On Rational Points of Varieties over Complete Local Fields with Algebraically Closed Residue Field

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Contents

Introduction

In this thesis we study smooth and proper varieties over a complete local field K with algebraically closed residue field k. We examine them with regard to the existence of rational points, in particular by looking at a certain kind of not necessarily regular models of such varieties.
In general, how can one detect rational points of varieties over local fields? The special property of such varieties is that they are naturally equipped with models. A model of a K-variety X is an integral flat $\mathcal{O}_K$-scheme $\mathcal{X}$ such that its generic fiber is isomorphic to X. Here $\mathcal{O}_K$ the ring of integers of K. One can discuss the question whether X has K-rational points in terms of the geometry of the special fiber $\mathcal{X}_k$ of a model $\mathcal{X} \to S := \mathrm{Spec}(\mathcal{O}_K)$ of X. To begin with, there is a natural map $\mathcal{X}(\mathcal{O}_K) \to X(K)$, so that if $\mathcal{X} \to S$ has a section, then X has a K-rational point. There is also a specializing map $\mathcal{X}(\mathcal{O}_K) \to \mathcal{X}_k(k)$. As $\mathcal{O}_K$ is Henselian, this map is surjective if $\mathcal{X}$ is smooth over S. Hence we obtain that if the special fiber of the smooth locus of $\mathcal{X}$ over S is not empty, then X has a K-rational point.
If $\mathcal{X}$ is proper over S, then the natural map $\mathcal{X}(\mathcal{O}_K) \to X(K)$ is a bijection. If in addition $\mathcal{X}$ is regular, then every section of $\mathcal{X} \to S$ factors through the smooth locus of $\mathcal{X}$ over S, see [BLR90, Chapter 3.1, Proposition 2]. Therefore, if $\mathcal{X} \to S$ is a regular proper model of X, then X has a K-rational point if and only if the special fiber of the smooth locus of $\mathcal{X}$ over S is not empty. But if $\mathcal{X}$ is not regular, then there may exist sections through the singular locus of $\mathcal{X}$. To see this, consider the following example:

Example 0.1. Let k be an algebraically closed field of $\mathrm{char}(k) \neq 2$. Consider the complete local field $K = k(\!(t)\!)$ with ring of integers $k[\![t]\!]$, and the smooth projective K-scheme $X := V(tx_0x_1 - x_2^2) \subset \mathbb{P}^2_K$. The $k[\![t]\!]$-scheme $\mathcal{X} := V(tx_0x_1 - x_2^2) \subset \mathbb{P}^2_{k[\![t]\!]}$ is a proper model of X, and singular in $P = (0, [1 : 0 : 0])$. An affine neighborhood of P is given by $U = \mathrm{Spec}(k[\![t]\!][x_1, x_2]/(tx_1 - x_2^2))$. There is a section through P given on ring level by the quotient map

$$k[\![t]\!][x_1, x_2]/(tx_1 - x_2^2) \to (k[\![t]\!][x_1, x_2]/(tx_1 - x_2^2))/(x_1, x_2) \cong k[\![t]\!] \,.$$

The existence of weak Néron models plays an important role in the study of rational points. A weak Néron model $\mathcal{Z} \to S$ of a smooth K-variety X is a model of X, which is smooth over S, and has the property that the natural map from $\mathcal{Z}(\mathcal{O}_K)$ to $X(K)$ is a bijection. It is known that there exists a weak Néron model for every smooth proper K-variety, see [BLR90, Chapter 3.5, Theorem 2]. Note that if X has a weak Néron model $\mathcal{Z} \to S$, then X has a K-rational point if and only if the special fiber of $\mathcal{Z} \to S$ is not empty. For a regular proper model $\mathcal{X} \to S$ of a smooth proper K-variety X, the smooth locus of $\mathcal{X}$ over S is a weak Néron model of X. There is a way to obtain a weak Néron model from any proper model, the so called Néron smoothening, see [BLR90, Chapter 3]. The Néron smoothening is constructed by blowing up singular points having sections through them. But given a singular point, it is hard to decide a priori whether there is a section containing that point. Therefore, the Néron smoothening does not yield a concrete method of constructing a weak Néron model out of an arbitrary singular model.

In this work, we study a special kind of singular models, namely models which are quotients by tame cyclic group actions. More precisely, consider the following situation: Let X be a K-variety, and let L/K be a tame Galois extension. Then $G := \mathrm{Gal}(L/K)$ is cyclic of order prime to $\mathrm{char}(k)$, and acts on $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ such that $X_L/G \cong X$. Let $\mathcal{O}_L$ be the ring of integers of L, $T := \mathrm{Spec}(\mathcal{O}_L)$. Let $\mathcal{Y} \to T$ be a model of X_L with a good G-action which is compatible with this action on X_L. Then the quotient $\mathcal{X} := \mathcal{Y}/G$ is an S-scheme and in fact a model of X. We call $\mathcal{X} \to S$ a quotient model of X. Be aware of the fact that in general $\mathcal{X}$ will be singular.
Note that models of X_L with the required action really exist. We show in Theorem 1.30 that if X is a projective smooth K-variety, then there is always a quasi-projective weak Néron model of X_L extending the action on X_L. The model in Example 0.1 is a quotient model, too. To see this, consider the following example:

Example 0.2. Notation and assumptions as in Example 0.1. Set $L = k((s))$ with $s^2 = t$. L/K is a Galois extension of degree 2, and $\mathrm{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z}$ acts on L by sending s to $-s$. Consider the smooth and projective $k[\![s]\!]$-scheme $\mathbb{P}^1_{k[\![s]\!]}$, and let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{P}^1_{k[\![s]\!]}$ given by $g \in \mathrm{Aut}(\mathbb{P}^1_{k[\![s]\!]})$ with $g((s, [y_0 : y_1])) = (-s, [-y_0 : y_1])$. This action is compatible with the Galois action on $k[\![s]\!] \subset k((s))$. Computations show that actually $\mathcal{X} = \mathbb{P}^1_{k[\![s]\!]}/G$. In particular the quotient is singular.

Note that we can slightly generalize the notion of a quotient model: Let G be a extension of $\mathrm{Gal}(L/K)$, which is cyclic of order prime to $\mathrm{char}(k)$, and let $\mathcal{Y}$ be any integral $\mathcal{O}_L$-scheme with a good G-action compatible with the Galois action on $\mathcal{O}_L$. Consider the $\mathcal{O}_K$-scheme $\mathcal{X} := \mathcal{Y}/G$. We also call $\mathcal{X} \to S$ a quotient model of its generic fiber.
We study the relation between sections of $\mathcal{Y} \to T$ and $\mathcal{X} \to S$. For every fixed closed point y in the smooth locus of $\mathcal{Y}$ over T, we construct a section

of $\mathcal{X} \to S$ through the image x of y in $\mathcal{X}$, see Proposition 3.3 and Proposition 3.8. For the proof we use an explicit description of the group action on the complete local ring of the fixed point y proven in Lemma 2.11. In general, $\mathcal{X}$ will be singular in x, so in fact we construct sections through singular points. In Example 0.1 one can find a demonstration for such a section through a singular point.

If we assume in addition that $\mathcal{Y}$ is regular, and that $G = \mathrm{Gal}(L/K)$, then we can show that $\mathcal{X} \to S$ has a section if and only if there is a closed fixed point in the smooth locus of $\mathcal{Y}$ over T, see Theorem 3.6.

Now, let X be a smooth projective K-variety, and let L/K be a tame Galois extension. Furthermore, let $\mathcal{Y} \to T := \mathrm{Spec}(\mathcal{O}_L)$ be a quasi-projective model of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$, such that the smooth locus of $\mathcal{Y}$ over T is a weak Néron model of X_L. Let $G := \mathrm{Gal}(L/K)$ act on $\mathcal{Y}$, compatible with the Galois action on $\mathcal{O}_L$. Consider the quotient model $\mathcal{X} := \mathcal{Y}/G \to S$ of X.

Theorem. (Theorem 4.11) *There is a unique weak Néron model $\mathcal{Z} \to S$ of X with an S-morphism to $\mathcal{X}$, which is an isomorphism on the generic fibers, such that for all smooth quasi-projective integral S-schemes $\mathcal{V}$ every dominant map $\mathcal{V} \to \mathcal{X}$ factors through $\mathcal{Z}$.*

Note that the uniqueness of $\mathcal{Z} \to S$ with its properties is interesting, because in general a weak Néron model is not unique. In fact, $\mathcal{Z}$ is a subscheme of fixed points of the Weil restriction to S of the smooth locus of $\mathcal{Y}$ over T, see Construction 4.2. The construction goes back to [Edi92], where it is used in the context of abelian varieties and Néron models.

Having this explicit description of a weak Néron model $\mathcal{Z} \to S$ of X at hand, we can examine its special fiber $\mathcal{Z}_k$. Let $\mathrm{Sm}(\mathcal{Y}/T)^G$ be the scheme of fixed points of the smooth locus of $\mathcal{Y}$ over T. We show that there is a k-morphism $b : \mathcal{Z}_k \to \mathrm{Sm}(\mathcal{Y}/T)^G$, such that for every (not necessarily closed) point $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$ with residue field $\kappa(y)$, we obtain that $b^{-1}(y) \cong \mathbb{A}^s_{\kappa(y)}$, see Lemma 4.15. This implies that $[\mathrm{Sm}(\mathcal{Y}/T)^G] = [\mathcal{Z}_k] \in K_0(\mathrm{Var}_k)/(\mathbb{L}-1)$, where $K_0(\mathrm{Var}_k)$ is the Grothendieck ring of varieties over k, and $\mathbb{L}$ is the class of $\mathbb{A}^1_k$. To compute the fibers of b, we use an explicit description of the action of G on the complete local ring of y proven in Lemma 2.14.

Moreover, we study some motivic invariant, which one be attached to a smooth K-variety X over a complete local field with a weak Néron model $\mathcal{Z} \to S$; the motivic Serre invariant and the rational volume. These invariants are interesting in the context of rational points, because they vanish if X has no K-rational point.

The motivic Serre invariant $S(X)$ of a K-variety X is defined to be the class of the special fiber of a weak Néron model $\mathcal{Z} \to S$ of X in some quotient of the Grothendieck ring of varieties, namely in $K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L}-1)$, see Definition 5.2. Using the computation of the special fiber of our specific weak Néron model, see Lemma 4.15, we show:

Theorem. (Theorem 5.2) *Let X be a smooth projective K-variety, and let L/K be a tame Galois extension, $\mathcal{O}_L$ the ring of integers of L, and $T := \operatorname{Spec}(\mathcal{O}_L)$. Let $\mathcal{Y}$ be an integral, quasi-projective $\mathcal{O}_L$-scheme, and assume that the smooth locus of $\mathcal{Y} \to T$, $\operatorname{Sm}(\mathcal{Y}/T)$, is a weak Néron model of $X_L := X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(L)$. Let $G := \operatorname{Gal}(L/K)$ act on $\mathcal{Y}$, compatible with the Galois action on $\mathcal{O}_L$. Then $S(X) = [\operatorname{Sm}(\mathcal{Y}/T)^G] \in K_0^{\mathcal{O}_K}(\operatorname{Var}_k)/(\mathbb{L}-1)$.*

The rational volume $s(X)$ of a K-variety X is defined to be the Euler characteristic χ_c with proper support and coefficients in $\mathbb{Q}_l$, $l \neq \operatorname{char}(k)$, of the special fiber of a weak Néron model $\mathcal{Z} \to S$ of X.

Theorem. (Theorem 5.5) *Let X be a smooth projective K-variety, and let L/K be a tame Galois extension of degree q^r, $q \neq \operatorname{char}(k)$ a prime. Set $X_L := X \times_{Spec(K)} \operatorname{Spec}(L)$. Then $s(X) = s(X_L) \mod q$.*

The proof of this theorem uses the fact that there is always a weak Néron model of X_L with an action of $\operatorname{Gal}(L/K)$ extending the Galois action on X_L, see Theorem 1.30, as well as the equation for the Serre invariant, see Theorem 5.2. Moreover, we use the fact that for a scheme of finite type V over some field with a good action of a q-group G, we get that $\chi_c(V) = \chi_c(V^G)$ modulo q, which goes back to [Ser09, Section 7.2].
Finally, we can deduce the existence of rational points for some varieties with potential good reduction from the obtained result. A variety X over a local field K has potential good reduction if there is a Galois extension L/K such that $X_L := X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(L)$ has a smooth and proper model $\mathcal{Y} \to T := \operatorname{Spec}(\mathcal{O}_L)$.

Corollary. (Corollary 6.1) *Let X be a smooth projective K-variety with potential good reduction after a base change of order q^r, $q \neq \operatorname{char}(k)$ a prime. If the Euler characteristic of X with coefficients in $\mathbb{Q}_l$, $l \neq \operatorname{char}(k)$, does not vanish modulo q, then X has a K-rational point.*

Here we use our result regarding the rational volume, see Theorem 5.5, and the fact that the Euler characteristic with coefficients in $\mathbb{Q}_l$ is constant on fibers for a smooth and proper morphism $\mathcal{Y} \to T$.
Moreover, we obtain a similar result for the Euler characteristic with coefficients in the structure sheaf. Let X be a smooth proper K-variety with potential good reduction after a base change L/K of prime order $q \neq \operatorname{char}(k)$, and assume that the action of $G := \operatorname{Gal}(L/K)$ on $X_L := X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(L)$ induced by the Galois action on L extents to a good G-action on a smooth and proper model of X_L. We show that if the Euler characteristic of X with coefficients in the structure sheaf does not vanish modulo q, then X has a K-rational point, see Corollary 6.4. To prove this, we show that the G-action on the smooth and proper model $\mathcal{Y} \to T$ of X_L has a closed fixed point y. Then we use the section through the image of the closed fixed point y in $\mathcal{Y}/G$ constructed in Proposition 3.3 to deduce the claim.

Outline

In Chapter 1 we give the basic definitions concerning cyclic group actions we need later on, see Section 1.1, and prove some basic facts. Moreover, we consider models of varieties over local fields and study cyclic group actions on them, see Section 1.3. In Section 1.4 we study weak Néron models with cyclic group actions. The main result here is Theorem 1.30.
In Chapter 2 we examine cyclic group actions on regular complete local fields. We need the results obtained here for computations in Chapter 3 and Chapter 4.
The heart of this thesis consists of Chapter 3 and Chapter 4. In Chapter 3 we study sections of quotient models. The main result of this chapter is Theorem 3.6. In Chapter 4 we construct our specific weak Néron model, see in particular Theorem 4.11. In Section 4.3 we compute the special fiber of our specific weak Néron model.
In the last two chapters we give some applications of the results obtained so far. In Chapter 5 we deduce results for the Serre invariant, see Section 5.1, and the rational volume, see Section 5.2. In Chapter 6 we examine varieties with potential good reduction.

Conventions

A variety over a field F is a geometrically integral, separated F-scheme of finite type over F.
We assume that an integral scheme is connected.
All schemes are assumed to be noetherian.

Chapter 1

Group Actions and Equivariant Morphisms

1.1 Group Actions on Schemes

Let G be an abstract finite group.

Recall.

- Let X be a scheme, $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathrm{Spec}(\mathbb{Z})}(X)$ the abstract group of automorphisms of X. A *G-action* on X is given by a group homomorphism $\mu_X : G \to \mathrm{Aut}(X)$.
  Note that in order to know a group action it suffices to know the images of the generators of G in $\mathrm{Aut}(X)$. In particular giving a group action of the cyclic group $\mathbb{Z}/r\mathbb{Z}$ is the same as giving $g \in \mathrm{Aut}(X)$ with $g^r = \mathrm{id}$. We always fix a generator of $\mathbb{Z}/r\mathbb{Z}$. If we say that a $\mathbb{Z}/r\mathbb{Z}$-action is given by $g \in \mathrm{Aut}(X)$, then we mean that g is the image of this generator.
  If X is affine, i. e. $X = \mathrm{Spec}(A)$, a group action on X is also given by a group homomorphism $\mu_X^\# : G \to \mathrm{Aut}(A)$. If $G = \mathbb{Z}/r\mathbb{Z}$, the G-action is given by some $\alpha \in \mathrm{Aut}(A)$ with $\alpha^r = \mathrm{id}$.

- Let X, S be schemes with G-actions. We call a morphism of schemes $\varphi : X \to S$ *G-equivariant*, if for all $g \in G$ $\varphi \circ \mu_X(g) = \mu_S(g) \circ \varphi$.
  Note that it suffices to check this for a set of generators g_i of G. In particular, if $G \cong \mathbb{Z}/r\mathbb{Z}$, and the group action on X is given by $g \in \mathrm{Aut}(X)$, and that on S by $g_S \in \mathrm{Aut}(S)$, then φ is G-equivariant if and only if $\varphi \circ g = g_S \circ \varphi$.

- A G-action on a scheme X is called *good*, if every orbit is contained in an affine open subscheme of X. By [Gro63, Exposé V, Proposition 1.8] this is the same as requiring a cover of X by affine, open, G-invariant subschemes. So if X is affine, every G-action on X is good. As G is finite, every orbit of a group action of G on X is a finite set. If X is quasi-projective over an affine scheme, by [Liu02, Chapter 3, Proposition 3.36.b] every finite set is contained in an open affine subschemes of X, so every G-action on X is good.

Fact. [Gro63, Exposé V.1]
Let S be a scheme, and let X be an S-scheme with a good G-action, such that the structure map is G-equivariant for this G-action and the trivial action on S. Then there exists a quotient $\pi : X \to X/G$ in the category of S-schemes with the following properties:

- $\pi : X \to X/G$ is a scheme quotient, i.e. π is G-equivariant for the G-action on X and the trivial action on X/G, and for all S-schemes Z the map
$$\mathrm{Hom}_S(X/G, Z) \to \mathrm{Hom}_S(X, Z)^G;\ f \mapsto f \circ \pi$$
is bijective.
We have a universal property, namely for all S-schemes Z, and all S-morphisms $f : X \to Z$ which are G-equivariant for the trivial G-action on Z, there exists a unique S-morphism $f' : X/G \to Z$ such that $f' \circ \pi = f$.
- $\pi : X \to X/G$ is a topological quotient on the underlying topological spaces, in particular π is surjective, and the fibers are the orbits of G in X. Moreover, $\mathcal{O}_{X/G} \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.
- In general, π is integral, and if X is of finite type over S, then π is finite. If X is affine or normal, the same holds for X/G. If X is separated over S, then X/G is separated over S. If X is of finite type over S, then X/G is of finite type over S.

Lemma 1.1. *Let Y, T be schemes with good G-actions, let $\pi : Y \to X$, $\pi_T : T \to S$ be the quotients. Let $\varphi : Y \to T$ be a G-equivariant morphism. Then there exists a unique morphism $\varphi_G : X \to S$ making the following diagram commutative:*

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_G} \\ T & \xrightarrow[\pi_T]{} & S \end{array}$$

Assume that π_T is finite. If φ is of finite type, then π is finite and φ_G is of finite type. If φ is proper, then the same holds for φ_G.

Proof. Note that $\pi_T \circ \varphi$ is G-equivariant for the trivial action on S, hence $\pi : Y \to X$ is a quotient in the category of S-schemes. Let φ_G be the structure map of X. As π is an S-morphism, $\varphi_G \circ \pi = \pi_T \circ \varphi$.

Assume now that π_T is finite. If φ is of finite type, then $\varphi \circ \pi_T$ is of finite type, and therefore π is finite, and φ_G is of finite type.
Assume now that φ is proper. Note that π is surjective. Moreover, π_T is finite which implies $\varphi \circ \pi_T$ is proper. Note that φ_G is separated and of finite type, because this holds for $\pi_T \circ \varphi$. Hence by [GW10, Proposition 12.59] φ_G is proper. □

Definition 1.1. [Edi92, Section 3]
Let S be a scheme, and let X be an S-scheme with a G-action, such that the structure map is G-equivariant for this action and the trivial action on S. We define the *functor of fixed points* by

$$\begin{aligned} X^G : (\mathrm{Sch}\,/S) &\to (\mathrm{Sets}) \\ W &\mapsto X(W)^G = \mathrm{Hom}_S(W, X)^G \end{aligned}$$

Fact. [Edi92, Proposition 3.1]
X^G is represented by a subscheme of X. If X is a separated S-scheme, then X^G is represented by a closed subscheme of X.

Remark 1.2. If not otherwise specified, we view X as $\mathbb{Z}$-scheme.

Lemma 1.3. *Let X, S be schemes with G-actions. Let $\varphi : X \to S$ be a G-equivariant morphism. Then there is a morphism $\varphi^G : X^G \to S^G$ such that the following diagram commutes:*

$$\begin{array}{ccc} X^G & \hookrightarrow & X \\ \downarrow{\scriptstyle \varphi^G} & & \downarrow{\scriptstyle \varphi} \\ S^G & \hookrightarrow & S \end{array}$$

In particular, X^G is a subscheme of $X \times_S S^G$.
Assume that S and φ are separated. Then φ^G is separated, and if φ is of finite type, then the same holds for φ^G. If φ is smooth, and $\#(G)$ is invertible in X, φ^G is smooth, too.

Proof. As X^G is a subscheme of X, we may consider $\varphi|_{X^G}$. Take any $W \in (\mathrm{Sch}\,/\mathbb{Z})$, $w \in X^G(W)$. Then $\varphi|_{X^G}(W)(w) = \varphi \circ w \in S(W)$. As w is G-equivariant for the trivial action on W and the G-action on X, and φ is G-equivariant for the G-actions on X and S , $\varphi \circ w$ is G-equivariant for the trivial action on W and the G-action on S, i. e. $\varphi \circ w \in S^G(W)$. Hence $\varphi(X^G) \subset S^G$, which yields the commutative diagram.
If S and φ are separated, $X^G \hookrightarrow X$ and $S^G \hookrightarrow S$ are closed immersions, and therefore separated. So φ^G is separated. If φ is of finite type, then $\varphi|_{X^G}$ is of finite type, and therefore φ^G is of finite type by [GW10, Proposition 10.7]. If φ is smooth, and $\#(G)$ is invertible in X, then φ^G is smooth by [Edi92, Proposition 3.5]. □

1.2 Induced Group Actions and Equivariant Morphisms

Let Y, T be schemes with good G-actions, $\varphi : Y \to T$ a G-equivariant morphism of finite type. In this section we prove some basic lemmas concerning such G-equivariant morphisms which we will need later.
For simplicity, we assume in this section that G is cyclic, i. e. $G := \mathbb{Z}/r\mathbb{Z}$ for some $r \in \mathbb{N}$. Everything proven in this subsection can also be proven for general finite groups. But assuming G to be cyclic simplifies the proofs, and the general case is not needed for this thesis.

Notation. Let the G-action on Y be given by $g \in \mathrm{Aut}(Y)$, and that on T by $g_T \in \mathrm{Aut}(T)$.

Lemma 1.4. *Let* $\mathrm{Sm}(Y/T)$ *be the smooth locus of* φ. *Then the* G*-action on* Y *restricts to a* G*-action on* $\mathrm{Sm}(Y/T)$, *i. e. there is a* G*-action on* $\mathrm{Sm}(Y/T)$ *such that the open immersion* $\mathrm{Sm}(Y/T) \hookrightarrow Y$ *is* G*-equivariant.*

Proof. In order to show the claim, it suffices to show that $\mathrm{Sm}(Y/T) \subset Y$ is G-invariant, i. e. that $g(\mathrm{Sm}(Y/T)) \subset \mathrm{Sm}(Y/T)$. Therefore it suffices to show that $\varphi|_{g(\mathrm{Sm}(Y/T))}$ is smooth. Note that

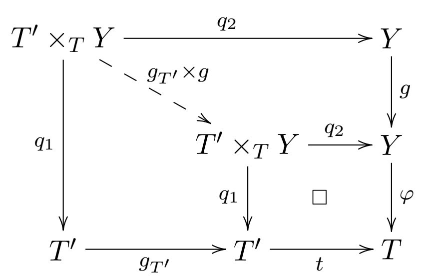
$$\varphi|_{g(\mathrm{Sm}(Y/T))} = g_T \circ \varphi \circ g^{-1}|_{g(\mathrm{Sm}(Y/T))} = g_T \circ \varphi|_{\mathrm{Sm}(Y/T)} \circ g^{-1}|_{g(\mathrm{Sm}(Y/T))}$$

The first equation holds, because φ is G-equivariant. The second equation holds, because g^{-1} maps $g(\mathrm{Sm}(Y/T))$ to $\mathrm{Sm}(Y/T)$. Note that $\mathrm{Sm}(Y/T) \subset Y$ is open, so $g(\mathrm{Sm}(Y/T)) \subset Y$ is open. Hence $g^{-1}|_{g(\mathrm{Sm}(Y/T))}$ is smooth, because open immersions and isomorphisms are smooth. Moreover, g_s is an isomorphism and hence smooth, and $\varphi|_{\mathrm{Sm}(Y/T)}$ is smooth by definition of $\mathrm{Sm}(Y/T)$. Altogether $\varphi|_{g(\mathrm{Sm}(Y/T))}$ is smooth, because it is the composition of smooth morphisms. □

Notation. With a slight abuse of notation, we use g also for $g|_{\mathrm{Sm}(Y/T)}$.

Lemma 1.5. *Let* T' *be a scheme with a* G*-action, and let* $t : T' \to T$ *be a* G*-equivariant map. Then there is a unique* G*-action on* $T' \times_T Y$, *such that the projection maps to* T *and* Y *are* G*-equivariant.*

Proof. Let the action on T' be given by $g_{T'} \in \mathrm{Aut}(T')$. Look at the following diagram (q_i are the projection maps):



As φ and t are G-equivariant, we have

$$t \circ g_{T'} \circ q_1 = g_T \circ t \circ q_1 = g_T \circ \varphi \circ q_2 = \varphi \circ g \circ q_2$$

i. e. the diagram commutes. In order to construct the G-action with the required properties we need to construct $g_{T'} \times g \in \mathrm{Aut}(T' \times_T Y)$ such that $(g_{T'} \times g)^r = \mathrm{id}$, $q_1 \circ (g_{T'} \times g) = g_{T'} \circ q_1$ and $q_2 \circ (g_{T'} \times g) = g \circ q_2$. Let $g_{T'} \times g$ be the unique morphism making the diagram commutative induced by the universal property of fiber product. By construction, the last two required equations hold. Moreover we have

$$q_1 \circ (g_{T'} \times g)^r = g_T^r \circ q_1 = q_1 \text{ and } q_2 \circ (g \times g_{T'})^r = q_2$$

By the universal property of the fiber product, id is the unique morphism with this properties, hence $(g_{T'} \times g)^r = \mathrm{id}$. $\square$

Notation. Denote the generator of the G-action on $T' \times_T Y$ constructed in Lemma 1.5 by $g_{T'} \times g \in \mathrm{Aut}(T' \times_T Y)$.
If $T = \mathrm{Spec}(R)$, $T' = \mathrm{Spec}(R')$ and $Y = \mathrm{Spec}(A)$ are affine, we use the following notation: $g^{\#} \otimes g_T^{\#} := (g \times g_T)^{\#} \in \mathrm{Aut}(A \otimes_R R')$.

Notation. Let $\pi : Y \to X$ and $\pi_T : T \to S$ be the quotients of the G-actions. By Lemma 1.1 we have the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_G} \\ T & \xrightarrow[\pi_T]{} & S \end{array}$$

Let $X_T := T \times_S X$ with cartesian diagram

$$\begin{array}{ccc} X_T & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & \square & \downarrow{\scriptstyle\varphi_G} \\ T & \xrightarrow[\pi_T]{} & S \end{array}$$

The next thing we do is examining the relation between X_T and Y. For simplicity, assume from now on that $T = \mathrm{Spec}(R)$ is affine.

