Universität Duisburg-Essen
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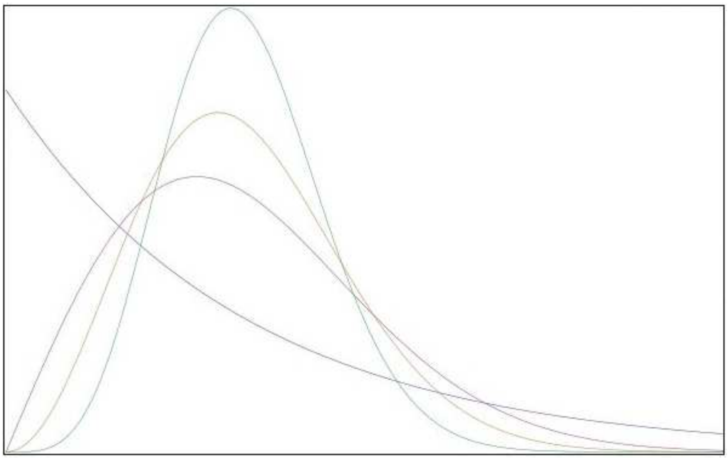
Supersymmetry in Random Matrix Theory



von

Mario Kieburg

geboren am 23.06.1981 in Zossen

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Jahrelang in einer Wüste
lebst du nun auf off'nem Meer,
erlebst ein paar neu' Gelüste
and die Angst fährt hinterher.

Jahrelang auf flachem Land
steigst du auf den höchsten Berg.
Die Sicht war einst dir unbekannt,
nun fühlst du dich wie ein Zwerg.

Jahrelang mit was Vertrautem,
jahrelang hast du nichts gesehen,
nach Jahren mit was Verbautem
hast du es schwer, neu zu verstehen.

Mario Kieburg

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Contents

Zusammenfassung

Die Untersuchung des Verhältnisses zwischen der Supersymmetrie und der Zufallsmatrixtheorie steht im Mittelpunkt dieser Arbeit. Es wird die Supersymmetriemethode verallgemeinert. Weiterhin werden drei neue Berechnungsmethoden von Eigenwertkorrelationsfunktionen entwickelt. Diese Korrelationsfunktionen sind Mittelwerte von Quotienten, welche aus charakteristischen Polynomen aufgebaut sind.

Im ersten Teil dieser Arbeit wird ein Zusammenhang zwischen Integralen über antikommutierenden Variablen (Grassmann–Variablen) und Differentialoperatoren hergeleitet. Die Differentialoperatoren wirken nur auf den kommutierenden Anteil der Variablen. Mittels dieses Zusammenhangs werden Cauchy–ähnliche Integraltheoreme verifiziert. Außerdem werden die Supermatrix–Bessel–Funktionen auf ein Produkt von zwei gewöhnlichen Matrix–Bessel–Funktionen zurückgeführt.

Im zweiten Teil wird die verallgemeinerte Hubbard–Stratonovich–Transformation auf beliebige rotationsinvariante Ensembles über den reell symmetrischen und hermitesch selbstdualen Matrizen angewandt. Somit wird ein Ansatz für die unitär rotationsinvarianten Matrixensembles erweitert. Es werden für die k–Punktkorrelationsfunktionen dieser Ensembles supersymmetrische Integralausdrücke in vereinheitlichter Form hergeleitet. Weiterhin wird gezeigt, dass die verallgemeinerte Hubbard–Stratonovich–Transformation mit der Superbosonisationsformel übereinstimmt. Ebenfalls wird eine alternative Abbildung von Integralen über gewöhnlichen Matrizen zu Integralen über Supermatrizen angegeben. Dabei werden explizite funktionale Ausdrücke für die Wahrscheinlichkeitsdichten über den Superräumen hergeleitet, welche man durch den Vergleich der Integralausdrücke mit den anderen beiden Supersymmetriemethoden erhält.

Wenn die Wahrscheinlichkeitsdichte über die Zufallsmatrizen faktorisiert, dann ergeben sich für die erzeugenden Funktionen Determinantenstrukturen oder Pfaff'sche Strukturen. Für einzelne Matrixensembles ist dies mit Hilfe von verschiedenen Berechnungsmethoden schon gezeigt worden. Hier wird gezeigt, dass diese Strukturen auf rein algebraische Weise enstehen. Die neue Methode nutzt Strukturen, die man ursprünglich in Superräumen findet. Für drei Arten von Integralen werden Determinantenausdrücke und Pfaff'sche Ausdrücke hergeleitet, ohne diese in einem Superraum abzubilden. Diese drei Integraltypen sind so allgemein, dass sie auf einer sehr grossen Klasse von Matrixensembles anwendbar sind.

Abstract

I study the applications of supersymmetry in random matrix theory. I generalize the supersymmetry method and develop three new approaches to calculate eigenvalue correlation functions. These correlation functions are averages over ratios of characteristic polynomials.

In the first part of this thesis, I derive a relation between integrals over anti-commuting variables (Grassmann variables) and differential operators with respect to commuting variables. With this relation I rederive Cauchy–like integral theorems. As a new application I trace the supermatrix Bessel function back to a product of two ordinary matrix Bessel functions.

In the second part, I apply the generalized Hubbard–Stratonovich transformation to arbitrary rotation invariant ensembles of real symmetric and Hermitian self-dual matrices. This extends the approach for unitarily rotation invariant matrix ensembles. For the k–point correlation functions I derive supersymmetric integral expressions in a unifying way. I prove the equivalence between the generalized Hubbard–Stratonovich transformation and the superbosonization formula. Moreover, I develop an alternative mapping from ordinary space to superspace. After comparing the results of this approach with the other two supersymmetry methods, I obtain explicit functional expressions for the probability densities in superspace.

If the probability density of the matrix ensemble factorizes, then the generating functions exhibit determinantal and Pfaffian structures. For some matrix ensembles this was already shown with help of other approaches. I show that these structures appear by a purely algebraic manipulation. In this new approach I use structures naturally appearing in superspace. I derive determinantal and Pfaffian structures for three types of integrals without actually mapping onto superspace. These three types of integrals are quite general and, thus, they are applicable to a broad class of matrix ensembles.

Chapter 1

Introduction

Random matrix theory is nowadays a large branch in statistical physics. In particular, it is an issue of multivariate statistical theory. It addresses problems which involve many stochastical variables arranged in one ore more matrices. One finds applications of it in physics as well as in mathematics and related areas.

Supersymmetry was originally developed in particle physics. In random matrix theory it is a mathematical tool which is indispensable for many problems. This approach is known as the supersymmetry method. It uses duality relations between ordinary and superspaces. The advantage of this approach is that it drastically reduces the number of integration variables.

In this thesis, we study the relation between supersymmetry and random matrix theory. We develop new approaches of the supersymmetry method and analyze structures found in random matrix theory as well as in supersymmetry.

In Sec. 1.1, we exemplarily discuss some applications of random matrix theory. In particular, we present examples which were of historical importance in the development of random matrix theory. In Sec. 1.2, we give a brief account of history of supersymmetry in physics. We show where other applications of supersymmetry can be found and what the connection to random matrix theory is. An outline of this thesis is given in Sec. 1.3.

1.1 Random matrix theory and its applications

Random matrices were first introduced by Wishart [6]. He considered a real, rectangular random matrix whose entries where independently, identically distributed by a Gaussian probability density. In 1955, Wigner applied random matrix theory to physics [7]. Since then there was an overwhelming wealth of applications. Quantum chaos and the related topic of many body quantum systems, quantum chromodynamics (QCD), representation theory of groups and econo physics are only a few of them.

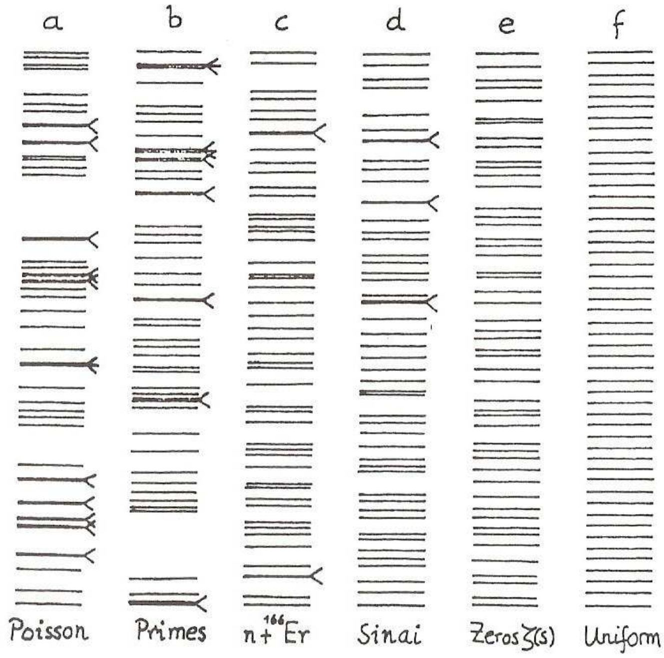

Figure 1.1: Sequences of 50 levels for some typical systems on the normalized scale of their local mean level spacing. The arrows indicate two or more levels whose spacing are smaller than 1/4. (a) Poisson process, random levels without any correlations. (b) The series of prime numbers. (c) Resonance levels of the Erbium (166) nucleus for slow neutrons. (d) The energy levels of a free particle in a Sinai billiard given by the area $\{(x, y) \in [-1, 1]^2 | x^2 + y^2 \geq R^2\}$. (e) Zeros of the Riemann ζ–function on the line $\mathrm{Re}\, z = 1/2$. (f) Uniform spectrum of a harmonic oscillator. Taken from Ref. [8].

Of particular interest are eigenvalue and eigenvector statistics of completely different systems in physics and related areas. In figure 1.1 we see six different spectra of physical and mathematical interest. Some spectra have the property that levels repell each other whereas others exhibit clusters of levels or seem to have no correlations at all. To understand this behavior in a unifying way one has to construct a theory which makes all these spectra comparable to each other.

Since in many physical systems the level density

$$\rho(E) = \lim_{N\to\infty} \sum_{n=1}^{N} \delta(E - E_n) \tag{1.1}$$

grows with increasing E, where E_n are the levels of a spectrum, we cannot directly compare different spectra. Also degeneracies in the spectra resulting from symmetries of the system make a direct comparison impossible. Another difficulty arises due to the different scales of the spectra for completely different systems. Hence one cannot expect that the level density has a universal behavior. Nevertheless, there are other quantities of spectra which have generic features.

As the first step to analyze experimental data, one has to choose a non-degenerate sub-spectrum of conserved quantum numbers such as chirality or spin. Then, we normalize all sub-spectra to a uniform scale. One example of such a normalization is to the local scale of the mean level spacing by the unfolding procedure. A detailed explanation of this procedure is given in Ref. [9]. The level spacing distribution $p(s)$ tells us that the probability of finding two neighboring levels separated by a distance in the range $[s, s+ds]$ is $p(s)ds$. The normalization to the local scale of the mean level spacing imposes that

$$\int_0^\infty sp(s)ds = 1\,. \tag{1.2}$$

Indeed, we lose specific information of the system under investigation. However, statistical information of the spectral fluctuations is retained.

To model the statistics of spectra one considers a model system which has the symmetry of a certain class of real system. The main object is a finite dimensional $N \times N$ matrix H whose entries are random variables. Since one is usually interested in generic features of a whole class of systems, one considers the limit $N \to \infty$. The eigenvalue and eigenvector statistics of such a matrix will then be compared to experimental data. Indeed, such a model is a drastic simplification for a real system. Yet it has proven quite powerful due to universalities of particular classes of systems. Experimental data such as figure 1.2 confirm these assumptions. Mathematical proofs of such universalities are given in Refs. [10, 11].

In the following subsections, we present two typical applications such as quantum chaos (subsection 1.1.1) and QCD (subsection 1.1.2).

1.1.1 Quantum chaos

The problem in quantum chaos is to specify how chaos shows itself in a quantum system and how chaotic such a system is. The notion of a Lyapunov exponent loses its meaning because we have no classical trajectories in phase space. However, the eigenvalue and the eigenvector statistics are meaningful quantities.

We consider a stationary wave equation

$$\mathcal{H}\psi_n = E_n\psi_n \tag{1.3}$$

with some boundary conditions, where E_n and ψ_n are an eigenvalue and an eigenvector of the Hamilton operator $\mathcal{H}$, respectively. Equation (1.3) can be the stationary Schrödinger equation or the stationary wave equation for elastic materials which are both equivalent to the Helmholtz equation. This shows that random matrix theory in quantum chaos is applicable to a large class of physical systems.

Wigner [12] showed that physical quantum systems with certain symmetries impose some conditions on the Hamilton operator $\mathcal{H}$. Such a condition is described by an anti-unitary operator T, i.e.

$$\langle T\psi_1|T\psi_2\rangle = (\langle\psi_1|\psi_2\rangle)^* = \langle\psi_2|\psi_1\rangle\,, \tag{1.4}$$

where ψ_i are two arbitrary vectors and "$*$" is the complex conjugation. There are three kinds of Hamilton operators whose corresponding quantum systems differ in the time-reversal and the rotation invariance.

(1) If the time-reversal as well as the rotation invariance is conserved, the Hamilton operator $\mathcal{H}$ and its corresponding random matrix H are real symmetric

$$H = H^T = H^*\,. \tag{1.5}$$

Here, "T" is the transposition of a matrix.

(2) Systems with broken time-reversal invariance impose no further symmetries on the Hermitian Hamilton operator $\mathcal{H}$ and its corresponding Hermitian random matrix H, i.e.

$$H = H^\dagger\,. \tag{1.6}$$

We introduce the adjunction "$\dagger$".

(3) For time-reversal invariant systems with broken rotation invariance and half integer spin, we have a Hermitian self-dual random matrix H modelling the Hamilton operator $\mathcal{H}$, i.e.

$$H = \begin{bmatrix} H_0 & H_1 \\ -H_1^* & H_0^* \end{bmatrix}, \tag{1.7}$$

where $H_0 = H_0^\dagger$ is a Hermitian matrix and $H_1 = -H_1^T$ is an anti-symmetric, complex matrix.

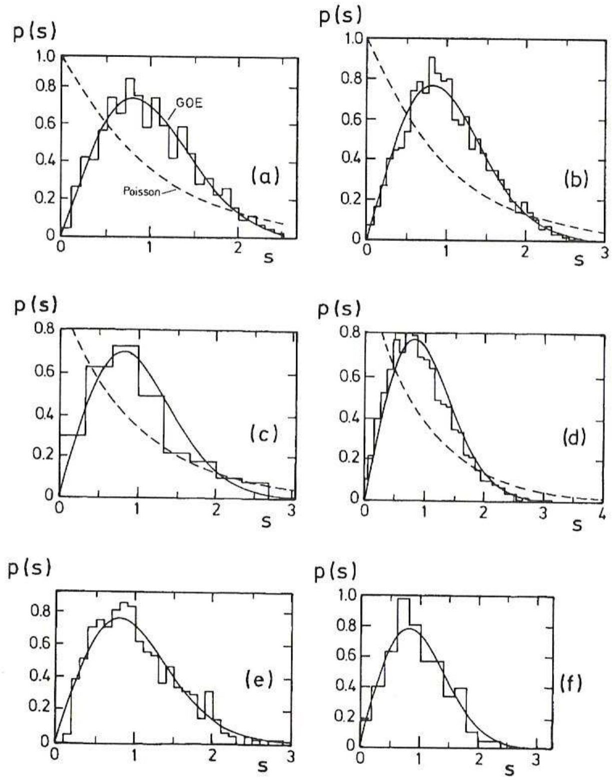

Figure 1.2: Level spacing distributions for some physical systems. (a) Sinai billiard [13]. (b) A hydrogen atom in a strong magnetic field [14]. (c) The excitation spectrum of an NO_2 molecule [15]. (d) The acoustic resonance spectrum of a quartz block with the shape of a Sinai billiard [16]. (e) The microwave spectrum of a three dimensional chaotic cavity [17]. (f) The vibration spectrum of a plate with the shape of a quarter stadium [18]. Taken from Ref. [19].

Since the flat ensembles of these matrices are not normalizable, they are weighted by probability densities $P(H)$. A common choice is that these matrices are drawn from a Gaussian orthogonal ensemble (GOE), a Gaussian unitary ensemble (GUE) and a Gaussian symplectical ensemble (GSE), according to the symmetries (1.5), (1.6) and (1.7). Hence the weight is

$$P(H) = c \exp\left[-\frac{\operatorname{tr} H^2}{v^2}\right] \tag{1.8}$$

with standard deviation v and normalization constant c. In a very early stage of random matrix theory these ensembles were investigated by Mehta et al. [20, 21, 22]. After the unfolding procedure, these Gaussian ensembles indeed describe experimental data, cf. figure 1.2, in the large N limit.

Random matrix theory was originally developed to describe systems with many degrees of freedom. Such systems are many body systems like molecules and atomic nuclei. Wigner wanted to interpret the giant resonance in nuclei with help of random matrices [23, 7]. He considered subsets of real symmetric matrices.

In the seventies, measurements in nuclear physics made it possible to compare the theoretical results with real data. This data set is known as the nuclear data ensemble.

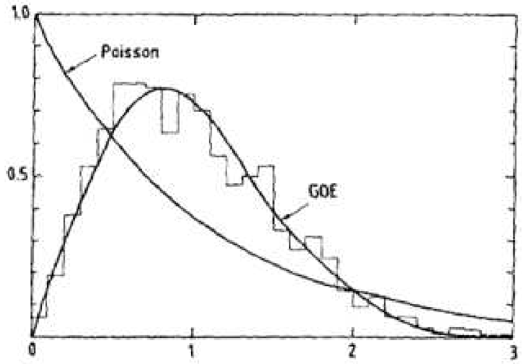

Figure 1.3: The histogram shows the level spacing distribution of the nuclear data ensemble. It contains 1726 nuclear energy levels of 36 sequences of 32 different nuclei. The solid lines are the level spacing distribution of Poisson statistics and of a GOE ensemble. Taken from Ref. [19].

As we see in figure 1.2.a, Gaussian matrix ensembles do not only describe many body systems but also systems with few degrees of freedom. This is due to the intimate connection of quantum chaos and random matrix theory. The simplest systems to consider are quantum billiards. In these systems one solves the free wave equation in a two-dimensional area bounded by hard walls, i.e. the wave function has to vanish at the boundary. Hence the whole dynamics is given by the shape of the area. For a regular shape like a rectangular billiard, see figure 1.4, one has an integrable system. The spectral statistics of quantum systems whose classical analog is integrable can be described by Poisson distributed random matrices, i.e. independently, identically distributed eigenvalues. This statement is apart from some exceptions, cf. harmonic oscillator, verified. A particular property of these systems is that their levels do not show any repulsion, i.e. the normalized level spacing distribution is

$$p(s) = \exp(-s)\,. \tag{1.9}$$

The levels are completely uncorrelated. This is the reason why such systems can be described by diagonal random matrices.

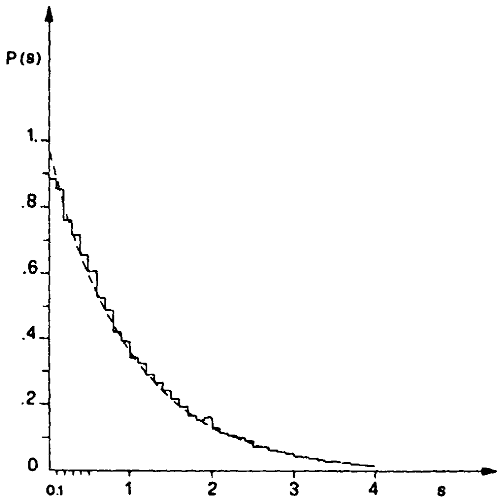

Figure 1.4: The histogram is the level spacing distribution for the first 100000 eigenvalues of a rectangular billiard. In comparison, the Poisson distribution is also shown (dotted line). Taken from Ref. [19].

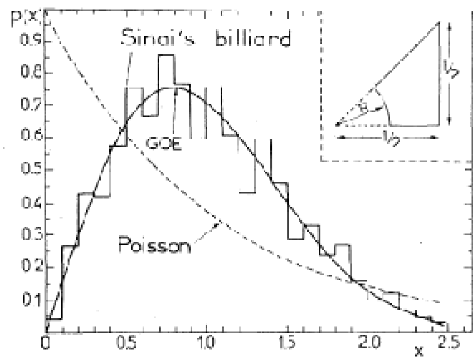

Figure 1.5: The histogram shows the level spacing distribution of the desymmetrized Sinai billiard shown in the upper right corner. It results from the analysis of 740 levels consisting of the 51st to 268th level for radius $R = 0.1$, 21st to 241st level for $R = 0.2$, 16th to 194th level for $R = 0.3$ and 11th to 132nd level for $R = 0.4$. Taken from Ref. [13].

The spectra of quantum billiards with irregular shape such as the Sinai billiard, see figure 1.5, cannot be modeled by Poisson ensembles. They show level repulsion. This repulsion linearly grows at small distances for time-reversal invariant systems. Since this behavior was seen for many physical

systems, cf. figure 1.2, Bohigas, Giannoni and Schmidt made the following conjecture [13]:
"*Spectra of time-reversal-invariant systems whose classical analogs are K systems show the same fluctuation properties as predicted by GOE*".
By K-systems they meant that in the classical system almost all neighboring phase trajectories exponentially disperse.

The level repulsion can be understood by a 2×2 Gaussian random matrix model. Such a random matrix model is due to its simple structures often used to model the level spacing behavior [24, 25, 26, 27, 28, 29]. We consider a real symmetric matrix

$$H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \tag{1.10}$$

where the real entries a, b and c are distributed according to Eq. (1.8). This matrix has two levels

$$\lambda_{\pm} = \frac{a+c\pm\sqrt{(a-c)^2+4b^2}}{2}. \tag{1.11}$$

Hence, the distance of both levels is

$$s = \lambda_+ - \lambda_- = \sqrt{(a-c)^2+4b^2}. \tag{1.12}$$

Due to the off-diagonal element b we have an additional distance between both levels implying the level repulsion. The normalized level spacing of this ensemble is the Wigner surmise

$$p(s) = \frac{\pi}{2} s \exp\left[-\frac{\pi}{4}s^2\right] \tag{1.13}$$

for the GOE. Although this model seems to be a strong approximation for the level spacing distribution, it models large N-results and experimental data very well, cf. figure 1.6.

