2})} dv$$

$$= \frac{e^{\sigma_{n-1}\sqrt{T_{n-1}}x/\sqrt{\alpha_{n-1}} - x^2/(2\alpha_{n-2})}}{x/\alpha_{n-2} - \sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}}}$$

$$- \int_x^\infty \frac{1/\alpha_{n-2}}{\left(\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}} - v/\alpha_{n-2}\right)^2} e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}} - v^2/(2\alpha_{n-2})} dv$$

and

$$\left| \int_x^\infty e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}}-v^2/(2\alpha_{n-2})}dv - \frac{e^{\sigma_{n-1}\sqrt{T_{n-1}}x/\sqrt{\alpha_{n-1}}-x^2/(2\alpha_{n-2})}}{x/\alpha_{n-2}-\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}}} \right|$$

$$= \int_x^\infty \frac{1/\alpha_{n-2}}{\left(\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}}-v/\alpha_{n-2}\right)^2} e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}}-v^2/(2\alpha_{n-2})}dv$$

$$\leq \frac{1/\alpha_{n-2}}{\left(\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}}-x/\alpha_{n-2}\right)^2} \int_x^\infty e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}}-v^2/(2\alpha_{n-2})}dv$$

$$= o\left(\int_x^\infty e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}}-v^2/(2\alpha_{n-2})}dv \right), \quad x \to \infty$$

meaning

$$\int_x^\infty e^{\sigma_{n-1}\sqrt{T_{n-1}}v/\sqrt{\alpha_{n-1}}-v^2/(2\alpha_{n-2})}dv \asymp \frac{e^{\sigma_{n-1}\sqrt{T_{n-1}}x/\sqrt{\alpha_{n-1}}-x^2/(2\alpha_{n-2})}}{x/\alpha_{n-2}-\sigma_{n-1}\sqrt{T_{n-1}}/\sqrt{\alpha_{n-1}}}, \quad x \to \infty.$$

From this it follows that

$$f_{n-2}(x) \asymp \exp\left(\frac{\sigma_{n-2}\sqrt{T_{n-2}}}{\sqrt{\alpha_{n-2}}}x\right), \quad x \to \infty.$$

Continuing backwards in this way, we derive

$$f_i(x) \asymp \exp\left(\frac{\sigma_i\sqrt{T_i}}{\sqrt{\alpha_i}}x\right), \quad x \to \infty$$

and

$$J_i(x) \asymp \frac{\alpha_i\sqrt{\alpha_{i+1}}e^{\sigma_{i+1}\sqrt{T_{i+1}}x/\sqrt{\alpha_{i+1}}-x^2/(2\alpha_i)}}{x\sqrt{\alpha_{i+1}}-\sigma_{i+1}\alpha_i\sqrt{T_{i+1}}}, \quad x \to +\infty \tag{A.2.11}$$

for all $i = 1, \ldots, n-1$.

Analogously, we have

$$f_i(x) \asymp \exp\left(-\sigma_i\sqrt{T_i}\frac{|x|}{\sqrt{\alpha_i}}\right), \quad x \to -\infty$$

as well as

$$J_i(x) \asymp A_i - \frac{\alpha_i\sqrt{\alpha_{i+1}}e^{-\sigma_{i+1}\sqrt{T_{i+1}}|x|/\sqrt{\alpha_{i+1}}-x^2/(2\alpha_i)}}{\sqrt{\alpha_{i+1}}\,|x| - \sigma_{i+1}\alpha_i\sqrt{T_{i+1}}}, \quad x \to -\infty \tag{A.2.12}$$

for all $i = 1, \dots, n-1$.

The asymptotic relations (A.2.11) and (A.2.12) indicate that the threshold functions

$$J_i^-(x) = \frac{\hat{c}_i e^{\sigma_{i+1}\sqrt{T_{i+1}}x/\sqrt{\alpha_{i+1}}-x^2/(2\alpha_i)}}{x\sqrt{\alpha_{i+1}} - \sigma_{i+1}\alpha_i\sqrt{T_{i+1}}}$$

and

$$J_i^+(x) = A_i - \frac{\check{c}_i e^{-\sigma_{i+1}\sqrt{T_{i+1}}|x|/\sqrt{\alpha_{i+1}}-x^2/(2\alpha_i)}}{\sqrt{\alpha_{i+1}}\,|x| - \sigma_{i+1}\alpha_i\sqrt{T_{i+1}}}$$

would satisfy the J_i^- and J_i^+ requirements of Theorem 2 for $\hat{c}_i > 0$ small enough and $\check{c}_i > 0$ big enough.

Note that technical Lemmas 5-6 hold due to the uniform boundedness of V_i' on $\mathbb{R}^+$, i.e.

$$\left|V_i'(z)\right| = \begin{cases} \exp\left(-\frac{\log^2(z)}{2\sigma_i^2 T_i} - l\left(\log(z)\right)\right)\frac{\log^2(z)+\sigma_i^2 T_i l'(\log(z))\log(z)}{\sigma_i^2 T_i z\log(z)}, & z > z_0, \\ \exp\left(-\frac{1}{2\sigma_i^2 T_i}\log^2\left(\frac{1}{z}\right) - l\left(\log\left(\frac{1}{z}\right)\right)\right)\frac{\log^2\left(\frac{1}{z}\right)+\sigma_i^2 T_i l'\left(\log\left(\frac{1}{z}\right)\right)\log\left(\frac{1}{z}\right)}{\sigma_i^2 T_i z\log\left(\frac{1}{z}\right)}, & 0 \le z \le z_0 \end{cases}$$

for some $z_0 > 0$.

The next lemma shows that with a proper choice of the thresholds $J_i^-(y)$ and $J_i^+(y)$ $V_i^{\leftarrow}(J_i(y))$ and $V_i^{\leftarrow}(\widehat{J}_i(y))$ are close, provided that $J_i(y)$ and $\widetilde{J}_i(y)$ are close.

Lemma 14. *For $x \in [0;A_i]$ set*

$$\mathcal{H}_i^-(x) := \frac{1}{1+\theta}\left[\theta + \frac{\sigma_i^2 T_i}{\sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}}\right],$$

$$\mathcal{H}_i^+(x) := \frac{1}{1+\theta}\left[\theta - \frac{\sigma_i^2 T_i}{\sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}}\right]$$

for some $\theta > \frac{\sigma_i^2 T_i}{\sqrt{k_i(x_0) - 2\sigma_i^2 T_i l(g(k_i(x_0)))}}$ and k_i, g, l as defined in Theorem 2 and Lemma 13. Then

$$\left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right| \leq \xi_i(y)\left(J_i(y) - \widetilde{J}_i(y)\right)^2,$$

where

$$\xi_i(y) := \left\{\frac{1}{2} + c(1+\theta)\right\} q_i(y)$$

for some constant $c > 0$ with

$$q_i(y) := \sup_{x \in [J_i^-(y), J_i^+(y)]} \left|(V_i^{\leftarrow})''(x)\right|,$$

provided that the thresholds satisfy

$$J_i^-(y) \leq J_i(y)\mathcal{H}_i^-(J_i(y)), \quad J_i(y) \in [0, x_0] \tag{A.2.13}$$

and

$$A_i - J_i^+(y) \leq (A_i - J_i(y))\mathcal{H}_i^+(A_i - J_i(y)), \quad J_i(y) \in (A_i - x_0, A_i] \tag{A.2.14}$$

respectively.

Before proceeding, we introduce a tool which makes it easier to check whether a real-valued, well-defined and continuously differentiable function for $x \in [0;x_0]$ ($x \geq x_0$) and some positive x_0 is slowly/regularly varying. Thus, the next lemma can be understood as a sufficient condition for a continuously differentiable function to be regularly varying and slowly varying respectively.

Lemma 15. *Any real-valued and well-defined, measurable function ψ which is positive, continuously differentiable for $x \in [0;x_0]$ ($x \geq x_0$), where x_0 is some positive constant, satisfying*

$$\frac{x\psi'(x)}{\psi(x)} \to 0 \tag{A.2.15}$$

as $x \to 0$ ($x \to \infty$) is slowly varying at zero (at infinity). Moreover, if the right hand side of (A.2.15) is more generally some finite number r, then ψ is said to be regularly varying and r is the so-called index of regular variation (see e.g. equation (1.4.3) in Bingham, Goldie, Teugels (1987)).

Lemma 16. *It holds for $j = 1,2$ and any fixed $0 < x_0 < A_i$*

$$\int_{\varepsilon}^{x_0} \left| \left(V_i^{\leftarrow}\right)^{(j)} \left(x\overline{\mathcal{H}}_i(x)\right) \right| dx \lesssim \left(\frac{1}{\varepsilon}\right)^{j-1} S_{i,j}^{-}\left(\frac{1}{\varepsilon}\right)$$

and

$$\int_{x_0}^{A_i-\varepsilon} \left| \left(V_i^{\leftarrow}\right)^{(j)} \left(A_i - (A_i - x)\overline{\mathcal{H}}_i(A_i - x)\right) \right| dx \lesssim \left(\frac{1}{\varepsilon}\right)^{j-1} S_{i,j}^{+}\left(\frac{1}{\varepsilon}\right)$$

as $\varepsilon \to 0$, where $S_{i,1}^{\pm}$ and $S_{i,2}^{\pm}$ are slowly varying functions.

