Kottwitz-Rapoport and p-rank strata in the reduction of Shimura varieties of PEL type

Dissertation
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)

von
Dipl.-Math. Philipp Hartwig
aus Freiburg im Breisgau

vorgelegt bei der Fakultät für
Mathematik der Universität
Duisburg-Essen (Campus Essen)

Essen 2012

Betreuer
Prof. Dr. Ulrich Görtz

Gutachter
Prof. Dr. Ulrich Görtz
Prof. Dr. Michael Rapoport

Datum der mündlichen Prüfung
30. August 2012

Vorsitz
Prof. Dr. Patrizio Neff

Meiner Mutter

Danksagung

Es ist mir ein Bedürfnis, an dieser Stelle meinem Betreuer und prospektiven Doktorvater Professor Dr. Ulrich Görtz von ganzem Herzen zu danken. Vom ersten Tag an hat er mich in allen Belangen uneingeschränkt unterstützt, er stand mir stets mit Rat und Tat zur Seite, und es kann nicht überschätzt werden, wie bedeutsam es für mich war, zu jedem Zeitpunkt sein vollstes persönliches Interesse an meiner Arbeit und an meinen Fortschritten zu spüren. Wann immer ich ein Anliegen hatte, nahm er sich praktisch umgehend die Zeit für ein Treffen mit mir, bei welchem er sich dann mit bewundernswerter Geduld jede meiner auch noch so technischen Fragen anhörte, um mir anschließend in der Regel direkt die entscheidende Hilfestellung zu geben. Und auch auf dem Fahrrad oder beim Mittagessen hatte er immer ein offenes Ohr für die Dinge und Fragen, die mir gerade durch den Kopf gingen. Dieses nie versiegende Interesse seinerseits hat mich stets in besonderem Maße motiviert und mir die Arbeit an meiner Dissertation auch dann zur Freude gemacht, wenn es gerade einmal nicht voranging. Und selbstredend war er es, der es mir insbesondere durch seine immer hilfreichen Erklärungen überhaupt erst ermöglicht hat, in diesem faszinierenden, aber durchaus schwierigen Gebiet der Mathematik einen Fuß auf den Boden zu bekommen. Herzlichen Dank für alles, ich hätte mir wahrlich keinen besseren Betreuer wünschen können!

Es war Professor Dr. Torsten Wedhorn, der mich durch eine Bemerkung während eines Aufenthaltes in Essen auf die Idee gebracht hat, den p-Rang mit der Newton-Abbildung in Verbindung zu bringen. Dafür will ich ihm hiermit danken. Auch danke ich Professor Dr. Michael Rapoport für hilfreiche Kommentare, und für all die Mathematik, die er mir insbesondere während meines Studiums in Bonn beigebracht hat.

Dass mir meine Zeit in Essen ohne Zweifel positiv in Erinnerung bleiben wird, ist nicht zuletzt der Verdienst der anderen Mitgliedern des Essener Seminars für Algebraische Geometrie und Arithmetik. Die fast schon familiäre Atmosphäre in den Seminaren und beim traditionellen gemeinsamen Mittagessen und Kaffeetrinken hat dafür gesorgt, dass ich mich an der Universität immer wohlgefühlt habe.

Allen voran bin ich Dr. Christian Kappen, Dr. Martin Kreidl und Dr. Ulrich Terstiege (freilich in keiner besonderen Reihenfolge) von Herzen dankbar. Sie sind während meiner Zeit in Essen zu wahren Freunden geworden, die das Leben auch außerhalb der Universität lebenswert gemacht haben. Danke für die vielen langen Gespräche, die wir geführt haben, für die gemeinsamen Unternehmungen zu jeder Tages- und Nachtzeit, und danke insbesondere für die vielen Einladungen und die großartigen Abende, die daraus entstanden sind.

Natürlich habe ich auch von so mancher mathematischer Diskussion mit Freunden und Kollegen profitiert, explizit will ich Dr. Martin Kreidl für einige Erklärungen zu Gittern über Ringen formaler Potenzreihen danken.

Nicht unerwähnt bleiben soll, dass mich die Universität Duisburg-Essen für knapp zwei Jahre mit einem wahrlich großzügigen Promotionsstipendium finanziert hat. Anschließend durfte ich mich glücklich schätzen, eine Stelle im Rahmen des SFB/Transregio 45, „Modulräume, Perioden und Arithmetik algebraischer Varietäten“, innezuhaben. Durch diese beiden Geldgeber wurde mir das Studium überhaupt erst ermöglicht und dafür danke ich hiermit allen Verantwortlichen.

Die abschließenden Worte gebühren meiner Mutter, die insbesondere während all der Jahre meines Studiums stets an meiner Seite stand und mich immer vorbehaltlos in allem unterstützt hat, was ich mir vorgenommen habe. Danke für alles!

KOTTWITZ-RAPOPORT AND p-RANK STRATA IN THE REDUCTION OF SHIMURA VARIETIES OF PEL TYPE

PHILIPP HARTWIG

Contents

1. Introduction

Fix a rational prime $p \neq 2$ and a PEL datum $\mathcal{B} = (B, *, V, (\cdot,\cdot), J)$ with auxiliary data $\mathcal{B}_p = (\mathcal{O}_B, \mathcal{L})$, see Section 3.1. The datum $\mathcal{B}$ gives rise to a reductive group G over $\mathbb{Q}$ and a conjugacy class h of homomorphisms $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$. Fix a compact open subgroup $C^p \subset G(\mathbb{A}_f^p)$. From C^p and $\mathcal{B}_p$ one obtains a compact open subgroup $C \subset G(\mathbb{A}_f)$ and thus a Shimura datum (G, h, C). In [32, Section 6], Rapoport and Zink construct from $\mathcal{B}$, $\mathcal{B}_p$ and C^p an integral model $\mathcal{A}_{C^p}$ of the Shimura variety associated with (G, h, C). Concretely $\mathcal{A}_{C^p}$ is defined as a moduli space of abelian schemes with additional structure.

In order to study properties of the scheme $\mathcal{A}_{C^p}$, Rapoport and Zink introduce the so-called local model M^{loc}. It is defined purely in terms of linear algebra and therefore easier to investigate than $\mathcal{A}_{C^p}$. The schemes $\mathcal{A}_{C^p}$ and M^{loc} are related via an intermediate object $\widetilde{\mathcal{A}}_{C^p}$ fitting into the so called *local model diagram*

$$\begin{tikzcd} & \widetilde{\mathcal{A}}_{C^p} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\ \mathcal{A}_{C^p} & & M^{\mathrm{loc}}. \end{tikzcd}$$

Étale locally on $\mathcal{A}_{C^p}$, there is a section $s : \mathcal{A}_{C^p} \to \widetilde{\mathcal{A}}_{C^p}$ of $\widetilde{\varphi}$ such that the composition $\mathcal{A}_{C^p} \xrightarrow{s} \widetilde{\mathcal{A}}_{C^p} \xrightarrow{\widetilde{\psi}} M^{\mathrm{loc}}$ is étale. Consequently $\mathcal{A}_{C^p}$ inherits any property from M^{loc} which is local for the étale topology. In particular, questions about singularities of $\mathcal{A}_{C^p}$ or the flatness of $\mathcal{A}_{C^p}$ can equivalently be studied for M^{loc}. Let us mention that recently a purely group-theoretic definition of the local model was given in [30] by Pappas and Zhu. This provides an intriguing new perspective on the local model diagram.

The PEL datum also gives rise to an affine smooth group scheme $\operatorname{Aut}(\mathcal{L})$, and $\operatorname{Aut}(\mathcal{L})$ acts on both $\tilde{A}$ and M^{loc}. The map $\widetilde{\varphi}$ is an $\operatorname{Aut}(\mathcal{L})$-torsor, while the map $\widetilde{\psi}$ is $\operatorname{Aut}(\mathcal{L})$-equivariant. Denote by $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$. Via the local model diagram, the decomposition of $M^{\mathrm{loc}}(\mathbb{F})$ into $\operatorname{Aut}(\mathcal{L})(\mathbb{F})$-orbits induces the *Kottwitz-Rapoport* (or KR) *stratification*

$$\mathcal{A}_{C^p}(\mathbb{F}) = \coprod_{x \in \operatorname{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_{C^p,x},$$

which was first introduced in [24]. The $\operatorname{Aut}(\mathcal{L})(\mathbb{F})$-orbits on $M^{\mathrm{loc}}(\mathbb{F})$ admit the following interesting description. In all cases considered thus far (cf. the discussion in [29, §3.3]), the special fiber $M^{\mathrm{loc}}_{\mathbb{F}}$ of M^{loc} embeds into the so-called *affine flag variety* $\mathcal{F}$, which is defined as a moduli space of lattice chains over the ring of formal power series $\mathbb{F}[\![u]\!]$. In analogy with the Bruhat decomposition of the classical flag variety, indexed by the finite Weyl group W, the affine flag variety admits the *Iwahori decomposition* $\mathcal{F}(\mathbb{F}) = \coprod_{x \in \widetilde{W}} \mathcal{F}_x$ into Schubert cells $\mathcal{F}_x$, indexed by the extended affine Weyl group $\widetilde{W}$. It then turns out that $M^{\mathrm{loc}}_{\mathbb{F}} \subset \mathcal{F}$ is a disjoint union of Schubert cells and that the decomposition $M^{\mathrm{loc}}(\mathbb{F}) = \coprod_{\mathcal{F}_x \subset M^{\mathrm{loc}}(\mathbb{F})} \mathcal{F}_x$ coincides with the decomposition of $M^{\mathrm{loc}}(\mathbb{F})$ into $\operatorname{Aut}(\mathcal{L})(\mathbb{F})$-orbits. As in the case of the

Bruhat decomposition, many properties of the Iwahori decomposition are easily expressed by combinatorial properties of the corresponding index element in $\widetilde{W}$. Notably, the dimension of $\mathcal{F}_x$ is given by the length $\ell(x)$ of x in $\widetilde{W}$, and the closure relation between Schubert cells is expressed by the Bruhat order on $\widetilde{W}$. We conclude that the same statements hold for the KR stratification on $\mathcal{A}_{C^p}(\mathbb{F})$.

Let us explain in detail one case in which this convenient combinatorial behavior of the KR stratification was fruitfully exploited. For $B = \mathbb{Q}$ and a complete lattice chain $\mathcal{L}$, the moduli problem $\mathcal{A}_{C^p}$ specializes to the Siegel moduli space $\mathcal{A}_I$ of principally polarized abelian varieties with Iwahori level structure. In [8], Görtz and Yu compute the dimension of the p-rank 0 locus in $\mathcal{A}_I$, and this computation was later generalized in [14] by Hamacher to the case of all p-rank strata. The method is the same in both cases: Determine all KR strata contained in a given p-rank stratum and compute the maximum of their dimensions. For this method to work one of course needs to know that a p-rank stratum is indeed the union of the KR strata contained in it. Thus both papers depend crucially on the result [24, Théorème 4.1] of Ngô and Genestier, which states that indeed the p-rank is constant on a KR stratum in $\mathcal{A}_I$, and also provides an explicit formula for the p-rank on a given KR stratum.

The subject of this paper is to generalize the result of Ngô and Genestier on the relationship between the KR and the p-rank stratification to more general PEL data. Let us give an outline of the structure of this paper and of the results that we have obtained.

In Section 2 we collect several facts that will be needed in the sequel. On the one hand we do this for fixing our notation and providing the reader with the necessary references. On the other hand there are quite a few statements which are "standard" or "well-known" and are used without further comment in the existing literature, but for which we were unable to find a suitable reference. We have decided to include complete proofs of these statements, even when they are elementary and/or easy.

The same remark also applies to Sections 3.1 through 3.3, where we recall the construction of the local model diagram. We then proceed to show the following result.

Theorem 1.0.1. *Let $\mathcal{B}$ be an arbitrary PEL datum. If $\mathcal{L}$ is* complete *(in the sense of Definition 3.5.1), the p-rank is constant on a KR stratum.*

In order to obtain a formula for the p-rank on a given KR stratum, we have to make the rather mild assumption that the group $\mathrm{GL}_{B\otimes\mathbb{Q}_p}(V \otimes \mathbb{Q}_p)$, considered as a reductive group over $\mathbb{Q}_p$, is quasi-split. We start by embedding the special fiber $M^{\mathrm{loc}}(\mathbb{F})$ into a (set-theoretic) mixed-characteristic analogue $\mathcal{F}$ of the affine flag variety. Concretely, $\mathcal{F}$ is defined as a set of self-dual lattice chains over the Witt ring $\mathcal{O}_K = W(\mathbb{F})$. Denote by K the fraction field of $\mathcal{O}_K$ and by σ the Frobenius on K. It turns out that the group $G(K)$ acts transitively on $\mathcal{F}$ and we denote by I the stabilizer of $\mathcal{L} \otimes \mathcal{O}_K$ for this action. Consequently we can identify $\mathcal{F}$ with the quotient $G(K)/I$ and thereby consider $M^{\mathrm{loc}}(\mathbb{F})$ as a subset of $G(K)/I$. On $G(K)/I$ we have the canonical left action of I. We show that $M^{\mathrm{loc}}(\mathbb{F})$ is I-stable, and that the I-orbits and the $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-orbits on $M^{\mathrm{loc}}(\mathbb{F})$ coincide. In particular we obtain an embedding $\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow I\backslash G(K)/I$ of the index set of the KR stratification into $I\backslash G(K)/I$.

Before being able to state our next result, we need some more notation. Denote by $\mathbb{D}$ the diagonalizable affine group with character group $\mathbb{Q}$ over K. For $g \in G(K)$ denote by $\nu_g : \mathbb{D} \to G_K$ the corresponding Newton map. By definition, the group G_K acts on V_K and thus ν_g gives rise to a representation of $\mathbb{D}$ on V_K. Consider the corresponding weight decomposition $V_K = \oplus_{\chi\in\mathbb{Q}} V_\chi$ and define

$$\nu_{g,0} := \dim_K V_0.$$

Also recall for $g \in G(K)$ and $x \in I\backslash G(K)/I$ the definition $X_x(g) := \{y \in G(K)/I \mid y^{-1}g\sigma(y) \in IxI\}$ of the affine Deligne-Lusztig variety associated with g and x.

Theorem 1.0.2. *Let $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F}) \subset I\backslash G(K)/I$ and let $g \in G(K)$. Assume that $X_x(g) \neq \emptyset$. Then the p-rank on $\mathcal{A}_{C^p,x}$ is equal to $\nu_{g,0}$.*

In the remaining subsections of Section 3, we then explain how the number $\nu_{g,0}$ can be computed in practice and thereby obtain a more concrete formula for the p-rank on a KR stratum.

In Sections 5 through 8 we turn to the aforementioned interpretation of the KR stratification in terms of the affine flag variety. Section 5 deals with the case of the symplectic group. Section 6 (resp. 7, resp. 8) deals with the case of a unitary group associated with a ramified (resp. inert, resp. split) quadratic extension. Let us note that the embedding of $M^{\mathrm{loc}}_{\mathbb{F}}$ into the affine flag variety has a long history and that most of our discussion has already appeared elsewhere. In particular we want to emphasize that we have greatly profited from the expositions by Pappas and Rapoport in [26], [27], [28], and by Smithling in [37], [36]. We justify the seeming duplication of material by the fact that we include complete proofs of all the statements.

Our discussion is quite similar in all cases. We begin with describing in detail the PEL datum at hand, including the Hodge decomposition and the resulting determinant morphism. We proceed by making explicit the definition of the local model and investigate its base-change to $\mathbb{F}$. We then recall the definition of the affine flag variety in terms of lattice chains and prove in detail that it can also be described as a suitable quotient of loop groups. We conclude the discussion of the local model by embedding it into the affine flag variety and prove that the $\mathrm{Aut}(\mathcal{L})$-orbits on $M^{\mathrm{loc}}(\mathbb{F})$ are precisely the Schubert cells contained in $M^{\mathrm{loc}}(\mathbb{F})$. Turning to the moduli problem $\mathcal{A}_{C^p}$, we make it explicit, and then state the formula for the p-rank on a KR stratum in the special case at hand.

To illustrate our results, we look in Section 5 at the case of the Hilbert-Blumenthal modular varieties. Without any additional work, we obtain the following result.

Theorem 1.0.3. *Let $g \geq 2$ and let $\mathcal{A}_{C^p}$ be the Hilbert-Blumenthal modular variety associated with a totally real extension of degree g of $\mathbb{Q}$. Denote by $\mathcal{A}^{(0)}_{C^p} \subset \mathcal{A}_{C^p,\mathbb{F}}$ and $\mathcal{A}^{(g)}_{C^p} \subset \mathcal{A}_{C^p,\mathbb{F}}$ the subsets where the p-rank of the underlying abelian variety is equal to 0 and g, respectively. Then*

$$\mathcal{A}_{C^p,\mathbb{F}} = \mathcal{A}^{(0)}_{C^p} \amalg \mathcal{A}^{(g)}_{C^p}.$$

$\mathcal{A}^{(g)}_{C^p}$ is the union of only two KR strata $\mathcal{A}_{C^p,x_1}$ and $\mathcal{A}_{C^p,x_2}$. Consequently we have

$$\mathcal{A}_{C^p,\mathbb{F}} = \overline{\mathcal{A}}_{C^p,x_1} \cup \overline{\mathcal{A}}_{C^p,x_2} \cup \mathcal{A}^{(0)}_{C^p}.$$

Here $\overline{\mathcal{A}}_{C^p,x}$ denotes the closure of the KR stratum $\mathcal{A}_{C^p,x}$ in $\mathcal{A}_{C^p,\mathbb{F}}$.

Each of $\mathcal{A}_{C^p,\mathbb{F}}, \overline{\mathcal{A}}_{C^p,x_1}, \overline{\mathcal{A}}_{C^p,x_2}$ and $\mathcal{A}^{(0)}_{C^p}$ is equidimensional of dimension 2^g. We conclude that the ordinary locus $\mathcal{A}^{(g)}_{C^p}$ is not dense in $\mathcal{A}_{C^p,\mathbb{F}}$.

Furthermore, we have

$$\overline{\mathcal{A}}_{C^p,x_1} \cap \overline{\mathcal{A}}_{C^p,x_2} \subset \mathcal{A}^{(0)}_{C^p}.$$

Taking $g = 2$, we recover the result [40, Theorem 2 (p. 408)] of Stamm.

As a second application, we obtain in Section 8, by copying the approach of Görtz and Yu mentioned above, the following result.

Theorem 1.0.4. *Assume that G is the unitary group of signature $(r, n-r)$ associated with an imaginary quadratic extension of $\mathbb{Q}$ in which p splits. Denote by $\mathcal{A}^{(0)}_{C^p} \subset \mathcal{A}_{C^p}(\mathbb{F})$ the subset where the p-rank of the underlying abelian variety is equal to 0. Then*

$$\dim \mathcal{A}^{(0)}_{C^p} = \min\big((r-1)(n-r), r(n-r-1)\big).$$

In the appendix we include a discussion of isogenies of abelian schemes over arbitrary bases, proving the results needed in the definition of the moduli problem $\mathcal{A}_{C^p}$.

2. Preliminaries

We fix once and for all a rational prime $p \neq 2$ and an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Let $n \in \mathbb{N}_{\geq 1}$.

2.1. **Some notation.** For elements $x_1, \dots, x_n$ of some set and $k_1, \dots k_n \in \mathbb{N}$, we denote by $(x_1^{(k_1)}, \dots, x_n^{(k_n)})$ the tuple

$$(\underbrace{x_1, \dots, x_1}_{k_1\text{-times}}, \dots, \underbrace{x_n, \dots, x_n}_{k_n\text{-times}}).$$

For a tuple $x = (x_1, \dots, x_n)$, we denote by $x(i)$ its i-th entry x_i.

2.1.1. *Modules and base change.* Let R be a ring and let M be an R-module. We denote by $M^\vee = M^{\vee,R} := \mathrm{Hom}_R(M, R)$ the dual of M.

Let $\varphi : R \to R'$ be an R-algebra. We will often write $M_{R'}$ instead of $M \otimes_R R'$. Similarly for morphisms or bilinear maps of R-modules. If M is free with basis $\mathfrak{B}$, we denote by $\mathfrak{B}_{R'}$ the basis $(b \otimes 1)_{b \in \mathfrak{B}}$ of $M_{R'}$ over R'. If there is no risk of confusion we write simply $\mathfrak{B}$ instead of $\mathfrak{B}_{R'}$.

If F is a functor on the category of R-algebras, we denote by $F \otimes_R R'$ or simply $F_{R'}$ the induced functor on the category of R'-algebras, given on objects by $F_{R'}(R' \to S) = F(R \xrightarrow{\varphi} R' \to S)$. Similarly for morphisms of functors. If there is no risk of confusion we write simply F instead of $F_{R'}$.

If G is a functor on the category of R'-algebras, we denote by $\mathrm{Res}_{R'/R}\, G$ the functor on the category of R-algebras given on objects by $(\mathrm{Res}_{R'/R}\, G)(S) = G(S \otimes_R R')$.

If $f \in R[T_1, \dots, T_n]$ is a polynomial, we denote by $f^\varphi \in R'[T_1, \dots, T_n]$ the polynomial obtained by applying φ to the coefficients of f.

Let S be an R-algebra with an R-linear involution $*$ and let $\langle\cdot,\cdot\rangle : M\times N \to L$ be a $*$-sesquilinear form of S-modules (i.e. $\langle am, bn\rangle = ab^* \langle m,n\rangle$ for $m \in M, n \in N$ and $a, b \in S$). For S-submodules $M_0 \subset M$ and $N_0 \subset N$, we write

$$M_0^{\perp} = M_0^{\perp,\langle\cdot,\cdot\rangle} := \{n \in N \mid \forall m \in M_0 :\ \langle m,n\rangle = 0\},$$
$$N_0^{\perp} = N_0^{\perp,\langle\cdot,\cdot\rangle} := \{m \in M \mid \forall n \in N_0 :\ \langle m,n\rangle = 0\}.$$

Note that for an R-algebra R', we obtain by base-change a $*_{R'}$-sesquilinear form $\langle\cdot,\cdot\rangle_{R'} : M_{R'} \times N_{R'} \to L_{R'}$ of $S_{R'}$-modules.

2.1.2. *The general linear group.* For a ring R, we denote by $\mathrm{GL}_n(R)$ the group of invertible $(n \times n)$-matrices over R and by $D_n(R) \subset B(R)$ its subgroups of diagonal and upper triangular matrices, respectively. We consider D_n, B and GL_n as functors on the category of rings. We denote by $\mathbb{G}_m = \mathrm{GL}_1$ the multiplicative group.

If A is a (not necessarily commutative) ring and V is a left A-module, we will also denote by $\mathrm{GL}_A(V)$ the group of invertible A-linear maps $V \to V$.

Denote by S_n the symmetric group on n letters. For $w \in S_n$ we denote by $A_w \in \mathrm{GL}_n(R)$ the matrix with $(A_w)_{ij} = \delta_{iw(j)}$. The map

$$S_n \to \mathrm{GL}_n(R), \quad w \mapsto A_w \tag{2.1.1}$$

is a group homomorphism.

Finally we introduce the matrices $\widetilde{I}_n \in \mathrm{GL}_n(R)$ and $\widetilde{J}_{2n} \in \mathrm{GL}_{2n}(R)$ given by

$$\widetilde{I}_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \quad \text{and} \quad \widetilde{J}_{2n} = \begin{pmatrix} 0 & \widetilde{I}_n \\ -\widetilde{I}_n & 0 \end{pmatrix}.$$

2.1.3. *The symplectic group.* Let R be a $\mathbb{Z}[\frac{1}{2}]$-algebra. The *standard symplectic form* on R^{2n} is the bilinear form $(\cdot,\cdot) : R^{2n} \times R^{2n} \to R$ described by the matrix $\widetilde{J}_{2n}$ with respect to the standard basis $(e_1,\dots,e_{2n})$ of R^{2n}.

Denote by $\mathrm{Sp} = \mathrm{Sp}_{2n}$ and $\mathrm{GSp} = \mathrm{GSp}_{2n}$ the functors on the category of $\mathbb{Z}[\frac{1}{2}]$-algebras with

$$\mathrm{Sp}(R) = \{g \in \mathrm{GL}_{2n}(R) \mid \forall x, y \in R^{2n} : (gx, gy) = (x,y)\}$$

and

$$\mathrm{GSp}(R) = \{g \in \mathrm{GL}_{2n}(R) \mid \exists c = c(g) \in R^{\times} \forall x, y \in R^{2n} : (gx, gy) = c\,(x,y)\}.$$

For $g \in \mathrm{GSp}(R)$, the scalar $c(g) \in R^{\times}$ is called the *factor of similitude* of g.

2.1.4. *The unitary group.* Let K/K_0 be a quadratic extension of fields of characteristic not equal to 2. Denote by $*$ the non-trivial element of the Galois group $\mathrm{Gal}(K/K_0)$. The *standard hermitian form* on K^n is the $*$-hermitian form $\langle\cdot,\cdot\rangle : K^n \times K^n \to K$ described by the matrix $\widetilde{I}_n$ with respect to the standard basis $(e_1,\dots,e_n)$ of K^n.

Denote by $\mathrm{U} = \mathrm{U}_n$ and $\mathrm{GU} = \mathrm{GU}_n$ the functors on the category of K_0-algebras with

$$\mathrm{U}(R) = \{g \in \mathrm{GL}_n(K \otimes_{K_0} R) \mid \forall x, y \in (K \otimes_{K_0} R)^n : \langle gx, gy\rangle_R = \langle x,y\rangle_R\}$$

and

$$\mathrm{GU}(R) = \left\{ g \in \mathrm{GL}_n(K \otimes_{K_0} R) \mid \exists c = c(g) \in R^\times \begin{pmatrix} \forall x, y \in (K \otimes_{K_0} R)^n \\ \langle gx, gy \rangle_R = c \, \langle x, y \rangle_R \end{pmatrix} \right\}.$$

For $g \in \mathrm{GU}(R)$, the scalar $c(g) \in R^\times$ is called the *factor of similitude* of g.

2.2. **Morphisms of affine spaces.** Let R be a ring. We denote by $\mathbb{A}^n_R$ the affine space of dimension n over R. We identify a morphism $\alpha : \mathbb{A}^n_R \to \mathbb{A}^1_R$ with the corresponding natural transformation $\alpha(S) : S^n \to S$ of functors on the category of R-algebras S. By Yoneda's lemma, the following map is a bijection.

$$\begin{aligned} F_R : \mathrm{Hom}(\mathbb{A}^n_R, \mathbb{A}^1_R) &\xrightarrow{\sim} R[T_1, \dots, T_n], \\ \alpha &\mapsto \alpha(R[T_1, \dots, T_n])(T_1, \dots, T_n). \end{aligned} \tag{2.2.1}$$

For any R-algebra S and any $\underline{s} \in S^n$, one has $\alpha(S)(\underline{s}) = F_R(\alpha)(\underline{s})$, and this equality characterizes $F_R(\alpha)$.

If $\varphi : R \to R'$ is an R-algebra, one has $F_{R'}(\alpha_{R'}) = (F_R(\alpha))^\varphi$. Thus if $\varphi : R \subset R'$ is injective, a morphism $\beta' : \mathbb{A}^n_{R'} \to \mathbb{A}^1_{R'}$ is defined over R if and only if $F_{R'}(\beta')$ has coefficients in R, and there is at most one morphism $\beta : \mathbb{A}^n_R \to \mathbb{A}^1_R$ with $\beta_{R'} = \beta'$.

Proposition 2.2.2. *Let $K \subset L$ be infinite fields and let $\alpha : \mathbb{A}^n_L \to \mathbb{A}^1_L$ be a morphism. Then α is defined over K if and only if $\alpha(L)(K^n) \subset K$.*

Proof. We need to show that $F_L(\alpha)$ has coefficients in K. This is achieved by the next lemma. $\square$

Lemma 2.2.3. *Let $K \subset L$ be infinite fields and let $f \in L[T_1, \dots, T_n]$ be a polynomial. Assume that $f(K^n) \subset K$. Then f has coefficients in K.*

Proof. By induction on n. Let $n = 1$ and say $f = \sum_{i=0}^m a_i T_1^i$, where $m \in \mathbb{N}$ and $a_0, \dots, a_m \in L$. Let $x_0, \dots, x_m \in K$. We have an equality

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^m \\ 1 & x_1 & x_1^2 & \dots & x_1^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^m \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{pmatrix}.$$

If the x_i are pairwise different, the matrix on the left is invertible and we can deduce that $(a_0, \dots, a_m) \in K^{m+1}$.

Now let $n \geq 2$. There exist $m \in \mathbb{N}$ and polynomials $a_0, \dots, a_m \in L[T_1, \dots, T_{n-1}]$ such that $f = \sum_{i=0}^m a_i T_n^i$. We need to show that $a_i \in K[T_1, \dots, T_{n-1}]$ for every i. By the induction hypothesis, it suffices to show that $a_i(K^{n-1}) \subset K$, so let $(y_1, \dots, y_{n-1}) \in K^{n-1}$. Then

$$g = \sum_{i=0}^m a_i(y_1, \dots, y_{n-1}) T_n^i$$

lies in $L[T_n]$ and satisfies $g(K) \subset K$. By the case $n = 1$ treated above, we deduce that g has coefficients in K, as desired. $\square$

2.3. **Determinants.** Let R be a ring, M a finite locally free R-module and $\alpha : M \to M$ an endomorphism. We denote by $\mathrm{tr}_R(\alpha)$ the trace, by $\det_R(\alpha)$ the determinant and by $\chi_R(\alpha) = \det_{R[T]}(T \cdot \mathrm{id}_{M\otimes_R R[T]} - \alpha_{R[T]})$ the characteristic polynomial of α.

Let R be a ring, A a (not necessarily commutative) R-algebra and let M be a left A-module which is finite locally free as an R-module. For $x \in A$, we write

$$\det_R(x|M) = \det_R(M \xrightarrow{x\cdot} M),$$
$$\chi_R(x|M) = \chi_R(M \xrightarrow{x\cdot} M).$$

Denote by $V = V_A$ the functor on the category of R-algebras with $V(S) = A \otimes_R S$. We define a morphism $\det_{M,A} = \det_M : V \to \mathbb{A}^1_R$ on S-valued points by

$$A \otimes_R S \to S, \quad x \mapsto \det_S(x|M_S).$$

If A is finite free over R, the choice of a basis $(a_1, \dots, a_n)$ of A over R provides us with a functorial isomorphism

$$S^n \xrightarrow{\sim} A \otimes_R S, \quad (x_1, \dots, x_n) \mapsto a_1 \otimes x_1 + \dots a_n \otimes x_n$$

in the R-algebra S. Thus V is representable by $\mathbb{A}^n_R$, and the morphism $\det_M$ corresponds under the bijection (2.2.1) to the polynomial

$$\det_{R[T_1,\dots,T_n]}(a_1 \otimes T_1 + \dots + a_n \otimes T_n | M \otimes R[T_1, \dots, T_n]).$$

Remark 2.3.1. *Let N be a second left A-module which is finite locally free as an R-module. Consider the condition*

$$\det_M = \det_N . \tag{2.3.2}$$

Let us note that (2.3.2) *holds if M and N are isomorphic as A-modules.*

On the other hand, consider the condition

$$\forall x \in A : \det_R(x|M) = \det_R(x|N). \tag{2.3.3}$$

We note that in general condition (2.3.2) *is stronger than condition* (2.3.3), *as can for example be seen by taking $A = R = \mathbb{F}_p$, $M = \mathbb{F}_p$ and $N = (\mathbb{F}_p)^p$.*

Lemma 2.3.4. *Let A be a (not necessarily commutative) R-algebra and let M and N be A-modules which are finite locally free as R-modules. Assume that $\det_M = \det_N$. Then $\mathrm{rk}_R\, M = \mathrm{rk}_R\, N$.*

Proof. For $f \in R$, we have $\det_{M_f, A_f} = \det_M \otimes_R R_f$, and similarly for N. Consequently we may assume that M and N are free over R. The equality $\det_M = \det_N$ implies in particular that we have an equality

$$\det_{R[T]}(T|M \otimes_R R[T]) = \det_{R[T]}(T|N \otimes_R R[T]).$$

The left-hand side is equal to $T^{\mathrm{rk}_R M}$, and the right-hand side is equal to $T^{\mathrm{rk}_R N}$. $\square$

Proposition 2.3.5. *Let B be a finite-dimensional (not necessarily commutative) $\mathbb{Q}$-algebra and let $\mathcal{O}_B \subset B$ be a $\mathbb{Z}$-order. Let V be a finitely generated left $B \otimes_{\mathbb{Q}} \mathbb{C}$-module.*

(1) The morphism $\det_V : V_{B\otimes_{\mathbb{Q}}\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ *is defined over the ring of integers* $\mathcal{O}_E$ *of the number field*

$$E = \mathbb{Q}(\mathrm{tr}_{\mathbb{C}}(b\otimes 1|V);\ b\in B).$$

(2) Let R *be an* $\mathcal{O}_E$*-algebra and let* M *be a* $\mathcal{O}_B\otimes R$ *modules which is finite locally free over* R*. Assume that* $\det_M = \det_V \otimes_{\mathcal{O}_E} R$*. Then* $\mathrm{rk}_R M = \dim_{\mathbb{C}} V$*.*

Proof. The $B\otimes_{\mathbb{Q}}\mathbb{C}$-module structure on V is given by a morphism of $\mathbb{Q}$-algebras $\varphi : B \to \mathrm{End}_{\mathbb{C}}(V)$. By choosing a basis of V over $\mathbb{C}$, we can identify $\mathrm{End}_{\mathbb{C}}(V)$ with $M^{n\times n}(\mathbb{C})$ for some $n\in\mathbb{N}$. As B is finite-dimensional over $\mathbb{Q}$, there is a finite field extension $E'/\mathbb{Q}$ such that φ factors as $B \xrightarrow{\varphi_0} M^{n\times n}(E') \hookrightarrow M^{n\times n}(\mathbb{C})$. Set $V_0 = E'^n$ with the $B\otimes_{\mathbb{Q}} E'$-module structure given by φ_0. Note that $V_0\otimes_{E'}\mathbb{C}$ is isomorphic to V as a $B\otimes_{\mathbb{Q}}\mathbb{C}$-module.

Let $\mathfrak{E}$ be a basis of V_0 over E' and let Λ_0 be the $\mathcal{O}_B\otimes_{\mathbb{Z}}\mathcal{O}_{E'}$-module generated by $\mathfrak{E}$. Then $\Lambda_0\otimes_{\mathcal{O}_{E'}} E'$ is isomorphic to V_0 as a $B\otimes_{\mathbb{Q}} E'$-module. Clearly Λ_0 is finite and torsion-free over $\mathcal{O}_{E'}$, and as $\mathcal{O}_{E'}$ is a Dedekind domain this implies that Λ_0 is finite locally free over $\mathcal{O}_{E'}$. The morphism $\det_{\Lambda_0} : V_{\mathcal{O}_B\otimes_{\mathbb{Z}}\mathcal{O}_{E'}} \to \mathbb{A}^1_{\mathcal{O}_{E'}}$ satisfies $\det_{\Lambda_0}\otimes_{\mathcal{O}_{E'}}\mathbb{C} = \det_V$.

(1) We have just seen that $\det_V$ is defined over $\mathcal{O}_{E'}$. As $E\cap\mathcal{O}_{E'} = \mathcal{O}_E$, it remains to show that $\det_V$ is defined over E. By Proposition 2.2.2, it suffices to show that $\det_{\mathbb{C}}(x|V)\in E$ for all $x\in B\otimes_{\mathbb{Q}} E$. By the definition of E we have $\mathrm{tr}_{\mathbb{C}}(x|V)\in E$ for all $x\in B\otimes_{\mathbb{Q}} E$. Lemma 2.3.6 below allows us to conclude.
(2) As the morphism $\mathrm{Spec}\,\mathcal{O}_{E'} \to \mathrm{Spec}\,\mathcal{O}_E$ is surjective, it suffices to compute the rank of $M\otimes_{\mathcal{O}_E}\mathcal{O}_{E'}$ over $R\otimes_{\mathcal{O}_E}\mathcal{O}_{E'}$. As $\det_{M\otimes_{\mathcal{O}_E}\mathcal{O}_{E'}} = \det_{\Lambda_0\otimes_{\mathcal{O}_E} R}$, the claim follows from Lemma 2.3.4 and the equality $\mathrm{rk}_{\mathcal{O}_{E'}}\Lambda_0 = \dim_{\mathbb{C}} V$.

□

Lemma 2.3.6. *Let* K *be a field of characteristic 0 and let* f *be an endomorphism of a finite-dimensional* K*-vector space* V*. Say* $\chi_K(f) = \sum_{i=0}^n s_{n-i}T^i$ *for* $n = \dim_K V$ *and* $s_i\in K$*. Then there are polynomials* $P_0,\dots,P_n\in\mathbb{Q}[T_1,\dots,T_n]$ *with*

$$s_i = P_i(\mathrm{tr}_K(f),\mathrm{tr}_K(f^2),\dots,\mathrm{tr}_K(f^n)),\quad 0\le i\le n.$$

Proof. Denote by $p_i\in\mathbb{Z}[X_1,\dots,X_n]$ the i-th power sum and by $e_i\in\mathbb{Z}[X_1,\dots,X_n]$ the i-th elementary symmetric polynomial. Let $\lambda_1,\dots,\lambda_n$ be the zeros of $\chi_K(f)$ in an algebraic closure of K. Then

$$\mathrm{tr}_K(f^i) = p_i(\lambda_1,\dots,\lambda_n),\quad 1\le i\le n$$

and

$$s_i = (-1)^i e_i(\lambda_1,\dots,\lambda_n),\quad 0\le i\le n.$$

The claim follows from the fact that there are polynomials $Q_0,\dots,Q_n\in\mathbb{Q}[T_1,\dots,T_n]$ with

$$e_i = Q_i(p_1,\dots,p_n),\quad 0\le i\le n,$$

see [21, I.2.12]. □

The following lemma shows that the determinant condition can be interpreted naively in terms of characteristic polynomials.

Proposition 2.3.7. *Let A be a (not necessarily commutative) R-algebra and let M and N be A-modules which are finite locally free over R. Let $A_0 \subset A$ be a generating set of A as an R-module. Then $\det_M = \det_N$ if (and only if) for all $a \in A_0$ we have $\chi_R(a|M) = \chi_R(a|N)$.*

Proof. Assume that $\chi_R(a|M) = \chi_R(a|N)$ for all $a \in A_0$. Let $\varphi : R \to R'$ be an R-algebra. Then $(a \otimes 1)_{a \in A_0}$ generates $A \otimes_R R'$ over R', and we have $\chi_{R'}(a \otimes 1|M_{R'}) = (\chi_R(a|M))^{\varphi}$ and $\chi_{R'}(a \otimes 1|N_{R'}) = (\chi_R(a|N))^{\varphi}$. Hence the assumption is universally satisfied and it suffices to show that $\det_R(a|M) = \det_R(a|N)$ for all $a \in A$. This however is clear by the existence of Amitsur's formula, which expresses the characteristic polynomial of a linear combination of endomorphisms in terms of the characteristic polynomials of the summands, see [1, Theorem A]. □

For future reference we note the following trivial lemmata.

Lemma 2.3.8. *Let R be a ring and A a (not necessarily commutative) R-algebra. Let M and N be left A-modules which are finite locally free over R. Assume that A decomposes as an R-algebra into a finite product $A = \prod_{i=1}^m A_i$ of R-algebras A_i. Let $M = \prod_i M_i$ and $N = \prod_i N_i$ be the corresponding decompositions. Then $\det_M = \det_N$ if and only if for all $1 \leq i \leq m$ the equality $\det_{M_i} = \det_{N_i}$ holds.*

Lemma 2.3.9. *Let $e \in \mathbb{N}$, let R be a* reduced *ring and let M be an $R[u]/u^e$-module which is finite free over R. Then $\chi_R(p|M) = \big(T - p(0)\big)^{\mathrm{rk}_R M}$ for all $p \in R[u]/u^e$.*

Proof. Immediate by passing to the residue fields of R. □

2.4. **de Rham cohomology of abelian schemes.** Let R be a ring and let X/R be a scheme. Define (for each $r \in \mathbb{N}$) the *sheaf of relative differentials of degree r* of X over R by $\Omega^r_{X/R} := \bigwedge^r \Omega^1_{X/R}$. In [12, 16.6.2] a differential $d : \Omega^r_{X/R} \to \Omega^{r+1}_{X/R}$ is constructed and we denote by $\Omega_{X/R}$ the resulting complex. We define an R-module $H^1_{dR}(X/R)$ by

$$H^1_{dR}(X/R) = H^1(R^+ f_*(\Omega_{X/R})).$$

Setting

$$H_1^{dR}(X/R) = (H^1_{dR}(X/R))^\vee,$$

we obtain a covariant functor H_1^{dR} from the category of R-schemes to the category of R-modules. If G/R is a group scheme, we denote by $e_G : \operatorname{Spec} R \to G$ its identity section. We define $\omega_G = e_G^* \Omega^1_{G/R}$. We denote by $\mathrm{Lie}(G) = \mathfrak{L}\mathrm{ie}(G)(R)$ the Lie algebra of G, see [4, II, §4]. It is an R-module. There is a canonical functorial isomorphism

$$\mu_G : \omega_G^\vee \xrightarrow{\sim} \mathrm{Lie}(G), \tag{2.4.1}$$

see [4, II, §4, 3.6].

Let A/R be an abelian scheme. We denote by $A^\vee$ the dual abelian scheme of A and by $j = j_A : A \xrightarrow{\sim} (A^\vee)^\vee$ the biduality isomorphism.

Proposition 2.4.2 ([2, Section 2.5])**.** *There is a canonical short exact sequence*

$$0 \to \omega_A \to H^1_{dR}(A/R) \to \mathrm{Lie}(A^\vee) \to 0, \tag{2.4.3}$$

which is functorial in A and commutes with base-change. All terms of (2.4.3) *are finite locally free R-modules. We have $\mathrm{rk}_R\, H^1_{dR}(A/R) = 2\dim_R A$ and $\mathrm{rk}_R \mathrm{Lie}(A^\vee) = \mathrm{rk}_R\, \omega_A = \dim_R A$.*

Proposition 2.4.4. [2, 5.1.10] *There is a functorial isomorphism $\Phi_A : H^1_{dR}(A/R)^\vee \xrightarrow{\sim} H^1_{dR}(A^\vee/R)$ fitting into the following commutative diagram.*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(A^\vee)^\vee \arrow[r] \arrow[d, "\mu^\vee_{A^\vee}"] & H^1_{dR}(A/R)^\vee \arrow[r] \arrow[d, "\Phi_A"] & \omega_A^\vee \arrow[r] \arrow[d, "-\mathrm{Lie}(j)\circ\mu_A"] & 0 \\ 0 \arrow[r] & \omega_{A^\vee} \arrow[r] & H^1_{dR}(A^\vee/R) \arrow[r] & \mathrm{Lie}(A^{\vee\vee}) \arrow[r] & 0. \end{tikzcd} \tag{2.4.5}$$

Here the first row is obtained from (2.4.3) *by dualizing and the second row is obtained by applying* (2.4.3) *to $A^\vee$.*

Remark 2.4.6. *We will always consider $\omega_{A^\vee}$ as a submodule of $H_1^{dR}(A/R)$ via the left-hand square in* (2.4.5)*.*

Let A and B be abelian schemes over R and let $\lambda : A \to B^\vee$ be a homomorphism. We obtain a morphism $f_\lambda : H_1^{dR}(A/R) \xrightarrow{H_1^{dR}(\lambda)} H_1^{dR}(B^\vee/R) \xrightarrow{\Phi_B^\vee} H_1^{dR}(B/R)^\vee$, which corresponds to a pairing

$$(\cdot,\cdot)_\lambda : H_1^{dR}(A/R) \times H_1^{dR}(B/R) \to R.$$

Similarly the composition $B \xrightarrow{j} (B^\vee)^\vee \xrightarrow{\lambda^\vee} A^\vee$ gives rise to a pairing $(\cdot,\cdot)_{\lambda^\vee} : H_1^{dR}(B/R) \times H_1^{dR}(A/R) \to R$.

Lemma 2.4.7. *Let $a \in H_1^{dR}(A/R), b \in H_1^{dR}(B/R)$. Then*

$$(a,b)_\lambda = -(b,a)_{\lambda^\vee}.$$

Proof. The claim is equivalent to the equation $f_\lambda = -(f_{\lambda^\vee\circ j})^\vee$. First $(f_{\lambda^\vee\circ j})^\vee = H_1^{dR}(j)^\vee \circ H_1^{dR}(\lambda^\vee)^\vee \circ \Phi_A$. The naturality of the isomorphism Φ_A implies that the diagram

$$\begin{tikzcd} H_1^{dR}(A/R) \arrow[r, "\Phi_A"] \arrow[d, "H_1^{dR}(\lambda)"'] & H_1^{dR}(A^\vee/R)^\vee \arrow[d, "H_1^{dR}(\lambda^\vee)^\vee"] \\ H_1^{dR}(B^\vee/R) \arrow[r, "\Phi_{B^\vee}"] & H_1^{dR}(B^{\vee\vee}/R)^\vee \end{tikzcd}$$

commutes. Thus $(f_{\lambda^\vee\circ j})^\vee = H_1^{dR}(j)^\vee \circ \Phi_{B^\vee} \circ H_1^{dR}(\lambda)$. Comparing this expression with f_λ, we need to see that $H_1^{dR}(j)^\vee \circ \Phi_{B^\vee} = -\Phi_B^\vee$. But this is precisely the content of the diagram [2, 5.1.5.1]. □

Lemma 2.4.8. *The submodules $\omega_{A^\vee} \subset H_1^{dR}(A/R)$ and $\omega_{B^\vee} \subset H_1^{dR}(B/R)$ pair to zero under $(\cdot,\cdot)_\lambda$.*

Proof. By the definition of $(\cdot,\cdot)_\lambda$, the claim is equivalent to the composition

$$\mathrm{Lie}(A^\vee)^\vee \to H^1_{dR}(A/R)^\vee \xrightarrow{H^1_{dR}(\lambda)^\vee} H^1_{dR}(B^\vee/R)^\vee \xrightarrow{\Phi_B^\vee} H^1_{dR}(B/R) \to \mathrm{Lie}(B^\vee)$$

being zero. Consider the following diagram.

$$\begin{tikzcd}
\mathrm{Lie}(A^\vee)^\vee \arrow[r, "\mathrm{Lie}(\lambda^\vee)^\vee"] \arrow[d] & \mathrm{Lie}(B^{\vee\vee})^\vee \arrow[r, "-\mu_B^\vee\circ\mathrm{Lie}(j)^\vee"] \arrow[d] & \omega_B \arrow[d] & \\
H^1_{dR}(A/R)^\vee \arrow[r, "H^1_{dR}(\lambda)^\vee"] & H^1_{dR}(B^\vee/R)^\vee \arrow[r, "\Phi_B^\vee"] & H^1_{dR}(B/R) \arrow[r] & \mathrm{Lie}(B^\vee).
\end{tikzcd}$$

The left-hand square commutes by the functoriality of (2.4.3), and the right-hand square commutes by Proposition 2.4.5. By (2.4.3) the composition $\omega_B \to H^1_{dR}(B/R) \to \mathrm{Lie}(B^\vee)$ is zero. □

2.5. $\mathbb{Z}_{(p)}$**-isogenies.** For a ring R denote by $\mathfrak{A}_R$ the additive category of abelian schemes over R. For $A, B \in \mathrm{Ob}\,\mathfrak{A}_R$ we write $\mathrm{Hom}_R(A,B) = \mathrm{Hom}_{\mathfrak{A}_R}(A,B)$. Let $\Gamma \subset \mathbb{Q}$ be a subring. We denote by $\mathfrak{A}_R \otimes \Gamma$ the following category.

$$\mathrm{Ob}(\mathfrak{A}_R \otimes \Gamma) = \mathrm{Ob}\,\mathfrak{A}_R,$$
$$\mathrm{Hom}_{\mathfrak{A}_R\otimes\Gamma}(A,B) = \mathrm{Hom}_R(A,B) \otimes_{\mathbb{Z}} \Gamma \quad \text{for } A, B \in \mathrm{Ob}(\mathfrak{A}_R \otimes \Gamma).$$

If $\Gamma \subset \Gamma' \subset \mathbb{Q}$ are subrings, $\mathfrak{A}_R \otimes \Gamma$ is a subcategory of $\mathfrak{A}_R \otimes \Gamma'$ by Corollary A.11. A morphism in $\mathfrak{A}_R \otimes \Gamma$ is called a Γ-*isogeny* if it is an isomorphism when considered as a morphism in $\mathfrak{A}_R \otimes \mathbb{Q}$. By Proposition A.13 the $\mathbb{Z}$-isogenies are precisely the isogenies in the usual sense (see Definition A.5).

Remark 2.5.1.
- *For each subring $\Gamma \subset \mathbb{Q}$, the additive functor $\mathfrak{A}_R \to \mathfrak{A}_R$, $A \mapsto A^\vee$ extends uniquely to a functor $\mathfrak{A}_R \otimes \Gamma \to \mathfrak{A}_R \otimes \Gamma$.*
- *By Lemma A.15 the functor $\mathfrak{A}_R \to \mathfrak{C}_R$, $A \mapsto A[p^\infty]$ extends uniquely to $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$. Here we use the notation of the beginning of Appendix A.*
- *Assume that R is a $\mathbb{Z}_{(p)}$-algebra. Then any additive functor $F : \mathfrak{A}_R \to \mathfrak{M}_R$ extends uniquely to $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$. Here $\mathfrak{M}_R$ denotes the category of R-modules. We will use this without further comment for the functors introduced in the previous section.*

Definition 2.5.2. *Let $f : A \to B$ be a $\mathbb{Z}_{(p)}$-isogeny over R. By functoriality f induces a morphism $f' : A[p^\infty] \to B[p^\infty]$. We call* $\ker f := \ker f'$ *the* kernel of f. *By Proposition A.14 the kernel* $\ker f$ *is a finite locally free group scheme over R. We call* $\deg f := \mathrm{rk}_R(\ker f)$ *the* degree of f.

Remark 2.5.3. *If we represent f a as a quotient $f = g/n$ for some $g \in$* $\mathrm{Hom}_R(A,B)$ *and some $n \in \mathbb{N}_{\geq 1}$ coprime to p, we have* $\ker f = (\ker g)[p^\infty]$. *In particular the kernel of $[p]_A$ as an isogeny coincides with its kernel as a $\mathbb{Z}_{(p)}$-isogeny. This will be used below without further comment.*

2.6. **Finite commutative group schemes over $\mathbb{F}$.** Denote by F the category of finite commutative group schemes over $\mathbb{F}$. It is an abelian category by [4, III, §3, 7.4]. By loc. cit. the epimorphisms in F are faithfully flat, and hence F is even an abelian subcategory of $\mathfrak{C}_{\mathbb{F}}$ (notation of the beginning of Appendix A).

Recall from [4, IV, §3.5] that F is the product of four subcategories

$$F = F^{e,m} \times F^{e,u} \times F^{i,m} \times F^{i,u},$$

with

- $F^{e,m}$ the subcategory of étale multiplicative groups,
- $F^{e,u}$ the subcategory of étale unipotent groups,
- $F^{i,m}$ the subcategory of infinitesimal multiplicative groups,
- $F^{i,u}$ the subcategory of infinitesimal unipotent groups.

For $G \in \mathrm{Ob}\, F$ we denote by $G = G^{e,m} \times G^{e,u} \times G^{i,m} \times G^{i,u}$ the corresponding decomposition. These four subcategories can be described rather explicitly as follows.

Denote by $\mathfrak{Ab}_{\mathrm{fin}}$ the category of finite abelian groups and by $\mathfrak{Ab}_{\mathrm{fin},p}$ (resp. $\mathfrak{Ab}_{\mathrm{fin},p'}$) its subcategory of groups of p-torsion (resp. of groups without p-torsion), so that $\mathfrak{Ab}_{\mathrm{fin}} = \mathfrak{Ab}_{\mathrm{fin},p} \times \mathfrak{Ab}_{\mathrm{fin},p'}$. For $A \in \mathrm{Ob}\, \mathfrak{Ab}_{\mathrm{fin}}$ denote by $A_{\mathbb{F}} \in \mathrm{Ob}\, F$ the corresponding constant group. The functor $A \mapsto A_{\mathbb{F}}$ induces equivalences of categories

$$A_{\mathrm{fin},p} \xrightarrow{\sim} F^{e,u} \quad \text{and} \quad A_{\mathrm{fin},p'} \xrightarrow{\sim} F^{e,m}.$$

Denote by $D : F \to F$ the (anti-)auto-equivalence given by Cartier duality (see [4, II, §1, 2.10]). It restricts to auto-equivalences of $F^{e,m}$ and $F^{i,u}$, while inducing an equivalence $F^{e,u} \to F^{i,m}$. This provides us with an explicit understanding of the subcategory $F^{i,m}$.

Finally denote by $\alpha_p \in \mathrm{Ob}\, F$ the group given on R-valued points by $\alpha_p(R) = \{r \in R \mid r^p = 0\}$. Then every object of $F^{i,u}$ is a successive extension of copies of α_p.

This discussion implies that for $G \in \mathrm{Ob}\, F$ we have

$$G[p^\infty] = G^{e,u} \times G^{i,m} \times G^{i,u}. \tag{2.6.1}$$

Define $\mathrm{rk}^{e,u}(G) := \mathrm{rk}(G^{e,u})$.

