Pricing Energy, Weather and Emission Derivatives under Future Information

Von der Fakultät für Mathematik der Universität Duisburg-Essen
zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigte

D I S S E R T A T I O N

vorgelegt von

MARKUS HESS

geb. in Trier

Referent: Prof. Dr. Rüdiger Kiesel, Universität Duisburg-Essen
Korreferenten: Prof. Dr. Denis Belomestny, Universität Duisburg-Essen
Prof. Dr. Thorsten Rheinländer, London School of Economics

Tag der mündlichen Prüfung: 14. Dezember 2012

Dedicated to my grandfather A. H.

who taught me to hold on also in difficult periods

and not to become discouraged.

Markus Hess, D-54341 Fell, **2012**

E-Mail: Markus-Hess@gmx.net

Doctoral Thesis

Pricing Energy, Weather and Emission Derivatives under Future Information

Markus Hess

Summary

The aim of this thesis mainly consists in the computation of risk-neutral option prices for energy, weather, emission and commodity derivatives, whereas we innovatively take future information – which we assume to be available to well-informed market insiders – into account via several customized enlargements of the underlying information filtrations. In this regard, we inter alia derive European as well as exotic option price formulas for electricity derivatives such as traded at the European Energy Exchange EEX, for example, but yet under the incorporation of forward-looking information about possible future electricity spot price behavior. Furthermore, we provide both utility-maximizing anticipating portfolio selection procedures and optimal liquidation strategies for electricity futures portfolios yielding minimal expected trading costs under forward-looking price impact considerations. Moreover, we correlate electricity spot prices with outdoor temperature and treat a related electricity derivatives pricing problem even under additional temperature forecasts. In this insider trading context, we also derive explicit expressions for different types of temperature futures indices such as usually traded at the Chicago Mercantile Exchange CME, for instance, and provide various pricing formulas for options written on the latter. Additionally, we construct optimal positions in a temperature futures portfolio under forecasted weather information in order to hedge against both temporal and spatial temperature risk adequately. Further on, we treat the pricing of carbon emission allowances, such as commonly traded in the European Union Emission Trading Scheme EU ETS, but under supplementary insider information on the future market zone net position. In this context, we propose two improved arithmetic multi-state approaches to model the 'length of the market net position' more realistically than in existing models. By the way, throughout this work we frequently discuss customized martingale compensators under enlarged filtrations and related information premia associated to our specific insider trading frameworks. Finally, we invent nonlinear double-jump stochastic filtering techniques for generalized Lévy-type processes in order to (theoretically) calibrate the emerging incomplete market models.

Zusammenfassung

Der Fokus dieser Dissertation liegt auf der Bereitstellung von risikoneutralen Optionspreis-Formeln für Energie-, Wetter- und Emissions-Derivate insbesondere unter Berücksichtigung von verfügbaren Prognosen beispielsweise bezüglich des zukünftigen Elektrizitätspreis-Levels oder der zu erwartenden Außentemperatur. Hierbei wird die jeweils zusätzlich verfügbare Information mathematisch durch maßgeschneiderte Vergrößerungen der zugrunde liegenden Filtrationen modelliert. In diesem Zusammenhang leiten wir diverse Preis-Formeln sowohl für europäische als auch für exotische Optionen auf Elektrizitätsderivate her. Ferner untersuchen wir die Auswahl nutzenmaximierender Elektrizitätsmarkt-Portfolios und optimaler Liquidierungsstrategien in Futures-Märkten unter Preis-Impact Zukunfts-Informationen. Des Weiteren korrelieren wir den Elektrizitäts-Spotpreis mit der Außentemperatur und behandeln ein hiermit eng verbundenes Preis-Kalkulationsproblem unter Temperatur-Vorhersagen. In diesem Insider-Handelsansatz geben wir anschließend explizite Ausdrücke für verschiedene Temperatur-Indizes nebst Preisformeln für hierauf abgeschlossene Optionen an. Ferner konstruieren wir optimale Hedging-Positionen in einem Temperatur-Indizes-Portfolio unter Berücksichtigung von Wettervorhersagen, womit sich beispielsweise Energieversorger sowohl gegen zeitliches als auch räumliches Temperaturrisiko absichern können. Darüber hinaus untersuchen wir die Preismodellierung im CO2-Emissionsmarkt *European Union Emission Trading Scheme* (EU ETS) erstmals unter Berücksichtigung von Insider-Informationen über die zukünftig anzunehmende Netzposition einer Marktzone. Parallel diskutieren wir die konkreten Formen diverser Martingal-Kompensatoren unter den auftauchenden vergrößerten Filtrationen und die hiermit eng verbundenen Informations-Prämien in den zugrunde liegenden Märkten. Abschließend entwickeln wir nicht-lineare stochastische Filter-Techniken für generalisierte Lévy-Typ Prozesse, welche letztlich zur (theoretischen) Kalibrierung der behandelten unvollständigen Modelle benutzt werden.

Abstract

In the **first chapter** we give a short introduction to energy, weather, emission and commodity markets. Simultaneously, we prepare and motivate our upcoming derivative pricing issues under forward-looking information in these markets which actually constitute the main topics of this thesis.

Next, **Chapter 2** designates our mathematical toolbox providing some helpful results on Lévy processes which embody one of our main modeling tools throughout the present work. In particular, we introduce appropriate martingale compensators under enlarged information filtrations therein.

It is a well-known fact that electricity markets exhibit several striking key characteristics such as a seasonal spiky price behavior due to the non-storability of the underlying *flow commodity* along with a strong mean-reversion to a periodic trend-line showing slow stochastic variation, a lack of arbitrage opportunities, extremely high price volatilities, heavy-tailed empirical return distributions, incompleteness and a nearly monopolistic structure with only a few *big players* as market participants acting on separated regional markets. In accordance to the just enumerated electricity market features, the aim of the **third chapter** consists in the computation of risk-neutral option prices for both plain-vanilla and exotic electricity derivatives on the basis of several multi-factor Ornstein-Uhlenbeck setups, whereas we newly take forward-looking information – which we assume to be available to well-informed traders – into account via numerous tailor-made enlargements of the underlying information-filtrations. In this context, we also correlate the electricity spot price with outdoor temperature and treat a related pricing problem under supplementary temperature forecasts. Our arithmetic approaches neither trouble an exponential function (to ensure positivity of the prices) nor are there Brownian motion terms involved in the appearing pure-jump electricity spot price models. Yet, we derive *information premia* associated to electricity futures contracts explicitly including a delivery period whereas we also examine utility-maximizing portfolio selection in electricity markets.

Since electrical-energy markets usually are dominated by a few *big players* only whose individual trading activities may shift prices essentially, the question of how to optimally liquidate an existing electricity futures portfolio over a fixed time horizon under the constraint of minimizing unfavorable market impact effects is of striking relevance for portfolio managers trading at e.g. the European Energy Exchange (EEX). Thus, in the **forth chapter** we invent a tractable price impact model for electricity futures, whereas we derive optimal liquidation strategies with respect to different target functions such as conditioned expected trading costs, for example. Moreover, we newly take supplementary anticipating information about future electricity swap price behavior into account via a rigorous exploitation of enlargement-of-filtration methods. Finally, we derive optimal liquidation strategies under this insider trading machinery as well.

In **Chapter 5** we deduce risk-neutral option prices for temperature derivatives on the basis of a mean-reverting Ornstein-Uhlenbeck temperature model admitting seasonality both in the mean-level and volatility, whereas multiple pure-jump Lévy-type processes as driving noises allow for seasonal dependent jump-amplitudes and frequencies. Moreover, we take relevant forecasts about future weather conditions into account via an adequate enlargement of the underlying information filtration. In this insider trading context, we exemplarily derive expressions for temperature indices like cumulative average temperature (CAT) futures and cooling degree day (CDD) futures whereas we provide a forward-looking pricing formula for a European call option written on the former. In addition, we propose a jump-diffusion temperature model (including both Brownian motion and pure-jump terms as driving noises) and hereafter price a CAT option related to this *mixed* approach even under temperature forecasts. Ultimately, we construct optimal positions in a temperature futures portfolio under forecasted weather information to hedge against both *temporal* and *spatial* temperature risk.

In the **sixth chapter** we derive risk-neutral prices for carbon emission allowances (EUAs) as commonly traded in the European Union Emission Trading Scheme (EU ETS), whereas we newly take forward-looking information about the *market zone net position* into account via a rigorous exploitation of enlargement-of-filtration methods. In this insider trading framework, we model the market zone net position as a linear combination of multiple real-valued compound Poisson processes, which – in contrast to e.g. a two-state Markov chain – yet may indicate how *long* or *short* the overall position of the EU ETS market precisely is. Consequently, we need to apply customized multi-dimensional Fourier transform techniques when it comes to related pricing purposes of EUA contracts. Moreover, we discuss the concept of minimum relative entropy in order to find a concrete equivalent martingale measure in our incomplete modeling approach. Eventually, we propose a continuous (Brownian motion driven) market zone net position model of Ornstein-Uhlenbeck type and derive EUA prices also for this mean-reverting setup.

The aim of **Chapter 7** consists in the computation of risk-neutral option prices for commodity derivatives on the basis of an extended Heath-Jarrow-Morton (HJM) approach, whereas the presence of random jumps in the underlying forward rate model requires the use of Fourier transform techniques. By the way, we derive an extended HJM drift restriction connected to our jump-diffusion setup, whereas the concepts of Esscher transforms and minimum relative entropy are adjusted to our purposes in order to determine a concrete equivalent martingale measure out of the large class of offering pricing probabilities in the present incomplete commodity market model. In this context, we particularly adapt a *successive Lagrange-approach* to the requirements of the underlying relative entropy minimization procedure.

Finally, **Chapter 8** is dedicated to the topic of nonlinear stochastic filtering which deals with the problem of estimating a dynamical system out of perturbed or incomplete observations, whereas a direct measurement of the underlying intriguing signal is only partially or even not possible. Innovatively, we model the signal variable as well as the observation process via fairly general jump-diffusion Lévy-type stochastic processes simultaneously. Afterwards, we derive extended Zakai- and Kushner-Stratonovic-Equations, the latter representing our optimal filter in the least-squares sense. Ultimately, several selected applications taken from financial, electricity and emission markets are presented.

Acknowledgements

First of all, I would like to express my deepest thanks to my supervisor Prof. Rüdiger Kiesel for giving me the possibility to finish my research as an external doctoral student at his Chair for Energy Trading and Financial Services at the University Duisburg-Essen. I am very grateful for his help and advice while writing this thesis. In addition, I would like to thank Prof. Denis Belomestny and Prof. Thorsten Rheinländer for acting as referents. Likewise, many thanks are due to Prof. Kiesel's research group for interesting seminars, discussions and talks.

Moreover, in no particular order, I would like to thank Prof. Fred Espen Benth and Prof. Bernt Øksendal (Oslo), Prof. Jörn Sass (Kaiserslautern), Prof. Alexander Schied (Mannheim), Prof. Jean Jacod (Paris) and Prof. Ernst Eberlein (Freiburg) for interesting and valuable e-mail conversation. Particularly, I am very grateful to Prof. Harald Luschgy (Trier) for introducing me to the topic of stochastic processes and mathematical statistics along with related financial applications.

Last but not least, I would like to thank my parents Anne and Bernd; I am aware of what they have done for the education of their children; my sisters, Eva and Ina, and my whole family for their love and support during my entire life. Especially, it has been very nice to be able to work in 'my calm office'. Very special thanks go to my girl-friend Jenny for her love, patience and numerous beautiful moments; and for that she always has supported and encouraged me – also in difficult periods. I would also like to thank Rita and Stefan Eberhard for their support during the last years. Finally, I am especially grateful to all my friends from different places and times for sharing uncountable many nice and funny moments.

Contents

Chapter 1

On Energy, Weather, Emission and Commodity Markets

1.1 Introduction to electricity markets

The creation of competitive power markets such as the European Energy Exchange (EEX) [38] or the Scandinavian Power Exchange *Nord Pool* [73], where e.g. electrical energy is traded as a commodity, has brought up new mathematical challenges concerning the risk-neutral pricing of available power derivatives. Starting off, we shortly enumerate a selection of the most important key characteristics of electricity markets – not at least, to get an idea about the most striking differences in contrast to ordinary financial stock markets.

Briefly summing up the most relevant findings in [3], [7], [8], [10], [13], [14], [23], [37], [48], [59], [66], [68] and [72], we announce right at the beginning that electrical energy exchanges/prices exhibit a seasonal spiky price behavior due to the non-storability of the underlying *flow commodity* along with a strong mean-reversion to a periodic trend-line showing slow stochastic variation itself, a lack of arbitrage opportunities, extremely high price volatilities, heavy-tailed empirical return distributions, incompleteness and a nearly monopolistic structure with only a few *big players* as market participants acting on separated regional markets. Moreover, one suspects electricity prices to be strongly correlated with outdoor temperature and other commodity prices such as of gas, oil or coal, for instance [13]. In what follows, we want to provide more detailed explanations concerning our recent announcements, whereas we particularly motivate a necessary incorporation of forward-looking information into mathematical (option) pricing issues in electricity (and related weather and emission) markets, which constitutes the main topic of this thesis. In this regard, we now present a selection of the most convincing arguments that strongly count in favor for the incorporation of future information in electricity markets and partly have been given by Benth and Meyer-Brandis [10], originally.

First of all, electricity depicts a commodity which is non-storable or has at least very limited storage possibilities – except from indirect ones like in water reservoirs [10], [13]. This lack of storability causes a collapse of conventional *cash-and-carry* or *buy-and-hold strategies* and is responsible for calling electricity a *flow commodity* [10], [13]. Since consumers cannot buy for storage, there is no reason why *today's* spot prices (and the corresponding *backward-looking* sigma-algebra which solely is generated by the spot price noises up to the present) should reflect public knowledge about *future* events like e.g. the introduction of carbon-dioxide-emission costs next year or simply noise-afflicted weekly weather forecasts [10]. Obviously, the *present* electricity spot price is unaffected by *future* market information, whereas it is a result of *today's* supply and demand situation only [10]. Hence, forward-looking information about *future* market conditions evidently is *not* incorporated in *today's* prices what makes the usual assumption *"The available market information only affects <u>today's</u> price behavior.*" no longer acceptable in markets for non-storable commodities [10].
Catching up another plausible example in [10], let us exemplarily study the outage of a major power plant during the next month which indisputably constitutes some worthy additional information that is available at least to well-informed market participants, so-called *market insiders*. In this case, the energy supply side will be reduced significantly so that one should expect an increase of the *future* (but not of the *present*) price level, since traders/consumers cannot buy for storage and thus, *today's* prices should *not* be affected [10]. As a consequence, the calculation mechanisms for power forward prices or, in particular, for related option prices written on electricity forwards/futures, should adequately take this additional knowledge into account which, mathematically spoken, may culminate in an enlargement of the underlying information filtration (such as proposed in [10]). Hence, throughout this thesis we will be confronted within a rigorous discussion concerning the pricing of electricity futures contracts based upon *enlarged* information filtrations – not at least to avoid "*information miss-specification*" (see pp. 3 and 6 in [10]) in our underlying electricity market models.
Moreover, the previously mentioned non-storability of electrical energy is responsible for a division of electricity markets into several regional trading territories excluding any arbitrage opportunities [13]. In contrast to energy derivatives associated to other commodities such as gas, coal or oil, *electricity futures/swap*[1] contracts posses the distinctive feature of yielding a delivery during a future time *span*, the so-called *delivery period*, rather than at a fixed maturity date such as known from forward contracts, respectively from forward rate theory (see Chapter 7 in the context of commodity forwards pricing); therefore, the basic products in electricity markets are options written on electricity spot or futures prices whose delivery is settled over a future *period* of time [13].

Due to inelastic demand, electricity prices show very impressive spikes – in contrast to common price histories observed at classical financial stock markets [8], [13]. As an example, altering weather conditions such as a significant dropping of temperature (in connection with the non-storability) often lead to a sudden increase of demand what results in strong upward jumps of electricity prices [8], [13], [14], [37], [48], [72]. However, after those sudden changes one most likely can observe a scenario wherein spot prices tend rather rapidly back to their mean-level [8]. More precise, the violent upward jumps are usually followed by a quick return to about the former level so that electricity prices yet are detected to be mean-reverting to a periodic trend-line which merely exhibits slow stochastic variations itself [8], [48], [72]. Therefore, mean-reverting stochastic processes of Ornstein-Uhlenbeck type present themselves to be the first choice in order to have realistic models for electricity price formation. Finally, let us emphasize the seasonal behavior of all involved magnitudes such as the jump sizes, the jump intensities and the mean-level as well in yearly, weekly and daily circles [8], [72].

[1] In accordance to the notation in [13], and particularly to the overall market jargon, throughout this work we associate *futures*, respectively *swap* contracts to admit a delivery *period*, whereas we understand *forward* contracts to deliver/mature at a fixed maturity *date*, on the contrary.

Likewise, Eberlein and Stahl [37] attest substantial price changes in energy markets even during a few days which happen with higher frequencies and greater magnitudes in contrast to time series observed at classical financial markets: Typically, energy price volatilities exceed the usually observed levels in ordinary stock markets by several orders due to sudden imbalances in supply and demand. Observing these "*low-probability large-amplitude*" spikes which clearly cannot arise in a Brownian motion (BM) framework (see page 8 in [72]), also Meyer-Brandis and Tankov [72] underline the outstanding non-Gaussian character along with a heavy-tailed empirical return distribution of electricity prices. From all this it is clear that we will need rather sophisticated mathematical models admitting heavy-tailed return distributions, violent price spikes, seasonality and mean-reversion in order to model electricity price behavior in an adequate manner.

Note that the just mentioned features obviously are neither met by geometric Brownian motion approaches, nor by simple Brownian Bachelier models, for example. Anyway, in [8] Benth, Kallsen and Meyer-Brandis present an arithmetic mean-reverting multi-factor pure-jump model with seasonality which seems to be extremely suitable to derive an adequate description of electricity price behavior (compare Fig.1 and Fig.2 in [8] in this context). For this reason, we catch up their arithmetic pure-jump onset in this thesis, whereas we actually use a slightly extended version of the latter as our main electricity spot price modeling tool in Chapter 3. All in all, the main objective of the third chapter has been to combine the two excellent articles [8] and [10], while examining diverse derivation methods for electricity *option* price formulas under additionally available forward-looking information (modeled by enlarged filtrations) in depth – a topic which has (to the best of our knowledge) not been treated in the literature in a comparable way.

Ultimately, we claim that the electrical energy industry worldwide possesses a rather monopolistic structure [37], whereas almost all electricity markets are dominated by a few *big players* merely whose individual trading activities may shift prices essentially. Hence, a precise mathematical modeling of the feedback/influence that individual trading activities have on the underlying electricity prices (so-called *price impact effects*) should be of a large interest particularly for portfolio managers trading at the EEX, for example. In Chapter 4 we thus present market impact considerations newly dedicated to electricity futures prices under forward-looking information for the first time in the literature (at least to the best of our knowledge).

1.2 Introduction to weather markets

During the last decades competitive weather markets like the Chicago Mercantile Exchange (CME) [28], wherein options on weather indices are traded, have been created all over the world. In such market places, numerous indices associated to different non-tradable underlyings like e.g. outdoor temperature, rainfall, snowfall, sunshine, wind or even the number of frost-days etc. are traded somewhat similar to financial products in ordinary stock markets [13].
However, there exist close connections between energy and weather markets, since outdoor temperature and the prices for electricity, gas, oil, coal etc. are strongly correlated (for further reading on this topic see [10], [13], [29], [50]). Especially in geographical regions in which there is a need for heating in the winter and air-conditioning in the summer season, electricity spot prices often turn out to be extremely sensitive to forecasts of unusually warm or cool weather [29]. Therefore, both energy producers as well as consumers should be interested in financial contracts that can be utilized to manage weather risk adequately [13]. In what follows, we search for obvious parallels in between temperature and electricity markets: Comparing energy prices for non-storable commodities like electricity on the one hand and outdoor temperature dynamics on the other, we primarily note clear

evidences of mean-reversion to a seasonally varying trend-line in both cases [11] – [13]. Since neither temperature nor electricity is storable, both corresponding markets turn out to be *incomplete* in the sense that hedging/replicating by using the particular "asset" itself as securing underlying is impossible [13]. (Recall the *second fundamental theorem of asset pricing* in this context.) In addition, temperature futures contracts are usually written on some temperature index measured over a time *span*, the so-called *measurement period*, which may be associated with the delivery period steadily appearing in electricity swap contracts [13]. In other words, temperature futures "*deliver*" the underlying "*asset*" over an entire period rather than at a fixed maturity date such as commonly known from forward contracts (see section 1.3 in [13]). In conclusion, weather options lend themselves to hedge against unfortunate weather risk especially in electricity markets.

Further, a common *backward-looking* information filtration approach (modeling the information flow in a weather market) does not at all reflect public knowledge about *future* weather conditions or, in particular, temperature *forecasts* [10]. Therefore, a pricing onset for weather contracts based upon a conditional expectation, given *past* and *current* information merely, actually sounds rather unrealistic – especially since omnipresent weather forecasts are completely neglected [10]. Certainly, weather forecasts do not provide *exact* but at least some useful additional (stochastic) information reducing the uncertainty about future weather behavior [10]. However, when it comes to pricing purposes of temperature derivatives, one clearly should take *all* available knowledge into account adhering to some kind of *information yield concept* [10]. Mathematically spoken, the just mentioned idea requires a rigorous enlargement of the underlying information filtration (also recall [10], [15] and [50] along with our former arguing in section 1.1 in this context). For this reason, in the corresponding paragraph 5.3 we will be confronted within a persistent discussion concerning the exploitation of insider trading principles also for the pricing of temperature contracts yet under enlarged information filtrations.

Actually, the new *asset classes* that are made up by temperature derivatives can be used to hedge against unfortunate weather conditions also in financial energy risk management. We now discuss this feature in more detail by following (partly) the argumentation given in [29]: Firstly, note that a business with weather exposure (such as e.g. gas, electricity or heating-oil retailers) may choose to buy or sell a futures contract written on outdoor temperature to secure itself against low sales figures induced by unfortunate weather conditions. More precisely, if a heating-oil retailer feels that the upcoming winter is going to be very cold, the merchant simultaneously expects high revenues and thus might sell a *heating degree day* (HDD) *call* which pays its holder a certain amount of money whenever the daily average temperature lies under a predetermined threshold [29] – see (5.1.3) below for a precise mathematical definition of an *accumulated HDD temperature index*. Consequently, if the winter is actually *not* going to be particularly cold afterwards, the heating oil retailer would keep the premium on the call at least [29]. Vice versa, if the winter is going to be very cold indeed, the retailer could easily rebalance the call option payoff with his extraordinary high revenues descending from higher-than-normal heating-oil sales [29]. In conclusion, our fictive heating-oil retailer has completely secured himself/herself against unfortunate weather conditions. Additionally, there exist many other branches in which weather conditions have a major impact on the gained revenues: For example, agricultural commodities like wine or corn simultaneously may be hedged with so-called *cross-commodity weather contracts* which simply couple weather risk to agricultural price risk [29].

1.3 Introduction to carbon emission markets

In the context of global warming, the issue of reducing carbon dioxide (CO2) emissions during the next decades can be found on almost every political agenda nowadays [47]. In order to comply with the Kyoto protocol the European Union Emission Trading Scheme (EU ETS) has been constructed which constitutes the largest emission trading market world-wide [25]. The latter organizes trade in emission permits which hence become a traded commodity [25]. More precisely, political regulators have determined target emission levels for all participating firms, whereas a penalty is levied for each unit of pollutant emitted outside the fixed limits during a given compliance period [47]. In this regard, installations may either reduce their actual carbon emissions or buy additional credits [25]. However, the payment of a penalty does not release the company from the obligation to surrender the precise number of allowances equal to its former emissions [25]. For a more rigorous reading on carbon emission trading schemes and the EU ETS framework, the interested reader is advised to paragraph 1.4 in [13], Chapter 1 in [25], references [27] and [39], or Chapter 1 in [47].

Anyway, in the present thesis we discuss risk-neutral pricing issues for CO2 emission allowances even under supplementary *forward-looking* information on the market zone net position. Parallel to [25], our modeling is done under the assumption of *no banking* of carbon allowances, whereas we newly discuss the effects that insider-information concerning the future market zone net position has on emission allowance (EUA) prices. As mentioned on p.2 in [25], we recall that "*the impact of the release of sensitive information regarding the ETS net position [on emission prices] can be dramatic*": In April/May 2006 prices extremely dropped during a few days after it has become common knowledge that the suggested emission levels "*had been too generous to have a significant impact on [the] emission practice*".[2] In conclusion, it sounds reasonable to consider intermediate announcements concerning the actually verified market zone net position as responsible for periodical strong jumps in emission prices [25]. In particular, a similar causality should be valid for forward-looking information about the (most likely) future market zone net position – at least from an insider's point of view. Thus, in section 6.5 we innovatively treat the pricing of carbon emission allowances under enlarged filtrations, whereas in the forthcoming Remark 6.2.4 we precisely explain why our improved multi-state EU ETS model reasonably incorporates (actually *time-delayed*) jumps in the underlying EUA prices.

1.4 Introduction to commodity markets

In competitive commodity exchanges, derivatives on gas, oil, coal etc. are traded similar to financial contracts in ordinary stock markets. One of the most popular products in those markets are options written on *commodity forwards*, respectively *power forwards*, which guarantee the buyer of such an instrument the either physical or financial delivery of a predetermined amount of a certain commodity at the maturity time against the payment of a contractually specified fixed strike price [13], [21], [59]. Yet, the problem of pricing commodity derivatives turns out to be more challenging than computing option prices for usual financial assets. Nevertheless, we will exploit obvious similarities with respect to interest rate theory (i.e. to forward rate modeling) and draw the corresponding conclusions for the creation of an appropriate commodity forward market model in Chapter 7. In this context, we use a Heath-Jarrow-Morton (HJM) approach (as firstly introduced in [49]) taken from interest rate theory which seems to be appropriate for the modeling of commodity forward contracts delivering at a fixed maturity time. For a more rigorous reading concerning possible applications of HJM-approaches to commodity pricing see e.g. Chapter 6 in [13], respectively the references [51] and [59].

[2] Also see Figure 1 on p.4 in [25] for a succeeded visualization of this issue. We further refer to Chapter 5 in [25] to read more about the effects/impact of intermediate announcements concerning the market zone net position in the EU ETS market.

Chapter 2

Some Useful Mathematical Results from Stochastic Calculus

2.1 Selected results for Lévy-processes

Throughout this work, Lévy-processes will be one of our main modeling tools. In this regard, we start off within a precise definition of this very tractable class of stochastic processes – cf. e.g. [1], [30], [32], [79], [80].

Definition 2.1.1 *(Lévy-process)*

We say that the d-dimensional stochastic process $X \coloneqq (X_t)_{t\in[0,T]}$ defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{F}$-adapted Lévy-process, if $X_0 = \mathbf{0}$ holds $\mathbb{P}$-a.s., X_t is $\mathcal{F}_t$-measurable for every $t \in [0,T]$ and all components of X admit càdlàg paths and stationary and independent increments.

The next result provides us with an explicit representation formula for Lévy-processes.

Theorem 2.1.2 *(Lévy-Itô decomposition)*

If $X \coloneqq (X_t)_{t\in[0,T]} \in \mathbb{R}^d$ is a Lévy-process, then there exists a constant vector $\alpha \in \mathbb{R}^d$, a Brownian motion $W \in \mathbb{R}^d$ with covariance matrix $A \in \mathbb{R}^{d\times d}$ and an independent Poisson-Random-Measure (PRM) N defined on $[0,T] \times \mathbb{R}^d \setminus \{\mathbf{0}\}$ such that, for each $t \in [0,T]$, we have the decomposition

(2.1.1)

$$X_t = \alpha t + W_t + \int_0^t \int_{\|x\|\geq 1} x \, dN(s,x) + \int_0^t \int_{0<\|x\|<1} x \, d\widetilde{N}_{\mathbb{P}}(s,x).$$

Herein, $\widetilde{N}_{\mathbb{P}}$ stands for the d-dimensional $\mathbb{P}$-compensated integer-valued PRM which is given through

(2.1.2)

$$d\widetilde{N}_{\mathbb{P}}(s,x) \coloneqq dN(s,x) - d\nu(x)\, ds$$

for a Lévy-measure ν being a positive and finite Borel-random-measure on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ that fulfills the condition

(2.1.3)

$$\int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} 1 \wedge \|x\|^2 \, d\nu(x) < \infty.$$

Finally, we call (A, ν, α) the characteristic triplet of the Lévy-process X.

Proof See Theorem 2.4.16 in [1]. ■

When it comes to option pricing purposes in Lévy driven energy market models (being one of the main topics of the present work actually), we very often have to deal with expectations of the stylized type

$$\mathbb{E}_{\mathbb{P}}[e^{X_t}]$$

wherein X_t is a Lévy (-type) process. In this context, the next result turns out to be extremely helpful as it tells us how such expectations can be computed more explicitly.

Theorem 2.1.3 (*Lévy-Khinchin formula)*

For a Lévy-process $X \in \mathbb{R}^d$ as given in (2.1.1) the corresponding characteristic function reads as

(2.1.4)

$$\Phi_{X_t}(z) \coloneqq \mathbb{E}_{\mathbb{P}}\left[e^{i\langle z, X_t\rangle}\right] = e^{t\,\psi(z)}$$

within a characteristic exponent

(2.1.5)

$$\psi(z) \coloneqq i\,\langle \alpha, z\rangle - \tfrac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d \setminus \{\mathbf{0}\}} \left[e^{i\,\langle z,y\rangle} - 1 - i\,\langle z, y\rangle\, \mathbb{I}_{\|y\|<1}\right] d\nu(y).$$

Herein, the brackets $\langle \cdot,\cdot\rangle$ designate the standard inner product on the space $\mathbb{R}^d$.

Proof See Corollary 2.4.20 in [1]. ■

Further on, somewhat similar to the above theorem, the following statement can be applied yet for expectations of the type

$$\mathbb{E}_{\mathbb{P}}[(X_t)^n]$$

called the n-th moment of the Lévy-process X_t, where $n \in \mathbb{N}$.

Lemma 2.1.4 *(Moment generating function of a Lévy-process)*

For a Lévy-process X as given in (2.1.1) the moment generating function $\mathfrak{M}_n(X_t) := \mathbb{E}_{\mathbb{P}}[(X_t)^n]$ *can be computed via*

$$\mathfrak{M}_n(X_t) = i^{-n} \frac{d^n}{dz^n} \left(\Phi_{X_t}(z) \right)_{z=0}$$

where the characteristic function $\Phi_{X_t}(z)$ *is such as defined in (2.1.4).*

Proof See section 1.1.6 in [1]. ■

Frequently, we will need a representation for the product of two stochastic (Lévy-) processes. For instance, if we discount an asset with a bond/bank account (as required by the risk-neutral pricing theory), the resulting discounted price process obviously possesses a product structure. The next lemma provides us with the related mathematical background.

Lemma 2.1.5 *(Itô's product rule)*

For (Lévy-) processes $X := (X_t)_{t\in[0,T]}$ *and* $Y := (Y_t)_{t\in[0,T]}$ *we have*

$$X_t \, Y_t = X_0 \, Y_0 + \int_0^t X_{s-} \, dY_s + \int_0^t Y_{s-} \, dX_s + [X, Y]_t$$

for all $t \in [0, T]$*. Herein, the term* $[X, Y]_t$ *depicts the quadratic co-variation of the processes* X *and* Y*.*

Proof See Theorem 4.4.13 in [1]. ■

The following theorem, which firstly has been proven by the Japanese mathematician Kiyoshi Itô, undisputedly constitutes one of the most celebrated results in the field of stochastic calculus. Essentially, it provides a representation for a transformed stochastic process of the form $f(X_t)$ where f is a real mapping and X_t depicts a Lévy-process. (However, *Itô's formula* can be extended to more general processes like semi-martingales, for instance; see Ch. II in [78].) As we will see later, a very common choice (descending from e.g. geometrical Lévy-models such as introduced in Ch. 7) in this context is to take $f(x)$ as the Euler function e^x. For notational reasons, we present the one-dimensional version of Itô's formula here, remarking that the multi-dimensional analogue can be found in section 2.5 in [13] or Th. 9.5 in [32]. The precise (one-dimensional) result reads as follows.

Theorem 2.1.6 *(Itô's formula)*

For a (one-dimensional) Lévy-process X *and a function* $(t, x) \longmapsto f(t, x)$ *mapping* $f: [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$*, which is once continuously differentiable in its first variable and twice continuously differentiable in its second variable, in symbols* $f \in C^{1,2}([0, T] \times \mathbb{R})$*, we have the representation*

$$f(t, X_t) = f(0, X_0) + \int_0^t f_s(s, X_s) \, ds + \int_0^t f_x(s, X_{s-}) \, dX_s + \frac{1}{2} \int_0^t f_{xx}(s, X_s) \, d[X^c]_s$$
$$+ \sum_{0 \le s \le t} [f(s, X_s) - f(s, X_{s-}) - \Delta X_s \, f_x(s, X_{s-})].$$

Herein, we have denoted the jump size of the process X at time t by $\Delta X_t := X_{t+} - X_{t-} = X_t - X_{t-}$, whereas $[X^c]_t$ stands for the quadratic variation of the continuous part of X.

Proof See Prop. 8.19 in [30] or Th. 9.4 in [32]. ■

Remark 2.1.7 *If we integrate a bounded and deterministic function $g: \mathbb{R}^+ \longrightarrow \mathbb{C}^d$ with respect to the Lévy-process X such as introduced in Definition 2.1.1, then the resulting stochastic integral*

$$I_t := \int_0^t g(s)\, dX_s$$

in general is not a Lévy-process again, but instead an additive or Sato-process admitting independent but not necessarily stationary increments. (For further reading on Sato-processes see [30] or [80].) ■

2.2 Girsanov's Change-of-Measure theorem for jump-diffusions

According to Chapter 3 in [26], we now introduce an (with respect to the probability measure $\mathbb{P}$) equivalent (martingale) measure $\mathbb{Q}$ in order to derive adequate option price formulas later. Thus, for *previsible/predictable*[3] and integrable stochastic processes G_t, $h(t,x)$ and $H(t,x) := exp\{h(t,x)\}$ with $t \in [0,T]$ and $x \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, we define the strictly positive Radon-Nikodym density process

(2.2.1)

$$Z_t := exp\left\{\int_0^t G_s\, dW_s - \frac{1}{2}\int_0^t G_s^2\, ds + \int_0^t \int_{\mathbb{R}_0} h(s-,x)\, d\widetilde{N}_{\mathbb{P}}(s,x) - \int_0^t \int_{\mathbb{R}_0} [H(s,x) - 1 - h(s,x)]\, d\nu(x)\, ds\right\}.$$

Note in passing that if $h(s,x)$ equals zero, then there obviously are no jumps occurring in (2.2.1) so that the latter transforms into the well-known *continuous* Doléans-Dade exponential

(2.2.2) $$Z_t = \mathfrak{E}(G_\cdot \circ W_\cdot)_t := exp\left\{\int_0^t G_s\, dW_s - \frac{1}{2}\int_0^t G_s^2\, ds\right\}.$$

However, in order to ensure the $\mathbb{P}$-martingale property of Z_t such as given in (2.2.1), we first apply Theorem 2.1.6 on the latter yielding the following integral representation with vanishing drift

(2.2.3) $$Z_t = 1 + \int_0^t Z_s\, G_s\, dW_s + \int_0^t \int_{\mathbb{R}_0} Z_{s-}[H(s-,x) - 1]\, d\widetilde{N}_{\mathbb{P}}(s,x).$$

Hence, Z_t designates a *local* $\mathbb{P}$-martingale. By the way, in differential notation (2.2.3) reads as

(2.2.4) $$\frac{dZ_t}{Z_{t-}} = G_t\, dW_t + \int_{\mathbb{R}_0} [H(t-,x) - 1]\, \widetilde{N}_{\mathbb{P}}(t,dx).$$

---

[3] In accordance to page 163 in [32], an integrable stochastic process is called *previsible/predictable*, if the process is measurable, $\mathcal{F}$-adapted and left-continuous (i.e. *càg*, French: *continue à gauche*) with respect to t. Also see Chapter 3 in [26] along with Def. 2.5 and Def. 2.8 in [13] for more details on this terminology.

At this step, we see very clear how the continuous Girsanov density process alters due to the introduction of a random jump component. Further on, we assume the processes G, H and h appearing inside (2.2.1) to be chosen such that both integrals in (2.2.3) constitute $\mathbb{P}$-martingales (with vanishing $\mathbb{P}$-expectation) for all t, resp. such that G, H and h fulfill a *Novikov condition* like announced in Th. 12.21 in [32]. Each of these presumptions implies $\mathbb{E}_{\mathbb{P}}[Z_t] = 1 \ \forall t \in [0, T]$ so that – in accordance to Th. 5.2.4 in [1] – the exponential Z in (2.2.1) even is declared as a true $\mathbb{P}$-martingale. (From now on, whenever necessary, we assume a Novikov condition to be in force, resp. the corresponding Radon-Nikodym process to constitute a true $\mathbb{P}$-martingale, when we work with Girsanov's theorem in this thesis.) Presuming either of the above presumptions to be in force, we state the following result.

Proposition 2.2.1 *(Girsanov's theorem for jump-diffusions)*

Let $\mathbb{Q}$ be an equivalent probability measure on $\mathcal{F}_t$ with respect to $\mathbb{P}$. Then for all $t \in [0, T]$ the density

(2.2.5)
$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := Z_t > 0$$

exists, wherein (the $\mathbb{P}$-martingale) Z_t is such as announced in (2.2.1). Therewith, the process

(2.2.6)
$$\widetilde{W}_t := W_t - \int_0^t G_s \, ds$$

indicates a standard Brownian motion (BM) under the equivalent martingale measure (EMM) $\mathbb{Q}$. Moreover, the positive and finite compensating (Lévy-) measure under $\mathbb{Q}$ is given by

(2.2.7)
$$d\tilde{v}(t, x) := H(t, x) \, dv(x) \, dt$$

which is such that the $\mathbb{Q}$-compensated integer-valued (Poisson-) random-measure (PRM)

(2.2.8)
$$d\widetilde{N}_{\mathbb{Q}}(t, x) := dN(t, x) - H(t, x) \, dv(x) \, dt$$

forms a $\mathbb{Q}$-martingale integrator on $[0, T] \times \mathbb{R} \setminus \{0\}$.

Proof See Theorem 3.2 in [26], respectively Problem 9.5 and Theorem 12.21 in [32].[4] ■

2.3 Martingale compensators under enlarged filtrations

One of the most innovative topics in the present thesis consists in the provision of sophisticated derivation methods concerning option price formulas for energy, weather and emission derivatives under additional forward-looking information modeled by enlarged filtrations. In this context, it is often necessary to know the precise structures of the involved martingale compensators for different types of Lévy-processes yet adapted to those *enlarged* filtrations. We remark that the theory of enlargement-of-filtration has been initiated by Itô (see reference "[121] in [32]"), whereas it has been extended and applied (especially with an insider-trading background) by several authors, for instance in [10], [15], [32], [78] and also in the references "[6], [49], [63], [65], [77], [78], [90], [92], [93], [116], [130], [179], [181] in [32]". All in all, the following results are extremely helpful when it comes to option pricing purposes with respect to additional future information.

[4] Actually, there is a factor $1/2$ missing in front of the second integral appearing inside the definition of $Z(t)$ in Theorem 12.21 on page 197 in [32].

Proposition 2.3.1 *(Itô's enlarged filtration result)*

Let $X := (X_t)_{t\in[0,T]}$ be a $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,T]}$-adapted Lévy-process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as introduced above. Moreover, we implement the enlarged filtration $\mathcal{E}_t := \mathcal{F}_t \vee \sigma\{X_\tau\}$ for a future time index $\tau \leq T$. Then the stochastic process

(2.3.1)

$$\left(X_t - \int_0^t \frac{X_\tau - X_s}{\tau - s} ds \right)_{t\in[0,\tau[}$$

constitutes a $\mathcal{E}_t$-adapted martingale under $\mathbb{P}$.

Proof See the first lines of the proof of Proposition 16.52 in [32]. (Also compare Theorem 3 in Chapter VI of [78].) ■

The main message of Proposition 2.3.1 easily can be specialized to Brownian motions or compound Poisson processes, as the two latter merely constitute special cases of general Lévy-processes. The precise results read as follows.

Corollary 2.3.2

***(a)** For a $\mathcal{F}$-adapted $\mathbb{P}$-Brownian-motion (BM) $W := (W_t)_{t\in[0,T]}$ and an enlarged filtration*

$$\hat{\mathcal{E}}_t := \mathcal{F}_t \vee \sigma\{W_\tau\}$$

the stochastic process

(2.3.2)

$$\left(\widehat{W}_t := W_t - \int_0^t \frac{W_\tau - W_s}{\tau - s} ds \right)_{t\in[0,\tau[}$$

embodies a $(\hat{\mathcal{E}}_t, \mathbb{P})$-martingale.[5]

***(b)** For a $\mathcal{F}$-adapted $\mathbb{P}$-compensated compound Poisson process $Q := (Q_t)_{t\in[0,T]}$ with*

$$Q_t := \int_0^t \int_{\mathbb{R}_0} z \, d\widetilde{N}_{\mathbb{P}}(s, z)$$

and an enlarged filtration $\check{\mathcal{E}}_t := \mathcal{F}_t \vee \sigma\{Q_\tau\}$ the stochastic process

(2.3.3)

$$\left(\check{Q}_t := Q_t - \int_0^t \frac{Q_\tau - Q_s}{\tau - s} ds \right)_{t\in[0,\tau[}$$

embodies a $(\check{\mathcal{E}}_t, \mathbb{P})$-martingale.

---

[5] Referring to Theorem 8.1 in [32], respectively to instance (16.158) on page 316 in [32], we deduce that $\widehat{W}_t$ not only depicts a $(\hat{\mathcal{E}}_t, \mathbb{P})$-*martingale*, but even a $(\hat{\mathcal{E}}_t, \mathbb{P})$-*Brownian motion*.

Proof Both statements immediately follow from Proposition 2.3.1. ■

The next result is of fundamental importance for our upcoming considerations concerning the pricing of energy, temperature and emission derivatives under additional forward-looking information, since the latter mostly will be modeled by *intermediate* enlarged filtrations in this thesis. For the sake of completeness and to become familiar with the most important mathematical backgrounds, we will give a full proof for the first part of the following Proposition 2.3.3, actually sticking to similar verification arguments as in the proof of Proposition 16.52 in [32]. However, we here give some additional explanatory words along with auxiliary computations in our proving procedure. Nevertheless, at this early step we remark that it seems to be hardly possible to apply the results of Proposition 2.3.3 *instantly* when it comes to option pricing purposes under enlarged filtrations. We will return to this complex and for our purposes rather delicate topic in section 3.3 later and therein also provide an easy but in this context (at least to the best of our knowledge) new key idea to overcome the appearing problems descending from *too general* intermediate filtrations (such as e.g. $\mathcal{H}_t$ introduced below).

Proposition 2.3.3 *For $\mathbb{P}$-independent stochastic processes W and Q (such as introduced in Corollary 2.3.2) and an enlarged filtration*

$$\mathcal{K}_t := \mathcal{F}_t \vee \sigma\{W_\tau, Q_\tau\}$$

we define the intermediate filtration $\mathcal{H}_t$ via

$$\mathcal{F}_t \subset \mathcal{H}_t \subset \mathcal{K}_t.$$

Then the stochastic process

(2.3.4)

$$\left(\widetilde{W}_t := W_t - \int_0^t \frac{\mathbb{E}_{\mathbb{P}}(W_\tau|\mathcal{H}_s) - W_s}{\tau - s}\, ds\right)_{t\in[0,\tau[}$$

depicts a $\mathcal{H}_t$-adapted martingale under the measure $\mathbb{P}$. As above, we denote this fact by writing $(\mathcal{H}_t, \mathbb{P})$-martingale shortly. [By the way, letting $\vartheta \to 0^+$ in Lemma 3.3.5 (a) below, we have that $\widetilde{W}$ such as given in (2.3.4) even indicates a $(\mathcal{H}_t, \mathbb{P})$-Brownian motion.]

Moreover, the stochastic process

(2.3.5)

$$\left(\tilde{Q}_t := Q_t - \int_0^t \frac{\mathbb{E}_{\mathbb{P}}(Q_\tau|\mathcal{H}_s) - Q_s}{\tau - s}\, ds\right)_{t\in[0,\tau[}$$

constitutes a $(\mathcal{H}_t, \mathbb{P})$-martingale.

Proof We aim to show that the conditional expectation $\mathbb{E}_{\mathbb{P}}\big(\widetilde{W}_t - \widetilde{W}_r \big| \mathcal{H}_r\big)$ vanishes for all time indices $0 \le r \le t$ which initially would prove the proposition for the Brownian motion case.

Starting off, by using the tower property for conditional expectations we immediately obtain

(2.3.6)

$$\mathbb{E}_{\mathbb{P}}\big(\widetilde{W}_t - \widetilde{W}_r \big| \mathcal{H}_r\big) = \mathbb{E}_{\mathbb{P}}\left(W_t - W_r - \int_r^t \frac{\mathbb{E}_{\mathbb{P}}(W_\tau - W_s|\mathcal{H}_s)}{\tau - s} ds \right| \left. \mathcal{H}_r \right)$$
$$= \mathbb{E}_{\mathbb{P}}(W_t - W_r|\mathcal{H}_r) - \int_r^t \frac{\mathbb{E}_{\mathbb{P}}(W_\tau - W_s|\mathcal{H}_r)}{\tau - s} ds$$
$$= \mathbb{E}_{\mathbb{P}}\left(W_t - W_r - \int_r^t \frac{W_\tau - W_s}{\tau - s} ds \right| \left. \mathcal{H}_r \right).$$

Since the inclusion $\mathcal{H}_r \subset \mathcal{K}_r$ is valid, applying the tower property for conditional expectations again, we may rewrite the latter equation as

(2.3.7)
$$\mathbb{E}_{\mathbb{P}}\big(\widetilde{W}_t - \widetilde{W}_r \big| \mathcal{H}_r\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}\big(\widehat{W}_t \big| \mathcal{K}_r\big) - \mathbb{E}_{\mathbb{P}}\big(\widehat{W}_r \big| \mathcal{K}_r\big) \big| \mathcal{H}_r\big)$$

wherein $\widehat{W}$ is such as defined in (2.3.2). Next, from Corollary 2.3.2 (a) we know that $\widehat{W}_t$ constitutes a $\mathcal{K}_t$-measurable martingale under $\mathbb{P}$ [shortly, a $(\mathcal{K}_t, \mathbb{P})$-martingale] so that

(2.3.8)
$$\mathbb{E}_{\mathbb{P}}\big(\widetilde{W}_t - \widetilde{W}_r \big| \mathcal{H}_r\big) = \mathbb{E}_{\mathbb{P}}\big(\widehat{W}_r - \widehat{W}_r \big| \mathcal{H}_r\big) = 0$$

follows, which ultimately proves the proposition for the BM-case.

The argumentation for the $(\mathcal{H}, \mathbb{P})$-compensated compound Poisson process $\tilde{Q}$ can be done in a similar manner. ■

2.4 The Leibniz-formula and Fourier-transforms

Particularly in the Heath-Jarrow-Morton framework investigated in Chapter 7 we will need the following Leibniz-formula yielding an explicit representation for the differential (with respect to the time parameter t) of a generalized Lebesgue-integral, wherein the parameter t is allowed to appear both inside the integrand and in the integration bounds. By the way, note that such special cases cannot be treated by Itô's formula directly.

Lemma 2.4.1 *(Leibniz-formula for parameter integrals)*

For differentiable, respectively integrable functions $x(t)$, $y(t)$ and $f(t,u)$ we have

(2.4.1)

$$\frac{d}{dt}\left(\int_{x(t)}^{y(t)} f(t,u)\, du \right) = \int_{x(t)}^{y(t)} \frac{d}{dt} f(t,u)\, du + f(t, y(t))\, y'(t) - f(t, x(t))\, x'(t).$$

Proof See the proof of Lemma 1 in [51], for instance. ■

Moreover, when it comes to option pricing issues in stochastic models with jumps one frequently decides to apply *Fourier transform methods* (see e.g. [8], [13], [22]) in order to treat the appearing conditional expectations descending from the risk-neutral pricing formula. This becomes necessary since in the most cases at hand the distributional properties of the underlying jump processes are not known explicitly – in contrast to simple log-normally distributed geometrical Brownian motion models (such as dealt with in Theorem 7.4.1, for instance) or even normally distributed (BM driven) Bachelier models, for example, which both can be handled by standard measure transformation arguments on the opposite [see (3.3.150) and (3.3.158) – (3.3.161)]. However, we next provide a precise definition for the Fourier transform of a real function and simultaneously for its inverse. Before doing so, let us remark that throughout the literature there can be found numerous slightly different definitions for Fourier transforms which might cause some confusion from time to time. Nevertheless, in this work we will consistently stick to the definitions given below.

Definition 2.4.2 *(multi-dimensional Fourier transform)*

For a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ *with* $f \in \mathcal{L}^1(\mathbb{R}^d, \lambda^d)$ *we define its Fourier transform via*

(2.4.2)

$$\hat{f}(y) := \int_{\mathbb{R}^d} f(x)\, e^{-i\,\langle x,y\rangle} d\lambda^d(x)$$

whereby the square brackets $\langle \cdot,\cdot \rangle$ *denote the standard inner product in the space* $\mathbb{R}^d$ *and* λ^d *constitutes the* d*-dimensional Lebesgue-measure. Moreover, its inverse is then given through*

(2.4.3)

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(y)\, e^{i\,\langle x,y\rangle} d\lambda^d(y).$$

Chapter 3

Forward-Looking Multi-Factor Ornstein-Uhlenbeck Models for Pricing Electricity Risk

3.1 A short chapter overview

The aim of this chapter mainly consists in the computation of risk-neutral option prices for both plain-vanilla and exotic electricity derivatives on the basis of an arithmetic multi-factor Ornstein-Uhlenbeck spot price model, whereas we take forward-looking information – which we assume to be available to well-informed market insiders – into account via numerous tailor-made enlargements of the underlying information filtrations. In this insider trading context, we also correlate electricity spot prices with outdoor-temperature and treat a related electricity derivatives pricing problem under supplementary temperature forecasts. However, it does not seem to be possible to derive *analytical* option price formulas whenever we assume future information on the driving *jump* noises to be available. In these cases we apply customized approximation procedures involving techniques from Complex analysis. Contrarily, whenever the historical filtration is enlarged with respect to *Brownian* noise, we fortunately obtain more explicit option price formulas. Finally, we stress that our *arithmetic* approaches do not trouble an exponential function (such as commonly appearing in *geometrical* models; compare e.g. Chapter 7) to ensure positivity of the prices. Having mentioned the most important key characteristics of electricity markets in section 1.1 already, we are now asked to draw the corresponding conclusions for the creation of an adequate *forward-looking* market model.

The remainder of the present chapter is organized as follows: In section 3.2 our initial pure-jump multi-factor electricity spot price model of mean-reverting Ornstein-Uhlenbeck type is introduced in detail. Applying Girsanov's Change-of-Measure theorem, we hereafter obtain a representation for the electricity futures price under an *equivalent martingale measure* (EMM)[6] and moreover, deduce related option prices for electricity derivatives within a rigorous utilization of Fourier transform methods. In the main paragraph 3.3 we focus on the pricing of electricity contracts again but yet under the incorporation of additional future information. Due to this insider trading machinery, we introduce the *information premium* measuring some kind of supplementary information gain with respect to common backward-looking scenarios and, in particular, tailor enlarged information filtrations to the requirements of our electricity market framework. This procedure finally culminates in the provision of electricity swap and connected option prices under complementary information on the future behavior of electricity spot prices or outdoor-temperature, for instance, actually representing some of the most innovative results in this thesis. Additionally, we propose a *mixed* electricity spot/futures price model including both Brownian motion along with pure-jump terms and deduce a corresponding *mixed* call option price formula. Hereafter, we compare our former approaches with an alternative forward-looking measure change method while introducing the notion of a *cross premium*. In section 3.4 the most important conclusions a drawn, whereas the closing paragraph 3.5 inter alia contains verification arguments for some of the results that have been used throughout Chapter 3.

3.2 Modeling electricity spot and futures prices

We start off with the description of the mathematical basis of our model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space, where the monotone *backward-looking* information filtration $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is assumed both to include a priori all $\mathbb{P}$-null-sets and to be *cad* (French: continue à droite), that is, $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ holds for all $t \in [0, T]$ within a fixed time horizon $T < \infty$.

3.2.1 A pure-jump multi-factor electricity spot price model

Right from the beginning, we devote our attention towards the mathematical modeling of the upcoming electricity spot price dynamics. Inspired by [8], we will make use of an arithmetic mean-reverting approach to achieve a tractable description of the spot price. More precise, we rigorously take the seasonality of the trend-line and particularly of the price spikes into account, allowing for seasonal dependent jump-amplitudes and frequencies. Hence, in accordance to [8], we model the spot price dynamics directly by a periodic deterministic function plus a weighted sum of mean-reverting Ornstein-Uhlenbeck (OU) processes, each of which reverting to zero with a different speed and having pure-jump processes as driving noises. This *arithmetic* onset yields the advantage of (semi-) analytical pricing formulas for electricity options [8], [72] (also see Prop. 3.2.4 below). In contrast to common *geometrical* setups (as appearing in e.g. [14], [48], [59]) there is no exponential function involved in the present *additive* model, whereas in return the positivity of the prices has to be ensured by allowing *positive* jumps only [8]. Yet, arithmetic multi-factor models of pure-jump OU-type capture the stylized facts of electricity spot prices, especially the seasonal features and the mean-reverting property, extremely well as imposingly proved by Benth, Kallsen and Meyer-Brandis [8] (see Ch. 2 therein for more details and compare Fig.1 with Fig.2 in [8]). Thus, a thorough analysis of those arithmetical models in connection with energy market applications is of steadily growing interest, presently.

[6] Since electricity is non-storable and thus, neither a traded *asset*, it does, however, not make sense to require the (discounted) electricity spot price to form a martingale under an *EMM*. Hence, whenever we speak of an equivalent *martingale* measure in an electricity market context (respectively, in any market with non-tradable underlying), we actually think of an (with respect to $\mathbb{P}$) equivalent *risk-neutral* probability measure, merely. Note that *any* measure, equivalent to $\mathbb{P}$, constitutes an admissible candidate – compare subsection 4.1.1 in [13].

Referring to [8], we assume the electricity spot price $S := (S_t)_{t\in[0,T]}$ to follow the additive structure

(3.2.1)
$$S_t := \mu(t) + Y_t$$

wherein $\mu(t)$ represents a deterministic and periodic seasonality function, i.e. the non-random part of the stochastic trend-line. Note that the function $\mu(t)$ does *not* constitute what we understand as the mean-level of the spot, as one could think on a first sight; instead, μ rather indicates the *floor* or *lower bound* of the spot price [8], [13]. Further, the summand Y_t is responsible for interspersing random price fluctuations. As in [8], the latter is supposed to be a weighted sum of stochastic processes

(3.2.2)
$$Y_t := \sum_{k=1}^{n} w_k X_t^k$$

with constant weights $w_k \geq 0$ and zero-reverting pure-jump OU components X_t^k obeying

(3.2.3)
$$dX_t^k = -\lambda_k X_t^k \, dt + \sigma_k(t) \, dL_t^k.$$

Herein, $\lambda_k > 0$ denote constant mean-reversion speeds and $\sigma_k(t) > 0$ are deterministic and bounded volatility functions controlling the seasonal variation of the jump-sizes. Moreover, the integrable, increasing and pair-wise independent pure-jump processes L_t^k regulate the price fluctuations including both small daily volatile variations on the one hand and violent large-amplitude price spikes on the other. In addition, we assume the noises L_t^k to be *càdlàg* (French: continue à droite avec des limites à gauche) and to admit independent increments. Finally, the deterministic initial values are given by $X_0^k := x_k$ with $x_1 := [S_0 - \mu(0)]/w_1$ and $x_k := 0$ for $k = 2, \ldots, n$. Since equation (3.2.1) implies $Y_t = S_t - \mu(t)$, we may interpret Y_t as the *deseasonalized* spot price [72]. Slightly deviating from [8], for all $k = 1, \ldots, n$ we next announce the concrete form of the monotone-increasing, finite-variation (see Th. 2.4.25 in [1]) and, for each z, $\mathcal{F}_t$-adapted (Lévy-type) Sato-processes L_t^k via

(3.2.4)
$$L_t^k := \int_0^t \int_{D_k} z \, dN_k(s, z)$$

for a set $D_k \subseteq \,]0, \infty[\, \subset \mathbb{R}$ and time indices $t \in [0, T]$ (cf. p.3 in [8]). In the latter equation N_k depicts a one-dimensional integer-valued Poisson-Random-Measure (PRM) on $[0, T] \times \,]0, \infty[$ for each index k. Similar to [8], we further assume the PRMs $dN_k(s, z)$ to have deterministic predictable (and in particular, time-dependent) $\mathbb{P}$-compensators of the form $\rho_k(s) \, d\nu_k(z) \, ds$ such that the objects

(3.2.5)
$$d\widetilde{N}_k^{\mathbb{P}}(s, z) := dN_k(s, z) - \rho_k(s) \, d\nu_k(z) \, ds$$

indicate $\mathbb{P}$-martingale integrators for every k. By the way, if we choose $\rho_k(s) \equiv \rho_k$ to be constant (resp. time-independent), then L_t^k such as given in (3.2.4) is a (increasing) *Lévy*-process (a so-called *subordinator*) admitting independent *and stationary* increments (cf. e.g. [1], [8], [30], [80]). However, in (3.2.5) the deterministic (time-inhomogeneous) functions $\rho_k(s) \geq 0$ control the seasonal variation of the jump-intensities and the one-dimensional Lévy-measures ν_k declare positive and finite Borel-random-measures on D_k for all $k = 1, \ldots, n$ obeying one of the following equivalent conditions

$$\int_{D_k} 1 \wedge z^2 \, d\nu_k(z) < \infty \quad \Leftrightarrow \quad \int_{D_k} \frac{z^2}{1+z^2} \, d\nu_k(z) < \infty.$$

Further on, as proposed in [8], it will turn out convenient to separate the sum (3.2.2) as follows:

- In our context, we ought to use the first $\#l$ $(< n)$ components $X_t^1, \dots, X_t^l$ to model the long-term level of the spot price, i.e. the daily volatile stochastic variations of the deterministic trend-line $\mu(t)$. Hence, the processes $X_t^1, \dots, X_t^l$ should permit (Brownian-motion-like) small fluctuations only with jump-sizes in a set $D_k :=]0, \varepsilon_k[$ for a small number $\varepsilon_k > 0$ along with a slow mean-reversion velocity λ_k for $k = 1, \dots, l$.

- In return, the remaining $\#(n-l)$ components $X_t^{l+1}, \dots, X_t^n$ are exploited to model the short-term spiky variations, i.e. the large price jumps. Thus, we may choose $D_k := [\varepsilon_k, \infty[$ for an arbitrary (maybe large) number $\varepsilon_k > 0$ together with a high speed of mean-reversion λ_k for every $k = l+1, \dots, n$. A high mean-reversion speed is important here to ensure that after a large-amplitude jump the spot price rapidly turns back to about the same level as before – being a characteristic property of price *spikes*, such as announced in section 1.1 formerly.

By the way, since in (3.2.4) we have permitted *positive* jumps only, the mean-reverting nature of the components X_t^k is necessary to guarantee that the spot price (3.2.1) is not only increasing but also decreasing [8], [13]. Taking (3.2.4) into account, the integral form of (3.2.3) reads as

(3.2.6)

$$X_t^k = x_k - \lambda_k \int_0^t X_s^k \, ds + \int_0^t \int_{D_k} z \, \sigma_k(s) \, dN_k(s,z).$$

Standard arguments from stochastic calculus purvey the explicit (Sato-type-) solution of (3.2.3) as

(3.2.7)

$$X_t^k = x_k \, e^{-\lambda_k t} + \int_0^t \sigma_k(s) \, e^{-\lambda_k (t-s)} \, dL_s^k = x_k \, e^{-\lambda_k t} + \int_0^t \int_{D_k} z \, \sigma_k(s) \, e^{-\lambda_k (t-s)} \, dN_k(s,z).$$

In our proceedings we will frequently make use of the iterated version

(3.2.8)

$$X_u^k = X_t^k e^{-\lambda_k (u-t)} + \int_t^u \sigma_k(s) \, e^{-\lambda_k (u-s)} \, dL_s^k$$

for time indices $0 \le t \le u \le T$. Merging (3.2.2) and (3.2.7) into (3.2.1), we receive a more explicit representation for the electricity spot price

(3.2.9)

$$S_t = \mu(t) + \sum_{k=1}^n w_k X_t^k = \mu(t) + \sum_{k=1}^n w_k \left(x_k \, e^{-\lambda_k t} + \int_0^t \sigma_k(s) \, e^{-\lambda_k (t-s)} \, dL_s^k \right).$$

At this step, let us pick up our introductory remark dedicated to (3.2.1) concerning the mean-level of the spot price S_t: Slightly deviating from [8] (for details see Remark 3.2.1 below), we yet define the *randomized trend-line of the electricity spot price* via

(3.2.10)

$$\eta_t := S_t - \sum_{k=l+1}^{n} w_k X_t^k = \mu(t) + \sum_{k=1}^{l} w_k X_t^k.$$

Hence, looking at the last member in (3.2.10), we state that the mean-level of the spot, namely η_t, is basically given through the positive, deterministic and periodic floor-function $\mu(t)$ which is randomly perturbed by a sum of slowly varying weighted noises $X_t^1, \dots, X_t^l$. Interpreting the first equality inside (3.2.10), we claim that, in order to get an idea about the shape of the stochastic trend-line η_t, we first have to filter out the large-amplitude spiky price variations of the spot S that are generated by the ingredients $X_t^{l+1}, \dots, X_t^n$. By the way, it is a rather difficult task to determine which price variations are caused by violent jumps (spikes) and which ones have their origin in the usual (Brownian-motion-like) price variations – cf. p. 14 in [72]. As claimed in [72], the latter constitutes an always present problem when it comes to the calibration/estimation of a spot price model of the above type. We will return to this challenging topic in section 8.4.4 while applying stochastic filtering techniques as invented in [53] to estimate the spiky components $X_t^{l+1}, \dots, X_t^n$, given the observed deseasonalized spot price Y_t.

Remark 3.2.1 *The authors of [8] prefer to define the mean-level $\bar{\eta}(t)$ of the spot S_t by the expected or averaged price after having taken out the large-amplitude spikes, namely*

(3.2.11) $$\bar{\eta}(t) := \mathbb{E}_{\mathbb{P}}\left[S_t - \textstyle\sum_{k=l+1}^{n} w_k X_t^k\right] = \mu(t) + \textstyle\sum_{k=1}^{l} w_k\, \mathbb{E}_{\mathbb{P}}\left[X_t^k\right]$$

depicting a positive function yet. Taking (3.2.5) and (3.2.7) into account, the latter equation becomes

(3.2.12)

$$\bar{\eta}(t) = \mu(t) + \sum_{k=1}^{l} w_k \left(x_k\, e^{-\lambda_k t} + \int_0^t \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k (t-s)}\, \rho_k(s)\, d\nu_k(z)\, ds \right).$$

As it seems to be reasonable (see p.4 in [8]) to assume constant daily volatilities $\sigma_k(s) \equiv \sigma_k > 0$ and constant daily jump-intensities $\rho_k(s) \equiv \rho_k \geq 0$ (both for $k = 1, \dots, l$), eq. (3.2.12) next simplifies to

(3.2.13)

$$\bar{\eta}(t) = \mu(t) + \sum_{k=1}^{l} w_k \left(x_k\, e^{-\lambda_k t} + \sigma_k\, \rho_k \frac{1 - e^{-\lambda_k t}}{\lambda_k} \int_{D_k} z\, d\nu_k(z) \right).$$

At this point, we recall that if we take constant *long-term/daily jump-intensities $\rho_1, \dots, \rho_l$ in our model, then the corresponding processes $L^1, \dots, L^l$ become (subordinating)* Lévy*-processes (see above). In this context, let us finally quote from p.14 in [72]: "Especially when spike behavior [...] appears in a non-stationary seasonal way [...] the model requires the separation of the non-stationary spikes path from the stationary path of the remaining daily variations.".* ■

Summing up, we can associate the electricity spot price (3.2.1) as decomposed as follows

(3.2.14)

$$S_t = \underbrace{\mu(t) + \sum_{k=1}^{l} w_k X_t^k}_{\substack{\text{stochastic mean level with a} \\ \text{slow mean-reversion velocity} \\ \text{and small jump-amplitudes}}} + \underbrace{\sum_{k=l+1}^{n} w_k X_t^k}_{\substack{\text{price spikes with a high} \\ \text{rate of mean-reversion}}}.$$

3.2.2 Switching to an equivalent risk-neutral measure

According to page 8 in [8] and our former announcements in section 2.2, we now introduce an (with respect to $\mathbb{P}$) equivalent risk-neutral measure $\mathbb{Q}$ in order to derive adequate pricing formulas for electricity derivatives later. Thus, the present subsection is dedicated to the required switching of probability measures. For deterministic and integrable functions $h_k(s,z)$ $(k=1,\dots,n)$, we initially define the (strictly positive) Radon-Nikodym derivative

(3.2.15)

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \prod_{k=1}^{n} \mathfrak{E}\left(M^k\right)_t$$

with (local) $\mathbb{P}$-martingale ingredients

(3.2.16)

$$M_t^k := \int_0^t \int_{D_k} \left[e^{h_k(s,z)} - 1\right] d\widetilde{N}_k^{\mathbb{P}}(s,z)$$

and discontinuous Doléans-Dade exponentials

(3.2.17)

$$\mathfrak{E}\left(M^k\right)_t := exp\left\{M_t^k - \frac{1}{2}\left[\left(M^k\right)^c\right]_t\right\} \times \prod_{0\le s\le t} \left(1 + \Delta M_s^k\right) e^{-\Delta M_s^k}$$

for $k=1,\dots,n$ and $t\in[0,T]$. Merging (3.2.16) into (3.2.17), we immediately receive

(3.2.18)

$$\mathfrak{E}\left(M^k\right)_t = exp\left\{\int_0^t \int_{D_k} h_k(s,z)\, d\widetilde{N}_k^{\mathbb{P}}(s,z) - \int_0^t \int_{D_k} \left[e^{h_k(s,z)} - 1 - h_k(s,z)\right] \rho_k(s)\, d\nu_k(z)\, ds\right\}$$

whereas an application of Itô's formula [see Theorem 2.1.6] on (3.2.18) yields the (local) $\mathbb{P}$-martingale representation[7]

(3.2.19)

$$\mathfrak{E}\left(M^k\right)_t = 1 + \int_0^t \int_{D_k} \mathfrak{E}\left(M^k\right)_{s-} \left[e^{h_k(s,z)} - 1\right] d\widetilde{N}_k^{\mathbb{P}}(s,z).$$

Since in addition $\mathbb{E}_{\mathbb{P}}\left[\mathfrak{E}\left(M^k\right)_t\right] = 1$ holds for all $t\in[0,T]$ and $k=1,\dots,n$, the exponentials $\mathfrak{E}\left(M^k\right)$ even are declared as true $\mathbb{P}$-martingales (cf. p.8 in [8]). Troubling Proposition 2.2.1, we may state that

(3.2.20)

$$d\widetilde{N}_k^{\mathcal{F},\mathbb{Q}}(s,z) := d\widetilde{N}_k^{\mathbb{Q}}(s,z) := dN_k(s,z) - e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds$$

depict compensated PRMs under $\mathbb{Q}$ [and thus, $\mathcal{F}$-adapted $\mathbb{Q}$-martingale integrators] for all $k=1,\dots,n$.

[7] By the way, coupling the two first equalities of section 5.1 in [1], we immediately obtain the representation (3.2.19) without any further application of Itô's formula.

3.2.3 Electricity futures prices under the multi-factor approach

In this subsection we concentrate on the computation of spot-dependent electricity futures/swap prices which will be denoted by $F_t(\tau_1,\tau_2)$ frequently. Right at the beginning, let us exemplarily consider a swap contract which promises the delivery of one unit of electrical energy, say 1 MWh, over the future delivery period $[\tau_1,\tau_2]$. Hence, the underlying time partition for our upcoming analysis reads as $0 \le t \le \tau_1 < \tau_2 \le T$. According to Chapter 3 in [8] and paragraph 4.1 in [13], we interpret the electricity delivery as a flow of rate $S_t/(\tau_2-\tau_1)$ obviously yielding *cumulated costs* of the form

(3.2.21) $$\bar{c}(\tau_1,\tau_2) := \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} S_u\, du.$$

Therewith, we define the *electricity futures price* at time $t \in [0,\tau_1]$ via

(3.2.22)

$$F_t^{\mathcal{F},\mathbb{Q}}(\tau_1,\tau_2) := F_t(\tau_1,\tau_2) := \mathbb{E}_{\mathbb{Q}}(\bar{c}(\tau_1,\tau_2)|\mathcal{F}_t) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{F}_t)\, du.$$

Similarly to the announcements on the top of page 8 in [8], for $t \in [0,u]$ and $u \in [\tau_1,\tau_2]$ we might interpret $f(t,u) := \mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{F}_t)$ as "*a continuous stream of forward contracts with fixed delivery times*" u spread over the delivery period $[\tau_1,\tau_2]$. However, the forthcoming result actually corresponds to Proposition 3.1 in [8], whereas it is adapted to our notations. It yields an explicit representation for the electricity futures price under the risk-neutral probability $\mathbb{Q}$.

Proposition 3.2.2 *The price of an electricity futures $F_t := F_t(\tau_1,\tau_2)$ at time $t \in [0,\tau_1]$ with delivery period $[\tau_1,\tau_2]$ is given by the $\mathcal{F}_t$-adapted (local) $\mathbb{Q}$-martingale*[8]

(3.2.23)

$$F_t(\tau_1,\tau_2) = F_0(\tau_1,\tau_2) + \sum_{k=1}^{n}\int_0^t\int_{D_k} z\, \Lambda_k(s,\tau_1,\tau_2)\, d\tilde{N}_k^{\mathbb{Q}}(s,z)$$

within a deterministic and positive (time-inhomogeneous) volatility function

(3.2.24)
$$\Lambda_k(s,\tau_1,\tau_2) := \frac{w_k\, \sigma_k(s)}{(\tau_2-\tau_1)\,\lambda_k}\left[e^{-\lambda_k(\tau_1-s)} - e^{-\lambda_k(\tau_2-s)}\right]$$

[note that Λ_k even is strictly positive whenever $w_k \neq 0$] and a deterministic initial condition

(3.2.25) $$F_0(\tau_1,\tau_2) :=$$

$$\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\left[\mu(u) + \sum_{k=1}^{n} w_k\left(x_k\, e^{-\lambda_k u} + \int_0^u\int_{D_k} z\,\sigma_k(s)\, e^{-\lambda_k(u-s)}\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds\right)\right] du.$$

[8] Note that $F_t(\tau_1,\tau_2)$, such as given in (3.2.23), is not a Lévy-process but an *additive* or *Sato-process* (see [80] for details; also compare our former Remark 2.1.7), meaning that F possesses $\mathcal{F}$*-independent* but not necessarily *stationary* increments under $\mathbb{Q}$. Further, recall that F_t constitutes a $(\mathcal{F}_t,\mathbb{Q})$-martingale by definition.

Proof Similarly to the proof of Proposition 3.1 in [8], we put (3.2.9) into (3.2.22) and obtain

(3.2.26)

$$F_t(\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\left[\mu(u) + \sum_{k=1}^{n} w_k\, x_k\, e^{-\lambda_k u} + \sum_{k=1}^{n} w_k\, \mathbb{E}_{\mathbb{Q}}\left(\int_0^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{F}_t\right)\right] du.$$

Recalling (3.2.4), (3.2.20) and the formerly assumed independent increment property of the Sato-noises L_t^k, the conditional expectation in (3.2.26) becomes

$$\mathbb{E}_{\mathbb{Q}}\left(\int_0^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{F}_t\right) =$$

$$\int_0^t \int_{D_k} z\,\sigma_k(s)\, e^{-\lambda_k(u-s)}\, d\tilde{N}_k^{\mathbb{Q}}(s,z) + \int_0^u \int_{D_k} z\,\sigma_k(s)\, e^{-\lambda_k(u-s)}\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds.$$

Substituting the latter expression into (3.2.26) along with an interchange of the integration order, we get the desired result. ■

Remark 3.2.3 *Note that the volatility (3.2.24) is decreasing in* τ_1 *what easily follows from ordinary calculus (for a full proof see Lemma 3.5.2 below). Hence, if the delivery period starts very far in the future, the present arrival of new information has a more or less negligible influence on the futures price (3.2.23) – cf. p.111 in [13]. On the top of p.112 in [13] the authors argue that in this (long-term) case – contrarily to soonly maturing (short-term) contracts – "the market has [much] time [left] to adjust before delivery takes place". They further conclude that in such a long-term scenario the electricity futures price is "less sensitive to changes in the spot [price]" and thus, in any relevant model the futures price volatility should decrease with an increasing time to maturity. This characteristic feature, which obviously is met by our futures price model too, frequently is called (averaged) Samuelson effect throughout economists and finally stands in line with the findings in [13] (particularly see the pages 111, 122 and 126 therein).* ■

Adhering to Proposition 3.2 in [8], similar computations as in the proof of Proposition 3.2.2 above [but yet using (3.2.1), (3.2.2) and (3.2.8) instead of (3.2.9)] lead us to an expression for the electricity futures price $F_t(\tau_1,\tau_2)$ in terms of the Ornstein-Uhlenbeck noises $X_t^1,\dots,X_t^n$, reading

(3.2.27)

$$F_t(\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\left[\mu(u) + \sum_{k=1}^{n} w_k \int_t^u \int_{D_k} z\,\sigma_k(s)\, e^{-\lambda_k(u-s)}\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds\right] du + \sum_{k=1}^{n} \frac{\Lambda_k(t,\tau_1,\tau_2)}{\sigma_k(t)} X_t^k.$$

3.2.4 Electricity futures multi-factor call option prices

Before we turn to our main topic concerning electricity derivatives pricing under additional forward-looking information in the next paragraph, we extend the put option result of Proposition 4.1 in [8] yet to European *call* option prices in the current preparatory subsection – not at least to get familiar with Fourier transform techniques and a specific exponential damping argument such as presented in subsection 9.1.2 in [13], respectively in [14], [22]. In this regard, we now introduce a *risk-free asset*

(3.2.28)
$$\beta_t := \beta_0\, e^{rt}$$

which can be regarded as a *bond* (i.e. a *bank account*) with constant interest rate $r > 0$ and initial value $\beta_0 > 0$. As usual, we define the European call option payoff with strike price $K > 0$ via

(3.2.29)
$$C_T := C_T(K, \tau_1, \tau_2) := [F_T(\tau_1, \tau_2) - K]^+ := max\{0, F_T(\tau_1, \tau_2) - K\}.$$

Moreover, let us recall the *risk-neutral pricing formula* (see e.g. subsection 5.2.4 in [83]) reading

(3.2.30)
$$\frac{C_t}{\beta_t} = \mathbb{E}_{\mathbb{Q}}\left(\frac{C_T}{\beta_T}\middle|\mathcal{F}_t\right)$$

for $t \in [0, T]$. Verbalizing, the ($\mathcal{F}$-adapted) discounted call option price is required to form a (local) martingale under an EMM $\mathbb{Q}$. Anyway, within (3.2.28) and (3.2.29), the latter equation becomes

(3.2.31)
$$C_t = e^{-r(T-t)}\, \mathbb{E}_{\mathbb{Q}}([F_T(\tau_1, \tau_2) - K]^+|\mathcal{F}_t).$$

Following Definition 2.4.2, we meanwhile obtain a representation for the one-dimensional Fourier transform $\hat{f}$ associated to a real function $f \in \mathcal{L}^1(\mathbb{R})$, namely

(3.2.32)
$$\hat{f}(y) = \int_{\mathbb{R}} f(x)\, e^{-iyx} dx.$$

With respect to (2.4.3), its inverse is then given by

(3.2.33)
$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y)\, e^{iyx} dy.$$

Unfortunately, we are facing the fact $g(x) := [x - K]^+ \notin \mathcal{L}^1(\mathbb{R}^+)$, whereas for the exponentially-damped function $q(x) := e^{-ax} g(x) \in \mathcal{L}^1(\mathbb{R}^+)$ is valid within a real damping parameter $0 < a < \infty$ (compare the bottom of p. 248 in [13], resp. [14], [22]). By the way, we even observe $q(x) \in \mathcal{L}^1(\mathbb{R})$. However, starting with (3.2.32), a straightforward calculation delivers

(3.2.34)
$$\hat{q}(y) = \int_K^\infty e^{-(a+iy)x}\, [x - K]\, dx = \frac{e^{-(a+iy)K}}{(a+iy)^2}.$$

Herein, we have just used the fact $|e^{-iyx}| = 1$.

Now we are able to derive the risk-neutral price for a European call option written on the electricity futures (3.2.23) with delivery period $[\tau_1, \tau_2]$. We remark that there is no call option result (comparing to Proposition 3.2.4 below) given in [8].

Proposition 3.2.4 *Denoting the risk-free interest rate by r, the EURO price $C_t := C_t(K, \tau_1, \tau_2)$ at time $t \in [0, \tau_1]$ of a European electricity call option with strike price $K > 0$ (in EURO) written on an electricity futures with delivery period $[\tau_1, \tau_2]$ is given by*

(3.2.35)

$$C_t = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}^+} \frac{e^{(a+iy)\{F_t(\tau_1,\tau_2)-K\}}}{(a+iy)^2} \prod_{k=1}^{n} e^{\psi_k(y,t,T)} \, dy$$

within a characteristic exponent

(3.2.36)

$$\psi_k(y,t,T) := \int_t^T \int_{D_k} \left[e^{(a+iy)\, z\, \Lambda_k(s,\tau_1,\tau_2)} - 1 - (a+iy)\, z\, \Lambda_k(s,\tau_1,\tau_2) \right] e^{h_k(s,z)} \rho_k(s) \, d\nu_k(z) \, ds.$$

Proof According to (3.2.31) and the definition of $q(\cdot)$, we obtain

(3.2.37)

$$C_t = e^{-r(T-t)} \, \mathbb{E}_{\mathbb{Q}}\left(e^{aF_T(\tau_1,\tau_2)} \, q\left(F_T(\tau_1,\tau_2)\right) \middle| \mathcal{F}_t \right)$$

whereas (3.2.33) immediately yields

(3.2.38)

$$C_t = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}^+} \hat{q}(y) \, \mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy)\, F_T(\tau_1,\tau_2)} \middle| \mathcal{F}_t \right) dy.$$

Exploiting the independent increment property of F, the above conditional expectation transforms into

(3.2.39)

$$\mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy)\, F_T(\tau_1,\tau_2)} \middle| \mathcal{F}_t \right) = e^{(a+iy)\, F_t(\tau_1,\tau_2)} \, \mathbb{E}_{\mathbb{Q}}\left[e^{(a+iy)\, \{F_T(\tau_1,\tau_2) - F_t(\tau_1,\tau_2)\}} \right].$$

Substituting (3.2.23) into (3.2.39), with pair-wise independent PRMs $\widetilde{N}_k^{\mathbb{Q}}$ $(k = 1, \ldots, n)$ we deduce

(3.2.40)

$$\mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy)\, F_T(\tau_1,\tau_2)} \middle| \mathcal{F}_t \right) = e^{(a+iy)\, F_t(\tau_1,\tau_2)} \prod_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[e^{\int_t^T \int_{D_k} (a+iy)\, z\, \Lambda_k(s,\tau_1,\tau_2)\, d\widetilde{N}_k^{\mathbb{Q}}(s,z)} \right].$$

If we rewrite the $\mathbb{Q}$-compensated jump-integral in (3.2.40) via (3.2.20) as "Lévy-Itô-decomposed", then, with respect to (3.2.36), the deterministic abbreviation $\theta_k(s) := (y - ia)\, \Lambda_k(s, \tau_1, \tau_2) \in \mathbb{C}$ and Proposition 2.1 in [13] combined with Prop. 1.9 in [65] (resp. with Prop. 8 in [35]), we receive

(3.2.41)

$$\mathbb{E}_{\mathbb{Q}}\left[exp\left(\int_t^T \int_{D_k} (a+iy)\, z\, \Lambda_k(s,\tau_1,\tau_2)\, d\widetilde{N}_k^{\mathbb{Q}}(s,z) \right) \right] = \mathbb{E}_{\mathbb{Q}}\left[exp\left(i \int_t^T \int_{D_k} z\, \theta_k(s)\, d\widetilde{N}_k^{\mathbb{Q}}(s,z) \right) \right]$$

$$= exp\left\{ \int_t^T \int_{D_k} \left[e^{i\, z\, \theta_k(s)} - 1 - i\, z\, \theta_k(s) \right] e^{h_k(s,z)} \rho_k(s) \, d\nu_k(z) \, ds \right\} = e^{\psi_k(y,t,T)}.$$

Putting (3.2.34), (3.2.40) and (3.2.41) into (3.2.38), the claimed result is verified.[9] ■

3.3 Modeling electricity risk under future information

In this section we devote our attention towards the pricing of electricity swap contracts with respect to some additional future information that informed market participants might have knowledge of. That is, as proposed in [10], in our upcoming considerations we will take forward-looking information into account via adequate enlargements of the underlying information filtrations in order to develop extended pricing mechanisms for electricity market insiders. In this context, once more, we emphasize that the previously introduced filtration $\mathcal{F}_t$ does only *look into the past* whereby all available information coming from market observations up to time t is stored in this monotone-increasing family of *retro* sigma-algebras. Mathematically spoken, the filtration $\mathcal{F}_t$ is (as usual) supposed to be generated by the spot price noises up to time t, in symbols

(3.3.1) $$\mathcal{F}_t \coloneqq \sigma\{S_u: 0 \leq u \leq t\} \coloneqq \sigma\{L_u^1, \dots, L_u^n: 0 \leq u \leq t\}.$$

Unfortunately, this traditional financial approach (which, by the way, neither might be appropriate for most *asset* pricing problems) does not at all reflect the case at hand when we are concerned with pricing applications in an energy market for a *non-storable* commodity such as electricity [10]. To be precise, one could think of a market situation wherein some additional (but still stochastic) future information is available: For instance, the market participants might know that a specific future event will take place with certainty but the exact effects remain random [10]. Taking such forward-looking knowledge into account, as a consequence, we would have to *enlarge* the information filtration $\mathcal{F}_t$ [10]. Hence, similar to [10], we now introduce the flow of additionally available market information at time t including forward-looking events by an enlarged filtration $\mathcal{G}_t$ with $\mathcal{G}_t \supset \mathcal{F}_t$. Exemplarily, the above mentioned *future event* could consist of political decisions like the planned introduction of CO2-emission costs next year, an outage of a major power plant during the next month, the building of new connecting cables to other electricity markets or simply noise-afflicted weekly weather forecasts [10]. All these future events have one thing in common, as they altogether reduce the uncertainty concerning future energy price behavior whereas the exact effects still remain random [10].

3.3.1 The information premium in electricity markets

In accordance to (3.2.22), we newly define the *electricity futures price* at time t associated to a delivery of a certain predetermined amount of electrical energy over the period $[\tau_1, \tau_2]$ within a time partition $0 \leq t \leq \tau_1 < \tau_2 \leq T$ yet under the enlarged filtration $\mathcal{G}_t$ by (the $\mathcal{G}_t$-adapted $\mathbb{Q}$-martingale)

(3.3.2) $$F_t^{\mathcal{G},\mathbb{Q}}(\tau_1, \tau_2) \coloneqq \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \mathbb{E}_{\mathbb{Q}}(S_u | \mathcal{G}_t)\, du.$$

[9] Note that on the bottom of p.12 in [8] the expression for $\tilde{\psi}_{t,T}^i(\theta)$ is derived – similarly to (3.2.41) – by an application of the *generalized* Lévy-Khinchin formula allowing for *time-dependent* deterministic functions $\theta(s)$. However, there is an inaccuracy in the use of $\tilde{\psi}_{t,T}^i\big(y\, \Sigma_i(\cdot; T_1, T_2)\big)$ in Prop. 4.1 in [8], as on the bottom of p.12 $\tilde{\psi}_{t,T}^i$ is associated with L_i, whereas in the second last line of the proof of Prop. 4.1, at the same time, $\tilde{\psi}_{t,T}^i$ is associated with the $\mathbb{Q}$-compensated (and thus, drifted) process $\bar{L}_i$. More precisely, in the second last line of the proof of Prop. 4.1 in [8] there is a summand of the form $-iyz\, \Sigma_i(s; T_1, T_2)$ missing inside the contained $\hat{\nu}_i(dz, ds)$-integral. Anyway, the correct put option price formula corresponding to Prop. 4.1 in [8] can be obtained similarly to our proof of Proposition 3.2.4 above (even with $a = 0$ therein).

Similar to before, we may introduce the *forward* price $f^{\mathcal{G},\mathbb{Q}}(t,u) := \mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{G}_t)$ which obviously designates a $(\mathcal{G}_t,\mathbb{Q})$-martingale in $t \in [0,\tau_1]$ for every fixed $u \in [\tau_1,\tau_2]$, but not necessarily a $(\mathcal{F}_t,\mathbb{Q})$-martingale. Moreover, slightly deviating from [10][10], we define the *information premium* by

(3.3.3) $$\mathfrak{I}_t^{\mathcal{G},\mathcal{F},\mathbb{Q}}(\tau_1,\tau_2) := \mathfrak{I}_t^{\mathbb{Q}}(\tau_1,\tau_2) := F_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2) - F_t^{\mathcal{F},\mathbb{Q}}(\tau_1,\tau_2).$$

With respect to (3.2.22) and (3.3.2), we immediately obtain

(3.3.4)
$$\mathfrak{I}_t^{\mathbb{Q}}(\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \left[\mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{G}_t) - \mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{F}_t)\right] du.$$

Parallel to page 6 in [10], the information premium (3.3.4) may be associated with an orthogonal projection of the random variable $\mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{G}_t)$ on the space $\mathcal{L}^2(\mathcal{F}_t,\mathbb{Q})$, since the tower property yields

$$\mathbb{E}_{\mathbb{Q}}\big(\mathfrak{I}_t^{\mathbb{Q}}(\tau_1,\tau_2)\big|\mathcal{F}_t\big) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\left[\mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{G}_t)\big|\mathcal{F}_t\big) - \mathbb{E}_{\mathbb{Q}}(S_u|\mathcal{F}_t)\right] du = 0$$

for all $0 \le t \le \tau_1$. Roughly speaking, the information premium measures the supplementary information gain which is contained in $\mathcal{G}_t$ compared with that in $\mathcal{F}_t$ [10], respectively the *information asymmetry* in between $\mathcal{G}_t$- and $\mathcal{F}_t$-electricity futures prices.

Following [10], we suppose the market participants to have knowledge of some insider information about the spot price behavior at a fixed future time τ. More precise, we now assume that the traders have an idea about the (still stochastic) future mean-level (3.2.10) of the electricity spot price. Exemplarily, it should sound reasonable to expect the introduction of CO2-emission costs at a future time τ to essentially influence the mean-level η_τ (compare the arguing in section 2.2 in [10]). Since the seasonality function μ remains deterministic, this additional knowledge merely affects the jump noises $L_\tau^1, \dots, L_\tau^l$ $(l < n)$ which are responsible for a random perturbation of the floor-function μ (as explained in subsection 3.2.1). Slightly extending the setting on p.12 in [10], we introduce the *overall filtration*

(3.3.5) $$\mathcal{H}_t := \mathcal{F}_t \vee \sigma\{X_\tau^1, \dots, X_\tau^l\} := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^l\}$$

representing *complete* market information at time t about the long-term level of the spot price, where *τ-forward-looking* events are included. That is, the enlarged sigma-algebra $\mathcal{H}$ can be associated with *exhaustive* knowledge of the future electricity spot price trend-line η_τ. Parallel to [10], we next assume

(3.3.6) $$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t$$

for $0 \le t < \tau$, whereas $\mathcal{F}_t = \mathcal{G}_t$ holds for all $t \ge \tau$. For later purposes, we recall the explicit notation

(3.3.7) $$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^l\}$$

perhaps being more intuitive, since the upper and lower *bounds* in between which $\mathcal{G}_t$ is situated can be read off directly. Following the above algebraic structures, the filtration $\mathcal{K}_t := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^n\}$ would hence correspond to exhaustive/exact knowledge (at time t) concerning the accurate spot price value at time τ, namely S_τ, including both long- and short-term variations.

In order to examine the information premium (3.3.3) more precisely and to derive option price

[10] On the contrary, Benth and Meyer-Brandis [10] consider the *information premium* under the measure $\mathbb{P}$. In addition, they restrict their examinations to electricity *forward* contracts, whereas we newly treat the electricity *futures/swap* case here, explicitly allowing for a *delivery period* instead of a fixed *maturity date*, merely.

formulas for electricity derivatives later, we need the following results which are closely connected to our former announcements in paragraph 2.3, actually. In other words, the upcoming Lemma 3.3.1 inter alia provides customized $\mathbb{Q}$-martingale compensators (so-called *information yields*; see Def. 2.4 in [10]) associated to the enlarged filtrations $\mathcal{H}$ and $\mathcal{G}$ such as introduced in (3.3.5), respectively (3.3.7). With respect to the information yield context, let us quote from page 7 in [10]: "*The extra information added to the filtration coming from knowledge of future states of the market leads thus to essentially the same result as changing a probability measure, namely introducing a drift* [which we will call information yield in this thesis].". This statement explains the correspondence between a changing of probability measure and an enlargement of filtration in a very comprehensive manner.

Condition A *Combining (3.2.4) with (3.2.20), we see that L_t^k is (not a Lévy- but) a Sato- or additive process under $\mathbb{Q}$, admitting independent but non-stationary increments. As the results of section 2.3 hold for* Lévy-*processes only, we ought to ensure that $L^1, \dots, L^l$ also have* stationary *increments, if we work with enlarged filtrations of the type (3.3.5) – (3.3.7). Thus, whenever we model additional future information as in the latter equations in this thesis, then for $k = 1, \dots, l$ we simultaneously assume $\rho_k(s) \equiv \rho_k \geq 0$ and $h_k(s,z) := h_k(z)$ inside (3.2.20) to be time-independent from now on (which is not necessary for $k = l+1, \dots, n$ actually), such that $L^1, \dots, L^l$ become* Lévy-*processes under $\mathbb{Q}$. Fortunately, choosing* constant *long-term/daily jump-intensities $\rho_1, \dots, \rho_l$ does not at all stand in contrast to the economical practice as described on page 4 in [8] (also see Remark 3.2.1 above).* ■

Lemma 3.3.1 ***(a)*** *For an intermediate filtration $\mathcal{G}_t$ as given in (3.3.7) the stochastic processes*

(3.3.8)
$$\hat{L}_t^k := L_t^k - \int_0^t \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k \middle| \mathcal{G}_s\right)}{\tau - s} ds$$

depict $(\mathcal{G}_t, \mathbb{Q})$-martingales for all $k = 1, \dots, l$ and $t \in [0, \tau[$.[11]

(b) *For an overall filtration $\mathcal{H}_t$ as given in (3.3.5) the stochastic processes*

(3.3.9)
$$\tilde{L}_t^k := L_t^k - \int_0^t \frac{L_\tau^k - L_s^k}{\tau - s} ds$$

constitute $(\mathcal{H}_t, \mathbb{Q})$-martingales for all $k = 1, \dots, l$ and $t \in [0, \tau[$.

(c) *For all $k = 1, \dots, l$ and time indices $0 \leq t \leq s < \tau$ we have the equality*

(3.3.10)
$$\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k \middle| \mathcal{G}_t\right) = \frac{\tau - s}{\tau - t} \, \mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right).$$

(d) *For $k = 1, \dots, l$ the (stochastic) $(\mathcal{G}, \mathbb{Q})$-compensator of $dN_k(s,z)$ is given by*

(3.3.11)
$$d\nu_k^{\mathcal{G},\mathbb{Q}}(s,z) := \frac{1}{\tau - s} \, \mathbb{E}_{\mathbb{Q}}\left(\int_{u=s}^{u=\tau} dN_k(u,z) \middle| \mathcal{G}_s\right) ds$$

whereas the $(\mathcal{G}, \mathbb{Q})$-compensated random measure[12] *is thus of the form*

(3.3.12)
$$d\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(s,z) := dN_k(s,z) - d\nu_k^{\mathcal{G},\mathbb{Q}}(s,z).$$

---

[11] Note that for <u>one fixed</u> $k^* \in \{1, \dots, l\}$ and $\hat{\mathcal{G}}_t$ with $\mathcal{F}_t \subset \hat{\mathcal{G}}_t \subset \mathcal{F}_t \vee \sigma\{L_\tau^{k^*}\}$ the stochastic process $\hat{L}_t^{k^*}$ such as given in (3.3.8) even depicts a $(\hat{\mathcal{G}}_t, \mathbb{Q})$-martingale, being a stronger conclusion [as long as we (reasonably) presume $\sigma\{L_\tau^{k^*}\} \subset \sigma\{L_\tau^1, \dots, L_\tau^l\}$ and likewise, $\hat{\mathcal{G}}_t \subset \mathcal{G}_t \subset \mathcal{H}_t$] than the one actually given in Lemma 3.3.1 (a). Similarly, $\tilde{L}_t^{k^*}$ [see (3.3.9)] not only is a $\mathcal{H}_t$-martingale then, but even a $(\mathcal{H}_t \supset)\ \mathcal{F}_t \vee \sigma\{L_\tau^{k^*}\}$-martingale under $\mathbb{Q}$.

[12] Since (3.3.11) is stochastic, (3.3.12) is not a (compensated) *Poisson* random measure in the classical sense – compare the sequel of Example 16.38 in [32]. Moreover, all conditional expectations $\mathbb{E}(\dots|\mathcal{H}_s)$ inside "(16.166) – (16.168), (16.170), (16.172) and Cor. 16.41 in [32]" are unnecessary; this is not the case in Prop. 16.53 in [32].

Proof **(a)** See Lemma 3.3 in [10] – additionally, compare Proposition 16.52 in [32].

(b) See Proposition 2.3.1 above. [By the way, the statement in (b) trivially follows from (a) if we replace the intermediate filtration $\mathcal{G}_s$ $(\supset \mathcal{F}_s)$ in (3.3.8) by the global filtration $\mathcal{H}_s$ $(\supset \mathcal{F}_s)$ and hereafter apply the *taking out what is known rule* for conditional expectations.]

(c) See Proposition A.3 in [10] with $g(u) := \frac{1}{\tau-u}$ and $f(u) \equiv 1$, together with Remark A.4 in [10] with $L := L^k$. Also compare p.13 in [10].