A.2.1 Proof of Lemma 12

If we assume the market values are explicitly given by the Black-Scholes-type formula with a constant instantaneous volatility $\sigma_i(t) = \sigma_i$, then the price at time zero for a digital caplet is

$$V_i(z) = \mathcal{D}_{0,T_{i+1}} \Phi(d_i(z)),$$

where Φ is the cumulative normal distribution function and thus

$$\Phi\left(d_i(z)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i(z)} \exp\left(-\frac{t^2}{2}\right) dt = 1 - \frac{1}{\sqrt{2\pi}} \int_{d_i(z)}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \tag{A.2.16}$$

with

$$d_i(z) = \frac{\log\left(L_0^i/z\right)}{\sigma_i\sqrt{T_i}} - \frac{1}{2}\sigma_i\sqrt{T_i} \lesssim -\frac{\log(z)}{\sigma_i\sqrt{T_i}} \tag{A.2.17}$$

and $\mathcal{D}_{0,T_{i+1}}$ is quoted on the market (see e.g. equation (12) in Hunt, Kennedy, Pelsser (2000)).

To obtain some information about the asymptotic behaviour of $V_i(z), z \to 0$ and $V_i(z), z \to \infty$, we begin by considering the asymptotic approximation to $\int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt, x \to \infty$ using the technique of integration-by-parts:

$$\int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt = \int_x^\infty \left(-\frac{1}{t}\right)\left(-t\exp\left(-\frac{t^2}{2}\right)\right) dt$$

$$= \left(-\frac{1}{t}\right)\left(\exp\left(-\frac{t^2}{2}\right)\right)\Big|_x^\infty - \int_x^\infty \frac{\exp\left(-\frac{t^2}{2}\right)}{t^2} dt$$

$$= \frac{\exp\left(-\frac{x^2}{2}\right)}{x} - \int_x^\infty \frac{\exp\left(-\frac{t^2}{2}\right)}{t^2} dt.$$

This implies that

$$\left|\int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt - \frac{\exp\left(-\frac{x^2}{2}\right)}{x}\right| = \int_x^\infty \frac{\exp\left(-\frac{t^2}{2}\right)}{t^2} dt$$

$$\leq \frac{1}{x^2}\int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt$$

$$= o\left(\int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt\right), \quad x \to \infty.$$

Hence,

$$\int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt \asymp \frac{\exp\left(-\frac{x^2}{2}\right)}{x}, \quad x \to \infty, \tag{A.2.18}$$

as well as

$$\int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right) dt \asymp \frac{\exp\left(-\frac{x^2}{2}\right)}{|x|}, \quad x \to -\infty. \tag{A.2.19}$$

Next, it follows with (A.2.17) that

$$d_i(z) \to -\infty \text{ as } z \to \infty,$$

as well as

$$d_i(z) \to \infty \text{ as } z \to 0.$$

Combining this with (A.2.16), (A.2.18) and (A.2.19), we get

$$\Phi(d_i(z)) \asymp \frac{1}{\sqrt{2\pi}} \frac{\exp\left(-\frac{1}{2}d_i^2(z)\right)}{|d_i(z)|}, \quad z \to \infty$$

and

$$\Phi(d_i(z)) \asymp 1 - \frac{1}{\sqrt{2\pi}} \frac{\exp\left(-\frac{1}{2}d_i^2(z)\right)}{d_i(z)}, \quad z \to 0.$$

□

A.2.2 Proof of Lemma 13

We split the determination of the asymptotic inverse function $V_i^{\leftarrow}(x)$ into two parts. First, we consider the case $x \to 0$, and afterwards the case $x \to A_i$.

CASE $z \to \infty$ AND $x \to 0$: Put $\log(z) =: \breve{z}$ and $t := \frac{1}{x}$. Thus

$$\frac{\exp\left(-\frac{1}{2\sigma_i^2 T_i}\log^2(z)\right)}{\exp\left(l\left(\log(z)\right)\right)} = x, \quad x \to 0$$

⇔

$$\exp\left(l\left(\breve{z}\right)\right)\exp\left(\frac{1}{2\sigma_i^2 T_i}\breve{z}^2\right) = t, \quad t \to \infty$$

⇔

$$\frac{1}{2\sigma_i^2 T_i}\breve{z}^2 = \log t - l\left(\breve{z}\right), \quad t \to \infty. \tag{A.2.20}$$

Due to $t \to \infty$ we may assume $t > e$. With $z \to \infty$ we get $\breve{z} \to \infty$ and so $l(\breve{z}) \overset{\breve{z}\to\infty}{\longrightarrow} \infty$. Using (A.2.20) we have

$$\frac{1}{2\sigma_i^2 T_i}\breve{z}^2 < \log t, \quad t \to \infty \tag{A.2.21}$$

and thus $\breve{z} < \sqrt{2\sigma_i^2 T_i \log t}, t \to \infty$. From this it follows that

$$l(\breve{z}) = O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)}\right)\right), \quad x \to 0.$$

Iterating, we get

$$\frac{1}{2\sigma_i^2 T_i}\breve{z}^2 = \log\left(\frac{1}{x}\right) + O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)}\right)\right), \quad x \to 0,$$

so

$$\breve{z} = \sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) + O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)}\right)\right)}, \quad x \to 0.$$

With a similar argumentation one can get

$$l(\breve{z}) = l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) + O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)}\right)\right)}\right), \quad x \to 0.$$

Using (A.2.20) we have

$$\frac{1}{2\sigma_i^2 T_i}\breve{z}^2 = \log\left(\frac{1}{x}\right) - l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) + O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)}\right)\right)}\right), \quad x \to 0$$

and so

$$\breve{z} = \sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) - 2\sigma_i^2 T_i l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) + O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)}\right)\right)}\right)}, \quad x \to 0.$$

It follows that as $x \longrightarrow 0$ we get

$$z = \exp\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) - 2\sigma_i^2 T_i l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) + O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)}\right)\right)}\right)}\right).$$

CASE $z \longrightarrow 0$ AND $x \longrightarrow A_i$: Denote $\tilde{z} := \frac{1}{z}$ and $\tilde{x} := A_i - x$. Now, analogous to the first case we have to decide on the behaviour of $\tilde{z}$ as a solution of

$$\frac{\exp\left(-\frac{1}{2\sigma_i^2 T_i}\log^2(\tilde{z})\right)}{\exp\left(l\left(\log \tilde{z}\right)\right)} = \tilde{x}, \quad \tilde{x} \longrightarrow 0$$

which is equivalent to

$$\frac{1}{2\sigma_i^2 T_i}\log^2(\tilde{z}) = \log\left(\frac{1}{\tilde{x}}\right) - l\left(\log \tilde{z}\right), \quad \tilde{x} \longrightarrow 0.$$

Due to $\tilde{z} \longrightarrow \infty$ we can assume $\tilde{z} > e$. Further, $l\left(\log \tilde{z}\right) \longrightarrow \infty$ as $\tilde{z} \longrightarrow \infty$. Thus,

$$1 < \log(\tilde{z}) < \sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)}, \quad x \longrightarrow A_i$$

$\Leftrightarrow$

$$\exp(1) < \tilde{z} < \exp\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)}\right), \quad x \longrightarrow A_i.$$

We use this result as a first approximation of $\tilde{z}$ and continue iterating. We have

$$\log\left(\frac{1}{z}\right) < \sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)}, \quad x \longrightarrow A_i,$$

and using

$$-\frac{1}{2\sigma_i^2 T_i}\log^2\left(\frac{1}{z}\right) = \log(A_i - x) + O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)}\right)\right), \quad x \longrightarrow A_i$$

we get

$$\log\left(\frac{1}{z}\right) = \sqrt{-2\sigma_i^2 T_i \log(A_i - x) + O\left(l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)}\right)\right)}, \quad x \longrightarrow A_i.$$

Hence, we obtain in similar manner as before

$$-\frac{1}{2\sigma_i^2 T_i}\log^2\left(\frac{1}{z}\right) = \log(A_i - x)$$

$$+ l\left(\sqrt{-2\sigma_i^2 T_i \log(A_i - x) + O\left(\log\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)}\right)}\right), \quad x \to A_i,$$

so

$$\log\left(\frac{1}{z}\right) =$$

$$= \sqrt{-2\sigma_i^2 T_i\left(\log(A_i - x) + l\left(\sqrt{-2\sigma_i^2 T_i \log(A_i - x) + O\left(\log\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)}\right)}\right)\right)},$$

and it follows that

$$z = \frac{1}{\exp\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right) - 2\sigma_i^2 T_i l\left(\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right) + O\left(\log\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)}\right)}\right)}}.$$

As a result,

$$V_i^{\leftarrow}(x) = \begin{cases} \exp\sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(m_i(k_i(x))\right)}, & 0 \le x < x_0, \\ \exp\left(-\sqrt{k_i(A_i - x) - 2\sigma_i^2 T_i l\left(m_i(k_i(A_i - x))\right)}\right), & x_0 \le x \le A_i, \end{cases}$$

where $m_i(k_i(x)) := \sqrt{k_i(x) + O\left(l\left(\sqrt{k_i(x)}\right)\right)}$ and $k_i(x) := 2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)$.