Lemma 2.6.2. *Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be $\mathbb{Z}_{(p)}$-isogenies of abelian varieties over $\mathbb{F}$. Then*

$$\mathrm{rk}^{e,u} \ker(\beta \circ \alpha) = (\mathrm{rk}^{e,u} \ker \beta) \cdot (\mathrm{rk}^{e,u} \ker \alpha).$$

Proof. By Proposition A.14 we obtain a short exact sequence $0 \to \ker \alpha \to A[p^\infty] \to B[p^\infty] \to 0$. It gives rise to a short exact sequence

$$0 \to \ker \alpha \to \ker(\beta \circ \alpha) \to \ker \beta \to 0,$$

which induces a short exact sequence

$$0 \to (\ker \alpha)^{e,u} \to (\ker(\beta \circ \alpha))^{e,u} \to (\ker \beta)^{e,u} \to 0.$$

The statement follows from the fact that the rank function on F is multiplicative in short exact sequences, see [3, Corollary II.6.4]. $\square$

Lemma 2.6.3. *Let $f : A \to B$ be a $\mathbb{Z}_{(p)}$-isogeny of abelian varieties over $\mathbb{F}$. Then $\ker f$ is étale if and only if the induced map $\omega_B \to \omega_A$ is surjective.*

Proof. Write $f = g/n$ with $g \in \mathrm{Hom}_{\mathbb{F}}(A, B)$ and $n \in \mathbb{N}$ coprime to p. By (2.6.1) we have $\ker g = \ker f \times (\ker g)^{e,m}$ and hence $\ker f$ is étale if and only if $\ker g$ is étale. Thus we may assume that $f = g$.

Let $K = \ker g$ with structure morphism $\alpha : K \to \operatorname{Spec} \mathbb{F}$. Pulling back the usual exact sequence $g^*\Omega^1_{B/\mathbb{F}} \to \Omega^1_{A/\mathbb{F}} \to \Omega^1_{A/B} \to 0$ under the identity section $e_A : \operatorname{Spec} \mathbb{F} \to A$ gives an exact sequence $\omega_B \to \omega_A \to \omega_K \to 0$. Thus $\omega_B \to \omega_A$ is surjective if and only if $\omega_K = 0$. From the equalities $\omega_K = e_K^*\Omega^1_{K/\mathbb{F}}$ and $\Omega^1_{K/\mathbb{F}} = \alpha^*\omega_K$, see [4, II, §4, 3.4], we conclude that $\omega_K = 0$ if and only if $\Omega^1_{K/\mathbb{F}} = 0$, which in turn is equivalent to K being étale. $\square$

Corollary 2.6.4. *Let $f : A \to B$ be a $\mathbb{Z}_{(p)}$-isogeny of abelian varieties over $\mathbb{F}$. Then $\ker f$ is étale if and only if the equality*

$$H_1^{dR}(B) = \operatorname{im} H_1^{dR}(f) + \omega_{B^\vee}$$

holds.

Proof. From (2.4.3) we obtain the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & \omega_{A^\vee} \arrow[r] \arrow[d] & H_1^{dR}(A/\mathbb{F}) \arrow[r] \arrow[d, "H_1^{dR}(f)"] & \omega_A^\vee \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \omega_{B^\vee} \arrow[r] & H_1^{dR}(B/\mathbb{F}) \arrow[r] & \omega_B^\vee \arrow[r] & 0. \end{tikzcd} \tag{2.6.5}$$

By Lemma 2.6.3 we know that $\ker f$ is étale if and only if $\omega_B \to \omega_A$ is surjective. As A and B are isogenous, they have the same dimension and hence ω_A and ω_B are $\mathbb{F}$-vector spaces of the same dimension. Consequently $\omega_B \to \omega_A$ is surjective if and only if $\omega_A^\vee \to \omega_B^\vee$ is surjective. In view of (2.6.5) this is the case if and only if $H_1^{dR}(B/\mathbb{F}) = \operatorname{im} H_1^{dR}(f) + \omega_{B^\vee}$. $\square$

2.7. **Morphisms of (polarized) multichains.** For the convenience of the reader we want to make explicit three definitions which are only implicit in [32, Section 3]. We assume that $V \neq 0$ in the notation of loc. cit.

We make the following addition to [32, Definition 3.6], freely using its notation.

Definition 2.7.1. *Let R be a $\mathbb{Z}_p$-algebra and let*

$$\mathcal{M} = (M_\Lambda, \rho_{\Lambda',\Lambda} : M_\Lambda \to M_{\Lambda'}, \theta_{\Lambda,b} : M_\Lambda^b \xrightarrow{\sim} M_{b\Lambda}),$$
$$\mathcal{M}' = (M'_\Lambda, \rho'_{\Lambda',\Lambda} : M'_\Lambda \to M'_{\Lambda'}, \theta'_{\Lambda,b} : M'^b_\Lambda \xrightarrow{\sim} M'_{b\Lambda})$$

be chains of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules of type $(\mathcal{L})$. A morphism $\mathcal{M} \to \mathcal{M}'$ *is a tuple $\varphi = (\varphi_\Lambda)_{\Lambda \in \mathcal{L}}$ of isomorphisms of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules $\varphi_\Lambda : M_\Lambda \to M'_\Lambda$ such that the diagrams*

$$\begin{tikzcd} M_\Lambda \arrow[r, "\rho_{\Lambda',\Lambda}"] \arrow[d, "\varphi_\Lambda"'] & M_{\Lambda'} \arrow[d, "\varphi_{\Lambda'}"] \\ M'_\Lambda \arrow[r, "\rho'_{\Lambda',\Lambda}"] & M_{\Lambda'}, \end{tikzcd} \qquad \begin{tikzcd} M^b_\Lambda \arrow[r, "\theta_{\Lambda,b}"] \arrow[d, "\varphi_\Lambda"'] & M_{b\Lambda} \arrow[d, "\varphi_{b\Lambda}"] \\ M'^b_\Lambda \arrow[r, "\theta'_{\Lambda,b}"] & M'_{b\Lambda} \end{tikzcd} \tag{2.7.2}$$

commute for all $\Lambda \subset \Lambda'$ in $\mathcal{L}$ and all $b \in B^\times$ that normalize $\mathcal{O}_B$.[1]

[1]Note that the first diagram merely states that φ is a natural transformation $\mathcal{M} \to \mathcal{M}'$ of functors on $\mathcal{L}$.

We make the following addition to [32, Definition 3.10] and the discussion following it, freely using its notation.

Definition 2.7.3. *Let R be a $\mathbb{Z}_p$-algebra and let $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m)$ and $\mathcal{M}' = (\mathcal{M}'_1, \ldots, \mathcal{M}'_m)$ be multichains of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules of type $(\mathcal{L})$. For $\Lambda \in \mathcal{L}$ denote by M_Λ (resp. M'_Λ) the $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-module associated with Λ by $\mathcal{M}$ (resp. by $\mathcal{M}'$).*

A morphism *$\mathcal{M} \to \mathcal{M}'$ is a tuple $\varphi = (\varphi_1, \ldots, \varphi_m)$ of morphism $\varphi_i : \mathcal{M}_i \to \mathcal{M}'_i$ of chains of $\mathcal{O}_{B_i} \otimes_{\mathbb{Z}_p} R$-modules of type $(\mathcal{L}_i)$.*

If $\Lambda \in \mathcal{L}$ has the decomposition $\Lambda = \Lambda_1 \times \ldots \Lambda_m$, $\Lambda_i \in \mathcal{L}_i$, we set $\varphi_\Lambda = \varphi_{1,\Lambda_1} \times \cdots \times \varphi_{m,\Lambda_m} : M_\Lambda \to M'_\Lambda$. In this way φ becomes a natural transformation $\mathcal{M} \to \mathcal{M}'$ of functors on $\mathcal{L}$.

We denote by $\operatorname{Isom}(\mathcal{M}, \mathcal{M}')$ the functor on the category of R-algebras with $\operatorname{Isom}(\mathcal{M}, \mathcal{M}')(R')$ the set of morphisms $\mathcal{M} \otimes_R R' \to \mathcal{M}' \otimes_R R'$ of multichains of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R'$-modules of type $(\mathcal{L})$. We also write $\operatorname{Aut}(\mathcal{M}) = \operatorname{Isom}(\mathcal{M}, \mathcal{M})$.

We make the following addition to [32, Definition 3.14], freely using its notation.

Definition 2.7.4. *Let R be a $\mathbb{Z}_p$-algebra and let $\mathcal{M}_0$ and $\mathcal{M}'_0$ be multichains of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules of type $(\mathcal{L})$. For $\Lambda \in \mathcal{L}$ denote by M_Λ (resp. M'_Λ) the $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-module associated with Λ by $\mathcal{M}_0$ (resp. by $\mathcal{M}'_0$). Let $\mathcal{E} = (\mathcal{E}_\Lambda : M_\Lambda \times M_{\Lambda^*} \to R)$ and $\mathcal{E}' = (\mathcal{E}'_\Lambda : M'_\Lambda \times M'_{\Lambda^*} \to R)$ be polarizations of $\mathcal{M}_0$ and $\mathcal{M}'_0$, respectively, so that $\mathcal{M} = (\mathcal{M}_0, \mathcal{E})$ and $\mathcal{M}' = (\mathcal{M}'_0, \mathcal{E}')$ are polarized multichains of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules of type $(\mathcal{L})$*

A similitude *$\mathcal{M} \to \mathcal{M}'$ is a morphism $\varphi = (\varphi_\Lambda) : \mathcal{M}_0 \to \mathcal{M}'_0$ such that there exists a $c = c(\varphi) \in R^\times$ with*

$$\forall \Lambda \in \mathcal{L} \forall (x,y) \in M_\Lambda \times M_{\Lambda^*} : \ \mathcal{E}'_\Lambda(\varphi_\Lambda(x), \varphi_{\Lambda^*}(y)) = c\mathcal{E}_\Lambda(x,y). \tag{2.7.5}$$

A morphism *$\mathcal{M} \to \mathcal{M}'$ is a similitude $\varphi : \mathcal{M} \to \mathcal{M}'$ with $c(\varphi) = 1$.*

We denote by $\operatorname{Isom}(\mathcal{M}, \mathcal{M}')$ (resp. $\operatorname{Sim}(\mathcal{M}, \mathcal{M}')$) the functor on the category of R-algebras with $\operatorname{Isom}(\mathcal{M}, \mathcal{M}')(R')$ (resp. $\operatorname{Sim}(\mathcal{M}, \mathcal{M}')(R)$) the set of morphisms (resp. similitudes) $\mathcal{M} \otimes_R R' \to \mathcal{M}' \otimes_R R'$ of polarized multichains of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R'$-modules of type $(\mathcal{L})$. We also write $\operatorname{Aut}(\mathcal{M}) = \operatorname{Isom}(\mathcal{M}, \mathcal{M})$ and $\operatorname{Sim}(\mathcal{M}) = \operatorname{Sim}(\mathcal{M}, \mathcal{M})$.

Note that for a similitude φ, the unit $c(\varphi) \in R^\times$ is indeed uniquely determined in view of the assumption $V \neq 0$ and the perfectness of the $\mathcal{E}_\Lambda$, justifying the notation.

Denote by K the completion of the maximal unramified extension of $\mathbb{Q}_p$ and by $\mathcal{O}_K$ the valuation ring of K.

Proposition 2.7.6. *Let $\mathcal{M}$ be a polarized multichain of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_K$-modules of type $(\mathcal{L})$. Then the $\mathcal{O}_K$-group schemes $\operatorname{Aut}(\mathcal{M})$ and $\operatorname{Sim}(\mathcal{M})$ are smooth and affine.*

Proof. Clearly $\operatorname{Aut}(\mathcal{M})$ and $\operatorname{Sim}(\mathcal{M})$ are affine group schemes of finite type over $\mathcal{O}_K$. By a theorem of Cartier (see for example [4, II, §6, 1.1]), $\operatorname{Aut}(\mathcal{M})$ is smooth at all points of its generic fiber. By [32, Theorem 3.16] the base-change $\operatorname{Aut}(\mathcal{M}) \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^n$ is smooth over $\mathcal{O}_K/p^n$ for every $n \in \mathbb{N}$. Using [12, Proposition 17.14.2] this implies that $\operatorname{Aut}(\mathcal{M})$ is also smooth at all points lying over $(p) \in \operatorname{Spec} \mathcal{O}_K$. Hence $\operatorname{Aut}(\mathcal{M})$ is smooth everywhere.

Consider the sequence $1 \to \mathrm{Aut}(\mathcal{M}) \to \mathrm{Sim}(\mathcal{M}) \xrightarrow{c} \mathbb{G}_{m,\mathcal{O}_K} \to 1$ of étale sheaves on $\mathcal{O}_K$. By definition it is exact on the left. Let R be an $\mathcal{O}_K$-algebra and let $x \in R^\times$. As $p \neq 2$, étale locally on R there is an element $y \in R^\times$ with $y^2 = x$. Then $c\left(\mathcal{M} \xrightarrow{\cdot y} \mathcal{M}\right) = x$, so that the sequence is also exact on the right. The smoothness of $\mathrm{Sim}(\mathcal{M})$ then follows from the following lemma. □

Lemma 2.7.7. *Let S be a scheme, let F, G, H be group schemes over S and let $0 \to F \xrightarrow{\varphi} G \xrightarrow{\psi} H \to 0$ be a short exact sequence of étale sheaves of groups on S. Let $\boldsymbol{P}$ be a property of morphisms of schemes over S which is stable under base-change and local on the target for the étale topology. Then $\psi : G \to H$ has $\boldsymbol{P}$ if and only if the structural morphism $F \to S$ has $\boldsymbol{P}$.*

Proof. As $\psi : G \to H$ is surjective, there is an étale covering $(U_i \xrightarrow{h_i} H)_{i \in I}$ such that for each $i \in I$ there is a morphism $g_i : U_i \to G$ with $\psi \circ g_i = h_i$. Denote by $\alpha : G \times_S F \to G$ the map with $\alpha(g, f) = g \cdot \varphi(f)$. For each $i \in I$ the following diagrams commute and are cartesian.

$$\begin{tikzcd}
U_i \times_S F \arrow[r, "g_i \times \mathrm{id}_F"] \arrow[d, "\mathrm{pr}_{U_i}"'] & G \times_S F \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}_G"] & G \arrow[d, "\psi"] \\
U_i \arrow[r, "g_i"] \arrow[rr, bend right, "h_i"'] & G \arrow[r, "\psi"] & H,
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d] & G \arrow[d, "\psi"] \\
S \arrow[r, "e_H"] & H.
\end{tikzcd}$$

From this the statement is obvious. □

We also have the following related statement.

Proposition 2.7.8. *Let R be a $\mathbb{Z}_p$-algebra which is separated and complete for the p-adic topology on R. Let $\mathcal{M}$ and $\mathcal{M}'$ be multichains (resp. polarized multichains) of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules of type $(\mathcal{L})$. The canonical map $\mathrm{Isom}(\mathcal{M}, \mathcal{M}')(R) \to \mathrm{Isom}(\mathcal{M}, \mathcal{M}')(R/p^n)$ is surjective for all $n \in \mathbb{N}_{\geq 1}$.*

Proof. Set $\mathcal{I} = \mathrm{Isom}(\mathcal{M}, \mathcal{M}')$. Clearly $\mathcal{I}$ is representable by an affine scheme over R. By [32, Theorem 3.11] (resp. [32, Theorem 3.16]) we know that the base-change $\mathcal{I} \otimes_R R/p^n$ is in particular formally smooth over R/p^n for every $n \in \mathbb{N}$. This easily implies the statement in view of $\mathcal{I}(R) = \lim_{n \in \mathbb{N}} \mathcal{I}(R/p^n)$. □

3. The general case

We assume that the reader is familiar with at least the definitions of [32, 3.1-3.27] and [32, 6.1-6.9]. The required results on orders in semisimple algebras can all be found in Reiner's excellent [33]. In Sections 3.1 through 3.3 we recall from [32] the general setup of integral models of PEL-type Shimura varieties and their local models. We want to emphasize that all of the results proven in these sections are standard and well-known.

3.1. **PEL data.**

A *PEL datum* consists of the following objects.

(1) A finite-dimensional semisimple $\mathbb{Q}$-algebra B.

(2) A positive[2] involution $*$ on B.
(3) A finitely generated left B-module V. We assume that $V \neq 0$.
(4) A symplectic form $(\cdot,\cdot) : V \times V \to \mathbb{Q}$ on the underlying $\mathbb{Q}$-vector space of V, such that for all $v, w \in V$ and all $b \in B$ the relation
$$(bv, w) = (v, b^* w)$$
is satisfied.
(5) An element $J \in \mathrm{End}_{B\otimes\mathbb{R}}(V \otimes \mathbb{R})$ with $J^2 + 1 = 0$ such that the bilinear form $(\cdot, J\cdot)_{\mathbb{R}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ is symmetric and positive definite.

We also fix the following data.

(a) A $\mathbb{Z}$-order $\mathcal{O}_B$ in B such that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is a maximal $\mathbb{Z}_p$-order in $B \otimes \mathbb{Q}_p$. We assume that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is stable under $*$.
(b) A self-dual multichain $\mathcal{L}$ of $\mathcal{O}_B \otimes \mathbb{Z}_p$-lattices in $V \otimes \mathbb{Q}_p$.

Denote by G the group on the category of $\mathbb{Q}$-algebras with
$$G(R) = \left\{ g \in \mathrm{GL}_{B\otimes R}(V \otimes R) \mid \exists c = c(g) \in R^\times \begin{pmatrix} \forall x, y \in V \otimes R \\ (gx, gy)_R = c\,(x, y)_R \end{pmatrix} \right\}.$$
Note that for $g \in G(R)$, the unit $c(g) \in R^\times$ is indeed uniquely determined in view of the assumption $V \neq 0$ and the perfectness of $(\cdot,\cdot)$, justifying the notation. We also denote by $c : G \to \mathbb{G}_{m,\mathbb{Q}}$ the resulting morphism.

Let $\Lambda \in \mathcal{L}$. We deviate slightly from the notation of [32] in writing $\Lambda^\vee = \{x \in V_{\mathbb{Q}_p} \mid (x, \Lambda)_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ (in loc. cit. the notation Λ^* is used instead). We denote by $(\cdot,\cdot)_\Lambda : \Lambda \times \Lambda^\vee \to \mathbb{Z}_p$ the restriction of $(\cdot,\cdot)_{\mathbb{Q}_p}$. It is a perfect pairing and induces an isomorphism $\Lambda^\vee \xrightarrow{\sim} \Lambda^{\vee,\mathbb{Z}_p}$ of $\mathcal{O}_B \otimes \mathbb{Z}_p$-modules, justifying the notation. For $\Lambda \subset \Lambda'$ in $\mathcal{L}$ we denote by $\rho_{\Lambda',\Lambda} : \Lambda \to \Lambda'$ the inclusion. For $b \in (B \otimes \mathbb{Q}_p)^\times$ in the normalizer of $\mathcal{O}_B \otimes \mathbb{Z}_p$ let $\vartheta_{\Lambda,b} : \Lambda^b \to b\Lambda$ be the isomorphism given by multiplication with b. Then $(\Lambda, \rho_{\Lambda',\Lambda}, \vartheta_{\Lambda,b}, (\cdot,\cdot)_\Lambda)$ is a polarized multichain of $\mathcal{O}_B \otimes \mathbb{Z}_p$-modules of type $(\mathcal{L})$ which, by abuse of notation, we also denote denote by $\mathcal{L}$.

Let $B \otimes \mathbb{Q}_p = B_1 \times \cdots \times B_m$ be the decomposition into simple factors. It induces a decomposition
$$\mathcal{O}_B \otimes \mathbb{Z}_p = \mathcal{O}_{B_1} \times \cdots \times \mathcal{O}_{B_m} \tag{3.1.1}$$
and each O_{B_i} is a maximal $\mathbb{Z}_p$-order in B_i.

We also get a decomposition $V \otimes \mathbb{Q}_p = V_1 \times \cdots \times V_m$ into left B_i-modules V_i. Denote by $\mathcal{L}_i$ the projection of $\mathcal{L}$ to V_i. It is a chain of $\mathcal{O}_{B_i}$-lattices in V_i. For $\Lambda \in \mathcal{L}$ we denote by $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m,\ \Lambda_i \in \mathcal{L}_i$ the corresponding decomposition.

Denote by $V_{\mathbb{C},\pm i}$ the $(\pm i)$-eigenspace of $J_{\mathbb{C}}$. Complex conjugation induces an isomorphism $V_{\mathbb{C},i} \to V_{\mathbb{C},-i}$ and consequently
$$\dim_{\mathbb{C}} V_{\mathbb{C},i} = \dim_{\mathbb{C}} V_{\mathbb{C},-i} = \frac{1}{2} \dim_{\mathbb{Q}} V. \tag{3.1.2}$$
As $V_{\mathbb{C},-i}$ is a $B \otimes \mathbb{C}$-module we get a morphism $\det_{V_{\mathbb{C},-i}} : V_{B\otimes\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$. Consider the reflex field $E = \mathbb{Q}(\mathrm{tr}_{\mathbb{C}}(b \otimes 1 | V_{\mathbb{C}});\ b \in B)$. From Proposition

[2] By this we mean that the involution on $B \otimes \mathbb{R}$ arising from $*$ via base-change is a positive involution in the sense of [18, §2].

2.3.5 we know that $\det_{V_{\mathbb{C},-i}}$ is defined over $\mathcal{O}_E$. Fix a place $\mathcal{Q}$ of $\mathcal{O}_E$ lying over p.

3.2. **Self-dual $\mathcal{L}$-sets of abelian varieties.** Let R be a ring. Let us recall from [32, 6.3] the category AV_R of pairs (A,κ), where $A \in \mathrm{Ob}(\mathfrak{A}_R \otimes \mathbb{Z}_{(p)})$ and $\kappa : \mathcal{O}_B \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_{\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}}(A_\Lambda)$ is an action. A morphism $(A,\kappa) \to (A',\kappa')$ in AV_R is a morphism $A \to B$ in $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$ compatible with κ and κ'.

In loc. cit. a morphism $(A,\kappa) \to (A,\kappa')$ is called an "isogeny in AV_R" if the underlying morphism $A \to A'$ is a $\mathbb{Z}_{(p)}$-isogeny in our sense. Implicit in loc. cit. is the notion of a "quasi-isogeny $(A,\kappa) \to (A',\kappa')$ in AV_R", which in our terminology is a $\mathbb{Q}$-isogeny $A \to A'$ compatible with κ and κ'. To avoid any confusion we will always spell out our terminology instead of using these two terms.

We make the following addition to [32, Definition 6.5] and the discussion following it, freely using its notation.[3]

Definition 3.2.1. *Let R be a ring. A* self-dual *$\mathcal{L}$-set of abelian varieties over R consists of an $\mathcal{L}$-set $(A_\Lambda, \varrho_{\Lambda',\Lambda})$ of abelian varieties over R together with a system $(\lambda_\Lambda)_{\Lambda\in\mathcal{L}}$ of isomorphisms $\lambda_\Lambda : A_\Lambda \to A^\vee_{\Lambda^\vee}$ in AV_R satisfying the following conditions for all $\Lambda \subset \Lambda'$ in $\mathcal{L}$.*

(1) The diagram

$$\begin{tikzcd} A_\Lambda \arrow[r, "\varrho_{\Lambda',\Lambda}"] \arrow[d, "\lambda_\Lambda"'] & A_{\Lambda'} \arrow[d, "\lambda_{\Lambda'}"] \\ A^\vee_{\Lambda^\vee} \arrow[r, "\varrho^\vee_{\Lambda^\vee,\Lambda'^\vee}"] & A^\vee_{\Lambda'^\vee} \end{tikzcd}$$

commutes.

(2) The morphism $A_\Lambda \xrightarrow{\lambda_\Lambda} A^\vee_{\Lambda^\vee} \xrightarrow{\varrho^\vee_{\Lambda^\vee,\Lambda}} A^\vee_\Lambda$ is symmetric.

Let $(A_\Lambda, \varrho_{\Lambda',\Lambda}, \lambda_\Lambda)$ and $(A'_\Lambda, \varrho'_{\Lambda',\Lambda}, \lambda'_\Lambda)$ be self-dual $\mathcal{L}$-sets of abelian varieties over R. A morphism *$(A_\Lambda, \varrho_{\Lambda',\Lambda}, \lambda_\Lambda) \to (A'_\Lambda, \varrho'_{\Lambda',\Lambda}, \lambda'_\Lambda)$ is a tuple $(\varphi_\Lambda)_{\Lambda\in\mathcal{L}}$ of isomorphisms $\varphi_\Lambda : A_\Lambda \to A'_\Lambda$ in AV_R such that the following diagrams commute for all $\Lambda \subset \Lambda'$ in $\mathcal{L}$.*

$$\begin{tikzcd} A_\Lambda \arrow[r, "\varrho_{\Lambda',\Lambda}"] \arrow[d, "\varphi_\Lambda"'] & A_{\Lambda'} \arrow[d, "\varphi_{\Lambda'}"] \\ A'_\Lambda \arrow[r, "\varrho'_{\Lambda',\Lambda}"] & A'_{\Lambda'}, \end{tikzcd} \qquad \begin{tikzcd} A_\Lambda \arrow[r, "\varphi_\Lambda"] \arrow[d, "\lambda_\Lambda"'] & A'_\Lambda \arrow[d, "\lambda'_\Lambda"] \\ A^\vee_{\Lambda^\vee} & A'^\vee_{\Lambda^\vee}. \arrow[l, "\varphi^\vee_{\Lambda^\vee}"'] \end{tikzcd}$$

Remark 3.2.2. *Let R be a ring and let $(A_\Lambda, \varrho_{\Lambda',\Lambda})$ be a (self-dual) $\mathcal{L}$-set of abelian varieties over R. Given $b \in B^\times \cap (\mathcal{O}_B \otimes \mathbb{Z}_{(p)})$ normalizing $\mathcal{O}_B \otimes \mathbb{Z}_{(p)}$ and $\Lambda \in \mathcal{L}$, there is by definition a periodicity isomorphism $\theta_{\Lambda,b} : A^b_\Lambda \xrightarrow{\sim} A_{b\Lambda}$ in AV_R such that the composition $\varrho_{\Lambda,b\Lambda} \circ \theta_{\Lambda,b}$ is equal to $A^b_\Lambda \xrightarrow{b} A_\Lambda$. Here $A^b_\Lambda \xrightarrow{b} A_\Lambda$ denotes the morphism given by multiplication with b via the action of $\mathcal{O}_B \otimes \mathbb{Z}_{(p)}$ on A_Λ. The morphism $\theta_{\Lambda,b}$ is a priori only assumed to exist, but working in $\mathfrak{A}_R \otimes \mathbb{Q}$ one immediately sees that it is in fact uniquely determined.*

[3]But recall that we denote the dual of an abelian scheme A by $A^\vee$, whereas in loc. cit. it is denoted by $A^\wedge$.

Definition 3.2.3. *Let R be an $\mathcal{O}_{E_{\mathbb{Q}}}$-algebra. We say that a (self-dual) $\mathcal{L}$-set of abelian varieties $(A_\Lambda, \varrho_{\Lambda',\Lambda})$ over R is* of determinant $\det_{V_{\mathbb{C},-i}}$ *if for all $\Lambda \in \mathcal{L}$ we have an equality*

$$\det_{\operatorname{Lie} A_\Lambda} = \det_{V_{\mathbb{C},-i}} \otimes_{\mathcal{O}_E} R$$

of morphisms $V_{\mathcal{O}_B \otimes R} \to \mathbb{A}^1_R$.

Remark 3.2.4. *Let R be an $\mathcal{O}_{E_{\mathbb{Q}}}$-algebra and let $A \in \mathrm{Ob}(AV_R)$. Assume that $\det_{\operatorname{Lie} A} = \det_{V_{\mathbb{C},-i}} \otimes_{\mathcal{O}_E} R$. Then $\dim_R A = \dim_{\mathbb{C}} V_{\mathbb{C},-i}$ by Lemma 2.3.4.*

Definition 3.2.5. *We denote by $\mathcal{A}$ the functor on the category of $\mathcal{O}_{E_{\mathbb{Q}}}$-algebras with $\mathcal{A}(R)$ the set of isomorphism classes of self-dual $\mathcal{L}$-sets of abelian varieties of determinant $\det_{V_{\mathbb{C},-i}}$ over R.*

Proposition 3.2.6. *Let R be an $\mathcal{O}_{E_{\mathbb{Q}}}$-algebra and let $A = (A_\Lambda, \varrho_{\Lambda',\Lambda}, \lambda_\Lambda) \in \mathcal{A}(R)$. For $\Lambda \in \mathcal{L}$ let $\mathcal{E}_\Lambda : H_1^{dR}(A_\Lambda) \times H_1^{dR}(A_{\Lambda^\vee}) \to R$ be the pairing denoted by $(\cdot,\cdot)_{\lambda_\Lambda}$ in Section 2.4. Then $(H_1^{dR}(A_\Lambda))_\Lambda$, equipped with the pairings $(\mathcal{E}_\Lambda)_\Lambda$, is a polarized multichain of $\mathcal{O}_B \otimes R$-modules of type $(\mathcal{L})$.*

Proof. Let $\Lambda \subset \Lambda'$ in $\mathcal{L}$ be neighbors. The existence of $\mathcal{O}_B \otimes R$-isomorphisms $H_1^{dR}(A_\Lambda) \simeq \Lambda_R$ and $H_1^{dR}(A_{\Lambda'})/\operatorname{im} H_1^{dR}(\varrho_{\Lambda',\Lambda}) \simeq (\Lambda'/\Lambda)_R$ locally on R is shown in [32, 3.23 c)-d)]. Using an approximation argument one sees that the periodicity isomorphisms of A induce periodicity isomorphisms of $(H_1^{dR}(A_\Lambda))_\Lambda$. Consequently $(H_1^{dR}(A_\Lambda))_\Lambda$ is a multichain of $\mathcal{O}_B \otimes R$-modules of type $(\mathcal{L})$.

By definition the composition $\lambda_{\Lambda^\vee} \circ \varrho_{\Lambda^\vee,\Lambda}$ is symmetric, so that

$$\varrho^\vee_{\Lambda^\vee,\Lambda} \circ \lambda^\vee_{\Lambda^\vee} = (\lambda_{\Lambda^\vee} \circ \varrho_{\Lambda^\vee,\Lambda})^\vee = \lambda_{\Lambda^\vee} \circ \varrho_{\Lambda^\vee,\Lambda} = \varrho^\vee_{\Lambda^\vee,\Lambda} \circ \lambda_\Lambda.$$

As $\varrho^\vee_{\Lambda^\vee,\Lambda}$ is a $\mathbb{Q}$-isogeny we deduce, using Proposition A.10, that $\lambda^\vee_{\Lambda^\vee} = \lambda_\Lambda$. By Lemma 2.4.7 we therefore have

$$\mathcal{E}_\Lambda(m, m') = -\mathcal{E}_{\Lambda^\vee}(m', m), \; m \in H_1^{dR}(A_\Lambda), m' \in H_1^{dR}(A_{\Lambda^\vee}).$$

The other properties required for making $(\mathcal{E}_\Lambda)_\Lambda$ into a polarization of the multichain $(H_1^{dR}(A_\Lambda))_\Lambda$ follow easily from analogous properties of the morphisms λ_Λ. □

Definition 3.2.7. *We denote by $\widetilde{\mathcal{A}}$ the functor on the category of $\mathcal{O}_{E_{\mathbb{Q}}}$-algebras with $\widetilde{\mathcal{A}}(R)$ the set of isomorphism classes of pairs (A, γ), where A is a self-dual $\mathcal{L}$-set of abelian varieties of determinant $\det_{V_{\mathbb{C},-i}}$ over R and*

$$\gamma : H_1^{dR}(A) \xrightarrow{\sim} \mathcal{L} \otimes R$$

is an isomorphism of polarized multichains of $\mathcal{O}_B \otimes R$-modules of type $(\mathcal{L})$.[4]

Denote by $\widetilde{\varphi} : \widetilde{\mathcal{A}} \to \mathcal{A}$ the morphism given on R-valued points by $\widetilde{\mathcal{A}}(R) \to \mathcal{A}(R)$, $(A, \gamma) \mapsto A$.

$\mathrm{Aut}(\mathcal{L})$ acts from the left on $\widetilde{\mathcal{A}}$ via $g \cdot (A, \gamma) = (A, g \circ \gamma)$ and $\widetilde{\varphi}$ is invariant for this action.

[4]Here two pairs (A, γ) and (A', γ') are considered to be isomorphic if there is an isomorphism $\gamma : A \to A'$ of self-dual $\mathcal{L}$-sets of abelian varieties over R such that $\vartheta = \vartheta' \circ H_1^{dR}(\gamma)$.

Proposition 3.2.8. *The morphism $\widetilde{\varphi}_{\mathbb{F}} : \widetilde{\mathcal{A}}_{\mathbb{F}} \to \mathcal{A}_{\mathbb{F}}$ is an $\mathrm{Aut}(\mathcal{L})_{\mathbb{F}}$-torsor for the étale topology. In particular $\widetilde{\varphi}(\mathbb{F})$ is an $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-torsor in the set-theoretic sense.*

Proof. We use the criterion [4, III, §4, 1.7(ii)]. That $\widetilde{\varphi}$ is surjective for the étale topology follows from [32, Theorem 3.14], which implies that for any $\mathbb{F}$-algebra R, any two polarized multichains of $\mathcal{O}_B \otimes R$-modules of type $(\mathcal{L})$ are isomorphic étale locally on R. The other property required in loc. cit. is trivially satisfied. □

Remark 3.2.9. *The preceding statement holds more generally, see* [25, Theorem 2.2].

3.3. **The local model diagram and the KR stratification.**

Lemma 3.3.1. *Let R be an $\mathcal{O}_{E_{\mathcal{Q}}}$-algebra, let $\Lambda \in \mathcal{L}$ and let*

$$0 \to t_\Lambda \to \Lambda_R \to t'_\Lambda \to 0,$$
$$0 \to t_{\Lambda^\vee} \to (\Lambda^\vee)_R \to t'_{\Lambda^\vee} \to 0$$

be short exact sequences of $\mathcal{O}_B \otimes R$-modules. Assume that t'_Λ and $t'_{\Lambda^\vee}$ are finite locally free over R and that we have equalities

$$\det_{t'_\Lambda} = \det_{V_{\mathbb{C},-i}} \otimes_{\mathcal{O}_E} R = \det_{t'_{\Lambda^\vee}} .$$

Denote by $\psi : (\Lambda^\vee)_R \xrightarrow{\sim} (\Lambda_R)^{\vee,R}$ the isomorphism corresponding to $(\cdot,\cdot)_{\Lambda,R}$. Then the following statements are equivalent.

(1) The composition $(t'_\Lambda)^{\vee,R} \to (\Lambda_R)^{\vee,R} \xrightarrow{\psi^{-1}} (\Lambda^\vee)_R \to t'_{\Lambda^\vee}$ is zero.
(2) The submodules t_Λ and $t_{\Lambda^\vee}$ pair to zero under $(\cdot,\cdot)_{\Lambda,R}$.
(3) $t_\Lambda^{\perp,(\cdot,\cdot)_{\Lambda,R}} = t_{\Lambda^\vee}$.

Proof. Consider the following diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & (t'_\Lambda)^{\vee,R} \arrow[r] & (\Lambda_R)^{\vee,R} \arrow[r] & t_\Lambda^{\vee,R} \arrow[r] & 0 \\ 0 \arrow[r] & t_{\Lambda^\vee} \arrow[r] & (\Lambda^\vee)_R \arrow[r] \arrow[u, "\simeq"', "\psi"] & t'_{\Lambda^\vee} \arrow[r] & 0 \end{tikzcd}$$

Condition (1) is equivalent to $\psi^{-1}((t'_\Lambda)^{\vee,R}) \subset t_{\Lambda^\vee}$, condition (2) is equivalent to $\psi(t_{\Lambda^\vee}) \subset (t'_\Lambda)^{\vee,R}$ and condition (3) is equivalent to ψ inducing an isomorphism $t_{\Lambda^\vee} \xrightarrow{\sim} (t'_\Lambda)^{\vee,R}$.

Now $(t'_\Lambda)^{\vee,R}$ and $t_{\Lambda^\vee}$ are locally direct summands in $(\Lambda_R)^{\vee,R}$ and $(\Lambda^\vee)_R$, respectively. By (3.1.2) and Proposition 2.3.5(2) both have the same rank over R. This implies the desired equivalences. □

We will use the following obvious variant of [32, Definition 3.27], adding one condition which seems natural to us and which will be used below.

Definition 3.3.2. *The local model M^{loc} is the functor on the category of $\mathcal{O}_{E_{\mathcal{Q}}}$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_\Lambda)_{\Lambda\in\mathcal{L}}$ of $\mathcal{O}_B \otimes R$-submodules $t_\Lambda \subset \Lambda_R$ satisfying the following conditions for all $\Lambda \subset \Lambda'$ in $\mathcal{L}$.*

(1) *We have* $\rho_{\Lambda',\Lambda,R}(t_\Lambda) \subset t_{\Lambda'}$, *so that we get a commutative diagram*

$$\begin{tikzcd} t_\Lambda \arrow[r] \arrow[d] & t_{\Lambda'} \arrow[d] \\ \Lambda_R \arrow[r, "\rho_{\Lambda',\Lambda,R}"] & \Lambda'_R. \end{tikzcd}$$

(2) *The quotient* Λ_R/t_Λ *is a finite locally free* R*-module.*

(3) *We have an equality*

$$\det_{\Lambda_R/t_\Lambda} = \det_{V_{\mathbb{C},-i}} \otimes_{\mathcal{O}_E} R$$

of morphisms $V_{\mathcal{O}_B\otimes R} \to \mathbb{A}^1_R$.

(4) *Under the pairing* $(\cdot,\cdot)_{\Lambda,R} : \Lambda_R \times \Lambda_R^\vee \to R$, *the submodules* t_Λ *and* $t_{\Lambda^\vee}$ *pair to zero.*

(5) *We have* $\vartheta_{\Lambda,b,R}(t_\Lambda^b) = t_{b\Lambda}$ *for all* $b \in (B \otimes \mathbb{Q}_p)^\times$ *that normalize* $\mathcal{O}_B \otimes \mathbb{Z}_p$.

Remark 3.3.3. *By definition,* M^{loc} *is a closed subscheme of a finite product of Grassmannians. In particular* M^{loc} *is a projective scheme over* $\operatorname{Spec} \mathcal{O}_{E_Q}$.

Remark 3.3.4. *Let* R *be an* $\mathcal{O}_{E_Q}$*-algebra and* $(t_\Lambda)_\Lambda \in M^{\mathrm{loc}}(R)$. *For* $\Lambda \in \mathcal{L}$ *the decomposition* $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$ *induces a decomposition* $t_\Lambda = t_{\Lambda,1} \times \cdots \times t_{\Lambda,m}$ *into* $\mathcal{O}_{B_i} \otimes R$*-submodules* $t_{\Lambda,i} \subset \Lambda_{i,R}$. *Let* $i \in \{1,\dots,m\}$ *and let* $\Lambda \subset \Lambda'$ *in* $\mathcal{L}$ *with* $\Lambda_i = \Lambda'_i$. *From condition 3.3.2(1) we conclude that* $t_{\Lambda,i} \subset t_{\Lambda',i}$. *From condition 3.3.2(3) we conclude (using Lemma 2.3.8 and Lemma 2.3.4) that* $t_{\Lambda,i}$ *and* $t_{\Lambda',i}$ *both have the same rank over* R. *Thus* $t_{\Lambda,i} = t_{\Lambda',i}$ *in view of 3.3.2(2). Consequently we may unambiguously write* $t_{\Lambda_i} = t_{\Lambda,i}$.

We conclude that the family $(t_\Lambda)_{\Lambda\in\mathcal{L}}$ *is determined by to the tuple of families*

$$\big((t_{\Lambda_1})_{\Lambda_1\in\mathcal{L}_1},\dots,(t_{\Lambda_m})_{\Lambda_m\in\mathcal{L}_m}\big).$$

In view of Lemma 2.3.8, all conditions of Definition 3.3.2 with the exception of condition (4) translate into independent conditions on the individual (t_{Λ_i}).

Lemma 3.3.5. *Let* R *be an* $\mathcal{O}_{E_Q}$*-algebra and let* $((A_\Lambda),(\gamma_\Lambda)) \in \widetilde{\mathcal{A}}(R)$. *For* $\Lambda \in \mathcal{L}$ *define* $t_\Lambda := \gamma_\Lambda(\omega_{A_\Lambda^\vee}) \subset \Lambda_R$ *(see Remark 2.4.6). Then* $(t_\Lambda)_\Lambda \in M^{\mathrm{loc}}(R)$.

Proof. Conditions 3.3.2(1) and 3.3.2(5) are fulfilled as the inclusion $\omega_{A^\vee} \subset H_1^{dR}(A/R)$ is functorial in the abelian scheme A/R. From Section 2.4 we know that $H_1^{dR}(A/R)/\omega_{A^\vee} \simeq \operatorname{Lie}(A)$, so that conditions 3.3.2(2) and 3.3.2(3) are fulfilled in view of the assumption that $(A_\Lambda)_\Lambda$ is of determinant $\det_{V_{\mathbb{C},-i}}$. Finally condition 3.3.2(4) is satisfied by Lemma 2.4.8. □

Definition 3.3.6. *Denote by* $\widetilde{\psi} : \widetilde{\mathcal{A}} \to M^{\mathrm{loc}}$ *the morphism given on* R*-valued points by*

$$\widetilde{\mathcal{A}}(R) \to M^{\mathrm{loc}}(R),$$
$$((A_\Lambda),(\gamma_\Lambda)) \mapsto (\gamma_\Lambda(\omega_{A_\Lambda^\vee}))_\Lambda.$$

$\operatorname{Aut}(\mathcal{L})$ acts from the left on M^{loc} via $(\varphi_\Lambda)\cdot(t_\Lambda) = (\varphi_\Lambda(t_\Lambda))$ and $\widetilde{\psi}$ is equivariant for this action.

Definition 3.3.7. *The diagram*

$$\begin{tikzcd} & \widetilde{\mathcal{A}} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\ \mathcal{A} & & M^{\mathrm{loc}} \end{tikzcd}$$

is called the local model diagram*.*

Consider the decomposition

$$M^{\mathrm{loc}}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} M_x^{\mathrm{loc}}$$

into $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-orbits.

Remark 3.3.8. *Let* $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$*. By* [32, Theorem 3.16] *the* $\mathbb{F}$*-group* $\mathrm{Aut}(\mathcal{L})_{\mathbb{F}}$ *is smooth and affine. This implies that the subset* $M_x^{\mathrm{loc}} \subset M^{\mathrm{loc}}(\mathbb{F})$ *is locally closed, and we equip it with the reduced scheme structure. Thus* M_x^{loc} *is a smooth quasi-projective variety over* $\mathbb{F}$*.*

For $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$, set $\widetilde{\mathcal{A}}_x = \widetilde{\psi}(\mathbb{F})^{-1}(M_x^{\mathrm{loc}})$ and $\mathcal{A}_x = \widetilde{\varphi}(\mathbb{F})(\widetilde{\mathcal{A}}_x)$. We claim that we obtain set-theoretic decompositions

$$\widetilde{\mathcal{A}}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \widetilde{\mathcal{A}}_x, \qquad \mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x.$$

First $\widetilde{\varphi}(\mathbb{F})$ is surjective by Proposition 3.2.8, so that

$$\mathcal{A}(\mathbb{F}) = \bigcup_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x.$$

Let $x, y \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$ and assume that $A \in \mathcal{A}_x \cap \mathcal{A}_y$. If $(A, \gamma_x) \in \widetilde{\mathcal{A}}_x$ and $(A, \gamma_y) \in \widetilde{\mathcal{A}}_y$ are preimages of A under $\widetilde{\varphi}(\mathbb{F})$, there is a $g \in \mathrm{Aut}(\mathcal{L})(\mathbb{F})$ with $g \cdot (A, \gamma_x) = (A, \gamma_y)$. Then

$$\widetilde{\psi}(\mathbb{F})((A, \gamma_y)) = \widetilde{\psi}(\mathbb{F})(g \cdot (A, \gamma_x)) = g \cdot \widetilde{\psi}(\mathbb{F})((A, \gamma_x)).$$

As the left-hand side is contained in M_y^{loc} and the right-hand side is contained in M_x^{loc}, we deduce that $x = y$.

Definition 3.3.9. *The decomposition*

$$\mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x$$

is called the Kottwitz-Rapoport *(or KR)* stratification *on* $\mathcal{A}$*.*

3.4. **Digression on simple algebras.** Let $D/\mathbb{Q}_p$ be a finite division algebra. We denote by $\mathcal{O}_D \subset D$ its unique maximal $\mathbb{Z}_p$-order, see [33, Theorem 12.8]. Denote by $\mathfrak{p} \subset \mathcal{O}_D$ the unique maximal ideal and by $k = \mathcal{O}_D/\mathfrak{p}$ the corresponding residue field, see [33, Theorem 13.2]. Every simple left (resp. right) $\mathcal{O}_D$-module is isomorphic to k.

Let $A/\mathbb{Q}_p$ be a finite simple algebra and let $\mathcal{O}_A \subset A$ be a maximal $\mathbb{Z}_p$-order. By [33, Theorem 17.3] there exist a finite division algebra $D/\mathbb{Q}_p$, an integer $n \in \mathbb{N}$ and an isomorphism $A \simeq M^{n \times n}(D)$ inducing an isomorphism

$\mathcal{O}_A \simeq M^{n\times n}(\mathcal{O}_D)$. By [33, 17.8] every simple left (resp. right) $M^{n\times n}(\mathcal{O}_D)$-module is isomorphic to $k^n = M^{n\times 1}(k)$ (resp. $k^n = M^{1\times n}(k)$). We note the following consequence for reference below.

Remark 3.4.1. *Let $A/\mathbb{Q}_p$ be a finite simple algebra and let $\mathcal{O}_A \subset A$ be a maximal $\mathbb{Z}_p$-order. Then there is an integer N such that all simple left $\mathcal{O}_A$-modules and all simple right $\mathcal{O}_A$-modules have cardinality N.*

3.5. The p-rank on a KR stratum.

Definition 3.5.1. *The multichain $\mathcal{L}$ is called* complete *if for any two neighbors $\Lambda \subset \Lambda'$ in $\mathcal{L}$, the quotient Λ'/Λ is a simple $\mathcal{O}_B \otimes \mathbb{Z}_p$-module.*

Let R be a ring and let $(A,\kappa),(A',\kappa') \in AV_R$. By functoriality and Remark A.16, the action κ induces an action $\mathcal{O}_B\otimes\mathbb{Z}_p \to \mathrm{End}(A[p^\infty])$. Let $\varrho : A \to A'$ be a morphism in AV_R which is a $\mathbb{Z}_{(p)}$-isogeny. Again by functoriality, the action $\mathcal{O}_B \otimes \mathbb{Z}_p \to \mathrm{End}(A[p^\infty])$ induces an action $\mathcal{O}_B \otimes \mathbb{Z}_p \to \mathrm{End}(\ker\varrho)$. Consequently (3.1.1) induces a decomposition $\ker\varrho = (\ker\varrho)_1 \times\cdots\times (\ker\varrho)_m$ into closed subschemes. In fact the $(\ker\varrho)_i$ are finite locally free group schemes by Lemma A.3. The action $\mathcal{O}_B \otimes \mathbb{Z}_p \to \mathrm{End}(\ker\varrho)$ decomposes into actions $\mathcal{O}_{B_i} \to \mathrm{End}((\ker\varrho)_i)$.

Lemma 3.5.2. *Assume that $\mathcal{L}$ is complete. Let $(A_\Lambda, \varrho_{\Lambda',\Lambda})$ be an $\mathcal{L}$-set of abelian varieties over $\mathbb{F}$ and let $\Lambda \subset \Lambda'$ be neighbors in $\mathcal{L}$. Then $\ker\varrho_{\Lambda',\Lambda}$ is either étale unipotent or infinitesimal multiplicative or infinitesimal unipotent.*

Proof. As Λ and Λ' are neighbors, there is a unique $i_0 \in \{1,\dots,m\}$ with $\Lambda_{i_0} \supsetneq \Lambda'_{i_0}$ and as $\mathcal{L}$ is complete we know that $\Lambda'_{i_0}/\Lambda_{i_0}$ is a simple left $\mathcal{O}_{B_{i_0}}$-module. Let $N = |\Lambda'_{i_0}/\Lambda_{i_0}|$. By the definition of an $\mathcal{L}$-set of abelian varieties we know that $(\ker\varrho_{\Lambda',\Lambda})_i = 0$ for $i \neq i_0$ and that $G := (\ker\varrho_{\Lambda',\Lambda})_{i_0}$ has rank N over $\mathbb{F}$.

The action $\mathcal{O}_{B_{i_0}} \to \mathrm{End}\,G$ induces on $G(\mathbb{F})$ the structure of a left $O_{B_{i_0}}$-module and as $|G(\mathbb{F})| \le \mathrm{rk}\,G = N$, Remark 3.4.1 implies $|G(\mathbb{F})| \in \{0,N\}$. As $|G(\mathbb{F})| = \mathrm{rk}(G^{e,u})$, we conclude that $G^{e,u} \in \{0,G\}$.

We also obtain on $D(G)(\mathbb{F})$ the structure of a right $O_{B_{i_0}}$-module and we analogously obtain that $D(G)^{e,u} \in \{0, D(G)\}$. As $D(G)^{e,u} = D(G^{i,m})$, it follows that also $G^{i,m} \in \{0,G\}$. □

Definition 3.5.3 ([22, p. 146-147])**.** *Let $A/\mathbb{F}$ be an abelian variety. The integer $\log_p \mathrm{rk}_{e,u} A[p] = \log_p \mathrm{rk}_{i,m} A[p]$ is called the* p-rank *of A.*

Proposition 3.5.4. *Assume that $\mathcal{L}$ is complete. Let $(A_\Lambda, \varrho_{\Lambda',\Lambda})$ be an $\mathcal{L}$-set of abelian varieties over $\mathbb{F}$. Let $\Lambda \in \mathcal{L}$ and choose a sequence $p^{-1}\Lambda = \Lambda^{(0)} \supsetneq \Lambda^{(1)} \supsetneq \cdots \supsetneq \Lambda^{(k)} = \Lambda$ of neighbors $\Lambda^{(j-1)} \supsetneq \Lambda^{(j)}$ in $\mathcal{L}$. Define*

$$J_{e,u} = \{j \in \{1,\dots,k\} \mid \ker\varrho_{\Lambda^{(j-1)},\Lambda^{(j)}} \text{ is étale}\}.$$

The p-rank of A_Λ is equal to

$$\sum_{j\in J_{e,u}} \log_p |\Lambda^{(j-1)}/\Lambda^{(j)}|.$$

Proof. Denote by $\theta_{p^{-1}\Lambda,p} : A_{p^{-1}\Lambda} \xrightarrow{\sim} A_\Lambda$ the periodicity isomorphism, see Remark 3.2.2, so that

$$[p]_{A_\Lambda} = \theta_{p^{-1}\Lambda,p} \circ \prod_{j=1}^{k} \varrho_{\Lambda^{(j-1)},\Lambda^{(j)}}.$$

Using Lemma 2.6.2 this implies

$$\mathrm{rk}_{e,u}\, A_\Lambda[p] = \prod_{j=1}^{k} \mathrm{rk}_{e,u} \ker \varrho_{\Lambda^{(j-1)},\Lambda^{(j)}}.$$

Lemma 3.5.2 and the definition of an $\mathcal{L}$-set of abelian varieties yield

$$\mathrm{rk}_{e,u} \ker \varrho_{\Lambda^{(j-1)},\Lambda^{(j)}} = \begin{cases} |\Lambda^{(j-1)}/\Lambda^{(j)}| & \text{if } j \in J_{e,u}, \\ 1 & \text{otherwise.} \end{cases}$$

□

Proposition 3.5.5. *Let $A = (A_\Lambda, \varrho_{\Lambda',\Lambda}) \in \mathcal{A}(\mathbb{F})$, choose a lift $A' \in \widetilde{\mathcal{A}}(\mathbb{F})$ of A under $\widetilde{\varphi}(\mathbb{F})$ and let $(t_\Lambda) = \widetilde{\psi}(\mathbb{F})(A') \in M_x^{\mathrm{loc}}$. Let $\Lambda \subset \Lambda'$ in $\mathcal{L}$. Then*

$$\begin{aligned} &\ker \varrho_{\Lambda',\Lambda} \text{ is multiplicative} \\ &\Leftrightarrow \rho_{\Lambda',\Lambda,\mathbb{F}}(t_\Lambda) = t_{\Lambda'} \end{aligned} \tag{3.5.6}$$

and

$$\begin{aligned} &\ker \varrho_{\Lambda',\Lambda} \text{ is étale} \\ &\Leftrightarrow \Lambda'_{\mathbb{F}} = \mathrm{im}\, \rho_{\Lambda',\Lambda,\mathbb{F}} + t_{\Lambda'}. \end{aligned} \tag{3.5.7}$$

Proof. Write $\varrho = \varrho_{\Lambda',\Lambda}$. By [22, §15, Theorem 1] we have an equality $\ker(\varrho^\vee) = D(\ker \varrho)$. Consequently $\ker \varrho$ is multiplicative if and only if $\ker(\varrho^\vee)$ is étale, which by Lemma 2.6.3 is the case if and only if the induced map $\omega_{A_\Lambda^\vee} \to \omega_{A_{\Lambda'}^\vee}$ is surjective. Using the functoriality of the inclusion $\omega_{A_\Lambda^\vee} \subset H_1^{dR}(A_\Lambda)$ and the definition of $\widetilde{\psi}$, we obtain (3.5.6). Furthermore (3.5.7) follows at once from Corollary 2.6.4. □

Corollary 3.5.8. *Let $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$ and $(A_\Lambda, \varrho_{\Lambda',\Lambda}), (A'_\Lambda, \varrho'_{\Lambda',\Lambda}) \in \mathcal{A}_x$. Let $\Lambda \subset \Lambda'$ in $\mathcal{L}$. Then $\ker \varrho_{\Lambda',\Lambda}$ is étale if and only if $\ker \varrho'_{\Lambda',\Lambda}$ is étale.*

Proof. For $(t_\Lambda) \in M^{\mathrm{loc}}(\mathbb{F})$, the condition $\Lambda'_{\mathbb{F}} = \mathrm{im}\, \rho_{\Lambda',\Lambda,\mathbb{F}} + t_{\Lambda'}$ is clearly invariant under the $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-action on $M^{\mathrm{loc}}(\mathbb{F})$. The claim therefore follows from (3.5.7). □

Theorem 3.5.9. *Assume that $\mathcal{L}$ is complete. Let $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$ and $(A_\Lambda, \varrho_{\Lambda',\Lambda}), (A'_\Lambda, \varrho'_{\Lambda',\Lambda}) \in \mathcal{A}_x$. Let $\Lambda, \Lambda' \in \mathcal{L}$. Then the p-ranks of A_Λ and $A'_{\Lambda'}$ coincide. In other words, the p-rank is constant on a KR stratum.*

Proof. The p-rank of an abelian variety is an isogeny invariant by [22, p. 147], so that it suffices to treat the case $\Lambda = \Lambda'$. The statement then follows from Proposition 3.5.4 and Corollary 3.5.8. □

3.6. **Digression on local fields.** Denote by K' the maximal unramified extension of $\mathbb{Q}_p$ and by K the completion of K'. Denote by $\mathcal{O}_{K'}$ and $\mathcal{O}_K$ the valuation ring of K and K', respectively. We identify the residue field of both K' and K with $\mathbb{F}$. We denote by σ the Frobenius automorphism on both K' and K, inducing the usual Frobenius $\mathbb{F} \to \mathbb{F}, x \mapsto x^p$ on the residue field.