Of particular interest are periodically driven models as the kicked rotator showing chaotic behavior. Then, a stroboscopic picture at the times T, $2T$, $3T$,... is appropriate, where T is the period of the driving force. The evolution is described by the Floquet operator

$$\mathcal{U} = \exp[-\imath\mathcal{H}]. \tag{1.14}$$

This operator is indeed unitary. Thus, we can describe chaotic quantum systems periodically driven by unitary subsets of matrices. Dyson [30, 31, 32, 33] studied such random matrix ensembles as two dimensional Coloumb gases in a one dimensional circular wire. He found out that they can also be classified by the symmetries of a physical system as it was done for the Hamilton operator. These random matrix ensembles are known as the circular ensembles. Apart from the level densities both kinds of ensembles, the Gaussian and the circular ones, lead to the same eigenvalue statistics in the large N limit.

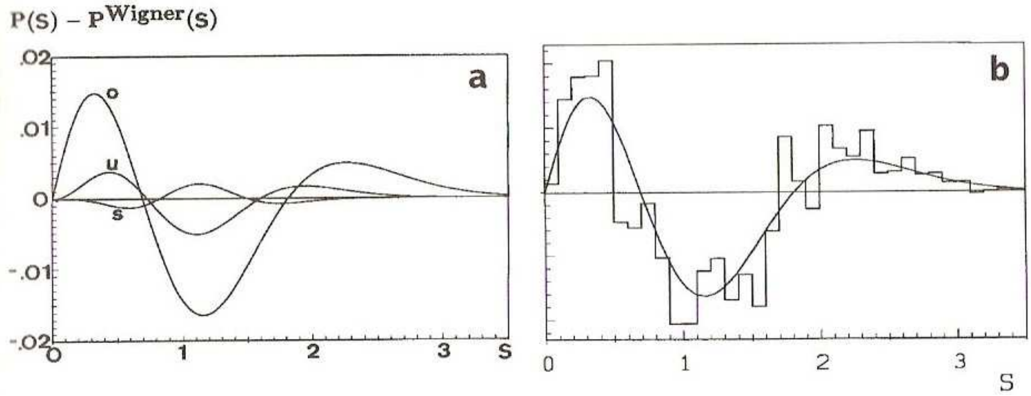

Figure 1.6: (a) The deviations of the Wigner surmise for the orthogonal (o), unitary (u) and unitary symplectic (s) symmetry class from the respective level spacing distributions of asymptotic ($N \to \infty$) Gaussian or circular matrix ensembles. (b) Numerical results of the deviation for a kicked top pertaining to the orthogonal universality class. One should notice that the deviations are the scale of one till ten percents. Taken from Ref. [26].

1.1.2 Quantum chromodynamics

In the nineties, random matrix theory was introduced in QCD by Shuryak and Verbaarschot [34, 35]. They modeled the Euclidian Dirac operator $\mathcal{D}$ by a chiral random matrix D. Such a random matrix has to fulfill the property [36]

$$D = -pDp^{-1} = -D^\dagger \quad , \quad p^2 = pp^\dagger = 1\,. \tag{1.15}$$

The constant matrix p can be diagonalized and has only ± 1 as eigenvalues. The signature of p yields the number ν of the generic zero eigenvalues of the Dirac operator. It corresponds to the difference of the number of eigenstates with positive chirality and of those with negative chirality is. This quantity is also known as the topological charge because it is an invariant and classifies the systems described by the Dirac operator. Hence, we have to replace

$$\imath\mathcal{D} + \imath m \longrightarrow \imath D + \imath m = \left[\begin{array}{cc} \imath m & W \\ W^\dagger & \imath m \end{array}\right] , \tag{1.16}$$

where W is a $N \times (N + \nu)$ rectangular matrix and m the mass of a fermion. The symmetries of real physical systems impose conditions on W. Thus, W can have real, complex or quaternionic entries according to the symmetries of cases (1)-(3) described in the previous subsection. The probability density of these matrices is given by

$$P(D) \sim \prod_{n=1}^{N_f} \det(D + m_n) \exp(-N \mathrm{tr}\, W W^\dagger)\,, \tag{1.17}$$

where N_f is the number of flavors with masses m_n. These random matrix models are known as chiral or Laguerre ensembles.

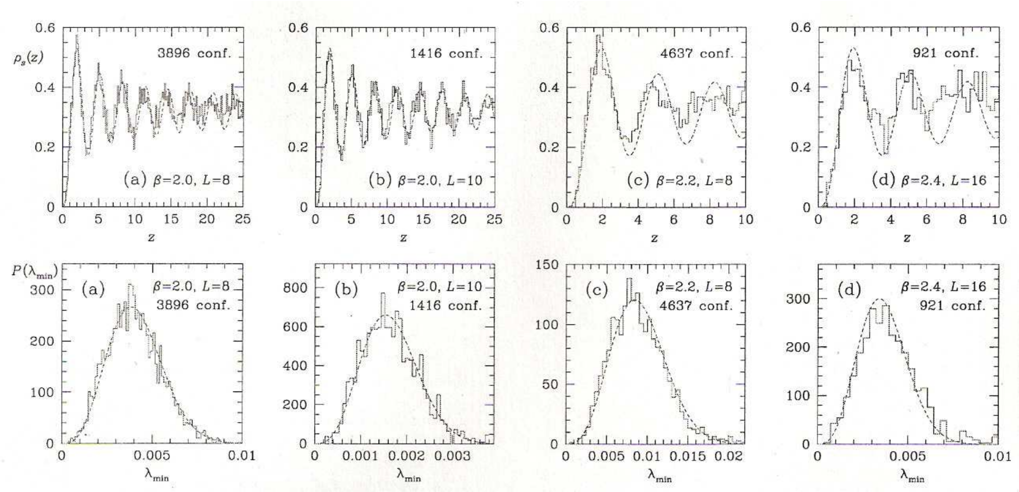

Figure 1.7: In the upper row we see microscopic spectral density whereas in the lower row we have the smallest eigenvalue distribution. The theoretical predictions for $N_f = \nu = 0$ (dashed curves) are compared to lattice data for the gauge group SU(2) (histogram). The quantity $2/\sqrt{\beta}$ is the coupling constant and $V = L^4$ is the four-dimensional volume of the lattice. Taken from Ref. [37].

The authors of Ref. [34] have shown that in the large N limit the random matrix model for complex W indeed yields the Leutwyler–Smilga sum rules for QCD in the low energy limit and small volumes. These sum rules are the mean values of the squared inverse non-zero eigenvalues of the Dirac operator. Due to the universality of these quantities it was conjectured that other quantities such as the microscopic spectral density are universal. In figure 1.7 we see that this conjecture agrees with QCD lattice data. The microscopic spectral density is given by

$$\rho_s(x) = \lim_{V\to\infty} \frac{1}{V|\langle\overline{\psi}\psi\rangle|}\rho\left(\frac{x}{V|\langle\overline{\psi}\psi\rangle|}\right)\,, \tag{1.18}$$

where V is the four-dimensional volume and ρ is the averaged spectral density of D. The chiral condensate

$$\langle\overline{\psi}\psi\rangle = -\lim_{m\to 0}\lim_{V\to\infty}\frac{\pi}{V}\rho(0) \tag{1.19}$$

is an order parameter and indicates how strong the axial flavor symmetry is broken.

In recent studies one considers QCD with non-zero chemical potential. The Dirac operator $\imath\mathcal{D}$ is then shifted by a complex matrix $\mu_{\rm chem}$. Hence its eigenvalues become complex. These models exhibit the so called "sign problem" for QCD lattice simulation [38] because the probability density (1.17) becomes complex, too. Strictly speaking it is not a probability density anymore. Nevertheless, the random matrix ensembles modelling these systems exist [39, 40, 41].

1.2 Supersymmetry in random matrix theory

Supersymmetry deals with the unified algebra of a commuting

$$\varphi_a\varphi_b = \varphi_b\varphi_a \tag{1.20}$$

and an anti-commuting

$$\psi_a\psi_b = -\psi_b\psi_a \tag{1.21}$$

algebra. This graded Lie algebra closes under commutations and anti-commutation relations. In second quantization the fields φ_a are known as bosons whereas the fields ψ_a are the fermions. Since one considers the whole algebra in supersymmetry instead of only the direct product, transformations between bosons and fermions are possible. It seems to be that Martin [42] was the first who has introduced anti-commuting variables fulfilling Eq. (1.21). His aim was to construct an analog theory for classical fermionic particles to the Hamilton formalism.

In subsection 1.2.1, we give a brief historical summary of supersymmetry and show where supersymmetry is applied outside of random matrix theory. As a related topic to random matrix theory, we explain how supersymmetry is applied to disordered systems in subsection 1.2.2. Also, we summarize the historical background and recent problems of the relation between supersymmetry and random matrix theory.

1.2.1 Supersymmetry in other branches of physics

Originally supersymmetry was introduced in particle physics. One reason was to generalize the internal symmetries of a particle theory to a unified one between integer and half-integer spins. In the sixties supersymmetry was developed in a series of articles on particle theory especially on string theory such as Refs. [43, 44, 45]. After the work of Wess and Zumino [46] supersymmetry became popular. They extended the Poincaré algebra by the superalgebra generators transforming bosons into fermions and vice versa. Unfortunately they had not realized that this extension was already made by Gol'fand and Likhtman [47].

Nowadays, some problems in particle physics would be solved if every particle had a superpartner. This means that every boson has a fermionic partner and vice versa. A review on this topic is given in Ref. [48]. Supersymmetry in particle physics is not only a mathematical tool but a fundamental symmetry in physical systems. However, up to now there is no experimental evidence that this symmetry really exists.

Supergravity is another application where supersymmetry has a physical interpretation. One of the first who considered a unification of a gravitational theory and supersymmetry were Arnowitt, Nath and Zumino [49, 50]. They added to the four space-time coordinates x^μ four anti-commuting spinor coordinates θ^α. Hence, one deals with an eight-dimensional superspace. A theory built on this background has to be covariant under all diffeomorphic transformations on this superspace. This also includes transformations like

$$(\tilde{x}^\mu, \tilde{\theta}^\alpha) = (\tilde{x}^\mu(x^\nu, \theta^\beta), \tilde{\theta}^\alpha(x^\nu, \theta^\beta)) . \tag{1.22}$$

The supersymmetric gravitational theory was analogously constructed to Einstein's. The fundamental field is the metric tensor on the superspace. In the usual way one constructs an analog to the Ricci–tensor and to the Einstein equations.

There are also other approaches to supergravity such as the one by Freedman, van Nieuwenhuizen and Ferrara [51]. All these approaches differ from the starting Lagrangian and, thus, are not always equivalent to each other. A good introduction to supergravity is given in Ref. [52]. This textbook also includes introductions to supersymmetry in particle physics.

Supersymmetric quantum mechanics evolved from particle physics where one investigated the breaking of supersymmetry [53, 54]. Very soon people realized that it has an existence in its own right. Supersymmetry is a helpful mathematical tool in quantum mechanics which has in some but not in all cases a physical interpretation. It helps to categorize the analytically solvable potential problems. One has to understand supersymmetric quantum mechanics as an extension of Infelds and Hulls factorization method [55]. This method generalizes the idea of the ladder operators for the quantum harmonic oscillator. For example, we consider a Hamilton operator $\mathcal{H}$ in a

Schrödinger equation with positive energy spectrum and with zero energy of the bounded ground state. Then one can find an operator A such that

$$\mathcal{H} = AA^\dagger \,. \tag{1.23}$$

The operator A is a linear polynomial in the derivatives such that Eq. (1.23) agrees with $\mathcal{H}$ which is a polynomial of order two in the derivatives. When A and $A^\dagger$ do not commute with each other the change of the position in the product (1.23),

$$\widetilde{\mathcal{H}} = A^\dagger A \,, \tag{1.24}$$

yields another Hamilton operator $\widetilde{\mathcal{H}}$. The spectrum and the eigenstates of both operators $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are related by supersymmetry. In particular for one-dimensional Schrödinger equations this method is very powerful. The authors of the textbook [56] give a very good introduction to this technique.

1.2.2 The supersymmetry method in the theory of disordered systems and in random matrix theory

Parisi and Sourlas [57] introduced supersymmetric techniques in condensed matter physics to study ferromagnets in a random magnetic field. They used anti-commuting fields as auxiliary fields to rewrite determinants as Gaussian integrals. Efetov established this technique a few years later in disordered systems [58]. He considered the electron transport in a sample with small volume and at low temperature. Then one is in the mesoscopic regime and the Drude model for the electron transport breaks down. The phase coherence length of the electron becomes about the same scale or larger than the scale of the sample and, thus, important. Hence one has to take into account quantum corrections when calculating physical quantities such as conductance and electric susceptibility.

One of the main questions in disordered systems is: When does the sample behave as an insulator and when as a conductor? In mesoscopic physics it is equivalent to the question of whether the one particle wave function is localized or is extended along the whole sample. The answers to these questions depend on the strength of disorder. Disorder is theoretically modeled by a Hamilton operator consisting of two components,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \,. \tag{1.25}$$

The Hamilton operator $\mathcal{H}_0$ comprises the kinetic energy and non-random interactions such as a homogene magnetic field or a non-random potential. This Hamilton operator is perturbed by a random Hamiltonian $\mathcal{H}_1$ which can be a random potential $V(r)$ with the space vector r. It describes random impurities of the sample. Usually such a random potential is considered to be a Gaussian random process

$$\langle V(r)\rangle = 0 \quad , \quad \langle V(r)V(r')\rangle = c\delta(r-r')\,, \tag{1.26}$$

where $\langle . \rangle$ denotes the ensemble average and $c = 1/(2\pi\nu\tau)$ is the variance depending on the density of single-particle states per volume ν and the elastic mean free time τ. The autocorrelation functions of the electron densities and the currents are a good measure for the phase coherence. These quantities are related to the advanced and retarded Green functions

$$G^{\mathrm{R/A}}(E,r,r') = \sum_n \frac{\phi_n(r)\phi_n^*(r')}{E-E_n \pm \imath\varepsilon}\,, \tag{1.27}$$

where ϕ_n are the eigenfunctions to the energies E_n of the Hamilton operator $\mathcal{H}$. Efetov managed to map the average over the random potential (1.26) of products of Green functions onto an integral over a supermatrix field $Q(r)$. In section 2.2 we explain what a supermatrix is. In the limit of weak disorder the elastic mean free time τ is large. This allowed Efetov to make a saddlepoint approximation in superspace. Since the result is similar to those in particle physics, it is also known as the non-linear σ-model.

In zero dimensions the model described above is an ordinary random matrix model. Hence, it was soon clear that the supersymmetry method in the theory of disordered systems carries over to random matrix theory [59]. One maps integrals over ratios of characteristic polynomials,

$$\det(H-x) = \prod_{n=1}^{N}(E_n - x)\,, \tag{1.28}$$

for an ordinary $N \times N$ matrix H onto integrals over supermatrices whose dimensions are independent of N. This allows one to discuss the large N limit by a saddlepoint approximation. Due to the drastic reduction of integration variables the supersymmetry technique is a powerful method in random matrix theory and disordered systems. However, it has no physical interpretation as in particle physics. It is only a useful bookkeeping tool.

Averages over ratios of characteristic polynomials play an important role in the investigation of random matrix ensembles. The matrix Green function can be generated by one characteristic polynomial in the denominator and one in the numerator [60, 61]. For the calculation of the free energy, one may use the replica trick [62]. The moments of the Riemann ζ-function are also related to these averages [63, 64]. Mathematicians are interested in averages over ratios of characteristic polynomials because of the connection to Weyl's character formula [65, 66]. In models for QCD [67, 39] and in the analysis of the sign problem [38], one employs mean values of characteristic polynomials, too.

For a long time it was thought that the supersymmetry method is applicable to Gaussian probability densities only [58, 62, 59, 68]. Due to universality on the local scale of the mean level spacing [69, 70, 10, 9, 11], this restriction was not a limitation for calculations in quantum chaos and disordered systems. The result for Gaussian ensembles are indeed identical to those for other invariant matrix ensembles with large matrix dimensions on the scale of the local mean level spacing. In the Wigner–Dyson theory [71] and its corrections for systems with diffusive dynamics [72], Gaussian ensembles are sufficient.

There are, however, situations in which one cannot simply resort to Gaussian random matrix ensembles. The level densities in high–energy physics [73] and finance [74] are needed for non-Gaussian ensembles. But the level densities strongly depend on the matrix ensemble. In particular they are not universal. Other examples are bound–trace and fixed–trace ensembles [75], which are both norm-dependent ensembles [61], as well as ensembles derived from a non-extensive entropy principle [76, 77, 78]. In all these cases one is interested in the non-universal behavior on special scales.

In a series of works, the supersymmetry method was extended to general rotation invariant probability densities [79, 10, 80, 81, 61, 82, 83, 84]. Currently, there are two approaches. The first one is the generalized Hubbard–Stratonovich transformation [61]. With help of an appropriate Dirac–distribution in superspace [85] integrals over rectangular supermatrices are mapped to a supermatrix integral with non-compact domain in the fermion–fermion block, see Chap. 8. The second approach is the superbosonization formula [83, 84] mapping the same integral over rectangular matrices as before to a supermatrix integral with compact domain in the fermion–fermion block, see Sec. 10.1.

While the supersymmetry method solves some problems others arise. For example, it is known [86] that the extension of the integrals over ordinary matrices to integrals over supermatrices is not unique. In Secs. 7.5, 9.3 and 11.3, we will discuss this problem. When one has to average over too many characteristic polynomials (1.28) then the supersymmetry method does not work in the conventional way [86], see also in Secs. 10.3 and 11.3. Another problem is that one cannot reconstruct the Pfaffian structures for GOE and GSE after the mapping to superspace. It was only possible to use these structures in combination with the supersymmetry method [87]. All three problems are subject of current research. For the first two problems one knows solutions, some of them are presented in this thesis. The last problem is still unsolved due to the lack of knowledge for some group integrals [88, 89, 90].

1.3 Outline

The discussions in this thesis belong to mathematical physics. The methods developed in the following parts are applied to problems known in random matrix theory. However they are not compared to any experimental data.

The thesis is divided into three main parts. In every single part a method is presented. All of them have their own applications. Indeed, one can combine these approaches. This yields powerful methods to calculate eigenvalue correlations. For example, the supersymmetry method (part II) in combination with the algebraic rearrangement (part III) drastically reduces the number of integrations which can be counted on the fingers of one hand.

In Chap. 2 of part I, I introduce some basic quantities in supersymmetry such as Grassmann variables and superfunctions. I explain their properties and what an integration over a Grassmann variable is defined by. In Chap. 3, I give a theorem which generalizes an approach of Wegner [91]. I employ rather general projection properties of superfunctions on superspaces to derive compact expressions for integrals over Grassmann variables. These expressions only involve derivatives with respect to the commuting variables. As particular examples I consider rotation invariant superfunctions in Chap. 4. I rederive and extend Cauchy–like integral theorems for these superfunctions [91, 58, 92, 93, 68]. Related to these theorems we show how Efetov–Wegner terms [94, 61, 82] can be obtained as boundary terms of integrations by parts. Efetov-Wegner terms appear by changing coordinates in superspace and have no analog in ordinary space. As another example I trace the supermatrix Bessel functions back to a product of ordinary matrix Bessel functions.

In part II, I explain what the supersymmetry method is. In Chap. 6, I give a brief introduction to the quantities I am interested in and sketch which steps of the supersymmetry method have to be made. The mapping from ordinary to superspace is explained in Chap. 7. I apply this mapping to arbitrary rotation invariant ensembles of real symmetric, Hermitian and Hermitian self-dual matrices. Thereby I extend the generalized Hubbard–Stratonovich transformation from the unitary case [61] to the other two cases, in Chap. 8. In Chap. 9, I calculate the k-point correlation function of these ensembles for finite N. Furthermore, I show that some ambiguities appearing in the generalized Hubbard–Stratonovich transformation do not affect the result. In

Chap. 10, the equivalence of this approach to the superbosonization formula [83, 84] is proven. Also, I present an alternative mapping from ordinary onto superspace. It yields in combination with the other two approaches of the supersymmetry method explicit formulas for the probability densities in superspace.

In part III, I study the determinantal and Pfaffian structures of the generating functions for factorizing probability densities. I outline our method in Chap. 13 and show that one finds supersymmetric structures, such as those found by Basor and Forrester [95], without actually mapping onto superspace. Here, I establish the link to supersymmetry. To the best of my knowledge this connection has not been observed before. The supersymmetric structures are derived in Chap. 14. In Chap. 15, I use these structures to derive determinantal and Pfaffian structures of three integral types in a quite general and direct way. This is done by purely algebraic manipulations. Hence, these types of integrals can be applied to a broad class of matrix ensembles shown in Chap. 16. Further, I calculate the k–point correlation function of an arbitrary unitarily invariant matrix ensemble in the presence of an external source.

Summaries of every single part are given in chapters 5, 12 and 17, respectively. In Chap. 18, I give an outlook on which open problems in random matrix theory the presented methods can be applied to. The appendices are also structured into three parts corresponding to the main parts of the thesis.

Part I

Harmonical analysis in superspaces

Chapter 2

Definitions and notations

The notations and definitions for a superspace are based on the descriptions in Berezin's book [96]. A brief and comprehensible introduction into the basic calculations is given in the appendices A and B of Ref. [59]. But also in the standard textbooks of Efetov [68] and Haake [26] the authors introduce the basic objects in supersymmetry theory and show some applications to random matrix theory.

In this chapter we recall some definitions and notations of the fundamental objects in superspace and their properties. In Sec. 2.1, we introduce the notion of a Grassmann variable and an arbitrary variable in a Grassmann algebra. Due to the commutation behavior of Grassmann variables a Grassmann algebra has a natural grading which allows us to construct superspaces. In Sec. 2.2, we define particular superspaces, more precisely supervectors and supermatrices according to a real, complex and quaternionic structure.

Our main interest in superspaces is less the physical nature of supersymmetry and its interpretation as it is in quantum field theory. For us it is a mathematical tool to map integrals over many variables in an ordinary space onto integrals with few variables in a superspace. Thus, we integrate at the end of the day over all Grassmann variables. In Sec. 2.3, we define the integration over Grassmann variables and give a standard example for Gaussian integrals in superspace.

2.1 The Grassmann algebra

We consider a complex Grassmann algebra $\Lambda = \bigoplus_{j=0}^{2L} \Lambda_j$. This algebra has $L \in \mathbb{N}$ pairs of complex Grassmann variables $\{\eta_j, \eta_j^*\}_{1\leq j\leq L} \subset \Lambda_1$. They differ from ordinary complex numbers, $\Lambda_0 = \mathbb{C}$, in their commutation behavior.

Grassmann variables are anticommuting variables, i.e.

$$[\eta_m, \eta_n]_+ = [\eta_m, \eta_n^*]_+ = [\eta_m^*, \eta_n^*]_+ = 0\,, \tag{2.1}$$

where $[.,.]_+$ is the anti-commutator. It follows from Eq. (2.1), that all Grassmann variables are nilpotent. Hence, they have no inverse and have no representation as numbers. We define in a canonical way the space of even $\Lambda^0 = \bigoplus_{j=0}^{L} \Lambda_{2j}$ and odd $\Lambda^1 = \bigoplus_{j=0}^{L-1} \Lambda_{2j+1}$ variables. The set Λ_n comprises all terms which are homogeneous of order n in these Grassmann variables. We recall that elements in Λ^0 commute with each other whereas elements in Λ^1 anti–commute.