Next, let us focus on the derivatives of $V_i^{\leftarrow}$. We begin by considering the representation of $V_i^{\leftarrow}(x)$, $x \le x_0$. Set $y := k_i(x)$. Then

$$V_i^{\leftarrow}(y) = \exp\sqrt{y - 2\sigma_i^2 T_i l\left(\sqrt{y + O\left(l\left(\sqrt{y}\right)\right)}\right)}, \quad y \to \infty.$$

With g, f as in Lemma 13 $V_i^{\leftarrow}(y)$ has the following asymptotic representation:

$$V_i^{\leftarrow}(y) = \exp\sqrt{y - 2\sigma_i^2 T_i l\,(g(y))}$$

$$= \exp\sqrt{y - 2\sigma_i^2 T_i l\left(\sqrt{y}\sqrt{1+f(y)}\right)}, \quad y \to \infty.$$

Next, we get

$$\left(V_i^{\leftarrow}\right)'(y) = \exp\sqrt{y - 2\sigma_i^2 T_i l\,(g(y))}\,\frac{1 - 2\sigma_i^2 T_i l'\,(g(y))\,g'(y)}{2\sqrt{y - 2\sigma_i^2 T_i l\,(g(y))}}$$

$$= \frac{\exp\sqrt{y - 2\sigma_i^2 T_i l\left((y(f(y)+1))^{1/2}\right)}}{2\sqrt{y - 2\sigma_i^2 T_i l\left((y(f(y)+1))^{1/2}\right)}} \times$$

$$\times\left(1 - \sigma_i^2 T_i \frac{l'\,(g(y))\,g(y)}{l(g(y))} l(g(y)) \frac{yf'(y) + 1 + f(y)}{y(f(y)+1)}\right)$$

and

$$\left(V_i^{\leftarrow}\right)''(y) = \frac{\exp\sqrt{y - 2\sigma_i^2 T_i l\,(g(y))}}{4\left(y - 2\sigma_i^2 T_i l\,(g(y))\right)^{3/2}}\left(\sqrt{y - 2\sigma_i^2 T_i l\,(g(y))} - 1\right)\left(1 - 2\sigma_i^2 T_i l'\,(g(y))\,g'(y)\right)^2$$

$$- \frac{\exp\sqrt{y - 2\sigma_i^2 T_i l\,(g(y))}}{\left(y - 2\sigma_i^2 T_i l\,(g(y))\right)^{1/2}}\left(\sigma_i^2 T_i l''\,(g(y))\,(g'(y))^2 + \sigma_i^2 T_i l'\,(g(y))\,g''(y)\right),$$

where

$$g'(y) = \frac{yf'(y) + 1 + f(y)}{2\,(y(f(y)+1))^{1/2}}$$

and

$$g''(y) = \frac{2y(f(y)+1)(yf''(y)+2f'(y)) - (yf'(y)+f(y)+1)^2}{4(y(f(y)+1))^{3/2}}$$

$$= \frac{-1-2f(y)-f^2(y)+2yf'(y)+2yf(y)f'(y)-y^2(f'(y))^2+2y^2f''(y)+2y^2f(y)f''(y)}{4(y(f(y)+1))^{3/2}}.$$

Since

$$\frac{d}{dx}(V_i^{\leftarrow})\left(2\sigma_i^2T_i\log\left(\frac{1}{x}\right)\right) = \frac{-2\sigma_i^2T_i}{x}(V_i^{\leftarrow})'\left(2\sigma_i^2T_i\log\left(\frac{1}{x}\right)\right) \tag{A.2.22}$$

and

$$\begin{aligned}\frac{d^2}{dx^2}(V_i^{\leftarrow})\left(2\sigma_i^2T_i\log\left(\frac{1}{x}\right)\right) = &\left(\frac{2\sigma_i^2T_i}{x}\right)^2(V_i^{\leftarrow})''\left(2\sigma_i^2T_i\log\left(\frac{1}{x}\right)\right)\\ &+\frac{2\sigma_i^2T_i}{x^2}(V_i^{\leftarrow})'\left(2\sigma_i^2T_i\log\left(\frac{1}{x}\right)\right)\end{aligned}$$

we obtain

$$\frac{x(V_i^{\leftarrow})''(x)}{(V_i^{\leftarrow})'(x)} = -1 - \frac{2\sigma_i^2T_i(V_i^{\leftarrow})''\left(2\sigma_i^2T_i\log\left(\frac{1}{x}\right)\right)}{(V_i^{\leftarrow})'\left(2\sigma_i^2T_i\log\left(\frac{1}{x}\right)\right)}, \quad x\to 0. \tag{A.2.23}$$

Continuing, we use the fact that

$$\frac{f'(y)}{f(y)} \to \frac{-2}{\sqrt{y}}, \quad y\to\infty \tag{A.2.24}$$

and

$$\frac{f''(y)}{f(y)} \to \frac{6}{y}, \quad y\to\infty, \tag{A.2.25}$$

as well as

$$\frac{g(y)l'(g(y))}{l(g(y))} \to 0, \quad y\to\infty \tag{A.2.26}$$

and

$$\frac{g^2(y)l''(g(y))}{l(g(y))} \to 0, \quad y\to\infty. \tag{A.2.27}$$

Thus,

$$\left(V_i^{\leftarrow}\right)'(y) = \frac{\exp\sqrt{y - 2\sigma_i^2 T_i l\left(\left(y(f(y)+1)\right)^{1/2}\right)}}{2\sqrt{y - 2\sigma_i^2 T_i l\left(\left(y(f(y)+1)\right)^{1/2}\right)}}, \quad y \to \infty. \tag{A.2.28}$$

Hence,

$$l(g(y))\frac{yf'(y)+1+f(y)}{y(f(y)+1)} \overset{y\to\infty}{\to} l(g(y))\frac{f(y)\left(1-2\sqrt{y}\right)+1}{y(f(y)+1)} \leq \frac{l(g(y))}{y} \overset{y\to\infty}{\to} 0. \tag{A.2.29}$$

Combining (A.2.22) with (A.2.28) and the fact that $y = k_i(x)$, we get

$$\left(V_i^{\leftarrow}\right)'(x) = \frac{-\sigma_i^2 T_i}{x} \cdot \frac{\exp\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) - 2\sigma_i^2 T_i l\left(g\left(2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)\right)\right)}}{\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{x}\right) - 2\sigma_i^2 T_i l\left(g\left(2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)\right)\right)}}, \quad x \to 0.$$

Let us continue by focusing on $\frac{\left(V_i^{\leftarrow}\right)''(y)}{\left(V_i^{\leftarrow}\right)'(y)}, y \to \infty$. First, we have

$$\begin{aligned}
\frac{\left(V_i^{\leftarrow}\right)''(y)}{\left(V_i^{\leftarrow}\right)'(y)} &= \frac{1 - 2\sigma_i^2 T_i l'(g(y))\, g'(y)}{2\left(y - 2\sigma_i^2 T_i l(g(y))\right)^{1/2}} \\
&\quad - \sigma_i^2 T_i \frac{\exp\sqrt{y - 2\sigma_i^2 T_i l(g(y))}\left(l''(g(y))\,[g'(y)]^2 + l'(g(y))g''(y)\right)}{\left(V_i^{\leftarrow}\right)'(y)\left(y - 2\sigma_i^2 T_i l(g(y))\right)^{1/2}} \\
&\quad - \frac{\exp\sqrt{y - 2\sigma_i^2 T_i l(g(y))}\left(1 - 2\sigma_i^2 T_i l'(g(y))\, g'(y)\right)^2}{4\left(V_i^{\leftarrow}\right)'(y)\left(y - 2\sigma_i^2 T_i l(g(y))\right)^{3/2}}.
\end{aligned}$$

Now, switching to the asymptotic reconsideration we use $l'(g(y))\,g'(y) \overset{y\to\infty}{\longrightarrow} 0$ by (A.2.29) and (A.2.28) to get

$$\frac{(V_i^{\leftarrow})''(y)}{(V_i^{\leftarrow})'(y)} = \frac{1}{2\left(y-2\sigma_i^2 T_i l\left(g(y)\right)\right)^{1/2}}$$

$$-2\sigma_i^2 T_i\left(l''(g(y))\left[g'(y)\right]^2 + l'(g(y))g''(y)\right)$$

$$-\frac{1}{2\left(y-2\sigma_i^2 T_i l\left(g(y)\right)\right)}, \quad y\to\infty.$$

Let us focus on the detailed analysis of the second element of the last asymptotic equation

$$l''(g(y))\left[g'(y)\right]^2 + l'(g(y))g''(y) = \frac{g^2(y)l''(g(y))}{l(g(y))}\frac{l(g(y))\left(yf'(y)+1+f(y)\right)^2}{4g^4(y)}$$

$$+\frac{g(y)l'(g(y))}{l(g(y))}\frac{l(g(y))g''(y)}{g(y)}, \quad y\to\infty.$$

Using (A.2.29) we get

$$\frac{l(g(y))\left(yf'(y)+1+f(y)\right)^2}{4g^4(y)} = O\left(\frac{l(y)}{y^2}\right), \quad y\to\infty.$$

Thus, the first element of the equation in focus tends to zero as $y\to\infty$. To be able to say a little bit more about the second element consider $\frac{g''(y)}{g(y)}, y\to\infty$:

$$\frac{g''(y)}{g(y)} = \frac{2y(f(y)+1)\left(6f(y)-\frac{4}{\sqrt{y}}f(y)\right)-\left(-2\sqrt{y}f(y)+f(y)+1\right)^2}{4g^4(y)}$$

$$= \frac{f(y)(3y-2\sqrt{y})}{y^2(1+f(y))} - \frac{1}{4y^2} + \frac{f(y)}{y^{3/2}(1+f(y))} - \frac{f^2(y)}{y(1+f(y))^2}, \quad y\to\infty.$$

It follows that

$$\frac{g''(y)}{g(y)} \le \frac{3f(y)}{y(1+f(y))} + \frac{f(y)}{y^{3/2}(1+f(y))}$$

$$= \frac{1}{g(y)}\frac{f(y)}{\sqrt{y}}\left(\frac{3}{\sqrt{f(y)+1}} + \frac{1}{\sqrt{y}\sqrt{f(y)+1}}\right), \quad y\to\infty.$$

Thus,

$$l(g(y))\frac{g''(y)}{g(y)} \to 0, \quad y \to \infty$$

and so

$$l''(g(y))\left[g'(y)\right]^2 + l'(g(y))g''(y) \to 0, \quad y \to \infty.$$

Moreover,

$$\frac{(V_i^{\leftarrow})''(y)}{(V_i^{\leftarrow})'(y)} \asymp \frac{1}{2\left(y - 2\sigma_i^2 T_i l\left(g(y)\right)\right)^{1/2}}, \quad y \to \infty. \tag{A.2.30}$$