Let $F/\mathbb{Q}_p$ be a finite extension. We will always denote by $\mathcal{O}_F$ the valuation ring of F. Let F' be the maximal unramified extension of $\mathbb{Q}_p$ in F. Denote by Ξ the set of all $\mathbb{Q}_p$-embeddings $F' \hookrightarrow K'$. For $\xi \in \Xi$ we define $L_\xi = F \otimes_{F',\xi} K$. Then L_ξ is a finite field extension of K, with valuation ring $\mathcal{O}_{L_\xi} = \mathcal{O}_F \otimes_{\mathcal{O}_{F'},\xi} \mathcal{O}_K$. There are canonical isomorphisms

$$F \otimes K = \prod_{\xi \in \Xi} L_\xi, \qquad \mathcal{O}_F \otimes \mathcal{O}_K = \prod_{\xi \in \Xi} \mathcal{O}_{L_\xi}. \tag{3.6.1}$$

Let $n \in \mathbb{N}$ and consider the simple $\mathbb{Q}_p$-algebra $A = M^{n\times n}(F)$ and the maximal $\mathbb{Z}_p$-order $\mathcal{O}_A = M^{n\times n}(\mathcal{O}_F)$ in A. Write $A_\xi = M^{n\times n}(L_\xi)$ and $\mathcal{O}_{A_\xi} = M^{n\times n}(\mathcal{O}_{L_\xi})$, so that (3.6.1) induces decompositions

$$A \otimes K = \prod_{\xi \in \Xi} A_\xi, \qquad \mathcal{O}_A \otimes \mathcal{O}_K = \prod_{\xi \in \Xi} \mathcal{O}_{A_\xi}. \tag{3.6.2}$$

In particular we see that $\mathcal{O}_A \otimes \mathcal{O}_K$ is a maximal $\mathcal{O}_K$-order in the K-algebra $A \otimes K$.

Lemma 3.6.3. *Let W be a finite left $A \otimes K$-module and let Λ and Λ' be $\mathcal{O}_A \otimes \mathcal{O}_K$-lattices in W. Then $\Lambda \simeq \Lambda'$ as $\mathcal{O}_A \otimes \mathcal{O}_K$-modules.*

Proof. In view of (3.6.2) we may assume that $A \otimes K = M^{n\times n}(L)$ and $\mathcal{O}_A \otimes \mathcal{O}_K = M^{n\times n}(\mathcal{O}_L)$ for some finite extension L/K with valuation ring $\mathcal{O}_L$.

As Λ is also an $\mathcal{O}_L$-lattice in W, it is a free $\mathcal{O}_L$-module of rank $N = \dim_L W$. Let λ be the $\mathcal{O}_L$-module corresponding to the $M^{n\times n}(\mathcal{O}_L)$-module Λ under Morita equivalence (see [33, §§16]). There is an isomorphism $\Lambda \simeq \lambda^{n\times 1}$ of $M^{n\times n}(\mathcal{O}_L)$-modules, so that λ is free of rank N/n over $\mathcal{O}_L$. Consequently the isomorphism type of λ is determined by the integer N, and by Morita equivalence the same is therefore true for Λ. $\square$

3.7. **Embedding the local model into a p-adic flag set.** Denote by G' the group on the category of $\mathbb{Q}$-algebras with $G'(R) = \mathrm{GL}_{B\otimes R}(V \otimes R)$. By definition we have $G(R) \subset G'(R)$ and we denote by $\iota : G \hookrightarrow G'$ the inclusion. Clearly G' is a connected reductive group.

Recall the various decompositions induced by (3.1.1).

Lemma 3.7.1. *The base-change $G'_{\mathbb{Q}_p}$ is quasi-split if and only if for each $1 \le i \le m$ with $V_i \ne \{0\}$, there exist a finite extension $F_i/\mathbb{Q}_p$ and an integer $n_i \in \mathbb{N}$ such that $B_i \simeq M^{n_i\times n_i}(F_i)$.*

Proof. Let $1 \le i \le m$. There are a finite division algebra D_i over $\mathbb{Q}_p$ and an integer $n_i \in \mathbb{Z}$ such that $B_i \simeq M^{n_i\times n_i}(D_i)$. Let v_i be the left D_i-module corresponding under Morita equivalence to the left B_i-module V_i. Then $\mathrm{GL}_{B_i\otimes R}(V_i \otimes R) = \mathrm{GL}_{D_i\otimes R}(v_i \otimes R)$ for any $\mathbb{Q}_p$-algebra R. As $G'_{\mathbb{Q}_p}$

decomposes into the product of these groups, it suffices to show the following statement.

Claim. Let $D/\mathbb{Q}_p$ be a finite division algebra and let $n \in \mathbb{N}_{\geq 1}$. Then the connected reductive $\mathbb{Q}_p$-group H with $H(R) = \mathrm{GL}_n(D \otimes_{\mathbb{Q}_p} R)$ is quasi-split if and only if D is a field.

Denote by K the center of D and by $F \supset K$ a maximal subfield of D. Consider the connected reductive K-group H' with $H'(R) = \mathrm{GL}_n(D \otimes_K R)$, and the torus $T' = \mathrm{Res}_{F/K} D_{n,F}$ in H'. Using that $\dim_K D = (\dim_K F)^2$ (see [33, 7.15]), one checks immediately (by passing to an algebraic closure of K) that T' is a maximal torus in H'. Then $T := \mathrm{Res}_{K/\mathbb{Q}_p} T'$ is a maximal torus in $H = \mathrm{Res}_{K/\mathbb{Q}_p} H'$. By [39, Lemma 16.2.7.] we know that $S := D_{n,\mathbb{Q}_p}$ is a maximal split torus in H. Consequently H is quasi-split if and only if T equals the centralizer $C_H(S)$ of S in H, see [39, §16.2]. As $C_H(S)(\mathbb{Q}_p)$ certainly contains all matrices of the form dI_n, $d \in D^\times$, we see that the equality $C_H(S) = T$ implies $D = F$, as desired. □

Unless explicitly stated otherwise, we assume for the rest of Section 3 that $G'_{\mathbb{Q}_p}$ is quasi-split.

Remark 3.7.2. *This is a rather mild restriction: The possible simple factors of the pair $(B \otimes \mathbb{Q}_p, *)$ fall into the following four cases, see* [32, A.6].

- *(I) $M^{n\times n}(D) \times M^{n\times n}(D)^{opp}$ for a finite division algebra $D/\mathbb{Q}_p$, equipped with the involution $(x, y) \mapsto (y, x)$.*
- *(II) $M^{n\times n}(F)$ for a finite extension $F/\mathbb{Q}_p$, equipped with an involution of the first kind.*
- *(III) $M^{n\times n}(F)$ for a finite extension $F/\mathbb{Q}_p$, equipped with an involution of the second kind.*
- *(IV) $M^{n\times n}(D)$ for a quaternion division algebra D over some finite extension of $\mathbb{Q}_p$, equipped with an involution of the first kind.*

Thus our assumption imposes a restriction in case (I) and excludes case (IV).

We use the notation of Section 3.6. For an $\mathcal{O}_B \otimes \mathcal{O}_K$-lattice Λ in V_K we write $\Lambda^\vee = \{x \in V_K \mid (x, \Lambda)_K \subset \mathcal{O}_K\}$.

Definition 3.7.3.
- *Let $1 \leq i \leq m$. We denote by $\mathcal{F}'_i$ the set of all tuples $(M_\Lambda)_{\Lambda \in \mathcal{L}_i}$ of $\mathcal{O}_{B_i} \otimes \mathcal{O}_K$-lattices M_Λ in $V_i \otimes K$ satisfying the following conditions for all neighbors $\Lambda \subset \Lambda'$ in $\mathcal{L}_i$ and all $b \in B_i^\times$ normalizing $\mathcal{O}_{B_i}$.*
  - *(1) We have $M_\Lambda \subset M_{\Lambda'}$.*
  - *(2) There is an isomorphism $M_{\Lambda'}/M_\Lambda \simeq (\Lambda'/\Lambda) \otimes \mathcal{O}_K$ of $\mathcal{O}_{B_i} \otimes \mathcal{O}_K$-modules.*
  - *(3) We have $M_{b\Lambda} = bM_\Lambda$.*
- *Let $\mathcal{F}' = \prod_{i=1}^m \mathcal{F}'_i$. For $((M_{i,\Lambda})_{\Lambda \in \mathcal{L}_i})_{i=1}^m \in \mathcal{F}'$ and $\Lambda \in \mathcal{L}$ we define $M_\Lambda = M_{1,\Lambda_1} \times \cdots \times M_{m,\Lambda_m}$. In this way we consider the elements of $\mathcal{F}'$ as tuples $(M_\Lambda)_{\Lambda \in \mathcal{L}}$ of $\mathcal{O}_B \otimes \mathcal{O}_K$-lattices M_Λ in $V \otimes K$.*

- *For $n \in \mathbb{Z}$ denote by $\mathcal{F}^{(n)} \subset \mathcal{F}'$ be the subset of those $(M_\Lambda)_{\Lambda\in\mathcal{L}}$ satisfying $M_\Lambda^\vee = p^n M_{\Lambda^\vee}$ for all $\Lambda \in \mathcal{L}$. We define*
$$\mathcal{F} = \bigcup_{n\in\mathbb{Z}} \mathcal{F}^{(n)}.$$

In particular $\mathcal{L} \otimes \mathcal{O}_K \in \mathcal{F}^{(0)}$.

Remark 3.7.4. *Let $1 \le i \le m$ and $(M_\Lambda)_{\Lambda\in\mathcal{L}_i} \in \mathcal{F}_i'$. For $\Lambda \subset \Lambda'$ in $\mathcal{L}_i$ denote by $\varrho_{\Lambda',\Lambda} : M_\Lambda \to M_{\Lambda'}$ the inclusion. For $b \in B_i^\times$ normalizing $\mathcal{O}_{B_i}$ denote by $\theta_{\Lambda,b} : M_\Lambda^b \to M_{b\Lambda}$ the isomorphism given by multiplication with b. In view of Lemma 3.6.3, the resulting family $(M_\Lambda, \varrho_{\Lambda',\Lambda}, \theta_{\Lambda,b})$ is a chain of $\mathcal{O}_{B_i} \otimes \mathcal{O}_K$-modules of type $(\mathcal{L}_i)$. Consequently we can consider an element of $\mathcal{F}'$ as a multichain of $\mathcal{O}_B \otimes \mathcal{O}_K$-modules of type $(\mathcal{L})$.*

Let $(M_\Lambda)_\Lambda \in \mathcal{F}^{(0)}$ and denote by $\mathcal{E}_\Lambda : M_\Lambda \times M_{\Lambda^\vee} \to \mathcal{O}_K$ the restriction of $(\cdot,\cdot)_K$. Then $(\mathcal{E}_\Lambda)_\Lambda$ is a polarization of $(M_\Lambda)_\Lambda$ and in this way we consider an element of $\mathcal{F}^{(0)}$ as a polarized multichain of $\mathcal{O}_B \otimes \mathcal{O}_K$-modules of type $(\mathcal{L})$.

Lemma 3.7.5. *Let $(M_\Lambda), (M_\Lambda') \in \mathcal{F}^{(0)}$, considered as polarized multichains of $\mathcal{O}_B \otimes \mathcal{O}_K$-modules of type $(\mathcal{L})$. Let $(\varphi_\Lambda) : (M_\Lambda) \to (M_\Lambda')$ be an isomorphism. Then there is a $g \in G(K)$ such that the morphism $g : V \otimes K \to V \otimes K$ restricts to $\varphi_\Lambda : M_\Lambda \to N_\Lambda$ for each $\Lambda \in \mathcal{L}$.*

Proof. Let $\Lambda \subset \Lambda'$ in $\mathcal{L}$. We have $M_\Lambda \otimes_{\mathcal{O}_K} K = V \otimes K = M_\Lambda' \otimes_{\mathcal{O}_K} K$, and the inclusions $M_\Lambda \subset M_{\Lambda'}$ and $M_\Lambda' \subset M_{\Lambda'}'$ induce the identity on $V \otimes K$ after base-change to K. In view of (2.7.2) we conclude that $\varphi_\Lambda \otimes_{\mathcal{O}_K} K = \varphi_{\Lambda'} \otimes_{\mathcal{O}_K} K$. The morphism $g := \varphi_\Lambda \otimes_{\mathcal{O}_K} K$ has the desired properties. □

The proof of the following result is similar to and therefore based on the proof of [27, Theorem 4.1].

Proposition 3.7.6. *The group $G(K)$ acts transitively on $\mathcal{F}$ via $g \cdot (M_\Lambda)_\Lambda = (gM_\Lambda)_\Lambda$. Denote by $I \subset G(K)$ the stabilizer of $\mathcal{L} \otimes \mathcal{O}_K$. We obtain a bijection*

$$G(K)/I \xrightarrow{\sim} \mathcal{F}, \ g \mapsto g \cdot (\mathcal{L} \otimes \mathcal{O}_K). \tag{3.7.7}$$

Proof. Let $g \in G(K)$ and let Λ be an $\mathcal{O}_B \otimes \mathcal{O}_K$-lattice in V_K. A short computation shows that

$$(g\Lambda)^\vee = c(g)^{-1} g\Lambda^\vee. \tag{3.7.8}$$

There are $n \in \mathbb{Z}$ and $u \in \mathcal{O}_K^\times$ such that $c(g) = p^n u$. Thus the action in question is well-defined.

Let $\mathcal{M} = (M_\Lambda)_{\Lambda\in\mathcal{L}} \in \mathcal{F}^{(n)}$ for some $n \in \mathbb{Z}$. By Lemma 3.7.9 below there is an $h \in G(K)$ with $c(h) = p^n$, so that $(hM_\Lambda)^\vee = hM_{\Lambda^\vee}$. Then $\mathcal{N} := h\mathcal{M}$ lies in $\mathcal{F}^{(0)}$ and we consider $\mathcal{N}$ as a polarized multichain of $\mathcal{O}_B \otimes \mathcal{O}_K$-modules of type $(\mathcal{L})$, using Remark 3.7.4.

Let $\mathcal{I} = \mathrm{Isom}(\mathcal{L} \otimes \mathcal{O}_K, \mathcal{N})$. By [32, Theorem 3.16] we know that $\mathcal{I}(\mathbb{F}) \neq \emptyset$. By Proposition 2.7.8 the canonical map $\mathcal{I}(\mathcal{O}_K) \to \mathcal{I}(\mathbb{F})$ is surjective. Hence $\mathcal{I}(\mathcal{O}_K) \neq \emptyset$, i.e. there is an isomorphism $\mathcal{L} \otimes \mathcal{O}_K \xrightarrow{\sim} \mathcal{N}$. By Lemma 3.7.5 such an isomorphism is given by multiplication with a single $g \in G(K)$ and consequently $(h^{-1}g) \cdot (\mathcal{L} \otimes \mathcal{O}_K) = \mathcal{M}$, as required. □

Lemma 3.7.9. *We do not assume that $G'_{\mathbb{Q}_p}$ is quasi-split. The morphism $c(K) : G(K) \to K^\times$ is surjective.*

Proof. Let X be an affine algebraic group over a field. We denote the connected component of the identity in X by X^0. Recall that the formation of X^0 commutes with extension of the base-field.

Let $c' : G^0 \to \mathbb{G}_m$ be the restriction of c and $H' \subset G^0$ the kernel of c'. Denote by $\overline{K}$ an algebraic closure of K. We claim that we obtain a short exact sequence $1 \to H'(\overline{K}) \to G^0(\overline{K}) \to \mathbb{G}_m(\overline{K}) \to 1$ and that H' is a connected linear algebraic group. The statement of the Lemma will then follow from the exact sequence of pointed sets $1 \to H'(K) \to G^0(K) \to \mathbb{G}_m(K) \to H^1(\mathrm{Gal}(\overline{K}/K), H'(\overline{K}))$, see [34, VII, Annexe, Proposition 1], and the fact that $H^1(\mathrm{Gal}(\overline{K}/K), H'(\overline{K}))$ is trivial by [35, III, 2.3, Théorème 1']. Note that loc. cit. indeed applies to K, as K is C_1 by Lang's theorem [20, Theorem 10], so that $\dim(K) \leq 1$ by [35, II, 3.2, Corollaire].

The surjectivity of the map $c'(\overline{K}) : G^0(\overline{K}) \to \mathbb{G}_m(\overline{K})$ does not pose a problem: Scalar multiplication of $\mathbb{Q}$ on V defines a morphism $\mathbb{G}_{m,\mathbb{Q}} \to G$. It factors through G^0 and the composition $\mathbb{G}_m \to G^0 \xrightarrow{c'} \mathbb{G}_m$ is equal to $x \mapsto x^2$, which alone induces a surjection on $\overline{K}$-valued points.

Let us show that H' is connected. For this we may work over $\mathbb{C}$. For the rest of this proof we identify a smooth affine scheme over $\mathbb{C}$ with its set of $\mathbb{C}$-valued points, considered as a classical affine variety over $\mathbb{C}$. This convention applies in particular to the base-change of any affine algebraic group over $\mathbb{Q}$.

Denote by $H \subset G$ the kernel of c, so that $H'_{\mathbb{C}} = H_{\mathbb{C}} \cap G^0_{\mathbb{C}}$. Denote by H^0 the connected component of H. Assume that we can find an affine algebraic group $\tilde{G} \supset G_{\mathbb{C}}$ over $\mathbb{C}$ such that $H_{\mathbb{C}} \cap \tilde{G}^0$ is connected. We then have the obvious chain of inclusions $H^0_{\mathbb{C}} \subset H_{\mathbb{C}} \cap G^0_{\mathbb{C}} \subset H_{\mathbb{C}} \cap \tilde{G}^0 \subset H^0_{\mathbb{C}}$, so that $H_{\mathbb{C}} \cap G^0_{\mathbb{C}} = H^0_{\mathbb{C}}$ is connected.

It remains to show that such a $\tilde{G}$ exists. Consider the semisimple $\mathbb{Q}$-algebra $C = \mathrm{End}_B(V)$ and denote by τ the adjoint involution for $(\cdot, \cdot)$ on C, so that for $\varphi \in C$ we have $(\varphi(x), y) = (x, \varphi^\tau(y))\,,\ x, y \in V$. By [18, p. 375], the pair $(C_{\mathbb{C}}, \tau)$ decomposes as a product $(C_{\mathbb{C}}, \tau) = (C_1, \tau) \times \cdots \times (C_r, \tau)$ of $\mathbb{C}$-algebras with involution (C_i, τ), such that each (C_i, τ) is of one of the following three types.

(1) $M^{n\times n}(\mathbb{C}) \times M^{n\times n}(\mathbb{C})^{opp}$ with the involution $(x, y) \mapsto (y, x)$.
(2) $M^{2n\times 2n}(\mathbb{C})$ with the involution $x \mapsto \widetilde{J}^{-1} x^t \widetilde{J}$, where $\widetilde{J} = \widetilde{J}_{2n}$.
(3) $M^{2n\times 2n}(\mathbb{C})$ with the involution $x \mapsto x^t$.

By definition we have $G_{\mathbb{C}} = \{x \in (C \otimes \mathbb{C})^\times \mid xx^\tau \in \mathbb{C}^\times\}$ and $H_{\mathbb{C}} = \{x \in (C \otimes \mathbb{C})^\times \mid xx^\tau = 1\}$. Define $G_i = \{x \in C_i^\times \mid xx^\tau \in \mathbb{C}^\times\}$ and $H_i = \{x \in C_i^\times \mid xx^\tau = 1\}$, considered as classical affine varieties. Consequently $H_{\mathbb{C}} = H_1 \times \cdots \times H_r$. Let us show that for $\tilde{G} := G_1 \times \cdots \times G_r$, the intersection $H_{\mathbb{C}} \cap \tilde{G}^0$ is indeed connected.

We may do this one factor at a time. According to the type (1), (2) or (3) of (C_i, τ), the groups G_i and H_i admit the following description.

(1) There is an obvious isomorphism $G_i \xrightarrow{\sim} \mathrm{GL}_{n,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}},\ (x,y) \mapsto (x,xy)$, identifying H_i with the first factor. Consequently G_i and H_i are both connected.
(2) In this case $G_i = \mathrm{GSp}_{2n,\mathbb{C}}$ and $H_i = \mathrm{Sp}_{2n,\mathbb{C}}$. Both of these groups are connected: This is clear for $\mathrm{Sp}_{2n,\mathbb{C}}$, as it is generated by transvections. For $\mathrm{GSp}_{2n,\mathbb{C}}$, note that the map $\mathbb{G}_{m,\mathbb{C}} \to \mathrm{GSp}_{2n,\mathbb{C}},\ x \mapsto \mathrm{diag}(x^{(n)}, 1^{(n)})$ is a splitting of the short exact sequence $1 \to \mathrm{Sp}_{2n,\mathbb{C}} \to \mathrm{GSp}_{2n,\mathbb{C}} \xrightarrow{c} \mathbb{G}_{m,\mathbb{C}} \to 1$. It thus induces an isomorphism $\mathrm{GSp}_{2n,\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}} \ltimes \mathrm{Sp}_{2n,\mathbb{C}}$.
(3) For a $\mathbb{C}$-vector space $W \neq 0$ and a non-degenerate symmetric bilinear form Φ on W, we define groups $\mathrm{GO}_\Phi = \{g \in \mathrm{GL}(W) \mid \exists c = c(g) \in \mathbb{C}^\times \forall x,y \in W : \Phi(gx,gy) = c\Phi(v,w)\}$ and $\mathrm{O}_\Phi = \{g \in \mathrm{GO}_\Phi \mid c(g) = 1\}$. We consider both GO_Φ and O_Φ as classical affine varieties over $\mathbb{C}$. If Φ is the standard symmetric bilinear form $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C},\ (x,y) \mapsto x^t y$ on $\mathbb{C}^n$, we also write $\mathrm{GO}_{n,\mathbb{C}}$ instead of GO_Φ and $\mathrm{O}_{n,\mathbb{C}}$ instead of O_Φ.

With this notation we have $G_i = \mathrm{GO}_{2n,\mathbb{C}}$ and $H_i = \mathrm{O}_{2n,\mathbb{C}}$. We claim that the short exact sequence $1 \to \mathrm{O}_{2n,\mathbb{C}} \to \mathrm{GO}_{2n,\mathbb{C}} \xrightarrow{c} \mathbb{G}_{m,\mathbb{C}} \to 1$ splits (non-canonically). Let Φ be the bilinear form described by the matrix $\widetilde{I}_{2n}$ with respect to the standard basis of $\mathbb{C}^{2n}$. Then a section $\mathbb{G}_{m,\mathbb{C}} \to \mathrm{GO}_\Phi$ of c is given by $x \mapsto \mathrm{diag}(x^{(n)}, 1^{(n)})$. As $\mathbb{C}$ is algebraically closed, any two non-degenerate symmetric bilinear forms on $\mathbb{C}^{2n}$ are equivalent, proving the claim.

Consequently we obtain an isomorphism $\mathrm{GO}_{2n} \simeq \mathbb{G}_{m,\mathbb{C}} \ltimes \mathrm{O}_{2n}$ inducing the identity on O_{2n}. It identifies GO_{2n}^0 with $(\mathbb{G}_{m,\mathbb{C}} \ltimes \mathrm{O}_{2n})^0 = \mathbb{G}_{m,\mathbb{C}} \ltimes \mathrm{O}_{2n}^0$. This shows that $\mathrm{O}_{2n}^0 = \mathrm{O}_{2n} \cap \mathrm{GO}_{2n}^0$, as required. $\square$

For $\Lambda \in \mathcal{L}$, denote by $\alpha_\Lambda : \Lambda \otimes \mathcal{O}_K \to \Lambda \otimes \mathbb{F}$ the morphism induced by the residue morphism $\mathcal{O}_K \to \mathbb{F}$.

Proposition 3.7.10. *Let $(t_\Lambda)_\Lambda \in M^{\mathrm{loc}}(\mathbb{F})$. For $\Lambda \in \mathcal{L}$ define $M_\Lambda = \alpha_\Lambda^{-1}(t_\Lambda) \subset \Lambda \otimes \mathcal{O}_K$. Then $(M_\Lambda)_\Lambda \in \mathcal{F}^{(-1)}$. Consequently we obtain an embedding*

$$\alpha : M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \mathcal{F},\ \alpha((t_\Lambda)_\Lambda) = (\alpha_\Lambda^{-1}(t_\Lambda))_\Lambda.$$

Proof. Assume we have shown that $(M_\Lambda)_\Lambda \in \mathcal{F}'$. Let $\Lambda \in \mathcal{L}$. From $(t_\Lambda, t_{\Lambda^\vee})_{\Lambda,\mathbb{F}} = 0$, we deduce that $(M_\Lambda, M_{\Lambda^\vee})_{\Lambda,\mathcal{O}_K} \subset p\mathcal{O}_K$ and hence that $p^{-1}M_{\Lambda^\vee} \subset (M_\Lambda)^\vee$.

From $p\Lambda \otimes \mathcal{O}_K \subset M_\Lambda$ on the other hand we deduce $pM_\Lambda^\vee \subset (\Lambda \otimes \mathcal{O}_K)^\vee = \Lambda^\vee \otimes \mathcal{O}_K$. By definition we know that $(M_\Lambda, pM_\Lambda^\vee)_{\Lambda,\mathcal{O}_K} \subset p\mathcal{O}_K$, which implies that $(t_\Lambda, \alpha_{\Lambda^\vee}(pM_\Lambda^\vee))_{\Lambda,\mathbb{F}} = 0$. By 3.3.2(4) and Lemma 3.3.1 we know that $t_\Lambda^{\perp,(\cdot,\cdot)_{\Lambda,\mathbb{F}}} = t_{\Lambda^\vee}$. Consequently $\alpha_{\Lambda^\vee}(pM_\Lambda^\vee) \subset t_{\Lambda^\vee}$, which shows that $pM_\Lambda^\vee \subset M_{\Lambda^\vee}$. Hence also $M_\Lambda^\vee \subset p^{-1}M_{\Lambda^\vee}$.

We now prove that $(M_\Lambda)_\Lambda \in \mathcal{F}'$. Using Remark 3.3.4 we may assume that $B \otimes \mathbb{Q}_p$ is simple. That M_Λ is an $\mathcal{O}_B \otimes \mathcal{O}_K$-lattice in V_K follows from the inclusions $p\Lambda \otimes \mathcal{O}_K \subset M_\Lambda \subset \Lambda \otimes \mathcal{O}_K$. Condition 3.7.3(1) follows from 3.3.2(1) and condition 3.7.3(3) follows from 3.3.2(5).

We need to verify 3.7.3(2). We have $B \otimes \mathbb{Q}_p = M^{n\times n}(F)$ and $\mathcal{O}_B \otimes \mathbb{Z}_p = M^{n\times n}(\mathcal{O}_F)$ for some $n \in \mathbb{N}$ and some finite extension $F/\mathbb{Q}_p$. Noting that

(3.6.2) induces compatible decompositions of all the objects in question (and keeping in mind Lemma 2.3.8) we may assume that $F/\mathbb{Q}_p$ is totally ramified.

Let $\pi \in \mathcal{O}_F$ be a uniformizer and let $\Lambda, \Lambda' \in \mathcal{L}$ with $\pi\Lambda' \subset \Lambda \subset \Lambda'$. Considering an $\mathcal{O}_K$-module annihilated by p as an $\mathbb{F}$-vector space, we have (3.7.11)

$$\dim_{\mathbb{F}}(\Lambda_{\mathcal{O}_K}/M_\Lambda) + \dim_{\mathbb{F}}(\Lambda'_{\mathcal{O}_K}/\Lambda_{\mathcal{O}_K}) = \dim_{\mathbb{F}}(M_{\Lambda'}/M_\Lambda) + \dim_{\mathbb{F}}(\Lambda'_{\mathcal{O}_K}/M_{\Lambda'}),$$

as both sides equal the length of the $\mathcal{O}_K$-module $\Lambda'_{\mathcal{O}_K}/M_\Lambda$. From condition 3.3.2(3) we conclude, using Lemma 2.3.4, that $\dim_{\mathbb{F}}(\Lambda_{\mathcal{O}_K}/M_\Lambda) = \dim_{\mathbb{F}}(\Lambda'_{\mathcal{O}_K}/M_{\Lambda'})$. Consequently (3.7.11) amounts to $\dim_{\mathbb{F}}(\Lambda'_{\mathcal{O}_K}/\Lambda_{\mathcal{O}_K}) = \dim_{\mathbb{F}}(M_{\Lambda'}/M_\Lambda)$. Both $\Lambda'_{\mathcal{O}_K}/\Lambda_{\mathcal{O}_K}$ and $M_{\Lambda'}/M_\Lambda$ are annihilated by π and are therefore modules over $(\mathcal{O}_B \otimes \mathcal{O}_K)/(\pi) = M^{n\times n}(\mathbb{F})$. We now conclude exactly as in the proof of Lemma 3.6.3. $\square$

Remark 3.7.12. *If we do not assume that $G'_{\mathbb{Q}_p}$ is quasi-split, it is in general not true that $M^{\mathrm{loc}}(\mathbb{F})$ can be embedded into the $G(K)$-orbit of $\mathcal{L} \otimes \mathcal{O}_K$. This is due to the fact that in general the order $\mathcal{O}_B \otimes \mathcal{O}_K$ in $B \otimes K$ is not maximal (but merely hereditary, see* [15, Theorem 4]*).*

Let us give an easy example falling into case (I) of Remark 3.7.2. By duality this case amounts to a linear (i.e. non-polarized) situation over the first factor, compare [32, A.8]. *Take for D the quaternion division algebra over $\mathbb{Q}_p$ with maximal order O_D and uniformizer π. Take the canonical left D-module D and the complete chain of lattices $(\pi^k \mathcal{O}_D)_{k\in\mathbb{Z}}$ in D. As above, $M^{\mathrm{loc}}(\mathbb{F})$ can be canonically identified with the set of $\mathcal{O}_D \otimes \mathcal{O}_K$-submodules M in $D\otimes K$ satisfying $p\mathcal{O}_D \otimes \mathcal{O}_K \subset M \subset \mathcal{O}_D \otimes \mathcal{O}_K$ and such that $\det_{\mathcal{O}_D\otimes\mathcal{O}_K/M} : V_{\mathcal{O}_D\otimes\mathbb{F}} \to \mathbb{A}^1_{\mathbb{F}}$ is equal to some prescribed morphism d.*

There is an isomorphism $D \otimes K \simeq M^{2\times 2}(K)$ identifying $\mathcal{O}_D \otimes \mathcal{O}_K$ with the ring of matrices

$$\begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K \end{pmatrix}.$$

Thus for suitable d, the set $M^{\mathrm{loc}}(\mathbb{F})$ consists of the single element

$$M = \begin{pmatrix} p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K \end{pmatrix}.$$

One checks that M does not lie in the $\mathrm{GL}_{D\otimes K}(D \otimes K)$-orbit of $\mathcal{O}_D \otimes \mathcal{O}_K$.

The residue morphism $\mathcal{O}_K \to \mathbb{F}$ induces a map $\mathrm{Aut}(\mathcal{L})(\mathcal{O}_K) \to \mathrm{Aut}(\mathcal{L})(\mathbb{F})$ and we thereby extend the $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-action on $M^{\mathrm{loc}}(\mathbb{F})$ to an $\mathrm{Aut}(\mathcal{L})(\mathcal{O}_K)$-action.

Lemma 3.7.13. *Let $x \in M^{\mathrm{loc}}(\mathbb{F})$. Then $\mathrm{Aut}(\mathcal{L})(\mathcal{O}_K) \cdot x = \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \cdot x$.*

Proof. The map $\mathrm{Aut}(\mathcal{L})(\mathcal{O}_K) \to \mathrm{Aut}(\mathcal{L})(\mathbb{F})$ is surjective by Proposition 2.7.8. $\square$

Define $I_0 = \{g \in I \mid c(g) = 1\}$.

Lemma 3.7.14. *We have $I = \mathcal{O}_K^\times I_0$. In particular, any $g \in I$ satisfies $c(g) \in \mathcal{O}_K^\times$.*

Proof. Let $g \in I$ and $\Lambda \in \mathcal{L}$. Then Λ is in particular an $\mathcal{O}_K$-lattice in the K-vector space V_K and the fact that g restricts to an automorphism of Λ implies that $\det(g) \in \mathcal{O}_K^\times$. The equation $\det(g)^2 = c(g)^{\dim_{\mathbb{Q}} V}$ then yields $c(g) \in \mathcal{O}_K^\times$.

As $\mathcal{O}_K$ is strictly Henselian of residue characteristic different from 2, there is an $x \in \mathcal{O}_K^\times$ with $x^2 = c(g)$. Then $x^{-1}g \in I_0$, as desired. □

Lemma 3.7.15. *Let $g \in I$. Then g restricts to an automorphism $g_\Lambda : \Lambda \otimes \mathcal{O}_K \to \Lambda \otimes \mathcal{O}_K$ for each $\Lambda \in \mathcal{L}$. The assignment $g \mapsto (g_\Lambda)_\Lambda$ defines an isomorphism $I \xrightarrow{\sim} \mathrm{Sim}(\mathcal{L})(\mathcal{O}_K)$, which restricts to an isomorphism $I_0 \xrightarrow{\sim} \mathrm{Aut}(\mathcal{L})(\mathcal{O}_K)$.*

Proof. We indeed have $(g_\Lambda)_\Lambda \in \mathrm{Sim}(\mathcal{L})(\mathcal{O}_K)$, as $c(g) \in \mathcal{O}_K^\times$ by Lemma 3.7.14. An inverse to this map $I \to \mathrm{Sim}(\mathcal{L})(\mathcal{O}_K)$ is provided by Lemma 3.7.5. □

Consider the decomposition $\mathcal{F} = \coprod_{x \in I \backslash \mathcal{F}} \mathcal{F}_x$ into I-orbits.

Theorem 3.7.16. *Let $t \in M^{\mathrm{loc}}(\mathbb{F})$. Then α induces a bijection*

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \cdot t \xrightarrow{\sim} I \cdot \alpha(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow I \backslash \mathcal{F}.$$

If, by abuse of notation, we also denote this embedding by α, (3.7.16) *can be restated as*

$$\alpha(M_x^{\mathrm{loc}}) = \mathcal{F}_{\alpha(x)}, \quad x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F}).$$

Proof. The map α is equivariant for the $\mathrm{Aut}(\mathcal{L})(\mathcal{O}_K)$-action on $M^{\mathrm{loc}}(\mathbb{F})$, the I_0-action on $\mathcal{F}$ and the isomorphism $\mathrm{Aut}(\mathcal{L})(\mathcal{O}_K) \xrightarrow{\sim} I_0$ of Lemma 3.7.15. It therefore induces a bijection $\mathrm{Aut}(\mathcal{L})(\mathcal{O}_K) \cdot t \xrightarrow{\sim} I_0 \cdot \alpha(x)$. We conclude by applying Lemmata 3.7.13 and 3.7.14. □

Also consider the decomposition $G(K)/I = \coprod_{x \in I \backslash G(K)/I} (G(K)/I)_x$ into I-orbits. Let $\alpha' : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{\alpha} \mathcal{F} \xrightarrow{(3.7.7)} G(K)/I$.

Corollary 3.7.17. *Let $t \in M^{\mathrm{loc}}(\mathbb{F})$. Then α' induces a bijection*

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \cdot t \xrightarrow{\sim} I \cdot \alpha'(t). \tag{3.7.18}$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow I \backslash G(K)/I. \tag{3.7.19}$$

If, by abuse of notation, we also denote this embedding by α', (3.7.18) *can be restated as*

$$\alpha'(M_x^{\mathrm{loc}}) = (G(K)/I)_{\alpha'(x)}, \quad x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F}).$$

Proof. Clear from Theorem 3.7.16, as the isomorphism (3.7.7) is in particular I-equivariant. □

Remark 3.7.20. *Even though it does not influence the results below, we want to highlight a subtlety related to the appearance of the group I in* (3.7.19)*.*

The base-change $\mathcal{L} \otimes K$ is the constant chain with value $V \otimes K$. Consequently the action of $G(K)$ on $V \otimes K$ defines an isomorphism $G(K) \xrightarrow{\sim}$

$\mathrm{Sim}(\mathcal{L})(K)$. *In view of Proposition 2.7.6, it follows that* $\mathcal{G} = \mathrm{Sim}(\mathcal{L} \otimes \mathcal{O}_K)$ *is the Bruhat-Tits group scheme associated with* $\mathcal{L} \otimes \mathcal{O}_K$ *(see* [13, §3.2]*). Denote by* $\mathcal{G}^0$ *the connected component of the identity in* $\mathcal{G}$.

From a group-theoretic point of view, it would be more natural to work with the group $I^0 = \mathcal{G}^0(\mathcal{O}_K) \subset G(K)$*, the Iwahori subgroup associated with* $\mathcal{L} \otimes \mathcal{O}_K$ *(see* [27, Appendix]*), instead of* I*. The question whether* I *already equals* I^0 *seems to require a case-by-case analysis.*

If G *is the unitary group associated with a ramified quadratic extension, this question is answered in* [28, §1.2]. *It turns out that if* $\mathcal{L}$ *is complete, which is the case of interest in this paper, the two groups agree. If* $\mathcal{L}$ *is not necessarily complete, one may have* $I^0 \subsetneq I$*. This phenomenon and its implications to flag varieties and local models are discussed in detail in* [36, §6].

If G *is an orthogonal group associated with an even-dimensional vector space, it is shown in* [38, §4.3] *that again the groups* I *and* I^0 *agree whenever* $\mathcal{L}$ *is complete.*

These results should suffice to conclude that the completeness of $\mathcal{L}$ *always implies the equality* $I = I^0$ *(though admittedly we have not verified this in detail). This would be convenient, as the embedding* (3.7.19) *would then amount to an embedding of* $\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$ *into* $I^0\backslash G(K)/I^0$*, and thus*[5] *to an embedding of* $\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$ *into a suitable extended affine Weyl group of* G *(see* [27, Appendix, Proposition 8]*).*

3.8. **A formula for the p-rank on a KR stratum.** Recall the inclusion $\iota : G \subset G'$ of $\mathbb{Q}$-groups and the Frobenius σ on K. By abuse of notation we also denote by σ both the morphism $G(\sigma) : G(K) \to G(K)$ and the morphism $G'(\sigma) : G'(K) \to G'(K)$.

We continue to assume that $G'_{\mathbb{Q}_p}$ is quasi-split. Furthermore we assume for the rest of Section 3 that $\mathcal{L}$ is complete.

Definition 3.8.1. *Let* $g \in G(K)$ *and* $x \in I\backslash G(K)/I$*. The* affine Deligne-Lusztig variety associated with g and x *is defined by*

$$X_x(g) = \{y \in G(K)/I \mid y^{-1}g\sigma(y) \in IxI\}.$$

Remark 3.8.2. *Affine Deligne-Lusztig varieties are normally defined using an Iwahori subgroup of* $G(K)$*, so that our definition is potentially nonstandard. But as explained in Remark 3.7.20, it should in fact be the case that* I *is an Iwahori subgroup of* $G(K)$.

Denote by $\mathbb{D}$ the diagonalizable affine group with character group $\mathbb{Q}$ over K. Let $g \in G(K)$. We denote by $\nu_g : \mathbb{D} \to G_K$ the corresponding Newton map, defined in [17, 4.2].[6] The morphism ν_g makes V_K into a representation of $\mathbb{D}$ and we consider the corresponding weight decomposition $V_K = \oplus_{\chi\in\mathbb{Q}} V_\chi$. We define

$$\nu_{g,0} := \dim_K V_0.$$

In complete analogy we also obtain a morphism $\nu_{g'} : \mathbb{D} \to G'_K$ and a natural number $\nu_{g',0}$ for each $g' \in G'(K)$.

[5] at least if G is connected

[6] Note that the discussion in loc. cit. still remains valid for not necessarily connected reductive groups over $\mathbb{Q}_p$.

Theorem 3.8.3. *Let $x \in \operatorname{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F}) \subset I\backslash G(K)/I$ (see (3.7.19)) and let $g \in G(K)$. Assume that $X_x(g) \neq \emptyset$. Then the p-rank on $\mathcal{A}_x$ is equal to $\nu_{g,0}$.*

Remark 3.8.4. *Let us make the trivial observation that we certainly have $X_x(g) \neq \emptyset$ whenever $g \in IxI$. In particular we can use any element g of IxI to compute the number $\nu_{g,0}$ and thereby the p-rank on $\mathcal{A}_x$. In Section 3.9 below we show that one often has a canonical representative of the double coset IxI and we explain how to compute the number $\nu_{g,0}$ for this canonical representative.*

Proof of Theorem 3.8.3. By assumption there is a $y \in G(K)$ with $yg\sigma(y)^{-1} \in IxI$. We know that $\nu_{yg\sigma(y)^{-1}} = \operatorname{Int}(y) \circ \nu_g$, where $\operatorname{Int}(y) : G(K) \to G(K)$, $h \mapsto yhy^{-1}$, and consequently $\nu_{g,0} = \nu_{yg\sigma(y)^{-1},0}$. Hence we may assume that $g \in IxI$. The assumption $x \in \operatorname{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$ then implies

$$\forall \Lambda \in \mathcal{L} : \; g(\Lambda \otimes \mathcal{O}_K) \subset \Lambda \otimes \mathcal{O}_K. \tag{3.8.5}$$

From the discussion in [17, 4.2], it is clear that ν_g and $\nu_{\iota(g)}$ make V_K into the *same* representation of $\mathbb{D}$. Consequently $\nu_{g,0} = \nu_{\iota(g),0}$. In view of Propositions 3.5.4 and 3.5.5, it therefore suffices to show Proposition 3.8.6 below in order to prove Theorem 3.8.3. □

Proposition 3.8.6. . *Denote by $\mathcal{H} \subset G'(K)$ the subset of those $g \in G'(K)$ satisfying (3.8.5). Let $g \in \mathcal{H}$, $\Lambda \in \mathcal{L}$ and choose a sequence $\Lambda = \Lambda^{(0)} \supsetneq \Lambda^{(1)} \supsetneq \cdots \supsetneq \Lambda^{(k)} = p\Lambda$ of neighbors $\Lambda^{(j-1)} \supsetneq \Lambda^{(j)}$ in $\mathcal{L}$. Define*

$$J_{e,u} = \{j \in \{1, \ldots, k\} \mid \Lambda^{(j-1)} \otimes \mathcal{O}_K = g(\Lambda^{(j-1)} \otimes \mathcal{O}_K) + \Lambda^{(j)} \otimes \mathcal{O}_K\}.$$

Then

$$\nu_{g,0} = \sum_{j \in J_{e,u}} \log_p |\Lambda^{(j-1)}/\Lambda^{(j)}|.$$

Proof. All the objects in question respect the decomposition of $B \otimes \mathbb{Q}_p$ into simple factors and we may therefore assume that $B \otimes \mathbb{Q}_p$ itself is simple. Consequently we have $B \otimes \mathbb{Q}_p = M^{n\times n}(F)$ and $\mathcal{O}_B \otimes \mathbb{Z}_p = M^{n\times n}(\mathcal{O}_F)$ for some $n \in \mathbb{N}$ and some finite extension $F/\mathbb{Q}_p$. We use the notation of Section 3.6, in particular (3.6.1). We write $B_\xi = M^{n\times n}(L_\xi)$ and $\mathcal{O}_{B_\xi} = M^{n\times n}(\mathcal{O}_{L_\xi})$, so that (3.6.1) induces decompositions

$$B \otimes K = \prod_{\xi\in\Xi} B_\xi, \qquad \mathcal{O}_B \otimes \mathcal{O}_K = \prod_{\xi\in\Xi} \mathcal{O}_{B_\xi}.$$

For $\xi \in \Xi$, we set $\sigma_\xi = \mathrm{id}_F \otimes \sigma : L_\xi \to L_{\sigma\circ\xi}$. Under (3.6.1) the morphism $\mathrm{id}_F \otimes \sigma : F \otimes K \to F \otimes K$ decomposes into the morphisms σ_ξ.

Also (3.6.1) induces a decomposition $V \otimes K = \prod_{\xi\in\Xi} V_\xi$ into left B_ξ-modules $V_\xi = V_{\mathbb{Q}_p} \otimes_F L_\xi$. The morphism $\mathrm{id}_V \otimes \sigma : V \otimes K \to V \otimes K$ decomposes into the morphisms $\mathrm{id}_{V_{\mathbb{Q}_p}} \otimes_F \sigma_\xi : V_\xi \to V_{\sigma\circ\xi}$.

Let $\Lambda \in \mathcal{L}$. We also obtain a decomposition $\Lambda \otimes \mathcal{O}_K = \prod_{\xi\in\Xi} \Lambda_\xi$ into $\mathcal{O}_{B_\xi}$-lattices $\Lambda_\xi = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{L_\xi}$ in V_ξ. Trivially

$$(\mathrm{id}_V \otimes \sigma)(\Lambda \otimes \mathcal{O}_K) = \Lambda \otimes \mathcal{O}_K, \tag{3.8.7}$$

and hence

$$\forall \xi \in \Xi : \; (\mathrm{id}_{V_{\mathbb{Q}_p}} \otimes_F \sigma_\xi)(\Lambda_\xi) = \Lambda_{\sigma\circ\xi}. \tag{3.8.8}$$

Let $h \in G'(K)$. Then h restricts to B_ξ-endomorphisms h_ξ of V_ξ and the map $G'(K) \to \prod_\xi \mathrm{GL}_{B_\xi}(V_\xi),\ h \mapsto (h_\xi)_\xi$ is an isomorphism. Note for use below the following commutative diagram.

$$\begin{tikzcd} V_\xi \arrow[r, "h_\xi"] \arrow[d, "\mathrm{id}_{V_{\mathbb{Q}_p}} \otimes \sigma_\xi"'] & V_\xi \arrow[d, "\mathrm{id}_{V_{\mathbb{Q}_p}} \otimes \sigma_\xi"] \\ V_{\sigma\circ\xi} \arrow[r, "\sigma(h)_{\sigma\circ\xi}"] & V_{\sigma\circ\xi}. \end{tikzcd} \tag{3.8.9}$$

For $h \in \mathcal{H}$ and $\Lambda' \subset \Lambda''$ in $\mathcal{L}$, we denote by

$$r_{h,\Lambda'',\Lambda'} : (\Lambda'' \otimes \mathcal{O}_K)/(\Lambda' \otimes \mathcal{O}_K) \to (\Lambda'' \otimes \mathcal{O}_K)/(\Lambda' \otimes \mathcal{O}_K)$$

the morphism induced by h and further for $\xi \in \Xi$ by

$$r_{h,\Lambda'',\Lambda',\xi} : \Lambda''_\xi/\Lambda'_\xi \to \Lambda''_\xi/\Lambda'_\xi$$

the morphism induced by h_ξ.

Let $s \geq 1$ be an integer. We write

$$g^{(s)} = g \circ \sigma(g) \circ \sigma^2(g) \circ \cdots \circ \sigma^{s-1}(g).$$

Note that $g^{(s)} \in \mathcal{H}$ by (3.8.7).

Lemma 3.8.10. *Let $\Lambda' \subset \Lambda''$ be neighbors in $\mathcal{L}$. Then for any $s \geq 1$ we have*

$$\begin{aligned} &\Lambda'' \otimes \mathcal{O}_K = g(\Lambda'' \otimes \mathcal{O}_K) + \Lambda' \otimes \mathcal{O}_K \\ \Leftrightarrow\ & r_{g^{(s)},\Lambda'',\Lambda'} \text{ is an isomorphism.} \end{aligned}$$

Furthermore for any $s \geq |\Xi|$ we have

$$\begin{aligned} &\Lambda'' \otimes \mathcal{O}_K \neq g(\Lambda'' \otimes \mathcal{O}_K) + \Lambda' \otimes \mathcal{O}_K \\ \Leftrightarrow\ & r_{g^{(s)},\Lambda'',\Lambda'} = 0. \end{aligned}$$

Proof. The assumption that $\Lambda' \subset \Lambda''$ are neighbors implies that $\Lambda''_\xi/\Lambda'_\xi$ is a simple $\mathcal{O}_{B_\xi}$-module for each $\xi \in \Xi$. Consequently any $\mathcal{O}_{B_\xi}$-linear endomorphism of $\Lambda''_\xi/\Lambda'_\xi$ is either an isomorphism or equal to zero.

From this observation we obtain the equivalences

$$\begin{aligned} &\Lambda'' \otimes \mathcal{O}_K = g(\Lambda'' \otimes \mathcal{O}_K) + \Lambda' \otimes \mathcal{O}_K \\ \Leftrightarrow\ & \forall \xi \in \Xi :\ r_{g,\Lambda'',\Lambda',\xi} \text{ is an isomorphism} \end{aligned} \tag{3.8.11}$$

and

$$\begin{aligned} &\Lambda'' \otimes \mathcal{O}_K \neq g(\Lambda'' \otimes \mathcal{O}_K) + \Lambda' \otimes \mathcal{O}_K \\ \Leftrightarrow\ & \exists \xi \in \Xi :\ r_{g,\Lambda'',\Lambda',\xi} = 0. \end{aligned} \tag{3.8.12}$$

Let $h \in \mathcal{H}$ and $\xi \in \Xi$. By (3.8.8), the morphism $\mathrm{id}_{V_{\mathbb{Q}_p}} \otimes_F \sigma_\xi$ induces an (additive) bijection $\varphi_\xi : \Lambda''_\xi/\Lambda'_\xi \to \Lambda''_{\sigma\circ\xi}/\Lambda'_{\sigma\circ\xi}$ and from (3.8.9) we obtain the commutative diagram

$$\begin{tikzcd} \Lambda''_\xi/\Lambda'_\xi \arrow[r, "r_{h,\Lambda'',\Lambda',\xi}"] \arrow[d, "\varphi_\xi"'] & \Lambda''_\xi/\Lambda'_\xi \arrow[d, "\varphi_\xi"] \\ \Lambda''_{\sigma\circ\xi}/\Lambda'_{\sigma\circ\xi} \arrow[r, "r_{\sigma(h),\Lambda'',\Lambda',\sigma\circ\xi}"] & \Lambda''_{\sigma\circ\xi}/\Lambda'_{\sigma\circ\xi}. \end{tikzcd}$$

Thus we see that $r_{h,\Lambda'',\Lambda',\xi}$ is an isomorphism if and only if $r_{\sigma(h),\Lambda'',\Lambda',\sigma\circ\xi}$ is an isomorphism.