(d) The proof of Proposition 16.53 in [32] (with z replaced by any arbitrary bounded and deterministic function $f_k(z)$ which vanishes in any small environment of zero and that determines a measure on each set D_k with weight zero whenever $z \longrightarrow 0^+$, respectively with z replaced by $f_k(z)$ invertible and integrable with respect to $\nu_k^{\mathcal{G},\mathbb{Q}}$) here applies equally. Nevertheless, we want to give the following additional explanatory comments: Merging (3.2.4) into the $(\mathcal{G}_t, \mathbb{Q})$-martingale (3.3.8) while identifying (3.3.11) and (3.3.12), we get the claimed result via

$$L_t^k - \int_0^t \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k \middle| \mathcal{G}_s\right)}{\tau - s} ds = \int_0^t \int_{D_k} z \, dN_k(s,z) - \int_0^t \mathbb{E}_{\mathbb{Q}}\left(\int_{u=s}^{u=\tau} \int_{D_k} \frac{z}{\tau - s} \, dN_k(u,z) \middle| \mathcal{G}_s \right) ds$$

$$= \int_0^t \int_{D_k} z \left[dN_k(s,z) - \frac{1}{\tau - s} \mathbb{E}_{\mathbb{Q}}\left(\int_{u=s}^{u=\tau} dN_k(u,z) \middle| \mathcal{G}_s \right) ds \right] = \int_0^t \int_{D_k} z \, d\widetilde{N}_k^{\mathcal{G},\mathbb{Q}}(s,z) \, . \blacksquare$$

Remark 3.3.2 *Note that for $k = l+1, \ldots, n$ the PRMs $\widetilde{N}_k^{\mathcal{F},\mathbb{Q}}(t,z)$ in (3.2.20) are (not only $\mathcal{F}_t$-adapted but also) $\mathcal{G}_t$-adapted $\mathbb{Q}$-martingale-integrators, since $\mathcal{F}_t \subset \mathcal{G}_t$ holds true for all* $0 \leq t < \tau$. ■

For $0 \leq t \leq \tau_1 < \tau \leq \tau_2$ the information premium can be computed further by applying standard transformation rules for conditional expectations. Having (3.3.1) and (3.3.7) in mind, we concretely argue as follows: Since (by assumption) the stochastic components $X_u^{l+1}, \ldots, X_u^n$ are $\mathbb{Q}$-independent of $L_u^1, \ldots, L_u^l$, conditioning the sum $\sum_{k=l+1}^n w_k \, X_u^k$ on $\mathcal{G}_t$ coincides with conditioning the latter on $\mathcal{F}_t$. Moreover, as every X_t^k is $\mathcal{F}_t$-adapted by definition, each X_t^k simultaneously is $\mathcal{G}_t$-adapted, since X_t^k depicts a measurable mapping, say $X_t^k \colon (\Omega, \mathcal{F}_t) \longrightarrow (\mathfrak{X}, \mathfrak{B})$, whereas $\left(X_t^k\right)^{-1}(\mathcal{B}) \in \mathcal{F}_t \subset \mathcal{G}_t$ holds for all Borel-sets $\mathcal{B} \in \mathfrak{B}$ and time indices $0 \leq t \leq \tau_1$. Thus, merging (3.2.14) into (3.3.4), we initially receive

(3.3.13)

$$\mathfrak{I}_t^{\mathbb{Q}}(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \sum_{k=1}^{l} w_k \left[\mathbb{E}_{\mathbb{Q}}\left(X_u^k \middle| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{Q}}\left(X_u^k \middle| \mathcal{F}_t\right) \right] du.$$

Appealing to (3.2.8), the difference of conditional expectations inside (3.3.13) transforms into

(3.3.14)

$$\mathbb{E}_{\mathbb{Q}}\left(X_u^k \middle| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{Q}}\left(X_u^k \middle| \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s) \, e^{-\lambda_k(u-s)} \, dL_s^k \middle| \mathcal{G}_t \right) - \mathbb{E}_{\mathbb{Q}}\left[\int_t^u \sigma_k(s) \, e^{-\lambda_k(u-s)} \, dL_s^k \right].$$

In the following, we presume $u < \tau$ (more precisely $\tau_1 \leq u < \tau \leq \tau_2$) which induces $t \leq s \leq u < \tau$. Consequently, we may apply Lemma 3.3.1 (a) and (c) what leads us to

(3.3.15)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{G}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k \middle| \mathcal{G}_s\right)}{\tau - s}\, ds \middle| \mathcal{G}_t\right)$$
$$= \int_t^u \frac{\sigma_k(s)\, e^{-\lambda_k(u-s)}}{\tau - s}\, \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k \middle| \mathcal{G}_s\right) \middle| \mathcal{G}_t\right) ds$$
$$= \int_t^u \frac{\sigma_k(s)\, e^{-\lambda_k(u-s)}}{\tau - s}\, \mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k \middle| \mathcal{G}_t\right) ds = \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right)}{\tau - t} \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, ds.$$

Moreover, with respect to (3.2.4), (3.2.20) and Condition A, the usual expectation in (3.3.14) becomes

(3.3.16)

$$\mathbb{E}_{\mathbb{Q}}\left[\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k\right] = \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)}\, e^{h_k(z)}\, \rho_k\, d\nu_k(z)\, ds.$$

Substituting (3.3.15) and (3.3.16) into (3.3.14), we deduce

(3.3.17)

$$\mathbb{E}_{\mathbb{Q}}\left(X_u^k \middle| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{Q}}\left(X_u^k \middle| \mathcal{F}_t\right) =$$
$$\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)} \left[\frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right)}{\tau - t} - \int_{D_k} z\, e^{h_k(z)}\, \rho_k\, d\nu_k(z)\right] ds.$$

Hence, merging (3.3.17) into (3.3.13), for $0 \leq t \leq \tau_1 < \tau \leq \tau_2$ the information premium points out as

(3.3.18)

$$\mathfrak{I}_t^{\mathbb{Q}}(\tau_1, \tau_2) =$$
$$\frac{1}{\tau_2 - \tau_1} \sum_{k=1}^{l} w_k \int_{\tau_1}^{\tau_2} \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)} \left[\frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right)}{\tau - t} - \rho_k \int_{D_k} z\, e^{h_k(z)}\, d\nu_k(z)\right] ds\, du.$$

Actually, (3.3.18) extends equality "(3.3) in [10]" yet to a consideration under a *risk-neutral* measure $\mathbb{Q}$ and to the case of electricity *futures* explicitly permitting a delivery period.

Recalling definition (3.2.24) while setting $\mathfrak{m} := \mathfrak{m}(s) := max\{s, \tau_1\}$, within an application of the *stochastic Fubini-Tonelli theorem* (similarly to the argumentation in e.g. the proof of Proposition 4.14 in [13]; also see reference "*Folland (1984)* in [13]") we ultimately derive

(3.3.19)

$$\mathfrak{I}_t^{\mathbb{Q}}(\tau_1, \tau_2) = \sum_{k=1}^{l} \int_t^{\tau_2} \frac{\tau_2 - \mathfrak{m}(s)}{\tau_2 - \tau_1}\, \Lambda_k(s, \mathfrak{m}(s), \tau_2) \left[\frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right)}{\tau - t} - \rho_k \int_{D_k} z\, e^{h_k(z)} d\nu_k(z)\right] ds$$

whenever $0 \leq t \leq \tau_1 < \tau \leq \tau_2$.

On the contrary, for $\tau \le \tau_1$ the information premium can be computed as follows: For a time partition $0 \le t \le \tau \le \tau_1 \le u \le \tau_2$ we firstly recall the inclusions $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}_\tau = \mathcal{F}_\tau$. Next, parallel to the arguing on page 14 in [10], within a simple application of the tower property the difference of conditional expectations inside equation (3.3.13) yet can be transformed into

(3.3.20)
$$\mathbb{E}_{\mathbb{Q}}\big(X_u^k\big|\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{Q}}\big(X_u^k\big|\mathcal{F}_t\big) =$$
$$\mathbb{E}_{\mathbb{Q}}\big(X_\tau^k\big|\mathcal{G}_t\big) + \mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}\big(X_u^k - X_\tau^k\big|\mathcal{F}_\tau\big)\big|\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{Q}}\big(X_\tau^k\big|\mathcal{F}_t\big) - \mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}\big(X_u^k - X_\tau^k\big|\mathcal{F}_\tau\big)\big|\mathcal{F}_t\big).$$

Further on, from (3.2.8) we deduce

$$\mathbb{E}_{\mathbb{Q}}\big(X_u^k - X_\tau^k\big|\mathcal{F}_\tau\big) = X_\tau^k\big[e^{-\lambda_k(u-\tau)} - 1\big] + \mathbb{E}_{\mathbb{Q}}\left[\int_\tau^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k\right].$$

Merging the latter expression into (3.3.20), we receive

(3.3.21)
$$\mathbb{E}_{\mathbb{Q}}\big(X_u^k\big|\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{Q}}\big(X_u^k\big|\mathcal{F}_t\big) = e^{-\lambda_k(u-\tau)}\big[\mathbb{E}_{\mathbb{Q}}\big(X_\tau^k\big|\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{Q}}\big(X_\tau^k\big|\mathcal{F}_t\big)\big].$$

Implanting (3.3.21) inside (3.3.13) while identifying (3.2.24), we finally end up with

(3.3.22)
$$\mathfrak{I}_t^{\mathbb{Q}}(\tau_1, \tau_2) = \sum_{k=1}^{l} \frac{\Lambda_k(\tau, \tau_1, \tau_2)}{\sigma_k(\tau)}\big[\mathbb{E}_{\mathbb{Q}}\big(X_\tau^k\big|\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{Q}}\big(X_\tau^k\big|\mathcal{F}_t\big)\big]$$

yielding the information premium for $\tau \le \tau_1$. Actually, property (3.3.22) extends equation “(3.4) in [10]” to the ($\mathbb{Q}$-*risk-neutral*) electricity *futures* case.

Remark 3.3.3 *Since $\mathcal{F}_t = \mathcal{G}_t$ whenever $t \ge \tau$, we observe that the information premium vanishes for all time indices $t \ge \tau$ [recall (3.3.4) or (3.3.22) to see this]. The latter fact seems quite natural from an economical point of view, as in this case the additional information either consists of present $(t = \tau)$ or of past $(t > \tau)$ information, none being any longer relevant in the context of forward-looking insider trading, of course (cf. p.12 in [10]). Yet, parallel to p.14 in [10], the information premium (3.3.22) tends to zero as τ_2 approaches infinity (for fixed τ and τ_1). Hence, for $\tau \le \tau_1$ the supplementary information impact [given through (3.3.22)] approximately vanishes for futures contracts with long delivery periods ending far in the future (i.e. $\tau_2 \to \infty$), which also sounds economically reasonable. At this step, we recall that a vanishing information premium corresponds to an equality between $\mathcal{F}$- and $\mathcal{G}$-futures prices [compare (3.3.3)]. In conclusion, also for our specific $\mathbb{Q}$-risk-neutral futures setup (admitting delivery periods) the above observations widely stand in line with the findings in [10], which themselves are related to forward contracts under $\mathbb{P}$, on the opposite.* ■

The risk premium In addition to the above *information premium* examinations, we likewise aim to study the so-called *risk premium* for our current pure-jump electricity market model. Thus, slightly extending Definition 2.2 in [10], for $0 \le t \le \tau_1 < \tau \le \tau_2$ we define the *risk premium* via

$$\mathfrak{R}_t^{\mathcal{G}}(\tau_1, \tau_2) := \mathfrak{R}_t^{\mathcal{G},\mathbb{P},\mathbb{Q}}(\tau_1, \tau_2) := F_t^{\mathcal{G},\mathbb{Q}}(\tau_1, \tau_2) - F_t^{\mathcal{G},\mathbb{P}}(\tau_1, \tau_2).$$

Verbalizing, the *risk premium* measures the difference between two kinds of electricity futures prices, once computed with respect to the true market measure $\mathbb{P}$, and once with respect to the risk-neutral measure $\mathbb{Q}$, while both futures prices are taken under the enlarged filtration $\mathcal{G}$. Loosely speaking, comparing the *risk premium* with the *information premium* [such as defined in (3.3.3)], one might declare the former to measure a certain kind of ($\mathcal{G}$-forward-looking) $\mathbb{P}$-$\mathbb{Q}$-difference, whereas the latter captures a ($\mathbb{Q}$-risk-neutral) $\mathcal{F}$-$\mathcal{G}$-difference in the underlying electricity futures prices. By the way, these observations entirely justify the used vocabulary. In what follows, let us devote our attention to some further calculations concerning the risk premium.
Initially, taking (3.2.1), (3.2.2), (3.2.8) and (3.3.2) into account, we obtain

$$\mathfrak{R}_t^{\mathcal{G}}(\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\left\{\mathbb{E}_{\mathbb{Q}}\left(\sum_{k=1}^{n} w_k \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \,\middle|\, \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{P}}\left(\sum_{k=1}^{n} w_k \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \,\middle|\, \mathcal{G}_t\right)\right\} du.$$

As above, we presume $0 \le t \le \tau_1 \le u < \tau \le \tau_2$ in the following (what induces $0 \le t \le s < \tau$). However, we treat the conditional expectations appearing inside the latter equation separately: Using (3.2.4), (3.2.20), (3.3.8) and (3.3.10), the first object therein transforms into

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}&\left(\sum_{k=1}^{n} w_k \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \,\middle|\, \mathcal{G}_t\right)\\ &= \sum_{k=1}^{l} \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k \middle| \mathcal{G}_t\right) - L_t^k}{\tau - t} \int_t^u w_k\, \sigma_k(s)\, e^{-\lambda_k(u-s)} ds\\ &\quad + \sum_{k=l+1}^{n} \int_t^u \int_{D_k} w_k\, z\, \sigma_k(s)\, e^{-\lambda_k(u-s)}\, e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds.\end{aligned}$$

Similar computations (but under $\mathbb{P}$ yet) yield for the second conditional expectation

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}&\left(\sum_{k=1}^{n} w_k \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \,\middle|\, \mathcal{G}_t\right) =\\ &= \sum_{k=1}^{l} \frac{\mathbb{E}_{\mathbb{P}}\left(L_\tau^k \middle| \mathcal{G}_t\right) - L_t^k}{\tau - t} \int_t^u w_k\, \sigma_k(s)\, e^{-\lambda_k(u-s)} ds\\ &\quad + \sum_{k=l+1}^{n} \int_t^u \int_{D_k} w_k\, z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} \rho_k(s)\, d\nu_k(z)\, ds.\end{aligned}$$

Collecting the two latter decompositions, the risk premium actually points out as

$$\begin{aligned}\mathfrak{R}_t^{\mathcal{G}}(\tau_1,\tau_2) &= \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\int_t^u \left\{\sum_{k=1}^{l} w_k\, \sigma_k(s)\, e^{-\lambda_k(u-s)}\, \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k \middle| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{P}}\left(L_\tau^k \middle| \mathcal{G}_t\right)}{\tau - t}\right.\\ &\quad \left. + \sum_{k=l+1}^{n} \int_{D_k} w_k\, z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} \left[e^{h_k(s,z)} - 1\right] \rho_k(s)\, d\nu_k(z)\right\} ds\, du.\end{aligned}$$

Applying the Fubini-Tonelli theorem [parallel to our former argumentation in (3.3.19)] and hereafter, identifying the volatility function (3.2.24), we ultimately receive

$$\mathfrak{R}_t^{\mathcal{G}}(\tau_1,\tau_2) = \int_t^{\tau_2} \frac{\tau_2 - \mathfrak{m}(s)}{\tau_2-\tau_1}\left(\sum_{k=1}^{l} \Lambda_k(s,\mathfrak{m}(s),\tau_2)\ \frac{\mathbb{E}_{\mathbb{Q}}\big(L_\tau^k\big|\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{P}}\big(L_\tau^k\big|\mathcal{G}_t\big)}{\tau - t} + \sum_{k=l+1}^{n} \Lambda_k(s,\mathfrak{m}(s),\tau_2)\,\rho_k(s)\int_{D_k} z\left[e^{h_k(s,z)}-1\right] d\nu_k(z)\right) ds$$

wherein we have just made use of the abbreviation $\mathfrak{m}(s) \coloneqq max\{s,\tau_1\}$. Examining the final risk premium representation in more depth, we see very clear how the formerly announced measuring of ($\mathcal{G}$-forward-looking) $\mathbb{P}$-$\mathbb{Q}$-differences of electricity futures prices works in detail, namely: the first $\#l$ stochastic processes involved in our electricity spot price model (3.2.1) generate the differences $\mathbb{E}_{\mathbb{Q}}\big(L_\tau^k\big|\mathcal{G}_t\big) - \mathbb{E}_{\mathbb{P}}\big(L_\tau^k\big|\mathcal{G}_t\big)$ $(k=1,\dots,l)$, whereas the remaining jump noises which were indexed by $k = l+1,\dots,n$ lead to the deterministic risk premium 'ingredients' $e^{h_k(s,z)}-1$ on the opposite. Note that if the random components L_τ^k $(k=1,\dots,l)$ were $\mathcal{G}_t$-measurable, then the risk premium would become deterministic. For $\mathcal{G}_t$ replaced by $\mathcal{H}_t$ [as defined in (3.3.5)] this would be the case, actually. However, in the light of Lemma 3.3.1 (b) this fact not at all constitutes a surprising observation.

3.3.2 Electricity swap prices under future information

In the present work we aim to compute option prices for electricity derivatives under supplementary forward-looking information – a procedure which, to the best of our knowledge, has not been done yet throughout the literature in a comparable way. Hence, in order to evaluate options that, in particular, are written on the electricity futures price (3.3.2), it appears worthwhile to provide a representation for $F_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2)$ initially. For this purpose, we put (3.2.14) into (3.3.2) yielding

(3.3.23)

$$F_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\left\{\mu(u) + \mathbb{E}_{\mathbb{Q}}\left(\sum_{k=1}^{l} w_k X_u^k\,\middle|\,\mathcal{G}_t\right) + \mathbb{E}_{\mathbb{Q}}\left(\sum_{k=l+1}^{n} w_k X_u^k\,\middle|\,\mathcal{G}_t\right)\right\} du$$

where $t \in [0,\tau_1]$. In the following, we will treat the two conditional expectations appearing in (3.3.23) separately: Remembering (3.2.8), the first object therein can be transformed into

(3.3.24)

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{k=1}^{l} w_k X_u^k\,\middle|\,\mathcal{G}_t\right) = \sum_{k=1}^{l} w_k\left\{X_t^k\, e^{-\lambda_k(u-t)} + \mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k\,\middle|\,\mathcal{G}_t\right)\right\}.$$

From now on, we suppose $u < \tau$ [more precisely $\tau_1 \le u < \tau \le \tau_2$]. Then, successively applying Lemma 3.3.1 (a), the Fubini-Tonelli theorem and the tower property, equation (3.3.24) becomes

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{k=1}^{l} w_k X_u^k\,\middle|\,\mathcal{G}_t\right) = \sum_{k=1}^{l} w_k\left\{X_t^k\, e^{-\lambda_k(u-t)} + \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\ \frac{\mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_s^k\big|\mathcal{G}_t\big)}{\tau - s}\, ds\right\}.$$

Finally, an application of Lemma 3.3.1 (c) delivers [note in passing that $0 \leq t \leq s \leq u < \tau$ holds]

(3.3.25)

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{k=1}^{l} w_k X_u^k \middle| \mathcal{G}_t\right) = \sum_{k=1}^{l} w_k \left\{X_t^k \, e^{-\lambda_k(u-t)} + \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right)}{\tau - t} \int_t^u \sigma_k(s) \, e^{-\lambda_k(u-s)} \, ds\right\}.$$

Moreover, the second conditional expectation in (3.3.23) can be rewritten as

(3.3.26)

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{k=l+1}^{n} w_k X_u^k \middle| \mathcal{G}_t\right) = \sum_{k=l+1}^{n} w_k \, \mathbb{E}_{\mathbb{Q}}\left(X_u^k \middle| \mathcal{F}_t\right)$$

since conditioning the appearing integrand on $\mathcal{G}_t$ coincides with conditioning it on $\mathcal{F}_t$ [as precisely explained in the sequel of Remark 3.3.2]. Hence, within (3.2.4), (3.2.8) and (3.2.20) we obtain

(3.3.27)

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{k=l+1}^{n} w_k X_u^k \middle| \mathcal{G}_t\right) = \sum_{k=l+1}^{n} w_k \left\{X_t^k \, e^{-\lambda_k(u-t)} + \int_t^u \int_{D_k} z \, \sigma_k(s) \, e^{-\lambda_k(u-s)} \, e^{h_k(s,z)} \, \rho_k(s) \, d\nu_k(z) \, ds\right\}.$$

Substituting (3.3.25) and (3.3.27) into our futures price equality (3.3.23) while identifying (3.2.24), we ultimately end up with the laborious expression

(3.3.28)

$$\begin{aligned} F_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2) = & \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2 - \tau_1} du + \sum_{k=1}^{n} \frac{\Lambda_k(t,\tau_1,\tau_2)}{\sigma_k(t)} X_t^k \\ & + \sum_{k=1}^{l} \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right)}{\tau - t} \, \frac{w_k}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \int_t^u \sigma_k(s) \, e^{-\lambda_k(u-s)} \, ds \, du \\ & + \sum_{k=l+1}^{n} \frac{w_k}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \int_t^u \int_{D_k} z \, \sigma_k(s) \, e^{-\lambda_k(u-s)} \, e^{h_k(s,z)} \rho_k(s) \, d\nu_k(z) \, ds \, du. \end{aligned}$$

Once more, within (3.2.24) and the abbreviation $\mathfrak{m} \coloneqq \mathfrak{m}(s) \coloneqq max\{s, \tau_1\}$, an application of the stochastic Fubini-Tonelli theorem [as in (3.3.19) above] yet on (3.3.28) yields

$$F_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2 - \tau_1} du + \sum_{k=1}^{n} \frac{\Lambda_k(t,\tau_1,\tau_2)}{\sigma_k(t)} X_t^k$$

$$+ \int_t^{\tau_2} \frac{\tau_2 - \mathfrak{m}}{\tau_2 - \tau_1} \left[\sum_{k=1}^{l} \Lambda_k(s,\mathfrak{m},\tau_2) \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right)}{\tau - t} + \sum_{k=l+1}^{n} \Lambda_k(s,\mathfrak{m},\tau_2) \int_{D_k} z \, e^{h_k(s,z)} \rho_k(s) \, d\nu_k(z)\right] ds.$$

Furthermore, introducing the deterministic functions

(3.3.29)

$$\Gamma(t) := \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2-\tau_1} du - \sum_{k=l+1}^{n} \int_{\tau_1}^{\tau_2} \int_{u}^{t} \int_{D_k} \frac{w_k\, \sigma_k(s)}{\tau_2-\tau_1} z\, e^{-\lambda_k(u-s)}\, e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds\, du,$$

$$\Psi_k(t) := \frac{\Lambda_k(t,\tau_1,\tau_2)}{\sigma_k(t)}, \qquad \Phi_k(t) := \int_{\tau_1}^{\tau_2} \int_{u}^{t} \frac{w_k\, \sigma_k(s)\, e^{-\lambda_k(u-s)}}{(t-\tau)(\tau_2-\tau_1)} ds\, du \left(= \int_{\tau_2}^{t} \frac{\Lambda_k(s,\mathfrak{m}(s),\tau_2)}{t-\tau} ds \right)$$

and the random variables

(3.3.30)
$$Z_t^k := \mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_t^k \big| \mathcal{G}_t\big)$$

for $t \le \tau_1 < \tau$ the electricity futures price disposition (3.3.28) can be written in shorthand notation as

(3.3.31)

$$F_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2) = \Gamma(t) + \sum_{k=1}^{n} \Psi_k(t)\, X_t^k + \sum_{k=1}^{l} \Phi_k(t)\, Z_t^k.$$

Denoting the derivation with respect to t by an inverted comma, Itô's product rule leads us to

(3.3.32)

$$dF_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2) = \left[\Gamma'(t) + \sum_{k=1}^{n} \Psi_k'(t)\, X_t^k + \sum_{k=1}^{l} \Phi_k'(t)\, Z_t^k\right] dt + \sum_{k=1}^{n} \Psi_k(t)\, dX_t^k + \sum_{k=1}^{l} \Phi_k(t)\, dZ_t^k$$

whereas (3.3.29) delivers [note in passing that $\mathfrak{m}(t) := t \vee \tau_1 = \tau_1$, as $t \in [0,\tau_1]$ by assumption]

(3.3.33)

$$\Psi_k'(t) = \lambda_k\, \Psi_k(t), \qquad \Phi_k'(t) = \frac{\Lambda_k(t,\tau_1,\tau_2) - \Phi_k(t)}{t-\tau},$$

$$\Gamma'(t) = -\sum_{k=l+1}^{n} \int_{D_k} z\, \Lambda_k(t,\tau_1,\tau_2)\, e^{h_k(t,z)} \rho_k(t)\, d\nu_k(z).$$

Merging (3.2.3), (3.3.29) and (3.3.33) into (3.3.32), we further receive

(3.3.34)

$$dF_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2) = \left[\Gamma'(t) + \sum_{k=1}^{l} \Phi_k'(t)\, Z_t^k\right] dt + \sum_{k=1}^{n} \Lambda_k(t,\tau_1,\tau_2)\, dL_t^k + \sum_{k=1}^{l} \Phi_k(t)\, dZ_t^k.$$

In our next step, we aim to express the dynamics (3.3.34) in terms of $(\mathcal{G}_t,\mathbb{Q})$-martingale integrators. For this purpose, we state the following result.

Lemma 3.3.4 *For Z_t^k such as defined in (3.3.30), the stochastic process $\big(Z_t^k/(\tau-t)\big)_{t\in[0,\tau[}$ embodies a $(\mathcal{G}_t,\mathbb{Q})$-martingale for all indices $k = 1,\dots,l$ and time parameters $t \in [0,\tau[$.*

Proof Obviously, $\frac{Z_t^k}{\tau-t} \in \mathcal{L}^1(\mathcal{G}_t, \mathbb{Q})$ for all $t \in [0,\tau[$, i.e. Z_t^k is $\mathbb{Q}$-integrable and $\mathcal{G}_t$-adapted. Further, for a partition $0 \le t \le s < \tau$ we deduce the claimed result within the tower property and (3.3.10) via

(3.3.35)

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{Z_s^k}{\tau-s}\middle|\mathcal{G}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\frac{L_\tau^k - L_s^k}{\tau-s}\middle|\mathcal{G}_{s\wedge t}\right) = \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k\middle|\mathcal{G}_t\right)}{\tau-s} = \frac{Z_t^k}{\tau-t}. \blacksquare$$

Moreover, Itô's product rule leads us to the stochastic differential equation

(3.3.36)

$$dZ_t^k = (\tau-t)\, d\left(\frac{Z_t^k}{\tau-t}\right) - \frac{Z_t^k}{\tau-t}\, dt.$$

Substituting (3.2.4), (3.2.20), (3.3.8), (3.3.33) and (3.3.36) into the dynamics (3.3.34), we eventually obtain the $\mathcal{G}$-*forward-looking* electricity futures price $\mathbb{Q}$-representation [associated to the case $u < \tau$]

(3.3.37)

$$dF_t^{\mathcal{G},\mathbb{Q}}(\tau_1,\tau_2) = \sum_{k=1}^{l} \Lambda_k(t,\tau_1,\tau_2)\left\{dL_t^k - \frac{Z_t^k}{\tau-t}dt\right\} + \sum_{k=l+1}^{n} \Lambda_k(t,\tau_1,\tau_2) \int_{D_k} z\, \widetilde{N}_k^{\mathcal{F},\mathbb{Q}}(t,dz)$$
$$+ \sum_{k=1}^{l} \Phi_k(t)\,(\tau-t)\, d\left(\frac{Z_t^k}{\tau-t}\right), \qquad t \in [0,\tau_1],\ \ \tau_1 < \tau,$$

with vanishing drift. Hence, with respect to Lemma 3.3.1 (a), Remark 3.3.2 and Lemma 3.3.4, the futures price (3.3.37) constitutes a $\mathcal{G}_t$-adapted (local) martingale under $\mathbb{Q}$. In the light of its definition in (3.3.2), this fact, however, is not a surprising result. In conclusion, (3.3.37) essentially extends our former *backward-looking* futures price representation (3.2.23), respectively Proposition 3.1 in [8], and (to the best of our knowledge) cannot be found in the literature elsewhere.

At this step, we recall that in the sequel of (3.3.24) we have presumed $u < \tau$. Complementarily, we now examine the case $u \ge \tau$. To claim our findings right at the beginning, we announce that also for $u \ge \tau$ we receive a $\mathcal{G}$-forward-looking futures price representation that very closely resembles the one in (3.3.37), actually. Starting off, we presently assume a time partition $0 \le t \le \tau_1 < \tau \le u \le \tau_2$. Next, we decompose the conditional expectation appearing on the right hand side of (3.3.24) via

$$\mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{G}_t\right)$$
$$= \mathbb{E}_{\mathbb{Q}}\left(\int_t^{\tau-} \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{G}_t\right) + \mathbb{E}_{\mathbb{Q}}\left(\int_\tau^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{G}_t\right) =: \mathfrak{S}_1 + \mathfrak{S}_2$$

where $k = 1,\dots,l$. Note that inside $\mathfrak{S}_1$ we have $t \le s < \tau$ so that we evidently are allowed to apply Lemma 3.3.1 (a) and (3.3.10) here. Vice versa, inside $\mathfrak{S}_2$ we observe time parameters $t < \tau \le s \le u$ inducing the inclusions $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{G}_\tau = \mathcal{F}_\tau$. As a consequence of the latter observations, we are led to

$$\mathfrak{S}_1 = \frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k\middle|\mathcal{G}_t\right)}{\tau-t} \int_t^\tau \sigma_k(s)\, e^{-\lambda_k(u-s)}\, ds,$$

$$\mathfrak{S}_2 = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(\int_\tau^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{F}_\tau\right)\middle| \mathcal{G}_t\right) = \mathbb{E}_{\mathbb{Q}}\left[\int_\tau^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k\right].$$

Referring to (3.2.4), (3.2.20) and Condition A, for $k = 1, \dots, l$ we further conclude

$$\mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{G}_t\right) =$$

$$\frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t\right)}{t-\tau}\int_\tau^t \sigma_k(s)\, e^{-\lambda_k(u-s)}\, ds + \int_\tau^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)}\, e^{h_k(z)}\, \rho_k\, d\nu_k(z)\, ds$$

while the last summand is deterministic and independent of t, remarkably. Moreover, similar computations as in (3.3.23) – (3.3.37) currently lead us to exactly the same futures price dynamics [but yet associated to the case $u \geq \tau$] as formerly given in (3.3.37) – except from a new function

$$\widetilde{\Phi}_k(t) := \int_{\tau_1}^{\tau_2} \int_\tau^t \frac{w_k\, \sigma_k(s)\, e^{-\lambda_k(u-s)}}{(t-\tau)(\tau_2-\tau_1)}\, ds\, du \quad \left(= \int_\tau^t \frac{\Lambda_k(s,\tau_1,\tau_2)}{t-\tau}\, ds \right)$$

instead of $\Phi_k(t)$. Evidently, the only difference between $\widetilde{\Phi}_k(t)$ and $\Phi_k(t)$ [as defined in (3.3.29)] can be detected in the lower integration bound of the inner integral. Thus, with respect to our upcoming option pricing purposes, we conclude that it is not really necessary to differ between the cases $u < \tau$ and $u \geq \tau$, since both instances induce (essentially) the same futures price dynamics. For the sake of notational simplicity, we will always assume $u < \tau$ in our proceedings, unless stated otherwise.

Nevertheless, the representation (3.3.37) is not really suitable for electricity derivatives pricing, since it is not at all clear whether the contained random variables Z_t^k still are Sato-processes under $\mathbb{Q}$ and thus, possess independent increments with respect to $\mathcal{G}_t$. In fact, the latter depicts a convenient property when it comes to the evaluation of conditional expectations in the context of risk-neutral option pricing. Moreover, it does not seem to be possible to provide an explicit representation neither for the dynamics dZ_t^k in terms of L_t^k and thus, nor for $d\left(Z_t^k/(\tau - t)\right)$. Unfortunately, we hence cannot hope for any explicit pricing formula, as long as we do not permit some additional structure to the (actually *non-explicit*) intermediate filtration $\mathcal{G}_t$. Looking at (3.3.6), we poorly know that $\mathcal{G}_t$ contains a bit more information than $\mathcal{F}_t$ and a bit less than $\mathcal{H}_t$ (which, by the way, also might cause some difficulties concerning the modeling of available future information in practical applications). In this regard, we announce the following *key idea* which actually has originated from Lemma 3.3.1 (b): If we replace (the *non-explicit* filtration) $\mathcal{G}_t$ by another intermediate filtration $\mathcal{G}_t^*$ which *explicitly* consists of a subfamily of the components appearing in $\mathcal{H}_t$, concretely defining

(3.3.38) $$\mathcal{G}_t^* := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^p\}$$

with $1 \leq p \leq l < n$, then $\mathcal{F}_t \subset \mathcal{G}_t^* \subset \mathcal{H}_t$ for $t < \tau$ and $\mathcal{G}_t^* = \mathcal{F}_t$ for $t \geq \tau$ still hold true. Putting $p = l$ yet would correspond to $\mathcal{G}_t^* = \mathcal{H}_t$ and thus to – in the sense of (3.3.5) – *complete* or *exhaustive* knowledge of the electricity spot price mean-level at a future time τ. In contrast, the case $p < l$ represents a scenario wherein the market participants merely have access to some restricted additional knowledge about the future long-term spot price behavior, sounding more realistically.

More importantly, in accordance to Condition A and Prop. 2.3.1, resp. Lemma 3.3.1 (b), the process

(3.3.39)

$$L_t^k - \int_0^t \frac{L_\tau^k - L_s^k}{\tau - s}\, ds$$

constitutes a $(\mathcal{G}_t^*, \mathbb{Q})$-martingale for all $k = 1, \dots, p$ and $t \in [0, \tau[$. By the way, substituting (3.2.4) into (3.3.39) and hereafter applying the stochastic Fubini-Tonelli theorem, we get the decomposition

$$L_t^k - \int_0^t \frac{L_\tau^k - L_s^k}{\tau - s}\, ds = [1 - ln(\tau - t)]\, L_t^k + ln(\tau - t)\, L_\tau^k - \int_t^\tau \int_{D_k} z\, ln(\tau - s)\, dN_k(s, z)$$

for all $t \in [0, \tau[$. Furthermore, taking (3.2.8) and (3.3.29) into account, similar arguments as in (3.3.23) – (3.3.28) yield the following electricity futures price expression (yet under $\mathcal{G}_t^*$) reading

(3.3.40)

$$\begin{aligned} F_t^{\mathcal{G}^*, \mathbb{Q}}(\tau_1, \tau_2) &= \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \left\{ \mu(u) + \sum_{k=1}^{p} w_k\, \mathbb{E}_{\mathbb{Q}}\big(X_u^k \big| \mathcal{G}_t^*\big) + \sum_{k=p+1}^{n} w_k\, \mathbb{E}_{\mathbb{Q}}\big(X_u^k \big| \mathcal{F}_t\big) \right\} du \\ &= \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2 - \tau_1}\, du + \sum_{k=1}^{p} \int_{\tau_1}^{\tau_2} \frac{w_k}{\tau_2 - \tau_1}\, \mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{G}_t^* \right) du \\ &\quad + \sum_{k=p+1}^{n} \int_{\tau_1}^{\tau_2} \frac{w_k}{\tau_2 - \tau_1}\, \mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{F}_t \right) du + \sum_{k=1}^{n} \Psi_k(t)\, X_t^k. \end{aligned}$$

In what follows, we presume $u < \tau$. Then, with respect to (3.3.10) [but for $\mathcal{G}_t^*$ now; also compare Lemma 3.5.1 in this context] and (3.3.39), the first conditional expectation becomes

(3.3.41)

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{G}_t^* \right) &= \int_t^u \frac{\sigma_k(s)\, e^{-\lambda_k(u-s)}}{\tau - s}\, \mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_s^k \big| \mathcal{G}_t^*\big)\, ds \\ &= \mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_t^k \big| \mathcal{G}_t^*\big) \int_t^u \frac{\sigma_k(s)\, e^{-\lambda_k(u-s)}}{\tau - t}\, ds \end{aligned}$$

($k = 1, \dots, p$), whereas – somewhat similar to (3.3.16) – the second one transforms into

(3.3.42)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k \middle| \mathcal{F}_t \right) = \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds$$

($k = p + 1, \dots, n$). Merging (3.3.41) and (3.3.42) into (3.3.40), with respect to (3.3.29) we deduce

(3.3.43)

$$F_t^{\mathcal{G}^*, \mathbb{Q}}(\tau_1, \tau_2) = \tilde{\Gamma}(t) + \sum_{k=1}^{n} \Psi_k(t)\, X_t^k + \sum_{k=1}^{p} \Phi_k(t)\, \mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_t^k \big| \mathcal{G}_t^*\big)$$

wherein we have just introduced the shorthand notation

(3.3.44)

$$\tilde{\Gamma}(t) := \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2-\tau_1} du - \sum_{k=p+1}^{n} \int_{\tau_1}^{\tau_2} \int_{u}^{t} \int_{D_k} \frac{w_k\, z\, \sigma_k(s)\, e^{-\lambda_k(u-s)}}{\tau_2-\tau_1} e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds\, du.$$

We recall that L_t^k is $\mathcal{G}_t^*$-measurable for every index $k \in \{1, \dots, p\}$, since $\mathcal{F}_t \subset \mathcal{G}_t^*$ holds true. Similarly, every L_τ^k is $\mathcal{G}_t^*$-measureable for all indices $k = 1, \dots, p$, since L_τ^k trivially is $\sigma\{L_\tau^k\}$-measureable and, in addition, for a certain (fixed) index $k \in \{1, \dots, p\}$ the inclusions

$$\sigma\{L_\tau^k\} \subset \sigma\{L_\tau^1, \dots, L_\tau^p\} \subset \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^p\} = \mathcal{G}_t^*$$

are valid. Hence, taking out what is known, for each $k = 1, \dots, p$ we obtain

(3.3.45) $$\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t^*\right) = L_\tau^k - L_t^k = \int_t^\tau \int_{D_k} z\, dN_k(s,z).$$

With respect to the derivation methodology of property (3.3.37) – but now using (3.3.39), (3.3.43) and (3.3.45) – within $t \in [0, \tau_1]$, $\tau_1 < \tau$, we eventually get the dynamics

(3.3.46) $$dF_t^{\mathcal{G}^*,\mathbb{Q}}(\tau_1,\tau_2) =$$

$$\sum_{k=1}^{p} [\Lambda_k(t,\tau_1,\tau_2) - \Phi_k(t)] \left(dL_t^k - \frac{L_\tau^k - L_t^k}{\tau - t} dt \right) + \sum_{k=p+1}^{n} \Lambda_k(t,\tau_1,\tau_2) \int_{D_k} z\, \tilde{N}_k^{\mathcal{F},\mathbb{Q}}(t,dz)$$

depicting a $\mathcal{G}_t^*$-adapted martingale under $\mathbb{Q}$, obviously.[13] Comparing (3.3.37) with (3.3.46), we claim that the latter representation not only yields a better overview, but also that $\mathcal{G}^*$ should be more appropriate than $\mathcal{G}$ for practical applications (as mentioned above). Anyway, in accordance to (3.3.39) and Lemma 3.3.1 (d), the $(\mathcal{G}^*, \mathbb{Q})$-compensator of $dN_k(s,z)$ for $k = 1, \dots, p$ yet is given by

(3.3.47)

$$d\nu_k^{\mathcal{G}^*,\mathbb{Q}}(s,z) := \frac{1}{\tau - s} \int_{u=s}^{u=\tau} dN_k(u,z)\, ds$$

whereas the $(\mathcal{G}^*, \mathbb{Q})$-compensated random measure (RM)[14] is thus of the form

(3.3.48) $$d\tilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(s,z) := dN_k(s,z) - d\nu_k^{\mathcal{G}^*,\mathbb{Q}}(s,z).$$

In conclusion, combining (3.3.39) with (3.3.47) and (3.3.48), we finally obtain the linking equality

(3.3.49)

$$L_t^k - \int_0^t \frac{L_\tau^k - L_s^k}{\tau - s} ds = \int_0^t \int_{D_k} z\, d\tilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(s,z).$$

---

[13] Recall that (3.3.46) is related to $u < \tau$ [compare the prolog of (3.3.41)]. On the contrary, for $u \geq \tau$ we may split the integral inside (3.3.41) [such as shown in the sequel of (3.3.37)] what leads us to exactly the same dynamics as claimed in (3.3.46) but with $\Phi_k(t)$ therein replaced by $\tilde{\Phi}_k(t)$ [as defined previously to (3.3.38)].

[14] Compare the footnote dedicated to (3.3.12) in this context. Also verify that we presently are in a very similar setting as presented in Example 16.38 in [32].

3.3.3 Forward-looking electricity call option prices

Let us now concentrate on the derivation of risk-neutral *forward-looking* prices for electricity options written on the futures (3.3.46). In accordance to (3.2.29), we define the $\mathcal{G}^*$-*forward-looking* call option payoff at time T (which is *not* identical with the T introduced in sect. 3.2) with strike price $K > 0$ by

$$C_T^{\mathcal{G}^*} \coloneqq C_T^{\mathcal{G}^*}(K,\tau_1,\tau_2) \coloneqq \left[F_T^{\mathcal{G}^*,\mathbb{Q}}(\tau_1,\tau_2) - K\right]^+ \coloneqq max\left\{0, F_T^{\mathcal{G}^*,\mathbb{Q}}(\tau_1,\tau_2) - K\right\}.$$

However, similarly to (3.2.31), for $0 \le t \le T$ the adjusted risk-neutral pricing formula now reads as

$$C_t^{\mathcal{G}^*} = e^{-r(T-t)}\, \mathbb{E}_{\mathbb{Q}}\left(\left[F_T^{\mathcal{G}^*,\mathbb{Q}}(\tau_1,\tau_2) - K\right]^+ \middle| \mathcal{G}_t^*\right). \tag{3.3.50}$$

Within a real function $q(x) \coloneqq e^{-ax}\,[x - K]^+ \in \mathcal{L}^1(\mathbb{R}^+)$ [as formerly introduced in subsection 3.2.4] and a shorthand notation $F_t^{\mathcal{G}^*,\mathbb{Q}} \coloneqq F_t^{\mathcal{G}^*,\mathbb{Q}}(\tau_1,\tau_2)$, parallel to (3.2.37) we consequently deduce

$$C_t^{\mathcal{G}^*} = e^{-r(T-t)}\, \mathbb{E}_{\mathbb{Q}}\left(e^{aF_T^{\mathcal{G}^*,\mathbb{Q}}} q\left(F_T^{\mathcal{G}^*,\mathbb{Q}}\right) \middle| \mathcal{G}_t^*\right) \tag{3.3.51}$$

whereas the inverse Fourier transform (3.2.33) further yields

$$C_t^{\mathcal{G}^*} = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}^+} \hat{q}(y)\, e^{(a+iy)F_t^{\mathcal{G}^*,\mathbb{Q}}}\, \mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy)\left[F_T^{\mathcal{G}^*,\mathbb{Q}} - F_t^{\mathcal{G}^*,\mathbb{Q}}\right]} \middle| \mathcal{G}_t^*\right) dy \tag{3.3.52}$$

with $\hat{q}(y)$ as announced in (3.2.34). Setting $\Lambda_k(s) \coloneqq \Lambda_k(s,\tau_1,\tau_2)$ and $\Xi_k(s,z) \coloneqq z\,[\Lambda_k(s) - \Phi_k(s)]$ while using the decomposition (3.3.46) along with the linking equality (3.3.49), we next obtain

$$\mathbb{E}_{\mathbb{Q}}\left(exp\left\{(a+iy)\left[F_T^{\mathcal{G}^*,\mathbb{Q}} - F_t^{\mathcal{G}^*,\mathbb{Q}}\right]\right\} \middle| \mathcal{G}_t^*\right) = \tag{3.3.53}$$

$$\mathbb{E}_{\mathbb{Q}}\left(exp\left\{(a+iy)\left[\sum_{k=1}^{p}\int_t^T\int_{D_k} \Xi_k(s,z)\, d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(s,z) + \sum_{k=p+1}^{n}\int_t^T\int_{D_k} z\,\Lambda_k(s)\, d\widetilde{N}_k^{\mathcal{F},\mathbb{Q}}(s,z)\right]\right\} \middle| \mathcal{G}_t^*\right).$$

Unfortunately, this conditional expectation does not reduce to a usual one, since the involved price process $F^{\mathcal{G}^*,\mathbb{Q}}$ does not possess independent increments with respect to $\mathcal{G}^*$ – neither for $(t <)\, \tau \le T$ (which is obvious), nor for $\tau > T$ (remarkably). Actually, the appearance of $\widetilde{N}^{\mathcal{G}^*,\mathbb{Q}}$ in (3.3.53) avoids the latter property, since for each $k \in \{1,\dots,p\}$ both $\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}$ and $\mathcal{G}^*$ contain L_τ^k. To verify this, we ought to remind (3.3.49) and then compare (3.3.46) with (3.3.38). In conclusion, (3.3.53) neither reduces to a usual expectation, nor can be handled by the Lévy-Khinchin formula (similarly to our arguing in the proof of Prop. 3.2.4). By the way, this neither would be possible, even if (3.3.53) reduced to a usual expectation, as the contained $(\mathcal{G}^*,\mathbb{Q})$-compensator $\nu^{\mathcal{G}^*,\mathbb{Q}}$, such as given in (3.3.47), [in contrast to the deterministic $(\mathcal{F},\mathbb{Q})$-compensator in (3.2.20)] is stochastic yet. Thus, the precise *analytical* treatment (if there is any appropriate at all) of the conditional expectation in (3.3.53) presently is a standing problem. Anyway, in section 3.5.1 we present a possible handling of *usual* expectations of the type

$$\mathbb{E}_{\mathbb{Q}}\left[exp\left\{(a+iy)\left[F_T^{\mathcal{G}^*,\mathbb{Q}} - F_t^{\mathcal{G}^*,\mathbb{Q}}\right]\right\}\right] \tag{3.3.54}$$

in more detail. Yet, in order to treat the much more challenging *conditional* expectation in (3.3.53), we propose two (in this context new) methods involving results from Complex analysis.

Excursus A: On the evaluation of forward-looking conditional expectations using complex power series expansions and linear approximation schemes

In this excursus we aim to examine forward-looking conditional expectations – such as appearing in (3.3.51), for example – in more depth, whereas we provide a customized evaluation method involving complex power series expansions and linear approximation schemes, innovatively. To begin with, we recall that the most challenging problem in (3.3.51) obviously consists in finding a proper handling of the conditional expectation

(A.1)
$$\mathbb{E}_{\mathbb{Q}}\left(e^{aF_T^{\mathcal{G}^*,\mathbb{Q}}}q\left(F_T^{\mathcal{G}^*,\mathbb{Q}}\right)\Big|\mathcal{G}_t^*\right)$$

within an enlarged filtration $\mathcal{G}_t^*$ such as implemented in (3.3.38), a deterministic real function $q(x) := e^{-ax}\,[x - K]^+ \in \mathcal{L}^1(\mathbb{R}^+)$ and a real damping parameter $0 < a < \infty$. Moreover, in accordance to (3.3.46) and (3.3.49), the forward-looking electricity futures price process $F_t^* := F_t^{\mathcal{G}^*,\mathbb{Q}}(\tau_1,\tau_2)$ satisfies the following $(\mathcal{G}_t^*,\mathbb{Q})$-martingale decomposition[15]

(A.2)
$$F_t^* = F_0^* + \sum_{k=1}^{p}\int_0^t\int_{D_k}\Xi_k(s,z)\,d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(s,z) + \sum_{k=p+1}^{n}\int_0^t\int_{D_k}\Lambda_k(s)\,z\,d\widetilde{N}_k^{\mathcal{F},\mathbb{Q}}(s,z)$$

where $0 \le t \le T$. Herein, we presume $\Lambda_k(s) := \Lambda_k(s,\tau_1,\tau_2)$ and $\Xi_k(s,z) := z\,[\Lambda_k(s) - \Phi_k(s)]$, as before. Appealing to (3.2.25), we further assume the initial value F_0^* to be deterministic. Next, applying (3.2.33) on (A.1), [somewhat similar to (3.3.52)] we derive the equality

(A.3)
$$\mathbb{E}_{\mathbb{Q}}\left(e^{aF_T^*}\,q(F_T^*)\middle|\mathcal{G}_t^*\right) = \frac{1}{2\pi}\int_{\mathbb{R}^+}\hat{q}(y)\,\mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy)F_T^*}\middle|\mathcal{G}_t^*\right)dy.$$

Hence, instead of (A.1), we may equivalently examine the conditional expectation on the right hand side of (A.3) in the sequel. Since the ingredients a, y and F_T^* altogether are real-valued, we declare the object $z_T := a_T + iy_T := aF_T^* + iyF_T^* = (a+iy)F_T^*$ to designate a (stochastic) complex number, i.e. $z_T \in \mathbb{C}$. Further, we introduce a *holomorphic*[16] function $f: \mathbb{C} \longrightarrow \mathbb{C}$ via $f(\zeta) := e^{\zeta}$ which can be developed into a power series/Taylor series due to

(A.4)
$$f(\zeta) = \sum_{\nu=0}^{\infty}\frac{\zeta^\nu}{\nu!}$$

with convergence radius

(A.5)
$$\mathfrak{R} = \left(\lim\,sup_{\nu\to\infty}\,\frac{1}{\sqrt[\nu]{\nu!}}\right)^{-1} = \infty.$$

[15] At this step, it appears interesting to compare equation (A.2) with the corresponding *backward-looking* futures price representation in (3.2.23), respectively with equality "(3.2) in [8]", as it may help us to understand how forward-looking information (modeled by enlarged filtrations) is weaved into futures prices, actually.

[16] A complex function $f: \mathbb{C} \longrightarrow \mathbb{C}$ is called *holomorphic* (respectively, *analytic*) at $z_0 \in \mathbb{C}$, if and only if there exists an open environment of z_0, say $U(z_0)$, wherein f is differentiable. If f is holomorphic over the whole complex plane, then it is frequently called an *entire function* (compare page 118 in [42]).

Thus, the representation (A.4) is valid on the whole complex plane $\mathbb{C}$. (For further reading on complex power series see e.g. Chapter V in [42] or Chapter II in [67].[17]) Returning to our original topic, we utilize (A.4) to obtain the approximation

(A.6) $$\mathcal{C}_{\mathbb{Q}}^{\mathcal{G}^*}(F_T^*; t, a, y) := \mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy)F_T^*}\middle|\mathcal{G}_t^*\right) = \mathbb{E}_{\mathbb{Q}}(f(z_T)|\mathcal{G}_t^*) =$$

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{\nu=0}^{\infty}\frac{(z_T)^\nu}{\nu!}\middle|\mathcal{G}_t^*\right) = \sum_{\nu=0}^{\infty}\frac{(a+iy)^\nu}{\nu!}\mathbb{E}_{\mathbb{Q}}((F_T^*)^\nu|\mathcal{G}_t^*) = \lim_{c\to\infty}\sum_{\nu=0}^{c}\frac{(a+iy)^\nu}{\nu!}\mathbb{E}_{\mathbb{Q}}((F_T^*)^\nu|\mathcal{G}_t^*) \approx \mathcal{T}_d + \mathcal{R}_d$$

wherein the d-th order Taylor-polynomial is given by

(A.7)
$$\mathcal{T}_d := \mathcal{T}_d(F_T^*; a, y, \mathcal{G}_t^*, \mathbb{Q}) := 1 + (a+iy)\,F_t^* + \sum_{\nu=2}^{d}\frac{(a+iy)^\nu}{\nu!}\mathbb{E}_{\mathbb{Q}}((F_T^*)^\nu|\mathcal{G}_t^*)$$

and the Lagrange-type approximation error (the so-called *remainder term*) possesses the structure

(A.8)
$$\mathcal{R}_d := \mathcal{R}_d(F_T^*; a, y, \xi) := \frac{e^\xi}{(d+1)!}\,(a+iy)^{d+1}\,(F_T^*)^{d+1}.$$

Herein, $\xi \in \mathbb{C}$ is such as $|\xi| \le F_T^*\sqrt{a^2+y^2}$ holds $\mathbb{Q}$-a.s. Additionally, we presume $0 < F_T^* \le M < \infty$ to be valid $\mathbb{Q}$-a.s. from now on[18], whereby M depicts a strictly positive constant. Hence, we observe $|\xi| \le M\sqrt{a^2+y^2}$ and, more importantly, $|\mathcal{R}_d| \to 0$ for $d \to \infty$. Anyway, regarding (A.7), we should devote our attention towards the computation of $\mathbb{E}_{\mathbb{Q}}((F_T^*)^\nu|\mathcal{G}_t^*)$ for indices $\nu = 2, \dots, d$ in the following. For this purpose, we firstly introduce a family of bounded and real-valued polynomials

$$\{g_\nu(x) := x^\nu \mid x \in [0, M] \subset \mathbb{R}^+; \nu = 0,1,\dots,d\}.$$

However, in order to treat the objects $\mathbb{E}_{\mathbb{Q}}((F_T^*)^\nu|\mathcal{G}_t^*)$ appearing in (A.7), one might propose to apply Itô's formula on $g_\nu(F_T^*)$ $(\nu = 2, \dots, d)$ in order to derive a representation of the latter in terms of stochastic integrals. Unfortunately, neither the incoming infinite sum nor the remaining conditional expectation seems to be analytically tractable, as the underlying dynamics (A.2) are rather demanding. Nevertheless, from (A.2) we know that F^* designates a $(\mathcal{G}^*, \mathbb{Q})$-martingale so that, by definition,

$$\mathbb{E}_{\mathbb{Q}}(F_T^*|\mathcal{G}_t^*) = F_t^*$$

is valid for all $0 \le t \le T$. Inspired by this observation, we yet propose a *linear* interpolation scheme to approximate $\mathbb{E}_{\mathbb{Q}}((F_T^*)^\nu|\mathcal{G}_t^*)$ for indices $\nu = 2, \dots, d$ adequately. To begin with, we implement a (not necessarily equidistant) partition of the real interval $[0, M]$ via $\mathfrak{P} := \{0 = x_0 < x_1 < \dots < x_m = M\}$, whereas we define the mesh Δ due to

$$\Delta := \Delta(\mathfrak{P}) := \max_{0\le j\le m-1}|x_{j+1} - x_j|.$$

[17] Most of the Complex analysis results used in Excursus A and B are taken from several lectures (which the author has attended) on complex function theory, power series theory and Complex analysis held by Prof. Wolfgang Luh at the University of Trier in 2004 – 2006.

[18] Having simulated various electricity futures price paths $(F_t^*: 0 \le t \le T)$ in practice, for an applicant it should not cause any further trouble to choose a reasonable (sufficiently large) upper bound M so that $F_T^* \in]0, M]$ is met within a probability close to one.

Actually, our key idea is to approximate the convex polynomial functions $g_\nu(x)$ $(\nu = 2, \dots, d)$ in each interval $[x_j, x_{j+1}]$ $(j = 0, \dots, m-1)$ by its particular *secants*

(A.9)

$$s_j^\nu(x) = \frac{x_{j+1}^\nu - x_j^\nu}{x_{j+1} - x_j}(x - x_j) + x_j^\nu.$$

Then, for $\nu = 2, \dots, d$ and $j = 0, \dots, m-1$ the approximation error ε_j^ν in each interval $[x_j, x_{j+1}]$ is given by the difference $\varepsilon_j^\nu \coloneqq \varepsilon_j^\nu(x) \coloneqq s_j^\nu(x) - g_\nu(x)$ which is bounded through

(A.10)
$$0 \le \varepsilon_j^\nu \le \frac{\nu\,(\nu-1)\,(x_{j+1} - x_j)^2}{8}\, x_{j+1}^{\nu-2}.$$

Obviously, the right hand side of (A.10) vanishes, as the mesh becomes finer, i.e. as Δ approaches zero. Consequently, we likewise deduce $\varepsilon_j^\nu \to 0$, whenever $\Delta \to 0$. In conclusion, we announce $s_j^\nu(x) \to g_\nu(x)$ for $\Delta \to 0$, whenever $x \in [x_j, x_{j+1}]$. These observations lead us to the approximation

(A.11)

$$g_\nu(x) = x^\nu \approx \sum_{j=0}^{m-1} s_j^\nu(x)\, \mathbb{1}_{]x_j, x_{j+1}]}(x)$$

wherein $x \in]0, M]$ and $\nu = 2, \dots, d$. Thus, taking (A.9), (A.11) and the $(\mathcal{G}^*, \mathbb{Q})$-martingale property of F^* into account, for $\nu = 2, \dots, d$ and $j = 0, \dots, m-1$ we may estimate [with vanishing approximation error, as $\Delta \to 0$] the conditional expectations appearing on the right hand side of (A.7) via

(A.12)

$$\mathbb{E}_{\mathbb{Q}}((F_T^*)^\nu | \mathcal{G}_t^*) \approx \frac{x_{j+1}^\nu - x_j^\nu}{x_{j+1} - x_j}(F_t^* - x_j) + x_j^\nu$$

whenever $x_j < F_T^* \le x_{j+1}$ $\mathbb{Q}$-a.s. Collecting (A.6), (A.7) and (A.12), we obtain the approximation

(A.13)

$$\mathcal{C}_{\mathbb{Q}}^{\mathcal{G}^*}(F_T^*; t, a, y) \approx 1 + (a + iy)\, F_t^* + \sum_{\nu=2}^{d} \frac{(a + iy)^\nu}{\nu!} \left[\frac{x_{j+1}^\nu - x_j^\nu}{x_{j+1} - x_j}(F_t^* - x_j) + x_j^\nu\right] =: \mathcal{A}_j^d(y; a, F_t^*)$$

whenever $x_j < F_T^* \le x_{j+1}$ $\mathbb{Q}$-a.s. Finally, referring to (3.3.51), (A.3), (A.6) and (A.13), we receive the following $\mathcal{G}^*$-forward-looking electricity futures call option price estimate

(A.14)

$$C_t^{\mathcal{G}^*} \approx \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}^+} \hat{q}(y)\, \mathcal{A}_j^d(y; a, F_t^*)\, dy$$

whenever $x_j < F_T^* \le x_{j+1}$ $(j = 0, \dots, m-1)$ is valid $\mathbb{Q}$-a.s. Herein, $\hat{q}(y)$ and $\mathcal{A}_j^d(y; a, F_t^*)$ are such as defined in (3.2.34) and (A.13), respectively. Actually, the forward-looking electricity futures price process F_t^* appearing in (A.14) has to be simulated numerically by using the dynamics (A.2). ■

Reasoning about Excursus A, one ultimately might wonder why we have not directly applied our secant approximation techniques on the argument inside the forward-looking conditional expectation appearing on the right hand side of (A.3), namely $e^{(a+iy)F_T^*}$, instead of developing the latter into a complex power series, initially. The point here is that $e^{(a+iy)F_T^*}$ obviously constitutes a *complex* function involving F_T^*, whereas the Taylor-polynomial in (A.7) merely contains *real* polynomials involving F_T^*, namely $(F_T^*)^\nu$, on the opposite, which may be approximated by a *real* linear interpolation approach, as presented. In a closing remark we recall that, in the absence of any appropriate *analytical* computation method for $\mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy)F_T^*}\big|\mathcal{G}_t^*\big)$, our goal in Excursus A actually was to establish a *linear* estimation scheme in order to exploit the $(\mathcal{G}^*, \mathbb{Q})$-martingale property of F^*.

Ultimately, referring to (3.2.34), (A.9), (A.13) and (A.14), we establish the following approximation for our forward-looking electricity call option price (3.3.51) yet reading

(3.3.55)

$$C_t^{\mathcal{G}^*} \approx \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}^+} \frac{e^{-(a+iy)K}}{(a+iy)^2} \left[1 + (a+iy)\, F_t^* + \sum_{\nu=2}^{d} \frac{(a+iy)^\nu}{\nu!}\, s_j^\nu(F_t^*)\right] dy$$

whenever $x_j < F_T^* \leq x_{j+1}$ $(j = 0, \dots, m-1)$ is valid $\mathbb{Q}$-a.s.

Excursus B: Computing forward-looking conditional expectations with Cauchy's integral formula – a Complex analysis approach

Catching up equality (A.3), we now propose to treat the contained forward-looking conditional expectation $\mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy)F_T^*}\big|\mathcal{G}_t^*\big)$ with Cauchy's integral formula, alternatively. Sticking to similar designations as in Excursus A [particularly compare (A.6)], we initially remind

(B.1)

$$\mathcal{C}_{\mathbb{Q}}^{\mathcal{G}^*}(F_T^*; t, a, y) = \mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy)F_T^*}\big|\mathcal{G}_t^*\big) = \mathbb{E}_{\mathbb{Q}}\big(f(z_T)\big|\mathcal{G}_t^*\big)$$

wherein $f(\zeta) \coloneqq e^\zeta$ embodies a holomorphic function on $\mathbb{C}$ (i.e. an *entire* function). To proceed in our arguing, we recall the following fundamental theorem from the field of Complex analysis.

Theorem B.1 *(Cauchy's integral formula; CIF)*

Suppose G is an (arbitrary) open subset of the complex plane $\mathbb{C}$ and $f: G \longrightarrow \mathbb{C}$ constitutes a holomorphic function on G. Additionally, let $\mathcal{K} \subset G$ be a closed, rectifiable, positive-oriented Jordan curve with winding number equal to 'one' about $z_0 \in I(\mathcal{K})$, whereby $I(\mathcal{K}) \subset G$ with $\mathcal{K} \cap I(\mathcal{K}) = \emptyset$ denotes the interior of $\mathcal{K}$. Then for all $z_0 \in I(\mathcal{K})$ we have the representation

(B.2)

$$f(z_0) = \frac{1}{2\pi i} \int_{\mathcal{K}} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta.$$

Moreover, we obtain $f(z_0) = 0$, whenever $z_0 \in \mathbb{C} \setminus \overline{I(\mathcal{K})} = \mathbb{C} \setminus \{\mathcal{K} \cup I(\mathcal{K})\}$.

Finally, we remark that for $z_0 \in I(\mathcal{K})$ *the complex function*

$$\mathfrak{K}(\zeta, z_0) := \frac{1}{2\pi i} \frac{1}{\zeta - z_0}$$

frequently is called 'Cauchy kernel' with overall mass 'one', since $\int_{\mathcal{K}} \mathfrak{K}(\zeta, z_0)\, d\zeta = 1$ *holds true.*

Proof See e.g. Chapter IV.4 in [42]. ■

Regarding our original problem (B.1), we now aim to express $f(z_T)$ due to (B.2) in the following. For this purpose, we choose $G = \mathbb{C}$ and $\mathcal{K}$ to be a circle line around the origin with radius $\varrho > 0$, i.e. $\mathcal{K} := \mathcal{K}_\varrho(0) := \{\zeta \in \mathbb{C}: |\zeta| = \varrho\} = \{\zeta \in \mathbb{C}: \zeta = \varrho\, e^{iu},\ u \in [0,2\pi[\}$. Furthermore, we yet presume ϱ to be *sufficiently large* in the sense of obeying $\varrho > M\sqrt{a^2 + y^2}$, whereby M constitutes a strictly positive constant, as above. Then the complex stochastic number $z_T := (a + iy)F_T^*$ $\mathbb{Q}$-a.s. is an element of $I\left(\mathcal{K}_\varrho(0)\right) = \{\zeta \in \mathbb{C}: |\zeta| < \varrho\}$, as long as we assume $0 < F_T^* \leq M$ to be valid $\mathbb{Q}$-a.s. [parallel to our assumption in Excursus A]. Particularly, note that the representation (B.2) is valid for an *arbitrary* curve $\mathcal{K}$ [as long as $\mathcal{K}$ fulfills the announced requirements of Theorem B.1] and thus, the actual value (B.2) does neither depend on the length of $\mathcal{K}$, nor on the radius ϱ in our current arrangement.[19] Hence, it is indeed possible to choose ϱ *sufficiently large* in the above sense. However, applying Cauchy's integral formula on (B.1) while referring to our recent assumptions, we obtain

(B.3)

$$C_{\mathbb{Q}}^{\mathcal{G}^*}(F_T^*; t, a, y) = \frac{1}{2\pi i} \int\limits_{|\zeta| = \varrho} e^{\zeta}\, \mathbb{E}_{\mathbb{Q}}\left(\frac{1}{\zeta - z_T}\middle|\mathcal{G}_t^*\right) d\zeta$$

whereas $|z_T| < \varrho$ holds $\mathbb{Q}$-a.s. Further on, the conditional expectation on the right hand side of (B.3) obviously can be rewritten as[20]

(B.4)

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{1}{\zeta - z_T}\middle|\mathcal{G}_t^*\right) = \frac{1}{\zeta}\, \mathbb{E}_{\mathbb{Q}}\left(\frac{1}{1 - \frac{a+iy}{\zeta} F_T^*}\middle|\mathcal{G}_t^*\right).$$

Since for $q_T := \frac{a+iy}{\zeta} F_T^* \in \mathbb{C}$ with $\zeta \in \mathcal{K}_\varrho(0)$ we observe $|q_T| \leq \frac{M}{\varrho}\sqrt{a^2 + y^2} < 1$ $\mathbb{Q}$-a.s., it is possible to develop the holomorphic function $A(q_T) := 1/(1 - q_T)$ for $|q_T| < 1$ into a geometric power series[21] (with *compact convergence* in the open disk $\{|q_T| < 1\} \subset \mathbb{C}$). Consequently, the conditional expectation in (B.4) may be transformed into

(B.5)

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{1}{\zeta - z_T}\middle|\mathcal{G}_t^*\right) = \frac{1}{\zeta}\, \mathbb{E}_{\mathbb{Q}}(A(q_T)|\mathcal{G}_t^*) = \frac{1}{\zeta} \sum_{\nu=0}^{\infty} \mathbb{E}_{\mathbb{Q}}((q_T)^\nu|\mathcal{G}_t^*) = \sum_{\nu=0}^{\infty} \frac{(a + iy)^\nu}{\zeta^{\nu+1}}\, \mathbb{E}_{\mathbb{Q}}((F_T^*)^\nu|\mathcal{G}_t^*).$$

---

[19] This remarkable property of Cauchy integrals frequently is called *path independency* in the literature.

[20] Alternatively, one might apply Itô's formula on $f(X_T) := 1/X_T$ with $X_T := \zeta - (a + iy)F_T^*$ where F_T^* follows the dynamics given in (A.2). Unfortunately, the incoming infinite sum is difficult to handle and not very useful for further computations. Thus, we instead make use of a geometric power series expansion here.

[21] By the way, we recall that the function $A(z) := 1/(1 - z)$ is differentiable in the unit circle $\{|z| < 1\} \subset \mathbb{C}$ with k-th order derivative $A^{(k)}(z) = k!\,(1 - z)^{-(k+1)}$ where $k \in \mathbb{N}_0$.

Hence, merging (B.1), (B.3) and (B.5) into (A.3), we immediately receive

(B.6)

$$\mathbb{E}_{\mathbb{Q}}\big(e^{aF_T^*}\, q(F_T^*)\big|\mathcal{G}_t^*\big) = \frac{1}{4\pi^2 i} \sum_{\nu=0}^{\infty} \mathbb{E}_{\mathbb{Q}}((F_T^*)^{\nu}|\mathcal{G}_t^*) \int_{\mathbb{R}^+} \hat{q}(y)\,(a+iy)^{\nu} \int_{\mathcal{K}_{\varrho}(0)} \frac{e^{\zeta}}{\zeta^{\nu+1}}\, d\zeta\, dy$$

whereby standard arguments from Complex analysis declare the remaining $d\zeta$-integral to equal $2\pi i/(\nu!)$. Ultimately, substituting (3.2.34) and (B.6) into (3.3.51), we deduce the following forward-looking electricity futures call option price formula

(B.7)

$$C_t^{\mathcal{G}^*} = \frac{e^{-r(T-t)}}{2\pi} \sum_{\nu=0}^{\infty} \mathbb{E}_{\mathbb{Q}}((F_T^*)^{\nu}|\mathcal{G}_t^*) \int_{0+}^{\infty} \frac{(a+iy)^{\nu-2}}{\nu!}\, e^{-(a+iy)K}\, dy$$

wherein F_T^* is given through (A.2) with $t = T$. Anyway, the terms $\mathbb{E}_{\mathbb{Q}}((F_T^*)^{\nu}|\mathcal{G}_t^*)$ appearing inside (B.7) may be approximated similarly to (A.12), whereas we ought to use Taylor-polynomial estimates again and thus, only choose finitely many summands in (B.5), respectively in (B.7), i.e. $\nu = 0, \dots, d$ with $d < \infty$, like in Excursus A. In this case, the resulting approximation for (B.7) possesses a similar structure as in (A.14). ■

3.3.4 Forward-looking electricity put option prices

According to our argumentation in paragraph 3.3.3, within a real function

(3.3.56)

$$p(x) := [K - x]^+ \in \mathcal{L}^1(\mathbb{R}^+)$$

the *forward-looking electricity put option price* at time $t \in [0, T]$, $t < \tau$, with strike price $K > 0$ written on the futures price (3.3.46) is given by

(3.3.57)

$$P_t^{\mathcal{G}^*} = e^{-r(T-t)}\, \mathbb{E}_{\mathbb{Q}}\left(p\left(F_T^{\mathcal{G}^*,\mathbb{Q}}\right)\middle|\mathcal{G}_t^*\right).$$

Applying Fourier transform techniques as before, we immediately deduce

(3.3.58)

$$P_t^{\mathcal{G}^*} = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}^+} \frac{1 - iyK - e^{-iyK}}{y^2}\, \mathbb{E}_{\mathbb{Q}}\left(e^{iyF_T^{\mathcal{G}^*,\mathbb{Q}}}\middle|\mathcal{G}_t^*\right) dy$$

wherein the appearing conditional expectation can be treated similar to the one in (3.3.51), respectively to $\mathcal{C}_{\mathbb{Q}}^{\mathcal{G}^*}(F_T^*; t, 0, y)$ such as introduced in (A.6). Alternatively, the *Put-Call-Parity*

$$[K - x]^+ = [x - K]^+ - [x - K]$$

along with (3.3.50) immediately delivers

(3.3.59)

$$P_t^{\mathcal{G}^*} = C_t^{\mathcal{G}^*} + e^{-r(T-t)}\left[K - F_t^{\mathcal{G}^*,\mathbb{Q}}(\tau_1, \tau_2)\right].$$

3.3.5 Forward-looking average-type electricity option prices

In the present paragraph we show that the just invented forward-looking pricing methodology under enlarged filtrations also is suitable for the treatment of more sophisticated *exotic* electricity derivatives such as average-type or Asian options written on the spot price (3.2.1). Referring to subsection 9.2.2 in [13], we initially assume the underlying Asian option contract to promise a payoff $f\left(\int_{\tau_1}^{\tau_2} S_u\,du\right)$ at the maturity date τ_2 within an arbitrary real function $f \in \mathcal{L}^1(\mathbb{R}^+)$. Extending this classical setup to our future information background, we newly define the $\mathcal{G}^*$-forward-looking price of an Asian option at time $t \in [0,\tau_1]$ paying $f\left(\int_{\tau_1}^{\tau_2} S_u\,du\right) \in \mathcal{L}^1(\mathcal{G}^*,\mathbb{Q})$ at maturity τ_2 $(> \tau_1 \geq t)$ via

(3.3.60)
$$A_t^{\mathcal{G}^*} := A_t^{\mathcal{G}^*}(\tau_1,\tau_2) := e^{-r(\tau_2-t)}\,\mathbb{E}_{\mathbb{Q}}\left(f\left(\int_{\tau_1}^{\tau_2} S_u\,du\right)\Big|\mathcal{G}_t^*\right).$$

A straightforward application of the inverse Fourier transform (3.2.33) delivers

(3.3.61)
$$A_t^{\mathcal{G}^*} = \frac{e^{-r(\tau_2-t)}}{2\pi}\int_{\mathbb{R}^+}\hat{f}(y)\,\mathbb{E}_{\mathbb{Q}}\left(e^{iy\int_{\tau_1}^{\tau_2} S_u\,du}\Big|\mathcal{G}_t^*\right)dy$$

(cf. the beginning of the proof of Prop. 9.8 in [13]). Next, recalling (3.2.1), (3.2.2), (3.2.8), (3.2.24) and (3.3.29), an interchange of the integration order [parallel to our arguing in (3.3.19)] yields

(3.3.62)
$$\int_{\tau_1}^{\tau_2} S_u\,du = \int_{\tau_1}^{\tau_2}\mu(u)\,du + \sum_{k=1}^{n}(\tau_2-\tau_1)\,\Psi_k(t)\,X_t^k + \sum_{k=1}^{n}\int_t^{\tau_2}(\tau_2-\mathfrak{m})\,\Lambda_k(s,\mathfrak{m},\tau_2)\,dL_s^k$$

with $\mathfrak{m} := \mathfrak{m}(s) := max\{s,\tau_1\}$ $(\leq \tau_2)$. Hence, the conditional expectation in (3.3.61) factors into

(3.3.63)
$$\mathbb{E}_{\mathbb{Q}}\left(e^{iy\int_{\tau_1}^{\tau_2} S_u\,du}\Big|\mathcal{G}_t^*\right) =$$
$$exp\left\{iy\left[\int_{\tau_1}^{\tau_2}\mu(u)\,du + \sum_{k=1}^{n}(\tau_2-\tau_1)\,\Psi_k(t)\,X_t^k\right]\right\} \times \mathbb{E}_{\mathbb{Q}}\left(exp\left\{\sum_{k=1}^{n}\int_t^{\tau_2}\eta_k(s)\,dL_s^k\right\}\Bigg|\mathcal{G}_t^*\right)$$

wherein we have just introduced the deterministic and complex function $\eta_k(s) := iy\,\pi_k(s)$ with $\pi_k(s) := (\tau_2-\mathfrak{m})\,\Lambda_k(s,\mathfrak{m},\tau_2) \geq 0$. Note that – similar to the formerly described situation in the sequel of (3.3.53) – for $\tau \leq \tau_2$ (which, by the way, constitutes the economically relevant case) the conditional expectation on the right hand side of (3.3.63) does not reduce to a usual one, unfortunately.[22] For this reason, we now apply approximation techniques as invented in Excursus A.

[22] Actually, there is a slight difference between (3.3.53) and (3.3.63), as the former equation contains $\widetilde{N}^{\mathcal{G}^*,\mathbb{Q}}$ at the place of L [respectively, of N] inside the latter. Hence, if we presume $(t <)\,\tau \leq \tau_2$ in (3.3.63) [being the economically relevant case that we aim to investigate], then the conditional expectation therein does not reduce to a usual one, since $\int_t^{\tau_2}\eta_k(s)\,dL_s^k$ for $k = 1,\dots,p$ is not $\mathbb{Q}$-independent of $\mathcal{G}_t^* := \mathcal{F}_t \vee \sigma\{L_\tau^1,\dots,L_\tau^p\}$ so that we have to apply approximation techniques. On the other hand, if we suppose $\tau > \tau_2$ [being the economically irrelevant scenario], then for all $k = 1,\dots,n$ the integral $\int_t^{\tau_2}\eta_k(s)\,dL_s^k$ becomes $\mathbb{Q}$-independent of $\mathcal{G}_t^*$. In this convenient case, the conditional expectation in (3.3.63) [equally well may be conditioned under $\mathcal{F}_t$ and thus] reduces to a usual one which trivially can be handled by the Lévy-Khinchin formula parallel to (3.2.41).

To ease the notation, for times $0 \le t < \tau_2$ we first establish the (real-valued) stochastic process

$$0 \le H_{t,\tau_2} := \sum_{k=1}^{n} \int_{t}^{\tau_2} \pi_k(s)\, dL_s^k$$

and the complex function $h(\zeta) := e^{\zeta}$. Parallel to Excursus A, we moreover presume $0 \le H_{t,\tau_2} \le M$ to be valid $\mathbb{Q}$-a.s. for all $0 \le t < \tau_2$ within a strictly positive constant M. Consequently, the conditional expectation on the right hand side of (3.3.63) can be expressed as

$$\mathbb{E}_{\mathbb{Q}}\left(exp\left\{ \sum_{k=1}^{n} \int_{t}^{\tau_2} \eta_k(s)\, dL_s^k \right\} \middle| \mathcal{G}_t^* \right) = \mathbb{E}_{\mathbb{Q}}\big(h(iy\, H_{t,\tau_2}) \big| \mathcal{G}_t^*\big).$$

Recalling (A.6), (A.7) and (A.9), we approximate the holomorphic function $h(\cdot)$ inside the latter equation by its (complex) d-th order Taylor-polynomial what leads us to

(3.3.64)

$$\mathbb{E}_{\mathbb{Q}}\left(exp\left\{ \sum_{k=1}^{n} \int_{t}^{\tau_2} \eta_k(s)\, dL_s^k \right\} \middle| \mathcal{G}_t^* \right) \approx$$

$$\sum_{\nu=0}^{d} \frac{(iy)^\nu}{\nu!} \mathbb{E}_{\mathbb{Q}}\big(s_j^\nu(H_{t,\tau_2}) \big| \mathcal{G}_t^*\big) = \sum_{\nu=0}^{d} \frac{(iy)^\nu}{\nu!} \left(\frac{x_{j+1}^\nu - x_j^\nu}{x_{j+1} - x_j} \left\{ \mathbb{E}_{\mathbb{Q}}\big(H_{t,\tau_2} \big| \mathcal{G}_t^*\big) - x_j \right\} + x_j^\nu \right)$$

whenever $x_j < H_{t,\tau_2} \le x_{j+1}$ $(j = 0, \dots, m-1)$ is valid $\mathbb{Q}$-a.s.[23] Furthermore, with respect to (3.3.38) and the definition of H_{t,τ_2}, the last conditional expectation inside (3.3.64) may be decomposed as

$$\mathbb{E}_{\mathbb{Q}}\big(H_{t,\tau_2} \big| \mathcal{G}_t^*\big) = \sum_{k=1}^{p} \mathbb{E}_{\mathbb{Q}}\left(\int_{t}^{\tau-} \pi_k(s)\, dL_s^k \middle| \mathcal{G}_t^* \right) + \sum_{k=1}^{p} \mathbb{E}_{\mathbb{Q}}\left(\int_{\tau}^{\tau_2} \pi_k(s)\, dL_s^k \middle| \mathcal{G}_t^* \right)$$
$$+ \sum_{k=p+1}^{n} \mathbb{E}_{\mathbb{Q}}\left(\int_{t}^{\tau_2} \pi_k(s)\, dL_s^k \middle| \mathcal{F}_t \right).$$

Meanwhile, we recall that $\mathcal{F}_t \subset \mathcal{G}_t^* \subset \mathcal{G}_\tau^* = \mathcal{F}_\tau$ is valid for $t < \tau\ (\le \tau_2)$ [cf. the sequel of (3.3.38)]. Hence, taking (3.3.39), (3.3.45), Lemma 3.5.1 and the tower property into account, we get

$$\mathbb{E}_{\mathbb{Q}}\big(H_{t,\tau_2} \big| \mathcal{G}_t^*\big) = \sum_{k=1}^{p} \frac{L_\tau^k - L_t^k}{\tau - t} \int_{t}^{\tau} \pi_k(s)\, ds + \sum_{k=1}^{p} \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(\int_{\tau}^{\tau_2} \pi_k(s)\, dL_s^k \middle| \mathcal{F}_\tau \right) \middle| \mathcal{G}_t^* \right)$$
$$+ \sum_{k=p+1}^{n} \mathbb{E}_{\mathbb{Q}}\left[\int_{t}^{\tau_2} \pi_k(s)\, dL_s^k \right].$$

[23] We remark that the secants in (A.9) originally have been defined for $\nu = 2, \dots, d$. Nevertheless, we may extend the setting of Excursus A to $\nu = 0, \dots, d$ without any further restrictions. Moreover, we recall that H_{t,τ_2} (playing the role of F_T^* in Excursus A) may become zero yet, while F_T^* has been strictly positive by definition. Hence, we have to respect the additional (but trivial) instance $H_{t,\tau_2} = x_0 \equiv 0$ now, whereas (3.3.64) is equal to one in this case. We conclude that it is indeed possible to extend the presumption of Excursus A yet to $0 \le H_{t,\tau_2} \le M$.

Referring to (3.2.4) and (3.2.20), the previous equation points out as

$$\mathbb{E}_{\mathbb{Q}}\left(H_{t,\tau_2}\middle|\mathcal{G}_t^*\right)=\sum_{k=1}^{p}\frac{L_\tau^k-L_t^k}{\tau-t}\int_t^\tau \pi_k(s)\,ds\ +\sum_{k=1}^{p}\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left[\int_\tau^{\tau_2}\pi_k(s)\,dL_s^k\right]\middle|\mathcal{G}_t^*\right)$$
$$+\sum_{k=p+1}^{n}\int_t^{\tau_2}\int_{D_k} z\,\pi_k(s)\,e^{h_k(s,z)}\,\rho_k(s)\,d\nu_k(z)\,ds.$$

In accordance to Condition A [the latter adjusted to $\mathcal{G}^*$], we introduce the deterministic abbreviations

$$\beta_k(t,\tau):=\int_t^\tau \frac{\pi_k(s)}{\tau-t}ds,\qquad \tilde{\xi}_k(t,v):=\int_t^v\int_{D_k} z\,\pi_k(s)\,e^{h_k(z)}\,\rho_k\,d\nu_k(z)\,ds,$$
$$\xi_k(t,v):=\int_t^v\int_{D_k} z\,\pi_k(s)\,e^{h_k(s,z)}\,\rho_k(s)\,d\nu_k(z)\,ds$$

and therewith receive the decomposition

$$\mathbb{E}_{\mathbb{Q}}\left(H_{t,\tau_2}\middle|\mathcal{G}_t^*\right)=\sum_{k=1}^{p}\beta_k(t,\tau)\left\{L_\tau^k-L_t^k\right\}+\sum_{k=1}^{p}\tilde{\xi}_k(\tau,\tau_2)+\sum_{k=p+1}^{n}\xi_k(t,\tau_2).$$

In conclusion, the estimation in (3.3.64) can be rewritten as

$$\mathbb{E}_{\mathbb{Q}}\left(exp\left\{\sum_{k=1}^{n}\int_t^{\tau_2}\eta_k(s)\,dL_s^k\right\}\middle|\mathcal{G}_t^*\right)\approx$$
$$\sum_{\nu=0}^{d}\frac{(iy)^\nu}{\nu!}\left(\frac{x_{j+1}^\nu-x_j^\nu}{x_{j+1}-x_j}\left\{-x_j+\sum_{k=1}^{p}\beta_k(t,\tau)\left\{L_\tau^k-L_t^k\right\}+\sum_{k=1}^{p}\tilde{\xi}_k(\tau,\tau_2)+\sum_{k=p+1}^{n}\xi_k(t,\tau_2)\right\}+x_j^\nu\right)$$
$$=:\mathfrak{M}_t^j(\tau,\tau_2;y,d;\eta).$$

At this step, we recall that the involved future values $L_\tau^1,\dots,L_\tau^p$ are already *known* from (3.3.38). In other words, an applicant should have *guessed*, respectively *established*, these values previously so that a proper numerical evaluation of $\mathfrak{M}_t^j(\tau,\tau_2;y,d;\eta)$ yet should not cause any further difficulties. The same is valid for the appearing deterministic functions β_k, $\tilde{\xi}_k$ and ξ_k by the way. However, having simulated multiple paths of the stochastic process H_{t,τ_2}, an applicant also should be able to determine a reasonable upper bound M so that the above presumption $0\le H_{t,\tau_2}\le M$ for times $0\le t<\tau_2$ is fulfilled within a probability close to one. Similarly, the constraint "*whenever* $x_j<H_{t,\tau_2}\le x_{j+1}$" ought to be implementable into a numerical simulation algorithm without any additional trouble, since H_{t,τ_2} anyway has to be simulated and thus, it is clear which particular intervals $]x_j,x_{j+1}]$ are hit by the actually realized trajectory of H_{t,τ_2}. Finally, note that our present reasoning about (practical) numerical application issues is valid for the evaluation of the call option price formula (3.3.55), likewise.

For the sake of completeness, we claim that in the economically irrelevant case $\tau>\tau_2$ we receive

$$\mathbb{E}_{\mathbb{Q}}\left(H_{t,\tau_2}\middle|\mathcal{G}_t^*\right)=\sum_{k=1}^{p}\tilde{\xi}_k(t,\tau_2)+\sum_{k=p+1}^{n}\xi_k(t,\tau_2)$$

for time indices $0 \leq t < \tau_2$. Herein, we have used (3.2.20) along with Condition A [for $l := p$ now]. In this context, we recall that supplementary (τ-forward-looking) future information about a selection of the electricity spot price driving noises, namely $L_\tau^1, \dots, L_\tau^p$, evidently becomes irrelevant for an (Asian) option that matures at τ_2 where $\tau > \tau_2$. This observation entirely stands in line with the structure of the latter formula, actually.
Summing up, our Asian option price in (3.3.61) finally may be approximated via

(3.3.65)

$$A_t^{\mathcal{G}^*} \approx \frac{e^{-r(\tau_2 - t)}}{2\pi} \int_{\mathbb{R}^+} \hat{f}(y)\, \mathfrak{M}_t^j(\tau, \tau_2; y, d; \eta)\, exp\left\{ iy \left[\int_{\tau_1}^{\tau_2} \mu(u)\, du + \sum_{k=1}^{n} (\tau_2 - \tau_1)\, \Psi_k(t)\, X_t^k \right] \right\} dy$$

which, by the way, extends Proposition 9.8 in [13] essentially, as available forward-looking information on the future spot price behavior currently has been taken into account.

3.3.6 Pricing electricity contracts under future information about correlated temperature

If we consider the Scandinavian energy market *Nord Pool* [73], we may state that the main driver of electricity demand is outdoor temperature [10]. Since low temperatures imply high prices (due to an increasing electricity demand for heating), we expect a negative correlation between temperature and electricity spot prices [10]. (Nevertheless, there of course exist geographical areas where high temperatures also lead to an increase of electricity demand due to the necessity of air conditioning.) Inspired by section 3.2 in [10], in our upcoming analysis we are going to weave additional forward-looking information about future temperature behavior into our electricity derivatives pricing framework. To be precise, we yet assume that the electricity market participants have access to weather forecasts and thus, to some information about outdoor temperature at a future time τ additionally to the information obtained from observing historical electricity price development.

Appealing to the discussion in [10] – [13], we initially suppose the temperature process Θ to follow the (slightly extended) seasonal *multi-factor* Ornstein-Uhlenbeck disposition

(3.3.66)

$$d\Theta_t = dm(t) + \vartheta\, [m(t) - \Theta_t]\, dt + \sum_{k=1}^{l} \xi_k\, dB_t^k$$

which should be suitable to describe the stylized facts of empirical temperature behavior reasonably well. In the latter equation the mean-reversion speed ϑ is assumed to be a positive constant, whereas the bounded, continuous and deterministic function $m(t)$ indicates the seasonal mean-level. Additionally, we presume the volatility components ξ_k to constitute positive constants and the stochastic processes B_t^k to embody standard Brownian motions (BMs) under $\mathbb{P}$ for every $k = 1, \dots, l$. The latter diffusion processes further are assumed to be both pair-wise independent and independent of the pure-jump noises $L_t^1, \dots, L_t^n$ driving the electricity spot price (recall subsection 3.2.1). Next, Itô's product rule yields the $\mathbb{P}$-solution of (3.3.66) reading

(3.3.67)

$$\Theta_t = m(t) + e^{-\vartheta t}\, [\Theta_0 - m(0)] + \sum_{k=1}^{l} \int_0^t \xi_k\, e^{-\vartheta(t-s)} dB_s^k.$$

Moreover, we here take the base components $X_t^1, \dots, X_t^l$ (which formerly have been used to model the long-term level of the spot price) to be *connected*[24] with outdoor temperature via a constant *adjusting screw* $\zeta \in [-1,1]$. Hence, referring to equality "(3.6) in [10]", for all indices $k = 1, \dots, l$ we newly replace our former equation (3.2.3) yet through

(3.3.68)
$$dX_t^k = -\lambda_k X_t^k \, dt + \sigma_k(t) \left[\zeta \, dB_t^k + \sqrt{1-\zeta^2} \, dL_t^k\right]$$

whereas for $k = l+1, \dots, n$ property (3.2.3) remains untouched. Consequently, for $k = 1, \dots, l$ and time indices $0 \leq t \leq u \leq T$ the iterated solution of (3.3.68) is of the form

(3.3.69)
$$X_u^k = X_t^k \, e^{-\lambda_k(u-t)} + \zeta \int_t^u \sigma_k(s) \, e^{-\lambda_k(u-s)} dB_s^k + \sqrt{1-\zeta^2} \int_t^u \sigma_k(s) \, e^{-\lambda_k(u-s)} dL_s^k.$$

In order to price electricity options written on the spot price (3.2.14) [but yet with extended base components therein such as given in (3.3.68)], we need to switch to an (with respect to $\mathbb{P}$) equivalent probability measure, say $\overline{\mathbb{Q}}$. For this purpose, we slightly modify the Radon-Nikodym derivative (3.2.15), instead defining

(3.3.70)
$$\left.\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}\right|_{\bar{\mathcal{F}}_t} := \prod_{k=1}^{l} \mathfrak{E}\left(G_k \circ B^k\right)_t \times \prod_{k=1}^{n} \mathfrak{E}\left(M^k\right)_t$$

with deterministic and time-dependent real functions $G_k(t)$, continuous Doléans-Dade exponentials

(3.3.71)
$$\mathfrak{E}\left(G_k \circ B^k\right)_t := exp\left\{\int_0^t G_k(s) \, dB_s^k - \frac{1}{2}\int_0^t G_k(s)^2 \, ds\right\}$$

and an (actually *backward-looking*) initial filtration

(3.3.72)
$$\bar{\mathcal{F}}_t := \sigma\{L_r^1, \dots, L_r^n, B_r^1, \dots, B_r^l : 0 \leq r \leq t\}.$$

Troubling Girsanov's theorem (compare Proposition 2.2.1), we declare

(3.3.73)
$$\bar{B}_t^{k,\bar{\mathcal{F}},\overline{\mathbb{Q}}} := \bar{B}_t^k := B_t^k - \int_0^t G_k(s) \, ds$$

to constitute a $\bar{\mathcal{F}}_t$-adapted BM under $\overline{\mathbb{Q}}$ for every index $k = 1, \dots, l$. However, merging (3.3.73) into (3.3.67), we immediately obtain the $\overline{\mathbb{Q}}$-dynamics

(3.3.74)
$$\Theta_t = m(t) + e^{-\vartheta t} \left[\Theta_0 - m(0)\right] + \sum_{k=1}^{l} \int_0^t \xi_k \, G_k(s) \, e^{-\vartheta(t-s)} ds + \sum_{k=1}^{l} \int_0^t \xi_k \, e^{-\vartheta(t-s)} d\bar{B}_s^k.$$

[24] Regarding (3.3.68), we are obviously not facing the classical setting with *correlated* Brownian motions here. We recall that the associated co-variation vanishes, i.e. $d[B^k, L^k]_t = 0$ for every $k = 1, \dots, l$ and $0 \leq t \leq T$. Thus, ζ does not play the role of a common correlation parameter. Instead, the noise term $\zeta dB_t^k + \sqrt{1-\zeta^2} dL_t^k$ in (3.3.68) is defined regardless of any classical correlation approaches.

Similarly, putting (3.3.73) into (3.3.69), for $k = 1, \dots, l$ and time indices $0 \leq t \leq u \leq T$ we receive the iterated $\overline{\mathbb{Q}}$-representation

(3.3.75)

$$X_u^k = X_t^k \, e^{-\lambda_k(u-t)} + \zeta \int_t^u \sigma_k(s)\, G_k(s)\, e^{-\lambda_k(u-s)} ds + \zeta \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)} d\bar{B}_s^k + \sqrt{1-\zeta^2} \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)} dL_s^k.$$

Further on, regarding (3.3.67) and (3.3.74), we introduce the overall filtration

(3.3.76)

$$\bar{\mathcal{H}}_t := \bar{\mathcal{F}}_t \vee \sigma\{\Theta_\tau\} := \bar{\mathcal{F}}_t \vee \sigma\left\{\int_0^\tau e^{\vartheta r} dB_r^k : k = 1, \dots, l\right\} = \bar{\mathcal{F}}_t \vee \sigma\left\{\int_0^\tau e^{\vartheta r} d\bar{B}_r^k : k = 1, \dots, l\right\}$$

[recall (3.3.73) for the last equality] which might be associated with *complete* or *exhaustive* knowledge of the future temperature value at time τ (cf. eq. "(3.7) in [10]"). Similar to before, we come up with a *non-explicit* intermediate filtration $\breve{\mathcal{G}}_t$ obeying (cf. eq. "(3.9) in [10]")

(3.3.77)
$$\bar{\mathcal{F}}_t \subset \breve{\mathcal{G}}_t \subset \bar{\mathcal{H}}_t$$

for $0 \leq t < \tau$, whereas $\bar{\mathcal{F}}_t = \breve{\mathcal{G}}_t$ holds for all $t \geq \tau$. Here, $\breve{\mathcal{G}}_t$ represents the *effective* information about future temperature at time τ that we assume the informed traders to have knowledge of. Parallel to our former *key idea* [such as precisely described in connection with (3.3.38)], we implement an *explicit* intermediate filtration $\bar{\mathcal{G}}_t$ consisting of a subfamily of the components in $\bar{\mathcal{H}}_t$, namely

(3.3.78)

$$\bar{\mathcal{G}}_t := \bar{\mathcal{F}}_t \vee \sigma\left\{\int_0^\tau e^{\vartheta r} d\bar{B}_r^k : k = 1, \dots, d; (d \leq l)\right\}$$

which also satisfies the inclusions $\bar{\mathcal{F}}_t \subset \bar{\mathcal{G}}_t \subset \bar{\mathcal{H}}_t$ whenever $0 \leq t < \tau$. Furthermore, for time indices $0 \leq s < \tau$ we notify the deterministic function

(3.3.79)
$$a(s) := \frac{2\vartheta e^{\vartheta s}}{e^{2\vartheta\tau} - e^{2\vartheta s}}$$

in order to formulate the following statements.

Lemma 3.3.5

(a) *Let $\breve{\mathcal{G}}_t$ as in (3.3.77) and $a(s)$ as in (3.3.79), then the stochastic process*

(3.3.80)

$$\bar{B}_t^{k,\breve{\mathcal{G}},\overline{\mathbb{Q}}} := \bar{B}_t^k - \int_0^t a(s)\, \mathbb{E}_{\overline{\mathbb{Q}}}\left(\int_s^\tau e^{\vartheta r} d\bar{B}_r^k \,\middle|\, \breve{\mathcal{G}}_s\right) ds$$

depicts a $(\breve{\mathcal{G}}_t, \overline{\mathbb{Q}})$-Brownian motion for all $k = 1, \dots, l$ and $t \in [0, \tau[$.

***(b)** Let $\bar{\mathcal{G}}_t$ as in (3.3.78) and $a(s)$ as in (3.3.79), then the stochastic process*

(3.3.81)

$$\bar{B}_t^{k,\bar{\mathcal{G}},\bar{\mathbb{Q}}} := \bar{B}_t^k - \int_0^t a(s) \int_s^\tau e^{\vartheta r} d\bar{B}_r^k \, ds$$

constitutes a $(\bar{\mathcal{G}}_t, \bar{\mathbb{Q}})$-Brownian motion for all $k = 1, \dots, d$ and $t \in [0, \tau[$.