The complex conjugation operator $(.)^* : \Lambda \to \Lambda$ can be generalized in two different ways. Since we want to extend the notion of a positive length, it has to fulfill the restriction

$$(\eta_m \eta_m^*)^* = \eta_m \eta_m^*\,. \tag{2.2}$$

Hence, we have the choice between the conjugation of the first kind

$$(\alpha\beta)^* = \beta^*\alpha^* \quad \forall \beta, \alpha \in \Lambda^1 \quad \Leftrightarrow \quad (\alpha^*)^* = \alpha \quad \forall \alpha \in \Lambda^1 \tag{2.3}$$

and the conjugation of the second kind

$$(\alpha\beta)^* = \alpha^*\beta^* \quad \forall \beta, \alpha \in \Lambda^1 \quad \Leftrightarrow \quad (\alpha^*)^* = -\alpha \quad \forall \alpha \in \Lambda^1\,. \tag{2.4}$$

For our purposes, both choices are equally good. Since the reader should recognize results known in the scientific community of random matrix theory, we restrict ourselves to the conjugation of the second kind throughout the thesis.

2.2 Supervectors and supermatrices and their symmetries

A supermatrix σ is build up of four blocks

$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_\eta \\ \sigma_\chi & \sigma_2 \end{bmatrix}\,. \tag{2.5}$$

The entries of the boson–boson block σ_1 and of the fermion–fermion block σ_2 are elements in Λ^0 whereas the entries of the off-diagonal blocks σ_η and σ_χ are in Λ^1. If σ is diagonal then the diagonal elements of σ_1 and σ_2 are called bosonic and fermionic eigenvalues, respectively.

The ordinary transposition is denoted by "T" and should not be mistaken for the supersymmetric one. However, the adjoint "$\dagger$" is the complex conjugation with the supersymmetric transposition "$T_{\rm S}$",

$$\sigma^{T_{\rm S}} = \left[\begin{array}{cc} \sigma_1^T & \sigma_\chi^T \\ -\sigma_\eta^T & \sigma_2^T \end{array}\right] . \tag{2.6}$$

This definition guarantees

$$(\sigma_1\sigma_2)^{T_{\rm S}} = \sigma_2^{T_{\rm S}}\sigma_1^{T_{\rm S}} \tag{2.7}$$

for two arbitrary supermatrices σ_1 and σ_2. For the ordinary transposition is Eq. (2.7) not true.

The extension of the trace and the determinant to supermatrices are the supertrace

$$\mathrm{Str}\,\sigma = \mathrm{tr}\,\sigma_1 - \mathrm{tr}\,\sigma_2 \tag{2.8}$$

and the superdeterminant

$$\mathrm{Sdet}\,\sigma = \frac{\det(\sigma_1 - \sigma_\eta\sigma_2^{-1}\sigma_\chi)}{\det\sigma_2} . \tag{2.9}$$

Though the determinant is defined for all matrices, the fermion–fermion block must be invertible for the superdeterminant to exist. Both definitions (2.8) and (2.9) are chosen in such a way that one has cyclic invariance

$$\mathrm{Str}\,\sigma_1\sigma_2 = \mathrm{Str}\,\sigma_2\sigma_1 , \tag{2.10}$$

see also App. B.1, and factorization

$$\mathrm{Sdet}\,\sigma_1\sigma_2 = \mathrm{Sdet}\,\sigma_1\mathrm{Sdet}\,\sigma_2 \tag{2.11}$$

for two arbitrary matrices σ_1 and σ_2.

Let β be the Dyson index, i.e $\beta = 1$ for the real number field, $\beta = 2$ for the complex one and $\beta = 4$ for the quaternionic one. We use the complex representation of the quaternionic numbers $\mathbb{H}$, i.e. in terms of Pauli matrices. For the relation between the single representations, we refer to a work by Jiang [97]. It is helpful to define the three parameters

$$\gamma_1 = \left\{\begin{array}{cl} 1 & , \ \beta\in\{2,4\} \\ 2 & , \ \beta = 1 \end{array}\right. \quad , \quad \gamma_2 = \left\{\begin{array}{cl} 1 & , \ \beta\in\{1,2\} \\ 2 & , \ \beta = 4 \end{array}\right. \tag{2.12}$$

and $\tilde{\gamma} = \gamma_1\gamma_2$.

As in the ordinary case of vectors and of matrices, we find three underlying structures resulting from the number fields which are responsible for the symmetries of supervectors and of supermatrices. We introduce the rectangular $(\gamma_2 c+\gamma_1 d)\times(\gamma_2 a+\gamma_1 b)$ supermatrix V on the complex Grassmann algebra $\Lambda=\bigoplus_{j=0}^{2(ad+bc)}\Lambda_j$. Such a supermatrix

$$V=\left(\Psi_{11}^{(\mathrm{C})},\ldots,\Psi_{a1}^{(\mathrm{C})},\Psi_{12}^{(\mathrm{C})},\ldots\Psi_{b2}^{(\mathrm{C})}\right)=\left(\Psi_{11}^{(\mathrm{R})*}\ldots,\Psi_{c1}^{(\mathrm{R})*},\Psi_{12}^{(\mathrm{R})*},\ldots\Psi_{d2}^{(\mathrm{R})*}\right)^{T_{\mathrm{S}}} \tag{2.13}$$

is defined by its columns

$$\Psi_{j1}^{(\mathrm{C})\dagger}=\begin{cases}\left(\{x_{jn}\}_{1\leq n\leq c},\{\chi_{jn},\chi_{jn}^*\}_{1\leq n\leq d}\right) & ,\ \beta=1,\\ \left(\{z_{jn}\}_{1\leq n\leq c},\{\chi_{jn}\}_{1\leq n\leq d}\right) & ,\ \beta=2,\\ \left(\left\{\begin{array}{cc} z_{jn1} & -z_{jn2}^* \\ z_{jn2} & z_{jn1}^*\end{array}\right\}_{1\leq n\leq c},\left\{\begin{array}{c}\chi_{jn}\\ \chi_{jn}^*\end{array}\right\}_{1\leq n\leq d}\right) & ,\ \beta=4,\end{cases} \tag{2.14}$$

$$\Psi_{j2}^{(\mathrm{C})\dagger}=\begin{cases}\left(\left\{\begin{array}{c}\zeta_{jn}\\ \zeta_{jn}^*\end{array}\right\}_{1\leq n\leq c},\left\{\begin{array}{cc}\tilde{z}_{jn1} & -\tilde{z}_{jn2}^*\\ \tilde{z}_{jn2} & \tilde{z}_{jn1}^*\end{array}\right\}_{1\leq n\leq d}\right) & ,\ \beta=1,\\ \left(\{\zeta_{jn}\}_{1\leq n\leq c},\{\tilde{z}_{jn}\}_{1\leq n\leq d}\right) & ,\ \beta=2,\\ \left(\{\zeta_{jn},\zeta_{jn}^*\}_{1\leq n\leq c},\{y_{jn}\}_{1\leq n\leq d}\right) & ,\ \beta=4,\end{cases} \tag{2.15}$$

or by its rows

$$\Psi_{j1}^{(\mathrm{R})\dagger}=\begin{cases}\left(\{x_{nj}\}_{1\leq n\leq a},\{\zeta_{nj}^*,-\zeta_{nj}\}_{1\leq n\leq b}\right) & ,\ \beta=1,\\ \left(\{z_{nj}^*\}_{1\leq n\leq a},\{\zeta_{nj}^*\}_{1\leq n\leq b}\right) & ,\ \beta=2,\\ \left(\left\{\begin{array}{cc} z_{nj1}^* & z_{nj2}^*\\ -z_{nj2} & z_{nj1}\end{array}\right\}_{1\leq n\leq a},\left\{\begin{array}{c}\zeta_{nj}^*\\ -\zeta_{nj}\end{array}\right\}_{1\leq n\leq b}\right) & ,\ \beta=4,\end{cases} \tag{2.16}$$

$$\Psi_{j2}^{(\mathrm{R})\dagger}=\begin{cases}\left(\left\{\begin{array}{c}-\chi_{nj}^*\\ \chi_{nj}\end{array}\right\}_{1\leq n\leq a},\left\{\begin{array}{cc}\tilde{z}_{nj1}^* & \tilde{z}_{nj2}^*\\ -\tilde{z}_{nj2} & \tilde{z}_{nj1}\end{array}\right\}_{1\leq n\leq b}\right) & ,\ \beta=1,\\ \left(\{-\chi_{nj}^*\}_{1\leq n\leq a},\{\tilde{z}_{nj}^*\}_{1\leq n\leq b}\right) & ,\ \beta=2,\\ \left(\{-\chi_{nj}^*,\chi_{nj}\}_{1\leq n\leq a},\{y_{nj}\}_{1\leq n\leq b}\right) & ,\ \beta=4\end{cases} \tag{2.17}$$

which are real, complex and quaternionic supervectors. We use the complex Grassmann variables χ_{mn} and ζ_{mn} and the real numbers x_{mn} and y_{mn}. Further, we introduce the complex numbers z_{mn}, $\tilde{z}_{mn}$, z_{mnl} and $\tilde{z}_{mnl}$. The set of matrices with the structure (2.13) is denoted by $\mathrm{Mat}_\beta^0(c\times a/d\times b)$. The set $\mathrm{Mat}_\beta(c\times a/d\times b)$ is the set of matrices which are analytically extended in their entries to the full graded structure of Λ. If the supermatrix is quadratic, i.e. $a=c$ and $b=d$, then we use the simplified notation $\mathrm{Mat}_\beta^0(a/b)$ and $\mathrm{Mat}_\beta(a/b)$.

The rectangular supermatrix above fulfills the property

$$V^* = Y_{cd} V Y_{ab}^T \,, \tag{2.18}$$

where

$$Y_{pq}\big|_{\beta=1} = \left[\begin{array}{cc} \mathbb{1}_p & 0 \\ 0 & Y_s \otimes \mathbb{1}_q \end{array}\right] \quad , \quad Y_{pq}\big|_{\beta=4} = \left[\begin{array}{cc} Y_s \otimes \mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{array}\right] \tag{2.19}$$

and $Y_{pq}\big|_{\beta=2} = \mathbb{1}_{p+q}$. We use the symplectic unit

$$Y_s = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \tag{2.20}$$

and the $N \times N$ unit matrix $\mathbb{1}_N$.

We define the supergroup

$$\mathrm{U}^{\,(\beta)}(p/q) = \left\{\begin{array}{lll} \mathrm{UOSp}^{\,(+)}(p/2q) & , & \beta = 1 \\ \mathrm{U}\,(p/q) & , & \beta = 2 \\ \mathrm{UOSp}^{\,(-)}(2p/q) & , & \beta = 4 \end{array}\right. \subset \mathrm{Mat}_{\,\beta}(p/q) \tag{2.21}$$

and use the notation of Refs. [98] for the representations $\mathrm{UOSp}^{\,(\pm)}$ of the supergroup UOSp. These representations are related to the classification of Riemannian symmetric superspaces by Zirnbauer [99]. The index "+" in Eq. (2.21) refers to real entries in the boson–boson block and to quaternionic entries in the fermion–fermion block and "−" indicates the other way around. Then, the supermatrix set $\mathrm{Mat}_{\,\beta}(c \times a/d \times b)$ is invariant under the action $\mathrm{U}^{\,(\beta)}(c/d) \times \mathrm{U}^{\,(\beta)}(a/b)$ of left and right multiplication.

The construction of a symmetric superspace is easily done by adding the self–adjointness to the set $\mathrm{Mat}_{\,\beta}(p/q)$, i.e.

$$\Sigma_{\beta,p/q} = \{\sigma \in \mathrm{Mat}_{\,\beta}(p/q) | \sigma^\dagger = \sigma\} \,. \tag{2.22}$$

The matrices lying in $\Sigma_{\beta,p/q}$ are referred to as $\mathrm{U}^{\,(\beta)}(p/q)$–symmetric supermatrices. The restriction of this set to $\Lambda_0 \oplus \Lambda_1$ is $\Sigma^0_{\beta,p/q}$.

We introduce a generalized Wick–rotation $e^{\imath\psi}$, $\psi \in]0,\pi[$, to guarantee the convergence of the supermatrix integrals in the ensuing sections . The usual choice of a Wick–rotation is $e^{\imath\psi} = \imath$ for investigations of Gaussian probability densities [58, 59, 61]. Here, general Wick–rotations are also of interest. Probability densities which lead to superfunctions as $\exp\left(-\mathrm{Str}\,\sigma^4\right)$ do not converge with the choice $\imath$. The Wick–rotated set is $\Sigma^\psi_{\beta,p/q} = \widehat{\Pi}_\psi \Sigma^0_{\beta,p/q} \widehat{\Pi}_\psi$ with the matrix $\widehat{\Pi}_\psi = \mathrm{diag}\left(\mathbb{1}_{\gamma_2 p}, e^{\imath\psi}\mathbb{1}_{\gamma_1 q}\right)$.

In the rest of our work, we restrict the calculations to a particular class of superfunctions. These superfunctions have a Wick–rotation such that the integrals are convergent. We have not explicitly analyzed the class of such functions. However, this class is very large and sufficient for physical interests. We consider the probability distribution

$$P(\sigma) = f(\sigma)\exp(-\mathrm{Str}\,\sigma^{2m}), \tag{2.23}$$

where $m \in \mathbb{N}$ and f is a superfunction which does not increase so fast as $\exp(\mathrm{Str}\,\sigma^{2m})$ in the infinity. In particular, there is an integer $m \in \mathbb{N}$ such that

$$\left(\lim_{\epsilon\to\infty}\exp\left(-\epsilon\mathrm{Str}\,\sigma^{2m}\right) = 0 \Rightarrow \lim_{\epsilon\to\infty} P\left(\epsilon\sigma\right) = 0\right) \quad \forall\sigma \in \Sigma^{\alpha}_{\beta,p/q} \tag{2.24}$$

for every angle $\alpha \in [0, 2\pi]$. Then, a Wick–rotation exists for P.

Let Herm (β, N) be either the set of $N\times N$ real symmetric $(\beta = 1)$, $N\times N$ Hermitian $(\beta = 2)$ or $2N \times 2N$ self-dual $(\beta = 4)$ matrices, according to the Dyson–index β. To visualize the block structure of the matrices in $\Sigma^{\psi}_{\beta,p/q}$, we give examples for $\beta = 1$

$$\sigma = \begin{bmatrix} \sigma_1 & e^{\imath\psi/2}\sigma_\eta & e^{\imath\psi/2}\sigma_\eta^* \\ -e^{\imath\psi/2}\sigma_\eta^\dagger & e^{\imath\psi}\sigma_{21} & e^{\imath\psi}\sigma_{22} \\ e^{\imath\psi/2}\sigma_\eta^T & -e^{\imath\psi}\sigma_{22}^* & e^{\imath\psi}\sigma_{21}^* \end{bmatrix}, \tag{2.25}$$

for $\beta = 2$

$$\sigma = \begin{bmatrix} \sigma_1 & e^{\imath\psi/2}\sigma_\eta \\ -e^{\imath\psi/2}\sigma_\eta^\dagger & e^{\imath\psi}\sigma_2 \end{bmatrix}, \tag{2.26}$$

and for $\beta = 4$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & e^{\imath\psi/2}\sigma_\eta \\ -\sigma_{12}^* & \sigma_{11}^* & e^{\imath\psi/2}\sigma_\eta^* \\ -e^{\imath\psi/2}\sigma_\eta^\dagger & e^{\imath\psi/2}\sigma_\eta^T & e^{\imath\psi}\sigma_2 \end{bmatrix}. \tag{2.27}$$

The boson–boson block σ_1 lies in Herm (β, p) and the fermion–fermion block σ_2 is an element in Herm $(4/\beta, q)$.

2.3 Integration in superspaces

Following Berezin [96], the integration of complex Grassmann variables is formally defined by

$$\int\limits_{\Lambda_1} \eta_i^n d\eta_j = \int\limits_{\Lambda_1} (\eta_i^*)^n d\eta_j^* = \frac{\delta_{n1}\delta_{ij}}{\sqrt{2\pi}} \qquad n \in \{0, 1\}\,. \tag{2.28}$$

The constant on the right hand side can be arbitrarily chosen. We fix it with the constant $1/\sqrt{2\pi}$ which is a common choice. The differentials $\{d\eta_j, d\eta_j^*\}$ are up to a constant equal to the partial derivatives $\{\partial/\partial\eta_j, \partial/\partial\eta_j^*\}$ and, also, build a Grassmann algebra.

Let M be a real p-dimensional differentiable, flat manifold. We are interested in functions on the superspace $\Lambda(p, 2L)$ with the base M and a sheaf of algebras $\mathfrak{A}$ [96]. Let $\mathfrak{U}$ be an open subset of M then $\mathfrak{A}(\mathfrak{U})$ is the algebra of functions on $\mathfrak{U}$ with values in Λ. We split $\Lambda(p, 2L)$ into a direct sum $\Lambda^0(p, 2L) \oplus \Lambda^1(p, 2L)$ corresponding to the $\mathbb{Z}_2$ grading of Λ. Functions on $\mathfrak{L}(p, 2L) = (\Lambda^0(p, 2L))^p \times (\Lambda^1(p, 2L))^{2L}$ with values in Λ can be represented as a finite power series in the generators of Λ

$$f(x,\eta) = \sum_{j_1,j_2\in\mathbb{I}} f_{j_1 j_2}(x) \left(\prod_{n=1}^{L} (\eta_n^*)^{j_{1n}} \eta_n^{j_{2n}} \right) \tag{2.29}$$

where j_1 and j_2 are multiple indices in the set $\mathbb{I} = \{0,1\}^L$ and $\prod_{n=1}^{L} A_n = A_1 A_2 \dots A_n$ is an ordered product. We call functions which have the representation (2.29) superfunctions.

Assuming the integral over M of $f_{1,\dots,1}$ exists, then integrals over superfunctions are defined by

$$\int\limits_{\mathfrak{L}(p,2L)} f(x,\eta) d[x,\eta] = (2\pi)^{-L} \int\limits_{M} f_{1,\dots,1}(x) d[x] \,, \tag{2.30}$$

where $d[x,\eta] = d[x]d[\eta] = \prod_{j=1}^{p} dx_j \prod_{i=1}^{L} d\eta_i d\eta_i^*$.

When changing coordinates under the integral (2.30), i.e.

$$\{x_j, \eta_i\} = \{x_j(y_m, \chi_n), \eta_i(y_m, \chi_n)\} \,, \quad g(y,\chi) = f(x,\eta) \,, \tag{2.31}$$

one has to guarantee that the definition (2.28) is true for the old Grassmann variables η_i as well as for the new ones χ_i. This yields

$$\int\limits_{\mathfrak{L}(p,2L)} f(x,\eta) d[x,\eta] = \int\limits_{\mathfrak{L}(p,2L)} g(y,\chi) \mathrm{Ber}_{p,2L} \left(\frac{\partial(x_j,\eta_i)}{\partial(y_m,\chi_n)} \right) d[y,\chi] + \text{b.t.} \tag{2.32}$$

Here, the Berezinian $\mathrm{Ber}_{p,2L}$ is defined as the superdeterminant of the $(p + 2L) \times (p + 2L)$ dimensional Jacobi matrix

$$\frac{\partial(x_j,\eta_i)}{\partial(y_m,\chi_n)} = \begin{bmatrix} \dfrac{\partial x_a}{\partial y_b} & \dfrac{\partial x_a}{\partial \chi_b} \\ \dfrac{\partial \eta_a}{\partial y_b} & \dfrac{\partial \eta_a}{\partial \chi_b} \end{bmatrix} \tag{2.33}$$

which is a supermatrix. The boundary terms "b.t." are referred to as Efetov–Wegner terms [94, 100, 82] and vanish if the Berezinian is constant or the boundary is empty. It is no surprise that such terms appear, since the integrals over the Grassmann variables are equivalent to the derivatives with respect to them. Some time ago, Rothstein [101] found that a change of variables when integrating over a superspace leads to differential operators, which are incorporated into the invariant measure. These differential operators are exponential functions of vector fields.

As a simple application of an integration in superspace, we consider a Gaussian integral over the set $\mathrm{Mat}\,_\beta^0(c\times a/d\times b)$ with broken rotation symmetry due to the matrices $\sigma\in\Sigma_{\beta,a/b}^0$ and $\rho\in\Sigma_{\beta,c/d}^0$. We find

$$\begin{aligned}&\int\limits_{\mathrm{Mat}\,_\beta^0(c\times a/d\times b)}\exp\left[\imath\mathrm{Str}\,V^\dagger V\sigma-\imath\mathrm{Str}\,\rho VV^\dagger\right]d[V]\\ =\ &\mathbf{K}_{ab}^{cd}\,\mathrm{Sdet}\,^{-1/\tilde{\gamma}}\left(\sigma\otimes\mathbb{1}_{\gamma_2c+\gamma_1d}-\mathbb{1}_{\gamma_2a+\gamma_1b}\otimes\rho\right)\end{aligned}\tag{2.34}$$

with the constant

$$\mathbf{K}_{ab}^{cd}=\left(\frac{\imath\pi}{\gamma_2}\right)^{\beta ac/2}\left(\frac{-\imath\pi}{\gamma_1}\right)^{4bd/\beta}\left(\frac{\imath\tilde{\gamma}}{2\pi}\right)^{ad+bc}.\tag{2.35}$$

The measure of V is

$$d[V]=\prod_{\substack{1\leq m\leq a\\1\leq n\leq c\\1\leq l\leq\beta}}dx_{mnl}\prod_{\substack{1\leq m\leq b\\1\leq n\leq d\\1\leq l\leq 4/\beta}}dy_{mnl}\prod_{\substack{1\leq m\leq b\\1\leq n\leq c}}d\zeta_{mn}d\zeta_{mn}^*\prod_{\substack{1\leq m\leq a\\1\leq n\leq d}}d\chi_{mn}d\chi_{mn}^*\,.\tag{2.36}$$

Here, x_{mna} and y_{mna} are the independent real components of the real, complex and quaternionic numbers of the supervectors $\Psi_{j1}^{(\mathrm{R})}$ and $\Psi_{j2}^{(\mathrm{R})}$, respectively. In particular for $b=c=0$, $a=d=1$, $\beta=2$ and $\rho=0$, we have

$$\int\limits_{\Lambda_2}\exp\left[\imath\sigma\chi^*\chi\right]d\chi d\chi^*=\frac{\imath\sigma}{2\pi}\,.\tag{2.37}$$

This indeed agrees with the expansion

$$\exp\left[\imath\sigma\chi^*\chi\right]=1+\imath\sigma\chi^*\chi\tag{2.38}$$

in combination with the definition (2.28). We notice that the variable σ stands in the numerator in Eq. (2.37) whereas such a constant is in the denominator for Gaussian integrals over commuting variables,

$$\int\limits_{\mathbb{C}}\exp\left[-\sigma|z|^2\right]dzdz^*=\frac{-2\pi\imath}{\sigma}\quad,\quad\sigma>0\,.\tag{2.39}$$

The simple equations (2.37) and (2.39) as well as the more general formula (2.34) are the point of contact between random matrix theory and supersymmetry. Thus, we will often come back to the integral (2.34).