Lemma 1.6. *The projection map $p_2 : X_T \to X$ is the quotient for the good G-action on X_T given by $g_T \times \mathrm{id} \in \mathrm{Aut}(X_T)$, and there is a unique surjective G-equivariant T-morphism $f : Y \to X_T$ with $p_2 \circ f = \pi$.*

$$\begin{array}{ccc} Y & & \\ {\scriptstyle f}\downarrow & \searrow^{\pi} & \\ X_T & \xrightarrow[p_1]{} & X \end{array}$$

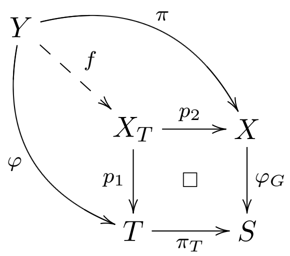
Proof. $T = \mathrm{Spec}(R)$, hence $S = \mathrm{Spec}(R^G)$. Let X be covered by $V_i = \mathrm{Spec}(A_i)$. Taking into account how the fiber product is constructed, X_T is covered by the $\mathrm{Spec}(R \otimes_{R^G} A_i)$, and

$$(g_T \times \mathrm{id})^{\#} : R \otimes_{R^G} A_i \to R \otimes_{R^G} A_i;\ \ s \otimes a \mapsto {g_T}^{\#}(s) \otimes a$$

Note that the $\mathrm{Spec}(R \otimes_{R^G} A_i)$ are G-invariant open affine subsets of X_T covering it, so the G-action on X_T is good. The quotient is given locally by

$$A_i = R^G \otimes_{R^G} A_i = (R \otimes_{R^G} A_i)^G \hookrightarrow R \otimes_{R^G} A_i$$

Therefore, $p_2 : X_T \to X$ is the quotient for the constructed action on X_T. Moreover we have the following commutative diagram:



By the universal property of the fiber product, we get a unique T-morphism $f : Y \to X_T$ with $p_2 \circ f = \pi$. Note that we have

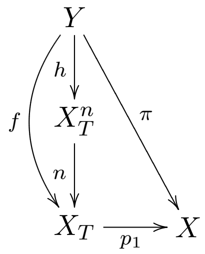
$$p_1 \circ ((g_T \times \mathrm{id})^{-1} \circ f \circ g) = g_T^{-1} \circ p_1 \circ f \circ g = g_T^{-1} \circ \varphi \circ g = \varphi$$

because p_2 and φ are G-equivariant, and

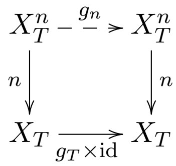
$$p_2 \circ ((g_T \times \mathrm{id})^{-1} \circ f \circ g) = p_2 \circ f \circ g = \pi \circ g = \pi$$

because π and p_1 are quotients. But since f is the unique morphism with this property, $(g_T \times \mathrm{id})^{-1} \circ f \circ g = f$, i. e. f is G-equivariant.
Assume that f is not surjective. Then there exists a point $y \in X_T \backslash f(Y)$. As f is G-equivariant, the orbit $G(y)$ of y lies in $X_T \backslash f(Y)$, and f has to map a point in $\pi^{-1}(p_2(y)) \subset Y$ to an element in $G(y)$, hence $\pi^{-1}(p_1(y)) = \emptyset$. But as $\pi : Y \to X$ is a quotient, it is surjective, which is a contradiction. Therefore f is surjective. □

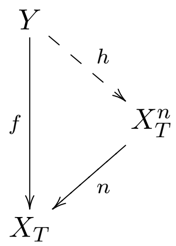
Lemma 1.7. *Assume that Y and X_T are integral. Let X_T^n be the normalization of X_T, and $n : X_T^n \to X_T$ the normalization map. Then there is a good G-action on X_T^n, such that n is G-equivariant for this G-action and the action on X_T given by $g_T \times \mathrm{id} \in \mathrm{Aut}(X_T)$. If Y is normal, there is a unique surjective G-equivariant morphism $h : Y \to X_T^n$ with $n \circ h = f$, f as* *in Lemma 1.6.*



Proof. Look at the following diagram:



Let g_n be the unique morphism making the diagram commutative, which is induced by the universal property of the normalization [GW10, Proposition 12.44] using that X_T^n is normal, and that $(g_T \times \mathrm{id}) \circ n$ is surjective. Moreover, $n \circ g_n^r = (g_T \times \mathrm{id})^r \circ n = n$, and as id is unique with this property, $g_n^r = \mathrm{id}$. So g_n defines a G-action on X_T^n. By construction, n is G-invariant. Note that the action on X_T^n is good: $V_i := n^{-1}(U_i)$ is affine for all affine open $U_i \subset X_T$, and $n\,|_{V_i}\colon V_i \to U_i$ is the normalization of the U_i by [GW10, Proposition 12.43]. So the preimages of a cover of X_T by open affine G-equivariant subsets give a similar cover of X_T^n.
Assume now that Y is normal. Let f be as in Lemma 1.6. Consider the following commutative diagram:



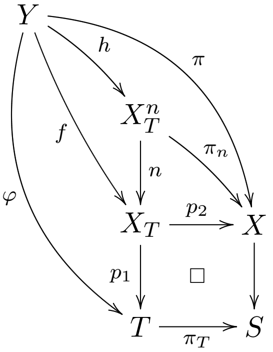
As Y is normal and integral, and f is surjective, there is a unique $h : Y \to X_T^n$ such that $n \circ h = f$. Note that

$$n \circ (g_n^{-1} \circ h \circ g) = (g_T \times \mathrm{id})^{-1} \circ n \circ h \circ g = (g_T \times \mathrm{id})^{-1} \circ f \circ g = f$$

This holds, because n and f are G-equivariant. But h is the unique morphism with $n \circ h = f$, hence $g_n^{-1} \circ h \circ g = h$, i. e. h is G-equivariant. We can use the same argument as in Lemma 1.6 to show that h is surjective. □

Lemma 1.8. *Assume that π_T is finite of degree r, that Y is integral and normal, and that X_T is integral. Then h as constructed in Lemma 1.7 is an isomorphism.*

Proof. Let the notation be as in Lemma 1.6 and Lemma 1.7. Set $\pi_n := p_2 \circ n$. Altogether we have the following commutative diagram:



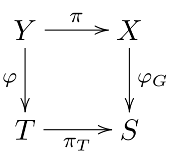
We need to show that h is an isomorphism. As $\pi : Y \to X$ is the quotient of a G-action, it is finite of degree smaller or equal $r = \#(G)$. As π_T is finite of degree r, the same holds for p_2, because this property is stable under base change. As n is the normalization map, it is finite of degree 1. Therefore, π_n is finite of degree r. As $\pi = \pi_n \circ h$ with π_n and π finite, by [GW10, Proposition 12.11] h is finite. Note that X_T^n is integral, because X_T is integral. Y is integral by assumption and X is integral, because it is the quotient of an integral scheme. Hence it makes sense to consider the function fields $\kappa(X_T^n)$, $\kappa(Y)$ and $\kappa(X)$. As h, π_n and π are finite morphisms, and $\pi = \pi_n \circ h$, we get:

$$r \geq \deg(\pi) = [\kappa(Y) : \kappa(X)] = [\kappa(Y) : \kappa(X_T^n)][\kappa(X_T^n) : \kappa(X)]$$

$[\kappa(X_T^n) : \kappa(X)] = \deg(\pi_n) = r$, hence $[\kappa(Y) : \kappa(X_T^n)] = 1$. So by [GW10, Lemma 9.33] h is birational. Altogether $h : Y \to X_T^n$ is a finite birational morphism between integral normal schemes. That means, by [GW10, Corollary 12.88], h is an open immersion. As h is surjective, it is an isomorphism. □

Lemma 1.9. *Let Y be an integral normal scheme. If $\pi_T : T \to S$ is finite of degree r and étale, the following diagram is cartesian:*

(1.1)



Moreover, the G-action on Y is given by $g_T \times \mathrm{id} \in \mathrm{Aut}(Y) = \mathrm{Aut}(X_T)$.

Proof. Let the notation be as in Lemma 1.6 and Lemma 1.7. By Lemma 1.8, we only need to show that $X_T = X_T^n$, and that $n = \mathrm{id}$. As Y is irreducible, the same holds for X_T, because f is surjective. By assumption Y is normal. X is normal, because it is the quotient by a finite group action of the normal scheme Y, see [GW10, Example 12.48]. As π_T is étale, the same holds for p_2, because this property is stable under base change. So we get from [Mil80, Chapter I, Proposition 3.17] that X_T is normal, too. Therefore $X_T = X_T^n$, and $n = \mathrm{id}$. □

1.3 Group Actions on Models of Varieties over Complete Local Fields

In this chapter we study models of varieties over complete local fields and cyclic group actions on them. Let K be a complete local field with ring of integers $\mathcal{O}_K$, $S := \mathrm{Spec}(\mathcal{O}_K)$, and residue field k. Assume that k is algebraically closed.

Definition 1.2. Let X be a K-variety. A *model* of X is an integral S-scheme $\mathcal{X}$ of finite type over S such that $\mathcal{X} \times_S \mathrm{Spec}(K) \cong X$.

Remark 1.10. Let X be a non-empty K-variety, and let $\mathcal{X} \to S$ be any model of X. Then $\mathcal{X}$ dominates S, so by [Har77, Chapter III, Proposition 9.7] $\mathcal{X}$ is flat over S.

Example 1.11. Let X be any K-variety. View X as an S-scheme via $\mathrm{Spec}(K) \hookrightarrow S$. This is a model of X.

Example 1.12. Let X be a projective K-variety, i. e. X is a closed subset of $\mathbb{P}^N_K$ for some $N \in \mathbb{N}$. Then the closure of X in $\mathbb{P}^N_S$ (with reduced scheme structure) is a model of X.

Example 1.13. Let X be a normal K-variety, and let $\mathcal{X} \to S$ be any model of X. Then the normalization of $\mathcal{X}$ is a model of X, too.

Now fix a Galois extension L/K with Galois group $\mathrm{Gal}(L/K)$. Let $\mathcal{O}_L$ be the ring of integers of L, $T := \mathrm{Spec}(\mathcal{O}_L)$. Note that k is the residue field of L. For a general introduction to local fields and their Galois extensions we refer to [Ser79]. Here we just give the following fact, already modified for the case that k is algebraically closed:

Fact. [Ser79, Chapter IV, Corollary 2 and Corollary 4]
If $\mathrm{char}(k) = 0$, then $\mathrm{Gal}(L/K)$ is cyclic.
If $\mathrm{char}(k) = p \neq 0$, then $\mathrm{Gal}(L/K)$ is the semi-direct product of a cyclic group of order prime to p with a normal subgroup whose order is a power of p.

Definition 1.3. A Galois extension L/K is called *tame*, if the order of $\mathrm{Gal}(L/K)$ is prime to $\mathrm{char}(k)$.

Remark 1.14. Note that the Galois group of a tame Galois extension L/K is always cyclic.

From now on, assume that L/K is tame.

Lemma 1.15. *Let X be a K-variety, $\mathcal{X} \to S$ be a model of X. Then $\mathcal{X}_T := \mathcal{X} \times_S T \to T$ is model of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$.*

Proof. As $\mathcal{X} \to S$ is a model of X, we have

$$\mathcal{X}_T \times_T \mathrm{Spec}(L) = \mathcal{X} \times_S \mathrm{Spec}(K) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) = X_L$$

As $\mathcal{X}$ is of finite type over S, $\mathcal{X}_T$ is of finite type over T. It remains to check that $\mathcal{X}_T$ is integral. By Remark 1.10, $\mathcal{X}$ is flat over S, therefore $\mathcal{X}_T$ is flat over T, because flatness is stable under base change. Hence there cannot be a connected component of $\mathcal{X}_T$ only supported on the special fiber. But the generic fiber X_L of $\mathcal{X}_T$ is connected, hence $\mathcal{X}_T$ is connected. Therefore, we can check that $\mathcal{X}_T$ is integral locally. We may assume that $\mathcal{X}_T = \mathrm{Spec}(A)$ is affine. Note that $X_L = \mathrm{Spec}(A \otimes_{\mathcal{O}_L} L)$ is integral, because it is the base change of the K-variety X, which is assumed to be geometrically integral. Take any $a, b \in A$ such that $ab = 0$. Look at $(a \otimes 1)(b \otimes 1) = (ab \otimes 1) \in A \otimes_{\mathcal{O}_L} L$. As $A \otimes_{\mathcal{O}_L} L$ is integral, $a \otimes 1 = 0$ or $b \otimes 1 = 0$. Without loss of generality, let $a \otimes 1 = 0$. Then there is an $N \in \mathbb{N}$ such that $at^N = 0$, with t the image of the uniformizer in $\mathcal{O}_L$. Now $\mathcal{O}_L$ is flat, which implies that t is not a zero divisor, hence $a = 0$. So there is no zero divisor in A. Altogether, $\mathcal{X}_T$ is integral. □

Remark 1.16. By definition of the Galois group, $\mathrm{Gal}(L/K)$ acts on L, and $K = L^{\mathrm{Gal}(L/K)}$. The action of $\mathrm{Gal}(L/K)$ can be restricted to $\mathcal{O}_L$, and $\mathcal{O}_L^{\mathrm{Gal}(L/K)} = \mathcal{O}_K$. We call this action the *Galois action* on $\mathcal{O}_L$. Note that $\mathrm{Spec}(L) \hookrightarrow T$ is $\mathrm{Gal}(L/K)$-equivariant for these actions.

Remark 1.17. Let X be a K-variety. Let the action of the Galois group on L be given by $g_L \in \mathrm{Aut}(\mathrm{Spec}(L))$. Then, by Lemma 1.6, $\mathrm{Gal}(L/K)$ acts on $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ given by $\mathrm{id} \times g_L \in \mathrm{Aut}(X_L)$, and the structure map $X_L \to \mathrm{Spec}(L)$ is $\mathrm{Gal}(L/K)$-equivariant. We call this action the *Galois action* on X_L. Note that $X_L / \mathrm{Gal}(L/K) = X$.

Lemma 1.18. *Let X be a K-variety. Furthermore let $\varphi : \mathcal{Y} \to T$ be a model of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ with a good $\mathrm{Gal}(L/K)$-action. Assume that $X_L \hookrightarrow \mathcal{Y}$ is $\mathrm{Gal}(L/K)$-equivariant for the action on $\mathcal{Y}$ and the Galois action on X_L as in Remark 1.17.*
Then φ is $\mathrm{Gal}(L/K)$-equivariant for this action and the Galois action on T. Moreover, $\mathcal{X} := \mathcal{Y}/\mathrm{Gal}(L/K) \to S$ is a model of X.

Proof. Let the $\mathrm{Gal}(L/K)$-action on $\mathcal{Y}$ be given by $g \in \mathrm{Aut}(\mathcal{Y})$, and that on T by $g_T \in \mathrm{Aut}(T)$. To show that φ is $\mathrm{Gal}(L/K)$-equivariant, we need to show that $g_T \circ \varphi \circ g^{-1} = \varphi$. We have $g_T \circ \varphi \circ g^{-1} |_{X_L} = \varphi |_{X_L}$, because $X_L \hookrightarrow \mathcal{Y}$, $X_L \to \mathrm{Spec}(L)$, and $\mathrm{Spec}(L) \hookrightarrow T$ are $\mathrm{Gal}(L/K)$-equivariant. As $X_L \subset \mathcal{Y}$ is open and dense, $\mathcal{Y}$ is reduced, and T is separated, [GW10, Corollary 9.9] implies that $g_T \circ \varphi \circ g^{-1} = \varphi$.
Note that $\mathcal{X} := \mathcal{Y}/\mathrm{Gal}(L/K)$ is an S-scheme of finite type by Lemma 1.1. As it is a quotient by a finite group of the integral scheme $\mathcal{Y}$, it is integral, too. As $\mathrm{Spec}(L) \hookrightarrow T$ is flat, by [Gro63, Exposé V, Proposition 1.9] we obtain

$$\mathcal{X} \times_S \mathrm{Spec}(K) \cong \mathcal{Y} \times_T \mathrm{Spec}(L)/\mathrm{Gal}(L/K) = X_L/\mathrm{Gal}(L/K) = X$$

Altogether, $\mathcal{X} \to S$ is a model of X. □

Example 1.19. Let X be a K-variety, and let $\mathcal{X} \to S$ be any model of X. Then $\mathcal{X}_T := \mathcal{X} \times_S T$ is a model of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ by Lemma 1.15. Let the Galois action on T be given by $g_T \in \mathrm{Aut}(T)$, hence $\mathrm{Gal}(L/K)$ acts on $\mathcal{X}_T$ given by $\mathrm{id} \times g_T \in \mathrm{Aut}(\mathcal{X}_T)$ by Lemma 1.5. Note that $X_L \hookrightarrow \mathcal{X}_T$ is $\mathrm{Gal}(L/K)$-equivariant by construction.

Example 1.20. Let X be a smooth K-variety, $\mathcal{Y} \to T$ be a model of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ with a $\mathrm{Gal}(L/K)$-action such that $\mathcal{X}_L \hookrightarrow \mathcal{Y}$ is $\mathrm{Gal}(L/K)$-equivariant. Let $\mathcal{Y}^n$ be the normalization of $\mathcal{Y}$. Then, as X_L is normal, $\mathcal{Y}^n$ is a model of X_L. By Lemma 1.7, $\mathrm{Gal}(L/K)$ acts on $\mathcal{Y}^n$, and $X_L \hookrightarrow \mathcal{Y}^n$ is still $\mathrm{Gal}(L/K)$-equivariant.

Definition 1.4. Let X be a K-variety. We call a model $\mathcal{X} \to S$ of X a *quotient model*, if the following properties are satisfied: L/K is a Galois extension, and G is a extension of $\mathrm{Gal}(L/K)$. $\mathcal{Y}$ is an integral $\mathcal{O}_L$-scheme of finite type with a good G-action which is compatible with the Galois action on $\mathcal{O}_L$, and $\mathcal{X} = \mathcal{Y}/G$.

Remark 1.21. In this thesis, we will only consider tame Galois extensions L/K, hence $\mathrm{Gal}(L/K)$ is cyclic of order prime to $\mathrm{char}(k)$, and extensions G of $\mathrm{Gal}(L/K)$ which are cyclic of order prime to $\mathrm{char}(k)$.

In fact, if $G = \mathrm{Gal}(L/K)$, we are in the case of Lemma 1.18, as the following lemma shows:

Lemma 1.22. *Let $\mathcal{Y}$ be an integral $\mathcal{O}_L$-scheme of finite type with a good $G = \mathrm{Gal}(L/K)$-action which is compatible with the Galois action on $\mathcal{O}_L$. Assume that $\mathcal{Y}_L := \mathcal{Y} \times_S \mathrm{Spec}(L)$ is normal. Then $\mathcal{Y}_L \cong \mathcal{Y}/G \times_S \mathrm{Spec}(L)$ as L-schemes, and the G-action on $\mathcal{Y}$ restricts to $\mathcal{Y}_L$ and coincides with the Galois action described in Remark 1.17.*

Proof. Note that G acts on $\mathrm{Spec}(L)$ such that $\mathrm{Spec}(L) \hookrightarrow T$ is $G = \mathrm{Gal}(L/K)$-equivariant, and $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is the quotient, see Remark 1.16. Therefore G acts on $\mathcal{Y}_L$ such that $\mathcal{Y}_L \hookrightarrow \mathcal{Y}$ and $\mathcal{Y}_L \to \mathrm{Spec}(L)$ are G-equivariant, see Lemma 1.5.
As L/K is Galois, $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is étale. Therefore, by Lemma 1.9 the following commutative diagram as given in Lemma 1.1 is cartesian:

$$\begin{array}{ccc} \mathcal{Y}_L & \longrightarrow & \mathcal{Y}_L/G \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(K) \end{array}$$

Let the G-action on $\mathrm{Spec}(L)$ be given by $g_L \in \mathrm{Aut}(\mathrm{Spec}(L))$. The isomorphism $f : \mathcal{Y}_L \to \mathcal{Y}_L/G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ is G-equivariant for the G-action on $\mathcal{Y}_L$, and the G-action on $\mathcal{Y}_L/G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ given by $id \times g_L$, see Lemma 1.6. As $\mathrm{Spec}(L) \hookrightarrow T$ is flat, by [Gro63, Exposé V, Proposition 1.9] we obtain $\mathcal{Y}_L/G \cong \mathcal{Y}/G \times_S \mathrm{Spec}(K)$ as K-schemes. Altogether we get over $\mathrm{Spec}(L)$:

$$\begin{aligned} \mathcal{Y}/G \times_S \mathrm{Spec}(L) &\cong \mathcal{Y}/G \times_S \mathrm{Spec}(K) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \\ &\cong \mathcal{Y}_L/G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \cong \mathcal{Y}_L \end{aligned}$$

□

1.4 Group Actions on Weak Néron Models

In this section we define weak Néron models of smooth varieties over complete local fields. We show that for a smooth projective variety X over a local field K, and a tame Galois extension L/K, there exists a quasi-projective weak Néron model of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ such that the Galois action on X_L as in Remark 1.17 extends to an action of $\mathrm{Gal}(L/K)$ on this model.
Fix a complete local field K with ring of integers $\mathcal{O}_K$, $S := \mathrm{Spec}(\mathcal{O}_K)$. Let k be the residue field of $\mathcal{O}_K$. Assume that k is algebraically closed.

Definition 1.5. Let X be a smooth K-variety. A *weak Néron model* of X is a smooth and separated model $\mathcal{X} \to S$ of X, such that the natural map $\mathcal{X}(\mathcal{O}_K) \to \mathcal{X} \times_S \mathrm{Spec}(K)(K)$ is a bijection.

Remark 1.23. Let X be a smooth K-variety attached with a weak Néron model $\mathcal{X} \to S$. Then $X(K) = \emptyset$ if and only if the special fiber $\mathcal{X}_k$ of $\mathcal{X} \to S$ is empty. This is true because by definition the natural map $\mathcal{X}(\mathcal{O}_K) \to X(K)$ is a bijection, and moreover by [BLR90, Chapter 2.3, Proposition 5] the specializing map $\mathcal{X}(\mathcal{O}_K) \to \mathcal{X}_k(k)$ is surjective, because $\mathcal{O}_K$ is Henselian and $\mathcal{X} \to S$ is smooth.

Remark 1.24. Note that a weak Néron model is not unique. Take any weak Néron model, blow up a point in the special fiber, and then take the smooth locus of the obtained scheme. This again is a weak Néron model.

Example 1.25. If X is a smooth K-variety and $X(K) = \emptyset$, then X viewed as an S-scheme via $\mathrm{Spec}(K) \hookrightarrow S$ is a weak Néron model of X.

Example 1.26. Let X be a proper smooth K-variety, $\mathcal{X} \to S$ a proper model of X. Let $\mathrm{Sm}(\mathcal{X}/S)$ be the smooth locus of $\mathcal{X}$ over S. If $\mathcal{X}$ is a regular scheme, $\mathrm{Sm}(\mathcal{X}/S) \to S$ is a weak Néron model of X. This holds, because, as $\mathcal{X} \to S$ is proper, $\mathcal{X}(\mathcal{O}_K) \to \mathcal{X}_K(K)$ is a bijection, and as $\mathcal{X}$ is regular, [BLR90, Chapter 3.1,Proposition 2] implies that $\mathcal{X}(\mathcal{O}_K) = \mathrm{Sm}(\mathcal{X}/S)(\mathcal{O}_K)$.

Remark 1.27. A weak Néron model does not exist for all smooth K-varieties X. It follows from [BLR90, Chapter 3.5, Theorem 2] that a weak Néron model exists if X is proper over K.

The main tool of showing that weak Néron models actually exist is the so called Néron smoothening.

Definition 1.6. Let X be a smooth K-variety, and let $\mathcal{X} \to S$ be a model of X. A *Néron smoothening* of $\mathcal{X}$ is a proper S-morphism $f : \mathcal{X}' \to \mathcal{X}$ such that f is an isomorphism on the generic fibers, and the canonical map $\mathrm{Sm}(\mathcal{X}'/S)(S) \to \mathcal{X}(S)$ is bijective. Here $\mathrm{Sm}(\mathcal{X}'/S)$ is the smooth locus of $\mathcal{X}'$ over S.

Fact. [BLR90, Chapter 3.1, Theorem 3]
Let X be a smooth K-variety, and let $\mathcal{X} \to S$ be a model of X. Then $\mathcal{X}$ admits a Néron smoothening $f : \mathcal{X}' \to \mathcal{X}$. In fact, f can be constructed as a finite sequence of blowups with centers in the special fibers.

Remark 1.28. Note that a Néron smoothening is not necessarily a resolution of singularities, because only singular points with sections through them need to be resolved. A Néron smoothening exists also in positive characteristic.

We will show that there is a Néron smoothening which is compatible with cyclic group actions as examined in the previous section.
Let L/K be a tame Galois extension, and let $\mathcal{O}_L$ be the ring of integers of L, $T := \mathrm{Spec}(\mathcal{O}_L)$. Set $G := \mathrm{Gal}(L/K)$.

Proposition 1.29. *Let Y be a smooth L-variety, let $\mathcal{Y} \to T$ be a model of Y with a good G-action, and assume that the structure map $\varphi : \mathcal{Y} \to T$ is G-equivariant for this action and the Galois action on T. Then there exists a projective Néron smoothening $f : \mathcal{Y}' \to \mathcal{Y}$, and a G-action on $\mathcal{Y}'$ such that f is G-equivariant.*

Proof. By [BLR90, Chapter 3.1, Theorem 3] there exists a projective Néron smoothening $f : \mathcal{Y}' \to \mathcal{Y}$, which consists of a finite sequence of blowups with centers in the special fibers. We need to construct a G-action on $\mathcal{Y}'$ such that f is G-equivariant.
Note that if we blow up an integral scheme U with a good G-action in a closed G-invariant subscheme $V \subset U$, and denote by $u : U' \to U$ the blowup, then there is a G-action on U' such that u is G-equivariant. The reason for this is the following: The G-action on U is given by a morphism $g_U \in \mathrm{Aut}(U)$ with $g_U^r = id$, and $g_U(V) = V$. So by the universal property of blowup, see [Har77, Chapter II, Corollary 7.15], there exists a unique $g_{U'} \in \mathrm{Aut}(U')$ making the following diagram commutative:

$$\begin{array}{ccc} U' & \overset{g_{U'}}{\dashrightarrow} & U' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ U & \underset{g_U}{\longrightarrow} & U \end{array}$$

Note that $g_{U'}$ defines the required group action on U', and u is G-equivariant by construction.
Now consider f. Note that f is a sequence of blowups, i. e. we have

$$\mathcal{Y}' =: \mathcal{Y}_m \xrightarrow{f_{m-1}} \mathcal{Y}_{m-1} \xrightarrow{f_{m-2}} \cdots \xrightarrow{f_1} \mathcal{Y}_1 \xrightarrow{f_0} \mathcal{Y}_0 := \mathcal{Y}, \qquad f : \mathcal{Y}_m \to \mathcal{Y}_0$$

with f_i the blowup of some closed subscheme $V_i \subset \mathcal{Y}_i$. One checks in the proof of [BLR90, Chapter 3.4, Theorem 2] that all the V_i are obtained using the same construction. Hence it suffices to show that $V := V_0 \subset \mathcal{Y}$ is G-invariant. Then we obtain a G-action on $\mathcal{Y}_1$ such that f_0 is G-equivariant, hence $\varphi \circ f_0$ is G-equivariant, and we can conclude inductively on the length of the sequence of the blowups.
One can check in [BLR90, Chapter 3.4, Theorem 2] that V is constructed as follows: Let $E \subset \mathcal{Y}(\mathcal{O}_L)$ be the subset of all $\sigma \in \mathcal{Y}(\mathcal{O}_L)$ not factoring through $\mathrm{Sm}(\mathcal{Y}/T)$, and let $\mathcal{Y}_k := \mathcal{Y} \times_S \mathrm{Spec}(k)$, $s : \mathcal{Y}(\mathcal{O}_L) \to \mathcal{Y}_k(k)$ be the specializing map. Set $F^1 := E$. Let V^i be the Zariski closure of $s(F^i)$ in $\mathcal{Y}$, and let $U^i \subset V^i$ the largest open subset, such that U^i is smooth over k, and that $\Omega^1_{\mathcal{Y}/T}|_{V^i}$ is locally free over U^i. Set $E^i := \{a \in F^i \mid s(a) \in U^i\}$, and $F^{i+1} := F^i \backslash E^i$. Note that there is a $t \in \mathbb{N}$ such that $F^{t+1} = \emptyset$. Set $V = V^t$.
Let the G-action on $\mathcal{Y}$ be given by $g \in \mathrm{Aut}(\mathcal{Y})$. The action of G on $\mathcal{Y}$ induces a G-action on $\mathcal{Y}(\mathcal{O}_L)$, and, as φ is G-equivariant, and hence $g(\mathcal{Y}_k) \subset \mathcal{Y}_k$, a G-action on $\mathcal{Y}_k(k)$. Note that s is G-equivariant.
We now show by induction that F^i is G-invariant for all i.
Using Lemma 1.4, E is G-invariant, and therefore F^1 is G-invariant, too. Hence we may assume that F^i is G-invariant for some i.