From this observation and the first paragraph, we obtain for fixed $\xi_0 \in \Xi$ the equivalences

$$\begin{aligned} &r_{g^{(s)},\Lambda'',\Lambda',\xi_0} \text{ is an isomorphism} \\ \Leftrightarrow\ &\forall i \in \{0,\dots,s-1\}:\ r_{g,\Lambda'',\Lambda',\sigma^{-i}\circ\xi_0} \text{ is an isomorphism} \end{aligned} \tag{3.8.13}$$

and

$$\begin{aligned} &r_{g^{(s)},\Lambda'',\Lambda',\xi_0} = 0 \\ \Leftrightarrow\ &\exists i \in \{0,\dots,s-1\}:\ r_{g,\Lambda'',\Lambda',\sigma^{-i}\circ\xi_0} = 0. \end{aligned} \tag{3.8.14}$$

(3.8.11) and (3.8.13) imply the first claim. (3.8.12) and (3.8.14) imply the second claim in view of the following observation: If $s \geq |\Xi|$, then $\{\sigma^{-i}\circ\xi_0 \mid i \in \{0,\dots,s-1\}\} = \Xi$. □

Lemma 3.8.15. *Let V be a finite-dimensional $\mathbb{F}$-vector space and let $V = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq \dots V_k = \{0\}$ be a flag of subspaces. Let $(f_i)_{i\geq 1}$ be a sequence of endomorphisms of V. Assume that $f_i(V_j) \subset V_j$ for all i, j and denote by $r_{f_i,j} : V_j/V_{j+1} \to V_j/V_{j+1}$ the morphism induced by f_i. Assume there is a subset $J \subset \{0,\dots,k-1\}$ such that for all $i \geq 1$, the map $r_{f_i,j}$ is an isomorphism if $j \in J$ and is the zero map if $j \notin J$. Then for all $s \geq k$ we have*

$$\mathrm{rk}_{\mathbb{F}}(f_1 \circ \cdots \circ f_s) = \sum_{j\in J} \dim_{\mathbb{F}} V_j/V_{j+1}.$$

Proof. We use induction on $k \geq 1$ and assume that the statement is true for flags of length $k-1$. If $0 \in J$, it suffices to apply the following obvious claim and invoke the induction hypothesis: Let $h : V \to V$ be an endomorphism with $h(V_1) \subset V_1$ and such that the map $V/V_1 \to V/V_1$ induced by h is an isomorphism. Denote by $h' : V_1 \to V_1$ the restriction of h. Then $\mathrm{rk}_{\mathbb{F}}\, h = \mathrm{rk}_{\mathbb{F}}\, h' + \dim_{\mathbb{F}} V/V_1$.

Assume that $0 \notin J$, which means that $f_i(V) \subset V_1$ for all i. Denote by $f'_i : V_1 \to V_1$ the restriction of f_i. Then $\mathrm{im}\, f'_1\circ\cdots\circ f'_s \subset \mathrm{im}\, f_1\circ\cdots\circ f_s \subset \mathrm{im}\, f'_1\circ\cdots\circ f'_{s-1}$ and consequently $\mathrm{rk}_{\mathbb{F}}\, f'_1\circ\cdots\circ f'_s \leq \mathrm{rk}_{\mathbb{F}}\, f_1\circ\cdots\circ f_s \leq \mathrm{rk}_{\mathbb{F}}\, f'_1\circ\cdots\circ f'_{s-1}$. By induction hypothesis, the outer two terms agree with the desired value if $s-1 \geq k-1$. □

Corollary 3.8.16. *We have*

$$\sum_{j\in J_{e,u}} \log_p |\Lambda^{(j)}/\Lambda^{(j-1)}| = \mathrm{rk}_{\mathbb{F}}\, r_{g^{(s|\Xi|)},\Lambda,p\Lambda}$$

for all $s \geq k$.

Proof. We apply Lemma 3.8.15 to the filtration

$$\frac{\Lambda\otimes\mathcal{O}_K}{p\Lambda\otimes\mathcal{O}_K} \supsetneq \frac{\Lambda^{(1)}\otimes\mathcal{O}_K}{p\Lambda\otimes\mathcal{O}_K} \supsetneq \cdots \supsetneq \frac{\Lambda^{(k)}\otimes\mathcal{O}_K}{p\Lambda\otimes\mathcal{O}_K} = \{0\}$$

and the endomorphisms $f_i = r_{\sigma^{(i-1)|\Xi|}(g^{(|\Xi|)}),\Lambda,p\Lambda}$. Note that the assumptions of Lemma 3.8.15 are indeed satisfied in view Lemma 3.8.10. □

Let $s \in \mathbb{N}$. Recall from [16, I] the notion of a σ^s-F-crystal. Denote by $\Phi : \Lambda \otimes \mathcal{O}_K \to \Lambda \otimes \mathcal{O}_K$ the restriction of the σ-linear endomorphism $g \circ (\mathrm{id}_V \otimes \sigma)$ of $V \otimes K$. The pair $(\Lambda \otimes \mathcal{O}_K, \Phi^s)$ is a σ^s-F-crystal. Recall from loc. cit. the corresponding Newton function $\mathrm{N}_{\Phi^s} : [0, r] \to \mathbb{R}_{\geq 0}$ and the corresponding Hodge function $\mathrm{H}_{\Phi^s} : [0, r] \to \mathbb{R}_{\geq 0}$, where $r = \mathrm{rk}_{\mathcal{O}_K}(\Lambda \otimes \mathcal{O}_K)$. By the very definition of the map ν_g , we have

$$[0, \nu_{g,0}] = \mathrm{N}_{\Phi}^{-1}(\{0\}), \tag{3.8.17}$$

as both $\nu_{g,0}$ and $\max \mathrm{N}_{\Phi}^{-1}(\{0\})$ equal the K-dimension of the isotypical component of slope 0 of the σ-F-isocrystal $(V \otimes K, g \circ (\mathrm{id}_V \otimes \sigma))$ (terminology of [31, Example 1.10]).

Lemma 3.8.18. *Let $s \geq 1$. Then* $[0, \mathrm{rk}_{\mathbb{F}}\, r_{g^{(s)}, \Lambda, p\Lambda}] = \mathrm{H}_{\Phi^s}^{-1}(\{0\})$.

Proof. Note that Φ^s is the restriction of $g^{(s)} \circ (\mathrm{id}_V \otimes \sigma^s)$ to $\Lambda \otimes \mathcal{O}_K$. Let $p^{a_1} \leq p^{a_2} \leq \cdots \leq p^{a_r}$, $a_i \in \mathbb{N}$, be the elementary divisors of the restriction of $g^{(s)}$ to $\Lambda \otimes \mathcal{O}_K$. By definition, $\max \mathrm{H}_{\Phi^s}^{-1}(\{0\}) = |\{1 \leq i \leq r \mid a_i = 0\}|$. From this the statement is clear. $\square$

By [16, p. 121], we know that $\mathrm{N}_{\Phi} = \frac{1}{s}\mathrm{N}_{\Phi^s}$. Therefore [16, Theorem 1.4.1] implies $\mathrm{N}_{\Phi} \geq \frac{1}{s}\mathrm{H}_{\Phi^s}$. From (3.8.17), Lemma 3.8.18 and Corollary 3.8.16 we conclude that $\nu_{g,0} \leq \sum_{j \in J_{e,u}} \log_p |\Lambda^{(j)}/\Lambda^{(j-1)}|$.

By [16, Corollary 1.4.4], we have $\mathrm{N}_{\Phi} = \lim_{s \to \infty} \frac{1}{s}\mathrm{H}_{\Phi^s}$ pointwise. By Lemma 3.8.18 and Corollary 3.8.16, the subsequence $(\frac{1}{s|\Xi|}\mathrm{H}_{\Phi^{s|\Xi|}})_{s \geq k}$ vanishes identically on $[0, \sum_{j \in J_{e,u}} \log_p |\Lambda^{(j)}/\Lambda^{(j-1)}|]$. Therefore the same is true for its limit N_{Φ}, so that $\sum_{j \in J_{e,u}} \log_p |\Lambda^{(j)}/\Lambda^{(j-1)}| \leq \nu_{g,0}$ by (3.8.17). This concludes the proof of Proposition 3.8.6. $\square$

3.9. **Computing the number $\nu_{g,0}$.** In this section we want to explain how the number $v_{g,0}$ from Theorem 3.8.3 can actually be computed in practice.

3.9.1. *A combinatorial lemma.* Fix $n \in \mathbb{N}$ and let $w \in S_n, \lambda \in \mathbb{Z}^n$. Denote by $w\lambda = w(\lambda)$ the canonical left action of S_n on $\mathbb{Z}^n$, so that $(w\lambda)(i) = \lambda(w^{-1}(i))$, $1 \leq i \leq n$. We consider the corresponding semidirect product $\widetilde{W} := S_n \ltimes \mathbb{Z}^n$. To avoid confusion of the action of S_n on $\mathbb{Z}^n$ and the product in $S_n \ltimes \mathbb{Z}^n$, we denote the element of $S_n \ltimes \mathbb{Z}^n$ corresponding to $\lambda \in \mathbb{Z}^n$ by u^{λ}.

Note that

$$(wu^{\lambda})^N = w^N \prod_{k=0}^{N-1} u^{w^{-k}\lambda}$$

for all $N \in \mathbb{N}$. In particular there is an $N \in \mathbb{N}_{\geq 1}$ such that $(wu^{\lambda})^N \in \mathbb{Z}^n$.

Let Ξ be a finite cyclic group of order f with generator σ. We have the shift $\prod_{\xi \in \Xi} \widetilde{W} \to \prod_{\xi \in \Xi} \widetilde{W}$, $(x_\xi)_\xi \mapsto (x_{\sigma^{-1}\xi})_\xi$. By abuse of notation, we simply denote it by σ.

Let $x = (x_\xi)_\xi \in \prod_{\xi \in \Xi} \widetilde{W}$. Then

$$\prod_{k=0}^{Nf-1} \sigma^k(x) = \left(\prod_{k=0}^{f-1} \sigma^k(x)\right)^N$$

for all $N \in \mathbb{N}$. In particular there is an $N \in \mathbb{N}_{\geq 1}$ such that $\prod_{k=0}^{Nf-1} \sigma^k(x) \in \prod_{\xi\in\Xi} \mathbb{Z}^n$.

Lemma 3.9.1. *Let $(w_\xi)_\xi \in \prod_{\xi\in\Xi} S_n$ and $(\lambda_\xi)_\xi \in \prod_{\xi\in\Xi} \mathbb{Z}^n$. Assume that for all $\xi \in \Xi$ and all $1 \leq i \leq n$, the following statement holds.*

$$\lambda_\xi(i) \geq 0 \quad \text{and} \quad (\lambda_\xi(i) = 0 \Rightarrow w_\xi(i) \leq i). \tag{3.9.2}$$

Let $x = (w_\xi u^{\lambda_\xi})_\xi \in \prod_{\xi\in\Xi} \widetilde{W}$. Choose $N \in \mathbb{N}_{\geq 1}$ such that $\prod_{k=0}^{Nf-1} \sigma^k(x) \in \prod_{\xi\in\Xi} \mathbb{Z}^n$. Consider the element

$$\nu = (\nu_\xi)_\xi := \frac{1}{Nf} \prod_{k=0}^{Nf-1} \sigma^k(x)$$

of $\prod_{\xi\in\Xi} \mathbb{Q}_{\geq 0}$. Then for each $1 \leq i \leq n$, the following statements are equivalent.

(1) $\exists \xi \in \Xi : \ \nu_\xi(i) = 0$.
(2) $\forall \xi \in \Xi : \ \nu_\xi(i) = 0$.
(3) $\forall \xi \in \Xi : \ (w_\xi(i) = i \wedge \lambda_\xi(i) = 0)$.

Proof. We leave the easy, but notationally somewhat tedious proof to the reader. ☐

3.9.2. *The Newton vector.* Let us explain how Lemma 3.9.1 is related to the Newton map. Unless explicitly stated otherwise, we drop the notation of the preceding sections. Let $F/\mathbb{Q}_p$ be a finite extension and let $n \in \mathbb{N}$. We fix once and for all a uniformizer π of $\mathcal{O}_F$. Denote by $(e_1, \dots, e_n)$ the standard basis of $V = F^n$ over F.

Let $0 \leq i < n$. We denote by Λ_i the $\mathcal{O}_F$-lattice in V with basis $(\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n)$. For $k \in \mathbb{Z}$ we further define $\Lambda_{nk+i} = \pi^{-k}\Lambda_i$. Then $\mathcal{L} = (\Lambda_i)_i$ is a complete chain of $\mathcal{O}_F$-lattices in V. Consider the $\mathbb{Q}_p$-group $G = \operatorname{Res}_{F/\mathbb{Q}_p} \mathrm{GL}_{n,F}$.

We use the notation of Section 3.6. In particular

$$G_K = \prod_{\xi\in\Xi} \operatorname{Res}_{L_\xi/K} \mathrm{GL}_{n,L_\xi}.$$

For $\xi \in \Xi$, we denote by k_ξ the residue field of L_ξ. Denote by $I_\xi \subset \mathrm{GL}_n(\mathcal{O}_{L_\xi})$ the preimage of $B(k_\xi)$ under the reduction map $\mathrm{GL}_n(\mathcal{O}_{L_\xi}) \to \mathrm{GL}_n(k_\xi)$.

Definition 3.9.3 (Definition 3.7.3(1))**.** *We denote by $\mathcal{F}'$ the set of all tuples $(M_i)_{i\in\mathbb{Z}}$ of $\mathcal{O}_F \otimes \mathcal{O}_K$-lattices M_i in $V \otimes K$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

(1) $M_i \subset M_{i+1}$.
(2) There is an isomorphism $M_{i+1}/M_i \simeq (\Lambda_{i+1}/\Lambda_i) \otimes \mathcal{O}_K$ of $\mathcal{O}_F \otimes \mathcal{O}_K$-modules.
(3) $M_{n+i} = \pi^{-1}M_i$.

As before, we have $\mathcal{L} \otimes \mathcal{O}_K \in \mathcal{F}'$. The group $G(K)$ acts on $\mathcal{F}'$ via $g \cdot (M_i)_i = (gM_i)_i$ and we denote by $I \subset G(K)$ the stabilizer of $\mathcal{L} \otimes \mathcal{O}_K$. An easy computation shows the following statement.

Lemma 3.9.4. *We have $I = \prod_{\xi\in\Xi} I_\xi$.*

Note that the image $\pi \otimes 1$ of π in $\mathcal{O}_{L_\xi}$ is again a uniformizer of $\mathcal{O}_{L_\xi}$, which we denote by π_ξ. For $\lambda \in \mathbb{Z}^n$, we define a matrix π_ξ^λ in $\mathrm{GL}_n(L_\xi)$ by $\pi_\xi^\lambda = \mathrm{diag}(\pi_\xi^{\lambda(1)}, \dots, \pi_\xi^{\lambda(n)})$. Consider the map $\upsilon_\xi : S_n \ltimes \mathbb{Z}^n \to \mathrm{GL}_n(L_\xi)$, $(w, \lambda) \mapsto A_w \pi_\xi^\lambda$. We consider $\widetilde{W} = S_n \ltimes \mathbb{Z}^n$ as a subgroup of $\mathrm{GL}_n(L_\xi)$ via υ_ξ.

Lemma 3.9.5. *The canonical projection* $\mathrm{GL}_n(L_\xi) \to I_\xi \backslash \mathrm{GL}_n(L_\xi)/I_\xi$ *induces a bijection* $\widetilde{W} \xrightarrow{\sim} I_\xi \backslash \mathrm{GL}_n(L_\xi)/I_\xi$.

Proof. This is the well-known Iwahori decomposition. □

Corollary 3.9.6. *The canonical projection* $G(K) \to I\backslash G(K)/I$ *induces a bijection* $\prod_{\xi \in \Xi} \widetilde{W} \xrightarrow{\sim} I\backslash G(K)/I$.

Proof. Combine Lemmata 3.9.4 and 3.9.5. □

Thus we always find a canonical representative of an I-double coset in $G(K)$, and in view of Remark 3.8.4 we can use this representative to compute the p-rank on a KR stratum.

Let $g \in G(K)$. As before, let $\nu_g : \mathbb{D} \to G_K$ be the corresponding Newton map and $V_K = \oplus_{\chi \in \mathbb{Q}} V_\chi$ the corresponding weight decomposition. We write $\nu_{g,0} = \dim_K V_0$.

Proposition 3.9.7. *Let* $x = (x_\xi) \in \prod_{\xi \in \Xi} \widetilde{W}$ *and assume that*

$$\forall i \in \mathbb{Z} : \ x(\Lambda_i \otimes \mathcal{O}_K) \subset \Lambda_i \otimes \mathcal{O}_K. \tag{3.9.8}$$

Write $x_\xi = w_\xi \pi_\xi^{\lambda_\xi}$ *with* $w_\xi \in S_n$ *and* $\lambda_\xi \in \mathbb{Z}^n$. *Then*

$$v_{x,0} = [F : \mathbb{Q}_p] \cdot |\{1 \le i \le n \mid \forall \xi \in \Xi (w_\xi(i) = i \wedge \lambda_\xi(i) = 0)\}|.$$

Here $[F : \mathbb{Q}_p]$ *denotes the degree of the extension* $F/\mathbb{Q}_p$.

Proof. Using the explicit description of the Newton map from [17, 4.3], it follows from Lemma 3.9.1 that the weight space V_0 of ν_x is free of rank $|\{1 \le i \le n \mid \forall \xi \in \Xi (w_\xi(i) = i \wedge \lambda_\xi(i) = 0)\}|$ over $F \otimes K$. This implies the statement. □

Finally let $F_1, \dots, F_m$ be finite extensions of $\mathbb{Q}_p$ and let $n_1, \dots, n_m \in \mathbb{N}$. Let $V_i = F^{n_i}$ and $G_i = \mathrm{Res}_{F_i/\mathbb{Q}_p} \mathrm{GL}_{n_i, F_i}$. Let $V = V_1 \times \cdots \times V_m$ and let $G = \prod_{i=1}^m G_i$. For $g_i \in G_i(K)$ we obtain as before the Newton map $v_{g_i} : \mathbb{D} \to G_{i,K}$, the weight decomposition $V_{i,K} = \oplus_{\chi \in \mathbb{Q}} V_{i,\chi}$ and the integer $v_{g_i,0}$. Also for $g \in G(K)$ we have the Newton map $v_g : \mathbb{D} \to G_K$, the weight decomposition $V_K = \oplus_{\chi \in \mathbb{Q}} V_\chi$ and the integer $v_{g,0}$. The following lemma is trivial.

Lemma 3.9.9. *Let* $g = (g_1, \dots, g_m) \in G(K) = \prod_{i=1}^m G_i(K)$. *Then* $v_{g,0} = \sum_{i=1}^m v_{g_i,0}$.

4. Preliminaries II

4.1. **Some more notation.** Let R be a ring. We denote by $R[\![u]\!]$ the ring of formal power series and by $R(\!(u)\!) = R[\![u]\!][\frac{1}{u}]$ the ring of formal Laurent series with coefficients in R. If F is a functor on the category of $R(\!(u)\!)$-algebras (resp. $R[\![u]\!]$-algebras), we denote by $\mathrm{L}\, F = \mathrm{L}_u\, F$ (resp. $\mathrm{L}^+ F = \mathrm{L}_u^+ F$)

the functor on the category of R-algebras with $\mathrm{L}\,F(S) = F(S(\!(u)\!))$ (resp. $\mathrm{L}^+\,F(S) = F(S[\![u]\!])$).

Denote by $I(R) \subset \mathrm{GL}_n(R[\![u]\!])$ the preimage of $B(R)$ under the reduction map $\mathrm{GL}_n(R[\![u]\!]) \to \mathrm{GL}_n(R),\ u \mapsto 0$. Note that $I \subset \mathrm{L}\,\mathrm{GL}_n$ is a subfunctor.

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. Define a matrix u^λ in $\mathrm{GL}_n(R(\!(u)\!))$ by $u^\lambda = \mathrm{diag}(u^{\lambda(1)}, \dots, u^{\lambda(n)})$.

4.2. **Switching between different pairings.** The following lemma is entirely trivial, but it will be used throughout the following sections.

Lemma 4.2.1. *Assume that we are given the following objects.*

- *A finite locally free ring extension $R \to S$.*
- *An R-linear involution $\alpha : S \to S$.*
- *A free S-module $\mathfrak{S}$ of rank 1 with an isomorphism of S-modules $\psi : S \to \mathfrak{S}$.*
- *An R-linear map $\varphi : \mathfrak{S} \to R$, such that the induced map*

$$\mathfrak{S} \xrightarrow{\varphi^*} S^{\vee,R},$$
$$m \longmapsto (s \mapsto \varphi(sm))$$

  is an isomorphism.
- *An α-sesquilinear form $\Phi' : M \times N \to \mathfrak{S}$ of S-modules.*

Define an R-bilinear form

$$\Phi : M \times N \to R,\ \Phi = \varphi \circ \Phi'$$

and an α-sesquilinear form

$$\widetilde{\Phi} : M \times N \to S,\ \widetilde{\Phi} = \psi^{-1} \circ \Phi'.$$

Let $R \to R'$ be a ring extension.

(1) Let $L \subset M_{R'}$ be an $S \otimes_R R'$-submodule. Then

$$L^{\perp,\Phi_{R'}} = L^{\perp,\widetilde{\Phi}_{R'}}.$$

(2) Let $f : M_{R'} \to M_{R'}$ and $g : N_{R'} \to N_{R'}$ be $S \otimes_R R'$-linear endomorphisms. Then the following two statements are equivalent.

$$\forall (x, y) \in M_{R'} \times N_{R'} :\ \Phi_{R'}(fx, gy) = \Phi_{R'}(x, y), \tag{4.2.2}$$

$$\forall (x, y) \in M_{R'} \times N_{R'} :\ \widetilde{\Phi}_{R'}(fx, gy) = \widetilde{\Phi}_{R'}(x, y). \tag{4.2.3}$$

4.3. **Extensions of local fields.** Let $F/\mathbb{Q}_p$ be a finite extension. We will always denote by $\mathcal{O}_F$ the valuation ring of F. Denote by $\mathcal{P}$ its maximal ideal, by $k = k_F = \mathcal{O}_F/\mathcal{P}$ its residue field and by $f = [k : \mathbb{F}_p]$ the inertia degree of $F/\mathbb{Q}_p$. Denote by e the ramification index of $F/\mathbb{Q}_p$, so that we have $p\mathcal{O}_F = \mathcal{P}^e$. Fix a uniformizer π of $\mathcal{O}_F$.

Let F' be the maximal unramified extension of $\mathbb{Q}_p$ in F. The extension $F'/\mathbb{Q}_p$ is unramified of degree f and the extension F/F' is totally ramified of degree e. By [34, Proposition I.18] the minimal polynomial f_π of π over F' is an Eisenstein polynomial of degree e over $\mathcal{O}_{F'}$, and the map $\mathcal{O}_{F'}[T]/f_\pi \to \mathcal{O}_F,\ T \mapsto \pi$ is an isomorphism. It induces an isomorphism

$$\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathbb{F}_p = k[T]/(T^e). \tag{4.3.1}$$

We therefore get an isomorphism of $\mathbb{F}$-algebras

$$\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathbb{F} = \prod_{\sigma:k\hookrightarrow\mathbb{F}} \mathbb{F}[T]/(T^e) \tag{4.3.2}$$

such that the diagram

$$\begin{tikzcd}
\mathcal{O}_F \arrow[rr] \arrow[d] & & \mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathbb{F} \arrow[dd, "\simeq"', "(4.3.2)"] \\
\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathbb{F}_p \arrow[d, "(4.3.1)"', "\simeq"] & & \\
k[T]/(T^e) \arrow[rr, "p\mapsto(p^\sigma)_\sigma"] & & \prod_{\sigma:k\hookrightarrow\mathbb{F}} \mathbb{F}[T]/(T^e)
\end{tikzcd}$$

commutes.

4.4. **Extensions of number fields.** Let $K/\mathbb{Q}$ be a number field. We will always denote by $\mathcal{O}_K$ the ring of integers of K. If $\mathcal{P}$ is a nonzero prime of $\mathcal{O}_K$, we will always denote by $k_{\mathcal{P}} = \mathcal{O}_K/\mathcal{P}$ its residue field and by $\rho_{\mathcal{P}} : \mathcal{O}_K \to k_{\mathcal{P}}$ the corresponding residue morphism. We further denote by $K_{\mathcal{P}}$ the completion of K with respect to $\mathcal{P}$ and by $\mathcal{O}_{K_{\mathcal{P}}}$ the valuation ring of $K_{\mathcal{P}}$.

For the rest of this section, we fix a number field $K_0/\mathbb{Q}$ and we assume that $p\mathcal{O}_{K_0} = \mathcal{P}_0^{e_0}$ for a single prime $\mathcal{P}_0$ of $\mathcal{O}_{K_0}$ and some $e_0 \in \mathbb{N}$.

Denote by Σ_0 the set of all embeddings $K_0 \hookrightarrow \mathbb{C}$. Fix a finite Galois extension $L/\mathbb{Q}$ with $K_0 \subset L$ and write $G = \mathrm{Gal}(L/\mathbb{Q})$ and $H_0 = \mathrm{Gal}(L/K_0)$. Fix a prime $\mathcal{Q}$ of $\mathcal{O}_L$ lying over $\mathcal{P}_0$ and denote by $G_{\mathcal{Q}} \subset G$ the corresponding decomposition group.

Proposition 4.4.1. *There is a unique map $\gamma_0 = \gamma_{\mathcal{P}_0} : \Sigma_0 \to \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$ making commutative the diagram*

$$\begin{tikzcd}
G_{\mathcal{Q}} \arrow[r] \arrow[d, "\cdot|_{K_0}"'] & \mathrm{Gal}(k_{\mathcal{Q}}/\mathbb{F}_p) \arrow[d, "\cdot|_{k_{\mathcal{P}_0}}"] \\
\Sigma_0 \arrow[r, "\gamma_0"] & \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p).
\end{tikzcd}$$

The map γ_0 is surjective and all its fibers have cardinality e_0. It satisfies

$$\forall \sigma \in \Sigma_0 \forall a \in \mathcal{O}_{K_0} : \quad \rho_{\mathcal{Q}}(\sigma(a)) = \gamma_0(\sigma)(\rho_{\mathcal{P}_0}(a)). \tag{4.4.2}$$

Proof. The restriction map $\alpha_0 : G \to \Sigma_0,\ \tau \mapsto \tau\big|_{K_0}$ is surjective and it induces a bijection $G/H_0 \xrightarrow{\sim} \Sigma_0$. By [23, p. 55] the set $H_0\backslash G/G_{\mathcal{Q}}$ is in canonical bijection to the set of primes of K_0 lying above p, so that our assumptions imply $G = G_{\mathcal{Q}} \cdot H_0$. Consequently α_0 induces a bijection $G_{\mathcal{Q}}/(H_0 \cap G_{\mathcal{Q}}) \xrightarrow{\sim} \Sigma_0$. This immediately implies the existence and uniqueness of the map γ_0. That it has the stated properties is easily verified. $\square$

Now let $K_0 \subset K \subset L$ be a second number field and denote by Σ the set of all embeddings $K \hookrightarrow \mathbb{C}$. Write $H = \mathrm{Gal}(L/K)$.

Lemma 4.4.3. *Assume that there is only a single prime $\mathcal{P}$ of $\mathcal{O}_K$ lying over $\mathcal{P}_0$. Consider the corresponding maps $\gamma = \gamma_{\mathcal{P}} : \Sigma \to \mathrm{Gal}(k_{\mathcal{P}}/\mathbb{F}_p)$ and $\gamma_0 = \gamma_{\mathcal{P}_0} : \Sigma_0 \to \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$.*

(1) The diagram

$$\begin{tikzcd} \Sigma \arrow[r, "\gamma_{\mathcal{P}}"] \arrow[d, "\cdot|_{K_0}"'] & \mathrm{Gal}(k_{\mathcal{P}}/\mathbb{F}_p) \arrow[d, "\cdot|_{k_{\mathcal{P}_0}}"] \\ \Sigma_0 \arrow[r, "\gamma_{\mathcal{P}_0}"] & \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p) \end{tikzcd} \tag{4.4.4}$$

commutes.

(2) Assume furthermore that K/K_0 is Galois and that $\mathcal{P}_0$ is inert in K, so that $\mathcal{P} = \mathcal{P}_0\mathcal{O}_K$. Then for each $\sigma_0 \in \Sigma_0$, the map $\gamma_{\mathcal{P}}$ induces a bijection

$$\{\sigma \in \Sigma \mid \sigma\big|_{K_0} = \sigma_0\} \xrightarrow{\sim} \{\sigma \in \mathrm{Gal}(k_{\mathcal{P}}/\mathbb{F}_p) \mid \sigma\big|_{k_{\mathcal{P}_0}} = \gamma_{\mathcal{P}_0}(\sigma_0)\}$$

of the fibers of the vertical maps in (4.4.4)*.*

4.4.1. *Split quadratic extensions.* We keep the notation of the previous section. From now on we assume that K/K_0 is a *quadratic* extension. Denote by $*$ the non-trivial element of $\mathrm{Gal}(K/K_0)$. Assume that $\mathcal{P}_0\mathcal{O}_K = \mathcal{P}_+\mathcal{P}_-$ for two distinct primes $\mathcal{P}_+, \mathcal{P}_-$ of $\mathcal{O}_K$, say $\mathcal{Q} \cap \mathcal{O}_K = \mathcal{P}_+$. Consequently $\mathcal{P}_- = \mathcal{P}_+^*$. Denote by $\alpha : G \to \Sigma$ the restriction map. Fix a lift $\tau_* \in G$ of $*$ under α. Note that the inclusion $\mathcal{O}_{K_0} \subset \mathcal{O}_K$ induces isomorphisms $k_{\mathcal{P}_0} \xrightarrow{\sim} k_{\mathcal{P}_\pm}$.

Again by [23, p. 55] the set $H\backslash G/G_{\mathcal{Q}}$ has two elements, corresponding to $\mathcal{P}_+$ and $\mathcal{P}_-$, respectively. Consequently we find that

$$G = G_{\mathcal{Q}}H \amalg G_{\mathcal{Q}}\tau_*H.$$

Define subsets $\Sigma_\pm \subset \Sigma$ by $\Sigma_+ = \alpha(G_{\mathcal{Q}}H)$ and $\Sigma_- = \alpha(G_{\mathcal{Q}}\tau_*H)$. Then $\Sigma = \Sigma_+ \amalg \Sigma_-$. The map $\Sigma \to \Sigma$, $\sigma \mapsto \sigma \circ *$ induces a bijection $\Sigma_+ \xrightarrow{\sim} \Sigma_-$. This implies that the restriction map $\Sigma \to \Sigma_0$ restricts to bijections $\Sigma_\pm \xrightarrow{\sim} \Sigma_0$. Consequently there are unique maps $\gamma_\pm : \Sigma_\pm \to \mathrm{Gal}(k_{\mathcal{P}_+}/\mathbb{F}_p)$ such that the diagram

$$\begin{tikzcd} \Sigma_\pm \arrow[r, "\gamma_\pm"] \arrow[d, "\cdot|_{K_0}"', "\simeq"] & \mathrm{Gal}(k_{\mathcal{P}_+}/\mathbb{F}_p) \arrow[d, "\simeq"', "\cdot|_{k_{\mathcal{P}_0}}"] \\ \Sigma_0 \arrow[r, "\gamma_0"] & \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p) \end{tikzcd}$$

commutes.[7] The commutative diagrams

$$\begin{tikzcd} G_{\mathcal{Q}} \arrow[r] \arrow[d, "\cdot|_K"'] & \mathrm{Gal}(k_{\mathcal{Q}}/\mathbb{F}_p) \arrow[d, "\cdot|_{k_{\mathcal{P}_+}}"] \\ \Sigma_+ \arrow[r, "\gamma_+"] & \mathrm{Gal}(k_{\mathcal{P}_+}/\mathbb{F}_p), \end{tikzcd} \qquad \begin{tikzcd} G_{\mathcal{Q}} \arrow[r] \arrow[d, "\cdot|_K \circ *"'] & \mathrm{Gal}(k_{\mathcal{Q}}/\mathbb{F}_p) \arrow[d, "\cdot|_{k_{\mathcal{P}_+}}"] \\ \Sigma_- \arrow[r, "\gamma_-"] & \mathrm{Gal}(k_{\mathcal{P}_+}/\mathbb{F}_p) \end{tikzcd}$$

then imply that the maps γ_+ and γ_- satisfy

$$\forall \sigma \in \Sigma_+ \forall a \in \mathcal{O}_K : \rho_{\mathcal{Q}}(\sigma(a)) = \gamma_+(\sigma)(\rho_{\mathcal{P}_+}(a)), \tag{4.4.5}$$

$$\forall \sigma \in \Sigma_- \forall a \in \mathcal{O}_K : \rho_{\mathcal{Q}}(\sigma(a)) = \gamma_-(\sigma)(\rho_{\mathcal{P}_+}(a^*)). \tag{4.4.6}$$

[7] The asymmetry is intended.

If we treat the isomorphisms $k_{\mathcal{P}_0} \xrightarrow{\sim} k_{\mathcal{P}_\pm}$ as identities, we have $\rho_{\mathcal{P}_+}(a^*) = \rho_{\mathcal{P}_-}(a),\ a \in \mathcal{O}_K$. Thus we can unify (4.4.5) and (4.4.6) into the following statement.

$$\forall \sigma \in \Sigma_\pm \forall a \in \mathcal{O}_K : \ \rho_{\mathcal{Q}}(\sigma(a)) = \gamma_0(\sigma|_{K_0})(\rho_{\mathcal{P}_\pm}(a)). \tag{4.4.7}$$

4.5. **Lattices.** Let R be a ring and let $n \in \mathbb{N}$.

Definition 4.5.1. *A* lattice *in $R(\!(u)\!)^n$ is an $R[\![u]\!]$-submodule $L \subset R(\!(u)\!)^n$ satisfying the following conditions for some $N \in \mathbb{N}$.*

(1) $u^N R[\![u]\!]^n \subset L \subset u^{-N} R[\![u]\!]^n$.

(2) $u^{-N} R[\![u]\!]^n / L$ is a finite locally free R-module.

Lemma 4.5.2. *Let A be a ring, and let $L \subset M \subset N$ be A-modules. Assume that N/M is projective. Then M/L is projective if and only if N/L is projective. Furthermore N/L is finitely generated if and only if both M/L and N/M are finitely generated.*

Proof. Clear, as $0 \to M/L \to N/L \to N/M \to 0$ splits by assumption. □

Corollary 4.5.3. *Let L be a lattice in $R(\!(u)\!)^n$. Condition 4.5.1(2) is satisfied for every $N \in \mathbb{N}$ with $L \subset u^{-N} R[\![u]\!]^n$.*

Corollary 4.5.4. *Let $L \subset L'$ be lattices in $R(\!(u)\!)^n$. Then L'/L is a finite locally free R-module.*

Proposition 4.5.5. *Let L be a lattice in $R(\!(u)\!)^n$. Then L is a finite locally free $R[\![u]\!]$-module of rank n.*

Proof. By Corollary 4.5.4, the quotient L/uL is a finite locally free R-module. We need the following result.

Lemma 4.5.6. *Let $N \in \mathbb{N}$. Let M be an $R[u]/u^N$-module such that multiplication by u^k induces an isomorphism $M/uM \to u^k M/u^{k+1} M$ for every $0 \le k \le N-1$. Assume that M/uM is finite free over R. Then M is finite free over $R[u]/u^N$. More precisely, if $e_1, \dots, e_m \in M$ are lifts of a basis of M/uM over R, then $(e_1, \dots, e_m)$ is a basis of M over $R[u]/u^N$.*

Proof. Easy, left to the reader. □

This easily implies that $L/u^N L$ is finite locally free over $R[u]/u^N$ for every $N \in \mathbb{N}$. We conclude using the following lemma.

Lemma 4.5.7. *Let A be a ring, $I \subset A$ an ideal and assume that A is separated and complete for the I-adic topology. Let M be a finite A-module. Assume that $M/I^N M$ is finite locally free over A/I^N for every $N \in \mathbb{N}$, and that the canonical map $M \to \lim_N M/I^N$ is an isomorphism. Then M is finite locally free over A.*

Proof. Choose an A-linear surjection $\pi : A^m \twoheadrightarrow M$. One shows that π splits by constructing a compatible system of splittings modulo I^N. See the proof of [19, Lemma 1.10] for details. □

□

Corollary 4.5.8. *Let L be a lattice in $R(\!(u)\!)^n$. Then Zariski locally on R, the $R[\![u]\!]$-module L is free of rank n.*

Proof. If $f_1, \dots, f_k \in R[\![u]\!]$ generate the unit ideal in $R[\![u]\!]$, then $f_1(0), \dots, f_k(0)$ generate the unit ideal in R. Furthermore, for each $f \in R[\![u]\!]$, the canonical map $R[\![u]\!] \to R_{f(0)}[\![u]\!]$ extends to $R[\![u]\!]_f \to R_{f(0)}[\![u]\!]$. Thus if $L \otimes_{R[\![u]\!]} R[\![u]\!]_f$ is free over $R[\![u]\!]_f$, then $L \otimes_{R[\![u]\!]} R_{f(0)}[\![u]\!]$ is free over $R_{f(0)}[\![u]\!]$. □

Definition 4.5.9. *The affine Grassmannian $\mathcal{G}$ is the functor on the category of rings with $\mathcal{G}(R)$ the set of lattices in $R(\!(u)\!)^n$.*

Denote by $\widetilde{\Lambda}_0 = R[\![u]\!]^n$ the *standard lattice*. Clearly $\mathrm{L}\,\mathrm{GL}_n(R)$ acts on $\mathcal{G}(R)$ by multiplication from the left, and the stabilizer of $\widetilde{\Lambda}_0$ for this action is given by $\mathrm{L}^+\,\mathrm{GL}_n(R)$. Consequently we get an injective map

$$\begin{aligned} \phi(R) : \mathrm{L}\,\mathrm{GL}_n(R)/\,\mathrm{L}^+\,\mathrm{GL}_n(R) &\to \mathcal{G}(R) \\ g &\mapsto g\widetilde{\Lambda}_0. \end{aligned}$$

It is equivariant for the left action by $\mathrm{L}\,\mathrm{GL}_n$.

Proposition 4.5.10. *The map ϕ identifies $\mathcal{G}$ with both the Zariski and the fpqc sheafification of the presheaf $\mathrm{L}\,\mathrm{GL}_n/\,\mathrm{L}^+\,\mathrm{GL}_n$.*

Proof. By Corollary 4.5.8 it is clear that any lattice lies in the image of ϕ Zariski locally on R. It follows that ϕ is the Zariski sheafification of the presheaf $\mathrm{L}\,\mathrm{GL}_n/\,\mathrm{L}^+\,\mathrm{GL}_n$. The fact that $\mathcal{G}$ is already an fpqc sheaf implies formally that ϕ is also the fpqc sheafification of the presheaf $\mathrm{L}\,\mathrm{GL}_n/\,\mathrm{L}^+\,\mathrm{GL}_n$. □

Definition 4.5.11. *A (complete, periodic)* lattice chain *in $R(\!(u)\!)^n$ is a tuple $(L_i)_{i\in\mathbb{Z}}$ of lattices L_i in $R(\!(u)\!)^n$ satisfying the following conditions for each $i \in \mathbb{Z}$.*

(1) $L_i \subset L_{i+1}$.
(2) (completeness) L_{i+1}/L_i is a locally free R-module of rank 1.
(3) (periodicity) $L_{n+i} = u^{-1}L_i$.

Definition 4.5.12. *The* affine flag variety *$\mathcal{F}$ is the functor on the category of rings with $\mathcal{F}(R)$ the set of (complete, periodic) lattice chains in $R(\!(u)\!)^n$.*

Denote by $(e_1, \dots, e_n)$ the standard basis of $R(\!(u)\!)^n$ over $R(\!(u)\!)$. For $0 \le i < n$ we denote by $\widetilde{\Lambda}_i$ the lattice in $R(\!(u)\!)^n$ with basis

$$\widetilde{\mathfrak{E}}_i = \left\langle u^{-1}e_1, \dots, u^{-1}e_i, e_{i+1}, \dots, e_n \right\rangle.$$

For $k \in \mathbb{Z}$ we further define $\widetilde{\Lambda}_{nk+i} = u^{-k}\widetilde{\Lambda}_i$ and we denote by $\widetilde{\mathfrak{E}}_{nk+i}$ the corresponding basis obtained from $\widetilde{\mathfrak{E}}_i$. Then $\widetilde{\mathcal{L}} = (\widetilde{\Lambda}_i)_i$ is a (complete, periodic) lattice chain in $R(\!(u)\!)^n$, called the *standard lattice chain*.

Remark 4.5.13. *The group $\mathrm{L}\,\mathrm{GL}_n(R)$ acts on $\mathcal{F}(R)$ via $g \cdot (L_i)_i = (gL_i)_i$. The stabilizer of $\widetilde{\mathcal{L}}$ for this action is given by $I(R)$. We will discuss the relationship between $\mathcal{F}$ and the quotient $\mathrm{L}\,\mathrm{GL}_n/I$ in Section 7.5 below.*

For later use we include the following easy lemmata.

Lemma 4.5.14. *Let R be a ring and let $a \in R(\!(u)\!)^\times$. Then Zariski locally on R, there are integers $n \le n_0$, nilpotent elements $a_n, a_{n+1}, \dots, a_{n_0-1} \in R$, a unit $a_{n_0} \in R^\times$ and elements $a_{n_0+1}, a_{n_0+2}, \dots \in R$ such that $a = \sum_{i=n}^{\infty} a_i u^i$.*

If $\operatorname{Spec} R$ is connected, such integers and elements exist globally on R.

Proof. Let $a \in R(\!(u)\!)^\times$. Pick $n \in \mathbb{Z}$ and $a_n, a_{n+1}, \dots \in R$ with $a = \sum_{i=n}^{\infty} a_i u^i$. Define a function $f_a : \operatorname{Spec}(R) \to \mathbb{Z}$ by

$$f_a(\mathfrak{p}) = \min\{i \in \mathbb{Z} \mid a_i \notin \mathfrak{p}\}.$$

Note that if we denote by $\mathrm{val}_{u,\mathfrak{p}}$ the canonical valuation on the discrete valuation ring $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})(\!(u)\!)$ and by $\overline{a} \in (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})(\!(u)\!)$ the reduction of a, we have $f_a(\mathfrak{p}) = \mathrm{val}_{u,\mathfrak{p}}(\overline{a})$ for any $\mathfrak{p} \in \operatorname{Spec}(R)$.

It suffices to see that f_a is locally constant. Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $f_a(\mathfrak{q}) \le f_a(\mathfrak{p})$ for all $\mathfrak{q} \in \operatorname{Spec}(R)$ with $a_{f_a(\mathfrak{p})} \notin \mathfrak{q}$. This shows that for each $k \in \mathbb{Z}$, the set $f_a^{-1}((-\infty, k])$ is open. Now $f_{a^{-1}} = -f_a$ and consequently also $f_a^{-1}([k,\infty)) = f_{a^{-1}}^{-1}((-\infty,-k])$ is open. Thus $f_a^{-1}(\{k\}) = f_a^{-1}((-\infty,k]) \cap f_a^{-1}([k,\infty))$ is open. □

Lemma 4.5.15. *Let L be a lattice in $R(\!(u)\!)^n$. Let $a \in R(\!(u)\!)^\times$ and assume that $aL = L$. Then $a \in R[\![u]\!]^\times$.*

Proof. We need to show properties of the coefficients of a which can be verified Zariski locally on R. By Corollary 4.5.8 we may therefore assume that L is a free $R[\![u]\!]$-module and then the statement is trivial. □

5. The symplectic case

5.1. **The PEL datum.** Let $g, n \in \mathbb{N}_{\ge 1}$. We start with the PEL datum consisting of the following objects.

(1) A totally real field extension $F/\mathbb{Q}$ of degree g.
(2) The identity involution id_F on F.
(3) A $2n$-dimensional F-vector space V.
(4) The symplectic form $(\cdot,\cdot) : V \times V \to \mathbb{Q}$ on the underlying $\mathbb{Q}$-vector space of V constructed as follows: Fix once and for all a symplectic form $(\cdot,\cdot)' : V \times V \to F$ and a basis $\mathfrak{E}' = (e_1', \dots, e_{2n}')$ of V such that $(\cdot,\cdot)'$ is described by the matrix $\widetilde{J}_{2n}$ with respect to $\mathfrak{E}'$. Define $(\cdot,\cdot) = \mathrm{tr}_{F/\mathbb{Q}} \circ (\cdot,\cdot)'$.
(5) The $F \otimes \mathbb{R}$-endomorphism J of $V \otimes \mathbb{R}$ described by the matrix $-\widetilde{J}_{2n}$ with respect to $\mathfrak{E}'$.

Remark 5.1.1. *Denote by $\mathrm{GSp}_{(\cdot,\cdot)'}$ the F-group given on R-valued points by $\mathrm{GSp}_{(\cdot,\cdot)'}(R) = \{g \in \mathrm{GL}_R(V \otimes_F R) \mid \exists c = c(g) \in R^\times \forall x, y \in V \otimes_F R : (gx, gy)_R' = c\,(x,y)_R'\}$. Then the reductive $\mathbb{Q}$-group G associated with the above PEL datum fits into the following cartesian diagram.*

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "c"] & \operatorname{Res}_{F/\mathbb{Q}} \mathrm{GSp}_{(\cdot,\cdot)'} \arrow[d, "c"] \\ \mathbb{G}_{m,\mathbb{Q}} \arrow[r, hook] & \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}. \end{tikzcd}$$

We assume that $p\mathcal{O}_F = \mathcal{P}^e$ for a single prime $\mathcal{P}$ of $\mathcal{O}_F$. Denote by $f = [k_\mathcal{P} : \mathbb{F}_p]$ the corresponding inertia degree, so that $g = ef$. We have $F \otimes \mathbb{Q}_p = F_\mathcal{P}$ and $\mathcal{O}_F \otimes \mathbb{Z}_p = \mathcal{O}_{F_\mathcal{P}}$. Fix once and for all a uniformizer π of $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$.

Denote by $\mathfrak{C} = \mathfrak{C}_{\mathcal{O}_{F_\mathcal{P}}|\mathbb{Z}_p}$ the inverse different of the extension $F_\mathcal{P}/\mathbb{Q}_p$. Fix a generator δ of $\mathfrak{C}$ over $\mathcal{O}_{F_\mathcal{P}}$ and define a basis $(e_1, \dots, e_{2n})$ of $V_{\mathbb{Q}_p}$ over $F_\mathcal{P}$ by $e_i = e_i'$, $e_{n+i} = \delta e_{n+i}'$, $1 \le i \le n$.

Let $0 \leq i < 2n$. We denote by Λ_i the $\mathcal{O}_{F_\mathcal{P}}$-lattice in $V_{\mathbb{Q}_p}$ with basis

$$\mathfrak{E}_i = (\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_{2n}).$$

For $k \in \mathbb{Z}$ we further define $\Lambda_{2nk+i} = \pi^{-k}\Lambda_i$ and we denote by $\mathfrak{E}_{2nk+i}$ the corresponding basis obtained from $\mathfrak{E}_i$. Then $\mathcal{L} = (\Lambda_i)_i$ is a complete chain of $\mathcal{O}_{F_\mathcal{P}}$-lattices in V. For $i \in \mathbb{Z}$, the dual lattice $\Lambda_i^\vee := \{x \in V_{\mathbb{Q}_p} \mid (x, \Lambda_i)_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ of Λ_i is given by Λ_{-i}. Consequently the chain $\mathcal{L}$ is self-dual.

Let $i \in \mathbb{Z}$. We denote by $\rho_i : \Lambda_i \to \Lambda_{i+1}$ the inclusion, by $\vartheta_i : \Lambda_{2n+i} \to \Lambda_i$ the isomorphism given by multiplication with π and by $(\cdot,\cdot)_i : \Lambda_i \times \Lambda_{-i} \to \mathbb{Z}_p$ the restriction of $(\cdot,\cdot)_{\mathbb{Q}_p}$. Then $(\Lambda_i, \rho_i, \vartheta_i, (\cdot,\cdot)_i)_i$ is a polarized chain of $\mathcal{O}_{F_\mathcal{P}}$-modules of type $(\mathcal{L})$, which, by abuse of notation, we also denote by $\mathcal{L}$.

Denote by $\langle\cdot,\cdot\rangle_i : \Lambda_i \times \Lambda_{-i} \to \mathcal{O}_{F_\mathcal{P}}$ the restriction of the pairing $\delta^{-1}(\cdot,\cdot)'_{\mathbb{Q}_p}$. It is the perfect pairing described by the matrix $\widetilde{J}_{2n}$ with respect to the bases $\mathfrak{E}_i$ and $\mathfrak{E}_{-i}$.

5.2. **The determinant morphism.** Denote by Σ the set of all embeddings $F \hookrightarrow \mathbb{C}$. The canonical isomorphism

$$F \otimes \mathbb{C} = \prod_{\sigma \in \Sigma} \mathbb{C} \tag{5.2.1}$$

induces a decomposition $V \otimes \mathbb{C} = \prod_{\sigma\in\Sigma} V_\sigma$ into $\mathbb{C}$-vector spaces V_σ, and the morphism $J_\mathbb{C}$ decomposes into the product of $\mathbb{C}$-linear maps $J_\sigma : V_\sigma \to V_\sigma$. Each J_σ induces a decomposition $V_\sigma = V_{\sigma,i} \oplus V_{\sigma,-i}$, where $V_{\sigma,\pm i}$ denotes the $\pm i$-eigenspace of J_σ. From the explicit description of J in terms of $\mathcal{B}$ above one sees that both $V_{\sigma,i}$ and $V_{\sigma,-i}$ have dimension n over $\mathbb{C}$.

The $(-i)$-eigenspace V_{-i} of $J_\mathbb{C}$ is given by $V_{-i} = \prod_{\sigma\in\Sigma} V_{-i,\sigma}$. As $\dim_\mathbb{C} V_{-i,\sigma} = n$ for all σ, there is an isomorphism $V_{-i} \simeq (\prod_\sigma \mathbb{C})^n$ of $\prod_\sigma \mathbb{C}$-modules and hence the $\mathcal{O}_F \otimes \mathbb{C}$-module corresponding to V_{-i} under (5.2.1) is isomorphic to $\mathcal{O}_F^n \otimes \mathbb{C}$. In particular, the morphism $\det_{V_{-i}} : V_{\mathcal{O}_F\otimes\mathbb{C}} \to \mathbb{A}^1_\mathbb{C}$ is defined over $\mathbb{Z}$, and we also denote by $\det_{V_{-i}}$ the corresponding morphism over $\mathbb{Z}$.

5.3. **The local model.** For the chosen PEL datum, Definition 3.3.2 amounts to the following.

Definition 5.3.1. *The local model M^{loc} is the functor on the category of $\mathbb{Z}_p$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_i)_{i\in\mathbb{Z}}$ of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

(a) $\rho_{i,R}(t_i) \subset t_{i+1}$.
(b) The quotient $\Lambda_{i,R}/t_i$ is a finite locally free R-module.
(c) We have an equality

$$\det_{\Lambda_{i,R}/t_i} = \det_{V_{-i}} \otimes R$$

of morphisms $V_{\mathcal{O}_F\otimes R} \to \mathbb{A}^1_R$.
(d) Under the pairing $(\cdot,\cdot)_{i,R} : \Lambda_{i,R} \times \Lambda_{-i,R} \to R$, the submodules t_i and t_{-i} pair to zero.
(e) $\vartheta_i(t_{2n+i}) = t_i$.

Corollary 5.3.2. *Condition 5.3.1(d) can be equivalently replaced by the following condition.*

(d') $t_i^{\perp,\langle\cdot,\cdot\rangle_{i,R}} = t_{-i}$.

Proof. By Lemma 3.3.1 we know that in the presence of conditions 5.3.1(b) and 5.3.1(c), condition 5.3.1(d) is equivalent to

$$t_i^{\perp,(\cdot,\cdot)_{i,R}} = t_{-i}.$$

Lemma 4.2.1(1), applied to the extension $\mathbb{Z}_p \to \mathcal{O}_{F_\mathcal{P}}$, the involution $\mathrm{id}_{\mathcal{O}_{F_\mathcal{P}}}$, the $\mathcal{O}_{F_\mathcal{P}}$-module $\mathfrak{C}$ with the isomorphism $\mathcal{O}_{F_\mathcal{P}} \xrightarrow{\cdot\delta} \mathfrak{C}$, the map $\mathrm{tr}_{F_\mathcal{P}/\mathbb{Q}_p}\big|_\mathfrak{C} : \mathfrak{C} \to \mathbb{Z}_p$ and the restriction $\Lambda_i \times \Lambda_{-i} \to \mathfrak{C}$ of $(\cdot,\cdot)'_{\mathbb{Q}_p}$, gives

$$t_i^{\perp,(\cdot,\cdot)_{i,R}} = t_i^{\perp,\langle\cdot,\cdot\rangle_{i,R}}.$$

□

5.4. **The special fiber of the local model.** For $i \in \mathbb{Z}$, denote by $\overline{\Lambda}_i$ the $\mathbb{F}[u]/u^e$-module $(\mathbb{F}[u]/u^e)^{2n}$ and by $\overline{\mathfrak{E}}_i$ its canonical basis. Denote by $\overline{\langle\cdot,\cdot\rangle}_i : \overline{\Lambda}_i \times \overline{\Lambda}_{-i} \to \mathbb{F}[u]/u^e$ the pairing described by the matrix $\widetilde{J}_{2n}$ with respect to $\overline{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_{-i}$. Denote by $\overline{\vartheta}_i : \overline{\Lambda}_{2n+i} \to \overline{\Lambda}_i$ the identity morphism. For $k \in \mathbb{Z}$ and $0 \le i < 2n$, let $\overline{\rho}_{2n+i} : \overline{\Lambda}_{2n+i} \to \overline{\Lambda}_{2n+i+1}$ be the morphism described by the matrix $\mathrm{diag}(1^{(i)}, u, 1^{(2n-i-1)})$ with respect to $\overline{\mathfrak{E}}_{2nk+i}$ and $\overline{\mathfrak{E}}_{2nk+i+1}$.