***(c)** For all $k = 1, \dots, d$ and time indices $0 \leq t \leq s < \tau$ we have*

(3.3.82)

$$\mathbb{E}_{\bar{\mathbb{Q}}}\left(\int_s^\tau e^{\vartheta r} d\bar{B}_r^k \middle| \bar{\mathcal{G}}_t \right) = \mathbb{E}_{\bar{\mathbb{Q}}}\left(\int_t^\tau e^{\vartheta r} d\bar{B}_r^k \middle| \bar{\mathcal{G}}_t \right) \frac{e^{2\vartheta\tau} - e^{2\vartheta s}}{e^{2\vartheta\tau} - e^{2\vartheta t}}.$$

Proof **(a)** This follows from Proposition 3.4 in [10].[25]

(b) This follows from part (a): If we replace $\breve{\mathcal{G}}_s$ in (a) by $\bar{\mathcal{G}}_s$ and hereafter decompose the integral

$$\int_s^\tau e^{\vartheta r} d\bar{B}_r^k = \int_0^\tau e^{\vartheta r} d\bar{B}_r^k - \int_0^s e^{\vartheta r} d\bar{B}_r^k$$

$(k = 1, \dots, d)$, we get the claimed result by applying the *taking out what is known rule* for conditional expectations, since $\int_0^\tau e^{\vartheta r} d\bar{B}_r^k$ is $\bar{\mathcal{G}}_s$-measurable [compare (3.3.78)] and $\int_0^s e^{\vartheta r} d\bar{B}_r^k$ is $\bar{\mathcal{F}}_s$-measurable [see (3.3.73)] and thus, the latter is $(\bar{\mathcal{F}}_s \subset)\ \bar{\mathcal{G}}_s$-measurable, too.

(c) This follows from Prop. A.3 in [10] with $g(t) := a(t)$, $f(u) := e^{\vartheta u}$, $B := \bar{B}^k$ and $T_1 := \tau$.[26] ■

Next, for notational reasons let us introduce the [$\bar{\mathcal{G}}_s$-adapted] Brownian $(\bar{\mathcal{G}}, \bar{\mathbb{Q}})$-*information yield*

(3.3.83) $$\bar{\theta}_s^k := a(s) \int_s^\tau e^{\vartheta r} d\bar{B}_r^k.$$

Further, appealing to (3.3.2), we define the *temperature-forecast electricity futures price* under $\bar{\mathcal{G}}$ by

(3.3.84)

$$\bar{F}_t := F_t^{\Theta,\bar{\mathcal{G}},\bar{\mathbb{Q}}}(\tau_1, \tau_2) := \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \mathbb{E}_{\bar{\mathbb{Q}}}(S_u | \bar{\mathcal{G}}_t) \, du.$$

Merging (3.2.1) and (3.2.2) into (3.3.84), within (3.3.78) we instantaneously derive the decomposition

(3.3.85)

$$\bar{F}_t = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \left\{ \mu(u) + \sum_{k=1}^d w_k \, \mathbb{E}_{\bar{\mathbb{Q}}}\left(X_u^k \middle| \bar{\mathcal{G}}_t\right) + \sum_{k=d+1}^l w_k \, \mathbb{E}_{\bar{\mathbb{Q}}}\left(X_u^k \middle| \bar{\mathcal{F}}_t\right) + \sum_{k=l+1}^n w_k \, \mathbb{E}_{\bar{\mathbb{Q}}}\left(X_u^k \middle| \bar{\mathcal{F}}_t\right) \right\} du.$$

[25] There is a notational error in Proposition 3.4 in [10], since the filtration $\mathcal{G}_t$ therein actually should be assumed to be such as defined in "(3.9) in [10]" and *not* as in "(3.2) in [10]".

[26] By the way, using a similar *taking out what is known* argument as in the proof of Lemma 3.3.5 (b), from (3.3.82) we deduce that $\frac{\int_t^\tau e^{\vartheta r} d\bar{B}_r^k}{e^{2\vartheta\tau} - e^{2\vartheta t}}$ designates a $(\bar{\mathcal{G}}_t, \bar{\mathbb{Q}})$-martingale in t for all $k = 1, \dots, d$ and $t \in [0, \tau[$.

[Reminding (3.3.78) while applying similar arguments as in the sequel of Remark 3.3.2, we indeed are allowed to condition the second and third expectation in (3.3.85) equally well under $\bar{\mathcal{F}}_t$ instead of under $\bar{\mathcal{G}}_t$.] In the following, we compute the three conditional expectations in (3.3.85) in their order of appearance:

Substituting (3.3.75) into the first expectation, we get

(3.3.86)
$$\mathbb{E}_{\bar{\mathbb{Q}}}\big(X_u^k\big|\bar{\mathcal{G}}_t\big) =$$
$$X_t^k\, e^{-\lambda_k(u-t)} + \zeta \int_t^u \sigma_k(s)\, G_k(s)\, e^{-\lambda_k(u-s)} ds + \zeta\, \mathbb{E}_{\bar{\mathbb{Q}}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)} d\bar{B}_s^k \middle| \bar{\mathcal{G}}_t\right)$$
$$+ \sqrt{1-\zeta^2}\, \mathbb{E}_{\bar{\mathbb{Q}}}\left(\int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)} dL_s^k \middle| \bar{\mathcal{F}}_t\right).$$

With respect to Lemma 3.3.5 (b), (3.2.4) and (3.3.83), [for $u < \tau$] the latter equation turns into

(3.3.87)
$$\mathbb{E}_{\bar{\mathbb{Q}}}\big(X_u^k\big|\bar{\mathcal{G}}_t\big) =$$
$$X_t^k\, e^{-\lambda_k(u-t)} + \zeta \int_t^u \sigma_k(s)\, G_k(s)\, e^{-\lambda_k(u-s)} ds + \zeta \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)} \mathbb{E}_{\bar{\mathbb{Q}}}\big(\bar{\theta}_s^k\big|\bar{\mathcal{G}}_t\big) ds$$
$$+ \sqrt{1-\zeta^2}\, \mathbb{E}_{\bar{\mathbb{Q}}}\left[\int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} dN_k(s,z)\right].$$

Taking (3.2.20), (3.3.79), (3.3.82) and (3.3.83) into account, equality (3.3.87) finally becomes

(3.3.88)
$$\mathbb{E}_{\bar{\mathbb{Q}}}\big(X_u^k\big|\bar{\mathcal{G}}_t\big) =$$
$$X_t^k\, e^{-\lambda_k(u-t)} + \zeta \int_t^u \sigma_k(s)\, G_k(s)\, e^{-\lambda_k(u-s)} ds + 2\vartheta\zeta \int_t^u \sigma_k(s)\, e^{\vartheta s}\, e^{-\lambda_k(u-s)}\, \frac{\int_t^\tau e^{\vartheta r} d\bar{B}_r^k}{e^{2\vartheta\tau} - e^{2\vartheta t}} ds$$
$$+ \sqrt{1-\zeta^2} \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds.$$

Merging (3.2.4) and (3.3.75) into the second conditional expectation in (3.3.85), we obtain

(3.3.89)
$$\mathbb{E}_{\bar{\mathbb{Q}}}\big(X_u^k\big|\bar{\mathcal{F}}_t\big) = X_t^k\, e^{-\lambda_k(u-t)} +$$
$$\zeta \int_t^u \sigma_k(s)\, G_k(s)\, e^{-\lambda_k(u-s)} ds + \sqrt{1-\zeta^2} \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds.$$

Implanting (3.2.8) into the third conditional expectation of (3.3.85), we ultimately deduce

(3.3.90)
$$\mathbb{E}_{\bar{\mathbb{Q}}}\big(X_u^k\big|\bar{\mathcal{F}}_t\big) = X_t^k e^{-\lambda_k(u-t)} + \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds.$$

Collecting (3.3.88), (3.3.89) and (3.3.90), within (3.2.24) our temperature-forecast electricity futures price equation (3.3.85) can be rearranged in shorthand notation as

(3.3.91)

$$\bar{F}_t = \bar{\Gamma}(t) + \sum_{k=1}^{n} \Psi_k(t)\, X_t^k + \sum_{k=1}^{d} \Pi_k(t)\, W_t^k$$

with $\Psi_k(t)$ as formerly defined in (3.3.29) and new abbreviations

(3.3.92)

$$\begin{aligned}\bar{\Gamma}(t) := \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2-\tau_1}\, du &+ \zeta \sum_{k=1}^{l} \int_{\tau_1}^{\tau_2}\int_{t}^{u} \frac{w_k \sigma_k(s)}{\tau_2-\tau_1} e^{-\lambda_k(u-s)} G_k(s)\, ds\, du \\ &+ \sqrt{1-\zeta^2} \sum_{k=1}^{l} \int_{\tau_1}^{\tau_2}\int_{t}^{u}\int_{D_k} \frac{w_k \sigma_k(s)}{\tau_2-\tau_1} z\, e^{-\lambda_k(u-s)} e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds\, du \\ &+ \sum_{k=l+1}^{n} \int_{\tau_1}^{\tau_2}\int_{t}^{u}\int_{D_k} \frac{w_k \sigma_k(s)}{\tau_2-\tau_1} z\, e^{-\lambda_k(u-s)} e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds\, du,\end{aligned}$$

$$\Pi_k(t) := \frac{2\vartheta\zeta}{e^{2\vartheta\tau} - e^{2\vartheta t}} \int_{\tau_1}^{\tau_2}\int_{t}^{u} \frac{w_k \sigma_k(s)}{\tau_2-\tau_1} e^{\vartheta s} e^{-\lambda_k(u-s)} ds\, du\,, \qquad W_t^k := \int_{t}^{\tau} e^{\vartheta r} d\bar{B}_r^k.$$

Applying Itô's product rule on (3.3.91) while recalling (3.2.3), (3.2.4), (3.2.20), (3.2.24), (3.3.29), (3.3.33), (3.3.68), (3.3.73), (3.3.79), (3.3.81), (3.3.83) and (3.3.92), within a long-winded computation [similar to the derivation methodology of property (3.3.37) before] we ultimately receive the (local) $(\bar{\mathcal{G}}_t, \bar{\mathbb{Q}})$-Sato-martingale dynamics[27]

(3.3.93)

$$\begin{aligned} d\bar{F}_t = \sum_{k=1}^{d} \left[\Lambda_k(t,\tau_1,\tau_2)\,\zeta - e^{\vartheta t}\, \Pi_k(t)\right] d\bar{B}_t^{k,\bar{\mathcal{G}},\bar{\mathbb{Q}}} &+ \zeta \sum_{k=d+1}^{l} \Lambda_k(t,\tau_1,\tau_2)\, d\bar{B}_t^{k,\bar{\mathcal{F}},\bar{\mathbb{Q}}} \\ + \sqrt{1-\zeta^2} \sum_{k=1}^{l} \int_{D_k} z\, \Lambda_k(t,\tau_1,\tau_2)\, \tilde{N}_k^{\bar{\mathcal{F}},\bar{\mathbb{Q}}}(t,dz) &+ \sum_{k=l+1}^{n} \int_{D_k} z\, \Lambda_k(t,\tau_1,\tau_2)\, \tilde{N}_k^{\bar{\mathcal{F}},\bar{\mathbb{Q}}}(t,dz). \end{aligned}$$

---

[27] Recall that $\bar{F}$ indeed possesses independent increments with respect to $\bar{\mathcal{G}}$; particularly, compare (3.3.78) and Lemma 3.3.5 (b) with (3.3.93) to verify this. Further, note that the dynamics (3.3.93) not necessarily is strictly positive any more, since the involved BM-terms may become negative and thus, drive the futures price $\bar{F}$ to negative values, too. Actually, it is only possible to compute the probability for the occurrence of negative prices in a continuous BM-case [compare Lemma 3.4 in [13] or (6.6.15) below]. However, an applicant might choose μ and σ_k contained in (3.3.84) in such a way that negative values for both the spot S and the futures $\bar{F}$ only appear with negligible probability. Also, an adequate jump-size distribution (jumps are strictly positive here) may help to avoid negative spot/futures prices. In particular, on pp. 74-75 in [13] it is argued that *"[...] an arithmetic model apparently allows for negative prices, a phenomenon which sounds odd in any normal market, since this means that the buyer [...] receives money rather than pays. However, in the electricity market [...] it can be more costly for a producer to switch off the generators than to pay someone to consume electricity in the case of more supply than demand. Thus, electricity is given away along with a payment. In fact, in almost all [...] electricity markets, negative prices occur from time to time, although very rarely.*". In conclusion, this citation strongly defends our choice of an arithmetic multi-factor Brownian-motion-driven electricity spot/futures model which actually may generate negative prices (but possibly within a very small probability only).

Moreover, for $0 \leq t \leq T$ equality (3.3.93) immediately delivers the integral representation

(3.3.94)

$$\bar{F}_T - \bar{F}_t = \sum_{k=1}^{d} \int_t^T \left[\Lambda_k(s,\tau_1,\tau_2)\, \zeta - e^{\vartheta s}\, \Pi_k(s) \right] d\bar{B}_s^{k,\bar{\mathcal{G}},\bar{\mathbb{Q}}} + \zeta \sum_{k=d+1}^{l} \int_t^T \Lambda_k(s,\tau_1,\tau_2)\, d\bar{B}_s^k$$
$$+ \sum_{k=1}^{n} \int_t^T \int_{D_k} z\, \Lambda_k(s,\tau_1,\tau_2) \left[1 + \left(\sqrt{1-\zeta^2} - 1 \right) \mathbb{1}_{\{1,\ldots,l\}}(k) \right] d\widetilde{N}_k^{\bar{\mathcal{F}},\bar{\mathbb{Q}}}(s,z).$$

Furthermore, let us concentrate on the computation of the price for an electricity call option written on the futures price (3.3.84) under additional knowledge about future temperature behavior. In accordance to our former designations and derivation procedure in subsection 3.3.3, for $t \leq T$ the temperature-forecast electricity call option price written on $\bar{F}$ yet under $\bar{\mathcal{G}}$ reads as

(3.3.95)

$$C_t^{\bar{\mathcal{G}}} := C_t^{\bar{\mathcal{G}}}(K,\tau_1,\tau_2) = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \hat{q}(y)\, e^{(a+iy)\bar{F}_t}\, \mathbb{E}_{\bar{\mathbb{Q}}}\left(e^{(a+iy)[\bar{F}_T - \bar{F}_t]} \middle| \bar{\mathcal{G}}_t \right) dy.$$

Taking the independent increment property of the $\bar{\mathcal{G}}$-adapted Sato-process $\bar{F}$ [such as given in (3.3.94)] into account, the above conditional expectation transforms into a product which consists of three categories of usual expectations, namely

(3.3.96)

$$\mathbb{E}_{\bar{\mathbb{Q}}}\left(e^{(a+iy)[\bar{F}_T - \bar{F}_t]} \middle| \bar{\mathcal{G}}_t \right) =$$
$$\mathbb{E}_{\bar{\mathbb{Q}}}\left[exp\left\{ \sum_{k=1}^{d} \int_t^T (a+iy)\left[\Lambda_k(s,\tau_1,\tau_2)\, \zeta - e^{\vartheta s}\, \Pi_k(s) \right] d\bar{B}_s^{k,\bar{\mathcal{G}},\bar{\mathbb{Q}}} + \sum_{k=d+1}^{l} \int_t^T (a+iy)\, \Lambda_k(s,\tau_1,\tau_2)\, \zeta\, d\bar{B}_s^k \right.\right.$$
$$\left.\left. + \sum_{k=1}^{n} \int_t^T \int_{D_k} (a+iy)\, z\, \Lambda_k(s,\tau_1,\tau_2) \left[1 + \left(\sqrt{1-\zeta^2} - 1 \right) \mathbb{1}_{\{1,\ldots,l\}}(k) \right] d\widetilde{N}_k^{\bar{\mathcal{F}},\bar{\mathbb{Q}}}(s,z) \right\} \right]$$
$$=: \prod_{k=1}^{d} \mathfrak{I}_1^k \times \prod_{k=d+1}^{l} \mathfrak{I}_2^k \times \prod_{k=1}^{n} \mathfrak{I}_3^k$$

with multipliers [note in passing that $\mathfrak{I}_1^k$ and $\mathfrak{I}_2^k$ merely differ by an additive *information drift*]

(3.3.97)

$$\mathfrak{I}_1^k := \mathbb{E}_{\bar{\mathbb{Q}}}\left[exp\left\{ \int_t^T (a+iy)\left[\Lambda_k(s,\tau_1,\tau_2)\, \zeta - e^{\vartheta s}\, \Pi_k(s) \right] d\bar{B}_s^{k,\bar{\mathcal{G}},\bar{\mathbb{Q}}} \right\} \right]$$
$$= exp\left\{ \int_t^T \frac{(a+iy)^2}{2} \left[\Lambda_k(s,\tau_1,\tau_2)\, \zeta - e^{\vartheta s}\, \Pi_k(s) \right]^2 ds \right\} =: e^{b_k(y,t,T)},$$

$$\mathfrak{I}_2^k := \mathbb{E}_{\bar{\mathbb{Q}}}\left[exp\left\{ \int_t^T (a+iy)\, \Lambda_k(s,\tau_1,\tau_2)\, \zeta\, d\bar{B}_s^k \right\} \right] = exp\left\{ \int_t^T \frac{(a+iy)^2}{2} \Lambda_k(s,\tau_1,\tau_2)^2\, \zeta^2 ds \right\}$$
$$=: e^{c_k(y,t,T)},$$

$$\mathfrak{I}_3^k := \mathbb{E}_{\overline{\mathbb{Q}}}\left[exp\left\{\int_t^T \int_{D_k} \Upsilon_k(s,z)\, d\widetilde{N}_k^{\bar{\mathcal{F}},\overline{\mathbb{Q}}}(s,z)\right\}\right]$$
$$= exp\left\{\int_t^T \int_{D_k} \left[e^{\Upsilon_k(s,z)} - 1 - \Upsilon_k(s,z)\right] e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds\right\} =: e^{\chi_k(y,t,T)}$$

and a deterministic function

$$\Upsilon_k(s,z) := (a+iy)\, z\, \Lambda_k(s,\tau_1,\tau_2)\left[1 + \left(\sqrt{1-\zeta^2} - 1\right)\mathbb{I}_{\{1,\dots,l\}}(k)\right].$$

Herein, we have used *Itô's isometry* (see eq. "(2.8) in [13]") twice along with the extended Lévy-Khinchin formula [as in (3.2.41) before]. Merging (3.2.34), (3.3.96) and (3.3.97) into (3.3.95), we finally end up with our innovative *temperature-forecast electricity futures call option price formula*

(3.3.98)

$$C_t^{\bar{\mathcal{G}}} = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \frac{e^{(a+iy)[\bar{F}_t - K]}}{(a+iy)^2} \times \prod_{k=1}^{d} e^{b_k(y,t,T)} \times \prod_{k=d+1}^{l} e^{c_k(y,t,T)} \times \prod_{k=1}^{n} e^{\chi_k(y,t,T)}\, dy.$$

Essentially, (3.3.98) exhibits three different classes of product terms which may be interpreted as follows: Firstly, $\mathfrak{I}_1^k = e^{b_k(y,t,T)}$ $(k = 1,\dots,d)$ is closely linked with *risk-reducing* temperature forecasts. Secondly, $\mathfrak{I}_2^k = e^{c_k(y,t,T)}$ $(k = d+1,\dots,l)$ descends from the remaining uncertainty concerning future temperature behavior. Thirdly, $\mathfrak{I}_3^k = e^{\chi_k(y,t,T)}$ $(k = 1,\dots,n)$ represents omnipresent electricity (spot) price risk originating from other risk sources than temperature (like e.g. carbon emission permit prices). Reasoning about the current paragraph, we eventually cherish that is has been possible to compute the conditional expectation in (3.3.96) explicitly which has not been the case in our former anticipative pure-jump setups. To the best of our knowledge, in the literature there is no comparable result to (3.3.98) available which extensively provides an (semi-) explicit electricity futures call option price formula under forward-looking information about *correlated* temperature.

3.3.7 Correlating electricity spot prices with carbon emission allowance prices

Reasoning about subsection 3.3.6, it sounds economically reasonable to assume (not only outdoor temperature but also) carbon emission allowance (EUA) prices to have a major impact on electricity prices as well. More precisely, one should suspect a strong positive correlation in between carbon emission allowance prices and electricity spot, forward, futures or even option prices: Indeed, convincing empirical evidences which manifest the above proposition have been detected by Benth and Meyer-Brandis – see subsection 2.2 in [10]. Fortunately, our above modeling framework appears suitable for an incorporation of such sophisticated dependency structures between electricity prices and carbon permit prices, since we may correlate the EUA price, A^0 say, possibly obeying

(3.3.99) $$dA_t^0 = A_t^0\left[\alpha\, dt + \textstyle\sum_{k=1}^{l} \beta_k\, dW_t^k\right]$$

(with constants $\alpha \in \mathbb{R}$, $\beta_k > 0$ and standard BMs W_t^k), with the electricity spot price (3.2.1) in an analogous way as described for the temperature case in the previous subsection 3.3.6 by replacing the temperature noises B_t^k in equation (3.3.68) through the EUA price noises W_t^k for $k = 1,\dots,l$. Moreover, the forward-looking machinery with enlarged filtrations then might be applied simultaneously as in our former paragraph 3.3.6, whereas forward-looking insider information about future carbon emission allowance prices might not be as commonly (and neither as permanently) available as public temperature forecasts, admittedly. However, we will return to the topic of pricing carbon emission allowances in Chapter 6 later, wherein we will indeed take future information about the European Union Emission Trading Scheme (EU ETS) market zone net position into account.

3.3.8 Forward-looking electricity floor option prices

Empirical studies (see Chapter 8 in [13] for an overview and related references) have revealed that in geographical regions in which there is a need for air-conditioning in the summer and for heating in the winter, electricity prices often rally in spring and autumn. Hence, in order to protect against low electricity (spot) prices during a pre-specified time period $[\tau_1, \tau_2]$, an electricity retailer might enter an electricity floor contract [14]. Following section 5.1 in [14], a *floor option* designates a European-type contract which ensures a cash flow at intensity $[K - S_u]^+$ with strike price $K > 0$ at *arbitrary* time $u \in [\tau_1, \tau_2]$. Thus, denoting the constant interest rate by $r > 0$, the fair price of an electricity floor option at any time t yet under our forward-looking information filtration $\mathcal{G}_t^*$ is given by

(3.3.100)

$$Floor^*(t) := Floor^{\mathcal{G}^*}(t; K, \tau_1, \tau_2) := \mathbb{E}_{\mathbb{Q}}\left(\int_{t\vee\tau_1}^{\tau_2} e^{-r(u-t)} \, [K - S_u]^+ du \middle| \mathcal{G}_t^* \right).$$

Next, with $g(x) := [K - x]^+ \in \mathcal{L}^1(\mathbb{R}^+)$ and (3.2.33), the Fubini-Tonelli theorem yields

(3.3.101)

$$Floor^*(t) = \frac{1}{2\pi} \int_{t\vee\tau_1}^{\tau_2} \int_{\mathbb{R}^+} e^{-r(u-t)} \hat{g}(y) \, \mathbb{E}_{\mathbb{Q}}\left(e^{iyS_u} \middle| \mathcal{G}_t^*\right) dy \, du.$$

Taking (3.2.1), (3.2.2) and (3.2.8) into account, the conditional expectation in (3.3.101) factors into

(3.3.102)

$$\mathbb{E}_{\mathbb{Q}}\left(e^{iyS_u} \middle| \mathcal{G}_t^*\right) = exp\left\{ iy \left[\mu(u) + \sum_{k=1}^{n} w_k \, X_t^k \, e^{-\lambda_k(u-t)} \right] \right\} \times \mathbb{E}_{\mathbb{Q}}\left(exp\left\{ \sum_{k=1}^{n} \int_t^u \varepsilon_k(s) \, dL_s^k \right\} \middle| \mathcal{G}_t^* \right)$$

wherein we have just introduced the complex and deterministic function $\varepsilon_k(s) := iy \, \tilde{\pi}_k(s)$ with $\tilde{\pi}_k(s) := \tilde{\pi}_k(s, u) := w_k \, \sigma_k(s) \, e^{-\lambda_k(u-s)} \geq 0$. We stress that the conditional expectation on the right hand side of equality (3.3.102) may be treated similarly to the one in (3.3.64), whereas η_k yet has to be replaced by ε_k, π_k by $\tilde{\pi}_k$, respectively τ_2 by u. Hence, parallel to our arguing in subsection 3.3.5, for a partition $0 \leq t < \tau \leq u \leq \tau_2$ we deduce the approximation

$$\mathbb{E}_{\mathbb{Q}}\left(exp\left\{ \sum_{k=1}^{n} \int_t^u \varepsilon_k(s) \, dL_s^k \right\} \middle| \mathcal{G}_t^* \right) \approx \mathfrak{M}_t^j(\tau, u; y, d; \varepsilon).$$

Moreover, with respect to (3.2.32) we obtain the inverse Fourier transform

(3.3.103)

$$\hat{g}(y) = \frac{1 - iyK - e^{-iyK}}{y^2}.$$

Consequently, our pricing formula (3.3.101) may be approximated via

(3.3.104)

$$Floor^{*}(t) \approx \int_{t\vee\tau_1}^{\tau_2} \frac{e^{-r(u-t)}}{2\pi} \int_{\mathbb{R}^+} \hat{g}(y)\, \mathfrak{M}_t^j(\tau,u;y,d;\varepsilon)\, exp\left\{iy\left[\mu(u)+\sum_{k=1}^{n} w_k\, X_t^k\, e^{-\lambda_k(u-t)}\right]\right\} dy\, du$$

yielding an electricity floor option price estimate at time $t \in [\tau_1, \tau_2]$ under additional forward-looking information modeled by the enlarged filtration $\mathcal{G}_t^*$. Eventually, we underline that the corresponding section 5.1 in [14], on the contrary, is concerned with the evaluation of electricity floor option prices under a common *backward-looking* information filtration approach. In addition, Biagini et al. [14] utilize a *geometrical* Heath-Jarrow-Morton setup to model the underlying electricity *forward* price dynamics, whereas we have made use of an *arithmetic multi-factor* electricity *futures* price Ornstein-Uhlenbeck disposition under an *enlarged filtration* on the opposite.

3.3.9 A mixed model for electricity spot, futures and option prices

Regarding the previous subsections, it seems to be impossible to compute expectations of the type (3.3.53) analytically. To overcome this problem, we now propose a *mixed* electricity spot price model including both Brownian motion (BM) and pure-jump terms. In accordance to our former explanations concerning the splitting of the spot price noises into small and large-amplitude jump components [such as claimed previously to (3.2.6)], we yet suppose to replace the equations in (3.2.3) through

(3.3.105) $$dX_t^k = -\lambda_k\, X_t^k\, dt + \sigma_k\, dB_t^k \qquad (k = 1, \dots, l)$$

(with strictly positive and constant[28] volatilities $\sigma_1, \dots, \sigma_l$ along with standard $\mathbb{P}$-BMs $B_t^1, \dots, B_t^l$) and

(3.3.106) $$dX_t^k = -\lambda_k\, X_t^k\, dt + \sigma_k(t)\, dL_t^k \qquad (k = l+1, \dots, n)$$

with pure-jump Sato-noises L_t^k such as defined in (3.2.4). Hence, the small-amplitude fluctuations of the long-term level of the spot reasonably are modeled by Brownian motions yet, whereas the short-term spiky variation components remain untouched [compare (3.2.3) with (3.3.106)]. Similar to before, we assume all involved random processes also in our current *mixed model* (3.3.105) – (3.3.106) such as $B_t^1, \dots, B_t^l, L_t^{l+1}, \dots, L_t^n$ to be pair-wise $\mathbb{P}$-independent. Further on, we recall that the associated *mixed* electricity spot price (compare subsection 3.2.2 in [13] at this step) reading

(3.3.107)

$$S_t = \mu(t) + \sum_{k=1}^{l} w_k X_t^k + \sum_{k=l+1}^{n} w_k X_t^k$$

now may become *negative* which, by the way, not at all constitutes a serious matter[29].

[28] On the bottom of page 4 in [8] it is argued that the assumption of *constant* (daily) long-term volatilities $\sigma_k(t) \equiv \sigma_k > 0$ $(k = 1, \dots, l)$ seems to be economically reasonable. Presently, we catch up this assumption which will turn out convenient in the context of modeling electricity prices under enlarged filtrations, as we will see later on.

[29] This statement is entirely justified by our former explanation concerning *negative* electricity prices which has been given in the context of the dynamics (3.3.93).

Slightly deviating from (3.3.70), we next define the *mixed* Radon-Nikodym derivative by

(3.3.108)

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\tilde{\mathcal{F}}_t} := \prod_{k=1}^{l} \mathfrak{E}\big(G_k \circ B^k\big)_t \times \prod_{k=l+1}^{n} \mathfrak{E}\big(M^k\big)_t$$

with multipliers $\mathfrak{E}\big(G_k \circ B^k\big)_t$ as in (3.3.71), $\mathfrak{E}\big(M^k\big)_t$ as in (3.2.18) and a new initial filtration

(3.3.109) $$\tilde{\mathcal{F}}_t := \sigma\{B_r^1, \dots, B_r^l, L_r^{l+1}, \dots, L_r^n : 0 \le r \le t\}.$$

Similarly to (3.3.73), we declare

(3.3.110) $$\tilde{B}_t^k := \tilde{B}_t^{k,\tilde{\mathcal{F}},\widetilde{\mathbb{Q}}} := B_t^k - \int_0^t G_k(s)\, ds$$

to constitute a $\tilde{\mathcal{F}}_t$-adapted Brownian motion under $\widetilde{\mathbb{Q}}$ for all $k = 1, \dots, l$. Moreover, for time indices $0 \le t \le u \le T$ the solution of (3.3.105) yet points out as

(3.3.111)

$$X_u^k = X_t^k e^{-\lambda_k(u-t)} + \sigma_k \int_t^u e^{-\lambda_k(u-s)}\, dB_s^k$$

$(k = 1, \dots, l)$, whereas (3.3.106) is solved by the Ornstein-Uhlenbeck-type Sato-process

(3.3.112)

$$X_u^k = X_t^k e^{-\lambda_k(u-t)} + \int_t^u \sigma_k(s)\, e^{-\lambda_k(u-s)}\, dL_s^k$$

$(k = l + 1, \dots, n)$. Referring to (3.3.110), equation (3.3.111) further yields the $\widetilde{\mathbb{Q}}$-representation

(3.3.113)

$$X_u^k = X_t^k e^{-\lambda_k(u-t)} + \sigma_k \int_t^u e^{-\lambda_k(u-s)}\, G_k(s)\, ds + \sigma_k \int_t^u e^{-\lambda_k(u-s)}\, d\tilde{B}_s^k$$

$(k = 1, \dots, l)$. In accordance to (3.3.76), we next introduce the overall/global filtration

(3.3.114)

$$\tilde{\mathcal{H}}_t := \tilde{\mathcal{F}}_t \vee \sigma\left\{\int_0^\tau e^{\lambda_k r} dB_r^k : k = 1, \dots, l\right\} = \tilde{\mathcal{F}}_t \vee \sigma\left\{\int_0^\tau e^{\lambda_k r} d\tilde{B}_r^k : k = 1, \dots, l\right\}$$

whereas, parallel to (3.3.78), we implement an associated *explicit* intermediate filtration via

(3.3.115)

$$\tilde{\mathcal{G}}_t := \tilde{\mathcal{F}}_t \vee \sigma\left\{\int_0^\tau e^{\lambda_k r} d\tilde{B}_r^k : k = 1, \dots, d; (d \le l)\right\}.$$

Actually, we observe $\tilde{\mathcal{F}}_t \subset \tilde{\mathcal{G}}_t \subset \tilde{\mathcal{H}}_t$ for $0 \le t < \tau$ and $\tilde{\mathcal{F}}_t = \tilde{\mathcal{G}}_t$ for $t \ge \tau$.

Furthermore, for notational convenience we put

(3.3.116)

$$a_k(s) := \frac{2\,\lambda_k\, e^{\lambda_k s}}{e^{2\lambda_k \tau} - e^{2\lambda_k s}}\,, \qquad \tilde{\theta}_s^k := a_k(s) \int\limits_s^\tau e^{\lambda_k r} d\tilde{B}_r^k.$$

Then, with respect to Lemma 3.3.5 (b), we deduce that

(3.3.117) $$\tilde{B}_t^{k,\tilde{\mathcal{G}},\tilde{\mathbb{Q}}} := \tilde{B}_t^k - \int_0^t \tilde{\theta}_s^k\, ds$$

constitutes a $(\tilde{\mathcal{G}}_t, \tilde{\mathbb{Q}})$-BM for all $k = 1, \dots, d$ and $t \in [0, \tau[$. Additionally, Lemma 3.3.5 (c) yields

(3.3.118)

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int\limits_s^\tau e^{\lambda_k r} d\tilde{B}_r^k \,\middle|\, \tilde{\mathcal{G}}_t\right) = \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int\limits_t^\tau e^{\lambda_k r} d\tilde{B}_r^k \,\middle|\, \tilde{\mathcal{G}}_t\right) \frac{e^{2\lambda_k \tau} - e^{2\lambda_k s}}{e^{2\lambda_k \tau} - e^{2\lambda_k t}}$$

for all $k = 1, \dots, d$ and time indices $0 \le t \le s < \tau$.

In accordance to (3.3.2), for $t \in [0, \tau_1]$ we yet define the ($\tilde{\mathcal{G}}$-forward-looking) *mixed electricity futures price* associated to our recent jump-diffusion electricity spot price model by dint of

(3.3.119)

$$\tilde{F}_t := F_t^{\tilde{\mathcal{G}},\tilde{\mathbb{Q}}}(\tau_1, \tau_2) := \frac{1}{\tau_2 - \tau_1} \int\limits_{\tau_1}^{\tau_2} \mathbb{E}_{\tilde{\mathbb{Q}}}\big(S_u \big| \tilde{\mathcal{G}}_t\big)\, du.$$

Merging (3.3.107) into the latter equation, we immediately deduce

(3.3.120)

$$\tilde{F}_t = \frac{1}{\tau_2 - \tau_1} \int\limits_{\tau_1}^{\tau_2} \left[\mu(u) + \sum_{k=1}^{d} w_k\, \mathbb{E}_{\tilde{\mathbb{Q}}}\big(X_u^k \big| \tilde{\mathcal{G}}_t\big) + \sum_{k=d+1}^{l} w_k\, \mathbb{E}_{\tilde{\mathbb{Q}}}\big(X_u^k \big| \tilde{\mathcal{F}}_t\big) + \sum_{k=l+1}^{n} w_k\, \mathbb{E}_{\tilde{\mathbb{Q}}}\big(X_u^k \big| \tilde{\mathcal{F}}_t\big)\right] du.$$

In what follows, we compute the three conditional expectations in (3.3.120) in their order of appearance: Using (3.3.113) and (3.3.116) – (3.3.118), [for $u < \tau$] the first object therein becomes

(3.3.121)

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\big(X_u^k \big| \tilde{\mathcal{G}}_t\big) = X_t^k e^{-\lambda_k(u-t)} + \sigma_k \int\limits_t^u e^{-\lambda_k(u-s)}\, G_k(s)\, ds + 2\,\lambda_k\, \sigma_k\, \frac{\int_t^\tau e^{\lambda_k r} d\tilde{B}_r^k}{e^{2\lambda_k \tau} - e^{2\lambda_k t}} \int\limits_t^u e^{\lambda_k(2s-u)} ds.$$

Utilizing (3.3.113) again, the second conditional expectation in (3.3.120) turns out as

(3.3.122)

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\big(X_u^k \big| \tilde{\mathcal{F}}_t\big) = X_t^k e^{-\lambda_k(u-t)} + \sigma_k \int\limits_t^u e^{-\lambda_k(u-s)}\, G_k(s)\, ds.$$

Finally, referring to (3.2.4), (3.2.20) and (3.3.112), we observe the third expectation to be of the form

(3.3.123)

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\left(X_u^k \middle| \tilde{\mathcal{F}}_t\right) = X_t^k e^{-\lambda_k(u-t)} + \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds.$$

Hence, substituting (3.3.121) – (3.3.123) into (3.3.120), we obtain the shorthand representation

(3.3.124)

$$\tilde{F}_t = \hat{\Gamma}(t) + \sum_{k=1}^{n} \widehat{\Psi}_k(t)\, X_t^k + \sum_{k=1}^{d} \widetilde{\Pi}_k(t)\, \widetilde{W}_t^k$$

with abbreviations

(3.3.125)

$$\hat{\Gamma}(t) := \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2 - \tau_1} du + \sum_{k=1}^{l} \frac{w_k\, \sigma_k}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \int_t^u e^{-\lambda_k(u-s)}\, G_k(s)\, ds\, du$$
$$+ \sum_{k=l+1}^{n} \frac{w_k}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k(u-s)} e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds\, du,$$

$$\widehat{\Psi}_k(t) := \frac{w_k}{\tau_2 - \tau_1}\, \frac{e^{-\lambda_k(\tau_1 - t)} - e^{-\lambda_k(\tau_2 - t)}}{\lambda_k},$$

$$\widetilde{\Pi}_k(t) := \frac{2\, w_k\, \lambda_k\, \sigma_k}{e^{2\lambda_k t} - e^{2\lambda_k \tau}} \int_{\tau_1}^{\tau_2} \int_u^t \frac{e^{\lambda_k(2s-u)}}{\tau_2 - \tau_1} ds\, du, \qquad \widetilde{W}_t^k := \int_t^\tau e^{\lambda_k r} d\tilde{B}_r^k.$$

From (3.3.125) we immediately deduce the derivatives (with respect to t)

(3.3.126) $$\widehat{\Psi}_k'(t) = \lambda_k\, \widehat{\Psi}_k(t), \qquad \widetilde{\Pi}_k'(t) = a_k(t)\left[\widetilde{\Pi}_k(t)\, e^{\lambda_k t} - \sigma_k\, \widehat{\Psi}_k(t)\right],$$

$$\hat{\Gamma}'(t) = -\sum_{k=1}^{l} \sigma_k\, G_k(t)\, \widehat{\Psi}_k(t) - \sum_{k=l+1}^{n} \int_{D_k} z\, \sigma_k(t)\, \widehat{\Psi}_k(t)\, e^{h_k(t,z)} \rho_k(t)\, d\nu_k(z).$$

Next, applying Itô's product rule on (3.3.124) and hereafter using (3.2.4), (3.2.20), (3.2.24), (3.3.105), (3.3.106), (3.3.110), (3.3.116), (3.3.117), (3.3.125) and (3.3.126), [within a similar computation as for (3.3.93)] we obtain the $\widetilde{\mathbb{Q}}$-dynamics

(3.3.127)

$$d\tilde{F}_t = \sum_{k=1}^{d} \left[\sigma_k\, \widehat{\Psi}_k(t) - \widetilde{\Pi}_k(t)\, e^{\lambda_k t}\right] d\tilde{B}_t^{k,\tilde{\mathcal{G}},\widetilde{\mathbb{Q}}} + \sum_{k=d+1}^{l} \sigma_k\, \widehat{\Psi}_k(t)\, d\tilde{B}_t^{k,\tilde{\mathcal{F}},\widetilde{\mathbb{Q}}}$$
$$+ \sum_{k=l+1}^{n} \int_{D_k} z\, \Lambda_k(t, \tau_1, \tau_2)\, \widetilde{N}_k^{\tilde{\mathcal{F}},\widetilde{\mathbb{Q}}}(t, dz).$$

Parallel to our former announcements in the footnote dedicated to (3.3.93), we remark that the electricity futures price $\tilde{F}$ such as given in (3.3.127) may become *negative* in our present mixed spot price model. Yet, examining the dynamics (3.3.127) in more depth, we declare $\tilde{F}_t$ to constitute a $\tilde{\mathcal{G}}_t$-adapted (local) martingale under $\widetilde{\mathbb{Q}}$ which is, in the light of (3.3.119), not a surprising observation. In addition, comparing (3.3.115) with (3.3.127), we classify $\tilde{F}$ to possess independent increments with respect to the enlarged filtration $\tilde{\mathcal{G}}$ and thus, to designate a $(\tilde{\mathcal{G}}, \widetilde{\mathbb{Q}})$-Sato-martingale, even.[30]

In what follows, we aim to price a European-type option written on our *mixed* electricity futures (3.3.127). For this purpose, we yet adopt the notations of subsection 3.3.3 to our current setup. Hence, with respect to (3.3.52), we introduce the $\tilde{\mathcal{G}}$-forward-looking call option price at time $t \leq T$ via

(3.3.128)

$$\tilde{C}_t := C_t^{\tilde{\mathcal{G}}}(K, \tau_1, \tau_2) = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \hat{q}(y)\, e^{(a+iy)\tilde{F}_t}\, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(e^{(a+iy)[\tilde{F}_T - \tilde{F}_t]} \big| \tilde{\mathcal{G}}_t\big)\, dy$$

where $\hat{q}(y)$ is like in (3.2.34). Taking (3.3.127) into account while introducing the shorthand notation $\theta_k(s) := (a+iy)\, \Lambda_k(s, \tau_1, \tau_2)$, the conditional expectation in (3.3.128) factors into

(3.3.129)

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(e^{(a+iy)[\tilde{F}_T - \tilde{F}_t]} \big| \tilde{\mathcal{G}}_t\big) =$$

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[exp\left\{ (a+iy) \left(\sum_{k=1}^{d} \int_t^T \left[\sigma_k\, \widehat{\Psi}_k(s) - \widetilde{\Pi}_k(s)\, e^{\lambda_k s}\right] d\tilde{B}_s^{k,\tilde{\mathcal{G}},\widetilde{\mathbb{Q}}} + \sum_{k=d+1}^{l} \int_t^T \sigma_k\, \widehat{\Psi}_k(s)\, d\tilde{B}_s^k \right.\right.\right.$$

$$\left.\left.\left. + \sum_{k=l+1}^{n} \int_t^T \int_{D_k} z\, \Lambda_k(s, \tau_1, \tau_2)\, d\widetilde{N}_k^{\tilde{\mathcal{F}},\widetilde{\mathbb{Q}}}(s,z) \right) \right\} \right] =: \prod_{k=1}^{d} P_1^k \times \prod_{k=d+1}^{l} P_2^k \times \prod_{k=l+1}^{n} P_3^k$$

with multipliers

(3.3.130)

$$P_1^k := \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[exp\left\{ \int_t^T (a+iy)\left[\sigma_k\, \widehat{\Psi}_k(s) - \widetilde{\Pi}_k(s)\, e^{\lambda_k s}\right] d\tilde{B}_s^{k,\tilde{\mathcal{G}},\widetilde{\mathbb{Q}}} \right\} \right]$$

$$= exp\left\{ \int_t^T \frac{(a+iy)^2}{2} \left[\sigma_k\, \widehat{\Psi}_k(s) - \widetilde{\Pi}_k(s)\, e^{\lambda_k s}\right]^2 ds \right\},$$

$$P_2^k := \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[exp\left\{ \int_t^T (a+iy)\, \sigma_k\, \widehat{\Psi}_k(s)\, d\tilde{B}_s^k \right\} \right] = exp\left\{ \int_t^T \frac{(a+iy)^2}{2}\, \sigma_k^2\, \widehat{\Psi}_k(s)^2\, ds \right\},$$

$$P_3^k := \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[exp\left\{ i \int_t^T \int_{D_k} (y - ia)\, z\, \Lambda_k(s, \tau_1, \tau_2)\, d\widetilde{N}_k^{\tilde{\mathcal{F}},\widetilde{\mathbb{Q}}}(s,z) \right\} \right]$$

$$= exp\left\{ \int_t^T \int_{D_k} \left[e^{z\,\theta_k(s)} - 1 - z\,\theta_k(s)\right] e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds \right\}.$$

[30] Note that for *any arbitrarily enlarged* filtration this is not necessarily true, but under (3.3.115) it is.

Herein, for the computation of P_1^k and P_2^k we have made use of Itô's isometry[31], whereas for the treatment of P_3^k we have exploited (3.2.20) along with Prop. 2.1 in [13] and Prop. 1.9 in [65]. Finally, referring to (3.2.34) and (3.3.129), our *mixed* electricity futures call option price (3.3.128) becomes

(3.3.131)

$$\tilde{C}_t = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \frac{e^{(a+iy)[\tilde{F}_t - K]}}{(a+iy)^2} \times \prod_{k=1}^{d} P_1^k \times \prod_{k=d+1}^{l} P_2^k \times \prod_{k=l+1}^{n} P_3^k \, dy$$

where P_1^k, P_2^k and P_3^k are such as claimed in (3.3.130). Regarding the structure of our innovative pricing formula (3.3.131), we essentially recognize three different classes of risk terms: Firstly, P_1^k is closely connected with risk-reducing $\tilde{\mathcal{G}}$-forward-looking information on a selection of the Brownian noises driving the (stochastic) mean-level of the underlying electricity spot price. Secondly, the terms P_2^k can be associated to some kind of *remaining risk* with respect to the long-term level of the spot. Roughly speaking, the difference between P_1^k and P_2^k (which evidently consists in an additive *information drift*, merely) describes to what extend the (explicit) intermediate filtration $\tilde{\mathcal{G}}_t$ is smaller than the overall filtration $\tilde{\mathcal{H}}_t$. Yet, we observe that if $\tilde{\mathcal{G}}_t = \tilde{\mathcal{H}}_t$ (*exhaustive knowledge*) and hence, if $d = l$, then the factors P_2^k would equal P_1^k so that the product in (3.3.129) would simplify to

(3.3.132) $$\prod_{k=1}^{l} P_1^k \times \prod_{k=l+1}^{n} P_3^k.$$

In other words, if the index d appearing in (3.3.115) is far from l (and thus, close to one), then there is not much supplementary information on the future behavior of the long-term level of the spot price available. As a consequence, the (actually *non*-forward-looking) members P_2^k in this case have a major impact on the resulting option price which sounds economically reasonable. Vice versa, if d is close to l, then there indeed is some worthy future information on the majority of the long-term level driving noises available what reasonably emphasizes the impact of the multipliers P_1^k, which themselves have been associated with additional insider information on the future electricity spot price mean level. Thirdly, the terms P_3^k appearing inside (3.3.131) originate from the omnipresent risk of an occurrence of electricity spot price *spikes*, i.e. violent upward jumps followed by a quick return to about the same level which periodically appear due to sudden imbalances in supply and demand – cf. [8]. Moreover, let us remark that the price of a *put* option written on $\tilde{F}_t$ easily can be obtained from (3.3.131) by exploiting the *Put-Call-Parity*, parallel to our argumentation in subsection 3.3.4 before. Ultimately, we emphasize that in (3.3.130) it fortunately has been possible to compute the appearing expectations *analytically* which, on the contrary, seemed to be impossible in our former (forward-looking) pure-jump cases [cf. e.g. section 3.3.3]. It also appears worthwhile to compare (3.3.131) with (3.2.35).

The information premium in a mixed electricity market model As a closing remark we want to examine the information premium also for our recent *mixed* electricity spot price model. Note that in the previous subsection 3.3.1 the information premium actually has been introduced in connection with a *pure-jump* multi-factor model. Right now, we are situated in a completely different model setup, since the available additional information currently is taken with respect to *Brownian* noises [compare (3.3.115)]. However, adhering to (3.3.3), we newly define the information premium via

(3.3.133) $$\mathfrak{I}_t^{\tilde{\mathcal{G}},\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) := F_t^{\tilde{\mathcal{G}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) - F_t^{\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2).$$

[31] Note that P_1^k and P_2^k in (3.3.130) actually may be computed further by exploiting the definitions in (3.3.125).

Taking (3.3.107), (3.3.110), (3.3.113) and (3.3.119) into account, the latter can be rewritten as

(3.3.134)
$$\mathfrak{I}_t^{\tilde{\mathcal{G}},\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) =$$
$$\sum_{k=1}^{d} \frac{w_k}{\tau_2-\tau_1} \int_{\tau_1}^{\tau_2} \left[\mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_u^k \middle| \tilde{\mathcal{G}}_t\right) - \mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_u^k \middle| \tilde{\mathcal{F}}_t\right)\right] du = \sum_{k=1}^{d} \frac{w_k\,\sigma_k}{\tau_2-\tau_1} \int_{\tau_1}^{\tau_2} \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^u e^{-\lambda_k(u-s)}\, d\tilde{B}_s^k \middle| \tilde{\mathcal{G}}_t \right) du.$$

Next, appealing to (3.3.116) – (3.3.118), [with $\tau_1 \le u < \tau \le \tau_2$ and thus, $0 \le t \le s < \tau$] we deduce

(3.3.135)
$$\mathfrak{I}_t^{\tilde{\mathcal{G}},\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) = \frac{2}{\tau_2-\tau_1} \sum_{k=1}^{d} \lambda_k\, w_k\, \sigma_k\, \frac{\int_t^\tau e^{\lambda_k r} d\tilde{B}_r^k}{e^{2\lambda_k \tau} - e^{2\lambda_k t}} \int_{\tau_1}^{\tau_2} \int_t^u e^{\lambda_k(2s-u)} ds\, du$$

yielding the information premium for $(t \le)\ \tau_1 < \tau$. We remark that the remaining integral on the right hand side of (3.3.135) can be straightforwardly computed by interchanging the integration order due to Fubini's theorem.

In order to treat the case $\tau \le \tau_1$, we apply a similar iterated-conditioning procedure such as presented in (3.3.20) – (3.3.22): At first, we observe the inclusions $\tilde{\mathcal{F}}_t \subset \tilde{\mathcal{G}}_t \subseteq \tilde{\mathcal{G}}_\tau = \tilde{\mathcal{F}}_\tau$ $(0 \le t \le \tau \le \tau_1 \le \tau_2)$ to be valid. Hereafter, referring to (3.3.113) [but with t replaced by τ therein], we claim[32]

(3.3.136)
$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_u^k \middle| \tilde{\mathcal{G}}_t\right) - \mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_u^k \middle| \tilde{\mathcal{F}}_t\right)$$
$$= \mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_\tau^k \middle| \tilde{\mathcal{G}}_t\right) + \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_u^k - X_\tau^k \middle| \tilde{\mathcal{F}}_\tau\right) \middle| \tilde{\mathcal{G}}_t\right) - \mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_\tau^k \middle| \tilde{\mathcal{F}}_t\right) - \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_u^k - X_\tau^k \middle| \tilde{\mathcal{F}}_\tau\right) \middle| \tilde{\mathcal{F}}_t\right)$$
$$= e^{-\lambda_k(u-\tau)} \left[\mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_\tau^k \middle| \tilde{\mathcal{G}}_t\right) - \mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_\tau^k \middle| \tilde{\mathcal{F}}_t\right)\right].$$

Combining this with (3.3.134) while identifying (3.2.24), for $0 \le t \le \tau \le \tau_1 \le \tau_2$ we finally get

(3.3.137)
$$\mathfrak{I}_t^{\tilde{\mathcal{G}},\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) = \sum_{k=1}^{d} \frac{\Lambda_k(\tau,\tau_1,\tau_2)}{\sigma_k(\tau)} \left[\mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_\tau^k \middle| \tilde{\mathcal{G}}_t\right) - \mathbb{E}_{\tilde{\mathbb{Q}}}\left(X_\tau^k \middle| \tilde{\mathcal{F}}_t\right)\right].$$

Additionally, substituting (3.3.113) [but yet with $u := \tau > t$ therein] into (3.3.137), we receive

$$\mathfrak{I}_t^{\tilde{\mathcal{G}},\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) = \sum_{k=1}^{d} \sigma_k \frac{\Lambda_k(\tau,\tau_1,\tau_2)}{\sigma_k(\tau)}\, \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^{\tau-} e^{-\lambda_k(\tau-s)}\, d\tilde{B}_s^k \middle| \tilde{\mathcal{G}}_t \right).$$

Again, we take (3.3.116) – (3.3.118) along with the Fubini-Tonelli theorem into account and obtain

$$\mathfrak{I}_t^{\tilde{\mathcal{G}},\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) = 2 \sum_{k=1}^{d} \lambda_k\, \sigma_k \frac{\Lambda_k(\tau,\tau_1,\tau_2)}{\sigma_k(\tau)} \frac{\int_t^\tau e^{\lambda_k r} d\tilde{B}_r^k}{e^{2\lambda_k \tau} - e^{2\lambda_k t}} \int_t^\tau e^{\lambda_k(2s-\tau)} ds$$

yielding the information premium for $t < \tau \le \tau_1$. (Yet, the last integral can be computed explicitly.)

[32] Recall that $\tau_1 \le u \le \tau_2$ and $\tau \le \tau_1$ hold true in our current setting. Thus, we have $\tau \le u$.

3.3.9.1 A Brownian single-factor electricity option price formula

In this subsection we study a very simple subclass of the *mixed* model presented in paragraph 3.3.9. More accurately, we do no longer permit jump components now and restrict our approach to merely *one* Brownian motion driving the (actually no longer *mixed*) electricity spot price, i.e. we choose $l = 1$ and $w_{l+1} = \cdots = w_n = 0$ in (3.3.107). Hence, putting $w_1 \coloneqq 1$, our *single*-factor electricity spot price model trivially boils down to $S_t = \mu(t) + X_t^1$. For notational reasons, we will omit all indices, just writing $X, \lambda, B, G(s)$ etc. instead of $X^1, \lambda_1, B^1, G_1(s)$, in the following. Consequently, we observe

(3.3.138)
$$S_t = \mu(t) + X_t$$

wherein X_t is such as given in (3.3.105). Note that $\mathbb{P}(\{S_t < 0\}) > 0$. Recalling the main statements in [8], [14], [72] and our introductory section 1.1 which altogether count in favor for the modeling of electricity spot prices with jump processes, our recent approach in (3.3.138) appears rather unrealistic. However, we aim to examine how option pricing works in this trivial case, hoping for a simple pricing formula in the present Brownian Bachelier model. To claim our findings right at the beginning, we state that, on the one hand, option pricing becomes (as expected) very simple, as we now may apply measure-transformation arguments (instead of Fourier transform techniques). On the other hand, the price one has to pay for this convenience is not only to have an unrealistic spot model (which cannot generate price spikes), but also the modeling of additional information concerning future price behavior turns out to be problematic, since we neither may work with an *explicit* intermediate filtration [like in (3.3.115)] for the following reason: Obviously, we are now caught in a single-factor model with $l = d = 1$ and hence – using the vocabulary of paragraph 3.3.9 – we observe $\tilde{\mathcal{G}}_t = \widetilde{\mathcal{H}}_t$. Consequently, we ought to work within a *non-explicit* intermediate filtration $\mathcal{G}_t$ obeying

(3.3.139)
$$\tilde{\mathcal{F}}_t \subset \mathcal{G}_t \subset \widetilde{\mathcal{H}}_t$$

for $0 \leq t < \tau$ (recall that $\tilde{\mathcal{F}}_t = \mathcal{G}_t = \widetilde{\mathcal{H}}_t$ for $t \geq \tau$, anyway), whereas we presume

(3.3.140)
$$\widetilde{\mathcal{H}}_t \coloneqq \tilde{\mathcal{F}}_t \vee \sigma\left\{\int_0^\tau e^{\lambda r} dB_r\right\} = \tilde{\mathcal{F}}_t \vee \sigma\left\{\int_0^\tau e^{\lambda r} d\tilde{B}_r\right\}$$

and $\tilde{\mathcal{F}}_t \coloneqq \sigma\{B_s : 0 \leq s \leq t\}$. Furthermore, similar computations as in (3.3.119) – (3.3.124) lead us to the following representation for the (single-factor) electricity futures price $F_t^{\mathcal{G}} \coloneqq F_t^{\mathcal{G},\tilde{\mathbb{Q}}}(\tau_1, \tau_2)$ [defined parallel to (3.3.119), but yet associated to (3.3.138) – (3.3.139)] reading

(3.3.141)
$$F_t^{\mathcal{G}} = \Gamma(t) + \Psi(t)\, X_t + \Pi(t)\, Z_t$$

with shorthand notations

(3.3.142)
$$\Gamma(t) \coloneqq \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2 - \tau_1} du - \frac{\sigma}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \int_u^t e^{-\lambda(u-s)}\, G(s)\, ds\, du, \qquad \Psi(t) \coloneqq \frac{e^{-\lambda(\tau_1 - t)} - e^{-\lambda(\tau_2 - t)}}{\lambda\,(\tau_2 - \tau_1)},$$

$$\Pi(t) \coloneqq \frac{2\,\lambda\,\sigma}{e^{2\lambda t} - e^{2\lambda\tau}} \int_{\tau_1}^{\tau_2} \int_u^t \frac{e^{\lambda(2s-u)}}{\tau_2 - \tau_1} ds\, du, \qquad Z_t \coloneqq \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^\tau e^{\lambda r} d\tilde{B}_r \,\middle|\, \mathcal{G}_t\right).$$

We remark that during the derivation procedure of (3.3.141) we have used (an adapted version of) Lemma 3.3.5 (a) and (c), meanwhile. Further, in order to obtain the stochastic dynamics of $F_t^{\mathcal{G}}$, we provide the following result which, by the way, slightly resembles Lemma 3.3.4 above. In particular, we ought to remind the footnote dedicated to the proof of Lemma 3.3.5 (c) at this step.

Lemma 3.3.6 *For Z_t as defined in (3.3.142), the stochastic process*

$$\left(\frac{Z_t}{e^{2\lambda\tau} - e^{2\lambda t}}\right)_{t\in[0,\tau[}$$

designates a $\mathcal{G}_t$-adapted martingale under the risk-neutral pricing measure $\widetilde{\mathbb{Q}}$.

Proof Let $0 \le t \le s < \tau$. With respect to (3.3.118) and (3.3.142), we deduce

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\frac{Z_s}{e^{2\lambda\tau} - e^{2\lambda s}}\middle|\mathcal{G}_t\right) = \frac{1}{e^{2\lambda\tau} - e^{2\lambda s}}\ \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\int_s^\tau e^{\lambda r} d\tilde{B}_r \middle| \mathcal{G}_{s\wedge t}\right) = \frac{Z_t}{e^{2\lambda\tau} - e^{2\lambda t}}. \blacksquare$$

Anyway, Lemma 2.1.5 immediately leads us to the decomposition

(3.3.143)

$$dZ_t = \left(e^{2\lambda\tau} - e^{2\lambda t}\right) d\left(\frac{Z_t}{e^{2\lambda\tau} - e^{2\lambda t}}\right) - \frac{2\,\lambda\, e^{2\lambda t}}{e^{2\lambda\tau} - e^{2\lambda t}}\, Z_t\, dt.$$

Next, applying Itô's product rule on (3.3.141) while taking (3.3.105), (3.3.110), (3.3.142), (3.3.143), [an adapted version of] Lemma 3.3.5 (a) and Lemma 3.3.6 into account, we finally end up with the $(\mathcal{G}_t, \widetilde{\mathbb{Q}})$-martingale dynamics

(3.3.144)

$$dF_t^{\mathcal{G}} = \sigma\,\Psi(t)\, d\tilde{B}_t^{\mathcal{G},\widetilde{\mathbb{Q}}} + \Pi(t)\left[e^{2\lambda\tau} - e^{2\lambda t}\right] d\left(\frac{Z_t}{e^{2\lambda\tau} - e^{2\lambda t}}\right)$$

(with vanishing drift) wherein

$$\tilde{B}_t^{\mathcal{G},\widetilde{\mathbb{Q}}} := \tilde{B}_t - \int_0^t a(s)\, Z_s\, ds := \tilde{B}_t - \int_0^t \frac{2\,\lambda\, e^{\lambda s}}{e^{2\lambda\tau} - e^{2\lambda s}}\ \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\int_s^\tau e^{\lambda r} d\tilde{B}_r \middle| \mathcal{G}_s\right) ds$$

constitutes a $(\mathcal{G}_t, \widetilde{\mathbb{Q}})$-Brownian motion [what follows from Lemma 3.3.5 (a)]. Unfortunately, the distribution of (3.3.144) is not obvious. Thus, in order to price options written on the electricity futures price $F^{\mathcal{G}}$, we replace $\mathcal{G}$ by $\widetilde{\mathcal{H}}$ in our proceedings what gives rise to the notation $F^{\widetilde{\mathcal{H}}}$. Initially, we recall that both Brownian integrals $\int_0^\tau e^{\lambda r} d\tilde{B}_r$ and $\int_0^s e^{\lambda r} d\tilde{B}_r$ are $\widetilde{\mathcal{H}}_s$-measurable – compare (3.3.114) along with Theorem 4.3.1 (ii) in [83] to verify this. Hence, the corresponding $(\widetilde{\mathcal{H}}, \widetilde{\mathbb{Q}})$-Brownian motion, denoted by $\tilde{B}^{\widetilde{\mathcal{H}},\widetilde{\mathbb{Q}}}$, possesses the decomposition

(3.3.145)

$$\tilde{B}_t^{\widetilde{\mathcal{H}},\widetilde{\mathbb{Q}}} := \tilde{B}_t - \int_0^t a(s) \int_s^\tau e^{\lambda r} d\tilde{B}_r\, ds.$$

Moreover, under $\widetilde{\mathcal{H}}$, the dynamics (3.3.144) transforms into

(3.3.146)

$$dF_t^{\widetilde{\mathcal{H}}} = \sigma\,\Psi(t)\,d\tilde{B}_t^{\widetilde{\mathcal{H}},\widetilde{\mathbb{Q}}} + \Pi(t)\left[e^{2\lambda\tau} - e^{2\lambda t}\right] d\left(\frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\int_t^\tau e^{\lambda r}\,d\tilde{B}_r\,\middle|\widetilde{\mathcal{H}}_t\right)}{e^{2\lambda\tau} - e^{2\lambda t}}\right)$$

whereas the second differential on the right hand side of (3.3.146) actually can be handled with Itô's product rule leading us to

$$d\left(\frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\int_t^\tau e^{\lambda r}\,d\tilde{B}_r\,\middle|\widetilde{\mathcal{H}}_t\right)}{e^{2\lambda\tau} - e^{2\lambda t}}\right) = d\left(\frac{\int_\tau^t e^{\lambda r}\,d\tilde{B}_r}{e^{2\lambda t} - e^{2\lambda\tau}}\right) = \frac{e^{\lambda t}}{e^{2\lambda t} - e^{2\lambda\tau}}\,d\tilde{B}_t^{\widetilde{\mathcal{H}},\widetilde{\mathbb{Q}}}.$$

Therewith, (3.3.146) may be rearranged as

(3.3.147) $$dF_t^{\widetilde{\mathcal{H}}} = \Sigma(t)\,d\tilde{B}_t^{\widetilde{\mathcal{H}},\widetilde{\mathbb{Q}}}$$

which evidently designates a [not necessarily strictly positive] Brownian $(\widetilde{\mathcal{H}}_t, \widetilde{\mathbb{Q}})$-martingale of single-factor Bachelier-type. Herein, we have just introduced the deterministic function

$$\Sigma(t) := \sigma\,\Psi(t) - \Pi(t)\,e^{\lambda t}.$$

Further on, the price of a European call option written on the $\widetilde{\mathcal{H}}$-forward-looking electricity futures (3.3.147) actually may be calculated by two different techniques: Firstly, we might apply Fourier transform methods as presented in paragraph 3.3.9, for instance. Secondly, the more straightforward way is to trouble standard measure-transformation arguments, since the distribution of $F^{\widetilde{\mathcal{H}}}$ is currently known explicitly. [Particularly, note that $F^{\widetilde{\mathcal{H}}}$ possesses $\widetilde{\mathbb{Q}}$-independent increments with respect to $\widetilde{\mathcal{H}}$.] Although we will investigate the mentioned measure-transformation approach in the following, we initially claim the corresponding Fourier transform call option price formula by adapting (3.3.128) – (3.3.131) to our present single-factor model. In this regard, we observe

(3.3.148)

$$C_t^{\widetilde{\mathcal{H}}} = e^{-r(T-t)}\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\left[F_T^{\widetilde{\mathcal{H}}} - K\right]^+\middle|\widetilde{\mathcal{H}}_t\right) = \frac{e^{-r(T-t)}}{2\pi}\int_{\mathbb{R}} \frac{e^{(a+iy)\left[F_t^{\widetilde{\mathcal{H}}} - K + \frac{a+iy}{2}\int_t^T \Sigma^2(s)ds\right]}}{(a+iy)^2}\,dy.$$

Alternatively, we present the second approach now: Taking (3.3.147) into account, we obtain

(3.3.149) $$C_t^{\widetilde{\mathcal{H}}} = e^{-r(T-t)}\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\left[F_t^{\widetilde{\mathcal{H}}} - K + \Upsilon_{t,T}\right]^+\middle|\widetilde{\mathcal{H}}_t\right)$$

whereas the (real-valued) random variable

$$\Upsilon_{t,T} := \int_t^T \Sigma(s)\,d\tilde{B}_s^{\widetilde{\mathcal{H}},\widetilde{\mathbb{Q}}}$$

is $\widetilde{\mathbb{Q}}$-independent of $\widetilde{\mathcal{H}}_t$ and moreover, normally distributed under $\widetilde{\mathbb{Q}}$ with zero mean and variance

$$\vartheta^2(t,T) := \int_t^T \Sigma^2(s)\,ds.$$

Hence, conditioning on $\varrho := F_t^{\widetilde{\mathcal{H}}}$, from (3.3.149) we receive the call option price formula

(3.3.150) $$C_t^{\widetilde{\mathcal{H}}} := C_t^{\widetilde{\mathcal{H}},\widetilde{\mathbb{Q}}} =$$

$$e^{-r(T-t)} \int_{K-\varrho}^{\infty} [x-(K-\varrho)]\, d\widetilde{\mathbb{Q}}^{\Upsilon_{t,T}}(x)\Big|_{\varrho:=F_t^{\widetilde{\mathcal{H}}}} = e^{-r(T-t)} \left[\vartheta(t,T)\, \Phi'(\xi_{t,T}) + \left(F_t^{\widetilde{\mathcal{H}}} - K\right) \Phi(\xi_{t,T})\right]$$

with stochastic arguments $\xi_{t,T} := \left(F_t^{\widetilde{\mathcal{H}}} - K\right)/\vartheta(t,T)$. Herein, Φ designates the standard normal distribution function. Actually, (3.3.150) closely resembles Theorem 2 in [6][33], whereas we remark that the latter stems from a non-explicit *intermediate* filtration approach, while (3.3.150) is associated to the *global* filtration $\widetilde{\mathcal{H}}$ on the contrary. Moreover, both call option price formulas succumb to completely different risk-neutral pricing measures. Finally, the information premium for our current single-factor model can be calculated parallel to the argumentation at the end of subsection 3.3.9.

3.3.9.2 A forward-looking pricing measure for electricity options

The motivation for the current subsection is twofold: Firstly, we aim to extend Theorem 2 in [6] yet to a (possibly more appropriate) *multi*-factor version and, secondly, to compare the approach in [6] with ours in 3.3.9 and 3.3.9.1 – particularly, we do this with respect to the different underlying risk-neutral pricing measures. To begin with, we catch up our arithmetic electricity spot price model presented in paragraph 3.3.9 but yet with weights $w_{l+1} = \cdots = w_n = 0$. That is, there are no jumps occurring in (3.3.107) which, by the way, should be regarded as true disadvantage (cf. e.g. Ch. 1 and 2 in [8] or Ch. 2 in [72] in this context). However, we actually aim to focus on an alternative *forward-looking change of probability measure* which firstly has been proposed by Protter for the Brownian motion case (see reference "[31] in [6]"). Initially, we introduce the notion of *dualism* which we establish with reference to subsection 3.3.9: Therein, we *firstly* switched to a risk-neutral probability measure [actually without touching the filtration; see (3.3.108)] and *secondly* took an enlargement of the underlying filtration into account [see (3.3.114) – (3.3.115)]. Whenever these two steps are done *separately*, we will call this *dualism* from now on. (By the way, except from the present paragraph, we always work with the dualism concept in this thesis; especially, in section 3.3.6 the dualism setup is very evident.) On the contrary, in the mentioned approach proposed by Protter both dualism steps are done *simultaneously*. Extending [6], we now adjust Protter's method to a BM-driven *multi*-factor electricity spot price model and hereafter, compare the associated option pricing results with those of section 3.3.9 and 3.3.9.1. Starting off, we refer to sect. 3.2 in [6] and to our former definition (3.3.108) in order to implement a customized $\mathcal{G}$-*forward-looking* risk-neutral probability measure $\widehat{\mathbb{Q}} := \widehat{\mathbb{Q}}(\mathcal{G})$ via

(3.3.151)

$$\left.\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{G}_t} := \prod_{k=1}^{l} \mathfrak{E}\left(-\theta_{\cdot}^{k,\mathcal{G}} \circ B_{\cdot}^{k,\mathcal{G},\mathbb{P}}\right)_t := \prod_{k=1}^{l} exp\left\{-\int_0^t \theta_s^{k,\mathcal{G}} dB_s^{k,\mathcal{G},\mathbb{P}} - \frac{1}{2}\int_0^t \left(\theta_s^{k,\mathcal{G}}\right)^2 ds\right\}.$$

Yet, the underlying filtrations are defined due to

$$\mathcal{F}_t := \sigma\{B_s^1, \dots, B_s^l : 0 \le s \le t\}, \qquad \mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t := \mathcal{F}_t \vee \sigma\{B_\tau^1, \dots, B_\tau^l\}.$$

[33] There are some notational inaccuracies in Theorem 2 in [6]: $F_{\mathcal{F}}$ therein has to be replaced by $F_{\mathcal{G}}$ while $d_2^{\mathcal{G}}$ must be replaced by $d_1^{\mathcal{G}}$ which is not "*defined as in equation (23)*", but defined as announced previously to Theorem 2.

Moreover, for $k = 1, \dots, l$ and $0 \le t < \tau$ we presume

(3.3.152)

$$B_t^{k,\mathcal{G},\mathbb{P}} := B_t^k - \int_0^t \theta_s^{k,\mathcal{G}}\, ds$$

to constitute a $(\mathcal{G}_t, \mathbb{P})$-Brownian motion (BM). Herein, B_t^k designates a $(\mathcal{F}_t, \mathbb{P})$-BM and $\theta_t^{k,\mathcal{G}}$ represents a $\mathcal{G}_t$-adapted stochastic process, the so-called *information yield* [10], which we assume to fulfill the Novikov condition (cf. sect. 2.2 above). By the way, Prop. 2.3.3 yields the explicit form

(3.3.153)

$$\theta_s^{k,\mathcal{G}} = \frac{\mathbb{E}_{\mathbb{P}}\big(B_\tau^k \big| \mathcal{G}_s\big) - B_s^k}{\tau - s}.$$

More importantly, from (3.3.151) and Girsanov's theorem we further deduce that

(3.3.154)

$$\hat{B}_t^k := \hat{B}_t^{k,\mathcal{G},\widehat{\mathbb{Q}}} := B_t^{k,\mathcal{G},\mathbb{P}} + \int_0^t \theta_s^{k,\mathcal{G}}\, ds$$

depicts a $\big(\mathcal{G}_t, \widehat{\mathbb{Q}}\big)$-BM for all $k = 1, \dots, l$. Combining (3.3.152) and (3.3.154), we remarkably observe

(3.3.155)

$$\hat{B}_t^{k,\mathcal{G},\widehat{\mathbb{Q}}} - \int_0^t \theta_s^{k,\mathcal{G}}\, ds = B_t^{k,\mathcal{G},\mathbb{P}} = B_t^k - \int_0^t \theta_s^{k,\mathcal{G}}\, ds$$

to be valid $\big(\mathbb{P} \equiv \widehat{\mathbb{Q}}\big)$-almost-sure (i.e. $\mathbb{P}$-a.s. and also $\widehat{\mathbb{Q}}$-a.s., as both measures are equivalent) for each t and $k = 1, \dots, l$. Thus, $\hat{B}^k$ is a *modification* of B^k for each k. This means that, for each t and k, there exist sets $\mathcal{N}_t^k \subset \Omega$ with $\mathbb{P}\big(\mathcal{N}_t^k\big) = \widehat{\mathbb{Q}}\big(\mathcal{N}_t^k\big) = 0$ such that $\mathbb{P}\big(\big\{\omega \in \Omega \setminus \mathcal{N}_t^k \big| \hat{B}_t^k(\omega) = B_t^k(\omega)\big\}\big) = \widehat{\mathbb{Q}}\big(\big\{\omega \in \Omega \setminus \mathcal{N}_t^k \big| \hat{B}_t^k(\omega) = B_t^k(\omega)\big\}\big) = 1$, respectively such that $\big\{\omega \in \Omega \big| \hat{B}_t^k(\omega) \neq B_t^k(\omega)\big\} \subset \mathcal{N}_t^k$. Further, Theorem 2 in Chapter I of [78] induces that $\hat{B}^k$ and B^k even are *indistinguishable*, meaning that $\big(\mathbb{P} \equiv \widehat{\mathbb{Q}}\big)$-a.s., for all t and k, we have $\hat{B}_t^k = B_t^k$. Moreover, Lemma 1.4.8 in [1] together with its short prolog declares the (Lévy-) processes $\hat{B}^k$ and B^k to possess the same characteristics and thus, also the same (actually Gaussian-) distribution. In other words, for all indices $k = 1, \dots, l$ the $(\mathcal{F}, \mathbb{P})$-Brownian motions B^k simultaneously are $\big(\mathcal{G}, \widehat{\mathbb{Q}}\big)$-Brownian motions. However, the essence of this result firstly has been announced by Protter – compare "[31] in [6]". We stress that, in order to obtain (3.3.155), we trivially have to choose $G := -\theta^{k,\mathcal{G}}$ and $W := B^{k,\mathcal{G},\mathbb{P}}$ (and $h \equiv 0$) in Prop. 2.2.1. Alternatively to [6], we next take the $\widehat{\mathbb{Q}}$-risk-neutral $\mathcal{G}$-forward-looking electricity futures price

(3.3.156) $$\hat{F}_t := F_t^{\mathcal{G},\widehat{\mathbb{Q}}}(\tau_1, \tau_2) := \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \mathbb{E}_{\widehat{\mathbb{Q}}}(S_u | \mathcal{G}_t)\, du$$

as the starting point of our current setup.[34]

[34] On the contrary, in [6] the futures price initially is introduced under $(\mathcal{F}, \mathbb{P})$; compare equation "(11) in [6]".

Then, in accordance to (3.3.107), (3.3.111), (3.3.125), (3.3.155) and (3.3.156), we obtain

$$F_t^{\mathcal{G},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2-\tau_1}\,du + \sum_{k=1}^{l} \widehat{\Psi}_k(t)\, X_t^k.$$

Moreover, using (3.3.105), (3.3.126), (3.3.155) and Lemma 2.1.5, the latter decomposition leads us to the Brownian $(\mathcal{G}_t,\widehat{\mathbb{Q}})$-martingale dynamics[35]

(3.3.157)

$$dF_t^{\mathcal{G},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) = \sum_{k=1}^{l} \sigma_k\, \widehat{\Psi}_k(t)\, d\widehat{B}_t^{k,\mathcal{G},\widehat{\mathbb{Q}}}$$

which extends property "(17) in [6]" to the multi-factor case. We highlight that (3.3.157) [combined with (3.3.155)] further delivers

$$dF_t^{\mathcal{G},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) = \sum_{k=1}^{l} \sigma_k\, \widehat{\Psi}_k(t)\, dB_t^k = dF_t^{\mathcal{F},\mathbb{P}}(\tau_1,\tau_2).$$

Thus, sticking to the vocabulary introduced in the sequel of (3.3.155), the electricity futures price processes $F^{\mathcal{G},\widehat{\mathbb{Q}}}$ and $F^{\mathcal{F},\mathbb{P}}$ are detected to be *indistinguishable* which might sound a bit strange from an economical point of view. In this context, we remark that also if we had taken

$$F_t^{\mathcal{F},\mathbb{P}}(\tau_1,\tau_2) := \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \mathbb{E}_{\mathbb{P}}(S_u|\mathcal{F}_t)\,du$$

[instead of (3.3.156)] as our starting point, we would yet have received $dF_t^{\mathcal{F},\mathbb{P}}(\tau_1,\tau_2) = dF_t^{\mathcal{G},\widehat{\mathbb{Q}}}(\tau_1,\tau_2)$, similar to above. By the way, the latter $(\mathcal{F},\mathbb{P})$-approach is presented in [6].

Nevertheless, we may calculate the price of a European call option written on (3.3.157) as follows: Parallel to (3.3.149) – (3.3.150), we claim the $\mathcal{G}$-forward-looking call price as

(3.3.158) $$\hat{C}_t^{\mathcal{G},\widehat{\mathbb{Q}}} = e^{-r(T-t)}\, \mathbb{E}_{\widehat{\mathbb{Q}}}\left(\left[P_{t,T} - \left(K-\hat{F}_t\right)\right]^+ \Big| \mathcal{G}_t\right)$$

within a linear combination of $(\mathcal{G}_t,\widehat{\mathbb{Q}})$-independent Brownian-martingale increments

(3.3.159)

$$P_{t,T} := \hat{F}_T - \hat{F}_t = \sum_{k=1}^{l} \sigma_k \int_t^T \widehat{\Psi}_k(s)\, d\widehat{B}_s^{k,\mathcal{G},\widehat{\mathbb{Q}}}.$$

Applying standard measure-transformation arguments on (3.3.158), we ultimately end up with the $(\mathcal{G},\widehat{\mathbb{Q}})$-call price formula

(3.3.160) $$\hat{C}_t^{\mathcal{G},\widehat{\mathbb{Q}}} = e^{-r(T-t)}\left[v(t,T)\,\Phi'\left(\zeta_{t,T}\right) + \left(\hat{F}_t - K\right)\Phi\left(\zeta_{t,T}\right)\right]$$

[35] Actually, the futures price (3.3.157) is no longer strictly positive which may be regarded as a disadvantage. However, in this context we remind the footnote dedicated to (3.3.93) – particularly, the explained influence which *positive jumps* may have on spot/futures prices. With respect to our current setup, we remark that it is easily possible to compute probabilities like $\widehat{\mathbb{Q}}\left(\left\{F_t^{\mathcal{G},\widehat{\mathbb{Q}}} > 0\right\}\right) = \mathbb{P}\left(\left\{F_t^{\mathcal{F},\mathbb{P}} > 0\right\}\right)$ or $\widehat{\mathbb{Q}}\left(\left\{F_t^{\mathcal{G},\widehat{\mathbb{Q}}} \le 0\right\}\right)$.

wherein Φ designates the standard normal distribution function and

(3.3.161)

$$\zeta_{t,T} := \frac{\hat{F}_t - K}{v(t,T)}, \quad v^2(t,T) := \mathbb{Var}_{\hat{\mathbb{Q}}}[P_{t,T}] = \sum_{k=1}^{l} \sigma_k^2 \int_t^T \hat{\Psi}_k(s)^2 \, ds, \quad \hat{F}_t = \hat{F}_0 + \sum_{k=1}^{l} \sigma_k \int_0^t \hat{\Psi}_k(s) \, d\hat{B}_s^{k,\mathcal{G},\hat{\mathbb{Q}}}.$$

If we choose $l = 1$ in (3.3.157), then the corresponding *single*-factor pricing formula (3.3.160) – not surprisingly – possesses exactly the same structure as the one announced in Theorem 2 in [6]. More accurately speaking, for $l = 1$ (and $r \equiv 0$) we obviously are in the setup presented in [6]. Anyway, our multi-factor extension of [6] only partly can be seen as the motivation for the current paragraph. Instead, the more interesting task is to compare the *dualism* call option price formula in (3.3.150) with the one in (3.3.160)[36], although both pricing formulas appear very similar on a superficial view (which itself is a remarkable observation). On a closer look, we firstly recognize a striking difference concerning the underlying pricing measures: More precisely, in section 3.3.9.1 we worked under the *backward*-looking *dualism*-measure $\tilde{\mathbb{Q}} := \tilde{\mathbb{Q}}(\tilde{\mathcal{F}})$ defined due to (3.3.108) [with $l = n = 1$ therein], whereas in paragraph 3.3.9.2 we utilized the *forward*-looking risk-neutral measure $\hat{\mathbb{Q}} := \hat{\mathbb{Q}}(\mathcal{G})$ such as established in (3.3.151). Secondly, in 3.3.9.1 the *global* (respectively *overall*) filtration $\tilde{\mathcal{H}}$ enlarges $\tilde{\mathcal{F}}$ by a whole Brownian *integral* [see (3.3.140)], whereas its counterpart in 3.3.9.2 merely adds Brownian *values* to $\mathcal{F}$ [recall the sequel of (3.3.151)]. Hence, there are different martingale compensators (i.e. *information yields*) involved [compare (3.3.145) with (3.3.153)]. More importantly, the call price in (3.3.150) actually succumbs to *complete* or *exhaustive* knowledge about future spot price behavior, as it is derived under the *overall* filtration $\tilde{\mathcal{H}}$. Contrarily, the call price in (3.3.160) is closely linked with the (*non*-explicit) *intermediate* filtration $\mathcal{G}$ which, at least in this instance, portrays a more realistic setup. In this context, we recall that the dynamics (3.3.144) did not seem to be suitable for a proper derivation of related option prices, so that we finally proposed to work under $\tilde{\mathcal{H}}$. In conclusion, the dualism concept along with *non-explicit* intermediate filtrations [like $\mathcal{G}$ in (3.3.139)] does not appear very useful for option pricing issues in electricity markets.[37]

Nevertheless, an *explicit* intermediate filtration approach such as presented in paragraph 3.3.9 [compare (3.3.115)] even should be *more* appropriate for electricity futures option pricing purposes than the one presented in 3.3.9.2, respectively in [6], mainly for the three following reasons:
Firstly, jump terms do not have to be omitted in the forward-looking dualism approach of subsection 3.3.9, whereas in [6] there are neither jump components in the electricity spot price permitted, nor is there a seasonality function present, on the opposite.[38]
Secondly, an *explicitly* enlarged filtration like in (3.3.115) ought to be more suitable for practical applications, as the available forward-looking information on the future spot price behavior can be established more accurately than in the *non*-explicit setup of subsection 3.3.9.2, respectively of [6].[39]
All in all, regarding the (Black-Scholes-type) option price formulas in (3.3.160) or in [6], one might doubt that the underlying model is able to capture the sophisticated properties of electricity price behavior sufficiently well. In other words, it seems difficult to believe that (3.3.160) portrays an adequate tool to derive option prices for, particularly, *electricity* derivatives under *future information*.

---

[36] In our comparison we neglect the fact of having a different number of noises involved in both models, i.e. the reader may assume $l = 1$ in (3.3.160) for a moment in order to concentrate on the more essential differences.
[37] Also recall our former announcements stated previously to (3.3.38), particularly the formulated *key idea*.
[38] Admittedly, the presence of a seasonality function does not influence option prices. Compare e.g. the equations (3.3.138) and (3.3.150) to verify this.
[39] Also in [10] the authors mostly work with *non*-explicit intermediate filtrations; particularly recall the equalities "(3.2), (3.9) and (3.23) in [10]" in this context.

Thirdly, the most striking curiosity throughout subsection 3.3.9.2 possibly is embodied by the fact that we obtain precisely the same pricing formula as in (3.3.160), even if we *neither* change the probability measure, *nor* work under an enlarged filtration at all! We now explain this peculiarity in more detail: Since $F^{\mathcal{G},\widehat{\mathbb{Q}}}$ and $F^{\mathcal{F},\mathbb{P}}$ are indistinguishable, the same is valid for the corresponding call option price processes $\hat{C}^{\mathcal{G},\widehat{\mathbb{Q}}}$ and $\hat{C}^{\mathcal{F},\mathbb{P}}$. This fact may be interpreted as if additional forward-looking information on future spot price behavior was *irrelevant* for both electricity futures and related option prices which sounds odd – not only with respect to the convincing argumentation in [10] counting in favor for a strong relevance of future information especially in electricity markets.
Admittedly, we rather should compare $\hat{C}^{\mathcal{G},\widehat{\mathbb{Q}}}$ with $\hat{C}^{\mathcal{F},\widehat{\mathbb{Q}}}$ [instead of $\hat{C}^{\mathcal{G},\widehat{\mathbb{Q}}}$ with $\hat{C}^{\mathcal{F},\mathbb{P}}$] in order to examine how (or if at all) additional future information is incorporated into a certain enlarged filtration model (reasonably). Anyway, the just explained oddity between $(\mathcal{F},\mathbb{P})$- and $(\mathcal{G},\widehat{\mathbb{Q}})$-prices certainly has to be classified as the non-standard case, as the futures prices $F^{\mathcal{G},\widehat{\mathbb{Q}}}$ and $F^{\mathcal{F},\widehat{\mathbb{Q}}}$ [in contrast to $F^{\mathcal{G},\widehat{\mathbb{Q}}}$ and $F^{\mathcal{F},\mathbb{P}}$; compare the sequel of (3.3.157)] usually differ by an additive information yield/drift[40] which, by the way, sounds economically reasonable. In this regard, we emphasize that for Protter's measure change in (3.3.151) we do not at all have an associated $(\mathcal{F},\widehat{\mathbb{Q}})$-Brownian motion available, unfortunately, and thus, neither know the dynamics of $F^{\mathcal{F},\widehat{\mathbb{Q}}}$, nor a formula for $\hat{C}^{\mathcal{F},\widehat{\mathbb{Q}}}$. For this reason, we only may compare $\hat{C}^{\mathcal{G},\widehat{\mathbb{Q}}}$ with $\hat{C}^{\mathcal{F},\mathbb{P}}$ leading us to the mentioned curiosity of indistinguishable option prices. Consequently, it is neither possible to compute the *information premium* under $\widehat{\mathbb{Q}}$ for the setup in [6], resp. in 3.3.9.2 (merely under $\mathbb{P}$, as presented in [6]), which might be regarded as another disadvantage of Protter's measure change, since futures prices are commonly defined under a risk-neutral measure what makes $\mathfrak{I}^{\mathcal{G},\mathcal{F},\widehat{\mathbb{Q}}}$ become the more interesting object, not $\mathfrak{I}^{\mathcal{G},\mathcal{F},\mathbb{P}}$. This problem fortunately does not arise in the setup of paragraph 3.3.9 [recall (3.3.110) and (3.3.133) ff. to verify this additional advantage of our dualism approach]. In order to prove the mentioned oddity in connection with $(\mathcal{F},\mathbb{P})$- and $(\mathcal{G},\widehat{\mathbb{Q}})$-call option prices, we finally claim that in the $(\mathcal{F},\mathbb{P})$-case we derive

$$\hat{C}_t^{\mathcal{F},\mathbb{P}} = e^{-r(T-t)}\left[v(t,T)\,\Phi'\big(\tilde{\zeta}_{t,T}\big) + \big(F_t^{\mathcal{F},\mathbb{P}} - K\big)\,\Phi\big(\tilde{\zeta}_{t,T}\big)\right],$$

$$\tilde{\zeta}_{t,T} := \frac{F_t^{\mathcal{F},\mathbb{P}} - K}{v(t,T)}, \qquad F_t^{\mathcal{F},\mathbb{P}} = F_0^{\mathcal{F},\mathbb{P}} + \sum_{k=1}^{l} \sigma_k \int_0^t \widehat{\Psi}_k(s)\, dB_s^k$$

instead of (3.3.160) and (3.3.161), respectively. Herein, the deterministic function $v(t,T)$ is like in (3.3.161). By the way, choosing $l = 1$ and $r = 0$ inside the latter equations, we get precisely the same pricing formula as announced in Theorem 1 in [6]. Since the futures prices $\hat{F} := F^{\mathcal{G},\widehat{\mathbb{Q}}}$ and $F^{\mathcal{F},\mathbb{P}}$ are indistinguishable, the stochastic argument processes $\zeta_{t,T}$ and $\tilde{\zeta}_{t,T}$ are so, too. Consequently, the related call option price processes $\hat{C}^{\mathcal{G},\widehat{\mathbb{Q}}}$ and $\hat{C}^{\mathcal{F},\mathbb{P}}$ are indistinguishable, likewise.
All in all, appealing to the just enumerated shortcomings of Protter's forward-looking measure change method, the latter does not at all seem to be appropriate for option pricing purposes under future information in electricity markets.

The risk premium for Protter's measure change In what follows, we aim to compute the *risk premium* for the just presented approach related to Protter's measure change, as it seems to be interesting to study the altitude between electricity futures prices that stem from $\mathbb{P}$ on the one hand, and from $\widehat{\mathbb{Q}}$ on the other (both under $\mathcal{G}$), in more detail. We recall that the mentioned magnitude in fact can be measured by the *risk premium* which we define through [cf. subsection 3.3.1 above]

[40] Actually, this is the case in subsection 3.3.9 [and also in e.g. paragraph 3.3.3; compare (3.2.23) with (A.2) to see this]. Moreover, we strongly refer to our quotation given previously to Condition A in this context.

$$\mathfrak{R}_t^{\mathcal{G},\mathbb{P},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) := F_t^{\mathcal{G},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) - F_t^{\mathcal{G},\mathbb{P}}(\tau_1,\tau_2) = \sum_{k=1}^{l} \frac{w_k}{\tau_2-\tau_1} \int_{\tau_1}^{\tau_2} \left\{ \mathbb{E}_{\widehat{\mathbb{Q}}}\left(X_u^k \middle| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{P}}\left(X_u^k \middle| \mathcal{G}_t\right) \right\} du.$$

In accordance to (3.3.111), (3.3.152) and (3.3.155), [for $u < \tau$] we immediately obtain

$$\mathbb{E}_{\widehat{\mathbb{Q}}}\left(X_u^k \middle| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{P}}\left(X_u^k \middle| \mathcal{G}_t\right) = -\sigma_k \int_t^u e^{-\lambda_k(u-s)}\, \mathbb{E}_{\mathbb{P}}\left(\theta_s^{k,\mathcal{G}} \middle| \mathcal{G}_t\right) ds.$$

Next, with respect to (3.3.153), the latter equation becomes

$$\mathbb{E}_{\widehat{\mathbb{Q}}}\left(X_u^k \middle| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{P}}\left(X_u^k \middle| \mathcal{G}_t\right) = -\sigma_k \int_t^u e^{-\lambda_k(u-s)}\, \frac{\mathbb{E}_{\mathbb{P}}\left(B_\tau^k - B_s^k \middle| \mathcal{G}_t\right)}{\tau - s} ds.$$

Further, note that Prop. A.3 in [10] with $g(u) := 1/(\tau - u)$, $f(u) \equiv 1$, $T_1 := \tau$ and $B := B^k$ yields

$$\mathbb{E}_{\mathbb{P}}\left(B_\tau^k - B_s^k \middle| \mathcal{G}_t\right) = \frac{\tau - s}{\tau - t}\, \mathbb{E}_{\mathbb{P}}\left(B_\tau^k - B_t^k \middle| \mathcal{G}_t\right)$$

for $0 \le t \le s < \tau$ and $k = 1, \dots, l$. Therewith, we receive

$$\mathbb{E}_{\widehat{\mathbb{Q}}}\left(X_u^k \middle| \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{P}}\left(X_u^k \middle| \mathcal{G}_t\right) = \sigma_k \frac{\mathbb{E}_{\mathbb{P}}\left(B_\tau^k - B_t^k \middle| \mathcal{G}_t\right)}{\tau - t} \frac{e^{-\lambda_k(u-t)} - 1}{\lambda_k}.$$

Hence, the *risk premium* turns out as

$$\mathfrak{R}_t^{\mathcal{G},\mathbb{P},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) = \sum_{k=1}^{l} \delta_k(t;\tau,\tau_1,\tau_2)\, \mathbb{E}_{\mathbb{P}}\left(B_\tau^k - B_t^k \middle| \mathcal{G}_t\right)$$

within deterministic functions

$$\delta_k(t;\tau,\tau_1,\tau_2) := w_k\, \sigma_k \frac{e^{-\lambda_k(\tau_1-t)} - e^{-\lambda_k(\tau_2-t)} - \lambda_k\,(\tau_2-\tau_1)}{\lambda_k^2\,(\tau_2-\tau_1)\,(\tau-t)}.$$

Verbalizing, the risk premium $\mathfrak{R}^{\mathcal{G},\mathbb{P},\widehat{\mathbb{Q}}}$ associated to Protter's measure change in (3.3.151) is given by a linear combination of deterministically weighted $\mathcal{G}$-adapted random variables under $\mathbb{P}$. Remarkably, there is no longer a $\widehat{\mathbb{Q}}$-dependency present.

The cross premium Reasoning about our above discussion concerning the striking differences between Protter's change-of-measure approach on the one hand, and our dualism concept on the other, we yet recommend to introduce a tailor-made mathematical indicator which is able to portray our previous observations adequately. However, it seems to be reasonable to measure – besides the *risk-* and *information premium* – also the altitude between $F^{\mathcal{G},\widehat{\mathbb{Q}}}$ and $F^{\mathcal{F},\mathbb{P}}$. In this context, let us remind that the *risk premium* describes the difference in futures prices with respect to the underlying probability measures, whereas the *information premium* captures the difference with respect to the involved information filtrations, in return. Innovatively, we now introduce a mixture of these magnitudes which we will call *cross premium* in our proceedings, as it measures the 'crossed' difference between $(\mathcal{F},\mathbb{P})$- and $(\mathcal{G},\widehat{\mathbb{Q}})$-futures prices. More accurately speaking, we define the *cross premium* via

$$d\mathcal{X}_t^{\mathcal{F},\mathcal{G},\mathbb{P},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) := dF_t^{\mathcal{G},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) - dF_t^{\mathcal{F},\mathbb{P}}(\tau_1,\tau_2)$$

wherein the differentials are taken with respect to the time parameter t.[41] Furthermore, taking (3.3.157) along with (3.3.155) into account, we remarkably receive

$$d\mathcal{X}_t^{\mathcal{F},\mathcal{G},\mathbb{P},\widehat{\mathbb{Q}}}(\tau_1,\tau_2) = 0$$

$(\mathbb{P} \equiv \widehat{\mathbb{Q}})$-almost-sure for all t. At this step, we emphasize that a vanishing cross premium adequately symbolizes the formerly described curiosity in connection with Protter's measure change, particularly the resulting indistinguishable futures (and also call option) prices.

In what follows, we aim to compute the cross premium also for our *dualism* setup presented in paragraph 3.3.9. To begin with, we adjust the above definition and announce

$$d\mathcal{X}_t^{\tilde{\mathcal{F}},\tilde{\mathcal{G}},\mathbb{P},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) = dF_t^{\tilde{\mathcal{G}},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) - dF_t^{\tilde{\mathcal{F}},\mathbb{P}}(\tau_1,\tau_2).$$

Next, similar computations as in (3.3.119) – (3.3.124) [but under $\tilde{\mathcal{F}}$ and $\mathbb{P}$ now] yield

$$F_t^{\tilde{\mathcal{F}},\mathbb{P}}(\tau_1,\tau_2) = \Gamma(t) + \sum_{k=1}^{n} \widehat{\Psi}_k(t)\, X_t^k$$

where $\widehat{\Psi}_k(t)$ is such as defined in (3.3.125) and

$$\Gamma(t) := \int_{\tau_1}^{\tau_2} \frac{\mu(u)}{\tau_2-\tau_1}\,du - \sum_{k=l+1}^{n} \frac{w_k}{\tau_2-\tau_1} \int_{\tau_1}^{\tau_2}\int_{u}^{t}\int_{D_k} z\,\sigma_k(s)\, e^{-\lambda_k(u-s)}\,\rho_k(s)\,d\nu_k(z)\,ds\,du.$$

Further on, using (3.2.4), (3.2.5), (3.2.24), (3.3.105), (3.3.106), (3.3.125) and (3.3.126), we obtain

$$dF_t^{\tilde{\mathcal{F}},\mathbb{P}}(\tau_1,\tau_2) = \sum_{k=1}^{l} \sigma_k\, \widehat{\Psi}_k(t)\, dB_t^k + \sum_{k=l+1}^{n} \int_{D_k} z\,\Lambda_k(t,\tau_1,\tau_2)\, \widetilde{N}_k^{\tilde{\mathcal{F}},\mathbb{P}}(t,dz)$$

which evidently constitutes a $(\tilde{\mathcal{F}}_t,\mathbb{P})$-martingale. Finally, the latter representation along with (3.3.127) claims the cross premium associated to the *dualism* setup of subsection 3.3.9 to be of the form

$$d\mathcal{X}_t^{\tilde{\mathcal{F}},\tilde{\mathcal{G}},\mathbb{P},\tilde{\mathbb{Q}}}(\tau_1,\tau_2) = \sum_{k=1}^{d}\left[\sigma_k\,\widehat{\Psi}_k(t) - \tilde{\Pi}_k(t)\, e^{\lambda_k t}\right] d\tilde{B}_t^{k,\tilde{\mathcal{G}},\tilde{\mathbb{Q}}} + \sum_{k=d+1}^{l} \sigma_k\,\widehat{\Psi}_k(t)\, d\tilde{B}_t^k$$

$$-\sum_{k=1}^{l} \sigma_k\,\widehat{\Psi}_k(t)\, dB_t^k + \sum_{k=l+1}^{n} \int_{D_k} z\,\Lambda_k(t,\tau_1,\tau_2)\left[1-e^{h_k(t,z)}\right]\rho_k(t)\,d\nu_k(z)\,dt$$

wherein we have just used (3.2.5) and (3.2.20). Regarding the latter equation, we recognize that, due to the structure of (3.3.115), the formerly involved jump noises have canceled out completely. Ultimately, we remark that, in general, the cross premium indicator associated to any *dualism* approach never vanishes – except from trivial and not relevant cases.