Consider $Z_i := \cap_{h\in G} h(V^i)$. By construction, Z_i is closed in $\mathcal{Y}$, and G-invariant, and $Z_i \subset V^i$. As F^i is G-invariant by assumption, and s is G-equivariant, $s(F^i)$ is G-invariant, and hence $s(F^i) \subset h(V^i)$ for all $h \in G$, so $s(F^i) \subset Z_i$. So by definition of the Zariski closure, $V^i = Z_i$, and therefore V^i is G-invariant.
Let $\mathrm{Sm}(V^i)$ be the smooth locus of V^i over k. Note that $U^i = \mathrm{Sm}(V^i) \cap W^i$, with $W^i \subset V^i$ the largest open subset over which $\Omega^1_{\mathcal{Y}/T}|_{V^i}$ is locally free. So in order to show that U^i is G-invariant, it suffices to show that the same holds for $\mathrm{Sm}(V^i)$ and W^i.
By Lemma 1.4, $\mathrm{Sm}(V^i)$ is G-invariant. Hence it remains to show that W^i is G-invariant. Consider the following commutative diagram:

$$\begin{array}{ccccccc} g^*(\varphi^*(\Omega^1_T)) & \longrightarrow & g^*(\Omega^1_{\mathcal{Y}}) & \longrightarrow & g^*(\Omega^1_{\mathcal{Y}/T}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \varphi^*(\Omega^1_T) & \longrightarrow & \Omega^1_{\mathcal{Y}} & \longrightarrow & \Omega^1_{\mathcal{Y}/T} & \longrightarrow & 0 \end{array} \tag{1.2}$$

The rows of this diagram are exact by [Har77, Chapter II, Proposition 8.11], and the fact that g is flat. The fact that g is an automorphism of $\mathcal{Y}$ implies that the map $g^*(\Omega^1_{\mathcal{Y}}) \to \Omega^1_{\mathcal{Y}}$ is an isomorphism. Let the action on T be given by $g_T \in \mathrm{Aut}_S(T)$. By assumption, φ is G-equivariant, i. e. $g \circ \varphi = \varphi \circ g_T$, and hence we obtain that $g^*(\varphi^*(\Omega^1_T)) = \varphi^*(g_T^*(\Omega^1_T))$. As g_T is an automorphism of T, $g^*(\varphi^*(\Omega^1_T)) \to \varphi^*(\Omega^1_T)$ is an isomorphism, too. So considering diagram (1.2), we get that $g^*(\Omega^1_{\mathcal{Y}/T}) \to \Omega^1_{\mathcal{Y}/T}$ is an isomorphism, and therefore, as V^i is G-invariant, $g^*(\Omega^1_{\mathcal{Y}/T})|_{V^i} \to \Omega^1_{\mathcal{Y}/T}|_{V^i}$ is an isomorphism, too. Altogether we obtain:

$$\Omega^1_{\mathcal{Y}/T}|_{V^i \cap W^i} \cong g^*(\Omega^1_{\mathcal{Y}/T})|_{V^i \cap W^i} = g^*(\Omega^1_{\mathcal{Y}/T}|_{V^i \cap g^{-1}(W^i)})$$

As the first is locally free by definition of W^i, $g^*(\Omega^1_{\mathcal{Y}/T}|_{V^i \cap g^{-1}(W^i)})$ is locally free, too. As g is an automorphism of $\mathcal{Y}$, $\Omega^1_{\mathcal{Y}/T}|_{V^i \cap g^{-1}(W^i)}$ is locally free. Hence by definition of W^i, $g^{-1}(W^i) \subset W^i$, i. e. W^i is G-invariant.
Choose any $a \in E^i$. So $a \in F^i$, which implies that $g(a) \in F^i$, because F^i is G-invariant. Additionally, $s(a) \in U^i$, hence $s(g(a)) \in U^i$, because U^i is G-invariant and s is G-equivariant. Altogether $g(a) \in E^i$, i. e. E^i is G-invariant. Hence F^{i+1} is G-invariant, as both F^i and E^i are G-invariant. So it follows by induction that for all i, F^i is G-invariant, in particular F^t is G-invariant. By the same argument as in the induction, we can show that $V^t = V$ is G-invariant, and this is what we wanted to show. □

In [Nic12] the following similar theorem in the context of formal schemes is proven:

Theorem. *Any generically smooth, flat, separated formal $\mathcal{O}_L$-scheme X_∞, topologically of finite type over $\mathcal{O}_L$, endowed with a good G-action compatible with the G-action on $\mathcal{O}_L$, admits a G-equivariant Néron smoothening.*

Now, for a given projective and smooth K-variety X, and a tame Galois extension L/K, we investigate Proposition 1.29 to construct a quasi-projective weak Néron model of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ with an action of the Galois group.

Theorem 1.30. *Let X be a smooth projective K-variety. Then there is a quasi-projective weak Néron model $\varphi : \mathcal{Y} \to T$ of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ with the following properties: There is a G-action on $\mathcal{Y}$ such that φ is G-equivariant for this action and the Galois action on T. Moreover, X is isomorphic to $\mathcal{Y}/G \times_S \mathrm{Spec}(K)$ over K.*

Proof. As X is projective, X is a closed subscheme of $\mathbb{P}^N_K$ for some $N \in \mathbb{N}$. Consider $X \subset \mathbb{P}^N_K \subset \mathbb{P}^N_{\mathcal{O}_K}$. Let $\mathcal{X}$ be the Zariski closure of X in $\mathbb{P}^N_{\mathcal{O}_K}$ (with reduced scheme structure). Let Φ be the composition of the inclusion of $\mathcal{X}$ into $\mathbb{P}^N_{\mathcal{O}_K}$ and the projection to S. By construction Φ is projective, and as X is closed in $\mathbb{P}^N_K$, the following diagram is cartesian:

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \Phi \\ \mathrm{Spec}(K) & \hookrightarrow & S \end{array}$$

Hence $\Phi : \mathcal{X} \to S$ is a projective model of X. Set $\mathcal{X}_T := \mathcal{X} \times_S T$, and look at the defining cartesian diagram:

$$\begin{array}{ccc} \mathcal{X}_T & \longrightarrow & \mathcal{X} \\ \Phi_T \downarrow & \square & \downarrow \Phi \\ T & \longrightarrow & S \end{array}$$

Note that Φ_T is projective, because this property is stable under base change. By Lemma 1.6, G acts on $\mathcal{X}_T$ such that Φ_T is G-equivariant for this action and the Galois action on T, and $\mathcal{X}_T/G \cong \mathcal{X}$ as S-schemes. Moreover, by Lemma 1.15, $\Phi_T : \mathcal{X}_T \to T$ is a model of X_L. By Proposition 1.29 there exists a projective Néron smoothening $f : \mathcal{Y}' \to \mathcal{X}_T$, such that G acts on $\mathcal{Y}'$, and f is G-equivariant. Let $\mathcal{Y} \subset \mathcal{Y}'$ be the smooth locus of $\Phi_T \circ f$. Set $\varphi := \Phi_T \circ f \mid_{\mathcal{Y}}$. Note that φ is quasi-projective, because $\mathcal{Y} \subset \mathcal{Y}'$ is open, and both f and Φ_T are projective. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & \mathcal{Y}' \\ & \searrow & \downarrow f \\ & \varphi & \mathcal{X}_T \\ & & \downarrow \Phi_T \\ & & T \end{array}$$

Note that X_L is smooth, because this property is stable under base change. As f is a Néron smoothening, $\mathcal{Y}' \times_T \operatorname{Spec}(L) = \mathcal{X}_T \times_T \operatorname{Spec}(L) = X_L$, hence $\mathcal{Y}' \times_T \operatorname{Spec}(L)$ is in particular smooth over $\operatorname{Spec}(L) \hookrightarrow T$, and hence $\mathcal{Y} \times_T \operatorname{Spec}(L) = \mathcal{Y}' \times_T \operatorname{Spec}(L) = X_L$. As $\mathcal{X}_T$ is integral, $\mathcal{Y}'$ and $\mathcal{Y}$ are integral, too. Hence $\varphi : \mathcal{Y} \to T$ is a quasi-projective model of X_L. As Φ_T and f are projective, by the valuative criterion of properness the natural map $\mathcal{Y}'(\mathcal{O}_L) \to X_L(L)$ is a bijection. As f is a Néron smoothing, we obtain that $\mathcal{Y}'(\mathcal{O}_L) = \mathcal{Y}(\mathcal{O}_L)$. Moreover, φ is smooth by construction. So $\varphi : \mathcal{Y} \to T$ is a quasi-projective weak Néron model of X_L.
By Lemma 1.4 G acts on $\mathcal{Y}$ such that $\mathcal{Y} \hookrightarrow \mathcal{Y}'$ is G-equivariant for this G-action and the G-action on $\mathcal{Y}'$. So φ is G-equivariant for the G-action on $\mathcal{Y}$ and that on T. Using [Gro63, Exposé V, Proposition 1.9] twice, we obtain

$$\begin{aligned}
\mathcal{Y}/G \times_S \operatorname{Spec}(K) &= (\mathcal{Y} \times_T \operatorname{Spec}(L))/G \\
&= (\mathcal{X}_T \times_T \operatorname{Spec}(L))/G \\
&= \mathcal{X}_T/G \times_S \operatorname{Spec}(K) \\
&= \mathcal{X} \times_S \operatorname{Spec}(K) = X
\end{aligned}$$

□

In [EN11, Proposition 4.5] the following similar statement is proven:

Proposition. *Let G be any finite group, X a smooth and proper K-variety, endowed with a good G-action. Then there is a weak Néron model $\mathcal{X} \to S$ of X endowed with a good G-action, such that $X \hookrightarrow \mathcal{X}$ is G-equivariant.*

Note that the induced action on S is trivial in this case.

Chapter 2

Cyclic Group Actions on Regular Complete Local Rings

In this chapter we examine tame actions of a cyclic group G on regular complete local rings, not necessarily of equal characteristic. The main result is Lemma 2.2, saying that if a cyclic group acts on a regular complete local ring such that the residual action is trivial, then there is a regular system of parameters on which G acts by multiplying with some roots of unity. This result should be known to the experts; a similar statement can be found in [Ser68]. But we could not find a reference covering the topic in full generality.

In Section 2.2 we apply the result from Section 2.1 to the relative case of a G-equivariant morphism $R \to A$ of local rings coming from some geometric situation. We will need these results to construct G-invariant sections of models through fixed points, see Proposition 3.3 and Proposition 3.8, and to compute the special fiber of the the weak Néron model constructed in Chapter 4, see Lemma 4.15. In particular in the latter, we need the notations defined in Section 2.2.

Throughout the chapter, let $G := \mathbb{Z}/r\mathbb{Z}$.

2.1 Absolute Case

Lemma 2.1. *Let A be a complete local ring, k its residue field containing a primitive r-th root of unity μ, r prime to* $\operatorname{char}(k)$. *Then μ lifts to an r-th root of unity in A.*

Proof. Complete local rings are Henselian, i. e. Hensel's Lemma holds for A, see [Eis95, Theorem 7.3]. Consider the polynomial $p(x) := x^r - 1 \in A[x]$. Note that $p(\mu) = 0 \in k$, and $p'(\mu) = r\mu^{r-1} \neq 0 \in k$, because $r \neq 0 \in k$. So

Hensel's Lemma gives us a $\tilde{\mu} \in A$, such that $\tilde{\mu} = \mu \mod \mathfrak{m}$, and $p(\tilde{\mu}) = 0$, i. e. $\tilde{\mu}$ is a lift of μ, and $\tilde{\mu}^r = 1$. □

From now on, let A be a regular complete local ring of dimension n with maximal ideal $\mathfrak{m}$, such that its residue field k is a field of $\mathrm{char}(k) \nmid r$ containing all r-th roots of unity, and let $\alpha \in \mathrm{Aut}(A)$ with $\alpha^r = \mathrm{id}$, such that the residual map on k is trivial. Note that α defines an action of G on A.

Lemma 2.2. *There exists a regular system of parameters $x_1, \dots, x_n \in \mathfrak{m}$, such that*

$$\alpha(x_i) = \mu^{\ell_i} x_i$$

with $\mu \in A$ a primitive r-th root of unity, and $\ell_i \in \{0, \dots, r-1\}$.

Proof. Fix a primitive r-th root of unity $\mu \in A$. Such μ exists by Lemma 2.1. Identify μ with its image in k under the residue map. As $\alpha \in \mathrm{Aut}(A)$, and A is a local ring, $\alpha(\mathfrak{m}) = \mathfrak{m}$. So for every $l \in \mathbb{N}$ we get a morphism $\bar{\alpha} : A/\mathfrak{m}^l \to A/\mathfrak{m}^l$ such that $\bar{\alpha}(\mathfrak{m}/\mathfrak{m}^l) = \mathfrak{m}/\mathfrak{m}^l$, and $\bar{\alpha}^r = \mathrm{id}$.
First, we prove by induction on l that there exist $\ell_1, \dots, \ell_n \in \mathbb{N}$, such that for all $l \geq 2$ there exist $x_{1,l}, \dots, x_{n,l} \in \mathfrak{m}/\mathfrak{m}^l$ with $x_{i,l} = x_{i,l+1} \mod \mathfrak{m}^l$, $\bar{\alpha}(x_{i,l}) = \mu^{\ell_i} x_{i,l}$ (μ identified with its image in $A/\mathfrak{m}^l$), and $\{x_{1,2}, \dots, x_{n,2}\}$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$.
Start with $l = 2$. As A is a regular local ring of dimension n with residue field k, $\mathfrak{m}/\mathfrak{m}^2$ is an n-dimensional k-vector space. As the morphism on $A/\mathfrak{m} = k$ induced by α is trivial, $\bar{\alpha} : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is a k-linear map. For some algebraic closure $\bar{k}$ of k, there exists a basis of $\mathfrak{m}/\mathfrak{m}^2 \otimes_k \bar{k}$, such that the matrix corresponding to $\bar{\alpha}$ has Jordan normal form. As $\bar{\alpha}^r = \mathrm{id}$, all eigenvalues are r-th roots of unity, i. e. powers of μ, and as $r \neq 0 \in k \subset \bar{k}$, the matrix is already diagonal. But all r-th roots of unity are assumed to be in k, so there is a basis $\{x_{1,2}, \dots, x_{n,2}\}$ of $\mathfrak{m}/\mathfrak{m}^2$, such that $\bar{\alpha}(x_{i,2}) = \mu^{\ell_i} x_{i,2}$ for some $\ell_i \in \{0, \dots, r-1\}$.
Now assume that for all $j \in \{2, \dots, l-1\}$, there exist $x_{1,j}, \dots, x_{n,j} \in \mathfrak{m}/\mathfrak{m}^j$ with the property that $x_{i,j-1} = x_{i,j} \mod \mathfrak{m}^{j-1}$, and $\bar{\alpha}(x_{i,j}) = \mu^{\ell_i} x_{i,j}$, and that the $x_{i,2}$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Choose any $y_1, \dots, y_n \in \mathfrak{m}/\mathfrak{m}^l$ such that $y_i \mod \mathfrak{m}^{l-1} = x_{i,l-1}$. Look at $\bar{\alpha} : \mathfrak{m}/\mathfrak{m}^l \to \mathfrak{m}/\mathfrak{m}^l$. By assumption

$$\bar{\alpha}(y_i) - \mu^{\ell_i} y_i \in \mathfrak{m}^{l-1}/\mathfrak{m}^l$$

Note that $\mathfrak{m}^{l-1}/\mathfrak{m}^l$ is a k-vector space generated by $y_1^{s_1} \dots y_n^{s_n}$, $s_1 + \ldots + s_n = l-1$, and the $y_1^{s_1} \dots y_n^{s_n}$ do not depend on the choice of the y_i. So we get

$$\bar{\alpha}(y_i) - \mu^{\ell_i} y_i = \sum_{s_1 + \cdots + s_n = l-1} a_{i,s_1 \dots s_n} y_1^{s_1} \dots y_n^{s_n} \in \mathfrak{m}^{l-1}/\mathfrak{m}^l$$

for some $a_{i,s_1 \dots s_n} \in k$. Define:

$$x_{i,l} := y_i + \sum_{s_1 + \cdots + s_n = l-1} \tilde{a}_{i,s_1 \dots s_n} y_1^{s_1} \dots y_n^{s_n} \in \mathfrak{m}/\mathfrak{m}^l$$

with

$$\tilde{a}_{i,s_1\dots s_n} := \begin{cases} \frac{a_{i,s_1\dots s_n}}{\mu^{\ell_i}-\mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}} & \mu^{\ell_i} \neq \mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n} \\ 0 & \mu^{\ell_i} = \mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n} \end{cases} \in k \tag{2.1}$$

Note that if $0 \neq \mu^{\ell_i}-\mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}$, then $\mu^{\ell_i}-\mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}$ is invertible in k, and hence the $\tilde{a}_{i,s_1\dots s_n}$ are well defined. Moreover, $x_{i,l} \mod \mathfrak{m}^{l-1} = x_{i,l-1}$, because $y_i - x_{i,l} \in \mathfrak{m}^{l-1}/\mathfrak{m}^l$. Note furthermore that

$$\bar{\alpha}^r(y_i) = \mu^{\ell_i r} y_i + \sum_{s_1+\dots+s_n=l-1} a_{i,s_1\dots s_n} \sum_{k=1}^{r} (\mu^{\ell_i})^{r-k}(\mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n})^{k-1} y_1^{s_1}\dots y_n^{s_n}$$

As $\bar{\alpha}^r = \mathrm{id}$, we have that $\bar{\alpha}^r(y_i) = y_i$. Moreover $\mu^{\ell_i r} = 1$. Comparing coefficients yields for all $s_1 + \dots + s_n = l-1$:

$$0 = a_{i,s_1\dots s_n} \sum_{k=1}^{r} (\mu^{\ell_i})^{r-k}(\mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n})^{k-1} \in k$$

In the case that $\mu^{\ell_i} = \mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}$ we get:

$$0 = a_{i,s_1\dots s_n} \sum_{k=1}^{r} (\mu^{\ell_i})^{r-k}(\mu^{\ell_i})^{k-1} = r a_{i,s_1\dots s_n} (\mu^{\ell_i})^{r-1}$$

As $r \neq 0$ and $(\mu^{l_i})^{r-1} \neq 0$, we obtain that $a_{i,s_1\dots s_n} = 0$ in this case. Here we use again our assumption on the characteristic of k. We obtain:

$$\begin{aligned} &\mu^{\ell_i}\tilde{a}_{i,s_1\dots s_n} - \mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}\tilde{a}_{i,s_1\dots s_n} \\ &= \begin{cases} \frac{\mu^{\ell_i}-\mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}}{\mu^{\ell_i}-\mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}} a_{i,s_1\dots s_n} & \mu^{\ell_i} \neq \mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n} \\ \mu^{l_i}0 - \mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}0 & \mu^{\ell_i} = \mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n} \end{cases} \\ &= a_{i,s_1\dots s_n} \end{aligned}$$

Hence in $\mathfrak{m}^{l-1}/\mathfrak{m}^l$ we get:

$$\begin{aligned} &\bar{\alpha}(x_{i,l}) - \mu^{\ell_i}x_{i,l} \\ &= \alpha(x_i) - \mu^{\ell_i}x_i + \alpha(\sum_{s_1+\dots+s_n=l-1} \tilde{a}_{i,s_1\dots s_n} y_1^{s_1}\dots y_n^{s_n}) - \mu^{l_i}(\sum_{s_1+\dots+s_n=l-1} \tilde{a}_{i,s_1\dots s_n} y_1^{s_1}\dots y_n^{s_n}) \\ &= \sum_{s_1+\dots+s_n=l-1} a_{i,s_1\dots s_n} y_1^{s_1}\dots y_n^{s_n} - \sum_{s_1+\dots+s_n=l-1} (\mu^{\ell_i}\tilde{a}_{i,s_1\dots s_n} - \mu^{\ell_1 s_1}\dots\mu^{\ell_n s_n}\tilde{a}_{i,s_1\dots s_n}) y_1^{s_1}\dots y_n^{s_n} \\ &= 0 \end{aligned}$$

So the $x_{i,l}$ have the required properties, and the proof of the claim follows by induction.
As A is a complete local ring,

$$A = \hat{A} \cong \{(a_1, a_2, \dots) \in \prod_j A/\mathfrak{m}^j \mid a_{j+1} = a_j \mod \mathfrak{m}^j\}$$

and α maps $(a_1, a_2, \dots)$ to $(\bar{\alpha}(a_1), \bar{\alpha}(a_2), \dots)$. Set $x_i := (0, x_{i,2}, x_{i,3}, \dots)$ for $i \in \{1, \dots, n\}$. One observes that $x_i \in A$, because $x_{i,l} = x_{i,l+1} \mod \mathfrak{m}^l$, and $x_{i,j} \in A/\mathfrak{m}^j$. Moreover,

$$\alpha(x_i) = (\bar{\alpha}(0), \bar{\alpha}(x_{i,2}), \bar{\alpha}(x_{i,3}), \dots) = (0, \mu^{\ell_i} x_{i,2}, \mu^{\ell_i} x_{i,3}, \dots) = \mu^{\ell_i} x_i$$

Note that the $x_i \mod \mathfrak{m}^2 = x_{i,2}$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$, i.e. the x_i are a regular system of parameters. □

If we do not assume that r is prime to $\mathrm{char}(k)$, Lemma 2.2 is wrong. To see this, look at the following example:

Example 2.3. Let k be an algebraically closed field, and assume furthermore that $\mathrm{char}(k) = 2$. $A := k[\![x, y]\!]$ is a complete local ring with maximal ideal $\mathfrak{m} := (x, y) \subset A$. Let $\alpha \in \mathrm{Aut}(A)$ given by $\alpha(P(x, y)) = P(x, x + y)$ for all $P(x, y) \in A$. We have that $\alpha^2(P(x, y)) = P(x, 2x + y) = P(x, y)$, because $\mathrm{char}(k) = 2$, hence $\alpha^2 = \mathrm{id}$. Note that $\bar{\alpha} : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is not diagonalizable.

Lemma 2.4. *Let $z_1, \dots, z_s \in \mathfrak{m} \subset A$, such that $\alpha(z_i) = \mu^{\ell_i} z_i$ for some $\ell_i \in \{0, \dots, r-1\}$, $\mu \in A$ a primitive r-th root of unity, and assume that the $\bar{z}_1, \dots, \bar{z}_s \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent.*
Then there exist $x_{s+1}, \dots, x_n \in \mathfrak{m}$ with $\alpha(x_i) = \mu^{\ell_i} x_i$, $\ell_i \in \{0, \dots, r-1\}$, and $z_1, \dots, z_s, x_{s+1}, \dots, x_n$ is a regular system of parameters of A.

Proof. To prove this lemma, we just need to modify the proof of Lemma 2.2 such that the $x_1, \dots, x_s$ coincide with $y_1, \dots, y_s$.
As $\bar{\alpha} : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is diagonalizable, $\mathfrak{m}/\mathfrak{m}^2$ decomposes into eigenspaces E_j. By assumption, for all i there exists a j such that $\bar{z}_i \in E_j$. Note that for all j one can choose a basis B_j of E_j such that for all i, $\bar{z}_i \in \cup B_j$. This uses the fact that the $\bar{z}_i$ are linearly independent. Set $x_{i,2} = \bar{z}_i$ for $i \leq s$, $\{x_{s+1,2}, \dots x_{n,2}\} := \cup B_j \setminus \{\bar{z}_1, \dots, \bar{z}_s\}$. Now, for any l one can choose the y_i to be the images of z_i in $\mathfrak{m}/\mathfrak{m}^\ell$ for $i \leq s$. By assumption, $\bar{\alpha}(y_i) - \mu^{\ell_i} y_i = 0$, i.e. $x_{i,l} = y_i \in \mathfrak{m}/\mathfrak{m}^\ell$. Altogether we get $x_i = z_i$ for $i \leq s$. □

Lemma 2.5. *For all $s \in k$ there exists a lift $\tilde{s} \in A$ with $\alpha(\tilde{s}) = \tilde{s}$.*

Proof. The proof of this Lemma works similar to the proof of Lemma 2.2. Note that it suffices to show that for all $s \in k$ there exist lifts $s_l \in A/\mathfrak{m}^l$ of s for all $l \geq 1$, such that $\bar{\alpha}(s_l) = s_l$.
Set $s_1 = s \in A/\mathfrak{m} = k$. By assumption on α, $\bar{\alpha}(s) = s$. So we may assume that the claim holds for $l - 1$, i.e. there is a lift $s_{l-1} \in A/\mathfrak{m}^{l-1}$ of s, and $\bar{\alpha}(s_{l-1}) = s_{l-1}$. Let $y \in A/\mathfrak{m}^l$ be any lift of s_{l-1}. By assumption on s_{l-1}, $\bar{\alpha}(y) - y \in \mathfrak{m}^{l-1}/\mathfrak{m}^l$. Let $x_1, \dots, x_n \in A$ be a system of parameters with $\bar{\alpha}(x_i) = \mu^{\ell_i} x_i$ for some $\mu \in A$, $\ell_i \in \mathbb{N}$, which we obtain from Lemma 2.2.

Hence $\mathfrak{m}^{l-1}/\mathfrak{m}^l$ is a k vector space generated by monomial in the x_i of degree l, and

$$\bar{\alpha}(y) - y = \sum_{s_1+\dots+s_n=l} a_{s_1\dots s_n} x_1^{s_1} \dots x_n^{s_n}$$

with $a_{s_1\dots s_n} \in k$. Set

$$s_l := y + \sum_{s_1+\dots+s_n=l} \tilde{a}_{s_1\dots s_n} x_1^{s_1} \dots x_n^{s_n}$$

with $\tilde{a}_{s_1\dots s_n} \in k$ defined analog to equation (2.1). With the same calculations as in the proof of Lemma 2.2, one can show that $s_l \in A/\mathfrak{m}^l$ is a well-defined lift of s, and that $\bar{\alpha}(s_l) = s_l$. The claim follows by induction. □

Definition 2.1.

$$A^G := \{a \in A \mid \alpha(a) = a\}$$

is the *ring of invariants.*

Lemma 2.6. *$A^G \subset A$ is a subring, which is local and has the same residue field as A.*

Proof. Take any $a_1, a_2 \in A^G$. As α is a ring homomorphism, we have that $\alpha(a_1a_2) = \alpha(a_1)\alpha(a_2) = a_1a_2$, i.e. $a_1a_2 \in A^G$. Moreover $\alpha(0) = 0$ and $\alpha(1) = 1$, i.e. $0, 1 \in A^G$. So A^G is a subring of A. Note that $\mathfrak{m}^G := \mathfrak{m} \cap A^G$ is an ideal in A^G. Take any $a \in A^G \backslash \mathfrak{m}^G$. Then $a \in A \backslash \mathfrak{m}$, so there is a unique $b \in A$ such that $ab = 1$. As $a \in A^G$, we get that

$$a\alpha(b) = \alpha(a)\alpha(b) = \alpha(ab) = \alpha(1) = 1$$

So $\alpha(b) = b$, i.e. $b \in A^G$, and a is invertible in A^G. So $\mathfrak{m}^G \subset A^G$ is maximal, and every proper ideal $\mathfrak{a} \subset A^G$ is contained in $\mathfrak{m}^G$. Therefore A^G is a local ring. We still need to show that A and A^G have the same residue field. As $\mathfrak{m}^G \subset \mathfrak{m}$, we get the following commutative diagram:

$$\begin{array}{ccc} A^G & \hookrightarrow & A \\ {\scriptstyle r^G}\downarrow & & \downarrow{\scriptstyle r} \\ k' & \hookrightarrow & k \end{array}$$

Here k' is the residue field of A^G, and r^G, r are the residue maps. By Lemma 2.5 there exists a lift $\tilde{s} \in A$ of s for all $s \in k$ with $\alpha(\tilde{s}) = \tilde{s}$, i.e. $\tilde{s} \in A^G$. Hence $k' = k$. □

Lemma 2.7. *G acts on $k \otimes_{A^G} A$ given by $\mathrm{id} \otimes \alpha \in \mathrm{Aut}(k \otimes_{A^G} A)$, such that the canonical maps $\rho_1 : A \to k \otimes_{A^G} A$ and $\rho_2 : k \to k \otimes_{A^G} A$ are G-equivariant for this G-action, and the given G-action on A and the trivial*

G-action on k, respectively. Moreover, $k \otimes_{A^G} A \cong k[x_1, \dots, x_m]/\mathfrak{I}$, $m \leq n$, and

$$(\mathrm{id} \otimes \alpha)(p(x_1, \dots, x_m)) = p(\mu^{\ell_0} x_1, \dots, \mu^{\ell_m} x_m)$$

for some $\ell_i \in \{1, \dots, r-1\}$, $p(x_1, \dots, x_m) \in k \otimes_{A^G} A$, $\mu \in k$ a primitive r-th root of unity, and $\mathfrak{I} \subset k[x_1, \dots, x_m]$ is the ideal generated by monomials of the form $x_1^{s_1} \dots x_m^{s_m}$ with $s_1\ell_1 + \dots + s_m\ell_m = sr$, $s \in \mathbb{N}$.