By (A.2.28) it follows that

$$(V_i^{\leftarrow})''(y) = \frac{\exp\sqrt{y - 2\sigma_i^2 T_i l\left(g(y)\right)}}{4\left(y - 2\sigma_i^2 T_i l\left(g(y)\right)\right)}, \quad y \to \infty,$$

and so using $k_i(x) = 2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)$

$$(V_i^{\leftarrow})''(x) = \left(\frac{\sigma_i^2 T_i}{x}\right)^2 \frac{\exp\sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}}{\left(k_i(x) - 2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)\right)}$$

$$+\frac{\sigma_i^2 T_i}{x^2} \cdot \frac{\exp\sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}}{\sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}}, \quad x \to 0.$$

Combining (A.2.30) with (A.2.23) we have

$$\frac{x\,(V_i^{\leftarrow})''(x)}{(V_i^{\leftarrow})'(x)} \asymp -1 - \frac{\sigma_i^2 T_i}{\sqrt{k_i(x) - 2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}}, \quad x \to 0.$$

Hence,

$$\frac{(V_i^{\leftarrow})'(x)}{x\,(V_i^{\leftarrow})''(x)} \asymp \frac{1}{-1 - \frac{\sigma_i^2 T_i}{\sqrt{k_i(x) - 2\sigma_i^2 T_i l(g(k_i(x)))}}}, \quad x \to 0. \tag{A.2.31}$$

Rewriting the right hand side of (A.2.31):

$$\frac{1}{-1-\frac{\sigma_i^2 T_i}{\sqrt{k_i(x)-2\sigma_i^2 T_i l(g(k_i(x)))}}} = -1+\frac{\sigma_i^2 T_i}{\sqrt{k_i(x)-2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}+\sigma_i^2 T_i}$$

we use

$$\frac{\sigma_i^2 T_i}{\sqrt{k_i(x)-2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}+\sigma_i^2 T_i} \asymp \frac{\sigma_i^2 T_i}{\sqrt{k_i(x)-2\sigma_i^2 T_i l\left(g\left(k_i(x)\right)\right)}}, \quad x\to 0$$

to get (A.2.3) as claimed.

Next, assume $x_0 \le x \le A_i$ for some $x_0 \in (0, A_i)$ and let us do the same procedure as above. Redefine $y := 2\sigma^2 T_i \log\left(\frac{1}{A_i - x}\right)$, hence

$$V_i^{\leftarrow}(y) = \exp\left(-\sqrt{y-2\sigma_i^2 T_i l\left(\sqrt{y+O\left(l\left(\sqrt{y}\right)\right)}\right)}\right), \quad y\to\infty.$$

Once again, with g, f as in Lemma 13 $V_i^{\leftarrow}(y)$ has the following asymptotic representation:

$$V_i^{\leftarrow}(y) = \exp\left(-\sqrt{y-2\sigma_i^2 T_i l\left(g(y)\right)}\right)$$

$$= \exp\left(-\sqrt{y-2\sigma_i^2 T_i l\left(\sqrt{y}\sqrt{1+f(y)}\right)}\right), \quad y\to\infty.$$

Then

$$\left(V_i^{\leftarrow}\right)'(y) = \exp\left(-\sqrt{y-2\sigma_i^2 T_i l\left(g(y)\right)}\right)\frac{1-2\sigma_i^2 T_i l'\left(g(y)\right)g'(y)}{-2\sqrt{y-2\sigma_i^2 T_i l\left(g(y)\right)}}$$

$$= \frac{\exp\left(-\sqrt{y-2\sigma_i^2 T_i l\left((y(f(y)+1))^{1/2}\right)}\right)}{-2\sqrt{y-2\sigma_i^2 T_i l\left((y(f(y)+1))^{1/2}\right)}}\times$$

$$\times\left(1-\sigma_i^2 T_i \frac{l'\left(g(y)\right)g(y)}{l(g(y))} l(g(y)) \frac{yf'(y)+1+f(y)}{y(f(y)+1)}\right)$$

and

$$\left(V_i^{\leftarrow}\right)''(y) = \frac{\exp\left(-\sqrt{y-2\sigma_i^2 T_i l\left(g(y)\right)}\right)}{4\left(y-2\sigma_i^2 T_i l\left(g(y)\right)\right)^{3/2}}\left(\sqrt{y-2\sigma_i^2 T_i l\left(g(y)\right)}+1\right)\times$$

$$\times\left(1-2\sigma_i^2 T_i l'\left(g(y)\right)g'(y)\right)^2 + \frac{\exp\left(-\sqrt{y-2\sigma_i^2 T_i l\left(g(y)\right)}\right)}{\left(y-2\sigma_i^2 T_i l\left(g(y)\right)\right)^{1/2}}\times$$

$$\times\left(\sigma_i^2 T_i l''\left(g(y)\right)\left(g'(y)\right)^2+\sigma_i^2 T_i l'\left(g(y)\right)g''(y)\right).$$

Since

$$\frac{d}{dx}\left(V_i^{\leftarrow}\right)\left(2\sigma_i^2 T_i \log\left(\frac{1}{A_i-x}\right)\right) = \frac{2\sigma_i^2 T_i}{A_i-x}\left(V_i^{\leftarrow}\right)'\left(2\sigma_i^2 T_i \log\left(\frac{1}{A_i-x}\right)\right)$$

and

$$\frac{d^2}{dx^2}\left(V_i^{\leftarrow}\right)\left(2\sigma_i^2 T_i \log\left(\frac{1}{A_i-x}\right)\right) = \left(\frac{2\sigma_i^2 T_i}{A_i-x}\right)^2\left(V_i^{\leftarrow}\right)''\left(2\sigma_i^2 T_i \log\left(\frac{1}{A_i-x}\right)\right)$$

$$+\frac{2\sigma_i^2 T_i}{(A_i-x)^2}\left(V_i^{\leftarrow}\right)'\left(2\sigma_i^2 T_i \log\left(\frac{1}{A_i-x}\right)\right),$$

we obtain

$$\frac{(A_i-x)\left(V_i^{\leftarrow}\right)''(x)}{\left(V_i^{\leftarrow}\right)'(x)} = 1+\frac{2\sigma_i^2 T_i\left(V_i^{\leftarrow}\right)''\left(2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)\right)}{\left(V_i^{\leftarrow}\right)'\left(2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)\right)}, \quad x\to A_i. \tag{A.2.32}$$

Continuing, we use (A.2.29) to get

$$l'\left(g(y)\right)g'(y)\overset{y\to\infty}{\to}0 \tag{A.2.33}$$

and thus

$$\left(V_i^{\leftarrow}\right)'(y) = \frac{\exp\left(-\sqrt{y-2\sigma_i^2 T_i l\left(\left(y(f(y)+1)\right)^{1/2}\right)}\right)}{-2\sqrt{y-2\sigma_i^2 T_i l\left(\left(y(f(y)+1)\right)^{1/2}\right)}}, \quad y\to\infty. \tag{A.2.34}$$

Using $k_i(x) = 2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)$ we have

$$\left(V_i^{\leftarrow}\right)'(x) = \frac{-\sigma_i^2 T_i}{A_i - x} \frac{\exp\left(-\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)}\right)}{\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)}}, \quad x \to A_i.$$

Focusing on $\frac{\left(V_i^{\leftarrow}\right)''(y)}{\left(V_i^{\leftarrow}\right)'(y)}, y \to \infty$, we continue by using (A.2.34) and (A.2.33) to get

$$\frac{\left(V_i^{\leftarrow}\right)''(y)}{\left(V_i^{\leftarrow}\right)'(y)} = \frac{1}{-2\left(y - 2\sigma_i^2 T_i l\left(g(y)\right)\right)^{1/2}}$$

$$-2\sigma_i^2 T_i \left(l''(g(y))\left[g'(y)\right]^2 + l'(g(y))g''(y)\right)$$

$$-\frac{1}{2\left(y - 2\sigma_i^2 T_i l\left(g(y)\right)\right)}, \quad y \to \infty.$$

As we already know from the first case

$$l''(g(y))\left[g'(y)\right]^2 + l'(g(y))g''(y) \to 0, \quad y \to \infty,$$

as well as

$$\left[l''(g(y))\left[g'(y)\right]^2 + l'(g(y))g''(y)\right]\left(y - 2\sigma_i^2 T_i l\left(g(y)\right)\right)^{1/2} \to 0, \quad y \to \infty$$

and so

$$\frac{\left(V_i^{\leftarrow}\right)''(y)}{\left(V_i^{\leftarrow}\right)'(y)} \asymp \frac{1}{-2\left(y - 2\sigma_i^2 T_i l\left(g(y)\right)\right)^{1/2}}, \quad y \to \infty. \tag{A.2.35}$$

By (A.2.34) we get

$$\left(V_i^{\leftarrow}\right)''(y) = \frac{\exp\left(-\sqrt{y - 2\sigma_i^2 T_i l\left(g(y)\right)}\right)}{4\left(y - 2\sigma_i^2 T_i l\left(g(y)\right)\right)}, \quad y \to \infty,$$

and so using $k_i(A_i - x) = 2\sigma_i^2 T_i \log\left(\frac{1}{A_i - x}\right)$

$$(V_i^{\leftarrow})''(x) = \left(\frac{\sigma_i^2 T_i}{A_i - x}\right)^2 \cdot \frac{\exp\left(-\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)}\right)}{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)}$$

$$- \frac{\sigma_i^2 T_i}{(A_i - x)^2} \cdot \frac{\exp\left(-\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)}\right)}{\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)}}, \quad x \to A_i.$$

Combining (A.2.35) with (A.2.32) we have

$$\frac{(A_i - x)(V_i^{\leftarrow})''(x)}{(V_i^{\leftarrow})'(x)} \asymp 1 - \frac{\sigma_i^2 T_i}{\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)}}, \quad x \to A_i.$$

Hence,

$$\frac{(V_i^{\leftarrow})'(x)}{(A_i - x)(V_i^{\leftarrow})''(x)} \asymp \frac{1}{1 - \frac{\sigma_i^2 T_i}{\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l(g(k_i(A_i-x)))}}}, \quad x \to A_i. \tag{A.2.36}$$

Rewriting the right hand side of (A.2.36):

$$\frac{1}{1 - \frac{\sigma_i^2 T_i}{\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l(g(k_i(A_i-x)))}}} = 1 + \frac{\sigma_i^2 T_i}{\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)} - \sigma_i^2 T_i}$$

we use

$$\frac{\sigma_i^2 T_i}{\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)} - \sigma_i^2 T_i} \asymp \frac{\sigma_i^2 T_i}{\sqrt{k_i(A_i-x) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i-x)\right)\right)}}, \quad x \to A_i$$

to get (A.2.4) as claimed. □

A.2.3 Proof of Lemma 14

Due to the definition of $\mathcal{H}_i^{\pm}(J_i(y))$ we have $J_i^- \le J_i \le J_i^+$. Next, consider 3 cases.