[41] Obviously, we likewise might have defined the *cross premium* without differentials – similar to our former definitions of the risk- and information premium. However, the actually presented differential version yields a *vanishing* cross premium for Protter's measure change which we believe is very suggestive.

Protter's forward-looking change-of-measure methodology for jump processes

As a closing remark, we want to examine the question whether it is possible to establish an alternative forward-looking measure change (equivalent to the one described above for the Brownian motion case) but yet for jump processes. Having equation (3.2.4) in mind, we initially introduce a $\mathcal{F}_t$-adapted, càdlàg, pure-jump, finite-variation[42], increasing (compound Poisson-type) Lévy process via

(3.3.162)

$$L_t := \int_0^t \int_D z \, dN(s,z)$$

within a real set $D \subset]0,\infty[$. Next, for a $(\mathcal{F},\mathbb{P})$-compensated Poisson-random-measure (PRM)

(3.3.163) $$d\tilde{N}^{\mathcal{F},\mathbb{P}}(s,z) := dN(s,z) - d\nu^{\mathcal{F},\mathbb{P}}(s,z) := dN(s,z) - d\nu(z)\, ds$$

(with Lévy-measure ν) we implement a related $(\mathcal{F}_t,\mathbb{P})$-martingale due to

(3.3.164)

$$L_t^{\mathcal{F},\mathbb{P}} := \int_0^t \int_D z \, d\tilde{N}^{\mathcal{F},\mathbb{P}}(s,z)$$

being a Lévy process again which, by the way, plays the role of B_t^k above. Further on, for time indices $0 \leq t < \tau$ we define the filtrations

(3.3.165) $$\mathcal{F}_t := \sigma\{L_s^{\mathcal{F},\mathbb{P}} : 0 \leq s \leq t\}, \qquad \mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t := \mathcal{F}_t \vee \sigma\{L_\tau^{\mathcal{F},\mathbb{P}}\}$$

whereby $\mathcal{F}_t = \mathcal{G}_t = \mathcal{H}_t$ is valid whenever $t \geq \tau$. Parallel to (3.3.152), we currently presume the existence of a $\mathcal{G}_t$-adapted stochastic process γ_t (the so-called *information yield* [10]) which induces the $(\mathcal{G}_t,\mathbb{P})$-martingale

(3.3.166)

$$L_t^{\mathcal{G},\mathbb{P}} := L_t^{\mathcal{F},\mathbb{P}} - \int_0^t \gamma_s \, ds.$$

Consequently, Proposition 2.3.3 along with (3.3.162) – (3.3.164) yields

(3.3.167)

$$\gamma_s = \frac{\mathbb{E}_{\mathbb{P}}\left(L_\tau^{\mathcal{F},\mathbb{P}} \middle| \mathcal{G}_s\right) - L_s^{\mathcal{F},\mathbb{P}}}{\tau - s} = \frac{\mathbb{E}_{\mathbb{P}}(L_\tau | \mathcal{G}_s) - L_s}{\tau - s} - \int_D z \, d\nu(z) = \int_D z \left\{ \frac{\mathbb{E}_{\mathbb{P}}\left(\int_{u=s}^{u=\tau} dN(u,z) \middle| \mathcal{G}_s\right)}{\tau - s} - d\nu(z) \right\}.$$

Hence, merging (3.3.163), (3.3.164) and (3.3.167) into (3.3.166), we receive

(3.3.168)

$$L_t^{\mathcal{G},\mathbb{P}} = \int_0^t \int_D z \, d\tilde{N}^{\mathcal{G},\mathbb{P}}(s,z) := \int_0^t \int_D z \left\{ dN(s,z) - d\nu^{\mathcal{G},\mathbb{P}}(s,z) \right\}$$

wherein the (stochastic) $(\mathcal{G},\mathbb{P})$-martingale compensator $\nu^{\mathcal{G},\mathbb{P}}$ is explicitly given through

[42] In accordance to Theorem 2.4.25 in [1], we hence assume z to be ν-integrable on the set $\{z \in \mathbb{R} : 0 < z < 1\}$.

$$d\nu^{\mathcal{G},\mathbb{P}}(s,z) := \frac{\mathbb{E}_{\mathbb{P}}\left(\int_{u=s}^{u=\tau} dN(u,z)\,\middle|\mathcal{G}_s\right)}{\tau - s}\,ds.$$

Note in passing that $L^{\mathcal{G},\mathbb{P}}$ no longer is a Lévy process, as its characteristics are not deterministic anymore. Combining the latter equation with (3.3.167), we instantly obtain

$$\gamma_s = \int_D z\left\{\frac{d\nu^{\mathcal{G},\mathbb{P}}(s,z)}{ds} - d\nu(z)\right\}.$$

Adhering to (3.3.151), we further introduce a $\mathbb{P}$-equivalent $\mathcal{G}$-forward-looking risk-neutral probability measure $\mathbb{Q} := \mathbb{Q}(\mathcal{G})$ via the Radon-Nikodym derivative

(3.3.169)

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_t} := \mathfrak{E}(M.)_t$$

within a local $(\mathcal{G}_t, \mathbb{P})$-martingale

(3.3.170)

$$M_t := \int_0^t \int_D \left[e^{h(s-,z)} - 1\right] d\tilde{N}^{\mathcal{G},\mathbb{P}}(s,z)$$

and a (strictly positive) discontinuous Doléans-Dade exponential

(3.3.171)

$$\begin{aligned}\mathfrak{E}(M.)_t &:= exp\left\{M_t - \frac{1}{2}[M^c_\cdot]_t\right\} \times \prod_{0\le s\le t}(1+\Delta M_s)\,e^{-\Delta M_s} \\ &= exp\left\{\int_0^t \int_D h(s-,z)\,d\tilde{N}^{\mathcal{G},\mathbb{P}}(s,z) - \int_0^t \int_D \left[e^{h(s,z)} - 1 - h(s,z)\right] d\nu^{\mathcal{G},\mathbb{P}}(s,z)\right\}.\end{aligned}$$

Herein, $h(s,z)$ constitutes a $\big(\mathcal{G}_s \otimes \mathfrak{B}(D)\big)$-previsible stochastic process which is supposed to be chosen such that $\mathbb{E}_{\mathbb{P}}[\mathfrak{E}(M.)_t] = 1$ holds for all $t \geq 0$ what implies that $\mathfrak{E}(M.)_t$ designates a ($\mathcal{G}_t$-adapted) *true* $\mathbb{P}$-martingale – also compare section 2.2 in this context. By the way, from Theorem 5.1.3 in [1] – or alternatively, from Theorem 2.1.6 above applied on (3.3.171) – we deduce that the exponential in (3.3.171) obeys the martingale representation

(3.3.172) $$\mathfrak{E}(M.)_t = 1 + \int_0^t \mathfrak{E}(M.)_{s-}\,dM_s.$$

Furthermore, we presume the $(\mathcal{G}, \mathbb{Q})$-compensated random measure to be of the form

(3.3.173) $$d\tilde{N}^{\mathcal{G},\mathbb{Q}}(s,z) := dN(s,z) - d\nu^{\mathcal{G},\mathbb{Q}}(s,z)$$

which gives rise to the $(\mathcal{G}_t, \mathbb{Q})$-martingale

(3.3.174)

$$L_t^{\mathcal{G},\mathbb{Q}} := \int_0^t \int_D z\,d\tilde{N}^{\mathcal{G},\mathbb{Q}}(s,z).$$

Exercise 3.3.7 (a) *Does Proposition 2.2.1 together with (3.3.169) – (3.3.173) allow to deduce the equality* $d\nu^{\mathcal{G},\mathbb{Q}}(s,z) = e^{h(s,z)}\, d\nu^{\mathcal{G},\mathbb{P}}(s,z)$*? [Hint: It might be helpful to consider the case* $h(s,z) := z$ *first. Also recall that the object* $\nu^{\mathcal{G},\mathbb{P}}$ *appearing inside (3.3.171) presently is not deterministic.]* ■

With respect to our former Brownian motion framework, we meanwhile remark that (3.3.167) corresponds to (3.3.153), (3.3.168) to (3.3.152) and (3.3.174) to (3.3.154), respectively. More importantly, we remind that our objective actually consists in an establishment of a jump-analogy to (3.3.155). For this purpose, we have to ensure that the jump processes $L^{\mathcal{F},\mathbb{P}}$ and $L^{\mathcal{G},\mathbb{Q}}$ become indistinguishable. Due to Theorem 2 in Chapter I of [78], we equivalently may guarantee that $L^{\mathcal{F},\mathbb{P}}$ and $L^{\mathcal{G},\mathbb{Q}}$ become modifications, i.e. $L_t^{\mathcal{F},\mathbb{P}} = L_t^{\mathcal{G},\mathbb{Q}}$ $\mathbb{P}$-a.s. (respectively, $\mathbb{Q}$-a.s.)[43] for all t. This, together with the presumed càdlàg-property of both involved processes, would imply that $L^{\mathcal{F},\mathbb{P}}$ and $L^{\mathcal{G},\mathbb{Q}}$ likewise are indistinguishable. Evidently, if $L^{\mathcal{G},\mathbb{Q}}$ was a modification of (the Lévy process) $L^{\mathcal{F},\mathbb{P}}$, then by definition

(3.3.175)

$$\int_0^t \int_D z \left\{ d\nu^{\mathcal{G},\mathbb{Q}}(s,z) - d\nu^{\mathcal{F},\mathbb{P}}(s,z) \right\} = 0$$

would hold $[\mathbb{P} \equiv \mathbb{Q}]$ $\forall t$. Further on, for $t > 0$ [$t = 0$ is trivial] equality (3.3.175) would lead us to

(3.3.176)
$$d\nu^{\mathcal{G},\mathbb{Q}}(s,z) = d\nu^{\mathcal{F},\mathbb{P}}(s,z) \;\; (= d\nu(z)\, ds)$$

$[\mathbb{P} \equiv \mathbb{Q}]$ for all $0 \le s \le t \; (< \tau)$ and $z \in D \subset\,]0,\infty[$. In addition, Lemma 1.4.8 in [1] would yield that $L^{\mathcal{G},\mathbb{Q}}$ also was a *Lévy* process, moreover with the same characteristics as $L^{\mathcal{F},\mathbb{P}}$ [which entirely stood in line with (3.3.176)]. Vice versa, we eventually might trouble Corollary 2.4.21 in [1] to deduce that (3.3.176) in return implies the modification property of (the underlying *Lévy* processes) $L^{\mathcal{G},\mathbb{Q}}$ and $L^{\mathcal{F},\mathbb{P}}$.

Exercise 3.3.7 (b) *Combining Exercise 3.3.7 (a) and (3.3.176), we finally ask: Does the choice*

$$h(s,z) := ln\left[\frac{d\nu^{\mathcal{G},\mathbb{Q}}(s,z)}{d\nu^{\mathcal{G},\mathbb{P}}(s,z)}\right] = ln\left[\frac{d\nu^{\mathcal{F},\mathbb{P}}(s,z)}{d\nu^{\mathcal{G},\mathbb{P}}(s,z)}\right] = ln\left[\frac{\tau - s}{\mathbb{E}_{\mathbb{P}}\left(\int_{u=s}^{u=\tau} dN(u,z) \,\middle|\, \mathcal{G}_s\right)}\, d\nu(z)\right]$$

$[\mathbb{P} \equiv \mathbb{Q}]$ $\forall (s,z) \in [0,\tau[\times D$ *imply indistinguishable Lévy processes* $L^{\mathcal{F},\mathbb{P}}$ *and* $L^{\mathcal{G},\mathbb{Q}}$*?* ■

3.4 Conclusions

In order to model electricity spot prices in a realistic manner, we have utilized an arithmetic multi-factor Ornstein-Uhlenbeck setup which originally has been proposed in [8]. The latter approach represents an appropriate alternative for modeling electricity spot price dynamics without sticking to an exponential onset such as known from e.g. the popular "*Lucia and Schwartz model*" (compare the notation in section 3.2.1 in [13] and the mentioned references therein). Having derived traditional, i.e. *backward-looking*, electricity futures and related call option prices for our underlying multi-factor framework, we subsequently have turned our attention towards the pricing of electricity derivatives under additional future information. Dealing with this subject, we have taken *forward-looking* insider information into account via different customized enlargements of the underlying information

[43] As $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, both probability measures possess the same null-sets. Hence, it does not matter whether we write $\mathbb{P}$-a.s. or $\mathbb{Q}$-a.s., actually. However, for the sake of notational simplicity, we will use the symbol $[\mathbb{P} \equiv \mathbb{Q}]$ to denote both cases simultaneously from now on.

filtrations. In this regard, our most innovative results consist in the provision of numerous derivation methodologies for both plain-vanilla and exotic option price formulas associated to electricity futures contracts – but yet under supplementary future information about the long-term level of the spot price or about correlated outdoor-temperature, for example. As we have seen in paragraph 3.3 for several times, in order to evaluate risk-neutral forward-looking electricity option prices, there is a strong need for approximation techniques and numerical pricing methods. On the contrary, for the *mixed* models such as presented in section 3.3.6 and 3.3.9 it fortunately has been possible to provide more explicit option price formulas than in our former forward-looking pure-jump cases [compare e.g. (3.3.55) with (3.3.131)]. Reasoning about subsection 3.3.2 and 3.3.3, we finally recognize that working with the (practical) *explicit* intermediate filtration $\mathcal{G}^*$ [instead of $\mathcal{G}$; compare the *key idea* described previously to (3.3.38)] not at all has generated independent increments in the corresponding futures price dynamics (3.3.46), unfortunately. Hence, the approximation techniques presented in Excursus A might have directly been applied as well on (3.3.37). On the other hand, in paragraph 3.3.6, for example, it has been crucial to work under $\bar{\mathcal{G}}$ (instead of under $\breve{\mathcal{G}}$), since W_t^k in (3.3.92) otherwise would have been to be replaced by a conditional expectation such as appearing in (3.3.80). In this case, it would not have been clear whether the corresponding futures price, say $\breve{F}$, in contrast to (3.3.93), constituted a Sato-process. In addition, the appearing differential with respect to t, reading $d\mathbb{E}_{\bar{\mathbb{Q}}}\big(\int_t^\tau e^{\vartheta r} d\bar{B}_r^k \,|\breve{\mathcal{G}}_t\big)$, would be unknown. Anyway, in subsection 3.3.9.2 we have compared our former *dualism* approaches with an alternative forward-looking measure change method such as proposed by Protter. In this context, we also have introduced the notion of a *cross premium*. Finally, comparing section 3.3.6 with 3.3.9, we propose to correlate the BM-driven base components in (3.3.105) [generating the mean-level fluctuations of the electricity spot price] with the temperature dynamics (3.3.66) – parallel to the procedure presented in (3.3.68). Then the involved BMs can be *correlated* in the common sense – possibly appearing more appropriate than (3.3.68) – while our option pricing methods still work.

3.5 Appendix

At first, we want to prove a result that has been used in the context of (3.3.41) formerly.

Lemma 3.5.1 *Let $\mathcal{G}_t^*$ as defined in (3.3.38). Then for all $k = 1, \dots, p$ and $0 \le t \le s < \tau$ we have*

$$\mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_s^k \big| \mathcal{G}_t^*\big) = \frac{\tau - s}{\tau - t}\, \mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_t^k \big| \mathcal{G}_t^*\big).$$

Proof Assume $k \in \{1, \dots, p\}$ and $0 \le t \le s < \tau$. Then, combining Proposition A.3 with Remark A.4 in [10] while taking (3.2.4), (3.3.47), (3.3.48) and Condition A (with $l := p$) into account, we deduce

(3.5.1) $$\mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_s^k \big| \mathcal{G}_t^*\big) - \mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_t^k \big| \mathcal{G}_t^*\big) =$$

$$-\mathbb{E}_{\mathbb{Q}}\left(\int_t^s \int_{D_k} z\, dN_k(r,z) \,\middle|\, \mathcal{G}_t^*\right) = -\mathbb{E}_{\mathbb{Q}}\left(\int_t^s \int_{D_k} z\, d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(r,z) \,\middle|\, \mathcal{G}_t^*\right) - \mathbb{E}_{\mathbb{Q}}\left(\int_t^s \int_{D_k} z\, d\nu_k^{\mathcal{G}^*,\mathbb{Q}}(r,z) \,\middle|\, \mathcal{G}_t^*\right)$$

$$= -\mathbb{E}_{\mathbb{Q}}\left(\int_{r=t}^{r=s} \int_{D_k} \int_{u=r}^{u=\tau} \frac{z}{\tau - r}\, dN_k(u,z)\, dr \,\middle|\, \mathcal{G}_t^*\right)$$

$$= -\int_{r=t}^{r=s} \frac{1}{\tau - r}\, \mathbb{E}_{\mathbb{Q}}\left(\int_{u=r}^{u=\tau} \int_{D_k} z\, dN_k(u,z) \,\middle|\, \mathcal{G}_t^*\right) dr = -\int_t^s \frac{\mathbb{E}_{\mathbb{Q}}\big(L_\tau^k - L_r^k \big| \mathcal{G}_t^*\big)}{\tau - r}\, dr.$$

Introducing the abbreviation

(3.5.2)

$$V_s^k := \mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k \middle| \mathcal{G}_t^*\right)$$

property (3.5.1) easily can be rewritten as

$$V_s^k = V_t^k - \int_t^s \frac{V_r^k}{\tau - r} dr.$$

Consequently, in differential notation the latter equality points out as

(3.5.3)

$$dV_s^k = \frac{V_s^k}{s - \tau} ds.$$

As in [10], a straightforward application of the *separation of the variables technique* from ordinary calculus leads us to the solution of (3.5.3) which at the same time proves the claimed result. ∎

In addition, combining (3.3.45) with Lemma 3.5.1, we announce that the stochastic process $\left(L_\tau^k - L_t^k\right)/(\tau - t)$ constitutes a $(\mathcal{G}_t^*, \mathbb{Q})$-martingale (in t) for all $0 \leq t < \tau$ and $1 \leq k \leq p$.
The following lemma is closely connected with our former announcements in Remark 3.2.3 concerning the (averaged) Samuelson effect.

Lemma 3.5.2 *Let $k \in \{1, \dots, n\}$. Then for fixed s and τ_2 the (strictly) positive volatility function $\Lambda_k(s, \tau_1, \tau_2)$ such as defined in equation (3.2.24) is decreasing in its second argument τ_1.*

Proof Differentiating (3.2.24) with respect to τ_1, (for fixed s and τ_2) we initially obtain

(3.5.4)

$$\frac{\partial \Lambda_k(s, \tau_1, \tau_2)}{\partial \tau_1} = w_k\, \sigma_k(s)\ \frac{e^{-\lambda_k(\tau_1 - s)}[1 - \lambda_k(\tau_2 - \tau_1)] - e^{-\lambda_k(\tau_2 - s)}}{\lambda_k\,(\tau_2 - \tau_1)^2}.$$

Further, we observe that the object $\mathcal{t}(\tau_2 - \tau_1) := 1 - \lambda_k(\tau_2 - \tau_1)$ just embodies the tangent on the function $\mathcal{f}(\tau_2 - \tau_1) := e^{-\lambda_k(\tau_2 - \tau_1)}$ in the point (0,1) what leads us to the inequality

(3.5.5)

$$1 - \lambda_k(\tau_2 - \tau_1) < e^{-\lambda_k(\tau_2 - \tau_1)}$$

which is valid for all $\tau_1 < \tau_2$. Next, (3.5.5) is equivalent to

(3.5.6)

$$e^{-\lambda_k(\tau_1 - s)}[1 - \lambda_k(\tau_2 - \tau_1)] < e^{-\lambda_k(\tau_2 - s)} \ \Leftrightarrow\ \frac{\partial \Lambda_k(s, \tau_1, \tau_2)}{\partial \tau_1} < 0$$

which proves our lemma. ∎

3.5.1 A numerical evaluation scheme for forward-looking expectations

Although the conditional expectation in (3.3.53) does not reduce to a usual one (since the contained stochastic process does not possess independent increments with respect to the conditioning filtration), it is interesting to study the corresponding usual expectation (3.3.54) in more depth. On a first sight, one might suspect that an analytical handling of (3.3.54) – in contrast to (3.3.53) – is easily possible. Actually, this is not the case, as we will see in our proceedings. Since the evaluation of the $\widetilde{N}_k^{\mathcal{F},\mathbb{Q}}$-part ($k = p+1, \dots, n$) contained in (3.3.54) can be done similarly to our former arguing in the proof of Proposition 3.2.4 [compare (3.2.41)], we yet concentrate on the more challenging objects like

(3.5.7)

$$\mathcal{E}_{\mathbb{Q}}^{\mathcal{G}^*}\left(t, T, D_k; y, a, \Xi_k; \widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}; p\right) := \mathbb{E}_{\mathbb{Q}}\left[exp\left\{\sum_{k=1}^{p}\int_t^T\int_{D_k}(a+iy)\,\Xi_k(s,z)\,d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(s,z)\right\}\right]$$

within a complex number $(a+iy)$ and deterministic, continuous functions $\Xi_k(s,z)$. Anyway, for the sake of notational simplicity we currently examine (forward-looking) expectations of the stylized form

(3.5.8)

$$\mathcal{E}_{\mathbb{Q}}^{\mathcal{G}^*}(k) := \mathcal{E}_{\mathbb{Q}}^{\mathcal{G}^*}\left(0, T, D_k; \delta_k; \widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}\right) := \mathbb{E}_{\mathbb{Q}}\left[exp\left\{\int_0^T\int_{D_k}\delta_k(s)\,z\,d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(s,z)\right\}\right]$$

$(k = 1, \dots, p)$ within arbitrary real sets $D_k \subseteq \mathbb{R} \setminus \{0\}$, deterministic and continuous (maybe complex) functions $\delta_k(s)$ and $(\mathcal{G}^*, \mathbb{Q})$-compensated random measures $\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}$ such as introduced in (3.3.48). To begin with, we decompose (3.5.8) due to (3.3.48) as follows

(3.5.9)

$$\mathcal{E}_{\mathbb{Q}}^{\mathcal{G}^*}(k) = \mathbb{E}_{\mathbb{Q}}\Bigg[exp\Bigg\{\int_0^T\int_{D_k}\mathbb{1}_{|z|<1}\,\delta_k(s)\,z\,d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(s,z) + \int_0^T\int_{D_k}\mathbb{1}_{|z|\geq 1}\,\delta_k(s)\,z\,dN_k(s,z) - \int_0^T\int_{D_k}\mathbb{1}_{|z|\geq 1}\,\delta_k(s)\,z\,d\nu_k^{\mathcal{G}^*,\mathbb{Q}}(s,z)\Bigg\}\Bigg].$$

On a superficial view, one now could think that it was possible to apply the extended Lévy-Khinchin formula on (3.5.9) – similar to our previous arguing in (3.2.41). Unfortunately, we are not allowed to do this here, since the $(\mathcal{G}^*, \mathbb{Q})$-compensating "*Lévy-measure*" $\nu_k^{\mathcal{G}^*,\mathbb{Q}}$ extraordinarily contains some randomness [compare its definition in (3.3.47)] in our current enlarged filtration case study. Hence, a (semi-) *analytical* evaluation – if there is any appropriate at all – of forward-looking expectations of the type (3.5.8) presently is a standing problem (at least to the best of our knowledge). At this step, we recall that the same is valid for conditional expectations of the type (3.3.53).

Nevertheless, exploiting our linking equation (3.3.49), property (3.5.8) yet may be rewritten as

(3.5.10)

$$\mathcal{E}_{\mathbb{Q}}^{\mathcal{G}^*}(k) = \mathbb{E}_{\mathbb{Q}}\left[exp\left\{\int_0^T\delta_k(s)\left(dL_s^k - \frac{L_\tau^k - L_s^k}{\tau - s}\,ds\right)\right\}\right].$$

Furthermore, putting $\gamma_k(s) := \delta_k(s)/(\tau - s)$, (3.5.10) can be rearranged as

(3.5.11)

$$\mathcal{E}_{\mathbb{Q}}^{\mathcal{G}^*}(k) = \mathbb{E}_{\mathbb{Q}}\left[exp\left\{\int_0^T \delta_k(s)\, dL_s^k - L_\tau^k \int_0^T \gamma_k(s)\, ds + \int_0^T \gamma_k(s)\, L_s^k\, ds\right\}\right].$$

In what follows, we treat the three integral terms on the right hand side of (3.5.11) in their order of appearance: Firstly, for all $k = 1, \dots, p$ we observe

(3.5.12)

$$\int_0^T \delta_k(s)\, dL_s^k = \sum_{0 \le s \le T} \delta_k(s)\, \Delta L_s^k = \sum_{0 \le s \le T} \delta_k(s)\, \left(L_s^k - L_{s-}^k\right)$$

whereby the involved *càdlàg* pure-jump *Lévy* noises L^k (recall Condition A with $l := p$ therein) such as implemented in (3.2.4) [but yet with $D_k \subseteq \mathbb{R} \setminus \{0\}$] have to be simulated numerically.[44] Secondly, the values L_τ^k for $k = 1, \dots, p$ are *known* from $\mathcal{G}_t^*$ [see (3.3.38)], so that an applicant (having access to some insider information) already should have established/guessed those values formerly.[45] Consequently, the evaluation of the inner integral in (3.5.11) should not cause any further problems: Either $\int_0^T \gamma_k(s) ds$ can be computed analytically at once, or it has to be handled by standard numerical integration methods for Riemann-integrals such as presented in section 19.3 in [19], for instance. Thirdly, for a (not necessarily equidistant) partition $\mathfrak{P} := \{0 = t_0 < t_1 < \cdots < t_m = T\}$ the last integral term in (3.5.11) can be approximated with the following Euler-Maruyama scheme (see [64])

(3.5.13)

$$\int_0^T \gamma_k(s)\, L_s^k\, ds \approx \sum_{j=1}^m \gamma_k(t_{j-1})\, L_{t_{j-1}}^k\, (t_j - t_{j-1})$$

with $k = 1, \dots, p$, wherein the (finite-variation) Lévy processes L^k have to be simulated numerically again, as explained above.

---

[44] Note that in practical applications of our multi-factor OU-model the processes L^k $(k = 1, \dots, n)$ anyway have to be simulated right from the beginning in order to obtain electricity spot price paths and related futures and option prices, even in the *non*-forward-looking framework presented in subsection 3.2.1, 3.2.3 and 3.2.4. Hence, an applicant already should have been confronted with such simulation issues, so that a proper evaluation of (3.5.12) yet should not cause any further trouble. However, Example 2.1 in [8] provides some helpful comments on a suitable numerical simulation of arithmetic multi-factor electricity spot price trajectories and thus, of the noises L^k likewise.

[45] Let us recall that we always assume a market *insider* to have access to some additional information about e.g. future electricity prices or outdoor temperature behavior at a future time τ. Thus, exemplarily referring to the *correlated temperature* context of subsection 3.3.6 now, within appropriately chosen ingredients ϑ, $m(t)$ and ξ_k $(k = 1, \dots, l)$ we may deduce suitable noise integral-processes $\int_0^\tau e^{\vartheta s} dB_s^k$ $(k = 1, \dots, l)$ from the forecasted temperature value Θ_τ. In other words, we may *invert* equality (3.3.67) to receive proper Brownian noise integral-processes from the predicted temperature Θ_τ. Obviously, for $l > 1$ an inversion in the common sense is impossible, while in this case suitable integral values have to be *guessed*, respectively *established*. Of course, weather forecasts do not tell us the precise noise values for our specific model but the (most likely) magnitude for Θ_τ is announced instead. Hence, an inversion/establishment in the just described sense naturally becomes necessary in practical applications.

Introducing the mesh $\Delta(\mathfrak{P}) := max_{1\le j\le m}(t_j - t_{j-1})$, for every $k = 1, \dots, p$ we assume

(3.5.14)

$$\int_0^T \gamma_k(s)\, L_s^k\, ds = \lim_{\Delta(\mathfrak{P})\to 0} \sum_{j=1}^m \gamma_k(t_{j-1})\, L_{t_{j-1}}^k\, (t_j - t_{j-1}) < \infty.$$

Eventually, for $k = 1, \dots, p$ let us denote the *simulated* Lévy noises by $\overline{L^k}$, i.e. the (deterministic) realization of L^k is denoted by $\overline{L^k}$ from now on. We further suppose these simulated noises to have (exponentially distributed) jump-times in the non-empty set $\mathcal{J} \subset [0,T]$. Then, with respect to (3.5.12) and (3.5.13), our forward-looking expectation (3.5.11) trivially can be approximated through

(3.5.15)

$$\mathcal{E}_{\mathbb{Q}}^{\mathcal{G}^*}(k) \approx exp\left\{\sum_{s\in\mathcal{J}} \delta_k(s)\left(\overline{L_s^k} - \overline{L_{s-}^k}\right) - L_\tau^k \int_0^T \gamma_k(s)\, ds + \sum_{j=1}^m \gamma_k(t_{j-1})\, \overline{L_{t_{j-1}}^k}\, (t_j - t_{j-1})\right\}$$

whereby we have assumed the values L_τ^k to be known, respectively established already (and thus, they do not have to be simulated anymore). However, we conclude that equation (3.5.15) ought to be suitable for a numerical evaluation of forward-looking (usual) expectations of the type (3.5.7).

We finally remark that our forward-looking *mixed* model such as implemented in subsection 3.3.6 fortunately admits a more explicit evaluation of the corresponding expectations [see (3.3.96) and (3.3.97)], since (3.3.81) not only constitutes a $(\bar{\mathcal{G}}, \overline{\mathbb{Q}})$-martingale, but even a $(\bar{\mathcal{G}}, \overline{\mathbb{Q}})$-Brownian-motion which possesses independent increments with respect to $\bar{\mathcal{G}}$. For this reason, the conditional expectation in (3.3.96) firstly reduced to a usual one, whereas $\mathfrak{I}_1^k$ in (3.3.97) secondly could be computed further by a straightforward application of Itô's isometry. However, the same statements keep valid for our *mixed* electricity spot price model presented in paragraph 3.3.9 (and also for our upcoming *mixed* temperature model in paragraph 5.3.4).

3.5.2 The information premium under $\mathcal{G}^*$

Parallel to our former considerations under $\mathcal{G}$ in subsection 3.3.1, we again examine the *information premium* but yet associated to the explicit intermediate filtration $\mathcal{G}^*$ such as introduced in (3.3.38). To this end, adapting definition (3.3.3), we initially put

(3.5.16)
$$\mathfrak{I}_t^{\mathcal{G}^*,\mathcal{F},\mathbb{Q}}(\tau_1,\tau_2) := F_t^{\mathcal{G}^*,\mathbb{Q}}(\tau_1,\tau_2) - F_t^{\mathcal{F},\mathbb{Q}}(\tau_1,\tau_2).$$

With similar computations as in paragraph 3.3.1 [but now using (3.3.39), (3.3.45) and Lemma 3.5.1], we obtain for $0 \le t \le \tau_1 < \tau \le \tau_2$

(3.5.17)
$$\mathfrak{I}_t^{\mathcal{G}^*,\mathcal{F},\mathbb{Q}}(\tau_1,\tau_2) =$$

$$\sum_{k=1}^p \int_t^{\tau_2} \frac{\tau_2 - \mathfrak{m}(s)}{\tau_2 - \tau_1}\, \Lambda_k(s, \mathfrak{m}(s), \tau_2) \left[\frac{L_\tau^k - L_t^k}{\tau - t} - \rho_k \int_{D_k} z\, e^{h_k(z)} d\nu_k(z)\right] ds$$

which actually corresponds to (3.3.19). Herein, we have set $\mathfrak{m}(s) := max\{s, \tau_1\}$, as before.

On the opposite, for $0 \leq t \leq \tau \leq \tau_1 \leq \tau_2$ and $\mathcal{F}_t \subset \mathcal{G}_t^* \subseteq \mathcal{G}_\tau^* = \mathcal{F}_\tau$ we apply similar conditioning techniques as in (3.3.20) – (3.3.22) what leads us to

(3.5.18)
$$\mathfrak{I}_t^{\mathcal{G}^*,\mathcal{F},\mathbb{Q}}(\tau_1,\tau_2) = \sum_{k=1}^{p} \frac{\Lambda_k(\tau,\tau_1,\tau_2)}{\sigma_k(\tau)} \left[\mathbb{E}_{\mathbb{Q}}\left(X_\tau^k \middle| \mathcal{G}_t^*\right) - \mathbb{E}_{\mathbb{Q}}\left(X_\tau^k \middle| \mathcal{F}_t\right)\right]$$

which designates the $\mathcal{G}^*$-counterpart of (3.3.22). Again, we observe $\mathfrak{I}_\tau^{\mathcal{G}^*,\mathcal{F},\mathbb{Q}}(\tau_1,\tau_2) = 0$.

Remark 3.5.3 *We finally underline that a proper computation, respectively approximation, of option prices (for e.g. electricity derivatives) under enlarged filtrations portrays a much more challenging task in contrast to the pure derivation of information premia. This fact immediately becomes clear if we compare the sophisticated derivation methodologies for option price formulas under enlarged filtrations on the one hand, and the less-demanding ones for information premia on the other. Exemplarily, we justify this statement while referring to our proceedings in paragraph 3.3.9:*
In connection with our derivations in (3.3.133) – (3.3.137) [dedicated to the information premium] we actually used precisely the same techniques that previously had been applied during our option pricing examinations in (3.3.119) – (3.3.131). Hence, option pricing implies the corresponding information premia, obviously. In conclusion, one could regard information premia as an actual by-product of related option prices under forward-looking information. For this reason, future research should rather concentrate on the derivation of option prices under enlarged filtrations, than on information premia. ■

3.5.3 Optimal electricity futures portfolio selection under forward-looking information

In this subsection we aim to examine the question of how to determine an *optimal* (in the sense of maximizing a certain utility functional) electricity futures investment strategy – particularly under supplementary knowledge on future price behavior. Right at the beginning, we announce that the current paragraph has been motivated by the sections 8.1, 8.6, 16.5 and 16.6 in [32] which extensively deal with portfolio analysis for an insider in a financial stock market. In what follows, we want to adapt some of the techniques presented in [32] to our electricity market framework. Starting off, we assume that there are two investment possibilities[46] in the underlying electricity market, namely:

- a bond/bank account β_t, $t \in [0,\tau_1]$, obeying

(3.5.19)
$$d\beta_t = r\,\beta_t\,dt$$

within a constant interest rate $r > 0$ and a deterministic initial value $\beta_0 > 0$ [recall (3.2.28)].

- an electricity futures $F_t := F_t(\tau_1,\tau_2)$, $t \in [0,\tau_1]$, such as given in Proposition 3.2.2, obeying

(3.5.20)
$$dF_t = \textstyle\sum_{k=1}^{n} \int_{D_k} z\,\Lambda_k(t)\,\widetilde{N}_k^{\mathcal{F},\mathbb{Q}}(t,dz)$$

where $\Lambda_k(t) := \Lambda_k(t,\tau_1,\tau_2) \geq 0$ is like defined in (3.2.24).

[46] In the electricity market practice there of course are various futures contracts with different delivery periods available. In this regard, note that our model easily can be extended to *multiple* futures investment possibilities. Nevertheless, we here illustrate the *one-bond-one-futures case* for the sake of notational simplicity.

Similar to above, we introduce the filtrations $\mathcal{F}_t := \sigma\{F_s : 0 \le s \le t\} := \sigma\{L_s^1, \dots, L_s^n : 0 \le s \le t\}$ and $\mathcal{K}_t := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^n\}$ while we presume the (non-explicit) intermediate filtration $\mathcal{G}_t$ to fulfill

$$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{K}_t \tag{3.5.21}$$

for all $0 \le t < \tau$ where $\tau_1 < \tau$. Consequently, the statements of (Condition A and) Lemma 3.3.1 (a) and (d) likewise apply in our current setting – even for indices $k = 1, \dots, n$ yet. Next, in accordance to [32], we implement the *set of admissible (forward-looking) portfolios* due to

$$\mathcal{A}(\mathcal{G}) := \left\{ \pi = (\pi_t)_{t \in [0,\tau_1]} \middle| \, \pi \text{ càglàd}, \ \mathcal{G} - adapted, \ \int_0^{\tau_1} \pi_t^2 \, dt < \infty \ [\mathbb{Q}] \right\}.$$

Parallel to the (stock market) setup presented on p.129 in [32], for a portfolio $\pi \in \mathcal{A}(\mathcal{G})$ we suppose the stochastic value π_t to denote the *fraction of the total wealth X_t^π invested in the electricity futures $F_t(\tau_1, \tau_2)$ at time $t \in [0, \tau_1]$*. In other words, we here think of a fictive electricity market participant (equipped with some additional insider knowledge modeled by the enlarged filtration $\mathcal{G}$) who wants to create an optimal portfolio with respect to his/her *individual* future information (which other traders do not have). Yet, we suppose such a trader to be a *small investor* and thus, to act as a *price taker*; that is, his/her transactions do not have a remarkable impact on the overall price dynamics – compare Remark 8.22 in [32]. For this reason, we presently do *not* involve the futures price $F^{\mathcal{G},\mathbb{Q}}$ inside (3.5.20), but $F := F^{\mathcal{F},\mathbb{Q}}$ instead, as the latter designates the reference price for a *small* investor (even if he/she personally has access to some future information). Vice versa, whenever the available future information consists of *public* knowledge or if we consider a *large investor* (who may influence prices by his/her individual transactions; also compare Chapter 4 below), then we ought to work with $F^{\mathcal{G},\mathbb{Q}}$ in (3.5.20). Nevertheless, our fictive insider indeed may choose a portfolio with respect to his/her individual future knowledge and hence, the portfolio π is allowed to be $\mathcal{G}$-adapted. Furthermore, we assume all portfolios $\pi \in \mathcal{A}(\mathcal{G})$ to be *self-financing* in the spirit of equality "(4.17) in [32]", i.e. we presume the corresponding wealth process X_t^π to fulfill the *forward SDE* (recall Def. 15.7 in [32])

$$d^- X_t^\pi = \theta_0^\pi(t) \, d\beta_t + \theta_1^\pi(t-) \, d^- F_t \tag{3.5.22}$$

with deterministic initial wealth $X_0^\pi = x > 0$ and coefficient processes $\theta_0^\pi(t) := [1 - \pi_t] \, X_t^\pi$ and $\theta_1^\pi(t) := \pi_t \, X_t^\pi$. At this step, we stress that F_t is $\mathcal{F}_t$-adapted while $\theta_1^\pi(t)$ is $\mathcal{G}_t$-adapted. Unfortunately, this special case is not captured by Itô's integration theory. In other words, the object $\int_0^t \theta_1^\pi dF$ is not well-defined as an Itô integral. Thus, we have to work with *forward integration* in (3.5.22) whereas we assume the coefficient θ_1^π to be *forward-integrable* with respect to F, respectively $\widetilde{N}_k^{\mathcal{F},\mathbb{Q}}$ $(k = 1, \dots, n)$, while we understand the symbol $d^- F_t$ in the sense of Definition 15.1 in [32].[47] Anyway, if either θ_1^π was $\mathcal{F}$-adapted or if F was replaced by $F^{\mathcal{G},\mathbb{Q}}$, then we would not need forward integration: In fact, for integrands and integrators adapted to the *same* filtration the forward and Itô integral coincide – see Remark 8.5, Lemma 8.9 and Corollary 8.10 in [32]. Conversely, in our case the integrand θ_1^π is *not* adapted to the filtration generated by its integrator. Further on, throughout section 8.1 and 8.6 in [32] the underlying risky asset is modeled by a *geometric* BM, while we are facing an *arithmetic* pure-jump electricity futures price disposition recently. Hence, our forward equation (3.5.22) slightly deviates from the basic scheme of "(8.1) and (8.27) in [32]". In addition, the portfolio analysis in [32] is done under the measure $\mathbb{P}$, whereas we work under a risk-neutral measure $\mathbb{Q}$ [recall (3.5.20)]. Finally, note that our current approach addresses the questions (1) and (3) on p.131 in [32]. However, taking (3.5.19) and (3.5.20) into account, the forward equation (3.5.22) becomes

[47] A reader not familiar with *forward integration* firstly might investigate paragraph 8.2 in [32], particularly Definition 8.3 therein (dedicated to the BM-case), before switching to section 15.1 in [32].

(3.5.23)

$$d^{-}X_t^{\pi} = X_{t-}^{\pi}\left[(1-\pi_t)\, r\, \beta_t\, dt + \pi_t \sum_{k=1}^{n} \int_{D_k} z\, \Lambda_k(t)\, \tilde{N}_k^{\mathcal{F},\mathbb{Q}}(d^{-}t, dz)\right].$$

Herein, we require the integrands to fulfill analogous conditions as in "(16.175) – (16.178) in [32]" with $\mathbb{H} \coloneqq \mathcal{G}$, $\theta(t,z) \coloneqq z\,\Lambda_k(t)$. As a consequence of the Itô formula[48] (see Th. 15.8 in [32]), we get

(3.5.24)

$$X_t^{\pi} =$$

$$x\, exp\left\{\beta_t - \beta_0 - \int_0^t \pi_s\left(r\,\beta_s + \sum_{k=1}^{n} \Lambda_k(s)\,\rho_k \int_{D_k} z\, e^{h_k(z)} d\nu_k(z)\right) ds + \sum_{k=1}^{n} \int_0^t \int_{D_k} ln(1 + z\,\Lambda_k(s)\,\pi_s)\, N_k(d^{-}s, dz)\right\}$$

wherein we have just used (3.2.20) along with Condition A. By the way, note that (3.5.24) corresponds to the equalities "(8.28) and (16.132) in [32]". Further on, we introduce a utility function $U\colon (0,\infty) \longrightarrow (-\infty,\infty)$ which we presume to be non-decreasing, concave and once continuously differentiable on $(0,\infty)$. Next, appealing to "(8.29) in [32]", we examine an optimization problem of the type

(3.5.25)

$$sup_{\pi\in\mathcal{A}(\mathcal{G})}\, \mathbb{E}_{\mathbb{Q}}\left[U\left(X_{\tau_1}^{\pi}\right)\right]$$

which requires us to find an optimal portfolio, say π^*, that maximizes the $\mathbb{Q}$-expected utility related to the final total wealth $X_{\tau_1}^{\pi^*}$ among all admissible and self-financing portfolios in $\mathcal{A}(\mathcal{G})$, in symbols

$$\mathbb{E}_{\mathbb{Q}}\left[U\left(X_{\tau_1}^{\pi^*}\right)\right] \geq \mathbb{E}_{\mathbb{Q}}\left[U\left(X_{\tau_1}^{\pi}\right)\right] \quad \forall\, \pi \in \mathcal{A}(\mathcal{G}).$$

Nevertheless, we refer to subsection 8.6.2 in [32] – particularly, recall (8.50) and (8.52) along with Example 8.33 therein – and penalize large trading volumes in our insider's portfolio by adding a penalty term to (3.5.25). Hence, instead of the latter, we newly consider the value/utility functional

(3.5.26)

$$V_{\mathcal{G}}(x) \coloneqq sup_{\pi\in\mathcal{A}(\mathcal{G})}\, \mathbb{E}_{\mathbb{Q}}\left[U\left(X_{\tau_1}^{\pi}\right) - \tfrac{1}{2}\int_0^{\tau_1} \overline{w}(s)^2\, \pi_s^2\, ds\right]$$

within a deterministic weight function $\overline{w}(s) > 0$. Merging (3.3.11), (3.3.12) and (3.5.24) into (3.5.26) while presuming a logarithmic utility function, we derive [remind "(16.168) in [32]" at this step]

(3.5.27)

$$V_{\mathcal{G}}(x) = ln(x) + \beta_{\tau_1} - \beta_0 +$$

$$\underset{\pi\in\mathcal{A}(\mathcal{G})}{sup}\, \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1}\left(\sum_{k=1}^{n}\int_{D_k} \frac{ln(1 + z\,\Lambda_k(s)\,\pi_s)}{\tau - s}\, \mathbb{E}_{\mathbb{Q}}\left(\int_{u=s}^{u=\tau} dN_k(u,z)\,\middle|\,\mathcal{G}_s\right) - \pi_s^2\,\frac{\overline{w}(s)^2}{2} - \pi_s\, r\,\beta_0\, e^{rs} - \pi_s \sum_{k=1}^{n} \Lambda_k(s)\,\rho_k \int_{D_k} z\, e^{h_k(z)} d\nu_k(z)\right) ds\right].$$

[48] The *Itô formula for forward integrals* essentially possesses the same structure as its counterpart for common Itô integrals – except from forward integrals appearing at the place of Itô integrals otherwise. We here refer to the very comprehensive (Brownian motion) case studies of section 8.3 in [32]; particularly, see Theorem 8.12 and Remark 8.13 therein. Moreover, Example 8.15 in [32] might be considered in the context of (3.5.24).

To find the optimal portfolio π^* which solves (3.5.27), we maximize (point-wise) the functional

(3.5.28)

$$\mathfrak{F}(\pi_s) := \sum_{k=1}^{n} \int_{D_k} \frac{ln(1 + z\,\Lambda_k(s)\,\pi_s)}{\tau - s}\, \mathbb{E}_{\mathbb{Q}}\left(\int_{u=s}^{u=\tau} dN_k(u,z) \,\middle|\, \mathcal{G}_s \right) - \pi_s^2\, \frac{\overline{w}(s)^2}{2} - \pi_s\, r\, \beta_0\, e^{rs} - \pi_s \sum_{k=1}^{n} \Lambda_k(s)\, \rho_k \int_{D_k} z\, e^{h_k(z)} d\nu_k(z)$$

with respect to π_s (for fixed $s \in [0, \tau_1]$; see the proof of Th. 16.54 in [32]) leading us to the condition

(3.5.29)

$$\overline{w}(s)^2\, \pi_s + r\, \beta_0\, e^{rs} = \sum_{k=1}^{n} \int_{D_k} z\, \Lambda_k(s) \left(\frac{\mathbb{E}_{\mathbb{Q}}\big(\int_{u=s}^{u=\tau} dN_k(u,z)\, |\mathcal{G}_s\big)}{[\tau - s][1 + z\,\Lambda_k(s)\,\pi_s]} - e^{h_k(z)}\, \rho_k\, d\nu_k(z) \right).$$

As $\partial^2 \mathfrak{F}(\pi_s)/\partial \pi_s^2 < 0$ is valid for all self-financing portfolios $\pi_s \in \mathcal{A}(\mathcal{G})$, the ($\mathcal{G}_s$-adapted) solution π_s^* of (3.5.29) indeed gives the *maximum* of $\mathfrak{F}(\cdot)$. Like in Corollary 16.41, Theorem 16.50 or Theorem 16.54 in [32], the optimality condition (3.5.29) neither can be analytically solved for π_s. Thus, numerical evaluation methods ought to be used in order to derive the optimal portfolio $\pi_s^* \in \mathcal{A}(\mathcal{G})$. However, we leave this topic for future work. Instead, we recall that Chapter 8 in [32] contains portfolio examinations (but for financial stock markets and geometric BM-approaches) wherein the maximization procedures can be done *analytically* – see "(8.4), (8.5), (8.9), (8.10), (8.44)[49] and the sequel of Theorem 8.34 in [32]". In this regard, we conclude with the following exercise.

Exercise 3.5.4 *Instead of (3.5.20) assume that the electricity futures price obeys the dynamics*

(3.5.30)
$$dF_t = \textstyle\sum_{k=1}^{n} \Lambda_k(t)\, dW_t^k$$

with pair-wise $\mathbb{Q}$-independent $(\mathcal{F}_t, \mathbb{Q})$-BMs $W_t^1, \dots, W_t^n$ and let $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{F}_t \vee \sigma\{W_\tau^1, \dots, W_\tau^n\}$ where $\mathcal{F}_t := \sigma\{F_s : 0 \le s \le t\}$. Show that the corresponding ($\mathcal{G}_t$-adapted) wealth process fulfills

(3.5.31)
$$X_t^\pi = x\, exp\left\{ \beta_t - \beta_0 - \int_0^t \pi_s \left(r\,\beta_s - \sum_{k=1}^{n} \Lambda_k(s) \frac{\mathbb{E}_{\mathbb{Q}}\big(W_\tau^k - W_s^k | \mathcal{G}_s\big)}{\tau - s} + \pi_s \sum_{k=1}^{n} \frac{\Lambda_k(s)^2}{2} \right) ds + \sum_{k=1}^{n} \int_0^t \pi_s\, \Lambda_k(s)\, d^- \widehat{W}_s^k \right\}$$

with $(\mathcal{G}_t, \mathbb{Q})$-Brownian motions $\widehat{W}_t^1, \dots, \widehat{W}_t^n$. [Hint: use Prop. 2.3.3.][50]

---

[49] By the way, the factor $1/(2\sigma^2)$ inside equality "(8.10) in [32]" must be replaced by $1/2$ simply. Moreover, the last summand on the right side of "(8.44) in [32]" actually has a term $\sigma^2(t)$ in its denominator.

[50] Admittedly, (3.5.30) constitutes a rather unrealistic model, as the futures price may become negative. However, the *analytical* optimization works in this case at least. Further, note that the last integral inside the exponent of (3.5.31) actually is a Brownian *Itô integral* again, i.e. we may write $d\widehat{W}_s^k$ instead of $d^-\widehat{W}_s^k$ therein.

Now presume logarithmic utility and show that the corresponding ($\mathcal{G}_s$-adapted) optimal portfolio with respect to the utility functional (3.5.26) is given by

(3.5.32)

$$\pi_s^* = \left[-r\,\beta_s + \sum_{k=1}^{n} \frac{\Lambda_k(s)}{\tau - s}\mathbb{E}_{\mathbb{Q}}\big(W_\tau^k - W_s^k \big| \mathcal{G}_s\big)\right] \Big/ \left[\overline{w}(s)^2 + \sum_{k=1}^{n} \Lambda_k(s)^2\right]. \blacksquare$$

Optimal electricity futures portfolio selection for a large investor with insider knowledge

In the following, we investigate the situation where the additionally available future information consists of *public* knowledge, respectively when the electricity market insider is a *large* investor. As explained above, in these cases we ought to replace (3.5.20) by a suitable *anticipating* electricity futures price disposition. Actually, we assume that – besides the bank account (3.5.19) – a market insider may invest – instead of (3.5.20) – into an electricity futures with price dynamics such as given in equality (A.2) but with $p \coloneqq n$ therein. In other words, our insider no longer is a *price taker* yet, but has his/her own reference price, namely F^*, explicitly depending on the known future noise values $L_\tau^1, \dots, L_\tau^n$. In this context, for a time partition $0 \le t \le \tau_1$, $t < \tau$, we put $F_t^* \coloneqq F_t^{\mathcal{K},\mathbb{Q}}(\tau_1, \tau_2)$ while we presume the enlarged filtration $\mathcal{K}$ to be like defined in the context of (3.5.21). More accurately speaking, putting $p = n$ in (3.3.38), we obtain $\mathcal{G}^* = \mathcal{K}$ which corresponds to complete/exhaustive knowledge of the future electricity price at time τ. In this case, inside the *pure-anticipating* futures price dynamics (A.2) there only appear integrals with respect to the forward-looking random measures $\widetilde{N}_1^{\mathcal{K},\mathbb{Q}}, \dots, \widetilde{N}_n^{\mathcal{K},\mathbb{Q}}$. Further on, the total wealth equation (3.5.22) currently translates into

(3.5.33)

$$d^- X_t^\pi = X_{t-}^\pi\,[(1 - \pi_t)\,d\beta_t + \pi_t\,d^- F_t^*], \qquad X_0^\pi = x > 0.$$

This time, we assume the class of admissible, self-financing (anticipating) portfolios $\mathcal{A}(\mathcal{K})$ to consist of all càglàd, $\mathcal{K}_t$-adapted and $\mathbb{Q}$-almost-sure Lebesgue-square-integrable stochastic processes π_t, $t \in [0, \tau_1]$, which fulfill the following conditions

- $\pi_t\,\Xi_k(t,z)$ is (forward-) integrable with respect to the $(\mathcal{K}, \mathbb{Q})$-compensated random measure $\widetilde{N}_k^{\mathcal{K},\mathbb{Q}}$ for all $(t,z) \in [0,\tau_1] \times D_k$ and $k = 1, \dots, n$,
- $\pi_t\,\Xi_k(t,z) > -1$ for $\mathbb{Q}$-almost-all $(t,z) \in [0,\tau_1] \times D_k$ $(k = 1, \dots, n)$,
- $\pi_t\,X_t^\pi$ is (forward-) integrable with respect to F_t^* for all $t \in [0, \tau_1]$.

Furthermore, we presently suppose the stochastic value π_t to denote the fraction of the total wealth X_t^π invested into the electricity futures F_t^* at time $t \in [0, \tau_1]$. Next, appealing to (3.5.19), (3.3.47), (3.3.48) and (A.2) [the three latter equations taken with $p = n$ and $\mathcal{G}^*$ replaced by $\mathcal{K}$], the ($\mathcal{K}_t$-adapted) solution of (3.5.33) points out as

(3.5.34)

$$X_t^\pi =$$

$$x\,exp\left\{\beta_t - \beta_0 + \int_0^t \left(-\pi_s\,r\,\beta_s + \sum_{k=1}^{n} \int_{D_k} \frac{ln(1 + \Xi_k(s,z)\,\pi_s) - \Xi_k(s,z)\,\pi_s}{\tau - s} \int_{u=s}^{u=\tau} dN_k(u,z)\right) ds \right.$$
$$\left. + \sum_{k=1}^{n} \int_0^t \int_{D_k} ln(1 + \Xi_k(s,z)\,\pi_s)\,\widetilde{N}_k^{\mathcal{K},\mathbb{Q}}(d^- s, dz)\right\}.$$

At this step, let us remind that x, β, r and Ξ_k in (3.5.34) altogether are deterministic. Also note that (3.5.34) directly corresponds to (3.5.24) and that the appearing *forward* integrals with respect to $\widetilde{N}_k^{\mathcal{K},\mathbb{Q}}$ $(k = 1, \dots, n)$ actually depict *common* (Itô-Lévy-type) stochastic integrals. In particular, we announce that the last sum inside the exponent of (3.5.34) designates a $(\mathcal{K}, \mathbb{Q})$-martingale. Parallel to (3.5.26), we further introduce the target functional

(3.5.35)

$$V_{\mathcal{K}}(x) := \sup_{\pi \in \mathcal{A}(\mathcal{K})} \mathbb{E}_{\mathbb{Q}}\left[ln\left(X_{\tau_1}^{\pi}\right) - \frac{1}{2}\int_0^{\tau_1} \overline{w}(s)^2\, \pi_s^2\, ds\right].$$

Merging (3.5.34) into (3.5.35), we receive

(3.5.36)

$$V_{\mathcal{K}}(x) = ln(x) + \beta_{\tau_1} - \beta_0 + \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1} \sup_{\pi \in \mathcal{A}(\mathcal{K})} \left\{-\pi_s\, r\, \beta_s - \frac{\overline{w}(s)^2}{2}\pi_s^2 + \sum_{k=1}^{n} \int_{D_k} \frac{ln(1 + \Xi_k(s,z)\,\pi_s) - \Xi_k(s,z)\,\pi_s}{\tau - s} \int_{u=s}^{u=\tau} dN_k(u,z)\right\} ds\right].$$

Thus, a (point-wise) maximization of the function

(3.5.37)

$$\overline{\mathfrak{F}}(\pi_s) := -\pi_s\, r\, \beta_s - \frac{\overline{w}(s)^2}{2}\pi_s^2 + \sum_{k=1}^{n} \int_{D_k} \frac{ln(1 + \Xi_k(s,z)\,\pi_s) - \Xi_k(s,z)\,\pi_s}{\tau - s} \int_{u=s}^{u=\tau} dN_k(u,z)$$

with respect to π_s (for fixed $s \in [0, \tau_1]$) yields the (necessary) optimality condition

(3.5.38)

$$\overline{w}(s)^2\, \pi_s + r\, \beta_0\, e^{rs} = \sum_{k=1}^{n} \int_{D_k} \frac{\pi_s\, \Xi_k(s,z)^2}{[s - \tau][1 + \Xi_k(s,z)\,\pi_s]} \int_{u=s}^{u=\tau} dN_k(u,z).$$

Obviously, $\partial^2 \overline{\mathfrak{F}}(\pi_s)/\partial \pi_s^2 < 0$ is valid for all portfolios $\pi_s \in \mathcal{A}(\mathcal{K})$. Hence, $\overline{\mathfrak{F}}(\cdot)$ is concave and the solution of (3.5.38), say $\overline{\pi}_s$, indeed constitutes the *maximum* of (3.5.37). In analogy to our former explanations in the sequel of (3.5.29) it is neither possible to compute the critical value $\overline{\pi}_s$ analytically. But in any case, from (3.5.38) we deduce that $\overline{\pi}$ remains $\mathcal{K}$-adapted, as desired.

Yet, it appears worthwhile to compare (3.5.38) with (3.5.29) in more depth. For this purpose, we initially remind $\Xi_k(s,z) := z\,[\Lambda_k(s) - \Phi_k(s)]$ wherein Λ_k and Φ_k are such as defined in (3.2.24) and (3.3.29), respectively. Thus, Φ_k might be interpreted as some kind of (deterministic) subtractive information drift which affects the number of futures holdings in the utility-maximizing insider portfolio $\overline{\pi}$ essentially. Remarkably, from (3.5.38) we derive $\overline{\pi}_s \neq 0$ for all $s \in [0, \tau_1]$, since otherwise (3.5.38) would contradictory simplify to $r\beta_0 e^{rs} = 0$. Interpreting this, we state that (no matter how $\overline{\pi}$ really looks like) the *optimal fraction of the total wealth invested in the (anticipating) electricity futures* F^*, namely $\overline{\pi}$, *always* is different from zero and hence, an insider with portfolio $\overline{\pi}$ *never* invests into the bank account β *solely* [remind (3.5.33) in this context]. On the opposite, in (3.5.29) the mathematical theory interestingly does *not* a priori exclude the instance $\pi_s^* = 0$. The two latter observations can be interpreted economically: Although having access to some risk-reducing *individual* future information, for a *small* investor the overall futures price F such as given in (3.5.20) actually embodies a (risk-afflicted) '*big unknown*' which, in particular, may be negatively influenced by other *large* traders in the market. Hence, for a *small* investor acting as a *price taker* with portfolio π^* [given as the solution of (3.5.29)] it temporarily might be optimal to invest into the risk-less bank account β *solely* [in which case $\pi_s^* = 0$ holds]. On the other hand, a *large* investor with portfolio $\bar{\pi}$ (whose trading decisions depend on the information flow modeled by $\mathcal{K}$ and who may essentially influence the overall price development by individual transactions) can maximize his/her utility if he/she *permanently* holds a *non-vanishing* number of electricity futures with price dynamics F^*, in symbols $\bar{\pi}_s \neq 0 \ \forall s \in [0, \tau_1]$.

In what follows, we complete our current portfolio selection investigations in electricity futures markets with an examination of the non-anticipating case wherein there is no additional future information available. To this end, we adhere to the setting of (3.5.19) – (3.5.20) but assume the portfolio π to be $\mathcal{F}$-adapted now which, by the way, makes any forward-integration (and also Condition A) become superfluous. Evidently, the latter assumption gives rise to a replacement of $\mathcal{A}(\mathcal{G})$, respectively $\mathcal{A}(\mathcal{K})$, by $\mathcal{A}(\mathcal{F})$. Hence, presuming $z\, \Lambda_k(s)\, \pi_s > -1$ for $\mathbb{Q}$-almost-all $(s, z) \in [0, \tau_1] \times D_k$ $(k = 1, \dots, n)$, equality (3.5.27), respectively (3.5.36), translates into

(3.5.39)
$$V_{\mathcal{F}}(x) = ln(x) + \beta_{\tau_1} - \beta_0 +$$
$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1} \sup_{\pi \in \mathcal{A}(\mathcal{F})} \left\{ -\pi_s\, r\, \beta_s - \frac{\bar{w}(s)^2}{2} \pi_s^2 + \sum_{k=1}^{n} \int_{D_k} [ln(1 + z\, \Lambda_k(s)\, \pi_s) - z\, \Lambda_k(s)\, \pi_s]\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z) \right\} ds\right].$$

The latter equation directly leads us to the $\mathcal{F}$-associated (utility-maximizing) optimality condition

(3.5.40)
$$\bar{w}(s)^2\, \pi_s + r\, \beta_0\, e^{rs} = \sum_{k=1}^{n} \int_{D_k} \frac{-\pi_s\, \Lambda_k(s)^2\, z^2}{1 + \pi_s\, \Lambda_k(s)\, z}\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z).$$

We stress that – similarly to (3.5.29) and (3.5.38) – property (3.5.40) neither can be analytically solved for π_s. Interestingly, if we (unrealistically) presume a *constant* electricity futures price and thus, choose $n = 0$ [or alternatively, $\Lambda_k \equiv 0$ and $\Xi_k \equiv 0$] inside (3.5.29), (3.5.32), (3.5.38) or (3.5.40), then in each case we observe the utility-maximizing portfolio to be of the form

$$\pi_s = -\frac{r\, \beta_0\, e^{rs}}{\bar{w}(s)^2} = \frac{-1}{\bar{w}(s)^2} \frac{d\beta_s}{ds}$$

which is strictly negative for all $s \in [0, \tau_1]$. Furthermore, if there exists a time point $\tilde{s} \in [0, \tau_1]$ where transactions in the insider's portfolio (approximately) are non-penalized [recall (3.5.26)], in symbols $\bar{w}(s) \to 0^+$ $(s \to \tilde{s})$, then $\pi_s \to -\infty$ $(s \to \tilde{s})$ what does not constitute an admissible scenario with respect to $\mathcal{A}(\mathcal{F})$ [which is defined similarly to $\mathcal{A}(\mathcal{K})$ above].
In the sequel, we provide optimal consumption rates for electricity futures market insiders who we suppose to be *large* traders. The following section has been motivated by [74] and §16.4 in [32].

A utility maximizing consumption rate for an anticipating electricity futures market insider

In [74] the problem of finding an optimal consumption rate (which maximizes a given utility functional) in a financial stock market is solved, while there is no a priori assumption made whether the considered agent has more or less information available than what can be obtained from observing the price development in the underlying market. In what follows, we aim to do similar examinations but yet for electricity futures market insiders. Combining equality (3.5.33) above with "(4.6) in [74]", for $0 \leq t \leq \tau_1$ we introduce the consumption-portfolio cash flow/amount $X_t := X_t^{c,\pi}$ due to

(3.5.41) $$d^- X_t = [1 - \pi_t]\, X_t\, d\beta_t + \pi_t\, X_{t-}\, d^- F_t^* - c_t\, dt$$

with deterministic initial wealth $X_0 := x > 0$. Herein, the stochastic component $c_t \geq 0$ constitutes the rate of consumption/dividends that the considered agent is free to take out of the cash amount at any time $t \in [0, \tau_1]$ – compare pp. 2 and 14 in [74] –, whereas β and F^* preliminarily are such as defined in (3.5.19) and (A.2), respectively. However, we presume $\pi_t \equiv 1$ from now on what makes it easier to concentrate on optimal *consumption* rates. In addition, we put $p := n$ inside (A.2) and (3.3.38) leading us to $\mathcal{G}_t^* = \mathcal{K}_t := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^n\}$ for $t < \tau$. In other words, we assume that the trader's decisions are based upon the information filtration $\mathcal{K}$ which corresponds to complete/exhaustive knowledge of the electricity futures price at the future time τ $(> \tau_1)$. However, we further suppose the consumption rate c_t to be $\mathcal{K}_t$-adapted. Hence, the agent's consumption decision at time t actually depends upon the information obtained from observing the cash flow history $\{X_s : 0 \leq s \leq t\}$ up to time t, namely $\mathcal{F}_t$, plus the additional anticipating noise values $L_\tau^1, \dots, L_\tau^n$. Moreover, appealing to p.10 in [74], we choose to represent the consumption rate c_t at any time t by its fraction λ_t of the total wealth X_t defining

(3.5.42) $$\lambda_t := c_t / X_t$$

while we call λ_t the *relative* consumption rate. Yet, we suppose λ_t to be $\mathcal{K}_t$-adapted likewise. Entirely taking the above assumptions into account, the cash flow equality (3.5.41) consequently points out as

(3.5.43)

$$d^- X_t^\lambda = X_{t-}^\lambda \left[\sum_{k=1}^{n} \int_{D_k} \Xi_k(t,z)\, \widetilde{N}_k^{\mathcal{K},\mathbb{Q}}(d^- t, dz) - \lambda_t\, dt \right], \qquad X_0^\lambda = x,$$

$(0 \leq t \leq \tau_1)$ where Ξ_k is like defined in the sequel of (A.2), while (3.3.47) and (3.3.48) deliver

(3.5.44) $$\widetilde{N}_k^{\mathcal{K},\mathbb{Q}}(d^- t, dz) := N_k(d^- t, dz) - \frac{1}{\tau - t} \int_{u=t}^{u=\tau} dN_k(u,z)\, dt.$$

Innovatively, we allow the coefficients $\Xi_k(t,z)$ to depend on the relative consumption rate λ_t as well, i.e. in the following we replace these ingredients through $\theta_k(t,z,\lambda_t)$. Thus, (3.5.43) becomes

(3.5.45)

$$d^- X_t^\lambda = X_{t-}^\lambda \left[\sum_{k=1}^{n} \int_{D_k} \theta_k(t,z,\lambda_t)\, \widetilde{N}_k^{\mathcal{K},\mathbb{Q}}(d^- t, dz) - \lambda_t\, dt \right], \qquad X_0^\lambda = x.$$

The latter cash flow equation appears suitable to model the situation where the considered agent is a *large* investor with reference price $F^{\mathcal{K},\mathbb{Q}}$ depending on both the trader's forward-looking information $\mathcal{K}$ and, in particular, on the trader's individual ($\mathcal{K}$-adapted) relative consumption rate λ. In this case, one should expect a negative correlation between the futures price $F^{\mathcal{K},\mathbb{Q}}$ and the consumption rate λ, since the higher the agent's consumption, the lower might be his/her investments in electricity futures and thus, the futures prices might decrease.[51] Admittedly, the concrete choice of $\theta_k(t,z,\lambda_t)$ as a function of λ_t requires some further examinations which, however, are left for future research. Nevertheless, we emphasize that throughout the financial stock market investigations in [74] the appearing cash flow coefficients do – contrarily to above – not depend on the underlying consumption rate. Next, in accordance to Def. 3.1 in [74], we introduce the set of admissible relative consumption rates, $\mathfrak{C} := \mathfrak{C}(\mathcal{K})$ say, by all càglàd and $\mathcal{K}$-adapted stochastic processes $\lambda = (\lambda_t)_{t\in[0,\tau_1]}$ which fulfill

$$\lambda_t > 0,\ X_t^\lambda > 0 \ \ \forall t \in [0,\tau_1], \qquad \int_0^{\tau_1} \lambda_t \, dt < \infty \ [\mathbb{Q}], \qquad \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1} |ln(\lambda_t)| \, dt\right] < \infty.$$

At this step, let us remark that – in analogy to (3.5.42) and [74] – to each $\lambda \in \mathfrak{C}$ we associate the consumption/dividend rate $c_t = \lambda_t X_t^\lambda$. More importantly, adapting "(1.4), (1.5), (2.3), Problem 3.2 and Problem 4.2 in [74]" to our purposes, we now consider the exercise of maximizing the $\mathbb{Q}$-expected accumulated discounted logarithmic utility of the realized consumption rate c, in symbols

(3.5.46)

$$J(\lambda) := J_{\mathcal{K}}(\lambda, x, \tau_1) := \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1} e^{\delta(t)} \, ln\big(\lambda_t X_t^\lambda\big) \, dt\right] \longrightarrow \max_{\lambda\in\mathfrak{C}(\mathcal{K})} !$$

wherein $\delta(t) \leq 0,\ t \in [0,\tau_1]$, designates a deterministic discounting exponent. From now on, we further assume $\theta_k(t,z,\lambda_t) > -1$ $(k = 1,\dots,n)$ for $\mathbb{Q}$-almost-all $(t,z,\lambda_t) \in [0,\tau_1] \times D_k \times (0,\infty)$ whereas $\lambda \in \mathfrak{C}(\mathcal{K})$. In addition, we denote the derivative of θ_k with respect to λ_t (for fixed t) by θ_k' while we presume that θ_k and θ_k'' possess the same sign, that is, $\theta_k \theta_k'' \geq 0$. [The latter presumption ensures that $J(\lambda)$ is concave.] Eventually, the (strictly positive) solution of (3.5.45) is given by

(3.5.47)

$$X_t^\lambda = x \, exp\left\{\int_0^t \left(-\lambda_s + \sum_{k=1}^n \int_{D_k} \frac{ln\big(1+\theta_k(s,z,\lambda_s)\big) - \theta_k(s,z,\lambda_s)}{\tau - s} \int_{u=s}^{u=\tau} dN_k(u,z)\right) ds \right.$$
$$\left. + \sum_{k=1}^n \int_0^t \int_{D_k} ln\big(1+\theta_k(s,z,\lambda_s)\big) \, d\widetilde{N}_k^{\mathcal{K},\mathbb{Q}}(s,z)\right\}.$$

Note in passing that we do no longer need to work with forward integrals in (3.5.47), since the appearing integrands are adapted to the filtration generated by their integrators. On the opposite, in [74] the situation is fundamentally different, as there are no a priori adaptation assumptions made on the coefficient processes in "(2.1), (4.1) and (4.2) in [74]" – except from the (not very restrictive) presumption that they are $\mathcal{F}_\infty$-measurable (compare the top of p.8 along with "(3.1) and (4.3) in [74]"). In particular, the forward-integrators B and $\widetilde{N}$ in [74] associate to $\mathcal{F}$, whereas our $\widetilde{N}_k^{\mathcal{K},\mathbb{Q}}$'s above associate to $\mathcal{K}$ instead. These facts constitute – besides the λ-dependency of θ_k and our electricity market insider trading context – the most striking differences between [74] and our current case study. Further, let us remind that the compensated random measures in (3.5.44) designate $(\mathcal{K},\mathbb{Q})$-martingale

[51] In this context, we must not forget that we always presume a *large* trader to be a '*big player*' who has a significant influence on the overall electricity price dynamics. Also recall that the considered (large) trader in our current model setup possesses *exhaustive* knowledge about the future electricity price at time τ (compare the definition of $\mathcal{K}$) so that it should not at all sound absurd to assume the trader's individual consumption to have a remarkable impact on the overall price development in the market.

integrators for all $k = 1, \dots, n$. Thus, substituting (3.5.47) into (3.5.46) while interchanging the integration order of the iterated martingale-integrals due to Fubini's theorem, we instantly derive

(3.5.48)

$$J(\lambda) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1} e^{\delta(t)}\, ln(x\,\lambda_t)\, dt - \int_0^{\tau_1}\int_0^t e^{\delta(t)}\, \lambda_s\, ds\, dt \right.$$
$$\left. + \sum_{k=1}^n \int_0^{\tau_1}\int_0^t \int_{D_k} e^{\delta(t)}\, \frac{ln\big(1+\theta_k(s,z,\lambda_s)\big) - \theta_k(s,z,\lambda_s)}{\tau - s} \int_{u=s}^{u=\tau} dN_k(u,z)\, ds\, dt\right].$$

Similar to the argumentation on the top of page 12 in [74], we again apply Fubini's theorem yet on (3.5.48) [and hereafter rename/interchange the integration variables] leading us to

(3.5.49)

$$J(\lambda) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1} e^{\delta(t)}\, ln(x\,\lambda_t)\, dt - \int_0^{\tau_1}\int_t^{\tau_1} e^{\delta(s)} ds\, \lambda_t\, dt \right.$$
$$\left. + \sum_{k=1}^n \int_0^{\tau_1}\int_{D_k}\int_t^{\tau_1} e^{\delta(s)} ds\, \frac{ln\big(1+\theta_k(t,z,\lambda_t)\big) - \theta_k(t,z,\lambda_t)}{\tau - t} \int_{u=t}^{u=\tau} dN_k(u,z)\, dt\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1}\left(e^{\delta(t)}\, ln(x\,\lambda_t) - \alpha(t,\tau_1)\,\lambda_t \right.\right.$$
$$\left.\left. + \sum_{k=1}^n \int_{D_k} \alpha(t,\tau_1)\, \frac{ln\big(1+\theta_k(t,z,\lambda_t)\big) - \theta_k(t,z,\lambda_t)}{\tau - t} \int_{u=t}^{u=\tau} dN_k(u,z)\right) dt\right]$$

whereby we have just introduced the deterministic function $\alpha(t,\tau_1) := \int_t^{\tau_1} e^{\delta(s)} ds \geq 0$ in the last equality. Referring to (3.5.46) and (3.5.49), we have to maximize (point-wise) the target functional

(3.5.50)

$$f(\lambda_t) :=$$
$$e^{\delta(t)}\, ln(x\,\lambda_t) - \alpha(t,\tau_1)\,\lambda_t + \alpha(t,\tau_1)\sum_{k=1}^n \int_{D_k} \frac{ln\big(1+\theta_k(t,z,\lambda_t)\big) - \theta_k(t,z,\lambda_t)}{\tau - t} \int_{u=t}^{u=\tau} dN_k(u,z)$$

with respect to $\lambda_t \in \mathfrak{C}(\mathcal{K})$ (for fixed $t \in [0,\tau_1]$) yielding the (necessary) optimality condition

(3.5.51)

$$\frac{e^{\delta(t)}}{\lambda_t} = \alpha(t,\tau_1)\left(1 + \sum_{k=1}^n \int_{D_k} \frac{\theta_k(t,z,\lambda_t)\,\theta_k'(t,z,\lambda_t)}{[1+\theta_k(t,z,\lambda_t)][\tau - t]} \int_{u=t}^{u=\tau} dN_k(u,z)\right)$$

which evidently cannot be solved analytically for λ_t – at least not, as long as we have not chosen concrete coefficient functions θ_k. Nevertheless, we observe $\partial^2 f(\lambda_t)/\partial\lambda_t^2 < 0$ for all $\lambda_t \in \mathfrak{C}(\mathcal{K})$ and $t \in [0,\tau_1]$ so that the solution of (3.5.51), say λ_t^*, indeed embodies the *maximum* of f, respectively J. At this step, we underline that θ_k has to be chosen such that λ_t^* remains $\mathcal{K}_t$-adapted, finite and strictly positive. Regarding the optimality condition (3.5.51) more accurately, we finally emphasize that – in contrast to [74] – our utility-maximizing relative consumption rate λ_t^* reasonably depends (not only on the discounting exponent δ as it is the case in "(2.5) and Theorem 3.3 in [74]" but also) on the electricity futures price volatility coefficients $\theta_1, \dots, \theta_n$ (and their derivatives, respectively). However, from the author's point of view it sounds a bit odd when the optimal relative consumption rate such as given in equality "(3.9) in [74]" does not depend on any of the coefficients μ, σ and θ involved in the underlying cash flow equation "(2.1) in [74]". Especially, this fact becomes striking in the financial stock market application presented in Chapter 4 of [74]. Herein, the optimal relative consumption-portfolio-rate provided in Theorem 4.3 and Corollary 4.4 not at all depends on any of the cash flow coefficients ϱ, α, β and ξ emerging in "(4.1), (4.2) and (4.6) in [74]" and thus, neither on the underlying bond price S_0 nor on the risky asset S_1. Particularly from an economical perspective, this feature might appear a bit strange, since it does not constitute the scenario one would expect intuitively. Typically, electricity market participants' consumption rates are suspected to be strongly linked with (bond and) electricity price development. In this regard, we remind that in our recent approach the optimal relative consumption rate λ_t^* [given as the solution of (3.5.51)] obviously stands in close connection with the $\mathcal{K}$-forward-looking electricity futures price $F^{\mathcal{K},\mathbb{Q}}$ due to the appearance of θ_k inside (3.5.51) being an economically reasonable linking. By the way, if we worked with (3.5.43) at the place of (3.5.45), then the corresponding optimal consumption rate would read as $\lambda_t^* = e^{\delta(t)}/\alpha(t,\tau_1)$, $t < \tau_1$, which stands in strong analogy to "(3.9) in [74]" (to see this, choose $\tau \equiv \tau_1$, $\gamma \equiv 0$ and δ deterministic therein). Further on, for illustrational purposes we assume $\delta(t) := -\delta_0 t$ within a constant $\delta_0 > 0$ for a moment. Then, for $0 \le t \le \tau_1$ property (3.5.51) can be rewritten as

$$\text{(3.5.52)} \qquad \frac{1-e^{-\delta_0(\tau_1-t)}}{\delta_0} = \left\{\lambda_t + \sum_{k=1}^n \int_{D_k} \frac{\lambda_t\, \theta_k(t,z,\lambda_t)\, \theta_k'(t,z,\lambda_t)}{[1+\theta_k(t,z,\lambda_t)][\tau-t]} \int_{u=t}^{u=\tau} dN_k(u,z)\right\}^{-1}$$

whereby the left hand side of (3.5.52) converges towards 0^+ when t approaches τ_1. Consequently, the appearing (t-dependent) coefficients θ_k have to be established in such a way that the sum (which we presume to be different from zero for each t) inside the curly brackets on the right hand side of (3.5.52) converges to plus infinity when $t \to \tau_1^-$. This feature constitutes another condition concerning the practical choice of the coefficients θ_k in addition to our former requirements $\theta_k > -1$, $\theta_k \theta_k'' \ge 0$ and the $\mathcal{K}$-adaptivity, finitude and strict positivity of λ^*. In order to make similar convergence observations in (3.5.51) [as for (3.5.52) but without defining $\delta(t) := -\delta_0 t$], it suffices to presume in (3.5.51) that either $e^{\delta(t)}$ is bounded from zero in an environment of τ_1 or $\delta(t) \nrightarrow -\infty$ for $t \to \tau_1^-$. However, if $\delta(t) \to -\infty$ $(t \to \tau_1^-)$, then $\delta'(t) \to -\infty$ $(t \to \tau_1^-)$ likewise, so that L'Hôpital's rule yields $\alpha(t,\tau_1)/e^{\delta(t)} \to 0^+$ for $t \to \tau_1^-$, similar to the corresponding case related to (3.5.52).

Ultimately, we remark that we also could investigate other utility functionals than the one studied in (3.5.46), of course. For instance, one might do similar examinations as above but for a target function as proposed in "(2.3) in [74]" (although this case should not provide valuable new insights in addition to our recent findings). Yet, in certain situations it might be worthwhile to consider accumulating utility functionals of the type $\mathbb{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} e^{\delta(t)} \ln\left(\lambda_t X_t^\lambda\right) dt\right]$ or $\mathbb{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} \int_0^u e^{\delta(t)} \ln\left(\lambda_t X_t^\lambda\right) dt\, du\right]$. Moreover, we could presume the discounting exponent δ to be an either $\mathcal{F}$ or $\mathcal{K}$-adapted stochastic process, while the choice $p := n$ in (3.3.38) and (A.2) could be replaced by $0 < p < n$. Consequently, the corresponding explicit intermediate filtration setup then would involve forward integration theory. Anyway, we would obtain another interesting exercise, if we did not a priori suppose $\pi_t \equiv 1$ in (3.5.41) and instead studied the related *consumption-portfolio-problem* [requiring a two-dimensional optimization subject to the vector (c,π)]. In this context, it even might be reasonable to link the discounting exponent δ with the bond (3.5.19), for example setting $\delta(t) := -\int_0^t r_s\, ds$ within a stochastic interest rate $r_s > 0$.

Chapter 4

Optimal Liquidation of Electricity Futures Portfolios under Market Impact

4.1 A short introduction to market impact modeling

It is a well-known fact that the electrical-energy industry worldwide possesses a rather monopolistic structure [37], whereas almost all electricity markets are dominated by a few *big players* merely whose individual trading activities may shift prices essentially (also recall our former announcements in section 1.1 in this context). Hence, an in-depth analysis of price impact effects for electricity markets should be of large interest for portfolio managers trading at the European Energy Exchange (EEX) [38] or the Scandinavian Power Exchange *Nord Pool* [73], for instance. In other words, the question of how to optimally liquidate an existing electricity futures portfolio over a fixed time horizon under the constraint of minimizing unfavorable market impact effects undisputably is of steadily growing relevance for energy risk management. In this chapter we thus invent a tractable price impact model for electricity futures, whereas we derive optimal liquidation strategies with respect to different target functions such as *conditional expected trading costs*, for example. In accordance to our previous motivating argumentation in section 1.1 and 3.3, we newly take supplementary forward-looking information about future electricity price behavior into account via a rigorous exploitation of enlargement-of-filtration methods also in our upcoming price impact examinations. Consequently, we derive optimal liquidation strategies for electricity futures portfolios under this insider trading machinery as well – a topic which has not at all been studied extensively in the literature, yet.

Generally speaking, market impact models describe the feedback that individual trading activities in a particular market have on the underlying prices [82]. More precisely, on page 1 in [82] the authors declare market impact risk as "*a specific kind of liquidity risk*", i.e. "*the risk of not being able to execute a trade at the currently quoted price*", since the trade itself feeds back in a negative manner on the underlying price process. All in all, market impact effects thus depict one of the basic price formation arguments [82]. Eventually, the main message throughout price impact literature is that liquidity costs associated to large trades can be reduced significantly by splitting the execution in question into a sequence of smaller ones, so-called "*child orders*", which are spread over a fixed time interval – compare p.4 in [45], p.1 in [81] or p.1 in [82].

To the best of our knowledge, there is not a single work in the market impact literature (comparable to the present discussion) dealing with optimal liquidation strategies particularly for *electricity futures* portfolios under *enlarged filtrations*. Nevertheless, in [43], [44], [45], [81] and [82] market impact models for ordinary financial stock markets are treated rigorously. In accordance to the just mentioned references, we first want to give a short introduction to some basic (stock) market impact material in the current paragraph, whereas in the forthcoming sections of this chapter we then propose a possible transformation of these stock market impact considerations to electricity markets. More accurately speaking, in this chapter we again consider electricity futures prices under enlarged filtrations (while slightly deviating from the basics presented in Ch. 3 actually) and subsequently turn to related *forward-looking* electricity market impact purposes. Obviously, relevant supplementary knowledge about *future* electricity price behavior should be taken into account when liquidating an existing electricity *futures* portfolio over a *future* time span. Therefore, in this chapter we newly provide *optimal* (in the sense of admitting *minimal expected trading costs*) liquidation strategies for electricity futures portfolios under enlarged filtrations associated to additional insider information.
Returning to our main topic, we here essentially address the question of how to optimally liquidate a given electricity futures portfolio throughout a finite time horizon under the constraint of minimizing the expected costs of trading over a certain class of admissible liquidation strategies. In other words, we aim to present a solution to the exercise of "*how to optimally split up a large [electricity futures] trade so as to minimize*" unfavorable price impact effects (see p.1 in [44]). As explained in [45], [81] and [82], under price impact considerations, trades are commonly not executed all at once but gradually over a predetermined time range. Hence, in accordance to [45], a sell (buy) order of x shares executed over the time interval $[0,T]$ in general corresponds to a non-increasing (non-decreasing) asset position $X := (X_t)_{t\in[0,T]}$ obeying the side conditions $X_0 = x$ and $X_T = 0$ ($X_0 = 0$ and $X_T = x$), as "*it is beneficial to split up large trades into a sequence of smaller ones that are spread over*" an entire time *span*, instead of executing a big single trade *at once* (see page 4 in [45]).

The remainder of the present chapter is organized as follows: In section 4.2 we introduce the set of admissible trading strategies and moreover, come up within a suitable market impact model for electricity futures prices. In this context, we derive expressions for the associated costs of trading and hereafter, parameterize the class of admissible trading strategies adequately. In section 4.3 we discuss different target functions of *conditioned-expected-cost type* and innovatively provide (semi-) explicit optimal liquidation strategies for electricity futures portfolios, both under the historical filtration and under forward-looking information flows modeled by enlarged filtrations. In this framework, we explicitly show to what extent our computed optimal liquidation strategies deviate from the so-called *most natural trading strategy* which is simply given by a deterministic and linear liquidation over the underlying fixed time interval. Ultimately, the most important conclusions are drawn in section 4.4 wherein, in addition, some accompanying future research topics are mentioned briefly.

4.2 A market impact model for electricity futures prices

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered and complete probability space which we assume to fulfill the *usual conditions/hypotheses* (compare p.3 in [78]). With respect to our upcoming *forward-looking* electricity market impact considerations under enlarged filtrations, we stress that $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,T]}$ constitutes the common *historical* (respectively, *retro* or *backward-looking*) information filtration.

First of all, we refer to p.4 in [45] and define a suitable class of admissible trading strategies. For this purpose, we initially denote the number of shares in a trader's portfolio at time $t \in [0,T]$ by a stochastic process X_t. Next, without loss of generality, we restrict ourselves to *sell* orders (compare the explanation at the end of paragraph 4.1 above), that is, a position of $x > 0$ shares shall be *liquidated* over the fixed time interval $[0,T]$. Hence, we require the stochastic process X to solve the boundary conditions $X_0 = x$ and $X_T = 0$. Moreover, we assume our fictive *uninitiated traders* only to have access at time $t \in [0,T]$ to an information flow modeled by the *retro* sigma-algebra $\mathcal{F}_t$. Summing up the latter assumptions, we now give the following definition.

Definition 4.2.1 *We define the* <u>*class of admissible trading/liquidation strategies*</u> $\mathcal{A} := \mathcal{A}(\mathcal{F})$ *by all càdlàg (French: continue à droite avec des limites à gauche),* $\mathcal{F}$*-adapted and time-differentiable stochastic processes* $X := (X_t)_{t\in[0,T]}$ *with finite total variation on* $[0,T]$ *that fulfill the side conditions* $X_0 = x$ *and* $X_T = 0$. *By the way, we denote the derivative of* X_t *with respect to* t *by* $\dot{X}_t$ *frequently.* ■

With view on Definition 4.2.1, we recall that the *most natural trading strategy* in $\mathcal{A}$ is embodied by the following *linear deterministic liquidation strategy* (cf. the top of p.3 in [44])

(4.2.1)
$$\hat{X}_t := x - \frac{x}{T}t.$$

Moreover, slightly deviating from (3.2.22), we define the *electricity futures price* (yet under $\mathbb{P}$) at time $t \in [0,\tau_1]$ associated to a swap contract which promises the delivery of one unit of electrical energy, say 1 MWh, over the future delivery period $[\tau_1,\tau_2]$ via

(4.2.2)
$$F_t := F_t^{\mathcal{F},\mathbb{P}}(\tau_1,\tau_2) := \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \mathbb{E}_{\mathbb{P}}(S_u|\mathcal{F}_t)\,du$$

wherein, similarly to (3.2.9), the electricity spot price is assumed to obey

(4.2.3)
$$S_t = \mu(t) + \sum_{k=1}^{n} w_k \left(x_k\, e^{-\lambda_k t} + \int_0^t \sigma_k(s)\, e^{-\lambda_k(t-s)} dL_s^k \right).$$

Anyway, in Remark 4.2.2 we will give some justifying comments on our extraordinary $\mathbb{P}$-choice inside equation (4.2.2). In particular, we therein explain why (respectively, in which cases or under which assumptions) we are allowed to deviate from the risk-neutral pricing standards [and thus, from our former $\mathbb{Q}$-definition in (3.2.22)], actually introducing the electricity futures price in (4.2.2) under the true market measure $\mathbb{P}$. But previously, let us do some further computations on the electricity futures price (4.2.2) leading us to a representation of the latter in terms of $(\mathcal{F},\mathbb{P})$-compensated stochastic jump integrals. Concretely, we argue as follows:

Parallel to the proof of Proposition 3.2.2, a straightforward calculation [but under $\mathbb{P}$ yet] using (3.2.4), (3.2.5), (3.2.9) and (4.2.2) yields the integral equation

(4.2.4)

$$F_t = F_0 + \sum_{k=1}^{n} \int_0^t \int_{D_k} z\, \Lambda_k(s)\, d\tilde{N}_k^{\mathbb{P}}(s,z)$$

within a deterministic initial value

(4.2.5)

$$F_0 := \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \left[\mu(u) + \sum_{k=1}^{n} w_k \left(x_k\, e^{-\lambda_k u} + \int_0^u \int_{D_k} z\, \sigma_k(s)\, e^{-\lambda_k (u-s)} \rho_k(s)\, d\nu_k(z)\, ds \right) \right] du$$

and a deterministic volatility function $\Lambda_k(s) := \Lambda_k(s, \tau_1, \tau_2) \geq 0$ such as given in (3.2.24). Herein, the compensated PRMs $\tilde{N}_k^{\mathbb{P}}$ are like defined in (3.2.5). Evidently, the electricity futures price F_t in (4.2.4) constitutes a $\mathcal{F}_t$-adapted (local) Sato-martingale under $\mathbb{P}$ which is, in the light of its definition in (4.2.2), not a surprising result.

Remark 4.2.2 *Originally, the electricity futures price (4.2.2) is defined under a "risk-neutral (equivalent) martingale measure", say $\mathbb{Q}$, in the electricity market literature and not under the "true market measure" $\mathbb{P}$, as above. However, standard assumptions in the market impact literature (see e.g. [44], footnote 1 in [45] or page 2 in [82]) require the unaffected (stock) price process to form a $(\mathcal{F}, \mathbb{P})$-martingale. Obviously, our futures price $F^{\mathcal{F},\mathbb{P}}$ as implemented in (4.2.2) forms a $(\mathcal{F}, \mathbb{P})$-martingale by definition, whereas the risk-neutral counterpart $F^{\mathcal{F},\mathbb{Q}}$ as defined in (3.2.22) does not meet this feature in general. Fortunately, in accordance to the footnote 1 in [45], we also may ignore drift effects (which a measure change actually would induce) in our electricity market framework, if we assume short trading horizons reaching from a few hours to a few days merely. Additionally, we defend our extraordinary $\mathbb{P}$-choice in (4.2.2) by remarking that the latter simultaneously (as commonly under $\mathbb{Q}$ only) excludes the existence of any arbitrage opportunities and free lunches, if one assumes – as just explained – short and thus, drift-unaffecting trading horizons. Similarly to the arguing in [45], we here mainly aim to focus on the quantitative effects of electricity price impact descending from specific trading strategies under the constraint of minimizing a certain cost criterion, instead of examining drift-effects associated to probability measure changes.* ■

Transferring the basic concepts of Chapter 2 in [44] to our electricity futures market framework, we define the (trading-size dependent) *perturbed electricity futures price* $\tilde{F}_t := \tilde{F}_t(X)$ by the $\mathcal{F}_t$-adapted stochastic process

(4.2.6)
$$\tilde{F}_t := F_t + \eta\, \dot{X}_t + \gamma\, [X_t - x]$$

wherein $t \in [0, T]$ while η and γ are assumed to constitute positive constants. Herein, the member F_t firstly embodies the *unperturbed*, respectively *unaffected electricity futures price* as defined in (4.2.2), secondly, the term $\eta \dot{X}_t$ describes the *temporary impact* of trading $\dot{X}_t dt$ shares at time $t \in [0, T]$, while the summand $\gamma [X_t - x]$ finally designates the *permanent impact* accumulated over all transactions in the time interval $[0, t]$, which affects all (that is, both current and future) trades equally [82].

Note in passing that the price impact part of (4.2.6) obviously can be interpreted as a simple functional, $\mathfrak{I}$ say, of the liquidation strategy $X \in \mathcal{A}$, in symbols

$$\mathfrak{I}: \mathcal{A} \longrightarrow \mathbb{R}, \qquad \mathfrak{I}(\cdot) := \eta * \frac{\partial}{\partial t}(\cdot) + \gamma * [\cdot - x].$$

Moreover, for $X_t \equiv x$ the price impact part vanishes, i.e. $\mathfrak{I}(x) = 0$, and thus, we get $\tilde{F}_t = F_t$ in this case what indeed appears economically reasonable (at least from the trader's individual point of view).

Further on, in accordance to Chapter 2 in [44], for a liquidation strategy $X \in \mathcal{A}$ we introduce the ($\mathcal{F}_T$-adapted) *cumulated costs* arising from trading dX_s shares at price $\tilde{F}_s$ over the time range $[0, T]$ via

(4.2.7)

$$\mathcal{C}_T^X := \int_0^T \tilde{F}_s \, dX_s = \int_0^T \tilde{F}_s \, \dot{X}_s \, ds.$$

Substituting (4.2.4) and (4.2.6) into (4.2.7), an integration by parts delivers

(4.2.8)

$$\mathcal{C}_T^X = \frac{\gamma}{2} x^2 - x\, F_0 + \eta \int_0^T \dot{X}_s^2 \, ds - \sum_{k=1}^{n} \int_0^T \int_{D_k} z\, X_s\, \Lambda_k(s)\, d\tilde{N}_k^{\mathbb{P}}(s, z).$$

Remark 4.2.3 *At this step, we underline that, similarly to the argumentation in Chapter 2 in [44], the linear deterministic liquidation strategy $\hat{X}_t$ as defined in (4.2.1) turns out to be optimal for the specific risk criterion "minimum expected trading costs" also in our electricity market context, since*

(4.2.9) $$\mathbb{E}_{\mathbb{P}}[\mathcal{C}_T^X] \geq \mathbb{E}_{\mathbb{P}}\big[\mathcal{C}_T^{\hat{X}}\big] \quad \forall X \in \mathcal{A}$$

holds true. The latter not at all is a surprising observation, if one compares our representation (4.2.8) with the corresponding equation in the middle of page 2 in [44]. ■

Actually, we aim to examine the specific risk criterion $\mathcal{F}_t$*-conditioned expected trading costs* in the sequel. Thus, in order to provide optimal liquidation strategies later, for time indices $0 \leq t \leq T$ we parameterize our trading strategies via

(4.2.10)

$$X_t^\alpha := \frac{T - t}{T} \left(x + \alpha \int_0^t F_u \, du \right)$$

within an arbitrary parameter $\alpha \in \mathbb{R}$. Obviously, X_t^α embodies an *admissible* strategy in the sense of Definition 4.2.1, i.e. $X_t^\alpha \in \mathcal{A}$. Moreover, for $\alpha = 0$ we obtain the linear deterministic liquidation strategy as implemented in (4.2.1). Hence, the *adjusting screw* α appearing in (4.2.10) controls the deviation of the actually realized liquidation activity X_t^α from the *most natural trading strategy* $\hat{X}_t$ in a very simple and practical way. We remark that the concrete choice (4.2.10) has been motivated by equation "(14) in [44]" which itself is closely connected with the specific risk criterion "(7) in [44]", admittedly.