Proof. Set $\tilde{A} := k \otimes_{A^G} A$. Let $i^G : A^G \hookrightarrow A$ be the inclusion, and $r^G : A^G \to k$ be the residue map. $\tilde{A}$ is defined by the following cocartesian diagram:

$$\begin{array}{ccc} \tilde{A} & \xleftarrow{\rho_1} & A \\ {\scriptstyle \rho_2}\uparrow & & \uparrow{\scriptstyle i^G} \\ k & \xleftarrow[r^G]{} & A^G \end{array}$$

As $\rho_2 \circ r^G = \rho_1 \circ i^G = \rho_1 \circ \alpha \circ i^G$, we get a unique $(\mathrm{id} \otimes \alpha) : \tilde{A} \to \tilde{A}$ with $(\mathrm{id} \otimes \alpha) \circ \rho_2 = \rho_2$ and $(\mathrm{id} \otimes \alpha) \circ \rho_1 = \rho_1 \circ \alpha$. Moreover, $(\mathrm{id} \otimes \alpha)^r \circ \rho_2 = \rho_2$ and $(\mathrm{id} \otimes \alpha)^r \circ \rho_1 = \rho_1 \circ \alpha^r = \rho_1$, and as $\mathrm{id}_{\tilde{A}}$ is the unique morphism with these properties, $(\mathrm{id} \otimes \alpha)^r = \mathrm{id}_{\tilde{A}}$. So the required action of G on $\tilde{A}$ is given by $(\mathrm{id} \otimes \alpha)$. Note that $(\mathrm{id} \otimes \alpha)(q \otimes a) = q \otimes \alpha(a)$ for $q \in k$, $a \in A$.
Now consider

$$\rho_1 : A \to \tilde{A};\ a \mapsto 1 \otimes a$$

Note that ρ_1 is surjective, because for any $q \in k$ and $a \in A$, using Lemma 2.5, we can choose a lift $q' \in A^G$ of q. Hence we get $\rho_1(q'a) = 1 \otimes q'a = q \otimes a$. Now we want to compute the kernel of ρ_1. Note that $0 = \rho_1(a) = 1 \otimes a$ for some $a \in A$ if and only if we can write $a = a_1 a_2$ for some $a_1 \in A^G$, $a_2 \in A$, and $r^G(a_1) = 0$, i. e. $a_1 \in \mathfrak{m}^G := \mathfrak{m} \cap A^G$. This implies that

$$\ker(\rho_1) = A\,\mathfrak{m}^G$$

By Lemma 2.2 there exists a system of parameters $y_1, \dots, y_n \in A$ with $\alpha(y_i) = \tilde{\mu}^{\ell_i} y_i$, $\ell_i \in \{0, \dots, r\}$, $\tilde{\mu} \in A$ a primitive r-th root of unity, which is a lift of $\mu \in k$. So $A\,\mathfrak{m}^G \subset A$ is the ideal generated by monomials of the form $y_1^{s_1} \dots y_n^{s_n}$ with $s_1\ell_1 + \dots + s_n\ell_n = sr$, $s \in \mathbb{N}$. Note that $\mathfrak{m}^{nr} \subset A\,\mathfrak{m}^G$: As $\mathfrak{m}^{nr}$ is generated by monomials of degree nr in the y_i, all generators are divisible by y_i^r for at least one i. Note that for all i, $y_i^r \in \mathfrak{m}^G$. Hence $\mathfrak{m}^{nr} \subset A\,\mathfrak{m}^G$.
Set $N := nr$. So $\tilde{A} \cong k \otimes_{A^G} (A/\mathfrak{m}^N)$. We show by induction that this is generated as a k-algebra by the images of the y_i. The induction assumption is clear, because in this case $k \otimes_{A^G} (A/\mathfrak{m}^1) \cong k$. Assume that $k \otimes_{A^G} (A/\mathfrak{m}^l)$ is generated as a k-algebra by the images of the y_i. Let $\tilde{A}_{l+1}$ be the subalgebra of $k \otimes_{A^G} (A/\mathfrak{m}^{l+1})$ generated by the images of the y_i. Take any $1 \otimes a$ in $k \otimes_{A^G} (A/\mathfrak{m}^{l+1})$. By induction assumption there is an $\tilde{a} \in A/\mathfrak{m}^{l+1}$, such that $a - \tilde{a} \in \mathfrak{m}^l/\mathfrak{m}^{l+1}$, and $1 \otimes \tilde{a} \in \tilde{A}_{l+1}$. Note that $\mathfrak{m}^l\,/\,\mathfrak{m}^{l+1}$ is a k-vector

space generated by monomials of degree l in the y_i. So $1 \otimes (\tilde{a} - a) \in \tilde{A}_{l+1}$, and therefore the same holds for $1 \otimes a = 1 \otimes \tilde{a} + 1 \otimes (a - \tilde{a})$. Altogether, $\tilde{A}$ is generated as a k-algebra by the images of the y_i.
Let $x_1, \dots, x_m$ be the images of those y_i with $\ell_i \neq 0$. Note that, if $\ell_i = 0$, $y_i \in \mathfrak{m}^G \subset ker(\rho_1)$, i. e. $\rho_1(y_i) = 0$. Hence the x_i generate $\tilde{A}$ as a k-algebra. Renumbering the y_i, we may assume that $\rho_1(y_i) = x_i$. We have

$$(\mathrm{id} \otimes \alpha)(x_i) = (\mathrm{id} \otimes \alpha)(1 \otimes y_i) = 1 \otimes \alpha(y_i) = 1 \otimes \tilde{\mu}^{\ell_j} y_i = \mu^{\ell_i} x_i$$

Moreover, using $\ker(\rho_1) = A\,\mathfrak{m}^G$, we obtain that

$$\tilde{A} \cong k[x_1, \dots, x_m]/\mathfrak{I}$$

with $\mathfrak{I}$ generated by $x_1^{s_1} \dots x_m^{s_m}$ with $s_1\ell_1 + \dots + s_m\ell_m = sr$, $s \in \mathbb{N}$. As $(\mathrm{id} \otimes \alpha)$ is a k-morphism, $(\mathrm{id} \otimes \alpha)(p(x_1, \dots, x_m)) = p(\mu^{\ell_1} x_1, \dots, \mu^{\ell_m} x_m)$ with $\ell_i \in \{1, \dots, r\}$ for $p(x_1, \dots, x_m) \in \tilde{A}$. □

Remark 2.8. Note that if A is of mixed characteristic, it is not a k-algebra, but $A \otimes_{A^G} k$ is. As we tensor over A^G, we somehow keep the information of the G-action on A.

2.2 Relative Case

In this section we examine the relative case coming from a special geometric case. More precisely, we have the following situation:
Let R be a complete discrete valuation ring with residue field k. Assume that k is algebraically closed, and that r is prime to $\mathrm{char}(k)$. Let G act non-trivially on R given by $\alpha_R \in \mathrm{Aut}(R)$ with $\alpha_R^r = \mathrm{id}$, such that the residual action on k is trivial. Let $\mathcal{Y}$ be an R-scheme of finite type endowed with a good G-action, such that the structure map $\varphi : \mathcal{Y} \to T := \mathrm{Spec}(R)$ is G-equivariant. Let $\mathrm{Sm}(\mathcal{Y}/T) \subset \mathcal{Y}$ be the smooth locus of φ, and let $y \in \mathrm{Sm}(\mathcal{Y}/T)^G \subset \mathcal{Y}$ be any point.

Remark 2.9. Let $\mathfrak{m} = (t) \subset R$ be the maximal ideal. By Lemma 2.2 we may assume, that $\alpha_R(t) = \mu^\ell t$, with $\mu \in R$ a primitive r-th root of unity, and $\ell \in \{1, \dots, r-1\}$. By Lemma 2.7 we have the following cocartesian diagram:

$$\begin{array}{ccc} k \otimes_{R^G} R & \xleftarrow{\rho_R} & R \\ \uparrow{\scriptstyle \rho_k} & & \uparrow{\scriptstyle i_R^G} \\ k & \xleftarrow[r_R^G]{} & R^G \end{array}$$

with $r_R : R^G \to k$ the residue map, and $i_R^G : R^G \hookrightarrow R$ the inclusion. By Lemma 2.7, $k \otimes_{R^G} R \cong k[t]/(t^r)$, and $(\mathrm{id} \otimes \alpha_R)(p(t)) = p(\mu^\ell t)$ for all $p(t) \in k[t]/(t^r)$.

Remark 2.10. Let n be the relative dimension of $\varphi|_{\mathrm{Sm}(\mathcal{Y}/T)}$. Then $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is a regular complete local ring of dimension smaller or equal n with residue field $\kappa(y)$. Let $r_y : \hat{\mathcal{O}}_{\mathcal{Y},y} \to \kappa(y)$ be the residue map.
By Lemma 1.3, $\mathrm{Sm}(\mathcal{Y}/T)^G$ is a $T^G = \mathrm{Spec}(k)$-scheme, so $\kappa(y)$ contains k. Denote by $i_k : k \hookrightarrow \kappa(y)$ the inclusion. As k is algebraically closed, $\kappa(y)$ contains all r-th roots of unity.
Let $j : \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}) \to \mathcal{Y}$ be the natural map, hence $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is an R-module via $\beta_y := (\varphi \circ j)^\# : R \to \hat{\mathcal{O}}_{\mathcal{Y},y}$. As $y \in \mathrm{Sm}(\mathcal{Y}/T)$, β_y is an injective. Moreover, there is the following commutative diagram:

$$\begin{array}{ccc} R & \overset{\beta_y}{\hookrightarrow} & \hat{\mathcal{O}}_{\mathcal{Y},y} \\ {\scriptstyle r_R}\downarrow & & \downarrow{\scriptstyle r_y} \\ k & \underset{i_k}{\hookrightarrow} & \kappa(y) \end{array} \tag{2.2}$$

As $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$, G acts on $\hat{\mathcal{O}}_{\mathcal{Y},y}$ given by some $\alpha_y \in \mathrm{Aut}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ with $\alpha_y^r = \mathrm{id}$, such that j is G-equivariant, and the residual action on $\kappa(y)$ is trivial. As φ and j are G-equivariant, the same holds for β_y.

Lemma 2.11. *Let $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$ be a closed point, i. e. $\kappa(y) = k$. Then there exist $x_1, \dots, x_n \in \hat{\mathcal{O}}_{\mathcal{Y},y}$ such that $\hat{\mathcal{O}}_{\mathcal{Y},y} \cong R[\![x_1, \dots, x_n]\!]$ as R-modules, and*

$$\alpha_y(x_i) = \mu^{\ell_i} x_i$$

for $\mu \in R \subset \hat{\mathcal{O}}_{\mathcal{Y},y}$ a primitive r-th root of unity, and some $\ell_i \in \mathbb{N}$.

Proof. As y lies in the smooth locus of φ, and the residue field of R is equal to the residue field of $\hat{\mathcal{O}}_{\mathcal{Y},y}$, hence [Gro67, Proposition 17.5.3] implies that $\hat{\mathcal{O}}_{\mathcal{Y},y} \cong R[\![\tilde{x}_1, \dots, \tilde{x}_n]\!]$ as R-module for some $\tilde{x}_1, \dots, \tilde{x}_n \in \hat{\mathcal{O}}_{\mathcal{Y},y}$. Note that $t, \tilde{x}_1, \dots, \tilde{x}_n$ form a regular system of parameters of $\hat{\mathcal{O}}_{\mathcal{Y},y}$.
As $\alpha_y(t) = \alpha_R(t) = \mu^\ell t$, by Lemma 2.4 we may choose a system of parameters $x_0, \dots, x_n$ with $\alpha_y(x_i) = \mu^{\ell_i} x_i$ for $\mu \in R$ a primitive r-th root of unity, and some $\ell_i \in \mathbb{N}$, such that $x_0 = t$. So $\hat{\mathcal{O}}_{\mathcal{Y},y} \cong R[\![\tilde{x}_1, \dots, \tilde{x}_n]\!] \cong R[\![x_1, \dots, x_n]\!]$ as R-modules, and $\alpha_y(x_i) = \mu^{\ell_i} x_i$. □

Remark 2.12. Let $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$ be any fixed point, $\kappa(y)$ the residue field of $\hat{\mathcal{O}}_{\mathcal{Y},y}$. Let $i_y^G : \hat{\mathcal{O}}_{\mathcal{Y},y}^G \hookrightarrow \hat{\mathcal{O}}_{\mathcal{Y},y}$ be the inclusion. By Lemma 2.6 we have a residue map $r_y^G : \hat{\mathcal{O}}_{\mathcal{Y},y}^G \to \kappa(y)$, so we can look at the following cocartesian diagram:

$$\begin{array}{ccc} \kappa(y) \otimes_{\hat{\mathcal{O}}_{\mathcal{Y},y}^G} \hat{\mathcal{O}}_{\mathcal{Y},y} & \overset{\rho_1}{\longleftarrow} & \hat{\mathcal{O}}_{\mathcal{Y},y} \\ {\scriptstyle \rho_2}\uparrow & & \uparrow{\scriptstyle i_y^G} \\ \kappa(y) & \underset{r_y^G}{\longleftarrow} & \hat{\mathcal{O}}_{\mathcal{Y},y}^G \end{array}$$

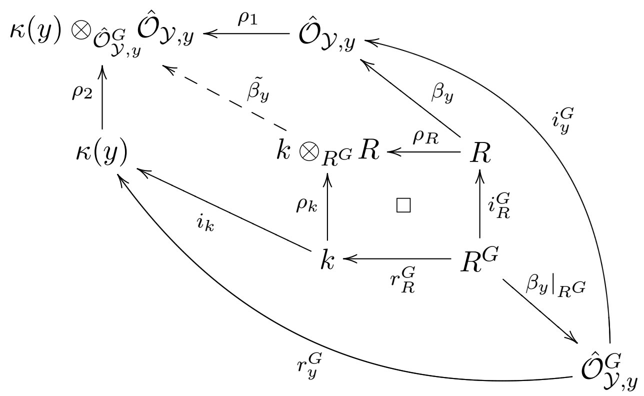
G acts on $\kappa(y) \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \hat{\mathcal{O}}_{\mathcal{Y},y}$ given by $\mathrm{id} \otimes \alpha_y$, and ρ_1 and ρ_2 are G-equivariant, see Lemma 2.7. Note that $\kappa(y) \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \hat{\mathcal{O}}_{\mathcal{Y},y}$ is a k-algebra via $\rho_2 \circ i_k$, with i_k as in Remark 2.10.

Lemma 2.13. *There is a unique k-morphism*

$$\tilde{\beta}_y : k \otimes_{R^G} R \to \kappa(y) \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \hat{\mathcal{O}}_{\mathcal{Y},y}$$

such that $\tilde{\beta}_y \circ \rho_R = \rho_1 \circ \beta_y$.

Proof. As β_y is G-equivariant, it maps R^G to $\hat{\mathcal{O}}^G_{\mathcal{Y},y}$. Consider the following diagram:



It is clear that $i^G_y \circ \beta_y|_{R^G} = \beta_y \circ i^G_R$, and by commutativity of diagram (2.2), $r^G_y \circ \beta_y|_{R^G} = i_k \circ r^G_R$. By Remark 2.12, $\rho_2 \circ r^G_y = \rho_1 \circ i^G_y$. Altogether we have

$$\rho_2 \circ (i_k \circ r^G_R) = \rho_1 \circ (\beta_y \circ i^G_R)$$

Hence the universal property of tensor product induces a unique morphism $\tilde{\beta}_y$ with the required properties. $\square$

Lemma 2.14.

$$\kappa(y) \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \hat{\mathcal{O}}_{\mathcal{Y},y} \cong \kappa(y)[x_0, x_1, \ldots, x_m]/\mathfrak{I}$$

with $m \leq n$, and for all $p(x_0, \ldots, x_m) \in \kappa(y)[x_0, x_1, \ldots, x_m]/\mathfrak{I}$

$$(\mathrm{id} \otimes \alpha_y)(p(x_0, x_1, \ldots, x_m)) = p(\mu^{\ell} x_0, \mu^{\ell_1} x_1, \ldots, \mu^{\ell_m} x_m)$$

with $\mu \in \kappa(y)$ a primitive r-th root of unity, and $\ell_i \in \{1, \ldots, r-1\}$, and $\mathfrak{I}$ is the ideal generated by monomials of the form $x_0^{s_0} x_1^{s_1} \ldots x_m^{s_m}$ such that $s_0 \ell + s_1 \ell_1 + \cdots + s_m \ell_m = rs$, $s \in \mathbb{N}$. Moreover, $\tilde{\beta}_y$ is the k-morphism mapping $t \in k[t]/(t^r) \cong k \otimes_{R^G} R$ to $x_0 \in \kappa(y)[x_0, x_1, \ldots, x_m]/\mathfrak{I}$.

Proof. We need to find a regular system of parameters $y_0, \ldots, y_{m'}$ in $\hat{\mathcal{O}}_{\mathcal{Y},y}$ with $\alpha_y(y_i) = \mu^{\ell_i} y_i$, $\ell_i \in \{0, \ldots, r-1\}$, $\ell_0 = \ell$, and $x_0 = \beta_y(t)$. Without loss of generality we may assume that there is an $m \leq m'$ such that $\ell_i \in \{1, \ldots, r-1\}$ for $i \leq m$, and $\ell_i = 0$ for $i > m$. Set $x_i := \rho_1(y_i)$. Then, using Lemma 2.7 and its proof, we get

$$\kappa(y) \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \hat{\mathcal{O}}_{\mathcal{Y},y} \cong \kappa(y)[t, x_1, \ldots, x_m]/\mathfrak{I}$$

with the required properties. By Lemma 2.13, $\tilde{\beta_y}$ is a k-morphism, and

$$\tilde{\beta_y}(t) = \tilde{\beta_y} \circ \rho_R(t) = \rho_1 \circ \beta_y(t) = \rho_1(y_0) = x_0$$

Set $\tilde{t} := \beta_y(t)$. As

$$\alpha_y(\tilde{t}) = \beta_y(\alpha_R(t)) = \mu^\ell \tilde{t}$$

using Lemma 2.4 it suffices to show that $\tilde{t} \in \mathfrak{m}_y$, and $\tilde{t} \neq 0 \mod \mathfrak{m}_y^2$ for the maximal ideal $\mathfrak{m}_y \subset \hat{\mathcal{O}}_{\mathcal{Y},y}$. As y lies in $\mathrm{Sm}(\mathcal{Y}/T)^G \subset \mathcal{Y}_k$, $\tilde{t} \in \mathfrak{m}_y$. Let $U = \mathrm{Spec}(C) \subset \mathrm{Sm}(\mathcal{Y}/T) \subset \mathcal{Y}$ be an affine neighborhood of y, $\mathfrak{p} \subset C$ the defining prime ideal of y. Choose a maximal ideal $\mathfrak{m} \subset C$ with $\mathfrak{p} \subset \mathfrak{m}$, let y' be the corresponding closed point. By [Gro67, Proposition 17.5.3], $\hat{\mathcal{O}}_{\mathcal{Y},y'} \cong R[\![\tilde{y}_1, \ldots, \tilde{y}_n]\!]$ as R-module. So $\tilde{t} \neq 0 \in \hat{\mathcal{O}}_{\mathcal{Y},y'}/\mathfrak{m}^2 \cong \mathcal{O}_{\mathcal{Y},y'}/\mathfrak{m}^2$. As $\mathfrak{p} \subset \mathfrak{m}$, $\tilde{t} \neq 0 \in \mathcal{O}_{\mathcal{Y},y'}/\mathcal{O}_{\mathcal{Y},y'}\mathfrak{p}^2$. As $\mathcal{O}_{\mathcal{Y},y'}/\mathcal{O}_{\mathcal{Y},y'}\mathfrak{p}^2 \subset \mathcal{O}_{\mathcal{Y},y}/\mathfrak{m}_y^2 \cong \hat{\mathcal{O}}_{\mathcal{Y},y}/\mathfrak{m}_y^2$ (as R-modules), we have $\tilde{t} \neq 0 \mod \mathfrak{m}_y^2$.

□

Chapter 3

Sections of Quotient Models

The aim of this chapter is to find necessary and sufficient conditions for the existence of sections of a given quotient model.
We consider the following situation: Let $G := \mathbb{Z}/r\mathbb{Z}$. Let R be a complete discrete valuation ring with residue field k. Assume that k is algebraically closed, and that r is prime to $\mathrm{char}(k)$. Let G act on R such that the residue action on k is trivial. Let $\mathcal{Y}$ be an integral R-scheme of finite type with a good G-action compatible with the G-action on R.

Notation. Set $T := \mathrm{Spec}(R)$. Let the G-action on R be given by $g_T \in \mathrm{Aut}(T)$ and $\alpha_R \in \mathrm{Aut}(R)$, respectively. By Lemma 2.2 there is a generator t of the maximal ideal in R, such that $\alpha_R(t) = \mu^\ell t$, with $\mu \in R$ a primitive r-th root of unity, and $\ell \in \{0, \dots, r-1\}$. Let $\pi_T : T \to S := \mathrm{Spec}(R^G)$ be the quotient.
Let the G-action on $\mathcal{Y}$ be given by $g \in \mathrm{Aut}(\mathcal{Y})$, and let $\pi : \mathcal{Y} \to \mathcal{X} := \mathcal{Y}/G$ be the quotient. As the G-action is compatible with the R-structure, the structure map $\varphi : \mathcal{Y} \to T$ is G-equivariant. Let $\mathrm{Sm}(\mathcal{Y}/T) \subset \mathcal{Y}$ be the smooth locus of φ. Let $\varphi_G : \mathcal{X} \to S$ be the unique map with $\varphi_G \circ \pi = \pi_T \circ \varphi$ as in Lemma 1.1:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_G} \\ T & \xrightarrow[\pi_T]{} & S \end{array}$$

Let L be the field of fractions of R, and let K be the field of fractions of R^G. Set $Y := \mathcal{Y} \times_T \mathrm{Spec}(L)$, $X := \mathcal{X} \times \mathrm{Spec}(K)$.

Now we examine the connection between sections of φ and sections of φ_G. Note that if Y is an L-variety, then $\varphi_G : \mathcal{X} \to S$ is a quotient model of the K-variety $X = Y/G$. Using the universal property of the fiber product, every section of φ_G yields a K-point of X. Hence, if φ_G has a section, X has a K-rational point. If φ is proper, the same holds for φ_G, see Lemma 1.1. So we can use the valuative criterion of properness, and obtain that X has a K-point if and only if φ_G has a section.

3.1 G-invariant sections

In this section we line out the connection between G-invariant sections of φ and sections of φ_G.

Definition 3.1. A section σ of φ is *G-invariant*, if $g \circ \sigma = \sigma \circ g_T$, i. e. if σ is a G-equivariant morphism.

Notation.

$$\mathrm{Hom}_T(T, \mathcal{Y})^G := \{\sigma \in \mathrm{Hom}_T(T, \mathcal{Y}) \mid \sigma \text{ is } G\text{-invariant.}\}$$

Definition 3.2. A section σ of φ *descends* to a section σ_G of φ_G, if the following diagram commutes:

(3.1)
$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} \\ {\scriptstyle\sigma}\uparrow & & \uparrow{\scriptstyle\sigma_G} \\ T & \xrightarrow[\pi_T]{} & S \end{array}$$

Remark 3.1. Let $y \in \mathcal{Y}$ be a closed point. If a section σ of φ through y (i. e. $y \in \sigma(T)$) descends to a section σ_G of φ_G, then σ_G goes through $x := \pi(y)$ (i. e. $x \in \sigma(S)$).

Proposition 3.2. *Every G-invariant section σ of φ descends to exactly one section σ_G of φ_G, in particular there is a unique map*

$$d : Hom_T(T, \mathcal{Y})^G \to Hom_S(S, \mathcal{X})$$

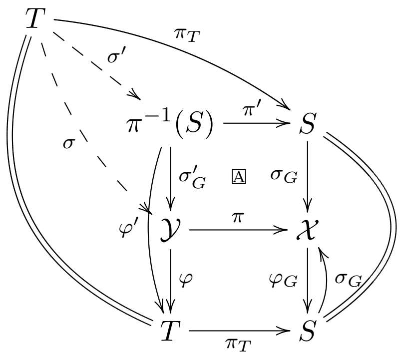
with $d(\sigma) \circ \pi_T = \pi \circ \sigma$ for all $\sigma \in Hom_T(T, \mathcal{Y})^G$.
Assume that $G \cong \mathrm{Aut}_S(T)$ and Y is an integral normal scheme, or that $Y \cong \mathcal{X} \times_S \mathrm{Spec}(L)$ as L-schemes. Then d is a bijection.

Proof. Note that the quotient map $\pi_T : T \to S$ is an epimorphism. Let Z be any scheme. If there are $h_i : S \to Z$, $i \in \{1; 2\}$, such that $h_1 \circ \pi_T = h_2 \circ \pi_T$, then $h_i \circ \pi_T \circ g_T = h_i \circ \pi_T$, because the quotient map is G-equivariant for the trivial group action on S. By the universal property of the quotient, there exists a unique $h : S \to Z$ such that $h \circ \pi_T = h_i \circ \pi_T$, hence $h_1 = h = h_2$.
Take any G-invariant section σ of φ. We construct a unique section σ_G of φ_G such that diagram (3.1) commutes. As σ is G-invariant, and as π is a quotient map, $\pi \circ \sigma \circ g_T = \pi \circ g \circ \sigma = \pi \circ \sigma$, so by the universal property of the quotient $\pi_T : T \to S$, there exists a unique $\sigma_G : S \to \mathcal{X}$ such that $\pi \circ \sigma = \sigma_G \circ \pi_T$, i. e. diagram (3.1) commutes. Furthermore,

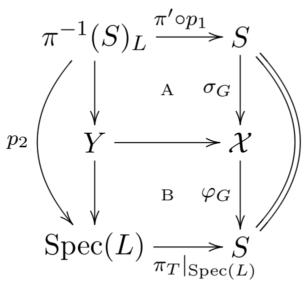
$$\varphi_G \circ \sigma_G \circ \pi_T = \varphi_G \circ \pi \circ \sigma = \pi_T \circ \varphi \circ \sigma = \pi_T \circ id_T = \pi_T$$

As π_T is an epimorphism, $\varphi_G \circ \sigma_G = id_S$, i.e. σ_G is a section of φ_G. So we showed that there is a unique map d as in the claim.

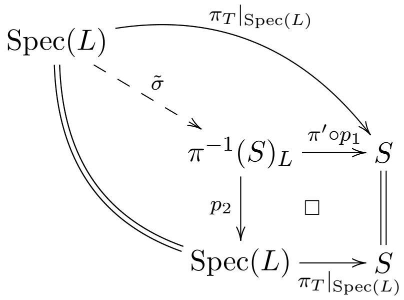
Now assume that $G \cong \mathrm{Aut}_S(T)$, and that Y is an integral normal scheme, or that $Y \cong \mathcal{X} \times_S \mathrm{Spec}(L)$ as L-schemes. In order to show that d is bijective, we show that for all $\sigma_G \in \mathrm{Hom}_S(S, \mathcal{X})$ there is a unique $\sigma \in \mathrm{Hom}_T(T, \mathcal{Y})$ descending to it, and then that σ is G-invariant. Take any section σ_G of φ_G, and consider the following commutative diagram:



Here $\pi^{-1}(S) = \mathcal{Y} \times_{\mathcal{X}} S$, and σ'_G is the projection map to the first factor, and π' is the projection map to the second factor. Set $\varphi' := \varphi \circ \sigma'_G$. By the universal property of the fiber product of A, we have a one to one correspondence of sections σ of φ with $\pi \circ \sigma = \sigma_G \circ \pi_T$, and sections σ' of φ' with $\pi' \circ \sigma' = \pi_T$. Set $\pi^{-1}(S)_L := \pi^{-1}(S) \times_T \mathrm{Spec}(L)$, and let $p_1 : \pi^{-1}(S)_L \to \pi^{-1}(S)$ and $p_2 : \pi^{-1}(S)_L \to \mathrm{Spec}(L)$ be the projection maps. Consider the following commutative diagram:



Note that (B) is cartesian by Lemma 1.22 or by assumption, respectively, and that (A) is cartesian by definition. Therefore the following diagram is cartesian:



By the universal property of the fiber product, we get a unique section $\tilde{\sigma}$ of p_2, such that $\pi_T \mid_{\mathrm{Spec}(L)} = \pi' \circ p_1 \circ \tilde{\sigma}$. To construct σ', look at the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(L) & \xrightarrow{p_1 \circ \tilde{\sigma}} & \pi^{-1}(S) \\ \big\downarrow & \overset{\sigma'}{\nearrow} & \big\downarrow \varphi' \\ T & = & T \end{array}$$

Note that $\pi_T \circ \varphi' = \pi'$. Moreover, π_T is separated, because it is a morphism between affine schemes, and π' is proper, because it is the base change of π, which is finite as it is a quotient map. Therefore φ' is proper by [GW10, Proposition 12.58]. Hence the valuative criterion of properness induces a unique section σ' of φ' such that $\sigma' \mid_{\mathrm{Spec}(L)} = p_1 \circ \tilde{\sigma}$. We show now that $\pi \circ \sigma' = \pi_T$. As S is separated, by the valuative criterion of separateness, it suffices to show that $\pi \circ \sigma' \mid_{\mathrm{Spec}(L)} = \pi_T \mid_{\mathrm{Spec}(L)}$. But this equation holds. Now, take any section $\hat{\sigma}$ of φ' with $\pi' \circ \hat{\sigma} = \pi_T$. Then $\hat{\sigma} \mid_{\mathrm{Spec}(L)}$ is a section of p_2, and $\pi' \circ p_1 \circ \hat{\sigma} \mid_{\mathrm{Spec}(L)} = \pi_T \mid_{\mathrm{Spec}(L)}$. But $\tilde{\sigma}$ is unique with these properties, i. e. $\tilde{\sigma} = \hat{\sigma} \mid_{\mathrm{Spec}(L)}$. As φ' is separated, $\hat{\sigma} = \sigma'$ by the valuative criterion of separateness.
Set $\sigma := \sigma'_G \circ \sigma'$. So far we showed that σ is the unique section of φ descending to σ_G. We still need to show that σ is G-invariant, i. e. that $g \circ \sigma = \sigma \circ g_T$, which is equivalent to $\sigma = g \circ \sigma \circ {g_T}^{-1}$. We have:

$$\varphi \circ (g \circ \sigma \circ {g_T}^{-1}) = g_T \circ (\varphi \circ \sigma) \circ {g_T}^{-1} = g_T \circ {g_T}^{-1} = id_T$$

Because of this, $g \circ \sigma \circ {g_T}^{-1}$ is a section of φ. Using that π and π_T are quotient maps, and that σ descends to σ_G, we obtain:

$$\pi \circ g \circ \sigma = \pi \circ \sigma = \varphi_G \circ \pi_T = \varphi_G \circ \pi_T \circ g_T$$

Therefore, $\pi \circ (g \circ \sigma \circ {g_T}^{-1}) = \varphi_G \circ \pi_T$, i. e. $g \circ \sigma \circ {g_T}^{-1}$ descends to σ_G. But σ is the only section of φ with this property. Hence σ is G-invariant. □

3.2 Case $\mathrm{Aut}_S(T) \cong G$

Throughout this subsection assume that $G \cong \mathrm{Aut}_S(T)$, or equivalently that $G \cong \mathrm{Gal}(L/K)$. Moreover, we assume that Y is an integral normal K-scheme. As a consequence, $Y \cong X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$, and the action on Y is given by the Galois action as described in Remark 1.17, see Lemma 1.22.