CASE $J_i^-(y) \le \widetilde{J}_i(y) \le J_i^+(y)$:

$$\left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widetilde{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \frac{1}{2}\left|(V_i^{\leftarrow})''(x)\right|\left|J_i(y) - \widetilde{J}_i(y)\right|^2, x \in (J_i(y), \widetilde{J}_i(y))$$

$$\le \frac{1}{2}\left|(V_i^{\leftarrow})''(x)\right|\left|J_i(y) - \widetilde{J}_i(y)\right|^2, x \in (J_i^-(y), J_i^+(y))$$

$$\le \frac{q_i(y)}{2}\left|J_i(y) - \widetilde{J}_i(y)\right|^2.$$

CASE $\widetilde{J}_i(y) < J_i^-(y)$: Due to the definition of $\mathcal{H}_i^-$ and from (A.2.3) we obtain

$$J_i(y) - J_i^-(y) \ge J_i(y)\left[1 - \mathcal{H}_i^-(J_i(y))\right]$$

$$= \frac{J_i(y)}{1+\theta}\left[1 - \frac{\sigma_i^2 T_i}{\sqrt{k_i(J_i(y)) - 2\sigma_i^2 T_i l\left(g\left(k_i(J_i(y))\right)\right)}}\right]$$

$$\asymp \frac{1}{1+\theta}\left|\frac{(V_i^{\leftarrow})'(J_i(y))}{(V_i^{\leftarrow})''(J_i(y))}\right|, \quad J_i(y) \to 0.$$

Due to the continuity of the function $\frac{(V_i^{\leftarrow})'(x)}{(V_i^{\leftarrow})''(x)}$ there is a constant $\check{c} > 0$ satisfying

$$\left|(V_i^{\leftarrow})'(J_i(y))\right| = \left|(V_i^{\leftarrow})''(J_i(y))\right|\left|\frac{(V_i^{\leftarrow})'(J_i(y))}{(V_i^{\leftarrow})''(J_i(y))}\right|$$

$$\le \check{c}(1+\theta)\left|(V_i^{\leftarrow})''(J_i(y))\right|\left(J_i(y) - J_i^-(y)\right).$$

Henceforth, we get

$$\begin{aligned}
&\left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right| \\
&= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^-(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right| \\
&= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^-(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i^-(y) - \widetilde{J}_i(y) + J_i(y) - J_i^-(y)\right)\right| \\
&\leq \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^-(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - J_i^-(y)\right)\right| \\
&+ \left|(V_i^{\leftarrow})'(J_i(y))\right|\left(J_i^-(y) - \widetilde{J}_i(y)\right) \\
&\leq q_i(y)\left(\frac{1}{2} + \check{c}(1+\theta)\right)\left|J_i(y) - \widetilde{J}_i(y)\right|^2,
\end{aligned}$$

using $(J_i^-(y), J_i(y)) \subseteq (J_i^-(y), J_i^+(y))$ and $(J_i^-(y), J_i(y)) \subseteq (\widetilde{J}_i(y), J_i(y))$ in the last inequality.

CASE $\widetilde{J}_i(y) > J_i^+(y)$: Due to the definition of $\mathcal{H}_i^+$ and (A.2.4) we have

$$\begin{aligned}
J_i^+(y) - J_i(y) &= J_i^+(y) - A_i - (J_i(y) - A_i) \\
&\geq (J_i(y) - A_i)\left[\mathcal{H}_i^+(A_i - J_i(y)) - 1\right] \\
&= \frac{A_i - J_i(y)}{1+\theta}\left[1 + \frac{\sigma_i^2 T_i}{\sqrt{k_i(A_i - J_i(y)) - 2\sigma_i^2 T_i l\left(g\left(k_i(A_i - J_i(y))\right)\right)}}\right] \\
&\asymp \frac{1}{1+\theta}\left|\frac{(V_i^{\leftarrow})'(J_i(y))}{(V_i^{\leftarrow})''(J_i(y))}\right|, \quad J_i(y) \to A_i.
\end{aligned}$$

By the continuity of the function $\frac{(V_i^{\leftarrow})'(x)}{(V_i^{\leftarrow})''(x)}$ there is a constant $\hat{c} > 0$ such that

$$\left|(V_i^{\leftarrow})'(J_i(y))\right| = \left|(V_i^{\leftarrow})''(J_i(y))\right| \left|\frac{(V_i^{\leftarrow})'(J_i(y))}{(V_i^{\leftarrow})''(J_i(y))}\right|$$

$$\leq \hat{c}(1+\theta)\left|(V_i^{\leftarrow})''(J_i(y))\right| \left(J_i^{+}(y) - J_i(y)\right).$$

Henceforth, we get

$$\left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(\widehat{J}_i(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^{+}(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - \widetilde{J}_i(y)\right)\right|$$

$$= \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^{+}(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i^{+}(y) - \widetilde{J}_i(y) + J_i(y) - J_i^{+}(y)\right)\right|$$

$$\leq \left|V_i^{\leftarrow}(J_i(y)) - V_i^{\leftarrow}(J_i^{+}(y)) - (V_i^{\leftarrow})'(J_i(y))\left(J_i(y) - J_i^{+}(y)\right)\right|$$

$$+ \left|(V_i^{\leftarrow})'(J_i(y))\right| \left|J_i^{+}(y) - \widetilde{J}_i(y)\right|$$

$$\leq q_i(y)\left(\frac{1}{2} + \hat{c}(1+\theta)\right)\left|J_i(y) - \widetilde{J}_i(y)\right|^2,$$

using $(J_i(y), J_i^{+}(y)) \subseteq (J_i^{-}(y), J_i^{+}(y))$; $(J_i(y), J_i^{+}(y)) \subseteq (J_i(y), \widetilde{J}_i(y))$ and $(J_i^{+}(y), \widetilde{J}_i(y)) \subseteq (J_i(y), \widetilde{J}_i(y))$ in the last inequality. Finally, take $c = \max\{\hat{c}, \check{c}\}$. □

A.2.4 Proof of Lemma 15

Let us prove the case of ψ being slowly varying at zero only because the other statements can be verified in the same manner. The main idea is to use (A.2.15) matching the representation of a slowly/regularly varying function (see e.g. Section 1.3, equation (1.3.1') in Bingham, Goldie, Teugels (1987)).

Define $\varepsilon(x) := \frac{x\psi'(x)}{\psi(x)}$. Thus, ε is a measurable and continuous function on $(0, x_0]$ for some positive x_0 satisfying $\varepsilon(x) \to 0$ as $x \to 0$. Denote

$$\frac{\varepsilon(x)}{x} = \frac{\psi'(x)}{\psi(x)} = [\log(\psi(x))]', \quad x \leq x_0.$$

By integrating over the interval $[x;x_0]$, we get for $x \leq x_0$

$$\int_x^{x_0} \frac{\varepsilon(t)}{t} dt = \log(\psi(x_0)) - \log(\psi(x))$$

and so

$$\log(\psi(x)) = \log(\psi(x_0)) - \int_x^{x_0} \frac{\varepsilon(t)}{t} dt, \quad x \leq x_0.$$

As a result,

$$\psi(x) = \exp\left\{\log(\psi(x_0)) - \int_x^{x_0} \frac{\varepsilon(t)}{t} dt\right\}, \quad x \leq x_0.$$

□

A.2.5 Proof of Lemma 16

Let us prove the first equation only because the other one can be done in the same manner. Let ψ_i be a non-negative, real-valued function defined as

$$\psi_i(x) := x\overline{\mathcal{H}}_i(x).$$

Using (4.3.2) $\psi_i(x)$ is a monotone increasing and varying function with index 1 at $x = 0$. Furthermore, $\psi_i(x)$ is two times continuously differentiable as $x \to 0$ satisfying $\psi_i(0) = 0$. Moreover, with $\frac{x\overline{\mathcal{H}}_i'(x)}{\overline{\mathcal{H}}_i(x)} \to 0, x \to 0$ we get

$$\psi_i'(x) = x\overline{\mathcal{H}}_i'(x) + \overline{\mathcal{H}}_i(x) \asymp \overline{\mathcal{H}}_i(x), \quad x \to 0.$$

In particular, $\psi_i'(x)$ is an asymptotic equivalent of $\overline{\mathcal{H}}_i(x)$ as $x \to 0$ and satisfies

$$\psi_i'(x) \to 0, \quad x \to 0, \tag{A.2.37}$$

as well as

$$\frac{x\psi_i''(x)}{\psi_i'(x)} \to 0, \quad x \to 0 \tag{A.2.38}$$

and

$$\frac{x\psi_i'(x)}{\psi_i(x)} = \frac{x\overline{\mathcal{H}}_i'(x)}{\overline{\mathcal{H}}_i(x)} + 1 \to 1, \quad x \to 0. \tag{A.2.39}$$