Chapter 3

Integrations in superspaces and Cauchy–like theorems

Various integral theorems exist in superanalysis which have no counterpart in ordinary analysis. Parisi and Sourlas [57] were the first to give such a theorem as a dimensional reduction. They related this feature to an invariance of the integrand with respect to a superrotation, which preserves the length of a supervector. Efetov [58] also obtained such a theorem for functions on the set of $\mathrm{U}^{(2)}(1/1)$–symmetric supermatrices which are rotation invariant and have zero boundary condition at infinity. He discussed that these integral theorems are also true for an integration over superfunctions which are invariant under the action of more general groups and applied these theorems for his calculations [68].

The equivalence of a Grassmann integration with the action of a differential operator is well known. If the superfunction is expanded in a Taylor series, any symmetry must manifest itself in the coefficients of this series. For an integration over a rotation invariant superfunction on symmetric supermatrices, one can change from Cartesian integration variables to eigenvalue–angle coordinates, i.e. the diagonalization of supermatrices. Then the superfunction is independent of the angles. Thus, only differential operators remain which stem from the transformation in the sense of Rothstein [101]. However, as the vector fields of such a coordinate transformation are very difficult to calculate, we do not pursue this route.

Another approach is due to Wegner [91] (worked out in Refs. [92, 93]), who generalized Efetov's result to the case of an integration over functions on sets of supermatrices which are invariant under the action of a discrete subgroup of a classical Lie supergroup. In their studies, the integration of the Grassmann variables gives a differential operator with respect to the ordinary variables. The integration over these commuting variables is performed in

Ref. [93] and leads to a Cauchy–like integral theorem.

In Sec. 3.1, we map the integration over all Grassmann variables of such a function onto the action of a differential operator with respect to the commuting variables. This differential operator is uniquely defined by the invariance class of the function. The derivation of this differential operator is rather general and applies to a wide class of functions.

Our result leads to integral theorems for supervectors and for supermatrices, as explained in Sec. 3.2 and Sec. 3.3, respectively. We also extend results obtained for the supergroup $\mathrm{U}^{(2)}(k/k)$ in Ref. [93] to the supergroups $\mathrm{U}^{(\beta)}(p/q)$. In Sec. 3.4, we show that for superfunctions invariant under $\mathrm{UOSp}\,(1/2)$ no Cauchy–like integral theorem exists.

3.1 Integration of Grassmann variables over invariant functions

Let $\mathfrak{L}(p,2L)$ be a Riemannian superspace with metric g. This metric is assumed to be diagonal and constant. The inner product of two elements in $\mathfrak{L}(p,2L)$ is

$$g((a,\alpha),(b,\beta)) = \sum_{n=1}^{p} g_n a_n b_n + \sum_{m=1}^{L} h_m(\alpha_m^*\beta_m + \beta_m^*\alpha_m)\,, \tag{3.1}$$

$g_n \in \mathbb{R}^+$ and $h_m \in \mathbb{C}$. We take the variables a_n, b_n and α_m, β_m from Λ^0 and Λ^1, respectively.

The idea of Wegner [91] (worked out in Ref. [93]) is the following. Consider the superfunction f which is invariant under a discrete subgroup of a Lie group acting on $\mathfrak{L}(p,2L)$, or equivalently, which is invariant under all transformations which connect the commuting with the anticommuting variables. Then all $f_{j_1j_2}$ in Eq. (2.29) are functionally dependent on the body $f_{0,0} = f(x,0) = f(x)$. In this spirit, we assume that there exists a differentiable map $\phi:\ \mathfrak{L}(p,2L) \to \mathfrak{L}(p,2L)$ with the properties $\phi(a,\alpha) = (r(a,\alpha),0)$ and $f(x,\eta) = f(r(x,\eta),0) = f(r(x,\eta))$ where $\{r_j\}_{1\le j\le p}$ are mappings onto Λ^0. The image $\mathcal{N} = \{(r(x,\eta),0)|(x,\eta)\in\mathfrak{L}(p,2L)\} \subset (\Lambda^0(p,2L))^p$ is a differentiable sub-supermanifold of $(\Lambda^0(p,2L))^p$. The dimension of this image tells us how many independent variables are needed to describe the resulting supermanifold in terms of commuting variables. These variables are referred to as radial variables. The remaining variables which parametrize the complement of this submanifold with respect to $\mathfrak{L}(p,2L)$ are referred to as angular variables.

Theorem 3.1.1 (integration–differentiation operator identity)
Let f be a differentiable superfunction on $\mathfrak{L}(p,2L)$ which is invariant under a differentiable map $\phi: \mathfrak{L}(p,2L) \to \mathfrak{L}(p,2L)$ with $\phi(a,\alpha) = (r(a,\alpha),0)$, that is $f(x,\eta) = f(r(x,\eta))$. Define the integral of f with respect to all Grassmann variables contained in f by $\int f(x,\eta)d[\eta]$. Then there exists a differential operator $D_{C,S}$ with respect to the real variables such that

$$\int\limits_{\Lambda_{2L}} f(x,\eta)d[\eta] = D_{C,S}(r)f(r,0) \tag{3.2}$$

holds. The differential operator $D_{C,S}$ is explicitly given by

$$D_{C,S}(r) = \frac{1}{L!(4\pi)^L}\left(\prod_{m=1}^{L} h_m\right)\sum_{n=0}^{L}\binom{L}{n}\Delta_C^{L-n}\left(-\Delta_{S,r}(r)\right)^n \ , \tag{3.3}$$

where

$$\Delta_C = \sum_{n=1}^{p}\frac{1}{g_n}\frac{\partial^2}{\partial x_n^2} \quad \text{and} \quad \Delta_S = \Delta_C + 2\sum_{m=1}^{L}\frac{1}{h_m}\frac{\partial^2}{\partial\eta_m\partial\eta_m^*} \tag{3.4}$$

are the Laplace–Beltrami operators of the pure commuting part and the whole superspace, respectively. Thus, the index C refers to a differential operator acting only on the space $(\Lambda^0(p,2L))^p$ and the index S denotes differential operators acting on the whole space $\mathfrak{L}(p,2L)$. The Laplace–Beltrami operator $\Delta_{S,r}(r)$ is the radial part of the differential operator Δ_S according to the mapping ϕ and comprises only the radial coordinates r and partial derivatives thereof.

The theorem is proven in App. A.1.

Theorem 3.1.1 connects the Berezin integral with the Laplace–Beltrami operators which is indeed the other way around as in Refs. [102, 103]. In these works the authors first define the Berezin integrals over the sphere and the ball in a flat superspace by differential operators. Then they extend this definition to an integration over the full superspace. Although their Laplace–Beltrami operators are presented in flat coordinates and their superfunctions are polynomials times a Gaussian function, one notices the intimate connection between their approach and ours.

The formula (3.2) will be applied to particular cases in the ensuing sections. We will thereby also re-derive some results of Refs. [91, 92, 93]. However, in particular in the matrix case to be discussed in Sec. 3.3 the operator $D_{C,S}(r)$ becomes very complex and a handier expression is thus highly desirable. We therefore first rewrite it using a transformation akin to the Baker–Campbell–Hausdorff formula. A full proof of the following lemma is given in App. A.2.

Lemma 3.1.2
The operator $D_{C,S}(r)$ is a differential operator of order L and can be written as

$$D_{C,S}(r) = \frac{1}{L!(4\pi)^L}\left(\prod_{m=1}^{L} h_m\right)\mathrm{IAd}[\Delta_C, \Delta_C - \Delta_{S,r}]^L(\boldsymbol{1}) \qquad (3.5)$$

where

$$\begin{aligned} \mathrm{IAd}[A,B](H) &= [A,H]_- + HB \quad \text{and} \\ \mathrm{IAd}[A,B]^N(H) &= [A,\mathrm{IAd}[A,B]^{N-1}(H)]_- + \mathrm{IAd}[A,B]^{N-1}(H)B \end{aligned} \qquad (3.6)$$

for three arbitrary linear operators A, B and H. The operator $\boldsymbol{1}$ is the identity operator. Moreover, the operator $D_{C,S}(r)$ is a differential operator of order L.

We use the symbol IAd to indicate a similarity to the linear operator $\mathrm{Ad}[A](B) = [A,B]_- = AB - BA$ which is the adjoint representation of a linear operator A in a Lie algebra.

3.2 Integral theorems for invariant functions on supervectors

As a first example for theorem 3.1.1, we consider functions on a space of supervectors which are invariant under the action of $\mathrm{U}^{(1)}(p/L)$. The invariance condition on a supervector v of the form (2.16) for $\beta = 1$ is

$$f(v) = f(Uv) \qquad (3.7)$$

for all $U \in \mathrm{U}^{(1)}(p/L)$. In this case the metric (3.1) is given by $g_n = h_m = 1$. The function f only depends on the invariant length $r = \sqrt{v^\dagger v}$. Eq. (3.2) takes the form

$$\int\limits_{\Lambda_{2L}} f(x,\eta)d[\eta] \qquad (3.8)$$

$$= \frac{1}{L!(4\pi)^L}\sum_{n=0}^{L}\binom{L}{n}\left(\frac{1}{r^{p-1}}\frac{\partial}{\partial r}r^{p-1}\frac{\partial}{\partial r}\right)^{L-n}\left(-\frac{1}{r^{p-1-2L}}\frac{\partial}{\partial r}r^{p-1-2L}\frac{\partial}{\partial r}\right)^{n} f(r)\,.$$

The differential operator on the right hand side of (3.8) is independent of p due to the invariance of f with respect to the orthogonal group in the commuting part. Hence, the commuting variables on the left hand side of

(3.8) can be written in terms of the radial coordinates only. We calculate the integral over f on an effective superspace $\Lambda^0(1,L)$ and obtain

$$\begin{aligned}
\int\limits_{\Lambda_{2L}} f(x,\eta)d[\eta] &= \frac{1}{L!(4\pi)^L}\sum_{n=0}^{L}\binom{L}{n}\left(\frac{\partial^2}{\partial r^2}\right)^{L-n}\left(-r^{2L}\frac{\partial}{\partial r}\frac{1}{r^{2L}}\frac{\partial}{\partial r}\right)^n f(r)\\
&= \left(\frac{1}{2\pi}\right)^L\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^L f(r)\\
&= D_r^{(1,L)}f(r)\,.
\end{aligned} \tag{3.9}$$

The second equality is true due to the particular structure of the length of a supervector which is shown in appendix A.3.

In particular, we obtain for small dimension L

$L=1$:

$$\int\limits_{\Lambda_2} f(x,\eta)d[\eta] = \frac{1}{2\pi}\frac{1}{r}\frac{\partial}{\partial r}f(r)\ , \tag{3.10}$$

$L=2$:

$$\int\limits_{\Lambda_4} f(x,\eta)d[\eta] = \left(\frac{1}{2\pi}\right)^2\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}-\frac{1}{r^3}\frac{\partial}{\partial r}\right)f(r)\ . \tag{3.11}$$

In general we can formulate the following integral theorem. This theorem generalizes the theorem of Wegner [91] worked out in theorem 4.1 of Ref. [93], which focusses on complex supervectors, to the case of real supervectors. We give a complete proof in appendix A.3.

Theorem 3.2.1 (real supervectors)
Let f be a differentiable function of supervectors v and of their adjoints $v^\dagger$ of the form (2.16) for $\beta=1$. Let f be invariant under the action of $\mathrm{U}^{(1)}(p/L)$ and have zero boundary condition at infinity. Then, we have

$$\int\limits_{\mathrm{Mat}_1^0(p\times 1/L\times 0)} f(x,\eta)d[x,\eta] = \tag{3.12}$$

$$= \begin{cases}
\imath^p\left(\dfrac{1}{2\pi r}\dfrac{\partial}{\partial r}\right)^{L-p/2} f(r)\Bigg|_{r=0} & ,\ p<2L\ \wedge\ p\in(2\mathbb{N}_0)\\
\imath^{p-1}\displaystyle\int\limits_{\mathbb{R}}\left(\dfrac{1}{2\pi\tilde{x}}\dfrac{\partial}{\partial\tilde{x}}\right)^{L-(p-1)/2} f(\tilde{x})d\tilde{x} & ,\ p<2L\ \wedge\ p\in(2\mathbb{N}_0+1)\\
(-1)^L f(0) & ,\ p=2L\\
(-1)^L\displaystyle\int\limits_{\mathbb{R}^{p-2L}} f(\tilde{x})d[\tilde{x}] & ,\ p>2L
\end{cases}$$

where $\tilde{x}$ *refers to the canonical embedding of the lower dimensional integration set in* $\mathrm{Mat}\,_1^0(p \times 1/L \times 0)$.

This theorem can easily be extended to functions $f(v, v^\dagger)$ of a complex supervector v and of its adjoint $v^\dagger$ of the form (2.16) for $\beta = 2$, which are invariant under the action of the group $\mathrm{U}\,^{(2)}(p/L)$. The radial variable is the length $\sum_{n=1}^{p}(a_n^2 + b_n^2) + \sum_{m=1}^{L} \alpha_m^* \alpha_m$ of the supervector. The metric g, Eq. (3.1), is defined by $g_n = 1$ and $h_m = 1/2$. We find

$$\begin{aligned} \int\limits_{\Lambda_{2L}} f(x, y, \eta) d[\eta] &= \left(\frac{1}{4\pi}\right)^L \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^L f(r) \\ &= D_r^{(2,L)} f(r)\,. \end{aligned} \tag{3.13}$$

An integration theorem follows rightaway.

Theorem 3.2.2 (complex supervectors)
Let f *be a differentiable function on supervectors* v *and of their adjoints* $v^\dagger$ *of the form* (2.16) *for* $\beta = 2$, *which is invariant under the action of* $\mathrm{U}\,^{(2)}(p/L)$ *and which has zero boundary condition at infinity, then*

$$\int\limits_{\mathrm{Mat}\,_2^0(p\times 1/L\times 0)} f(z, \eta) d[z, \eta] = \begin{cases} \left(-\frac{1}{2}\right)^L \left(\frac{-1}{2\pi r}\frac{\partial}{\partial r}\right)^{L-p} f(r)\Big|_{r=0} & , \ p < L \\ \left(-\frac{1}{2}\right)^L f(0) & , \ p = L \\ \left(-\frac{1}{2}\right)^L \int\limits_{\mathbb{C}^{p-L}} f(\tilde{z}) d[\tilde{z}] & , \ p > L \end{cases} \tag{3.14}$$

where $d[z] = \prod_{n=1}^{p} d\mathrm{Re}\ z_n d\mathrm{Im}\ z_n$ *and* $\tilde{z}$ *refers to the canonical embedding of the lower dimensional integration set in* $\mathrm{Mat}\,_2^0(p \times 1/L \times 0)$.

For $p = L$, this integral theorem coinides with the integral theorem 4.1 of Ref. [93].

We now turn to functions $f(v, v^\dagger)$ of vectors v of the form (2.16) for $\beta = 4$. We assume these function to be invariant under $\mathrm{U}\,^{(4)}(p/L)$. Hence, these functions only depend on the quaternionic matrix $v^\dagger v$ which is diagonal and self-adjoint. The corresponding metric is defined by $g_n = 2$ and $h_m = 1$. This leads to the differential operator

$$\begin{aligned} \int\limits_{\Lambda_{2L}} f(x, y, \eta) d[\eta] &= \left(\frac{1}{4\pi}\right)^L \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^L f(r) \\ &= D_r^{(4,L)} f(r)\,, \end{aligned} \tag{3.15}$$

implying the integral theorem

Theorem 3.2.3 (quaternionic supervectors)
Let f be a differentiable function on $(2p+L)\times 2$ supervectors of the form (2.16) for $\beta=4$ which is invariant under the action of $\mathrm{U}^{(4)}(p/L)$ and has zero boundary condition at infinity. Then, we have

$$\int\limits_{\mathrm{Mat}_4^0(p\times 1/L\times 0)} f(A,\eta)d[A,\eta] = \begin{cases} \dfrac{1}{2^L}\left(\dfrac{1}{2\pi r}\dfrac{\partial}{\partial r}\right)^{L-2p} f(r)\Bigg|_{r=0} & ,\ 2p<L \\ \dfrac{1}{2^L}f(0) & ,\ 2p=L \\ \left(-\dfrac{1}{2}\right)^{L}\displaystyle\int\limits_{\mathbb{H}^{p-L/2}} f(\tilde{A})d[\tilde{A}] & ,\ 2p>L \end{cases} \tag{3.16}$$

where $d[A]=\prod_{n=1}^{p} d\mathrm{Re}\ z_{1n}d\mathrm{Im}\ z_{1n}d\mathrm{Re}\ z_{2n}d\mathrm{Im}\ z_{2n}$ and $A_n=\begin{bmatrix} z_{n1} & z_{n2} \\ -z_{n2}^* & z_{n1}^*\end{bmatrix}$ and $\tilde{A}$ refers to the canonical embedding of the lower dimensional integration set in $\mathrm{Mat}_4^0(p\times 1/L\times 0)$. For odd L in the case $2p>L$, we integrate over a vector depending on $p-(L-1)/2$ quaternions and one diagonal quaternion.

The three theorems given here are crucial for the proof of the ensuing integral theorems for invariant functions on supermatrix spaces.

As a useful application of these three theorems, we consider a rectangular supermatrix V of the form (2.13). To guarantee the convergence of the integrals below, let $V_\psi=\widehat{\Pi}_\psi V$ and $V_{-\psi}^\dagger=V^\dagger\widehat{\Pi}_\psi$ be the Wick–rotated matrices. The supermatrix

$$B_\psi=\frac{1}{\tilde{\gamma}}V_\psi V_{-\psi}^\dagger \tag{3.17}$$

is a $\mathrm{U}^{(\beta)}(c/d)$–symmetric supermatrices which is an element in $\Sigma_{\beta,a/b}^{\psi}$. Due to the similarity to ordinary Wishart matrices, we refer to it as supersymmetric Wishart matrix. Considering a function f on the set of supersymmetric Wishart matrices, we give a lemma and a corollary which are of equal importance for the superbosonization formula and the generalized Hubbard–Stratonovich transformation given in part II. This lemma was proven in Ref. [84] by representation theory. Here, we only state it.

Lemma 3.2.4
Let f be a superfunction on rectangular supermatrices of the form (2.13) and invariant under

$$f(V_\psi,V_{-\psi}^\dagger)=f\left(V_\psi U^\dagger,UV_{-\psi}^\dagger\right)\,, \tag{3.18}$$

for all V and $U \in \mathrm{U}^{(\beta)}(a/b)$. Then there is a superfunction F on the $\mathrm{U}^{(\beta)}(c/d)$–symmetric supermatrices with

$$F(B_\psi) = f(\widehat{V}_\psi, \widehat{V}_{-\psi}^\dagger)\,. \tag{3.19}$$

The invariance condition (3.18) implies that f only depends on the rows of V_ψ by $\Psi_{nr}^{(\mathrm{R})\dagger}\Psi_{ms}^{(\mathrm{R})}$ for arbitrary n, m, r and s. These scalar products are the entries of the supermatrix B_ψ which leads to the statement.

The corollary below states that an integration over supersymmetric Wishart matrices can be reduced to integrations over supersymmetric Wishart matrices consisting of a lower dimensional rectangular supermatrix. We assume that $\tilde{a} = a - 2(b - \tilde{b})/\beta \geq 0$ with

$$\tilde{b} = \left\{ \begin{array}{cl} 1 & ,\ \beta = 4 \text{ and } b \in 2\mathbb{N}_0 + 1 \\ 0 & ,\ \text{else} \end{array} \right. . \tag{3.20}$$

Corollary 3.2.5

Let F be the superfunction of lemma 3.2.4, analytic in its real independent entries and a Schwartz function on the Wick–rotated real axis. Then, we find

$$\int\limits_{\mathrm{Mat}_\beta^0(c\times a/d\times b)} F(B_\psi)d[V] = C \int\limits_{\mathrm{Mat}_\beta^0(c\times \tilde{a}/d\times \tilde{b})} F(\widetilde{B}_\psi)d[\widetilde{V}]\,, \tag{3.21}$$

where $\widetilde{B}_\psi$ is defined as B_ψ The constant is

$$C = \left[-\frac{\gamma_1}{2}\right]^{(b-\tilde{b})c} \left[\frac{\gamma_2}{2}\right]^{(a-\tilde{a})d} \tag{3.22}$$

and the measure is the one of Eq. (2.36). The $(\gamma_2 c + \gamma_1 d) \times (\gamma_2 \tilde{a} + \gamma_1 \tilde{b})$ supermatrix $\widetilde{V}$ and its measure $d[\widetilde{V}]$ is defined analogous to V and $d[V]$, respectively.

Proof:

We integrate F over all supervectors $\Psi_{j1}^{(\mathrm{R})}$ and $\Psi_{j2}^{(\mathrm{R})}$ except $\Psi_{11}^{(\mathrm{R})}$. Then,

$$\int\limits_{\mathrm{Mat}_\beta^0((c-1)\times a/d\times b)} F(B_\psi)d[V_{\neq 11}] \tag{3.23}$$

only depends on $\Psi_{11}^{(\mathrm{R})\dagger}\Psi_{11}^{(\mathrm{R})}$. The measure $d[V_{\neq 11}]$ is $d[V]$ without the measure for the supervector $\Psi_{11}^{(\mathrm{R})}$. With help of the theorems 3.2.1, 3.2.2 and 3.2.3, see also Ref. [91, 92, 93], the integration over $\Psi_{11}^{(\mathrm{R})}$ is up to a constant equivalent to an integration over a supervector $\widetilde{\Psi}_{11}^{(\mathrm{R})}$. This supervector is equal to $\Psi_{11}^{(\mathrm{R})}$ in the first $\tilde{a}$ entries and else zero. We repeat this procedure for all other supervectors reminding that we only need the invariance under the supergroup action $\mathrm{U}^{(\beta)}(b-\tilde{b}/b-\tilde{b})$ on f as in Eq. (3.18) embedded in $\mathrm{U}^{(\beta)}(a/b)$. This invariance is preserved in each step due to the zero entries in the new supervectors. □

3.3 Integral theorems for invariant functions on supermatrices

To begin with, we consider functions on $\Sigma_{2,p/q}^{\psi}$ invariant under the action of $\mathrm{U}^{(2)}(p/q)$. In this case the integral theorem is equivalent to the one obtained in Refs. [91, 93]. However, there the authors did not derive the differential operator we will present here.