Proposition 3.3. *If $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$ is a closed point, then there exists a G-invariant section of φ through y, and therefore a section of φ_G through $x := \pi(y)$.*

Proof. By Remark 2.10 G acts on $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ given by some $\alpha_y \in \mathrm{Aut}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ with $\alpha_y^r = \mathrm{id}$, such that the natural map $j : \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}) \to \mathcal{Y}$ is G-equivariant. Moreover, $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is an R-module via $\beta_y := (\varphi \circ j)^{\#}$, and β_y is G-equivariant. By Lemma 2.11, $\hat{\mathcal{O}}_{\mathcal{Y},y} \cong R[\![x_1, \ldots, x_n]\!]$ as R-modules, and

$$\alpha_y(x_i) = \mu^{\ell_i} x_i$$

for some $\ell_i \in \mathbb{N}$, $\mu \in R \subset \hat{\mathcal{O}}_{\mathcal{Y},y}$ a primitive r-th root of unity. Here n is the relative dimension of $\mathrm{Sm}(\mathcal{Y}/T)$ over T. Let $I \subset \hat{\mathcal{O}}_{\mathcal{Y},y}$ be the ideal generated by $x_1, \ldots, x_n$. Note that $\alpha_y(I) \subset I$. So the quotient map

$$\hat{\sigma} : \hat{\mathcal{O}}_{\mathcal{Y},y} \to \hat{\mathcal{O}}_{\mathcal{Y},y}/I = R[\![x_1, \ldots, x_n]\!]/(x_1, \ldots, x_n) \cong R$$

is a G-equivariant retraction of β_y. Therefore $\hat{\sigma}^{\#}$ is a section of $\varphi \circ j$, and $\sigma := j \circ \hat{\sigma}^{\#}$ is a section of φ. As both $\hat{\sigma}$ and j are G-equivariant, the same holds for σ.
By Proposition 3.2 every G-invariant section of φ through a closed point $y \in \mathcal{Y}$ descends to a section of φ_G through $\pi(y) = x$, hence there exists a section of φ_G through x. □

Note that in general the image under π of a closed fixed point $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$ is a singular point of $\mathcal{X}$, so in fact we construct sections through singular points. Here is an example for such a section through a singular point:

Example 3.4. Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ act on the smooth $k[\![t]\!]$-scheme $\mathbb{A}^1_{k[\![t]\!]} = \mathrm{Spec}(k[\![t]\!][x])$ given by

$$\alpha : k[\![t]\!][x] \to k[\![t]\!][x];\ P(t,x) \to P(-t,-x)$$

This action is compatible with the G-action on $k[\![t]\!]$ given by

$$\alpha_R : k[\![t]\!] \to k[\![t]\!];\ P(t) \to P(-t)$$

Note that the closed point $Q = (0,0)$ is fixed, and the $k[\![t]\!]^G = k[\![t^2]\!]$-scheme $\mathbb{A}^1_{k[\![t]\!]}/G \cong \mathrm{Spec}(k[\![t^2]\!][tx, x^2]) \cong \mathrm{Spec}(k[\![t^2]\!][b,c]/(t^2c - b^2))$ is singular in the image $Q' = (0,0,0)$ of Q under the quotient map. Proposition 3.3 implies that there is a section σ_G of $\mathbb{A}^1_{k[\![t]\!]}/G \to \mathrm{Spec}(k[\![t^2]\!])$ through Q'. Such a section is for example given by

$${\sigma_G}^{\#}(P(t^2, a, b)) = P(t^2, 0, 0) \in k[\![t^2]\!]$$

Note that the G-invariant section σ of φ which descends to φ_G is given by

$$\sigma^{\#}(P(t,x)) = P(t,0)$$

Proposition 3.5. *Let $y \in \mathcal{Y}$ be a closed non-fixed point, or let $y \in \mathcal{Y}$ be a closed point such that there is no section of φ through y. Then there exists no section of φ_G through $x := \pi(y)$.*

Proof. Assume that there is a section σ_G of φ_G through x. By Proposition 3.2 there is a G-equivariant section σ of φ descending to σ_G, in particular σ is a section through y. Let $y \in \mathcal{Y}$ be given by $j_y : \mathrm{Spec}(k) \to \mathcal{Y}$, hence $\sigma\,|_{\mathrm{Spec}(k)} = j_y$. As σ is G-equivariant, and $g_T\,|_{\mathrm{Spec}(k)} = \mathrm{id}$, we get

$$j_y = \sigma\,|_{\mathrm{Spec}(k)} = g \circ \sigma \circ g_T^{-1}\,|_{\mathrm{Spec}(k)} = g \circ \sigma\,|_{\mathrm{Spec}(k)} = g \circ j_y$$

In particular, y is fixed. □

Theorem 3.6. *Let $\mathcal{Y}$ be regular, and $\mathrm{Aut}_S(T) \cong G$. Let $y \in \mathcal{Y}$ be a closed point. Then there exists a section of $\varphi_G : \mathcal{X} \to S$ through $\pi(y)$ if and only if $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$. In particular, φ_G has a section if and only if $\mathrm{Sm}(\mathcal{Y}/T)^G \neq \emptyset$.*

Proof. As $\mathcal{Y}$ is regular, $\mathrm{Hom}_T(T, \mathcal{Y}) = \mathrm{Hom}_T(T, \mathrm{Sm}(\mathcal{Y}/T))$ by [BLR90, Chapter 3.1, Proposition 2]. Therefore the claim follows from Proposition 3.3 and Proposition 3.5. □

In the case that $\mathcal{Y}$ is quasi-projective, Theorem 3.6 follows also from Lemma 4.15, see Corollary 4.16.
If $\mathcal{Y}$ is is not regular, Theorem 3.6 is wrong. To see this, consider the following example:

Example 3.7. Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ act on the singular $k[\![t]\!]$-scheme $\mathcal{Y} := \mathrm{Spec}(k[\![t]\!][b, c]/(tc - b^2))$ given by

$$\alpha : k[\![t]\!][b, c]/(tc - b^2) \to k[\![t]\!][b, c](tc - b^2);\ P(t, b, c) \to P(-t, b, -c)$$

This action is compatible with the G-action on $k[\![t]\!]$ given by

$$\alpha_R : k[\![t]\!] \to k[\![t]\!];\ P(t) \to P(-t)$$

The closed point $Q = (0, 0, 0)$ is singular, and the only fixed point, hence $\mathrm{Sm}(\mathcal{Y}/\mathrm{Spec}(k[\![t]\!]))^G = \emptyset$.
The $k[\![t]\!]^G = k[\![t^2]\!]$-scheme $\mathcal{Y}/G \cong \mathrm{Spec}(k[\![t^2]\!][b, c^2]/(t^2c^2 - b^4))$ is the quotient. There is a section σ_G of $\mathcal{Y}/G \to \mathrm{Spec}(k[\![t^2]\!])$ through the image of Q given by

$$\sigma_G{}^{\#}(P(t^2, b, c^2)) = P(t^2, 0, 0) \in k[\![t^2]\!]$$

3.3 General Case

In this section we do not have any special assumption on G. This is more complicated than the case $G = \mathrm{Aut}_S(T)$, but the general ideas are the same. Note that if G acts trivially on T, for every section σ of φ, $\pi \circ \sigma$ is a section of φ_G, hence this case is not interesting. Therefore we may assume that G does not act trivially on T.

Proposition 3.8. *Let $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$ be a closed point. Then there exists a section of φ_G through $x := \pi(y)$.*

Proof. Note that there is a surjective group homomorphism $G \to \mathrm{Aut}_S(T)$. Let G' be its kernel, hence $\mathrm{Aut}_S(T) \cong G/G'$. Note that G' acts on $\mathcal{Y}$, and this action is good. Moreover, G/G' acts on $\mathcal{Y}/G'$, the quotient of $\mathcal{Y}$ by G'. This action is good, and $(\mathcal{Y}/G')/(G/G') \cong \mathcal{X}$. Let $\pi' : \mathcal{Y} \to \mathcal{Y}/G'$ and $\pi'' : \mathcal{Y}/G' \to \mathcal{X}$ be the quotient maps, and let $\varphi_{G'}$ the unique map with $\varphi = \varphi_{G'} \circ \pi'$. Note that $\varphi_{G'}$ is G/G'-equivariant. We obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\pi'} & \mathcal{Y}/G' & \xrightarrow{\pi''} & \mathcal{X} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_{G'}} & & \downarrow{\scriptstyle \varphi_G} \\ T & = & T & \xrightarrow{\pi_T} & S \end{array} \qquad (\pi = \pi'' \circ \pi' : \mathcal{Y} \to \mathcal{X})$$

Note that G acts on $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ given by some $\alpha_y \in \mathrm{Aut}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ with $\alpha_y^r = \mathrm{id}$ such that the natural map $j : \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}) \to \mathcal{Y}$ is G-equivariant. Moreover, $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is an R-module via $\beta_y := (\varphi \circ j)^\#$, and β_y is G-equivariant. By Lemma 2.11, $\hat{\mathcal{O}}_{\mathcal{Y},y} \cong R[\![x_1, \dots, x_n]\!]$ as R-modules, and $\alpha_y(x_i) = \mu^{\ell_i} x_i$ for some $\ell_i \in \mathbb{N}$, $\mu \in R \subset \hat{\mathcal{O}}_{\mathcal{Y},y}$ a primitive r-th root of unity. As $G' \subset G$ is a subgroup, it acts on $\hat{\mathcal{O}}_{\mathcal{Y},y}$ given by $\alpha_y^s \in \mathrm{Aut}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ for some $s \in \mathbb{N}$. By construction of G', we have $\alpha_y^s|_R = \mathrm{id}$. Moreover, $\alpha_y^s(x_i) = \mu^{\ell_i s} x_i$. Set $y' := \pi'(y)$. Note that G/G' acts on $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y}/G',y'})$ given by some $\alpha_{y'} \in \mathrm{Aut}(\hat{\mathcal{O}}_{\mathcal{Y}/G',y'})$ with $\alpha_{y'}^r = \mathrm{id}$, such that the natural map $j' : \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y}/G',y'}) \to \mathcal{Y}/G'$ is G/G'-equivariant. Moreover, $\hat{\mathcal{O}}_{\mathcal{Y}/G',y'}$ is an R-module via $\beta_{y'} := (\varphi \circ j')^\#$, and $\beta_{y'}$ is G/G'-equivariant.

We construct a G/G'-invariant section of $\varphi_{G'}$ through y', hence by Proposition 3.2 we get a section of φ_G through $\pi''(y') = \pi'' \circ \pi'(y) = \pi(y) = x$. Note that in order to construct a G/G'-invariant section of $\varphi_{G'}$ through y', it suffices to construct a G/G'-equivariant retraction of $\beta_{y'}$.

We now use that $\hat{\mathcal{O}}_{\mathcal{Y}/G',y'} \cong \hat{\mathcal{O}}_{\mathcal{Y},y}^{G'}$ as R-modules, which holds by the proof of [Mum08, Chapter 7, Theorem]. Hence $\hat{\mathcal{O}}_{\mathcal{Y}/G',y'} \cong R[\![c_i]\!]/J$ as R-modules with c_i monomials in $x_1, \dots, x_n$, and J the ideal generated by the relation of the c_i. Let $I \subset \hat{\mathcal{O}}_{\mathcal{Y},y}^{G'}$ be the ideal generated by the c_i. Taking into account how G and G' act on $\mathcal{O}_{\mathcal{Y},y}$, we get that $\alpha_{y'}(I) \subset I$. Hence a G/G'-equivariant retraction of $\beta_{y'}$ is given by the quotient map

$$\hat{\mathcal{O}}_{\mathcal{Y},y}^{G'} \to \hat{\mathcal{O}}_{\mathcal{Y},y}^{G'}/I \cong R$$

Note that this yields a G/G'-invariant section of φ_G through y', and a section of φ_G through x. □

Proposition 3.9. *Let $y \in \mathcal{Y}$ be a closed point, and let $Stab_y(G)$ be the stabilizer of y. If the induced map $\phi_y : Stab_y(G) \to \mathrm{Aut}_S(T)$ is not surjective, then there exists no section of φ_G through $x := \pi(y)$.*

Proof. Take a closed point $y \in \mathcal{Y}$ as in the claim. Set $\tilde{G} := \mathrm{Stab}_y(G)$. Note that $\tilde{G}$ acts on $\hat{\mathcal{O}}_{\mathcal{Y},y}$ given by some $\tilde{\alpha}_y \in \mathrm{Aut}(\hat{\mathcal{O}}_{\mathcal{Y},y})$, and from the proof of [Mum08, Chapter 7, Theorem], we get that $\hat{\mathcal{O}}_{\mathcal{X},x} \cong \hat{\mathcal{O}}^{\tilde{G}}_{\mathcal{Y},y}$. In particular, the morphism $\hat{\mathcal{O}}_{\mathcal{X},x} \to \hat{\mathcal{O}}_{\mathcal{Y},y}$ induced by π is given by $\hat{\mathcal{O}}^{\tilde{G}}_{\mathcal{Y},y} \hookrightarrow \hat{\mathcal{O}}_{\mathcal{Y},y}$. Consider the natural morphisms $j : \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}) \to \mathcal{Y}$ and $j_G : \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{X},x}) \to \mathcal{X}$. Set $\beta_y := (j \circ \varphi)^{\#}$ and $\beta_x := (j_G \circ \varphi_G)^{\#}$. Note that β_y is $\tilde{G}$-equivariant. We get the following commutative diagram:

$$\begin{array}{ccccc} \hat{\mathcal{O}}_{\mathcal{Y},y} & \hookleftarrow & \hat{\mathcal{O}}^{\tilde{G}}_{\mathcal{Y},y} & & \\ \uparrow{\scriptstyle \beta_y} & & \uparrow{\scriptstyle \beta_x} & \searrow{\scriptstyle \rho} & \\ R & \hookleftarrow & R^G & & \end{array}$$

Assume that there is a section of φ_G through x, which yields a retraction ρ of β_x.
G acts on R given by α_R with $\alpha_R(t) = \mu^{\ell} t$. As μ is a primitive r-th root of unity, μ^{ℓ} is a primitive r'-th root of unity, and $r' \leq r$. Therefore $t^{r'}$ lies in R^G, but no nontrivial root of $t^{r'}$ does. As $\tilde{G} \subset G$ is a subgroup, $\tilde{G}$ acts on R given by $\tilde{\alpha}_R = \alpha_R^s$, $s \in \mathbb{N}$, hence $\tilde{\alpha}_R(t) = \mu^{\ell s} t$. Note that $\mu^{\ell s}$ is a primitive q-th root of unity, and q divides r'. As ϕ_y is not surjective, $q < r'$.
Set $z := \beta_y(t)$. As β_y is $\tilde{G}$-invariant, $\tilde{\alpha}_y(z) = \mu^{\ell s} z$. So $z^q \in \hat{\mathcal{O}}^{\tilde{G}}_{\mathcal{Y},y}$. Consider $\rho(z^q) \in R^G$. We have

$$\rho(z^q)^{\frac{r'}{q}} = \rho(z)^{r'} = t^{r'}$$

As there are no nontrivial roots of $t^{r'}$ in R^G, $\frac{r'}{q} = 1$, i.e. $q = r'$. This contradicts to the fact that $q < r'$. Hence there cannot exist a section of φ_G through x. □

If $G \neq \mathrm{Aut}_S(T)$, we are not able to describe all sections of φ_G in terms of the sections of φ. Let $y \in \mathcal{Y}$ be a closed point through which there is no section of φ. In general there will not be sections of φ_G through $x := \pi(y)$, but it is possible that there is a section of φ_G through x. To see this, consider the following example:

Example 3.10. Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$, let the regular scheme $\mathbb{A}^1_{k[\![s]\!]} = \mathrm{Spec}(k[\![s]\!][x])$ be a $k[\![t]\!]$-scheme by

$$\varphi^{\#} : k[\![t]\!] \to k[\![s]\!][x]; P(t) \mapsto P(s^2)$$

and let $G = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{A}^1_{k[\![s]\!]}$ given by

$$\alpha : k[\![s]\!][x] \to k[\![s]\!][x]; P(s,x) \to P(-s,x)$$

This action is compatible with the trivial G-action on $k[\![t]\!]$. Note that there is no point in the special fiber lying in the smooth locus of $\mathbb{A}^1_{k[\![s]\!]}$ over $\mathrm{Spec}(k[\![t]\!])$, so there is no section of this map.
$\mathbb{A}^1_{k[\![s]\!]}/G \cong \mathrm{Spec}(k[\![s^2]\!][x])$ is a $k[\![t]\!]^G = k[\![t]\!]$-scheme by

$$\varphi_G^{\#} : k[\![t]\!] \to k[\![s^2]\!][x]; P(t) \mapsto P(s^2)$$

So $\mathbb{A}^1_{k[\![s]\!]}/G \cong \mathbb{A}^1_{k[\![t]\!]}$ as $k[\![t]\!]$-scheme, i. e. $\mathbb{A}^1_{k[\![s]\!]}/G \to \mathrm{Spec}(k[\![t]\!])$ has a section through every closed point.

3.4 Remarks on the Assumptions

One might wonder what happens if one weakens the assumptions made in this chapter. Here are some remarks concerning this question:

- It should be possible to replace R by an Henselian discrete valuation ring with algebraically closed residue field k. In Proposition 3.2 for example it is not necessary that R is complete. But in Proposition 3.3 we need that R is complete, because otherwise we do not get that $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is isomorphic to $R[\![x_1, \ldots, x_n]\!]$ as R-modules, i. e. Lemma 2.11 does not hold. Hence we cannot prove Theorem 3.6 for general Henselian R with the methods of this chapter.

- It should be possible to weaken the assumptions on k, but one has to be careful. For example, if k is not algebraically closed, $\mathrm{Sm}(\mathcal{Y}/T)^G \neq \emptyset$ does not imply that there is a k-point in $\mathrm{Sm}(\mathcal{Y}/T)^G$. Note that at a minimum, one has to assume that k contains all r-th roots of unity, because otherwise Lemma 2.11 does not hold.

- If we do not assume that the considered G-action is tame, i. e. that r is prime to $\mathrm{char}(k)$, we will run into trouble. For example Lemma 2.2 is wrong in this case, see Example 2.3.
  It would be interesting to know what happens in the case of wild actions.

Chapter 4

A Canonical Weak Néron Model

In this chapter we construct, for a given quotient model $\mathcal{X}$ of a smooth variety X over a complete local field, a smooth model $\mathcal{Z}$ of X with a map to $\mathcal{X}$, such that the induced map on sections is a bijection. If we assume that $\mathcal{X}$ is coming from a model $\mathcal{Y}$ with some additional properties, we show that $\mathcal{Z}$ is in fact a weak Néron model of X endowed with some universal property. The construction of $\mathcal{Z}$ is taken from [Edi92], where it is used in the context of abelian varieties and Néron models.
In Section 4.1 we explain the construction of $\mathcal{Z}$. In Section 4.2 we prove some properties of $\mathcal{Z}$, the central result in this section is Theorem 4.11. In Section 4.3 we examine the special fiber of $\mathcal{Z}$. The heart of this section consists of the computation in the proof of Lemma 4.15.
Let $G := \mathbb{Z}/r\mathbb{Z}$, and let R be a complete discrete valuation ring with algebraically closed residue field k such that r is prime to $\mathrm{char}(k)$. Let G act on R such that the residue action on k is trivial, and no subgroup of G acts trivially. Let $\mathcal{Y}$ be a quasi-projective integral R-scheme of finite type with an action of G compatible with the G-action on R.
We need to assume that $\mathcal{Y}$ is quasi-projective, because otherwise the Weil restrictions of $\mathcal{Y}$, which we need to construct $\mathcal{Z}$, might not be representable.

Notation. Set $T := \mathrm{Spec}(R)$. Let the G-action on R be given by $g_T \in \mathrm{Aut}(T)$ and $\alpha_R \in \mathrm{Aut}(R)$, respectively. By Lemma 2.2 we may assume that there is a generator t of the maximal ideal of R such that $\alpha_R(t) = \mu t$, $\mu \in R$ a primitive r-th root of unity. Let $\pi_T : T \to S := \mathrm{Spec}(R^G)$ be the quotient. Note that $G \cong Aut_T(S)$.
Let the G-action on $\mathcal{Y}$ be given by some $g \in \mathrm{Aut}(\mathcal{Y})$, and let $\pi : \mathcal{Y} \to \mathcal{X}$ be the quotient. As the G-action is compatible with the R-structure, the structure map $\varphi : \mathcal{Y} \to T$ is G-equivariant. Let $\varphi_G : \mathcal{X} \to S$ be the unique map with $\varphi_G \circ \pi = \pi_T \circ \varphi$ as in Lemma 1.1. We have the following commutative

diagram:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_G} \\ T & \xrightarrow[\pi_T]{} & S \end{array}$$

Let $\mathrm{Sm}(\mathcal{Y}/T) \subset \mathcal{Y}$ be the smooth locus of φ.
Let L be the field of fractions of R, and let K be the field of fractions of R^G. Set $Y := \mathcal{Y} \times_T \mathrm{Spec}(L)$, $\mathcal{Y}_k := \mathcal{Y} \times_T \mathrm{Spec}(k)$, $X := \mathcal{X} \times_S \mathrm{Spec}(K)$, $\mathcal{X}_k := \mathcal{X} \times_S \mathrm{Spec}(k)$.

4.1 Construction

Definition 4.1. The *Weil restriction* of a T-scheme U to S is defined as the functor

$$\begin{aligned} \mathrm{Res}_{T/S}(U) : (\mathrm{Sch}/S) &\to (\mathrm{Sets}) \\ W &\mapsto \mathrm{Hom}_T(W \times_S T, U) \end{aligned}$$

Fact. [BLR90, Chapter 7.6, Theorem 4]
If U is quasi-projective over T, $\mathrm{Res}_{T/S}(U)$ is representable.

Construction 4.1. [Edi92, Construction 2.4]
Let U be a T-scheme with a G-action given by $g \in \mathrm{Aut}(U)$, such that the structure map $\varphi : U \to T$ is G-equivariant for this G-action and the given G-action on T. Then G acts on $\mathrm{Res}_{T/S}(U)$ given by $\tilde{g}$ which maps $f \in \mathrm{Hom}_T(W \times_S T, U)$ to $g \circ f \circ (\mathrm{id}_W \times g_T)^{-1}$ for every $W \in (\mathrm{Sch}/S)$.

Remark 4.1. It is easy to see that $\tilde{g}$ is an S-morphism. Therefore the structure map $\mathrm{Res}_{T/S}(U) \to S$ is G-equivariant for the G-action on $\mathrm{Res}_{T/S}(U)$ and the trivial G-action on S.

Construction 4.2. [Edi92, Theorem 4.2]
By Lemma 1.4 the G-action on $\mathcal{Y}$ induces a G-action on $\mathrm{Sm}(\mathcal{Y}/T)$. So we get an action of G on $\mathrm{Res}_{T/S}(\mathrm{Sm}(\mathcal{Y}/T))$ using Construction 4.1. Define

$$\begin{aligned} \mathcal{Z} : (\mathrm{Sch}/S) &\to (\mathrm{Sets}) \\ W &\mapsto (\mathrm{Res}_{T/S}(\mathrm{Sm}(\mathcal{Y}/T)))^G(W) = \mathcal{H}om_T(W \times_S T, \mathrm{Sm}(\mathcal{Y}/T))^G \end{aligned}$$

Notation. $\mathcal{Z}_K := \mathcal{Z} \times_S \mathrm{Spec}(K)$, $\mathcal{Z}_k := \mathcal{Z} \times_S \mathrm{Spec}(k)$.

Proposition 4.2. *$\mathcal{Z}$ is represented by a quasi-projective smooth S-scheme.*

Proof. As $\mathrm{Sm}(\mathcal{Y}/T)$ is open in $\mathcal{Y}$, $Sm(\mathcal{Y}/T)$ is quasi-projective. Moreover, π_T is proper and flat. So by [Edi92, Remark 2.1], $\mathrm{Res}_{T/S}(\mathrm{Sm}(\mathcal{Y}/T))$ is

represented by a quasi-projective smooth S-scheme. By [Edi92, Proposition 3.1], $(\mathrm{Res}_{T/S}(\mathrm{Sm}(\mathcal{Y}/T)))^G$ is represented by a closed subscheme $\mathcal{Z}$ of $\mathrm{Res}_{T/S}(\mathrm{Sm}(\mathcal{Y}/T))$. In particular $\mathcal{Z}$ is quasi-projective.
As $\mathrm{Res}_{T/S}(\mathrm{Sm}(\mathcal{Y}/T))$ is smooth over S, $\mathcal{Z}$ is smooth over S by [Edi92, Proposition 3.4]. □

Remark 4.3. Note that assuming $\mathcal{Y}$ is quasi-projective ensures that the Weil restriction is representable. In addition, all G-actions on $\mathcal{Y}$ and on $\mathrm{Sm}(\mathcal{Y}/T)$ are automatically good, so quotients exist.

4.2 Properties

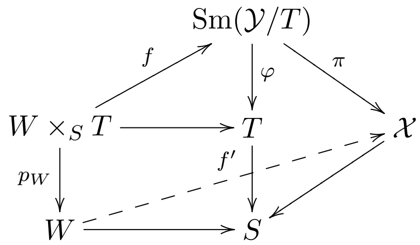
Lemma 4.4. *There is an S-morphism $\Phi : \mathcal{Z} \to \mathcal{X}$.*

Proof. We construct an S-morphism $\Phi : \mathcal{Z} \to \mathcal{X}$ by constructing maps

$$\Phi_W : \mathcal{Z}(W) = \mathrm{Hom}_T(W \times_S T, \mathrm{Sm}(\mathcal{Y}/T))^G \to \mathcal{X}(W)$$

for all $W \in (\mathrm{Sch}\,/S)$, and show that they are functorial.
For every $f \in \mathcal{Z}(W)$ we obtain a commutative diagram with $\pi = \pi\mid_{\mathrm{Sm}(\mathcal{Y}/T)}$, $\varphi = \varphi\mid_{\mathrm{Sm}(\mathcal{Y}/T)}$ as follows:

(4.1)



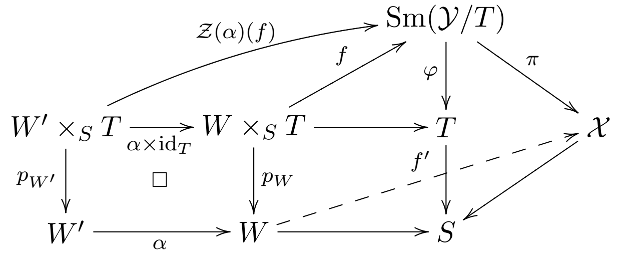
By Lemma 1.6 the projection map $p_W : W \times_S T \to W$ is the quotient of the G-action on $W \times_S T$ given by $\mathrm{id}_W \times g_T$. As $f = g \circ f \circ (\mathrm{id}_W \times g_T)^{-1}$, and π is G-equivariant for the G-action on $\mathrm{Sm}(\mathcal{Y}/T) \subset \mathcal{Y}$ and the trivial action on $\mathcal{X}$, we get

$$(\pi \circ f) \circ (\mathrm{id}_W \times g_T) = \pi \circ g \circ f = \pi \circ f$$

Hence $\pi \circ f$ is G-equivariant for the G-action on $W \times_S T$ and the trivial action on $\mathcal{X}$, and therefore, by the universal property of the quotient $p_W : W \times_S T \to W$, we obtain a unique $f' \in \mathcal{X}(W)$ making the whole diagram commutative. We set $\Phi_W(f) := f'$.
One still needs to check functoriality. So take any $W' \in (\mathrm{Sch}\,/S)$, and any $\alpha \in \mathrm{Hom}_S(W', W)$. We need to show that for all $f \in \mathcal{Z}(W)$ the following equation holds:

$$(\phi_{W'} \circ \mathcal{Z}(\alpha))(f) = (\mathcal{X}(\alpha) \circ \Phi_W)(f) \tag{4.2}$$

Note that $(\mathcal{X}(\alpha) \circ \Phi_W)(f) = f' \circ \alpha$, f' as constructed above, and that $\mathcal{Z}(\alpha)(f) = f \circ (\alpha \times \mathrm{id}_T)$. We have the following commutative diagram:



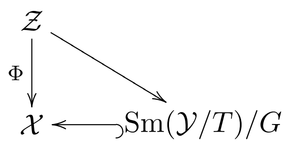
Note that $\phi_{W'}(\mathcal{Z}(\alpha)(f))$ is the unique morphism with

$$\phi_{W'}(\mathcal{Z}(\alpha)(f)) \circ p_{W'} = \mathcal{Z}(\alpha)(f) \circ \pi$$

At the same time, $f' \circ \alpha \circ p_{W'} = \mathcal{Z}(\alpha)(f) \circ \pi$, which yields equation (4.2). $\square$

Notation. From now on denote the morphism constructed in Lemma 4.4 by Φ.