Nevertheless, the proposed structure of X_t^α seems to be a reasonable generalization of the linear deterministic trading strategy $\hat{X}_t$ – not only for the specific risk criterion examined in [44]. Note that, unfortunately, it appears necessary to restrict the rather general class of admissible trading strategies $\mathcal{A}$ yet to strategies that are (for example) of the form (4.2.10) in order to make fruitful statements concerning *optimal* liquidation strategies also in our challenging electricity futures market framework.

We recall that e.g. in [44] an optimal trading strategy is found without sticking to a restrictive presumption such as introduced in (4.2.10) above, whereas the authors therein have to deal with a very sophisticated stochastic control problem in return – although they have merely started with a rather simple geometric Brownian motion model with constant coefficients (compare equality "(3) in [44]") describing the dynamics of the underlying *stock* price. By the way, throughout the majority of market impact literature there are remarkable often very simple (sometimes inadequate) setups for the underlying stock price process chosen (such as geometric Brownian motion or even Brownian Bachelier models with constant coefficients), whenever the main objective consists in the provision of *explicit* optimal trading strategies. Since in our electricity market framework we are facing a rather complex multi-factor jump-term dynamics for the underlying futures price (4.2.4), a restrictive choice such as made in (4.2.10) appears convenient in order to provide (at least semi-explicit) optimal liquidation strategies yet for electricity futures portfolios. Otherwise, one is left within a very complicated stochastic control problem descending from a multi-factor jump model. Nevertheless, some further research might deal with the derivation of optimal liquidation strategies for electricity futures portfolios without sticking to the constraint (4.2.10).

Returning to our main topic, we substitute (4.2.4) into (4.2.10) and hereafter apply the stochastic Fubini-Tonelli theorem [similarly to our arguing in (3.3.19) formerly] which leads us to the ($\mathcal{F}_t$-adapted) representation

(4.2.11)

$$X_t^\alpha = \frac{T-t}{T}\left(x + \alpha\, F_0\, t + \alpha \sum_{k=1}^{n} \int_0^t \int_{D_k} z\, \Lambda_k(s)\, (t-s)\, d\widetilde{N}_k^{\mathbb{P}}(s,z) \right).$$

Moreover, from (4.2.10) we immediately receive the ($\mathcal{F}_t$-adapted) derivative

(4.2.12)

$$\dot{X}_t^\alpha = -\frac{1}{T}\left(x + \alpha \int_0^t F_u\, du + (t-T)\, \alpha\, F_t \right) = \frac{X_t^\alpha}{t-T} + \frac{T-t}{T}\, \alpha\, F_t$$

describing the instantaneous alteration in the electricity futures holdings.

4.3 Optimal liquidation strategies

In the present paragraph we devote our attention towards the derivation of optimal liquidation strategies for electricity futures portfolios with respect to different risk criterions: Firstly, we treat the *$\mathcal{F}_t$-conditioned expected costs case* in subsection 4.3.1. Afterwards, we introduce *forward-looking* electricity futures price representations and consider optimal trading strategies under additional insider information on the future electricity price behavior which is modeled by enlarged filtrations.

4.3.1 A trading strategy admitting minimal $\mathcal{F}$-conditioned expected costs

Starting off, for $0 \leq t \leq T$ we introduce the risk criterion $\mathcal{F}_t$*-conditioned expected costs* due to

(4.3.1)
$$\Re^{\mathcal{F}}(\alpha) := \Re^{\mathcal{F},\mathbb{P}}(\alpha, t, T) := \mathbb{E}_{\mathbb{P}}\big(C_T^{X^\alpha} \big| \mathcal{F}_t\big).$$

Our objective yet consists in finding the precise parameter α (and simultaneously the corresponding liquidation strategy X^α) which *minimizes* the above target function $\Re^{\mathcal{F}}$. Putting (4.2.8) [but with $X := X^\alpha$ such as given in (4.2.10) therein] into (4.3.1), we instantly derive

(4.3.2)
$$\Re^{\mathcal{F}}(\alpha) = \frac{\gamma}{2}x^2 - x\,F_0 + \eta \int_0^t \dot{X}_s^\alpha\, dX_s^\alpha + \eta\, \mathbb{E}_{\mathbb{P}}\left(\int_t^T \dot{X}_s^\alpha\, dX_s^\alpha \middle| \mathcal{F}_t \right) - \sum_{k=1}^n \int_0^t \int_{D_k} z\, X_s^\alpha\, \Lambda_k(s)\, d\widetilde{N}_k^{\mathbb{P}}(s,z).$$

Further on, using (4.2.12), the remaining conditional expectation in (4.3.2) becomes

(4.3.3)
$$\mathbb{E}_{\mathbb{P}}\left(\int_t^T \dot{X}_s^\alpha\, dX_s^\alpha \middle| \mathcal{F}_t \right) = \int_t^T \mathbb{E}_{\mathbb{P}}\left(\left(\dot{X}_s^\alpha\right)^2 \middle| \mathcal{F}_t \right) ds$$
$$= \frac{1}{T^2} \int_t^T \mathbb{E}_{\mathbb{P}}\left(\left[x + \alpha \left(\int_0^s F_u\, du + (s-T)\, F_s \right) \right]^2 \middle| \mathcal{F}_t \right) ds.$$

Merging (4.2.10), (4.2.12) and (4.3.3) into (4.3.2), we immediately deduce

(4.3.4)
$$\Re^{\mathcal{F}}(\alpha) = \frac{\gamma}{2}x^2 - x\,F_0 + \frac{\eta}{T^2}\int_0^t (x + \alpha\,\xi_s)^2\, ds + \frac{\eta}{T^2}\int_t^T \mathbb{E}_{\mathbb{P}}([x + \alpha\,\xi_s]^2 | \mathcal{F}_t)\, ds$$
$$- \frac{1}{T}\sum_{k=1}^n \int_0^t \int_{D_k} z\,\Lambda_k(s)\,(T-s)\left(x + \alpha \int_0^s F_u\, du \right) d\widetilde{N}_k^{\mathbb{P}}(s,z)$$

wherein we have just introduced the short hand notation

$$\xi_s := \int_0^s F_u\, du + (s-T)\, F_s.$$

Differentiating (4.3.4) with respect to α, we next get

(4.3.5)
$$\frac{d\Re^{\mathcal{F}}(\alpha)}{d\alpha} = \frac{2\eta}{T^2}\int_0^t (x + \alpha\,\xi_s)\,\xi_s\, ds + \frac{2\eta}{T^2}\int_t^T \mathbb{E}_{\mathbb{P}}([x + \alpha\,\xi_s]\,\xi_s | \mathcal{F}_t)\, ds$$
$$- \frac{1}{T}\sum_{k=1}^n \int_0^t \int_{D_k} z\,\Lambda_k(s)\,(T-s) \int_0^s F_u\, du\; d\widetilde{N}_k^{\mathbb{P}}(s,z).$$

Yet, the first order condition $d\mathfrak{R}^{\mathcal{F}}(\alpha)/d\alpha = 0$ yields the *minimizing*[52] coefficient

(4.3.6)

$$\hat{\alpha} := \frac{C - D - E}{A + B}$$

with abbreviations

(4.3.7) $A := \int_0^t \xi_s^2 \, ds \geq 0, \qquad B := \int_t^T \mathbb{E}_{\mathbb{P}}(\xi_s^2 | \mathcal{F}_t) \, ds \geq 0, \qquad D := x \int_0^t \xi_s \, ds,$

$$E := x \int_t^T \mathbb{E}_{\mathbb{P}}(\xi_s | \mathcal{F}_t) \, ds \,, \qquad C := \frac{T}{2\eta} \sum_{k=1}^n \int_0^t \int_{D_k} \int_0^s z \, F_u \, \Lambda_k(s) \, (T-s) \, du \, d\widetilde{N}_k^{\mathbb{P}}(s,z).$$

Applying the stochastic Fubini-Tonelli theorem several times while exploiting the fact that F_t such as given in (4.2.4) constitutes a $(\mathcal{F}_t, \mathbb{P})$-martingale, we moreover observe the equality $E = -D$ to be valid (see section 4.5 for a full proof). Hence, (4.3.6) further simplifies to $\hat{\alpha} = C/(A+B)$. Ultimately, the precise optimal liquidation strategy yielding minimal $\mathcal{F}_t$-conditioned expected costs reads as

(4.3.8)

$$X_t^{\hat{\alpha}} = \frac{T-t}{T} \left(x + \hat{\alpha} \int_0^t F_u \, du \right)$$

within a *liquidation intensity coefficient* $\hat{\alpha} = C/(A+B)$. In practice, the electricity futures price process F appearing inside (4.3.8) must be simulated numerically by using the dynamics (4.2.4).

4.3.2 The electricity futures price under an enlarged filtration

In this subsection we implement the flow of supplementary information about future electricity price behavior by an *enlarged* filtration. More precise, we assume a fictive *informed trader* to have an idea about the jump noise values $L_\tau^1, \dots, L_\tau^p$ within a time partition $0 \leq t < \tau \leq T$ and $1 \leq p \leq n$. At this step, we emphasize that $\tau \leq T$ constitutes the interesting case (that we want to investigate in the following), while $\tau > T$ embodies the economically irrelevant scenario, rather. Hence, similarly to (3.3.38), we introduce the (explicitly) *enlarged* filtration $\mathcal{G}_t^*$ due to

(4.3.9) $$\mathcal{G}_t^* := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^p\}$$

whereas we recall that $\mathcal{F}_t \subset \mathcal{G}_t^*$ is valid whenever $t < \tau$ and $\mathcal{F}_t = \mathcal{G}_t^*$ holds for all time indices $t \geq \tau$. Further on, for $0 \leq t < \tau$ and $k = 1, \dots, p$ we come up with the ($\mathcal{G}_t^*$-adapted) information yield

(4.3.10) $$Y_t^k := \frac{L_\tau^k - L_t^k}{\tau - t}.$$

Then, in accordance to Condition A [with $l := p$] and (3.3.39) [but both under $\mathbb{P}$ now], the process

(4.3.11) $$L_t^k - \int_0^t Y_s^k \, ds$$

designates a $(\mathcal{G}_t^*, \mathbb{P})$-martingale for all $t \in [0, \tau[$ and $k = 1, \dots, p$.

[52] Note that $d^2\mathfrak{R}^{\mathcal{F}}(\alpha)/d\alpha^2 > 0$ is valid for all α.

Moreover, referring to Lemma 3.3.1 (d) [respectively to (3.3.47) and (3.3.48)], for $k = 1, \dots, p$ the (stochastic) $(\mathcal{G}^*, \mathbb{P})$-compensator of the PRM $dN_k(s,z)$ is given by

(4.3.12)

$$d\nu_k^{\mathcal{G}^*,\mathbb{P}}(s,z) := \frac{1}{\tau - s} \int_{u=s}^{u=\tau} dN_k(u,z)\, ds$$

whereas the $(\mathcal{G}^*, \mathbb{P})$-compensated random measure is thus of the form

(4.3.13) $$d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{P}}(s,z) := dN_k(s,z) - d\nu_k^{\mathcal{G}^*,\mathbb{P}}(s,z).$$

Appealing to (3.3.45) and Lemma 3.5.1, for all $k = 1, \dots, p$ and $0 \le t \le s < \tau$ we get

(4.3.14)

$$\mathbb{E}_{\mathbb{P}}\big(L_\tau^k - L_s^k \big| \mathcal{G}_t^*\big) = \frac{\tau - s}{\tau - t}\, \mathbb{E}_{\mathbb{P}}\big(L_\tau^k - L_t^k \big| \mathcal{G}_t^*\big) = \frac{\tau - s}{\tau - t}\, \big[L_\tau^k - L_t^k\big].$$

Parallel to our former argumentation in Remark 3.3.2 [but speaking for the $\mathbb{P}$-case now], we further remind that for $k = p+1, \dots, n$ the compensated PRMs $\widetilde{N}_k^{\mathcal{F},\mathbb{P}}(t,z)$ are (not only $\mathcal{F}_t$-adapted but also) $\mathcal{G}_t^*$-adapted $\mathbb{P}$-martingale integrators, since $\mathcal{F}_t \subset \mathcal{G}_t^*$ is valid for all $t < \tau$.
Corresponding to (4.2.2), we yet define the electricity futures price under the enlarged filtration $\mathcal{G}^*$ by

(4.3.15)

$$F_t^* := F_t^{\mathcal{G}^*,\mathbb{P}}(\tau_1, \tau_2) := \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \mathbb{E}_{\mathbb{P}}(S_u | \mathcal{G}_t^*)\, du.$$

By the way, note that F_t^* equals F_t for $t \ge \tau$. Furthermore, similar arguments as in (3.3.40) – (3.3.46) [but with respect to the measure $\mathbb{P}$ now] yield the $(\mathcal{G}_t^*, \mathbb{P})$-martingale representation

(4.3.16)

$$F_t^* = F_0^* + \sum_{k=1}^{p} \int_0^t \int_{D_k} [\Lambda_k(s) - \Phi_k(s)]\, z\, d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{P}}(s,z) + \sum_{k=p+1}^{n} \int_0^t \int_{D_k} \Lambda_k(s)\, z\, d\widetilde{N}_k^{\mathcal{F},\mathbb{P}}(s,z)$$

within a deterministic initial value F_0^* along with deterministic functions $\Lambda_k(s) := \Lambda_k(s, \tau_1, \tau_2)$ and $\Phi_k(s)$ such as defined in (3.2.24) and (3.3.29), respectively. Herein, the $(\mathcal{F}, \mathbb{P})$-compensated PRMs $\widetilde{N}_k^{\mathcal{F},\mathbb{P}}$ are such as implemented in (3.2.5). Finally, a rigorous comparison of (4.3.16) with (4.2.4) shows us to what extend additional future information about the jump noises $L_\tau^1, \dots, L_\tau^p$ influences the electricity futures price dynamics under $\mathbb{P}$.

4.3.3 A trading strategy admitting minimal $\mathcal{G}^*$-conditioned expected costs

In accordance to (4.2.6), for a *forward-looking* electricity futures price process F_t^* such as introduced in (4.3.15) and a trading strategy X_t^α such as given in (4.2.10) [but F_u therein replaced by F_u^* now], we define the *perturbed electricity futures price under* $\mathcal{G}^*$ by dint of

(4.3.17) $$\tilde{F}_t^* := F_t^* + \eta\, \dot{X}_t^\alpha + \gamma\, [X_t^\alpha - x].$$

Referring to (4.2.7) and (4.2.8), we currently take (4.3.16) and (4.3.17) into account and thus obtain cumulated costs associated to the trading strategy X_t^α but yet under $\mathcal{G}^*$ reading

(4.3.18)
$$C_T^{X^\alpha,\mathcal{G}^*} := \int_0^T \tilde{F}_s^* \, dX_s^\alpha$$
$$= \frac{\gamma}{2} x^2 - x\, F_0^* + \eta \int_0^T \left(\dot{X}_s^\alpha\right)^2 ds - \sum_{k=1}^{p} \int_0^T \int_{D_k} z\, X_s^\alpha \, [\Lambda_k(s) - \Phi_k(s)] \, d\tilde{N}_k^{\mathcal{G}^*,\mathbb{P}}(s,z)$$
$$- \sum_{k=p+1}^{n} \int_0^T \int_{D_k} z\, X_s^\alpha \, \Lambda_k(s) \, d\tilde{N}_k^{\mathcal{F},\mathbb{P}}(s,z).$$

Moreover, we now choose the $\mathcal{G}_t^*$*-conditioned expected costs*

(4.3.19)
$$\mathfrak{R}^{\mathcal{G}^*}(\alpha) := \mathfrak{R}^{\mathcal{G}^*,\mathbb{P}}(\alpha,t,T) := \mathbb{E}_{\mathbb{P}}\left(C_T^{X^\alpha,\mathcal{G}^*} \middle| \mathcal{G}_t^* \right)$$

as our risk criterion. Parallel to the derivation methodology in section 4.3.1, by putting (4.3.18) into (4.3.19) while appealing to our former comments on the PRMs given in the sequel of (4.3.14), we get

(4.3.20)
$$\mathfrak{R}^{\mathcal{G}^*}(\alpha) = \frac{\gamma}{2} x^2 - x\, F_0^* + \eta \int_0^t \left(\dot{X}_s^\alpha\right)^2 ds + \eta\, \mathbb{E}_{\mathbb{P}}\left(\int_t^T \dot{X}_s^\alpha \, dX_s^\alpha \middle| \mathcal{G}_t^* \right)$$
$$- \sum_{k=1}^{p} \int_0^t \int_{D_k} z\, X_s^\alpha \, [\Lambda_k(s) - \Phi_k(s)] \, d\tilde{N}_k^{\mathcal{G}^*,\mathbb{P}}(s,z)$$
$$- \sum_{k=p+1}^{n} \int_0^t \int_{D_k} z\, X_s^\alpha \, \Lambda_k(s) \, d\tilde{N}_k^{\mathcal{F},\mathbb{P}}(s,z).$$

For the sake of notational simplicity, we presume $0 \le t \le \tau = T$ in the following. That is, we suppose market insiders to have knowledge (respectively, an idea) about the future electricity price behavior particularly concerning the end of the liquidation interval. Then, similar to (4.3.3), we take (4.2.12) [but with F^* instead of F therein] into account what leads us to

(4.3.21)
$$\mathbb{E}_{\mathbb{P}}\left(\int_t^T \dot{X}_s^\alpha \, dX_s^\alpha \middle| \mathcal{G}_t^* \right) = \int_t^T \mathbb{E}_{\mathbb{P}}\left(\left(\dot{X}_s^\alpha\right)^2 \middle| \mathcal{G}_t^* \right) ds$$
$$= \frac{1}{T^2} \int_t^T \mathbb{E}_{\mathbb{P}}\left(\left[x + \alpha \left(\int_0^s F_u^* \, du + (s-T)\, F_s^* \right) \right]^2 \middle| \mathcal{G}_t^* \right) ds.$$

Minimizing $\mathfrak{R}^{\mathcal{G}^*}(\alpha)$ [as given in (4.3.20) – (4.3.21)] with respect to α, we receive the optimal value

(4.3.22)
$$\alpha^* := \frac{C_1^* + C_2^* - D^* - E^*}{A^* + B^*}$$

wherein we have just set

(4.3.23)

$$A^* := \int_0^t \xi_s^{*2}\, ds\,, \qquad B^* := \int_t^T \mathbb{E}_{\mathbb{P}}\big(\xi_s^{*2} \big| \mathcal{G}_t^*\big)\, ds\,, \qquad D^* := x\int_0^t \xi_s^*\, ds\,, \qquad E^* := x\int_t^T \mathbb{E}_{\mathbb{P}}(\xi_s^* | \mathcal{G}_t^*)\, ds,$$

$$C_1^* := \frac{T}{2\eta}\sum_{k=1}^{p}\int_0^t \int_{D_k}\int_0^s z\, F_u^*\, (T-s)\, [\Lambda_k(s) - \Phi_k(s)]\, du\, d\widetilde{N}_k^{\mathcal{G}^*,\mathbb{P}}(s,z),$$

$$C_2^* := \frac{T}{2\eta}\sum_{k=p+1}^{n}\int_0^t \int_{D_k}\int_0^s z\, F_u^*\, (T-s)\, \Lambda_k(s)\, du\, d\widetilde{N}_k^{\mathcal{F},\mathbb{P}}(s,z)\,, \qquad \xi_s^* := \int_0^s F_u^*\, du + (s-T)\, F_s^*.$$

Applying the Fubini-Tonelli theorem several times while exploiting the fact that F_t^* [as given in (4.3.16)] designates a $(\mathcal{G}_t^*, \mathbb{P})$-martingale, we finally obtain $E^* = -D^*$ [see section 4.5 for a proving sketch]. Therefore, (4.3.22) further simplifies to

$$\alpha^* = \frac{C_1^* + C_2^*}{A^* + B^*}.$$

Again, we observe $d^2\mathfrak{R}^{\mathcal{G}^*}(\alpha)/d\alpha^2 > 0$ to hold for all α. Hence, the liquidation parameter α^* indeed *minimizes* the ($\mathcal{G}^*$-forward-looking) conditional expected liquidation costs $\mathfrak{R}^{\mathcal{G}^*}(\alpha)$. In conclusion, the specific liquidation strategy

(4.3.24)

$$X_t^{\alpha^*} = \frac{T-t}{T}\left(x + \alpha^* \int_0^t F_u^*\, du \right) \in \mathcal{A}^* := \mathcal{A}(\mathcal{G}^*)$$

with liquidation intensity $\alpha^* = (C_1^* + C_2^*)/(A^* + B^*)$ is optimal in the sense of minimizing our cost criterion (4.3.19). Here, $\mathcal{A}^*$ is defined parallel to $\mathcal{A}$ in Def. 4.2.1. Finally, comparing (4.3.6) with (4.3.22), respectively (4.3.7) with (4.3.23), we notice an outstanding similarity, whereas the previously appearing electricity futures price process F recently has been replaced by its insider trading counterpart F^*. Actually, the most striking difference between (4.3.6) and (4.3.22) consists in the decomposition of the former summand C yet into $C_1^* + C_2^*$. Herein, the term C_2^* closely resembles C, whereby additional insider information is weaved into our (forward-looking) optimal liquidation strategy (4.3.24) via the innovative summand C_1^*. Unfortunately, any further (analytical) treatment of the conditional expectations such as appearing inside the coefficients B and B^* seems to be an extremely challenging issue and thus, ought to be examined in some separated future research. Nevertheless, in subsection 4.5.1 below we propose an initial approach related to a linear interpolation scheme in order to approximate the coefficient B, while similar estimation techniques can be used to deal with B^* likewise. Maybe, it is moreover possible to modify (4.2.10) in such a way that X_t^α possesses independent increments with respect to $\mathcal{F}_t$, respectively $\mathcal{G}_t^*$. If this was the case, then the conditional expectations in (4.3.3) and (4.3.21) would conveniently reduce to usual ones, at least.

Remark 4.3.1 *As a closing remark, we aim to underline the following connections between, firstly, insider trading models in electricity markets with (initially) enlarged filtrations, related price impact considerations and, thirdly, the theory of backward stochastic differential equations (BSDEs). In our* *opinion, this triumvirate embodies a challenging future research area both from a mathematical as well as from an economical point of view. To begin with, we stress that so-called 'insiders' (in any market) are usually supposed to be 'large traders' so that it should sound legitimate to assume them to be able to manipulate the underlying price dynamics by their individual transactions, respectively trading behavior. As discussed above, this fact immediately gives rise to price impact examinations also in electricity markets. Now, remind that most enlarged filtration approaches throughout this thesis (and also in the literature) essentially possess the coarse structure* "historical sigma-algebra $\mathcal{F}$ plus a sigma-algebra generated by future noise/price values" *[compare e.g. (4.3.9)], while large investors (due to influential individual transactions) ought to be able to achieve/establish a certain desired price level at a fixed future time (or drive the price close to such a level at least). This is where the theory of 'degenerated' [as in general $p < n$ in (4.3.9)] BSDEs enters the scene, whereas we simultaneously have detected a beneficial method to establish the future noise values [such as $L_\tau^1, \dots, L_\tau^p$ in (4.3.9)] in practice. By the way, for a large investor it a priori sounds reasonable to assume p close to n. In this context, we recall that if $p = n$ in (4.3.9), then the futures price $F_\tau^*(\tau_1, \tau_2)$ in (4.3.15) entirely is determined by the noise values $L_\tau^1, \dots, L_\tau^n$. Evidently, this instance is closely connected with the common BSDE-case. Moreover, previously to (4.3.21) we have supposed market insiders to have knowledge about future electricity price behavior particularly concerning the end of the liquidation interval (i.e. $\tau = T$) which also fits well into the just described framework.* ■

4.4 Conclusions

In this chapter we have proposed a suitable market impact model for electricity futures prices both under common knowledge and under supplementary forward-looking information about future electricity price behavior. More precisely, we have derived optimal liquidation strategies for electricity futures portfolios by minimizing different target functions of conditional expected cost type over a suitably parameterized class of admissible trading strategies. A challenging related research topic might consist in a numerical study of the invented electricity futures price impact models. Especially, it would be interesting to visualize the precise market impact effects that descend from additionally available future information by numerical simulations. Moreover, a comparison of our *forward-looking* optimal liquidation strategy (4.3.24) with the (actually *backward-looking*) optimal liquidation strategy (4.3.8) certainly would yield some worthy new (visualized) insight concerning the undisputable advantages of insider information. Last but not least, a proper optimization of our conditional cost criterions without sticking to the (admittedly rather restrictive) parameterization of admissible trading strategies should embody another reasonable extension of our model, whereas the handling of the incoming multi-factor jump-noise optimal control problem for this much more general case definitely bears a challenging issue.

4.5 Appendix

In this paragraph we show why $E = -D$ holds in (4.3.7). Firstly, from the definition of E and ξ we get

(4.5.1)

$$E = x \int_t^T \left(\int_0^t \mathbb{E}_{\mathbb{P}}(F_u | \mathcal{F}_t)\, du + \int_t^s \mathbb{E}_{\mathbb{P}}(F_u | \mathcal{F}_t)\, du + (s - T)\, \mathbb{E}_{\mathbb{P}}(F_s | \mathcal{F}_t) \right) ds.$$

Since the futures price (4.2.4) embodies a $(\mathcal{F},\mathbb{P})$-martingale, equation (4.5.1) further simplifies to

(4.5.2)

$$E = x\int_t^T\left(\int_0^t \mathbb{E}_{\mathbb{P}}(F_u|\mathcal{F}_t)\,du + F_t\,(2s-t-T)\right)ds = x\int_t^T\int_0^t \mathbb{E}_{\mathbb{P}}(F_u|\mathcal{F}_t)\,du\,ds$$

$$= x\,(T-t)\,\mathbb{E}_{\mathbb{P}}\left(\int_0^t F_u\,du\,\middle|\,\mathcal{F}_t\right).$$

Substituting the representation (4.2.4) into the last expression in (4.5.2) and hereafter, applying the Fubini-Tonelli theorem, we finally end up with

(4.5.3)

$$E = x\,(T-t)\left[F_0\,t + \sum_{k=1}^{n}\int_0^t\int_{D_k} z\,\Lambda_k(s)\,(t-s)\,d\widetilde{N}_k^{\mathbb{P}}(s,z)\right].$$

On the opposite, from definition (4.3.7) we deduce

(4.5.4)

$$D = x\left(\int_0^t\int_0^s F_u\,du\,ds + \int_0^t (s-T)\,F_s\,ds\right) = x\left(\int_0^t (t-u)\,F_u\,du + \int_0^t (s-T)\,F_s\,ds\right)$$

$$= x\,(t-T)\int_0^t F_u\,du.$$

Merging (4.2.4) into (4.5.4), a straightforward application of the Fubini-Tonelli theorem yields

(4.5.5)

$$D = x\,(t-T)\left[F_0\,t + \sum_{k=1}^{n}\int_0^t\int_{D_k} z\,\Lambda_k(s)\,(t-s)\,d\widetilde{N}_k^{\mathbb{P}}(s,z)\right]$$

which finally leads us to $D=-E$, as desired. The argumentation for $E^* = -D^*$ can be done in a similar manner, yet using the dynamics (4.3.16) instead of (4.2.4).

4.5.1 A linear interpolation scheme for a particular liquidation cost term

To begin with, we claim that equation (4.3.2) can be rewritten as

(4.5.6)

$$\mathfrak{R}^{\mathcal{F}}(\alpha) = \frac{\gamma}{2}x^2 - x\,F_0 + \eta\,\mathfrak{B}^{\mathcal{F},\mathbb{P}}(t,T,X^\alpha) - \sum_{k=1}^{n}\int_0^t\int_{D_k} z\,X_s^\alpha\,\Lambda_k(s)\,d\widetilde{N}_k^{\mathbb{P}}(s,z)$$

wherein we have just set

(4.5.7)

$$\mathfrak{B} := \mathfrak{B}^{\mathcal{F},\mathbb{P}}(t,T,X^{\alpha}) := \mathbb{E}_{\mathbb{P}}\left(\int_0^T \left(\dot{X}_s^{\alpha}\right)^2 ds \,\middle|\, \mathcal{F}_t\right).$$

Further on, regarding the progress of paragraph 4.3.1, we recognize that $\mathfrak{B}$ in (4.5.7) and B in (4.3.7) are closely linked, as the latter directly originates from the former, obviously. However, we yet propose a linear approximation scheme to treat the object $\mathfrak{B}$ more accurately (than B in subsection 4.3.1 before). In accordance to (4.2.12), for $0 \le t \le T$ we initially claim the estimation

(4.5.8)

$$\left|\dot{X}_t^{\alpha}\right| \le \frac{x}{T} + \frac{|\alpha|}{T}\left(\int_0^t F_u \, du + (T-t)\, F_t\right) \le \frac{x}{T} + |\alpha| \max_{0 \le u \le t} F_u$$

$\mathbb{P}$-a.s. Additionally, for $\alpha \ne 0$ and a constant $M > \frac{x}{T} > 0$ we assume the boundary condition

$$\max_{0 \le u \le t} F_u < \frac{1}{|\alpha|}\left(M - \frac{x}{T}\right)$$

to be in force $\mathbb{P}$-a.s. Consequently, we receive $\left|\dot{X}_t^{\alpha}\right| < M$ $\mathbb{P}$-a.s. for all $0 \le t \le T$. Since we have restricted ourselves to *liquidation* strategies formerly, we actually obtain $-M < \dot{X}_t^{\alpha} \le 0$ $\mathbb{P}$-a.s. for all $0 \le t \le T$ (which, by the way, should not stand in contradiction to the economical practice). Parallel to Excursus A, we next implement a (not necessarily equidistant) partition of the interval $[-M, 0]$ via $\{-M = w_0 < w_1 < \cdots < w_m = 0\}$ in order to approximate the real function $h(w) := w^2$ in each partial section $\left]w_j, w_{j+1}\right]$ $(j = 0,1,\dots,m-1)$ by its particular secant

(4.5.9) $$s_j(w) = \left(w_{j+1} + w_j\right)\left(w - w_j\right) + w_j^2.$$

Hence, with respect to (4.5.7) and (4.5.9), we finally get the approximation

(4.5.10)

$$\mathfrak{B} = \mathbb{E}_{\mathbb{P}}\left(\int_0^T h\left(\dot{X}_s^{\alpha}\right) ds \,\middle|\, \mathcal{F}_t\right) \approx \mathbb{E}_{\mathbb{P}}\left(\int_0^T s_j\left(\dot{X}_s^{\alpha}\right) ds \,\middle|\, \mathcal{F}_t\right) = -\left(w_{j+1} + w_j\right) x - T\, w_j\, w_{j+1}$$

whenever $\dot{X}_s^{\alpha} \in \left]w_j, w_{j+1}\right]$ holds $\mathbb{P}$-a.s. for all $0 \le s \le T$ and $j = 0,1,\dots,m-1$. Reasonably, in the case $\dot{X}_s^{\alpha} = -M$, $s \in [0,T]$, we set $\mathfrak{B} := M^2\, T$.

Eventually, we remark that the other problematic coefficient B^* such as appearing in (4.3.23) can be treated by applying a similar approximation scheme as above.

Chapter 5

Pricing and Hedging Temperature Derivatives under Future Weather Information

5.1 A short introduction to temperature derivatives

The creation of competitive weather markets like the Chicago Mercantile Exchange (CME) [28], wherein options on weather indices are traded somehow similar to financial products at ordinary stock markets, has brought up new challenges concerning the stochastic modeling of those *non-tradable* underlyings such as temperature, rainfall, snowfall, sunshine, wind, or even the number of frost days in a certain location etc. – cf. [13]. However, in this chapter we particularly concentrate on suitable mathematical descriptions for different kinds of *temperature* indices. More precisely, in this work we derive risk-neutral option prices for plain-vanilla temperature derivatives on the basis of a mean-reverting Ornstein-Uhlenbeck temperature model allowing for seasonality both in its mean-level and its volatility, whereas multiple pure-jump Lévy-type processes as driving noises allow for seasonal dependent jump-amplitudes and frequencies. Especially, we take stochastic forecasts about future weather conditions that are available to well-informed traders into account via adequate enlargements of the underlying information filtrations. In this insider trading context, we derive expressions for forward-looking *cumulative average temperature* (CAT) futures and *cooling degree day* (CDD) futures, whereas we provide a pricing formula for a European call option written on the former. Ultimately, we construct optimal positions in a temperature futures portfolio under forecasted weather information to hedge against both *temporal* and *spatial* temperature risk simultaneously. Anyway, let us initially make some comments concerning the usual (actually *backward-looking*) mathematical definitions of temperature indices that presently can be found in the literature (see e.g. [12] or [13]).

As mentioned above, the Chicago Mercantile Exchange inter alia organizes trade in financial derivatives written on outdoor temperature. Hence, the market participants actually may trade in futures contracts written on temperature indices such as cumulative average temperature (CAT), Pacific rim (PRIM), heating degree days (HDD) or cooling degree days (CDD) [13], whose precise mathematical descriptions will be introduced in the following.

With reference to [13], we exemplarily study a CDD futures contract in detail now: To begin with, imagine that air-conditioners, for instance, often are switched on when the temperature increases above 18°C [13]. In this context, it makes sense to examine the stochastic process

(5.1.1)

$$CDD(t) := [\Theta_t - c]^+ := max\,\{\Theta_t - c, 0\}\ (\geq 0)$$

measuring the altitude (whenever it is positive) between the instantaneous daily average outdoor temperature Θ_t at time $t \geq 0$ and a constant threshold c (=18°C). Consequently, the stochastic process in (5.1.1) not only visualizes the time ranges during which air-conditioning might be switched on, but also gives us a feeling for the intensity of necessary cooling which itself strongly depends on the difference between Θ_t and c, evidently. The following definitions are taken from section 10.1 in [13]:

We introduce the *accumulated* CDD index over the measurement period $[\tau_1, \tau_2]$ via

(5.1.2)

$$CDD[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} CDD(u)\,du = \int_{\tau_1}^{\tau_2} [\Theta_u - c]^+ du\ \ (\geq 0)$$

whereas we obtain the corresponding HDD analogue by dint of

(5.1.3)

$$HDD[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} [c - \Theta_u]^+ du\ \ (\geq 0).$$

Finally, as in [13], we define the (real-valued) cumulative average temperature (CAT) index through

(5.1.4)

$$CAT[\tau_1, \tau_2] := \int_{\tau_1}^{\tau_2} \Theta_u\,du.$$

For the sake of completeness, we provide the definition of the Pacific rim temperature index (PRIM)

$$PRIM[\tau_1, \tau_2] := \frac{CAT[\tau_1, \tau_2]}{\tau_2 - \tau_1}$$

although we will not investigate the latter in any more detail, as it very closely resembles the CAT index (5.1.4), obviously. Note in passing that, in accordance to p. 278 in [13], we assume the above contracts altogether to be "*settled [financially] in terms of a currency with unit one*". That is, in contrast to the real CME practice, we do *not* multiply the above properties (5.1.2) – (5.1.4) with a factor like 20$ or 20£ to convert them into money [13]. (See Benth et al. [13], section 1.3 and 10.1 therein, to read more about this *money-converting context* and the real CME practice.)

Thus, as explained on p. 278 in [13], the buyer of a CDD futures contract will receive the precise amount of money as given in (5.1.2) at the end of the measurement period τ_2. In order to attain this right, the buyer has to pay the *CDD futures price* $F_{CDD}(t,\tau_1,\tau_2)$ at time t prior to the start of the measurement period τ_1 [13]. The *profit* from this trade then obviously is given by the difference

$$CDD[\tau_1,\tau_2] - F_{CDD}(t,\tau_1,\tau_2)$$

[13]. Further on, as announced in section 10.1 in [13], common *no-arbitrage arguments* yield the CDD futures price with respect to the accumulated past information which is stored in a filtration (generated by the temperature process Θ), say $\mathcal{F}_t$, as the $\mathcal{F}_t$-conditioned expected payoff under a (still to be determined) risk-neutral probability measure $\mathbb{Q}$, in symbols

(5.1.5)

$$F_{CDD}(t,\tau_1,\tau_2) \coloneqq \mathbb{E}_{\mathbb{Q}}(CDD[\tau_1,\tau_2]|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} [\Theta_u - c]^+ du \,\middle|\, \mathcal{F}_t \right) \ (\geq 0)$$

– compare eq. "(10.4) in [13]". Analogously, we find the *HDD futures price*

(5.1.6)

$$F_{HDD}(t,\tau_1,\tau_2) \coloneqq \mathbb{E}_{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} [c - \Theta_u]^+ du \,\middle|\, \mathcal{F}_t \right) \ (\geq 0)$$

and the (real-valued) *CAT futures price*

(5.1.7)

$$F_{CAT}(t,\tau_1,\tau_2) \coloneqq \mathbb{E}_{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} \Theta_u \, du \,\middle|\, \mathcal{F}_t \right)$$

– compare p. 279 in [13]. Herein, the increasing family of sigma-algebras $\mathcal{F}_t$ is supposed to be made up by all information coming from observing the temperature Θ up to time t. Hence, we presume

(5.1.8) $$\mathcal{F}_t \coloneqq \sigma\{\Theta_u : 0 \leq u \leq t\}.$$

At this step, let us catch up our former announcements in section 1.2 concerning the shortcomings associated to a temperature derivatives pricing approach which is simply based upon the *retro* sigma algebra $\mathcal{F}$. Essentially, the reader should be aware of the fact that a backward-looking filtration $\mathcal{F}_t$, such as defined in (5.1.8) above, does not at all reflect public knowledge about *future* weather conditions or, in particular, about omnipresent temperature *forecasts* [10]. In addition, such an ordinary backward-looking approach completely neglects the non-storability and non-tradability of temperature – compare the bottom of p.7 in [10].

In this regard, the rather popular pricing onsets for temperature contracts such as presented in [12] or [13], respectively in the above equations (5.1.5) – (5.1.7) which obviously are based on a conditional expectation given the *past* information $\mathcal{F}_t$ merely, actually appear rather unrealistic – primarily, since temperature *forecasts* are not taken into account adequately. Nevertheless, to the best of our knowledge, there is no work in the literature (comparable to the present discussion) dealing with option pricing purposes for temperature derivatives under enlarged filtrations. However, in this thesis we will innovatively treat the pricing of temperature derivatives under available future weather information by implementing a customized enlargement-of-filtration procedure.

The remainder of the current chapter is organized as follows: In section 5.2 our underlying mathematical basis is established, whereas the sophisticated daily average temperature model of mean-reverting Ornstein-Uhlenbeck type with pure-jump processes as driving noises is introduced in detail. Applying Girsanov's Change-of-Measure theorem, we hereafter obtain the associated temperature dynamics under a risk-neutral measure $\mathbb{Q}$. In paragraph 5.3 we invest some innovative effort concerning the construction of enlarged information filtrations tailored to the requirements of our temperature derivatives context, whereas we rigorously take additional information about future weather conditions (which is assumed to be available to well-informed traders) into account. This procedure finally culminates in the provision of CAT and CDD futures prices under complementary temperature forecasts and, more importantly, of a forward-looking pricing formula for a plain-vanilla option written on the former CAT index. In addition, we invent a forward-looking *mixed* temperature model including both Brownian motion and pure-jump terms and highlight a corresponding *mixed* CAT call option pricing formula. In subsection 5.4 we introduce a multi-dimensional temperature model in order to create an *optimal* temperature futures portfolio under complementary future weather information. In this framework, we present a CAT hedging strategy which minimizes both the spatial and temporal temperature risk. Ultimately, the most important conclusions are drawn in the closing section 5.5, whereas in addition some further research topics are presented.

5.2 Modeling temperature dynamics

We start off with the description of the mathematical basis of our stochastic temperature model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered and complete probability space, whereby the *backward-looking* information filtration $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_t$ as defined in (5.1.8) is presumed both to include a priori all $\mathbb{P}$-null-sets and to be *cad* (French: continue à droite)[53].

5.2.1 Temperature variations: a mean-reverting pure-jump approach

Slightly deviating from the setup presented in subsection 2.2 in [11], in this work we model the instantaneous outdoor temperature dynamics by a mean-reverting Ornstein-Uhlenbeck (OU) process driven by *multiple pure-jump* Lévy-type processes. To be precise, we presently assume the continuous-time dynamics of the *daily average*, respectively *mean* temperature variations[54] $\Theta := (\Theta_t)_{t\in[0,T]}$ to follow the stochastic differential equation (SDE)[55]

(5.2.1) $$d\Theta_t = d\mu(t) + \alpha\,[\mu(t) - \Theta_t]\,dt + \sum_{k=1}^{n} \sigma_k(t)\,dL_t^k$$

(cf. eq. "(2.5) in [11]"). Herein, $\mu(t)$ represents a deterministic, bounded, continuously-differentiable, periodic function modeling the trend and seasonality of the temperature variations. Exemplarily, it can be chosen as a truncated Fourier series like in [12] or alternatively, as a linear plus a trigonometric function as supposed in [11], for instance.

[53] See the beginning of section 3.2 for a precise definition of *cad* sigma-algebras.
[54] The *daily average* or *mean temperature* is defined as the arithmetic average of the maximum and minimum temperature during the 24 hours of the day in consideration [11]. As a consequence, the observed temperature variations actually possess a time-discrete nature. Nevertheless, we model the temperature dynamics as a continuous-time stochastic process in order to be able to profit from the power of stochastic calculus.
[55] To read more about the (with respect to common OU-models admittedly rather unusual) appearance of the differential $d\mu(t)$ on the right hand side of (5.2.1), we refer to the bottom of page 3 in [11]. In fact, from the solution of equality (5.2.1) [such as given in (5.2.3)] we reasonably deduce $|\Theta_u - \mu(u)| \to 0$ as $u \to \infty$.

Further on, the constant and strictly positive mean-reversion velocity is denoted by α, whereas $\sigma_k(t)$ depicts a deterministic, strictly positive, bounded and time-dependent (seasonal) volatility function for every index $k = 1, \dots, n$ which controls the seasonal variation of the jump-sizes. Moreover, for a real subset $D_k \subseteq \mathbb{R} \setminus \{0\}$, time indices $t \in [0, T]$ and $k = 1, \dots, n$ we introduce a family of integrable, pair-wise independent, (maybe Brownian-motion-like small-amplitude) pure-jump, càdlàg and finite-variation (see Theorem 2.4.25 in [1] in the finite-variation context) Sato processes via

(5.2.2) $$L_t^k := \int_0^t \int_{D_k} z \, dN_k(s, z)$$

being responsible for interspersing random temperature fluctuations[56] in (5.2.1). In the latter equation N_k constitutes a one-dimensional integer-valued Poisson-Random-Measure (PRM) actually living on the product space $[0, T] \times \mathbb{R} \setminus \{0\}$ for each index k. We further assume the PRMs $dN_k(s, z)$ to have predictable $\mathbb{P}$-compensators such as given in (3.2.5), but for *real-valued* non-zero jump-amplitudes now – instead of *strictly positive* ones as originally supposed in (3.2.4). Thus, (5.2.2) does not completely coincide with equation (3.2.4), as one could think on a first sight.
As pointed out in subsection 2.2 in [11], by choosing an appropriate distribution for the Lévy-type noises (5.2.2) admitting e.g. (semi-) heavy-tails or skewness, we may achieve a very precise description concerning the (possibly non-Gaussian) distributional properties of temperature variations also in our specific model setup (5.2.1). All in all, the above modeling proposal (5.2.1) within its tractable (small and large jump-amplitude) Lévy-type processes as driving noises should be able to describe empirical temperature variations in a better way than simple Brownian motion models, although the latter indisputably are much easier to handle from a mathematical point of view. Returning to our main topic, we use Itô's product rule which leads us to the solution of (5.2.1) reading

(5.2.3)

$$\Theta_u = \mu(u) + [\Theta_t - \mu(t)] \, e^{-\alpha(u-t)} + \sum_{k=1}^{n} \int_t^u \sigma_k(s) \, e^{-\alpha(u-s)} dL_s^k$$

for time indices $0 \leq t \leq u \leq T$.

5.2.2 Risk-neutral martingale measures in the temperature market

As the risk-neutral/arbitrage-free pricing theory requires the introduction of an (with respect to $\mathbb{P}$) equivalent martingale measure (EMM)[57], say $\mathbb{Q}$, we certainly will need a representation of (5.2.3) under $\mathbb{Q}$ when it comes to pricing issues associated to temperature derivatives. Also note in this context that the properties (5.1.5) – (5.1.7) likewise succumb to the measure $\mathbb{Q}$. Since the pure-jump Sato noises (5.2.2) are defined similarly to (3.2.4) [except from slightly different jump-amplitudes as explained in the sequel of (5.2.2) above], the measure change methodology of subsection 3.2.2 yet applies simultaneously. Nevertheless, we give the following remark explaining the obvious *incompleteness* of the present temperature derivatives market in more detail.

---

[56] Recall section 3.2.1 to read more about a possibly useful distinction between small- and large-amplitude jump noises. Further note that, in contrast to (3.2.4), in equation (5.2.2) we yet allow for *negative* jump-sizes, too, since temperature may become negative, of course. Moreover, in subsection 5.3.4 we will add Brownian motion noises to our model (5.2.1) and derive temperature futures prices under this *mixed* model approach likewise.

[57] Since the underlying (i.e. temperature) is neither storable nor tradable, we rather should call $\mathbb{Q}$ a *risk-neutral* (arbitrage-free) equivalent probability measure, instead of EMM, while it is not clear what should become a (discounted) local *martingale* under $\mathbb{Q}$ – cf. p. 22 in [13]. The same is valid for electricity markets, by the way.

Remark 5.2.1 *Note that an EMM by definition is a risk-neutral probability measure $\mathbb{Q}$ which, firstly, has to be equivalent to $\mathbb{P}$ on the sigma algebra $\mathcal{F}_t$ meaning that $\mathbb{P}$ and $\mathbb{Q}$ possess the same null-sets throughout $\mathcal{F}_t$. By the way, the latter property simultaneously implies the existence of a Radon-Nikodym density such as defined in (3.2.15). Secondly, all tradable assets in the considered market must designate (local) $\mathbb{Q}$-martingales after discounting (cf. p.95 in [13]). Since temperature is neither storable nor a "tradable asset", any arbitrary equivalent probability $\mathbb{Q}$ will instantly become a risk-neutral one ([11], [13]), whereas it is not a trivial question which $\mathbb{Q}$ to choose. Hence, in accordance to the second fundamental theorem of asset pricing, the present temperature market turns out to be highly incomplete. Moreover, the final determination of a precise EMM out of the huge class of offering pricing probabilities (using e.g. the concepts of Esscher transforms, minimum relative entropy or the Föllmer-Schweizer minimal measure) may embody a challenging future research topic.* ■

5.3 Temperature futures under enlarged filtrations

In this section we devote our attention towards the derivation of temperature futures prices with respect to some additional weather forecast information that well-informed market participants might have knowledge of. More precise, in our forthcoming considerations we will – as firstly proposed in [10] – take forward-looking information about future weather conditions into account via an adequate enlargement of the underlying information filtration. Initially, we emphasize that the filtration

(5.3.1) $$\mathcal{F}_t := \sigma\{\Theta_u : 0 \le u \le t\} := \sigma\{L_u^1, \dots, L_u^n : 0 \le u \le t\}$$

generated by the temperature process Θ does only *look into the past* and all available information coming from temperature observations up to time t is stored in this *retro* sigma algebra. As explained before, this traditional (financial) approach obviously does not at all reflect the case at hand when we are concerned with pricing applications for a non-storable *commodity* such as temperature.

Adapting ideas from Chapter 3 in [10], we now introduce the flow of additionally available market information at time t including temperature forecasts by the enlarged filtration $\mathcal{G}_t$ $(\supset \mathcal{F}_t)$ for times $0 \le t \le T$. In this regard, we assume that the traders have an idea about the temperature behavior at a future time τ or – converting the latter assumption into the language of our underlying mathematical model – have a feeling about most-likely values of the jump-noises at the future time τ driving the temperature dynamics (5.2.1). Consequently, we refer to [10] and introduce the *overall filtration*

(5.3.2) $$\mathcal{H}_t := \mathcal{F}_t \vee \sigma\{\Theta_\tau\} := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^n\} := \sigma\{L_s^k : s \in [0,t] \cup \{\tau\}, k = 1, \dots, n\}$$

representing *complete* or *exhaustive* information at time t about the future temperature behavior at time τ. As explained in Chapter 3, we thus *associate* knowledge about the future temperature Θ_τ with knowledge about the values of its driving noises $L_\tau^1, \dots, L_\tau^n$. Once more, we highlight that the enlarged sigma algebra $\mathcal{H}$ is closely linked with *exact* knowledge concerning the future temperature conditions at time τ. In this regard, as in [10], we next introduce the (non-explicit) *intermediate filtration* $\mathcal{G}_t$ via

(5.3.3) $$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t$$

for time points $0 \le t < \tau$, whereas (5.3.2) immediately delivers $\mathcal{F}_t = \mathcal{G}_t$ for all $t \ge \tau$. Similarly to (3.3.38), also in our current temperature derivatives framework we implement an *explicit* intermediate filtration $\mathcal{G}_t^*$ consisting of a subfamily of the components appearing in $\mathcal{H}_t$, namely

(5.3.4) $$\mathcal{G}_t^* := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^p\}$$

with $1 \leq p \leq n$ and $0 \leq t < \tau$.[58] Then $\mathcal{F}_t \subset \mathcal{G}_t^* \subset \mathcal{H}_t$ for $t < \tau$ and $\mathcal{G}_t^* = \mathcal{F}_t$ for $t \geq \tau$ likewise hold true. Putting $p := n$ would yet correspond to $\mathcal{G}_t^* = \mathcal{H}_t$ and thus, to *complete* or *exhaustive* knowledge of the precise temperature value at the future time τ. On the other hand, the case $p < n$ represents a scenario wherein the market participants merely have access to some *restricted* additional information concerning future temperature behavior, sounding more realistically.

5.3.1 Forward-looking CAT futures prices

The present subsection is dedicated to the derivation of explicit CAT futures price dynamics under additional temperature forecasts, i.e. under complementary knowledge about the values of a selection of the temperature process driving jump noises $L_\tau^1, \ldots, L_\tau^p$ at the future time τ. In accordance to (5.1.7), we newly define the (real-valued) CAT futures price under the enlarged filtration $\mathcal{G}_t^*$ by dint of

(5.3.5)
$$F_{CAT}^{\mathcal{G}^*}(t,\tau_1,\tau_2) := \mathbb{E}_{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} \Theta_u \, du \,\middle|\, \mathcal{G}_t^*\right).$$

Merging (5.2.3) into (5.3.5), we immediately receive

(5.3.6)
$$F_{CAT}^{\mathcal{G}^*}(t,\tau_1,\tau_2) =$$
$$\int_{\tau_1}^{\tau_2} \mu(u)\, du + [\Theta_t - \mu(t)] \int_{\tau_1}^{\tau_2} e^{-\alpha(u-t)} du + \int_{\tau_1}^{\tau_2} \mathbb{E}_{\mathbb{Q}}\left(\sum_{k=1}^{n} \int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} dL_s^k \,\middle|\, \mathcal{G}_t^*\right) du.$$

Next, we split the sum appearing inside the conditional expectation in (5.3.6) and obtain

(5.3.7)
$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{k=1}^{n} \int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} dL_s^k \,\middle|\, \mathcal{G}_t^*\right)$$
$$= \sum_{k=1}^{p} \mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} dL_s^k \,\middle|\, \mathcal{G}_t^*\right) + \sum_{k=p+1}^{n} \mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} dL_s^k \,\middle|\, \mathcal{F}_t\right).$$

Parallel to (3.3.45) and our former arguing in the sequel of (3.3.44), we use (5.2.2) and (5.3.4) such that – also in the present temperature derivatives context – we receive for all $k = 1, \ldots, p$

(5.3.8)
$$\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_t^k \middle| \mathcal{G}_t^*\right) = L_\tau^k - L_t^k = \int_t^\tau \int_{D_k} z\, dN_k(s,z).$$

Taking Cond. A, Lemma 3.5.1, (3.3.39), (4.3.10), (5.3.8) and the Fubini-Tonelli theorem into account, [for $u < \tau$] the first conditional expectation on the right hand side of equation (5.3.7) transforms into

(5.3.9)
$$\mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} dL_s^k \,\middle|\, \mathcal{G}_t^*\right) = Y_t^k \int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} ds.$$

[58] Note that neither the processes $L^1, \ldots, L^p$ nor the filtration $\mathcal{F}_t$ in (5.3.4) are the same ones as in (3.3.38). Hence, the intermediate filtration $\mathcal{G}_t^*$ in (5.3.4) does *not* coincide with (3.3.38), as one could suspect on a first sight.

Moreover, appealing to (3.2.20) and (5.2.2), for all $k = p+1, \ldots, n$ the second conditional expectation on the right hand side of (5.3.7) reduces to a usual expectation which can be computed as

(5.3.10)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} dL_s^k \middle| \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\alpha(u-s)}\, dN_k(s,z)\right]$$
$$= \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\alpha(u-s)}\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds.$$

Substituting (4.3.10), (5.3.7), (5.3.9) and (5.3.10) into (5.3.6), we ultimately end up with

(5.3.11)

$$F_{CAT}^{\mathcal{G}^*}(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \mu(u)\, du + \Theta_t \int_{\tau_1}^{\tau_2} e^{-\alpha(u-t)} du - \mu(t) \int_{\tau_1}^{\tau_2} e^{-\alpha(u-t)} du$$
$$+ \sum_{k=1}^{p} \frac{L_\tau^k - L_t^k}{\tau - t} \int_{\tau_1}^{\tau_2} \int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} ds\, du$$
$$+ \sum_{k=p+1}^{n} \int_{\tau_1}^{\tau_2} \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\alpha(u-s)}\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds\, du.$$

Furthermore, introducing the deterministic functions

(5.3.12)

$$\Gamma(t) := \int_{\tau_1}^{\tau_2} \mu(u)\, du - \mu(t)\, \Psi(t) - \sum_{k=p+1}^{n} \int_{\tau_1}^{\tau_2} \int_u^t \int_{D_k} z\, \sigma_k(s)\, e^{-\alpha(u-s)}\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds\, du,$$

$$\Psi(t) := \frac{e^{-\alpha(\tau_1 - t)} - e^{-\alpha(\tau_2 - t)}}{\alpha}, \qquad \Phi_k(t) := \int_{\tau_1}^{\tau_2} \int_u^t \frac{\sigma_k(s)\, e^{-\alpha(u-s)}}{t - \tau} ds\, du,$$

equation (5.3.11) can be rewritten in shorthand notation as

(5.3.13)

$$F_{CAT}^{\mathcal{G}^*}(t,\tau_1,\tau_2) = \Gamma(t) + \Psi(t)\, \Theta_t + \sum_{k=1}^{p} \Phi_k(t) \left\{L_\tau^k - L_t^k\right\}.$$

Next, denoting the derivation with respect to t by an inverted comma, Itô's product rule yields

(5.3.14)

$$dF_{CAT}^{\mathcal{G}^*}(t,\tau_1,\tau_2) = \left[\Gamma'(t) + \Psi'(t)\, \Theta_t + \sum_{k=1}^{p} \Phi_k'(t) \left\{L_\tau^k - L_t^k\right\}\right] dt + \Psi(t)\, d\Theta_t + \sum_{k=1}^{p} \Phi_k(t)\, d\left(L_\tau^k - L_t^k\right)$$

whereas (5.3.12) delivers

(5.3.15)

$$\Psi'(t) = \alpha\, \Psi(t), \qquad \Phi_k'(t) = \frac{\sigma_k(t)\, \Psi(t) - \Phi_k(t)}{t - \tau},$$

$$\Gamma'(t) = -\Psi(t) \sum_{k=p+1}^{n} \int_{D_k} z\, \sigma_k(t)\, e^{h_k(t,z)} \rho_k(t)\, d\nu_k(z) - \mu'(t)\, \Psi(t) - \mu(t)\, \Psi'(t).$$

Merging (5.2.1) and (5.3.15) into (5.3.14), with respect to (3.2.20) and (4.3.10) we get [as expected; compare definition (5.3.5)] the real-valued (local) $(\mathcal{G}_t^*, \mathbb{Q})$-martingale representation

(5.3.16)

$$dF_{CAT}^{\mathcal{G}^*}(t,\tau_1,\tau_2) = \sum_{k=1}^{p} [\sigma_k(t)\, \Psi(t) - \Phi_k(t)] \{dL_t^k - Y_t^k dt\} + \sum_{k=p+1}^{n} \sigma_k(t)\, \Psi(t) \int_{D_k} z\, \tilde{N}_k^{\mathcal{F},\mathbb{Q}}(t,dz)$$

with vanishing drift. In accordance to (3.3.39), (3.3.49) and (4.3.10), eq. (5.3.16) may be rewritten as

(5.3.17)

$$F_{CAT}^{\mathcal{G}^*}(t,\tau_1,\tau_2) = F_{CAT}^{\mathcal{G}^*}(0,\tau_1,\tau_2) + \sum_{k=1}^{p} \int_0^t \int_{D_k} z\, [\sigma_k(s)\, \Psi(s) - \Phi_k(s)]\, d\tilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}(s,z)$$
$$+ \sum_{k=p+1}^{n} \int_0^t \int_{D_k} z\, \sigma_k(s)\, \Psi(s)\, d\tilde{N}_k^{\mathcal{F},\mathbb{Q}}(s,z).$$

5.3.2 European options on CAT futures under temperature forecasts

In this paragraph we present a suitable pricing method for a European call option written on the *enlarged* CAT futures price (5.3.5). Adapting (3.2.29), we currently define the $\mathcal{G}^*$*-forward-looking* CAT futures call option payoff with exercise time $\tilde{T}$ and strike price $K > 0$ (in EURO) by dint of

(5.3.18)
$$C_{CAT}^{\mathcal{G}^*}(\tilde{T}) := C_{CAT}^{\mathcal{G}^*}(\tilde{T},K,\tau_1,\tau_2) := \left[F_{CAT}^{\mathcal{G}^*}(\tilde{T},\tau_1,\tau_2) - K\right]^+ := max\left\{0, F_{CAT}^{\mathcal{G}^*}(\tilde{T},\tau_1,\tau_2) - K\right\}.$$

Parallel to our previous announcements given in the sequel of (3.2.33), we recall that $g(x) := [x - K]^+ \notin \mathcal{L}^1(\mathbb{R})$, whereby on the other hand for the exponentially-damped function $q(x) := e^{-ax} g(x) \in \mathcal{L}^1(\mathbb{R})$ is valid within a real damping parameter $0 < a < \infty$. Next, for $t \leq \tilde{T}$ and a constant interest rate $r > 0$ the adjusted risk-neutral pricing formula [cf. (3.3.50)] reads as

(5.3.19) $$C_{CAT}^{\mathcal{G}^*}(t) = e^{-r(\tilde{T}-t)}\, \mathbb{E}_{\mathbb{Q}}\left(\left[F_{CAT}^{\mathcal{G}^*}(\tilde{T},\tau_1,\tau_2) - K\right]^+ \middle| \mathcal{G}_t^*\right).$$

Further, within a shorthand notation $F_{CAT}^{\mathcal{G}^*}(t) := F_{CAT}^{\mathcal{G}^*}(t,\tau_1,\tau_2)$ we consequently deduce

(5.3.20) $$C_{CAT}^{\mathcal{G}^*}(t) = e^{-r(\tilde{T}-t)}\, \mathbb{E}_{\mathbb{Q}}\left(e^{aF_{CAT}^{\mathcal{G}^*}(\tilde{T})}\, q\left(F_{CAT}^{\mathcal{G}^*}(\tilde{T})\right) \middle| \mathcal{G}_t^*\right)$$

whereas (3.2.33) immediately yields

(5.3.21)

$$C_{CAT}^{\mathcal{G}^*}(t) = \frac{e^{-r(\tilde{T}-t)}}{2\pi} \int_{\mathbb{R}} \hat{q}(y)\, \mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy)\, F_{CAT}^{\mathcal{G}^*}(\tilde{T})} \middle| \mathcal{G}_t^*\right) dy$$

with $\hat{q}(y)$ as announced in (3.2.34). At this step, we remind that $F_{CAT}^{\mathcal{G}^*} \in \mathbb{R}$ constitutes a $(\mathcal{G}^*, \mathbb{Q})$-martingale [see (5.3.16)] which unfortunately does *not* possess independent increments with respect to $\mathcal{G}^*$. [Our former argumentation in the sequel of (3.3.53) here applies equally.] Thus, the conditional expectation in (5.3.21) ought to be approximated as follows: Using (A.6) and (A.13), we derive

(5.3.22) $$\mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy)\, F_{CAT}^{\mathcal{G}^*}(\tilde{T})} \middle| \mathcal{G}_t^*\right) = \mathcal{C}_{\mathbb{Q}}^{\mathcal{G}^*}\left(F_{CAT}^{\mathcal{G}^*}(\tilde{T}); t, a, y\right) \approx \mathcal{A}_\kappa^d\left(y; a, F_{CAT}^{\mathcal{G}^*}(t)\right)$$

whenever $x_\kappa < F_{CAT}^{\mathcal{G}^*}(\tilde{T}) \leq x_{\kappa+1}$ $\mathbb{Q}$-a.s. (where $\kappa = 0, \dots, \widetilde{m}-1$).[59] In conclusion, the estimated price at time $t\ (\leq \tilde{T})$ of a European call option written on the CAT futures $F_{CAT}^{\mathcal{G}^*}$ under $\mathcal{G}^*$ with strike price $K > 0$ at exercise time $\tilde{T}\ (\leq \tau_1)$ and measurement period $[\tau_1, \tau_2]$ finally points out as

(5.3.23)

$$C_{CAT}^{\mathcal{G}^*}(t) \approx \frac{e^{-r(\tilde{T}-t)}}{2\pi} \int_{\mathbb{R}} \frac{e^{-(a+iy)K}}{(a+iy)^2}\, \mathcal{A}_\kappa^d\left(y; a, F_{CAT}^{\mathcal{G}^*}(t)\right) dy$$

whenever $x_\kappa < F_{CAT}^{\mathcal{G}^*}(\tilde{T}) \leq x_{\kappa+1}$ $\mathbb{Q}$-a.s. $(\kappa = 0, \dots, \widetilde{m}-1)$. Note in passing that (5.3.23) appears suitable for numerical pricing techniques. Ultimately, we recall that the enlarged *put* option price, say $P_{CAT}^{\mathcal{G}^*}$, written on the CAT futures $F_{CAT}^{\mathcal{G}^*}$ with strike price $K > 0$ easily can be obtained by exploiting the *Put-Call-Parity* (compare the end of subsection 3.3.4). Remembering that $F_{CAT}^{\mathcal{G}^*}$ as defined in (5.3.5) depicts a $\mathcal{G}^*$-adapted $\mathbb{Q}$-martingale, we may use (5.3.19) to receive [analogously to (3.3.59)]

(5.3.24) $$P_{CAT}^{\mathcal{G}^*}(t) = C_{CAT}^{\mathcal{G}^*}(t) + e^{-r(\tilde{T}-t)}\left[K - F_{CAT}^{\mathcal{G}^*}(t)\right].$$

5.3.3 Forward-looking CDD futures prices

In this subsection we derive CDD futures prices under temperature forecasts modeled by the enlarged filtration (5.3.4). Thus, in accordance to (5.1.5), we firstly define the CDD futures price under $\mathcal{G}_t^*$ via

(5.3.25) $$F_{CDD}^{\mathcal{G}^*}(t, \tau_1, \tau_2) := \mathbb{E}_{\mathbb{Q}}\left(\int_{\tau_1}^{\tau_2} [\Theta_u - c]^+ du \,\middle|\, \mathcal{G}_t^*\right) (\geq 0).$$

Applying the Fubini-Tonelli theorem on (5.3.25) while introducing similar functions as in the sequel of (5.3.18) [but replacing K inside the function g by c now], we obtain

[59] We recall that $F_{CAT}^{\mathcal{G}^*}$ (in contrast to the *electricity* futures F^* such as originally appearing in Excursus A) is not strictly positive (but real-valued), since the temperature process Θ may become negative. Thus, (A.13) does not apply instantly here, as it actually is connected with the former presumptions $0 < F_T^* \leq M$ and $x_j < F_T^* \leq x_{j+1}$. Nevertheless, we easily may extend the partition $\mathfrak{P}$ of Excursus A yet to $\mathcal{P} := \{-M = x_0 < x_1 < \cdots < x_{\widetilde{m}} = M\}$ which obviously stems from the new presumption $-M < F_{CAT}^{\mathcal{G}^*}(\tilde{T}) \leq M$ $\mathbb{Q}$-a.s., which we assume to be in force from now on. Consequently, in (5.3.22) we then have $\kappa = 0, \dots, \widetilde{m}-1$ (instead of $j = 0, \dots, m-1$). Apart from this slight extension of the underlying partition, the approximation techniques of Excursus A here apply equally.

(5.3.26) $$F_{CDD}^{\mathcal{G}^*}(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \mathbb{E}_{\mathbb{Q}}(e^{a\Theta_u}\, q(\Theta_u)|\mathcal{G}_t^*)\, du.$$

With respect to the inverse Fourier transform (3.2.33), the latter equation turns into

(5.3.27) $$F_{CDD}^{\mathcal{G}^*}(t,\tau_1,\tau_2) = \frac{1}{2\pi}\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}} \hat{q}(y)\, \mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy)\Theta_u}\big|\mathcal{G}_t^*\big)\, dy\, du$$

whereby (3.2.34) delivers

(5.3.28) $$\hat{q}(y) = \frac{e^{-(a+iy)c}}{(a+iy)^2}.$$

Moreover, substituting (5.2.3) and (5.3.28) into (5.3.27), we receive

(5.3.29) $$F_{CDD}^{\mathcal{G}^*}(t,\tau_1,\tau_2) =$$

$$\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}} \frac{e^{(a+iy)\{\mu(u)-c+[\Theta_t-\mu(t)]e^{-\alpha(u-t)}\}}}{2\pi\,(a+iy)^2}\, \mathbb{E}_{\mathbb{Q}}\left(exp\left\{\sum_{k=1}^{n}\int_t^u \chi_k(s)\, dL_s^k\right\}\middle|\mathcal{G}_t^*\right) dy\, du$$

wherein we have just set $\chi_k(s) \coloneqq (a+iy)\,\hat{\pi}_k(s)$ with $\hat{\pi}_k(s) \coloneqq \sigma_k(s)\, e^{-\alpha(u-s)} > 0$. Note that the sum of stochastic integrals in (5.3.29) is not necessarily positive yet, as we have permitted *real-valued* (non-zero) jump sizes in (5.2.2). Thus, we cannot apply the approximation techniques of section 3.3.5 instantly. Nevertheless, we refer to the footnote dedicated to (5.3.22) and introduce a similar *extension of partition* as described therein: More precisely, we now define the (real-valued) stochastic process

(5.3.30)
$$\widehat{H}_{t,u} \coloneqq \sum_{k=1}^{n}\int_t^u \hat{\pi}_k(s)\, dL_s^k = \sum_{k=1}^{n}\sum_{t\le s\le u} \hat{\pi}_k(s)\, \Delta L_s^k \in \mathbb{R}$$

while assuming $-M < \widehat{H}_{t,u} \le M$ $\mathbb{Q}$-a.s. for all $t < u$. Hence, appealing to our former argumentation in the sequel of (3.3.63), within an adjusted partition $\hat{\mathcal{P}} \coloneqq \{-M = x_0 < x_1 < \cdots < x_{\widehat{m}} = M\}$ we get

(5.3.31)
$$\mathbb{E}_{\mathbb{Q}}\left(exp\left\{\sum_{k=1}^{n}\int_t^u \chi_k(s)\, dL_s^k\right\}\middle|\mathcal{G}_t^*\right) \approx \widehat{\mathfrak{M}}_t^l(\tau,u;y,d;\chi)$$

whenever $x_l < \widehat{H}_{t,u} \le x_{l+1}$ $\mathbb{Q}$-a.s. $(l = 0,\dots,\widehat{m}-1)$. Herein, we have just set

(5.3.32) $$\widehat{\mathfrak{M}}_t^l(\tau,u;y,d;\chi) \coloneqq$$

$$\sum_{\nu=0}^{d}\frac{(a+iy)^\nu}{\nu!}\left(\frac{x_{l+1}^\nu - x_l^\nu}{x_{l+1}-x_l}\left\{-x_l + \sum_{k=1}^{p}\hat{\beta}_k(t,\tau,u)\,\{L_\tau^k - L_t^k\} + \sum_{k=1}^{p}\hat{\xi}_k^*(\tau,u) + \sum_{k=p+1}^{n}\hat{\xi}_k(t,u)\right\} + x_l^\nu\right),$$

$$\hat{\beta}_k(t,\tau,u) \coloneqq \int_t^\tau \frac{\hat{\pi}_k(s)}{\tau-t}\, ds, \qquad \hat{\xi}_k^*(\tau,u) \coloneqq \int_\tau^u\int_{D_k} z\, \hat{\pi}_k(s)\, e^{h_k(z)}\, \rho_k\, d\nu_k(z)\, ds,$$

$$\hat{\xi}_k(t,u) \coloneqq \int_t^u\int_{D_k} z\, \hat{\pi}_k(s)\, e^{h_k(s,z)}\, \rho_k(s)\, d\nu_k(z)\, ds.$$

[Recall Condition A (adjusted to $\mathcal{G}^*$) at this step.]

Finally, putting (5.3.29) and (5.3.31) together, we end up with the approximation

(5.3.33)

$$F_{CDD}^{\mathcal{G}^*}(t,\tau_1,\tau_2) \approx \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \frac{e^{(a+iy)\{\mu(u)-c+[\Theta_t-\mu(t)]e^{-\alpha(u-t)}\}}}{2\pi\,(a+iy)^2}\,\widehat{\mathfrak{M}}_t^l(\tau,u;y,d;\chi)\,dy\,du.$$

In conclusion, it seems to be a rather difficult task to derive (estimated) pricing formulas for options written on the CDD futures index (5.3.33) – somehow similar to our proceedings in subsection 5.3.2 for CAT futures –, since there is not an equally nice martingale representation [such as derived in (5.3.16) for the CAT-case] available for the CDD futures, unfortunately.

5.3.4 A mixed model for temperature dynamics

Referring to our former arguing in subsection 3.3.9, we now introduce a *mixed* temperature model including both Brownian motion (BM) and pure-jump processes as driving noises. For this purpose, we replace equality (5.2.1) yet through

(5.3.34)

$$d\Theta_t = d\mu(t) + \alpha\,[\mu(t)-\Theta_t]\,dt + \sum_{k=1}^{l} \sigma_k\,dB_t^k + \sum_{k=l+1}^{n} \sigma_k(t)\,dL_t^k$$

with strictly positive and constant volatilities $\sigma_1,\dots,\sigma_l$ along with standard $\mathbb{P}$-BMs $B_t^1,\dots,B_t^l$. Again, we assume the involved noises $B_t^1,\dots,B_t^l,L_t^{l+1},\dots,L_t^n$ to be pair-wise $\mathbb{P}$-independent. Consequently, the Ornstein-Uhlenbeck solution of (5.3.34) turns out as

(5.3.35)

$$\Theta_u = \mu(u) + [\Theta_t-\mu(t)]\,e^{-\alpha(u-t)} + \sum_{k=1}^{l} \sigma_k \int_t^u e^{-\alpha(u-s)} dB_s^k + \sum_{k=l+1}^{n} \int_t^u \sigma_k(s)\,e^{-\alpha(u-s)} dL_s^k$$

for time indices $0 \le t \le u \le T$. Fortunately, the properties (3.3.108) – (3.3.110) simultaneously apply in our recent temperature framework. Moreover, we implement both an *overall filtration* $\widetilde{\mathcal{H}}_t$ and an *explicit intermediate filtration* $\tilde{\mathcal{G}}_t$ which we suppose to be such as in (3.3.114), respectively (3.3.115), but with λ_k therein replaced by α now. Parallel to (3.3.116), we next come up with abbreviations

(5.3.36) $\quad a(s) := \frac{2\alpha e^{\alpha s}}{e^{2\alpha\tau}-e^{2\alpha s}}$ and $\tilde{\theta}_s^k := a(s)\int_s^\tau e^{\alpha r} d\tilde{B}_r^k$

whereby the $(\tilde{\mathcal{F}},\widetilde{\mathbb{Q}})$-compensated BMs $\tilde{B}_t^k := \tilde{B}_t^{k,\tilde{\mathcal{F}},\widetilde{\mathbb{Q}}}$ are like in (3.3.110). Consequently, we declare

(5.3.37)

$$\tilde{B}_t^{k,\tilde{\mathcal{G}},\widetilde{\mathbb{Q}}} := \tilde{B}_t^k - \int_0^t \tilde{\theta}_s^k\,ds$$

to constitute $(\tilde{\mathcal{G}}_t,\widetilde{\mathbb{Q}})$-BMs for all $k=1,\dots,d$ and $t \in [0,\tau[$. Finally, (3.3.118) still holds true in our current mixed temperature model, if we replace λ_k by α therein.

In accordance to (5.3.5), we newly define the CAT futures price under the enlarged filtration $\tilde{\mathcal{G}}$ by

(5.3.38)

$$\tilde{F}_{CAT}(t) := F_{CAT}^{\tilde{\mathcal{G}},\tilde{\mathbb{Q}}}(t,\tau_1,\tau_2) := \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_{\tau_1}^{\tau_2} \Theta_u \, du \,\middle|\, \tilde{\mathcal{G}}_t \right).$$

Substituting (5.3.35) into (5.3.38), we obtain

(5.3.39)

$$\begin{aligned}\tilde{F}_{CAT}(t) = \int_{\tau_1}^{\tau_2} \mu(u)\, du + [\Theta_t - \mu(t)]\, \Psi(t) + \sum_{k=1}^{d} \int_{\tau_1}^{\tau_2} \sigma_k \, \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^u e^{-\alpha(u-s)} dB_s^k \,\middle|\, \tilde{\mathcal{G}}_t \right) du \\ + \sum_{k=d+1}^{l} \int_{\tau_1}^{\tau_2} \sigma_k \, \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^u e^{-\alpha(u-s)} dB_s^k \,\middle|\, \tilde{\mathcal{F}}_t \right) du \\ + \sum_{k=l+1}^{n} \int_{\tau_1}^{\tau_2} \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} dL_s^k \,\middle|\, \tilde{\mathcal{F}}_t \right) du\end{aligned}$$

wherein $\Psi(t)$ is such as defined in (5.3.12). In what follows, we compute the three conditional expectations in (5.3.39) in their order of appearance: Firstly, troubling (3.3.110), (3.3.118), (5.3.36) and (5.3.37) [while appealing to a *dualism* concept], we actually receive for $k = 1, \ldots, d$

(5.3.40)

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^u e^{-\alpha(u-s)} dB_s^k \,\middle|\, \tilde{\mathcal{G}}_t \right) &= \int_t^u e^{-\alpha(u-s)}\, G_k(s)\, ds + \int_t^u e^{-\alpha(u-s)}\, \mathbb{E}_{\tilde{\mathbb{Q}}}\big(\tilde{\theta}_s^k \big| \tilde{\mathcal{G}}_t\big)\, ds \\ &= \int_t^u e^{-\alpha(u-s)}\, G_k(s)\, ds + 2\alpha\, \frac{\int_t^\tau e^{\alpha r} d\tilde{B}_r^k}{e^{2\alpha\tau} - e^{2\alpha t}} \int_t^u e^{\alpha(2s-u)} ds\end{aligned}$$

[whereby we have just assumed $u < \tau$; remind the epilog of (3.3.37) in this context]. Secondly, taking (3.3.110) into account, we simply get for $k = d+1, \ldots, l$

(5.3.41)

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^u e^{-\alpha(u-s)} dB_s^k \,\middle|\, \tilde{\mathcal{F}}_t \right) = \int_t^u e^{-\alpha(u-s)}\, G_k(s)\, ds.$$

Thirdly, with respect to (3.2.20) and (5.2.2), we deduce for $k = l+1, \ldots, n$

(5.3.42)

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left(\int_t^u \sigma_k(s)\, e^{-\alpha(u-s)} dL_s^k \,\middle|\, \tilde{\mathcal{F}}_t \right) = \int_t^u \int_{D_k} z\, \sigma_k(s)\, e^{-\alpha(u-s)} e^{h_k(s,z)} \rho_k(s)\, d\nu_k(z)\, ds.$$

Merging (5.3.40) – (5.3.42) into (5.3.39), we end up with the representation

(5.3.43)

$$\tilde{F}_{CAT}(t) = \tilde{\Gamma}(t) + \Psi(t)\,\Theta_t + \sum_{k=1}^{d} \tilde{\Phi}_k(t)\,W_t^k$$

wherein we have just introduced the shorthand notations

(5.3.44)

$$\tilde{\Gamma}(t) := \int_{\tau_1}^{\tau_2} \mu(u)\,du - \mu(t)\,\Psi(t) - \sum_{k=1}^{l} \int_{\tau_1}^{\tau_2} \int_{u}^{t} \sigma_k\, e^{-\alpha(u-s)}\, G_k(s)\,ds\,du$$
$$- \sum_{k=l+1}^{n} \int_{\tau_1}^{\tau_2} \int_{u}^{t} \int_{D_k} z\,\sigma_k(s)\, e^{-\alpha(u-s)} e^{h_k(s,z)} \rho_k(s)\,d\nu_k(z)\,ds\,du,$$

$$\tilde{\Phi}_k(t) := \frac{2\alpha\sigma_k}{e^{2\alpha t} - e^{2\alpha\tau}} \int_{\tau_1}^{\tau_2} \int_{u}^{t} e^{\alpha(2s-u)} ds\,du, \qquad W_t^k := \int_{t}^{\tau} e^{\alpha r} d\tilde{B}_r^k.$$

Next, from (5.3.44) we get the derivatives

(5.3.45)
$$\tilde{\Phi}_k'(t) = a(t)\left[e^{\alpha t}\,\tilde{\Phi}_k(t) - \sigma_k\,\Psi(t)\right],$$

$$\tilde{\Gamma}'(t) = -\,\Psi(t)\left[\mu'(t) + \alpha\,\mu(t) + \sum_{k=1}^{l} \sigma_k\,G_k(t) + \sum_{k=l+1}^{n} \int_{D_k} z\,\sigma_k(t)\,e^{h_k(t,z)} \rho_k(t)\,d\nu_k(z)\right].$$

Hence, applying Lemma 2.1.5 on (5.3.43) and hereafter using (3.2.20), (3.3.110), (5.2.2), (5.3.15), (5.3.34), (5.3.36), (5.3.37), (5.3.44) and (5.3.45), we obtain the Lévy-type $(\tilde{\mathcal{G}}, \tilde{\mathbb{Q}})$-martingale dynamics

(5.3.46)

$$d\tilde{F}_{CAT}(t) = \sum_{k=1}^{d} \left[\sigma_k\,\Psi(t) - e^{\alpha t}\,\tilde{\Phi}_k(t)\right] d\tilde{B}_t^{k,\tilde{\mathcal{G}},\tilde{\mathbb{Q}}} + \sum_{k=d+1}^{l} \sigma_k\,\Psi(t)\,d\tilde{B}_t^{k,\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}$$
$$+ \sum_{k=l+1}^{n} \int_{D_k} z\,\sigma_k(t)\,\Psi(t)\,\tilde{N}_k^{\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(t,dz).$$

In our proceedings, we aim to price a European call option written on the $\tilde{\mathcal{G}}$-forward-looking (mixed) CAT futures price (5.3.46). Thus, sticking to the notational framework of subsection 5.3.2 [particularly compare equations (5.3.20) and (5.3.21) therein], we assume the CAT call option price under $\tilde{\mathcal{G}}$ to obey

(5.3.47)
$$C_{CAT}^{\tilde{\mathcal{G}}}(t) = e^{-r(\tilde{T}-t)}\, \mathbb{E}_{\tilde{\mathbb{Q}}}\left(e^{a\tilde{F}_{CAT}(\tilde{T})}\, q\left(\tilde{F}_{CAT}(\tilde{T})\right) \middle| \tilde{\mathcal{G}}_t\right)$$

$$= \frac{e^{-r(\tilde{T}-t)}}{2\pi} \int_{\mathbb{R}} \hat{q}(y)\, e^{(a+iy)\tilde{F}_{CAT}(t)}\, \mathbb{E}_{\tilde{\mathbb{Q}}}\left(e^{(a+iy)[\tilde{F}_{CAT}(\tilde{T}) - \tilde{F}_{CAT}(t)]} \middle| \tilde{\mathcal{G}}_t\right) dy$$

whereby $\hat{q}(y)$ is such as given in (3.2.34).

Then, with respect to the independent increment property of the $(\tilde{\mathcal{G}}, \tilde{\mathbb{Q}})$-Sato-martingale (5.3.46)[60], the conditional expectation on the right hand side of (5.3.47) factors into

(5.3.48)
$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left(e^{(a+iy)[\tilde{F}_{CAT}(\tilde{T})-\tilde{F}_{CAT}(t)]}\middle|\tilde{\mathcal{G}}_t\right) =$$

$$\prod_{k=1}^{d} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[exp\left\{\int_t^{\tilde{T}} (a+iy)\left[\sigma_k\,\Psi(s) - e^{\alpha s}\,\tilde{\Phi}_k(s)\right] d\tilde{B}_s^{k,\tilde{\mathcal{G}},\tilde{\mathbb{Q}}}\right\}\right]$$
$$\times \prod_{k=d+1}^{l} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[exp\left\{\int_t^{\tilde{T}} (a+iy)\,\sigma_k\,\Psi(s)\,d\tilde{B}_s^{k,\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}\right\}\right]$$
$$\times \prod_{k=l+1}^{n} \mathbb{E}_{\tilde{\mathbb{Q}}}\left[exp\left\{i\int_t^{\tilde{T}}\int_{D_k} (y-ia)\,z\,\sigma_k(s)\,\Psi(s)\,d\tilde{N}_k^{\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}(s,z)\right\}\right]$$
$$=: \prod_{k=1}^{d} \mathfrak{I}_1^k \times \prod_{k=d+1}^{l} \mathfrak{I}_2^k \times \prod_{k=l+1}^{n} \mathfrak{I}_3^k$$

with multipliers

(5.3.49)
$$\mathfrak{I}_1^k = exp\left\{\int_t^{\tilde{T}} \frac{(a+iy)^2}{2}\left[\sigma_k\,\Psi(s) - e^{\alpha s}\,\tilde{\Phi}_k(s)\right]^2 ds\right\},$$

$$\mathfrak{I}_2^k = exp\left\{\int_t^{\tilde{T}} \frac{(a+iy)^2}{2}\,\sigma_k^2\,\Psi(s)^2 ds\right\}, \qquad \mathfrak{I}_3^k = e^{\psi_k(y,t,\tilde{T})},$$

wherein $\psi_k(y,t,\tilde{T})$ is such as defined in (3.2.41), but yet with $\theta_k(s) \coloneqq (y-ia)\,\sigma_k(s)\,\Psi(s)$.
By the way, comparing $\mathfrak{I}_1^k$ with $\mathfrak{I}_2^k$, we recognize that both factors merely differ by an additive *information drift* which originates from the presumed supplementary knowledge about future (mean-) temperature behavior and hence, reasonably affects the multipliers that are indexed by $k = 1, \dots, d$ solely [also recall the precise definition of $\tilde{\mathcal{G}}$ in this context].
However, appealing to (3.2.34) and (5.3.47) – (5.3.49), our *mixed* CAT call option price under additional forward-looking information modeled by the (explicit) intermediate filtration $\tilde{\mathcal{G}}$ reads as

(5.3.50)
$$C_{CAT}^{\tilde{\mathcal{G}}}(t) = \frac{e^{-r(\tilde{T}-t)}}{2\pi}\int_{\mathbb{R}} \frac{e^{(a+iy)[\tilde{F}_{CAT}(t)-K]}}{(a+iy)^2} \prod_{k=1}^{d} \mathfrak{I}_1^k \times \prod_{k=d+1}^{l} \mathfrak{I}_2^k \times \prod_{k=l+1}^{n} \mathfrak{I}_3^k\ dy.$$

Herein, the factors $\mathfrak{I}_1^k$ are closely connected with *risk-reducing* $\tilde{\mathcal{G}}$-forward-looking information on a selection of the Brownian noises driving the temperature mean-level. Secondly, the terms $\mathfrak{I}_2^k$ can be associated to some kind of *remaining risk* with respect to the future long-term level of the temperature variations (while weather forecasts never hold with exhaustive certainty). Finally, the multipliers $\mathfrak{I}_3^k$ originate from the omnipresent *risk of temperature jumps* (which possibly cannot be generated by Brownian noise). Regarding (5.3.48) – (5.3.50), we ultimately cherish that it has turned out possible to compute the appearing expectations more explicitly than in our former (anticipating) *pure-jump* approach presented in subsection 5.3.2 [particularly, compare equations (5.3.22) – (5.3.23) therein].

---

[60] Recall that $\tilde{B}^{k,\tilde{\mathcal{G}},\tilde{\mathbb{Q}}}$ is a $\tilde{\mathcal{G}}$-*Brownian motion*, while the integrators $\tilde{B}^{k,\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}$ and $\tilde{N}_k^{\tilde{\mathcal{F}},\tilde{\mathbb{Q}}}$ generate stochastic integrals that possess independent increments with respect to $\tilde{\mathcal{F}}$. Yet, the two latter integrator types are *not* affected by the enlargement of $\tilde{\mathcal{F}}$ to $\tilde{\mathcal{G}}$ at all! Consequently, $\tilde{F}_{CAT}$ indeed possesses independent increments with respect to $\tilde{\mathcal{G}}$.

5.4 Hedging temperature risk under weather forecasts

Inspired by [4], the present paragraph is dedicated to the construction of an *optimal* (in a sense to be determined) temperature futures portfolio including suitable market-traded temperature indices in order to hedge against both *spatial* and *temporal* temperature risk adequately. In this regard, imagine a company which would like to hedge against its temperature risk at a certain location or geographical area whereas, unfortunately, there merely are indices written on temperature in *surrounding* locations available [4]. In such a case it sounds convenient to construct a portfolio consisting of indeed *suboptimal located* temperature futures, which at least are *available* in the present market [4]. More precise, electricity producers or heating-oil retailers, for instance, often face a severe temperature risk, whereas their company, or their clients respectively, are not necessarily located in or nearby to one of the (nine European, eighteen North-American or two Japanese [13]) cities for which temperature futures are traded [4]. Possibly, there may neither be any temperature index with a measurement period as desired by the investor offered in the market (*temporal risk*), nor be any temperature futures for the locations of interest available (*spatial risk*) [4]. Hence, an investor's goal should be to minimize both the present temporal and spatial temperature risk by constructing a suitable portfolio out of (possibly *suboptimal* but at least) *available* temperature indices that approximately covers the desired (but *non-traded*) futures in a best possible manner [4].

Although interesting, in this work we will *not* stick to the random field framework presented in [4], wherein the time- and space-dependent temperature dynamics $dT(t,x)$ are driven by a *Gaussian random field*. Anyway, Barth et al. [4] model the coordinate (with respect to an appropriate cartesian coordinate system) of the city for which the temperature futures of interest is traded by a two-dimensional vector $x \in \mathbb{R}^2$. More precisely, the spatial temperature component x therein is assumed to be an element in a compact domain $\mathcal{D} \subset \mathbb{R}^2$ (with piecewise smooth boundary) representing a geographical area like the USA, Europe or Japan, respectively. In this context, the authors of [4] presume that there is trade on temperature indices in $\#n$ different cities which are located at the coordinates $x_1, \dots, x_n \in \mathcal{D} \subset \mathbb{R}^2$. However, in the following sections we will also concentrate on a *spatio-temporal temperature risk hedging problem*, whereby we deviate from the random field approach presented in [4]: Instead, we introduce a *multi-dimensional* temperature model in order to generate the temperature behavior at $\#m$ different locations/cities of interest simultaneously. In addition, we newly take forward-looking information about future temperature conditions [i.e. weather forecasts, whenever available, for each individual location of interest $i \in \{1, \dots, m\}$] into account and hence, examine a minimum variance temperature futures portfolio optimization problem with respect to generalized insider trading principles in the upcoming sections innovatively.

5.4.1 A space-dependent multi-dimensional temperature model

In accordance to (5.2.1), we now implement a m-dimensional random temperature vector

(5.4.1) $$\boldsymbol{\Theta}_t := (\Theta_t^1, \dots, \Theta_t^m) \in \mathbb{R}^m$$

with Ornstein-Uhlenbeck type entries

(5.4.2)
$$d\Theta_t^i = d\mu_i(t) + \alpha_i \left[\mu_i(t) - \Theta_t^i\right] dt + \sum_{k=1}^{n} \sigma_{ik}(t)\, dL_t^{ik}.$$

In our framework, the components Θ_t^i model the daily average outdoor temperature at time t in the location $i \in \{1, \ldots, m\}$. Similarly to (5.2.1), the deterministic ingredients $\mu_i(t)$, α_i and $\sigma_{ik}(t)$ yet represent the space- and time-dependent mean-level, mean-reversion speed and volatility of the temperature variations at location i, respectively. In accordance to (5.2.2), for each index $i \in \{1, \ldots, m\}$ we further introduce a family of $\#n$ pair-wise independent pure-jump finite-variation Lévy-type noises

(5.4.3)

$$L_t^{ik} := \int_0^t \int_{D_{ik}} z \, dN_{ik}(s, z)$$

with $k = 1, \ldots, n$ and $D_{ik} \subset \mathbb{R} \setminus \{0\}$, interspersing random fluctuations into the temperature dynamics given by (5.4.2). Philosophically speaking, one could interpret (5.4.3) as a $(m \times n)$-*matrix-valued Poisson random field*. Parallel to (5.1.4), let us moreover introduce a CAT index associated to the location/city with number $i \in \{1, \ldots, m\}$ and measurement period $[\tau_1^i, \tau_2^i]$ by dint of

(5.4.4)

$$CAT_i := CAT[\tau_1^i, \tau_2^i] := \int_{\tau_1^i}^{\tau_2^i} \Theta_u^i \, du.$$

Furthermore, in accordance to (5.2.3), we yet find the Sato-solution of (5.4.2) as

(5.4.5)

$$\Theta_u^i = \mu_i(u) + [\Theta_t^i - \mu_i(t)] \, e^{-\alpha_i(u-t)} + \sum_{k=1}^n \int_t^u \sigma_{ik}(s) \, e^{-\alpha_i(u-s)} dL_s^{ik}$$

with $0 \leq t \leq u$. Thus, substituting (5.4.5) into (5.4.4), we instantaneously obtain

(5.4.6)

$$CAT_i = \int_{\tau_1^i}^{\tau_2^i} \mu_i(u) \, du + [\Theta_t^i - \mu_i(t)] \int_{\tau_1^i}^{\tau_2^i} e^{-\alpha_i(u-t)} du + \sum_{k=1}^n \int_{\tau_1^i}^{\tau_2^i} \int_t^u \sigma_{ik}(s) \, e^{-\alpha_i(u-s)} dL_s^{ik} \, du.$$

Meanwhile, we introduce the abbreviation $\mathfrak{m}_i := \mathfrak{m}_i(s) := max\{s, \tau_1^i\}$ and the deterministic function

(5.4.7)

$$\Lambda(t, \tau_1^i, \tau_2^i) := \frac{e^{-\alpha_i(\tau_1^i - t)} - e^{-\alpha_i(\tau_2^i - t)}}{\alpha_i}.$$

Then, applying the Fubini-Tonelli theorem on (5.4.6), we end up with

(5.4.8)

$$CAT_i = \int_{\tau_1^i}^{\tau_2^i} \mu_i(u) \, du + [\Theta_t^i - \mu_i(t)] \, \Lambda(t, \tau_1^i, \tau_2^i) + \sum_{k=1}^n \int_t^{\tau_2^i} \sigma_{ik}(s) \, \Lambda(s, \mathfrak{m}_i, \tau_2^i) \, dL_s^{ik}.$$

5.4.2 Modeling space-dependent temperature forecasts

For the remainder of the present chapter we set

(5.4.9) $$\tilde{\mathcal{F}}_t := \sigma\{\Theta_u^i : 0 \le u \le t,\ i = 1, \dots, m\} := \sigma\{L_u^{i1}, \dots, L_u^{in} : 0 \le u \le t,\ i = 1, \dots, m\}$$

whereas, analogously to (5.3.4), we implement the enlarged filtration $\tilde{\mathcal{G}}_t \ (\supset \tilde{\mathcal{F}}_t)$ via

(5.4.10) $$\tilde{\mathcal{G}}_t := \tilde{\mathcal{F}}_t \vee \sigma\left\{L_\tau^{i1}, \dots, L_\tau^{ip_i} : i = 1, \dots, m\right\}$$

for $0 \le p_i \le n$ and $0 \le t < \tau$. Note in passing that $\tilde{\mathcal{G}}_t = \tilde{\mathcal{F}}_t$ still holds true for all $t \ge \tau$. Further, the choice $p_i = 0$ for a certain index $i \in \{1, \dots, m\}$ means that there are *no* temperature forecasts for the location i at the future time τ available. Vice versa, setting $p_i = n$ corresponds to having access to (rather unrealistically) *complete* or *exhaustive* knowledge of the future temperature at time τ in the location i.