Let σ and ρ be two $\mathrm{U}^{(2)}(p/q)$–symmetric supermatrices as given in equation (2.26), then the metric is defined through the supertrace $g(\sigma,\rho)=\mathrm{Str}\,\sigma\rho$. We notice that the body of $g(\sigma,\sigma)$ does not lie in $\mathbb{R}^+$ for an arbitrary Wick–rotation. However, the body of $g(\sigma,\sigma)$ lies for a $\mathrm{U}^{(2)}(p/q)$–symmetric supermatrix with Wick–rotation $\imath$ in $\mathbb{R}^+$. The metric is for such a choice

$$g_n=\begin{cases}1 & , \ n \text{ is a diagonal index,}\\ 2 & , \ n \text{ is an off-diagonal index}\end{cases} \quad \text{and} \quad h_m=\imath\ . \tag{3.24}$$

We continue this metric in an analytic way on the space of $\mathrm{U}^{(2)}(p/q)$–symmetric supermatrices with arbitrary Wick–rotation. We exchange the real numbers of the fermion–fermion block entries to $-\imath e^{\imath\psi}$ times the real numbers and substitute the Grassmann variables with $\sqrt{-\imath e^{\imath\psi}}$ times the same Grassmann variables.

Now let f be an invariant function on the space of $\mathrm{U}^{(2)}(p/q)$–symmetric supermatrices

$$f(\sigma)=f(U^{-1}\sigma U) \quad , \quad U\in\mathrm{U}^{(2)}(p/q)\,. \tag{3.25}$$

We identify the radial part of the space of $\mathrm{U}^{(2)}(p/q)$–symmetric supermatrices as the space of diagonal matrices $s=\mathrm{diag}\,(s_{1,1},\ldots,s_{p,1},e^{\imath\psi}s_{1,2},\ldots,e^{\imath\psi}s_{q,2})$.

Therefore, we can apply theorem 3.1.1 and find

$$\begin{aligned}
&\int\limits_{\Lambda_{2pq}} f(\sigma)d[\eta] \\
&= \frac{e^{\imath\psi pq}}{(pq)!(4\pi)^{pq}}\sum_{n=0}^{pq}\binom{pq}{n}\left(\Delta_{s_1}^{(2;p)} - e^{-2\imath\psi}\Delta_{s_2}^{(2;q)}\right)^{pq-n}\left(-\Delta_s^{(2,2;pq)}\right)^n f(s) \\
&= D_s^{(2;pq)} f(s) \qquad (3.26)
\end{aligned}$$

where

$$\Delta_s^{(2;k)} = \sum_{j=1}^{k}\frac{1}{\Delta_k^2(s)}\frac{\partial}{\partial s_j}\Delta_k^2(s)\frac{\partial}{\partial s_j} \qquad (3.27)$$

is the radial part of the Laplace operator on the space of ordinary Hermitian matrices. Here,

$$\Delta_k(s) = \prod_{1\leq a<b\leq k}(s_a - s_b) = (-1)^{k(k-1)/2}\det[s_a^{b-1}]_{1\leq a,b\leq k} \qquad (3.28)$$

is the Vandermonde determinant. We denote by $\Delta_s^{(2;pq)}$ the radial part of the Laplacian in the superspace of $\mathrm{U}^{(2)}(p/q)$–symmetric matrices. It was calculated in Refs. [94, 104] for a Wick–rotation with angle $\psi = \pi/2$

$$\Delta_s^{(2;pq)} = \frac{1}{B_{p/q}^{(2)}(s)}\left(\sum_{j=1}^{p}\frac{\partial}{\partial s_{j1}}B_{p/q}^{(2)}(s)\frac{\partial}{\partial s_{j1}} - e^{-2\imath\psi}\sum_{j=1}^{q}\frac{\partial}{\partial s_{j2}}B_{p/q}^{(2)}(s)\frac{\partial}{\partial s_{j2}}\right), \qquad (3.29)$$

where

$$V_{pq}(s) = \prod_{n=1}^{p}\prod_{m=1}^{q}(s_{n1} - e^{\imath\psi}s_{m2}) \qquad (3.30)$$

mixes bosonic and fermionic eigenvalues and

$$B_{p/q}^{(2)}(s) = \frac{\Delta_p^2(s_1)\Delta_q^2(e^{\imath\psi}s_2)}{V_{pq}^2(s)} \qquad (3.31)$$

is the Berezinian of the transformation from Cartesian to eigenvalue–angle coordinates [94, 105, 104]. The variables s_{n1}, $1 \leq n \leq p$, and s_{m2}, $1 \leq m \leq q$, are the eigenvalue bodies of the Hermitian matrix σ_1 in the boson–boson block, respectively of the Hermitian matrix σ_2 in the fermion–fermion block.

The operator $D_s^{(2;pq)}$ can be cast into a simpler form. Using the identities

$$\Delta_s^{(2;k)} = \frac{1}{\Delta_k(s)} \sum_{j=1}^{k} \frac{\partial^2}{\partial s_j^2} \Delta_k(s) \tag{3.32}$$

$$\Delta_s^{(2;pq)} = \frac{1}{\sqrt{B_{p/q}^{(2)}(s)}} \mathrm{Str} \frac{\partial^2}{\partial s^2} \sqrt{B_{p/q}^{(2)}(s)}\,, \tag{3.33}$$

we find

$$\begin{aligned} D_s^{(2;pq)} &= \frac{e^{\imath\psi pq}}{(pq)!(4\pi)^{pq}} \frac{1}{\Delta_p(s_1)\Delta_q(e^{\imath\psi} s_2)} \\ \times & \sum_{n=0}^{pq} \binom{pq}{n} \left(\mathrm{Str} \frac{\partial^2}{\partial s^2}\right)^{pq-n} V_{pq}(s) \left(-\mathrm{Str} \frac{\partial^2}{\partial s^2}\right)^{n} \sqrt{B_{p/q}^{(2)}(s)}\,, \end{aligned} \tag{3.34}$$

where we defined

$$\mathrm{Str} \frac{\partial^2}{\partial s^2} = \sum_{j=1}^{p} \frac{\partial^2}{\partial s_{j1}^2} - e^{-2\imath\psi} \sum_{j=1}^{q} \frac{\partial^2}{\partial s_{j2}^2}\,. \tag{3.35}$$

We obtain for the particular case $p = q = 1$ the well known result [92, 93, 59]

$$D_s^{(2;1,1)} = \frac{e^{\imath\psi}}{2\pi} \frac{1}{s_1 - e^{\imath\psi} s_2} \left(\frac{\partial}{\partial s_1} + e^{-\imath\psi} \frac{\partial}{\partial s_2}\right)\,. \tag{3.36}$$

We now state an integral theorem which is a generalization of the integral theorem due to Wegner [91] worked out in theorem 4.1 of Ref. [93] to a generalized Wick–rotation and to an arbitrary dimension of the supermatrix.

Theorem 3.3.1 ($\mathrm{U}^{(2)}(p/q)$–symmetric matrices)
Let f be a differentiable function of $\mathrm{U}^{(2)}(p/q)$–symmetric supermatrices of the form (2.26), which is invariant under the action of $\mathrm{U}^{(2)}(p/q)$ and which has zero boundary condition at infinity, then

$$\int\limits_{\Sigma_{2,p/q}^{\psi}} f(\sigma) d[\sigma] = \begin{cases} (-1)^{pq} 2^{-q(q-1)} \imath^{q^2} \left(\dfrac{e^{\imath\psi}}{2}\right)^{q(p-q)} \displaystyle\int\limits_{\mathrm{Herm}\,(2,p-q)} f(\tilde{\sigma}_1) d[\tilde{\sigma}_1] & ,p > q \\ (-\imath)^{k^2} 2^{-k(k-1)} f(0) & ,p = q = k \\ (-\imath)^{p^2} 2^{-p(p-1)} \left(\dfrac{e^{-\imath\psi}}{2}\right)^{p(q-p)} \displaystyle\int\limits_{\mathrm{Herm}\,(2,q-p)} f(e^{\imath\psi}\tilde{\sigma}_2) d[\tilde{\sigma}_2] & ,p < q \end{cases} \tag{3.37}$$

where $d[\sigma]=d[\eta]d[\sigma_1]d[\sigma_2]$ *and*

$$d[\sigma_j]=\prod_{n=1}^{k_j}d\sigma_{nnj}\prod_{1\leq n<m\leq k_j}d\mathrm{Re}\ \sigma_{nmj}d\mathrm{Im}\ \sigma_{nmj}\,. \tag{3.38}$$

$\tilde{\sigma}_1$ *refers to the canonical embedding in the boson–boson matrix block and* $\tilde{\sigma}_2$ *refers to the canonical embedding in the fermion–fermion matrix block.*

We prove this theorem in App. A.4.

We next consider supermatrices which are form-invariant under the action of $\mathrm{U}^{(\beta)}(p/q)$, $\beta\in\{1,4\}$. We first focus on the representation $\mathrm{U}^{(1)}(p/q)$ of the supergroup $\mathrm{UOSp}\,(p/2q)$ and later extend our results to $\mathrm{U}^{(4)}(p/q)$.

Let f be an invariant function on the space of supermatrices of the form (2.25) and

$$f(\Sigma)=f(U^{-1}\Sigma U)\quad,\quad U\in\mathrm{U}^{(1)}(p/q)\,. \tag{3.39}$$

The radial part of the space of supermatrices of the form (2.25) is the space of diagonal matrices $s=\mathrm{diag}\,(s_1,e^{\imath\psi}s_2)$ and

$$s_1=\mathrm{diag}\,(s_{11},\ldots,s_{p1})\ ,\ s_2\ =\ \mathrm{diag}\,(s_{12},\ldots,s_{q2})\otimes\mathbb{1}_2\,. \tag{3.40}$$

The metric is

$$g_n=\begin{cases}1 & ,\ n\text{ is a diagonal index in the boson–boson block,}\\ 2 & ,\ n\text{ is an off-diagonal index in the boson–boson block or}\\ & \ \ \text{a diagonal index in the fermion–fermion block,}\\ 4 & ,\ n\text{ is an off-diagonal index in the fermion–fermion block}\end{cases} \tag{3.41}$$

and $h_m=2\imath$. Applying theorem 3.1.1 yields for the integration over the Grassmann variables of an invariant superfunction f

$$\begin{aligned}&\int\limits_{\Lambda_{2pq}}f(\sigma)d[\eta]\\ =\ &\frac{e^{\imath\psi pq}}{(pq)!(4\pi)^{pq}}\sum_{n=0}^{pq}\binom{pq}{n}\left(\Delta_{s_1}^{(1;p)}-e^{-2\imath\psi}\Delta_{s_2}^{(4;q)}\right)^{pq-n}\left(-\Delta_s^{(1;pq)}\right)^n f(s)\\ =\ &D_s^{(1;pq)}f(s)\,.\end{aligned} \tag{3.42}$$

Here, we used the radial parts of the Laplacians in the space of symmetric matrices $\Delta_s^{(1;k)}$ and in the space of Hermitian self-dual matrices $\Delta_s^{(4;k)}$, i.e.

$$\Delta_s^{(1;k)}=\sum_{j=1}^{k}\frac{1}{\Delta_k(s)}\frac{\partial}{\partial s_j}\Delta_k(s)\frac{\partial}{\partial s_j}\quad\text{and}\quad\Delta_s^{(4;k)}=\frac{1}{2}\sum_{j=1}^{k}\frac{1}{\Delta_k^4(s)}\frac{\partial}{\partial s_j}\Delta_k^4(s)\frac{\partial}{\partial s_j}\,. \tag{3.43}$$

The radial part of the Laplacian in the superspace of $\mathrm{U}^{(1)}(p/q)$–symmetric supermatrices (3.40) reads

$$\Delta_s^{(1;pq)} = \frac{1}{B_{p/q}^{(1)}(s)}\left(\sum_{j=1}^{p}\frac{\partial}{\partial s_{j1}}B_{p/q}^{(1)}(s)\frac{\partial}{\partial s_{j1}} - \frac{e^{-2\imath\psi}}{2}\sum_{j=1}^{q}\frac{\partial}{\partial s_{j2}}B_{p/q}^{(1)}(s)\frac{\partial}{\partial s_{j2}}\right) \tag{3.44}$$

and

$$B_{p/q}^{(1)}(s) = \frac{\Delta_p(s_1)\Delta_q^4(e^{\imath\psi}s_2)}{V_{pq}^2(s)}\,, \tag{3.45}$$

see Ref. [104]. As in the $\beta = 2$ case, we can simplify $D_s^{(1;pq)}$ using the identities

$$\Delta_s^{(1;k)} = \frac{1}{\sqrt{|\Delta_k(s)|}}H_s^{(1;k)}\sqrt{|\Delta_k(s)|}\,, \tag{3.46}$$

$$\Delta_s^{(4;k)} = \frac{1}{\Delta_k^2(s)}H_s^{(4;k)}\Delta_k^2(s) \qquad \text{and} \tag{3.47}$$

$$\Delta_s^{(1;pq)} = \frac{1}{\sqrt{B_{p/q}^{(1)}(s)}}H_s^{(1;pq)}\sqrt{B_{p/q}^{(1)}(s)}\,, \tag{3.48}$$

where we introduced the operators

$$H_s^{(1;k)} = \sum_{j=1}^{k}\frac{\partial^2}{\partial s_j^2} + \frac{1}{2}\sum_{1\leq m<n\leq k}\frac{1}{(s_n - s_m)^2}\,, \tag{3.49}$$

$$H_s^{(4;k)} = \frac{1}{2}\sum_{j=1}^{k}\frac{\partial^2}{\partial s_j^2} - 2\sum_{1\leq m<n\leq k}\frac{1}{(s_n - s_m)^2}\,, \tag{3.50}$$

$$H_s^{(1;pq)} = H_{s_1}^{(1;p)} - e^{-2\imath\psi}H_{s_2}^{(4;q)} - \sum_{m=1}^{p}\sum_{n=1}^{q}\frac{1}{(s_{m1} - e^{\imath\psi}s_{n2})^2}\,. \tag{3.51}$$

As in the $\mathrm{U}^{(2)}(p/q)$ case, this transformation is useful because the Laplacians are represented in a Hamiltonian form. The ordinary matrix Bessel functions times the square root of the Vandermonde determinant and the supermatrix Bessel functions times the square root of the Berezinian are eigenfunctions of $H_r^{(\beta;k)}$ and $H_r^{(\beta;pq)}$, respectively. One can find the definition of matrix Bessel functions in Refs. [88, 89], see also Eqs. (4.4) and (4.3). In Chap. 4 we calculate an explicit formula for supermatrix Bessel functions depending on matrix Bessel functions in the ordinary space. We will show that the Hamiltonian expression is indeed a helpful representation to calculate the supermatrix Bessel function in the unitary case. Thus, we hope this representation might be helpful for the other cases as well.

For the differential operator (3.42) one obtains

$$
\begin{aligned}
D_s^{(1;pq)} = & \frac{e^{\imath\psi pq}}{(pq)!(4\pi)^{pq}} \frac{1}{\sqrt{\Delta_p(s_1)}\Delta_q^2(e^{\imath\psi}s_2)} \sum_{n=0}^{pq} \left[\binom{pq}{n} \right. \\
\times & \left. \left(H_{s_1}^{(1;p)} - e^{-2\imath\psi} H_{s_2}^{(4;q)}\right)^{pq-n} V_{pq}(s) \left(-H_s^{(1,4;pq)}\right)^n \sqrt{B_{p/q}^{(1)}(s)} \right] .
\end{aligned} \tag{3.52}
$$

We give the two simplest examples for $q = 1$ for illustrative purposes. For $p = 1$, we have

$$
D_s^{(1;1,1)} = \frac{e^{\imath\psi}}{4\pi} \frac{1}{s_1 - e^{\imath\psi}s_2} \left(2\frac{\partial}{\partial s_1} + e^{-\imath\psi}\frac{\partial}{\partial s_2} \right) , \tag{3.53}
$$

and for $p = 2$, we find

$$
\begin{aligned}
D_s^{(1;2,1)} = & \frac{e^{2\imath\psi}}{4\pi^2} \frac{1}{s_{11} - s_{21}} \\
& \times \left[\left(2\frac{\partial}{\partial s_{11}} + e^{-\imath\psi}\frac{\partial}{\partial s_2} \right) \frac{1}{s_{21} - e^{\imath\psi}s_2} \left(2\frac{\partial}{\partial s_{21}} + e^{-\imath\psi}\frac{\partial}{\partial s_2} \right) \right. \\
& \left. - \left(2\frac{\partial}{\partial s_{21}} + e^{-\imath\psi}\frac{\partial}{\partial s_2} \right) \frac{1}{s_{11} - e^{\imath\psi}s_2} \left(2\frac{\partial}{\partial s_{11}} + e^{-\imath\psi}\frac{\partial}{\partial s_2} \right) \right] .
\end{aligned} \tag{3.54}
$$

The second example is needed to prove the following integral theorem. The proof is given in appendix A.5.

Theorem 3.3.2 ($\mathrm{U}^{(1)}(p/q)$–symmetric matrices)
Let f be a differentiable function on $\mathrm{U}^{(1)}(p/q)$–symmetric supermatrices of the form (2.25), which is invariant under the action of $\mathrm{U}^{(1)}(p/q)$ and which has zero boundary condition at infinity. In addition, it fulfills the condition

$$
\left(2\frac{\partial}{\partial s_{n1}} + e^{-\imath\psi}\frac{\partial}{\partial s_{m2}} \right) f(s) \bigg|_{s_{n1}=s_{m2}=0} = 0 \tag{3.55}
$$

for all pairs of eigenvalues of the $\mathrm{U}^{(1)}(p/q)$–symmetric supermatrices, then

$$
\int\limits_{\Sigma_{1,p/q}^{\psi}} f(\sigma) d[\sigma] \tag{3.56}
$$

$$
= \begin{cases}
\left(2^{2-q}\imath e^{\imath\psi}\right)^q (-e^{\imath\psi})^{q(p-2q)} \int\limits_{\mathrm{Herm}\,(1,p-2q)} f(\tilde{\sigma}_1) d[\tilde{\sigma}_1] & ,\ p > 2q \\
\left(2\imath e^{\imath\psi}\right)^k 2^{-k(k-1)} f(0) & ,\ p/2 = q = k \\
\left(2\imath e^{\imath\psi}\right)^{p/2} 2^{-p(p/2-1)/2} \left(\dfrac{e^{-\imath\psi}}{2}\right)^{p(q-p/2)} & ,\ p < 2q \wedge \\
\times \int\limits_{\mathrm{Herm}\,(4,q-p/2)} f(e^{\imath\psi}\tilde{\sigma}_2) d[\tilde{\sigma}_2] & \ p \in (2\mathbb{N}_0) \\
\left(-2^{(5-p)/2}\imath e^{2\imath\psi}\right)^{(p-1)/2} \left(\dfrac{e^{-\imath\psi}}{2}\right)^{(p-1)(q-(p-1)/2)} & ,\ p < 2q \wedge \\
\times \int\limits_{\Sigma^{\psi}_{1,1/[q-(p-1)/2]}} f(\tilde{\sigma})\, d[\tilde{\sigma}] & \ p \in (2\mathbb{N}_0 + 1)
\end{cases}
$$

where $d[\sigma] = d[\eta] d[\sigma_1] d[\sigma_2]$ *and*

$$
d[\sigma_1] = \prod_{1 \leq n \leq m \leq p} d\sigma_{nm1} \,, \tag{3.57}
$$

$$
d[\sigma_2] = \prod_{n=1}^{q} d\sigma_{nn21} \prod_{1 \leq n < m \leq q} d\mathrm{Re}\ \sigma_{nm21} d\mathrm{Im}\ \sigma_{nm21} d\mathrm{Re}\ \sigma_{nm22} d\mathrm{Im}\ \sigma_{nm22} \,. \tag{3.58}
$$

The supermatrix $\tilde{\sigma}_1$ *refers to the canonical embedding in the boson–boson matrix block,* $\tilde{\sigma}_2$ *refers to the canonical embedding in the fermion–fermion matrix block and* $\tilde{\eta}$ *refers to the canonical embedding in the boson–fermion matrix block.*

The idea of the proof is to apply the recursive method of Wegner [91, 93] using the operator $D_s^{(1;pq)}$. We remark that the property (3.55) is not a strong restriction on the set of functions. For example, the class of functions which are C^1-differentiable at zero in their supertraces satisfy this condition. Furthermore, we remark here that there is no Cauchy–like integral theorem for $\mathrm{U}^{(1)}(1/1)$–symmetric supermatrices and thus there is no integral reduction as above for $\mathrm{U}^{(1)}(1/q)$, see Sec. 3.4.

The extension of these results to invariant functions of $\mathrm{U}^{(4)}(p/q)$–symmetric supermatrices is straightforward. The metric is

$$
g_n = \begin{cases}
1 & ,\ n \text{ is a diagonal index in the fermion–fermion block,} \\
2 & ,\ n \text{ is an off-diagonal index in the fermion–fermion block or} \\
 & \ \text{a diagonal index in the boson–boson block,} \\
4 & ,\ n \text{ is an off-diagonal index in the boson–boson block}
\end{cases} \tag{3.59}
$$

and $h_m = 2\imath$. The differential operator $D_s^{(4;pq)}$ obeys the following symmetry relation

$$D_s^{(4;pq)} f(s) \;=\; (-1)^{pq} D_{\tilde{s}}^{(1;qp)} f(s) \tag{3.60}$$

where $\tilde{s}$ is related to s by $\tilde{s} = \operatorname{diag}\left(-e^{\imath\psi}s_2, -s_1\right)$. The corresponding theorem follows directly from theorem 3.3.2.