Remark 4.5. The G-action on $\mathcal{Y}$ restricts to a G-action on $\mathrm{Sm}(\mathcal{Y}/T)$. Therefore, $\mathrm{Sm}(\mathcal{Y}/T)/G \subset \mathcal{X}$. Going through the construction of Φ, one observes that in fact $\Phi(\mathcal{Z}) \subset \mathrm{Sm}(\mathcal{Y}/T)/G$, i. e. we obtain the following commutative diagram:

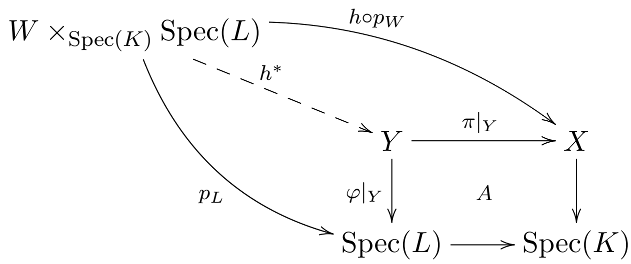


Lemma 4.6. *Assume that Y is smooth. Then Φ induces an isomorphism between $\mathcal{Z}_K$ and X.*

Proof. As Φ is an S-morphism, it maps $\mathcal{Z}_K$ to X. In order to show that $\Phi_{|\mathcal{Z}_K}$ is an isomorphism, we construct an inverse map $(\Phi_{|\mathcal{Z}_K})^{-1} : X \to \mathcal{Z}_K \subset \mathcal{Z}$, such that $(\Phi_{|\mathcal{Z}_K})^{-1} \circ \Phi\,|_{\mathcal{Z}_K} = \mathrm{id}\,|_{\mathcal{Z}_K}$ and $\Phi \circ (\Phi_{|\mathcal{Z}_K})^{-1} = \mathrm{id}\,|_X$.
Take any $W \in (\mathrm{Sch}\,/K)$. Note that $W \times_S T \cong W \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. So we have

$$\begin{aligned} \mathcal{Z}_K(W) &= Hom_T(W \times_S T, \mathrm{Sm}(\mathcal{Y}/T))^G \\ &= Hom_T(W \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L), \mathrm{Sm}(\mathcal{Y}/T))^G \\ &= Hom_L(W \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L), \mathrm{Sm}(\mathcal{Y}/T) \times_T \mathrm{Spec}(L))^G \\ &= Hom_L(W \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L), Y)^G \end{aligned}$$

In the last line we used the fact that Y is smooth.
Take any $h \in X(W)$, and consider the following diagram (p_L is the projection to $\operatorname{Spec}(L)$):



As there is no subgroup of G which acts trivially on T, $\operatorname{Aut}_S T \cong G$, so Lemma 1.22 implies that A is Cartesian. As h is a K-morphism, the diagram commutes, and hence the universal property of fiber product induces a unique $h^* \in \operatorname{Hom}_L(W \times_{\operatorname{Spec}(K)} \operatorname{Spec}(L), Y)$ with $\pi \circ h^* = h \circ p_W$. Set $g_L := g_T\,|_{\operatorname{Spec}(L)}$. Note that

$$\begin{aligned}\pi \circ (g \circ h^* \circ (\mathrm{id}_W \times g_L)^{-1}) &= \pi \circ h^* \circ (\mathrm{id}_W \times g_L)^{-1}\\ &= h \circ p_W \circ (\mathrm{id}_W \times g_L)^{-1} = h \circ p_W\end{aligned}$$

and

$$\begin{aligned}\varphi \circ (g \circ h^* \circ (\mathrm{id}_W \times g_L)^{-1}) &= g_L \circ \varphi \circ h^* \circ (\mathrm{id}_W \times g_L)^{-1}\\ &= g_L \circ p_L \circ (\mathrm{id}_W \times g_L)^{-1} = p_L\end{aligned}$$

As h^* is unique with this property, $h^* = g \circ h^* \circ (\mathrm{id}_W \times g_L)^{-1}$, i. e. h^* is G-equivariant. Altogether, $h^* \in \mathcal{Z}_K(W)$. Set $(\Phi_{|\mathcal{Z}_K})^{-1}_W(h) := h^*$. To check functoriality of $(\Phi_{|\mathcal{Z}_K})^{-1}$, let $W' \in (\mathrm{Sch}/K)$, $\alpha \in Hom_K(W, W')$. It suffices to show that for all $h \in \mathcal{Z}(W)$ the following equation holds:

$$((\Phi_{|\mathcal{Z}_K})^{-1}_{W'} \circ X(\alpha))(h) = (\mathcal{Z}_K(\alpha) \circ (\Phi_{|\mathcal{Z}_K})^{-1}_{W})(h) \tag{4.3}$$

Note that $X(\alpha)(h) = h \circ \alpha$, and $\mathcal{Z}_K(\alpha) \circ \Phi_W(h) = h^* \circ (\alpha \times \mathrm{id})$. We have

$$\varphi \circ (h^* \circ (\alpha \times \mathrm{id})) = p_L \circ (\alpha \times \mathrm{id}) = p_L$$

and

$$\pi \circ (h^* \circ (\alpha \times \mathrm{id})) = h \circ p_W \circ (\alpha \times \mathrm{id}) = h \circ p_{W'}$$

But $(\Phi_{|\mathcal{Z}_K})^{-1}_{W'}$ maps $h \circ \alpha$ to the unique morphism with these properties, so equation (4.3) holds.
It remains to check that $(\Phi_{|\mathcal{Z}_K})^{-1} \circ \Phi\,|_{\mathcal{Z}_K} = \mathrm{id}$, and $\Phi \circ (\Phi_{|\mathcal{Z}_K})^{-1} = \mathrm{id}$. Take any $f \in \mathcal{Z}_K(W)$. $\Phi(f)$ is the unique morphism f' with $\pi \circ f = f' \circ p_W$. $(\Phi_{|\mathcal{Z}_K})^{-1}$ sends f' to the unique $f'^* \in \mathcal{Z}_K(W)$ with $\pi \circ f'^* = f' \circ p_W$. As f has this property, $f'^* = f$, i. e. $(\Phi_{|\mathcal{Z}_K})^{-1} \circ (\Phi_{|\mathcal{Z}_K}) = \mathrm{id}$. Now take any $h \in X(W)$. $(\Phi_{|\mathcal{Z}_K})^{-1}$ sends h to h^* with $h \circ p_W = \pi_K \circ h^*$. Φ sends $h^{*\prime}$ to the unique morphism with $h^{*\prime} \circ p_W = \pi \circ h^*$. As h has this property, $h^{*\prime} = h$, i. e. $\Phi \circ (\Phi_{|\mathcal{Z}_K})^{-1} = \mathrm{id}$. □

Proposition 4.7. *Assume that Y is a smooth L-scheme, and moreover that* $\mathrm{Sm}(\mathcal{Y}/T) \to T$ *is a weak Néron model of Y. Then $\mathcal{Z} \to S$ is a weak Néron model of X.*

Proof. To show that $\mathcal{Z}$ is a weak Néron model of X, we need to check three conditions, namely that $\mathcal{Z} \to S$ is a model of X, that $\mathcal{Z}$ is smooth and separated over S, and that the natural map $\mathcal{Z}(S) \to \mathcal{Z}_K(K) \cong X(K)$ is a bijection. The second condition follows directly from Proposition 4.2. By Lemma 4.6, $X \cong \mathcal{Z}_K$, and X is integral, because it is the quotient of the integral scheme Y by a finite group. As $\mathcal{Z} \to S$ is smooth, it is of finite type and flat. Hence we can use the same proof as in Lemma 1.15 to obtain that $\mathcal{Z}$ is integral. This yields that $\mathcal{Z} \to S$ is a model of X, i. e. the first condition. To check the third condition, consider the natural map

$$\mathcal{Z}(S) = Hom_T(T, \mathrm{Sm}(\mathcal{Y}/T))^G \to \mathcal{Z}_K(K) = Hom_L(\mathrm{Spec}(L), Y)^G$$
$$\sigma \mapsto \sigma|_{\mathrm{Spec}(L)}$$

This map is injective, because $\mathrm{Sm}(\mathcal{Y}/T)$ is a separated T-scheme. We still need to show that it is surjective. Therefore, take any $\sigma' \in \mathcal{Z}_K(K)$. As $\mathrm{Sm}(\mathcal{Y}/T)$ is a weak Néron model of Y, the following map is a bijection:

$$\mathrm{Sm}(\mathcal{Y}/T)(T) = Hom_T(T, \mathrm{Sm}(\mathcal{Y}/T)) \to Y(L) = Hom_L(\mathrm{Spec}(L), Y)$$
$$\sigma \mapsto \sigma|_{\mathrm{Spec}(L)}$$

So there is a $\sigma \in Hom_T(T, \mathrm{Sm}(\mathcal{Y}/T))$ with $\sigma|_{\mathrm{Spec}(L)} = \sigma'$. It remains to show that σ is G-invariant. As g_T^{-1} maps $\mathrm{Spec}(L)$ to itself, we get using that $\sigma' \in \mathcal{Z}_K(K)$

$$g \circ \sigma \circ g_T^{-1}|_{\mathrm{Spec}(L)} = g \circ \sigma' \circ g_T^{-1}|_{\mathrm{Spec}(L)} = \sigma' = \sigma|_{\mathrm{Spec}(L)}$$

As $\mathrm{Sm}(\mathcal{Y}/T)$ is a separated T-scheme, $g \circ \sigma \circ g_T^{-1} = \sigma$, i. e. σ is G-invariant. Hence $\mathcal{Z}(S) \to \mathcal{Z}_K(K)$ is surjective. □

Lemma 4.8. *Let $\mathcal{Y}'$ be another quasi-projective integral T-scheme with a G-action, and let $\pi' : \mathcal{Y}' \to \mathcal{X}'$ be the quotient. Let $\gamma : \mathcal{Y}' \to \mathcal{Y}$ be a G-equivariant T-morphism, and let $\gamma_G : \mathcal{X}' \to \mathcal{X}$ be the S-morphism with $\gamma_G \circ \pi' = \pi \circ \gamma$ (see Lemma 1.1). Assume that $\gamma(\mathrm{Sm}(\mathcal{Y}'/T)) \subset \mathrm{Sm}(\mathcal{Y}/T)$, for the smooth locus $\mathrm{Sm}(\mathcal{Y}'/T)$ of $\mathcal{Y}'$ over T. Set $\mathcal{Z}' := \mathrm{Res}_{T/S}(\mathrm{Sm}(\mathcal{Y}'/T))^G$, and let $\Phi' : \mathcal{Z}' \to \mathcal{X}'$ be the map as constructed in Lemma 4.4. Then there is a unique S-morphism $\tilde{\gamma} : \mathcal{Z}' \to \mathcal{Z}$ making the following diagram commutative:*

(4.4)
$$\begin{array}{ccc} \mathcal{X}' & \xleftarrow{\Phi'} & \mathcal{Z}' \\ {\scriptstyle \gamma_G}\downarrow & & \downarrow{\scriptstyle \tilde{\gamma}} \\ \mathcal{X} & \xleftarrow[\Phi]{} & \mathcal{Z} \end{array}$$

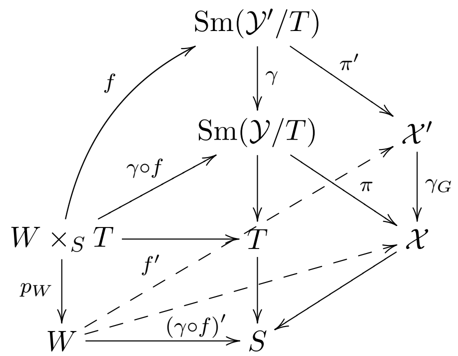
Proof. Set $X' := \mathcal{X}' \times_S \operatorname{Spec}(K)$, $\mathcal{Z}'_K := \mathcal{Z}' \times_S \operatorname{Spec}(K)$.
Assume $\tilde{\gamma}$ exists such that diagram (4.4) commutes. As Φ' is an S-morphism, it maps $\mathcal{Z}'_K$ to X'. As γ_G is an S-morphism, it maps X' to X. By Lemma 4.6, $\Phi'|_{\mathcal{Z}_K} \colon \mathcal{Z}_K \to X$ is an isomorphism with inverse map $(\Phi|_{\mathcal{Z}_K})^{-1}$. Altogether, we obtain

$$\tilde{\gamma}|_{\mathcal{Z}'_K} = (\Phi|_{\mathcal{Z}_K})^{-1} \circ \gamma \circ \Phi'|_{\mathcal{Z}'_K}$$

So $\tilde{\gamma}$ is uniquely determined on the open, dense subset $\mathcal{Z}'_K \subset \mathcal{Z}'$, and therefore it is unique by [GW10, Corollary 9.9], because $\mathcal{Z}$ is a separated S-scheme and $\mathcal{Z}'$ is reduced.
Now we construct $\tilde{\gamma}$. Take any $W \in (\operatorname{Sch}/S)$. We need to construct $\tilde{\gamma}(W) : \mathcal{Z}'(W) \to \mathcal{Z}(W)$, and then show functoriality. Hence take any $f \in \mathcal{Z}'(W) = \operatorname{Hom}_T(W \times_S T, \operatorname{Sm}(\mathcal{Y}'/T))^G$. Let the G-action on $\mathcal{Y}'$ be given by $g' \in \operatorname{Aut}(\mathcal{Y}')$. Then $g' \circ f \circ (\operatorname{id} \times g_T)^{-1} = f$. Set $\tilde{\gamma}(W)(f) := \gamma \circ f$. Note that $\gamma \circ f \in \operatorname{Hom}_T(W \times_S T, \operatorname{Sm}(\mathcal{Y}/T))$, because $\gamma(\operatorname{Sm}(\mathcal{Y}'/T)) \subset \operatorname{Sm}(\mathcal{Y}/T)$, and as γ is G-equivariant

$$g \circ (\gamma \circ f) \circ (\operatorname{id} \times g_T)^{-1} = \gamma \circ g' \circ f \circ (\operatorname{id} \times g_T)^{-1} = \gamma \circ f$$

Therefore $\gamma \circ f \in \mathcal{Z}(W)$. It is clear that this map is functorial, hence $\tilde{\gamma}$ is an S-morphism. We still need to check the commutativity of diagram (4.4), i. e. that $\gamma_G \circ \Phi'(f) = \Phi \circ \tilde{\gamma}(f)$ for any $W \in (\operatorname{Sch}/S)$ and any $f \in \mathcal{Z}'(W)$. Consider the following commutative diagram:



Considering the construction of Φ in the proof of Lemma 4.4 yields that Φ' sends f to f', and that Φ sends $\tilde{\gamma}(f) = \gamma \circ f$ to $(\gamma \circ f)'$, both unique making the diagram commutative. One observes that $(\gamma \circ f)' = \gamma_G \circ f'$, which we wanted to show. □

Remark 4.9. Let $\mathcal{C}$ be the category with

- objects: quasi-projective, integral, smooth T-schemes with a G-action, such that the structure map is G-equivariant
- morphisms: G-equivariant T-morphisms

Lemma 4.8 implies that

$$\begin{aligned} F_{\mathcal{Z}} : \mathcal{C} &\to (\operatorname{Sch}/S) \\ \mathcal{Y} &\mapsto (\operatorname{Res}_{T/S}(\mathcal{Y}))^G \end{aligned}$$

is a functor.

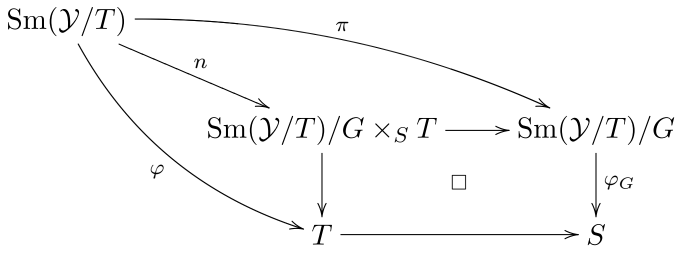
Proposition 4.10. *Assume that* Y *is a smooth* L*-scheme, and furthermore that* $\operatorname{Sm}(\mathcal{Y}/T) \to T$ *is a weak Néron model of* Y*. For every smooth quasi-projective integral* S*-scheme* $\mathcal{V}$ *with a dominant* S*-morphism* $\Psi : \mathcal{V} \to \mathcal{X}$*, there is a unique* S*-morphism* $\Psi' : \mathcal{V} \to \mathcal{Z}$ *making the following diagram commutative:*

(4.5)
$$\begin{array}{ccc} \mathcal{V} & & \\ & \searrow^{\Psi'} & \\ \Psi \downarrow & & \mathcal{Z} \\ & \swarrow_{\Phi} & \\ \mathcal{X} & & \end{array}$$

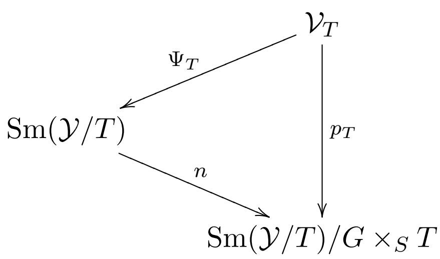
Proof. Assume that there exists a $\Psi' : \mathcal{V} \to \mathcal{Z}$ making diagram (4.5) commutative. Consider $\mathcal{V}_K := \mathcal{V} \times_S \operatorname{Spec}(K)$. As Ψ is an S-morphism, it maps $\mathcal{V}_K$ to X. By Lemma 4.6, $\Phi|_{\mathcal{Z}_K} : \mathcal{Z}_K \to X$ is an isomorphism with inverse map $(\Phi\,|_{\mathcal{Z}_K})^{-1}$. Therefore we have $\Psi'|_{\mathcal{V}_K} = (\Phi|_{\mathcal{Z}_K})^{-1} \circ \Psi|_{\mathcal{V}_K}$. As $\mathcal{V}_K$ is open and dense in $\mathcal{V}$, and $\mathcal{Z}$ is a separated S-scheme and $\mathcal{V}$ is reduced, Ψ' is unique on $\mathcal{V}$ by [GW10, Corollary 9.9].
Now we construct Ψ'. First we show that $\Psi(\mathcal{V}) \subset \operatorname{Sm}(\mathcal{Y}/T)/G \subset \mathcal{X}$. Assume that $\Psi(\mathcal{V}) \not\subset \operatorname{Sm}(\mathcal{Y}/T)/G$. As Y is smooth, $\mathcal{X} \backslash \operatorname{Sm}(\mathcal{Y}/T)/G \subset \mathcal{X}_k$. As Ψ is an S-morphism, $\Psi^{-1}(\mathcal{X}_k) = \mathcal{V}_k := \mathcal{V} \times_S \operatorname{Spec}(k)$. So there is a closed point $v \in \mathcal{V}_k \subset \mathcal{V}$, such that $\Psi(v) \in \mathcal{X} \backslash \operatorname{Sm}(\mathcal{Y}/T)$. As $\mathcal{V}$ is a smooth S-scheme, there is a section σ of $\varphi_G \circ \Psi$ by [BLR90, Chapter 2.3, Proposition 5]. That means that $\sigma \circ \Psi$ is a section of φ_G through $\Psi(v)$, and therefore it corresponds to a G-equivariant section of φ through $\pi^{-1}(\Psi(v))$ by Proposition 3.2. As $\operatorname{Sm}(\mathcal{Y}/T)$ is a weak Néron model of Y, there are only sections of φ through $\operatorname{Sm}(\mathcal{Y}/T)$, i. e. $\pi^{-1}(\Psi(v)) \in \operatorname{Sm}(\mathcal{Y}/T)$, and $\Psi(v) \in \operatorname{Sm}(\mathcal{Y}/T)/G$, which is a contradiction.
Now consider the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{V}_T & \xrightarrow{\pi_{\mathcal{V}}} & \mathcal{V} \\ {\scriptstyle p_T}\downarrow & \square & \downarrow{\scriptstyle \Psi} \\ \operatorname{Sm}(\mathcal{Y}/T)/G \times_S T & \longrightarrow & \operatorname{Sm}(\mathcal{Y}/T)/G \\ \downarrow & \square & \downarrow{\scriptstyle \varphi_G|_{\operatorname{Sm}(\mathcal{Y}/T)/G}} \\ T & \longrightarrow & S \end{array}$$

with $\mathcal{V}_T := \mathcal{V} \times_S T = \mathcal{V} \times_{\mathrm{Sm}(\mathcal{Y}/T)/G} (\mathrm{Sm}(\mathcal{Y}/T)/G \times_S T)$, $\pi_{\mathcal{V}}$ and p_T the projection maps.
As $\mathrm{Sm}(\mathcal{Y}/T)$ is smooth over T, it is normal. By Lemma 1.15 we obtain that $\mathrm{Sm}(\mathcal{Y}/T)/G \times_S T$ is integral, so we can use Lemma 1.9 to obtain that the following diagram commutes:



Moreover, $n : \mathrm{Sm}(\mathcal{Y}/T) \to \mathrm{Sm}(\mathcal{Y}/T)/G \times_S T$ is the normalization map, which is G-equivariant for the G-action on $\mathrm{Sm}(\mathcal{Y}/T)$ and the G-action on $\mathrm{Sm}(\mathcal{Y}/T)/G \times_S T$ given by $\mathrm{id} \times g_T$.
$\mathcal{V}$ is smooth over S, so $\mathcal{V}_T$ is smooth over T. In particular, $\mathcal{V}_T$ is normal. As Ψ is dominant, the same holds for p_T.
It might happen that $\mathcal{V}_T$ is not connected. In this case consider the G-action on $\mathcal{V}_T$ given by $\mathrm{id}_{\mathcal{V}} \times g_T$. Then $\pi_{\mathcal{V}} : \mathcal{V}_T \to \mathcal{V}$ is the quotient for this action by Lemma 1.6. Moreover, p_T is G-equivariant for the G-action on $\mathrm{Sm}(\mathcal{Y}/T)/G \times_S T$ given by $\mathrm{id} \times g_T$. Let $\mathcal{V}_T = U_1 \sqcup \cdots \sqcup U_m$, $U_i \subset \mathcal{V}_T$ a connected component. As $\mathcal{V}$ is connected, G acts transitively on the connected components. As p_T is dominant and $\mathrm{Sm}(\mathcal{Y}/T)/G \times_S T$ is connected, there exists at least one component U_i such that $p_T|_{U_i}$ is dominant. For any $j \in \{1, \ldots, m\}$, there exists an l such that $(\mathrm{id}_{\mathcal{V}} \times g_T)^l(U_i) = U_j$. As $(\mathrm{id}_{\mathcal{V}} \times g_T)^l$ is an isomorphism, $(\mathrm{id} \times g_T)^l p_T|_{U_i} = p_T \circ (\mathrm{id}_{\mathcal{V}} \times g_T)^l|_{U_i}$ is dominant. Hence $p_T|_{U_j}$ is dominant, i. e. p_T is dominant for every connected component. Hence for every component U_i of $\mathcal{V}_T$ there is a unique morphism $\Psi_T|_{U_i} : U_i \to \mathrm{Sm}(\mathcal{Y}/T)$ such that $n \circ \Psi_T|_{U_i} = p_T|_{U_i}$ by the universal property of normalization, using that U_i is normal for all i. This defines a morphism Ψ_T on all of $\mathcal{V}_T$ such that the following diagram commutes:



Now we will check that Ψ_T is G-equivariant for this G-action and the one on $\mathrm{Sm}(\mathcal{Y}/T)$. As p_T is G-equivariant for the G-action on $\mathcal{V}_T$ and that on

$\mathrm{Sm}(\mathcal{Y}/T)/G \times_S T$, we get

$$\begin{aligned} n \circ (g \circ \Psi_T \circ (\mathrm{id}_{\mathcal{V}} \times g_T)^{-1}) &= (\mathrm{id} \times g_T) \circ n \circ \Psi_T \circ (\mathrm{id}_{\mathcal{V}} \times g_T)^{-1} \\ &= (\mathrm{id} \times g_T) \circ p_T \circ (\mathrm{id}_{\mathcal{V}} \times g_T)^{-1} = p_T \end{aligned}$$

As Ψ_T is unique with this property, $\Psi_T = g \circ \Psi_T \circ g_{\mathcal{V}}{}^{-1}$, i.e. Ψ_T is G-equivariant.
Let $\mathcal{Z}_{\mathcal{V}} := \mathrm{Res}_{T/S}(\mathcal{V}_T)^G$ as in Construction 4.2. As $\mathcal{V}$ is quasi-projective, the same holds for $\mathcal{V}_T$, which implies that $\mathcal{Z}_V$ is actually representable. Consider $\phi_{\mathcal{V}} : \mathcal{Z}_{\mathcal{V}} \to \mathcal{V} = \mathcal{V}_T/G$ as constructed in Lemma 4.4. By Lemma 4.8 there is a unique S-morphism $\tilde{\Psi} : \mathcal{Z}_{\mathcal{V}} \to \mathcal{Z}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{V} & \xleftarrow{\Phi_{\mathcal{V}}} & \mathcal{Z}_{\mathcal{V}} \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\tilde{\Psi}} \\ \mathcal{X} & \xleftarrow[\Phi]{} & \mathcal{Z} \end{array}$$

Note that neither Lemma 4.4 nor Lemma 4.8 needs that $\mathcal{V}_T$ is connected. By Lemma 4.6, $\phi_{\mathcal{V}}|_{\mathcal{Z}_{\mathcal{V}} \times_S \mathrm{Spec}(K)}$ is an isomorphism, hence $\Phi_{\mathcal{V}}$ is birational. This Lemma does not use that $\mathcal{V}_T$ is connected, but it uses the fact that $\mathcal{V}_T \times_T \mathrm{Spec}(L) = \mathcal{V} \times_S \mathrm{Spec}(L)$, which holds by construction. Now take any $W \in (\mathrm{Sch}/S)$. Consider

$$\Phi_{\mathcal{V}}(W) : \mathcal{Z}_{\mathcal{V}}(W) = \mathrm{Hom}_T(W \times_S T, \mathcal{V} \times_S T)^G \to \mathcal{V}(W) = \mathrm{Hom}_S(W, \mathcal{V})$$

Note that $s \in \mathcal{Z}_{\mathcal{V}}(W)$ is mapped to the unique $s' \in \mathcal{V}(W)$ such that $\pi_{\mathcal{V}} \circ s = s' \circ p_W$. By the universal property of the fiber product, there exists a unique T-morphism $s : W \to \mathcal{V}$ with $s' \circ p_W = \pi_{\mathcal{V}} \circ s$ for all $s' \in \mathcal{V}(W)$. Hence $\Phi_{\mathcal{V}}(W)$ is bijective, and therefore $\Phi_{\mathcal{V}}$ is quasi-finite and surjective. Recall that $\mathcal{V}$ is smooth over S, hence it is in particular normal. So by [GW10, Lemma 12.88], $\Phi_{\mathcal{V}}$ is an isomorphism. This means that $\Psi' := \tilde{\Psi} \circ \Phi_{\mathcal{V}}^{-1}$ is the unique S-morphism making diagram (4.5) commutative. □

Theorem 4.11. *Assume that Y is a smooth L-scheme, and* $\mathrm{Sm}(\mathcal{Y}/T) \to T$ *is a weak Néron model of Y. Then $\mathcal{Z} \to S$ is a weak Néron model of X with a dominant S-morphism Φ to $\mathcal{X}$, such that for all smooth quasi-projective integral S-schemes $\mathcal{V}$, every dominant S-morphism from $\mathcal{V}$ to $\mathcal{X}$ factors uniquely through $\mathcal{Z}$. Moreover, $\mathcal{Z}$ is unique with its properties, up to a unique isomorphism over $\mathcal{X}$.*

Proof. By Proposition 4.7, $\mathcal{Z} \to S$ is a weak Néron model of X. By Proposition 4.10, $\Phi : \mathcal{Z} \to \mathcal{X}$ has the required properties. We still need to check that $\mathcal{Z}$ and Φ are unique. Assume there is a $\mathcal{Z}'$ and a morphism $\Phi' : \mathcal{Z}' \to \mathcal{X}$ having the same properties as $\mathcal{X}$ and Φ. So we get unique morphisms $\alpha : \mathcal{Z} \to \mathcal{Z}'$ and $\alpha' : \mathcal{Z}' \to \mathcal{Z}$ with $\Phi \circ \alpha' = \Phi'$ and $\Phi' \circ \alpha = \Phi$.

Note that $\Phi \circ (\alpha' \circ \alpha) = \Phi' \circ \alpha = \Phi$. But $\mathrm{id}_{\mathcal{Z}}$ is unique with $\Phi \circ \mathrm{id}_{\mathcal{Z}} = \Phi$, so $\alpha' \circ \alpha = \mathrm{id}_{\mathcal{Z}}$. Similarly one gets $\alpha \circ \alpha' = \mathrm{id}_{\mathcal{Z}'}$. So α is a unique isomorphism over $\mathcal{X}$ of $\mathcal{Z}$ and $\mathcal{Z}'$. □

In [Edi92, Theorem 4.2] the following statement is proven:

Theorem. *If Y and X are abelian varieties, and $\mathcal{Y} \to T$ is the Néron model of Y, then $\mathcal{Z}$ is the Néron model of X.*

Remark 4.12. Let $G\mathcal{C}_L$ be the category with

- objects: smooth L-varieties Y with weak Néron model $\mathcal{Y}$ with a G-action compatible with the G-action on R
- morphisms: G-equivariant T-morphisms of the weak Néron models.

Let $\mathcal{C}_K$ the category with

- objects: K-varieties X with weak Néron model $\mathcal{Z}$
- morphisms: S-morphisms of the weak Néron models

Theorem 4.11 and Lemma 4.8 imply that

$$\begin{aligned} F : G\mathcal{C}_L &\to \mathcal{C}_K \\ (Y, \mathcal{Y}) &\to (Y/G, \mathrm{Res}_{T/S}(\mathcal{Y})^G) \end{aligned}$$

is a functor.