Remark 5.4.1 *The properties (3.3.39), (3.3.47), (3.3.48), (3.3.49) and Lemma 3.5.1 simultaneously hold in our recent setup (5.4.9) – (5.4.10). In other words, we are allowed to replace $\mathcal{G}_t^*$ by $\tilde{\mathcal{G}}_t$, L_t^k by L_t^{ik}, N_k by N_{ik}, p by p_i etc. inside the former properties. Moreover, in accordance to (3.2.20), the PRMs $dN_{ik}(s,z)$ such as appearing in (5.4.3) possess (preliminarily speaking for the pure $\tilde{\mathcal{F}}$-case) deterministic predictable (time-inhomogeneous) $(\tilde{\mathcal{F}}, \mathbb{Q})$-compensators that are of the form*

$$e^{h_{ik}(s,z)}\, \rho_{ik}(s)\, d\nu_{ik}(z)\, ds$$

for $i = 1, \dots, m$ and $k = 1, \dots, n$. Nevertheless, under $\tilde{\mathcal{G}}$, we currently assume an adjusted version of Condition A to hold. ■

5.4.3 The residual hedging risk under enlarged filtrations

As explained beforehand, an investor may try to combine a selection of temperature futures traded for actually *suboptimal located* cities/regions into a portfolio which reflects his/her company's needs for temperature securities in a best possible way [4]. In this regard, extending the *backward-looking Gaussian* random field approach in [4], we now dedicate our attention to the derivation of an *optimal* hedging portfolio consisting of temperature futures contracts which, in particular, minimize the $\tilde{\mathcal{G}}$-conditioned variance within a certain *desired* (but actually *unavailable*) temperature index associated to a specific *location of interest* $\kappa \in \{1, \dots, m\}$. Thus, in the sequel we will create *optimal*, namely "*synthetic hedges*" (compare the notation in [4]) to cover those unavailable but desired temperature indices related to a specific location of interest.
Starting off, we first need a reasonable hedging error criterion. For this purpose, we come up with a tailor-made conditional variance measure which (as in [4]) will be called the "*residual risk*" (associated to an arbitrary temperature index $\mathfrak{I}$) from now on. Extending equation "(3.2) in [4]" to our innovative insider trading context, we concretely define the *residual risk* via

(5.4.11)

$$\tilde{\mathfrak{R}}(t, \boldsymbol{\xi}_t) := \mathbb{E}_{\mathbb{Q}}\left(\left\{\mathfrak{I}[\tau_1^\kappa, \tau_2^\kappa] - \sum_{\substack{i=1 \\ i \neq \kappa}}^{m} \xi_t^i\ \mathfrak{I}[\tau_1^i, \tau_2^i]\right\}^2 \middle| \tilde{\mathcal{G}}_t\right).$$

Herein, $\Im[\tau_1^\kappa,\tau_2^\kappa] := \Im_\kappa$ stands for a *desired* but actually *unavailable* either CAT, HDD or CDD index associated to a location/city with number $\kappa \in \{1,\dots,m\}$ and measurement period $[\tau_1^\kappa,\tau_2^\kappa]$ along with a time partition $\tau_1^\kappa \le \tau_1^i \le \tau_2^i \le \tau_2^\kappa$ for indices $i = 1,\dots,m;\ i \ne \kappa$. That is, $\Im[\tau_1^\kappa,\tau_2^\kappa]$ embodies the *temperature risk* connected with the location κ to which we assume our fictive agent to be exposed. Moreover, the multi-dimensional $\tilde{\mathcal{G}}_t$-adapted stochastic process (i.e. each component is $\tilde{\mathcal{G}}_t$-adapted)

(5.4.12) $$\boldsymbol{\xi}_t := (\xi_t^1,\dots,\xi_t^{\kappa-1},\xi_t^{\kappa+1},\dots,\xi_t^m) \in \mathbb{R}^{m-1}$$

describes the number of contracts invested in each of the (*available* but) *suboptimal located* temperature futures $\Im[\tau_1^i,\tau_2^i] := \Im_i$ $(i = 1,\dots,m;\ i \ne \kappa)$ at time t (cf. p.6 in [4]). Having the structure of (5.4.10) in mind, we emphasize that – since each investment decision ξ_t^i associated to the temperature index $\Im_i$ is assumed to be $\tilde{\mathcal{G}}_t$*-measurable* – the trading position (5.4.12) basically depends on the available *backward-looking* market information up to time t, namely $\tilde{\mathcal{F}}_t$, and possibly on some additional temperature forecasts for the location $i \in \{1,\dots,m\}\backslash\{\kappa\}$ at a future time τ. Verbalizing, the object $\widetilde{\Re}(t,\boldsymbol{\xi}_t)$ measures the $\mathbb{Q}$-expected (non-hedgeable) *remaining risk* at location κ associated to the temperature index $\Im_\kappa$ in the $\mathcal{L}^2$-sense conditioned on all available information $\tilde{\mathcal{G}}_t$ including temperature forecasts. Parallel to "(3.3) in [4]", our fictive agent's goal should be to find an *optimal* hedging strategy $\hat{\boldsymbol{\xi}}_t$ which minimizes the residual risk indicator (5.4.11) throughout all $\tilde{\mathcal{G}}_t$-adapted trading positions $\boldsymbol{\xi}_t$. Consequently, we are facing a *minimum variance hedging problem* of the form

(5.4.13)
$$\min_{\xi_t\ \tilde{\mathcal{G}}_t-adapted} \widetilde{\Re}(t,\boldsymbol{\xi}_t)$$

bearing the optimal position, say

$$\hat{\boldsymbol{\xi}}_t := \arg\min_{\xi_t\ \tilde{\mathcal{G}}_t-adapted} \widetilde{\Re}(t,\boldsymbol{\xi}_t).$$

5.4.4 Computing minimum variance hedging positions for CAT indices

In our proceedings, we choose (without loss of generality) the dummy temperature index $\Im$ to be a cumulative average temperature index, that is, $\Im := CAT$. In order to minimize the $\mathcal{L}^2(\tilde{\mathcal{G}},\mathbb{Q})$-distance (5.4.11) between the *desired* (but unfortunately *non-available*) futures contract $CAT_\kappa := CAT[\tau_1^\kappa,\tau_2^\kappa]$ and the linear combination of *available* CAT indices in the market, namely the *synthetic hedge*

(5.4.14)
$$\sum_{i=1;i\ne\kappa}^{m} \xi_t^i\, CAT[\tau_1^i,\tau_2^i],$$

under the constraint that our fictive agent enters the futures market at time t $(\le \tau_1^\kappa)$, we obviously have to solve the following first order optimality condition

(5.4.15)
$$\frac{\partial}{\partial \xi_t^j}\widetilde{\Re}(t,\boldsymbol{\xi}_t) = 0$$

for a running index $j \in \{1,\dots,m\}\backslash\{\kappa\}$ (compare the proof of Prop. 3.1 in [4]).

Substituting (5.4.11) into (5.4.15) along with an interchange of the differential operator and the conditional expectation, we receive

(5.4.16)

$$0 = \frac{\partial}{\partial \xi_t^j} \widetilde{\mathfrak{R}}(t, \boldsymbol{\xi}_t) = 2\, \mathbb{E}_{\mathbb{Q}}\left(\left[\sum_{\substack{i=1 \\ i \neq \kappa}}^{m} \xi_t^i\, CAT_i - CAT_\kappa \right] CAT_j \middle| \tilde{\mathcal{G}}_t \right).$$

Since each component of $\boldsymbol{\xi}_t$ formerly has been assumed to be $\tilde{\mathcal{G}}_t$-adapted [compare the sequel of (5.4.12)], equation (5.4.16) next transforms into

(5.4.17)

$$\sum_{\substack{i=1 \\ i \neq \kappa}}^{m} \xi_t^i\, \mathbb{E}_{\mathbb{Q}}\big(CAT_i\, CAT_j \big| \tilde{\mathcal{G}}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(CAT_\kappa\, CAT_j \big| \tilde{\mathcal{G}}_t\big).$$

In accordance to (5.3.5), we yet define the *component-wise* futures price at time t associated to a CAT index at location $i \in \{1, \dots, m\}$, i.e. CAT_i, with measurement period $[\tau_1^i, \tau_2^i]$ under the forward-looking information flow $\tilde{\mathcal{G}}_t$ by dint of

(5.4.18)
$$F_{CAT_i}^{\tilde{\mathcal{G}}}(t, \tau_1^i, \tau_2^i) := \mathbb{E}_{\mathbb{Q}}\left(\int_{\tau_1^i}^{\tau_2^i} \Theta_u^i\, du \middle| \tilde{\mathcal{G}}_t \right).$$

Therewith, we exemplarily compute the conditional expectation on the left hand side of (5.4.17) in detail now: Taking (5.4.4), (5.4.8) and (5.4.18) into account, we firstly deduce

(5.4.19)
$$\mathbb{E}_{\mathbb{Q}}\big(CAT_i\, CAT_j \big| \tilde{\mathcal{G}}_t\big) =$$

$$\left(\int_{\tau_1^i}^{\tau_2^i} \mu_i(u)\, du + \big[\Theta_t^i - \mu_i(t)\big]\, \Lambda\big(t, \tau_1^i, \tau_2^i\big) \right) \times F_{CAT_j}^{\tilde{\mathcal{G}}}\big(t, \tau_1^j, \tau_2^j\big)$$

$$+\, \mathbb{E}_{\mathbb{Q}}\left(CAT_j \sum_{k=1}^{n} \int_t^{\tau_2^i} \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)\, dL_s^{ik} \middle| \tilde{\mathcal{G}}_t \right).$$

Within analogous arguments, the remaining conditional expectation on the right hand side of equality (5.4.19) finally becomes

(5.4.20)

$$\mathbb{E}_{\mathbb{Q}}\left(CAT_j \sum_{k=1}^{n} \int_t^{\tau_2^i} \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)\, dL_s^{ik} \middle| \tilde{\mathcal{G}}_t \right)$$

$$= \left(\int_{\tau_1^j}^{\tau_2^j} \mu_j(u)\, du + \big[\Theta_t^j - \mu_j(t)\big]\, \Lambda\big(t, \tau_1^j, \tau_2^j\big) \right) \times \mathbb{E}_{\mathbb{Q}}\big(H_t^i \big| \tilde{\mathcal{G}}_t\big) + \mathbb{E}_{\mathbb{Q}}\big(H_t^i\, H_t^j \big| \tilde{\mathcal{G}}_t\big)$$

whereby we have just introduced the shorthand notation

(5.4.21)

$$H_t^i := \sum_{k=1}^{n} \int_t^{\tau_2^i} \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)\, dL_s^{ik}.$$

Remembering (5.4.3), (5.4.10), (5.4.21) and, in particular, Remark 5.4.1, within a similar derivation methodology as in (5.3.7) – (5.3.10) [while presuming $\tau_2^i < \tau \leq \tau_2^\kappa$] we presently derive

(5.4.22)

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\big(H_t^i \big| \tilde{\mathcal{G}}_t\big) &= \sum_{k=1}^{p_i} \mathbb{E}_{\mathbb{Q}}\left(\int_t^{\tau_2^i} \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big) dL_s^{ik} \middle| \tilde{\mathcal{G}}_t \right) + \sum_{k=p_i+1}^{n} \mathbb{E}_{\mathbb{Q}}\left(\int_t^{\tau_2^i} \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big) dL_s^{ik} \middle| \tilde{\mathcal{F}}_t \right) \\
&= \sum_{k=1}^{p_i} \int_t^{\tau_2^i} \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)\, \mathbb{E}_{\mathbb{Q}}\left(\frac{L_\tau^{ik} - L_s^{ik}}{\tau - s} \middle| \tilde{\mathcal{G}}_t \right) ds \\
&\quad + \sum_{k=p_i+1}^{n} \mathbb{E}_{\mathbb{Q}}\left[\int_t^{\tau_2^i} \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)\, dL_s^{ik} \right] \\
&= \sum_{k=1}^{p_i} \mathbb{E}_{\mathbb{Q}}\big(L_\tau^{ik} - L_t^{ik} \big| \tilde{\mathcal{G}}_t\big) \int_t^{\tau_2^i} \frac{\sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)}{\tau - t} ds \\
&\quad + \sum_{k=p_i+1}^{n} \int_t^{\tau_2^i} \int_{D_{ik}} z\, \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)\, e^{h_{ik}(s,z)}\, \rho_{ik}(s)\, d\nu_{ik}(z)\, ds \\
&= \Gamma_i(t) + \sum_{k=1}^{p_i} \Phi_{ik}(t)\, \big\{L_\tau^{ik} - L_t^{ik}\big\}
\end{aligned}$$

with abbreviations

(5.4.23)

$$\Phi_{ik}(t) := \int_t^{\tau_2^i} \frac{\sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)}{\tau - t}\, ds$$

and

(5.4.24)

$$\Gamma_i(t) := \sum_{k=p_i+1}^{n} \int_t^{\tau_2^i} \int_{D_{ik}} z\, \sigma_{ik}(s)\, \Lambda\big(s, \mathfrak{m}_i, \tau_2^i\big)\, e^{h_{ik}(s,z)}\, \rho_{ik}(s)\, d\nu_{ik}(z)\, ds.$$

Additionally, we find the decomposition

(5.4.25) $$\mathbb{E}_{\mathbb{Q}}\big(H_t^i\, H_t^j \big| \tilde{\mathcal{G}}_t\big) = \mathbb{Cov}_{\mathbb{Q}}\big(H_t^i, H_t^j \big| \tilde{\mathcal{G}}_t\big) + \mathbb{E}_{\mathbb{Q}}\big(H_t^i \big| \tilde{\mathcal{G}}_t\big)\, \mathbb{E}_{\mathbb{Q}}\big(H_t^j \big| \tilde{\mathcal{G}}_t\big).$$

Admittedly, the co-variance expression in (5.4.25) requires some further examinations, whereas the members $\mathbb{E}_{\mathbb{Q}}\big(H_t^i\big|\tilde{\mathcal{G}}_t\big)$ and $\mathbb{E}_{\mathbb{Q}}\big(H_t^j\big|\tilde{\mathcal{G}}_t\big)$ fortunately can be handled similar to (5.4.22). Finally, merging (5.4.20), (5.4.22) and (5.4.25) into (5.4.19), we get an expression for $\mathbb{E}_{\mathbb{Q}}\big(CAT_i\, CAT_j\big|\tilde{\mathcal{G}}_t\big)$ involved inside our optimality condition (5.4.17). Actually, the treatment of the object $\mathbb{E}_{\mathbb{Q}}\big(CAT_\kappa\, CAT_j\big|\tilde{\mathcal{G}}_t\big)$ such as appearing on the right hand side of (5.4.17) can be done in an analogous manner.

Remark 5.4.2 *Note in passing that equation (5.4.17) can be interpreted as a time-dependent stochastic matrix-vector-equation reading*

(5.4.26)
$$\boldsymbol{A}(t)\, \boldsymbol{\xi}_t = \boldsymbol{b}(t)$$

(compare Prop. 3.1 in [4] at this step) with matrix entries

(5.4.27)
$$A_{ji}(t) := \mathbb{E}_{\mathbb{Q}}\big(CAT_i\, CAT_j\big|\tilde{\mathcal{G}}_t\big)$$

and vector components

(5.4.28)
$$b_j(t) := \mathbb{E}_{\mathbb{Q}}\big(CAT_\kappa\, CAT_j\big|\tilde{\mathcal{G}}_t\big)$$

for indices $i, j \in \{1, \dots, m\}\backslash\{\kappa\}$. Since all entries $A_{ji}(t)$ by definition are pair-wise $\mathbb{Q}$-independent, the matrix $\boldsymbol{A}(t)$ possesses full rank and thus, $\mathbb{Q}$-almost-sure is invertible. ■

Returning to our optimization exercise (5.4.13), we recognize that the second order condition for a minimum is also fulfilled, since

(5.4.29)
$$\frac{\partial^2}{\partial\big(\xi_t^j\big)^2}\, \tilde{\mathfrak{R}}(t, \boldsymbol{\xi}_t) = 2\, \mathbb{E}_{\mathbb{Q}}\left(\big[CAT_j\big]^2\Big|\tilde{\mathcal{G}}_t\right) > 0$$

is valid whenever $CAT_j \neq 0$. [In fact, the trivial case $CAT_j \equiv 0$ only appears with negligible probability and thus, is not of any interest here – compare (5.4.4) to verify this proposition.]

In conclusion, by solving (5.4.17) for $\xi_t^1, \dots, \xi_t^{\kappa-1}, \xi_t^{\kappa+1}, \dots, \xi_t^m$ [respectively (5.4.26) for $\boldsymbol{\xi}_t$], we obtain the synthetic hedging position (5.4.14) [respectively $\hat{\boldsymbol{\xi}}_t$ as introduced in the sequel of (5.4.13)], which is *optimal* in the sense of minimizing the $\mathbb{Q}$-expected $\tilde{\mathcal{G}}$-conditioned squared distance in between the *desired* (but *unavailable*) CAT futures index CAT_κ and what can be hedged by investing in *suboptimal located* (*surrounding*) CAT indices, namely CAT_i with $i \in \{1, \dots, m\}\backslash\{\kappa\}$. Similar to [4], the optimal position $\hat{\boldsymbol{\xi}}_t = \boldsymbol{A}(t)^{-1}\, \boldsymbol{b}(t)$ designates a *dynamic* hedge as it changes with time. All in all, a company exposed to temperature risk in a certain location κ may combine the available surrounding indices to create a dynamic hedging portfolio which minimizes the residual temperature risk effectively.

5.4.5 A spatially correlated temperature model

Regarding our space-dependent multi-dimensional temperature model like introduced at the beginning of paragraph 5.4.1, we have to declare the lack of any spatial dependency structures as a remarkable disadvantage, as the temperature behavior in neighbored locations ought to be (more or less) correlated. Unfortunately, it is not evident how to incorporate such local dependency structures into our model (5.4.2), since the driving pure-jump Lévy-type components – in contrast to Brownian noises – hardly are to be correlated. To overcome the just described problem, for $i = 1, \ldots, m$ we might replace equality (5.4.2) through

(5.4.30)

$$d\hat{\Theta}_t^i = d\mu_i(t) + \alpha_i \left[\mu_i(t) - \hat{\Theta}_t^i\right] dt + \sum_{k=1}^{n} \sigma_{ik}\, dB_t^{ik}$$

with standard Brownian motions (BMs) B_t^{ik} and strictly positive volatility constants σ_{ik}. Therewith, we tend very closely to the *Gaussian* random field approach presented in [4], admittedly. Nevertheless, we still may apply enlargement-of-filtration techniques (yet concerning the Brownian noises, similarly to our framework proposed in subsection 5.3.4) in order to incorporate additional information on future temperature behavior – an issue which has not been taken into account in [4], on the contrary. More accurately speaking, we currently might define

(5.4.31)

$$\hat{\mathcal{G}}_t := \hat{\mathcal{F}}_t \vee \sigma\left\{\int_0^{\tau} e^{\alpha_i s} dB_s^{ik} : k = 1, \ldots, p_i; i = 1, \ldots, m\right\}$$

instead of (5.4.10), whereas we set

(5.4.32) $$\hat{\mathcal{F}}_t := \sigma\{B_s^{i1}, \ldots, B_s^{in} : 0 \le s \le t; i = 1, \ldots, m\}.$$

In addition, with respect to our modified approach (5.4.30), we now introduce the Brownian integrals

(5.4.33)

$$\hat{H}_t^i := \sum_{k=1}^{n} \sigma_{ik} \int_t^{\tau_2^i} \Lambda\left(s, \mathfrak{m}_i, \tau_2^i\right) dB_s^{ik}$$

instead of (5.4.21). Consequently, for indices $i, j \in \{1, \ldots, m\} \backslash \{\kappa\}$ the adjusted co-variance expressions

(5.4.34) $$\varrho_t^{ij} := \mathbb{C}\mathbb{o}\mathbb{v}_{\mathbb{Q}}\left(\hat{H}_t^i, \hat{H}_t^j \middle| \hat{\mathcal{G}}_t\right)$$

[such as originally appearing inside (5.4.25)] then can be interpreted as some kind of time-dependent $\hat{\mathcal{G}}$-conditioned co-variance matrix entries which explicitly allow for the desired spatial dependency structure, respectively spatial correlation, yet. Roughly speaking, the currently proposed model modification with correlated Brownian motions obviously admits both a *dependency in i-direction* along with an *independency in k-direction*, meaning that $B^{1k}, \ldots, B^{\kappa-1,k}, B^{\kappa+1,k}, \ldots, B^{mk}$ for each $k \in \{1, \ldots, n\}$ are pair-wise correlated via (5.4.34) and that $B^{i1}, \ldots, B^{in}$ are pair-wise independent for each fixed index $i \in \{1, \ldots, m\} \backslash \{\kappa\}$, what indeed sounds reasonable – especially from an applicant's point of view. All in all, the precise treatment of the related hedging procedure with suboptimal located temperature futures [yet associated to our modified Brownian motion model (5.4.30), particularly considered under (5.4.31)] is left to the reader – not at least, as the required techniques have already been presented in our former paragraphs 5.3.4 and 5.4.1 – 5.4.4.

5.5 Conclusions

In order to model the daily average outdoor temperature dynamics in an appropriate manner, we have suggested an alternative multi-factor *pure-jump* Ornstein-Uhlenbeck setup permitting mean-reversion to a periodic trend-line in addition to seasonal volatility. Subsequently, we have turned our attention towards the derivation of CAT and CDD futures prices under supplementary future weather information. Dealing with this innovative insider trading topic, we have rigorously taken temperature forecasts into account via an adequate enlargement of the underlying information filtration. As we have seen in connection with our forward-looking CAT call option price estimate in (5.3.23), in order to evaluate explicit risk-neutral temperature futures option prices under future information there is a strong need for approximation techniques and numerical pricing methods. A challenging related research topic might consist in the derivation of corresponding forward-looking price representations for European options written on CDD and HDD futures likewise (such as already mentioned at the end of subsection 5.3.3). Nevertheless, for our *mixed* temperature model (like introduced in paragraph 5.3.4) including both Brownian motion and pure-jump terms it fortunately has been possible to derive a more explicit CAT call option price formula [see equation (5.3.50) along with (5.3.49)] than in our former anticipative pure-jump approaches.
Ultimately, in section 5.4 we have constructed optimal positions for a CAT futures portfolio under forecasted temperature behavior. In this context, we have introduced a multi-dimensional temperature setup which has turned out suitable to model the temperature dynamics at different locations of interest simultaneously. Moreover, inspired by [4], we have implemented an appropriate risk indicator, namely the *residual risk* [see (5.4.11)], in order to hedge against both *temporal* and *spatial* temperature risk adequately by using a dynamic *synthetic hedge portfolio* for CAT futures.

Chapter 6

Pricing Carbon Emission Allowances under Future Information on the Market Zone Net Position

6.1 Extending the Markov-chain onset for the EU ETS net position

In the present chapter we derive risk-neutral prices for carbon emission allowances (EUAs) as commonly traded in the European Union Emission Trading Scheme (EU ETS), whereas we take forward-looking information about the future market zone net position into account via a rigorous exploitation of enlargement-of-filtration methods. In this insider trading framework, we model the market zone net position as a linear combination of multiple real-valued compound Poisson processes which – in contrast to the two-state Markov-chain onset proposed by Cetin and Verschuere [25] – may indicate the overall net position of the EU ETS market more precisely. Consequently, we need to apply tailor-made multi-dimensional Fourier transform techniques when it comes to pricing purposes of EUA contracts under our extended multi-factor pure-jump approach. Moreover, we discuss the concept of minimum relative entropy in order to find a concrete risk-neutral pricing measure in our incomplete EU ETS market model.

In addition to the above summarizing comments, we now want to present some detailed motivating aspects for our upcoming EU ETS examinations. More concretely speaking, in this work we extend the sophisticated carbon emission allowance pricing approach proposed by Cetin and Verschuere [25] in the following concerns:
Firstly, we newly model the market zone net position by a linear combination of several real-valued compound Poisson processes (CPPs) which – in contrast to the poor two-state Markov-chain onset in [25] – takes more than just the two values plus and minus one.

To motivate our idea, let us quote from the bottom of page 15 in [25]:

> "*Obviously, it is not possible to measure how long or short the zone is by a θ [net position] process with only two states. Therefore, a better fit to the data could be more easily achieved by introducing more states to the model.*" and "*The resemblance [between the simulated and empirical data] could be improved by extending the set of admissible states for θ from just two elements [being plus and minus one, actually].*".

Note in passing that on the one hand our innovative CPP onset yet is able to indicate how *long* or *short*[61] the overall position of the EU ETS market precisely is, whereas in return it requires customized multi-dimensional Fourier transform techniques and exponential dampening methods when it comes to pricing issues of carbon dioxide emission allowances, as we will see in section 6.3 later.
Secondly, we treat the carbon emission permits pricing under supplementary forward-looking information on the market zone net position, rigorously exploiting enlargement-of-filtration methods as examined in e.g. [10], [15], [32], [50]. Let us remark that this insider trading machinery constitutes the right opposite to the *incomplete* information setup presented in Chapter 4 of [25]. More precisely, the authors therein treat the pricing of carbon emission allowances under *partial*, respectively *incomplete* information concerning the market zone net position. For this purpose, they introduce some kind of *reduced* filtration, arguing that "*the EU ETS market participants typically do not observe [the net position process] θ continuously*" (see p.9 in [25]) but at randomized discrete time steps in reality. Admittedly, on a first view the *partial* information assumption in [25] might appear a bit more practical than our *forward-looking* considerations. Nevertheless, one easily may imagine a scenario wherein market insiders know that the future (e.g. steel-) production will increase and thus, there will be more CO2-emission certificates needed what should make the EU ETS market become *short*. Vice versa, an insider might expect the (steel-) demand/production to decrease (due to a particular reason), so that there will be less emission allowances needed in the future what should drive the EU ETS market *long*, rather. However, in section 6.5 we will give some additional motivating aspects that also count in favor for the consideration of EUAs under enlarged filtrations.
Thirdly, we discuss the concept of minimum relative entropy in order to determine a concrete risk-neutral measure out of the large class of offering pricing probabilities in our incomplete model, whereas in Chapter 3 in [25] the so-called *Föllmer-Schweizer minimal measure* (see [40] for details) is examined on the contrary.
Fourthly, in section 8.4.5 we apply stochastic filtering techniques as invented in [53] to (theoretically) estimate the unobservable market net position out of observable emission allowance prices.

The remainder of the current chapter is organized as follows: In section 6.2 we specify the carbon emission allowance prices and, in particular, the market zone net position mathematically. In this context, we provide a precise definition of the EU ETS market to be *long*, *short* or *in equilibrium*, respectively. Applying Girsanov's Change-of-Measure theorem, we next switch to an equivalent pricing measure. Due to the risk-neutral pricing theory, we hereafter obtain a proper drift-restriction which, as usually, ensures the martingale property of the discounted EUA prices. Afterwards, we invest some innovative effort concerning the pricing of carbon emission allowances under our new multi-factor compound Poisson modeling approach, whereas the subsequent paragraph 6.4 is dedicated to an appropriate minimum relative entropy procedure. In section 6.5 we construct enlarged information filtrations tailored to the requirements of our EU ETS framework and, in addition, derive EUA prices under this forward-looking information approach as well, extending our former results of section 6.3 essentially. Finally, in paragraph 6.6 a BM-driven mean-reverting Ornstein-Uhlenbeck

[61] For a precise definition of the EU ETS market being *long*, respectively *short*, see Definition 6.2.1.

market zone net position model is introduced, whereas the closing section 6.7 contains our conclusions along with some future research topics.

6.2 Modeling carbon emission allowance prices

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered and complete probability space, whereby the *backward-looking* (respectively *historical*) information filtration $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,T]}$ (within a fixed time horizon $T < \infty$) is assumed both to include a priori all $\mathbb{P}$-null-sets and to be *cad* [62]. However, we will give a more precise definition of the sigma algebra $\mathcal{F}_t$ in equality (6.2.10) later. In order to be able to focus on the main ideas, we correspond to Chapter 2 in [25] and likewise assume a stylized market setup wherein just *two* EU ETS carbon emission allowances (EUAs) are traded, namely: one EUA for the *current* year, which is denoted by EUA0, and one for the *next* year, notified by the abbreviation EUA1 (cf. p.5 in [25]). Hence, a EUA0 contract can be interpreted as some kind of emission allowances *spot price*, whereas a EUA1 contract may be considered as the corresponding *forward price* (cf. p.3 in [25]).

Slightly deviating from the setup in Ch. 2 of [25], we come up with the stochastic differential equation (SDE) fulfilled by the *forward price process* F_t (associated to the EUA1 contract) reading

$$(6.2.1) \qquad dF_t = \alpha(t, F_t, \theta_t)\, dt + \sigma(t, F_t, \theta_t)\, dW_t.$$

Herein, W_t designates a one-dimensional standard Brownian motion (BM) under $\mathbb{P}$, whereas we suppose the drift, respectively the volatility coefficient to reveal the tailor-made structures

$$(6.2.2) \qquad \alpha(t, F_t, \theta_t) := F_t\,(r + \mu - \theta_t), \qquad \sigma(t, F_t, \theta_t) := F_t\,\sigma(t)$$

for an arbitrary real constant μ, a risk-less interest rate $r > 0$, a deterministic and continuous volatility function $\sigma(t) > 0$ and a real-valued stochastic process θ_t modeling the *market zone net position*. Furthermore, we introduce a bank account β_t obeying

$$(6.2.3) \qquad d\beta_t = r\,\beta_t\, dt$$

with initial value $\beta_0 > 0$. Consequently, equation (6.2.3) possesses the solution

$$\beta_t = \beta_0\, e^{rt}.$$

As usual, we define the *discounted EUA1 forward price* $\hat{F}_t$ via

$$(6.2.4) \qquad \hat{F}_t := \frac{F_t}{\beta_t}$$

whereas Itô's product rule immediately yields the dynamics

$$(6.2.5) \qquad d\hat{F}_t = \hat{F}_t\,[(\mu - \theta_t)\, dt + \sigma(t)\, dW_t].$$

Note in passing that (6.2.5) essentially extends equation "(2.1) in [25]", as we newly permit both a *time-dependent* volatility coefficient $\sigma(t)$ and, more importantly, a *multi-state* (real-valued) net position process θ_t. In what follows, we characterize the EU ETS market to be *long*, *short* or *in equilibrium* respectively, by appealing to the sign of the market zone net position process θ (cf. p.3 and p.5 in [25]), whereas the net position itself will be precisely defined in (6.2.6) later.

[62] See the beginning of section 3.2 for a precise definition of *cad* sigma-algebras.

Definition 6.2.1 ***(a)*** *If $\theta_t > 0$ holds for a time index $t \in [0,T]$, then we say that the ETS market is long at time t. In this case the market faces a surplus situation of emission permits, meaning that there are some firms holding emission allowances that they actually do not need.*

(b) *If $\theta_t < 0$ holds for a time index $t \in [0,T]$, then we say that the ETS market is short at time t. In this (much more delicate) scenario the market faces a shortage situation of emission permits, meaning that there are some firms which still need emission allowances to cover their actual pollution level.*

(c) *If $\theta_t = 0$ holds for a time index $t \in [0,T]$, then we say that the ETS market is in equilibrium at time t. In this case there is a perfect match in between the overall issued emission allowances and the actually verified pollution level – a scenario which, by the way, might not be the most likely one.* ■

Remark 6.2.2 *Similarly to p.5 in [25], regarding eq. (6.2.5), we observe a negative relationship (correlation) between the sign of the market net position θ_t and the overall drift part $\mu - \theta_t$ of the EUA1 price $\hat{F}_t$. More precisely, in the case of emission permit shortage we expect an increase in the demand for EUA1 contracts and thus, in their prices [25]. Fortunately, this economical feature is met by our model, since for $\theta_t < 0$ we observe an upward shift of the drift in (6.2.5) what most likely makes forward prices increase. Vice versa, in the case of permit longness we expect a decrease in the demand for EUA1 contracts [25]. Yet, contrarily to above, for $\theta_t > 0$ we observe a downward shift of the drift in (6.2.5) and hence, forward prices should decrease. Finally, in the equilibrium case $\theta_t = 0$ the basic drift component μ in (6.2.5) reasonably is not affected, neither in upward nor in downward direction. In conclusion, the above features sound economically reasonable and agree with [25].* ■

In [25] the market zone net position θ_t is modeled as a *càdlàg* (French: continue à droite avec des limites à gauche) Markov-chain in continuous time merely taking values in the set $E := \{-1\} \cup \{1\}$ whereas the ETS market therein is called *short* for $\theta_t = -1$, and *long* for $\theta_t = 1$ respectively (cf. p.5 in [25]). Reminding our citations in section 6.1 concerning the striking shortcomings connected with such a *two-state* Markov-chain approach, in this work we will model the market zone net position as a linear-combination of $\#n$ real-valued compound Poisson processes. More accurately, we yet define

(6.2.6)

$$\theta_t := \theta_0 + \sum_{k=1}^{n} L_t^k$$

within an initial value θ_0 (which we assume without loss of generality to equal zero[63] in the remainder of this chapter, i.e. $\theta_0 = 0$) and a family of pair-wise independent, pure-jump, *càdlàg*, finite-variation compound Poisson (Lévy-) processes

(6.2.7) $$L_t^k := \int_0^t \int_{D_k} z \, dN_k(s,z)$$

for a real subset $D_k \subseteq \mathbb{R} \setminus \{0\}$ and time indices $t \in [0,T]$.

[63] Hence, with respect to Definition 6.2.1, we here suppose the ETS market to be *in equilibrium* at time $t = 0$. At this step, let us remark that in section 6.6 we alternatively propose a zero-reverting market zone net position model of Ornstein-Uhlenbeck type with multiple Brownian motion (BM) terms as driving noises [compare equation (6.6.1)]. Remarkably, in this BM-setup it is possible to compute the probability for a *short*, respectively *long* market zone net position [see equation (6.6.15)].

In the previous equation N_k constitutes a one-dimensional integer-valued Poisson-Random-Measure (PRM) on $[0,T] \times D_k$ for each index $k = 1, \dots, n$. We further assume the PRMs $dN_k(s,z)$ both to be $\mathbb{P}$-independent of the (EUA1 forward price driving) Brownian noise W and to have $\mathbb{P}$-compensators $d\nu_k(z)\,ds$ such that

(6.2.8)

$$d\widetilde{N}_k^{\mathbb{P}}(s,z) := dN_k(s,z) - d\nu_k(z)\,ds$$

designate $\mathcal{F}$-adapted martingale integrators under $\mathbb{P}$ for all $k = 1, \dots, n$. By the way, note that $\hat{F}$ and θ hence are assumed to be $\mathbb{P}$-independent which will become important later. Furthermore, the one-dimensional compensating Lévy-measures ν_k appearing in (6.2.8) declare positive and sigma-finite Borel-random-measures on D_k for all $k = 1, \dots, n$ obeying the usual condition

(6.2.9)

$$\int_{D_k} 1 \wedge z^2 \, d\nu_k(z) < \infty.$$

Remark 6.2.3 *Similarly to our former announcements in section 3.2.1, also in the current emission allowances framework it might turn out convenient to split the finite sum in (6.2.6) as follows: We ought to utilize the first* $\#l$ $(< n)$ *components* $L_t^1, \dots, L_t^l$ *to model (Brownian-motion-like) small fluctuations with jump-sizes in a set* $D_k := [-\varepsilon_k, \varepsilon_k] \setminus \{0\}$ *for a small number* $\varepsilon_k > 0$ *and* $k = 1, \dots, l$. *In return, the remaining* $\#(n-l)$ *components* $L_t^{l+1}, \dots, L_t^n$ *might be exploited to model the short-term spiky variations of the market zone net position process whereby we might choose* $D_k := \mathbb{R} \setminus [-\varepsilon_k, \varepsilon_k]$ *for an arbitrary (maybe large) number* $\varepsilon_k > 0$ *and* $k = l+1, \dots, n$. ■

In contrast to a two-state Markov-chain approach, our real-valued compound Poisson process θ_t [like implemented in (6.2.6)] yet can be regarded as a helpful indicator showing how long or short the overall net position of a certain EU ETS market zone precisely is (cf. p.5 in [25]). In this context, comparing fictive values like $\theta_t = -0.01$ and $\theta_t = -100$, for instance, we clearly notice some worthy additional information (interpreted with respect to the underlying jump-size distribution, of course) contained in the relation of the above values, although the market is simply said to be *short* in both cases. Actually, in [25] both previous scenarios trivially would have led to $\theta_t = -1$ so that a big part of information gets lost in the mentioned two-state setup, obviously.
Furthermore, on page 18 in [25] the authors propose another possible improvement of their model, arguing that one might allow for *non-symmetric* changes in the drift[64] of the EUA1 forward prices (6.2.5) in the following sense: News on *short* positions in the market should make the drift *increase*[65] more strongly than news on *long* positions make it *decrease* on the opposite, since (equilibriums or even) long positions of the regarded EU ETS zone evidently embody an extremely less delicate market scenario in contrast to 'trouble-making' permit shortages on the other hand.

[64] The setup in [25] merely allows for *symmetric* changes in the drift, namely either $\mu - \alpha$ in the case of the market being short or $\mu + \alpha$ in the case of the market being long, whereas μ is an arbitrary and α a negative constant.
[65] In [25] the authors write on page 18 „*…in the sense that news regarding short positions reduce the drift more strongly…*" which is wrong (or at least misleading/illogical), as they have assumed a *negative* constant α on the bottom of page 5 and thus, in their setup *short* positions (i.e. $\theta = -1$) make the drift *increase* instead.

Contrarily to [25], in our innovative multi-state CPP model such *asymmetrical* drift effects for $\hat{F}_t$ can easily be achieved by choosing a tailor-made jump-size distribution for the market zone net position process θ_t, that is, by implementing a customized (possibly asymmetric) distribution for the Lévy noises L_t^k appearing inside (6.2.6). More precisely, if one has an idea about the EU ETS market being more likely to turn into or to end up in a particular position, then one might adjust the jump-size distribution as follows:
If one exemplarily feels that the ETS market is more likely to end up *short*, then one ought to choose a jump-size distribution that is *positively skewed*, i.e. *skewed to the right* with respect to the vertical axis. In this case, *negative* jumps appear more likely which should drive the market zone net position process to *negative* values within a high probability. Vice versa, an opposite effect can be achieved for an expected *long* market scenario (also think of what happened in April/May 2006: recall section 1.3 in this context), if one implements a *negatively skewed*, i.e. *skewed to the left*, jump-size distribution on the contrary. Finally, if one suspects the market to turn into or to end up in a precise *equilibrium*, then a *symmetric* (i.e. symmetric to the vertical axis) jump-size distribution might be a proper choice.

In conclusion, the use of *hyperbolic distributions* (see e.g. section 5.6.7 in [1], section 2.6.2 in [13] or the references [30], [80]) for the Lévy noises L_t^k, admitting (heavy-tails and in particular) *skewness*, may yield a very precise description of the distributional properties of the market zone net position. Anyway, the examination of empirical evidences along with numerical simulation issues concerning the just described modeling idea with *asymmetrical jump-size distributions* (which actually are induced by our innovative use of multiple Lévy process noises instead of a simple Brownian motion approach or even a two-state Markov-chain onset for θ) may embody a challenging future research topic which has, to the best of our knowledge, not at all been treated in the context of carbon emission allowances modeling in the literature yet.

Remark 6.2.4 *From a modeling point of view, however, there is no obvious necessity to incorporate an additional random <u>jump</u> component in (6.2.5), respectively in (6.2.1), although one may suspect more or less strong jumps in EUA1 forward prices from time to time, e.g. as a consequence of intermediate announcements concerning the truly verified market zone net position (also compare the second half of the abstract in [25] to read more about a possible occurrence of jumps in emission prices). With respect to our improved CPP approach, we argue that this modeling job simultaneously is taken over by the pure-jump process θ_t appearing inside the drift of (6.2.5), since jumps in θ_t make the drift part jump at the same time (yet in opposite direction, actually) and thus, forward prices also should exhibit a more or less intensive de- or increase <u>during the sequel time range after a jump</u>. This <u>time-delay-feature</u> seems to be extremely reasonable, as one expects the ETS market to need some time to adjust in the case of an abruptly changing (i.e. jumping) net position (due to e.g. information release or other economical reasons). From an economical point of view, a <u>simultaneous</u> jump in the forward price path hence appears rather unrealistic, as it completely neglects a delaying adjustment period.* ■

Further on, with respect to (6.2.5) and (6.2.6), we yet specify the aforementioned filtration $\mathcal{F}_t$ via

(6.2.10) $$\mathcal{F}_t := \sigma\{\hat{F}_s, \theta_s : 0 \le s \le t\} := \sigma\{W_s, L_s^1, \dots, L_s^n : 0 \le s \le t\}.$$

In order to price carbon emission allowances later, we also need the following technical assumption concerning the market zone net position: For the remainder of this chapter we presume that $\theta_t \in [M_1, M_2] \subset \mathbb{R}$ holds true $\mathbb{P}$-almost-sure for all $t \in [0, T]$ with arbitrary constants $M_1 < 0$ and $M_2 > 0$.

In this regard, for $\theta_t = M_1$ we interpret the EU ETS market to be *maximal short* at time t. Vice versa, for $\theta_t = M_2$ the market is said to be *maximal long*.

Remark 6.2.5 *Alternatively to (6.2.6), it might be reasonable to model the market zone net position as a zero-reverting Ornstein-Uhlenbeck process of the form*

(6.2.11)

$$d\theta_t = -\alpha\, \theta_t\, dt \;+ \sum_{k=1}^{n} \zeta_k(t)\, dL_t^k$$

within a deterministic mean-reversion speed $\alpha > 0$, *deterministic volatility functions* $\zeta_k(t) > 0$ *and compound Poisson (Lévy-) processes* L_t^k *as defined in (6.2.7). Interpreting the latter economically motivated modeling proposal, one here expects the market zone net position* θ_t *to revert towards zero, meaning that the ETS market possesses a natural tendency of turning towards its equilibrium. In section 6.6 we will examine a model of the type (6.2.11) in more detail.* ■

6.2.1 Switching to an equivalent martingale measure

As the risk-neutral pricing theory requires the introduction of an (with respect to $\mathbb{P}$) equivalent martingale measure $\mathbb{Q}$, the present subsection is dedicated to a change of probability measures. Thus, for a square-integrable and $\mathcal{F}_t$-adapted process G_t we define the Radon-Nikodym derivative

(6.2.12)

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \tilde{Z}_t := \mathfrak{E}(G_\cdot \circ W_\cdot)_t \times \prod_{k=1}^{n} \mathfrak{E}(M_\cdot^k)_t > 0$$

within a continuous Doléans-Dade exponential $\mathfrak{E}(G_\cdot \circ W_\cdot)_t$ as defined in (2.2.2), (local) $\mathbb{P}$-martingale ingredients M_t^k as previously introduced in (3.2.16) [but with $h_k(s,z) := z$ now] and discontinuous Doléans-Dade exponentials $\mathfrak{E}(M_\cdot^k)_t$ as given in (3.2.17). Moreover, the former representation (3.2.18) likewise holds in our recent setup, if we replace $h_k(s,z) := z$ and $\rho_k(s) \equiv 1$ therein. Actually, Itô's formula yields the local $\mathbb{P}$-martingale representation

(6.2.13)

$$\mathfrak{E}(G_\cdot \circ W_\cdot)_t \;= 1 + \int_0^t \mathfrak{E}(G_\cdot \circ W_\cdot)_s\, G_s\, dW_s.$$

In addition, equation (3.2.19) simultaneously holds, but with $h_k(s,z) := z$ therein now. Assuming the presumptions of paragraph 2.2 to be in force, by means of independency we find $\mathbb{E}_{\mathbb{P}}[\tilde{Z}_t] = 1$ for all t such that $\tilde{Z}$ embodies a true $\mathbb{P}$-martingale. Troubling Proposition 2.2.1, we consequently state that

(6.2.14)

$$\widetilde{W}_t := W_t - \int_0^t G_s\, ds$$

indicates a $\mathcal{F}_t$-adapted BM under the equivalent martingale measure (EMM) $\mathbb{Q}$ [compare (2.2.6)].

Furthermore, with respect to (2.2.8) we claim that

(6.2.15)
$$d\widetilde{N}_k^{\mathcal{F},\mathbb{Q}}(s,z) := d\widetilde{N}_k^{\mathbb{Q}}(s,z) := dN_k(s,z) - e^z \, d\nu_k(z) \, ds$$

depict $(\mathcal{F},\mathbb{Q})$-compensated PRMs and thus, $\mathcal{F}$-adapted $\mathbb{Q}$-martingale integrators for all $k = 1, \dots, n$.

Remark 6.2.6 *In accordance to the risk-neutral pricing theory, we recall that all* <u>*tradable*</u> *"assets" in the considered market must designate local $\mathbb{Q}$-martingales after discounting (cf. e.g. subsection 4.1.1 in [13]). Since there are only two "tradable assets" in our current model, namely the EUA0 contract and the EUA1 forward with price process $\hat{F}$ (whereas the market net position is non-tradable, of course), we are facing $\#(n+2)$ sources of risk (one coming from the EUA0 contract and one from the EUA1 forward $\hat{F}$ along with $\#n$ associated to θ) what, in accordance to the second fundamental theorem of asset pricing, declares our underlying EU ETS market model to be highly incomplete (parallel to p.6 in [25]). Thus, the determination of a precise equivalent martingale measure using the approximation concept of 'minimum relative entropy' will lead us to some sophisticated but necessary examinations in section 6.4 later. (Also remind Remark 5.2.1 at this step.)* ■

Returning to our main topic, we substitute (6.2.14) into (6.2.5) and hence obtain the $\mathbb{Q}$-representation

(6.2.16)
$$d\hat{F}_t = \hat{F}_t \left[(\mu - \theta_t + \sigma(t) \, G_t) \, dt + \sigma(t) \, d\widetilde{W}_t \right].$$

With respect to Remark 6.2.6, we require the discounted EUA1 forward price $\hat{F}_t$ to form a local martingale under $\mathbb{Q}$, what leads us to the *drift-restriction*

(6.2.17)
$$G_t = \frac{\theta_t - \mu}{\sigma(t)}.$$

Assuming the latter drift-restriction to be in force, equality (6.2.16) simplifies to

(6.2.18)
$$d\hat{F}_t = \hat{F}_t \, \sigma(t) \, d\widetilde{W}_t$$

bearing the explicit (continuous) Doléans-Dade $\mathbb{Q}$-solution

(6.2.19)
$$\hat{F}_t = \hat{F}_0 \, \mathfrak{E}\big(\sigma(\cdot) \circ \widetilde{W}_\cdot\big)_t = \hat{F}_0 \, exp\left\{ \int_0^t \sigma(s) \, d\widetilde{W}_s - \frac{1}{2} \int_0^t \sigma(s)^2 \, ds \right\}.$$

Herein, we assume the initial value

$$\hat{F}_0 = \frac{F_0}{\beta_0} > 0$$

to be deterministic.

6.3 Pricing carbon emission allowances

In this paragraph we invent an appropriate method involving multi-dimensional Fourier transforms in order to derive explicit pricing formulas for carbon emission allowances as commonly traded in the EU ETS but newly under our improved multi-state market zone net position approach. We initially do this under the ordinary assumption that the market participants' knowledge is such as modeled in (6.2.10). In other words, at time t we suppose these *uninitiated traders* solely to have an idea about the histories of $\hat{F}_s$ and θ_s during a time range $0 \leq s \leq t$. On the contrary, in section 6.5 we will show how these pricing formulas alter if one presumes some additional forward-looking insider information about the market zone net position at a future time τ available to so-called *informed traders*, respectively *market insiders*.

As described at the end of Chapter 2 in [25], we firstly should notice that if the EU ETS market ends up *long* $(\theta_T > 0)$ at the expiry date T (meaning that there are still some firms holding carbon emission allowances that they do not need any more), then – under the assumption of *no banking* – EUA0 contracts will become worthless what drives their prices towards zero immediately. Theoretically, we assume the same to be valid for the rather unlikely but of course possible case wherein the market ends up in a precise *equilibrium* $(\theta_T = 0)$. Contrarily, if the market ends up *short* $(\theta_T < 0)$ at the final time T (meaning that there is a shortage situation throughout carbon emission allowances in the market), then – in this much more delicate market scenario – a EUA0 contract will *not* become worthless, since the latter easily can be turned into a EUA1 contract by paying an imposed penalty K (cf. p.6 in [25]).

Hence, slightly deviating from "(2.5) in [25]", we yet define the *price of a EUA0 contract* via

(6.3.1)
$$C_T := \left(\hat{F}_T + K\right) \mathbb{1}_{\{\theta_T < 0\}} = \begin{cases} 0, & \theta_T \geq 0 \\ \hat{F}_T + K, & \theta_T < 0 \end{cases}$$

within a constant penalty value $K > 0$ (given in EURO). Thus, the EUA0 price C_T in (6.3.1) can be interpreted as a generalized contingent claim, respectively as a non-standard option on $\hat{F}$ [25]. In other words, a EUA0 contract may be considered as some kind of *exotic* option on a EUA1 contract.

As explained before, in [25] the market zone net position θ_t is modeled as a two-state Markov-chain merely taking values in the set $E := \{-1,1\}$, whereby the case $\theta_T = 1$ $(\theta_T = -1)$ in [25] corresponds to $\theta_T \geq 0$ $(\theta_T < 0)$ in our setup. For this reason, in the Cetin-Verschuere-approach (see the top of page 8 in [25]) the contingent claim C_T easily can be written in the extremely convenient form

(6.3.2)
$$C_T = \left(\hat{F}_T + K\right) \mathbb{1}_{\{\theta_T = -1\}} = \frac{1 - \theta_T}{2} \left(\hat{F}_T + K\right).$$

At this step, we notice a (preliminary) disadvantage of our innovative CPP approach (although the latter possibly appears more realistic from a modeling point of view), as the very tractable representation (6.3.2) unfortunately is no longer valid if we model the market zone net position by a linear combination of compound Poisson processes taking arbitrary many values in the compact set $[M_1, M_2] \subset \mathbb{R}$, instead of a net position with values in $E := \{-1,1\}$, solely. In fact, regarding the structure of our contingent claim (6.3.1), it seems hardly possible to express the latter within a similar notational form as in (6.3.2). However, the claim (6.3.1) not at all reveals a European-type structure (as commonly known from popular plain-vanilla options), though on a superficial sight rather similar, since inside the indicator function not the process $\hat{F}$ itself appears, but another process, namely θ.

Our key idea to overcome the just formulated problem reads as follows: We newly associate the contingent claim C_T as given in (6.3.1) within a customized real function f mapping

$$f\colon (0,\infty) \times [M_1, M_2] \longrightarrow [0,\infty)$$

whereas we concretely define

(6.3.3)
$$f(x_1, x_2) := (x_1 + K)\, \mathbb{1}_{\{x_2<0\}}$$

so that $f(\hat{F}_T, \theta_T) = C_T$ proves true instantly. Unfortunately, the function f is *not* integrable on the set $\mathcal{M} := (0,\infty) \times [M_1, M_2] \subset \mathbb{R}^2$ with respect to the two-dimensional Lebesgue measure λ^2, in symbols $f(x_1, x_2) \notin \mathcal{L}^1(\mathcal{M}, \lambda^2)$. However, for later purposes we introduce the exponentially damped function

(6.3.4)
$$q(x_1, x_2) := e^{-ax_1}\, f(x_1, x_2)$$

defined on $\mathcal{M}$ as well, within a real dampening parameter $a > 0$, yet obeying $q(x_1, x_2) \in \mathcal{L}^1(\mathcal{M}, \lambda^2)$ on the opposite.

6.3.1 Pricing EUA0 contracts with Fourier transform methods

In this subsection we derive risk-neutral prices for EUA0 contracts such as traded in the EU ETS market by applying a customized Fourier transform procedure (also compare paragraph 3.2.4 above). Starting off, with respect to (6.3.1), (6.3.3) and (6.3.4), we immediately obtain

(6.3.5)
$$C_T = f(\hat{F}_T, \theta_T) = e^{a\hat{F}_T}\, q(\hat{F}_T, \theta_T).$$

Moreover, in accordance to (2.4.3) [but with $d = 2$ therein], we receive

(6.3.6)
$$q(\hat{F}_T, \theta_T) = \frac{1}{(2\pi)^2} \int_{\mathcal{M}} \hat{q}(y_1, y_2)\, e^{i(y_1\hat{F}_T + y_2\theta_T)} d\lambda^2(y_1, y_2).$$

Next, for $0 \le t \le T$ the risk-neutral pricing formula [cf. (3.2.30)] can be written as

(6.3.7)
$$C_t = \frac{\beta_t}{\beta_T}\, \mathbb{E}_{\mathbb{Q}}(C_T|\mathcal{F}_t) = e^{-r(T-t)}\, \mathbb{E}_{\mathbb{Q}}\big(f(\hat{F}_T, \theta_T)\big|\mathcal{F}_t\big).$$

Appealing to (6.3.5), (6.3.6) and the Fubini-Tonelli theorem, the conditional expectation on the right hand side of (6.3.7) becomes

(6.3.8)
$$\mathbb{E}_{\mathbb{Q}}\big(f(\hat{F}_T, \theta_T)\big|\mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(e^{a\hat{F}_T}\, q(\hat{F}_T, \theta_T)\big|\mathcal{F}_t\big)$$

$$= \frac{1}{(2\pi)^2} \int_{\mathcal{M}} \hat{q}(y_1, y_2)\, \mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy_1)\hat{F}_T}\, e^{iy_2\theta_T}\big|\mathcal{F}_t\big)\, d\lambda^2(y_1, y_2).$$

Further on, the $\mathcal{F}_t$-measurability of $\hat{F}_t$ and θ_t together with the independent increment property of the two latter $\mathbb{Q}$-independent processes implies

(6.3.9) $\mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy_1)\hat{F}_T}\, e^{iy_2\theta_T}\big|\mathcal{F}_t\big) = e^{(a+iy_1)\hat{F}_t}\, e^{iy_2\theta_t}\, \mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy_1)\{\hat{F}_T-\hat{F}_t\}}\, e^{iy_2\{\theta_T-\theta_t\}}\big|\mathcal{F}_t\big)$

$$= e^{(a+iy_1)\hat{F}_t}\, e^{iy_2\theta_t}\, \mathbb{E}_{\mathbb{Q}}\big[e^{(a+iy_1)\{\hat{F}_T-\hat{F}_t\}}\big]\, \mathbb{E}_{\mathbb{Q}}\big[e^{iy_2\{\theta_T-\theta_t\}}\big] =: e^{(a+iy_1)\hat{F}_t}\, e^{iy_2\theta_t} \times \mathfrak{J}_1 \times \mathfrak{J}_2.$$

Meanwhile, we proceed with the computation of the Fourier transform $\hat{q}$ such as appearing inside (6.3.8): In accordance to (2.4.2), (6.3.3) and (6.3.4), we get

(6.3.10)

$$\hat{q}(y_1,y_2) = \int_{\mathcal{M}} (x_1+K)\, \mathbb{1}_{\{x_2<0\}}\, e^{-ax_1}\, e^{-i\{x_1y_1+x_2y_2\}}\, d\lambda^2(x_1,x_2)$$
$$= \int_{M_1}^{M_2} \mathbb{1}_{\{x_2<0\}}\, e^{-ix_2y_2} \int_0^{\infty} (x_1+K)\, e^{-(a+iy_1)x_1}\, dx_1\, dx_2$$
$$= \int_{M_1}^{0-} e^{-ix_2y_2}\, \frac{1+(a+iy_1)\,K}{(a+iy_1)^2}\, dx_2 = \frac{1+(a+iy_1)\,K}{(a+iy_1)^2} \times \frac{e^{-iM_1y_2}-1}{iy_2}.$$

Anyway, applying (6.2.6), (6.2.7), (6.2.15) and the Lévy-Khinchin formula [see Theorem 2.1.3], by common independency (and stationarity) arguments we next derive

(6.3.11)

$$\mathfrak{J}_2 := \mathbb{E}_{\mathbb{Q}}\big[e^{iy_2\{\theta_T-\theta_t\}}\big] = \prod_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[e^{iy_2 L^k_{T-t}}\right] = \prod_{k=1}^{n} e^{(T-t)\,\psi_k(y_2)}$$

within characteristic exponents

(6.3.12)

$$\psi_k(y_2) = \int_{D_k} \big[e^{iy_2z}-1\big]\, e^z\, d\nu_k(z).$$

What remains is the computation of the first multiplier $\mathfrak{J}_1$: Using (6.2.19) while exploiting standard conditioning methods (cf. e.g. the last equality of the proof of Prop. 10.4 in [13]), we deduce

(6.3.13)

$$\mathfrak{J}_1 := \mathbb{E}_{\mathbb{Q}}\big[e^{(a+iy_1)\{\hat{F}_T-\hat{F}_t\}}\big] = \mathbb{E}_{\mathbb{Q}}\left[e^{(a+iy_1)\left\{\frac{\hat{F}_T}{\hat{F}_t}-1\right\}\hat{F}_t}\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[exp\left\{(a+iy_1)\left[exp\left(\int_t^T \sigma(s)\, d\widetilde{W}_s - \frac{1}{2}\int_t^T \sigma(s)^2\, ds\right)-1\right]\varrho\right\}\right]_{\varrho:=\hat{F}_t}.$$

Furthermore, for $0 \le t \le T$ the stochastic process

(6.3.14)

$$X_{T-t} := \frac{\hat{F}_T}{\hat{F}_t} = exp\left\{\int_t^T \sigma(s)\, d\widetilde{W}_s - \frac{1}{2}\int_t^T \sigma(s)^2\, ds\right\}$$

is *log-normally* distributed under the EMM $\mathbb{Q}$ with *mean*

$$\mathbb{E}_{\mathbb{Q}}[X_{T-t}] = 1$$

and *variance*

$$\mathbb{V}\mathrm{ar}_{\mathbb{Q}}[X_{T-t}] = exp\left\{\int_t^T \sigma(s)^2 ds\right\} - 1.$$

As usual, we denote the latter fact by writing

$$\mathbb{Q}^{X_{T-t}} = LN\left(-\frac{\Sigma^2}{2}, \Sigma^2\right), \qquad \Sigma^2 := \int_t^T \sigma(s)^2 ds$$

in shorthand notation. Hence, adhering to similar measure-transformation/conditioning arguments as applied in e.g. the proof of Prop. 10.4 in [13], with respect to (6.3.13) and (6.3.14) we next obtain

(6.3.15)

$$\begin{aligned}\mathfrak{I}_1 = \mathbb{E}_{\mathbb{Q}}\left[e^{(a+iy_1)(X_{T-t}-1)\varrho}\right]_{\varrho:=\hat{F}_t} &= \int_{(0,\infty)} e^{(a+iy_1)(x-1)\varrho}\, d\mathbb{Q}^{X_{T-t}}(x)_{\varrho:=\hat{F}_t} \\ &= \int_{0+}^{\infty} \frac{e^{(a+iy_1)(x-1)\varrho}}{x\sqrt{2\pi\Sigma^2}} exp\left\{-\frac{1}{2}\left(\frac{ln(x)+\frac{\Sigma^2}{2}}{\Sigma}\right)^2\right\} dx_{\ \varrho:=\hat{F}_t} =: \delta\left(y_1, \hat{F}_t\right)\end{aligned}$$

which can be calculated further by standard numerical integration methods for Riemann integrals (see paragraph 19.3 in [19], for example). Merging (6.3.8), (6.3.9), (6.3.11) and (6.3.15) into (6.3.7), we finally end up with the expression

(6.3.16)

$$C_t = \frac{e^{-r(T-t)}}{(2\pi)^2} \int_{\mathcal{M}} \hat{q}(y_1, y_2)\, \delta\left(y_1, \hat{F}_t\right) e^{(a+iy_1)\hat{F}_t + iy_2\theta_t} \prod_{k=1}^{n} e^{(T-t)\psi_k(y_2)}\, d\lambda^2(y_1, y_2)$$

yielding the K-penalized EUA0 price at time t of a contingent claim paying C_T as given in (6.3.1) at the expiry date T, whereby $\hat{q}(y_1, y_2)$, $\psi_k(y_2)$ and $\delta\left(y_1, \hat{F}_t\right)$ are such as defined in (6.3.10), (6.3.12) and (6.3.15), respectively. In practical applications, the random processes $\hat{F}_t$ and θ_t appearing inside (6.3.16) naturally have to be simulated, whereas the two-dimensional $d\lambda^2$-integral over $\mathcal{M}$ must be evaluated numerically.

6.4 The minimum relative entropy measure

Due to the second fundamental theorem of asset pricing, in an incomplete market model we are facing several candidates for equivalent martingale measures and it is not a trivial question which one to choose (also recall Remark 6.2.6 in this context). Thus, our following considerations are dedicated to the *minimum relative entropy* methodology which provides an appropriate method to overcome (at least approximately) the above mentioned selection problem. We want to stress here that in [25] the so-called *Föllmer-Schweizer minimal martingale measure* (see [40] for details) is examined. Alternatively, in the present work we will stick to an adjusted relative entropy approach, whereas both onsets obviously succumb to a related minimization concept.

Following the idea of measuring some kind of *distance* between two equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$, parallel to the bottom of page 520 in [26] we define the so-called *relative entropy* of *any* measure $\mathbb{Q}$ with respect to the *fixed* market measure $\mathbb{P}$ as our approximation-error criterion via

(6.4.1)

$$\mathcal{E}\langle\mathbb{P}|\mathbb{Q}\rangle := \mathcal{E}\langle\mathbb{P}|\mathbb{Q}\rangle(t) := \mathbb{E}_{\mathbb{Q}}\left[ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}\right)\right].$$

Merging (2.2.2), (3.2.18) [but with $h_k(s,z) := z$ and $\rho_k(s) \equiv 1$ therein] and (6.2.12) into (6.4.1) while taking (6.2.8), (6.2.14) and (6.2.15) into account, we initially get

(6.4.2)

$$\mathcal{E}\langle\mathbb{P}|\mathbb{Q}\rangle = \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \left(\frac{G_s^2}{2} + \sum_{k=1}^{n} \int_{D_k} [z\, e^z + 1 - e^z]\, d\nu_k(z)\right) ds\right].$$

According to section 3.3 in [26] (see p.521), we announce right at the beginning that the problem of finding the precise EMM $\mathbb{Q}$ yielding *minimal* relative entropy forces us to minimize the above expression (6.4.2) with respect to the stochastic process G under the restrictive constraint (6.2.17). Roughly speaking, the measure of minimum relative entropy thus minimizes the distance between the true market measure $\mathbb{P}$ and its approximating candidate $\mathbb{Q}$ and therefore, delivers the [with respect to our specific error criterion (6.4.1)] *best possible* approach towards the verity $\mathbb{P}$ under the restrictive condition '*the discounted EUA1 forward price $\hat{F}$ has to form a local $\mathbb{Q}$-martingale*'.
Since $z\, e^z + 1 - e^z > 0$ is valid for all $z \in \mathbb{R} \setminus \{0\}$ while the Lévy-measures ν_k have been assumed to be positive on their supports $D_k \subseteq \mathbb{R} \setminus \{0\}$ for every index $k = 1, \dots, n$ anyway, the object

(6.4.3)

$$\Gamma := \sum_{k=1}^{n} \int_{D_k} [z\, e^z + 1 - e^z]\, d\nu_k(z)$$

appearing inside (6.4.2) designates a positive and deterministic constant which obviously is time-independent. Hence, an application of the Fubini-Tonelli theorem on (6.4.2) immediately yields

(6.4.4)

$$\mathcal{E}\langle\mathbb{P}|\mathbb{Q}\rangle = \int_0^t \left(\frac{\mathbb{E}_{\mathbb{Q}}[G_s^2]}{2}\right) ds + \Gamma t.$$

Further on, a substitution of the drift-condition (6.2.17) into (6.4.4) leads us to

(6.4.5)

$$\mathcal{E}\langle\mathbb{P}|\mathbb{Q}\rangle = \int_0^t \frac{\mathbb{E}_{\mathbb{Q}}[\theta_s^2] - 2\,\mu\,\mathbb{E}_{\mathbb{Q}}[\theta_s] + \mu^2}{2\,\sigma(s)^2} ds + \Gamma t.$$

In order to solve the upcoming minimization exercise explicitly, we assume the volatility function $\sigma(s)$ appearing inside (6.4.5) to be constant, i.e. $\sigma(s) \equiv \sigma > 0$, for the remainder of paragraph 6.4. Moreover, Theorem 2.1.3 and Lemma 2.1.4 together with (6.2.6), (6.2.7) and (6.2.15) [respectively, together with (6.3.11) and (6.3.12)] yield the first and second moments

(6.4.6)

$$\mathbb{E}_{\mathbb{Q}}[\theta_s] = s \sum_{k=1}^{n} \int_{D_k} z\, e^z d\nu_k(z),$$

$$\mathbb{E}_{\mathbb{Q}}[\theta_s^2] = -\frac{\partial^2}{\partial x^2}\left(\mathbb{E}_{\mathbb{Q}}\left[e^{ix\theta_s}\right]\right)_{x=0} = s \sum_{k=1}^{n} \int_{D_k} z^2 e^z d\nu_k(z) + s^2 \left(\sum_{k=1}^{n} \int_{D_k} z\, e^z d\nu_k(z)\right)^2.$$

Using (6.4.6) and the deterministic abbreviations

(6.4.7)

$$A := \sum_{k=1}^{n} \int_{D_k} z^2 e^z d\nu_k(z), \quad B := \sum_{k=1}^{n} \int_{D_k} z\, e^z d\nu_k(z),$$

for $0 \leq t \leq T$ property (6.4.5) presently transforms into

(6.4.8)

$$\mathcal{E}\langle \mathbb{P} | \mathbb{Q} \rangle = \frac{t}{2\sigma^2}\left[\frac{B^2}{3} t^2 + \frac{A - 2B\mu}{2} t + \mu^2\right] + \Gamma t$$

which may be interpreted as a deterministic function

$$(\mu, \sigma) \longmapsto \mathcal{E}\langle \mathbb{P} | \mathbb{Q} \rangle(\mu, \sigma) := \mathcal{E}(\mu, \sigma)$$

with arguments $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$. Hence, the minimum relative entropy framework forces us to *minimize* (6.4.8) with respect to μ and σ. Yet, the gradient vector of $\mathcal{E}$ reads as

(6.4.9)

$$\nabla\mathcal{E}(\mu, \sigma) := \left(\frac{\partial \mathcal{E}}{\partial \mu}, \frac{\partial \mathcal{E}}{\partial \sigma}\right)(\mu, \sigma) = \left(\frac{t}{2\sigma^2}[2\mu - Bt],\ \frac{t}{\sigma^3}\left[\frac{2B\mu - A}{2} t - \frac{B^2}{3} t^2 - \mu^2\right]\right)$$

so that (for $t > 0$) the first order condition $\nabla\mathcal{E}(\mu, \sigma) = \mathbf{0}$ yields the non-linear equality system

(6.4.10)

$$\mu = \frac{Bt}{2} \quad \wedge \quad \frac{2B\mu - A}{2} t - \frac{B^2}{3} t^2 - \mu^2 = 0$$

which unfortunately does not possess a unique solution for μ and neither for σ, as the latter has canceled out completely, anyway. In conclusion, our model does obviously not allow to minimize the function (6.4.8) over μ and σ *simultaneously*. [By the way, note that for $t = 0$ we have $\mathcal{E}(\mu, \sigma) = 0$ which can be associated with the *martingale modeling case* $\mathbb{P} = \mathbb{Q}$.] Nevertheless, for $t > 0$ and a *fixed* volatility coefficient σ the mapping $\mu \longmapsto \mathcal{E}(\mu)$ attains its minimum at the critical value

(6.4.11)

$$\mu^* := \frac{1}{2} \sum_{k=1}^{n} \int_0^t \int_{D_k} z\, e^z d\nu_k(z)\, ds$$

whereas $\mathcal{E}''(\mu^*) > 0$ holds whenever $0 < t \leq T$.

Putting (6.2.6), (6.2.7) and (6.4.11) into the drift condition (6.2.17), we ultimately end up with

(6.4.12)
$$G_t = G_t^* := \frac{1}{\sigma}\sum_{k=1}^{n}\int_0^t \int_{D_k} z\left\{dN_k(s,z) - \frac{e^z}{2} d\nu_k(z)\, ds\right\}$$

(yet for $0 \le t \le T$ again). In conclusion, the precise equivalent martingale measure (EMM), $\mathbb{Q}^*$ say, yielding *minimum* relative entropy (in the just discussed sense) is given through the Radon-Nikodym derivative (6.2.12) but within a stochastic process G^* such as defined in (6.4.12).

6.5 Emission allowances under enlarged filtrations

In this section we devote our attention towards the pricing of EUA0 contracts but yet with respect to some additional information about the market zone net position that an informed market insider may have knowledge of. More precise, in our upcoming considerations we will take forward-looking information about a selection of the jump-noises driving the stochastic net position process θ at some future time τ into account via an adequate enlargement of the underlying filtration. By the way, we emphasize that the just mentioned enlargement-of-filtration procedure constitutes the right opposite to the pricing framework under *incomplete* information such as treated in Chapter 4 in [25].

Once more, we here recall that the monotone increasing *retro* sigma algebras $\mathcal{F}_t$ such as defined in (6.2.10) only store *past* information that is available up to time t, actually. On the contrary, we now introduce the flow of supplementary *future* information concerning the market zone net position by an *enlarged* filtration: To be precise, we assume that a fictive *informed* EU ETS trader has an idea about the net position value θ_τ, respectively has guessed/established appropriate values for the pure-jump noises $L_\tau^1, \dots, L_\tau^n$ driving the net position (6.2.6) at the future time τ. [Note that it is *not* necessary to impose Condition A presently, since the jump noises in (6.2.7) are *Lévy* processes already, as we have chosen $h_k(s,z) := z$ and $\rho_k(s) \equiv 1$ in Ch. 6.] However, somewhat similar to our former approaches, for a time partition $0 \le t < \tau \le T$ we initially introduce an overall/global filtration $\mathcal{H}$ via

(6.5.1)
$$\mathcal{H}_t := \mathcal{F}_t \vee \sigma\{\theta_\tau\} := \mathcal{F}_t \vee \sigma\{L_\tau^1, \dots, L_\tau^n\}$$

representing the (rather unrealistic) case of having access to *complete* or *exhaustive* information at time $t\ (< \tau)$ about the market net position θ_τ where τ-forward-looking events are included. Slightly deviating from our main idea presented in subsection 3.3.2 [recall (3.3.38)], we next implement an explicit intermediate filtration $\mathcal{G}_t^*$ by using a subfamily of the components appearing in $\mathcal{H}_t$ defining

(6.5.2)
$$\mathcal{G}_t^* := \mathcal{F}_t \vee \sigma\{L_\tau^k : k \in \mathcal{N}\}$$

for an arbitrary index subset $\mathcal{N} \subseteq \{1, \dots, n\}$ and times $0 \le t < \tau \le T$. Then the properties

(6.5.3)
$$\mathcal{F}_t \subset \mathcal{G}_t^* \subset \mathcal{H}_t$$

for $t < \tau$ and $\mathcal{F}_t = \mathcal{G}_t^* = \mathcal{H}_t$ for $t \ge \tau$ are valid. The choice $\mathcal{N} = \{1, \dots, n\}$ in (6.5.2) corresponds to $\mathcal{G}_t^* = \mathcal{H}_t$ and thus, to *complete* knowledge of the market zone net position value at the future time τ. On the other hand, the case $\mathcal{N} \subset \{1, \dots, n\}$ actually represents a scenario wherein the EU ETS market insiders merely have access to some *restricted* additional information on the future net position behavior, sounding more realistically. Obviously, the trivial instance $\mathcal{N} = \emptyset$ implies $\mathcal{G}_t^* = \mathcal{F}_t$.

Eventually, we recall that the properties (3.3.39), (3.3.45), (3.3.47) – (3.3.49) and Lemma 3.5.1 [but altogether adjusted to the $\mathcal{N}$-notation yet] simultaneously hold true in our recent enlargement-of-filtration setup (6.5.1) – (6.5.3) associated to the EU ETS framework.

6.5.1 Pricing EUA0 contracts under future information on the market zone net position

In accordance to the pricing framework introduced in section 6.3 [particularly remind equality (6.3.7) therein], we define the K-penalized EUA0 price at time $t \in [0, T]$ of a contingent claim paying C_T as given in (6.3.1) at the (fixed) expiry date T yet under the filtration $\mathcal{G}_t^*$ [as implemented in (6.5.2)] via

(6.5.4)
$$C_t^* := e^{-r(T-t)} \, \mathbb{E}_{\mathbb{Q}}\big(f\big(\hat{F}_T, \theta_T\big)\big|\mathcal{G}_t^*\big).$$

By the way, referring to (6.3.7), (6.5.3) and (6.5.4), for $\tau \leq t \leq T$ we observe $C_t^* = C_t$, since $\mathcal{G}_t^* = \mathcal{F}_t$ holds whenever $t \geq \tau$. Thus, we merely need to treat (6.5.4) under the presumption $0 \leq t < \tau$ in the following. Applying Fourier transform methods like presented in subsection 6.3.1, we initially deduce

(6.5.5)
$$C_t^* = \frac{e^{-r(T-t)}}{(2\pi)^2} \int_{\mathcal{M}} \hat{q}(y_1, y_2) \, \mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy_1)\hat{F}_T + iy_2\theta_T}\big|\mathcal{G}_t^*\big) \, d\lambda^2(y_1, y_2)$$

within an inverse Fourier transform $\hat{q}$ such as claimed in (6.3.10).

At this step, let us remark that in general the market zone net position process θ does *not* possess independent increments with respect to $\mathcal{G}^*$. More precisely, $\theta_s - \theta_t$ and $\mathcal{G}_t^*$ are not necessarily $\mathbb{Q}$-independent for *arbitrary* time indices $0 \leq t < s \leq T$ (unless $\mathcal{N} = \emptyset$), as long as the instance $s = \tau$ is not excepted. (The latter fact directly becomes clear if we compare (6.2.6) with (6.5.2) wherein $\mathcal{N} \neq \emptyset$.) Fortunately, we have already presumed $t < \tau$ above, so that for a fixed expiry date T $(\neq \tau)$, $t \leq T$, the conditional expectation appearing inside (6.5.5) [similarly to (6.3.9)] yet factors into

$$\mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy_1)\hat{F}_T + iy_2\theta_T}\big|\mathcal{G}_t^*\big) = e^{(a+iy_1)\hat{F}_t + iy_2\theta_t} \times \delta\big(y_1, \hat{F}_t\big) \times \prod_{k \in \mathcal{N}} \mathfrak{I}_k^{\mathcal{G}^*} \times \prod_{k \in \mathcal{N}^c} \mathfrak{I}_k^{\mathcal{F}}$$

with $\mathcal{N}^c := \{1, \dots, n\} \setminus \mathcal{N}$ and multipliers

$$\mathfrak{I}_k^{\mathcal{G}^*} := \mathbb{E}_{\mathbb{Q}}\left[e^{iy_2\{L_T^k - L_t^k\}}\right], \qquad \mathfrak{I}_k^{\mathcal{F}} := \mathbb{E}_{\mathbb{Q}}\left[e^{iy_2 L_{T-t}^k}\right] = e^{(T-t)\,\psi_k(y_2)}$$

where $\psi_k(y_2)$ and $\delta\big(y_1, \hat{F}_t\big)$ are such as defined in (6.3.12), respectively in (6.3.15). Unfortunately, a proper *analytical* handling of $\mathfrak{I}_k^{\mathcal{G}^*}$ does not seem to be achievable, since the contained Lévy processes L^k $(k \in \mathcal{N})$ even inside those *usual* expectations (as some kind of 'heritage') still *associate* with $\mathcal{G}^*$: more precisely, the corresponding (actually stochastic) martingale compensators presently have to be taken with respect to $\mathcal{G}^*$ [remind (3.3.47) – (3.3.49) along with the epilog of (6.5.3) in this context]. In other words, the appearing objects $\mathfrak{I}_k^{\mathcal{G}^*} = \mathcal{E}_{\mathbb{Q}}^{\mathcal{G}^*}(t, T, D_k; iy_2; N_k)$ [recall definition (3.5.8)] essentially belong to the 'stubborn' class of forward-looking usual expectations such as examined in paragraph 3.5.1. Anyway, inspired by Excursus A, we now propose an alternative treatment of the conditional expectation inside (6.5.5) while applying an approximation technique involving complex Taylor-estimates. For notational convenience, we preliminarily introduce the complex stochastic process

(6.5.6)
$$H_t := (a + iy_1)\,\hat{F}_t + iy_2\,\theta_t.$$

Starting off, we make use of a (complex) Taylor-approximation of order one[66] to obtain

(6.5.7) $$\mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy_1)\hat{F}_T+iy_2\theta_T}\middle|\mathcal{G}_t^*\right) \approx$$

$$\sum_{\nu=0}^{1}\frac{\mathbb{E}_{\mathbb{Q}}((H_T)^\nu|\mathcal{G}_t^*)}{\nu!} = 1+(a+iy_1)\,\mathbb{E}_{\mathbb{Q}}\left(\hat{F}_T\middle|\mathcal{G}_t^*\right)+iy_2\,\mathbb{E}_{\mathbb{Q}}(\theta_T|\mathcal{G}_t^*).$$

Next, taking (6.5.2) and (6.2.18) into account, we receive

(6.5.8) $$\mathbb{E}_{\mathbb{Q}}\left(\hat{F}_T\middle|\mathcal{G}_t^*\right)=\mathbb{E}_{\mathbb{Q}}\left(\hat{F}_T\middle|\mathcal{F}_t\right)=\hat{F}_t.$$

Moreover, with respect to (6.2.6) and (6.5.2), we may decompose

(6.5.9)

$$\mathbb{E}_{\mathbb{Q}}(\theta_T|\mathcal{G}_t^*)=\mathbb{E}_{\mathbb{Q}}\left(\sum_{k\in\mathcal{N}}L_T^k\middle|\mathcal{G}_t^*\right)+\mathbb{E}_{\mathbb{Q}}\left(\sum_{k\in\mathcal{N}^c}L_T^k\middle|\mathcal{F}_t\right).$$

Further on, appealing to (3.3.45), (3.3.47) – (3.3.49), (6.2.7), (6.2.15), Lemma 3.5.1 and the tower property, the first conditional expectation on the right hand side of (6.5.9) transforms into

(6.5.10)

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(\sum_{k\in\mathcal{N}}L_T^k\middle|\mathcal{G}_t^*\right)&=\sum_{k\in\mathcal{N}}\mathbb{E}_{\mathbb{Q}}\left(L_T^k-L_t^k\middle|\mathcal{G}_t^*\right)+\sum_{k\in\mathcal{N}}L_t^k=\sum_{k\in\mathcal{N}}\mathbb{E}_{\mathbb{Q}}\left(\int_t^T\int_{D_k}z\,d\nu_k^{\mathcal{G}^*,\mathbb{Q}}(s,z)\middle|\mathcal{G}_t^*\right)+\sum_{k\in\mathcal{N}}L_t^k\\
&=\sum_{k\in\mathcal{N}}\int_t^{\tau-}\frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k-L_s^k\middle|\mathcal{G}_t^*\right)}{\tau-s}ds+\sum_{k\in\mathcal{N}}\int_\tau^T\frac{\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k-L_s^k\middle|\mathcal{G}_t^*\right)}{\tau-s}ds+\sum_{k\in\mathcal{N}}L_t^k\\
&=\sum_{k\in\mathcal{N}}\int_t^{\tau-}\frac{L_\tau^k-L_t^k}{\tau-t}ds-\sum_{k\in\mathcal{N}}\int_\tau^T\frac{\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(L_s^k-L_\tau^k\middle|\mathcal{F}_\tau\right)\middle|\mathcal{G}_t^*\right)}{\tau-s}ds+\sum_{k\in\mathcal{N}}L_t^k\\
&=\sum_{k\in\mathcal{N}}L_\tau^k-\sum_{k\in\mathcal{N}}\int_\tau^T\frac{\mathbb{E}_{\mathbb{Q}}[L_s^k-L_\tau^k]}{\tau-s}ds=\sum_{k\in\mathcal{N}}\left(L_\tau^k+(T-\tau)\int_{D_k}z\,e^z\,d\nu_k(z)\right)
\end{aligned}$$

wherein we have just used the fact $\mathcal{G}_t^*\subset\mathcal{G}_\tau^*=\mathcal{F}_\tau$ for $t<\tau\ (\le s\le T)$, $T\neq\tau$, in connection with the iterated conditioning step. On the other hand, referring to (6.2.7), (6.2.10) and (6.2.15), the second conditional expectation on the right hand side of equation (6.5.9) points out as

(6.5.11)

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{k\in\mathcal{N}^c}L_T^k\middle|\mathcal{F}_t\right)=\sum_{k\in\mathcal{N}^c}\left(L_t^k+(T-t)\int_{D_k}z\,e^z d\nu_k(z)\right).$$

[66] Recall Excursus A, particularly (A.6), at this step [although we will not apply secant estimations as presented therein now]. Admittedly, using Taylor-polynomials of higher orders (than one) certainly would yield a better approximation in (6.5.7), whereas the handling of the incoming conditional expectations then becomes extremely longwinded, unfortunately. However, the author has done the corresponding computations for a Taylor-polynomial of order two while applying Itô's formula on $(H_T)^2$. Nevertheless, the underlying techniques should become clearer in the *tangent-plane case* presented above, not at least as it yields a much better overview.

Hence, substituting (6.5.10) and (6.5.11) into (6.5.9), we derive

(6.5.12)

$$\mathbb{E}_{\mathbb{Q}}(\theta_T|\mathcal{G}_t^*) = \sum_{k\in\mathcal{N}} \left(L_\tau^k - \xi_k(\tau)\right) + \sum_{k\in\mathcal{N}^c} \left(L_t^k - \xi_k(t)\right) + \sum_{k=1}^{n} \xi_k(T)$$

wherein we have recently introduced the deterministic abbreviation

$$\xi_k(t) := t \int_{D_k} z\, e^z\, d\nu_k(z).$$

Finally, merging (6.5.8) and (6.5.12) into (6.5.7), we end up with the estimate

$$\mathbb{E}_{\mathbb{Q}}\left(e^{(a+iy_1)\hat{F}_T+iy_2\theta_T}\middle|\mathcal{G}_t^*\right) \approx V_t(\tau,T;y_1,y_2) \tag{6.5.13}$$

within a complex stochastic process

$$V_t := V_t(\tau,T;y_1,y_2) := \tag{6.5.14}$$

$$1 + (a+iy_1)\,\hat{F}_t + iy_2\left[\sum_{k\in\mathcal{N}} \left(L_\tau^k - \xi_k(\tau)\right) + \sum_{k\in\mathcal{N}^c} \left(L_t^k - \xi_k(t)\right) + \sum_{k=1}^{n} \xi_k(T)\right].$$

We remark that in practical applications the stochastic ingredients of V_t, such as $\hat{F}_t$ and L_t^k $(k \in \mathcal{N}^c)$, have to be simulated numerically, whereas the involved values L_τ^k $(k \in \mathcal{N})$ have to be *guessed*, respectively *established*, from the additionally available future information concerning the market zone net position at time τ, namely θ_τ [recall the precise definition of the intermediate filtration $\mathcal{G}_t^*$ in (6.5.2) and of the market zone net position process in (6.2.6) in this context].
All in all, the K-penalized $\mathcal{G}^*$-forward-looking EUA0 price at time t $(< \tau)$ of a contingent claim paying C_T such as given in (6.3.1) at the expiry date T can be estimated via

(6.5.15)

$$C_t^* \approx \frac{e^{-r(T-t)}}{(2\pi)^2} \int_{\mathcal{M}} \hat{q}(y_1,y_2)\, V_t(\tau,T;y_1,y_2)\, d\lambda^2(y_1,y_2).$$

Herein, the (deterministic) inverse Fourier transform $\hat{q}$ is defined like in (6.3.10), whereby the two-dimensional Lebesgue integral over $\mathcal{M}$ can be evaluated by standard numerical integration methods. Eventually, we propose a rigorous (numerical) comparison of (6.5.15) with (6.3.16) to examine the precise effects of supplementary future information on emission allowance prices in more detail.

6.6 A Brownian mean-reverting market zone net position model

In order to obtain an alternative EUA0 pricing formula descending from a continuous net position model (without jumps), we now propose a Brownian motion (BM) driven approach to describe the dynamics of the latter. To be precise, we innovatively model the market zone net position by a continuous zero-reverting multi-factor Ornstein-Uhlenbeck disposition, such as formerly supposed in Remark 6.2.5. Referring to (6.2.11), we thus replace equation (6.2.6) through

(6.6.1)
$$d\hat{\theta}_t = -\alpha\, \hat{\theta}_t\, dt + \sum_{k=1}^{n} \zeta_k\, dB_t^k$$

whereas we presume (without loss of generality) that the market zone net position is *in equilibrium* at time $t = 0$, i.e. $\hat{\theta}_0 = 0$. Moreover, the constant parameter $\alpha > 0$ appearing inside (6.6.1) denotes the mean-reversion velocity, $\zeta_1, \dots, \zeta_n$ are deterministic and strictly positive volatility coefficients and $B_t^1, \dots, B_t^n$ designate standard BMs under $\mathbb{P}$ which we assume both to be pair-wise independent and independent of the EUA1 price driving noise W_t. Again, we suppose the technical assumption $\hat{\theta}_t \in [M_1, M_2] \subset \mathbb{R}$ to be valid $\mathbb{P}$-almost-sure for all $t \in [0, T]$ (see eq. (6.6.10) ff. in this context). Somehow similar to (6.2.12), we next implement an EMM $\widetilde{\mathbb{Q}}$ due to the Radon-Nikodym derivative

(6.6.2)
$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\tilde{\mathcal{F}}_t} \coloneqq \mathfrak{E}\big(\hat{G}_{\cdot} \circ W_{\cdot}\big)_t \times \prod_{k=1}^{n} \mathfrak{E}\big(G_k(\cdot) \circ B_{\cdot}^k\big)_t$$

with multipliers $\mathfrak{E}\big(\hat{G}_{\cdot} \circ W_{\cdot}\big)_t$ as defined in (2.2.2) [but with G therein replaced by $\hat{G}$ to avoid a double-notation here], $\mathfrak{E}\big(G_k(\cdot) \circ B_{\cdot}^k\big)_t$ as introduced in (3.3.71) and a new initial filtration

(6.6.3)
$$\tilde{\mathcal{F}}_t \coloneqq \sigma\{W_s, B_s^1, \dots, B_s^n : 0 \le s \le t\}.$$

Further on, the solution of (6.6.1) under $\mathbb{P}$ obviously is given by

(6.6.4)
$$\hat{\theta}_t = \sum_{k=1}^{n} \zeta_k \int_0^t e^{-\alpha(t-s)}\, dB_s^k.$$

Moreover, taking Girsanov's Change-of-Measure theorem into account [cf. paragraph 2.2, respectively eq. (3.3.110)], the latter equality transforms into the $\widetilde{\mathbb{Q}}$-representation

(6.6.5)
$$\hat{\theta}_t = \sum_{k=1}^{n} \zeta_k \int_0^t e^{-\alpha(t-s)}\, G_k(s)\, ds + \sum_{k=1}^{n} \zeta_k \int_0^t e^{-\alpha(t-s)}\, d\tilde{B}_s^k$$

within $\big(\tilde{\mathcal{F}}, \widetilde{\mathbb{Q}}\big)$-BMs $\tilde{B}^1, \dots, \tilde{B}^n$. Referring to the notations and derivation methodologies of section 6.3 [particularly recall (6.3.7) – (6.3.9) therein], we announce the (actually *backward-looking*) K-penalized EUA0 price $\hat{C}_t$ of a contingent claim (6.3.1) with expiry date T, but yet associated to our recent BM-driven mean-reverting market zone net position model (6.6.1), to be of the form

(6.6.6)
$$\hat{C}_t = \frac{e^{-r(T-t)}}{(2\pi)^2} \int_{\mathcal{M}} \hat{q}(y_1, y_2)\, e^{(a+iy_1)\hat{F}_t}\, e^{iy_2\hat{\theta}_t} \times \mathfrak{N}_1 \times \mathfrak{N}_2\, d\lambda^2(y_1, y_2)$$

wherein $\hat{q}$ is such as defined in (6.3.10). In accordance to (6.3.13) and (6.3.15), we firstly observe

(6.6.7)
$$\mathfrak{N}_1 \coloneqq \mathbb{E}_{\widetilde{\mathbb{Q}}}\big[e^{(a+iy_1)\{\hat{F}_T - \hat{F}_t\}}\big] = \delta\big(y_1, \hat{F}_t\big).$$

Secondly, with respect to (6.6.5), we deduce

(6.6.8)

$$\mathfrak{N}_2 := \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{iy_2\{\hat{\theta}_T-\hat{\theta}_t\}}\right] = \Psi(t,T) \times \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\prod_{k=1}^{n} exp\left\{iy_2\,\zeta_k\left(\Upsilon(t,T)\int_0^t e^{\alpha s}\,d\tilde{B}_s^k + e^{-\alpha T}\int_t^T e^{\alpha s}\,d\tilde{B}_s^k\right)\right\}\right]$$

wherein we have just introduced the deterministic abbreviations $\Upsilon(t,T) := e^{-\alpha T} - e^{-\alpha t}$ and

$$\Psi(t,T) := exp\left\{iy_2\sum_{k=1}^{n}\zeta_k\left(\int_0^T e^{-\alpha(T-s)}\,G_k(s)\,ds - \int_0^t e^{-\alpha(t-s)}\,G_k(s)\,ds\right)\right\}.$$

Meanwhile, note that for every $k \in \{1,\dots,n\}$ the $d\tilde{B}_s^k$-integrals appearing on the right hand side of (6.6.8) are $\widetilde{\mathbb{Q}}$-independent of each other. Thus, the involved usual expectation factors what leads us [within an application of Itô's isometry] to

$$\mathfrak{N}_2 = \Psi(t,T) \times exp\left\{y_2^2\,\frac{(1-e^{2\alpha t})\,\Upsilon^2(t,T) + e^{2\alpha(t-T)} - 1}{4\alpha}\sum_{k=1}^{n}\zeta_k^2\right\} =: \eta(y_2,t,T).$$

In conclusion, equation (6.6.6) points out as

(6.6.9)

$$\hat{C}_t = \frac{e^{-r(T-t)}}{(2\pi)^2}\int_{0+}^{\infty}\delta(y_1,\hat{F}_t)\,e^{(a+iy_1)\hat{F}_t}\left(\int_{M_1}^{M_2}\hat{q}(y_1,y_2)\,\eta(y_2,t,T)\,e^{iy_2\hat{\theta}_t}\,dy_2\right)dy_1$$

yielding the (backward-looking) EUA0 price at time $t \in [0,T]$ associated to our innovative BM-driven Ornstein-Uhlenbeck net position model (6.6.1). Ultimately, we aim to compute the probability

(6.6.10)

$$\mathbb{P}\big(\hat{\theta}_t \in [M_1,M_2]\big) = \mathbb{P}\left(M_1 \le \sum_{k=1}^{n}\int_0^t \zeta_k\,e^{-\alpha(t-s)}dB_s^k \le M_2\right), \qquad t \in [0,T].$$

For this purpose, we recall that $\hat{\theta}_t$ is normally distributed under $\mathbb{P}$ with zero mean and variance

(6.6.11)

$$\Sigma^2(t) := \frac{1-e^{-2\alpha t}}{2\alpha}\sum_{k=1}^{n}\zeta_k^2$$

whereby we have used Itô's isometry and the pair-wise $\mathbb{P}$-independence of $B^1,\dots,B^n$ in (6.6.11). We denote the latter distributional property by writing $\hat{\theta}_t \sim N\big(0,\Sigma^2(t)\big)$. Hence, (6.6.10) becomes

(6.6.12)

$$\mathbb{P}\big(\hat{\theta}_t \in [M_1,M_2]\big) = \Phi\left(\frac{M_2}{\Sigma(t)}\right) - \Phi\left(\frac{M_1}{\Sigma(t)}\right)$$

wherein Φ constitutes the standard normal distribution function and $t \in [0,T]$.

In practice, equation (6.6.12) ought to be a rather helpful indicator for choosing appropriate bounds $M_1 < 0$ and $M_2 > 0$, since applicants might exemplarily require a probability like

$$\mathbb{P}\big(\hat{\theta}_t \in [M_1, M_2]\big) > 0.99 \;\; \forall t \in [0, T]$$

for their underlying model setup. Particularly, for our *Brownian* multi-factor mean-reverting net position model (6.6.1) it is moreover possible to compute the probability for the EU ETS market to end up *long* $\big(\hat{\theta}_T > 0\big)$, respectively *short* $\big(\hat{\theta}_T < 0\big)$, as we – not surprisingly – observe the symmetry

(6.6.13)
$$\mathbb{P}\big(\hat{\theta}_T > 0\big) = \mathbb{P}\big(\hat{\theta}_T < 0\big) = \Phi(0) = 0.5.$$

Consequently, there presently seems to be no possibility which allows for an adjustment of our recent model (6.6.1) in the sense of incorporating *asymmetrical* drift effects – such as discussed previously to Remark 6.2.4 in connection with our former pure-jump case study.

Nevertheless, if we permit a *non-vanishing* deterministic initial value $\hat{\theta}_0$ in (6.6.1), say $\hat{\theta}_0 > 0$ [i.e. we exemplarily presume the EU ETS market to start *long* in the following], then the $\mathbb{P}$-solution in (6.6.4) translates into

(6.6.14)
$$\hat{\theta}_t = \hat{\theta}_0 \, e^{-\alpha t} + \sum_{k=1}^{n} \zeta_k \int_0^t e^{-\alpha(t-s)} \, dB_s^k \;\sim\; N\left(\hat{\theta}_0 \, e^{-\alpha t}, \Sigma^2(t)\right)$$

what leads us to

(6.6.15)
$$\mathbb{P}\big(\hat{\theta}_t < 0\big) = 1 - \Phi\left(\frac{\hat{\theta}_0 \, e^{-\alpha t}}{\Sigma(t)}\right), \qquad \mathbb{P}\big(\hat{\theta}_t > 0\big) = \Phi\left(\frac{\hat{\theta}_0 \, e^{-\alpha t}}{\Sigma(t)}\right)$$

for all $t \in [0, T]$. Evidently, the two latter probabilities *differ* (not only for $\hat{\theta}_0 > 0$ but) whenever $\hat{\theta}_0 \neq 0$. Thus, *long* resp. *short* EU ETS market net positions (at time t) finally occur – in contrast to (6.6.13) – with different/asymmetrical probabilities. Furthermore, the probability for the *equilibrium event* $\big\{\hat{\theta}_t = 0\big\}$ may be approximated via the elementary De-Moivre-Laplace formula yielding

(6.6.16)
$$\mathbb{P}\big(\hat{\theta}_t = 0\big) \approx \frac{1}{\Sigma(t)} \, \varphi\left(\frac{\hat{\theta}_0 \, e^{-\alpha t}}{\Sigma(t)}\right)$$

wherein $\varphi(\cdot)$ denotes the standard normal density function.

Eventually, a proper application of *Strassen's invariance principle*, respectively *Strassen's iterated logarithm theorem* (see [5]: Chapter VII, §33, along with Chapter IX, §47 and section 3 in §51), should yield some further insight into the most likely behavior of the market zone net position process (6.6.14). Anyway, we leave such examinations for future work. Instead, we close with the following remark.

Remark 6.6.1 *We finally emphasize that throughout this thesis we essentially have been confronted with four different types of forward-looking conditional expectations associated to (option) pricing purposes under enlarged filtrations: The first kind appears in (3.3.53), (3.3.58) and (5.3.22), for which – in the lack of any independent increment property of the involved processes [which explicitly depend* *on the $(\mathcal{G}^*, \mathbb{Q})$-compensated random measures $\widetilde{N}_k^{\mathcal{G}^*,\mathbb{Q}}$] – we have proposed tailor-made approximation techniques such as presented in Excursus A and B. The second type can be found in (3.3.64), (3.3.102) and (5.3.31), wherein we particularly had to distinguish between different scenarios concerning the future information time parameter τ. Thirdly, in (3.3.96), (3.3.129), (5.3.48) and (6.6.8) we have dealt with rather convenient Brownian motion cases; the same is valid for the trivial instances (3.3.149) and (3.3.158), by the way. Fourthly, the situation in (6.5.5) actually resembles the second case (at least on a superficial sight) but – as we have seen above – has to be handled with more care, as there is a combination of Brownian and pure-jump noises involved. For the sake of completeness, we eventually remind (4.3.21), whereas this object does not possess any option pricing background and also exposes a completely different structure with respect to the aforementioned conditional expectations.* ■

6.7 Conclusions

In this chapter we have discussed the pricing of carbon emission allowances such as traded in the EU ETS market both under common knowledge and under supplementary forward-looking information about the market zone net position. In the second insider trading case, we have rigorously taken customized enlargement-of-filtration techniques into account. Additionally, we have modeled the net position of the ETS market by a linear combination of multiple compound Poisson processes taking values in a compact real interval. Thus, our approach essentially extends the two-state Markov-chain disposition presented in [25] and moreover, yields the reasonable opportunity of indicating *how long*, respectively *how short*, the EU ETS market overall net position precisely is. As a consequence, our innovative compound Poisson multi-state setup yet requires two-dimensional Fourier transform methods along with exponential dampening arguments when it comes to the pricing of EUA0 contracts. Fortunately, in our model it is easily possible to incorporate *asymmetrical* drift effects by choosing tailor-made jump-size distributions for the market zone net position process. At this point, we recall from [25] that those non-symmetrical drift changes may be utilized to emphasize 'delicate' market scenarios such as 'trouble-making' emission permit shortages more strongly in contrast to rather harmless surplus or equilibrium situations. Subsequently, we have turned our attention towards a minimum relative entropy procedure in order to determine a concrete EMM in our incomplete model.