Now, let us have a closer look at

$$\int_\varepsilon^{x_0} \left| (V_i^{\leftarrow})^{(j)} (\psi_i(x)) \right| dx, \quad j = 1, 2.$$

CASE $j = 1$:

$$\int_\varepsilon^{x_0} \left| (V_i^{\leftarrow})^{(1)} (\psi_i(x)) \right| dx = \int_\varepsilon^{x_0} \left| (V_i^{\leftarrow})' (\psi_i(x)) \right| \psi_i'(x) \frac{1}{\psi_i'(x)} dx$$

$$= -\int_\varepsilon^{x_0} (V_i^{\leftarrow})' (\psi_i(x)) \psi_i'(x) \frac{1}{\psi_i'(x)} dx$$

$$= \frac{(V_i^{\leftarrow})(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)} - \frac{(V_i^{\leftarrow})(\psi_i(x_0))}{\psi_i'(x_0)}$$

$$- \int_\varepsilon^{x_0} (V_i^{\leftarrow})(\psi_i(x)) \frac{\psi_i''(x)}{(\psi_i'(x))^2} dx.$$

Next,

$$\left| \int_\varepsilon^{x_0} (V_i^{\leftarrow})(\psi_i(x)) \frac{\psi_i''(x)}{(\psi_i'(x))^2} dx \right| \leq (V_i^{\leftarrow})(\psi_i(\varepsilon)) \left(\frac{1}{\psi_i'(\varepsilon)} - \frac{1}{\psi_i'(x_0)} \right)$$

so

$$\left|\int_{\varepsilon}^{x_0}\left|(V_i^{\leftarrow})^{(1)}(\psi_i(x))\right|dx-\frac{(V_i^{\leftarrow})(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}\right|\leq\frac{(V_i^{\leftarrow})(\psi_i(x_0))}{\psi_i'(x_0)}$$

$$+(V_i^{\leftarrow})(\psi_i(\varepsilon))\left(\frac{1}{\psi_i'(\varepsilon)}-\frac{1}{\psi_i'(x_0)}\right)$$

$$\asymp\frac{(V_i^{\leftarrow})(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)},\quad \varepsilon\to 0$$

due to $(V_i^{\leftarrow})(\psi_i(\varepsilon))\to\infty,\varepsilon\to 0$ and using (A.2.37). We obtain

$$\int_{\varepsilon}^{x_0}\left|(V_i^{\leftarrow})^{(1)}(\psi_i(x))\right|dx\lesssim\frac{(V_i^{\leftarrow})(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)},\quad \varepsilon\to 0.$$

Finally, if we assumed that $\tilde{S}_{i,1}(\varepsilon):=\frac{(V_i^{\leftarrow})(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}$ varies slowly at $\varepsilon=0$ then $\tilde{S}_{i,1}\left(\frac{1}{\varepsilon}\right)$ would be slowly varying at infinity. So we could find a slowly varying function at infinity, say $S_{i,1}^{-}$, that satisfies $S_{i,1}^{-}\left(\frac{1}{\varepsilon}\right)=\tilde{S}_{i,1}(\varepsilon),\varepsilon\to 0$, which would complete the proof for the case $j=1$. So let us prove that the function $\tilde{S}_{i,1}(\varepsilon)$ indeed varies slowly at zero. We have

$$\left(\tilde{S}_{i,1}\right)'(\varepsilon)=\frac{(\psi_i'(\varepsilon))^2(V_i^{\leftarrow})'(\psi_i(\varepsilon))-\psi_i''(\varepsilon)(V_i^{\leftarrow})(\psi_i(\varepsilon))}{(\psi_i'(\varepsilon))^2}$$

$$=(V_i^{\leftarrow})'(\psi_i(\varepsilon))-(V_i^{\leftarrow})(\psi_i(\varepsilon))\frac{\psi_i''(\varepsilon)}{(\psi_i'(\varepsilon))^2}$$

and so

$$\frac{\varepsilon\left(\tilde{S}_{i,1}\right)'(\varepsilon)}{\tilde{S}_{i,1}(\varepsilon)}=\varepsilon\psi_i'(\varepsilon)\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{(V_i^{\leftarrow})(\psi_i(\varepsilon))}-\frac{\varepsilon\psi_i''(\varepsilon)}{\psi_i'(\varepsilon)}.$$

Now, due to $\psi_i(\varepsilon)\to 0,\varepsilon\to 0$ and due to the monotony of $\psi_i(\varepsilon)$ we can use the representation of $(V_i^{\leftarrow})(\psi_i(\varepsilon)),(V_i^{\leftarrow})'(\psi_i(\varepsilon))$ for $0\leq\psi_i(\varepsilon)\leq\psi_i(x_0)$ and some $x_0\in(0,A_i)$ satisfying $0\leq\varepsilon\leq x_0$. With Lemma 13 we get

$$\frac{\varepsilon\left(\tilde{S}_{i,1}\right)'(\varepsilon)}{\tilde{S}_{i,1}(\varepsilon)}=-\frac{\varepsilon\psi_i'(\varepsilon)}{\psi_i(\varepsilon)}\frac{\sigma_i^2T_i}{\sqrt{k_i(\psi_i(\varepsilon))-2\sigma_i^2T_il\left(\sqrt{k_i(\psi_i(\varepsilon))+O\left(l\left(\sqrt{k_i(\psi_i(\varepsilon))}\right)\right)}\right)}}$$

$$-\frac{\varepsilon\psi_i''(\varepsilon)}{\psi_i'(\varepsilon)},\quad 0\leq\varepsilon\leq x_0.$$

Combining (A.2.39) and (A.2.38) with

$$\sqrt{k_i(\psi_i(\varepsilon)) - 2\sigma_i^2 T_i l\left(\sqrt{k_i(\psi_i(\varepsilon)) + O\left(l\left(\sqrt{k_i(\psi_i(\varepsilon))}\right)\right)}\right)} \to \infty, \quad \varepsilon \to 0$$

we get

$$\frac{\varepsilon\left(\tilde{S}_{i,1}\right)'(\varepsilon)}{\tilde{S}_{i,1}(\varepsilon)} \to 0, \quad \varepsilon \to 0$$

and we are done using Lemma 15.

CASE $j = 2$:

$$\int_\varepsilon^{x_0} \left|\left(V_i^{\leftarrow}\right)^{(2)}(\psi_i(x))\right| dx = \int_\varepsilon^{x_0} \left(V_i^{\leftarrow}\right)''(\psi_i(x))\psi_i'(x)\frac{1}{\psi_i'(x)}dx$$

$$= \frac{\left(V_i^{\leftarrow}\right)'(\psi_i(x_0))}{\psi_i'(x_0)} - \frac{\left(V_i^{\leftarrow}\right)'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}$$

$$- \int_\varepsilon^{x_0} \left(V_i^{\leftarrow}\right)'(\psi_i(x))\frac{-\psi_i''(x)}{\left(\psi_i'(x)\right)^2}dx.$$

Next, due to $\left|\left(V_i^{\leftarrow}\right)'(\psi_i(\varepsilon))\right| \to \infty, \varepsilon \to 0$ and (A.2.37) we get

$$\left|\int_\varepsilon^{x_0} \left|\left(V_i^{\leftarrow}\right)^{(2)}(\psi_i(x))\right| dx - \frac{\left|\left(V_i^{\leftarrow}\right)'(\psi_i(\varepsilon))\right|}{\psi_i'(\varepsilon)}\right|$$

$$= \left|\frac{\left(V_i^{\leftarrow}\right)'(\psi_i(x_0))}{\psi_i'(x_0)} + \int_\varepsilon^{x_0} \left(V_i^{\leftarrow}\right)'(\psi_i(x))\frac{\psi_i''(x)}{\left(\psi_i'(x)\right)^2}dx\right|$$

$$\leq \frac{\left|\left(V_i^{\leftarrow}\right)'(\psi_i(x_0))\right|}{\psi_i'(x_0)} + \left|\left(V_i^{\leftarrow}\right)'(\psi_i(\varepsilon))\right|\left(\frac{1}{\psi_i'(\varepsilon)} - \frac{1}{\psi_i'(x_0)}\right)$$

$$\asymp \frac{\left|\left(V_i^{\leftarrow}\right)'(\psi_i(\varepsilon))\right|}{\psi_i'(\varepsilon)}, \quad \varepsilon \to 0.$$

Hence,

$$\int_{\varepsilon}^{x_0} \left|(V_i^{\leftarrow})^{(2)}(\psi_i(x))\right| dx \lesssim \frac{\left|(V_i^{\leftarrow})'(\psi_i(\varepsilon))\right|}{\psi_i'(\varepsilon)} = -\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}, \quad \varepsilon \to 0.$$

Finally, let us verify $-\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}$ being a regularly varying function at 0 with index -1. Before proceeding, recall that due to (A.2.3) we have

$$\frac{x(V_i^{\leftarrow})''(x)}{(V_i^{\leftarrow})'(x)} \to -1, \quad x \to 0. \tag{A.2.40}$$

Hence, with (A.2.40) and due to $\psi_i(\varepsilon) \to 0, \varepsilon \to 0$ we get

$$\frac{\varepsilon\left[-\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}\right]'}{-\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}} = \frac{\varepsilon\left[\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}\right]'}{\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}}$$

$$= \varepsilon\psi_i'(\varepsilon)\frac{(V_i^{\leftarrow})''(\psi_i(\varepsilon))}{(V_i^{\leftarrow})'(\psi_i(\varepsilon))} - \frac{\varepsilon\psi_i''(\varepsilon)}{\psi_i'(\varepsilon)}$$

$$\to -\frac{\varepsilon\psi_i'(\varepsilon)}{\psi_i(\varepsilon)} - \frac{\varepsilon\psi_i''(\varepsilon)}{\psi_i'(\varepsilon)}, \quad \varepsilon \to 0.$$

Moreover, combining (A.2.39) and (A.2.38) with Lemma 15 we can see that $-\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}$ is regularly varying at 0 with index -1. Thus, a slowly varying function at zero exists, say $\tilde{S}_{i,2}$, satisfying

$$\frac{1}{\varepsilon}\tilde{S}_{i,2}(\varepsilon) = -\frac{(V_i^{\leftarrow})'(\psi_i(\varepsilon))}{\psi_i'(\varepsilon)}, \quad \varepsilon \to 0.$$

Hence, with the same argumentation as in the former case we can find some slowly varying function at infinity, say $S_{i,2}^{-}$, which satisfies $\tilde{S}_{i,2}(\varepsilon) = S_{i,2}^{-}\left(\frac{1}{\varepsilon}\right), \varepsilon \to 0$, and we are done. □

A.2.6 Proof of Theorem 2

Before proceeding, let us verify that Lemmas 9-11 still hold.