4.3 Geometry of the Special Fiber

In this section we describe the special fiber $\mathcal{Z}_k$ of $\mathcal{Z}$ by comparing it with $\mathrm{Sm}(\mathcal{Y}/T)^G$. We are going to use the result to compute some motivic invariants in the next chapter, see Theorem 5.2 and Theorem 5.5.

Remark 4.13. As $T^G = \mathrm{Spec}(k)$, by Lemma 1.3

$$\begin{aligned} \mathrm{Sm}(\mathcal{Y}/T)^G : (\mathrm{Sch}/k) &\to (\mathrm{Sets}) \\ W &\mapsto \mathrm{Hom}_T(W, \mathrm{Sm}(\mathcal{Y}/T))^G \end{aligned}$$

is represented by a smooth separated k-scheme of finite type with structure map φ^G, and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Sm}(\mathcal{Y}/T)^G & \hookrightarrow & \mathrm{Sm}(\mathcal{Y}/T) \\ \downarrow{\scriptstyle \varphi^G} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Spec}(k) & \hookrightarrow & T \end{array}$$

Lemma 4.14. *$\mathcal{Z}_k$ is a smooth k-scheme, and there is a k-morphism of finite type*

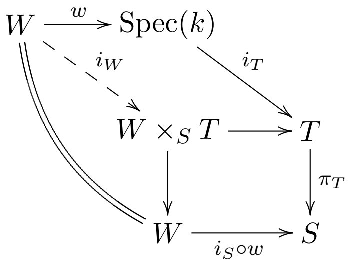
$$b : \mathcal{Z}_k \to \mathrm{Sm}(\mathcal{Y}/T)^G$$

such that $\pi \circ b = \Phi_{|\mathcal{Z}_k}$.

Proof. As $\mathcal{Z}$ is smooth over S, $\mathcal{Z}_k$ is smooth over k, because smoothness is stable under base change. To construct b, let $W \in (\mathrm{Sch}/k)$ be any k-scheme, $w : W \to \mathrm{Spec}(k)$ the structure map. Set

$$b_W := b(W) : \ \mathcal{Z}_k(W) = \mathrm{Hom}_T(W \times_S T, \mathrm{Sm}(\mathcal{Y}/T))^G \to \mathrm{Sm}(\mathcal{Y}/T)^G(W)$$
$$f \mapsto f \circ i_W$$

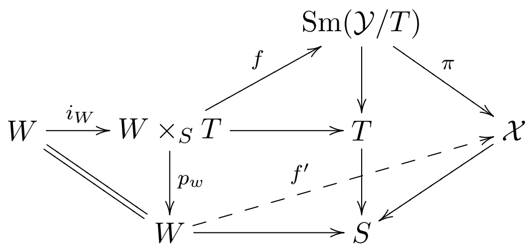
with $i_W : W \hookrightarrow W \times_S T$ the inclusion of the special fiber, i. e. the unique morphism i_W making the following diagram commutative:



Here $i_T : \mathrm{Spec}(k) \to T$ and $i_S : \mathrm{Spec}(k) \to S$ are the inclusion maps of the special points. We need to check that for all $f \in \mathcal{Z}_k(W)$ we have that $b_W(f) \in \mathrm{Sm}(\mathcal{Y}/T)^G(W)$. By construction, $b_W(f) \in Hom_T(W, \mathrm{Sm}(\mathcal{Y}/T))$, that means that we just need to check that $g \circ b_W(f) = b_W(f)$, i. e. that $b_W(f)$ is G-invariant. This holds, because

$$g \circ f \circ i_W = f \circ (\mathrm{id}_W \times g_T) \circ i_W = f \circ i_W$$

Here the first equation holds, because f is G-equivariant, the second by construction of i_W, and because $g_T\,|_{\mathrm{Spec}(k)}$ is trivial.
It is obvious that b is functorial, so we have the required k-morphism. By [GW10, Proposition 10.7], b is of finite type, because both $\mathcal{Z}_k$ and $\mathrm{Sm}(\mathcal{Y}/T)$ are separated k-schemes of finite type. Consider the following commutative diagram:



By construction of Φ, see Lemma 4.4, Φ maps f to the unique $f' \in \mathcal{X}(W)$ making the diagram commutative. The commutativity of the diagram gives us $(\pi \circ b)(f) = \pi \circ f \circ i_W = f' = \Phi(f)$, i. e. $\pi \circ b = \Phi\,|_{\mathcal{Z}_k}$. □

Lemma 4.15. *Let $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$ be any point, $\kappa(y)$ the residue field of y. Then $b^{-1}(y)$ is isomorphic to $\mathbb{A}^m_{\kappa(y)}$ as $\kappa(y)$-schemes for some $m \in \mathbb{N}$.*

Proof. Let $j_y : \mathrm{Spec}(\kappa(y)) \hookrightarrow \mathrm{Sm}(\mathcal{Y}/T)^G \subset \mathcal{Y}$ be the immersion of the point y. Note that $b^{-1}(y)$ is defined by the following cartesian diagram:

$$\begin{array}{ccc} b^{-1}(y) & \longrightarrow & \mathcal{Z}_k \\ \downarrow & \square & \downarrow b \\ \mathrm{Spec}(\kappa(y)) & \xrightarrow[j_y]{} & \mathrm{Sm}(\mathcal{Y}/T)^G \end{array}$$

Take any affine $\kappa(y)$-scheme $W = \mathrm{Spec}(A) \in (\mathrm{Sch}/\kappa(y))$ with structure map $\omega : W \to \mathrm{Spec}(\kappa(y))$. By the universal property of the fiber product we obtain

$$\begin{aligned} b^{-1}(y)(W) &= \{f \in \mathcal{Z}_k(W) \mid b \circ f = j_y \circ \omega\} \\ &= \{f \in \mathrm{Hom}_T(W \times_S T, \mathrm{Sm}(\mathcal{Y}/T))^G \mid f \circ i_W = j_y \circ \omega\} \end{aligned}$$

Here we used the construction of b as in Lemma 4.14 (same notation as there, and W is viewed as k-scheme with structure map $\varphi^G \circ j_y \circ \omega$). Recall that G acts on $\mathrm{Hom}_T(W \times_S T, \mathrm{Sm}(\mathcal{Y}/T))$ by sending $f \in \mathrm{Hom}_T(W \times_S T, \mathrm{Sm}(\mathcal{Y}/T))$ to $g \circ f \circ (\mathrm{id}_W \times g_T)^{-1}$. Note that

$$\begin{aligned} W \times_S T &= W \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k) \times_S T \\ &= W \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k \otimes_{R^G} R) \\ &= \mathrm{Spec}(A) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[t]/(t^r)) \\ &= \mathrm{Spec}(A[t]/(t^r)) \end{aligned}$$

In the third line of this computation we use Lemma 2.7. This lemma also implies that

$$\alpha := (\mathrm{id} \times g_T)^{\#} : A[t]/(t^r) \to A[t]/(t^r);\ p(t) \mapsto p(\mu t)$$

Note that we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{\omega} & \mathrm{Spec}(\kappa(y)) \\ i_W \downarrow & & \downarrow j_y \\ \mathrm{Spec}(A[t]/(t^r)) & \xrightarrow[f]{} & \mathrm{Sm}(\mathcal{Y}/T) \end{array} \tag{4.6}$$

Note that $r_W := i_W^{\#} : A[t]/(t^r) \to A;\ p(t) \mapsto p(0)$. One observes that f sends all points in $\mathrm{Spec}(A[t]/(t^r))$ to $y \in \mathrm{Sm}(\mathcal{Y}/T)$, so it factors uniquely through $\mathrm{Spec}(\mathcal{O}_{\mathcal{Y},y})$, i. e. there is the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{Spec}(\mathcal{O}_{\mathcal{Y},y}) \\ & \overset{f'}{\nearrow} & \downarrow \\ \mathrm{Spec}(A[t]/(t^r)) & \xrightarrow[f]{} & \mathrm{Sm}(\mathcal{Y}/T) \end{array}$$

Let $\mathfrak{m} \subset \mathcal{O}_{\mathcal{Y},y}$ be the maximal ideal. As diagram (4.6) commutes, we have that $f'^{\#}(\mathfrak{m}) \subset (t)$, and therefore $f'^{\#}(\mathfrak{m}^r) \subset (t^r) = (0) \subset A[t]/(t^r)$, hence f also factors uniquely through $\mathrm{Spec}(\mathcal{O}_{\mathcal{Y},y}/\,\mathfrak{m}^r) = \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}/\,\mathfrak{m}_y^r)$, with $\mathfrak{m}_y \subset \hat{\mathcal{O}}_{\mathcal{Y},y}$ the maximal ideal. Therefore f also factors uniquely through $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y})$, i. e. there is a unique morphism

$$\hat{f} : \mathrm{Spec}(A[t]/(t^r)) \to \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y})$$

such that the following diagram commutes:

$$\begin{array}{ccc} & & \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}) \\ & \overset{\hat{f}}{\nearrow} & \downarrow j \\ \mathrm{Spec}(A[t]/(t^r)) & \xrightarrow[f]{} & \mathrm{Sm}(\mathcal{Y}/T) \end{array}$$

Note that we also have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}) & \xleftarrow{\hat{i}_y} & \mathrm{Spec}(\kappa(y)) \\ j \downarrow & \overset{j_y}{\swarrow} & \\ \mathrm{Sm}(\mathcal{Y}/T) & & \end{array}$$

with $r_y := \hat{i}_y^{\#} : \hat{\mathcal{O}}_{\mathcal{Y},y} \to \kappa(y)$ the residue map. As $f \circ i_W = j_y \circ \omega$, we get that $j \circ \hat{f} \circ i_W = j \circ \hat{i}_y \circ \omega$, and the fact that j is a monomorphism implies that $\hat{f} \circ i_W = \hat{i}_y \circ \omega$. As y lies in $\mathrm{Sm}(\mathcal{Y}/T)^G$, there is an induced G-action on $\mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ given by some $\hat{g} \in \mathrm{Aut}(\mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}))$ with $\hat{g}^r = \mathrm{id}$ and $\alpha_y \in \mathrm{Aut}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ with $\alpha_y^r = \mathrm{id}$, respectively, such that j is G-equivariant, see Remark 2.10. As $f = g \circ f \circ (\mathrm{id}_W \times g_T)^{-1}$, we have

$$j \circ (\hat{g} \circ \hat{f} \circ (\mathrm{id}_W \times g_T)^{-1}) = g \circ j \circ \hat{f} \circ (\mathrm{id}_W \times g_T)^{-1} = f$$

As $\hat{f}$ is unique with this property, $\hat{g} \circ \hat{f} \circ (\mathrm{id}_W \times g_T)^{-1} = \hat{f}$.
Assume that $\hat{f} \in \mathrm{Hom}_T(\mathrm{Spec}(A[t]/(t^r)), \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}))$ with $\hat{f} \circ i_W = \hat{i}_y \circ \omega$, and $\hat{g} \circ \hat{f} \circ (\mathrm{id}_W \times g_T)^{-1} = \hat{f}$, then $f := j \circ \hat{f} \in \mathrm{Hom}_T(\mathrm{Spec}(A[t]/(t^r)), \mathrm{Sm}(\mathcal{Y}/T))$, and

$$g \circ f \circ (\mathrm{id}_W \times g_T)^{-1} = j \circ \hat{g} \circ \hat{f} \circ (\mathrm{id}_W \times g_T)^{-1} = j \circ \hat{f} = f$$

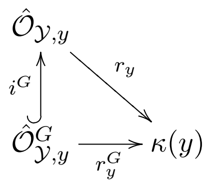
as well as

$$f \circ i_W = j \circ \hat{i}_y \circ \omega = j_y \circ \omega$$

Altogether, we obtain

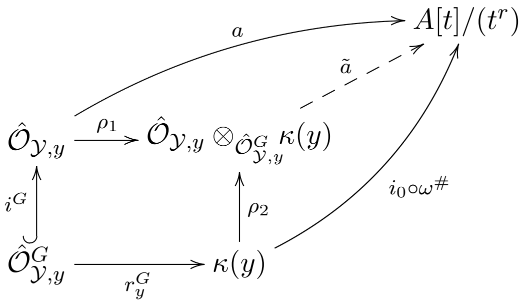
$$\begin{aligned} b^{-1}(y)(W) &= \{\hat{f} \in \mathrm{Hom}_T(\mathrm{Spec}(A[t]/(t^r)), \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y})) \\ &\qquad \mid \hat{f} \circ i_W = \hat{i}_y \circ \omega \text{ and } \hat{g} \circ \hat{f} \circ (\mathrm{id}_W \times g_T)^{-1} = \hat{f}\} \\ &= \{a \in \mathrm{Hom}_R(\hat{\mathcal{O}}_{\mathcal{Y},y}, A[t]/(t^r)) \\ &\qquad \mid r_W \circ a = \omega^{\#} \circ r_y \text{ and } \alpha^{-1} \circ a \circ \alpha_y = a\} \end{aligned}$$

By Lemma 2.6, $\hat{\mathcal{O}}^G_{\mathcal{Y},y} \subset \hat{\mathcal{O}}_{\mathcal{Y},y}$ is a subring which is local, and has the same residue field, and the following diagram commutes



Here $i^G : \hat{\mathcal{O}}^G_{\mathcal{Y},y} \hookrightarrow \hat{\mathcal{O}}_{\mathcal{Y},y}$ is the inclusion, and $r^G_y : \hat{\mathcal{O}}^G_{\mathcal{Y},y} \to \kappa(y)$ the residue map, see Remark 2.12. Consider the following diagram

(4.7)



Here $a \in b^{-1}(y)(W)$ as described before, ρ_1 and ρ_2 are the morphisms we get from the definition of tensor product, and

$$i_0 : A \to A[t]/(t^r);\ c \mapsto c$$

Note that $r_W \circ i_0 = \mathrm{id}$, and $i_0 \circ r_W\,|_{i_0(A)} = \mathrm{id}$. One observes that for every $u \in \hat{\mathcal{O}}^G_{\mathcal{Y},y}$, $(\alpha^{-1} \circ a)(u) = (\alpha^{-1} \circ a \circ \alpha_y)(u) = a(u)$. Now $a(u) = \sum_{i=0}^{r-1} a_i t^i$ for some $a_i \in A$, and $(\alpha^{-1} \circ a)(u) = \sum_{i=0}^{r-1} \mu^{-i} a_i t^i$, μ a primitive r-th root of unity. Comparing coefficients yields $a(u) = a_0$, i. e. $a(\hat{\mathcal{O}}^G_{\mathcal{Y},y}) \subset i_0(A)$. Using in addition that $r_W \circ a = \omega^{\#} \circ r_y$, and $i_0 \circ r_W\,|_{i_0(A)} = \mathrm{id}$, we obtain

$$\begin{aligned} i_0 \circ \omega^{\#} \circ r^G_y &= i_0 \circ \omega^{\#} \circ r_y \circ i^G = i_0 \circ r_W \circ a \circ i^G \\ &= i_0 \circ r_W\,|_{i_0(A)} \circ a \circ i^G = a \circ i^G \end{aligned}$$

So, by the universal property of tensor product there is a unique $\tilde{a}$ such that diagram (4.7) commutes.
Now, G acts on $\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \kappa(y)$ given by $\tilde{\alpha}_y := \alpha_y \otimes \mathrm{id} \in \mathrm{Aut}(\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \kappa(y))$, such that ρ_1 and ρ_2 are G-equivariant, see Remark 2.12. As $\alpha^{-1} \circ a \circ \alpha_y = a$, we get

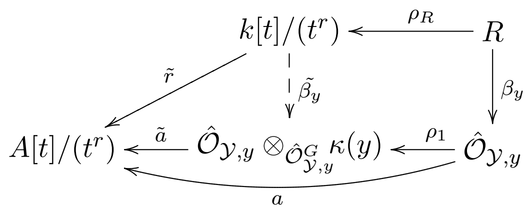
$$(\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y) \circ \rho_1 = \alpha^{-1} \circ \tilde{a} \circ \rho_1 \circ \alpha_y = a$$

and, using that G acts trivially on $i_0(A)$, we obtain

$$(\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y) \circ \rho_2 = \alpha^{-1} \circ \tilde{a} \circ \rho_2 = \alpha^{-1} \circ i_0 \circ \omega^{\#} = i_0 \circ \omega^{\#}$$

As $\tilde{a}$ is unique with these properties, $\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y = \tilde{a}$.
Denote by $\tilde{r} : k \otimes_{R^G} R \cong k[t]/(t^r) \to A \otimes_k k \otimes_{R^G} R \cong A[t]/(t^r)$ the canonical map given by the properties of the tensor product. We have $\tilde{r}(t) = t$. The R-structure of $A[t]/(t^r)$ is given by $\tilde{r} \circ \rho_R$. As a is an R-morphism, we obtain the following commutative diagram:

(4.8)



By Lemma 2.13 there exists a $\tilde{\beta}_y$ such that $\tilde{\beta}_y \circ \rho_R = \rho_1 \circ \beta_y$, hence

$$\tilde{r} \circ \rho_R = a \circ \beta_y = \tilde{a} \circ \rho_1 \circ \beta_y = \tilde{a} \circ \tilde{\beta}_y \circ \rho_R$$

As ρ_R is surjective, $\tilde{r} = \tilde{a} \circ \tilde{\beta}_y$, i. e. $\tilde{a}$ preserves the $k[t]/(t^r)$-structure on $\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \kappa(y)$ given by $\tilde{\beta}_y$, and that on $A[t]/(t^r)$ given by $\tilde{r}$.
Using the universal property of the tensor product, and that $r_y \circ i^G = r_y^G$, we get a unique morphism $\tilde{r}_y : \hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \kappa(y) \to \kappa(y)$, such that $\tilde{r}_y \circ \rho_1 = r_y$ and $\tilde{r}_y \circ \rho_2 = \mathrm{id}$. Using that $a = \tilde{a} \circ \rho_1$ and $\tilde{a} \circ \rho_2 = i_0 \circ \omega^{\#}$, and that $r_W \circ a = \omega^{\#} \circ r_y$, we get

$$(r_W \circ \tilde{a}) \circ \rho_1 = r_W \circ a \text{ and } (\omega^{\#} \circ \tilde{r}_y) \circ \rho_1 = \omega^{\#} \circ r_y = r_W \circ a$$
$$(r_W \circ \tilde{a}) \circ \rho_2 = r_W \circ i_0 \circ \omega^{\#} = \omega^{\#} \text{ and } (\omega^{\#} \circ \tilde{r}_y) \circ \rho_2 = \omega^{\#}$$

Moreover,

$$r_W \circ a \circ i^G = r_W \circ i_0 \circ \omega^{\#} \circ r_y^G = \omega^{\#} \circ r_y^G$$

hence by the universal property of tensor product there is a unique morphism $v : \hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \kappa(y) \to A$ such that $v \circ \rho_1 = r_W \circ a$ and $v \circ \rho_2 = \omega^{\#}$, so

$$r_W \circ \tilde{a} = v = \omega^{\#} \circ \tilde{r}_y$$

Note that for a given morphism $\tilde{a} \in \mathrm{Hom}_{k[t]/(t^r)}(\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \kappa(y), A[t]/(t^r))$, $a := \tilde{a} \circ \rho_1 \in Hom_R(\hat{\mathcal{O}}_{\mathcal{Y},y}, A[t]/(t^r))$. If $\alpha^{-1} \circ \tilde{\alpha} \circ \tilde{\alpha}_y = \tilde{a}$, then

$$\alpha^{-1} \circ a \circ \alpha_y = \alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y \circ \rho_1 = \tilde{a} \circ \rho = a$$

If we assume furthermore that $r_W \circ \tilde{a} = \omega^{\#} \circ \tilde{r}_y$, then

$$r_W \circ a = r_W \circ \tilde{a} \circ \rho_1 = \omega^{\#} \circ \tilde{r}_y \circ \rho_1 = \omega^{\#} \circ r_y$$

So altogether

$$\begin{aligned} b^{-1}(y)(W) = \{&\tilde{a} \in \mathrm{Hom}_{k[t]/(t^r)}(\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \kappa(y), A[t]/(t^r)) \\ &\mid \tilde{a} \circ \rho_2 = i_0 \circ \omega^{\#} \text{ and } r_W \circ \tilde{a} = \omega^{\#} \circ \tilde{r}_y \text{ and } \alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y = \tilde{a}\} \end{aligned}$$

Note that $\tilde{a} \circ \rho_2 = i_0 \circ \omega^{\#}$ is actually redundant.
By Lemma 2.14, $\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\hat{\mathcal{O}}^G_{\mathcal{Y},y}} \kappa(y) \cong \kappa(y)[x_0, x_1, \ldots, x_m]/\mathfrak{I}$,

$$\tilde{\alpha}_y(p(x_0, x_1, \ldots, x_m)) = p(\mu x_0, \mu^{\ell_1} x_1, \ldots, \mu^{\ell_m} x_m)$$

for $p(x_0, x_1, \ldots, x_m) \in \kappa(y)[x_0, x_1, \ldots, x_m]/\mathfrak{I}$, $\mu \in \kappa(y)$ a primitive r-th root of unity, $\ell_i \in \{1, \ldots, r-1\}$, $m \in \mathbb{N}$, and $\mathfrak{I} \subset \kappa(y)[x_0, x_1, \ldots, x_m]$ is the ideal generated by monomials of the form $x_0^{s_0} x_1^{s_1} \ldots x_m^{s_m}$ such that $s_0 + \ell_1 s_1 + \cdots + \ell_m s_m = rs$, $s \in \mathbb{N}$. Furthermore, $\tilde{\beta}_y$ is the k-morphism sending $t \in k[t]/(t^r)$ to x_0. Chose any $\tilde{a} \in b^{-1}(y)(W)$. For $j \in \{1, \ldots, m\}$ we have

$$\tilde{a}(x_j) = \sum_{i=0}^{r-1} a_{ij} t^i \in A[t]/(t^r)$$

for some $a_{ij} \in A$. Using $r_W \circ \tilde{a} = \omega^{\#} \circ \tilde{r}_y$, we obtain

$$a_{0j} = r_W(\tilde{a}(x_j)) = \omega^{\#}(\tilde{r}_y(x_j)) = \omega^{\#}(0) = 0$$

From $\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha} = \tilde{a}$ we get

$$\sum_{i=1}^{r-1} \mu^{\ell_j - i} a_{ij} t^i = (\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y)(x_j) = \tilde{a}(x_j) = \sum_{i=1}^{r-1} a_{ij} t^i$$

Comparing coefficients yields to either $a_{ij} = 0$, or $\mu^{\ell_j - i} = 1$. As i and ℓ_i lie in $\{1, \ldots, r-1\}$, the latter is equivalent to $i = \ell_j$. As $\tilde{a}$ preserves the $k[t]/(t^r)$-structure, i. e. $\tilde{a} \circ \tilde{\beta}_y = \tilde{r}$, we get that $\tilde{a}(x_0) = t$. So using that $\tilde{a}$ is a $\kappa(y)$-morphism, i. e. that $\tilde{a} \circ \rho_2 = i_0 \circ w^{\#}$, we get that

$$\tilde{a}(p(x_0, x_1, \ldots, x_m)) = p(t, a_1 t^{\ell_1}, \ldots, a_m t^{\ell_m}) \tag{4.9}$$

for all $p(x_0, x_1, \ldots, x_m) \in \kappa(y)[x_0, x_1, \ldots x_m]/\mathfrak{I}$, and for some $a_i \in A$. On the right site, p is viewed as a polynomial with coefficients in A.

Let $\tilde{a} : \kappa(y)[x_0, x_1, \dots, x_m] \to A[t]/(t^r)$ be defined by formula (4.9). For any generator $x_0^{s_0} x_1^{s_1} \dots x_m^{s_m}$ of $\mathfrak{I}$

$$\tilde{a}(x_0^{s_0} x_1^{s_1} \dots x_m^{s_m}) = t^{s_0 + \ell_1 s_1 + \dots + \ell_m s_m} = t^{rs} = 0 \in A[t]/(t^r)$$

This implies that $\mathfrak{I} \subset ker(\tilde{a})$. Therefore, we get a well-defined map

$$\tilde{a} : \kappa(y)[x_0, x_1, \dots, x_m]/\mathfrak{I} \to A[t]/(t^r)$$

Note that $\tilde{a}$ is a $\kappa(y)$-morphism, and preserves the $k[t]/(t^r)$-structure. For all $p(x_0, x_1, \dots, x_m) \in \kappa(y)[x_0, x_1, \dots x_m]/\mathfrak{I}$ we have

$$\begin{aligned}\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y(p(x_0, x_1, \dots, x_m)) &= \alpha^{-1} \circ \tilde{a}(p(\mu x_0, \mu^{\ell_1} x_1, \dots, \mu^{\ell_m} x_m)) \\ &= \alpha^{-1}(p(\mu t, a_1 \mu^{\ell_1} t^{\ell_1}, \dots, a_m \mu^{\ell_m} t^{\ell_m})) \\ &= p(\mu^{1-1} t, a_1 \mu^{\ell_1 - \ell_1} t^{\ell_1}, \dots, a_m \mu^{\ell_m - \ell_m} t^{\ell_m}) \\ &= \tilde{a}(p(x_0, x_1, \dots, x_m))\end{aligned}$$

and

$$\begin{aligned}r_W \circ \tilde{a}(p(x_0, x_1, \dots, x_m)) &= r_W(p(\mu t, a_1 t^{\ell_1}, \dots, a_m t^{\ell_m})) \\ &= p(0, \dots, 0) \\ &= w^{\#} \circ \tilde{r}_y(p(x_0, x_1, \dots, x_m))\end{aligned}$$

So $\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y = \tilde{a}$, and $r_W \circ \tilde{a} = w^{\#} \circ \tilde{r}_y$. Altogether, $\tilde{a} \in b^{-1}(y)(W)$ if and only if it is given by formula (4.9).
Now we are ready to construct a $\kappa(y)$-isomorphism

$$\beta : b^{-1}(y) \to \mathbb{A}^m_{\kappa(y)}$$

Note that

$$\mathbb{A}^m_{\kappa(y)}(W) = \mathrm{Hom}_{\kappa(y)}(W, \mathbb{A}^m_{\kappa(y)}) = \mathrm{Hom}_{\kappa(y)}(\kappa(y)[y_1, \dots, y_m], A)$$

It suffices to give bijective, functorial maps

$$\beta(W) : b^{-1}(y)(W) \to \mathbb{A}^m_{\kappa(y)}(W)$$

for all affine $W = \mathrm{Spec}(A) \in (\mathrm{Sch}/\kappa(y))$. Let $\beta(W)$ sent $\tilde{a} \in b^{-1}(W)$ given by

$$\begin{aligned}\tilde{a} :\ & \kappa(y)[x_0, x_1, \dots, x_m]/\mathfrak{I} \to A[t]/(t^r) \\ & p(x_0, x_1, \dots, x_m) \mapsto p(x_0, a_1 t^{\ell_1}, \dots, a_m t^{\ell_m})\end{aligned}$$

to $a' \in \mathbb{A}^m_{\kappa(y)}(W)$ with

$$\begin{aligned}a' :\ & \kappa(y)[y_1, \dots, y_m] \to A \\ & p(y_1, \dots, y_m) \mapsto p(a_1, \dots, a_m)\end{aligned}$$

It is easy to see that this map is bijective, because there is an obvious inverse map. It remains to check functoriality. Choose $W' = \mathrm{Spec}(A') \in (\mathrm{Sch}/\kappa(y))$ affine, $\gamma \in \mathrm{Hom}_{\mathrm{Spec}(\kappa(y))}(W', W)$, $\gamma^{\#} \in \mathrm{Hom}_{\kappa(y)}(A, A')$ the corresponding morphism of rings. We have to show that

$$\mathbb{A}^n_{\kappa(y)}(\gamma) \circ \beta(W) = \beta(W') \circ b^{-1}(y)(\gamma) \tag{4.10}$$

Note that $b^{-1}(y)(\gamma)$ sends $a \in b^{-1}(y)(W)$ with

$$a(p(x_0, x_1, \ldots, x_m)) = p(t, a_1 t^{\ell_1}, \ldots, a_m t^{\ell_m}), a_i \in A$$

to $b^{-1}(y)(\gamma)(a) \in b^{-1}(y)(W')$ with

$$b^{-1}(y)(\gamma)(a)(p(x_0, x_1, \ldots, x_m)) = p(t, \gamma^{\#}(a_1) t^{\ell_1}, \ldots, \gamma^{\#}(a_m) t^{\ell_m})$$

$\mathbb{A}^m_{\kappa(y)}(\gamma)$ sends $a' \in \mathbb{A}^m_{\kappa(y)}(W)$ with

$$a'(p(y_1, \ldots, y_m)) = p(a_1, \ldots, a_m), a_i \in A$$

to $\mathbb{A}^m_{\kappa(y)}(\gamma)(a') \in \mathbb{A}^m_{\kappa(y)}(W')$ with

$$\begin{aligned} \mathbb{A}^m_{\kappa(y)}(\gamma)(a')(p(y_1, \ldots, y_m)) &= \gamma^{\#}(p(a_1, \ldots, a_m)) \\ &= p(\gamma^{\#}(a_1), \ldots, \gamma^{\#}(a_m)) \end{aligned}$$