Some challenging related research topics might consist in a numerical study of our multi-state CPP model similar to the examinations in Chapter 6 and 7 in [25]. Especially, it would be interesting to visualize the effects of forward-looking information (about the market zone net position) on the EUA0 contract prices such as derived in (6.5.15) by numerical simulations. In this context, a proper comparison of simulated EUA0 price trajectories descending from equation (6.3.16) on the one hand, and from (6.5.15) on the other, should yield some new insights concerning the question of how forward-looking insider information may affect the price formation in the EU ETS market. Similarly, an accurate illustration of the expected *time-delay-feature* related to a possible market adjustment period – such as mentioned in Remark 6.2.4 before – by using simulative instruments should be very suggestive. In addition, it might be worthwhile to model the market zone net position as a zero-reverting Ornstein-Uhlenbeck process like proposed in section 6.6 but yet under the incorporation of future information concerning the involved Brownian motion noises (similarly to the procedure presented in e.g. paragraph 3.3.6 and 3.3.9) and hereafter, derivate the corresponding *forward-looking* pricing equations for the underlying (continuous) diffusion setup. Last but not least, the incorporation of emission permits *banking*, as "*allowed during and after the second phase of [the] Kyoto protocol*" (see p.17 in [25]), should embody another reasonable extension of our model.

Chapter 7

Explicit Pricing Measures for Commodity Forwards in a Heath-Jarrow-Morton-Framework with Jumps

7.1 A short chapter overview

The creation of competitive commodity exchanges, wherein gas, oil or coal (just to nominate a few) are traded somehow similar to financial products in ordinary stock markets, has brought up new mathematical challenges concerning the risk-neutral pricing of commodity derivatives. In the present chapter, we will exploit obvious similarities between commodity derivative contracts and forward rate theory (see [13], [17], [18], [33], [36], [49], [51], [59], [65], [79], [83]) and draw the corresponding conclusions for the creation of an appropriate commodity forward market model. More precisely, we here aim to compute risk-neutral option prices for commodity derivatives on the basis of an extended Heath-Jarrow-Morton (HJM) setup, whereas the presence of random jumps in our underlying forward rate onset requires Fourier transform techniques. By the way, we derive an extended HJM-drift-restriction dedicated to our jump-diffusion approach and the concepts of Esscher transforms and minimum relative entropy are adapted to our purposes in order to determine a concrete EMM out of the large class of offering pricing probabilities in the present incomplete market model.

As we have seen in section 1.1, respectively in Chapter 3, in contrast to energy derivatives associated to *storable* commodities like e.g. coal or oil, we remind that *electricity* futures contracts in particular possess the distinctive feature of yielding a delivery during a future time *span*, the so-called *delivery period*, rather than at a fixed maturity date, since electricity is non-storable. Hence, the basic products in electricity markets are options written on electricity futures/swap contracts which are settled over a future *period* of time [13]. Anyway, in the present chapter we use a Heath-Jarrow-Morton (HJM) approach (as firstly introduced in [49]) taken from interest rate theory which seems to be appropriate for the modeling of commodity forward contracts delivering at a *fixed maturity time* (cf. Ch. 6 in [13]).

The remainder of the current chapter is organized as follows: In section 7.2 our underlying mathematical basis is established and a selection of facts from interest rate theory is recalled. Yet, in order to derive the corresponding HJM-equations dedicated to our extended jump approach, we newly make use of the *Leibniz-rule for parameter integrals* in this work and therewith provide an alternative derivation procedure in contrast to the acquainted techniques presented in [17] and [18]. Applying Girsanov's Change-of-Measure theorem, we hereafter obtain an extended HJM-drift-restriction linked to our jump-diffusion case. Since in incomplete market models the equivalent martingale measure (EMM) cannot be uniquely chosen, we have to invest some additional effort concerning the manifestation of an (in a certain sense) *optimal* EMM. In this context, a generalized Esscher transform is introduced in the subsequent paragraph 7.3 and furthermore tailored to our requirements. Moreover, dealing with the relative entropy idea, we utilize a customized *successive Lagrange approach* to receive the explicit 'minimizing pair-process' leading to another *optimal* EMM (in the sense of yielding *minimal* relative entropy). In subsection 7.4 the price for a European commodity call option is obtained, whereas the occurrence of random jumps in the underlying power forward prices requires a rigorous application of Fourier transform techniques.

7.2 Modeling power forward prices

We start with the description of the mathematical disposition of our underlying commodity forward model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered and complete probability space, whereas the information filtration $\mathcal{F}$ (which we assume to include a priori all $\mathbb{P}$-null-sets) is assumed to be *cad* (French: continue à droite)[67]. Furthermore, all appearing stochastic processes in the following are assumed to be $\mathcal{F}$-adapted and to fulfill the usual integration assumptions.

7.2.1 The extended Heath-Jarrow-Morton approach with jumps

Right from the beginning, we devote our attention towards the description of the upcoming forward price dynamics. For this purpose, let us determine the time frame of an underlying commodity forward contract via $0 \le t \le \tau \le T$, where τ denotes the *exercise time*[68] and T represents the *delivery date* in return (also see the explanations dedicated to Th. 7.4.1 below). Further, in this chapter we will make use of an extended HJM-approach to model the *forward* prices $f_t(\tau)$ *directly*, instead of modeling the commodity *spot* price *first*, and deriving the corresponding forward price dynamics *afterwards* (cf. [13], [14]) – such as presented at the beginning of section 3.2.3 for the electricity futures case.
In this regard, the reader is advised to the first lines of Chapter 1 in [14], wherein two major disadvantages of (actually *electricity*) *spot* price models are presented: Firstly, the difficulty of giving a *precise* definition of (electricity) *spot* prices and, secondly, the lack of a straightforward connection in between the (electricity) spot price on the one hand, and forward/futures prices on the other.

---

[67] See the beginning of section 3.2 for a precise definition of *cad* sigma algebras.

[68] For *infinitely many* values $\tau \in [0, T]$ our upcoming HJM-jump-diffusion model actually would be *complete* in the sense of yielding a *unique* EMM, as the involved noises might be recovered by *arbitrary many* forward contracts $(f_.(\tau): \tau \in [0, T])$. Nevertheless, with respect to the common commodity exchange practice, we assume that there are only *finitely many* forwards traded in the time interval $[0, T]$, i.e. we are facing finitely many exercise times only, what makes our model become *incomplete* yet. We even might presume $\tau \in [0, T]$ to be fixed right from the beginning and hence, regard only *one* commodity forward contract in the following.

Moreover, on the bottom of p.4 in [10] it is argued that "*For the case of electricity, the exact relation between the spot and forward is not clear.*". Similarly, on p.32 in [13] Benth et al. declare the loss of the connection between a HJM-modeled (electricity) forward/futures price and the underlying spot price as a "*possibly undesirable consequence of the HJM approach for electricity futures price modeling*". On the same page they further announce "*Given an electricity futures price dynamics, one cannot trace back a spot price dynamics except in trivial and not relevant cases. This is a serious matter [...], since the spot is namely the reference index for the futures.*". To read more about the modeling of commodity forward contracts via HJM-approaches the interested reader particularly is advised to Ch. 6 in [13]. By the way, in section 6.3 therein the authors provide a possible extension of the HJM-*forward*-approach to *futures/swap* contracts (delivering the underlying over a time *span*) by simply integrating the forward price over the delivery period (cf. eq. "(6.8) in [13]").
Inspired by Björk, Di Masi, Kabanov and Runggaldier [17], we adjust their setting by adding two generalized compound Poisson process parts to the primitive Brownian HJM model in [49], whereas we differ between small and large jump-sizes in our approach. In contrast to [17] and [18], we newly receive the commodity forward price evolution equation by using the *Leibniz-rule for parameter integrals* (see Lemma 2.4.1 above), instead of troubling solely and elaborately the stochastic Fubini-Tonelli theorem (as on p.16 in [17], resp. on pp. 9-11 in [18]). Starting off, we suppose the stochastic differential equation (SDE) describing the *forward rates* $f_t(\tau)$ under the probability measure $\mathbb{P}$ to admit a *càdlàg* representation that possesses the following fairly general Lévy-type structure

(7.2.1)

$$df_t(\tau) = \alpha_t(\tau)\, dt + \sigma_t(\tau)\, dW_t + \int_{|x|\geq 1} \beta_{t-}(x,\tau)\, N(t,dx) + \int_{|x|<1} \gamma_{t-}(x,\tau)\, \widetilde{N}_{\mathbb{P}}(t,dx)$$

for times $0 \leq t \leq \tau \leq T$ (cf. eq. "(3) in [18]"). Consequently, the equivalent integral form reads as

(7.2.2)

$$f_t(\tau) = f_0(\tau) + \int_0^t \alpha_s(\tau)\, ds + \int_0^t \sigma_s(\tau)\, dW_s + \int_0^t \int_{|x|\geq 1} \beta_{s-}(x,\tau)\, dN(s,x) + \int_0^t \int_{|x|<1} \gamma_{s-}(x,\tau)\, d\widetilde{N}_{\mathbb{P}}(s,x)$$

wherein the initial condition $f_0(\tau)$ represents *today's* forward rate, W designates a one-dimensional standard Brownian motion (BM) under $\mathbb{P}$ and $\widetilde{N}_{\mathbb{P}}$ stands for a one-dimensional $\mathbb{P}$-compensated integer-valued Poisson-Random-Measure (PRM) on $[0,\tau] \times \mathbb{R}_0$ with $\mathbb{R}_0 \coloneqq \mathbb{R} \setminus \{0\}$.[69]

Remark 7.2.1 *For the sake of notational simplicity and to simplify a focusing on the main ideas, we restrict ourselves to the one-dimensional case here, remarking that the multi-dimensional modeling of the noises W and N is not essential to understand the theoretical and practical gain of our enlarged jump setup. The extension to higher-dimensional cases does not require any essential new ideas and is of technical character only.* ■

Further, the $\mathbb{P}$-compensator of $dN(s,x)$ is denoted by $d\nu(x)ds$ which is chosen such as

(7.2.3) $$d\widetilde{N}_{\mathbb{P}}(s,x) \coloneqq dN(s,x) - d\nu(x)\, ds$$

depicts a $\mathbb{P}$-martingale integrator.

[69] We denote the set $\{x \in \mathbb{R} \setminus \{0\}: |x| < 1\} = (-1,0) \cup (0,1)$ by writing $|x| < 1$ in (7.2.1) and (7.2.2) shortly. For the remainder of Chapter 7 we always suppose jump-sizes in the set $\mathbb{R}_0 \coloneqq \mathbb{R} \setminus \{0\}$, although we frequently will omit the null-exception for notational reasons, instead writing e.g. $|x| < 1$ merely.

Moreover, in our case the Lévy-measure ν appearing in (7.2.3) is supposed to be a positive and finite Borel-random-measure on $\mathbb{R}_0$ that fulfills the condition

$$\int_{\mathbb{R}_0} \frac{x^2}{1+x^2}\, d\nu(x) < \infty.$$

(For details concerning the integrability condition associated to the Lévy-measure ν see the beginning of subsection 1.2.4 in [1] or Chapter 3 in [30], for instance.)
Finally, we suppose the stochastic[70] coefficient processes

$$\begin{aligned} \alpha &: [0,\tau] \times [0,T] \longrightarrow \mathbb{R} \\ \sigma &: [0,\tau] \times [0,T] \longrightarrow \mathbb{R}^+ \\ \beta &: [0,\tau] \times \mathbb{R}_0 \times [0,T] \longrightarrow \mathbb{R} \\ \gamma &: [0,\tau] \times \mathbb{R}_0 \times [0,T] \longrightarrow \mathbb{R} \end{aligned} \tag{7.2.4}$$

such as appearing in (7.2.1) and (7.2.2) altogether to be integrable and bounded, so that all integrals in the two latter equations are well-defined.

7.2.2 The power forward price dynamics under the true market measure

In what follows, we will derive the corresponding power forward price dynamics associated to our extended model setup with jumps. As in [18], for $0 \le t \le \tau \le T$ we firstly define the *short rate* r_t by

$$r_t := f_t(t). \tag{7.2.5}$$

Interest rate theory (see e.g. Chapter 10 in [83] for an overview and motivating aspects; particularly, see section 10.3 therein for details on the *Brownian* HJM standards) allocates the following well-known relation between *(commodity) forward prices* $p_t(\tau)$ and forward rates $f_t(u)$, namely

$$p_t(\tau) = exp\left\{ - \int_t^\tau f_t(u)\, du \right\} \tag{7.2.6}$$

for $0 \le t \le \tau$ (cf. eq. "(5.1) in [17]", resp. "(10.3.3) in [83]"). Consequently, we observe $p_t(t) = 1$. By the way, taking the logarithm in (7.2.6) and hereafter differentiating with respect to τ, we receive

$$f_t(\tau) = -\frac{\partial}{\partial \tau}[ln\ p_t(\tau)]$$

(cf. eq. "(5.2) in [17]", resp. Def. 2.2 in [18]). Let us recall that our goal is the provision of the SDE fulfilled by the power forward price $p_t(\tau)$. Hence, in order to apply Itô's formula on (7.2.6), we obviously need a representation for the stochastic differential of the appearing exponent

$$U_t := U_t(\tau) := - \int_t^\tau f_t(u)\, du. \tag{7.2.7}$$

[70] Anyway, we will assume the coefficients (7.2.4) to be *deterministic* in some later paragraphs.

For the above special case it is *not* possible to provide the t-dynamics dU_t immediately (by applying Itô's formula, for example), since on the right hand side of (7.2.7) the time parameter t appears both inside the integrand and in the lower integration bound. To treat this extraordinary instance, we innovatively make use of the *Leibniz-rule for parameter integrals* in the following:

In accordance to the notations in Lemma 2.4.1, we presently have $x(t) := \tau$ and $y(t) := t$ leading us to the dynamics

(7.2.8)

$$dU_t = d\left(\int_\tau^t f_t(u)\, du\right) = f_t(t)\, dt - \int_t^\tau df_t(u)\, du$$

for time parameters $0 \leq t \leq \tau$. Merging (7.2.1) and (7.2.5) into (7.2.8), we get

$$dU_t = r_t\, dt - \int_t^\tau \alpha_t(u)\, dt\, du - \int_t^\tau \sigma_t(u)\, dW_t\, du - \int_t^\tau \int_{|x|\geq 1} \beta_{t-}(x,u)\, N(t,dx)\, du$$
$$- \int_t^\tau \int_{|x|<1} \gamma_{t-}(x,u)\, \widetilde{N}_{\mathbb{P}}(t,dx)\, du.$$

Referring to the stochastic Fubini-Tonelli theorem, we next derive

(7.2.9)

$$dU_t = [r_t - a_t(\tau)]\, dt - s_t(\tau)\, dW_t - \int_{|x|\geq 1} b_{t-}(x,\tau)\, N(t,dx) - \int_{|x|<1} g_{t-}(x,\tau)\, \widetilde{N}_{\mathbb{P}}(t,dx)$$

wherein – somehow parallel to eq. "(6) in [18]" – we have just introduced the shorthand notations

(7.2.10)

$$a_t(\tau) := \int_t^\tau \alpha_t(u)\, du\,, \qquad s_t(\tau) := \int_t^\tau \sigma_t(u)\, du \ \ (> 0),$$

$$b_t(x,\tau) := \int_t^\tau \beta_t(x,u)\, du\,, \qquad g_t(x,\tau) := \int_t^\tau \gamma_t(x,u)\, du.$$

As usual, we define the jump size of the *càdlàg* process U at time t by $\Delta U_t := U_{t+} - U_{t-} = U_t - U_{t-}$ whereas $[U^c]_t$ denotes the quadratic variation of the continuous part of U. Therewith, we are able to state the stochastic evolution equation for $p_t(\tau)$ under the measure $\mathbb{P}$ in the subsequent way:

Applying Itô's formula (see Theorem 2.1.6) for discontinuous semi-martingales on the Euler function $c(x) := e^x$, we receive

$$p_t(\tau) = c(U_t) = c(U_0) + \int_0^t c(U_{s-})\, dU_s + \frac{1}{2}\int_0^t c(U_s)\, d[U^c]_s + \sum_{0\leq s\leq t} [c(U_s) - c(U_{s-}) - \Delta U_s\, c(U_{s-})].$$

Substituting (7.2.9) into the latter equation, we immediately derive

(7.2.11)

$$p_t(\tau) = p_0(\tau) + \int_0^t p_s(\tau)\left(r_s - a_s(\tau) + \frac{s_s(\tau)^2}{2}\right) ds - \int_0^t p_s(\tau)\, s_s(\tau)\, dW_s$$
$$- \int_0^t \int_{|x|\geq 1} p_{s-}(\tau)\, b_{s-}(x,\tau)\, dN(s,x) - \int_0^t \int_{|x|<1} p_{s-}(\tau)\, g_{s-}(x,\tau)\, d\widetilde{N}_{\mathbb{P}}(s,x)$$
$$+ \sum_{0\leq s\leq t} p_{s-}(\tau)\, [e^{\Delta U_s} - 1 - \Delta U_s].$$

In the following, we will handle the infinite sum appearing in (7.2.11) separately: Remembering (7.2.9), we obtain

(7.2.12)

$$\sum_{0\leq s\leq t} p_{s-}(\tau)\, [e^{\Delta U_s} - 1 - \Delta U_s]$$
$$= \int_0^t \int_{|x|\geq 1} p_{s-}(\tau)\left[e^{-b_{s-}(x,\tau)} - 1 + b_{s-}(x,\tau)\right] dN(s,x)$$
$$+ \int_0^t \int_{|x|<1} p_{s-}(\tau)\left[e^{-g_{s-}(x,\tau)} - 1 + g_{s-}(x,\tau)\right] dN(s,x)$$

$$= \int_0^t \int_{\mathbb{R}_0} p_{s-}(\tau)\left[\left\{e^{-b_{s-}(x,\tau)} + b_{s-}(x,\tau)\right\} \mathbb{1}_{|x|\geq 1} + \left\{e^{-g_{s-}(x,\tau)} + g_{s-}(x,\tau)\right\} \mathbb{1}_{|x|<1} - 1\right] dN(s,x).$$

Putting the auxiliary calculation (7.2.12) into equation (7.2.11) while taking the compensator property (7.2.3) into account, we derive a representation for $p_t(\tau)$ under the true probability measure $\mathbb{P}$ reading

(7.2.13)

$$p_t(\tau) = p_0(\tau) + \int_0^t p_s(\tau)\left(r_s - a_s(\tau) + \frac{s_s(\tau)^2}{2} + \int_{|x|<1} g_s(x,\tau)\, d\nu(x)\right) ds - \int_0^t p_s(\tau)\, s_s(\tau)\, dW_s$$
$$+ \int_0^t \int_{\mathbb{R}_0} p_{s-}(\tau)\left[e^{-b_{s-}(x,\tau)}\, \mathbb{1}_{|x|\geq 1} + e^{-g_{s-}(x,\tau)}\, \mathbb{1}_{|x|<1} - 1\right] dN(s,x).$$

In differential form the latter equation becomes

(7.2.14)

$$\frac{dp_t(\tau)}{p_{t-}(\tau)} = \left(r_t - a_t(\tau) + \frac{s_t(\tau)^2}{2} + \int_{|x|<1} g_t(x,\tau)\, d\nu(x)\right) dt - s_t(\tau)\, dW_t + \int_{\mathbb{R}_0} \delta_{t-}(x,\tau)\, N(t,dx)$$

(which extends Prop. 2.4 (3) in [18]) whereby we have recently introduced the abbreviation

(7.2.15) $$\delta_t(x,\tau) := e^{-b_t(x,\tau)}\,\mathbb{1}_{|x|\geq 1} + e^{-g_t(x,\tau)}\,\mathbb{1}_{|x|<1} - 1.$$

However, for discounting we trouble a *bank account* with stochastic interest rate $r_s = f_s(s)$ defined by

(7.2.16)
$$B_t := B_0\, exp\left\{\int_0^t r_s\, ds\right\}$$

(cf. p.11 in [18]) within a deterministic initial value $B_0 > 0$, whereas

(7.2.17) $$dB_t = r_t B_t dt \qquad \text{and} \qquad d\left(\frac{1}{B_t}\right) = -\frac{r_t}{B_t}dt$$

result as trivial consequences. Next, for $0 \leq t \leq \tau$ we define the *discounted power forward price* via

(7.2.18) $$\hat{p}_t(\tau) := \frac{p_t(\tau)}{B_t}$$

(cf. Definition 3.3 (1) in [18]). Using (7.2.14), (7.2.17), (7.2.18) and Itô's product rule, we derive the $\mathbb{P}$-dynamics

(7.2.19)
$$\frac{d\hat{p}_t(\tau)}{\hat{p}_{t-}(\tau)} = \left(\frac{s_t(\tau)^2}{2} - a_t(\tau) + \int_{|x|<1} g_t(x,\tau)\, d\nu(x)\right)dt - s_t(\tau)\, dW_t + \int_{\mathbb{R}_0} \delta_{t-}(x,\tau)\, N(t,dx)$$

which corresponds to equality "(5.12) in [17]". Note that the short rate r_t has canceled out so far. By the way, we may compute the exact solution of the integro-SDE (7.2.19) which reads as

(7.2.20)
$$\hat{p}_t(\tau) = \hat{p}_0(\tau)\, exp\left\{\int_0^t\left(-a_s(\tau) + \int_{|x|<1} g_s(x,\tau)\, d\nu(x)\right)ds - \int_0^t s_s(\tau)\, dW_s \right.$$
$$\left. - \int_0^t \int_{\mathbb{R}_0} \left[b_{s-}(x,\tau)\,\mathbb{1}_{|x|\geq 1} + g_{s-}(x,\tau)\,\mathbb{1}_{|x|<1}\right] dN(s,x)\right\}.$$

However, we will save up the detailed derivation procedure of (7.2.20) for later investigations, since the underlying techniques will come into play while deriving the representation for the discounted power forward price under a forthcoming equivalent martingale measure in the next subsection.

7.2.3 The power forward price dynamics under an equivalent martingale measure

Let us turn back to the analysis of the discounted power forward price as introduced in (7.2.18) but under an equivalent martingale measure (EMM) now. Taking the results of Proposition 2.2.1 into account, we are able to state the evolution equation of $\hat{p}_t(\tau)$ under an EMM $\mathbb{Q}$ which itself we assume to be defined like in (2.2.5).

More precisely, substituting (2.2.6) and (2.2.8) into (7.2.19), we instantly derive the $\mathbb{Q}$-dynamics

(7.2.21)

$$\frac{d\hat{p}_t(\tau)}{\hat{p}_{t-}(\tau)} = \left(\frac{s_t(\tau)^2}{2} - a_t(\tau) + \int_{|x|<1} g_t(x,\tau)\, d\nu(x) - s_t(\tau)\, G_t + \int_{\mathbb{R}_0} \delta_t(x,\tau)\, H(t,x)\, d\nu(x) \right) dt$$
$$- s_t(\tau)\, d\widetilde{W}_t + \int_{\mathbb{R}_0} \delta_{t-}(x,\tau)\, \widetilde{N}_{\mathbb{Q}}(t,dx).$$

In accordance to the risk-neutral pricing theory, the discounted power forward price in (7.2.21) has to constitute a local $\mathbb{Q}$-martingale. For this reason, we have to require the following *extended HJM-drift-restriction* (cf. Prop. 5.3 and Prop. 5.6 (4), eq. (5.36), in [17]) corresponding to our jump-case, reading

(7.2.22)

$$0 = \frac{s_t(\tau)^2}{2} - a_t(\tau) - s_t(\tau)\, G_t + \int_{\mathbb{R}_0} \left[g_t(x,\tau)\, \mathbb{1}_{|x|<1} + \delta_t(x,\tau)\, H(t,x) \right] d\nu(x).$$

Remark 7.2.2 *Although looking completely different on a first glance, our extended HJM-drift-restriction in (7.2.22) can be traced back to the one received on p.9 in [59], which – adjusted to our current notations – on the contrary reads as*

(7.2.23) $$a_t(\tau) = \frac{s_t(\tau)^2}{2}.$$

To justify this statement, we argue as follows: Comparing (7.2.23) with our extended drift-condition (7.2.22), one should note that the authors in [59] make use of a "martingale modeling approach" (see the last line of p.6 in [59]), which in our framework a priori would imply $G \equiv 0$ in (2.2.6) and thus, also in (7.2.22). In addition, there are no jumps permitted in "(25) in [59]", what corresponds to choosing $\beta \equiv 0$ and $\gamma \equiv 0$ in (7.2.1), which would imply $b \equiv 0$ and $g \equiv 0$, and hence, $\delta \equiv 0$ in our setup. (To read more on "martingale modeling" the interested reader is advised to the prolog of Prop. 3.15 on pp. 19-20 in [18], while related model calibration particularities are discussed on page 158 in [13].) Anyway, differentiating (7.2.22) with respect to τ while using (7.2.10) and (7.2.15), we get

(7.2.24)

$$\alpha_t(\tau) = \sigma_t(\tau) \int_t^{\tau} \sigma_t(u)\, du - \sigma_t(\tau)\, G_t$$
$$+ \int_{\mathbb{R}_0} \left[\gamma_t(x,\tau) \left\{ 1 - H(t,x)\, e^{-g_t(x,\tau)} \right\} \mathbb{1}_{|x|<1} - \beta_t(x,\tau)\, H(t,x)\, e^{-b_t(x,\tau)}\, \mathbb{1}_{|x|\geq 1} \right] d\nu(x).$$

Yet, it appears worthwhile to compare the latter equation with "(28) in [18]". Next, let us recall that in the continuous (i.e. Brownian-) HJM-model descending from a martingale modeling approach (such as implemented in [59]) the volatility coefficient $\sigma_t(\tau)$ represents the only parameter (besides the initial condition $f_0(\tau)$ which can be obtained from the observed forward price $p_0(\tau)$ via $f_0(\tau) = -\partial \ln p_0(\tau)/\partial\tau$) that has to be chosen in order to develop the entire forward rate dynamics (cf. eq. "(28) in [59]" and the top of p.10 in [59]). In other words, once having selected a concrete volatility process $\sigma_t(\tau)$, the drift coefficient $\alpha_t(\tau)$ simultaneously is determined via the nice relationship

(7.2.25) $$\alpha_t(\tau) = \sigma_t(\tau) \int_t^\tau \sigma_t(u)\, du$$

which easily is deduced from (7.2.23) by differentiating the latter equation with respect to τ [59]. Comparing (7.2.24) with (7.2.25), the most striking difference consists in the additional $d\nu$-integral term appearing in (7.2.24) which comes in due to the admission of jumps in the present work. A parallel opening of our extended jump-diffusion approach also using a "martingale modeling onset" would culminate in choosing $G \equiv 0$ in (2.2.6) and $H(t,x) \equiv 1$ in (2.2.8) right from the beginning. ■

However, assuming the drift-restriction (7.2.22) to be in force, the $\mathbb{Q}$-dynamics (7.2.21) shortens to

(7.2.26)

$$\frac{d\hat{p}_t(\tau)}{\hat{p}_{t-}(\tau)} = -s_t(\tau)\, d\widetilde{W}_t + \int_{\mathbb{R}_0} \delta_{t-}(x,\tau)\, \widetilde{N}_{\mathbb{Q}}(t,dx)$$

extending equality "(31) in [59]". In order to work out the explicit solution of the latter integro stochastic differential equation (ISDE), let us define the local $\mathbb{Q}$-martingale

(7.2.27)

$$Y_t := -\int_0^t s_u(\tau)\, d\widetilde{W}_u + \int_0^t \int_{\mathbb{R}_0} \delta_{u-}(x,\tau)\, d\widetilde{N}_{\mathbb{Q}}(u,x)$$

for notational simplicity. Therewith, equality (7.2.26) shortly can be expressed as

(7.2.28)

$$d\hat{p}_t(\tau) = \hat{p}_{t-}(\tau)\, dY_t.$$

Next, standard arguments from stochastic calculus (see section 5.1 in [1], for instance) purvey the solution of (7.2.28) as a discontinuous stochastic Doléans-Dade exponential reading

(7.2.29)

$$\hat{p}_t(\tau) = \hat{p}_0(\tau)\, exp\left\{Y_t - \frac{1}{2}[Y^c]_t\right\} \prod_{0\le u\le t} (1+\Delta Y_u)\, e^{-\Delta Y_u}.$$

In the following, we handle the infinite product appearing in (7.2.29) separately: Recalling (7.2.15) and (7.2.27), we get

(7.2.30)

$$\begin{aligned}\prod_{0\le u\le t} (1+\Delta Y_u)\, e^{-\Delta Y_u} &= exp\left\{\sum_{0\le u\le t} [ln(1+\Delta Y_u) - \Delta Y_u]\right\} \\ &= exp\left\{\int_0^t \int_{\mathbb{R}_0} \left[ln\big(1+\delta_{u-}(x,\tau)\big) - \delta_{u-}(x,\tau)\right] dN(u,x)\right\} \\ &= exp\left\{-\int_0^t \int_{\mathbb{R}_0} \left[b_{u-}(x,\tau)\, \mathbb{I}_{|x|\ge 1} + g_{u-}(x,\tau)\, \mathbb{I}_{|x|<1} + \delta_{u-}(x,\tau)\right] dN(u,x)\right\}.\end{aligned}$$

Regarding (7.2.27), the first exponent in (7.2.29) turns out as

(7.2.31)

$$Y_t - \frac{1}{2}[Y^c]_t = -\int_0^t s_u(\tau)\, d\widetilde{W}_u + \int_0^t \int_{\mathbb{R}_0} \delta_{u-}(x,\tau)\, d\widetilde{N}_{\mathbb{Q}}(u,x) - \frac{1}{2}\int_0^t s_u(\tau)^2 du.$$

Merging (7.2.30), (7.2.31) and the compensator property (2.2.8) into (7.2.29), we finally end up with

(7.2.32)

$$\hat{p}_t(\tau) =$$

$$\hat{p}_0(\tau)\, exp\left\{-\int_0^t \left[\frac{s_u(\tau)^2}{2} + \int_{\mathbb{R}_0} \left\{b_u(x,\tau)\, \mathbb{1}_{|x|\geq 1} + g_u(x,\tau)\, \mathbb{1}_{|x|<1} + \delta_u(x,\tau)\right\} H(u,x)\, d\nu(x)\right] du \right.$$
$$\left. -\int_0^t s_u(\tau)\, d\widetilde{W}_u - \int_0^t \int_{\mathbb{R}_0} \left[b_{u-}(x,\tau)\, \mathbb{1}_{|x|\geq 1} + g_{u-}(x,\tau)\, \mathbb{1}_{|x|<1}\right] d\widetilde{N}_{\mathbb{Q}}(u,x)\right\}.$$

Remark 7.2.3 *Comparing (7.2.32) with the corresponding equation "(6.1) in [13]", we notice that our formula essentially possesses a similar structure. In this context, note that the authors of [13] utilize a "martingale modeling approach" and therewith provide the discounted power forward prices under the EMM immediately. Hence, their process* $f(t,\tau)$ *in (6.1) corresponds to our price process* $\hat{p}_t(\tau)$ *such as given in equation (7.2.32) above. However, our approach is much better motivated, since we have modeled the forward rates in (7.2.1) first, then derived the power forward prices via (7.2.6), respectively (7.2.14), discounted afterwards in (7.2.18) and finally switched to an EMM in the present subsection. Roughly speaking, Benth et al. [13] thus start at the very point where we have just come to (also see the top of p.20 in [18] in this context). The price one has to pay using such a (convenient) "martingale modeling onset" lies in the statistical problems associated with parameter estimation: In fact, the main drawback consists in the problem of calibrating the model, since in reality one does not observe commodity forward prices under the EMM* $\mathbb{Q}$ *but instead under the true market measure* $\mathbb{P}$ *(cf. p.158 in [13], resp. the top of p.21 in [18]).*[71] *Unfortunately, this fact has completely been neglected by Hinz et al. [59], as they do their maximum (log-) likelihood "Historical calibration" in Chapter 4 under the EMM falsely.* ■

7.3 Determining an optimal equivalent martingale measure

In accordance to the second fundamental theorem of asset pricing, in an incomplete market model we have to deal with several candidates for EMMs and it is not a trivial question, which one to choose. In this context, the following subsections are dedicated to the concepts of Esscher transforms and minimum relative entropy, both providing appropriate methods to overcome (at least approximately) the just mentioned selection problem. We start off within a rigorous examination of Esscher transforms tailored to the requirements of our HJM-approach with jumps in the subsequent paragraph.

[71] On p.158 in [13] the authors recognize this particularity of their *martingale modeling approach* and hence, propose to establish a measure change (yet in "opposite direction", i.e. from $\mathbb{Q}$ to $\mathbb{P}$) in order to describe their model under the true market measure $\mathbb{P}$ eventually. Nevertheless, also in *real* markets, *option* prices are observed under $\mathbb{Q}$ of course [13]. But, since in most power exchanges "*options are rather thinly traded*", the more liquid forward markets actually appear more appropriate (than option prices) for calibration issues, because forward markets usually permit a "*good access to reliable data under* $\mathbb{P}$" (see p.158 in [13]).

7.3.1 The Esscher transform

Since the EMM cannot be uniquely chosen in an incomplete market model, it might be useful to restrict the large class of potential risk-neutral pricing measures to a flexible subclass of parameterized arbitrage-free probabilities given through the *Esscher transform* (cf. sect. 4.1.1 in [13]). Referring to p.95 in [13], we may interpret the latter as an extension of the Girsanov transform for Brownian motions yet to jump processes in the following sense: On the one hand, the popular Girsanov transform for Wiener processes provides a measure change which "*preserves the normality of the [underlying] distribution*". On the other hand, the Esscher transform only slightly alters the distributional properties of the involved jump noises [in particular, we can force correspondence between (2.2.1) and (7.3.7) via (7.3.9), as we will see later on] and fully "*preserves the independent increment property*" of the driving (Lévy- or Sato-) processes (see p.95 in [13]). All in all, the upcoming *Esscher parameter* θ can be interpreted as the "*[market] price of jump risk*" (see p.98 in [13]). In order to define a version of the Esscher transform that fits our purposes, let us initially introduce the following $\mathbb{P}$-martingale

(7.3.1)

$$L_t := W_t + \int_0^t \int_{\mathbb{R}_0} x \, d\widetilde{N}_{\mathbb{P}}(s,x).$$

Using (7.2.3), we immediately deduce its Lévy-Itô decomposition under $\mathbb{P}$ reading

(7.3.2)

$$L_t = -t \int_{|x|\geq 1} x \, d\nu(x) + W_t + \int_0^t \int_{|x|<1} x \, d\widetilde{N}_{\mathbb{P}}(s,x) + \int_0^t \int_{|x|\geq 1} x \, dN(s,x)$$

(cf. "(2.3) and (2.4) in [26]"). Thus, L obviously depicts a Lévy process with characteristic triplet

$$\left(1, \nu, -\int_{|x|\geq 1} x \, d\nu(x)\right).$$

Parallel to p.96 in [13], for a deterministic Esscher parameter $\theta \in \mathbb{R}$ we further assume the condition

$$\int_{|x|\geq 1} e^{\theta x} \, d\nu(x) < \infty.$$

In what follows, let us denote the characteristic function of L_t by Φ_{L_t} (recall Theorem 2.1.3 above). Then, slightly deviating from the bottom of p.519 in [26], we define the *Esscher transform of* L_t by

(7.3.3)

$$\left.\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \frac{e^{-\theta L_t}}{\mathbb{E}_{\mathbb{P}}[e^{-\theta L_t}]} = \frac{e^{-\theta L_t}}{\Phi_{L_t}(i\theta)} = \frac{e^{-\theta L_t}}{e^{t\,\psi_L(i\theta)}} = exp\{-\theta L_t - t\,\psi_L(i\theta)\} > 0.$$

In (7.3.3) the function $\psi_L(\cdot)$ represents the characteristic exponent of the Lévy process (7.3.1), which in our case is explicitly given by the Lévy-Khinchin formula [see equality (2.1.5)] via

(7.3.4)

$$\psi_L(i\theta) = \frac{\theta^2}{2} + \int_{\mathbb{R}_0} \left[e^{-\theta x} - 1 + \theta x\right] d\nu(x).$$

Let us remark that if L simply is a standard BM, i.e. $L = W$ being a Lévy process with characteristic triplet $(1,0,0)$, then (7.3.3) becomes the familiar Girsanov density with constant drift parameter $\theta \in \mathbb{R}$ which is explicitly given through the continuous Doléans-Dade exponential

$$\left.\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathfrak{E}(-\theta \circ W_{\cdot})_t := exp\left\{-\theta W_t - \frac{\theta^2 t}{2}\right\}.$$

Returning to our original context, we have to consider a slightly generalized and, in particular, *time-dependent* Esscher transform in this work, since our underlying (discounted) commodity forward price process $\hat{p}_t(\tau)$ exhibits *time-dependent* coefficients [compare e.g. equality (7.2.19)]. Hence, inspired by equation "(3.19) in [26]", we claim the following more appropriate definition extending (7.3.3).

Definition 7.3.1 *For a Lévy process L_t such as given in (7.3.1) and a real, (square-) integrable, (finite), continuous, deterministic and time-dependent Esscher parameter/function θ_t we define the generalized Esscher transform due to*

(7.3.5)

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \Xi_t := exp\left\{-\int_0^t \theta_{s-}\, dL_s - \int_0^t \psi_L(i\theta_s)\, ds\right\} > 0$$

with time parameters $0 \le t \le \tau$. Herein, the object $\psi_L(\cdot)$ represents the characteristic exponent associated to the Lévy process L which, in accordance to (2.1.5), respectively to (7.3.4), is given by

$$\psi_L(i\theta_s) = \frac{\theta_s^2}{2} + \int_{\mathbb{R}_0}\left[e^{-\theta_s x} - 1 + \theta_s x\right] d\nu(x). \qquad (7.3.6)$$

Moreover, the probability measure $\widetilde{\mathbb{Q}}$ defined through (7.3.5) frequently will be referred to as the Esscher-EMM in the following. ■

We now show that Ξ is a (true) $\mathbb{P}$-martingale: Merging (7.3.1) and (7.3.6) into (7.3.5), we derive

(7.3.7)

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = exp\left\{-\int_0^t \theta_s\, dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t\int_{\mathbb{R}_0} \theta_{s-} x\, d\widetilde{N}_{\mathbb{P}}(s,x) - \int_0^t\int_{\mathbb{R}_0}\left[e^{-\theta_s x} - 1 + \theta_s x\right] d\nu(x)\, ds\right\}.$$

Applying Itô's formula on (7.3.7), we next obtain the alternative integral representation

(7.3.8)

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = 1 - \int_0^t \left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_s} \theta_s\, dW_s + \int_0^t\int_{\mathbb{R}_0} \left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_{s-}} \left[e^{-\theta_{s-} x} - 1\right] d\widetilde{N}_{\mathbb{P}}(s,x)$$

(with vanishing drift) declaring the density process Ξ in (7.3.5) as a *local* $\mathbb{P}$-martingale. Since the emerging Esscher parameter θ is deterministic, the integrals in (7.3.8) both are $\mathbb{P}$-martingales with expectation zero such that $\Xi > 0$ even constitutes a *true* $\mathbb{P}$-martingale (cf. the bottom of p.514 in [26]; In fact, strictly positive *local* martingales with expectation one are *true* martingales – see Theorem 5.2.4 in [1].). By the way, if θ was stochastic, in order to ensure the latter (true) $\mathbb{P}$-martingale property we might impose a *Novikov condition* on θ (cf. Th. 12.21 in [32]) such that the positive local $\mathbb{P}$-martingale Ξ also in this case would yield a $\mathbb{P}$-expectation equal to one (recall section 2.2 above).

Comparing (2.2.1) with (7.3.7) [or (2.2.3) with (7.3.8), alternatively], we recognize that we can force correspondence between (2.2.1) and (7.3.7) by setting

(7.3.9) $$G_s := -\theta_s\,, \quad h(s,x) := -\theta_s x\,, \quad H(s,x) := e^{h(s,x)}.$$

Moreover, substituting (7.3.9) into our extended HJM-drift-restriction (7.2.22), we receive

(7.3.10)

$$\int_{\mathbb{R}_0} \delta_t(x,\tau)\, e^{-\theta_t x}\, d\nu(x) = -s_t(\tau)\,\theta_t + a_t(\tau) - \frac{s_t(\tau)^2}{2} - \int_{|x|<1} g_t(x,\tau)\, d\nu(x)$$

which corresponds to "(3.20) in [26]". Yet, we assume the coefficients δ, s, a and g such as appearing in (7.3.10) to be deterministic in the following, whereas we emphasize that they do *not* depend on the Esscher parameter θ. Further, note that $s_t(\tau)$ is strictly positive and thus, the right hand side of equation (7.3.10) forms a decreasing linear function in θ_t for every fixed t (and τ). Hence, defining

$$\Gamma(\theta_t) := \int_{\mathbb{R}_0} \delta_t(x,\tau)\, e^{-\theta_t x}\, d\nu(x)$$

for notational reasons, equality (7.3.10) can be expresses as

(7.3.11) $$\Gamma(\theta_t) = k\theta_t + d$$

within a strictly negative and constant[72] coefficient k and an arbitrary constant d (both for fixed t and τ). At this point, the reader is invited to invest some further effort concerning the solvability of (7.3.11), whereas meanwhile a concrete form of both the Lévy-measure ν – descending from e.g. *generalized hyperbolic distributions* like the *normal inverse Gaussian distribution*, or *variance gamma distributions* such as the *Carr, Geman, Madan, Yor distribution* (compare subsection 2.6.2 and 2.6.3 in [13] for an overview and definitions) – and the appearing coefficients δ, s, a and g might have to be chosen in order to make fruitful statements. By the way, introducing the function

(7.3.12) $$\Lambda(\theta_t) := \frac{\Gamma(\theta_t) - d}{k}$$

equation (7.3.11) may be transferred into the *fix-point problem*[73]

$$\Lambda(\theta_t) = \theta_t.$$

However, once having computed a specific solution to (7.3.11), say $\tilde{\theta}_t$, the latter can be substituted into the density process (7.3.7) and thus, can be utilized for switching to an (in such a way determined) *Esscher-EMM* $\tilde{\mathbb{Q}} := \tilde{\mathbb{Q}}(\tilde{\theta})$, depending directly on the concrete solution $\tilde{\theta}$.

7.3.2 The measure of minimum relative entropy

In order to measure the *distance* between two equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$, we recall definition (6.4.1) introducing the *relative entropy* as an appropriate approximation-error criterion.

[72] In this context, the expression „*constant*" means „*not depending on* θ_t".

[73] Unfortunately, *Banach's fix-point theorem* does not apply here, since the function $\Lambda(\cdot)$, respectively $\Gamma(\cdot)$, appearing in (7.3.12) does not constitute a *contractive operator* on the complete metrical space $(\mathbb{R}, |\cdot|)$, as the inequality $|\Lambda(y) - \Lambda(z)| \le q\,|y - z|$ with $q \in (0,1)$ and $y, z \in [M_1, M_2] \subset \mathbb{R}$ does not seem to be in force.

Merging (2.2.1) and (2.2.5) into (6.4.1) while taking (2.2.6), (2.2.8) and (7.2.3) into account, we obtain

(7.3.13) $$\mathcal{E}\langle\mathbb{P}|\mathbb{Q}\rangle =$$

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^t G_s\,dW_s - \frac{1}{2}\int_0^t G_s^2\,ds + \int_0^t\int_{\mathbb{R}_0} h(s-,x)\,d\tilde{N}_{\mathbb{P}}(s,x) - \int_0^t\int_{\mathbb{R}_0}[H(s,x)-1-h(s,x)]\,d\nu(x)\,ds\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[\int_0^t G_s\,d\widetilde{W}_s + \frac{1}{2}\int_0^t G_s^2\,ds + \int_0^t\int_{\mathbb{R}_0} h(s-,x)\,d\tilde{N}_{\mathbb{Q}}(s,x)\right.$$
$$\left. + \int_0^t\int_{\mathbb{R}_0}[1+H(s,x)\,\{h(s,x)-1\}]\,d\nu(x)\,ds\right].$$

Remembering the martingale properties claimed in connection with Prop. 2.2.1, (7.3.13) shortens to

(7.3.14)

$$\mathcal{E}\langle\mathbb{P}|\mathbb{Q}\rangle = \mathbb{E}_{\mathbb{Q}}\left[\int_0^t\left(\frac{G_s^2}{2} + \int_{\mathbb{R}_0}[1+H(s,x)\,\{\ln H(s,x)-1\}]\,d\nu(x)\right)ds\right]$$

(parallel to the top of p.521 in [26]). As explained on p.521 in [26], the problem of finding the EMM yielding *minimum* relative entropy forces us to minimize (for a fixed time index s) the expression (7.3.14) with respect to G_s and $H(s,x)$, both being connected via the extended HJM-drift-condition (7.2.22). Thus, roughly speaking, the measure of *minimum* relative entropy in particular minimizes the 'distance' between the true market measure $\mathbb{P}$ and its approximating candidate $\mathbb{Q}$ and therefore delivers the [with respect to our error criterion (6.4.1)] *best* approach towards the verity $\mathbb{P}$ under the restrictive constraint (7.2.22) which requires the discounted power forward price $\hat{p}_t(\tau)$ to form a local $\mathbb{Q}$-martingale.

In accordance to p.521 in [26], we can equivalently minimize the *ds-integrand* in (7.3.14) merely, reducing our minimum relative entropy problem to the following optimization exercise:

For a fixed time index s, find processes G_s and $H(s,x)$ which are coupled via the constraint (7.2.22) and moreover solve the minimization exercise

(7.3.15)

$$\min_{\left\{G_s \underset{(7.2.22)}{\longleftrightarrow} H(s,x)\right\}}\left(\frac{G_s^2}{2} + \int_{\mathbb{R}_0}[1+H(s,x)\,\{\ln H(s,x)-1\}]\,d\nu(x)\right).$$

Once having derived a proper minimizing solution to (7.3.15), say G_s^* and $H^*(s,x)$, then this *minimizing pair-process* determines [via (2.2.1) and (2.2.5)] a concrete *minimum relative entropy EMM* $\mathbb{Q}^* := \mathbb{Q}^*(G^*,H^*)$ by dint of

$$\left.\frac{d\mathbb{Q}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t G^*dW - \frac{1}{2}\int_0^t (G^*)^2ds + \int_0^t\int_{\mathbb{R}_0}\ln(H^*)\,d\tilde{N}_{\mathbb{P}} - \int_0^t\int_{\mathbb{R}_0}[H^*-1-\ln(H^*)]\,d\nu\,ds\right\}.$$

In what follows, we devote our attention towards the precise minimization procedure of (7.3.15).

Of course, as mentioned on the bottom of p.521 in [26], one could use (7.2.22) to express G_s in terms of $H(s,x)$, substitute the resulting expression into (7.3.15) and finally minimize over $H(s,x)$ separately. However, the main drawback of this *G_s-eliminating approach* is the incoming squared $d\nu$-integral involving $H(s,x)$ which is difficult to manage [26] – at least, as long as we have not chosen a concrete form of the Lévy-measure ν, that is, a concrete density of ν with respect to e.g. the Lebesgue measure [such as descending from *generalized hyperbolic distributions*, for example].

Alternatively, in order to solve the optimization problem (7.3.15), in this work we firstly fix G_s, subsequently minimize over $H(s,x)$ and finally minimize (7.3.15) [with this optimal $H(s,x)$] over G_s, adapting the *successive Lagrange approach* suggested by Chan on p.522 in [26] to our requirements. For this purpose, we define the fitted *Lagrange function*[74]

(7.3.16)
$$L(H) := L(\lambda, H) := L\big(\lambda_s, H(s,x)\big) :=$$

$$\int_{\mathbb{R}_0} [1 + H(s,x)\{ln\, H(s,x) - 1\}]\, d\nu(x) + \lambda_s \int_{\mathbb{R}_0} \big[g_s(x,\tau)\, \mathbb{1}_{|x|<1} + H(s,x)\, \delta_s(x,\tau)\big]\, d\nu(x)$$

$$= \int_{\mathbb{R}_0} \big[1 + \lambda_s\, g_s(x,\tau)\, \mathbb{1}_{|x|<1} + H(s,x)\, \{\lambda_s\, \delta_s(x,\tau) - 1 + ln\, H(s,x)\}\big]\, d\nu(x)$$

within a Lagrange multiplier $\lambda_s := \lambda(G_s)$ being a differentiable function of G.

In order to attain a better overview, we suppress the subscripts and arguments appearing in (7.3.16) during our following considerations. Since both the process H and the Lévy-measure ν formerly have been supposed to be strictly positive, ordinary calculus delivers $L''(H) > 0$ which ensures the convexity of the Lagrange function $H \longmapsto L(\lambda, H)$. Therefore, as in [26], we merely have to require the following (actually *necessary*) minimum condition for a strictly positive process $F(s,x)$ and a scalar ϑ

(7.3.17)
$$0 = \left(\frac{\partial}{\partial\vartheta} L(\lambda, H + \vartheta F)\right)\Big|_{\vartheta=0} =$$

$$\left(\frac{\partial}{\partial\vartheta} \int_{\mathbb{R}_0} \big[1 + \lambda\, g\, \mathbb{1}_{|x|<1} + (H + \vartheta F\,)\, \{\lambda\, \delta - 1 + ln(H + \vartheta F)\}\big]\, d\nu(x)\right)\Bigg|_{\vartheta=0} .$$

An interchange of the differential and the integral operator in (7.3.17) yields

$$0 = \int_{\mathbb{R}_0} F(s,x)\, \big[\lambda_s\, \delta_s(x,\tau) + ln\big(H(s,x)\big)\big]\, d\nu(x).$$

Thus, since $F > 0$ holds while ν has already been assumed to be strictly positive formerly, within a similar argument as on page 522 in [26] we ν-almost-sure receive

$$0 = \lambda_s\, \delta_s(x,\tau) + ln\big(H(s,x)\big)$$

or equivalently

(7.3.18)
$$H(s,x) = exp\{-\lambda_s\, \delta_s(x,\tau)\} = exp\{-\lambda(G_s)\, \delta_s(x,\tau)\}$$

which represents the precise minimizing $H(s,x)$ for fixed G_s.

74 In contrast to Chan [26], we will motivate the structure of the Lagrange function (7.3.16) in more detail in the closing section 7.6.

What remains is the computation of the minimizing G_s corresponding to the *partial* solution (7.3.18): Substituting (7.3.18) into (7.3.15), the argument therein becomes

(7.3.19)

$$\frac{G_s^2}{2} + \int_{\mathbb{R}_0} \left[1 - \{1 + \lambda_s\, \delta_s(x,\tau)\}\, e^{-\lambda_s\, \delta_s(x,\tau)}\right] d\nu(x).$$

Again, let us omit the subscripts and arguments yet in (7.3.19) for a moment, defining the functional

(7.3.20)

$$\mu(G) := \frac{G^2}{2} + \int_{\mathbb{R}_0} \left[1 - \{1 + \lambda(G)\, \delta\}\, e^{-\lambda(G)\, \delta}\right] d\nu$$

which corresponds to "(3.23) in [26]". In order to receive its derivative with respect to G, once more, we interchange the differential and the integral operator yielding (cf. eq. "(3.24) in [26]")

$$\mu'(G) := \frac{\partial \mu}{\partial G} = G + \lambda(G)\, \lambda'(G) \int_{\mathbb{R}_0} \delta^2\, e^{-\lambda(G)\, \delta} d\nu.$$

Unfortunately, it is not straightforward to solve the *necessary* optimality condition $\mu'(G) = 0$ for G, since the function $\lambda(\cdot)$ is not known explicitly. Computing the second derivative of μ with respect to G, we can at least ensure convexity of μ, if we require the following *convexity-constraint*

$$\mu''(G) > 0 \;\Leftrightarrow\; \int_{\mathbb{R}_0} \delta^2\, e^{-\lambda(G)\, \delta}\, [\lambda'(G)^2\, \{1 - \lambda(G)\, \delta\} + \lambda''(G)\, \lambda(G)]\, d\nu > -1.$$

Recalling the positivity of the Lévy-measure ν, we could alternatively demand the stronger (and hence, *not best possible*) restriction

$$\lambda''(G)\, \lambda(G) > \{\lambda(G)\, \delta - 1\}\, \lambda'(G)^2$$

to guarantee positivity of μ'' and thus, convexity of μ likewise. If we assume the latter inequality to be in force, then – in order to ensure *minimality* – we solely have to require the condition

(7.3.21)

$$0 = \mu'(G) = G + \lambda(G)\, \lambda'(G) \int_{\mathbb{R}_0} \delta^2\, e^{-\lambda(G)\, \delta} d\nu.$$

On the other hand, substituting (7.3.18) into our extended drift-restriction (7.2.22), we get

$$0 = \frac{s_t(\tau)^2}{2} - a_t(\tau) - s_t(\tau)\, G_t + \int_{\mathbb{R}_0} \left[g_t(x,\tau)\, \mathbb{1}_{|x|<1} + \delta_t(x,\tau)\, e^{-\lambda(G_t)\, \delta_t(x,\tau)}\right] d\nu(x)$$

whereby a differentiation of the latter equation with respect to G_t (for fixed t) yields

(7.3.22)

$$0 = s_t(\tau) + \lambda'(G_t) \int_{\mathbb{R}_0} \delta_t(x,\tau)^2\, e^{-\lambda(G_t)\, \delta_t(x,\tau)}\, d\nu(x).$$

Comparing (7.3.21) with (7.3.22), [for $\lambda(G) \neq 0$] we finally end up with

(7.3.23) $$G_t = \lambda_t \, s_t(\tau).$$

Substituting the *minimizing pair* "(7.3.18) and (7.3.23)" into equation (7.2.22), the precise HJM-drift-restriction associated to *minimal relative entropy* ultimately possesses the structure

(7.3.24)
$$0 = \left(\frac{1}{2} - \lambda_t\right) s_t(\tau)^2 - a_t(\tau) + \int_{\mathbb{R}_0} \left[g_t(x,\tau) \, \mathbb{1}_{|x|<1} + \delta_t(x,\tau) \, e^{-\lambda_t \, \delta_t(x,\tau)} \right] d\nu(x)$$

which can be associated with eq. "(3.25) in [26]". In the closing section 7.6 we will provide some further examinations concerning the just discussed minimization procedure. In particular, we therein explain why the above *Lagrange function* precisely has been chosen like in (7.3.16).

7.3.3 Comparing the Esscher transform with the measure of minimum relative entropy

In order to force correspondence between the minimum relative entropy measure $\mathbb{Q}^* := \mathbb{Q}^*(G^*, H^*)$ and the EMM $\widetilde{\mathbb{Q}} := \widetilde{\mathbb{Q}}(\tilde{\theta})$ induced by the Esscher transform, (parallel to [26]) we have to compare the drift restrictions (7.3.10) and (7.3.24) yielding the subsequent *correspondence requests*

(7.3.25) $$\theta_t = -\lambda_t \, s_t(\tau) \quad \text{and} \quad \theta_t x = \lambda_t \, \delta_t(x,\tau).$$

Hence, in contrast to the announcements on the top of p.523 in [26], in our HJM-jump-diffusion model the measure of minimum relative entropy $\mathbb{Q}^*$ does (at least on a first glance) not directly correspond to the measure $\widetilde{\mathbb{Q}}$ induced by the Esscher transform. Nevertheless, by solving (7.3.25), we may obtain the precise Esscher parameter θ_t which leads us [via (7.3.5)] to a specific *Esscher EMM* $\widetilde{\mathbb{Q}}^*$ that *simultaneously* minimizes the relative entropy (6.4.1). To see this, we argue as follows: Eliminating θ_t inside (7.3.25), for $\lambda_t \neq 0$ we get the *correspondence condition*

(7.3.26) $$\delta_t(x,\tau) = -s_t(\tau) \, x.$$

Implanting (7.2.10) and (7.2.15) into the latter equation, for all $t \leq \tau$ we achieve

(7.3.27)
$$e^{-\int_t^\tau \beta_t(x,u)du} \, \mathbb{1}_{|x|\geq 1} + e^{-\int_t^\tau \gamma_t(x,u)du} \, \mathbb{1}_{|x|<1} = 1 - x \int_t^\tau \sigma_t(u) \, du.$$

Note that (for $t < \tau$) the right hand side of (7.3.27) denotes a decreasing linear function in x, since the volatility $\sigma_t(u)$ has been assumed to be strictly positive and thus, so is the integrated volatility $s_t(\tau)$. In what follows, we construct a case wherein the correspondence condition (7.3.26) is at least approximately fulfilled: Firstly, let us recall the definition of the Lévy-measure associated to $\hat{p}$ due to

(7.3.28) $$\nu(\mathcal{B}) := \nu_{\hat{p}}(\mathcal{B}; [0,\varrho]) := \mathbb{E}_{\mathbb{Q}}[\#\{t \in [0,\varrho]: \, \Delta\hat{p}_t(\tau) \neq 0, \, \Delta\hat{p}_t(\tau) \in \mathcal{B}\}]$$

for times $0 \leq \varrho \leq \tau \leq T$ (with τ fixed) and a Borel-set $\mathcal{B} \in \mathfrak{B}(\mathbb{R}_0)$. Verbalizing, $\nu(\mathcal{B})$ counts the $\mathbb{Q}$-expected number of jumps in the interval $[0,\varrho]$ with a (non-zero) jump-amplitude belonging to the set $\mathcal{B}$. As on p.524 in [26], we next presume that the discounted commodity forward price $\hat{p}$ mostly makes very small jumps under $\mathbb{Q}$, say $0 \neq |\Delta\hat{p}_t(\tau)| \leq \varepsilon$ for $0 \leq t \leq \tau$ and a small strictly positive number ε.

Thus, we expect a large quantity of jumps within a (non-zero) jump-size between $-\varepsilon$ and ε so that the set $[-\varepsilon,\varepsilon] \setminus \{0\} \subset \mathbb{R}_0$ will 'weigh heavily' under ν. Interpreting the latter, we certify the Lévy-measure ν to be "*concentrated around zero*" (see p.524 in [26]). Note in passing that in this case we only have to care about the instance $|x| < 1$ in (7.3.27). Hence, assuming the Lévy-measure ν to be *concentrated around zero* yet, we can use (7.3.27) to specify the coefficient $\gamma_t(x,u)$, given $\sigma_t(u)$, in the following way: For $|x| < 1$ and $0 \leq t \leq \tau$ equation (7.3.27) simplifies to

(7.3.29) $$\int_t^\tau \gamma_t(x,u)\,du = -ln\left(1 - x\int_t^\tau \sigma_t(u)\,du\right).$$

Differentiating (7.3.29) with respect to the upper integration bound τ, we next obtain

(7.3.30)

$$\gamma_t(x,\tau) = \frac{x\,\sigma_t(\tau)}{1 - x\int_t^\tau \sigma_t(u)\,du}.$$

Thus, similar to the original (Brownian) Heath-Jarrow-Morton model in [49], due to a given volatility $\sigma_t(\tau)$ all other model coefficients (besides the initial condition) are determined simultaneously (also recall Remark 7.2.2 in this context). Summing up, we may state that under the above *concentrated around zero assumption* the condition (7.3.26) is at least approximately fulfilled. Therefore, in accordance to our recent case study, the Esscher transform spawns a measure which even admits *approximately* minimum relative entropy in the sense of obeying (7.3.25) – cf. p.524 in [26]. Eventually, we remark that a similar case study can be done by assuming the Lévy-measure ν to be *bounded from below* instead, yet dealing with the instance $|\Delta\hat{p}_t(\tau)| > \varepsilon$, resp. with $|x| \geq 1$ in (7.3.27).

7.4 Pricing commodity forward options

Let us now turn to the topic of pricing commodity derivatives. At first, we assume that the discounted power forward price (7.2.32) is given in the same currency as the underlying strike price K, thus in EURO. Anyway, Hinz et al. model the (electricity) forward price $p_t(\tau)$ in MWh (cf. p.8 in [59]). For this reason, in their setup $\hat{p}_t(\tau)$ is given in MWh/EURO, since the bank account B_t bears in EURO. Hence, the problem of *intermingled currencies* in the European call option payoff

(7.4.1) $$C_\tau \coloneqq (\hat{p}_\tau(T) - K)^+ \coloneqq max\{0, \hat{p}_\tau(T) - K\}$$

$(\tau \leq T)$ arises, as the strike price K in [59] still is given in EURO. To overcome this problem, one has to invest a great effort in *Change-of-Numéraire techniques*[75] (as in Ch. 3 of [59]) to fit the different currencies, respectively to achieve an appropriate *currency change*. Furthermore, on p.11 in [59] a detaining *time-change property for continuous (semi-) martingales*[76] has to be troubled in order to compute the conditional expectation appearing inside the risk-neutral pricing formula

(7.4.2) $$\frac{C_t}{B_t} = \mathbb{E}_{\mathbb{Q}}\left(\frac{C_\tau}{B_\tau}\middle|\mathcal{F}_t\right), \quad t \in [0,\tau]$$

by terms of a usual expectation.

[75] For further reading on *Change-of-Numéraire techniques* see Chapter 9 in [83]. Additionally, the interested reader is advised to Chapter 2 in [14] or Chapter 3 in [51] to read more about a possible conversion of, particularly, *electricity* markets into money markets by Change-of-Numéraire techniques.
[76] See section 3.4 B, Theorem 4.6, in Karatzas and Shreve [62] for more details.

In the following, we aim to derive a pricing formula for a European commodity call option based upon our extended jump-diffusion HJM-model. Before doing so, we briefly present the continuous version (without jumps) related to a Black-Scholes setup in the next subsection (also compare paragraph 9.1.1 in [13]). We do this not only for the sake of completeness, but also to present an alternative way to proof the pricing formula declared in Prop. 1 in [59]. Actually, we suppose the coefficients in (7.2.4) [and thus, also those in (7.2.10) and (7.2.15)] to be deterministic for the remainder of section 7.4.

7.4.1 Commodity forward option prices in the continuous Black-Scholes case

When there are no jumps involved in the underlying (discounted) commodity forward price $\hat{p}_t(\tau)$, then the dynamics in (7.2.26) simply reads as

$$d\hat{p}_t(\tau) = -\hat{p}_t(\tau)\, s_t(\tau)\, d\widetilde{W}_t$$

bearing the continuous Doléans-Dade solution

$$\hat{p}_t(\tau) = \hat{p}_0(\tau)\, \mathfrak{E}\big(-s_\cdot(\tau) \circ \widetilde{W}_\cdot\big)_t := \hat{p}_0(\tau)\, exp\Big\{-\int_0^t s_u(\tau)\, d\widetilde{W}_u - \tfrac{1}{2}\int_0^t s_u(\tau)^2 du\Big\}. \tag{7.4.3}$$

Yet, we provide the following theorem which essentially corresponds to Proposition 9.1 in [13].

Theorem 7.4.1 *As above, let $0 \leq t \leq \tau \leq T$. In what follows, we denote the constant interest rate*[77] *by r while Φ stands for the cumulative standard normal distribution function. Then the European commodity call option EURO price C_t at time t (prior to the exercise time*[78] *τ) written on a commodity forward with delivery time T and strike price K (in EURO) is given by*

$$C_t := C_t(K,\tau,T) = e^{-r(\tau-t)}[\hat{p}_t(T)\, \Phi(d_1) - K\, \Phi(d_2)] \tag{7.4.4}$$

with arguments

$$d_{1,2} = \frac{ln\left(\frac{\hat{p}_t(T)}{K}\right) \pm \frac{1}{2}\int_t^\tau s_u(T)^2\, du}{\sqrt{\int_t^\tau s_u(T)^2\, du}}.$$

Proof Assuming a constant interest rate $r_t \equiv r$ in (7.2.16), equation (7.4.2) transforms into

$$C_t = e^{-r(\tau-t)}\, \mathbb{E}_{\mathbb{Q}}(C_\tau | \mathcal{F}_t) \tag{7.4.5}$$

where $0 \leq t \leq \tau$. Further, appealing to (7.4.1), the call option payoff C_τ can be expressed as

$$C_\tau = \left(\hat{p}_t(T)\frac{\hat{p}_\tau(T)}{\hat{p}_t(T)} - K\right)^+. \tag{7.4.6}$$

[77] Admittedly, assuming a *constant* interest rate $r_t \equiv r$ sounds somewhat contradictory with respect to the underlying HJM-framework, since we formerly have announced $r_t = f_t(t)$ in (7.2.5). However, we recall that in [13] and [59] the same assumption is supposed. Actually, we could easily get rid of the problematic *stochastic* discount factor B appearing in (7.4.2) by applying standard Change-of-Numéraire techniques (see e.g. [14], [51], [59], [83]). Yet, section 9.4.3 in [83] particularly deals with option pricing purposes under a *random* interest rate. We leave the workout of the details to the reader, as we instead want to focus on the topic of commodity forward option pricing with Fourier transforms under our extended jump-diffusion approach in the next section.

[78] The trading period ends at the *exercise time* τ, whereas the delivery takes place at the *maturity time* T $(\geq \tau)$.

Thus, linking (7.4.5) with (7.4.6) while implanting the representation (7.4.3), we receive

$$C_t = e^{-r(\tau-t)}\, \mathbb{E}_{\mathbb{Q}}\left(\left[\hat{p}_t(T)\, exp\left\{-\int_t^\tau s_u(T)\, d\widetilde{W}_u - \frac{1}{2}\int_t^\tau s_u(T)^2 du\right\} - K\right]^+ \middle| \mathcal{F}_t\right)$$

(cf. p.239 in [13]). Since $\hat{p}_t(T)$ is strictly positive and $\mathcal{F}_t$-adapted, the latter equality can be written as

(7.4.7)

$$C_t = e^{-r(\tau-t)}\, \hat{p}_t(T)\, \mathbb{E}_{\mathbb{Q}}\left(\left[e^{X_{\tau-t}} - \frac{K}{\hat{p}_t(T)}\right]^+ \middle| \mathcal{F}_t\right)$$

whereby [for a *deterministic* coefficient $s_u(T)$] we have currently introduced the abbreviation

$$X_{\tau-t} := X_{\tau-t}(T) := -\int_t^\tau s_u(T)\, d\widetilde{W}_u - \frac{1}{2}\int_t^\tau s_u(T)^2 du.$$

Note that $X_{\tau-t}$ depicts a drifted Brownian integral under $\mathbb{Q}$ which is *normally distributed* with *mean*

$$-\frac{1}{2}\int_t^\tau s_u(T)^2 du$$

and *variance*

$$\int_t^\tau s_u(T)^2 du.$$

Moreover, since $X_{\tau-t}$ and $\mathcal{F}_t$ are $\mathbb{Q}$-independent by definition, the conditional expectation in (7.4.7) reduces to a usual expectation which trivially can be computed by standard measure transformation arguments, ultimately yielding the desired result (similarly to the proof of Prop. 9.1 in [13]). ■

Remark 7.4.2 *Obviously, the pricing formula (7.4.4) reveals a very similar structure with respect to the celebrated Black-Scholes formula. Corresponding to subsection 9.1.1 in [13], we claim that in the above treated continuous case without jumps the risk of a possible loss can be hedged by holding a well defined number of commodity forwards given by the so-called* <u>*delta hedging strategy*</u>

(7.4.8)

$$\Delta_t := \Delta_t(K,\tau,T) := \frac{\partial C_t(K,\tau,T)}{\partial \hat{p}_t(T)}$$

(cf. equation "(9.2) in [13]"). Taking (7.4.4) into account, a straightforward application of the elementary product- and chain-rule from ordinary calculus delivers

(7.4.9)
$$\Delta_t = e^{-r(\tau-t)}\, \Phi\left(d_1\big(\hat{p}_t(T)\big)\right)$$

giving the precise number of commodity forwards an investor should hold at time $t\ (\leq \tau \leq T)$ *in his/her portfolio. We remark that equation (7.4.9) directly corresponds to Proposition 9.2 in [13].* ■

7.4.2 Commodity forward option prices in the case of jumps

As there is no explicit distribution function for the *discontinuous* commodity forward price (7.2.32) available, we cannot handle the conditional expectation appearing inside the risk-neutral pricing formula via a straightforward measure transformation (similarly to our arguing in the proof of Theorem 7.4.1 in the previous subsection) when it comes to commodity derivatives pricing in the jump-case (also see p.247 in [13]). On the contrary, inspired by the argumentation in subsection 9.1.2 in [13] (and moreover, by the results in [14], [22], [33] and [79]), we now adopt customized Fourier transform techniques to our mission. Instantly, let us introduce the real-valued local (Lévy-type-/Sato-) $\mathbb{Q}$-martingale

(7.4.10)

$$M_t(\tau) := -\int_0^t s_u(\tau)\, d\widetilde{W}_u - \int_0^t \int_{\mathbb{R}_0} k_{u-}(x,\tau)\, d\widetilde{N}_{\mathbb{Q}}(u,x)$$

within a (deterministic) integrand

(7.4.11) $$k_u(x,\tau) := \ b_u(x,\tau)\, \mathbb{1}_{|x|\geq 1} + g_u(x,\tau)\, \mathbb{1}_{|x|<1}.$$

In addition, we also assume the process H to be deterministic and define the strictly positive function

(7.4.12)

$$z_t(\tau) := \hat{p}_0(\tau)\, exp\left\{ -\int_0^t \left(\frac{s_u(\tau)^2}{2} + \int_{\mathbb{R}_0} [k_u(x,\tau) + \delta_u(x,\tau)]\, H(u,x)\, d\nu(x) \right) du \right\}.$$

Therewith, parallel to the arguing in subsection 9.1.2 in [13], we can express the $\mathbb{Q}$-dynamics (7.2.32) in the subsequent shorthand version

(7.4.13) $$\hat{p}_t(\tau) = z_t(\tau)\, e^{M_t(\tau)}$$

(cf. p.248 in [13]). Inserting (7.4.13) into the call price formula (7.4.1), we derive

(7.4.14)

$$C_\tau = \left[z_\tau(T)\, e^{M_\tau(T)} - K \right]^+.$$

Keeping the structure of (7.4.14) in mind, we next concentrate on the exponentially-damped call price function

(7.4.15) $$q(x) := q(x; K, a, \tau, T) := e^{-ax}\, [z_\tau(T)\, e^x - K]^+$$

within a real damping parameter $1 < a < \infty$ (compare Lemma 9.1 in [13] in this context). Note that $[z_\tau(T)\, e^x - K]^+ \notin \mathcal{L}^1(\mathbb{R})$, while $q(x) \in \mathcal{L}^1(\mathbb{R})$ holds true in return. Thus, the Fourier transform of $q(x)$ exists and can be computed via (3.2.32) yielding

$$\hat{q}(y) = \int_{\mathbb{R}} e^{-(a+iy)x}\, [z_\tau(T)\, e^x - K]^+\, dx = \int_{\mathbb{R}} z_\tau(T)\, e^{-(a+iy)x} \left[e^x - \frac{K}{z_\tau(T)} \right]^+ dx$$

$$= z_\tau(T) \int_{ln\left(\frac{K}{z_\tau(T)}\right)}^{\infty} e^{-(a+iy)x} \left[e^x - \frac{K}{z_\tau(T)} \right] dx.$$

Recalling the fact $|e^{-iyx}| = 1$, a straightforward calculation finally delivers

(7.4.16)

$$\hat{q}(y) = \frac{K}{(a+iy)(a-1+iy)} \left(\frac{z_\tau(T)}{K} \right)^{a+iy}$$

(cf. Lemma 9.1 in [13]). Now we are able to state the discontinuous analogue of Theorem 7.4.1 which actually corresponds to Proposition 9.4 in [13].[79]

Theorem 7.4.3 *Denoting the risk-less interest rate by r, the EURO price $C_t := C_t(K, \tau, T)$ at time $t\ (\leq \tau)$ of a European commodity call option with exercise time $\tau\ (\leq T)$ and strike price $K > 0$ (in EURO) written on a commodity forward maturing at the delivery time T is given by*

(7.4.17)

$$C_t = \frac{e^{-r(\tau-t)}}{2\pi} \int_{\mathbb{R}} \hat{q}(y)\, exp\left\{ (a+iy)\, M_t(T) + \frac{(a+iy)^2}{2} \int_t^\tau s_u(T)^2\, du + \psi(y,t,\tau,T) \right\} dy.$$

Herein, M and $\hat{q}$ are such as declared in (7.4.10) and (7.4.16) respectively, whereas the (deterministic) characteristic exponent $\psi(y,t,\tau,T)$ is explicitly given through

(7.4.18)

$$\psi(y,t,\tau,T) := \int_t^\tau \int_{\mathbb{R}_0} \left[e^{i\,\theta_u(x,y,T)} - 1 - i\,\theta_u(x,y,T) \right] H(u,x)\, d\nu(x)\, du$$

within a shorthand notation $\theta_u(x,y,T) := (ia - y)\, k_u(x,T)$.

Proof (Cf. the proof of Prop. 9.4 in [13].) Combining (7.4.14) with (7.4.15), we derive

$$C_\tau = e^{a\, M_\tau(T)} q\big(M_\tau(T)\big).$$

Recalling (3.2.33), the latter can be transformed into

$$C_\tau = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{q}(y)\, e^{(a+iy)\, M_\tau(T)} dy.$$

Therewith, the risk-neutral pricing formula (7.4.2) points out as

(7.4.19)

$$C_t = \frac{e^{-r(\tau-t)}}{2\pi} \int_{\mathbb{R}} \hat{q}(y)\, \mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy)\, M_\tau(T)} \big| \mathcal{F}_t \big)\, dy.$$

In the following, we handle the conditional expectation appearing in (7.4.19) separately: Since $M_\tau(T) - M_t(T)$ and $\mathcal{F}_t$ are $\mathbb{Q}$-independent, with respect to (7.4.10) we receive

[79] Note that in Prop. 9.4 in [13] the characteristic exponent (7.4.18) is not given explicitly on the contrary.

(7.4.20) $$\mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy)\,M_\tau(T)}\big|\mathcal{F}_t\big) = e^{(a+iy)\,M_t(T)}\,\mathbb{E}_{\mathbb{Q}}\big[e^{(a+iy)\,\{M_\tau(T)-M_t(T)\}}\big] =$$

$$e^{(a+iy)\,M_t(T)}\,\mathbb{E}_{\mathbb{Q}}\left[exp\left\{i\left(\int_t^\tau (ia-y)\,s_u(T)\,d\widetilde{W}_u + \int_t^\tau\int_{\mathbb{R}_0}(ia-y)\,k_{u-}(x,T)\,d\widetilde{N}_{\mathbb{Q}}(u,x)\right)\right\}\right].$$

As $\widetilde{W}$ and $\widetilde{N}_{\mathbb{Q}}$ have a priori been assumed to be $\mathbb{Q}$-independent, equality (7.4.20) factors into the triplet

(7.4.21) $$\mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy)\,M_\tau(T)}\big|\mathcal{F}_t\big) = e^{(a+iy)\,M_t(T)} \times \mathfrak{J}_1 \times \mathfrak{J}_2$$

with usual expectations

$$\mathfrak{J}_1 := \mathbb{E}_{\mathbb{Q}}\left[exp\left\{-\int_t^\tau (a+iy)\,s_u(T)\,d\widetilde{W}_u\right\}\right],$$

$$\mathfrak{J}_2 := \mathbb{E}_{\mathbb{Q}}\left[exp\left\{i\int_t^\tau\int_{\mathbb{R}_0}(ia-y)\,k_{u-}(x,T)\,d\widetilde{N}_{\mathbb{Q}}(u,x)\right\}\right].$$

Applying *Itô's isometry*, a straightforward calculation yields

(7.4.22)
$$\mathfrak{J}_1 = exp\left\{\frac{(a+iy)^2}{2}\int_t^\tau s_u(T)^2 du\right\}.$$

Further on, we recall our former definition $\theta_u(x,y,T) := (ia-y)\,k_u(x,T)$ and therewith, rewrite the compensated jump-integral appearing inside $\mathfrak{J}_2$ with respect to (2.2.8) as Lévy-Itô-decomposed

(7.4.23)
$$\mathfrak{J}_2 = \mathbb{E}_{\mathbb{Q}}\left[exp\left\{i\int_t^\tau\int_{\mathbb{R}_0}\theta_{u-}(x,y,T)\,d\widetilde{N}_{\mathbb{Q}}(u,x)\right\}\right] =$$

$$\mathbb{E}_{\mathbb{Q}}\left[exp\left\{i\left(-\int_t^\tau\int_{|x|\geq 1}\theta_u(x,y,T)\,H(u,x)\,d\nu(x)\,du + \int_t^\tau\int_{|x|<1}\theta_{u-}(x,y,T)\,d\widetilde{N}_{\mathbb{Q}}(u,x) + \int_t^\tau\int_{|x|\geq 1}\theta_{u-}(x,y,T)\,dN(u,x)\right)\right\}\right].$$

Similarly to our former arguing in (3.2.41), an application of the *generalized Lévy-Khinchin formula* (combine Prop. 8 in [35] or Prop. 1.9 in [65] with Prop. 2.1 in [13]) finally delivers

(7.4.24)
$$\mathfrak{J}_2 = exp\left\{\int_t^\tau\int_{\mathbb{R}_0}\left[e^{i\,\theta_u(x,y,T)} - 1 - i\,\theta_u(x,y,T)\right]H(u,x)\,d\nu(x)\,du\right\} = e^{\psi(y,t,\tau,T)}$$

within a characteristic exponent $\psi(y,t,\tau,T)$ such as defined in (7.4.18) above. Implanting (7.4.21), (7.4.22) and (7.4.24) into the pricing formula (7.4.19), we get the claimed result. ■

In order to evaluate the pricing formula (7.4.17) with respect to the EMM $\widetilde{\mathbb{Q}}$ induced by the Esscher transform (the so-called *Esscher-EMM*; compare subsection 7.3.1), we would first need an explicit (deterministic) solution of equation (7.3.11), say $\tilde{\theta}_t$, which would lead us via (7.3.9) to

(7.4.25)
$$\widetilde{H}(u,x) := e^{-\tilde{\theta}_u x}.$$

This specific coefficient $\widetilde{H}$ then could be weaved into (7.4.17) by dint of (7.4.10), (7.4.12), (7.4.16) and (7.4.18), while replacing H by $\widetilde{H}$ in the latter equalities. On the other hand, in order to work out the commodity call option price (7.4.17) under the minimum relative entropy EMM $\mathbb{Q}^*$ (compare subsection 7.3.2), we would first need to know the concrete minimizing solution G_u^* and $H^*(u,x)$ such as introduced in the sequel of (7.3.15). Ultimately, the proper coefficient H^* then could be merged into (7.4.17) – similar to the way described for $\widetilde{H}$ above – and thus, would yield the explicit commodity call option price related to the minimum relative entropy measure $\mathbb{Q}^*$ in return.

7.5 Conclusions

In order to model commodity forward prices adequately, we have suggested an extended Heath-Jarrow-Morton setup permitting random jumps at random time points via additive compound Poisson-type processes, while we have distinguished between small and large jump sizes. By the way, we have presented an alternative derivation modality for the involved forward price dynamics using the *Leibniz-rule for parameter integrals* instead of troubling the stochastic Fubini-Tonelli theorem laboriously as suggested in [17], respectively [18]. Having derived an extended HJM-drift-restriction associated to our jump-diffusion case, we subsequently have adapted generalized Esscher transform methods to our purposes. Moreover, with respect to our present incomplete market model, minimum relative entropy techniques have been elaborated to determine a specific equivalent martingale measure out of the large class of offering pricing probabilities. Dealing with this subject, we have invested some pursuing effort concerning the connected minimization procedure, while adapting the *successive Lagrange approach* supposed in [26] to our requirements. In order to compute risk-neutral European commodity call option prices, we finally have applied tailor-made Fourier transform techniques such as proposed in [13]. Regarding the resulting pricing formula (7.4.17), we remark that there obviously is a need for numerical pricing methods in order to evaluate explicit commodity option prices. Ultimately, we highlight that (7.4.17) descends from an *exponential* model – see (7.4.13). In this concern, it appears interesting to compare our argumentation in subsection 7.4.2 with that in paragraph 3.2.4, for example, while the latter is dedicated to an *arithmetical* approach on the contrary.

7.6 Appendix: The Lagrange function

In this closing section we want to give some additional explanatory comments on the structure of the Lagrange function (7.3.16) while exploring the connected minimization procedure in more detail. Referring to our minimization exercise (7.3.15), for a fixed time parameter s we initially declare

(7.6.1)
$$f(G,H) := f\big(G_s, H(s,x)\big) := \frac{G_s^2}{2} + \int_{\mathbb{R}_0} [1 + H(s,x)\,\{\ln H(s,x) - 1\}]\,d\nu(x)$$

with $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ as our *target function*. Next, in accordance to our extended drift-restriction (7.2.22), we introduce the functional $g: \mathbb{R}^2 \longrightarrow \mathbb{R}$ due to

(7.6.2)
$$g(G,H) := g\big(G_s, H(s,x)\big) :=$$
$$\frac{s_s(\tau)^2}{2} - a_s(\tau) - s_s(\tau)\, G_s + \int_{\mathbb{R}_0} \big[g_s(x,\tau)\, \mathbb{1}_{|x|<1} + \delta_s(x,\tau)\, H(s,x)\big]\, d\nu(x)$$

(again for a fixed time parameter s). Hence, with respect to (7.3.15), our objective yet consists in a proper minimization of the target function (7.6.1) under the constraint

(7.6.3)
$$g\big(G_s, H(s,x)\big) = 0.$$

In conclusion, the *Lagrange multiplier method* (see section 6.2.5.6 in [19]) then requires the equality

(7.6.4)
$$\nabla f(G,H) = \lambda\ \nabla g(G,H)$$

to be valid, wherein $\lambda \in \mathbb{R}$ embodies the *Lagrange multiplier* and ∇ denotes the *gradient operator*. Appealing to (7.6.1) and (7.6.2), in vectorial notation equation (7.6.4) turns out as

(7.6.5)
$$\begin{pmatrix} G_s \\ \int_{\mathbb{R}_0} \ln H(s,x)\, d\nu(x) \end{pmatrix} = \lambda \begin{pmatrix} -s_s(\tau) \\ \int_{\mathbb{R}_0} \delta_s(x,\tau)\, d\nu(x) \end{pmatrix}.$$

From this we obtain the equality system

(7.6.6)
$$G_s = -\lambda\, s_s(\tau) \quad \wedge \quad \int_{\mathbb{R}_0} [\ln H(s,x) - \lambda\, \delta_s(x,\tau)]\, d\nu(x) = 0$$

(with fixed time-parameter s), which ν-almost-sure is equivalent to

(7.6.7)
$$G_s = -\lambda\, s_s(\tau) \quad \wedge \quad H(s,x) = e^{\lambda\, \delta_s(x,\tau)}$$

since we have assumed ν to be positive in the sequel of (7.2.3) – also recall the arguing on p.522 in [26] at this step. Thus, comparing (7.6.7) with (7.3.18) and (7.3.23)[80], we achieve a better understanding why the *fitted Lagrange function* precisely has been chosen like in (7.3.16). Further on, referring to (7.6.2), (7.6.3) and (7.6.7), we have the *three* following equalities available

(7.6.8)
$$\begin{cases} \textbf{(I)} \ \dfrac{s_s(\tau)^2}{2} - a_s(\tau) - s_s(\tau)\, G_s^* + \displaystyle\int_{\mathbb{R}_0} \big[g_s(x,\tau)\, \mathbb{1}_{|x|<1} + \delta_s(x,\tau)\, H^*(s,x)\big]\, d\nu(x) = 0 \\ \textbf{(II)} \qquad G_s^* = -\lambda\, s_s(\tau) \\ \textbf{(III)} \qquad \ln H^*(s,x) = \lambda\, \delta_s(x,\tau) \end{cases}$$

(ν-almost-sure) in order to determine the *three* values G_s^*, $H^*(s,x)$ and λ.

[80] The opposite signs do not at all constitute a serious matter here, as we could equally well have considered (7.6.4) for λ replaced by $-\lambda$, obviously.

Parallel to our earlier notation in the sequel of (7.3.15), we here denote the *critical values* by G_S^* and $H^*(s,x)$ again. Adhering to (7.2.15), we reasonably assume the coefficient δ to be non-zero, so that equation (**III**) turns out to be equivalent to

(7.6.9) $$\lambda = \frac{\ln H^*(s,x)}{\delta_S(x,\tau)}.$$

Substituting (7.6.9) into equation (**II**), we immediately obtain

(7.6.10) $$G_S^* = -\frac{s_S(\tau)}{\delta_S(x,\tau)} \ln H^*(s,x).$$

Hence, putting (7.6.10) into (**I**), we finally end up with

(7.6.11)

$$s_S(\tau)^2 \left[\frac{1}{2} + \frac{\ln H^*(s,x)}{\delta_S(x,\tau)}\right] + \int_{\mathbb{R}_0} \left[g_S(x,\tau)\, \mathbb{1}_{|x|<1} + \delta_S(x,\tau)\, H^*(s,x)\right] d\nu(x) = a_S(\tau).$$

Examining (7.6.11) in more depth, we conclude that – after having chosen concrete coefficients s, a, δ, g and an appropriate form of the Lévy-measure ν (i.e. a density with respect to the Lebesgue-measure, for instance) – this equation may be solved numerically for $H^*(s,x)$ which simultaneously leads us to G_S^* via (7.6.10), ultimately. All in all, we underline that the equalities (7.6.10) and (7.6.11) indeed are suitable to derive the precise *minimizing pair-process* (G^*, H^*) which induces the minimum relative entropy measure $\mathbb{Q}^*(G^*, H^*)$ such as defined in the sequel of (7.3.15).

Chapter 8

Nonlinear Double-Jump Stochastic Filtering using Generalized Lévy-Type Processes

8.1 Introduction to stochastic filtering

Nonlinear stochastic filtering deals with the problem of estimating a dynamical signal system on the basis of perturbed or incomplete observations whereas a direct measurement of the underlying intriguing signal is only partially or even not possible (cf. e.g. [2], [70], [77]). In this chapter both the signal variable X and the observation process Y innovatively are modeled via fairly general jump-diffusion Lévy-type stochastic processes for the first time in the literature (at least to the best of our knowledge). Moreover, an extended Zakai- and Kushner-Stratonovic-Equation is derived, the latter representing our optimal filter in the least-squares sense. Finally, selected applications taken from electricity and emission markets are presented. Anyway, we start off by giving some introductory comments on the topic of stochastic filtering, simultaneously providing a short literature overview.

The problem of estimating a partially hidden dynamical signal system is in general a difficult task, since information about the state process X only can be obtained by extracting those contained in the noisy observation process Y using a challenging mathematical procedure provided by modern stochastic calculus (cf. p.1 in [70]). If any of the functions involved in the underlying filtering setup is nonlinear or "*if a jump term is present, then it is rarely possible to [derive] the conditional distribution [of the signal] by a 'finite [analytical] computation'*" (see p.559 in [76]). However, in the literature there are two fundamental ways to solve such a mentioned filtering problem: On the one hand, we have the *Change-of-Measure method*, which makes use of an adjusted version of Girsanov's theorem, and on the other, there is the *Innovation-Process approach* (see e.g. pp. 52 and 70 in [2]).

In this chapter we trouble the Change-of-Measure method wherein an equivalent martingale measure (EMM) is constructed under which the observation process Y becomes a martingale that is stochastically independent of the signal X. Further, the conditional distribution of the transformed state variable $f(X)$, given the measurement information Y, then possesses a representation in terms of an associated *non-normalized* process which satisfies a specific stochastic differential equation (SDE) known as the Zakai-Equation [70], [77]. Finally, Itô-calculus tells us how to derive the so-called Kushner-Stratonovic-Eq. representing the *normalized* conditional distribution of the signal in return.

Although stochastic filtering has several important applications in modern signal processing reaching from the biomedical area and chemistry to micro-electronics, mechanics and engineering, ending up in the economic and finance branch (just to nominate a few), to the best of our knowledge there is not a single work in the literature dealing with nonlinear filtering in a generalized Lévy-type *double*-jump diffusion case. Nevertheless, continuous-time diffusion models without jumps have rigorously been studied by Bain and Crisan [2] and Xiong [84], recently. Furthermore, Mandrekar, Meyer-Brandis and Proske [70] deal with a nonlinear filtering model wherein the observation process contains both general Gaussian noise as well as Cox noise whose jump-intensity explicitly depends on the drifted continuous signal itself. An analogous setting with the state X a diffusion and the observation process Y a generalized Lévy-type process can be found in Meyer-Brandis and Proske [71], where in addition financial filtering applications are treated (see Ch. 4 therein for the calibration of a jump-diffusion model via nonlinear filtering).

Recently, an arresting topic has been brought up by Calzolari, Florchinger and Nappo [20], as they have dealt with nonlinear filtering under *time-delayed* observations. More precise, in their framework a delaying time transformation modifies the time index of the observation process in a deterministic manner. However, this idea is related to the *stochastic* time-change procedure with *Lévy-subordinators* (see subsection 1.3.2 in [1] for further reading), as it can be seen as a special case of the latter but without any randomness 'under the new clock'. Yet, introducing *stochastic* time-changes to the filtering theory might bear a challenging future research topic. Note that, whenever multiple-sensor measurement scenarios cause information inputs at *random* (non-equidistant) time points (such as it is the case in [31]; see below), a *subordinated* filtering model might be appropriate. For more information about possible applications of Lévy-subordinators in stochastic finance (e.g. to model leverage effects) see Cont and Tankov [30], for instance. By the way, another inspiring example in [20] is embodied by the proposed time-delayed diffusion model, wherein the multi-dimensional observation process is available after a fixed delay τ only. To be precise, in this model setup no information is available during the time range $0 \leq t \leq \tau$, while after the *flashpoint* τ the *belated* observations eventually start off. Further on, a collateral case study in [20] deals with an information input that "*arrives by packets*" (cf. p.51 therein). Comparable situations arise in numerous applications, whenever an observer is confronted with measurements at *discrete* time steps merely: In between the observation points there is no new information available then, what makes the observed component look like a pure-jump compound Poisson process, i.e. some kind of randomized step function. Referring to the bottom of p.50 in [20], we claim that the above mentioned partially observable systems with delayed observations exemplarily appear in risk minimizing problems of financial models for incomplete markets.

Another challenging connection between nonlinear filtering and stochastic finance has been drawn by Cvitanic, Liptser and Rozovskii [31], as they have applied stochastic filtering techniques to extract the unobservable market volatility out of asset prices that have been observed at discrete random time points. Malcolm, James and Elliott [69] consider a "*risk-sensitive*" filtering model based upon a continuous signal process of mean-reverting Ornstein-Uhlenbeck type together with increasing standard Poisson process observations within a jump-intensity that explicitly depends on the signal. Additionally, a very detailed presentation of a nonlinear filtering problem has been published by Popa

and Sritharan [77], whereas the authors model the state process as a discontinuous semi-martingale with both diffusion and Cox noise while the observation process comes as a drifted Brownian motion (and thus without jumps, unfortunately).
As we have seen so far, there are very little publications in which *both* the signal *and* the observation process are *simultaneously* modeled as generalized Lévy-type jump-diffusions. To the best of our knowledge, the only approach that allows for a *double-jump* setting has been disclosed by Poklukar [76], wherein the signal indicates a jump-diffusion and the observation process comes as a standard Poisson process within a jump-rate depending on the signal itself and jump-times that are disjoint to the jumps of the latter. Thus, our aim in this chapter essentially consists in an extension of Poklukar's setting to a generalized Lévy-type approach with random jump-sizes at random jump-times in addition to Brownian observation noise *both* in the state *and* observation process. Particularly, we will permit complete Brownian *integrals* in the signal as well as in the observation process, instead of 'naked' Brownian motions solely such as appearing in the majority of the above mentioned filtering setups.

The remainder of the current chapter is organized as follows: In section 8.2 our extended double-jump nonlinear filtering problem is introduced mathematically while the associated stochastic evolution equations for the signal and the observation process are given explicitly. Afterwards, we apply a tailored version of Girsanov's Change-of-Measure theorem and switch to an equivalent probability measure $\mathbb{Q}$ under which the signal and the observation process become independent. The following paragraph 8.3 contains an extended representation for the non-normalized conditional distribution associated to the signal in terms of stochastic integrals, called extended Zakai-Equation throughout this work. Subsequently, an extended Kushner-Stratonovic-Equation is derived which constitutes our main result so far. Hereafter, in section 8.4 several selected applications taken from electricity and emission markets are presented, whereas we explicitly focus on suitable choices of the appearing coefficient processes. The most relevant conclusions are drawn in the closing section 8.5, wherein in addition some accompanying future research topics are mentioned briefly.

8.2 The nonlinear filtering problem

Generally speaking, nonlinear stochastic filtering deals with the estimation of a dynamical signal system, whereas a direct observation of the latter is not possible [77]. However, partial/perturbed measurements of the underlying signal often can be obtained in reality [77]. Hence, one may get information about the intriguing state X by observing (theoretically) the stochastic process Y which is – loosely speaking – a functional of X corrupted by additive noise (cf. p.1 in [77]). Yet, the aim of stochastic filtering lies in the derivation of the conditional distribution of the current state of the signal X, given the accumulated past information stored in the monotone increasing observation filtration

$$\mathcal{F}_t^Y := \sigma\{Y_s : 0 \le s \le t\}$$

(cf. e.g. [2], [70], [71], [84]). This *backward-looking* family of sigma-algebras contains all foretime information that has been collected by observing the process Y up to and including time t.

Remark 8.2.1 *At this early stage, we should spend a moment to think about a possible enlargement of the observation filtration $\mathcal{F}^Y$ also in the current filtering framework. We here recall that the increasing family of sigma-algebras $\mathcal{F}_t^Y$ does only "look into the past" while all available information coming from observing the process Y up to time t is stored in this "retro filtration". Actually, this is not always the case at hand when we are dealing with filtering applications in reality. To be precise, one could think of a measurement situation where some additional (but possibly stochastic) future* *information is available. Exemplarily, an observer might know that a specific future event will take place with certainty while the exact effects remain random. Taking such forward-looking knowledge into account, we would have to enlarge the observation filtration $\mathcal{F}^Y$ appropriately. Fitting the connections between stochastic filtering and such an enlargement of the information filtration as described above while taking forward-looking events into account portrays a rather challenging issue. Exemplarily, one might use stochastic filtering techniques to estimate the unobservable volatility in an electricity market model (state process), wherein the observable electricity spot price (observation process) might be modeled as a sum of non-Gaussian mean-reverting Ornstein-Uhlenbeck jump-processes under anticipative information as supposed in Ch. 3 formerly. In this case, one would have to derive the corresponding filtering equations (i.e. the Zakai- and Kushner-Stratonovic-Equation) for additive multi-factor mean-reverting (forward-looking) observations.* ■

To proceed in our arguing, let us recall the *best approximation property of conditional expectations* (for details see Xiong [84], Lemma 5.1 therein) which is crucial for the construction of our nonlinear filter. Precisely, we argue as follows: Suppose that the random variable U is square-integrable but not measureable with respect to the filtration $\mathcal{G}$. That is, the information cumulated in $\mathcal{G}$ does not completely determine the values of U. Then, the conditional expectation $V := \mathbb{E}(U|\mathcal{G})$ possesses the remarkable property that it yields the uniformly best approximation for U in the least-squares sense. Hence, the inequality

$$\mathbb{E}[(R-U)^2] \geq \mathbb{E}[(V-U)^2]$$

holds for all random variables R that are $\mathcal{G}$-measureable and square-integrable. Adhering to the above *least-square error criterion*, our nonlinear filtering problem comes down to the computation of the precise stochastic evolution equation that is fulfilled by the conditional expectation of $f(X_t)$, given the observation filtration $\mathcal{F}_t^Y$, under the probability measure $\mathbb{P}$, in symbols $\mathbb{E}_{\mathbb{P}}(f(X_t)|\mathcal{F}_t^Y)$ (cf. p.2 in [71] and p.1 in [77]). Herein, the test-function f allows for a (nearly) arbitrary transformation of the signal X_t whereas for most applications we can choose it as the identity. In this context, we introduce the shorthand notation

(8.2.1) $$P_t(f) := \mathbb{E}_{\mathbb{P}}(f(X_t)|\mathcal{F}_t^Y)$$

(cf. "(3.4) in [70]", resp. Def. 4.6 in [77]) which will be called the *normalized conditional distribution of X given Y* throughout this chapter. Therewith, our nonlinear stochastic filtering problem yet consists in the determination of the specific stochastic differential equation (SDE), namely the Kushner-Stratonovic-Equation, which is fulfilled by the (in the least-squares sense) *optimal* filter $P_t(f)$.