Definition 5.4.1. *Let $M^{e,n}$ be the functor on the category of $\mathbb{F}$-algebras with $M^{e,n}(R)$ the set of tuples $(t_i)_{i\in\mathbb{Z}}$ of $R[u]/u^e$-submodules $t_i \subset \overline{\Lambda}_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

(a) $\overline{\rho}_{i,R}(t_i) \subset t_{i+1}$.
(b) The quotient $\overline{\Lambda}_{i,R}/t_i$ is finite locally free over R.
(c) For all $p \in R[u]/u^e$, we have

$$\chi_R(p|\overline{\Lambda}_{i,R}/t_i) = \big(T - p(0)\big)^{ne}$$

in $R[T]$.
(d) $t_i^{\perp,\overline{\langle\cdot,\cdot\rangle}_{i,R}} = t_{-i}$.
(e) $\overline{\vartheta}_i(t_{2n+i}) = t_i$.

Corollary 5.4.2. *Assume that R is* reduced. *Then condition 5.4.1(c) is equivalent to the following condition.*

(c') $\mathrm{rk}_R\, t_i = ne$.

Proof. See Lemma 2.3.9. □

Denote by $\mathfrak{S}$ the set of all embedding $\sigma : k_\mathcal{P} \hookrightarrow \mathbb{F}$ and recall from Section 4.3 the canonical isomorphism

$$\mathcal{O}_F \otimes \mathbb{F} = \prod_{\sigma\in\mathfrak{S}} \mathbb{F}[u]/(u^e). \tag{5.4.3}$$

Let $i \in \mathbb{Z}$. From (5.4.3) we obtain an isomorphism

$$\Lambda_{i,\mathbb{F}} = \prod_{\sigma\in\mathfrak{S}} \overline{\Lambda}_i \tag{5.4.4}$$

by identifying the basis $\mathfrak{E}_{i,\mathbb{F}}$ with the product of the bases $\overline{\mathfrak{E}}_i$. Under this identification, the morphism $\rho_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\rho}_i$, the pairing $\langle\cdot,\cdot\rangle_{i,\mathbb{F}}$ decomposes into the pairings $\overline{\langle\cdot,\cdot\rangle}_i$ and the morphism $\vartheta_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\vartheta}_i$.

Let R be an $\mathbb{F}$-algebra and let $(t_i)_{i\in\mathbb{Z}}$ be a tuple of $\mathcal{O}_F\otimes R$-submodules $t_i\subset\Lambda_{i,R}$. Then (5.4.4) induces decompositions $t_i=\prod_{\sigma\in\mathfrak{S}}t_{i,\sigma}$ into $R[u]/u^e$-submodules $t_{i,\sigma}\subset\overline{\Lambda}_{i,R}$.

Proposition 5.4.5. *The morphism $M_{\mathbb{F}}^{\mathrm{loc}}\to\prod_{\sigma\in\mathfrak{S}}M^{e,n}$ given on R-valued points by*

$$M_{\mathbb{F}}^{\mathrm{loc}}(R)\to\prod_{\sigma\in\mathfrak{S}}M^{e,n}(R),\qquad (t_i)\mapsto\left((t_{i,\sigma})_i\right)_\sigma \tag{5.4.6}$$

is an isomorphism of functors on the category of $\mathbb{F}$-algebras.

Proof. It is clear that conditions 5.3.1(a), 5.3.1(b), 5.3.2(d') and 5.3.1(e) can be verified factorwise and correspond to the analogous parts of Definition 5.4.1. For condition 5.3.1(c) the same is true by Proposition 2.3.7. □

5.5. **The affine flag variety.** This section deals with the affine flag variety for the symplectic group. Our discussion loosely follows the one in [26, §10-11]. Note though that in loc. cit. there seems to be a (minor) problem with the definition of the notion of self-duality for lattice chains, see Remark 5.5.9 below. We have learned the correct formulation of this definition from [37, §4.2], which deals with the case of a ramified unitary group.

Let R be an $\mathbb{F}$-algebra. Let $\widetilde{\langle\cdot,\cdot\rangle}$ be the standard symplectic form on $R(\!(u)\!)^{2n}$. For a lattice Λ in $R(\!(u)\!)^{2n}$ we define $\Lambda^\vee:=\{x\in R(\!(u)\!)^{2n}\mid\widetilde{\langle x,\Lambda\rangle}\subset R[\![u]\!]\}$. Recall from Section 4.5 the standard lattice chain $\widetilde{\mathcal{L}}=(\widetilde{\Lambda}_i)_i$ in $R(\!(u)\!)^{2n}$. Note that $(\widetilde{\Lambda}_i)^\vee=\widetilde{\Lambda}_{-i}$ for all $i\in\mathbb{Z}$. We denote by $\widetilde{\langle\cdot,\cdot\rangle}_i:\widetilde{\Lambda}_i\times\widetilde{\Lambda}_{-i}\to R[\![u]\!]$ the restriction of $\widetilde{\langle\cdot,\cdot\rangle}$.

In complete analogy with [32, Definition 3.14] we make the following definition.

Definition 5.5.1. *Let R be an $\mathbb{F}[\![u]\!]$-algebra. A* polarized chain of R-modules *of type $(\widetilde{\mathcal{L}})$ is a tuple*

$$\mathcal{M}=(M_i,\varrho_i:M_i\to M_{i+1},\theta_i:M_{2n+i}\xrightarrow{\sim}M_i,\mathcal{E}_i:M_i\times M_{-i}\to R)_{i\in\mathbb{Z}},$$

where the M_i are R-modules, the ϱ_i are R-linear maps, the θ_i are R-linear isomorphisms and the $\mathcal{E}_i$ are perfect R-bilinear pairings, such that for all $i\in\mathbb{Z}$ the following conditions hold.

1. *Locally on R, there are isomorphisms $M_i\simeq\widetilde{\Lambda}_{i,R}$ and $M_{i+1}/\varrho_i(M_i)\simeq(\widetilde{\Lambda}_{i+1}/\widetilde{\Lambda}_i)_R$ of R-modules.*
2. *The diagram*

$$\begin{tikzcd} M_{2n+i} \arrow[r, "\varrho_{2n+i}"] \arrow[d, "\theta_i"'] & M_{2n+i+1} \arrow[d, "\theta_{i+1}"] \\ M_i \arrow[r, "\varrho_i"] & M_{i+1} \end{tikzcd}$$

commutes.

(3) *The composition* $M_{2n+i} \xrightarrow{\theta_i} M_i \xrightarrow{\prod_{j=0}^{2n-1} \varrho_{i+j}} M_{2n+i}$ *is multiplication by* u.
(4) $\mathcal{E}_i(a,b) = -\mathcal{E}_{-i}(b,a)$ *for all* $a \in M_i, b \in M_{-i}$.
(5) $\mathcal{E}_i(a, \varrho_{-i-1}(b)) = \mathcal{E}_{i+1}(\varrho_i(a), b)$ *for all* $a \in M_i, b \in M_{-i-1}$.
(6) $\mathcal{E}_i(a,b) = \mathcal{E}_{2n+i}(\theta_i^{-1}(a), \theta_{-2n-i}(b))$ *for all* $a \in M_i, b \in M_{-i}$.

Let

$$\mathcal{M} = (M_i, \varrho_i : M_i \to M_{i+1}, \theta_i : M_{2n+i} \xrightarrow{\sim} M_i, \mathcal{E}_i : M_i \times M_{-i} \to R)_i,$$
$$\mathcal{M}' = (M_i', \varrho_i' : M_i' \to M_{i+1}', \theta_i' : M_{2n+i}' \xrightarrow{\sim} M_i', \mathcal{E}_i' : M_i' \times M_{-i}' \to R)_i$$

be polarized chains of R-modules of type $(\widetilde{\mathcal{L}})$. *A* morphism $\varphi : \mathcal{M} \to \mathcal{M}'$ *is a tuple* $(\varphi_i)_{i\in\mathbb{Z}}$ *of isomorphisms of R-modules* $\varphi_i : M_i \to M_i'$ *satisfying the following conditions for all* $i \in \mathbb{Z}$.

(a) *The diagrams*

$$\begin{tikzcd} M_i \arrow[r, "\varrho_i"] \arrow[d, "\varphi_i"'] & M_{i+1} \arrow[d, "\varphi_{i+1}"] \\ M_i' \arrow[r, "\varrho_i'"] & M_{i+1}', \end{tikzcd} \qquad \begin{tikzcd} M_i \arrow[d, "\varphi_i"'] & M_{2n+i} \arrow[l, "\theta_i"'] \arrow[d, "\varphi_{2n+i}"] \\ M_i' & M_{2n+i}' \arrow[l, "\theta_i'"'] \end{tikzcd}$$

commute.

(b) $\forall (x,y) \in M_i \times M_{-i} :\ \mathcal{E}_i'(\varphi_i(x), \varphi_{-i}(y)) = \mathcal{E}_i(x,y)$.

We denote by $\mathrm{Isom}(\mathcal{M}, \mathcal{M}')$ *the functor on the category of R-algebras with* $\mathrm{Isom}(\mathcal{M}, \mathcal{M}')(R')$ *the set of morphisms* $\mathcal{M} \otimes_R R' \to \mathcal{M}' \otimes_R R'$ *of polarized chains of R'-modules of type* $(\widetilde{\mathcal{L}})$. *We also write* $\mathrm{Aut}(\mathcal{M}) = \mathrm{Isom}(\mathcal{M}, \mathcal{M})$.

Proposition 5.5.2. *Let R be an $\mathbb{F}[\![u]\!]$-algebra such that the image of u in R is nilpotent. Then any two polarized chains $\mathcal{M}, \mathcal{N}$ of R-modules of type $(\widetilde{\mathcal{L}})$ are isomorphic locally for the Zariski topology on R. Furthermore the functor* $\mathrm{Isom}(\mathcal{M}, \mathcal{N})$ *is representable by a smooth affine scheme over R.*

Proof. The proof of [32, Proposition A.21] carries over to this situation without any changes. □

Proposition 5.5.3. *Let R be an $\mathbb{F}$-algebra and let $\mathcal{M}, \mathcal{N}$ be polarized chains of $R[\![u]\!]$-modules of type $(\widetilde{\mathcal{L}})$. Then the canonical map* $\mathrm{Isom}(\mathcal{M}, \mathcal{N})(R[\![u]\!]) \to \mathrm{Isom}(\mathcal{M}, \mathcal{N})(R[\![u]\!]/u^m)$ *is surjective for all $m \in \mathbb{N}_{\geq 1}$. In particular $\mathcal{M}$ and $\mathcal{N}$ are isomorphic locally for the Zariski topology on R.*

Proof. The proof of the first part is analogous to the one of Proposition 2.7.8, referring to Proposition 5.5.2 in place of [32, Theorem 3.16]. By Proposition 5.5.2 we know that $\mathrm{Isom}(\mathcal{M}, \mathcal{N})(R) \neq \emptyset$ Zariski locally on R. Thus the first statement implies the second. □

The following definition is a straightforward variant of [37, §4.2].

Definition 5.5.4. *Let R be an $\mathbb{F}$-algebra and let $(L_i)_i$ be a lattice chain in* $R(\!(u)\!)^{2n}$.

(1) *Let $r \in \mathbb{Z}$. The chain $(L_i)_i$ is called* r-self-dual *if*

$$\forall i \in \mathbb{Z} :\ L_i^\vee = u^r L_{-i}.$$

Denote by $\mathcal{F}_{\mathrm{Sp}}^{(r)}$ *the functor on the category of* $\mathbb{F}$*-algebras with* $\mathcal{F}_{\mathrm{Sp}}^{(r)}(R)$ *the set of* r*-self-dual lattice chains in* $R(\!(u)\!)^{2n}$.

(2) *The chain* $(L_i)_i$ *is called* self-dual *if Zariski locally on* R *there is an* $a \in R(\!(u)\!)^\times$ *such that*

$$\forall i \in \mathbb{Z}: \; L_i^\vee = aL_{-i}. \tag{5.5.5}$$

We denote by $\mathcal{F}_{\mathrm{GSp}}$ *the functor on the category of* $\mathbb{F}$*-algebras with* $\mathcal{F}_{\mathrm{GSp}}(R)$ *the set of self-dual lattice chains in* $R(\!(u)\!)^{2n}$.

Note that $\widetilde{\mathcal{L}} \in \mathcal{F}_{\mathrm{Sp}}^{(0)}(R)$.

Remark 5.5.6. *Let* R *be a reduced* $\mathbb{F}$*-algebra such that* $\operatorname{Spec} R$ *connected. Then*

$$\mathcal{F}_{\mathrm{GSp}}(R) = \bigcup_{r\in\mathbb{Z}} \mathcal{F}_{\mathrm{Sp}}^{(r)}(R).$$

Proof. This follows easily from Lemma 4.5.14. □

Remark 5.5.7. *Let* R *be an* $\mathbb{F}$*-algebra and let* $(L_i)_i \in \mathcal{F}_{\mathrm{Sp}}^{(0)}(R)$. *For* $i \in \mathbb{Z}$ *denote by* $\varrho_i : L_i \to L_{i+1}$ *the inclusion, by* $\theta_i : L_{2n+i} \to L_i$ *the isomorphism given by multiplication with* u *and by* $\mathcal{E}_i : L_i \times L_{-i} \to R[\![u]\!]$ *the restriction of* $\widetilde{\langle\cdot,\cdot\rangle}$. *Then* $(L_i, \varrho_i, \theta_i, \mathcal{E}_i)$ *is a polarized chain of* $R[\![u]\!]$*-modules of type* $(\widetilde{\mathcal{L}})$. *Here we use Proposition 4.5.5 to verify condition 5.5.1(1).*

Recall from Section 2.1.2 the subfunctor $I \subset \mathrm{L}\,\mathrm{GL}_{2n}$. We define a subfunctor $I_{\mathrm{GSp}} = I_{\mathrm{GSp}_{2n}}$ of $\mathrm{L}\,\mathrm{GSp} = \mathrm{L}\,\mathrm{GSp}_{2n}$ by $I_{\mathrm{GSp}} = \mathrm{L}\,\mathrm{GSp}_{2n} \cap I$. We consider all of these functors as functors on the category of $\mathbb{F}$-algebras.

The proof of the following result is similar to and therefore based on the proof of [27, Theorem 4.1].

Proposition 5.5.8. *The natural action of* $\mathrm{L}\,\mathrm{GL}_{2n}$ *on* $\mathcal{F}$ *(cf. Remark 4.5.13) restricts to an action of* $\mathrm{L}\,\mathrm{GSp}$ *on* $\mathcal{F}_{\mathrm{GSp}}$. *Consequently we obtain an injective map*

$$\begin{aligned} \mathrm{L}\,\mathrm{GSp}(R)/I_{\mathrm{GSp}}(R) &\xrightarrow{\phi(R)} \mathcal{F}_{\mathrm{GSp}}(R), \\ g &\longmapsto g\cdot\widetilde{\mathcal{L}} \end{aligned}$$

for each $\mathbb{F}$*-algebra* R. *The morphism* ϕ *identifies* $\mathcal{F}_{\mathrm{GSp}}$ *with both the Zariski and the fpqc sheafification of the presheaf* $\mathrm{L}\,\mathrm{GSp}/I_{\mathrm{GSp}}$.

Proof. Let R be an $\mathbb{F}$-algebra. First note that the map $c : \mathrm{GSp}(R(\!(u)\!)) \to R(\!(u)\!)^\times$ is surjective, as it admits the section $a \mapsto \operatorname{diag}(a^{(n)}, 1^{(n)})$. Using Remark 5.5.7 and Proposition 5.5.3, one sees as in the proof of Proposition 3.7.6 that the action in question is well-defined and that any element of $\mathcal{F}_{\mathrm{GSp}}(R)$ lies in the image of $\phi(R)$ Zariski locally on R. As $\mathcal{F}_{\mathrm{GSp}}$ is clearly a Zariski sheaf, it follows that $\mathcal{F}_{\mathrm{GSp}}$ is indeed the Zariski sheafification of the presheaf $\mathrm{L}\,\mathrm{GSp}/I_{\mathrm{GSp}}$.

To see that $\mathcal{F}_{\mathrm{GSp}}$ is also the fpqc sheafification of $\mathrm{L}\,\mathrm{GSp}/I_{\mathrm{GSp}}$, it suffices to show that $\mathcal{F}_{\mathrm{GSp}}$ is an fpqc sheaf. Let $(L_i)_i$ be a lattice chain in $R(\!(u)\!)^{2n}$. Assume that fpqc locally on R there is an $a \in R(\!(u)\!)^\times$ such that (5.5.5) holds. By Lemma 4.5.15, the scalar a gives rise to a well-defined element of

$(\mathrm{L}\,\mathbb{G}_m/\,\mathrm{L}^+\,\mathbb{G}_m)_{\mathrm{fpqc}}(R)$, where $(\mathrm{L}\,\mathbb{G}_m/\,\mathrm{L}^+\,\mathbb{G}_m)_{\mathrm{fpqc}}$ denotes the fpqc sheafification of the presheaf $\mathrm{L}\,\mathbb{G}_m/\,\mathrm{L}^+\,\mathbb{G}_m$. By Proposition 4.5.10 any element of $(\mathrm{L}\,\mathbb{G}_m/\,\mathrm{L}^+\,\mathbb{G}_m)_{\mathrm{fpqc}}(R)$ can be represented in $\mathrm{L}\,\mathbb{G}_m(R)$ Zariski locally on R, so that the scalar a exists in fact Zariski locally on R. $\square$

Remark 5.5.9. *Let us note that there seems to exist a misconception surrounding the notion of self-duality for lattice chains. In the literature one finds the following definition: Let R be an $\mathbb{F}$-algebra. A lattice chain $(L_i)_i \in \mathcal{F}(R)$ is called* (naively) self-dual *if for each $i \in \mathbb{Z}$ there is a $j \in \mathbb{Z}$ such that $L_i^\vee = L_j$. It is then claimed that the fpqc local $\mathrm{L}\,\mathrm{GSp}$-orbit of $\widetilde{\mathcal{L}}$ (in the sense of Proposition 5.5.8) is precisely the set of (naively) self-dual lattice chains. This is wrong in both directions: There are elements of the $\mathrm{L}\,\mathrm{GSp}$-orbit of $\widetilde{\mathcal{L}}$ that are not (naively) self-dual, and there are (naively) self-dual lattice chains that are not (fpqc locally) contained in the $\mathrm{L}\,\mathrm{GSp}$-orbit of $\widetilde{\mathcal{L}}$. Let us give two easy examples for these phenomena.*

- *Let $n = 1$ and $a \in R(\!(u)\!)^\times$. The chain $(L_i)_i = a\widetilde{\mathcal{L}}$ satisfies $L_i^\vee = a^{-2}L_{-i}$, $i \in \mathbb{Z}$. Assume there is a $j \in \mathbb{Z}$ with $L_0^\vee = L_j$. Then $a^{-2}L_0 = L_j$ and hence $a^{-2}\widetilde{\Lambda}_0 = \widetilde{\Lambda}_j$. Projecting this equality inside $R(\!(u)\!)^2$ to its first components yields the existence of a $k \in \mathbb{Z}$ with $a^{-2}R\llbracket u\rrbracket = u^kR\llbracket u\rrbracket$, so that $u^ka^2 \in R\llbracket u\rrbracket^\times$. If for example $R = \mathbb{F}[x]/x^2$ and $a = 1 + xu^{-1}$, such a k does not exist.*
- *Conversely one easily sees that for $n \geq 2$, the (naively) self-dual chain $(\widetilde{\Lambda}_{i+1})_{i\in\mathbb{Z}}$ does not lie in the $\mathrm{L}\,\mathrm{GSp}(R)$-orbit of $\widetilde{\mathcal{L}}$ (unless $R = \{0\}$).*

5.6. **Embedding the local model into the affine flag variety.** Let R be an $\mathbb{F}$-algebra. We consider an $R[u]/u^e$-module as an $R\llbracket u\rrbracket$-module via the canonical projection $R\llbracket u\rrbracket \to R[u]/u^e$. For $i \in \mathbb{Z}$ denote by $\alpha_i : \widetilde{\Lambda}_i \to \overline{\Lambda}_{i,R}$ the morphism described by the identity matrix with respect to $\widetilde{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_i$. It induces an isomorphism $\widetilde{\Lambda}_i/u^e\widetilde{\Lambda}_i \xrightarrow{\sim} \overline{\Lambda}_{i,R}$. Clearly the following diagrams commute.

$$\begin{tikzcd}
\widetilde{\Lambda}_i \arrow[r, phantom, "\subset"] \arrow[d, "\alpha_i"'] & \widetilde{\Lambda}_{i+1} \arrow[d, "\alpha_{i+1}"] \\
\overline{\Lambda}_{i,R} \arrow[r, "\overline{\rho}_{i,R}"] & \overline{\Lambda}_{i+1,R},
\end{tikzcd}
\quad
\begin{tikzcd}
\widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} \arrow[r, "\widetilde{\langle\cdot,\cdot\rangle}_i"] \arrow[d, "\alpha_i\times\alpha_{-i}"'] & R\llbracket u\rrbracket \arrow[d] \\
\overline{\Lambda}_{i,R} \times \overline{\Lambda}_{-i,R} \arrow[r, "\overline{\langle\cdot,\cdot\rangle}_{i,R}"] & R[u]/u^e,
\end{tikzcd}
\quad
\begin{tikzcd}
\widetilde{\Lambda}_i \arrow[d, "\alpha_i"'] & \widetilde{\Lambda}_{2n+i} \arrow[l, "u\cdot"'] \arrow[d, "\alpha_{2n+i}"] \\
\overline{\Lambda}_{i,R} & \overline{\Lambda}_{2n+i,R}. \arrow[l, "\overline{\vartheta}_{i,R}"']
\end{tikzcd}$$

The following proposition allows us to consider $M^{e,n}$ as a subfunctor of $\mathcal{F}_{\mathrm{Sp}}^{(-e)}$.

Proposition 5.6.1 ([26, §11])**.** *There is an embedding $\alpha : M^{e,n} \hookrightarrow \mathcal{F}_{\mathrm{Sp}}^{(-e)}$ given on R-valued points by*

$$\begin{aligned} M^{e,n}(R) &\to \mathcal{F}_{\mathrm{Sp}}^{(-e)}(R), \\ (t_i)_i &\mapsto (\alpha_i^{-1}(t_i))_i. \end{aligned}$$

It induces a bijection from $M^{e,n}(R)$ onto the set of those $(L_i)_i \in \mathcal{F}_{\mathrm{Sp}}^{(-e)}(R)$ satisfying the following conditions for all $i \in \mathbb{Z}$.

(1) $u^e\widetilde{\Lambda}_i \subset L_i \subset \widetilde{\Lambda}_i$.

(2) For all $p \in R[u]/u^e$, we have

$$\chi_R(p|\widetilde{\Lambda}_i/L_i) = (T - p(0))^{ne}$$

in $R[T]$. Here $\widetilde{\Lambda}_i/L_i$ is considered as an $R[u]/u^e$-module using (1).

Proof. Let $(t_i)_i \in M^{e,n}(R)$ and set $(L_i)_i = (\alpha_i^{-1}(t_i))_i$. It is clear that this defines a periodic lattice chain in $R(\!(u)\!)^{2n}$. Let $i \in \mathbb{Z}$. We have

$$\mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/\widetilde{\Lambda}_i) + \mathrm{rk}_R(\widetilde{\Lambda}_i/L_i) = \mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/L_{i+1}) + \mathrm{rk}_R(L_{i+1}/L_i), \tag{5.6.2}$$

as both sides are equal to $\mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/L_i)$. We conclude from 5.4.1(c) that $\mathrm{rk}_R(\widetilde{\Lambda}_i/L_i) = ne = \mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/L_{i+1})$. Thus (5.6.2) amounts to the equation $\mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/\widetilde{\Lambda}_i) = \mathrm{rk}_R(L_{i+1}/L_i)$, so that the chain $(L_i)_i$ is complete. Exactly as in the proof of Proposition 3.7.10 one then shows that $L_i^\vee = u^{-e}L_{-i}$.

This proves the existence of the map α. Its injectivity as well as the characterization of its image are immediate. □

Note that $\overline{\mathcal{L}} = (\overline{\Lambda}_i, \overline{\rho}_i, \overline{\vartheta}_i, \overline{\langle\cdot,\cdot\rangle}_i)_i$ is a polarized chain of $\mathbb{F}[u]/u^e$-modules of type $(\widetilde{\mathcal{L}})$. In fact $\overline{\mathcal{L}} = \widetilde{\mathcal{L}} \otimes_{\mathbb{F}[\![u]\!]} \mathbb{F}[u]/u^e$. Let R be an $\mathbb{F}$-algebra. There is an obvious action of $\mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ on $M^{e,n}(R)$, given by $(\varphi_i)\cdot(t_i) = (\varphi_i(t_i))$. The canonical morphism $R[\![u]\!] \to R[u]/u^e$ induces a morphism $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ and we thereby extend this $\mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$-action on $M^{e,n}(R)$ to an $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!])$-action.

Lemma 5.6.3. *Let R be an $\mathbb{F}$-algebra and let $t \in M^{e,n}(R)$. We have $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \cdot t = \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e) \cdot t$.*

Proof. The map $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ is surjective by Proposition 5.5.3. □

Define a subfunctor $I_{\mathrm{Sp}} = I_{\mathrm{Sp}_{2n}}$ of $\mathrm{L}\,\mathrm{Sp}_{2n}$ by $I_{\mathrm{Sp}} = \mathrm{L}\,\mathrm{Sp}_{2n} \cap I_{\mathrm{GSp}}$.

Lemma 5.6.4. *We have $I_{\mathrm{GSp}}(\mathbb{F}) = \mathbb{F}[\![u]\!]^\times I_{\mathrm{Sp}}(\mathbb{F})$.*

Proof. Let $g \in I_{\mathrm{GSp}}(\mathbb{F})$. Clearly $c(g) \in \mathbb{F}[\![u]\!]^\times$. As $\mathrm{char}\,\mathbb{F} \neq 2$, there is an $x \in \mathbb{F}[\![u]\!]^\times$ with $x^2 = c(g)$. Then $x^{-1}g \in I_{\mathrm{Sp}}(\mathbb{F})$. □

Lemma 5.6.5. *Let $g \in I_{\mathrm{Sp}}(\mathbb{F})$. Then g restricts to an automorphism $g_i : \widetilde{\Lambda}_i \to \widetilde{\Lambda}_i$ for each $i \in \mathbb{Z}$. The assignment $g \mapsto (g_i)_i$ defines an isomorphism $I_{\mathrm{Sp}}(\mathbb{F}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!])$.*

Proof. Clear (cf. the proof of Lemma 3.7.15). □

Proposition 5.6.6. *Let $t \in M^{e,n}(\mathbb{F})$. Then α induces a bijection*

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \cdot t \xrightarrow{\sim} I_{\mathrm{GSp}}(\mathbb{F}) \cdot \alpha(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)\backslash M^{e,n}(\mathbb{F}) \hookrightarrow I_{\mathrm{GSp}}(\mathbb{F})\backslash \mathcal{F}_{\mathrm{GSp}}(\mathbb{F}).$$

Proof. The composition $M^{e,n}(\mathbb{F}) \xrightarrow{\alpha} \mathcal{F}_{\mathrm{Sp}}^{(-e)}(\mathbb{F}) \subset \mathcal{F}_{\mathrm{GSp}}(\mathbb{F})$ is equivariant for the $\mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!])$-action on $M^{e,n}(\mathbb{F})$, the $I_{\mathrm{Sp}}(\mathbb{F})$-action on $\mathcal{F}_{\mathrm{GSp}}(\mathbb{F})$ and the isomorphism $I_{\mathrm{Sp}}(\mathbb{F}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!])$ of Lemma 5.6.5. It therefore induces a bijection $\mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!]) \cdot t \xrightarrow{\sim} I_{\mathrm{Sp}}(\mathbb{F}) \cdot \alpha(t)$. We conclude by applying Lemmata 5.6.3 and 5.6.4. □

Consider $\alpha' : M^{e,n}(\mathbb{F}) \hookrightarrow \mathcal{F}_{\mathrm{GSp}}(\mathbb{F}) \xrightarrow{\phi(\mathbb{F})^{-1}} \mathrm{L}\,\mathrm{GSp}(\mathbb{F})/I_{\mathrm{GSp}}(\mathbb{F})$.

Proposition 5.6.7. *Let $t \in M^{e,n}(\mathbb{F})$. Then α' induces a bijection*

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \cdot t \xrightarrow{\sim} I_{\mathrm{GSp}}(\mathbb{F}) \cdot \alpha'(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)\backslash M^{e,n}(\mathbb{F}) \hookrightarrow I_{\mathrm{GSp}}(\mathbb{F})\backslash \mathrm{GSp}(\mathbb{F}(\!(u)\!))/I_{\mathrm{GSp}}(\mathbb{F}).$$

Proof. Clear from Proposition 5.6.6, as the isomorphism $\phi(\mathbb{F})$ is in particular $I_{\mathrm{GSp}}(\mathbb{F})$-equivariant. □

Let R be an $\mathbb{F}$-algebra and $(\varphi_i)_i \in \mathrm{Aut}(\mathcal{L})(R)$. The decomposition (5.4.4) induces for each i a decomposition of $\varphi_i : \Lambda_{i,R} \xrightarrow{\sim} \Lambda_{i,R}$ into the product of $R[u]/u^e$-linear automorphisms $\varphi_{i,\sigma} : \overline{\Lambda}_{i,R} \xrightarrow{\sim} \overline{\Lambda}_{i,R}$.

Proposition 5.6.8. *Let R be an $\mathbb{F}$-algebra. The following map is an isomorphism, functorial in R.*

$$\mathrm{Aut}(\mathcal{L})(R) \to \prod_{\sigma\in\mathfrak{S}} \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e),$$
$$(\varphi_i)_i \mapsto ((\varphi_{i,\sigma})_i)_\sigma.$$

Proof. Let $(\varphi_i)_i \in \mathrm{Aut}(\mathcal{L})(R)$. The duality condition (2.7.5) amounts to the following condition.

$$\forall (x,y) \in \Lambda_{i,R} \times \Lambda_{-i,R} : (\varphi_i(x), \varphi_{-i}(y))_{i,R} = (x,y)_{i,R}\,. \tag{5.6.9}$$

By Lemma 4.2.1(2), applied to the same setup as in the proof of Corollary 5.3.2, condition (5.6.9) is equivalent to

$$\forall (x,y) \in \Lambda_{i,R} \times \Lambda_{-i,R} : \ \langle \varphi_i(x), \varphi_{-i}(y)\rangle_{i,R} = \langle x,y\rangle_{i,R}\,.$$

From this the claim is obvious. □

Consider the composition

$$\tilde{\alpha} : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{(5.4.6)} \prod_{\sigma\in\mathfrak{S}} M^{e,n}(\mathbb{F}) \xrightarrow{\prod_\sigma \alpha'} \prod_{\sigma\in\mathfrak{S}} \mathrm{L}\,\mathrm{GSp}(\mathbb{F})/I_{\mathrm{GSp}}(\mathbb{F}).$$

For $\sigma \in \mathfrak{S}$ denote by $\tilde{\alpha}_\sigma : M^{\mathrm{loc}}(\mathbb{F}) \to \mathrm{L}\,\mathrm{GSp}(\mathbb{F})/I_{\mathrm{GSp}}(\mathbb{F})$ the corresponding component of $\tilde{\alpha}$.

Theorem 5.6.10. *Let $t \in M^{\mathrm{loc}}(\mathbb{F})$. Then $\tilde{\alpha}$ induces a bijection*

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \cdot t \xrightarrow{\sim} \prod_{\sigma\in\mathfrak{S}} I_{\mathrm{GSp}}(\mathbb{F}) \cdot \tilde{\alpha}_\sigma(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \prod_{\sigma\in\mathfrak{S}} I_{\mathrm{GSp}}(\mathbb{F})\backslash \mathrm{GSp}(F(\!(u)\!))/I_{\mathrm{GSp}}(\mathbb{F}).$$

Proof. The isomorphism $M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{(5.4.6)} \prod_{\sigma\in\mathfrak{S}} M^{e,n}(\mathbb{F})$ is equivariant for the $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$ action on $M^{\mathrm{loc}}(\mathbb{F})$, the $\prod_{\sigma\in\mathfrak{S}} \mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)$ action on $\prod_{\sigma\in\mathfrak{S}} M^{e,n}(\mathbb{F})$ and the isomorphism of Lemma 5.6.8. The statement thus follows from Proposition 5.6.7. □

5.7. **The extended affine Weyl group.** Let T be the maximal torus of diagonal matrices in GSp_{2n} and let N be its normalizer. We denote by $\widetilde{W} = N(\mathbb{F}(\!(u)\!))/T(\mathbb{F}[\![u]\!])$ the extended affine Weyl group of GSp with respect to T. Setting

$$W = \{w \in S_{2n} \mid \forall i \in \{1, \dots, 2n\} : \; w(i) + w(2n+1-i) = 2n+1\}$$

and

$$X = \{(a_1, \dots, a_{2n}) \in \mathbb{Z}^{2n} \mid a_1 + a_{2n} = a_2 + a_{2n-1} = \dots = a_n + a_{n+1}\},$$

the group homomorphism $\upsilon : W \ltimes X \to N(\mathbb{F}(\!(u)\!)),\ (w, \lambda) \mapsto A_w u^\lambda$ induces an isomorphism $W \ltimes X \xrightarrow{\sim} \widetilde{W}$. We use it to identify $\widetilde{W}$ with $W \ltimes X$ and consider $\widetilde{W}$ as a subgroup of $\mathrm{GSp}(\mathbb{F}(\!(u)\!))$ via υ.

To avoid any confusion of the product inside $\widetilde{W}$ and the canonical action of S_{2n} on $\mathbb{Z}^{2n}$, we will always denote the element of $\widetilde{W}$ corresponding to $\lambda \in X$ by u^λ.

Recall from [7, §2.5-2.6] the notion of an extended alcove $(x_i)_{i=0}^{2n-1}$ for GSp_{2n}. Also recall the standard alcove $(\omega_i)_{i=0}^{2n-1}$. As in loc. cit. we identify $\widetilde{W}$ with the set of extended alcoves by using the standard alcove as a base point.

Write $\mathbf{e} = (e^{(2n)})$.

Definition 5.7.1 (Cf. [7, Definition 2.4]). *An extended alcove $(x_i)_{i=0}^{2n-1}$ is called* permissible *if it satisfies the following conditions for all $i \in \{0, \dots, 2n-1\}$.*

(1) $\omega_i \le x_i \le \omega_i + \mathbf{e}$, where $\le$ is to be understood componentwise.
(2) $\sum_{j=1}^{2n} x_i(j) = ne - i$.

Denote by Perm *the set of all permissible extended alcoves.*

Proposition 5.7.2. *The inclusion $N(\mathbb{F}(\!(u)\!)) \subset \mathrm{GSp}(\mathbb{F}(\!(u)\!))$ induces a bijection $\widetilde{W} \xrightarrow{\sim} I_{\mathrm{GSp}}(\mathbb{F})\backslash \mathrm{GSp}(\mathbb{F}(\!(u)\!))/I_{\mathrm{GSp}}(\mathbb{F})$. In other words,*

$$\mathrm{GSp}(\mathbb{F}(\!(u)\!)) = \coprod_{x \in \widetilde{W}} I_{\mathrm{GSp}}(\mathbb{F}) x I_{\mathrm{GSp}}(\mathbb{F}).$$

Under this bijection, the subset

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)\backslash M^{e,n}(\mathbb{F}) \subset I_{\mathrm{GSp}}(\mathbb{F})\backslash \mathrm{GSp}(\mathbb{F}(\!(u)\!))/I_{\mathrm{GSp}}(\mathbb{F})$$

of Proposition 5.6.7 corresponds to the subset Perm $\subset \widetilde{W}$.

Proof. The first statement is the well-known Iwahori decomposition. The second statement follows easily from the explicit description of the image of α in Proposition 5.6.1, keeping in mind Corollary 5.4.2. □

Corollary 5.7.3. *Under the identifications of Theorem 5.6.10, the set* $\prod_{\sigma \in \mathfrak{S}}$ Perm *constitutes a set of representatives of* $\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$.

5.8. **The p-rank on a KR stratum.** We make Definitions 3.2.1 and 3.2.3 explicit for the chosen PEL datum.

Definition 5.8.1. *Let R be a $\mathbb{Z}_p$-algebra. A* self-dual $\mathcal{L}$-set of abelian varieties of determinant $\det_{V_{-i}}$ over R *is a commutative diagram*

$$\begin{tikzcd} \dots \arrow[r, "\varrho_{-2}"] & A_{-1} \arrow[r, "\varrho_{-1}"] \arrow[d, "\lambda_{-1}"] & A_0 \arrow[r, "\varrho_0"] \arrow[d, "\lambda_0"] & A_1 \arrow[r, "\varrho_1"] \arrow[d, "\lambda_1"] & \dots \\ \dots \arrow[r, "\varrho_1^\vee"] & A_1^\vee \arrow[r, "\varrho_0^\vee"] & A_0^\vee \arrow[r, "\varrho_{-1}^\vee"] & A_{-1}^\vee \arrow[r, "\varrho_{-2}^\vee"] & \dots \end{tikzcd}$$

in $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$ satisfying the following conditions for all $i \in \mathbb{Z}$.

(1) A_i is an abelian scheme over R equipped with an action $\kappa_i : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A_i) \otimes \mathbb{Z}_{(p)}$.
(2) $\varrho_i : A_i \to A_{i+1}$ is a $\mathbb{Z}_{(p)}$-isogeny of degree p^f, compatible with κ_i and κ_{i+1}.
(3) There is an isomorphism $\theta_i : A_{2n+i} \to A_i$ in $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$ such that the composition

$$A_{2n+i} \xrightarrow{\theta_i} A_i \xrightarrow{\prod_{j=0}^{2n-1} \varrho_{i+j}} A_{2n+i}$$

is equal to $\kappa_{2n+i}(\pi)$.
(4) $\lambda_i : A_i \to A_{-i}^\vee$ is an isomorphism in $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$, compatible with κ_i and $\kappa_{-i}^\vee$. Here $\kappa_i^\vee : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A_i^\vee) \otimes \mathbb{Z}_{(p)}$ is defined by $\kappa_i^\vee(x) = \kappa_i(x)^\vee,\ x \in \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$.
(5) λ_0 is symmetric.
(6) We have an equality

$$\det_{\mathrm{Lie}(A_i)} = \det_{V_{-i}} \otimes R$$

of morphisms $V_{\mathcal{O}_F \otimes R} \to \mathbb{A}^1_R$.

Remark 5.8.2. *Let R be a $\mathbb{Z}_p$-algebra and let A/R be an abelian scheme equipped with an action $\kappa : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A) \otimes \mathbb{Z}_{(p)}$. Assume that* $\det_{\mathrm{Lie}(A)} = \det_{V_{-i}} \otimes R$. *Then $\dim_R A_i = ng$ by Lemma 2.3.4.*

Recall from Section 3.3 the diagram

$$\begin{tikzcd} & \widetilde{\mathcal{A}} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\ \mathcal{A} & & M^{\mathrm{loc}} \end{tikzcd}$$

of functors on the category of $\mathbb{Z}_p$-algebras. Also recall the KR stratification $\mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x$. We have identified the occurring index set with $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$ in Corollary 5.7.3. We can then state the following result.

Theorem 5.8.3. *Let $x = (x_\sigma)_\sigma \in \prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$. Write $x_\sigma = w_\sigma u^{\lambda_\sigma}$ with $w_\sigma \in W,\ \lambda_\sigma \in X$. Then the p-rank on $\mathcal{A}_x$ is constant with value*

$$g \cdot |\{1 \le i \le 2n \mid \forall \sigma \in \mathfrak{S}(w_\sigma(i) = i \wedge \lambda_\sigma(i) = 0)\}|.$$

Remark 5.8.4. *For $F = \mathbb{Q}$, we recover the result* [24, Théorème 4.1] *of Ngô and Genestier.*

Proof of Theorem 5.8.3. See Theorem 3.8.3 and Proposition 3.9.7 (leaving the transition between the equal and mixed characteristic situations to the reader).

Let us also give a direct proof. Let $1 \leq i \leq 2n$ and $x \in \mathrm{Perm}$. Write $x = wu^{\lambda}$ with $w \in W, \lambda \in X$. By Propositions 3.5.4 and 3.5.5 it suffices to show the following equivalence.

$$\widetilde{\Lambda}_i = x(\widetilde{\Lambda}_i) + \widetilde{\Lambda}_{i-1} \Leftrightarrow (w(i) = i \wedge \lambda(i) = 0).$$

Consider the subset $\mathcal{S} = \{u^k e_j \mid k \in \mathbb{Z},\ 1 \leq j \leq 2n\}$ of $\mathbb{F}((u))^{2n}$. Then x induces a permutation of $\mathcal{S}$, namely $x(u^k e_j) = u^{\lambda(j)+k} e_{w(j)}$. We have $\widetilde{\Lambda}_i \cap \mathcal{S} = \widetilde{\Lambda}_{i-1} \cap \mathcal{S} \amalg \{u^{-1}e_i\}$, and $x \in \mathrm{Perm}$ implies $x(\widetilde{\Lambda}_{i-1} \cap \mathcal{S}) \subset \widetilde{\Lambda}_{i-1} \cap \mathcal{S}$. Consequently $u^{-1}e_i \in x(\widetilde{\Lambda}_i \cap \mathcal{S})$ if and only if $x(u^{-1}e_i) = u^{-1}e_i$, which in turn is equivalent to $w(i) = i \wedge \lambda(i) = 0$, as desired. □

5.9. **An explicit example: Hilbert-Blumenthal modular varieties.** In this section we use the explicit case of the Hilbert-Blumenthal modular varieties to illustrate how Theorem 5.8.3 and the KR stratification in general yield results about the geometry of the moduli spaces $\mathcal{A}$. We also compare these results to some of those obtained by Stamm in [40].

Assume from now on that p is *inert* in $\mathcal{O}_F$, so that we have $e = 1$ and $f = g$. Assume also that $\dim_F V = 2$, so that $n = 1$. Fix a compact open subgroup $C^p \subset G(\mathbb{A}_f^p)$.

Definition 5.9.1 ([32, Definition 6.9])**.** *We denote by $\mathcal{A}_{C^p}$ the functor on the category of $\mathbb{Z}_p$-algebras with $\mathcal{A}_{C^p}(R)$ the set of isomorphism classes of pairs $(A, \bar{\eta})$, where $A = (A_i, \varrho_i, \lambda_i) \in \mathcal{A}(R)$ is such that $\lambda_0 : A_0 \to A_0^\vee$ is a principal polarization in AV_R in the sense of* [32, §6.4]*, and where $\bar{\eta}$ is a C^p-level structure on A in the sense of* [32, Definition 6.9].

We assume that C^p is chosen sufficiently small, such that $\mathcal{A}_{C^p}$ is a quasi-projective variety over $\mathbb{Z}_p$. Exactly as in Section 3.2 we obtain the intermediate object $\widetilde{\mathcal{A}}_{C^p}$ and the local model diagram

$$\begin{tikzcd} & \widetilde{\mathcal{A}}_{C^p} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\ \mathcal{A}_{C^p} & & M^{\mathrm{loc}}, \end{tikzcd}$$

which induces the KR stratification

$$\mathcal{A}_{C^p}(\mathbb{F}) = \coprod_{\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}} \mathcal{A}_{C^p, x}.$$

Let us start with a discussion of the index set $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$. From Definition 5.7.1 one immediately obtains that the subset $\mathrm{Perm} \subset \widetilde{W}$ is given by $\mathrm{Perm} = \{u^{(1,0)}, u^{(0,1)}, (1,2)u^{(1,0)}\}$. To put this set into a group theoretic perspective, we recall the setup described in [7, §2.1] in this easy special case. Consider the elements $\tau = (1,2)u^{(1,0)}, s_1 = (1,2)$ and $s_0 = (1,2)u^{(1,-1)}$ of $\widetilde{W}$. The subgroup W_a of $\widetilde{W}$ generated by s_0 and s_1 is a Coxeter group on the generators s_0 and s_1, and we denote by $\leq$ and ℓ the corresponding Bruhat order and length function on W_a, respectively. Denoting by Ω the cyclic subgroup of $\widetilde{W}$ generated by τ, we have $\widetilde{W} = W_a \rtimes \Omega$. We extend $\leq$ and ℓ to $\widetilde{W}$ by setting $w'\tau' \leq w''\tau'' :\Leftrightarrow (w' \leq w'' \wedge \tau' = \tau'')$ and $\ell(w'\tau') := \ell(w')$, for $w', w'' \in W_a$ and $\tau', \tau'' \in \Omega$. Finally we extend $\leq$ and ℓ to $\prod_{\sigma\in\mathfrak{S}} \widetilde{W}$ by setting $(x_\sigma)_\sigma \leq (x'_\sigma)_\sigma :\Leftrightarrow (\forall \sigma \in \mathfrak{S} :\ x_\sigma \leq x'_\sigma)$ and $\ell((x_\sigma)_\sigma) = \sum_\sigma \ell(x_\sigma)$.

We see that

$$\mathrm{Perm} = \{s_1\tau, s_0\tau, \tau\} \subset W_a\tau.$$

The Bruhat order on Perm is determined by the non-trivial relations $\tau \leq s_1\tau$ and $\tau \leq s_0\tau$, while the length function on Perm is given by $\ell(\tau) = 0$ and $\ell(s_1\tau) = \ell(s_0\tau) = 1$.

Lemma 5.9.2. *Let $x \in \prod_{\sigma\in\mathfrak{S}} \mathrm{Perm}$. The subset $\mathcal{A}_{C^p,x} \subset \mathcal{A}_{C^p}(\mathbb{F})$ is locally closed and we equip it with the reduced scheme structure. Then $\mathcal{A}_{C^p,x}$ is a smooth variety over $\mathbb{F}$. It is equidimensional of dimension $\ell(x)$. Furthermore the closure $\overline{\mathcal{A}}_{C^p,x}$ of $\mathcal{A}_{C^p,x}$ in $\mathcal{A}_{C^p,\mathbb{F}}$ is given by*

$$\overline{\mathcal{A}}_{C^p,x} = \coprod_{y\leq x} A_{C^p,y}.$$

Proof. By [25, Theorem 2.2d)] there is, étale locally on $\mathcal{A}_{C^p}$, an étale morphism $\mathcal{A}_{C^p,x} \to M_x^{\mathrm{loc}}$. The first properties therefore follow from Remark 3.3.8. In Theorem 5.6.10, we have further identified M_x^{loc} with the Schubert cell $\mathcal{C}_x \subset \mathrm{GSp}_{2n}(\mathbb{F}(\!(u)\!))/I_{\mathrm{GSp}}(\mathbb{F})$ corresponding to x. The remaining statements therefore follow from well-known properties of Schubert cells once we know that all KR strata are non-empty. This is shown in [5, Theorem 2.5.2(1)]. ☐

Let us state Theorem 5.8.3 in this special case. Denote by $\mathcal{A}^{(0)}_{C^p} \subset \mathcal{A}_{C^p}(\mathbb{F})$ and $\mathcal{A}^{(g)}_{C^p} \subset \mathcal{A}_{C^p}(\mathbb{F})$ the subsets where the p-rank of the underlying abelian variety is equal to 0 and g, respectively.

Proposition 5.9.3. *We have*

$$\mathcal{A}_{C^p}(\mathbb{F}) = \mathcal{A}^{(0)}_{C^p} \amalg \mathcal{A}^{(g)}_{C^p}.$$

$\mathcal{A}^{(g)}_{C^p}$ is the union of only two KR strata, namely those corresponding to the elements $((s_1\tau)^{(g)}) = (s_1\tau, s_1\tau, \ldots, s_1\tau)$ and $((s_0\tau)^{(g)}) = (s_0\tau, s_0\tau, \ldots, s_0\tau)$ of $\prod_{\sigma\in\mathfrak{S}} \mathrm{Perm}$. The p-rank on all other KR strata is equal to 0.

Lemma 5.9.4. *The maximal elements in $\prod_{\sigma\in\mathfrak{S}} \mathrm{Perm}$ for the Bruhat order are precisely the elements of length 2^g in $\prod_{\sigma\in\mathfrak{S}} \mathrm{Perm}$. The set of these maximal elements is given by $\prod_{\sigma\in\mathfrak{S}}\{s_1\tau, s_0\tau\}$*

Proof. Clear. ☐

From the preceding results, we obtain without any additional work the following theorem.

Theorem 5.9.5. *Let $g \geq 2$. Then*

$$\mathcal{A}_{C^p,\mathbb{F}} = \overline{\mathcal{A}}_{C^p,((s_1\tau)^{(g)})} \cup \overline{\mathcal{A}}_{C^p,((s_0\tau)^{(g)})} \cup \mathcal{A}^{(0)}_{C^p}.$$

Each of $\overline{\mathcal{A}}_{C^p,((s_1\tau)^{(g)})}, \overline{\mathcal{A}}_{C^p,((s_0\tau)^{(g)})}$ and $\mathcal{A}^{(0)}_{C^p}$ is equidimensional of dimension 2^g, and hence so is $\mathcal{A}_{C^p,\mathbb{F}}$.

More precisely, $\mathcal{A}^{(0)}_{C^p}$ *is the union*

$$\mathcal{A}^{(0)}_{C^p} = \bigcup_{\substack{x \in \prod_{\sigma \in \mathfrak{S}} \{s_1\tau, s_0\tau\} \\ x \neq ((s_1\tau)^{(g)}), ((s_0\tau)^{(g)})}} \overline{\mathcal{A}}_{C^p, x}$$

of $2^g - 2$ *closed subsets, all equidimensional of dimension* 2^g*.*

Furthermore, we have

$$\overline{\mathcal{A}}_{C^p, ((s_1\tau)^{(g)})} \cap \overline{\mathcal{A}}_{C^p, ((s_0\tau)^{(g)})} \subset \mathcal{A}^{(0)}_{C^p}.$$

Taking $g = 2$, we recover [40, Theorem 2 (p. 408)]. Note that for $g = 2$, the set $\mathcal{A}^{(0)}_{C^p}$ is precisely the supersingular locus in $\mathcal{A}_{C^p, \mathbb{F}}$, because a 2-dimensional abelian variety is supersingular if and only if its p-rank is equal to zero.

Corollary 5.9.6. *Let* $g \geq 2$*. Then the ordinary locus* $\mathcal{A}^{(g)}_{C^p}$ *is not dense in* $\mathcal{A}_{C^p, \mathbb{F}}$*.*

6. The ramified unitary case

6.1. **The PEL datum.** Let $n \in \mathbb{N}_{\geq 1}$. We start with the PEL datum consisting of the following objects.

(1) An imaginary quadratic extension F/F_0 of a totally real extension $F_0/\mathbb{Q}$. Let $g_0 = [F_0 : \mathbb{Q}]$ and $g = [F : \mathbb{Q}]$, so that $g = 2g_0$.
(2) The non-trivial element $*$ of $\mathrm{Gal}(F/F_0)$.
(3) An n-dimensional F-vector space V.
(4) The symplectic form $(\cdot, \cdot) : V \times V \to \mathbb{Q}$ on the underlying $\mathbb{Q}$-vector space of V constructed as follows: Fix once and for all a $*$-skew-hermitian form $(\cdot, \cdot)' : V \times V \to F$ (i.e. $(av, bw)' = ab^* (v, w)'$ and $(v, w)' = -(w, v)'^*$ for $v, w \in V$, $a, b \in F$). Define $(\cdot, \cdot) = \mathrm{tr}_{F/\mathbb{Q}} \circ (\cdot, \cdot)'$.
(5) The element $J \in \mathrm{End}_{B \otimes \mathbb{R}}(V \otimes \mathbb{R})$ to be specified below.

Remark 6.1.1. *Denote by* $\mathrm{GU}_{(\cdot,\cdot)'}$ *the* F_0*-group given on* R*-valued points by* $\mathrm{GU}_{(\cdot,\cdot)'}(R) = \{g \in \mathrm{GL}_{F \otimes_{F_0} R}(V \otimes_{F_0} R) \mid \exists c = c(g) \in R^\times \forall x, y \in V \otimes_{F_0} R : (gx, gy)'_R = c\,(x, y)'_R\}$*. Then the reductive* $\mathbb{Q}$*-group* G *associated with the above PEL datum fits into the following cartesian diagram.*

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "c"'] & \mathrm{Res}_{F_0/\mathbb{Q}} \mathrm{GU}_{(\cdot,\cdot)'} \arrow[d, "c"] \\ \mathbb{G}_{m,\mathbb{Q}} \arrow[r, hook] & \mathrm{Res}_{F_0/\mathbb{Q}} \mathbb{G}_{m,F_0}. \end{tikzcd}$$

We assume that $p\mathcal{O}_{F_0} = (\mathcal{P}_0)^{e_0}$ for a single prime $\mathcal{P}_0$ of $\mathcal{O}_{F_0}$ and that $\mathcal{P}_0\mathcal{O}_F = \mathcal{P}^2$ for a prime $\mathcal{P}$ of $\mathcal{O}_F$. Write $e = 2e_0$, so that $p\mathcal{O}_F = \mathcal{P}^e$. Denote by $f = [k_{\mathcal{P}_0} : \mathbb{F}_p]$ the corresponding inertia degree, so that $g = ef$ and $g_0 = fe_0$. Fix once and for all uniformizers π_0 of $\mathcal{O}_{F_0} \otimes \mathbb{Z}_{(p)}$ and π of $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$, satisfying $\pi^2 = \pi_0$. We have $\pi^* = -\pi$.