Let δ satisfy $\delta \asymp L_{\gamma,\Delta}^{-1/2}, L_{\gamma,\Delta} \to \infty$. Then, using (A.2.1) we get with $k_i(x) = 2\sigma_i^2 T_i \log\left(\frac{1}{x}\right)$

$$V_i^{\leftarrow}(\delta\overline{\mathcal{H}}_i(\delta)) = \exp\sqrt{k_i(\delta\overline{\mathcal{H}}_i(\delta)) - 2\sigma_i^2 T_i l\left(\sqrt{k_i(\delta\overline{\mathcal{H}}_i(\delta)) + O\left(l\left(\sqrt{k_i(\delta\overline{\mathcal{H}}_i(\delta))}\right)\right)}\right)}$$

$$\asymp \exp\sqrt{2\sigma_i^2 T_i \log\left(\frac{1}{\delta\overline{\mathcal{H}}_i(\delta)}\right)}, \quad \delta \to 0.$$

Next,

$$K_{L,i} = K_{1,i} + (L-1)\Delta \asymp \exp\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}, \quad L_{\gamma,\Delta} \to \infty.$$

Due to

$$\frac{\frac{1}{\sqrt{L_{\gamma,\Delta}}}}{\overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right)} \to 0, \quad L_{\gamma,\Delta} \to \infty$$

we get

$$\frac{1}{\overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right)} < \sqrt{L_{\gamma,\Delta}}, \quad L_{\gamma,\Delta} \to \infty$$

and thus

$$\frac{1}{\delta\overline{\mathcal{H}}_i(\delta)} \asymp \frac{1}{\frac{1}{\sqrt{L_{\gamma,\Delta}}}\overline{\mathcal{H}}_i\left(\frac{1}{\sqrt{L_{\gamma,\Delta}}}\right)} < L_{\gamma,\Delta}, \quad L_{\gamma,\Delta} \to \infty.$$

Hence, using the monotony of the logarithmic and the exponential functions we have

$$K_{L,i} > V_i^{\leftarrow}(\delta\overline{\mathcal{H}}_i(\delta)), \quad L_{\gamma,\Delta} \to \infty$$

and so Lemmas 9 to 11 hold, provided that we use $C_{i,3}, C_{i,4}$ instead of $C_{i,1}, C_{i,2}$ respectively.

Introduce the events

$$\mathcal{D}_{i,L} := \left\{\mathfrak{R}_i(L) \leq c_1 \gamma_L^{-1}\right\}, \quad i = 1, \ldots, n.$$

Due to Lemma 11 $\rho_n \leq \widetilde{\mathcal{R}}_{n,0}$ with

$$\widetilde{\mathcal{R}}_{n,0} = A_n \left(K_{1,n} + \frac{K_{L,n}\mathcal{L}_\Phi}{h} \left[\mathfrak{R}_n(L) + \frac{\Delta \|V_n'\|_\infty}{2} \right] \right)$$

$$+ 2hC_{n,3} \left(S_{n,1}^- \left(\frac{1}{\delta}\right) + S_{n,1}^+ \left(\frac{1}{\delta}\right) \right) + 2(\delta + h)(A_n + K_{L,n})$$

$$\leq \hat{C}_n \left[K_{1,n} + \frac{K_{L,n}}{h} \max\{\gamma_L^{-1}, \Delta\} + h \left(S_{n,1}^- \left(\frac{1}{\delta}\right) + S_{n,1}^+ \left(\frac{1}{\delta}\right) \right) + (\delta + h)(A_n + K_{L,n}) \right]$$

$$\leq \check{C}_n \left[K_{1,n} + \frac{K_{1,n} + \Delta L}{\sqrt{L_{\gamma,\Delta}}} + \frac{1}{\sqrt{L_{\gamma,\Delta}}} \left(S_{n,1}^- \left(\sqrt{L_{\gamma,\Delta}}\right) + S_{n,1}^+ \left(\sqrt{L_{\gamma,\Delta}}\right) \right) + \frac{1}{\sqrt{L_{\gamma,\Delta}}} \right]$$

$$\leq \frac{\tilde{C}_n}{\sqrt{L_{\gamma,\Delta}}} \left[1 + \exp\sqrt{2\sigma_n^2 T_n \log\left(L_{\gamma,\Delta}\right)} + S_{n,1}^- \left(\sqrt{L_{\gamma,\Delta}}\right) + S_{n,1}^+ \left(\sqrt{L_{\gamma,\Delta}}\right) \right]$$

$$=: \frac{\mathcal{S}_{\rho,n}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty$$

on $\mathcal{D}_{n,L}$ for some slowly varying function at infinity $\mathcal{S}_{\rho,n}$ and some constants $\hat{C}_n, \check{C}_n, \tilde{C}_n$, where in the last equation we used the fact that products and sums of slowly varying functions are slowly varying and that the function $1 + \exp\sqrt{2\sigma_n^2 T_n \log\left(L_{\gamma,\Delta}\right)}$ varies slowly at infinity. Let us verify more generally that for any positive constant C and $i \leq n$ the function

$$I_i(L_{\gamma,\Delta}) := 1 + C\exp\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}$$

varies slowly at infinity. To see this, combine Lemma 15 with

$$\frac{L_{\gamma,\Delta} \frac{d}{dL_{\gamma,\Delta}} \left[1 + C\exp\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}\right]}{1 + C\exp\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}} = \frac{C\sigma_i^2 T_i \exp\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}}{\left(1 + C\exp\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}\right)\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}}$$

$$\to 0, \quad L_{\gamma,\Delta} \to \infty.$$

Since $\vartheta_n = 0$, it holds on $\mathcal{D}_{n,L} \cap \mathcal{D}_{n-1,L}$ with some constants $\tilde{C}_{n-1}$ and $\hat{C}_{n-1}$:

$$\rho_{n-1} \leq \widetilde{\mathcal{R}}_{n-1,0} + \mathcal{R}_{n-1,1}\varsigma_n\rho_n + 3\mathcal{R}_{n-1,2}\varsigma_n^2\rho_n^2$$

$$\leq \frac{\tilde{C}_{n-1}}{\sqrt{L_{\gamma,\Delta}}}\left[1+\exp\sqrt{2\sigma_{n-1}^2 T_{n-1}\log\left(L_{\gamma,\Delta}\right)} + S_{n-1,1}^{-}\left(\sqrt{L_{\gamma,\Delta}}\right) + S_{n-1,1}^{+}\left(\sqrt{L_{\gamma,\Delta}}\right)\right]$$

$$+ \frac{\varsigma_n}{\sqrt{L_{\gamma,\Delta}}}\mathcal{S}_{\rho,n}(L_{\gamma,\Delta})V_{n-1}^{\leftarrow}(h+\delta)$$

$$+ \hat{C}_{n-1}\varsigma_n^2\frac{\mathcal{S}_{\rho,n}^2(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}\left(S_{n-1,2}^{-}\left(\sqrt{L_{\gamma,\Delta}}\right) + S_{n-1,2}^{+}\left(\sqrt{L_{\gamma,\Delta}}\right)\right)$$
$$=: \frac{\mathcal{S}_{\rho,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty,$$

where in the last equation we used the same argumentation as in the previous step combined with the fact that $V_{n-1}^{\leftarrow}\left(\frac{2}{\sqrt{L_{\gamma,\Delta}}}\right)$ varies slowly as $L_{\gamma,\Delta} \to \infty$. With some more details, as $L_{\gamma,\Delta} \to \infty$ we have for $i = 1, \ldots, n$

$$V_i^{\leftarrow}(\delta+h) \asymp V_i^{\leftarrow}\left(\frac{2}{\sqrt{L_{\gamma,\Delta}}}\right).$$

Using Lemma 13 it follows that

$$\frac{\frac{2}{\sqrt{L_{\gamma,\Delta}}}V_i^{\leftarrow\prime}\left(\frac{2}{\sqrt{L_{\gamma,\Delta}}}\right)}{V_i^{\leftarrow}\left(\frac{2}{\sqrt{L_{\gamma,\Delta}}}\right)} = -\frac{\sigma_i^2 T_i}{\sqrt{k_i\left(\frac{2}{\sqrt{L_{\gamma,\Delta}}}\right) - 2\sigma_i^2 T_i l\left(\sqrt{k_i\left(\frac{2}{\sqrt{L_{\gamma,\Delta}}}\right)} + O\left(l\left(\sqrt{k_i\left(\frac{2}{\sqrt{L_{\gamma,\Delta}}}\right)}\right)\right)\right)}}$$
$$\to 0, \quad L_{\gamma,\Delta} \to \infty.$$

By Lemma 15 $V_{n-1}^{\leftarrow}\left(\frac{2}{\sqrt{L_{\gamma,\Delta}}}\right)$ varies slowly as $L_{\gamma,\Delta} \to \infty$ (as claimed).