Theorem 3.3.3 ($\mathrm{U}^{(4)}(p/q)$–symmetric matrices)
Let the measures be the same as in theorem 3.3.2. Let f be a differentiable function on $\mathrm{U}^{(4)}(p/q)$–symmetric supermatrices, which is invariant under the action of $\mathrm{U}^{(4)}(p/q)$ and which has zero boundary condition at infinity. In addition it fulfills the condition

$$\left.\left(\frac{\partial}{\partial s_{n1}} + 2e^{-\imath\psi}\frac{\partial}{\partial s_{m2}}\right) f(s)\right|_{s_{n1}=s_{m2}=0} = 0 \tag{3.61}$$

for all pairs of eigenvalues of the $\mathrm{U}^{(4)}(p/q)$–symmetric matrices then

$$\int\limits_{\Sigma^{\psi}_{4,p/q}} f(\sigma) d[\sigma]$$
$$= \begin{cases} \left(-2\imath e^{-\imath\psi}\right)^{q/2} 2^{-q(q/2-1)/2} \left(\dfrac{e^{\imath\psi}}{2}\right)^{q(p-q/2)} & ,\ 2p > q \wedge \\ \times \displaystyle\int\limits_{\mathrm{Herm}\,(4,p-q/2)} f(\tilde{\sigma}_1) d[\tilde{\sigma}_1] & \quad q \in (2\mathbb{N}_0) \\ \left(2^{(5-q)/2}\imath e^{-2\imath\psi}\right)^{(q-1)/2} \left(\dfrac{e^{\imath\psi}}{2}\right)^{(q-1)(p-(q-1)/2)} & ,\ 2p > q \wedge \\ \times \displaystyle\int\limits_{\Sigma^{\psi}_{4,[p-(q-1)/2]/1}} f\left(\tilde{\sigma}\right) d[\tilde{\sigma}] & \quad q \in (2\mathbb{N}_0+1) \\ \left(-2\imath e^{-\imath\psi}\right)^{k} 2^{-k(k-1)} f(0) & ,\ p = q/2 = k \\ \left(-2^{2-p}\imath e^{-\imath\psi}\right)^{p} e^{-\imath\psi p(q-2p)} \displaystyle\int\limits_{\mathrm{Herm}\,(1,q-2p)} f(e^{\imath\psi}\tilde{\sigma}_2) d[\tilde{\sigma}_2] & ,\ 2p < q \end{cases} \tag{3.62}$$

where $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ and $\tilde{\eta}$ have the same meaning as in theorem 3.3.2.

We now investigate the structure of the differential operators $D_s^{(\beta;pq)}$. The Laplacians in ordinary space Eqs. (3.27) and (3.43), respectively in superspace Eqs. (3.29) and (3.44), can be written as

$$\Delta_s^{(\beta;k)} = \gamma_2 \left(\sum_{n=1}^{k} \frac{\partial^2}{\partial s_n^2} + \sum_{1\leq m<n\leq k} \frac{\beta}{s_m - s_n} \left(\frac{\partial}{\partial s_m} - \frac{\partial}{\partial s_n} \right) \right) , \quad (3.63)$$

$$\begin{aligned} \Delta_s^{(\beta;pq)} &= \Delta_{s_1}^{(\beta;p)} - e^{-2\imath\psi} \Delta_{s_2}^{(4/\beta;q)} \\ &- \sum_{\substack{1\leq m\leq p \\ 1\leq n\leq q}} \frac{2}{s_{m1} - e^{\imath\psi} s_{n2}} \left(\gamma_2 \frac{\partial}{\partial s_{m1}} - \gamma_1 e^{-\imath\psi} \frac{\partial}{\partial s_{n2}} \right) . \end{aligned} \quad (3.64)$$

We introduce a set of operators $D^{(\mu,\nu)}(s_a, s_b)$

$$D^{(\beta,\beta)}(s_{nj}, s_{mj}) = \frac{1}{s_{nj} - s_{mj}} \left(\frac{\partial}{\partial s_{nj}} - \frac{\partial}{\partial s_{mj}} \right) , \ j \in \{1, 2\} , \quad (3.65)$$

$$D^{(\beta,4/\beta)}(s_{n1}, s_{m2}) = \frac{1}{s_{n1} - e^{\imath\psi} s_{m2}} \left(\gamma_2 \frac{\partial}{\partial s_{n1}} - \gamma_1 e^{-\imath\psi} \frac{\partial}{\partial s_{m2}} \right) \quad (3.66)$$

and notice that

$$\left[\mathrm{Str}\, \frac{\partial^2}{\partial s^2}, D^{(\mu,\nu)}(s_a, s_b) \right]_- = -2 \left(D^{(\mu,\nu)}(s_a, s_b) \right)^2 \quad (3.67)$$

for all operators $D^{(\mu,\nu)}(s_a, s_b)$, where

$$\mathrm{Str}\, \frac{\partial^2}{\partial s^2} = \gamma_2 \sum_{n=1}^{p} \frac{\partial^2}{\partial s_{n1}^2} - \gamma_1 e^{-2\imath\psi} \sum_{n=1}^{q} \frac{\partial^2}{\partial s_{n2}^2} . \quad (3.68)$$

Now, we recall Eq. (3.5) and combine it with (3.67). We see that $D_s^{(\beta;pq)}$ is homogeneous in the operators $D^{(\beta,\beta)}(s_{nj}, s_{mj})$ and $D^{(\beta,4/\beta)}(s_{n1}, s_{m2})$ of degree pq.

3.4 On certain integrals for functions invariant under UOSp(1/2)

As a counter example for a Cauchy–like integration theorem, we consider superfunctions rotation invariant under the supergroup UOSp(1/2). This example shows that it is not sufficient to have rotation invariance and the same amount of Grassmann variables and of ordinary variables.

The differential operator which results from an integration over the Grassmann variables in the matrix case of the supergroup UOSp(1/2) for the representation $\mathrm{U}^{(1)}(1/1)$ is given by Eq. (3.53). For the other representation we get a similar expression in which the factor of two stands in front of the fermionic partial derivative. We consider the integral

$$I[f,\alpha]=\int\limits_{\mathbb{R}^2}\frac{1}{x-e^{\imath\psi}y}\left(\frac{\partial}{\partial x}+\alpha e^{-\imath\psi}\frac{\partial}{\partial y}\right)f(x,y)dxdy \tag{3.69}$$

for $\mathrm{Re}\left(e^{2\imath\psi}/\alpha\right)<0$, where f is a Schwartz function along the Wick–rotated real axis, analytic and $I[f,\alpha]$ finite for all α which fulfills the first requirement. The functional I has the properties

$$\begin{aligned} I\left[\exp\left(-x^2+\frac{e^{2\imath\psi}}{\alpha}y^2\right),\alpha\right] &= -2\pi\imath e^{-\imath\psi}\sqrt{\alpha}=C_\alpha\,, && (3.70)\\ I[f,\alpha]+I[f,\beta] &= 2I\left[f,\frac{\alpha+\beta}{2}\right]\,, && (3.71)\\ I[f,1] &= -2\pi\imath e^{-\imath\psi}f(0)\,. && (3.72)\end{aligned}$$

The first and the third property are obvious. Due to the linearity of the integral, the second one is true if $I[f,\alpha]$ and $I[f,\beta]$ exist. Surprisingly, there is no Cauchy–like integral theorem for $\alpha\neq 1$. The integral $I[f,1]$ represents the $\mathrm{U}^{(2)}(1/1)$ case. More precisely, we have

Proposition 3.4.1
There is no $\alpha\neq 1$ with $\mathrm{Re}\left(e^{2\imath\psi}/\alpha\right)<0$ such that $I[f,\alpha]=\mathrm{const.}f(0)$ for all Schwartz functions along the Wick–rotated real axis which are analytic and possess a finite $I[f,\alpha]$.

Proof:
We assume that there exists an $\alpha\neq 1$ with the described requirements which fulfill $I[f,\alpha]=\mathrm{const.}f(0)$. Then, the constant is equal to C_α because the Gaussian function in Eq. (3.70) fulfills the requirements of the function in the proposition. Thus, we use Eqs. (3.70) and (3.72). We compute

$$I\left[f,\frac{\alpha+1}{2}\right]\overset{(3.71)}{=}\frac{1}{2}(I[f,\alpha]+I[f,1])=\frac{1}{2}(C_\alpha+C_1)f(0)\,. \tag{3.73}$$

Therefore, there exists a constant for $I\left[f,(\alpha+1)/2\right]$ and this constant is equal to $(C_\alpha+C_1)/2$. On the other hand, the constant is unique and, accordingly, equal to $C_{(\alpha+1)/2}$. We find

$$C_{(\alpha+1)/2}=\frac{1}{2}(C_\alpha+C_1) \tag{3.74}$$

which becomes after some calculation

$$(\alpha-1)^2=0\ . \tag{3.75}$$

As this contradicts the assumption, the theorem is proven. □

We give a counter example to illustrate this theorem. We consider the function

$$f(x,y) = \left(x - \frac{e^{\imath\psi}}{\alpha}y\right)^2 \exp\left[-x^2 + \frac{e^{2\imath\psi}}{\alpha}y^2\right] , \qquad (3.76)$$

which vanishes at zero. However, the integral is

$$I[f,\alpha] = -\imath\pi e^{-\imath\psi}\sqrt{\alpha}\left(1 - \frac{1}{\alpha}\right) \overset{\alpha\neq 1}{\neq} 0 \, . \qquad (3.77)$$

Consequently Efetov's method [58] to derive such integral theorems cannot be applied to all kinds of integrals over invariant functions on superspaces. Nevertheless, it is a mystery to us why this method works for $\mathrm{U}^{(\beta)}(\gamma_1 k/\gamma_2 k)$–symmetric matrices, see theorems 3.3.1, 3.3.2 and 3.3.3, but would fail for $\mathrm{UOSp}\,(2k-1/2k)$, even though there is the same number of anticommuting and commuting variables to integrate.

Chapter 4

Applications of theorem 3.1.1

In this chapter we give two examples for the usefulness of the formalism developed previously. It is well known in random matrix theory that the energy density $\rho(x)$ of a Gaussian random matrix ensemble can be expressed as the derivative with respect to a source term of a generating function $Z(x+J)$. This generating function has a representation as a matrix integral over certain spaces of supermatrices. For a GUE it is given by

$$Z(x_1, J_1) = \imath \int\limits_{\Sigma^{\psi}_{2,1/1}} \exp\left[-\mathrm{Str}\ (\sigma + J)^2\right] \mathrm{Sdet}^{\,-N}(\sigma - x^-) d[\sigma] \tag{4.1}$$

where σ is a $\mathrm{U}^{(2)}(1/1)$–symmetric supermatrix as presented in Eq. (2.26) for the Wick–rotation $e^{\imath\psi} = \imath$. The variable N is the level number, $x^- = (x_1 - \imath\varepsilon)\mathbb{1}_2$ and $J = \mathrm{diag}\,(-J_1, J_1)$ with $x_1, J_1 \in \mathbb{R}$.

For a GOE the generating function of the energy density is given by

$$Z(x_1, J_1) = -\frac{1}{2} \int\limits_{\Sigma^{\psi}_{1,2/1}} \exp\left[-\mathrm{Str}\ (\sigma + J)^2\right] \mathrm{Sdet}^{\,-N/2}(\sigma - x^-) d[\sigma] \tag{4.2}$$

where σ is a $\mathrm{U}^{(1)}(2/1)$–symmetric supermatrix as defined in Eq. (3.39) and $x = (x_1 - \imath\varepsilon)\mathbb{1}_4$. N is the level number and $J = \mathrm{diag}\,(-J_1, -J_1, J_1, J_1)$ with $J_1 \in \mathbb{R}$.

In our first example we show how these supermatrix integrals are efficiently evaluated within the present formalism. The second example concerns the calculation of supermatrix Bessel functions [89], defined as the supersymmetric group integral

$$\varphi^{(\beta)}_{p/q}(s, x) = \int\limits_{\mathrm{U}^{(\beta)}(p/q)} \exp\left[\mathrm{Str}\, sUxU^{\dagger}\right] d\mu(U) \tag{4.3}$$

for two diagonal $(p+q)\times(p+q)$ supermatrices s and x. The Haar–measure $d\mu(U)$ of the group $\mathrm{U}^{(\beta)}(p/q)$ cannot be normalized because some of the supergroups have zero volume, e.g. $\mathrm{U}^{(2)}(1/1)$. We re-derive within the present formalism the result, derived in Refs. [94, 105]. Furthermore, we trace the supermatrix Bessel function back to the ordinary matrix Bessel functions [88]

$$\varphi_p^{(\beta)}(s,x) = \int\limits_{\mathrm{U}^{(\beta)}(p)} \exp\left[\imath \mathrm{tr}\, sUxU^\dagger\right] d\mu(U)\,. \tag{4.4}$$

The Haar-measure $d\mu(U)$ for the ordinary groups

$$\mathrm{U}^{(\beta)}(p) = \begin{cases} \mathrm{O}(p) & ,\ \beta=1 \\ \mathrm{U}\,(p) & ,\ \beta=2 \\ \mathrm{USp}(2p) & ,\ \beta=4 \end{cases} \tag{4.5}$$

is normalized. Since these functions are only known for particular parameters β and p, we give an explicit expression for the unitary case ($\beta = 2$), in Sec. 4.2, whereas for the other cases a formula connecting the supermatrix Bessel functions with the ordinary ones has to be sufficient, see Sec. 4.3.

4.1 One–point correlation functions and supermatrix Bessel functions

We start with a GUE and want to calculate the integral (4.1). For this purpose we consider the integral

$$I(x_1,x_2) = \int\limits_{\Sigma^{\psi}_{2,1/1}} f(\sigma)\exp[\mathrm{Str}\,\sigma x]d[\eta]d[\sigma_2]d[\sigma_1]\,, \tag{4.6}$$

where $x = \mathrm{diag}(x_1,x_2)$ is a diagonal $(1+1)\times(1+1)$ supermatrix. Here, f is a rotation invariant superfunction on the $\mathrm{U}^{(2)}(1/1)$–symmetric supermatrices with zero boundary condition at infinity. Since Grassmannian variables are contained in f only, we can apply theorem 3.1.1 and find

$$I(x_1,x_2) = \int\limits_{\mathbb{R}^2} \exp\left[s_1x_1 - e^{\imath\psi}s_2x_2\right] D_s^{(2;1,1)} f(s) ds_2 ds_1\,. \tag{4.7}$$

Now, we perform an integration by parts and shift the differential operator onto the exponential function

$$\begin{aligned} I(x_1,x_2) = &\int\limits_{\mathbb{R}^2} \Big[D_s^{(2;1,1)} f(s)\exp\left(s_1x_1 - e^{\imath\psi}s_2x_2\right) \\ - &\quad f(s)D_s^{(2;1,1)}\exp\left(s_1x_1 - e^{\imath\psi}s_2x_2\right)\Big]\, ds_2ds_1\,. \end{aligned} \tag{4.8}$$

Due to the simple structure of $D_s^{(2;1,1)}$, we apply the Cauchy integral theorem and obtain

$$I(x_1,x_2) = -\imath f(0) - \frac{e^{\imath\psi}}{2\pi}\int\limits_{\mathbb{R}^2} f(s_1,s_2)\frac{x_1-x_2}{s_1-e^{\imath\psi}s_2}\exp\left(s_1x_1-e^{\imath\psi}s_2x_2\right)ds_2ds_1\,. \tag{4.9}$$

The integrand in the second term of $I(x)$ is a product of the invariant function f, the Berezinian $B_{1/1}^{(2)}(s)=(s_1-e^{\imath\psi}s_2)^{-2}$ and the supermatrix Bessel function which coincides with Ref. [94]. We remark that Eq. (4.9) agrees with the known general transformation from the Cartesian coordinates to the eigenvalue–angle coordinates for Wick–rotation $\imath$ [106]. Since the generating function Z, see Eq. (4.1), is exactly of the form (4.6), we can use Eq. (4.9) and find

$$Z(x+J) = 1+\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\exp[-\mathrm{Str}\,(s+J)^2]\mathrm{Sdet}^{\,-N}(s-x^-)\frac{4J_1}{s_1-\imath s_2}ds_2ds_1 \tag{4.10}$$

which is indeed the correct result [94, 61], see also Sec. 9.2. Furthermore, we identify the boundary term in Eqs. (4.10) and (4.9) as the Efetov–Wegner term [94, 61, 82]. This term guarantees the normalization of Z at $J_1=0$.

In analogy to the GUE case, we consider the following integral for the GOE

$$I(x_{11},x_{21},x_2) = \int\limits_{\Sigma_{1,2/1}^{\psi}} f(\sigma)\exp[\mathrm{Str}\,\sigma x]d[\sigma]\,, \tag{4.11}$$

where $x=\mathrm{diag}(x_{11},x_{21},x_2,x_2)$ is a diagonal $(2+2)\times(2+2)$ supermatrix. Now, f is a rotation invariant superfunction on the space of $\mathrm{U}^{\,(1)}(2/1)$–symmetric supermatrices with zero boundary condition at infinity. As in the unitary case we integrate over the Grassmann variables employing theorem 3.1.1. Integration over the group $\mathrm{U}^{\,(1)}(2)$ in the boson–boson block yields

$$\begin{aligned} & I(x_{11},x_{21},x_2) \qquad (4.12)\\ = \ & \pi\int\limits_{\mathbb{R}^3}|s_{11}-s_{21}|\varphi_2^{(1)}(s_{11},s_{21},x_{11},x_{21})\exp\left[-2e^{\imath\psi}s_2x_2\right]D_s^{(1;2,1)}f(s)d[s]\,, \end{aligned}$$

where $\varphi_2^{(1)}$ is the matrix Bessel function for $\mathrm{U}^{\,(1)}(2)$, see Eq. (4.4). This matrix Bessel function can be expressed with the standard Bessel function

[107], see Ref. [88],

$$\begin{aligned}\varphi_2^{(1)}(s_{11},s_{21},x_{11},x_{21}) &= \exp\left[\frac{(s_{11}+s_{21})(x_{11}+x_{21})}{2}\right]\\ &\times\ J_0\left(-\imath\frac{(s_{11}-s_{21})(x_{11}-x_{21})}{2}\right)\end{aligned}\tag{4.13}$$

with the normalization $J_0(0)=1$. We integrate by parts twice and define

$$\varphi(s,x)=\varphi_2^{(1)}(s_{11},s_{21},x_{11},x_{21})\exp\left[-2e^{\imath\psi}s_2x_2\right]\,.\tag{4.14}$$

We obtain

$$\begin{aligned}&I(x_{11},x_{21},x_2)=\frac{e^{2\imath\psi}}{2\pi}\int\limits_{\mathbb{R}^3}\frac{|s_{11}-s_{21}|}{s_{11}-s_{21}}\varphi(s,x)\\ &\times\left(2\frac{\partial}{\partial s_{11}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)\frac{1}{s_{21}-e^{\imath\psi}s_2}\left(2\frac{\partial}{\partial s_{21}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)f(s)d[s]\\ &=2\imath e^{\imath\psi}f(0)+2\imath e^{\imath\psi}\int\limits_{\mathbb{R}^+}\left[f(s)\left(\left(1-e^{-2\imath\psi}\right)\frac{\partial}{\partial z^*}+\frac{\partial}{\partial r}\right)\varphi(s,x)\right.\\ &\left.\left.-\varphi(s,x)\left(\left(1-e^{-2\imath\psi}\right)\frac{\partial}{\partial z^*}-\frac{\partial}{\partial r}\right)f(s)\right]\right|_{z=z^*=r}dr+\frac{e^{2\imath\psi}}{2\pi}\int\limits_{\mathbb{R}^3}\frac{|s_{11}-s_{21}|}{s_{11}-s_{21}}f(s)\\ &\times\left(2\frac{\partial}{\partial s_{21}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)\frac{1}{s_{21}-e^{\imath\psi}s_2}\left(2\frac{\partial}{\partial s_{11}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)\varphi(s,x)d[s]\,.\end{aligned}\tag{4.15}$$

We have used the permutation symmetry of both bosonic eigenvalues and the coordinate transformation of the proof of theorem 3.3.2. We are interested in the third summand of (4.15) because the integrand is the invariant function times the Berezinian $B_{2/2}^{(1)}(s_1,e^{\imath\psi}s_2)=|s_{11}-s_{21}|/\left((s_{11}-e^{\imath\psi}s_2)^2(s_{21}-e^{\imath\psi}s_2)^2\right)$ and another function. This additional function is the supermatrix Bessel function regarding the group $\mathrm{U}^{(1)}(2/1)$ [89],

$$\begin{aligned}\varphi_{2/1}^{(1)}(s,x) &= \frac{1}{2\pi}\frac{(s_{11}-e^{\imath\psi}s_2)^2(s_{21}-e^{\imath\psi}s_2)^2}{s_{11}-s_{21}}\\ &\times\ \left[\left(2\frac{\partial}{\partial s_{21}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)\frac{1}{s_{21}-e^{\imath\psi}s_2}\left(2\frac{\partial}{\partial s_{11}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)\right.\\ &-\ \left.\left(2\frac{\partial}{\partial s_{11}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)\frac{1}{s_{11}-e^{\imath\psi}s_2}\left(2\frac{\partial}{\partial s_{21}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)\right]\varphi(s,x)\end{aligned}$$

$$= \frac{1}{2\pi}\left[\left[\left(R-e^{\imath\psi}s_2\right)^2-r^2\right]\left[\left(\frac{\partial}{\partial R}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)^2-\frac{\partial^2}{\partial r^2}\right]-2\left[R-e^{\imath\psi}s_2\right]\right.$$
$$\left.\times\ \left[\frac{\partial}{\partial R}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right]-\frac{(R-e^{\imath\psi}s_2)^2+r^2}{r}\frac{\partial}{\partial r}\right]\varphi(R+r,R-r,s_2,x) \quad (4.16)$$

where we have changed the coordinates to $R=\frac{1}{2}(s_{11}+s_{21})$ and $r=\frac{1}{2}(s_{11}-s_{21})$. Now, we use the explicit representation (4.13) of the matrix Bessel function and the differential equation for the standard Bessel function J_0

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k^2\right)J_0(kr)=0\,. \quad (4.17)$$

Thus, we find for (4.16)

$$\varphi_{2/1}^{(1)}(s,x)=\frac{1}{2\pi}\exp\left[R(x_{11}+x_{21})-2e^{\imath\psi}s_2x_2\right]$$
$$\times\ \left[\left[\left(R-e^{\imath\psi}s_2\right)^2-r^2\right]\left[\mathrm{Str}^{\,2}x+\frac{1}{r}\frac{\partial}{\partial r}-(x_{11}-x_{21})^2\right]\right.$$
$$\left.-\ 2\left[R-e^{\imath\psi}s_2\right]\mathrm{Str}\,x-\frac{(R-e^{\imath\psi}s_2)^2+r^2}{r}\frac{\partial}{\partial r}\right]J_0(-\imath r(x_{11}-x_{21}))$$
$$=\ \frac{1}{2\pi}\exp\left[R(x_{11}+x_{21})-2e^{\imath\psi}s_2x_2\right]$$
$$\times\ \left[4V_{2,1}(s)V_{2,1}(x)-\mathrm{Str}\,s\ \mathrm{Str}\,x-2r\frac{\partial}{\partial r}\right]J_0(-\imath r(x_{11}-x_{21}))\,. \quad (4.18)$$

Indeed, this is up to a constant the same result for the supermatrix Bessel function for $\mathrm{U}^{(1)}(2/1)$ as in Ref. [89]. We remark that for the function

$$f(\sigma)=\exp(-\mathrm{Str}\,\sigma^2)\mathrm{Sdet}^{\,-N/2}(\sigma+\imath\epsilon\mathbb{1}_4) \quad (4.19)$$

its partial derivative

$$\left(2\frac{\partial}{\partial s_{j1}}+e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)f(s) \quad (4.20)$$

vanishes at $s_{j1}=s_2=0$ for $j\in\{1,2\}$. Moreover, the first derivative of the standard Bessel function is zero at the point zero. Hence, we have the following integral representation of the equation (4.2)

$$Z(x+J)\ =\ 1+\frac{1}{\pi}\int\limits_{\mathbb{R}^3}\exp\left[-\mathrm{Str}\,(s+J)^2\right]\mathrm{Sdet}^{\,-N/2}(s-x^-) \quad (4.21)$$
$$\times\ \left[16J_1^2V_{2,1}(s)+4J_1\mathrm{Str}\,s\right]\mathrm{Ber}_{2/1}^{(1)}(s)d[s]\,,$$

cf. the general result in Chap. 9. This is indeed the correct generator, see Refs. [89, 87, 108]. We notice that without the properties of the invariant function, vanishing at zero and fulfilling (3.55), we get two additional boundary terms, see Eq. (4.15). These terms are needed to regularize the integral (4.15) at zero. If the invariant function is well behaved, for example C^1 in their supertraces, then these terms disappear.