For typographical reasons, we denote the ring of integers in $F_{\mathcal{P}}$ by $\mathcal{O}_{\mathcal{P}}$ and the ring of integers in $(F_0)_{\mathcal{P}_0}$ by $\mathcal{O}_{\mathcal{P}_0}$. Denote by $\mathfrak{C} = \mathfrak{C}_{\mathcal{O}_{\mathcal{P}}|\mathbb{Z}_p}$, $\mathfrak{C}_0 = \mathfrak{C}_{\mathcal{O}_{\mathcal{P}_0}|\mathbb{Z}_p}$ and $\mathfrak{C}' = \mathfrak{C}_{\mathcal{O}_{\mathcal{P}}|\mathcal{O}_{\mathcal{P}_0}}$ the corresponding inverse differents. Then $\mathfrak{C}_0 = (\pi_0^{-k})$ for some $k \in \mathbb{N}$. The extension $F_{\mathcal{P}}/(F_0)_{\mathcal{P}_0}$ is tamely ramified, so that $\mathfrak{C}' = (\pi^{-1})$. The equality $\mathfrak{C} = \mathfrak{C}' \cdot \mathfrak{C}_0$ then implies that $\mathfrak{C} = (\pi^{-2k-1})$ and we denote by $\delta = \pi^{-2k-1}$ the corresponding generator of $\mathfrak{C}$. It satisfies $\delta^* = -\delta$. Consequently the form $\delta^{-1}(\cdot,\cdot)'_{\mathbb{Q}_p} : V_{\mathbb{Q}_p} \times V_{\mathbb{Q}_p} \to F_{\mathcal{P}}$ is $*$-hermitian and we assume that it *splits*, i.e. that there is a basis $(e_1,\dots,e_n)$ of $V_{\mathbb{Q}_p}$ over $F_{\mathcal{P}}$ such that $(e_i, e_{n+1-j})'_{\mathbb{Q}_p} = \delta\delta_{ij}$ for $1 \le i,j \le n$.

Let $0 \le i < n$. We denote by Λ_i the $\mathcal{O}_{\mathcal{P}}$-lattice in $V_{\mathbb{Q}_p}$ with basis

$$\mathfrak{E}_i = (\pi^{-1}e_1,\dots,\pi^{-1}e_i, e_{i+1},\dots,e_n).$$

For $k \in \mathbb{Z}$ we further define $\Lambda_{nk+i} = \pi^{-k}\Lambda_i$ and we denote by $\mathfrak{E}_{nk+i}$ the corresponding basis obtained from $\mathfrak{E}_i$. Then $\mathcal{L} = (\Lambda_i)_i$ is a complete chain of $\mathcal{O}_{\mathcal{P}}$-lattices in $V_{\mathbb{Q}_p}$. For $i \in \mathbb{Z}$, the dual lattice $\Lambda_i^\vee := \{x \in V_{\mathbb{Q}_p} \mid (x,\Lambda_i)_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ of Λ_i is given by Λ_{-i}. Consequently the chain $\mathcal{L}$ is self-dual.

Let $i \in \mathbb{Z}$. We denote by $\rho_i : \Lambda_i \to \Lambda_{i+1}$ the inclusion, by $\vartheta_i : \Lambda_{n+i} \to \Lambda_i$ the isomorphism given by multiplication with π and by $(\cdot,\cdot)_i : \Lambda_i \times \Lambda_{-i} \to \mathbb{Z}_p$ the restriction of $(\cdot,\cdot)_{\mathbb{Q}_p}$. Then $(\Lambda_i, \rho_i, \vartheta_i, (\cdot,\cdot)_i)_i$ is a polarized chain of $\mathcal{O}_{F_{\mathcal{P}}}$-modules of type $(\mathcal{L})$, which, by abuse of notation, we also denote by $\mathcal{L}$.

Denote by $\langle\cdot,\cdot\rangle_i : \Lambda_i \times \Lambda_{-i} \to \mathcal{O}_{\mathcal{P}}$ the restriction of the $*$-hermitian form $\delta^{-1}(\cdot,\cdot)'_{\mathbb{Q}_p}$, and by H_i the matrix describing $\langle\cdot,\cdot\rangle_i$ with respect to $\mathfrak{E}_i$ and $\mathfrak{E}_{-i}$. We have

$$H_i = \text{anti-diag}((-1)^{a_{i,1}},\dots,(-1)^{a_{i,n}}) \tag{6.1.2}$$

for some $a_{i,1},\dots,a_{i,n} \in \mathbb{Z}/2\mathbb{Z}$.

Denote by Σ_0 the set of all embeddings $F_0 \hookrightarrow \mathbb{R}$ and by Σ the set of all embeddings $F \hookrightarrow \mathbb{C}$. For each $\sigma \in \Sigma_0$, we denote by $\tau_{\sigma,1}, \tau_{\sigma,2} \in \Sigma$ the two embeddings with $\tau_{\sigma,j}\big|_{F_0} = \sigma$. Of course we have $\tau_{\sigma,2} = \tau_{\sigma,1} \circ *$.

We obtain isomorphisms

$$F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\sigma\in\Sigma_0} \mathbb{C}, \qquad F \ni x \mapsto (\tau_{\sigma,1}(x))_\sigma, \tag{6.1.3}$$

$$F \otimes_{\mathbb{Q}} \mathbb{C} = \prod_{\sigma\in\Sigma_0} \mathbb{C}\times\mathbb{C}, \qquad F \ni x \mapsto (\tau_{\sigma,1}(x), \tau_{\sigma,2}(x))_\sigma \tag{6.1.4}$$

of $\mathbb{R}$- and $\mathbb{C}$-algebras, respectively.

The isomorphism (6.1.3) induces a decomposition $V \otimes \mathbb{R} = \prod_{\sigma\in\Sigma_0} V_\sigma$ into $\mathbb{C}$-vector spaces V_σ and $(\cdot,\cdot)'_{\mathbb{R}}$ decomposes into the product of skew-hermitian forms $(\cdot,\cdot)'_\sigma : V_\sigma \times V_\sigma \to \mathbb{C}$, $\sigma \in \Sigma_0$. For each $\sigma \in \Sigma_0$, there are $r_\sigma, s_\sigma \in \mathbb{N}$ with $r_\sigma + s_\sigma = n$ and a basis $\mathfrak{B}_\sigma$ of V_σ over $\mathbb{C}$ such that $(\cdot,\cdot)'_\sigma$ is described by the matrix $D_\sigma = \text{diag}(i^{(r_\sigma)}, (-i)^{(s_\sigma)})$ with respect to $\mathfrak{B}_\sigma$. Denote by J_σ the endomorphism of V_σ described by the matrix D_σ with respect to $\mathfrak{B}_\sigma$. We complete the description of the PEL datum by defining $J := \prod_{\sigma\in\Sigma_0} J_\sigma \in \text{End}_{B\otimes\mathbb{R}}(V\otimes\mathbb{R})$.

6.2. **The determinant morphism.** The isomorphism (6.1.4) induces a decomposition $V \otimes \mathbb{C} = \prod_{\sigma\in\Sigma_0}(V_{\tau_{\sigma,1}} \times V_{\tau_{\sigma,2}})$ into $\mathbb{C}$-vector spaces $V_{\tau_{\sigma,j}}$. The basis $\mathfrak{B}_\sigma$ of V_σ induces bases $\mathfrak{B}_{\tau_{\sigma,j}}$ of $V_{\tau_{\sigma,j}}$ over $\mathbb{C}$, and the endomorphism $J_{\sigma,\mathbb{C}}$ decomposes into the product of endomorphisms $J_{\tau_{\sigma,j}}$ of $V_{\tau_{\sigma,j}}$. We find that $J_{\tau_{\sigma,1}}$ is described by the matrix D_σ with respect to $\mathfrak{B}_{\tau_{\sigma,1}}$, while $J_{\tau_{\sigma,2}}$ is described by the matrix $-D_\sigma$ with respect to $\mathfrak{B}_{\tau_{\sigma,2}}$.

Denote by V_{-i} the $(-i)$-eigenspace of $J_{\mathbb{C}}$. From the explicit description of the $J_{\tau_{\sigma,j}}$ with respect to the $\mathfrak{B}_{\tau_{\sigma,j}}$, one concludes that V_{-i} is the $\mathcal{O}_F \otimes \mathbb{C}$-module corresponding to the $\prod_{\sigma\in\Sigma_0} \mathbb{C}\times\mathbb{C}$-module $\prod_{\sigma\in\Sigma_0}\mathbb{C}^{s_\sigma}\times\mathbb{C}^{r_\sigma}$ under (6.1.4).

Let E' be the Galois closure of F inside $\mathbb{C}$ and choose a prime $\mathcal{Q}'$ of E' over $\mathcal{P}$. In absolute analogy to (6.1.4), we have a decomposition

$$F\otimes_{\mathbb{Q}} E' = \prod_{\sigma\in\Sigma_0} E'\times E'. \tag{6.2.1}$$

Let M be the $\mathcal{O}_F\otimes E'$-module corresponding to the $\prod_{\sigma\in\Sigma_0} E'\times E'$-module $\prod_{\sigma\in\Sigma_0}(E')^{s_\sigma}\times(E')^{r_\sigma}$ under (6.2.1). From the present discussion we obtain an identification $M\otimes_{E'}\mathbb{C} = V_{-i}$ of $\mathcal{O}_F\otimes\mathbb{C}$-modules. As in the proof of Proposition 2.3.5, we find an $\mathcal{O}_F\otimes\mathcal{O}_{E'}$-stable $\mathcal{O}_{E'}$-lattice M_0 in M. In particular, the morphism $\det_{V_{-i}} : V_{\mathcal{O}_F\otimes\mathbb{C}}\to\mathbb{A}^1_{\mathbb{C}}$ descends to the morphism $\det_{M_0} : V_{\mathcal{O}_F\otimes\mathcal{O}_{E'}}\to\mathbb{A}^1_{\mathcal{O}_{E'}}$.

6.3. **The special fiber of the determinant morphism.** We write $\mathfrak{S} = \mathrm{Gal}(k_{\mathcal{P}}/\mathbb{F}_p) = \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$. We fix once and for all an embedding $\iota_{\mathcal{Q}'} : k_{\mathcal{Q}'}\hookrightarrow\mathbb{F}$. We consider $\mathbb{F}$ as an $\mathcal{O}_{E'}$-algebra via the composition $\mathcal{O}_{E'}\xrightarrow{\rho_{\mathcal{Q}'}} k_{\mathcal{Q}'}\overset{\iota_{\mathcal{Q}'}}{\hookrightarrow}\mathbb{F}$. Also $\iota_{\mathcal{Q}'}$ induces an embedding $\iota_{\mathcal{P}} : k_P\hookrightarrow\mathbb{F}$ and thereby an identification of the set of all embeddings $k_{\mathcal{P}}\hookrightarrow\mathbb{F}$ with $\mathfrak{S}$. Consider the isomorphism

$$\mathcal{O}_F\otimes\mathbb{F} = \prod_{\sigma\in\mathfrak{S}}\mathbb{F}[u]/(u^e) \tag{6.3.1}$$

from Section 4.3.

Proposition 6.3.2. *Let $x\in\mathcal{O}_F$ and let $(p_\sigma)_\sigma\in\prod_{\sigma\in\mathfrak{S}}\mathbb{F}[u]/(u^e)$ be the element corresponding to $x\otimes 1$ under* (6.3.1)*. Then*

$$\chi_{\mathbb{F}}(x|M_0\otimes_{\mathcal{O}_{E'}}\mathbb{F}) = \prod_{\sigma\in\mathfrak{S}}\big(T-p_\sigma(0)\big)^{ne_0}$$

in $\mathbb{F}[T]$.

Proof. The definition of M_0 gives

$$\chi_{\mathcal{O}_{E'}}(x|M_0) = \prod_{\sigma\in\Sigma_0}\big(T-\tau_{\sigma,1}(x)\big)^{s_\sigma}\big(T-\tau_{\sigma,2}(x)\big)^{r_\sigma}.$$

Consider the maps $\gamma:\Sigma\to\mathfrak{S}$ and $\gamma_0:\Sigma_0\to\mathfrak{S}$ of Proposition 4.4.1. By Lemma 4.4.3(1) the map γ factors as $\Sigma\xrightarrow{\cdot|_{F_0}}\Sigma_0\xrightarrow{\gamma_0}\mathfrak{S}$ and consequently $\rho_{\mathcal{Q}'}(\tau_{\sigma,j}(x)) = \gamma_0(\sigma)(\rho_{\mathcal{P}}(x))$ for $\sigma\in\Sigma_0$, $j=1,2$. The fact that γ_0 is surjective with all fibers of cardinality e_0 therefore yields

$$(\chi_{\mathcal{O}_{E'}}(x|M_0))^{\rho_{\mathcal{Q}'}} = \prod_{\sigma\in\mathfrak{S}}\big(T-\sigma(\rho_{\mathcal{P}}(x))\big)^{ne_0}.$$

The claim then follows from the equality $p_\sigma(0) = (\iota_{\mathcal{P}}\circ\sigma)(\rho_{\mathcal{P}}(x))$. □

Denote by $E = \mathbb{Q}(\mathrm{tr}_{\mathbb{C}}(x\otimes 1|V_{-i});\ x\in F)$ the reflex field and define $\mathcal{Q} = \mathcal{Q}'\cap\mathcal{O}_E$. By Proposition 2.3.5 the morphism $\det_{V_{-i}}$ is defined over $\mathcal{O}_E$, and we also denote by $\det_{V_{-i}}$ the corresponding morphism over $\mathcal{O}_E$.

6.4. **The local model.** For the chosen PEL datum, Definition 3.3.2 amounts to the following.

Definition 6.4.1. *The local model M^{loc} is the functor on the category of $\mathcal{O}_{E_{\mathbb{Q}}}$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_i)_{i\in\mathbb{Z}}$ of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

(a) $\rho_{i,R}(t_i) \subset t_{i+1}$.
(b) *The quotient $\Lambda_{i,R}/t_i$ is a finite locally free R-module.*
(c) *We have an equality*

$$\det_{\Lambda_{i,R}/t_i} = \det_{V_{-i}} \otimes_{\mathcal{O}_E} R$$

of morphisms $V_{\mathcal{O}_F\otimes R} \to \mathbb{A}^1_R$.
(d) *Under the pairing $(\cdot,\cdot)_{i,R} : \Lambda_{i,R} \times \Lambda_{-i,R} \to R$, the submodules t_i and t_{-i} pair to zero.*
(e) $\vartheta_i(t_{n+i}) = t_i$.

Corollary 6.4.2. *Condition 6.4.1(d) can be equivalently replaced by the following condition.*

(d') $t_i^{\perp,\langle\cdot,\cdot\rangle_{i,R}} = t_{-i}$.

Proof. Analogous to the proof of Corollary 5.3.2. □

6.5. **The special fiber of the local model.** For $i \in \mathbb{Z}$, denote by $\overline{\Lambda}_i$ the free $\mathbb{F}[u]/u^e$-module $(\mathbb{F}[u]/u^e)^n$ and by $\overline{\mathfrak{E}}_i$ its canonical basis. Consider the $\mathbb{F}$-automorphism $\overline{*} : \mathbb{F}[u]/u^e \to \mathbb{F}[u]/u^e$, $u \mapsto -u$. Denote by $\overline{\langle\cdot,\cdot\rangle}_i : \overline{\Lambda}_i \times \overline{\Lambda}_{-i} \to \mathbb{F}[u]/u^e$ the $\overline{*}$-sesquilinear form described by the matrix H_i of (6.1.2) with respect to $\overline{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_{-i}$. Denote by $\overline{\vartheta}_i : \overline{\Lambda}_{n+i} \to \overline{\Lambda}_i$ the identity morphism. For $k \in \mathbb{Z}$ and $0 \leq i < n$, let $\overline{\rho}_{nk+i} : \overline{\Lambda}_{nk+i} \to \overline{\Lambda}_{nk+i+1}$ be the morphism described by the matrix $\mathrm{diag}(1^{(i)}, u, 1^{(n-i-1)})$ with respect to $\overline{\mathfrak{E}}_{nk+i}$ and $\overline{\mathfrak{E}}_{nk+i+1}$.

Definition 6.5.1. *Define a functor $M^{e,n}$ on the category of $\mathbb{F}$-algebras with $M^{e,n}(R)$ the set of tuples $(t_i)_{i\in\mathbb{Z}}$ of $R[u]/u^e$-submodules $t_i \subset \overline{\Lambda}_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

(a) $\overline{\rho}_{i,R}(t_i) \subset t_{i+1}$.
(b) *The quotient $\overline{\Lambda}_{i,R}/t_i$ is finite locally free over R.*
(c) *For all $p \in R[u]/u^e$, we have*

$$\chi_R(p|\overline{\Lambda}_{i,R}/t_i) = \big(T - p(0)\big)^{ne_0}$$

in $R[T]$.
(d) $t_i^{\perp,\overline{\langle\cdot,\cdot\rangle}_{i,R}} = t_{-i}$.
(e) $\overline{\vartheta}_i(t_{n+i}) = t_i$.

Corollary 6.5.2. *Assume that R is* reduced. *Then condition 6.5.1(c) is equivalent to the following condition.*

(c') $\mathrm{rk}_R\, t_i = ne_0$.

Proof. See Lemma 2.3.9. □

Let $i \in \mathbb{Z}$. From (6.3.1) we obtain an isomorphism

$$\Lambda_{i,\mathbb{F}} = \prod_{\sigma \in \mathfrak{S}} \overline{\Lambda}_i \tag{6.5.3}$$

by identifying the basis $\mathfrak{E}_{i,\mathbb{F}}$ with the product of the bases $\overline{\mathfrak{E}}_i$. Under this identification, the morphism $\rho_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\rho}_i$, the pairing $\langle \cdot, \cdot \rangle_{i,\mathbb{F}}$ decomposes into the pairings $\overline{\langle \cdot, \cdot \rangle}_i$ and the morphism $\vartheta_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\vartheta}_i$.

Let R be an $\mathbb{F}$-algebra and let $(t_i)_{i\in\mathbb{Z}}$ be a tuple of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$. Then (6.5.3) induces decompositions $t_i = \prod_{\sigma \in \mathfrak{S}} t_{i,\sigma}$ into $R[u]/u^e$-submodules $t_{i,\sigma} \subset \overline{\Lambda}_{i,R}$.

Proposition 6.5.4. *The morphism $M_{\mathbb{F}}^{\mathrm{loc}} \to \prod_{\sigma\in\mathfrak{S}} M^{e,n}$ given on R-valued points by*

$$\begin{aligned} M_{\mathbb{F}}^{\mathrm{loc}}(R) &\to \prod_{\sigma \in \mathfrak{S}} M^{e,n}(R), \\ (t_i) &\mapsto ((t_{i,\sigma})_i)_\sigma \end{aligned} \tag{6.5.5}$$

is an isomorphism of functors on the category of $\mathbb{F}$-algebras.

Proof. It is clear that conditions 6.4.1(a), 6.4.1(b), 6.4.2(d') and 6.4.1(e) can be verified factorwise and correspond to the analogous parts of Definition 6.5.1. For condition 6.4.1(c) the same is true by Propositions 2.3.7 and 6.3.2. □

6.6. **The affine flag variety.** This section deals with the affine flag variety for the ramified unitary group. Our discussion is based on and has greatly profited from [27], [28] and [37], [36].

Let R be an $\mathbb{F}$-algebra. Consider the extension $R[\![u]\!]/R[\![u_0]\!]$ with $u_0 = u^2$. Also consider the $R(\!(u_0)\!)$-automorphism $\widetilde{*} : R(\!(u)\!) \to R(\!(u)\!),\ u \mapsto -u$. Let $\widetilde{\langle \cdot, \cdot \rangle}$ be the $\widetilde{*}$-hermitian form on $R(\!(u)\!)^n$ described by the matrix $\widetilde{I}_n$ with respect to the standard basis of $R(\!(u)\!)^n$ over $R(\!(u)\!)$. For a lattice Λ in $R(\!(u)\!)^n$ we define $\Lambda^\vee := \{x \in R(\!(u)\!)^n \mid \widetilde{\langle x, \Lambda \rangle} \subset R[\![u]\!]\}$. Recall from Section 4.5 the standard lattice chain $\widetilde{\mathcal{L}} = (\widetilde{\Lambda}_i)_i$ in $R(\!(u)\!)^n$. Note that $(\widetilde{\Lambda}_i)^\vee = \widetilde{\Lambda}_{-i}$ for all $i \in \mathbb{Z}$. We denote by $\widetilde{\langle \cdot, \cdot \rangle}_i : \widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} \to R[\![u]\!]$ the restriction of $\widetilde{\langle \cdot, \cdot \rangle}$. It is the perfect $\widetilde{*}$-sesquilinear pairing described by the matrix H_i of (6.1.2) with respect to $\widetilde{\mathfrak{E}}_i$ and $\widetilde{\mathfrak{E}}_{-i}$.

Definition 6.6.1 (Cf. [32, Definition A.41]). *Let $\mathbb{F}[\![u_0]\!] \to R$ be an $\mathbb{F}[\![u_0]\!]$-algebra. A polarized chain of $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$-modules of type $(\widetilde{\mathcal{L}})$ is a tuple*

$\mathcal{M} =$

$$(M_i, \varrho_i : M_i \to M_{i+1}, \theta_i : M_{n+i} \xrightarrow{\sim} M_i, \mathcal{E}_i : M_i \times M_{-i} \to \mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R)_{i \in \mathbb{Z}},$$

where the M_i are $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$-modules, the ϱ_i are $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$-linear maps, the θ_i are $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$-linear isomorphisms and the $\mathcal{E}_i$ are perfect $\widetilde{*}_R$-sesquilinear pairings, such that for each $i \in \mathbb{Z}$ the following conditions hold.

(1) *Locally on R, there are $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$-linear isomorphisms $M_i \simeq \widetilde{\Lambda}_i \otimes_{\mathbb{F}[\![u_0]\!]} R$ and $M_{i+1}/\varrho_i(M_i) \simeq (\widetilde{\Lambda}_{i+1}/\widetilde{\Lambda}_i) \otimes_{\mathbb{F}[\![u_0]\!]} R$.*

(2) The diagram

$$\begin{tikzcd} M_{n+i} \arrow[r, "\varrho_{n+i}"] \arrow[d, "\theta_i"'] & M_{n+i+1} \arrow[d, "\theta_{i+1}"] \\ M_i \arrow[r, "\varrho_i"] & M_{i+1} \end{tikzcd}$$

commutes.

(3) The composition $M_{n+i} \xrightarrow{\theta_i} M_i \xrightarrow{\prod_{j=0}^{n-1} \varrho_{i+j}} M_{n+i}$ *is multiplication by* u.

(4) $\mathcal{E}_i(a,b) = \mathcal{E}_{-i}(b,a)^{\widetilde{*}_R}$ *for all* $a \in M_i, b \in M_{-i}$.

(5) $\mathcal{E}_i(a, \varrho_{-i-1}(b)) = \mathcal{E}_{i+1}(\varrho_i(a), b)$ *for all* $a \in M_i, b \in M_{-i-1}$.

(6) $\mathcal{E}_i(a,b) = -\mathcal{E}_{n+i}(\theta_i^{-1}(a), \theta_{-n-i}(b))$ *for all* $a \in M_i, b \in M_{-i}$.

Let

$$\mathcal{M} = (M_i, \varrho_i : M_i \to M_{i+1}, \theta_i : M_{n+i} \xrightarrow{\sim} M_i, \mathcal{E}_i : M_i \times M_{-i} \to R)_i,$$
$$\mathcal{M}' = (M_i', \varrho_i' : M_i' \to M_{i+1}', \theta_i' : M_{n+i}' \xrightarrow{\sim} M_i', \mathcal{E}_i' : M_i' \times M_{-i}' \to R)_i$$

be polarized chains of $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$*-modules of type* $(\widetilde{\mathcal{L}})$. *A morphism* $\varphi : \mathcal{M} \to \mathcal{M}'$ *is a tuple* $(\varphi_i)_{i\in\mathbb{Z}}$ *of isomorphisms of* $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$*-modules* $\varphi_i : M_i \to M_i'$ *satisfying the following conditions for all* $i \in \mathbb{Z}$.

(a) The diagrams

$$\begin{tikzcd} M_i \arrow[r, "\varrho_i"] \arrow[d, "\varphi_i"'] & M_{i+1} \arrow[d, "\varphi_{i+1}"] \\ M_i' \arrow[r, "\varrho_i'"] & M_{i+1}', \end{tikzcd} \qquad \begin{tikzcd} M_i \arrow[d, "\varphi_i"'] & M_{n+i} \arrow[l, "\theta_i"'] \arrow[d, "\varphi_{n+i}"] \\ M_i' & M_{n+i}' \arrow[l, "\theta_i'"'] \end{tikzcd}$$

commute.

(b) $\forall (x,y) \in M_i \times M_{-i} :\ \mathcal{E}_i'(\varphi_i(x), \varphi_{-i}(y)) = \mathcal{E}_i(x,y)$.

We denote by $\operatorname{Isom}(\mathcal{M}, \mathcal{M}')$ *the functor on the category of* R*-algebras with* $\operatorname{Isom}(\mathcal{M}, \mathcal{M}')(R')$ *the set of morphisms* $\mathcal{M} \otimes_R R' \to \mathcal{M}' \otimes_R R'$ *of polarized chains of* $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R'$*-modules of type* $(\widetilde{\mathcal{L}})$. *We also write* $\operatorname{Aut}(\mathcal{M}) = \operatorname{Isom}(\mathcal{M}, \mathcal{M})$.

Proposition 6.6.2. *Let* R *be an* $\mathbb{F}[\![u_0]\!]$*-algebra such that the image of* u_0 *in* R *is nilpotent. Then any two polarized chains* $\mathcal{M}, \mathcal{N}$ *of* $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$*-modules of type* $(\widetilde{\mathcal{L}})$ *are isomorphic locally for the étale topology on* R. *Furthermore the functor* $\operatorname{Isom}(\mathcal{M}, \mathcal{N})$ *is representable by a smooth affine scheme over* R.

Proof. The proof of [32, Proposition A.43] carries over to this situation without any changes. □

Proposition 6.6.3. *Let* R *be an* $\mathbb{F}$*-algebra and let* $\mathcal{M}, \mathcal{N}$ *be polarized chains of* $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R[\![u_0]\!]$*-modules of type* $(\widetilde{\mathcal{L}})$. *Then the canonical map* $\operatorname{Isom}(\mathcal{M}, \mathcal{N})(R[\![u_0]\!]) \to \operatorname{Isom}(\mathcal{M}, \mathcal{N})(R[\![u_0]\!]/u_0^m)$ *is surjective for all* $m \in \mathbb{N}_{\geq 1}$. *In particular* $\mathcal{M}$ *and* $\mathcal{N}$ *are isomorphic locally for the étale topology on* R.

Proof. Analogous to the proof of Proposition 5.5.3, referring to Proposition 6.6.2 in place of Proposition 5.5.2. □

Definition 6.6.4 ([37, §4.2], [36, §6.2]). *Let R be an $\mathbb{F}$-algebra and let $(L_i)_i$ be a lattice chain in $R(\!(u)\!)^n$.*

(1) Let $r \in \mathbb{Z}$. The chain $(L_i)_i$ is called r-self-dual *if*

$$\forall i \in \mathbb{Z}: \ L_i^\vee = u_0^r L_{-i}.$$

Denote by $\mathcal{F}_{\mathrm{U}}^{(r)}$ the functor on the category of $\mathbb{F}$-algebras with $\mathcal{F}_{\mathrm{U}}^{(r)}(R)$ the set of r-self-dual lattice chains in $R(\!(u)\!)^n$.

(2) The chain $(L_i)_i$ is called self-dual *if Zariski locally on R there is an $a \in R(\!(u_0)\!)^\times$ such that*

$$\forall i \in \mathbb{Z}: \ L_i^\vee = aL_{-i}. \tag{6.6.5}$$

Denote by $\mathcal{F}_{\mathrm{GU}}$ the functor on the category of $\mathbb{F}$-algebras with $\mathcal{F}_{\mathrm{GU}}(R)$ the set of self-dual lattice chains in $R(\!(u)\!)^n$.

Note that $\widetilde{\mathcal{L}} \in \mathcal{F}_{\mathrm{U}}^{(0)}(R)$.

Remark 6.6.6. *Let R be a reduced $\mathbb{F}$-algebra such that* $\operatorname{Spec} R$ *is connected. Then*

$$\mathcal{F}_{\mathrm{GU}}(R) = \bigcup_{r \in \mathbb{Z}} \mathcal{F}_{\mathrm{U}}^{(r)}(R).$$

Proof. This follows easily from Lemma 4.5.14. □

Remark 6.6.7. *Let R be an $\mathbb{F}$-algebra and let $(L_i)_i \in \mathcal{F}_{\mathrm{U}}^{(0)}(R)$. For $i \in \mathbb{Z}$ denote by $\varrho_i : L_i \to L_{i+1}$ the inclusion, by $\theta_i : L_{n+i} \to L_i$ the isomorphism given by multiplication with u and by $\mathcal{E}_i : L_i \times L_{-i} \to R[\![u]\!]$ the restriction of $\widetilde{\langle \cdot, \cdot \rangle}$. Then $(L_i, \varrho_i, \theta_i, \mathcal{E}_i)$ is a polarized chain of $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R[\![u_0]\!]$-modules of type $(\widetilde{\mathcal{L}})$. Here we use Proposition 4.5.5 to verify condition 6.6.1(1).*

Recall from Section 2.1.4 the $\mathbb{F}(\!(u_0)\!)$-groups GU and U associated with the quadratic extension $\mathbb{F}(\!(u)\!)/\mathbb{F}(\!(u_0)\!)$. Note that for an $\mathbb{F}$-algebra R, the canonical maps

$$\begin{gathered}\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R[\![u_0]\!] \to R[\![u]\!], \\ \mathbb{F}(\!(u)\!) \otimes_{\mathbb{F}(\!(u_0)\!)} R(\!(u_0)\!) \to R(\!(u)\!)\end{gathered}$$

are isomorphisms. Consequently we can consider $\mathrm{L}_{u_0}\mathrm{GU}$ and $\mathrm{L}_{u_0}\mathrm{U}$ as subfunctors of $\mathrm{L}_u \mathrm{GL}_n$. Recall from Section 2.1.2 the subfunctor $I \subset \mathrm{L}\,\mathrm{GL}_n$. We define a subfunctor I_{GU} of $\mathrm{L}_{u_0}\mathrm{GU}$ by $I_{\mathrm{GU}} = \mathrm{L}_{u_0}\mathrm{GU} \cap I$.

Proposition 6.6.8. *The natural action of $\mathrm{L}_u \mathrm{GL}_n$ on $\mathcal{F}$ (cf. Remark 4.5.13) restricts to an action of $\mathrm{L}_{u_0}\mathrm{GU}$ on $\mathcal{F}_{\mathrm{GU}}$. Consequently we obtain an injective map*

$$\begin{aligned}\mathrm{L}_{u_0}\mathrm{GU}(R)/I_{\mathrm{GU}}(R) &\xrightarrow{\phi(R)} \mathcal{F}_{\mathrm{GU}}(R), \\ g &\longmapsto g \cdot \widetilde{\mathcal{L}}\end{aligned}$$

for each $\mathbb{F}$-algebra R. The morphism ϕ identifies $\mathcal{F}_{\mathrm{GU}}$ with both the étale and the fpqc sheafification of the presheaf $\mathrm{L}_{u_0}\mathrm{GU}/I_{\mathrm{GU}}$.

Proof. Let R be an $\mathbb{F}$-algebra. We claim that every $a \in R(\!(u_0)\!)^\times$ lies in the image of the map $c : \mathrm{GU}(R(\!(u_0)\!)) \to R(\!(u_0)\!)^\times$ étale locally on R. Assuming this, one can proceed exactly as in the proof of Proposition 5.5.8, referring to

Remark 6.6.7 and Proposition 6.6.3 in place of Remark 5.5.7 and Proposition 5.5.3.

To prove the claim, first note that for $b \in R(\!(u)\!)$, the matrix $bI_n \in \mathrm{GU}(R(\!(u_0)\!))$ satisfies $c(bI_n) = b\widetilde{b^*}$. Lemma 4.5.14 implies that Zariski locally on R, the element a is of the form $a = u_0^k v(1+n)$ for some $k \in \mathbb{Z}$, a unit $v \in R[\![u_0]\!]^\times$ and a nilpotent element $n \in R(\!(u_0)\!)$. Consequently it suffices to show that each of u_0^k, v and $1+n$ is of the form $b\widetilde{b^*}$ for some $b \in R(\!(u)\!)$ étale locally on R.

As $2 \in R^\times$, one easily sees that v is a square in $R[\![u_0]\!]^\times$ whenever $v(0)$ is a square in $R^\times$, which is the case étale locally on R. For $b = \sqrt{-1}u^k$, one has $b\widetilde{b^*} = u_0^k$. Finally, $1+n$ is a square in $R(\!(u_0)\!)^\times$; this follows from the Taylor expansion of $\sqrt{1+x}$ if one notes that $\binom{1/2}{l} \in \mathbb{Z}[\frac{1}{2}]$ for all $l \in \mathbb{N}$. $\square$

6.7. **Embedding the local model into the affine flag variety.** Let R be an $\mathbb{F}$-algebra. We consider an $R[u]/u^e$-module as an $R[\![u]\!]$-module via the canonical projection $R[\![u]\!] \to R[u]/u^e$. For $i \in \mathbb{Z}$ denote by $\alpha_i : \widetilde{\Lambda}_i \to \overline{\Lambda}_{i,R}$ the morphism described by the identity matrix with respect to $\widetilde{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_i$. It induces an isomorphism $\widetilde{\Lambda}_i/u^e\widetilde{\Lambda}_i \xrightarrow{\sim} \overline{\Lambda}_{i,R}$. Clearly the following diagrams commute.

$$
\begin{array}{ccc}
\widetilde{\Lambda}_i & \subset & \widetilde{\Lambda}_{i+1} \\
{\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha_{i+1}} \\
\overline{\Lambda}_{i,R} & \xrightarrow{\overline{\rho}_{i,R}} & \overline{\Lambda}_{i+1,R},
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} & \xrightarrow{\widetilde{\langle\cdot,\cdot\rangle}_i} & R[\![u]\!] \\
{\scriptstyle \alpha_i\times\alpha_{-i}}\downarrow & & \downarrow \\
\overline{\Lambda}_{i,R} \times \overline{\Lambda}_{-i,R} & \xrightarrow{\overline{\langle\cdot,\cdot\rangle}_{i,R}} & R[u]/u^e,
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\Lambda}_i & \xleftarrow{u\cdot} & \widetilde{\Lambda}_{n+i} \\
{\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha_{n+i}} \\
\overline{\Lambda}_{i,R} & \xleftarrow{\overline{\vartheta}_{i,R}} & \overline{\Lambda}_{n+i,R}.
\end{array}
$$

The following proposition allows us to consider $M^{e,n}$ as a subfunctor of $\mathcal{F}_{\mathrm{U}}^{(-e_0)}$.

Proposition 6.7.1 ([28, §3.3],[37, §4.4-5.1], [36, §6.4-7.1])**.** *There is an embedding $\alpha : M^{e,n} \hookrightarrow \mathcal{F}_{\mathrm{U}}^{(-e_0)}$ given on R-valued by*

$$
\begin{aligned}
M^{e,n}(R) &\to \mathcal{F}_{\mathrm{U}}^{(-e_0)}(R), \\
(t_i)_i &\mapsto (\alpha_i^{-1}(t_i))_i.
\end{aligned}
$$

It induces a bijection from $M^{e,n}(R)$ onto the set of those $(L_i)_i \in \mathcal{F}_{\mathrm{U}}^{(-e_0)}(R)$ satisfying the following conditions for all $i \in \mathbb{Z}$.

(1) $u^e\widetilde{\Lambda}_i \subset L_i \subset \widetilde{\Lambda}_i$.

(2) For all $p \in R[u]/u^e$, we have

$$\chi_R(p|\widetilde{\Lambda}_i/L_i) = (T - p(0))^{ne_0}$$

in $R[T]$. Here $\widetilde{\Lambda}_i/L_i$ is considered as an $R[u]/u^e$-module using (1).

Proof. Identical to the proof of Proposition 5.6.1. $\square$

Note that $\overline{\mathcal{L}} = (\overline{\Lambda}_i, \overline{\rho}_i, \overline{\vartheta}_i, \overline{\langle\cdot,\cdot\rangle}_i)_i$ is a polarized chain of $\mathbb{F}[\![u]\!]\otimes_{\mathbb{F}[\![u_0]\!]}\mathbb{F}[u_0]/u_0^{e_0}$-modules of type $(\widetilde{\mathcal{L}})$. In fact $\overline{\mathcal{L}} = \widetilde{\mathcal{L}} \otimes_{\mathbb{F}[\![u_0]\!]} \mathbb{F}[u_0]/u_0^{e_0}$. Let R be an $\mathbb{F}$-algebra. There is an obvious action of $\mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0})$ on $M^{e,n}(R)$, given by $(\varphi_i)\cdot(t_i) = (\varphi_i(t_i))$. The canonical morphism $R[\![u_0]\!] \to R[u_0]/u_0^{e_0}$ induces a morphism $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u_0]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0})$ and we thereby extend this $\mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0})$-action on $M^{e,n}(R)$ to an $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u_0]\!])$-action.

Lemma 6.7.2. *Let R be an $\mathbb{F}$-algebra and let $t \in M^{e,n}(R)$. We have $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u_0]\!]) \cdot t = \mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e}) \cdot t$.*

Proof. The map $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u_0]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0})$ is surjective by Proposition 6.6.3. $\square$

Define a subfunctor I_{U} of $\mathrm{L}_{u_0}\,\mathrm{U}$ by $I_{\mathrm{U}} = \mathrm{L}_{u_0}\,\mathrm{U} \cap I_{\mathrm{GU}}$.

Lemma 6.7.3. *We have $I_{\mathrm{GU}}(\mathbb{F}) = \mathbb{F}[\![u_0]\!]^{\times} I_{\mathrm{U}}(\mathbb{F})$.*

Proof. Analogous to the proof of Lemma 5.6.4, noting that for $g \in I_{\mathrm{GU}}(\mathbb{F})$ one has $c(g) \in \mathbb{F}[\![u_0]\!]^{\times}$. $\square$

Lemma 6.7.4. *Let $g \in I_{\mathrm{U}}(\mathbb{F})$. Then g restricts to an automorphism $g_i : \widetilde{\Lambda}_i \xrightarrow{\sim} \widetilde{\Lambda}_i$ for each $i \in \mathbb{Z}$. The assignment $g \mapsto (g_i)_i$ defines an isomorphism $I_{\mathrm{Sp}}(\mathbb{F}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u_0]\!])$.*

Proof. Clear (cf. the proof of Lemma 3.7.15). $\square$

Proposition 6.7.5. *Let $t \in M^{e,n}(\mathbb{F})$. Then α induces a bijection*

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0}) \cdot t \xrightarrow{\sim} I_{\mathrm{GU}}(\mathbb{F}) \cdot \alpha(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0}) \backslash M^{e,n}(\mathbb{F}) \hookrightarrow I_{\mathrm{GU}}(\mathbb{F}) \backslash \mathcal{F}_{\mathrm{GU}}(\mathbb{F}).$$

Proof. Analogous to the proof of Proposition 5.6.6. $\square$

Consider $\alpha' : M^{e,n}(\mathbb{F}) \hookrightarrow \mathcal{F}_{\mathrm{GU}}(\mathbb{F}) \xrightarrow{\phi(\mathbb{F})^{-1}} \mathrm{L}_{u_0}\,\mathrm{GU}(\mathbb{F})/I_{\mathrm{GU}}(\mathbb{F})$.

Proposition 6.7.6. *Let $t \in M^{e,n}(\mathbb{F})$. Then α' induces a bijection*

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0}) \cdot t \xrightarrow{\sim} I_{\mathrm{GU}}(\mathbb{F}) \cdot \alpha'(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0}) \backslash M^{e,n}(\mathbb{F}) \hookrightarrow I_{\mathrm{GU}}(\mathbb{F}) \backslash \mathrm{GU}(\mathbb{F}(\!(u_0)\!))/I_{\mathrm{GU}}(\mathbb{F}).$$

Proof. Clear from Proposition 6.7.5, as the isomorphism $\phi(\mathbb{F})$ is in particular $I_{\mathrm{GU}}(\mathbb{F})$-equivariant. $\square$

Let R be an $\mathbb{F}$-algebra and $(\varphi_i)_i \in \mathrm{Aut}(\mathcal{L})(R)$. The decomposition (6.5.3) induces for each i a decomposition of $\varphi_i : \Lambda_{i,R} \xrightarrow{\sim} \Lambda_{i,R}$ into the product of $R[u]/u^e$-linear automorphisms $\varphi_{i,\sigma} : \overline{\Lambda}_{i,R} \xrightarrow{\sim} \overline{\Lambda}_{i,R}$.

Proposition 6.7.7. *Let R be an $\mathbb{F}$-algebra. The following map is an isomorphism, functorial in R.*

$$\mathrm{Aut}(\mathcal{L})(R) \to \prod_{\sigma \in \mathfrak{S}} \mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0}),$$
$$(\varphi_i)_i \mapsto ((\varphi_{i,\sigma})_i)_{\sigma \in \mathfrak{S}}.$$

Proof. Analogous to the proof of Proposition 5.6.8. $\square$

Consider the composition

$$\tilde{\alpha} : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{(6.5.5)} \prod_{\sigma \in \mathfrak{S}} M^{e,n}(\mathbb{F}) \xrightarrow{\prod_\sigma \alpha'} \prod_{\sigma \in \mathfrak{S}} \mathrm{L}_{u_0}\, \mathrm{GU}(\mathbb{F})/I_{\mathrm{GU}}(\mathbb{F}).$$

For $\sigma \in \mathfrak{S}$ denote by $\tilde{\alpha}_\sigma : M^{\mathrm{loc}}(\mathbb{F}) \to \mathrm{L}_{u_0}\, \mathrm{GU}(\mathbb{F})/I_{\mathrm{GU}}(\mathbb{F})$ the corresponding component of $\tilde{\alpha}$.

Theorem 6.7.8. *Let $t \in M^{\mathrm{loc}}(\mathbb{F})$. Then $\tilde{\alpha}$ induces a bijection*

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \cdot t \xrightarrow{\sim} \prod_{\sigma \in \mathfrak{S}} I_{\mathrm{GU}}(\mathbb{F}) \cdot \tilde{\alpha}_\sigma(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \prod_{\sigma \in \mathfrak{S}} I_{\mathrm{GU}}(\mathbb{F}) \backslash \mathrm{GU}(F(\!(u_0)\!))/I_{\mathrm{GU}}(\mathbb{F}).$$

Proof. Identical to the proof of Theorem 5.6.10. □

6.8. **The extended affine Weyl group.** As in [37, 3.2],[36, 3.2], we denote by S the standard diagonal maximal split torus in GU. Denote by T the centralizer and by N the normalizer of S in GU. By the discussion in [37, 3.4], [36, 5.4], the Kottwitz homomorphism for T is given by

$$\kappa_T : T(\mathbb{F}(\!(u_0)\!)) \to \mathbb{Z}^n, \quad \mathrm{diag}(x_1, \ldots, x_n) \mapsto (\mathrm{val}_u(x_1), \ldots, \mathrm{val}_u(x_n)).$$

Consequently the kernel $T\big(\mathbb{F}(\!(u_0)\!)\big)_1$ of κ_T is equal to $T(\mathbb{F}(\!(u_0)\!)) \cap D_n(\mathbb{F}[\![u]\!])$, with the intersection taking place in $\mathrm{GL}_n(\mathbb{F}(\!(u)\!))$. By definition, the extended affine Weyl group of GU with respect to S is given by $\widetilde{W} := N(\mathbb{F}(\!(u_0)\!))/T(\mathbb{F}(\!(u_0)\!))_1$.

Set

$$W = \{w \in S_n \mid \forall i \in \{1, \ldots, n\} : \ w(i) + w(n+1-i) = n+1\}$$

and

$$X = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \exists r \in \mathbb{Z} \forall i \in \{1, \ldots, n\} : \ x_i + x_{n+1-i} = 2r\}.$$

We identify W with a subgroup of $\mathrm{U}(\mathbb{F}(\!(u_0)\!))$ via $W \ni w \mapsto A_w$. One easily sees that $N(\mathbb{F}(\!(u_0)\!)) = W \ltimes T(\mathbb{F}(\!(u_0)\!))$. The Kottwitz homomorphism κ_T induces an isomorphism $T(\mathbb{F}(\!(u_0)\!))/T(\mathbb{F}(\!(u_0)\!))_1 \xrightarrow{\sim} X$ and we thereby identify $\widetilde{W}$ with $W \ltimes X$.

To avoid any confusion of the product inside $\widetilde{W}$ and the canonical action of S_n on $\mathbb{Z}^n$, we will always denote the element of $\widetilde{W}$ corresponding to $\lambda \in X$ by u^λ.

Recall from [7, §2.5] the notion of an extended alcove $(x_i)_{i=0}^{n-1}$ for GL_n. An *extended alcove for* GU is an extended alcove $(x_i)_{i=0}^{n-1}$ for GL_n such that

$$\exists r \in \mathbb{Z} \forall i \in \{0, \ldots, n\} \forall j \in \{1, \ldots, n\} : \ x_i(j) + x_{n-i}(n+1-j) = 2r-1.$$

Here $x_n = x_0 + (1^{(n)})$.

Also recall the standard alcove $(\omega_i)_{i=0}^{n-1}$. As in the linear case treated in loc. cit., we identify $\widetilde{W}$ with the set of extended alcoves for GU by using the standard alcove as a base point.

Write $\mathbf{e} = (e^{(n)})$.

Definition 6.8.1 (Cf. [7, Definition 2.4])**.** *An extended alcove* $(x_i)_{i=0}^{n-1}$ *for* GU *is called* permissible *if it satisfies the following conditions for all* $i \in \{0, \dots, n-1\}$.

(1) $\omega_i \leq x_i \leq \omega_i + \mathbf{e}$, *where* $\leq$ *is to be understood componentwise.*
(2) $\sum_{j=1}^{n} x_i(j) = ne_0 - i$.

Denote by Perm *the set of all permissible extended alcoves for* GU*.*

Proposition 6.8.2. *The inclusion* $N(\mathbb{F}(\!(u_0)\!)) \subset \mathrm{GU}(\mathbb{F}(\!(u_0)\!))$ *induces a bijection* $\widetilde{W} \xrightarrow{\sim} I_{\mathrm{GU}}(\mathbb{F})\backslash \mathrm{GU}(\mathbb{F}(\!(u_0)\!))/I_{\mathrm{GU}}(\mathbb{F})$*. In other words,*

$$\mathrm{GU}(\mathbb{F}(\!(u_0)\!)) = \coprod_{x \in \widetilde{W}} I_{\mathrm{GU}}(\mathbb{F}) x I_{\mathrm{GU}}(\mathbb{F}).$$

Under this bijection, the subset

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0})\backslash M^{e,n}(\mathbb{F}) \subset I_{\mathrm{GU}}(\mathbb{F})\backslash \mathrm{GU}(\mathbb{F}(\!(u_0)\!))/I_{\mathrm{GU}}(\mathbb{F})$$

of Proposition 6.7.6 corresponds to the subset Perm $\subset \widetilde{W}$*.*

Proof. The first statement is discussed in [37, 4.4], [36, 6.4]. The second statement follows easily from the explicit description of the image of α in Proposition 6.7.1, keeping in mind Corollary 6.5.2. $\square$

Corollary 6.8.3. *Under the identifications of Theorem 6.7.8, the set* $\prod_{\sigma \in \mathfrak{S}}$ Perm *constitutes a set of representatives of* $\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$*.*

6.9. **The p-rank on a KR stratum.** We make Definitions 3.2.1 and 3.2.3 explicit for the chosen PEL datum.

Definition 6.9.1. *Let R be an* $\mathcal{O}_{E_Q}$*-algebra. A* self-dual $\mathcal{L}$-set of abelian varieties of determinant $\det_{V_{-i}}$ over R *is a commutative diagram*

$$\begin{tikzcd} \dots \arrow[r, "\varrho_{-2}"] & A_{-1} \arrow[r, "\varrho_{-1}"] \arrow[d, "\lambda_{-1}"] & A_0 \arrow[r, "\varrho_0"] \arrow[d, "\lambda_0"] & A_1 \arrow[r, "\varrho_1"] \arrow[d, "\lambda_1"] & \dots \\ \dots \arrow[r, "\varrho_1^\vee"] & A_1^\vee \arrow[r, "\varrho_0^\vee"] & A_0^\vee \arrow[r, "\varrho_{-1}^\vee"] & A_{-1}^\vee \arrow[r, "\varrho_{-2}^\vee"] & \dots \end{tikzcd}$$

in $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$ *satisfying the following conditions for all* $i \in \mathbb{Z}$.

(1) A_i *is an abelian scheme over R equipped with an action* $\kappa_i : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A_i) \otimes \mathbb{Z}_{(p)}$.
(2) $\varrho_i : A_i \to A_{i+1}$ *is an* $\mathbb{Z}_{(p)}$*-isogeny of degree* p^f*, compatible with* κ_i *and* κ_{i+1}.
(3) *There is an isomorphism* $\theta_i : A_{n+i} \to A_i$ *in* $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$ *such that the composition*
$$A_{n+i} \xrightarrow{\theta_i} A_i \xrightarrow{\prod_{j=0}^{n-1} \varrho_{i+j}} A_{n+i}$$
is equal to $\kappa_{n+i}(\pi)$.
(4) $\lambda_i : A_i \to A_{-i}^\vee$ *is an isomorphism in* $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$*, compatible with* κ_i *and* $\kappa_{-i}^\vee$*. Here* $\kappa_i^\vee : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A_i^\vee) \otimes \mathbb{Z}_{(p)}$ *is defined by* $\kappa_i^\vee(x) = \kappa_i(x^*)^\vee$, $x \in \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$.
(5) λ_0 *is symmetric.*
(6) $\det_{\mathrm{Lie}(A_i)} = \det_{V_{-i}} \otimes_{\mathcal{O}_E} R$.

Remark 6.9.2. *Let R be an $\mathcal{O}_{E_\mathcal{Q}}$-algebra and let A/R be an abelian scheme equipped with an action $\kappa : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A) \otimes \mathbb{Z}_{(p)}$. Assume that $\det_{\mathrm{Lie}(A)} = \det_{V_{-i}} \otimes_{\mathcal{O}_E} R$. Then $\dim_R A_i = ng_0$ by Lemma 2.3.4.*

Recall from Section 3.3 the diagram

$$\begin{tikzcd} & \widetilde{\mathcal{A}} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\ \mathcal{A} & & M^{\mathrm{loc}} \end{tikzcd}$$

of functors on the category of $\mathcal{O}_{E_\mathcal{Q}}$-algebras. Also recall the KR stratification $\mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x$. We have identified the occurring index set with $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$ in Corollary 6.8.3 . We can then state the following result.

Theorem 6.9.3. *Let $x = (x_\sigma)_\sigma \in \prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$. Write $x_\sigma = w_\sigma u^{\lambda_\sigma}$ with $w_\sigma \in W$, $\lambda_\sigma \in X$. Then the p-rank on $\mathcal{A}_x$ is constant with value*

$$g \cdot |\{1 \le i \le n \mid \forall \sigma \in \mathfrak{S}(w_\sigma(i) = i \wedge \lambda_\sigma(i) = 0)\}|.$$

Proof. The proof is identical to the one of Theorem 5.8.3. □

7. The inert unitary case

7.1. **The PEL datum.** Let $n \in \mathbb{N}_{\geq 1}$. We start with the PEL datum consisting of the following objects.

(1) An imaginary quadratic extension F/F_0 of a totally real extension $F_0/\mathbb{Q}$. Let $g_0 = [F_0 : \mathbb{Q}]$ and $g = [F : \mathbb{Q}]$, so that $g = 2g_0$.
(2) The non-trivial element $*$ of $\mathrm{Gal}(F/F_0)$.
(3) An n-dimensional F-vector space V.
(4) The symplectic form $(\cdot,\cdot) : V \times V \to \mathbb{Q}$ on the underlying $\mathbb{Q}$-vector space of V constructed as follows: Fix once and for all a $*$-skew-hermitian form $(\cdot,\cdot)' : V \times V \to F$ (i.e. $(av, bw)' = ab^* (v,w)'$ and $(v,w)' = -(w,v)'^*$ for $v, w \in V$, $a, b \in F$). Define $(\cdot,\cdot) = \mathrm{tr}_{F/\mathbb{Q}} \circ (\cdot,\cdot)'$.
(5) The element $J \in \mathrm{End}_{B \otimes \mathbb{R}}(V \otimes \mathbb{R})$ to be specified below.

Remark 7.1.1. *The reductive $\mathbb{Q}$-group G associated with the above PEL datum is described in Remark 6.1.1.*

We assume that $p\mathcal{O}_{F_0} = (\mathcal{P}_0)^e$ for a single prime $\mathcal{P}_0$ of $\mathcal{O}_{F_0}$ and that $\mathcal{P}_0\mathcal{O}_F = \mathcal{P}$ for a single prime $\mathcal{P}$ of $\mathcal{O}_F$. Denote by $f_0 = [k_{\mathcal{P}_0} : \mathbb{F}_p]$ and $f = [k_\mathcal{P} : \mathbb{F}_p]$ the corresponding inertia degrees, so that $f = 2f_0$. We fix once and for all a uniformizer π of $\mathcal{O}_{F_0} \otimes \mathbb{Z}_{(p)}$. Then π is also a uniformizer of $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$.