8.2.1 The representation of the signal and the observation process

We start off with the description of the mathematical basis of our filtering setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered complete probability space, whereas the monotone information filtration $\mathcal{F} := (\mathcal{F}_t)_{t\geq 0}$ is assumed to be *cad* (French: continue à droite). In addition, sticking to common filtering notations (see e.g. p.5 in [70]), we denote the (augmented) sigma-algebra generated by the signal up to time t by

(8.2.2) $$\mathcal{F}_t^X := \sigma\{X_s : 0 \leq s \leq t\}$$

and the (augmented) sigma-algebra generated by the observation process up to time t by

(8.2.3) $$\mathcal{F}_t^Y := \sigma\{Y_s : 0 \leq s \leq t\}.$$

In this context, we assume that $\mathcal{F}_t^X$ and $\mathcal{F}_t^Y$ a priori include all $\mathbb{P}$-null-sets.[81] As usually, the *overall*, respectively *global filtration* is given by

(8.2.4) $$\mathcal{F}_t := \mathcal{F}_t^X \vee \mathcal{F}_t^Y.$$

Further on, the d-dimensional *signal process* $X := (X_t)_{t\geq 0}$ is modeled as a generalized Lévy-type stochastic process in this work. Since we suppose that a direct observation of the discontinuous semimartingale X is not possible, the latter frequently is referred to as the *unobservable component*. More precisely, we assume that X admits a *càdlàg* (French: continue à droite avec des limites à gauche) representation under the true probability measure $\mathbb{P}$ that is of the fairly general Lévy-type form

(8.2.5)

$$X_t = X_0 + \int_0^t \alpha(s, X_s)\, ds + \int_0^t \beta(s, X_s)\, dW_s + \int_0^t \int_D \gamma(s-, X_{s-}, x)\, d\widetilde{N}_X^{\mathbb{P}}(s, x)$$

(cf. e.g. "(2.1) in [77]") wherein the random variable X_0 denotes the d-dimensional initial condition, W stands for a p-dimensional Brownian motion (BM) under $\mathbb{P}$ with $p \leq d$ and $\widetilde{N}_X^{\mathbb{P}}$ designates the $\mathbb{P}$-compensated d-dimensional integer-valued Poisson-Random-Measure (PRM) on $\mathbb{R}^+ \times D$ associated to the signal X within jump-amplitudes in the set $D \subseteq \mathbb{R}^d \setminus \{\mathbf{0}\}$. Further, the $\mathbb{P}$-compensator of the PRM $dN_X(s, x)$ is denoted by $d\nu_X(x) ds$ which is such as

(8.2.6) $$d\widetilde{N}_X^{\mathbb{P}}(s, x) := dN_X(s, x) - d\nu_X(x)\, ds$$

forms a martingale integrator under $\mathbb{P}$. Herein, the d-dimensional Lévy-measure ν_X declares a positive and finite Borel-random-measure on D that fulfills one of the following equivalent assumptions

(8.2.7) $$\int_D \|x\|^2 \wedge 1 \;\; d\nu_X(x) < \infty \quad \Leftrightarrow \quad \int_D \frac{\|x\|^2}{1+\|x\|^2}\, d\nu_X(x) < \infty.$$

Ultimately, the coefficients appearing in (8.2.5) such as

(8.2.8) $$\alpha: \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad \beta: \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times p}, \qquad \gamma: \mathbb{R}^+ \times \mathbb{R}^d \times D \longrightarrow \mathbb{R}^{d\times d}$$

altogether are assumed to be integrable and bounded.

Furthermore, we suppose that the signal process X may be partially observed via the *observation process* $Y := (Y_t)_{t\geq 0}$ which comes as a fairly general m-dimensional Lévy-type jump-diffusion with $m \leq d$. More precisely, we presume the explicit form under the measure $\mathbb{P}$ as

(8.2.9) $$Y_t =$$

$$Y_0 + \int_0^t \varepsilon(s)\, h(s, X_s)\, ds - \int_0^t \int_M \delta(s, y)\, e^{g(s, X_s, y)} d\nu_Y(y)\, ds + \int_0^t \varepsilon(s)\, dB_s + \int_0^t \int_M \delta(s-, y)\, dN_Y(s, y)$$

(which essentially extends "(1.1) in [70]"). In the latter equation the random variable Y_0 denotes the m-dimensional initial condition, B depicts a q-dimensional BM under $\mathbb{P}$ with $q \leq m$ and N_Y stands for the m-dimensional integer-valued PRM on $\mathbb{R}^+ \times M$ associated to Y within jump-amplitudes in the set $M \subseteq \mathbb{R}^m \setminus \{\mathbf{0}\}$. The m-dimensional Lévy-measure ν_Y is assumed to fulfill analogous conditions as ν_X in (8.2.7), whereas in return it plays the role of the $\mathbb{P}$-compensator of N_Y, that is, the object

[81] To get an idea of what may happen if we do *not* impose this assumption, see Remark 2.3 in [2].

(8.2.10) $$d\tilde{N}_Y^{\mathbb{P}}(s,y) := dN_Y(s,y) - d\nu_Y(y)\,ds$$

presently indicates a $\mathbb{P}$-martingale integrator. Moreover, the coefficients appearing in (8.2.9) such as

(8.2.11) $$\varepsilon\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^{m\times q}, \qquad h\colon \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^q,$$

$$\delta\colon \mathbb{R}^+ \times M \longrightarrow \mathbb{R}^{m\times m}, \qquad g\colon \mathbb{R}^+ \times \mathbb{R}^d \times M \longrightarrow \mathbb{R}$$

altogether are supposed to be integrable and bounded, whereby the so-called *sensor function*

$$H\colon \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^m \quad \text{with} \quad H(s,X_s) := \varepsilon(s)\,h(s,X_s)$$

usually is exploited to model the specific sensor characteristics associated to the underlying measurement procedure.

Let us further assume that all appearing random components in (8.2.5) and (8.2.9) such as $X_0, Y_0, W, B, (N_X, \nu_X)$ and (N_Y, ν_Y) are $\mathbb{P}$-independent of one another and that the coefficients $\alpha, \beta, \gamma, \varepsilon, h, \delta$ and g fulfill the usual linear-growth and Lipschitz-continuity conditions, so that both the signal equation (8.2.5) and the observation equation (8.2.9) possess unique strong solutions. (For details on the existence and uniqueness of solutions to SDEs see for instance Theorem 1.19 in [75] or Chapter 6.2 in [1].) *At this early point, we can already state that our above double-jump filtering approach contains the complete amount of filtering settings mentioned in the introductory section 8.1 as subclasses. In addition, our model is flexible enough to manage pure-jump cases, continuous diffusion cases or a mixture of these by setting a selection of the appearing coefficients equal to zero. Nevertheless, the most powerful filtering onset for numerous applications should be the double-jump diffusion case which is the main topic of the present chapter.*

For the sake of notational simplicity and to be able to focus on the main ideas, we restrict ourselves to the one-dimensional case with $d = p = m = q = 1$ in the following, remarking that the multi-dimensionality of the state and the observation process is not essential to understand the additional gain of our *double-jump* diffusion setup. An extension to higher-dimensional cases, which might be important in practice, does not require any essential new ideas and is of technical character only.

8.2.2 Switching to an equivalent martingale measure

For $M \subseteq \mathbb{R} \setminus \{0\}$ let us introduce the pure-jump Doléans-Dade exponential $\Lambda := (\Lambda_t)_{t\geq 0}$ defined by

(8.2.12)

$$\Lambda_t := exp\left\{\int_0^t \int_M g(s-, X_{s-}, y)\,d\tilde{N}_Y^{\mathbb{P}}(s,y) - \int_0^t \int_M \left[e^{g(s,X_s,y)} - 1 - g(s,X_s,y)\right] d\nu_Y(y)\,ds\right\}$$

(also recall eq. “(3.3) in [70]”). Then Itô’s formula yields the associated integro-SDE (ISDE)

(8.2.13)

$$d\Lambda_t = \Lambda_{t-} \int_M \left[e^{g(t-,X_{t-},y)} - 1\right] \tilde{N}_Y^{\mathbb{P}}(t,dy)$$

which classifies Λ as a local martingale under $\mathbb{P}$. In addition, we presume g to be chosen such that $\mathbb{E}_{\mathbb{P}}[\Lambda_t] = 1$ for all $t \geq 0$ what declares the exponential Λ as a true $\mathbb{P}$-martingale (cf. sect. 2.2 above).

Analogously, we define the continuous Doléans-Dade exponential $\Lambda' := (\Lambda'_t)_{t\geq 0}$ via

(8.2.14)

$$\Lambda'_t := \mathfrak{E}(-h(\cdot, X.) \circ B.)_t := exp\left\{-\int_0^t h(s, X_s)\, dB_s - \frac{1}{2}\int_0^t h^2(s, X_s)\, ds\right\}$$

(recall "(4.2) in [77]", resp. p.6 in [70]) also yielding a stochastic differential with vanishing drift

(8.2.15)
$$d\Lambda'_t = -\Lambda'_t\, h(t, X_t)\, dB_t.$$

Thus, Λ'_t forms a local martingale under $\mathbb{P}$. Yet, we presume h to be chosen such that $\mathbb{E}_{\mathbb{P}}[\Lambda'_t\,] = 1$ holds for all $t \geq 0$ (resp. such that h fulfills a *Novikov condition*; see "(4.3) in [77]") what either declares Λ' as a true $\mathbb{P}$-martingale. Finally, we switch to the (with respect to $\mathbb{P}$) equivalent martingale measure (EMM) $\mathbb{Q}$ due to the Radon-Nikodym density process $Z := (Z_t)_{t\geq 0}$ which is defined via

(8.2.16)

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := Z_t := \Lambda'_t\, \Lambda_t$$

$$= exp\left\{-\int_0^t h(s, X_s)\, dB_s - \frac{1}{2}\int_0^t h^2(s, X_s)\, ds + \int_0^t \int_M g(s-, X_{s-}, y)\, d\widetilde{N}_Y^{\mathbb{P}}(s, y)\right.$$
$$\left. - \int_0^t \int_M \left[e^{g(s,X_s,y)} - 1 - g(s, X_s, y)\right] d\nu_Y(y)\, ds\right\}$$

(cf. (2.2.1), resp. Lemma 3.2 in [70]). Further, Itô's product rule yields the integral equation under $\mathbb{P}$

(8.2.17)

$$Z_t = 1 - \int_0^t Z_s\, h(s, X_s)\, dB_s + \int_0^t \int_M Z_{s-}\left[e^{g(s-,X_{s-},y)} - 1\right] d\widetilde{N}_Y^{\mathbb{P}}(s, y).$$

Hence, Z indicates a local $\mathbb{P}$-martingale. Appealing to our above presumptions, we recognize that $\mathbb{E}_{\mathbb{P}}[Z_t] = 1$ holds for all $t \geq 0$ such that Z (not surprisingly) features a $\mathbb{P}$-martingale even. Adjusting Girsanov's Change-of-Measure theorem (remind Prop. 2.2.1) to our recent setup, we may state that

(8.2.18)
$$d\widetilde{B}_t := dB_t + h(t, X_t)\, dt$$

depicts a BM under the EMM $\mathbb{Q}$ (also see Lemma 4.1 (1) in [77]) and that

(8.2.19)
$$d\widetilde{N}_Y^{\mathbb{Q}}(t, y) := dN_Y(t, y) - e^{g(t,X_t,y)} d\nu_Y(y)\, dt$$

constitutes the $\mathbb{Q}$-compensated random measure associated to Y [recall the first part of the footnote dedicated to (3.3.12) at this step]. Further note that, similar to [71], the $\mathbb{Q}$-compensator of dN_Y, namely

$$e^{g(t,X_t,y)} d\nu_Y(y)\, dt$$

explicitly depends on the hidden state variable X_t. Adapting/extending corresponding results/proofs in [2], [70], [76] and [77] (in particular, Proposition 3.13 in [2], Lemma 3.2 in [70], Lemma 1 and 2 in [76] and Lemma 4.1 in [77]; also see [84]) to our purposes, we derive the subsequent statements.

Lemma 8.2.2 *Defining the density process Z as in (8.2.16), we get the following:*

(a) *Under $\mathbb{Q}$, the process Y possesses a local martingale representation that is of the form*

(8.2.20)
$$Y_t = Y_0 + G_t + L_t$$

wherein

(8.2.21)
$$G_t := \int_0^t \varepsilon(s)\, d\tilde{B}_s$$

indicates a Brownian local $\mathbb{Q}$-martingale and

(8.2.22)
$$L_t := \int_0^t \int_M \delta(s-, y)\, d\tilde{N}_Y^{\mathbb{Q}}(s, y)$$

denotes a pure-jump local $\mathbb{Q}$-martingale.

(b) *Under $\mathbb{Q}$, the signal process X and the observation process Y are independent. Moreover, under $\mathbb{Q}$, X has the same distribution as under $\mathbb{P}$.*

(c) *Under $\mathbb{Q}$, G, L and X are independent.*

Proof

- **(a)** This follows from a straightforward substitution of (8.2.18) and (8.2.19) into equation (8.2.9).
- **(b)** Without loss of generality, we assume that $Y_0 = 0$ holds $\mathbb{P}$-almost-sure. If we further define

$$\Theta_t := \int_0^t \varepsilon(s)\, dB_s\,, \qquad \Gamma_t := \int_0^t \int_M \delta(s-, y)\, dN_Y(s, y),$$

then – due to (8.2.9) – we immediately deduce the vectorial decomposition

(8.2.23)
$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} 0 \\ \int_0^t \left[\varepsilon(s)\, h(s, X_s) - \int_M \delta(s, y)\, e^{g(s, X_s, y)}\, d\nu_Y(y)\right] ds \end{pmatrix} + \begin{pmatrix} X_t \\ \Theta_t + \Gamma_t \end{pmatrix}.$$

Moreover, an extension of Proposition 3.13 and Exercise 3.14 in [2] to our jump-diffusion case implies that, under the measure $\mathbb{P}$, the law of the pair process $(X_t, Y_t)^T$ is absolutely continuous with respect to the law of $(X_t, \Theta_t + \Gamma_t)^T$ whereas the corresponding Radon-Nikodym derivative is given by Z_t, in symbols

(8.2.24)
$$\frac{d\mathbb{P}^{\begin{pmatrix} X_t \\ \Theta_t + \Gamma_t \end{pmatrix}}}{d\mathbb{P}^{\begin{pmatrix} X_t \\ Y_t \end{pmatrix}}} = Z_t.$$

Thus, referring to (8.2.16) and (8.2.24), for any *arbitrary* measureable and bounded (test-) function $\xi: \mathbb{R}^2 \longrightarrow \mathbb{R}$ we next obtain the equality

(8.2.25)
$$\mathbb{E}_{\mathbb{Q}}\left[\xi\begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right] = \int_\Omega \xi\begin{pmatrix} X_t \\ Y_t \end{pmatrix} Z_t\, d\mathbb{P} = \int_{\mathbb{R}^2} \xi\begin{pmatrix} x \\ y \end{pmatrix} d\mathbb{P}^{\begin{pmatrix} X_t \\ \Theta_t + \Gamma_t \end{pmatrix}}(x, y) = \mathbb{E}_{\mathbb{P}}\left[\xi\begin{pmatrix} X_t \\ \Theta_t + \Gamma_t \end{pmatrix}\right].$$

Hence, the stochastic vector $(X_t, Y_t)^T$ possesses the same joint-distribution under $\mathbb{Q}$ as $(X_t, \Theta_t + \Gamma_t)^T$ has under $\mathbb{P}$, whereby a priori the ingredients Θ_t, Γ_t and X_t have been assumed to be $\mathbb{P}$-independent [compare the sequel of (8.2.11)]. Consequently, X and Y are finally detected to be $\mathbb{Q}$-independent. From (8.2.25) we moreover deduce that X has the same distribution under $\mathbb{Q}$ as under $\mathbb{P}$.

(c) This follows immediately from part (b). ■

Appealing to (8.2.16) (and p.7 in [70]), we next introduce the inverse density process via

(8.2.26)

$$Z_t^{-1} := (\Lambda_t' \, \Lambda_t)^{-1} := \frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} :=$$

$$exp\left\{\int_0^t h(s, X_s)\, dB_s + \frac{1}{2}\int_0^t h^2(s, X_s)\, ds - \int_0^t \int_M g(s-, X_{s-}, y)\, d\tilde{N}_Y^{\mathbb{P}}(s, y) + \int_0^t \int_M \left[e^{g(s, X_s, y)} - 1 - g(s, X_s, y)\right] d\nu_Y(y)\, ds\right\}.$$

Applying Itô’s formula on (8.2.26), we obtain the following stochastic integral representation under $\mathbb{P}$

(8.2.27)

$$Z_t^{-1} = 1 + \int_0^t Z_s^{-1}\, h^2(s, X_s)\, ds + 2\int_0^t \int_M Z_s^{-1} \{cosh[g(s, X_s, y)] - 1\}\, d\nu_Y(y)\, ds + \int_0^t Z_s^{-1}\, h(s, X_s)\, dB_s + \int_0^t \int_M Z_{s-}^{-1} \left[e^{-g(s-, X_{s-}, y)} - 1\right] d\tilde{N}_Y^{\mathbb{P}}(s, y).$$

In accordance to (8.2.26), the so-called *Kallianpur-Striebel-Formula* (cf. “(4.6) in [77]”) yet reads as

(8.2.28)

$$\mathbb{E}_{\mathbb{P}}(f(X_t)|\mathcal{F}_t^Y) = \frac{\mathbb{E}_{\mathbb{Q}}(f(X_t)\, Z_t^{-1}|\mathcal{F}_t^Y)}{\mathbb{E}_{\mathbb{Q}}(Z_t^{-1}|\mathcal{F}_t^Y)}$$

– for a full proof of (8.2.28) see Theorem 3.22 along with Theorem 5.3 in [84]. Further on, merging (8.2.10), (8.2.18) and (8.2.19) into equation (8.2.27), we receive the following $\mathbb{Q}$-representation

(8.2.29)

$$Z_t^{-1} = 1 + \int_0^t Z_s^{-1}\, h(s, X_s)\, d\tilde{B}_s + \int_0^t \int_M Z_{s-}^{-1} \left[e^{-g(s-, X_{s-}, y)} - 1\right] d\tilde{N}_Y^{\mathbb{Q}}(s, y).$$

Hence, Z_t^{-1} is classified as a local martingale under $\mathbb{Q}$. Similar to above, we require $\mathbb{E}_{\mathbb{Q}}[Z_t^{-1}] = 1$ for all $t \geq 0$ such that the inverse density process Z_t^{-1} even designates a true $\mathbb{Q}$-martingale.

In differential form, equation (8.2.29) points out as

(8.2.30)

$$\frac{dZ_t^{-1}}{Z_{t-}^{-1}} = dA_t := h(t,X_t)\,d\tilde{B}_t + \int_M \left[e^{-g(t-,X_{t-},y)} - 1\right] \tilde{N}_Y^{\mathbb{Q}}(t,dy)$$

whereby we have just introduced the $\mathbb{Q}$-martingale

(8.2.31)

$$A_t := \int_0^t h(s,X_s)\,d\tilde{B}_s + \int_0^t \int_M \left[e^{-g(s-,X_{s-},y)} - 1\right] d\tilde{N}_Y^{\mathbb{Q}}(s,y)$$

for notational reasons. Let us recall that the solution of (8.2.30) possesses the representation (cf. [1])

(8.2.32)

$$Z_t^{-1} = exp\left\{A_t - \frac{1}{2}[A^c]_t\right\} \prod_{0\le s\le t} (1+\Delta A_s)\, e^{-\Delta A_s}$$

wherein $[A^c]_t$ denotes the quadratic variation of the continuous part of A_t and $\Delta A_t := A_t - A_{t-}$ symbolizes the jump-magnitude of A at time t. Thus, we instantaneously derive

(8.2.33)

$$Z_t^{-1} = exp\left\{\int_0^t h(s,X_s)\,d\tilde{B}_s + \int_0^t \int_M \left[e^{-g(s-,X_{s-},y)} - 1\right] d\tilde{N}_Y^{\mathbb{Q}}(s,y) - \frac{1}{2}\int_0^t h^2(s,X_s)\,ds\right\}$$
$$\times \prod_{0\le s\le t} exp\{ln(1+\Delta A_s) - \Delta A_s\}.$$

In the following, we handle the infinite product in (8.2.33) separately: Utilizing (8.2.31), we get

(8.2.34)

$$\prod_{0\le s\le t} exp\{ln(1+\Delta A_s) - \Delta A_s\} = exp\left\{\sum_{0\le s\le t} [ln(1+\Delta A_s) - \Delta A_s]\right\}$$
$$= exp\left\{\int_0^t \int_M \left[1 - g(s-,X_{s-},y) - e^{-g(s-,X_{s-},y)}\right] dN_Y(s,y)\right\}.$$

Merging the latter equation into (8.2.33) while remembering (8.2.19), we finally end up with

(8.2.35)

$$Z_t^{-1} = exp\left\{\int_0^t h(s,X_s)\,d\tilde{B}_s - \frac{1}{2}\int_0^t h^2(s,X_s)\,ds - \int_0^t \int_M g(s-,X_{s-},y)\,d\tilde{N}_Y^{\mathbb{Q}}(s,y)\right.$$
$$\left. + \int_0^t \int_M \left[e^{g(s,X_s,y)} - 1 - g(s,X_s,y)\,e^{g(s,X_s,y)}\right] d\nu_Y(y)\,ds\right\}$$

yielding the explicit solution of (8.2.29), respectively of (8.2.30), with respect to the equivalent martingale measure $\mathbb{Q}$.[82] For the sake of completeness, we state the following result which extends the fourth and fifth instance of Lemma 4.1 in [77] to our double-jump filtering approach.

Lemma 8.2.3

(a) *Under the EMM* $\mathbb{Q}$, *we have* $\mathbb{E}_{\mathbb{Q}}(Z_t^{-1}|\mathcal{F}_t^X) = 1$.

(b) *The restrictions of* $\mathbb{Q}$ *and* $\mathbb{P}$ *to the filtration* $\mathcal{F}_t^X$ *are the same.*

Proof

(a) Using the representation (8.2.29), we derive

(8.2.36)

$$\mathbb{E}_{\mathbb{Q}}(Z_t^{-1}|\mathcal{F}_t^X) = \mathbb{E}_{\mathbb{Q}}\left(1 + \int_0^t Z_s^{-1}\, h(s,X_s)\, d\tilde{B}_s + \int_0^t \int_M Z_{s-}^{-1}\left[e^{-g(s-,X_{s-},y)} - 1\right] d\tilde{N}_Y^{\mathbb{Q}}(s,y)\middle|\mathcal{F}_t^X\right).$$

Next, Lemma 8.2.2 (c) implies that $\tilde{B}, \tilde{N}_Y^{\mathbb{Q}}$ and $\mathcal{F}^X$ are $\mathbb{Q}$-independent. Thus, the conditional expectation in (8.2.36) reduces to a usual expectation which, due to the $\mathbb{Q}$-martingale property of the appearing stochastic integrals in (8.2.36), yields the desired result.

(b) This immediately follows from (8.2.16) and Lemma 8.2.2 (b). ■

8.3 The filtering equations

In this paragraph we define the non-normalized (respectively, unnormalized) and the normalized conditional distribution of $f(X_t)$, given the observation filtration $\mathcal{F}_t^Y$, and hereafter derive the corresponding filtering SDEs, namely the extended Zakai- and Kushner-Stratonovic-Equation.

8.3.1 The extended Zakai-Equation

We start off by introducing some shorthand notations for terms appearing when we are dealing with Itô's formula for discontinuous semi-martingales. Although we have restricted ourselves to the one-dimensional case formerly, we will provide an adjusted *multi*-dimensional version (which originally can be found in [75]) of the upcoming jump-diffusion differential operator now.

Lemma 8.3.1 *For* $t \geq 0$, $x, z \in D \subseteq \mathbb{R}^d \setminus \{\mathbf{0}\}$ *and* $f: D \longrightarrow \mathbb{R}$, $f \in C_0^2(D)$ *(=* C^2*-class functions that vanish at infinity) the time and state dependent jump-diffusion differential operator* $(\mathcal{A}f)(t,\cdot)$ *associated to equation (8.2.5) exists and is given by*

[82] Yet, we could also have derived the solution (8.2.35) by putting (8.2.10), (8.2.18) and (8.2.19) into (8.2.26).

(8.3.1)

$$(\mathcal{A}f)(t,x) := \sum_{i=1}^{d} \alpha_i(t,x)\,\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{d} (\beta\beta^T)_{ij}\,(t,x)\,\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$
$$+ \int_D \sum_{i=1}^{d} \left[f\left(x + \gamma^{(i)}(t,x,z)\right) - f(x) - \gamma^{(i)}(t,x,z)\,\nabla f(x)\right]\, d(\nu_X)_i(z_i).$$

Herein, β^T denotes the transposed matrix of β, $\gamma^{(i)}$ stands for the i-th column of the $(d \times d)$-matrix γ, $\nabla f(x)$ symbols the gradient vector of $f(x)$ and finally, $(\nu_X)_i$ represents the i-th component of the d-dimensional Lévy-measure associated to the signal. (For further reading on jump-diffusion differential operators see e.g. Applebaum [1] or Øksendal and Sulem [75]; also recall Theorem 4.3 in [77].) ■

Remark 8.3.2 *In the one-dimensional case the operator (8.3.1) exhibits the familiar structure*

(8.3.2) $$(\mathcal{A}f)(t,x) =$$

$$\alpha(t,x)\,f_x(x) + \frac{1}{2}\,\beta^2(t,x)\,f_{xx}(x) + \int_D \left[f\big(x + \gamma(t,x,z)\big) - f(x) - \gamma(t,x,z)\,f_x(x)\right] d\nu_X(z)$$

for all $t \geq 0$ and $x, z \in D \subseteq \mathbb{R} \setminus \{0\}$. The latter equation further simplifies to

$$(\mathcal{A}f)(t,x) = \alpha(t,x)$$

for $f(x) := id(x) := x$. ■

Applying Itô's formula on the signal equation (8.2.5), we obtain the following $\mathbb{P}$-representation for $f(X_t)$ [again for $d = 1$ and thus, for $f: D \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$] reading

(8.3.3)

$$f(X_t) = f(X_0) + \int_0^t \Bigg[f_x(X_s)\,\alpha(s,X_s) + \frac{1}{2}\,\beta^2(s,X_s)\,f_{xx}(X_s)$$
$$+ \int_D \left[f\big(X_s + \gamma(s,X_s,x)\big) - f(X_s) - \gamma(s,X_s,x)\,f_x(X_s)\right] d\nu_X(x)\Bigg]\,ds$$
$$+ \int_0^t f_x(X_s)\,\beta(s,X_s)\,dW_s + \int_0^t \int_D \left[f\big(X_{s-} + \gamma(s-,X_{s-},x)\big) - f(X_{s-})\right] d\widetilde{N}_X^{\mathbb{P}}(s,x).$$

Identifying the differential operator (8.3.2) inside (8.3.3), the latter equation shortens to

(8.3.4) $$f(X_t) = f(X_0) +$$

$$\int_0^t (\mathcal{A}f)(s,X_s)\,ds + \int_0^t f_x(X_s)\,\beta(s,X_s)\,dW_s + \int_0^t \int_D \left[f\big(X_{s-} + \gamma(s-,X_{s-},x)\big) - f(X_{s-})\right] d\widetilde{N}_X^{\mathbb{P}}(s,x).$$

Note in passing that property (8.3.4) implies the following *Dynkin-Equation* (cf. e.g. p.562 in [76])

(8.3.5) $$\mathbb{E}_{\mathbb{P}}[P_t(f)] = \mathbb{E}_{\mathbb{P}}[f(X_0)] + \int_0^t \mathbb{E}_{\mathbb{P}}[(\mathcal{A}f)(s,X_s)]\,ds.$$

Definition 8.3.3 *The unnormalized conditional distribution of $f(X_t)$ given $\mathcal{F}_t^Y$ is defined by*

(8.3.6) $$p_t(f) := \mathbb{E}_{\mathbb{Q}}(f(X_t)\,Z_t^{-1}|\mathcal{F}_t^Y)$$

where Z_t^{-1} is such as announced in (8.2.26) – cf. Def. 4.4 in [77]. ■

Remark 8.3.4 ***(a)*** *Identifying (8.2.1) and (8.3.6) inside (8.2.28), we derive the shorthand notation*

(8.3.7)
$$P_t(f) = \frac{p_t(f)}{p_t(1)}$$

as an equivalence to our previous Kallianpur-Striebel-Formula (cf. eq. "(4.26) in [77]").

(b) *Further, $p_t(\cdot)$ depicts a linear functional, since*

(8.3.8) $$p_t(\lambda f + \eta g) = \lambda\, p_t(f) + \eta\, p_t(g)$$

holds for arbitrary constants λ, η and $f, g \in \mathfrak{D}(\mathcal{A}) := C_0^2(\mathbb{R}^d)$. Here, $\mathfrak{D}(\mathcal{A})$ denotes the domain of the operator $\mathcal{A}$ representing the space of all admissible functions. Since $p_t(\cdot)$ is linear, so is $P_t(\cdot)$. ■

Theorem 8.3.5 *(extended Zakai-Equation)*[83]

The unnormalized conditional distribution of $f(X_t)$ given $\mathcal{F}_t^Y$, as defined in (8.3.6), solves $\mathbb{Q}$-almost-sure for all $t \geq 0$ and $f \in \mathfrak{D}(\mathcal{A})$ the stochastic evolution equation

(8.3.9)
$$p_t(f) = p_0(f) + \int_0^t p_s(\mathcal{A}f)\,ds + \int_0^t p_s(fh)\,d\tilde{B}_s + \int_0^t \int_M p_{s-}(f \cdot \{e^{-g} - 1\})\,d\widetilde{N}_Y^{\mathbb{Q}}(s,y).$$

Proof Since $f(X_t)\,Z_t^{-1}$ impresses on the right hand side of (8.3.6), it should be worthwhile to study this product process more accurately (also see p.8 in [77]). Using Itô's product rule and hereafter implanting (8.2.29) and (8.3.4) into the resulting expression, we obtain

83 Note that our innovative representation (8.3.9) essentially extends equation "(15) in [76]" from *standard* yet to *compound* Poisson (Lévy-type) observations admitting *randomized* jump-amplitudes. Unfortunately, there are several mistakes in the proof of Theorem 1 in [76]: e.g. in (16) the initial value $X_0 Y_0$ is missing, whereas in the subsequent equality the term $f(x(0))$ should appear. Moreover, the quadratic variation on the top of page 563 actually vanishes, while on the right hand side of (24) instead of the capital Π a small π should appear. All in all, the left side limits in (15) do neither make sense for the Brownian $dy(s)$-integral, nor for the ds-integrals. Also in the proof of Theorem 4.5 in [77] there are a couple of errors, as in (4.11) there should stand $f(X(0))$, whereby in (4.15) and (4.17) $\widetilde{N}_j$ must be replaced by N_j. Finally, in (4.18) the integrator $dY(s)$ has to be replaced by ds.

(8.3.10) $$f(X_t)\,Z_t^{-1} = f(X_0)\,Z_0^{-1} + \int_0^t f(X_{s-})\,dZ_s^{-1} + \int_0^t Z_{s-}^{-1}\,df(X_s) + [f(X_.),Z_.^{-1}]_t =$$

$$f(X_0) + \int_0^t f(X_s)\,Z_s^{-1}\,h(s,X_s)\,d\tilde{B}_s + \int_0^t \int_M f(X_{s-})\,Z_{s-}^{-1}\left[e^{-g(s-,X_{s-},y)} - 1\right] d\tilde{N}_Y^{\mathbb{Q}}(s,y)$$

$$+ \int_0^t Z_s^{-1}\,(\mathcal{A}f)(s,X_s)\,ds + \int_0^t Z_s^{-1}\,f_x(X_s)\,\beta(s,X_s)\,dW_s$$

$$+ \int_0^t \int_D Z_{s-}^{-1}\left[f\big(X_{s-} + \gamma(s-,X_{s-},x)\big) - f(X_{s-})\right] d\tilde{N}_X^{\mathbb{P}}(s,x) + [f(X_.),Z_.^{-1}]_t.$$

Comparing the $\mathbb{P}$-representations (8.3.4) and (8.2.27), we state that the pair $f(X)$ and Z^{-1} does neither eject a quadratic co-variation coming from the continuous parts related to W and B nor one coming from the Poisson jump parts related to N_X and N_Y. Roughly speaking, $f(X)$ and Z^{-1} *have nothing in common*, since all appearing random components formerly have been assumed to be independent [compare the sequel of (8.2.11)] and therefore, generate a quadratic co-variation that equals zero

(8.3.11) $$[f(X_.),Z_.^{-1}]_t = 0.$$

Since our goal is the derivation of the specific SDE that is satisfied by $p_t(f)$, we ought to devote our attention towards the examination of the conditional expectation $\mathbb{E}_{\mathbb{Q}}(f(X_t)\,Z_t^{-1}|\mathcal{F}_t^Y)$ from now on. Putting (8.3.10) inside (8.3.6), the resulting conditional $\mathbb{Q}$-expectation turns out to consist of six separated summands. With respect to the derivation methodologies of the results in Theorem 1 in [76] and Lemma 5.4 in [84], we now treat these six additive components in their order of appearance:

By definition, for the first object we trivially receive

(8.3.12) $$\mathbb{E}_{\mathbb{Q}}(f(X_0)|\mathcal{F}_t^Y) = p_0(f).$$

Next, an application of the stochastic Fubini-Tonelli theorem yields for the second integrand

(8.3.13)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_0^t f(X_s)\,Z_s^{-1}\,h(s,X_s)\,d\tilde{B}_s\middle|\mathcal{F}_t^Y\right) = \int_0^t \mathbb{E}_{\mathbb{Q}}(f(X_s)\,Z_s^{-1}\,h(s,X_s)|\mathcal{F}_t^Y)\,d\tilde{B}_s.$$

Parallel to the proof of Theorem 1 on page 563 in [76], the latter term can equally well be conditioned on $\mathcal{F}_{s-}^Y$, since the filtration $\mathcal{F}_t^Y$ may be decomposed into

(8.3.14) $$\mathcal{F}_t^Y = \mathcal{F}_{s-}^Y \vee \sigma\{Y_u - Y_s : s \le u \le t\}$$

whereas the second sigma-algebra on the right hand side of (8.3.14) "*tells us nothing [new] about the integrand*" $f(X_s)\,Z_s^{-1}\,h(s,X_s)$ [for time indices $0 \le s \le t$] such as appearing on the right hand side of (8.3.13). Thus, identifying (8.3.6), equation (8.3.13) next becomes

(8.3.15)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_0^t f(X_s)\,Z_s^{-1}\,h(s,X_s)\,d\tilde{B}_s\middle|\mathcal{F}_t^Y\right) = \int_0^t \mathbb{E}_{\mathbb{Q}}(f(X_s)\,Z_s^{-1}\,h(s,X_s)|\mathcal{F}_{s-}^Y)\,d\tilde{B}_s = \int_0^t p_s(fh)\,d\tilde{B}_s.$$

Applying similar arguments as in (8.3.13) – (8.3.15), we receive for the third summand

(8.3.16)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_0^t \int_M f(X_{s-})\, Z_{s-}^{-1} \left[e^{-g(s-,X_{s-},y)} - 1\right] d\widetilde{N}_Y^{\mathbb{Q}}(s,y) \middle| \mathcal{F}_t^Y\right)$$
$$= \int_0^t \int_M \mathbb{E}_{\mathbb{Q}}\left(f(X_{s-})\, Z_{s-}^{-1} \left[e^{-g(s-,X_{s-},y)} - 1\right] \middle| \mathcal{F}_{s-}^Y\right) d\widetilde{N}_Y^{\mathbb{Q}}(s,y)$$
$$= \int_0^t \int_M p_{s-}(f \cdot \{e^{-g} - 1\})\, d\widetilde{N}_Y^{\mathbb{Q}}(s,y)$$

whereas the fourth additive component points out as

(8.3.17)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_0^t Z_s^{-1}\, (\mathcal{A}f)(s,X_s)\, ds \middle| \mathcal{F}_t^Y\right) = \int_0^t \mathbb{E}_{\mathbb{Q}}(Z_s^{-1}\, (\mathcal{A}f)(s,X_s) | \mathcal{F}_{s-}^Y)\, ds = \int_0^t p_s(\mathcal{A}f)\, ds.$$

Coming to the fifth conditional expectation, we have to remember the $\mathbb{Q}$-independence of X and Y provided by Lemma 8.2.2 (b), which implies the $\mathbb{Q}$-independence of X and $\mathcal{F}^Y$ likewise and thus, also of $(W_s)_{s\in[0,t]}$ and $\mathcal{F}_t^Y$ (cf. the top of p.10 in [77]). The latter properties yield

(8.3.18)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_0^t Z_s^{-1}\, f_x(X_s)\, \beta(s,X_s)\, dW_s \middle| \mathcal{F}_t^Y\right) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^t Z_s^{-1}\, f_x(X_s)\, \beta(s,X_s)\, dW_s\right].$$

Since under $\mathbb{P}$ the Brownian dW_s-integral in (8.3.18) is a martingale which is normally distributed with zero mean, we can use the second statement of Lemma 8.2.2 (b) which, parallel to "(23) in [76]" resp. "(4.16) in [77]", allows us to replace $\mathbb{Q}$ by $\mathbb{P}$ on the right hand side of (8.3.18) in order to obtain

(8.3.19)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_0^t Z_s^{-1}\, f_x(X_s)\, \beta(s,X_s)\, dW_s \middle| \mathcal{F}_t^Y\right) = 0.$$

Similar arguments tell us that the sixth and last summand vanishes, too: More precise, we now may exploit the $\mathbb{Q}$-independence of $\widetilde{N}_X^{\mathbb{P}}$ and $\mathcal{F}^Y$ provided by Lemma 8.2.2 (b). Therewith, we announce

(8.3.20)

$$\mathbb{E}_{\mathbb{Q}}\left(\int_0^t \int_D Z_{s-}^{-1} \left[f\big(X_{s-} + \gamma(s-,X_{s-},x)\big) - f(X_{s-})\right] d\widetilde{N}_X^{\mathbb{P}}(s,x) \middle| \mathcal{F}_t^Y\right)$$
$$= \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \int_D Z_{s-}^{-1} \left[f\big(X_{s-} + \gamma(s-,X_{s-},x)\big) - f(X_{s-})\right] d\widetilde{N}_X^{\mathbb{P}}(s,x)\right] = 0$$

whereby we have used the second statement of Lemma 8.2.2 (b) again for the last equality.

Collecting the properties (8.3.12), (8.3.15), (8.3.16), (8.3.17), (8.3.19) and (8.3.20), we ultimately get the desired result. ■

8.3.2 The extended Kushner-Stratonovic-Equation

Now we are able to provide the dynamics of our optimal filter. At first, let us recall property (8.2.1).

Definition 8.3.6 *The normalized conditional distribution of $f(X_t)$ given $\mathcal{F}_t^Y$ is defined by*

(8.3.21)
$$P_t(f) := \mathbb{E}_{\mathbb{P}}(f(X_t)|\mathcal{F}_t^Y). \blacksquare$$

The stochastic process $P_t(f)$ is sometimes called *(probability-) measure-valued process* or simply the *conditional distribution of X given Y* in the literature, since it can be interpreted as a transition- or Markov-kernel (see e.g. pp. 3, 13, 15, 27 and 229 ff. in [2] for more details).

Regarding our version of the Kallianpur-Striebel-Formula in (8.3.7), we should take care of the process $p_t(1)$ in our proceedings. Anyway, note that the jump-diffusion differential operator (8.3.1) vanishes for $f \equiv 1$. Thus, using (8.3.9), we immediately receive the local $\mathbb{Q}$-martingale representation

(8.3.22)
$$p_t(1) = 1 + \int_0^t p_s(h)\, d\tilde{B}_s + \int_0^t \int_M p_{s-}(e^{-g} - 1)\, d\tilde{N}_Y^{\mathbb{Q}}(s,y)$$

which, by the way, extends equality "(4.27) in [77]" to our double-jump diffusion case.

Lemma 8.3.7 *The inverse unnormalized conditional distribution $p_t(1)^{-1}$ solves $\mathbb{Q}$-almost-sure for all $t \geq 0$ the stochastic evolution equation*

(8.3.23)
$$p_t(1)^{-1} = 1 + \int_0^t \frac{P_s(h)^2}{p_s(1)} ds + \int_0^t \int_M \frac{1}{p_s(1)} \left[\frac{1}{P_s(e^{-g})} - 1 + P_s(e^{-g} - 1)\right] e^{g(s,X_s,y)} d\nu_Y(y)\, ds$$
$$- \int_0^t \frac{P_s(h)}{p_s(1)} d\tilde{B}_s + \int_0^t \int_M \frac{1}{p_{s-}(1)} \left[\frac{1}{P_{s-}(e^{-g})} - 1\right] d\tilde{N}_Y^{\mathbb{Q}}(s,y).$$

Proof Applying Itô's formula on (8.3.22), we immediately obtain

(8.3.24)
$$p_t(1)^{-1} = p_0(1)^{-1}$$
$$- \int_0^t \frac{1}{p_{s-}(1)^2}\, dp_s(1) + \int_0^t \frac{1}{p_s(1)^3}\, d[p.(1)^c]_s + \sum_{0 \leq s \leq t} \left[\frac{1}{\Delta p_s(1) + p_{s-}(1)} - \frac{1}{p_{s-}(1)} + \frac{\Delta p_s(1)}{p_{s-}(1)^2}\right].$$

Taking (8.3.22) and Remark 8.3.4 (b) into account, the above infinite sum translates into

(8.3.25)

$$\mathcal{S}_t := \sum_{0\le s\le t}\left[\frac{1}{\Delta p_s(1)+p_{s-}(1)}-\frac{1}{p_{s-}(1)}+\frac{\Delta p_s(1)}{p_{s-}(1)^2}\right]$$
$$=\int_0^t\int_M\left[\frac{1}{p_{s-}(e^{-g})}-\frac{1}{p_{s-}(1)}+\frac{p_{s-}(e^{-g}-1)}{p_{s-}(1)^2}\right]dN_Y(s,y).$$

Further, with respect to (8.3.7), the latter can be transformed into

(8.3.26)

$$\mathcal{S}_t=\int_0^t\int_M\frac{1}{p_{s-}(1)}\left[\frac{1}{P_{s-}(e^{-g})}-1+P_{s-}(e^{-g}-1)\right]dN_Y(s,y).$$

Remembering (8.2.19), we finally end up with

(8.3.27)

$$\mathcal{S}_t=\int_0^t\int_M\frac{1}{p_{s-}(1)}\left[\frac{1}{P_{s-}(e^{-g})}-1+P_{s-}(e^{-g}-1)\right]d\widetilde{N}_Y^{\mathbb{Q}}(s,y)$$
$$+\int_0^t\int_M\frac{1}{p_s(1)}\left[\frac{1}{P_s(e^{-g})}-1+P_s(e^{-g}-1)\right]e^{g(s,X_s,y)}d\nu_Y(y)\,ds.$$

Merging (8.3.22) into the first integral on the right hand side of equality (8.3.24), we receive

(8.3.28)

$$\int_0^t\frac{1}{p_{s-}(1)^2}\,dp_s(1)=\int_0^t\frac{P_s(h)}{p_s(1)}\,d\tilde{B}_s+\int_0^t\int_M\frac{P_{s-}(e^{-g}-1)}{p_{s-}(1)}\,d\widetilde{N}_Y^{\mathbb{Q}}(s,y).$$

What remains is the computation of the second integral in (8.3.24) which immediately turns out as

(8.3.29)

$$\int_0^t\frac{1}{p_s(1)^3}\,d[p_\cdot(1)^c]_s=\int_0^t\frac{P_s(h)^2}{p_s(1)}\,ds.$$

Substituting (8.3.27) – (8.3.29) into equality (8.3.24), we finally get the desired result. ■

Now we come to our main proposition throughout Chapter 8 culminating in the provision of the specific stochastic differential equation fulfilled by our optimal filter (8.3.21). More concretely speaking, the forthcoming Theorem 8.3.8 yields the extended Kushner-Stratonovic-Equation descending from our innovative *double-jump* diffusion (nonlinear) filtering approach with generalized Lévy-type signal and observation processes. By the way, it might be worthwhile to compare Theorem 8.3.8 below with Corollary 4.7 in [77].

Theorem 8.3.8 *(extended Kushner-Stratonovic-Equation; main result)*[84]

The normalized conditional distribution of $f(X_t)$ given $\mathcal{F}_t^Y$, as defined in (8.3.21), solves $\mathbb{Q}$-almost-sure for all $t \geq 0$ and $f \in \mathfrak{D}(\mathcal{A})$ the stochastic evolution equation

(8.3.30)

$$P_t(f) = P_0(f) + \int_0^t P_s(\mathcal{A}f)\,ds + \int_0^t [P_s(f)\,P_s(h) - P_s(fh)]\,P_s(h)\,ds$$
$$+ \int_0^t \int_M \frac{P_s(f \cdot e^{-g}) - P_s(f)\,P_s(e^{-g})}{P_s(e^{-g})}\,P_s(1 - e^{-g})\,e^{g(s,X_s,y)} d\nu_Y(y)\,ds$$
$$+ \int_0^t [P_s(fh) - P_s(f)\,P_s(h)]\,d\tilde{B}_s + \int_0^t \int_M \frac{P_{s-}(f \cdot e^{-g}) - P_{s-}(f)\,P_{s-}(e^{-g})}{P_{s-}(e^{-g})}\,d\tilde{N}_Y^{\mathbb{Q}}(s,y).$$

Proof Recalling (8.3.7) and Itô's product rule (see Lemma 2.1.5), we initially obtain

(8.3.31)

$$P_t(f) = \frac{p_t(f)}{p_t(1)} = p_0(f)\,p_0(1)^{-1} + \int_0^t p_{s-}(f)\,d(p_s(1)^{-1}) + \int_0^t p_{s-}(1)^{-1}\,dp_s(f) + [p_\cdot(f), p_\cdot(1)^{-1}]_t.$$

Taking (8.3.23) into account, for the first integral on the right hand side of (8.3.31) we deduce

(8.3.32)

$$\int_0^t p_{s-}(f)\,d(p_s(1)^{-1}) =$$
$$\int_0^t P_s(f)\,P_s(h)^2\,ds + \int_0^t \int_M P_s(f)\left[\frac{1}{P_s(e^{-g})} - 1 + P_s(e^{-g} - 1)\right] e^{g(s,X_s,y)} d\nu_Y(y)\,ds$$
$$- \int_0^t P_s(f)\,P_s(h)\,d\tilde{B}_s + \int_0^t \int_M P_{s-}(f)\left[\frac{1}{P_{s-}(e^{-g})} - 1\right] d\tilde{N}_Y^{\mathbb{Q}}(s,y).$$

Remembering (8.3.9), the second integral in (8.3.31) moreover becomes

(8.3.33)

$$\int_0^t p_{s-}(1)^{-1}\,dp_s(f) = \int_0^t P_s(\mathcal{A}f)\,ds + \int_0^t P_s(fh)\,d\tilde{B}_s + \int_0^t \int_M P_{s-}(f \cdot \{e^{-g} - 1\})\,d\tilde{N}_Y^{\mathbb{Q}}(s,y).$$

[84] We stress that Theorem 8.3.8 essentially extends Corollary 1 and 2 in [76]. More accurately speaking, (8.3.30) obviously includes the filters "(32) and (35) in [76]" as subclasses. Unfortunately, in the proof of Corollary 1 in [76] there are two mistakes: firstly, $[\sigma_\cdot(f), \sigma_\cdot(1)^{-1}]_t$ does *not* equal $\Pi_0(f)$, but instead $-\int_0^t \Pi_s(fh)\Pi_s(h)ds$ and, secondly, in the last line of page 565 there must stand a minus sign in front of the appearing integral.

Recalling (8.2.19), (8.3.9) and (8.3.23), we finally derive

(8.3.34)

$$[p.(f), p.(1)^{-1}]_t = [p.(f)^c, (p.(1)^{-1})^c]_t + \sum_{0 \le s \le t} \Delta p_s(f)\ \Delta(p_s(1)^{-1})$$
$$= -\int_0^t P_s(fh)\, P_s(h)\, ds + \int_0^t \int_M \left[\frac{P_{s-}(f \cdot \{e^{-g} - 1\})}{P_{s-}(e^{-g})} - P_{s-}(f \cdot \{e^{-g} - 1\})\right] d\tilde{N}_Y^{\mathbb{Q}}(s, y)$$
$$+ \int_0^t \int_M \left[\frac{P_s(f \cdot \{e^{-g} - 1\})}{P_s(e^{-g})} - P_s(f \cdot \{e^{-g} - 1\})\right] e^{g(s, X_s, y)} d\nu_Y(y)\, ds.$$

In conclusion, substituting (8.3.32) – (8.3.34) into equality (8.3.31) while taking the linearity of $P_t(\cdot)$ [provided by Remark 8.3.4 (b)] into account, we receive the claimed result. ■

8.4 Some practical filtering applications

In the following subsections we present a selection of practical filtering applications related to our double-jump setup whereby we particularly focus on suitable choices of the appearing coefficients.

8.4.1 Concrete choices of the signal process coefficients

If we define

(8.4.1) $$\alpha(s, X_s) := \alpha\, X_s, \quad \beta(s, X_s) := \beta\, X_s, \quad \gamma(s-, X_{s-}, x) := x\, X_{s-}$$

within $x \in \widetilde{D} \subseteq (-1, \infty)$ and arbitrary constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, then our signal equation (8.2.5) possesses the Doléans-Dade solution

(8.4.2)

$$X_t = X_0\, exp\left\{\left[\alpha - \frac{\beta^2}{2} - \int_{\widetilde{D}} x\, d\nu_X(x)\right] t + \beta\, W_t + \int_0^t \int_{\widetilde{D}} ln(1 + x)\, dN_X(s, x)\right\}.$$

Note that in a financial application (as proposed in [31], for example) the signal process X might be interpreted as an unobservable stochastic volatility process, i.e. $X := \sigma := (\sigma_t)_{t \ge 0}$. In this context, by referring to (8.4.2), the *volatility log-returns* then would be given through

(8.4.3)

$$ln\left(\frac{\sigma_t}{\sigma_0}\right) = \left[\alpha - \frac{\beta^2}{2} - \int_{\widetilde{D}} x\, d\nu_\sigma(x)\right] t + \beta\, W_t + \int_0^t \int_{\widetilde{D}} ln(1 + x)\, dN_\sigma(s, x).$$

In order to attain an Ornstein-Uhlenbeck (OU)-type signal process on the contrary, one might choose

(8.4.4) $$X_0 := x_0, \quad \alpha(s, X_s) := -\alpha\, X_s, \quad \beta(s, X_s) := 1, \quad \gamma(s-, X_{s-}, x) := -x$$

in (8.2.5) within $x \in D \subseteq \mathbb{R} \setminus \{0\}$, a constant initial condition $x_0 \in \mathbb{R}$ and a jump intensity $\alpha \in \mathbb{R}^+$.

Moreover, let us introduce the compound Poisson process (CPP)

(8.4.5)

$$Q_t := \int_0^t \int_D x \, dN_X(s,x) = \sum_{\substack{0 \le s \le t \\ 0 \neq \Delta X_s \in D}} \Delta X_s := \sum_{i=1}^{R_t} K_i$$

wherein, under $\mathbb{P}$, the standard Poisson process R_t is distributed via

$$R_t \sim Poi(\alpha t), \qquad \mathbb{P}(R_t = k) = \frac{(\alpha t)^k}{k!} e^{-\alpha t} \;\; (k = 0,1,2,\dots),$$

and $(K_i)_{i \in \mathbb{N}}$ constitutes a family of *independent* and *identically distributed* (iid) random variables with $K_i \in D \subseteq \mathbb{R} \setminus \{0\}$ $\mathbb{P}$-a.s. for all i bearing a constant mean value $\theta := \mathbb{E}_{\mathbb{P}}[K_1] \in \mathbb{R}$. Hence, from Theorem 11.3.1 in [83] we deduce that the compensated CPP

$$(Q_t - \theta \alpha t)_{t \ge 0}$$

depicts a $\mathbb{P}$-martingale. We further denote the distribution of K_1 under $\mathbb{P}$ by $\mathbb{P}^{K_1} := \varphi$ and the corresponding Lebesgue-density by ρ. Therewith, we derive

(8.4.6)
$$d\nu_X(x) = \alpha \, d\mathbb{P}^{K_1}(x) = \alpha \, d\varphi(x) = \alpha \, \rho(x) \, dx.$$

Taking (8.2.6), (8.4.5) and (8.4.6) into account, we immediately receive

(8.4.7)
$$\mathbb{E}_{\mathbb{P}}[Q_t] = \mathbb{E}_{\mathbb{P}}\left[\int_0^t \int_D x \, dN_X(s,x)\right] = t \int_D x \, d\nu_X(x) = \theta \alpha t.$$

If we moreover define the Lévy process L via

(8.4.8) $$L_t := W_t - Q_t$$

then, in accordance to the specific coefficient choices in (8.4.4), our signal equation (8.2.5) exhibits the following mean-reverting structure

(8.4.9) $$dX_t = \alpha \, (\theta - X_t) \, dt + dL_t$$

with mean-level $\theta \in \mathbb{R}$ and mean-reversion speed $\alpha \in \mathbb{R}^+$. The solution of (8.4.9) yet is given by

(8.4.10)

$$X_t = \theta + (x_0 - \theta) e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} \, dW_s - \int_0^t \int_D x \, e^{-\alpha(t-s)} \, dN_X(s,x).$$

Summing up, our innovative double-jump diffusion filtering model is flexible enough to deal with signal processes both of *geometric* Doléans-Dade type [see (8.4.2)] and of mean-reverting Ornstein-Uhlenbeck type [see (8.4.10)]. Note that, within a *positive* initial condition $X_0 > 0$, the geometric signal in (8.4.2) also is strictly positive (which is crucial for stochastic volatility process estimations, for instance), whereas the OU-signal (8.4.10) is not necessarily so.

8.4.2 Concrete choices of the observation process coefficients

In this subsection we newly allow the observation process coefficients to depend explicitly on Y itself which does not affect the validity of our invented filtering setup. In other words, our former derivation methodologies remain valid for an extended observation equation yet reading

(8.4.11)

$$Y_t = Y_0 + \int_0^t \varepsilon(s,Y_s)\, h(s,X_s,Y_s)\, ds - \int_0^t \int_M \delta(s,Y_s,y)\, e^{g(s,X_s,Y_s,y)} d\nu_Y(y)\, ds + \int_0^t \varepsilon(s,Y_s)\, dB_s + \int_0^t \int_M \delta(s-,Y_{s-},y)\, dN_Y(s,y).$$

Note in passing that equality (8.4.11) closely resembles (8.2.9). Moreover, if we choose coefficients

(8.4.12)

$$\varepsilon(s,Y_s) := Y_s\, \tilde{\varepsilon}(s), \quad h(s,X_s,Y_s) := \tilde{h}(s,X_s),$$

$$\delta(s,Y_s,y) := Y_s\, \tilde{\delta}(s,y), \quad g(s,X_s,Y_s,y) := \tilde{g}(s,X_s,y),$$

then (8.4.11) possesses the Doléans-Dade solution

(8.4.13)

$$Y_t = Y_0\, exp\left\{ \int_0^t \tilde{\varepsilon}(s) \left[\tilde{h}(s,X_s) - \frac{\tilde{\varepsilon}(s)}{2} \right] ds - \int_0^t \int_M \tilde{\delta}(s,y)\, e^{\tilde{g}(s,X_s,y)}\, d\nu_Y(y)\, ds + \int_0^t \tilde{\varepsilon}(s)\, dB_s + \int_0^t \int_M ln\left(1+\tilde{\delta}(s-,y)\right) dN_Y(s,y) \right\}.$$

If we elsewise specify

(8.4.14)

$$\varepsilon(s,Y_s) \equiv \varepsilon \in \mathbb{R}^+, \quad \delta(s,Y_s,y) := y, \quad g(s,X_s,Y_s,y) := \tilde{g}(s,X_s),$$

then (8.4.11) becomes

(8.4.15)

$$Y_t = Y_0 + \varepsilon \int_0^t h(s,X_s,Y_s)\, ds - \int_0^t e^{\tilde{g}(s,X_s)} \int_M y\, d\nu_Y(y)\, ds + \varepsilon\, B_t + \int_0^t \int_M y\, dN_Y(s,y).$$

Committing ourselves to similar definitions as in (8.4.5) – (8.4.7), the latter equation can be written as

(8.4.16)

$$Y_t = Y_0 + \int_0^t \left[\varepsilon\, h(s,X_s,Y_s) - \theta^* \alpha^* e^{\tilde{g}(s,X_s)} \right] ds + L_t^*$$

wherein we presume $\theta^* \in \mathbb{R}^-$ and $\alpha^* \in \mathbb{R}^+$ to be constant.

Herein,

$$L_t^* := \varepsilon\, B_t + Q_t^*$$

denotes a (jump-diffusion) Lévy process, whereas Q_t^* indicates a compound Poisson process (CPP) with random jump sizes in the set $M \subseteq \mathbb{R} \setminus \{0\}$. If we furthermore define

(8.4.17)

$$h(s, X_s, Y_s) := -\frac{\alpha^*}{\varepsilon} Y_s, \qquad \tilde{g}(s, X_s) := ln\left(-\frac{\mu(X_s)}{\theta^*}\right)$$

within a strictly positive function $\mu \in C^2$, then – in differential notation – (8.4.16) points out as

(8.4.18) $$dY_t = \alpha^* \,[\mu(X_t) - Y_t]\, dt + dL_t^*$$

constituting an Ornstein-Uhlenbeck type equality with constant mean-reversion velocity $\alpha^* \in \mathbb{R}^+$ and signal-dependent (and thus, *stochastic*) mean-reversion level $\mu(X)$.

For a deterministic initial value $Y_0 := y_0 \in \mathbb{R}$ the solution of (8.4.18) eventually is given through

(8.4.19)

$$Y_t = y_0\, e^{-\alpha^* t} + \alpha^* \int_0^t \mu(X_s)\, e^{-\alpha^*(t-s)}\, ds + \varepsilon \int_0^t e^{-\alpha^*(t-s)} dB_s + \int_0^t \int_M y\, e^{-\alpha^*(t-s)} dN_Y(s, y).$$

8.4.3 Estimating the stochastic mean-level of electricity spot prices

In an electricity market application the (OU-type) observation process (8.4.18) might be interpreted as the (indeed mean-reverting) electricity spot price. In addition, we may assume the signal process X to embody the stochastic driver of the randomized mean-level $\mu(X)$, i.e. the stochastically varying periodic trend-line towards which the spot price reverts. In this context, the reader should also recall our former suggestions in Chapter 3 concerning adequate Ornstein-Uhlenbeck modeling onsets for electricity spot prices. With this background information, the strictly positive function $\mu(\cdot) \in C^2$ appearing in (8.4.18) ought to be chosen as a bounded and periodic seasonality function. Hence, adhering to our recent filtering vocabulary, the estimation of the underlying *unobservable* randomized mean-level $\mu(X)$ should reasonably be based upon the *observable* electricity spot price Y. Yet taking (8.2.10), (8.2.18) and (8.2.19) into account, for a state component $\mu(X)$ and an observed spot price Y our optimal filter (8.3.30) can be expressed in terms of $\mathbb{P}$-ingredients reading

(8.4.20)

$$\begin{aligned}
P_t(\mu) = P_0(\mu) &+ \int_0^t P_s(\mathcal{A}\mu)\, ds + \int_0^t [P_s(\mu)\, P_s(h) - P_s(\mu h)]\, [P_s(h) - h(s, X_s, Y_s)]\, ds \\
&+ \int_0^t \int_M \frac{P_s(\mu \cdot e^{-g}) - P_s(\mu)\, P_s(e^{-g})}{P_s(e^{-g})} \left[1 - P_s(e^{-g})\, e^{g(s, X_s, Y_s, y)}\right] d\nu_Y(y)\, ds \\
&+ \int_0^t [P_s(\mu h) - P_s(\mu)\, P_s(h)]\, dB_s + \int_0^t \int_M \frac{P_{s-}(\mu \cdot e^{-g}) - P_{s-}(\mu)\, P_{s-}(e^{-g})}{P_{s-}(e^{-g})}\, d\widetilde{N}_Y^{\mathbb{P}}(s, y).
\end{aligned}$$

Further, recalling the specific choices of the coefficients g and h such as determined in (8.4.14), respectively in (8.4.17), while keeping Definition 8.3.6 in mind, we obtain

$$P_s(h) = -\frac{\alpha^*}{\varepsilon}\,Y_s, \qquad P_s(\mu h) = -\frac{\alpha^*}{\varepsilon}\,Y_s\,P_s(\mu), \qquad P_s(e^{-g}) = -\theta^*\,P_s(\mu^{-1}), \qquad P_s(\mu\cdot e^{-g}) = -\theta^*.$$

Therewith, equation (8.4.20) translates into

(8.4.21)

$$P_t(\mu) = P_0(\mu) + \int_0^t P_s(\mathcal{A}\mu)\,ds + \int_0^t\int_M \frac{1-P_s(\mu)\,P_s(\mu^{-1})}{P_s(\mu^{-1})}\,[1-P_s(\mu^{-1})\,\mu(X_s)]\,d\nu_Y(y)\,ds + \int_0^t\int_M \frac{1-P_{s-}(\mu)\,P_{s-}(\mu^{-1})}{P_{s-}(\mu^{-1})}\,d\widetilde{N}_Y^{\mathbb{P}}(s,y)$$

representing our optimal filter (i.e. the best estimator in the least-squares sense) for the stochastic mean-level $\mu(X)$, given the observed electricity spot price (8.4.18).

8.4.4 Filtering out the spikes of electricity spot prices

Inspired by [72], we now apply nonlinear double-jump stochastic filtering techniques as invented above in order to (theoretically) calibrate our multi-factor electricity spot price model like introduced in section 3.2.1. In this context, the main difficulty actually lies in the examination of the question:[85]

> *Which electricity spot price fluctuations are caused by jumps/spikes and which ones have their origin in the usual (BM-like) small-amplitude price variations? (Cf. p.14 in [72].)*

Since for $Y := (Y_t)_{t\in[0,T]}$ as defined in (3.2.2) there is no explicit distribution available (due to the occurrence of multiple jump noises), we ought to utilize *stochastic filtering* to detect the involved short-term OU-components $X_t^{l+1},\dots,X_t^n$ properly. To be precise, we now aim to estimate the spiky components $X_t^{l+1},\dots,X_t^n$ given the observed deseasonalized spot price Y_t (cf. p.15 in [72]).
To this end, let us first identify all relevant filtering ingredients such as the observation and the signal process along with the Kushner-Stratonovic-Equation under $\mathbb{P}$ in the following. Since we can observe the electricity spot price (3.2.1) – or the deseasonalized spot price (3.2.2) respectively – we declare $Y := (Y_t)_{t\in[0,T]}$ given in (3.2.2) as the observation process. From now on, we further assume the volatility functions appearing in (3.2.3) for all $k = 1,\dots,n$ to be constant, i.e. $\sigma_k(t) \equiv \sigma_k > 0$. In conclusion, the upcoming calibration exercise yet can be transformed into a stochastic filtering problem for a partially observed model with vectorial signal $\left(X_t^{l+1},\dots,X_t^n\right)_{t\in[0,T]}$ and observations $(Y_t)_{t\in[0,T]}$. Additionally, for the remainder of Chapter 8 we put $D_k := D \subseteq\,]0,\infty[$ for all $k = 1,\dots,n$.

To keep matters simple, we fit the dimensions as follows: Observing the one-dimensional process Y_t, we estimate the *one*-dimensional signals (that is, every single spiky component) X_t^k successively for $k = l+1,\dots,n$ [in accordance to (8.2.1)] via the (in the least-squares sense) *optimal* filter

[85] Also see the beginning of Chapter 4 in [72] (in particular, pp. 14 and 15 therein) in the context of detecting electricity price spikes via filtering. However, Meyer-Brandis and Tankov [72] do not pursue the stochastic filtering approach any further whereas they instead adopt methods from nonparametric statistics to filter out price spikes (see p.15 ff. in [72]).

(8.4.22)
$$P_t^k(f) := \mathbb{E}_{\mathbb{P}}\big(f(X_t^k)\big|\mathcal{F}_t^Y\big)$$

with $f(x) := id(x) := x$ and an observation filtration $\mathcal{F}_t^Y := \sigma\{Y_u: 0 \le u \le t\}$ (cf. the top of p.15 in [72]). Taking (3.2.1), (3.3.1) and (8.2.3) into account, we (unfortunately) recognize the identity

$$\mathcal{F}_t^Y = \mathcal{F}_t$$

to be valid for all $t \in [0, T]$. Since X_t^k is $\mathcal{F}_t$-measureable for every index $k = l + 1, \dots, n$, as an immediate consequence our optimal filter (8.4.22) trivially boils down to

(8.4.23)
$$P_t^k(id) = X_t^k$$

which does not make sense in any practical filtering application, of course. Hence, for our above electricity market framework the mathematical filtering theory tells us nothing else than:

If you monitor the deseasonalized spot price Y, then you simultaneously consider its driving noises $X^1, \dots, X^n$ [via (3.2.2)] anyway and thus, the signal $(X^{l+1}, \dots, X^n)$ does not at all need to be filtered!

Consequently, the main problem we are facing here actually lies in undesired dependency structures in between the signal driving noises $L^{l+1}, \dots, L^n$ and the observation process driving noises $L^1, \dots, L^n$, the former obviously being a subfamily of the latter. On the contrary, in common filtering setups the signal and observation noises *a priori* are assumed to be *independent* (see e.g. p.1 in [76]; also recall our announcements at the end of subsection 8.2.1 above).

In this regard, somewhat similar to (3.2.3), we newly suppose the (slightly modified) signal vector $(\hat{X}^{l+1}, \dots, \hat{X}^n)$ to admit components

(8.4.24)
$$d\hat{X}_t^k = -\lambda_k\, \hat{X}_t^k\, dt + \sigma_k\, dQ_t^k$$

$(k = l + 1, \dots, n)$ with initial values $\hat{X}_0^k := x_k$ and L_t^k-independent Lévy-type/Sato noises

(8.4.25)
$$Q_t^k := \int_0^t \int_D z\, dM_k(s, z)$$

along with N_k-independent $\mathbb{P}$-compensated PRMs

(8.4.26)
$$d\widetilde{M}_k^{\mathbb{P}}(s, z) := dM_k(s, z) - \hat{\rho}_k(s)\, d\hat{\nu}_k(z)\, ds.$$

Therewith, the [to (8.4.22)] *analogous* optimal filter reads as

(8.4.27)
$$\hat{P}_t^k(id) := \mathbb{E}_{\mathbb{P}}\big(\hat{X}_t^k\big|\mathcal{F}_t^Y\big)$$

$(k = l + 1, \dots, n)$ which, in contrast to before, does not simplify any further yet. Next, referring to (8.4.24) – (8.4.26) for $k = l + 1, \dots, n$ the one-dimensional signal components are given by

(8.4.28)
$$\hat{X}_t^k = x_k + \int_0^t \left[-\lambda_k\, \hat{X}_s^k + \sigma_k\, \hat{\rho}_k(s) \int_D z\, d\hat{\nu}_k(z)\right] ds + \sigma_k \int_0^t \int_D z\, d\widetilde{M}_k^{\mathbb{P}}(s, z).$$

Hence, comparing (8.4.28) with (8.2.5), in order to achieve correspondence we may choose

(8.4.29)

$$X_t := \hat{X}_t^k, \qquad \alpha(s, \hat{X}_s^k) := -\lambda_k \hat{X}_s^k + \sigma_k \hat{\rho}_k(s) \int_D z \, d\hat{\nu}_k(z),$$

$$\beta(s, \hat{X}_s^k) := 0, \qquad \gamma(s-, \hat{X}_{s-}^k, z) := \sigma_k z, \qquad \widetilde{N}_X^{\mathbb{P}} := \widetilde{M}_k^{\mathbb{P}}.$$

Appealing to (3.2.2) and (3.2.6), the one-dimensional observation process exhibits the structure

(8.4.30)

$$Y_t = \sum_{k=1}^{n} w_k x_k - \sum_{k=1}^{n} \int_0^t w_k \lambda_k X_s^k \, ds + \sum_{k=1}^{n} \int_0^t \int_D z \, w_k \sigma_k \, dN_k(s,z).$$

Additionally, for all $k = 1, \dots, n$ and $z \in D$ we assume the Lévy-measures ν_k to possess explicit Lebesgue-densities

(8.4.31) $$d\nu_k(z) = \pi_k(z) \, dz$$

with density functions $\pi_k(z)$ bearing the real mean-values/expectations

(8.4.32)

$$a_k := \int_D z \, \pi_k(z) \, dz.$$

Thus, choosing

(8.4.33) $$\varepsilon(s) := 0, \qquad h(s, \hat{X}_s^k) := 0, \qquad \delta(s,z) := z \, (w_1 \sigma_1, \dots, w_n \sigma_n)^T \in \mathbb{R}^n,$$

$$g(s, \hat{X}_s^k, z) := ln\left(\frac{\sum_{k=1}^n w_k \lambda_k \hat{X}_s^k}{\sum_{k=1}^n w_k \sigma_k a_k}\right), \qquad Y_0 := \sum_{k=1}^{n} w_k x_k,$$

$$\nu_Y := (\nu_1, \dots, \nu_n)^T \in \mathbb{R}^n, \qquad N_Y := (N_1, \dots, N_n)^T \in \mathbb{R}^n, \qquad M := D,$$

we can force correspondence between (8.4.30) and (8.2.9) yet.

Referring to (8.4.27), (8.4.33), Remark 8.3.2 and Theorem 8.3.8, for $k = l+1, \dots, n$ the components of our optimal filter under $\mathbb{P}$ with $f := id$ explicitly read as

(8.4.34) $$\hat{P}_t^k(id) =$$

$$\hat{P}_0^k(id) + \int_0^t \hat{P}_s^k(\alpha) \, ds + \int_0^t \int_D \left[\hat{P}_s^k(id) \, \hat{P}_s^k(e^{-g}) - \hat{P}_s^k(id \cdot e^{-g})\right] e^{g(s, \hat{X}_s^k, z)} \, d\nu_k(z) \, ds$$
$$+ \int_0^t \int_D \frac{\hat{P}_{s-}^k(id \cdot e^{-g}) - \hat{P}_{s-}^k(id) \, \hat{P}_{s-}^k(e^{-g})}{\hat{P}_{s-}^k(e^{-g})} \, dN_k(s,z)$$

whereby α and g are such as defined in (8.4.29), respectively in (8.4.33).

Herein, we easily find for the initial value

(8.4.35) $$\hat{P}_0^k(id) = \hat{X}_0^k = x_k$$

whereas (8.4.29) delivers

(8.4.36) $$\hat{P}_s^k(\alpha) = -\lambda_k \, \hat{P}_s^k(id) + \sigma_k \, \hat{\rho}_k(s) \, \hat{a}_k$$

with mean-values

(8.4.37)

$$\hat{a}_k := \int_D z \, \hat{\pi}_k(z) \, dz$$

defined similar to (8.4.32). Unfortunately, the term

(8.4.38)

$$\hat{P}_s^k(e^{-g}) = \mathbb{E}_{\mathbb{P}}\left(\left(\sum_{k=1}^n w_k \, \lambda_k \, \hat{X}_s^k\right)^{-1} \middle| \mathcal{F}_s^Y\right) \sum_{k=1}^n w_k \, \sigma_k \, a_k$$

cannot be simplified any further. The same is valid for $\hat{P}_s^k(id \cdot e^{-g})$, by the way. Finally, note that for $\mathcal{F}_t^Y = \mathcal{F}_t$ and thus, for $P_t^k(id) = X_t^k$ as explained in connection with (8.4.22) and (8.4.23), one would receive instead of (8.4.34) the following *Dynkin-equation*

(8.4.39)

$$X_t^k = P_t^k(id) = P_0^k(id) + \int_0^t P_s^k(\alpha) \, ds = x_k - \lambda_k \int_0^t X_s^k \, ds + \sigma_k \, a_k \int_0^t \rho_k(s) \, ds.$$

8.4.5 Estimating the market zone net position in the EU ETS market

In order to calibrate our incomplete carbon dioxide emission allowances model such as introduced in Chapter 6, we now apply stochastic filtering techniques as presented above. More precisely, we want to (theoretically) estimate the unobservable market zone net position θ out of public EUA1 forward prices $\hat{F}$. For this intention, we identify all relevant filtering ingredients such as the observation and signal process along with the Kushner-Stratonovic-Equation under $\mathbb{P}$ in the following.

Starting off, for $t \in [0, T]$ we declare $\hat{F}_t$ as given in (6.2.5) as the *observation process* and θ_t as defined in (6.2.6) as the underlying *signal process*. Moreover, with view on our upcoming filtering purposes, we ought to introduce the *observation filtration*

(8.4.40) $$\mathcal{F}_t^{\hat{F}} := \hat{\mathcal{F}}_t := \sigma\{\hat{F}_s : 0 \leq s \leq t\}.$$

Therewith, the normalized conditional distribution of the market zone net position is given by the filter

(8.4.41) $$P_t(f) := \mathbb{E}_{\mathbb{P}}\left(f(\theta_t) \middle| \hat{\mathcal{F}}_t\right)$$

[recall eq. (8.2.1)] whereas we choose $f(x) := id(x) := x$ inside (8.4.41) from now on.

Additionally, we put $D_k := D \subseteq \mathbb{R} \setminus \{0\}$ for all indices $k = 1, \dots, n$ in our proceedings. Then, in accordance to (6.2.6) – (6.2.8), the market zone net position process obviously obeys

(8.4.42)
$$\theta_t = t \sum_{k=1}^{n} \int_D z \, d\nu_k(z) + \sum_{k=1}^{n} \int_0^t \int_D z \, d\widetilde{N}_k^{\mathbb{P}}(s,z).$$

Hence, in order to achieve correspondence between (8.2.5) and (8.4.42), we may choose

(8.4.43)
$$X_t := \theta_t, \qquad X_0 := \theta_0 = 0, \qquad \alpha(s,\theta_s) := \sum_{k=1}^{n} \xi_k, \qquad \xi_k := \int_D z \, d\nu_k(z),$$
$$\beta(s,\theta_s) := 0, \qquad \gamma(s-,\theta_{s-},z) := (z,\dots,z) \in \mathbb{R}^n, \qquad \widetilde{N}_X^{\mathbb{P}} := \left(\widetilde{N}_1^{\mathbb{P}},\dots,\widetilde{N}_n^{\mathbb{P}}\right) \in \mathbb{R}^n.$$

On the contrary, taking (6.2.5) into account, the observation process moreover exhibits the structure

(8.4.44)
$$\widehat{F}_t = \widehat{F}_0 + \int_0^t \widehat{F}_s \, (\mu - \theta_s) \, ds \; + \int_0^t \widehat{F}_s \, \sigma(s) \, dW_s.$$

As a consequence, defining

(8.4.45)
$$Y_t := \widehat{F}_t, \qquad \varepsilon\big(s,\widehat{F}_s\big) := \widehat{F}_s \, \sigma(s), \qquad h\big(s,\theta_s,\widehat{F}_s\big) := \frac{\mu - \theta_s}{\sigma(s)},$$
$$\delta\big(s,\widehat{F}_s,y\big) := g\big(s,\theta_s,\widehat{F}_s,y\big) := 0, \qquad B := W,$$

we can force correspondence between (8.4.11) and (8.4.44) yet. Referring to (8.4.20) [but with $\mu := id$ therein[86]], (8.4.45) and Remark 8.3.2, our optimal filter under $\mathbb{P}$ explicitly becomes

(8.4.46)
$$P_t(id) = P_0(id) +$$
$$\int_0^t P_s(\alpha) \, ds + \int_0^t [P_s(id) \, P_s(h) - P_s(id \cdot h)] \left[P_s(h) - \frac{\mu - \theta_s}{\sigma(s)}\right] ds + \int_0^t [P_s(id \cdot h) - P_s(id) \, P_s(h)] \, dW_s.$$

Herein, we find for the initial value

(8.4.47)
$$P_0(id) = 0$$

whereas (8.4.43) delivers

(8.4.48)
$$P_s(\alpha) = \textstyle\sum_{k=1}^{n} \int_D z \, d\nu_k(z).$$

[86] Unfortunately, we are facing a double notation here: To be precise, we merely choose the C^2-function $\mu(\cdot)$ appearing in (8.4.20) to be the identity, whereas the constant drift coefficient $\mu \in \mathbb{R}$ associated with equation (6.2.5), respectively with (8.4.44), remains untouched. Further note that, in contrast to our former assumption in subsection 8.4.3, the C^2-function $\mu(\cdot)$ does no longer need to be strictly positive in our present EU ETS market framework so that the choice $\mu(x) := x$ ($x \in [M_1, M_2] \subset \mathbb{R}$) in (8.4.20) indeed is possible/admissible.

Moreover, with respect to (8.4.45) we deduce

(8.4.49)

$$P_s(h) = \frac{\mu - P_s(id)}{\sigma(s)}, \qquad P_s(id \cdot h) = \frac{\mu\, P_s(id) - P_s(id^2)}{\sigma(s)}.$$

Substituting (8.4.41) and (8.4.47) – (8.4.49) into (8.4.46), we ultimately obtain

(8.4.50)

$$\mathbb{E}_{\mathbb{P}}\big(\theta_t \big| \hat{\mathcal{F}}_t\big) = \int_0^t \left(\frac{\theta_s - P_s(id)}{\sigma(s)^2} \times \mathbb{Var}_{\mathbb{P}}\big(\theta_s \big| \hat{\mathcal{F}}_s\big) + \sum_{k=1}^{n} \int_D z\, d\nu_k(z) \right) ds - \int_0^t \frac{\mathbb{Var}_{\mathbb{P}}\big(\theta_s \big| \hat{\mathcal{F}}_s\big)}{\sigma(s)}\, dW_s$$

representing the (in the least-squares sense) optimal time-t estimate for the market zone net position process θ_t, given the observed (respectively, simulated) EUA1 forward price $\hat{F}_t$, under the historical market measure $\mathbb{P}$.

8.5 Conclusions

Based upon the *best approximation property of conditional expectations* in this chapter we have constructed an (in the least-squares sense) optimal filter associated to our underlying generalized Lévy-type filtering disposition. More precisely, in order to model various measurement procedures in a more realistic manner, we have suggested an extended *double-jump* diffusion filtering onset permitting jumps with random amplitudes at random time points via independent compound Poisson-type processes both in the signal as well as in the observation process dynamics along with Brownian diffusion noise. By the way, we have tailored enhanced measure change techniques to our mission and moreover, have rigorously occupied ourselves with the emerging jump-diffusion differential operators. As a first highlight, we have introduced an expanded derivation modality for our extended double-jump Zakai-Equation afterwards. Having derived the extended Kushner-Stratonovic-Equation associated to our innovative double-jump framework, we subsequently have presented some practical filtering applications dealing inter alia with the estimation of an unobservable (stochastically varying) trend-line of electricity spot prices out of noise-afflicted spot price histories, with the detection of price spikes out of observed electricity spot price dynamics and, last but not least, with an adequate calibration method for the unobservable market zone net position in the EU ETS market while referring to public emission allowance prices. Concentrating on these subjects, in paragraph 8.4 we finally have invested some pursuing effort concerning concrete choices of the emerging coefficient processes.

Throughout the just mentioned applications we particularly have cherished that our innovative Lévy-type filtering approach is flexible enough to generate geometric Doléans-Dade types as well as mean-reverting Ornstein-Uhlenbeck types both for the signal and the observation equation – not at least due to the novel appearance of an *entire Brownian integral* (instead of a *naked* Brownian motion solely as appearing in the majority of related work in the filtering literature) inside the observation dynamics (8.2.9), respectively inside (8.4.11). Note that classical Kalman-Bucy (particle) filtering techniques (see e.g. [2]) should mostly fail in connection with such elaborated emission or electricity market applications as discussed throughout this thesis, since we frequently have been confronted with highly non-Gaussian settings herein.

References

[1] Applebaum, D. (2009) *Lévy Processes and Stochastic Calculus*, Second Edition, Cambridge studies in advanced mathematics

[2] Bain, A., Crisan, D. (2009) *Fundamentals of Stochastic Filtering*, First Edition, Springer

[3] Barlow, M. (2002) *A Diffusion Model for Electricity Prices*, Mathematical Finance, Vol. 12, 287-298

[4] Barth, A., Benth, F., Potthoff, J. (2011) *Hedging of Spatial Temperature Risk with Market-Traded Futures*, Applied Mathematical Finance 18 (2), 93-117

[5] Bauer, H. (2001) *Wahrscheinlichkeitstheorie*, Fifth Edition, De Gruyter

[6] Benth, F., Biegler-König, R., Kiesel, R. (2012) *Electricity Options and Additional Information*, to appear in the proceedings of the Wolfgang Pauli Institute “Quantitative Energy Finance”, Editors Benth, Kholodnyi and Laurence, Springer

[7] Benth, F., Cartea, A., Kiesel, R. (2008) *Pricing Forward Contracts in Power Markets by the Certainty Equivalence Principle: Explaining the Sign of the Market Risk Premium*, Journal of Banking & Finance 32 (10), 2006-2021

[8] Benth, F., Kallsen, J., Meyer-Brandis, T. (2006) *A non-Gaussian Ornstein-Uhlenbeck Process for Electricity Spot Price Modeling and Derivatives Pricing*, Applied Mathematical Finance, Vol. 14 (2), 153-169

[9] Benth, F., Meyer-Brandis, T. (2005) *The Density Process of the Minimal Entropy Martingale Measure in a Stochastic Volatility Model with Jumps with Applications to Option Pricing*, Finance and Stochastics, Vol. 9 (4)

[10] Benth, F., Meyer-Brandis, T. (2009) *The Information Premium for Non-storable Commodities*, Journal of Energy Markets, Vol. 2 (3), 111-140, [Actually, in this work we always refer to the former version called *The Information Premium in Electricity Markets*.]

[11] Benth, F., Saltyte-Benth, J. (2005) *Stochastic Modeling of Temperature Variations with a View towards Weather Derivatives*, Applied Mathematical Finance, Vol. 12 (1), 53-85

[12] Benth, F., Saltyte-Benth, J. (2007) *The Volatility of Temperature and Pricing of Weather Derivatives*, Quantitative Finance, Vol. 7 (5), 553-561

[13] Benth, F., Saltyte-Benth, J., Koekebakker, S. (2008) *Stochastic Modeling of Electricity and Related Markets*, First Edition, World Scientific

[14] Biagini, F., Bregman, Y., Meyer-Brandis, T. (2009) *Pricing of Electricity Options*, (preprint)

[15] Biagini, F., Øksendal, B. (2005) *A General Stochastic Calculus Approach to Insider Trading*, Applied Mathematical Optimization, Vol. 52, 167-181

[16] Bingham, N., Kiesel, R. (2004) *Risk-Neutral Valuation, Pricing and Hedging of Financial Derivatives*, Second Edition, Springer

[17] Björk, T., Di Masi, G., Kabanov, Y., Runggaldier, W. (1997) *Towards a General Theory of Bond Markets*, Finance and Stochastics, Vol. 1 (2), 141-174

[18] Björk, T., Kabanov, Y., Runggaldier, W. (1997) *Bond Market Structure in the Presence of Marked Point Processes*, Mathematical Finance, Vol. 7 (2), 211-239

[19] Bronstein, I., Musiol, G., Mühlig, H., Semendjajew, I. (2001) *Taschenbuch der Mathematik*, Fifth Edition

[20] Calzolari, A., Florchinger, P., Nappo, G. (2009) *Nonlinear Filtering for Markov Systems with Delayed Observations*, International Journal of Applied Mathematical Computational Sciences, Vol. 19, No. 1, 49-57

[21] Carmona, R., Ludkovski, M. (2006) *Pricing Commodity Derivatives with Basis Risk and Partial Observations*, (preprint)

[22] Carr, P., Madan, D. (1999) *Option valuation using the fast Fourier transform*, Journal of Computational Finance, Vol. 2 (4), 61-73

[23] Cartea, A., Figueroa, M., Geman, H. (2007) *Modeling Electricity Prices with Forward Looking Capacity Constraints*, (WP), Birkbeck College, University of London

[24] Cetin, U., Jarrow, R., Protter, P. (2004) *Liquidity Risk and Arbitrage Pricing Theory*, Finance and Stochastics, Vol. 8

[25] Cetin U., Verschuere, M. (2009) *Pricing and Hedging in Carbon Emissions Markets*, International Journal of Theoretical and Applied Finance, Vol. 12 (7), 949-967

[26] Chan, T. (1999) *Pricing Contingent Claims on Stocks Driven by Lévy Processes*, Annals of Applied Probability, Vol. 9, No. 2, 504-528

[27] Chesney, M., Taschini, L. (2008) *The Endogenous Price Dynamics of Emission Allowances: An Application to CO2 Option Pricing*, Swiss Finance Institute Research Paper No. 08-02

[28] Chicago Mercantile Exchange *CME*, http://www.cmegroup.com

[29] Considine, G. *Introduction to Weather Derivatives*, Weather Derivatives Group, Aquila Energy

[30] Cont, R., Tankov, P. (2004) *Financial Modeling with Jump Processes*, First Edition, Chapman and Hall

[31] Cvitanic, J., Liptser, R., Rozovskii, B. (2006) *A Filtering Approach to Tracking Volatility from Prices Observed at Random Times*, The Annals of Applied Probability, Vol. 16, No. 3, 1633-1652

[32] Di Nunno, G., Øksendal, B., Proske, F. (2009) *Malliavin Calculus for Lévy Processes with Applications to Finance*, First Edition, Springer

[33] Eberlein, E., Glau, K., Papapantoleon, A. (2010) *Analysis of Fourier transform valuation formulas and applications*, Applied Mathematical Finance, Vol. 17 (3), 211-240

[34] Eberlein, E., Jacod, J., Raible, S. (2005) *Lévy term structure models: no-arbitrage and completeness*, Finance and Stochastics, Vol. 9, 67-88

[35] Eberlein, E., Kluge, W. (2005) *Exact pricing formulae for caps and swaptions in a Lévy term structure model*, Journal of Computational Finance, Vol. 9, 99-125

[36] Eberlein, E., Özkan, F. (2003) *The defaultable Lévy term structure: ratings and restructuring*, Mathematical Finance, Vol. 13, 277-300

[37] Eberlein, E., Stahl, G. (2003) *Both Sides of the Fence: A Statistical and Regulatory View of Electricity Risk*, Energy & Power Risk Management 09/2003

[38] European Energy Exchange *EEX*, http://www.eex.com

[39] Fehr, M., Hinz, J. (2007) *A Quantitative Approach to Carbon Price Risk Modeling*, (preprint)

[40] Föllmer, H., Schweizer, M. (1991) *Hedging of Contingent Claims Under Incomplete Information*, Applied Stochastic Analysis, Vol. 5, 389-414

[41] Frey, R., Runggaldier, W. (2001) *Nonlinear Filtering Techniques for Volatility Estimation with a View towards High Frequency Data*, International Journal of Theoretical and Applied Finance, Vol. 4, 271-300

[42] Gamelin, T. (2001) *Complex Analysis*, First Edition, Springer

[43] Gatheral, J. (2008) *No-Dynamic Arbitrage and Market Impact*, (WP)

[44] Gatheral, J., Schied, A. (2011) *Optimal Trade Execution under Geometric Brownian Motion in the Almgren and Chriss Framework*, International Journal of Theoretical and Applied Finance, Vol. 14 (3), 353-368

[45] Gatheral, J., Schied, A., Slynko, A. (2010) *Transient Linear Price Impact and Fredholm Integral Equations*, (to appear in Mathematical Finance)

[46] Godsill, S. (2007) *Particle Filters for Continuous-Time Jump Models in Tracking Applications*, ESAIM: Proceedings, Vol. 19, 39-52

[47] Grüll, G., Kiesel, R. (2009) *Pricing CO2 Permits Using Approximation Approaches*, (WP)

[48] Hambly, B., Howison, S., Kluge, T. (2009) *Modeling Spikes and Pricing Swing Options in Electricity Markets*, Quantitative Finance, Vol. 9 (8), 937-949

[49] Heath, D., Jarrow, R., Morton, A. (1992) *Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation*, Econometrica, Vol. 60 (1), 77-105

[50] Hess, M. (2011) *A Forward-Looking Multi-Factor Ornstein-Uhlenbeck Model for Pricing Electricity Risk*, (WP)

[51] Hess, M. (2008) *Energy Markets and Option Prices for Energy Derivatives*, Diploma Thesis, Mathematical Financial Market Analysis, University of Trier

[52] Hess, M. (2011) *Explicit Pricing Measures for Commodity Forwards in a HJM-Framework with Jumps*, (WP)

[53] Hess, M. (2010) *Nonlinear Double-Jump Stochastic Filtering Using Generalized Lévy-Type Processes*, (WP)

[54] Hess, M. (2010) *Nonlinear Filtering with Stochastic Differential Equations and Related Tracking and Sensor Data Fusion Applications*, Research report 06/2010, Fraunhofer Institute for Communication, Information Processing and Ergonomics, Division *Sensor Data and Information Fusion*

[55] Hess, M. (2011) *Optimal Liquidation of Electricity Futures Portfolios under Market Impact*, (WP)

[56] Hess, M. (2011) *Pricing and Hedging Temperature Derivatives under Future Weather Information*, (WP)

[57] Hess, M. (2011) *Pricing Carbon Emission Allowances under Future Information on the Market Zone Net Position*, (WP)

[58] Hess, M. (2010) *Pricing Energy Derivatives in the Presence of Jumps Using Fourier Transform Methods*, (WP)

[59] Hinz, J., von Grafenstein, L., Verschuere, M., Wilhelm, M. (2005) *Pricing Electricity Risk by Interest Rate Methods*, Quantitative Finance, Vol. 5 (1), 49-60

[60] Jacod, J., Shiryaev, A. (2003) *Limit Theorems for Stochastic Processes*, Second Edition, Springer

[61] Johannes, M., Polson, N., Stroud, J. (2002) *Nonlinear Filtering of Stochastic Differential Equations with Jumps*, (WP)

[62] Karatzas, I., Shreve, S. (1991) *Brownian Motion and Stochastic Calculus*, Second Edition, Springer

[63] Kliemann, W., Koch, G., Marchetti, F. (1990) *On the Unnormalized Solution of the Filtering Problem with Counting Process Observations*, IEEE Transactions on Information Theory, Vol. 36, No. 6, 1415-1425

[64] Kloeden, P., Platen, E. (1999) *Numerical Solution of Stochastic Differential Equations*, (Stochastic Modeling and Applied Probability), Fourth Edition, Springer

[65] Kluge, W. (2005) *Time-inhomogeneous Lévy Processes in Interest Rate and Credit Risk Models*, Doctoral Thesis, Albert-Ludwigs-University Freiburg

[66] Klüppelberg, C., Meyer-Brandis, T., Schmidt, A. (2009) *Electricity Spot Price Modeling with a View Towards Extreme Spike Risk*, (preprint)

[67] Lang, S. (1999) *Complex Analysis*, Fourth Edition, Springer

[68] Lucia, J., Torro, H. (2005) *Short-term Electricity Future Prices at Nordpool: Forecasting Power and Risk Premiums*, (WP)

[69] Malcolm, W., James, M., Elliott, R. (2000) *Risk Sensitive Filtering with Poisson Process Observations*, Applied Mathematical Optimization, Vol. 41, 387-402

[70] Mandrekar, V., Meyer-Brandis, T., Proske, F. (2007) *A Bayes Formula for Non-Linear Filtering with Gaussian and Cox-Noise*, (WP)

[71] Meyer-Brandis, T., Proske, F. (2004) *Explicit Solution of a Non-Linear Filtering Problem for Lévy Processes with Applications to Finance*, Applied Mathematical Optimization, Vol. 50, 119-134

[72] Meyer-Brandis, T., Tankov, P. (2008) *Multi-factor jump-diffusion models of electricity prices*, International Journal of Theoretical and Applied Finance, Vol. 11, No. 5, 503-528

[73] Nordic Power Exchange *Nord Pool*, http://www.nordpool.com

[74] Øksendal, B. (2006) *A Universal Optimal Consumption Rate for an Insider*, Mathematical Finance, Vol. 16, No. 1, 119-129

[75] Øksendal, B., Sulem, A. (2007) *Applied Stochastic Control of Jump Diffusions*, Second Edition, Springer

[76] Poklukar, D. (2006) *Nonlinear Filtering for Jump-Diffusions*, Journal of Computational and Applied Mathematics, No. 197, 558-567

[77] Popa, S., Sritharan, S. (2009) *Nonlinear Filtering of Itô-Lévy Stochastic Differential Equations with Continuous Observations*, (preprint)

[78] Protter, P. (2005) *Stochastic Integration and Differential Equations*, Second Edition, Springer

[79] Raible, S. (2000) *Lévy Processes in Finance: Theory, Numerics and Empirical Facts*, Doctoral Thesis, Mathematical Faculty at the Albert-Ludwigs-University Freiburg

[80] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*, Cambridge studies in advanced mathematics, No. 68

[81] Schied, A., Schöneborn, T., Tehranchi, M. (2010) *Optimal Basket Liquidation for CARA Investors is deterministic*, Applied Mathematical Finance, Vol. 17, 471-489

[82] Schied, A., Slynko, A. (2011) *Some Mathematical Aspects of Market Impact Modeling*, To appear in Surveys in Stochastic Processes, Proceedings of the 33rd SPA, EMS Series of Congress Reports

[83] Shreve, S. (2004) *Stochastic Calculus for Finance II, Continuous-Time Models*, Springer

[84] Xiong, J. (2008) *An Introduction to Stochastic Filtering Theory*, Oxford Graduate Texts in Mathematics, Oxford University Press

Erklärung zur Dissertation

Hiermit erkläre ich, Markus Hess, dass ich die vorliegende Dissertation selbstständig verfasst und dabei keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Desweiteren habe ich die aus fremden Quellen wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht. Die vorliegende Dissertation habe ich bisher in keinem anderen Promotionsverfahren in gleicher oder vergleichbarer Form vorgelegt. Ferner erkläre ich, dass ich keine vorausgegangenen Promotionsversuche im Fach Mathematik oder in einem anderen Fach unternommen habe, die endgültig gescheitert sind.

Ort, Datum: ________________________ Unterschrift: ________________________