4.2 Supermatrix Bessel functions for $\mathrm{U}^{(2)}(p/q)$

We consider the supermatrix Bessel functions (4.3) for $\beta = 2$ with a generalized Wick–rotation $e^{\imath\psi}$. The definition (4.3) is equivalent to the implicit definition

$$\begin{aligned}&\int\limits_{\Sigma^{\psi}_{2,p/q}} f(\sigma)\exp[\mathrm{Str}\,\sigma x]d[e^{-\imath\psi/2}\eta]d[e^{\imath\psi}\sigma_2]d[\sigma_1]\\ =&\int\limits_{\mathbb{R}^{p+q}} f(s)\varphi^{(2)}_{p/q}(s,x)B^{(2)}_{p/q}(s)d[e^{\imath\psi}s_2]d[s_1]+\mathrm{b.t.}\end{aligned} \tag{4.22}$$

for all rotational invariant functions f with zero boundary condition at infinity. Up to boundary terms (b.t.) in the eigenvalue manifold of $\mathrm{Herm}\,(2,p)\dot{\oplus}$ $\mathrm{Herm}\,(2,q)$ the integral on the left hand side of Eq. (4.22) is equal to the integral on the right hand side. The exponential term on the left hand side does not depend on Grassmann variables. Thus, we shift the integral over these variables and use theorem 3.1.1 and the operator $D_s^{(2;pq)}$ in Eq. (3.29). We obtain

$$\begin{aligned}&\int\limits_{\mathrm{Herm}\,(2,p)}\int\limits_{\mathrm{Herm}\,(2,q)}\exp[\mathrm{Str}\,\sigma x]D_s^{(2;pq)}f(s)d[e^{\imath\psi}\sigma_2]d[\sigma_1]\\ =&\int\limits_{\mathbb{R}^{p+q}} f(s)\varphi^{(2)}_{p/q}(s,x)B^{(2)}_{p/q}(s)d[e^{\imath\psi}s_2]d[s_1]+\mathrm{b.t.}\end{aligned} \tag{4.23}$$

We use the Itzykson–Zuber integral [109, 110] for the boson–boson and fermion–fermion block and rewrite $D_s^{(2;pq)}$ with Eq. (3.32). Then, we have

$$\frac{1}{\Delta_p(x_1)\Delta_q(x_2)}\int\limits_{\mathbb{R}^{p+q}}\det\left[\exp(s_{a1}x_{b1})\right]_{1\leq a,b\leq p}\det\left[\exp\left(e^{\imath\psi}s_{a2}x_{b2}\right)\right]_{1\leq a,b\leq q}$$

$$
\begin{aligned}
\times\ & \sum_{n=0}^{pq} \binom{pq}{n} \left(\mathrm{Str}\,\frac{\partial^2}{\partial s^2}\right)^{pq-n} V_{pq}(s) \left(-\mathrm{Str}\,\frac{\partial^2}{\partial s^2}\right)^{n} \left(\sqrt{B_{p/q}^{(2)}(s)}f(s)\right) d[s] \\
=\ & \frac{2^{pq}(pq)!p!q!\pi^{(p+q)/2}}{\pi^{(p-q)^2/2}} \int\limits_{\mathbb{R}^{p+q}} f(s)\varphi_{p/q}^{(2)}(s,x)B_{p/q}^{(2)}(s)d[s] + \text{b. t.}\,. \qquad (4.24)
\end{aligned}
$$

Due to the symmetry in the x variables, we omit the determinants and get a factor of $p!q!$. Then, we integrate by parts and perform the differential operators on the exponential functions. The emerging boundary terms are identified with these on the right hand side. Thus, we get

$$
\begin{aligned}
& \frac{1}{\Delta_p(x_1)\Delta_q(x_2)} \int\limits_{\mathbb{R}^{p+q}} \sqrt{B_{p/q}^{(2)}(s)}f(s) \\
& \times \sum_{n=0}^{pq} \binom{pq}{n} \left(\mathrm{Str}\,\frac{\partial^2}{\partial s^2}\right)^{pq-n} V_{pq}(s) \left(-\mathrm{Str}\,\frac{\partial^2}{\partial s^2}\right)^{n} \exp(\mathrm{Str}\,sx)d[s] \\
=& \frac{1}{\Delta_p(x_1)\Delta_q(x_2)} \int\limits_{\mathbb{R}^{p+q}} \sqrt{B_{p/q}^{(2)}(s)}f(s)\exp(\mathrm{Str}\,sx) \qquad (4.25) \\
& \times \left(\mathrm{Str}\,\frac{\partial^2}{\partial s^2} + 2\sum_{n=1}^{p} x_{n1}\frac{\partial}{\partial s_{n1}} + 2e^{-\imath\psi}\sum_{n=1}^{q} x_{n2}\frac{\partial}{\partial s_{n2}}\right)^{pq} V_{pq}(s)d[s] \\
=& \frac{2^{pq}(pq)!\pi^{(p+q)/2}}{\pi^{(p-q)^2/2}} \int\limits_{\mathbb{R}^{p+q}} f(s)\varphi_{p/q}^{(2)}(s,x)B_{p/q}^{(2)}(s)d[s]\,.
\end{aligned}
$$

The differential operator acts on a polynomial of order pq. An expansion of this operator leads to a differential operator which depends on a sum of derivatives of order pq to $2pq$. Therefore, the second derivatives do not contribute. The remaining operator acts on the polynomial and we find

$$
\begin{aligned}
& \int\limits_{\mathbb{R}^{p+q}} \sqrt{\frac{B_{p/q}^{(2)}(s)}{B_{p/q}^{(2)}(x)}}f(s)\exp(\mathrm{Str}\,sx)d[s] \qquad (4.26) \\
=& \frac{2^{pq}\pi^{(p+q)/2}}{\pi^{(p-q)^2/2}} \int\limits_{\mathbb{R}^{p+q}} f(s)\varphi_{p/q}^{(2)}(s,x)B_{p/q}^{(2)}(s)d[s]\,.
\end{aligned}
$$

Now, we analyze both integrals for all rotational invariant functions f. Thereby, we take notice of the invariance of f regarding the tensor product of the permutation group $\mathfrak{S}_p \otimes \mathfrak{S}_q$ acting on the boson–boson and the fermion–

fermion block. We get

$$\begin{aligned}\varphi_{p/q}^{(2)}(s,x) &= \frac{\pi^{(p-q)^2/2}}{2^{pq}\pi^{(p+q)/2}p!q!} \\ &\times \frac{\det\left[\exp\left(s_{m1}x_{b1}\right)\right]_{1\leq a,b\leq p}\det\left[\exp\left(e^{\imath\psi}s_{a2}x_{b2}\right)\right]_{1\leq a,b\leq q}}{\sqrt{B_{p/q}^{(2)}(s)B_{p/q}^{(2)}(x)}}\,.\end{aligned} \tag{4.27}$$

Indeed, this is for $p=q=k$, $e^{\imath\psi}=\imath$ and exchanging $s\rightarrow\imath s$ the correct result [105, 104, 61]. Also, we notice that the choice of the normalization constant in the measure $d\mu(U)$ arises in a natural way if we take Eq. (4.22) as the definition of the supermatrix Bessel functions [104, 61].

4.3 Supermatrix Bessel-functions for arbitrary Dyson index β

We consider the implicit definition

$$\begin{aligned}&\int\limits_{\Sigma_{\beta,p/q}^{\psi}} f(\sigma)\exp[\mathrm{Str}\,\sigma x]d[e^{-\imath\psi/2}\eta]d[e^{\imath\psi}\sigma_2]d[\sigma_1] \\ =&\int\limits_{\mathbb{R}^{p+q}} f(s)\varphi_{p/q}^{(\beta)}(s,x)B_{p/q}^{(\beta)}(s)d[e^{\imath\psi}s_2]d[s_1]+\text{b.t.}\end{aligned} \tag{4.28}$$

which generalizes the ansatz (4.22). The Berezinian is

$$B_{k_1/k_2}^{(\beta)}(s)=\frac{\Delta_{k_1}^{\beta}(s_1)\Delta_{k_2}^{4/\beta}(e^{\imath\psi}s_2)}{V_k^2(s)}\,, \tag{4.29}$$

see Eqs. (3.31) and (3.45). The superfunction f is again rotation invariant with zero boundary condition. Additionally, it has to fulfill the conditions (3.55) or (3.61) according to $\beta=1$ or $\beta=4$.

Following the same reasoning as in the previous section, we apply theorem 3.1.1 and find

$$\begin{aligned}&\int\limits_{\mathrm{Herm}\,(\beta,p)}\int\limits_{\mathrm{Herm}\,(4/\beta,q)} \exp[\mathrm{Str}\,\sigma x]D_s^{(\beta;pq)}f(s)d[e^{\imath\psi}\sigma_2]d[\sigma_1] \\ =&\int\limits_{\mathbb{R}^{p+q}} f(s)\varphi_{p/q}^{(\beta)}(s,x)B_{p/q}^{(\beta)}(s)d[e^{\imath\psi}s_2]d[s_1]+\text{b.t.}\end{aligned} \tag{4.30}$$

We have seen in the previous section that this normalization agrees with Refs. [105, 104, 89, 61]. The boundary terms (b.t.) are again referred to as Efetov–Wegner terms [94, 100, 82] and appear upon changing the integration variables [101] or, equivalently, upon partial integration, see Sec. 4.1.

We integrate over the ordinary groups $\mathrm{U}^{(\beta)}(p)$ and $\mathrm{U}^{(4/\beta)}(q)$ and use the definition of the ordinary matrix Bessel functions (4.4). This yields

$$\begin{aligned}&\int\limits_{\mathbb{R}^{p+q}} |\Delta_p(s_1)|^\beta \left|\Delta_q\left(e^{\imath\psi}s_2\right)\right|^{4/\beta} \varphi_p^{(\beta)}(-\imath s_1, x_1)\varphi_q^{(4/\beta)}(\imath e^{\imath\psi}s_2, x_2) D_s^{(\beta;pq)} f(s)d[s]\\ =&\frac{1}{\mathrm{FU}_p^{(\beta)}\mathrm{FU}_q^{(4/\beta)}}\int\limits_{\mathbb{R}^{p+q}} f(s)\varphi_{p/q}^{(\beta)}(s,x)B_{p/q}^{(\beta)}(s)d[s] + \text{b.t.}\,,\end{aligned} \tag{4.31}$$

where

$$\mathrm{FU}_n^{(\beta)} = \frac{1}{n!}\prod_{j=1}^{n}\frac{\pi^{\beta(j-1)/2}\Gamma\left(\beta/2\right)}{\Gamma\left(\beta j/2\right)}\,. \tag{4.32}$$

Due to the eigenvalue equation

$$\Delta_s^{(\beta;k)}\varphi_k^{(\beta)}(s,x) = -\mathrm{tr}\, x^2\varphi_k^{(\beta)}(s,x) \tag{4.33}$$

and the definition of $D_s^{(\beta;pq)}$, see Eqs. (3.26), (3.42) and (3.60), we obtain after an integration by parts

$$\begin{aligned}&\int\limits_{\mathbb{R}^{p+q}} B_{p/q}^{(\beta)}(s)f(s)\left[\mathrm{tr}\, x^2 - \Delta_s^{(\beta;pq)}\right]^{pq}\left[\varphi_p^{(\beta)}(-\imath s_1, x_1)\varphi_q^{(4/\beta)}(\imath e^{\imath\psi}s_2, x_2)V_{pq}^2(s)\right] d[s]\\ =&\frac{(4\pi)^{pq}(pq)!}{\mathrm{FU}_p^{(\beta)}\mathrm{FU}_q^{(4/\beta)}}\int\limits_{\mathbb{R}^{p+q}} f(s)\varphi_{p/q}^{(\beta)}(s,x)B_{p/q}^{(\beta)}(s)d[s]\,.\end{aligned} \tag{4.34}$$

The boundary terms are again identified with those which occur by shifting $D_s^{(\beta;pq)}$ to the product of the ordinary matrix Bessel functions.

We identify the left and the right hand side of Eq. (4.34) for arbitrary f and find

$$\begin{aligned}\varphi_{p/q}^{(\beta)}(s,x) =& \frac{\mathrm{FU}_p^{(\beta)}\mathrm{FU}_q^{(4/\beta)}}{(4\pi)^{pq}(pq)!}\\ &\times\left[\mathrm{Str}\, x^2 - \Delta_s^{(\beta;pq)}\right]^{pq}\left[\varphi_p^{(\beta)}(-\imath s_1, x_1)\varphi_q^{(4/\beta)}(\imath e^{\imath\psi}s_2, x_2)V_{pq}^2(s)\right]\,.\end{aligned} \tag{4.35}$$

Surprisingly, the supermatrix Bessel function is in the kernel of the same differential operator which generates these functions from the ordinary matrix Bessel functions,

$$\left(\mathrm{Str}\, x^2 - \Delta_s^{(\beta;pq)}\right)\varphi_{p/q}^{(\beta)}(s,x) = 0\,. \tag{4.36}$$

For $\beta = 2$, we obtain the correct result, see Eq. (4.27).

Chapter 5

Summary of part I

We derived a handy form for the differential operator with respect to commuting variables acting on an invariant superfunction. This operator is equivalent to integrating Grassmann variables over the same function. It is uniquely defined by the invariance class which the function fulfills. Detailed group theoretical considerations are not needed in our approach. We only used a mapping from the whole superspace to $(\Lambda^0(p,2L))^p$ which leaves the superfunction invariant.

There are various strong motivations for deriving formula (3.2). First of all, we aim at giving an explicit transformation formula, in contrast to Rothstein [101], of the change from Cartesian coordinates of a matrix to the eigenvalue–angle coordinates. The full account of the Efetov–Wegner terms is closely linked to this task and is also an aim of this work. As shown in many studies [111, 112, 113, 61], these terms guarantee the normalization or rather the reduction to a smaller integral if the function is whole or partly invariant under the action of a supergroup.

Moreover, we traced the supermatrix Bessel functions (4.3) back to a product of two ordinary ones with help of the differential operators $D_s^{(\beta;pq)}$, see Eqs. (3.26), (3.42) and (3.60). This simplifies the calculation of the supermatrix Bessel functions a lot. The integrations over the Grassmann variables are performed and only ordinary group integrals remain which are difficult enough to perform for $\beta \in \{1, 4\}$. Nevertheless, we rederived the supermatrix Bessel functions for the unitary supergroup in a straightforward way without using the heat equation [104] or Gelfand–Tzetlin coordinates [105, 89].

Part II

The supersymmetry method

Chapter 6

k-point correlation functions and the supersymmetry method

In Sec. 6.1, we pose the problem for k-point correlation functions of an arbitrary probability density over a subset of the Hermitian matrices. We give an outline of the supersymmetry method in Sec. 6.2. Thereby, we sketch the difference between the generalized Hubbard–Stratonovich transformation and the superbosonization formula, see chapters 8 and 10, respectively.

6.1 Posing the problem

We consider a sub-vector space $\mathfrak{M}_N$ of $\mathrm{Herm}\,(2,N)$. The object of interest is an arbitrary, sufficiently integrable probability density P on $\mathfrak{M}_N$. Later, we assume that P is an invariant function under the action of the group $\mathrm{U}^{(\beta)}(N)$, see Eq. (4.5) and $\mathfrak{M}_{\gamma_2 N} = \mathrm{Herm}\,(\beta,N)$.

We are interested in the k–point correlation functions

$$R_k(x) = \mathbf{d}^k \int\limits_{\mathfrak{M}_N} P(H) \prod_{p=1}^{k} \mathrm{tr}\,\delta(x_p \mathbb{1}_N - H) d[H] \tag{6.1}$$

with the k energies $x = \mathrm{diag}\,(x_1, \ldots, x_k)$. The measure $d[H]$ is defined as in Eqs. (3.38), (3.57) and (3.58), it is the product of all real and imaginary parts of the matrix entries. Here, $\mathbf{d}$ is the inverse averaged eigenvalue degeneracy of an arbitrary matrix $H \in \mathfrak{M}_N$. For example, we have $\mathbf{d} = 1/2$ for $\mathfrak{M}_{2N} = \mathrm{Herm}\,(4,N)$ and $\mathbf{d} = 1$ for no eigenvalue degeneracy as for $\mathfrak{M}_N = \mathrm{Herm}\,(\beta,N)$ with $\beta \in \{1,2\}$. We use in Eq. (6.1) the δ–distribution which is defined by the matrix Green function. The definition of the k–point correlation function (6.1) differs from Mehta's [114]. The two definitions can always be mapped onto each other as explained, for example, in Ref. [9].

We recall that it is convenient to consider the more general function

$$\widehat{R}_k\left(x^{(L)}\right) = \mathbf{d}^k \int\limits_{\mathfrak{M}_N} P(H) \prod_{p=1}^{k} \mathrm{tr}\,[(x_p + L_p\imath\varepsilon)\mathbb{1}_N - H]^{-1} d[H] \tag{6.2}$$

where we have suppressed the normalization constant. The quantities L_j in $x^{(L)} = \mathrm{diag}\,(x_1 + L_1\imath\varepsilon, \ldots, x_k + L_k\imath\varepsilon)$ are elements in $\{\pm 1\}$. We define $x^{\pm} = \mathrm{diag}\,(x_1 \pm \imath\varepsilon, \ldots, x_k \pm \imath\varepsilon)$. Considering the Fourier transform of (6.1) we have

$$\begin{aligned} r_k(t) &= (2\pi)^{-k/2} \int\limits_{\mathbb{R}^k} R_k(x) \prod_{p=1}^{k} \exp\left(\imath x_p t_p\right) d[x] \\ &= \left(\frac{\mathbf{d}}{\sqrt{2\pi}}\right)^k \int\limits_{\mathfrak{M}_N} P(H) \prod_{p=1}^{k} \mathrm{tr}\,\exp\left(\imath H t_p\right) d[H]\,. \end{aligned} \tag{6.3}$$

The Fourier transform of (6.2) yields

$$\begin{aligned} \widehat{r}_k(t) &= (2\pi)^{-k/2} \int\limits_{\mathbb{R}^k} \widehat{R}_k\left(x^{(L)}\right) \prod_{p=1}^{k} \exp\left(\imath x_p t_p\right) d[x] \\ &= \prod_{p=1}^{k} \left[-L_p\ 2\pi\imath\Theta(-L_p t_p) \exp\left(\varepsilon L_p t_p\right)\right] r_k(t)\,, \end{aligned} \tag{6.4}$$

where Θ is the Heavyside–distribution.

As in Ref. [61], the k–point correlation function is completely determined by Eq. (6.2) with $L_p = -1$ for all p if the Fourier transform (6.3) is entire in all entries, i.e. analytic in all entries with infinite radius of convergence. We obtain such a Fourier transform if the k–point correlation function R_k is a Schwartz function on $\mathbb{R}^k$ with the property

$$\int\limits_{\mathbb{R}^k} |R_k(x)| \prod_{p=1}^{k} \exp\left(\tilde{\delta} x_p\right) d[x] < \infty \quad , \quad \forall \tilde{\delta} \in \mathbb{R}\,. \tag{6.5}$$

This set of functions is dense in the set of Schwartz functions on $\mathbb{R}^k$ without this property. The notion *dense* refers to uniform convergence. This statement is true since every Schwartz function times a Gaussian distribution $\exp\left(-\epsilon \sum_{p=1}^{k} x_p^2\right)$, $\epsilon > 0$, is a Schwartz function and fulfills Eq. (6.5).

We proof that r_k, see Eq. (6.3), is indeed entire in all entries for such k–point correlation functions. To this end, we consider the function

$$r_{k\delta}(t) = \int\limits_{\mathfrak{B}_\delta} R_k(x) \prod_{p=1}^{k} \exp\left(\imath x_p t_p\right) d[x] \,, \tag{6.6}$$

where $\mathfrak{B}_\delta$ is the closed k-dimensional real ball with radius $\delta \in \mathbb{R}^+$. Due to the Paley–Wiener theorem [115], $r_{k\delta}$ is for all $\delta \in \mathbb{R}^+$ entire analytic. Let $\mathfrak{B}_{\tilde{\delta}}^{\mathbb{C}}$ be another k-dimensional complex ball with radius $\tilde{\delta} \in \mathbb{R}^+$. Then, we have

$$\lim_{\delta\to\infty} \sup_{t\in\mathfrak{B}_{\tilde{\delta}}^{\mathbb{C}}} |r_{k\delta}(t) - r_k(t)| \leq \lim_{\delta\to\infty} \int\limits_{\mathbb{R}^k\setminus\mathfrak{B}_\delta} |R_k(x)| \prod_{p=1}^{k} \exp\left(\tilde{\delta} x_p\right) d[x] = 0 \,. \tag{6.7}$$

The limit of $r_{k\delta}$ to r_k is uniform on every compact support on $\mathbb{C}^k$. Thus, r_k is entire analytic.

The modified correlation function $\widehat{R}_k$ for all choices of the L_p can be reconstructed by Eq. (6.4). In Sec. 9.4, we extend the results by a limit–value–process in a local convex way to non-analytic functions.

We derive $\widehat{R}_k\left(x^-\right)$ from the generating function

$$Z_k\left(x^- + J\right) = \int\limits_{\mathfrak{M}_N} P(H) \prod_{p=1}^{k} \frac{\det[H - (x_p^- + J_p)\mathbb{1}_N]}{\det[H - (x_p^- - J_p)\mathbb{1}_N]} d[H] \tag{6.8}$$

by differentiation with respect to the source variables [60]

$$\widehat{R}_k\left(x^-\right) = \left(\frac{\mathbf{d}}{2}\right)^k \frac{\partial^k}{\prod_{p=1}^{k} \partial J_p} Z_k\left(x^- + J\right)\Bigg|_{J=0} \tag{6.9}$$

where $x^- + J = x^- \otimes \mathbb{1}_4 + \operatorname{diag}\left(J_1, \ldots, J_k\right) \otimes \operatorname{diag}\left(-\mathbb{1}_2, \mathbb{1}_2\right)$. By definition, Z_k is normalized to unity at $J = 0$.