Denote by $\mathfrak{C} = \mathfrak{C}_{\mathcal{O}_{F_\mathcal{P}}|\mathbb{Z}_p}$ the corresponding inverse different. Choose a generator δ of $\mathfrak{C}$ satisfying $\delta^* = -\delta$. Consequently the form $\delta^{-1}(\cdot,\cdot)'_{\mathbb{Q}_p} : V_{\mathbb{Q}_p} \times V_{\mathbb{Q}_p} \to F_\mathcal{P}$ is $*$-hermitian and we assume that it *splits*, i.e. that there is a basis $(e_1, \ldots, e_n)$ of $V_{\mathbb{Q}_p}$ over $F_\mathcal{P}$ such that $(e_i, e_{n+1-j})'_{\mathbb{Q}_p} = \delta\delta_{ij}$ for $1 \le i, j \le n$.

Let $0 \le i < n$. We denote by Λ_i the $\mathcal{O}_{F_\mathcal{P}}$-lattice in $V_{\mathbb{Q}_p}$ with basis

$$\mathfrak{E}_i = (\pi^{-1}e_1, \ldots, \pi^{-1}e_i, e_{i+1}, \ldots, e_n).$$

For $k \in \mathbb{Z}$ we further define $\Lambda_{nk+i} = \pi^{-k}\Lambda_i$ and we denote by $\mathfrak{E}_{nk+i}$ the corresponding basis obtained from $\mathfrak{E}_i$. Then $\mathcal{L} = (\Lambda_i)_i$ is a complete chain of $\mathcal{O}_{F_\mathcal{P}}$-lattices in $V_{\mathbb{Q}_p}$. For $i \in \mathbb{Z}$, the dual lattice $\Lambda_i^\vee := \{x \in V_{\mathbb{Q}_p} \mid (x, \Lambda_i)_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ of Λ_i is given by Λ_{-i}. Consequently the chain $\mathcal{L}$ is self-dual.

Let $i \in \mathbb{Z}$. We denote by $\rho_i : \Lambda_i \to \Lambda_{i+1}$ the inclusion, by $\vartheta_i : \Lambda_{n+i} \to \Lambda_i$ the isomorphism given by multiplication with π and by $(\cdot,\cdot)_i : \Lambda_i \times \Lambda_{-i} \to \mathbb{Z}_p$ the restriction of $(\cdot,\cdot)_{\mathbb{Q}_p}$. Then $(\Lambda_i, \rho_i, \vartheta_i, (\cdot,\cdot)_i)_i$ is a polarized chain of $\mathcal{O}_{F_\mathcal{P}}$-modules of type $(\mathcal{L})$, which, by abuse of notation, we also denote by $\mathcal{L} = \mathcal{L}^{\text{inert}}$.

Denote by $\langle\cdot,\cdot\rangle_i : \Lambda_i \times \Lambda_{-i} \to \mathcal{O}_{F_\mathcal{P}}$ the restriction of the $*$-hermitian form $\delta^{-1}(\cdot,\cdot)'_{\mathbb{Q}_p}$. It is the $*$-sesquilinear form described by the matrix $\widetilde{I}_n$ with respect to $\mathfrak{E}_i$ and $\mathfrak{E}_{-i}$.

Denote by Σ_0 the set of all embeddings $F_0 \hookrightarrow \mathbb{R}$ and by Σ the set of all embeddings $F \hookrightarrow \mathbb{C}$. Also write $\mathfrak{S} = \mathrm{Gal}(k_\mathcal{P}/\mathbb{F}_p)$ and $\mathfrak{S}_0 = \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$.

Let E' be the Galois closure of F inside $\mathbb{C}$ and choose a prime $\mathcal{Q}'$ of E' over $\mathcal{P}$. Consider the maps $\gamma : \Sigma \to \mathfrak{S}$ and $\gamma_0 : \Sigma_0 \to \mathfrak{S}_0$ of Proposition 4.4.1. For each $\sigma \in \mathfrak{S}_0$ we denote by $\tau_{\sigma,1}, \tau_{\sigma,2} \in \mathfrak{S}$ the two elements with $\tau_{\sigma,j}\big|_{k_{\mathcal{P}_0}} = \sigma$.

Let $\sigma \in \Sigma_0$ and $j \in \{1,2\}$. By Lemma 4.4.3(2) there is a unique $\tau_{\sigma,j} \in \Sigma$ with $\tau_{\sigma,j}\big|_{F_0} = \sigma$ satisfying

$$\gamma(\tau_{\sigma,j}) = \tau_{\gamma_0(\sigma),j}. \tag{7.1.2}$$

Exactly as in Section 6, we define for each $\sigma \in \Sigma_0$ integers r_σ, s_σ with $r_\sigma + s_\sigma = n$, and using these the element $J \in \mathrm{End}_{B\otimes\mathbb{R}}(V \otimes \mathbb{R})$. Denote by V_{-i} the $(-i)$-eigenspace of $J_\mathbb{C}$. As before, we construct an $\mathcal{O}_F \otimes \mathcal{O}_{E'}$-module M_0 which is finite locally free over $\mathcal{O}_{E'}$, such that $M_0 \otimes_{\mathcal{O}_{E'}} \mathbb{C} = V_{-i}$ as $\mathcal{O}_F \otimes \mathbb{C}$-modules.

7.2. **The special fiber of the determinant morphism.** Let $\sigma \in \mathfrak{S}_0$. We define

$$\overline{r}_\sigma = \sum_{\sigma' \in \gamma_0^{-1}(\sigma)} r_{\sigma'} \quad \text{and} \quad \overline{s}_\sigma = \sum_{\sigma' \in \gamma_0^{-1}(\sigma)} s_{\sigma'}.$$

As the fibers of γ_0 have cardinality e, it follows that $\overline{r}_\sigma + \overline{s}_\sigma = ne$.

We fix once and for all an embedding $\iota_{\mathcal{Q}'} : k_{\mathcal{Q}'} \hookrightarrow \mathbb{F}$. We consider $\mathbb{F}$ as an $\mathcal{O}_{E'}$-algebra with respect to the composition $\mathcal{O}_{E'} \xrightarrow{\rho_{\mathcal{Q}'}} k_{\mathcal{Q}'} \overset{\iota_{\mathcal{Q}'}}{\hookrightarrow} \mathbb{F}$. Also $\iota_{\mathcal{Q}'}$ induces an embedding $\iota_\mathcal{P} : k_P \hookrightarrow \mathbb{F}$ and thereby an identification of the set of all embeddings $k_\mathcal{P} \hookrightarrow \mathbb{F}$ with $\mathfrak{S}$.

Consider the isomorphism

$$\mathcal{O}_F \otimes \mathbb{F} = \prod_{\sigma \in \mathfrak{S}_0} \mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e) \tag{7.2.1}$$

from Section 4.3. Here in the component $\mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e)$ corresponding to $\sigma \in \mathfrak{S}_0$, the first factor is supposed to correspond to $\tau_{\sigma,1}$ and the second factor is supposed to correspond to $\tau_{\sigma,2}$.

Proposition 7.2.2. *Let $x \in \mathcal{O}_F$ and let $\big((q_{\tau_{\sigma,1}}, q_{\tau_{\sigma,2}})\big)_\sigma \in \prod_{\sigma\in\mathfrak{S}_0} \mathbb{F}[u]/(u^e)\times \mathbb{F}[u]/(u^e)$ be the element corresponding to $x\otimes 1$ under* (7.2.1). *Then*

$$\chi_{\mathbb{F}}(x|M_0\otimes_{\mathcal{O}_{E'}}\mathbb{F}) = \prod_{\sigma\in\mathfrak{S}_0}\big(T-q_{\tau_{\sigma,1}}(0)\big)^{\overline{s}_\sigma}\big(T-q_{\tau_{\sigma,2}}(0)\big)^{\overline{r}_\sigma}$$

in $\mathbb{F}[T]$.

Proof. The definition of M_0 gives

$$\chi_{\mathcal{O}_{E'}}(x|M_0) = \prod_{\sigma\in\Sigma_0}\big(T-\tau_{\sigma,1}(x)\big)^{s_\sigma}\big(T-\tau_{\sigma,2}(x)\big)^{r_\sigma}.$$

Using (7.1.2) we obtain

$$(\chi_{\mathcal{O}_{E'}}(x|M_0))^{\rho_{\mathcal{Q}'}} = \prod_{\sigma\in\mathfrak{S}_0}\big(T-\tau_{\sigma,1}(\rho_{\mathcal{P}}(x))\big)^{\overline{s}_\sigma}\cdot\big(T-\tau_{\sigma,2}(\rho_{\mathcal{P}}(x))\big)^{\overline{r}_\sigma}.$$

The claim then follows from the equality $q_{\tau_{\sigma,j}}(0) = (\iota_{\mathcal{P}}\circ\tau_{\sigma,j})(\rho_{\mathcal{P}}(x)),\ \sigma\in\mathfrak{S}_0,\ j=1,2.$ ☐

Denote by $E = \mathbb{Q}(\mathrm{tr}_{\mathbb{C}}(x\otimes 1|V_{-i});\ x\in F)$ the reflex field and define $\mathcal{Q} = \mathcal{Q}'\cap\mathcal{O}_E$. By Proposition 2.3.5 the morphism $\det_{V_{-i}}$ is defined over $\mathcal{O}_E$, and we also denote by $\det_{V_{-i}}$ the corresponding morphism over $\mathcal{O}_E$.

7.3. **The local model.** For the chosen PEL datum, Definition 3.3.2 amounts to the following.

Definition 7.3.1. *The local model $M^{\mathrm{loc}} = M^{\mathrm{loc,inert}}$ is the functor on the category of $\mathcal{O}_{E_{\mathcal{Q}}}$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_i)_{i\in\mathbb{Z}}$ of $\mathcal{O}_F\otimes R$-submodules $t_i\subset\Lambda_{i,R}$, satisfying the following conditions for all $i\in\mathbb{Z}$.*

- *(a)* $\rho_{i,R}(t_i)\subset t_{i+1}$.
- *(b) The quotient $\Lambda_{i,R}/t_i$ is a finite locally free R-module.*
- *(c) We have an equality*

$$\det_{\Lambda_{i,R}/t_i} = \det_{V_{-i}}\otimes_{\mathcal{O}_E}R$$

  of morphisms $V_{\mathcal{O}_F\otimes R}\to\mathbb{A}^1_R$.
- *(d) Under the pairing $(\cdot,\cdot)_{i,R}:\Lambda_{i,R}\times\Lambda_{-i,R}\to R$, the submodules t_i and t_{-i} pair to zero.*
- *(e)* $\vartheta_i(t_{n+i}) = t_i$.

Corollary 7.3.2. *Condition 7.3.1(d) can be equivalently replaced by the following condition.*

- *(d')* $t_i^{\perp,\langle\cdot,\cdot\rangle_{i,R}} = t_{-i}$.

Proof. Identical to the proof of Corollary 5.3.2. ☐

7.4. **The special fiber of the local model.** For $i\in\mathbb{Z}$, denote by $\overline{\Lambda}_i$ the free $\mathbb{F}[u]/u^e$-module $(\mathbb{F}[u]/u^e)^n$ and by $\overline{\mathfrak{E}}_i$ its canonical basis. Denote by $\overline{\vartheta}_i:\overline{\Lambda}_{n+i}\to\overline{\Lambda}_i$ the identity morphism. Consider the map $\overline{*}:\mathbb{F}[u]/u^e\times\mathbb{F}[u]/u^e\to\mathbb{F}[u]/u^e\times\mathbb{F}[u]/u^e,\ (a,b)\mapsto(b,a)$. Let $\overline{\Lambda}_{i,1}$ and $\overline{\Lambda}_{i,2}$ be two copies of $\overline{\Lambda}_i$ and denote by $\overline{\langle\cdot,\cdot\rangle}_{i,1}:\overline{\Lambda}_{i,1}\times\overline{\Lambda}_{-i,2}\to\mathbb{F}[u]/u^e$ (resp. $\overline{\langle\cdot,\cdot\rangle}_{i,2}$:

$\overline{\Lambda}_{i,2} \times \overline{\Lambda}_{-i,1} \to \mathbb{F}[u]/u^e)$ the perfect bilinear map described by the matrix $\widetilde{I}_n$ with respect to $\overline{\mathfrak{E}}_{i,1}$ and $\overline{\mathfrak{E}}_{-i,2}$ (resp. $\overline{\mathfrak{E}}_{i,2}$ and $\overline{\mathfrak{E}}_{-i,1}$). Consider the pairing

$$\overline{\langle \cdot, \cdot \rangle}_i : (\overline{\Lambda}_{i,1} \times \overline{\Lambda}_{i,2}) \times (\overline{\Lambda}_{-i,1} \times \overline{\Lambda}_{-i,2}) \to \mathbb{F}[u]/u^e \times \mathbb{F}[u]/u^e,$$
$$\big((x_1, x_2), (y_1, y_2)\big) \mapsto \big(\langle x_1, y_2 \rangle_{i,1}, \langle x_2, y_1 \rangle_{i,2}\big).$$

It is a perfect $\overline{*}$-sesquilinear pairing.

For $k \in \mathbb{Z}$ and $0 \le i < n$, let $\overline{\rho}_{nk+i} : \overline{\Lambda}_{nk+i} \to \overline{\Lambda}_{nk+i+1}$ be the morphism described by the matrix $\operatorname{diag}(1^{(i)}, u, 1^{(n-i-1)})$ with respect to $\overline{\mathfrak{E}}_{nk+i}$ and $\overline{\mathfrak{E}}_{nk+i+1}$.

Definition 7.4.1. *Let $r, s \in \mathbb{N}$ with $r + s = ne$. Define a functor $M^{e,n,r}$ on the category of $\mathbb{F}$-algebras with $M^{e,n,r}(R)$ the set of tuples $(t_i)_{i\in\mathbb{Z}}$ of $R[u]/u^e$-submodules $t_i \subset \overline{\Lambda}_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

(a) $\overline{\rho}_{i,R}(t_i) \subset t_{i+1}$.
(b) The quotient $\overline{\Lambda}_{i,R}/t_i$ is a finite locally free R-module.
(c) For all $p \in R[u]/u^e$, we have

$$\chi_R(p|\overline{\Lambda}_{i,R}/t_i) = \big(T - p(0)\big)^s$$

in $R[T]$.
(d) $\overline{\vartheta}_i(t_{n+i}) = t_i$.

Corollary 7.4.2. *Assume that R is* reduced. *Then condition 7.4.1(c) is equivalent to the following condition.*

(c') $\operatorname{rk}_R \overline{\Lambda}_{i,R}/t_i = s$.

Proof. See Lemma 2.3.9. □

Let $i \in \mathbb{Z}$. From (7.2.1) we obtain an isomorphism

$$\Lambda_{i,\mathbb{F}} = \prod_{\sigma\in\mathfrak{S}_0} \overline{\Lambda}_{i,1} \times \overline{\Lambda}_{i,2} \tag{7.4.3}$$

by identifying the basis $\mathfrak{E}_{i,\mathbb{F}}$ with the product of the bases $\overline{\mathfrak{E}}_i$. Under this identification, the morphism $\rho_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\rho}_i$, the pairing $\langle \cdot, \cdot \rangle_{i,\mathbb{F}}$ decomposes into the pairings $\overline{\langle \cdot, \cdot \rangle}_i$ and the morphism $\vartheta_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\vartheta}_i$.

Let R be an $\mathbb{F}$-algebra and let $(t_i)_{i\in\mathbb{Z}}$ be a tuple of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$. Then (7.4.3) induces decompositions $t_i = \prod_{\sigma\in\mathfrak{S}_0} t_{i,\tau_{\sigma,1}} \times t_{i,\tau_{\sigma,2}}$ into $R[u]/u^e$-submodules $t_{i,\tau_{\sigma,j}} \subset \overline{\Lambda}_{i,j,R}$.

Proposition 7.4.4. *The morphism $\Phi_1 : M^{\mathrm{loc}}_{\mathbb{F}} \to \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}$ given on R-valued points by*

$$M^{\mathrm{loc}}_{\mathbb{F}}(R) \to \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}(R),$$
$$(t_i) \mapsto \big((t_{i,\tau_{\sigma,1}})_i\big)_\sigma$$

is an isomorphism of functors on the category of $\mathbb{F}$-algebras. M^{loc}

Proof. It follows from Propositions 2.3.7 and 7.2.2 that the map in question is well-defined. Conversely, Corollary 7.3.2 (and repeated applications of Lemma 4.2.1) imply that the map

$$\prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}(R) \to M^{\mathrm{loc}}_{\mathbb{F}}(R),$$
$$((t_{i,\sigma})_i)_\sigma \mapsto \left(\prod_{\sigma\in\mathfrak{S}_0}(t_{i,\sigma}\times t_{-i,\sigma}^{\perp,\overline{\langle\cdot,\cdot\rangle}_{-i,1,R}})\right)_i$$

is well-defined and inverse to the map in question. Details left to the reader. □

Remark 7.4.5. *For symmetry reasons, also the morphism* $\Phi_2 : M^{\mathrm{loc}}_{\mathbb{F}} \to \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}$ *given on* R*-valued points by*

$$M^{\mathrm{loc}}_{\mathbb{F}}(R) \to \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}(R),$$
$$(t_i) \mapsto \big((t_{i,\tau_{\sigma,2}})_i\big)_\sigma$$

is an isomorphism of functors on the category of $\mathbb{F}$*-algebras.*

The morphism $\prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{r}_\sigma} \to \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}$ *making commutative the diagram*

$$\begin{tikzcd} & \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{r}_\sigma} \arrow[dd] \\ M^{\mathrm{loc}}_{\mathbb{F}} \arrow[ur, "\Phi_1"] \arrow[dr, "\Phi_2"'] & \\ & \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{s}_\sigma} \end{tikzcd}$$

is given by on R*-valued points by*

$$\prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}(R) \to \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}(R), \tag{7.4.6}$$
$$((t_{i,\sigma})_i)_\sigma \mapsto ((t_{-i,\sigma}^{\perp,\overline{\langle\cdot,\cdot\rangle}_{-i,1,R}})_i)_\sigma.$$

7.5. **The affine flag variety.** This section deals with the affine flag variety for the general linear group. Our discussion loosely follows the one in [26, §4].

Let R be an $\mathbb{F}$-algebra. Recall from Section 4.5 the standard lattice chain $\widetilde{\mathcal{L}} = (\widetilde{\Lambda}_i)_i$ in $R(\!(u)\!)^n$.

Definition 7.5.1 (Cf. [32, p. 131])**.** *Let* $\mathbb{F}[\![u]\!] \to R$ *be an* $\mathbb{F}[\![u]\!]$*-algebra. A* chain of R-modules of type $(\widetilde{\mathcal{L}})$ *is a tuple*

$$\mathcal{M} = (M_i, \varrho_i : M_i \to M_{i+1}, \theta_i : M_{n+i} \xrightarrow{\sim} M_i)_{i\in\mathbb{Z}},$$

where the M_i *are* R*-modules, the* ϱ_i *are* R*-linear maps and the* θ_i *are* R*-linear isomorphisms, such that for all* $i\in\mathbb{Z}$ *the following conditions hold.*

1. *Locally on* R*, there are isomorphisms* $M_i \simeq \widetilde{\Lambda}_{i,R}$ *and* $M_{i+1}/\varrho_i(M_i) \simeq (\widetilde{\Lambda}_{i+1}/\widetilde{\Lambda}_i)_R$ *of* R*-modules.*

(2) *The diagram*

$$\begin{array}{ccc} M_{n+i} & \xrightarrow{\varrho_{n+i}} & M_{n+i+1} \\ \downarrow{\scriptstyle\theta_i} & & \downarrow{\scriptstyle\theta_{i+1}} \\ M_i & \xrightarrow{\varrho_i} & M_{i+1} \end{array}$$

commutes.

(3) *The composition* $M_{n+i} \xrightarrow{\theta_i} M_i \xrightarrow{\prod_{j=0}^{n-1} \varrho_{i+j}} M_{n+i}$ *is multiplication by* u.

Let

$$\mathcal{M} = (M_i, \varrho_i : M_i \to M_{i+1}, \theta_i : M_{n+i} \xrightarrow{\sim} M_i)_i,$$
$$\mathcal{M}' = (M'_i, \varrho'_i : M'_i \to M'_{i+1}, \theta'_i : M'_{n+i} \xrightarrow{\sim} M'_i)_i$$

be chains of R-modules of type $(\widetilde{\mathcal{L}})$. A morphism $\varphi : \mathcal{M} \to \mathcal{M}'$ *is a tuple* $(\varphi_i)_{i\in\mathbb{Z}}$ *of isomorphisms of R-modules* $\varphi_i : M_i \to M'_i$ *such that the following diagrams commute for all* $i \in \mathbb{Z}$.

$$\begin{array}{ccc} M_i & \xrightarrow{\varrho_i} & M_{i+1} \\ \downarrow{\scriptstyle\varphi_i} & & \downarrow{\scriptstyle\varphi_{i+1}} \\ M'_i & \xrightarrow{\varrho'_i} & M'_{i+1}, \end{array} \qquad \begin{array}{ccc} M_i & \xleftarrow{\theta_i} & M_{n+i} \\ \downarrow{\scriptstyle\varphi_i} & & \downarrow{\scriptstyle\varphi_{n+i}} \\ M'_i & \xleftarrow{\theta'_i} & M'_{n+i}. \end{array}$$

We denote by $\mathrm{Isom}(\mathcal{M}, \mathcal{M}')$ *the functor on the category of R-algebras with* $\mathrm{Isom}(\mathcal{M}, \mathcal{M}')(R')$ *the set of morphisms* $\mathcal{M} \otimes_R R' \to \mathcal{M}' \otimes_R R'$ *of chains of R'-modules of type $(\widetilde{\mathcal{L}})$. We also write* $\mathrm{Aut}(\mathcal{M}) = \mathrm{Isom}(\mathcal{M}, \mathcal{M})$.

Proposition 7.5.2. *Let R be an $\mathbb{F}[\![u]\!]$-algebra such that the image of u in R is nilpotent. Then any two chains $\mathcal{M}, \mathcal{N}$ of R-modules of type $(\widetilde{\mathcal{L}})$ are isomorphic locally for the Zariski topology on R. Furthermore the functor* $\mathrm{Isom}(\mathcal{M}, \mathcal{N})$ *is representable by a smooth affine scheme over R.*

Proof. The proof of [32, Proposition A.4] carries over to this situation without any changes. □

Proposition 7.5.3. *Let R be an $\mathbb{F}$-algebra and let $\mathcal{M}, \mathcal{N}$ be chains of $R[\![u]\!]$-modules of type $(\widetilde{\mathcal{L}})$. Then the canonical map* $\mathrm{Isom}(\mathcal{M}, \mathcal{N})(R[\![u]\!]) \to \mathrm{Isom}(\mathcal{M}, \mathcal{N})(R[\![u]\!]/u^m)$ *is surjective for all $m \in \mathbb{N}_{\geq 1}$. In particular $\mathcal{M}$ and $\mathcal{N}$ are isomorphic locally for the Zariski topology on R.*

Proof. Analogous to the proof of Proposition 5.5.3, referring to Proposition 7.5.2 in place of Proposition 5.5.2. □

Remark 7.5.4. *Let R be an $\mathbb{F}$-algebra and let $(L_i)_i \in \mathcal{F}(R)$. For $i \in \mathbb{Z}$ denote by $\varrho_i : L_i \to L_{i+1}$ the inclusion and by $\theta_i : L_{n+i} \to L_i$ the isomorphism given by multiplication with u. Then $(L_i, \varrho_i, \theta_i)$ is a chain of $R[\![u]\!]$-modules of type $(\widetilde{\mathcal{L}})$. Here we use Proposition 4.5.5 to verify condition 7.5.1(1).*

Proposition 7.5.5. *Let R be an $\mathbb{F}$-algebra. Recall from Remark 4.5.13 the injective map*

$$\begin{aligned} \mathrm{L}\,\mathrm{GL}_n(R)/I(R) &\xrightarrow{\phi(R)} \mathcal{F}(R), \\ g &\longmapsto g \cdot \widetilde{\mathcal{L}}. \end{aligned}$$

The morphism ϕ identifies $\mathcal{F}$ with both the Zariski and the fpqc sheafification of the presheaf $\mathrm{L}\,\mathrm{GL}_n/I$.

Proof. Similar to the proof of Proposition 5.5.8, referring to Remark 7.5.4 and Proposition 7.5.3 in place of Remark 5.5.7 and Proposition 5.5.3. ☐

7.6. **Embedding the local model into the affine flag variety.** Let R be an $\mathbb{F}$-algebra. We consider an $R[u]/u^e$-module as an $R[\![u]\!]$-module via the canonical projection $R[\![u]\!] \to R[u]/u^e$. For $i \in \mathbb{Z}$, denote by $\alpha_i : \widetilde{\Lambda}_i \to \overline{\Lambda}_{i,R}$ the morphism described by the identity matrix with respect to $\widetilde{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_i$. It induces an isomorphism $\widetilde{\Lambda}_i/u^e\widetilde{\Lambda}_i \xrightarrow{\sim} \overline{\Lambda}_{i,R}$. Clearly the following diagrams commute.

$$\begin{tikzcd} \widetilde{\Lambda}_i \arrow[r, phantom, "\subset"] \arrow[d, "\alpha_i"'] & \widetilde{\Lambda}_{i+1} \arrow[d, "\alpha_{i+1}"] \\ \overline{\Lambda}_{i,R} \arrow[r, "\overline{\rho}_{i,R}"] & \overline{\Lambda}_{i+1,R}, \end{tikzcd} \qquad \begin{tikzcd} \widetilde{\Lambda}_i \arrow[d, "\alpha_i"'] & \widetilde{\Lambda}_{n+i} \arrow[l, "u\cdot"'] \arrow[d, "\alpha_{n+i}"] \\ \overline{\Lambda}_{i,R} & \overline{\Lambda}_{n+i,R}. \arrow[l, "\overline{\vartheta}_{i,R}"'] \end{tikzcd}$$

Let $r, s \in \mathbb{N}$ with $r + s = ne$. The following proposition allows us to consider $M^{e,n,r}$ as a subfunctor of $\mathcal{F}$.

Proposition 7.6.1 ([26, §4])**.** *There is an embedding $\alpha : M^{e,n,r} \hookrightarrow \mathcal{F}$ given on R-valued points by*

$$\begin{aligned} M^{e,n,r}(R) &\to \mathcal{F}(R), \\ (t_i)_i &\mapsto (\alpha_i^{-1}(t_i))_i. \end{aligned}$$

It induces a bijection from $M^{e,n,r}(R)$ onto the set of those $(L_i)_i \in \mathcal{F}(R)$ satisfying the following conditions for all $i \in \mathbb{Z}$.

(1) $u^e\widetilde{\Lambda}_i \subset L_i \subset \widetilde{\Lambda}_i$.

(2) For all $p \in R[u]/u^e$, we have

$$\chi_R(p|\widetilde{\Lambda}_i/L_i) = \big(T - p(0)\big)^s$$

in $R[T]$. Here $\widetilde{\Lambda}_i/L_i$ is considered as an $R[u]/u^e$-module using (1).

Proof. Analogous to the proof of Proposition 5.6.1. ☐

Let R be an $\mathbb{F}$-algebra. Denote by $\widetilde{\langle \cdot, \cdot \rangle} : R(\!(u)\!)^n \times R(\!(u)\!)^n \to R(\!(u)\!)$ the bilinear form described by the matrix $\widetilde{I_n}$ with respect to the standard basis of $R(\!(u)\!)^n$ over $R(\!(u)\!)$. Further denote by $\widetilde{\langle \cdot, \cdot \rangle}_i : \widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} \to R[\![u]\!]$ the restriction of $\widetilde{\langle \cdot, \cdot \rangle}$. Note that the diagram

$$\begin{tikzcd} \widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} \arrow[rr, "\widetilde{\langle\cdot,\cdot\rangle}_i"] \arrow[d, "\alpha_i \times \alpha_{-i}"'] & & R[\![u]\!] \arrow[d] \\ \overline{\Lambda}_{i,1,R} \times \overline{\Lambda}_{-i,2,R} \arrow[rr, "\overline{\langle\cdot,\cdot\rangle}_{i,1,R}"] & & R[u]/u^e \end{tikzcd}$$

commutes. For a lattice Λ in $R(\!(u)\!)^n$ we define $\Lambda^\vee := \{x \in R(\!(u)\!)^n \mid \widetilde{\langle x, \Lambda\rangle} \subset R[\![u]\!]\}$.

As in Remark 7.6.11, the morphism $\Psi : M^{e,n,r} \to M^{e,n,s}$ given on R-valued points by

$$M^{e,n,r}(R) \to M^{e,n,s}(R),$$
$$(t_i)_i \mapsto (t_{-i}^{\perp, \overline{\langle\cdot,\cdot\rangle}_{-i,1,R}})_i$$

is an isomorphism.

Proposition 7.6.2. *The following diagram commutes.*

$$\begin{tikzcd} M^{e,n,r} \arrow[r, hook, "\alpha"] \arrow[d, "\Psi"'] & \mathcal{F} \arrow[d, "(L_i)_i \mapsto (u^e L_{-i}^\vee)_i"] \\ M^{e,n,s} \arrow[r, hook, "\alpha"] & \mathcal{F}. \end{tikzcd}$$

Proof. Similar to the proof of the duality statement in the proof of Proposition 3.7.10. □

Note that $\overline{\mathcal{L}} = (\overline{\Lambda}_i, \overline{\rho}_i, \overline{\vartheta}_i)$ is a chain of $\mathbb{F}[u]/u^e$-modules of type $(\widetilde{\mathcal{L}})$. In fact $\overline{\mathcal{L}} = \widetilde{\mathcal{L}} \otimes_{\mathbb{F}[\![u]\!]} \mathbb{F}[u]/u^e$. Let R be an $\mathbb{F}$-algebra. There is an obvious action of $\mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ on $M^{e,n,r}(R)$, given by $(\varphi_i) \cdot (t_i) = (\varphi_i(t_i))$. The canonical morphism $R[\![u]\!] \to R[u]/u^e$ induces a map $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ and we thereby extend this $\mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$-action on $M^{e,n,r}(R)$ to an $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!])$-action.

Lemma 7.6.3. *Let R be an $\mathbb{F}$-algebra and let $t \in M^{e,n,r}(R)$. We have $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \cdot t = \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e) \cdot t$.*

Proof. The map $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ is surjective by Proposition 7.5.3. □

Lemma 7.6.4. *Let $g \in I(\mathbb{F})$. Then g restricts to an automorphism $g_i : \widetilde{\Lambda}_i \xrightarrow{\sim} \widetilde{\Lambda}_i$ for each $i \in \mathbb{Z}$. The assignment $g \mapsto (g_i)_i$ defines an isomorphism $I(\mathbb{F}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!])$.*

Proof. Clear (cf. the proof of Lemma 3.7.15). □

Proposition 7.6.5. *Let $t \in M^{e,n,r}(\mathbb{F})$. Then α induces a bijection*

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \cdot t \xrightarrow{\sim} I(\mathbb{F}) \cdot \alpha(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n,r}(\mathbb{F}) \hookrightarrow I(\mathbb{F}) \backslash \mathcal{F}(\mathbb{F}).$$

Proof. Analogous to the proof of Proposition 5.6.7. □

Consider $\alpha' : M^{e,n,r}(\mathbb{F}) \hookrightarrow \mathcal{F}(\mathbb{F}) \xrightarrow{\phi(\mathbb{F})^{-1}} \mathrm{L}\,\mathrm{GL}_n(\mathbb{F})/I(\mathbb{F})$.

Proposition 7.6.6. *Let $t \in M^{e,n,r}(\mathbb{F})$. Then α' induces a bijection*

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \cdot t \xrightarrow{\sim} I(\mathbb{F}) \cdot \alpha'(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n,r}(\mathbb{F}) \hookrightarrow I(\mathbb{F}) \backslash \mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}). \tag{7.6.7}$$

Proof. Clear from Proposition 7.6.5, as the isomorphism $\phi(\mathbb{F})$ is in particular $I(\mathbb{F})$-equivariant. $\square$

Denote by τ the adjoint involution for $\widetilde{\langle\cdot,\cdot\rangle}$ on $\mathrm{GL}_n(\mathbb{F}(\!(u)\!))$, so that for $g \in \mathrm{GL}_n(\mathbb{F}(\!(u)\!))$ we have $\widetilde{\langle gx, y\rangle} = \widetilde{\langle x, g^\tau\rangle},\ x, y \in \mathbb{F}(\!(u)\!)^n$.

Proposition 7.6.8. *The vertical maps in the following diagram are well-defined bijections and the diagram commutes.*

$$\begin{tikzcd} \mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)\backslash M^{e,n,r}(\mathbb{F}) \arrow[r, hook, "(7.6.7)"] \arrow[d, "\Psi"'] & I(\mathbb{F})\backslash \mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}) \arrow[d, "g\mapsto u^e(g^\tau)^{-1}"] \\ \mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)\backslash M^{e,n,s}(\mathbb{F}) \arrow[r, hook, "(7.6.7)"] & I(\mathbb{F})\backslash \mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}). \end{tikzcd}$$

Proof. In view of Proposition 7.6.2 it suffices to note the following statement, which follows from a short computation: Let Λ be a lattice in $\mathbb{F}(\!(u)\!)^n$ and let $g \in \mathrm{GL}_n(\mathbb{F}(\!(u)\!))$. Then $(g\Lambda)^\vee = (g^\tau)^{-1}(\Lambda^\vee)$. $\square$

Let R be an $\mathbb{F}$-algebra and $\varphi = (\varphi_i)_i \in \mathrm{Aut}(\mathcal{L})(R)$. The decomposition (7.4.3) induces for each i a decomposition of $\varphi_i : \Lambda_{i,R} \xrightarrow{\sim} \Lambda_{i,R}$ into the product of $R[u]/u^e$-linear automorphisms $\varphi_{i,\tau_{\sigma,j}} : \overline{\Lambda}_{i,j,R} \xrightarrow{\sim} \overline{\Lambda}_{i,j,R}$.

Proposition 7.6.9. *Let R be an $\mathbb{F}$-algebra. The following map is an isomorphism, functorial in R.*

$$\mathrm{Aut}(\mathcal{L})(R) \to \prod_{\sigma\in\mathfrak{S}_0} \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e),$$
$$(\varphi_i)_i \mapsto ((\varphi_{i,\tau_{\sigma,1}})_i)_\sigma.$$

Proof. Similar to the proof of Proposition 5.6.8. Details left to the reader. $\square$

Consider the composition

$$\tilde{\alpha}_1 : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{\Phi_1} \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}(\mathbb{F}) \xrightarrow{\prod_\sigma \alpha'} \prod_{\sigma\in\mathfrak{S}_0} \mathrm{L}\,\mathrm{GL}_n(\mathbb{F})/I(\mathbb{F}).$$

For $\sigma \in \mathfrak{S}_0$ denote by $\tilde{\alpha}_{1,\sigma} : M^{\mathrm{loc}}(\mathbb{F}) \to \mathrm{L}\,\mathrm{GL}_n(\mathbb{F})/I(\mathbb{F})$ the corresponding component of $\tilde{\alpha}_1$.

Theorem 7.6.10. *Let $t \in M^{\mathrm{loc}}(\mathbb{F})$. Then $\tilde{\alpha}_1$ induces a bijection*

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \cdot t \xrightarrow{\sim} \prod_{\sigma\in\mathfrak{S}_0} I(\mathbb{F}) \cdot \tilde{\alpha}_{1,\sigma}(t).$$

Consequently we obtain an embedding

$$\iota_1 : \mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \prod_{\sigma\in\mathfrak{S}_0} I(\mathbb{F})\backslash \mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}).$$

Proof. Identical to the proof of Theorem 5.6.10. $\square$

Remark 7.6.11. *In the same way, the composition*

$$\tilde{\alpha}_2 : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{\Phi_2} \prod_{\sigma\in\mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}(\mathbb{F}) \xrightarrow{\prod_\sigma \alpha'} \prod_{\sigma\in\mathfrak{S}_0} \mathrm{L}\,\mathrm{GL}_n(\mathbb{F})/I(\mathbb{F})$$

induces an embedding

$$\iota_2 : \mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \prod_{\sigma\in\mathfrak{S}_0} I(\mathbb{F})\backslash \mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}).$$

By Proposition 7.6.8 the following diagram commutes.

$$\begin{tikzcd}
 & \prod_{\sigma\in\mathfrak{S}_0} I(\mathbb{F})\backslash \mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}) \arrow[dd, "(g_\sigma)_\sigma\mapsto(u^e(g_\sigma^\tau)^{-1})_\sigma"] \\
\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F}) \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & \\
 & \prod_{\sigma\in\mathfrak{S}_0} I(\mathbb{F})\backslash \mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F})
\end{tikzcd}$$

7.7. **The extended affine Weyl group.** Let $T = D_n$ be the maximal torus of diagonal matrices in GL_n and let N be its normalizer. We denote by $\widetilde{W} = N(\mathbb{F}(\!(u)\!))/T(\mathbb{F}[\![u]\!])$ the extended affine Weyl group of GL_n with respect to T. Setting $W = S_n$ and $X = \mathbb{Z}^n$, the group homomorphism $\upsilon : W \ltimes X \to N(\mathbb{F}(\!(u)\!))$, $(w, \lambda) \mapsto A_w u^\lambda$ induces an isomorphism $W \ltimes X \xrightarrow{\sim} \widetilde{W}$. We use it to identify $\widetilde{W}$ with $W \ltimes X$ and consider $\widetilde{W}$ as a subgroup of $\mathrm{GL}_n(\mathbb{F}(\!(u)\!))$ via υ.

To avoid any confusion of the product inside $\widetilde{W}$ and the canonical action of S_n on $\mathbb{Z}^n$, we will always denote the element of $\widetilde{W}$ corresponding to $\lambda \in X$ by u^λ.

Recall from [7, §2.5] the notion of an extended alcove $(x_i)_{i=0}^{n-1}$ for GL_n. Also recall the standard alcove $(\omega_i)_i$. As in loc. cit. we identify $\widetilde{W}$ with the set of extended alcoves by using the standard alcove as a base point.

Let $r, s \in \mathbb{N}$ with $r + s = ne$ and write $\mathbf{e} = (e^{(n)})$.

Definition 7.7.1 (Cf. [7, Definition 2.4])**.** *An extended alcove $(x_i)_{i=0}^{n-1}$ is called r-permissible if it satisfies the following conditions for all $i \in \{0, \ldots, n-1\}$.*

(1) $\omega_i \le x_i \le \omega_i + \mathbf{e}$, where $\le$ is to be understood componentwise.
(2) $\sum_{j=1}^{n} x_i(j) = s - i$.

Denote by Perm_r the set of all r-permissible extended alcoves.

Proposition 7.7.2. *The inclusion $N(\mathbb{F}(\!(u)\!)) \subset \mathrm{GL}_n(\mathbb{F}(\!(u)\!))$ induces a bijection $\widetilde{W} \xrightarrow{\sim} I(\mathbb{F})\backslash\mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F})$. In other words,*

$$\mathrm{GL}_n(\mathbb{F}(\!(u)\!)) = \coprod_{x\in\widetilde{W}} I(\mathbb{F})xI(\mathbb{F}).$$

Under this bijection, the subset

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)\backslash M^{e,n,r}(\mathbb{F}) \subset I(\mathbb{F})\backslash\mathrm{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F})$$

of (7.6.7) *corresponds to the subset* $\mathrm{Perm}_r \subset \widetilde{W}$.

Proof. The first statement is the well-known Iwahori decomposition. The second statement follows easily from the explicit description of the image of α in Proposition 7.6.1, keeping in mind Corollary 7.4.2. □

Corollary 7.7.3. *With respect to the embedding ι_1 of Theorem 7.6.10, the set* $\prod_{\sigma\in\mathfrak{S}_0}\mathrm{Perm}_{\bar{r}_\sigma}$ *constitutes a set of representatives of* $\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$.

The following lemma will be used below.

Lemma 7.7.4. *Let $x\in\widetilde{W}$. Write $x=wu^{\lambda}$ with $w\in W$, $\lambda\in X$. Define $w'\in W$ and $\lambda'\in X$ by*

$$w'(i)=n+1-w(n+1-i),\quad 1\le i\le n$$

and

$$\lambda'(i)=e-\lambda(n+1-i),\quad 1\le i\le n.$$

Let $x'=w'u^{\lambda'}$. Then $x'=u^e(x^\tau)^{-1}$.

Proof. This is an easy computation. □

7.8. **The p-rank on a KR stratum.** We make Definitions 3.2.1 and 3.2.3 explicit for the chosen PEL datum.

Definition 7.8.1. *Let R be an $\mathcal{O}_{E_Q}$-algebra. A* self-dual $\mathcal{L}$-set of abelian varieties of determinant $\det_{V_{-i}}$ over R *is a commutative diagram*

$$\begin{tikzcd}
\dots \arrow[r, "\varrho_{-2}"] & A_{-1} \arrow[r, "\varrho_{-1}"] \arrow[d, "\lambda_{-1}"] & A_0 \arrow[r, "\varrho_0"] \arrow[d, "\lambda_0"] & A_1 \arrow[r, "\varrho_1"] \arrow[d, "\lambda_1"] & \dots \\
\dots \arrow[r, "\varrho_1^\vee"] & A_1^\vee \arrow[r, "\varrho_0^\vee"] & A_0^\vee \arrow[r, "\varrho_{-1}^\vee"] & A_{-1}^\vee \arrow[r, "\varrho_{-2}^\vee"] & \dots
\end{tikzcd}$$

in $\mathfrak{A}_R\otimes\mathbb{Z}_{(p)}$ satisfying the following conditions for all $i\in\mathbb{Z}$.

(1) A_i is an abelian scheme over R equipped with an action $\kappa_i:\mathcal{O}_F\otimes\mathbb{Z}_{(p)}\to\mathrm{End}_R(A_i)\otimes\mathbb{Z}_{(p)}$.
(2) $\varrho_i:A_i\to A_{i+1}$ is a $\mathbb{Z}_{(p)}$-isogeny of degree p^f, compatible with κ_i and κ_{i+1}.
(3) There is an isomorphism $\theta_i:A_{n+i}\to A_i$ in $\mathfrak{A}_R\otimes\mathbb{Z}_{(p)}$ such that the composition

$$A_{n+i}\xrightarrow{\theta_i}A_i\xrightarrow{\prod_{j=0}^{n-1}\varrho_{i+j}}A_{n+i}$$

is equal to $\kappa_{n+i}(\pi)$.
(4) $\lambda_i:A_i\to A_{-i}^\vee$ is an isomorphism in $\mathfrak{A}_R\otimes\mathbb{Z}_{(p)}$, compatible with κ_i and $\kappa_{-i}^\vee$. Here $\kappa_i^\vee:\mathcal{O}_F\otimes\mathbb{Z}_{(p)}\to\mathrm{End}_R(A_i^\vee)\otimes\mathbb{Z}_{(p)}$ is defined by $\kappa_i^\vee(x)=\kappa_i(x^*)^\vee$, $x\in\mathcal{O}_F\otimes\mathbb{Z}_{(p)}$.
(5) λ_0 is symmetric.
(6) $\det_{\mathrm{Lie}(A_i)}=\det_{V_{-i}}\otimes_{\mathcal{O}_E}R$.

Remark 7.8.2. *Let R be an $\mathcal{O}_{E_Q}$-algebra and let A/R be an abelian scheme equipped with an action $\kappa:\mathcal{O}_F\otimes\mathbb{Z}_{(p)}\to\mathrm{End}_R(A)\otimes\mathbb{Z}_{(p)}$. Assume that* $\det_{\mathrm{Lie}(A)}=\det_{V_{-i}}\otimes_{\mathcal{O}_E}R$. *Then $\dim_R A_i=ng_0$ by Lemma 2.3.4.*

Recall from Section 3.3 the diagram

$$\begin{tikzcd}
 & \widetilde{\mathcal{A}} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\
\mathcal{A} & & M^{\mathrm{loc}}
\end{tikzcd}$$

of functors on the category of $\mathcal{O}_{E_\mathbb{Q}}$-algebras. Also recall the KR stratification $\mathcal{A}(\mathbb{F}) = \coprod_{x\in \mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x$. We have identified the occurring index set with $\prod_{\sigma\in\mathfrak{S}_0} \mathrm{Perm}_{\overline{r}_\sigma}$ in Corollary 7.7.3. We can then state the following result.

Theorem 7.8.3. *Let $x = (x_\sigma)_\sigma \in \prod_{\sigma\in\mathfrak{S}_0} \mathrm{Perm}_{\overline{r}_\sigma}$. Write $x_\sigma = w_\sigma u^{\lambda_\sigma}$ with $w_\sigma \in W$, $\lambda_\sigma \in X$ and define elements $w'_\sigma \in W$ and $\lambda'_\sigma \in X$ as in Lemma 7.7.4. Then the p-rank on $\mathcal{A}_x$ is constant with value*

$$g \cdot \left| \left\{ 1 \leq i \leq n \middle| \forall \sigma \in \mathfrak{S}_0 \begin{pmatrix} w_\sigma(i) = w'_\sigma(i) = i \wedge \\ \lambda_\sigma(i) = \lambda'_\sigma(i) = 0 \end{pmatrix} \right\} \right|.$$

Proof. In view of Proposition 7.6.8 and Lemma 7.7.4, the statement is contained in Theorem 3.8.3 and Proposition 3.9.7 (leaving the transition between the equal and mixed characteristic situations to the reader). Alternatively one can use Proposition 7.6.8 and Lemma 7.7.4 together with the arguments of the proof of Theorem 5.8.3 to obtain a direct proof of the statement. □

8. The split unitary case

8.1. **The PEL datum.** Let $n \in \mathbb{N}_{\geq 1}$. We start with the PEL datum consisting of the following objects.

(1) An imaginary quadratic extension F/F_0 of a totally real extension $F_0/\mathbb{Q}$. Let $g_0 = [F_0 : \mathbb{Q}]$ and $g = [F : \mathbb{Q}]$, so that $g = 2g_0$.
(2) The non-trivial element $*$ of $\mathrm{Gal}(F/F_0)$.
(3) An n-dimensional F-vector space V.
(4) The symplectic form $(\cdot,\cdot) : V \times V \to \mathbb{Q}$ on the underlying $\mathbb{Q}$-vector space of V constructed as follows: Fix once and for all a $*$-skew-hermitian form $(\cdot,\cdot)' : V \times V \to F$ (i.e. $(av, bw)' = ab^* (v, w)'$ and $(v, w)' = -(w, v)'^*$ for $v, w \in V$, $a, b \in F$). Define $(\cdot,\cdot) = \mathrm{tr}_{F/\mathbb{Q}} \circ (\cdot,\cdot)'$.
(5) The element $J \in \mathrm{End}_{B\otimes\mathbb{R}}(V \otimes \mathbb{R})$ to be specified below.

Remark 8.1.1. *The reductive $\mathbb{Q}$-group G associated with the above PEL datum is described in Remark 6.1.1.*

We assume that $p\mathcal{O}_{F_0} = (\mathcal{P}_0)^e$ for a single prime $\mathcal{P}_0$ of $\mathcal{O}_{F_0}$ and that $\mathcal{P}_0\mathcal{O}_F = \mathcal{P}_+\mathcal{P}_-$ for two distinct primes $\mathcal{P}_\pm$ of $\mathcal{O}_F$. Consequently $\mathcal{P}_- = (\mathcal{P}_+)^*$. Denote by $f_0 = [k_{\mathcal{P}_0} : \mathbb{F}_p]$ the corresponding inertia degree. We fix once and for all a uniformizer π_0 of $\mathcal{O}_{F_0} \otimes \mathbb{Z}_{(p)}$.

For typographical reasons, we denote the ring of integers in $(F_0)_{\mathcal{P}_0}$ by $\mathcal{O}_{\mathcal{P}_0}$. The inclusion $\mathcal{O}_{F_0} \hookrightarrow \mathcal{O}_F$ induces identifications

$$\begin{aligned} \mathcal{O}_F \otimes \mathbb{Z}_p &= \mathcal{O}_{\mathcal{P}_0} \times \mathcal{O}_{\mathcal{P}_0}, \\ F \otimes \mathbb{Q}_p &= (F_0)_{\mathcal{P}_0} \times (F_0)_{\mathcal{P}_0}. \end{aligned} \tag{8.1.2}$$

Here the first (resp. second) factor is always supposed to correspond to $\mathcal{P}_+$ (resp. $\mathcal{P}_-$). Under (8.1.2), the base-change $F \otimes \mathbb{Q}_p \to F \otimes \mathbb{Q}_p$ of $*$ takes the simple form $(F_0)_{\mathcal{P}_0} \times (F_0)_{\mathcal{P}_0} \to (F_0)_{\mathcal{P}_0} \times (F_0)_{\mathcal{P}_0}$, $(a, b) \mapsto (b, a)$.

The identification (8.1.2) further induces a decomposition $V \otimes \mathbb{Q}_p = V_+ \times V_-$ into $(F_0)_{\mathcal{P}_0}$-vector spaces $V_\pm$. The pairing $(\cdot,\cdot)'_{\mathbb{Q}_p}$ decomposes into its restrictions $(\cdot,\cdot)_\pm : V_\pm \times V_\mp \to (F_0)_{\mathcal{P}_0}$. Both $(\cdot,\cdot)_+$ and $(\cdot,\cdot)_-$

are perfect $(F_0)_{\mathcal{P}_0}$-bilinear pairings and they are related by the equation $(v,w)_+ = -(w,v)_-$, $v \in V_+, w \in V_-$.

Denote by $\mathfrak{C}_0 = \mathfrak{C}_{\mathcal{O}_{\mathcal{P}_0}|\mathbb{Z}_p}$ the corresponding inverse different and fix a generator δ_0 of $\mathfrak{C}_0$. We fix bases $(e_{1,\pm},\dots,e_{n,\pm})$ of $V_\pm$ over $(F_0)_{\mathcal{P}_0}$ such that $(e_{i,+}, e_{n+1-j,-})_+ = \delta_0\delta_{ij}$ for $1 \le i,j \le n$.

Let $0 \le i < n$. We denote by $\Lambda_{i,\pm}$ the $\mathcal{O}_{\mathcal{P}_0}$-lattice in $V_\pm$ with basis

$$\mathfrak{E}_{i,\pm} = (\pi_0^{-1}e_{1,\pm},\dots,\pi_0^{-1}e_{i,\pm}, e_{i+1,\pm},\dots,e_{n,\pm}). \tag{8.1.3}$$

For $k \in \mathbb{Z}$ we further define $\Lambda_{nk+i,\pm} = \pi_0^{-k}\Lambda_{i,\pm}$ and we denote by $\mathfrak{E}_{nk+i,\pm}$ the corresponding basis obtained from $\mathfrak{E}_{i,\pm}$. Then $\mathcal{L}_\pm = (\Lambda_{i,\pm})_i$ is a complete chain of $\mathcal{O}_{\mathcal{P}_0}$-lattices in $V_\pm$.

Let $i \in \mathbb{Z}$. We denote by $\rho_{i,\pm} : \Lambda_{i,\pm} \to \Lambda_{i+1,\pm}$ the inclusion and by $\vartheta_{i,\pm} : \Lambda_{n+i,\pm} \to \Lambda_{i,\pm}$ the isomorphism given by multiplication with π_0. Then $(\Lambda_{i,\pm}, \rho_{i,\pm}, \vartheta_{i,\pm})$ is a chain of $\mathcal{O}_{\mathcal{P}_0}$-modules of type $(\mathcal{L}_\pm)$ which, by abuse of notation, we also denote by $\mathcal{L}_\pm$.

For $(i,j) \in \mathbb{Z}\times\mathbb{Z}$ we define $\Lambda_{(i,j)} := \Lambda_{i,+}\times\Lambda_{j,-}$. Then $\Lambda_{(i,j)}$ is an $\mathcal{O}_F\otimes\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$. A basis $\mathfrak{E}_{(i,j)}$ of $\Lambda_{(i,j)}$ over $\mathcal{O}_F\otimes\mathbb{Z}_p$ is given by the diagonal in $\mathfrak{E}_{i,+}\times\mathfrak{E}_{j,-}$. Then $\mathcal{L} = (\Lambda_{i,j})_{(i,j)}$ is a complete multichain of $\mathcal{O}_F\otimes\mathbb{Z}_p$-lattices in $V_{\mathbb{Q}_p}$. For $(i,j) \in \mathbb{Z}\times\mathbb{Z}$ the dual lattice $\Lambda^\vee_{(i,j)} := \{x \in V_{\mathbb{Q}_p} \mid \big(x, \Lambda_{(i,j)}\big)_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ of $\Lambda_{(i,j)}$ is given by $\Lambda_{(-j,-i)}$. Consequently the multichain $\mathcal{L}$ is a self-dual.