Moreover,

$$\vartheta_{n-1} \leq \varsigma_n\rho_n = \frac{\varsigma_n}{\sqrt{L_{\gamma,\Delta}}}\mathcal{S}_{\rho,n-1}(L_{\gamma,\Delta})$$

$$=: \frac{\mathcal{S}_{\vartheta,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty.$$

On $\mathcal{D}_{n,L} \cap \mathcal{D}_{n-1,L} \cap \mathcal{D}_{n-2,L}$ we have with some constants $\hat{C}_{n-2}, \check{C}_{n-2}$:

$$\rho_{n-2} \leq \widetilde{\mathcal{R}}_{n-2,0} + \mathcal{R}_{n-2,1}\varsigma_{n-1}\rho_{n-1}$$

$$+ \mathcal{R}_{n-2,1}\left(1+\varsigma_{n-1}K_{L,n-1}\right)\vartheta_{n-1} + 3\mathcal{R}_{n-2,2}\left[\vartheta_{n-1}^2 + \varsigma_{n-1}^2\left(\rho_{n-1}^2 + K_{L,n-1}^2\vartheta_{n-1}^2\right)\right]$$

$$\leq \frac{\tilde{C}_{n-2}}{\sqrt{L_{\gamma,\Delta}}}\left[1+\exp\sqrt{2\sigma_{n-2}^2 T_{n-2}\log\left(L_{\gamma,\Delta}\right)} + S_{n-2,1}^{-}\left(\sqrt{L_{\gamma,\Delta}}\right) + S_{n-2,1}^{+}\left(\sqrt{L_{\gamma,\Delta}}\right)\right]$$

$$+ \frac{\varsigma_{n-1}}{\sqrt{L_{\gamma,\Delta}}}\mathcal{S}_{\rho,n-1}(L_{\gamma,\Delta})V_{n-2}^{\leftarrow}(h+\delta)$$

$$+ V_{n-2}^{\leftarrow}(h+\delta)\left(1+\varsigma_{n-1}\exp\sqrt{2\sigma_{n-1}^2 T_{n-1}\log\left(L_{\gamma,\Delta}\right)}\right)\frac{\mathcal{S}_{\vartheta,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}$$

$$+ \frac{\hat{C}_{n-2}\left(S_{n-2,2}^{-}\left(\sqrt{L_{\gamma,\Delta}}\right) + S_{n-2,2}^{+}\left(\sqrt{L_{\gamma,\Delta}}\right)\right)}{\sqrt{L_{\gamma,\Delta}}}\mathcal{S}_{\vartheta,n-1}^2(L_{\gamma,\Delta})$$

$$+ \frac{\hat{C}_{n-2}\left(S_{n-2,2}^{-}\left(\sqrt{L_{\gamma,\Delta}}\right) + S_{n-2,2}^{+}\left(\sqrt{L_{\gamma,\Delta}}\right)\right)}{\sqrt{L_{\gamma,\Delta}}}\varsigma_{n-1}^2\mathcal{S}_{\rho,n-1}^2(L_{\gamma,\Delta})$$

$$+ \frac{\hat{C}_{n-2}\left(S_{n-2,2}^{-}\left(\sqrt{L_{\gamma,\Delta}}\right) + S_{n-2,2}^{+}\left(\sqrt{L_{\gamma,\Delta}}\right)\right)}{\sqrt{L_{\gamma,\Delta}}}\mathcal{S}_{\vartheta,n-1}^2(L_{\gamma,\Delta})\varsigma_{n-1}^2 \times$$

$$\times \exp\left(2\sqrt{2\sigma_{n-1}^2 T_{n-1}\log(L_{\gamma,\Delta})}\right)$$

$$=: \frac{\mathcal{S}_{\rho,n-2}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty,$$

where in the last equation we used the same argumentation as in the previous step combined with the fact that for $i \leq n$ the function $\exp\left(2\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}\right)$ varies slowly as $L_{\gamma,\Delta} \to \infty$.

With some more details, we have

$$\frac{L_{\gamma,\Delta}\left[\exp\left(2\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}\right)\right]'}{\exp\left(2\sqrt{2\sigma_i^2 T_i \log\left(L_{\gamma,\Delta}\right)}\right)} = \frac{\sqrt{2}\sigma_i^2 T_i}{\sqrt{\sigma_i^2 T_i \log(L_{\gamma,\Delta})}} \to 0, \quad L_{\gamma,\Delta} \to \infty.$$

As a result, using Lemma 15, the function in focus turns out to be slowly varying as claimed. Furthermore,

$$\begin{aligned}
\vartheta_{n-2} &\leq \left(1+\varsigma_{n-1}K_{L,n-1}\right)\vartheta_{n-1} + \varsigma_{n-1}\rho_{n-1} \\
&\leq \left(1+\varsigma_{n-1}\exp\sqrt{2\sigma_{n-1}^2 T_{n-1}\log\left(L_{\gamma,\Delta}\right)}\right)\frac{\mathcal{S}_{\vartheta,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}} + \frac{\varsigma_{n-1}\mathcal{S}_{\rho,n-1}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}} \\
&=: \frac{\mathcal{S}_{\vartheta,n-2}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty.
\end{aligned}$$

Continuing in the same way we derive after the $n-(i-1)$th step on $\bigcap_{j=i}^{n}\mathcal{D}_{j,L}$ for some slowly varying functions $\mathcal{S}_{\rho,i}$ and $\mathcal{S}_{\vartheta,i}$:

$$\rho_i \leq \frac{\mathcal{S}_{\rho,i}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad \vartheta_i \leq \frac{\mathcal{S}_{\vartheta,i}(L_{\gamma,\Delta})}{\sqrt{L_{\gamma,\Delta}}}, \quad L_{\gamma,\Delta} \to \infty$$

with

$$\begin{aligned}
\mathcal{S}_{\vartheta,i}(L_{\gamma,\Delta}) = &\left(1+\varsigma_{i+1}\exp\sqrt{2\sigma_{i+1}^2 T_{i+1}\log\left(L_{\gamma,\Delta}\right)}\right)\mathcal{S}_{\vartheta,i+1}(L_{\gamma,\Delta}) \\
&+\varsigma_{i+1}\mathcal{S}_{\rho,i+1}(L_{\gamma,\Delta}), \quad L_{\gamma,\Delta} \to \infty.
\end{aligned}$$

Set $L_k := 2^k$ and introduce the events

$$\mathcal{A}_{i,k} := \left\{\max_{L_{k-1}<L\leq L_k}\gamma_L\mathfrak{R}_i(L) > c_1\right\}, \quad k \in \mathbb{N}.$$

For any $L>0$ there is a natural number k with

$$\overline{\mathcal{D}}_{j,L} \subseteq \mathcal{A}_{j,k}$$

yielding

$$\overline{\bigcap_{j=i}^{n} \mathcal{D}_{j,L}} \subseteq \bigcup_{j=i}^{n} \mathcal{A}_{j,k}.$$

Furthermore,

$$\mathbb{P}\left(\bigcup_{j=i}^{n} \mathcal{A}_{j,k}\right) \leq \sum_{j=1}^{n} \mathbb{P}\left(\mathcal{A}_{j,k}\right) \leq nc_2 L_k^{-r}.$$

As a result, using the limit comparison test we get

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\bigcup_{j=i}^{n} \mathcal{A}_{j,k}\right) < \infty$$

and

$$\mathbb{P}\left(\limsup_{L} \overline{\bigcap_{j=i}^{n} \mathcal{D}_{j,L}}\right) = \mathbb{P}\left(\limsup_{k} \bigcup_{j=i}^{n} \mathcal{A}_{j,k}\right) = 0.$$

□

Bibliography

Bingham, N. H. / Goldie, C. M. / Teugels, J. L. (1987). Regular variation: Encyclopedia of mathematics and its applications (27., edited by Rota, G. C.). Cambridge: University Press.

Chen, X. / Reiss, M. (2011). On rate optimality for ill-posed inverse problems in econometrics. Econometric Theory, 2011 (Vol. 27, Issue 3), 497–521.

Cont, R. / Tankov, P. (2004). Financial Modelling with Jump Processes. Financial Mathematics Series. New York: Chapman & Hall / CRC.

Cont, R. / Tankov, P. (2006). Retrieving Lévy Processes from Option Prices: Regularization of an Ill-posed Inverse Problem. SIAM Journal on Control and Optimization, 2006 (Vol. 45, No. 1) , 1–25.

Eggermont, P. P. B. / LaRiccia, V. N. (2009). Maximum Penalized Likelihood Estimation. Springer Series in Statistics (Volume II: Regression). New York: Springer.

Forster, O. (1983). Analysis 1: Differential– und Integralrechnung einer Veränderlichen (4., durchgesehene Auflage). Braunschweig: Friedr. Vieweg & Sohn.

Giné, E. / Guillou, A. (2001). On consistency of kernel density estimators for randomly censored data: rates holding uniformly over adaptive intervals. Ann. I. H. Poincaré, 2001 (Vol. 37, Issue 4), 503–522.

Hoffmann, M. / Reiss, M. (2008). Nonlinear estimation for linear inverse problems with error in the operator. Annals of Statistics, 2008 (Vol. 36, No. 1), 310–336.

Hunt, P. J. / Kennedy, J. E. (2000). Financial Derivatives in Theory and Practice. Chichester: John Wiley & Sons Ltd.

Hunt, P. / Kennedy, J. / Pelsser, A. (2000). Markov-Functional Interest Rate Models. Finance and Stochastics, 2000 (4), 391–408.

Montgomery-Smith, S. J. (1993). Comparison of Sums of independent Identically Distributed Random Variables. Prob. and Math. Stat., 1993 (14), 281–285.

Tsybakov, A. B. (2009). Introduction to Nonparametric Estimation. Springer Series in Statistics. New York: Springer Science+Business Media.

Wasserman, L. (2006). All of Nonparametric Statistics. Springer Texts in Statistics. New York: Springer Science+Business Media.