Using these formulas on sees immediately that equation (4.10) holds. □

Note that Lemma 4.15 implies that b is surjective, because no fiber of b is empty. In particular, Lemma 4.15 implies Theorem 3.6 for $\mathcal{Y}$ quasi-projective, i. e. we obtain the following Corollary:

Corollary 4.16. *Assume that $\mathcal{Y}$ is regular. Then there is a section of φ_G if and only if $\mathrm{Sm}(\mathcal{Y}/T)^G \neq \emptyset$.*

Proof. If $\mathrm{Sm}(\mathcal{Y}/T)^G \neq \emptyset$, then $\mathcal{Z}_k \neq \emptyset$. As k is algebraically closed, there is an k-point in $\mathcal{Z}_k$. As R^G is Henselian, and $\varphi_G \circ \Phi : \mathcal{Z} \to S$ is smooth, there is a section σ of $\varphi_G \circ \Phi$ through this point, see [BLR90, Chapter 2.3, Proposition 5]. Hence $\Phi \circ \sigma$ is a section of φ_G.
If there is a section of φ_G, there is a G-invariant section σ of φ, see Proposition 3.2. Hence we get a closed fixed point $y \in \mathcal{Y}$. As $\mathcal{Y}$ is regular by [BLR90, Chapter 3.1, Proposition 2] σ has to factor through $\mathrm{Sm}(\mathcal{Y}/T)$. Therefore, $y \in \mathrm{Sm}(\mathcal{Y}/T)^G$, i. e. $\mathrm{Sm}(\mathcal{Y}/T)^G \neq \emptyset$. □

Lemma 4.17. *Let $V \subset \mathrm{Sm}(\mathcal{Y}/T)^G$ be any closed subscheme. Then there exists an open subscheme $U \subset V$, such that $b^{-1}(U) \cong \mathbb{A}^m_U$ for some $m \in \mathbb{N}$.*

Proof. Choose any open subscheme $\mathrm{Spec}(A) = U \subset V$, which is affine, irreducible and has the generic point η given by $\mathrm{Spec}(\kappa(\eta)) \hookrightarrow U$, $\kappa(\eta)$ the residue field of η, and consider the following cartesian diagram:

$$\begin{array}{ccc} b^{-1}(\eta) & \longrightarrow & b^{-1}(U) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(\kappa(\eta)) & \hookrightarrow & U \end{array}$$

Lemma 4.15 implies that

$$b^{-1}(\eta) \cong \mathbb{A}^m_{\kappa(\eta)} = \mathrm{Spec}(\kappa(\eta)[x_1, \ldots, x_m]) \tag{4.11}$$

as $\kappa(\eta)$-schemes for some $m \in \mathbb{N}$. To show the claim, it suffices to construct $\mathrm{Spec}(A') = U' \subset U$ open, such that

$$b^{-1}(U') = b^{-1}(U) \times_U U' \cong \mathrm{Spec}(A'[x_1, \ldots, x_m]) = \mathbb{A}^m_{U'}$$

Choose a finite set of open affine $\mathrm{Spec}(B_i) = V_i \subset b^{-1}(U)$ covering $b^{-1}(U)$. This is possible, because b is of finite type. Then $b^{-1}(\eta)$ is covered by $W_i := \mathrm{Spec}(B_i \otimes_A \kappa(\eta))$. As b is a morphism of finite type, the B_i are generated by finitely many b_{ij} as A-modules, and $B_i \otimes_A \kappa(\eta)$ is generated by $b_{ij} \otimes 1$ as $\kappa(\eta)$-module. The isomorphism in equation (4.11) is given by compatible $\kappa(\eta)$-morphisms

$$f_i : B_i \otimes_A \kappa(\eta) \to \kappa(\eta)[x_1, \ldots, x_m]_{I_i} \ ; \ g_i : \kappa(\eta)[x_1, \ldots, x_m]_{I_i} \to B_i \otimes_A \kappa(\eta)$$

with $f_i \circ g_i = \mathrm{id}$ and $g_i \circ f_i = \mathrm{id}$. The $\mathrm{Spec}(\kappa(\eta)[x_1, \ldots, x_m]_{I_i})$ form a cover of $\mathrm{Spec}(\kappa(\eta)[x_1, \ldots, x_m])$. The I_i are finitely generated, i.e. $I_i = (h_{ij})$ with $h_{ij} = \sum h_{ij}^{i_1 \ldots i_s} x_1^{i_1} \ldots x_s^{i_s}$ The f_i and g_i agree on the intersections of the open subschemes on which they are defined. The f_i and g_i are given uniquely by the image of the $b_{ij} \otimes 1$ and the x_i, respectively. Set

$$f_i(b_{ij} \otimes 1) =: f_{ij} = \sum f_{ij}^{i_1 \ldots i_s} x_1^{i_1} \ldots x_m^{i_m} \in \kappa(\eta)[x_1, \ldots x_m]_{I_i}$$

and

$$g_i(x_j) =: g_{ij} = \sum b_{i1}^{i_1} \ldots b_{il}^{i_l} \otimes g_{ij}^{i_1 \ldots i_l} \in B_i \otimes_A \kappa(\eta)$$

Note that these sums are finite. Let $I \subset A$ be an ideal generated by the denominators of the $g_{ij}^{i_1 \ldots i_l}$, the denominators of $f_{ij}^{i_1 \ldots i_s}$ and the denominators of $h_{ij}^{i_1 \ldots i_s}$ in $\kappa(\eta) = \mathrm{Quot}(A)$. This is a well-defined ideal, because these are just finitely many generators. Set $U' := \mathrm{Spec}(A_I)$. This is open in $U = \mathrm{Spec}(A)$. The I_i are defined over A_I, so the $\mathrm{Spec}(A_I[x_1, \ldots, x_m]_{I_i})$ form a cover of $\mathrm{Spec}(A_I[x_1, \ldots, x_m])$. Moreover, we have $g_{ij} \in B_i \otimes_A A_I$ and $f_{ij} \in A_I[x_1, \ldots, x_s]_{I_i}$. So we get

$$f_i : B_i \otimes_A A_I \to A_I[x_1, \ldots, x_m]_{I_i} \ ; \ g_i : A_I[x_1, \ldots, x_m]_{I_i} \to B_i \otimes_A A_I$$

with $f_i(b_{ij} \otimes 1) = f_{ij} \in A_I[x_1, \ldots x_m]_{I_i}$ and $g_i(x_j) = g_{ij} \in B_i \otimes_A A_I$. By construction $f_i \circ g_i = \mathrm{id}$ and $g_i \circ f_i = \mathrm{id}$, and the f_i and g_i still glue together. Hence we obtain the required isomorphism. □

4.4 Remarks on the Assumptions

One might wonder what happens if one weakens the assumptions made in this chapter. Here are some remarks concerning this question:

- It should be possible to replace R by a Henselian discrete valuation ring with algebraically closed residue field k, but this has not been checked this carefully. It seems as if this would not cause problems in Section 4.1 and Section 4.2. In Section 4.3, the special fiber of $\mathcal{Z}$ only depends on $\mathcal{Y} \times_T (T \times_S \operatorname{Spec}(k)) = \mathcal{Y} \times_T \operatorname{Spec}(R \otimes_{R^G} k)$. Let $\hat{R}$ be the completion of R. We should have that $\hat{R} \otimes_{\hat{R}^G} k \cong R \otimes_{R^G} k$. Therefore, it should be possible to replace $\mathcal{Y}$ by $\mathcal{Y} \times_T \operatorname{Spec}(\hat{R})$ for computations concerning the special fiber of $\mathcal{Z}$.

- It should be possible to weaken the assumptions on k, but one needs to check the details. Note that one has to assume at least that k contains all r-th roots of unity, because otherwise Lemma 2.14 does not hold, which we need to compute the special fiber of our specific weak Néron model in Lemma 4.15.

- If we do not assume that the considered G-action is tame, i. e. that the order of G is prime to $\operatorname{char}(k)$, we will run into trouble. For example Lemma 2.2 is wrong in this case, see Example 2.3. Therefore Lemma 2.14 does not hold, which is needed to compute the special fiber of $\mathcal{Z}$, see Lemma 4.15. Moreover, if the order of G is not prime to $\operatorname{char}(k)$, we cannot show in Proposition 4.2 that $\mathcal{Z}$ is smooth, because then [Edi92, Proposition 3.4] does not hold. As a consequence, we cannot use Construction 4.2 to obtain a weak Néron model.
  Anyway, it would be interesting to know what happens in the case of wild actions.

- If we do not assume that $\mathcal{Y}$ is quasi-projective, the Weil restriction might not be representable, so this assumption is necessary.

Chapter 5

Motivic Invariants

In this chapter we investigate the weak Néron model $\mathcal{Z}$ constructed and examined in the previous chapter, and discuss some motivic invariants, namely the Serre invariant and the rational volume, with respect to tame Galois extensions of local fields. These motivic invariants can be attached to a smooth variety over a local field having a weak Néron model.
Let K be a complete local field, and $\mathcal{O}_K$ its ring of integers, $S := \mathrm{Spec}(\mathcal{O}_K)$. Assume that the residue field k of $\mathcal{O}_K$ is algebraically closed.

5.1 Serre Invariant

Definition 5.1. The *Grothendieck group of k-varieties* $K_0(\mathrm{Var}_k)$ is defined to be the abelian group with

- generators: isomorphism classes $[U]$ of separated k-schemes U of finite type
- relations: $[U] = [U \setminus V] + [V]$ for every closed immersion $V \hookrightarrow U$ (scissor relations)

The product $[U][V] = [U \times_{\mathrm{Spec}(k)} V]$ defines a ring structure on $K_0(\mathrm{Var}_k)$. We call this ring the *Grothendieck ring of k-varieties.*
Set $\mathbb{L} := [\mathbb{A}^1_k]$.
The *modified Grothendieck ring of k-varieties* $K_0^{\mathrm{mod}}(\mathrm{Var}_k)$ is the quotient of $K_0(\mathrm{Var}_k)$ by the ideal $\mathcal{I}$ generated by elements

$$[U] - [V]$$

where U and V are separated k-schemes of finite type such that there exists a finite, surjective, purely inseparable k-morphism $U \to V$.
Set again $\mathbb{L} := [\mathbb{A}^1_k]$.

$$K_0^{\mathcal{O}_K}(\mathrm{Var}_k) := \begin{cases} K_0(\mathrm{Var}_k) & \text{if } \mathcal{O}_K \text{ has equal characteristic} \\ K_0^{\mathrm{mod}}(\mathrm{Var}_k) & \text{if } \mathcal{O}_K \text{ has mixed characteristic} \end{cases}$$

Definition 5.2. Let X be a smooth K-variety with weak Néron model $\mathcal{X} \to T$. Then the *motivic Serre invariant* $S(X)$ is defined by

$$S(X) := [\mathcal{X}_k] \in K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L} - 1)$$

with $\mathcal{X}_k$ the special fiber of $\mathcal{X} \to T$.

Fact. [NS11b, Proposition-Definition 3.6]
The motivic Serre invariant does not depend on the choice of a weak Néron model.

Remark 5.1. Let X be a smooth separated K-variety without K-rational point. Then $S(X) = 0$. This holds, because in this case X viewed as S-scheme is a weak Néron model of X, i. e. the special fiber of the weak Néron model is empty. So if $S(X) \neq 0$, X has a K-rational point.

Theorem 5.2. *Let X be a smooth projective K-variety. Let L/K be a tame Galois extension, $\mathcal{O}_L$ the ring of integers of L. Let $\mathcal{Y}$ be an integral, quasi-projective $\mathcal{O}_L$-scheme, and assume that the smooth locus of $\mathcal{Y} \to T := \mathrm{Spec}(\mathcal{O}_L)$, $\mathrm{Sm}(\mathcal{Y}/T)$, is a weak Néron model of $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. Let $G := \mathrm{Gal}(L/K)$ act on $\mathcal{Y}$, compatible with the Galois action on $\mathcal{O}_L$, and assume furthermore that $\mathcal{Y}/G \times_S \mathrm{Spec}(K) \cong X$. Then*

$$S(X) = [\mathrm{Sm}(\mathcal{Y}/T)^G] \in K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L} - 1)$$

By Theorem 4.11 we know that $\mathcal{Z} \to S$ as constructed in Construction 4.2 is a weak Néron model of X. Let $\mathcal{Z}_k := \mathcal{Z} \times_S \mathrm{Spec}(k)$ be the special fiber of $\mathcal{Z}$. Hence by definition we have

$$S(X) = [\mathcal{Z}_k] \in K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L} - 1)$$

It follows that Theorem 5.2 is equivalent to the following statement:

Theorem 5.3. *Assumptions as in Theorem 5.2, and let $\mathcal{Z}_k$ be the special fiber of $\mathcal{Z} \to S$ as constructed in Construction 4.2. Then*

$$[\mathcal{Z}_k] = [\mathrm{Sm}(\mathcal{Y}/T)^G] \in K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L} - 1)$$

Proof. As $K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L} - 1)$ is a quotient of $K_0(\mathrm{Var}_k)/(\mathbb{L} - 1)$, it suffices to show

$$[\mathcal{Z}_k] = [\mathrm{Sm}(\mathcal{Y}/T)^G] \in K_0(\mathrm{Var}_k)/(\mathbb{L} - 1)$$

Consider $b : \mathcal{Z}_k \to \mathrm{Sm}(\mathcal{Y}/T)^G$ as constructed in Lemma 4.14. We can find $U_i \subset \mathrm{Sm}(\mathcal{Y}/T)^G$ such that

$$\mathrm{Sm}(\mathcal{Y}/T)^G = U_1 \sqcup \cdots \sqcup U_m,$$

$U_i \subset \mathrm{Sm}(\mathcal{Y}/T)^G \setminus (\cap_{j<i} U_j)$ is open, and $b^{-1}(U_i) \cong \mathbb{A}_k^{m_i} \times_{\mathrm{Spec}(k)} U_i$ for some $m_i \in \mathbb{N}$, by proceeding in the following way: By Lemma 4.17 we have

$U_1 \subset \mathrm{Sm}(\mathcal{Y}/T)^G$ open with $b^{-1}(U_1) \cong \mathbb{A}_k^{m_1} \times_{\mathrm{Spec}(k)} U_i$ for some $m_1 \in \mathbb{N}$. Suppose we already constructed U_i. Then $V_i := \mathrm{Sm}(\mathcal{Y}/T)^G \setminus \cap_{j\leq i} U_j$ is closed in $\mathrm{Sm}(\mathcal{Y}/T)^G$, and we can again use Lemma 4.17 to get $U_{i+1} \subset V_i$ open such that $b^{-1}(U_{i+1}) \cong \mathbb{A}_k^{m_{i+1}} \times_{\mathrm{Spec}(k)} U_{i+1}$ for some $m_{i+1} \in \mathbb{N}$. As $\mathrm{Sm}(\mathcal{Y}/T)^G$ is of finite type over k this process has to terminate. This proves the claim. Note that

$$[\mathbb{A}_k^{m_i}] = [\mathbb{A}_k^1 \times_{\mathrm{Spec}(k)} \cdots \times_{\mathrm{Spec}(k)} \mathbb{A}_k^1] = \mathbb{L}^{m_i} = 1^{m_i} = 1 \in K_0(\mathrm{Var}_k)/(\mathbb{L}-1)$$

This implies that

$$[b^{-1}(U_i)] = [\mathbb{A}_k^{m_i} \times_{\mathrm{Spec}(k)} U_i] = [\mathbb{A}_k^{m_i}][U_i] = [U_i] \in K_0(Var_k)/(\mathbb{L}-1)$$

Note that as $U_i \subset \mathrm{Sm}(\mathcal{Y}/T)^G \setminus (\cap_{j<i} U_j) = \cap_{j\geq i} U_j$ is open, $b^{-1}(U_i)$ is open in $b^{-1}(\cap_{j\geq i} U_j) = \cap_{j\geq i} b^{-1}(U_j)$. So using the scissor relations in the Grothendieck ring of k-varieties we get in $K_0(\mathrm{Var}_k)/(\mathbb{L}-1)$:

$$\begin{aligned}
[\mathcal{Z}_k] = [b^{-1}(\mathrm{Sm}(\mathcal{Y}/T)^G)] &= [b^{-1}(U_1) \sqcup \cdots \sqcup b^{-1}(U_m)] \\
&= [b^{-1}(U_1)] + [b^{-1}(U_2) \sqcup \cdots \sqcup b^{-1}(U_m)] \\
&= \ldots \\
&= \sum_{i=1}^{m} [b^{-1}(U_i)] \\
&= \sum_{i=1}^{m} [U_i] \\
&= [U_1] + \cdots + [U_{m-2}] + [U_{m-1} \sqcup U_m] \\
&= \ldots \\
&= [U_1 \sqcup \cdots \sqcup U_m] \\
&= [\mathrm{Sm}(\mathcal{Y}/T)^G]
\end{aligned}$$

This proves Theorem 5.3, which is equivalent to Theorem 5.2. □

5.2 Rational Volume

Fact. [NS11a, Example 4.3 and Corollary 4.14]
There exists a unique ring morphism (realization morphism)

$$\chi_c : K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L}-1) \to \mathbb{Z}$$

that sends a class of a separated k-scheme U of finite type to

$$\chi_c(U) = \sum_{i\geq 0} (-1)^i \dim H_c^i(U, \mathbb{Q}_l),$$

with l prime to $\mathrm{char}(k)$, the Euler characteristic with proper support. The map does not depend on the choice of l.

Definition 5.3. Let X be a smooth K-variety with weak Néron model. Then the *rational volume* of X is defined as follows:

$$s(X) := \chi_c(S(X)) \in \mathbb{Z}$$

Remark 5.4. Let X be a smooth separated K-variety without K-rational point. Then $s(X) = 0$. This holds, because by Remark 5.1 $S(X) = 0$, so in particular $s(X) = \chi_c(S(X)) = 0$. So if $s(X) \neq 0$, X has a K-rational point.

Theorem 5.5. *Let X be a smooth projective K-variety, and let L/K be a tame Galois extension, such that $G := \mathrm{Gal}(L/K)$ is an q-group, $q \neq \mathrm{char}(k)$ a prime. Set $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. Then*

$$s(X_L) = s(X) \mod q$$

In particular, if $s(X_L)$ does not vanish modulo q, then X has a rational point.

Proof. As $q \neq \mathrm{char}(k)$, G is cyclic. Let $\mathcal{O}_L$ be the ring of integers of L, $T := \mathrm{Spec}(\mathcal{O}_L)$. By Theorem 1.30 there is a quasi-projective weak Néron model $\varphi : \mathcal{Y} \to T$ of X_L with a G-action on $\mathcal{Y}$, which is compatible with the Galois action of G on $\mathcal{O}_L$, and $\mathcal{Y}/G \times_S \mathrm{Spec}(K) \cong X$. So Theorem 5.2 implies that

$$S(X) = [\mathcal{Y}^G] \in K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L} - 1)$$

As φ is G-equivariant, the action of G on $\mathcal{Y}$ restricts to $\mathcal{Y}_k := \mathcal{Y} \times_T \mathrm{Spec}(k)$. By [EN11, Proposition 5.4], for every variety U over a field F with a good G-action, $\chi_c(U) = \chi_c(U^G) \mod q$ holds. This Proposition is based on an argument in [Ser09, Section 7.2]. In our case we get

$$\chi_c(\mathcal{Y}_k) = \chi_c(\mathcal{Y}_k^G) \mod q$$

As $\mathcal{Y}^G \subset \mathcal{Y}_k$, see Lemma 1.3, $\mathcal{Y}^G = \mathcal{Y}_k^G$. As $\mathcal{Y}$ is a weak Néron model of X_L, by definition $S(X_L) = [\mathcal{Y}_k] \in K_0^{\mathcal{O}_K}(\mathrm{Var}_k)$. So altogether we obtain

$$\begin{aligned} s(X_L) = \chi_c(S(X_L)) = \chi_c(\mathcal{Y}_k) &= \chi_c(\mathcal{Y}^G) && \mod q \\ &= \chi_c(S(X)) && \mod q \\ &= s(X) && \mod q \end{aligned}$$

Assume now that $s(X_L) \neq 0 \mod q$. This implies that $s(X) \neq 0$. But the rational volume of a smooth K-variety without K-rational point vanishes, see Remark 5.4, hence $X(K) \neq \emptyset$, i. e. X has a K-rational point. □

Chapter 6

The Existence of Rational Points on Certain Varieties with Potential Good Reduction

In this chapter we use the results proved so far to show that certain varieties over complete local fields with potential good reduction have rational points. Note that in the proof of Corollary 6.1 we use Theorem 5.5, in the proof of the other corollaries we use Theorem 3.6. Therefore Corollary 6.1 only holds for smooth projective varieties, whereas the other corollaries hold for smooth proper varieties.
Let K be a complete local field with ring of integers $\mathcal{O}_K$. Assume that the residue field k of $\mathcal{O}_K$ is algebraically closed.

Definition 6.1. A smooth proper K-variety X has *potential good reduction after a base change of order* r, if there exists a Galois extension L/K of degree r, such that $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ has a smooth and proper model $\mathcal{Y} \to T := \mathrm{Spec}(\mathcal{O}_L)$.

$$\begin{array}{ccccc} \mathcal{Y} & \longleftarrow & X_L & \longrightarrow & X \\ \downarrow & \square & \downarrow & \square & \downarrow \\ T & \longleftarrow & \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(K) \end{array}$$

Definition 6.2. Let U be a K-variety.

$$\chi(U) := \sum_{i \geq 0} (-1)^i \dim H^i(U \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^s), \mathbb{Q}_l)$$

with K^s a separable closure of K, l prime to $\mathrm{char}(k)$.

Corollary 6.1. *Let X be a smooth projective K-variety, which has potential good reduction after a base change of order q^r, with $q \neq char(k)$ a prime. Then*

$$\chi(X) = s(X) \mod q$$

In particular, if $\chi(X)$ does not vanish modulo q, then X has a K-rational point.

Proof. Let L/K be the field extension of degree q^r, such that there is a smooth and proper model of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. Let $\mathcal{O}_L$ be the ring of integers of L, $T := \mathrm{Spec}(\mathcal{O}_L)$, and $\varphi : \mathcal{Y} \to T$ a smooth and proper model of X_L. In particular, $\varphi : \mathcal{Y} \to T$ is a weak Néron model of X_L, so by definition $s(X_L) = \chi_c(\mathcal{Y}_k)$ for the special fiber $\mathcal{Y}_k := \mathcal{Y} \times_T \mathrm{Spec}(k)$. Note that φ is proper, and $\mathcal{Y}_k$ is proper over k, hence the ordinary cohomology coincides with the cohomology with proper support, i. e. $\chi_c(\mathcal{Y}_k) = \chi(\mathcal{Y}_k)$. As φ is proper and smooth, by [Del77, Exposé V, Theorem 3.1] we get bijections between $H^i(\mathcal{Y}_k, \mathbb{Z}/n\mathbb{Z})$, and $H^i(\mathcal{Y}, \mathbb{Z}/n\mathbb{Z})$, and $H^i(X_L \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L^s), \mathbb{Z}/n\mathbb{Z})$ for all i, with L^s a separable closure of L. Therefore we have for all i that

$$\dim H^i(\mathcal{Y}_k, \mathbb{Q}_l) = \dim H^i(X_L \times_{\mathrm{Spec}(L)} L^s, \mathbb{Q}_l)$$

Note that $L^s = K^s$ for a separable closure K^s of K, because L/K is a tame Galois extension. So $X_L \times_{\mathrm{Spec}(L)} \mathrm{Spec}(L^s) = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^s)$, and we obtain

$$\begin{aligned}\chi(X) &= \sum_{i \geq 0} (-1)^i \dim H^i(X \times_{\mathrm{Spec}(K)} K^s, \mathbb{Q}_l) \\ &= \sum_{i \geq 0} (-1)^i \dim H^i(\mathcal{Y}_k, \mathbb{Q}_l) = \chi(\mathcal{Y}_k)\end{aligned}$$

This implies that $s(X_L) = \chi(X)$. By Theorem 5.5, $s(X_L) = s(X) \mod q$, which yields the corollary. □

Corollary 6.2. *Let X be a smooth proper K-variety. Assume that there is a tame Galois extension L/K of order r, r prime to* char(k)*, with the property that there is a smooth proper model $\varphi : \mathcal{Y} \to T := \mathrm{Spec}(\mathcal{O}_L)$ of $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$, such that there is a good $G := \mathrm{Gal}(L/K)$-action on $\mathcal{Y}$, compatible with the Galois action on X_L.*
Then X has a K-rational point if and only if $\mathcal{Y}^G \neq \emptyset$.

Proof. Set $\mathcal{X} := \mathcal{Y}/G$. By Lemma 1.1, $\varphi_G : \mathcal{X} \to S := \mathrm{Spec}(\mathcal{O}_K)$ is proper. By Lemma 1.18, $\varphi_G : \mathcal{X} \to S$ is a model of X. By the valuative criterion of properness, X has a K-rational point if and only if φ_G has a section. By Theorem 3.6, φ_G has a section if and only if $\mathcal{Y}^G \neq \emptyset$. □

Remark 6.3. The properties in Corollary 6.2 imply in particular that X has potential good reduction after a base change of order order r.

Corollary 6.4. *Notation and assumptions as in Corollary 6.2, and assume furthermore that the order r of L/K is a prime. If $\chi(X, \mathcal{O}_X)$ does not vanish modulo r, then X has a K-rational point.*

Proof. By Corollary 6.2 it suffices to show that $\mathcal{Y}^G \neq \emptyset$.
Note that $H^i(X_K, \mathcal{O}_{X_L}) = H^i(X_K, \mathcal{O}_X \otimes_K L) = H^i(X, \mathcal{O}_X) \otimes_K L$ holds for all $i \geq 0$. The first equation holds, because $\mathrm{Spec}(L)$ is affine, and the second follows - as L is a flat K-algebra - with [Mum08, Chapter 5, Corollary 5]. So in particular $\chi(X_L, \mathcal{O}_{X_L}) = \chi(X, \mathcal{O}_X)$.
Now consider the smooth and proper model $\varphi : \mathcal{Y} \to T$ of X_L on which G acts. As φ is smooth and proper, and T is connected, by [Gro61, Theorem 7.9.4.I] the Euler characteristic is constant on the fibers of φ, so in particular $\chi(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}) = \chi(X_L, \mathcal{O}_{X_L})$, for the special fiber $\mathcal{Y}_k := \mathcal{Y} \times_T \mathrm{Spec}(k)$ of $\mathcal{Y} \to T$. Altogether, $\chi(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k})$ does not vanish modulo r.
By Lemma 1.18 G acts on $\mathcal{Y}$ such that φ is G-equivariant. As $\mathrm{Spec}(k) \subset T$ is fixed, the G-action on $\mathcal{Y}$ restricts to a G-action on $\mathcal{Y}_k$. Let $f : \mathcal{Y}_k \to \mathcal{Y}_k/G$ be the quotient.
Assume that the action of G on $\mathcal{Y}$ has no fixed point, i. e. that $\mathcal{Y}^G = \emptyset$. Then the same holds for the action of G on $\mathcal{Y}_k$, because by Lemma 1.3 $\mathcal{Y}^G \subset \mathcal{Y}_k$. As r is a prime, this implies that the action of G on $\mathcal{Y}_k$ is free. So f is a finite étale morphism of degree r by [Gro63, Exposé V, Corollaire 2.3]. Note that $\mathcal{Y}_k$ is smooth and proper over k, because these properties are stable under base change, and φ is smooth and proper. As f is étale and finite, $\mathcal{Y}_k/G$ is smooth and proper over k, too.
As f is étale, $f^*(T_{\mathcal{Y}_k/G}) = T_{\mathcal{Y}_k}$, and therefore $f^*(\mathrm{td}(T_{\mathcal{Y}_k/G})) = \mathrm{td}(T_{\mathcal{Y}_k})$. Let $s : \mathcal{Y}_k \to \mathrm{Spec}(k)$ and $s' : \mathcal{Y}_k/G \to \mathrm{Spec}(k)$ be the structure maps. Note that $s = s' \circ f$. Using [Ful98, Corollary 15.2.2], and the projection formula in the third line, we obtain:

$$\begin{aligned}\chi(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}) &= s_*(\mathrm{ch}(\mathcal{O}_{\mathcal{Y}_k})\,\mathrm{td}(T_{\mathcal{Y}_k})) \\ &= s'_*(f_*(\mathrm{ch}(\mathcal{O}_{\mathcal{Y}_k}) f^*(\mathrm{td}(T_{\mathcal{Y}_k/G})))) \\ &= s'_*(f_*(\mathrm{ch}(\mathcal{O}_{\mathcal{Y}_k}))\,\mathrm{td}(T_{\mathcal{Y}_k/G})) \\ &= s'_*(\deg(f)\,\mathrm{ch}(\mathcal{O}_{\mathcal{Y}_k/G})\,\mathrm{td}(T_{\mathcal{Y}_k/G})) = r\ \chi(\mathcal{Y}_k/G, \mathcal{O}_{\mathcal{Y}_k/G})\end{aligned}$$

So $\chi(X, \mathcal{O}_X) = \chi(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}) = 0 \mod r$. This is a contradiction, hence $\mathcal{Y}^G \neq \emptyset$, and by Corollary 6.2 X has a K-rational point. ☐

Remark 6.5. Notation and assumptions as in Corollary 6.4. If for all $i > 0$ $H^i(X, \mathcal{O}_X)$ vanishes, then X has a K-rational point.

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