Let $(i,j) \in \mathbb{Z}\times\mathbb{Z}$. We denote by $\rho_{(i,j),+} : \Lambda_{(i,j)} \to \Lambda_{(i+1,j)}$, $\rho_{(i,j),-} : \Lambda_{(i,j)} \to \Lambda_{(i,j+1)}$ and $\rho_{(i,j)} : \Lambda_{(i,j)} \to \Lambda_{(i+1,j+1)}$ the inclusions. We denote by $\vartheta_{(i,j),+} : \Lambda_{(n+i,j)} \to \Lambda_{(i,j)}$ (resp. $\vartheta_{(i,j),-} : \Lambda_{(i,n+j)} \to \Lambda_{(i,j)}$, resp. $\vartheta_{(i,j)} : \Lambda_{(n+i,n+j)} \to \Lambda_{(i,j)}$) the isomorphism given by multiplication with π_0 in the first (resp. second, resp. first and second) component. We further denote by $(\cdot,\cdot)_{(i,j)} : \Lambda_{(i,j)} \times \Lambda_{(-j,-i)} \to \mathbb{Z}_p$ the restriction of $(\cdot,\cdot)_{\mathbb{Q}_p}$.

We find that $(\mathcal{L}_+, \mathcal{L}_-)$, equipped with $((\cdot,\cdot)_{(i,j)})_{(i,j)}$, is a polarized multichain of $\mathcal{O}_F\otimes\mathbb{Z}_p$-modules of type $(\mathcal{L})$, which, by abuse of notation, we also denote by $\mathcal{L} = \mathcal{L}^{\mathrm{split}}$.

Denote by $\langle\cdot,\cdot\rangle_{(i,j)} : \Lambda_{(i,j)} \times \Lambda_{(-j,-i)} \to \mathcal{O}_{\mathcal{P}_0}\times\mathcal{O}_{\mathcal{P}_0}$ the restriction of the $*$-hermitian form $(\delta_0^{-1}, -\delta_0^{-1})\,(\cdot,\cdot)'_{\mathbb{Q}_p}$. It is the $*$-sesquilinear form described by the matrix $\widetilde{I}_n$ with respect to $\mathfrak{E}_{(i,j)}$ and $\mathfrak{E}_{(-j,-i)}$.

Denote by Σ_0 the set of all embeddings $F_0 \hookrightarrow \mathbb{R}$ and by Σ the set of all embedding $F \hookrightarrow \mathbb{C}$. The inclusion $\mathcal{O}_{F_0} \hookrightarrow \mathcal{O}_F$ induces an identification of $k_{\mathcal{P}_\pm}/\mathbb{F}_p$ with $k_{\mathcal{P}_0}/\mathbb{F}_p$. We write $\mathfrak{S}_0 = \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$ and also identify $\mathrm{Gal}(k_{\mathcal{P}_\pm}/\mathbb{F}_p)$ with $\mathfrak{S}_0$. Let E' be the Galois closure of F inside $\mathbb{C}$ and choose a prime $\mathcal{Q}'$ of E' over $\mathcal{P}_+$. Consider the decomposition $\Sigma = \Sigma_+ \amalg \Sigma_-$ and the maps $\gamma_0 : \Sigma_0 \to \mathfrak{S}_0$, $\gamma_\pm : \Sigma_\pm \to \mathfrak{S}_0$ of Section 4.4.1. For $\sigma \in \Sigma_0$ we denote by $\tau_{\sigma,\pm}$ the unique lift of σ to $\Sigma_\pm$. Exactly as in Section 6, we define for each $\sigma \in \Sigma_0$ integers r_σ, s_σ with $r_\sigma + s_\sigma = n$,[8] and using these the element $J \in \mathrm{End}_{B\otimes\mathbb{R}}(V\otimes\mathbb{R})$. Denote by $V_{\mathbb{C},-i}$ the $(-i)$-eigenspace of $J_{\mathbb{C}}$. As before, we construct an $\mathcal{O}_F\otimes\mathcal{O}_{E'}$-module M_0 which is finite locally free over $\mathcal{O}_{E'}$, such that $M_0 \otimes_{\mathcal{O}_{E'}} \mathbb{C} = V_{\mathbb{C},-i}$ as $\mathcal{O}_F\otimes\mathbb{C}$-modules.

[8]In Section 6 we have written $\tau_{\sigma,1}$ and $\tau_{\sigma,2}$ instead of $\tau_{\sigma,+}$ and $\tau_{\sigma,-}$, respectively.

8.2. **The special fiber of the determinant morphism.** For $\sigma \in \mathfrak{S}_0$ we write

$$\overline{r}_\sigma = \sum_{\sigma' \in \gamma_0^{-1}(\sigma)} r_{\sigma'} \quad \text{and} \quad \overline{s}_\sigma = \sum_{\sigma' \in \gamma_0^{-1}(\sigma)} s_{\sigma'}.$$

As the fibers of γ_0 have cardinality e, it follows that $\overline{r}_\sigma + \overline{s}_\sigma = ne$.

We fix once and for all an embedding $\iota_{\mathcal{Q}'} : k_{\mathcal{Q}'} \hookrightarrow \mathbb{F}$. We consider $\mathbb{F}$ as an $\mathcal{O}_{E'}$-algebra with respect to the composition $\mathcal{O}_{E'} \xrightarrow{\rho_{\mathcal{Q}'}} k_{\mathcal{Q}'} \overset{\iota_{\mathcal{Q}'}}{\hookrightarrow} \mathbb{F}$. Also $\iota_{\mathcal{Q}'}$ induces an embedding $\iota_{\mathcal{P}_0} : k_{\mathcal{P}_0} \hookrightarrow \mathbb{F}$ and thereby an identification of the set of all embeddings $k_{\mathcal{P}_0} \hookrightarrow \mathbb{F}$ with $\mathfrak{S}_0$.

Consider the isomorphism

$$\mathcal{O}_F \otimes \mathbb{F} = \prod_{\sigma \in \mathfrak{S}_0} \mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e) \tag{8.2.1}$$

obtained from (8.1.2) and the isomorphism from Section 4.3.

Proposition 8.2.2. *Let $x \in \mathcal{O}_F$ and let $\big((q_{\sigma,+}, q_{\sigma,-})\big)_\sigma \in \prod_{\sigma \in \mathfrak{S}_0} \mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e)$ be the element corresponding to $x \otimes 1$ under* (8.2.1). *Then*

$$\chi_{\mathbb{F}}(x | M_0 \otimes_{\mathcal{O}_{E'}} \mathbb{F}) = \prod_{\sigma \in \mathfrak{S}_0} \big(T - q_{\sigma,+}(0)\big)^{\overline{s}_\sigma} \big(T - q_{\sigma,-}(0)\big)^{\overline{r}_\sigma}$$

in $\mathbb{F}[T]$.

Proof. The definition of M_0 gives

$$\chi_{\mathcal{O}_{E'}}(x | M_0) = \prod_{\sigma \in \Sigma_0} \big(T - \tau_{\sigma,+}(x)\big)^{s_\sigma} \big(T - \tau_{\sigma,-}(x)\big)^{r_\sigma}.$$

By (4.4.7) we have

$$\rho_{\mathcal{Q}'}(\tau_{\sigma,\pm}(x)) = \gamma_0(\sigma)(\rho_{\mathcal{P}_\pm}(x)).$$

Consequently

$$(\chi_{\mathcal{O}_{E'}}(x | M_0))^{\rho_{\mathcal{Q}'}} = \prod_{\sigma \in \mathfrak{S}_0} \big(T - \sigma(\rho_{\mathcal{P}_+}(x))\big)^{\overline{s}_\sigma} \cdot \big(T - \sigma(\rho_{\mathcal{P}_-}(x))\big)^{\overline{r}_\sigma}.$$

The claim then follows from the equality $q_{\sigma,\pm}(0) = (\iota_{\mathcal{P}_0} \circ \sigma)(\rho_{\mathcal{P}_\pm}(x))$, $\sigma \in \mathfrak{S}_0$. □

Denote by $E = \mathbb{Q}(\mathrm{tr}_{\mathbb{C}}(x \otimes 1 | V_{-i});\ x \in F)$ the reflex field and define $\mathcal{Q} = \mathcal{Q}' \cap \mathcal{O}_E$. By Proposition 2.3.5 the morphism $\det_{V_{-i}}$ is defined over $\mathcal{O}_E$, and we also denote by $\det_{V_{-i}}$ the corresponding morphism over $\mathcal{O}_E$.

8.3. **The local model.** For the chosen PEL datum, Definition 3.3.2 amounts to the following.

Definition 8.3.1. *The local model $M^{\mathrm{loc}} = M^{\mathrm{loc,split}}$ is the functor on the category of $\mathcal{O}_{E_{\mathcal{Q}}}$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_{(i,j)})_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}$ of $\mathcal{O}_F \otimes R$-submodules $t_{(i,j)} \subset \Lambda_{(i,j),R}$ satisfying the following conditions for all $(i,j) \in \mathbb{Z} \times \mathbb{Z}$.*

(a) *$\rho_{(i,j),+,R}(t_{(i,j)}) \subset t_{(i+1,j)}$ and $\rho_{(i,j),-,R}(t_{(i,j)}) \subset t_{(i,j+1)}$.*
(b) *The quotient $\Lambda_{(i,j),R}/t_{(i,j)}$ is a finite locally free R-module.*

(c) *We have an equality*

$$\det_{\Lambda_{(i,j),R}/t_{(i,j)}} = \det_{V_{-i}} \otimes_{\mathcal{O}_E} R$$

of morphisms $V_{\mathcal{O}_F \otimes R} \to \mathbb{A}^1_R$.

(d) *Under the pairing* $(\cdot,\cdot)_{(i,j),R} : \Lambda_{(i,j),R} \times \Lambda_{(-j,-i),R} \to R$, *the submodules* $t_{(i,j)}$ *and* $t_{(-j,-i)}$ *pair to zero.*

(e) $\vartheta_{(i,j),+,R}(t_{(n+i,j)}) = t_{(i,j)}$ *and* $\vartheta_{(i,j),-,R}(t_{(i,n+j)}) = t_{(i,j)}$.

Remark 8.3.2. *Let* R *be an* $\mathcal{O}_{E_\mathcal{Q}}$*-algebra and let* $(t_{(i,j)})_{(i,j)} \in M^{\mathrm{loc}}(R)$. *For* $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, *the decomposition* (8.1.2) *induces a decomposition* $t_{(i,j)} = t_{(i,j),+} \times t_{(i,j),-}$ *into* $\mathcal{O}_{\mathcal{P}_0} \otimes_{\mathbb{Z}_p} R$*-submodules* $t_{(i,j),+} \subset \Lambda_{i,+,R}$ *and* $t_{(i,j),-} \subset \Lambda_{j,-,R}$. *As in Remark 3.3.4 one sees that* $t_{(i,j),+}$ *(resp.* $t_{(i,j),-}$*) is independent of* j *(resp.* i*). Writing* $t_{i,+} = t_{(i,j),+}$ *and* $t_{j,-} = t_{(i,j),-}$, *the tuple* $(t_{(i,j)})_{(i,j)}$ *is determined by the pair of tuples* $((t_{i,+})_i, (t_{j,-})_j)$.

Recall from Section 7 the chain $\mathcal{L}^{\mathrm{inert}}$ and the functor $M^{\mathrm{loc,inert}}$. The identifications (7.2.1) and (8.2.1), together with our choices of bases, give rise to a canonical identification of the tuple $(\Lambda_{(i,i),\mathbb{F}}, \rho_{(i,i),\mathbb{F}}, \vartheta_{(i,i),\mathbb{F}}, (\cdot,\cdot)_{(i,i),\mathbb{F}})_i$ with the chain $\mathcal{L}^{\mathrm{inert}} \otimes_{\mathbb{Z}_p} \mathbb{F}$.

We can then state the following result.

Proposition 8.3.3. (1) *The morphism* $M^{\mathrm{loc,split}}_{\mathbb{F}} \to M^{\mathrm{loc,inert}}_{\mathbb{F}}$ *given on* R*-valued points by*

$$M^{\mathrm{loc,split}}_{\mathbb{F}}(R) \to M^{\mathrm{loc,inert}}_{\mathbb{F}}(R),$$
$$(t_{(i,j)})_{(i,j)} \mapsto (t_{(i,i)})_i$$

is an isomorphism.

(2) *The morphism* $\mathrm{Aut}(\mathcal{L}^{\mathrm{split}})_{\mathbb{F}} \to \mathrm{Aut}(\mathcal{L}^{\mathrm{inert}})_{\mathbb{F}}$ *given on* R*-valued points by*

$$\mathrm{Aut}(\mathcal{L}^{\mathrm{split}})_{\mathbb{F}}(R) \to \mathrm{Aut}(\mathcal{L}^{\mathrm{inert}})_{\mathbb{F}}(R),$$
$$(\varphi_{(i,j)})_{(i,j)} \mapsto (\varphi_{(i,i)})_i$$

is an isomorphism.

Proof. Clear in view of Remark 8.3.2 and Propositions 2.3.7, 7.2.2 and 8.2.2. □

Consequently all the statements about $M^{\mathrm{loc,inert}}_{\mathbb{F}}$ from Section 7 are also valid for $M^{\mathrm{loc,split}}_{\mathbb{F}}$.

8.4. **The p-rank on a KR stratum.** Let R be a $\mathbb{Z}_p$-algebra, let A and B be abelian schemes over R with actions $\kappa_A : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A) \otimes \mathbb{Z}_{(p)}$ and $\kappa_B : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(B) \otimes \mathbb{Z}_{(p)}$ and let $\varrho : A \to B$ be a $\mathbb{Z}_{(p)}$-isogeny compatible with κ_A and κ_B. As in Section 3.5, the identification (8.1.2) induces a decomposition $\ker \varrho = (\ker \varrho)_+ \times (\ker \varrho)_-$ into finite locally free group schemes $(\ker \varrho)_\pm$. We define

$$\deg_\pm \varrho := \mathrm{rk}(\ker \varrho)_\pm.$$

We make Definitions 3.2.1 and 3.2.3 explicit for the chosen PEL datum. To make the notation more concise, define for $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ tuples $(i,j)_+ = (i+1,j)$, $(i,j)_- = (i,j+1)$ and $-(i,j) = (-i,-j)$.

Definition 8.4.1. *Let R be an $\mathcal{O}_{E_{\mathbb{Q}}}$-algebra. A* self-dual $\mathcal{L}$-set of abelian varieties of determinant $\det_{V_{-i}}$ over R *is a tuple*

$$(A_{(i,j)}, \varrho_{(i,j),\pm}, \lambda_{(i,j)})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$$

satisfying the following conditions for all $(i,j) \in \mathbb{Z}\times\mathbb{Z}$.

(1) $A_{(i,j)}$ is an abelian scheme over R equipped with an action $\kappa_{(i,j)} : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A_{(i,j)}) \otimes \mathbb{Z}_{(p)}$,

(2) $\varrho_{(i,j),\pm} : A_{(i,j)} \to A_{(i,j)_\pm}$ is a $\mathbb{Z}_{(p)}$-isogeny with $\deg_\pm \varrho_{(i,j),\pm} = p^{f_0}$ and $\deg_\mp \varrho_{(i,j),\pm} = 1$, compatible with $\kappa_{(i,j)}$ and $\kappa_{(i,j)_\pm}$. Also the following diagram commutes.

$$\begin{tikzcd}
A_{(i,j)} \arrow[r, "\varrho_{(i,j),+}"] \arrow[d, "\varrho_{(i,j),-}"'] & A_{(i+1,j)} \arrow[d, "\varrho_{(i+1,j),-}"] \\
A_{(i,j+1)} \arrow[r, "\varrho_{(i,j+1),+}"] & A_{(i+1,j+1)}.
\end{tikzcd} \tag{8.4.2}$$

(3) Denote by $\nu_\pm$ the discrete valuation on F associated with $\mathcal{P}_\pm$. We require that for each $x \in \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$ there is an isomorphism $\theta_{(i,j),x} : A_{(i+\nu_+(x),j+\nu_-(x))} \to A_{(i,j)}$ in $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$, such that the composition

$$A_{(i+\nu_+(x),j+\nu_-(x))} \xrightarrow{\theta_{(i,j),x}} A_{(i,j)} \xrightarrow{\varrho} A_{(i+\nu_+(x),j+\nu_-(x))}$$

is equal to $\kappa_{(i+\nu_+(x),j+\nu_-(x))}(x)$. Here ϱ is an appropriate composition, independent of the ordering of the factors by the commutativity of (8.4.2)*.*

(4) $\lambda_{(i,j)} : A_{(i,j)} \to A^\vee_{(-j,-i)}$ is an isomorphism in $\mathfrak{A}_R \otimes \mathbb{Z}_{(p)}$, compatible with $\kappa_{(i,j)}$ and $\kappa^\vee_{(-j,-i)}$. Here $\kappa^\vee_{(i,j)} : \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_R(A^\vee_{(i,j)}) \otimes \mathbb{Z}_{(p)}$ is defined by $\kappa^\vee_{(i,j)}(x) = \kappa_{(i,j)}(x^*)^\vee$, $x \in \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$. Also the following diagram commutes.

$$\begin{tikzcd}
A_{(i,j)} \arrow[r, "\varrho_{(i,j),\pm}"] \arrow[d, "\lambda_{(i,j)}"'] & A_{(i,j)_\pm} \arrow[d, "\lambda_{(i,j)_\pm}"] \\
A^\vee_{-(j,i)} \arrow[r, "\varrho^\vee_{-((j,i)_\mp),\mp}"] & A^\vee_{-((j,i)_\mp)}.
\end{tikzcd}$$

(5) $\lambda_{(0,0)}$ is symmetric.

(6) $\det_{\mathrm{Lie}(A_{(i,j)})} = \det_{V_{-i}} \otimes_{\mathcal{O}_E} R$.

Recall from Section 3.3 the diagram

$$\begin{tikzcd}
& \widetilde{\mathcal{A}} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\
\mathcal{A} & & M^{\mathrm{loc}}
\end{tikzcd}$$

of functors on the category of $\mathcal{O}_{E_{\mathbb{Q}}}$-algebras. Also recall the KR stratification $\mathcal{A}(\mathbb{F}) = \coprod_{x\in \mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x$. We have identified the occurring index set with $\prod_{\sigma\in\mathfrak{S}_0} \mathrm{Perm}_{\overline{r}_\sigma}$ in Corollary 7.7.3. We can then state the following result.

Theorem 8.4.3. *Let $x = (x_\sigma)_\sigma \in \prod_{\sigma\in\mathfrak{S}_0} \mathrm{Perm}_{\overline{r}_\sigma}$. Write $x_\sigma = w_\sigma u^{\lambda_\sigma}$ with $w_\sigma \in W$, $\lambda_\sigma \in X$. Then the p-rank on $\mathcal{A}_x$ is constant with value*

$$\begin{aligned} &g_0 \cdot |\{1 \le i \le n \mid \forall\sigma \in \mathfrak{S}_0 (w_\sigma(i) = i \wedge \lambda_\sigma(i) = 0)\}| \\ &+ g_0 \cdot |\{1 \le i \le n \mid \forall\sigma \in \mathfrak{S}_0 (w_\sigma(i) = i \wedge \lambda_\sigma(i) = e)\}|. \end{aligned}$$

Proof. Define elements $w'_\sigma \in W$ and $\lambda'_\sigma \in X$ as in Lemma 7.7.4. In view of Proposition 7.6.8 and Lemma 7.7.4, applying Theorem 3.8.3, Proposition 3.9.7 and Lemma 3.9.9 (while again leaving the transition between the equal and mixed characteristic situations to the reader) yields that the p-rank on $\mathcal{A}_x$ is constant with value

$$\begin{aligned} &g_0 \cdot |\{1 \le i \le n \mid \forall\sigma \in \mathfrak{S}_0 (w_\sigma(i) = i \wedge \lambda_\sigma(i) = 0)\}| \\ &+ g_0 \cdot |\{1 \le i \le n \mid \forall\sigma \in \mathfrak{S}_0 (w'_\sigma(i) = i \wedge \lambda'_\sigma(i) = 0)\}|. \end{aligned}$$

This implies the statement in view of the explicit formulas of Lemma 7.7.4.

Let us also give a direct proof. Let $t = (t_{(i,j)})_{(i,j)} \in M^{\mathrm{loc}}(\mathbb{F})$ and let $(t_{i,\pm})_i$ be the two associated tuples of Remark 8.3.2. Let $(i,j) \in \mathbb{Z}\times\mathbb{Z}$. Then we have the following equivalences.

$$\begin{aligned} \Lambda_{(i,j),\mathbb{F}} = \operatorname{im}\rho_{(i-1,j),+,\mathbb{F}} + t_{(i,j)} &\Leftrightarrow \Lambda_{i,+,\mathbb{F}} = \operatorname{im}\rho_{i-1,+,\mathbb{F}} + t_{i,+}, \\ \Lambda_{(i,j),\mathbb{F}} = \operatorname{im}\rho_{(i,j-1),-,\mathbb{F}} + t_{(i,j)} &\Leftrightarrow \Lambda_{j,-,\mathbb{F}} = \operatorname{im}\rho_{j-1,-,\mathbb{F}} + t_{j,-}. \end{aligned}$$

Assume now that t lies in the $\mathrm{Aut}(\mathcal{L}^{\mathrm{inert}})(\mathbb{F})$-orbit corresponding to x under the identifications of Corollary 7.7.3. Consider the chain of neighbors

$$\Lambda_{(0,0)} \subset \Lambda_{(1,0)} \subset \cdots \subset \Lambda_{(n,0)} \subset \Lambda_{(n,1)} \subset \cdots \subset \Lambda_{(n,n)} = \pi_0^{-1}\Lambda_{(0,0)},$$

and let $1 \le i \le n$. By Propositions 3.5.4 and 3.5.5 the claim of the theorem follows from the following equivalences, noting the explicit formulas in Lemma 7.7.4.

$$\begin{aligned} \Lambda_{i,+,\mathbb{F}} = \operatorname{im}\rho_{i-1,+,\mathbb{F}} + t_{i,+} &\Leftrightarrow \forall\sigma \in \mathfrak{S}_0(w_\sigma(i) = i \wedge \lambda_\sigma(i) = 0), \\ \Lambda_{i,-,\mathbb{F}} = \operatorname{im}\rho_{i-1,-,\mathbb{F}} + t_{i,-} &\Leftrightarrow \forall\sigma \in \mathfrak{S}_0(w'_\sigma(i) = i \wedge \lambda'_\sigma(i) = 0). \end{aligned}$$

In view of Proposition 7.6.8 and Lemma 7.7.4, these equivalences are established in the proof of Theorem 5.8.3. □

8.5. **An application to the dimension of the p-rank 0 locus.** Assume from now on that $F_0 = \mathbb{Q}$, so that $F/\mathbb{Q}$ is an imaginary quadratic extension in which p splits. We write $r = r_{\mathrm{id}_\mathbb{Q}}$ and $s = s_{\mathrm{id}_\mathbb{Q}}$, so that $n = r + s$. Also write $I_n = \{1, \dots, n\}$.

Fix a sufficiently small compact open subgroup $C^p \subset G(\mathbb{A}_f^p)$. Exactly as in Section 5.9 we define a new moduli problem $\mathcal{A}_{C^p}$ by adding a polarization and a C^p-level structure to $\mathcal{A}$. We obtain a corresponding local model diagram

$$\begin{tikzcd} & \widetilde{\mathcal{A}}_{C^p} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\ \mathcal{A}_{C^p} & & M^{\mathrm{loc}} \end{tikzcd}$$

which induces the KR stratification

$$\mathcal{A}_{C^p}(\mathbb{F}) = \coprod_{x\in\mathrm{Perm}_r} \mathcal{A}_{C^p,x}.$$

Note that the moduli problem $\mathcal{A}_{C^p}$ is a special case of the "fake" unitary case considered in [13]. Concretely, the moduli problem defined in [13, §5.2] specializes to $\mathcal{A}_{C^p}$ for $D = F$.

Denote by $\ell : \widetilde{W} \to \mathbb{N}$ the length function defined in [7, §2.1].

Lemma 8.5.1. *Let $x \in \mathrm{Perm}_r$. The subset $\mathcal{A}_{C^p,x} \subset \mathcal{A}_{C^p}(\mathbb{F})$ is locally closed and we equip it with the reduced scheme structure. Then $\mathcal{A}_{C^p,x}$ is a smooth variety over $\mathbb{F}$. It is equidimensional of dimension $\ell(x)$.*

Proof. We know from [13, Lemma 13.1] that $\mathcal{A}_{C^p,x}$ is non-empty. The rest of the proof is identical to the one of Lemma 5.9.2, referring to Theorem 7.6.10 in place of Theorem 5.6.10. □

Let us state Theorem 8.4.3 in this special case.

Theorem 8.5.2. *Let $x \in \mathrm{Perm}_r$. Write $x = wu^\lambda$ with $w \in W, \lambda \in X$. Then the p-rank on $\mathcal{A}_x$ is constant with value $|\mathrm{Fix}(w)|$, where $\mathrm{Fix}(w) = \{i \in I_n \mid w(i) = i\}$.*

We want to use this result to compute the dimension of the p-rank 0 locus in $\mathcal{A}_{C^p,\mathbb{F}}$. We do this by copying the approach of [8, §8].

Denote by $\mathrm{Perm}_r^{(0)}$ the subset of those $x \in \mathrm{Perm}_r$ such that the p-rank on $\mathcal{A}_{C^p,x}$ is equal to 0. Denote by $W_{n,r}$ the subset of those $w \in W$ satisfying $\mathrm{Fix}(w) = \emptyset$ and

$$|\{i \in I_n \mid w(i) < i\}| = r. \tag{8.5.3}$$

Lemma 8.5.4 (Cf. [8, Lemma 8.1])**.** *The canonical projection $\widetilde{W} \to W$ induces a bijection $\mathrm{Perm}_r^{(0)} \to W_{n,r}$. Its inverse is given by $w \mapsto u^{\lambda(w)}w$ with*

$$\lambda(w)(i) = \begin{cases} 0, & \text{if } w^{-1}(i) > i \\ 1, & \text{if } w^{-1}(i) < i \end{cases}, \quad i \in I_n.$$

Proof. This is an easy combinatorial consequence of Theorem 8.5.2 and the interpretation of Perm_r in terms of extended alcoves, see Section 7.7. □

Define for $\sigma \in S_n$ the following sets and natural numbers.

$$\begin{aligned} A_\sigma &= \{(i,j) \in (I_n)^2 \mid i < j < \sigma(j) < \sigma(i)\}, && a_\sigma = |A_\sigma|, \\ \tilde{A}_\sigma &= \{(i,j) \in (I_n)^2 \mid \sigma(j) < \sigma(i) < i < j\}, && \tilde{a}_\sigma = |\tilde{A}_\sigma|, \\ B_\sigma &= \{(i,j) \in (I_n)^2 \mid \sigma(i) < i < j < \sigma(j)\}, && b_\sigma = |B_\sigma|, \\ \tilde{B}_\sigma &= \{(i,j) \in (I_n)^2 \mid i < \sigma(i) < \sigma(j) < j\}, && \tilde{b}_\sigma = |\tilde{B}_\sigma|, \\ N_\sigma &= a_\sigma + \tilde{a}_\sigma + b_\sigma + \tilde{b}_\sigma. \end{aligned}$$

Note that $N_\sigma = N_{\sigma^{-1}}$ in view of the obvious identities $a_\sigma = \tilde{a}_{\sigma^{-1}}$ and $b_\sigma = \tilde{b}_{\sigma^{-1}}$.

Proposition 8.5.5. *Let $u^\lambda w \in \mathrm{Perm}_r^{(0)}$, $w \in W, \lambda \in X$. Then $\ell(u^\lambda w) = N_w$.*

Proof. Denote by e_i the i-th standard basis vector of $\mathbb{Z}^n$. The positive roots $\beta > 0$ of GL_n are given by $\beta_{ij} = e_i - e_j,\ 1 \le i < j \le n$. Denote by $\langle \cdot, \cdot \rangle$ the standard symmetric pairing on $\mathbb{Z}^n$, determined by $\langle e_i, e_j \rangle = \delta_{ij}$. We use the following version of the Iwahori-Matsumoto formula to compute the length in question, see [8, (8.1)].

$$\ell(u^\lambda w) = \sum_{\substack{\beta>0 \\ w^{-1}\beta>0}} |\langle \beta, \lambda \rangle| + \sum_{\substack{\beta>0 \\ w^{-1}\beta<0}} |\langle \beta, \lambda \rangle + 1|. \tag{8.5.6}$$

By Lemma 8.5.4 we have

$$\langle \beta_{ij}, \lambda \rangle = \begin{cases} \langle (1,-1),(1,0) \rangle = 1, & \text{if } w^{-1}(i) < i \text{ and } w^{-1}(j) > j, \\ \langle (1,-1),(0,1) \rangle = -1, & \text{if } w^{-1}(i) > i \text{ and } w^{-1}(j) < j, \\ \langle (1,-1),(0,0) \rangle = 0, & \text{if } w^{-1}(i) > i \text{ and } w^{-1}(j) > j, \\ \langle (1,-1),(1,1) \rangle = 0, & \text{if } w^{-1}(i) < i \text{ and } w^{-1}(j) < j. \end{cases} \tag{8.5.7}$$

The first sum in (8.5.6) runs over those β_{ij} with $w^{-1}(i) < w^{-1}(j)$ and we see that we get a nonzero contribution of the summand corresponding to β_{ij} if and only if $(i,j) \in B_{w^{-1}} \cup \tilde{B}_{w^{-1}}$. Each of these contributions is equal to 1.

The second sum in (8.5.6) runs over those β_{ij} with $w^{-1}(i) > w^{-1}(j)$ and we see that we get a nonzero contribution of the summand corresponding to β_{ij} if and only if $(i,j) \in A_{w^{-1}} \cup \tilde{A}_{w^{-1}}$. Each of these contributions is equal to 1.

Thus $\ell(u^\lambda w) = N_{w^{-1}}$. The equality $N_{w^{-1}} = N_w$ has already been noted above. □

Define

$$N_{n,r} := \min\big((r-1)(n-r), r(n-r-1)\big) = \begin{cases} (r-1)(n-r), & \text{if } r \leq n/2, \\ r(n-r-1), & \text{if } r \geq n/2. \end{cases}$$

Proposition 8.5.8. *Let $\sigma \in W_{n,r}$. Then $N_\sigma \leq N_{n,r}$.*

Proof. Consider the set $M = \{(n,r,i_0) \in \mathbb{N}^3 \mid 1 \leq r \leq n-1,\ 2 \leq i_0 \leq n\}$ and equip it with the lexicographical ordering $<$, which is a well-ordering on M. For $(n,r,i_0) \in M$ we define $W_{n,r,i_0} = \{\sigma \in W_{n,r} \mid \min\{2 \leq i \leq n \mid \sigma(i) < i\} = i_0\}$. Denote by $\mathcal{P}(n,r,i_0)$ the following statement.

$$\forall \sigma \in W_{n,r,i_0} : \ N_\sigma \leq N_{n,r}.$$

We will prove it by induction on (n,r,i_0).

Let $(n,r,i_0) \in M$, $\sigma \in W_{(n,r,i_0)}$ and assume that $\mathcal{P}(n',r',i_0')$ is true for all $(n',r',i_0') \in M$ with $(n',r',i_0') < (n,r,i_0)$. Set $\sigma' = \sigma \circ (i_0 - 1, i_0)$. We distinguish four cases.

Case 1: $\sigma(i_0) < i_0 - 1$ and $\sigma(i_0 - 1) > i_0$.

In this case $\sigma' \in W_{(n,r,i_0-1)}$ and it is easily checked that $N_\sigma = N_{\sigma'}$, so that the required inequality holds by induction hypothesis.

Case 2: $\sigma(i_0) = i_0 - 1$ and $\sigma(i_0 - 1) > i_0$.

We read off the following identities.

$$\begin{aligned} a_\sigma &= a_{\sigma'}, \\ \tilde{a}_\sigma &= \tilde{a}_{\sigma'} + |\{j \in I_n \mid \sigma(j) < i_0 - 1 < i_0 < j\}|, \\ b_\sigma &= b_{\sigma'} + |\{j \in I_n \mid i_0 - 1 < i_0 < j < \sigma(j)\}|, \\ \tilde{b}_\sigma &= \tilde{b}_{\sigma'} + |\{i \in I_n \mid i < \sigma(i) < i_0 - 1 < i_0\}|. \end{aligned}$$

Identifying $\{1,\dots,\widehat{i_0-1},\dots,n\}$ with $\{1,\dots,n-1\}$, we consider the restriction $\sigma'\big|_{\{1,\dots,\widehat{i_0-1},\dots,n\}}$ as an element of $W_{n-1,r-1,j_0}$ for some j_0. By induction hypothesis we know that $N_{\sigma'} = N_{\sigma'|_{\{1,\dots,\widehat{i_0-1},\dots,n\}}} \le N_{n-1,r-1}$. In view of $N_{n,r} - N_{n-1,r-1} \ge n-r-1$ it therefore suffices to show that the following sum is bounded by $n-r-1$.

$$\begin{aligned} &|\{j \in I_n \mid \sigma(j) < i_0 - 1 < i_0 < j\}| \\ &+ |\{j \in I_n \mid i_0 < j < \sigma(j)\}| \\ &+ |\{i \in I_n \mid i < \sigma(i) < i_0 - 1\}|. \end{aligned} \tag{8.5.9}$$

First note that

$$|\{j \in I_n \mid \sigma(j) < i_0 - 1 < i_0 < j\}| + |\{i \in I_n \mid i < \sigma(i) < i_0 - 1\}| \le i_0 - 2,$$

as σ maps both sets in question into I_{i_0-2}. By the definition of i_0 we have $I_{i_0-1} \subset \{i \in I_n \mid i < \sigma(i)\}$, so that (8.5.3) implies

$$|\{j \in I_n \mid i_0 < j < \sigma(j)\}| \le n - r - (i_0 - 1).$$

Combining these two inequalities, we obtain the desired bound for (8.5.9).

Case 3: $\sigma(i_0) < i_0 - 1$ and $\sigma(i_0 - 1) = i_0$.

We read off the following identities.

$$\begin{aligned} a_\sigma &= a_{\sigma'} + |\{i \in I_n \mid i < i_0 - 1 < i_0 < \sigma(i)\}|, \\ \tilde{a}_\sigma &= \tilde{a}_{\sigma'}, \\ b_\sigma &= b_{\sigma'}, \\ \tilde{b}_\sigma &= \tilde{b}_{\sigma'} + |\{j \in I_n \mid i_0 - 1 < i_0 < \sigma(j) < j\}|. \end{aligned}$$

Identifying $\{1,\dots,\widehat{i_0},\dots,n\}$ with $\{1,\dots,n-1\}$, we consider $\sigma'\big|_{\{1,\dots,\widehat{i_0},\dots,n\}}$ as an element of W_{n-1,r,j_0} for some j_0. By induction hypothesis we know that $N_{\sigma'} = N_{\sigma'|_{\{1,\dots,\widehat{i_0},\dots,n\}}} \le N_{n-1,r}$. In view of $N_{n,r} - N_{n-1,r} \ge r - 1$ it suffices to show that

$$|\{i \in I_n \mid i < i_0 - 1 < i_0 < \sigma(i)\}| + |\{j \in I_n \mid i_0 < \sigma(j) < j\}| \le r - 1.$$

In view of (8.5.3) this is equivalent to

$$|\{i \in I_n \mid i < i_0 - 1 < i_0 < \sigma(i)\}| \le |\{j \in I_n \mid \sigma(j) \le i_0 - 1 < i_0 < j\}|.$$

We claim that even equality holds. Let us first look at the right-hand side. The following equalities hold.

$$\begin{aligned} |\{j \in I_n \mid \sigma(j) \le i_0 - 1 < i_0 < j\}| &= |\{i \in I_n \mid i \le i_0 - 1 < i_0 < \sigma^{-1}(i)\}| \\ &= |I_{i_0-1} - \sigma(\{i \in I_{i_0} \mid \sigma(i) \le i_0 - 1\})| \\ &= |I_{i_0-1}| - |\{i \in I_{i_0} \mid \sigma(i) \le i_0 - 1\}|. \end{aligned}$$

On the other hand we have

$$|\{i \in I_n \mid i < i_0 - 1 < i_0 < \sigma(i)\}| = |I_{i_0-1}| - |\{i \in I_{i_0-1} \mid \sigma(i) \le i_0\}|.$$

It thus suffices to note the equality

$$|\{i \in I_{i_0-1} \mid \sigma(i) \le i_0\}| = |\{i \in I_{i_0} \mid \sigma(i) \le i_0 - 1\}|.$$

Case 4: $\sigma(i_0) = i_0 - 1$ and $\sigma(i_0 - 1) = i_0$.

We read off the following identities.

$$\begin{aligned}
a_\sigma &= a_{\sigma'} + |\{i \in I_n \mid i < i_0 - 1 < i_0 < \sigma(i)\}|, \\
\tilde{a}_\sigma &= \tilde{a}_{\sigma'} + |\{j \in I_n \mid \sigma(j) < i_0 - 1 < i_0 < j\}|, \\
b_\sigma &= b_{\sigma'} + \{j \in I_n \mid i_0 - 1 < i_0 < j < \sigma(j)\}, \\
\tilde{b}_\sigma &= \tilde{b}_{\sigma'} + |\{j \in I_n \mid i_0 - 1 < i_0 < \sigma(j) < j\}| \\
&\qquad + |\{i \in I_n \mid i < \sigma(i) < i_0 - 1 < i_0\}|.
\end{aligned}$$

Identifying $\{1, \dots, \widehat{i_0 - 1, i_0}, \dots, n\}$ with $\{1, \dots, n-2\}$, we consider the restriction $\sigma'|_{\{1,\dots,\widehat{i_0-1,i_0},\dots,n\}}$ as an element of $W_{n-2,r-1,j_0}$ for some j_0. By induction hypothesis we know that $N_{\sigma'} = N_{\sigma'|_{\{1,\dots,\widehat{i_0-1,i_0},\dots,n\}}} \le N_{n-2,r-1}$. In view of $N_{n,r} - N_{n-2,r-1} = n - 2$ it therefore suffices to show that the following sum is bounded by $n-2$.

$$\begin{aligned}
&|\{i \in I_n \mid i < i_0 - 1 < i_0 < \sigma(i)\}| \\
+&|\{j \in I_n \mid \sigma(j) < i_0 - 1 < i_0 < j\}| \\
+&\{j \in I_n \mid i_0 < j < \sigma(j)\} \\
+&|\{j \in I_n \mid i_0 < \sigma(j) < j\}| \\
+&|\{i \in I_n \mid i < \sigma(i) < i_0 - 1\}|.
\end{aligned}$$

This is trivial as the sets in question are pairwise disjoint and their union is equal to $I_n - \{i_0 - 1, i_0\}$. ☐

Proposition 8.5.10. *We have*

$$\max_{\sigma \in W_{n,r}} N_\sigma = N_{n,r}.$$

Proof. It suffices to show that there is a $\sigma \in W_{n,r}$ satisfying $N_\sigma = N_{n,r}$. As $W_{n,r} \to W_{n,n-r},\ \sigma \mapsto \sigma^{-1}$ is a bijection and as $N_\sigma = N_{\sigma^{-1}}$, we may assume that $r \le n/2$. One easily checks that

$$\sigma = (1,2)(3,4)\cdots(2(r-1)-1, 2(r-1))(2r-1, 2r, 2r+1, \dots, n) \in W_{n,r}$$

satisfies $N_\sigma = (r-1)(n-r) = N_{n,r}$. ☐

Denote by $\mathcal{A}_{C^p}^{(0)} \subset \mathcal{A}_{C^p}(\mathbb{F})$ the subset where the p-rank of the underlying abelian variety is equal to 0. It is a closed subset and we equip it with the reduced scheme structure. From the discussion above we obtain the following result.

Theorem 8.5.11. $\dim \mathcal{A}_{C^p}^{(0)} = \min\big((r-1)(n-r), r(n-r-1)\big)$.

Appendix A. Isogenies of abelian schemes

Let S be any scheme. Denote by $\mathfrak{C}_S$ the category of abelian fppf sheaves on S. If R is a ring we write $\mathfrak{C}_R$ instead of $\mathfrak{C}_{\operatorname{Spec} R}$. Let $\mathcal{F} \in \operatorname{Ob} \mathfrak{C}_S$ and $n \in \mathbb{N}$. We denote by $[n]_\mathcal{F} : \mathcal{F} \to \mathcal{F}$ the multiplication by n. We set $\mathcal{F}[n] = \ker[n]_\mathcal{F}$ and $\mathcal{F}[n^\infty] = \operatorname{colim}_{k \in \mathbb{N}} \mathcal{F}[n^k]$, where the transition map $F[n^k] \to F[n^{k+1}]$ is the natural inclusion of subsheaves of $\mathcal{F}$.

All group schemes will be assumed to be commutative.

Theorem A.1 (Deligne, [41, Theorem, p.4])**.** *Let G/S be a finite locally free group scheme of rank n. Then $[n]_G = 0$.*

Proposition A.2. *Let G/S be a finite locally free group scheme with $[n]_G = 0$ for some $n \in \mathbb{N}$ and let $n = \prod_{i=1}^k p_i^{d_i}$ be the prime factorization of n. Then $G = \prod_{i=1}^k G[p_i^{d_i}]$ and each $G[p_i^{d_i}]$ is finite locally free of rank a p_i-power.*

Proof. G represents a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules and the decomposition $\mathbb{Z}/n\mathbb{Z} = \prod_{i=1}^k \mathbb{Z}/p_i^{d_i}\mathbb{Z}$ induces a decomposition $G = \prod_{i=1}^k G[p_i^{d_i}]$ into closed subschemes. That $G[p_i^{d_i}]$ is finite locally free follows from Lemma A.3 below and the statement about the rank follows from [3, II.9 Corollary]. $\square$

Lemma A.3 ("Serre's trick")**.** *Let X and Y be S-schemes. If $X \times_S Y$ is flat over S and if $Y(S) \neq \emptyset$, then X is flat over S.*

Let A and B be abelian schemes over a S. Let $f : A \to B$ be a homomorphism.

Lemma A.4. *f is proper and of finite presentation.*

Proof. As $A \to S$ is proper and $B \to S$ is separated, f is proper. As $A \to S$ is locally of finite presentation and $B \to S$ is locally of finite type, f is locally of finite presentation by [9, 1.4.3(v)]. $\square$

Definition A.5. *f is called an* isogeny *if it is finite and surjective.*

Proposition A.6. *f is an isogeny if and only if for all $s \in S$ the base-change f_s is an isogeny.*

Proof. As f is proper, it is finite if and only if it is quasi-finite, see [12, 18.12.4]. As f is of finite type, quasi-finiteness is a purely set-theoretic condition which can be verified on fibers. Of course the same is true for surjectivity. $\square$

Corollary A.7. *$[n]_A$ is an isogeny for each $n \in \mathbb{N}_{\geq 1}$.*

Proof. By the preceding result it suffices to prove this over an algebraically closed field. This is done in [22, §6]. $\square$

Lemma A.8. *If f is surjective, it is flat.*

Proof. By [11, 11.3.10] we may assume that S is an algebraically closed field. By [10, 6.9.1] the flat locus of f is non-empty and using translations one sees that f is flat everywhere. $\square$

Corollary A.9. *An isogeny is finite locally free.*

Proof. Follows from [6, Corollary 7.41] and [6, Proposition B.12]. $\square$

Proposition A.10. *An isogeny is both a mono- and an epimorphism in the category of abelian schemes.*

Proof. An isogeny is faithfully flat and locally of finite presentation and hence even an epimorphism in the category of schemes. Assume that $f : A \to B$ is an isogeny and let $g, h : C \to A$ be two homomorphisms with $f \circ g = f \circ h$. Replacing g by $g - h$ we may assume that $h = 0$. Then g factors through $K = \ker f$. Working Zariski locally on S, we may assume that K has constant rank n over S. Then $[n]_K = 0$ by Theorem A.1, so that $0 = [n]_K \circ g = g \circ [n]_C$. This implies $g = 0$, as $[n]_C$ is an isogeny and hence an epimorphism by what we have already seen. $\square$

Corollary A.11. *The canonical map* $\mathrm{Hom}_S(A,B) \to \mathrm{Hom}_S(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is injective.*

Proposition A.12. *Assume that* f *is an isogeny. Then it induces a short exact sequence*

$$0 \to \ker f \to A \xrightarrow{f} B \to 0$$

in $\mathfrak{C}_S$. *In other words,* f *identifies* B *with the fppf quotient* $A/\ker f$.

Proof. The sequence is exact on the right as f is faithfully flat and locally of finite presentation and thus a cover for the fppf topology. □

Proposition A.13. *Assume that* S *is quasi-compact. Then* f *is an isogeny if and only if there exist* $n \in \mathbb{N}_{\geq 1}$ *and a homomorphism* $g : B \to A$ *with* $f \circ g = [n]_B$ *and* $g \circ f = [n]_A$.

Proof. Assume that n and g as above exist. As $[n]_B$ is surjective, f is surjective. As $[n]_A$ has finite fibers, f has finite fibers.

Conversely assume that $f : A \to B$ is an isogeny and let $K = \ker f$. As S is quasi-compact, Theorem A.1 implies that there is an $n \in \mathbb{N}$ with $[n]_K = 0$. Thus $K \subset \ker[n]_A$ and consequently there is a morphism $A/K \to A$ such that

$$\begin{tikzcd} A \arrow[r, "{[n]_A}"] \arrow[d] & A \\ A/K \arrow[ur, dotted] & \end{tikzcd}$$

commutes. But $B = A/K$ by Proposition A.12, so that we have found a morphism $g : B \to A$ with $g \circ f = [n]_A$. Then also $(fg)f = f(gf) = f[n]_A = [n]_B f$, and as f is an epimorphism we deduce $f \circ g = [n]_B$. □

Proposition A.14. *Assume that* f *is an isogeny and let* $K = \ker f$. *Then the induced map* $A[p^\infty] \xrightarrow{f'} B[p^\infty]$ *is surjective and* $\ker f' = K[p^\infty]$ *is a finite locally free group scheme over* S.

Proof. As filtered colimits in $\mathfrak{C}_S$ are exact, it is clear that $\ker f' = K[p^\infty]$. We may work Zariski locally on S and therefore assume that K has constant rank n. Let $d \in \mathbb{N}$ be maximal with $p^d \mid n$, say $n = p^d n'$, $n' \in \mathbb{N}$. By Theorem A.1 and Proposition A.2 we get a decomposition $K = K[p^d] \times K[n']$, and $K[p^d]$ is finite locally free.

$[p]_{K[n']}$ is an isomorphism, as $K[n']$ represents a sheaf of $\mathbb{Z}/n'\mathbb{Z}$-modules. This first implies that $K[p^\infty] = K[p^d]$, which is therefore indeed a finite locally free scheme over S. Secondly, it implies that $\mathrm{im}([p^{d+k}]_K) = \mathrm{im}([p^d]_K)$ for all $k \in \mathbb{N}$. From this one quickly deduces that a section of $B[p^k]$ locally has a preimage in $A[p^{d+k}]$ under f'. □

We close with the following trivial statements.

Lemma A.15. *Let* $n, m \in \mathbb{N}$ *be coprime. Let* $\mathcal{F} \in \mathrm{Ob}\,\mathfrak{C}_S$ *and assume that* $\mathcal{F}[n^\infty] = \mathcal{F}$. *Then* $[m]_{\mathcal{F}}$ *is an isomorphism.*

Remark A.16. *Let* $\mathcal{F} \in \mathfrak{C}_S$ *with* $\mathcal{F} = \mathcal{F}[p^\infty]$. *There is the canonical morphism* $\mathbb{Z}_p \to \mathrm{End}(\mathcal{F})$, *given by* $\lim_n \mathbb{Z}/p^n\mathbb{Z} \ni (a_n)_n \mapsto \mathrm{colim}(\mathcal{F}[p^n] \xrightarrow{\cdot a_n} \mathcal{F}[p^n])$.

References

[1] S. A. Amitsur, *On the characteristic polynomial of a sum of matrices*, Linear and Multilinear Algebra **8** (1979/80), no. 3, 177–182. MR 560557 (82a:15014)

[2] P. Berthelot, L. Breen, and W. Messing, *Théorie de Dieudonné cristalline. II*, Lecture Notes in Mathematics, vol. 930, Springer-Verlag, Berlin, 1982. MR 667344 (85k:14023)

[3] M. Demazure, *Lectures on p-divisible groups*, Lecture Notes in Mathematics, Vol. 302, Springer-Verlag, Berlin, 1972. MR 0344261 (49 #9000)

[4] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970, Avec un appendice *Corps de classes local* par Michiel Hazewinkel. MR 0302656 (46 #1800)

[5] E. Goren and P. Kassaei, *Canonical subgroups over Hilbert modular varieties*, J. Reine Angew. Math., To appear.

[6] U. Görtz and T. Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010, Schemes with examples and exercises. MR 2675155 (2011f:14001)

[7] U. Görtz and C.-F. Yu, *Supersingular Kottwitz-Rapoport strata and Deligne-Lusztig varieties*, J. Inst. Math. Jussieu **9** (2010), no. 2, 357–390. MR 2602029 (2011c:14077)

[8] ———, *The supersingular locus of Siegel modular varieties with Iwahori level structure*, Math. Ann. **353** (2012), no. 2, 465–498.

[9] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Inst. Hautes Études Sci. Publ. Math. (1964), no. 20, 259. MR 0173675 (30 #3885)

[10] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 24, 231. MR 0199181 (33 #7330)

[11] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28, 255. MR 0217086 (36 #178)

[12] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 0238860 (39 #220)

[13] T. Haines, *Introduction to Shimura varieties with bad reduction of parahoric type*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 583–642. MR 2192017 (2006m:11085)

[14] P. Hamacher, *The p-rank stratification on the Siegel moduli space with Iwahori level structure*, `arXiv:1109.5061v2 [math.AG]`.

[15] G. J. Janusz, *Tensor products of orders*, J. London Math. Soc. (2) **20** (1979), no. 2, 186–192. MR 551444 (81g:16009)

[16] N. Katz, *Slope filtration of F-crystals*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I, Astérisque, vol. 63, Soc. Math. France, Paris, 1979, pp. 113–163. MR 563463 (81i:14014)

[17] R. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220. MR 809866 (87i:14040)

[18] ———, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444. MR 1124982 (93a:11053)

[19] M. Kreidl, *Spaces of Lattices in Equal and Mixed Characteristics*, (2010), Thesis.

[20] S. Lang, *On quasi algebraic closure*, Ann. of Math. (2) **55** (1952), 373–390. MR 0046388 (13,726d)

[21] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR 1354144 (96h:05207)

[22] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay, 1970. MR 0282985 (44 #219)

[23] J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. MR 1697859 (2000m:11104)

[24] B.C. Ngô and A. Genestier, *Alcôves et p-rang des variétés abéliennes*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 6, 1665–1680. MR 1952527 (2004b:14036)

[25] G. Pappas, *On the arithmetic moduli schemes of PEL Shimura varieties*, J. Algebraic Geom. **9** (2000), no. 3, 577–605. MR 1752014 (2001g:14042)

[26] G. Pappas and M. Rapoport, *Local models in the ramified case. II. Splitting models*, Duke Math. J. **127** (2005), no. 2, 193–250. MR 2130412 (2006a:11075)

[27] ———, *Twisted loop groups and their affine flag varieties*, Adv. Math. **219** (2008), no. 1, 118–198, With an appendix by T. Haines and M. Rapoport. MR 2435422 (2009g:22039)

[28] ———, *Local models in the ramified case. III. Unitary groups*, J. Inst. Math. Jussieu **8** (2009), no. 3, 507–564. MR 2516305 (2010h:11098)

[29] G. Pappas, M. Rapoport, and B. Smithling, *Local models of Shimura varieties, I. Geometry and combinatorics*, Handbook of Moduli, To appear.

[30] G. Pappas and X. Zhu, *Local models of Shimura varieties and a conjecture of Kottwitz*, `arXiv:1110.5588v3 [math.AG]`.

[31] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181. MR 1411570 (98c:14015)

[32] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996. MR 1393439 (97f:14023)

[33] I. Reiner, *Maximal orders*, London Mathematical Society Monographs. New Series, vol. 28, The Clarendon Press Oxford University Press, Oxford, 2003, Corrected reprint of the 1975 original, With a foreword by M. J. Taylor. MR 1972204 (2004c:16026)

[34] J.-P. Serre, *Corps locaux*, Hermann, Paris, 1968, Deuxième édition, Publications de l'Université de Nancago, No. VIII. MR 0354618 (50 #7096)

[35] ———, *Cohomologie galoisienne*, fifth ed., Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin, 1994. MR 1324577 (96b:12010)

[36] B. Smithling, *Topological flatness of local models for ramified unitary groups. II. The even dimensional case*, `arXiv:1002.3520v1 [math.AG]`.

[37] ———, *Topological flatness of local models for ramified unitary groups. I. The odd dimensional case*, Adv. Math. **226** (2011), no. 4, 3160–3190. MR 2764885 (2012a:14058)

[38] ———, *Topological flatness of orthogonal local models in the split, even case. I*, Math. Ann. **350** (2011), no. 2, 381–416. MR 2794915

[39] T. A. Springer, *Linear algebraic groups*, second ed., Progress in Mathematics, vol. 9, Birkhäuser Boston Inc., Boston, MA, 1998. MR 1642713 (99h:20075)

[40] H. Stamm, *On the reduction of the Hilbert-Blumenthal-moduli scheme with $\Gamma_0(p)$-level structure*, Forum Math. **9** (1997), no. 4, 405–455. MR 1457134 (98h:14030)

[41] J. Tate and F. Oort, *Group schemes of prime order*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 1–21. MR 0265368 (42 #278)

Universität Duisburg-Essen, Institut für experimentelle Mathematik, Ellernstr. 29, 45326 Essen, Germany

E-mail address: `philipp.hartwig@uni-due.de`

URL: `http://www.esaga.uni-due.de/philipp.hartwig`