6.2 Sketch of our approach

To provide a guideline through the detailed presentation to follow in the ensuing chapters, we briefly sketch the main ideas as in Ref. [61] and as further extended in the present thesis.

To express the generating function (6.8) as an integral in superspace, we write the determinants as Gaussian integrals over vectors of ordinary and Grassmann variables. We then perform the ensemble average which is equivalent to calculating the characteristic function

$$\Phi(K) = \int P(H) \exp(\imath \mathrm{tr}\, HK) d[H] \tag{6.10}$$

of the probability density. The rotation invariance of $P(H)$ carries over to $\Phi(K)$. The ordinary matrix K contains the abovementioned vectors of ordinary and Grassmann variables as dyadic matrices. It has a dual matrix B in superspace whose entries are all scalar products of these vectors. the supermatrix B consists of a rectangular supermatrix V as in Eq. (2.13), i.e. $B \sim VV^\dagger$. Thus, it is a supersymmetric Wishart matrix. The reduction in the degrees of freedom is fully encoded in this duality, as the dimensions of K and B scale with N and k, respectively. The crucial identity

$$\mathrm{tr}\, K^m = \mathrm{Str}\, B^m, \quad \forall m \in \mathbb{N}, \tag{6.11}$$

yields the supersymmetric extension of the rotation invariant characteristic function,

$$\Phi(K) = \Phi(\mathrm{tr}\, K, \mathrm{tr}\, K^2, ...) = \Phi(\mathrm{Str}\, B, \mathrm{Str}\, B^2, ...) = \Phi(B)\,, \tag{6.12}$$

which is now viewed as a function in ordinary and superspace.

We have two choices to exchange the integration over the supersymmetric Wishart matrix B by a symmetric supermatrix ρ. The first approach proposed in Ref. [79] and extended to arbitrary unitary rotational invariant ensembles in Ref. [61] is the generalized Hubbard–Stratonovich transformation. We rewrite Φ by inserting a proper Dirac distribution in superspace,

$$\begin{aligned} \Phi(B) &= \int \Phi(\rho)\delta(\rho - B) d[\rho] && (6.13) \\ &\sim \int\int \Phi(\rho) \exp[\imath \mathrm{Str}\,(\rho - B)\sigma] d[\rho] d[\sigma]\,, && (6.14) \end{aligned}$$

where the supermatrix ρ and σ are introduced as integration variables. The vectors of ordinary and Grassmann variables now appear as in the conventional Hubbard–Stratonovich transformation and can hence be integrated in the same way. We are left with the integrals over ρ and σ. If we do the integral over ρ we arrive at the result

$$Z_k\left(x^- + J\right) \sim \int Q(\sigma) \mathrm{Sdet}^{\,-N/\gamma_1}(\sigma - x^- - J) d[\sigma]\,. \tag{6.15}$$

for the generating function. The superfunction Q is the superspace Fourier transform of Φ and plays the role of a probability density in superspace,

$$Q(\sigma) = \int \Phi(\rho) \exp(\imath \mathrm{Str}\, \rho\sigma) d[\rho] \,. \tag{6.16}$$

If we choose to integrate over σ instead, we obtain another representation of the generating function

$$Z_k\left(x^- + J\right) \sim \int \Phi(\rho) I(\rho) \exp[-\imath \mathrm{Str}\, \rho(x^- + J)] d[\rho] \,, \tag{6.17}$$

which still contains the characteristic function. Both supermatrices ρ and σ are flat symmetric matrices. The distribution $I(\rho)$ appears. It is the supersymmetric version of the Ingham–Siegel integral. It is a rotation invariant function resulting from the Fourier transformation of the superdeterminant in Eq. (6.15).

The second approach is the superbosonization formula developed in Refs. [83, 84]. It is the integral identity

$$\begin{aligned} & \int \Phi(B) \exp[-\imath \mathrm{Str}\, B(x^- + J)] d[B] \\ = \ & \text{const.} \int \Phi(\rho) \exp[-\imath \mathrm{Str}\, \rho(x^- + J)] \mathrm{Sdet}^{\,(N+1)/\gamma_1 - 1/\gamma_2} \rho d[\rho] \,, \end{aligned} \tag{6.18}$$

where ρ equals to the one in Eq. (6.17) in the boson–boson block and the off-diagonal blocks. However, it differs in the fermion–fermion block which is in this approach drawn from the circular ensemble.

One way to proceed further is to diagonalize the supermatrix ρ and to integrate over the angles. We may omit Efetov–Wegner terms and have

$$Z_k\left(x^- + J\right) \sim \int \Phi(r) I(r) \varphi(-\imath r, x^- + J) d[r], \tag{6.19}$$

where φ is a supermatrix Bessel function. The differentiation with respect to J gives $\widehat{R}_k$. We can introduce other signatures of L by Fourier transformation of Eq. (6.17) and identification with Eq. (6.4). Eventually, we find the correlation functions R_k.

Chapter 7

From ordinary matrix space to superspace

In this chapter, we map the integral expression of the generating function from ordinary to superspace. We show that in principle every random matrix ensemble can be mapped into superspace, also those which are not rotational invariant. Nevertheless, the supersymmetry method only exhibits a drastic reduction in the number of integration variables for random matrix ensembles with certain symmetries.

In Sec. 7.1, we express the determinants in Eq. (6.8) as Gaussian integrals and introduce the characteristic function of the matrix ensemble. In Sec. 7.2, we qualitatively present the duality between ordinary and superspace which is quantitatively discussed in Sec. 7.3. Then, we restrict the matrix ensembles to the classical symmetry classes. In Sec. 7.4, we investigate the diagonalization of the dyadic matrix K appearing from the Gaussian integrals. The ambiguity of the supersymmetric extension of the characteristic function is discussed in Sec. 7.5. In Sec. 7.6, we present the symmetries of the appearing supermatrices.

7.1 Average over the ensemble and the characteristic function

To formulate the generating function as a supersymmetric integral, we consider a complex Grassmann algebra $\Lambda = \bigoplus_{j=0}^{2Nk} \Lambda_j$ with Nk-pairs $\{\zeta_{jp}, \zeta_{jp}^*\}_{j,p}$ of Grassmann variables. We define the k anticommuting vectors and their adjoint

$$\zeta_p = (\zeta_{1p}, \ldots, \zeta_{Np})^T \quad \text{and} \quad \zeta_p^\dagger = (\zeta_{1p}^*, \ldots, \zeta_{Np}^*) \ , \tag{7.1}$$

respectively. We also consider k N–dimensional complex vectors $\{z_p, z_p^\dagger\}_{1\leq p\leq k}$. In the usual way, we write the determinants as Gaussian integrals and find for Eq. (6.8)

$$Z_k(x^- + J) = (-\imath)^{Nk} \int\limits_{\mathfrak{M}_N} \int\limits_{\mathfrak{C}_{kN}} d[\zeta]d[z]d[H]P(H) \tag{7.2}$$
$$\times \quad \exp\left(\imath\sum_{p=1}^{k}\left\{\zeta_p^\dagger[H-(x_p^- + J_p)\mathbb{1}_N]\zeta_p + z_p^\dagger[H-(x_p^- - J_p)\mathbb{1}_N]z_p\right\}\right)$$

where $d[\zeta] = \prod_{p=1}^{k}\prod_{j=1}^{N} d\zeta_{jp}d\zeta_{jp}^*$, $d[z] = \prod_{p=1}^{k}\prod_{j=1}^{N} dz_{jp}dz_{jp}^*$ and $\mathfrak{C}_{kN} = \mathbb{C}^{kN} \times \Lambda_{2Nk}$. Using

$$\sum_{p=1}^{k}\left(\zeta_p^\dagger H\zeta_p + z_p^\dagger H z_p\right) = \mathrm{tr}\, H\widetilde{K} \tag{7.3}$$

with

$$\widetilde{K} = \sum_{p=1}^{k}\left(z_p z_p^\dagger - \zeta_p\zeta_p^\dagger\right) \tag{7.4}$$

leads to

$$Z_k(x^- + J) = (-\imath)^{Nk}\int\limits_{\mathfrak{C}_{kN}} \mathcal{F}P\left(\widehat{\pi}(\mathfrak{M}_N;\widetilde{K})\right) \tag{7.5}$$
$$\times \quad \exp\left(-\imath\sum_{p=1}^{k}\left[(x_p^- + J_p)\zeta_p^\dagger\zeta_p + (x_p^- - J_p)z_p^\dagger z_p\right]\right) d[\zeta]d[z]\,.$$

where the integration over H is the Fourier transform of the probability density P,

$$\mathcal{F}P\left(\widehat{\pi}(\mathfrak{M}_N;\widetilde{K})\right) = \int\limits_{\mathfrak{M}_N} P(H)\exp\left(\imath\mathrm{tr}\, H\widetilde{K}\right) d[H]\,. \tag{7.6}$$

This Fourier transform is called characteristic function and is denoted by Φ in Ref. [61] and in Eq. (6.10). The projection operator $\widehat{\pi}(\mathfrak{M}_N)$ onto the space $\mathfrak{M}_N$ is crucial. For $\mathfrak{M}_{\gamma_2 N} = \mathrm{Herm}\,(\beta,N)$ the projection operator is

$$\widehat{\pi}\left(\mathrm{Herm}\,(\beta,N);\widetilde{K}\right) = \frac{1}{2}\left[\widetilde{K} + \widetilde{Y}(\widetilde{K})\right] \tag{7.7}$$

with

$$\widetilde{Y}(\widetilde{K}) = \begin{cases} \widetilde{K}^T & , \ \beta = 1\,, \\ \widetilde{K} & , \ \beta = 2\,, \\ (Y_{\rm s} \otimes \leavevmode\hbox{\small1\kern-3.8pt\normalsize1}_N)\, \widetilde{K}^T \left(Y_{\rm s}^T \otimes \leavevmode\hbox{\small1\kern-3.8pt\normalsize1}_N\right) & , \ \beta = 4\,. \end{cases} \tag{7.8}$$

The transposition in Eq. (7.8) can also be replaced by the complex conjugation due to $\widetilde{K}^\dagger = \widetilde{K}$. The projection onto the set of diagonal matrices $\bigoplus_{j=1}^N \mathbb{R}$ is

$$\widehat{\pi}\left(\bigoplus_{j=1}^N \mathbb{R}; \widetilde{K}\right) = \operatorname{diag}\left(\widetilde{K}_{11}, \widetilde{K}_{22}, \ldots, \widetilde{K}_{NN}\right) . \tag{7.9}$$

7.2 Duality between ordinary and superspace

Is it always possible to find a supermatrix representation for the characteristic function $\mathcal{F}P$ such that Eq. (7.5) has an integral representation over supermatrices as it is known [61, 84] for rotation invariant P on $\mathfrak{M}_{\gamma_2 N} = \operatorname{Herm}(\beta, N)$? The integral (7.5) is an integral over the supervectors $v_j = (z_{j1}^*, \ldots, z_{jk}^*, -\zeta_{j1}^*, \ldots, -\zeta_{jk}^*)^T$ and their adjoints $v_j^\dagger = (z_{j1}, \ldots, z_{jk}, \zeta_{j1}, \ldots, \zeta_{jk})$. The entries of the matrix $\widetilde{K}$ are $v_n^\dagger v_m$. If we do not use any symmetry of the matrix ensemble, we can write these scalar products of supervectors as supertraces

$$v_n^\dagger v_m = \operatorname{Str} v_m v_n^\dagger \ . \tag{7.10}$$

Then, we can transform each of these supertraces with a Dirac distribution to an integral over a $(k + k) \times (k + k)$ supermatrix. We defined the Dirac distribution in superspace as in Refs. [85, 82]. The ambiguity discussed in Ref. [86] occurring by such a transformation is discussed in the Secs. 7.5 and 9.3.

The procedure above is tedious. Using the symmetries of the ensemble $(\mathcal{F}P, \mathfrak{M}_N)$, we can reduce the number of integrals in superspace. We will see that the number of commuting real integrals and of Grassmannian integrals is $2k^2 + 2k^2$ $(\beta = 2)$ or $4k^2 + 4k^2$ $(\beta \in \{1, 4\})$ for a rotation invariant matrix ensembles on $\operatorname{Herm}(\beta, N)$. If there is no symmetry, the number of integrals has not been reduced. One has to integrate over $N(N+1)$ ordinary Hermitian $k \times k$ matrices and their corresponding anticommuting parameters if the transformation above is used.

7.3 Analysis of the duality between ordinary and superspace

We consider an orthonormal basis $\{A_n\}_{1\leq n\leq d}$ of $\mathfrak{M}_N$ where d is the dimension of $\mathfrak{M}_N$. We use the trace $\operatorname{tr} A_n A_m = \delta_{nm}$ as the scalar product and recall that $\mathfrak{M}_N$ is a real vector space. Every element of this basis is represented as

$$A_n = \sum_{j=1}^{N} \lambda_{jn} e_{jn} e_{jn}^\dagger \quad \text{with} \quad \sum_{j=1}^{N} \lambda_{jn}^2 = 1 \ . \tag{7.11}$$

Here, e_{jn} are the normalized eigenvectors of A_n to the eigenvalues λ_{jn}. Then we construct every matrix $H \in \mathfrak{M}_N$ in this basis

$$H = \sum_{n=1}^{d} h_n A_n \ . \tag{7.12}$$

We find for the characteristic function

$$\begin{aligned} \mathcal{F}P\left(\widehat{\pi}(\mathfrak{M}_N;\widetilde{K})\right) &= \int\limits_{\mathfrak{M}_N} P\left(\sum_{n=1}^{d} h_n A_n\right) \exp\left(\imath \sum_{n=1}^{d} h_n \operatorname{tr} A_n \widetilde{K}\right) d[H] \\ &= \mathcal{F}P\left(\sum_{n=1}^{d} \operatorname{tr}\left(\widetilde{K} A_n\right) A_n\right) \ . \end{aligned} \tag{7.13}$$

With help of Eq. (7.11) and an equation analogous to (7.10), the characteristic function is

$$\mathcal{F}P\left(\widehat{\pi}(\mathfrak{M}_N;\widetilde{K})\right) = \mathcal{F}P\left(\sum_{n=1}^{d} \operatorname{Str}\left(\sum_{j=1}^{N} \lambda_{jn} \widehat{V} e_{jn} e_{jn}^\dagger \widehat{V}^\dagger\right) A_n\right) \tag{7.14}$$

with $\widehat{V} = (v_1, \ldots, v_N)$. We see that the matrix $\widetilde{K}$ is projected onto

$$\widehat{K} = \widehat{\pi}(\mathfrak{M}_N;\widetilde{K}) \tag{7.15}$$

where the projection is the argument of the characteristic function in Eq. (7.13). The matrices in the supertraces of (7.14) can be exchanged by $(k+k)\times(k+k)$ supermatrices with the Dirac distributions described above. If the ensemble has no symmetry then we have reduced the number of supermatrices to the dimension of $\mathfrak{M}_N$. Nevertheless, we can find a more compact supersymmetric expression of the matrix $\widehat{K}$ such that the number of the resulting integrals only depends on k but not on N. This is possible if $\widehat{K}$ is a dyadic matrix of vectors where the number of vectors is independent of N and the probability density only depends on invariants of H. The ensembles with $\mathfrak{M}_{\gamma_2 N} = \text{Herm}\,(\beta, N)$ and a probability density P invariant under the action of $\text{U}^{(\beta)}(N)$ fulfill these properties. It is known [61, 84] that these cases have a very compact supersymmetric expression. Furthermore, these ensembles are well analyzed for Gaussian distributions with help of the ordinary Hubbard–Stratonovitch transformation [58, 68, 59].

In the present context, the cases of interest are $\mathfrak{M}_{\gamma_2 N} = \text{Herm}\,(\beta, N)$ with a probability density P invariant under the action $\text{U}^{(\beta)}(N)$. We need this symmetry to simplify Eq. (7.14). Let $N \geq \gamma_1 k$. This restriction also appears in the superbosonization formula [84]. If $N < \gamma_1 k$, one has to modify the calculations below. For the superbosonization formula, Bunder, Efetov, Kravtsov, Yevtushenko, and Zirnbauer [86] presented such a modification. We give two other solutions for this problem in Secs. 10.3 and 11.3.

The symmetries of a function f carry over to its Fourier transform $\mathcal{F}f$. Thus, the characteristic function $\mathcal{F}P$ is invariant under the action of $\text{U}^{(\beta)}(N)$. Let $\widetilde{K}_0$ be an arbitrary ordinary Hermitian matrix in the Fourier transform (7.6) of the probability density. We assume that the characteristic function is analytic in the eigenvalues of $\widetilde{K}_0$. Then, we expand $\mathcal{F}P$ as a power series in these eigenvalues. Since the characteristic function is rotation invariant, every single polynomial in this power series of a homogeneous degree is permutation invariant. With help of the fundamental theorem of symmetric functions [116] we rewrite these polynomials in the basis of elementary polynomials. This is equivalent to writing these polynomials in the basis of the traces $\text{tr}\left[\widehat{\pi}\left(\text{Herm}\,(\beta, N), \widetilde{K}_0\right)\right]^m$, $m \in \mathbb{N}$. The analytic continuation of $\mathcal{F}P$ from $\widetilde{K}_0$ to $\widetilde{K}$ yields that the characteristic function in Eq. (7.6) only depends on $\text{tr}\left[\widehat{\pi}\left(\text{Herm}\,(\beta, N), \widetilde{K}\right)\right]^m$, $m \in \mathbb{N}$.

Defining the matrix

$$\widehat{V}^\dagger = (z_1, \ldots, z_k, \widehat{Y} z_1^*, \ldots, \widehat{Y} z_k^*, \zeta_1, \ldots, \zeta_k, \widehat{Y} \zeta_1^*, \ldots, \widehat{Y} \zeta_k^*) \tag{7.16}$$

and its adjoint

$$\widehat{V} = (z_1^*, \ldots, z_k^*, \widehat{Y} z_1, \ldots, \widehat{Y} z_k, -\zeta_1^*, \ldots, -\zeta_k^*, \widehat{Y} \zeta_1, \ldots, \widehat{Y} \zeta_k)^T \tag{7.17}$$

with

$$\widehat{Y} = \begin{cases} \mathbb{1}_N & , \ \beta = 1 \\ 0 & , \ \beta = 2 \\ Y_{\rm s}^T \otimes \mathbb{1}_N & , \ \beta = 4 \end{cases} , \tag{7.18}$$

we find

$$\widehat{K} = \widehat{\pi}\left(\text{Herm}\,(\beta, N); \widetilde{K}\right) = \frac{1}{\tilde{\gamma}} \widehat{V}^\dagger \widehat{V} \,. \tag{7.19}$$

The crucial identity

$$\mathrm{tr}\,(\widehat{V}^\dagger\widehat{V})^m = \mathrm{Str}\,(\widehat{V}\widehat{V}^\dagger)^m \tag{7.20}$$

holds for all β. It connects ordinary and superspace. For $\beta = 2$, a proof can be found in Ref. [61]. In App. B.1, we show that the equation

$$\mathrm{Str}\,V_1V_2 = \mathrm{Str}\,V_2V_1 \tag{7.21}$$

holds for all rectangular matrices of the form

$$V_1 = \left[\begin{array}{ccl} \overbrace{A_1}^{a} & \overbrace{B_1}^{b} & \}c \\ C_1 & D_1 & \}d \end{array}\right] \quad \text{and} \quad V_2 = \left[\begin{array}{ccl} \overbrace{A_2}^{c} & \overbrace{B_2}^{d} & \}a \\ C_2 & D_2 & \}b \end{array}\right] \tag{7.22}$$

where A_j and D_j have commuting entries and B_j and C_j anticommuting ones. This implies in particular that Eq. (7.20) holds for all β. Hence, we reduced the amount of supermatrices corresponding to $\widetilde{K}$ in Eq. (7.14) to one $(2k+2k)\times(2k+2k)$ supermatrix. In Ref. [61], the characteristic function Φ was, with help of Eq. (7.20), extended to superspace. We follow this idea.

7.4 Problems when diagonalizing $\widehat{K}$

In Ref. [61], two approaches of the duality relation between ordinary and superspace were presented. The first approach is the duality equation (7.20) for $\beta = 2$. In this thesis, we follow this idea. In the second approach, the matrix $\widehat{K}$ was diagonalized. With the eigenvalues of $\widehat{K}$, a projection operator was constructed for the definition of a reduced probability density according to the probability density P.

The latter approach fails because $\widehat{K}$ is only diagonalizable if it has no degeneracy larger than γ_2. Moreover, for diagonalizable $\widehat{K}$, one cannotfind an eigenvalue $\lambda = 0$. This is included in the following proposition which we prove in App. B.2.

Proposition 7.4.1 (Diagonalization of $\widehat{K}$)
Let $N, \widetilde{N} \in \mathbb{N}$, $H^{(0)} \in \mathrm{Herm}\,(\beta, N)$, $l \in \mathbb{R}^{\widetilde{N}}$ and $\{\tau_q\}_{1\leq q\leq\widetilde{N}}$ $\gamma_2 N$–dimensional vectors consisting of Grassmann variables $\tau_q = (\tau_q^{(1)},\ldots,\tau_q^{(\gamma_2 N)})^T$. Then, the matrix

$$H = H^{(0)} + \sum_{q=1}^{\widetilde{N}} l_q\left[\tau_q\tau_q^\dagger + \widetilde{Y}\left(\tau_q^*\tau_q^T\right)\right] \tag{7.23}$$

cannot be diagonalized $H = U\mathrm{diag}\,(\lambda_1,\ldots,\lambda_N)U^\dagger$ by a matrix U with the properties

$$U^\dagger U = UU^\dagger = 1\!\!1_N\ , \quad U^* = \widetilde{Y}(U) \tag{7.24}$$

and the body of U lies in $\mathrm{U}^{(\beta)}(N)$ iff $H^{(0)}$ has an eigenvalue with degeneracy larger than γ_2. Moreover, H has no eigenvalue $\lambda \in \mathbb{R} \subset \Lambda_0$.

In our particular case, $\widehat{K}$ cannot be diagonalized for $k < N-1$. Hence, we do not follow the second approach of Ref. [61]. We emphasize that none of the other results in Ref. [61] is affected as they are proven by the correct first approach which we pursue here.

7.5 Ambiguity of the characteristic function in the supersymmetric extension

We discuss the problem that the extension of the characteristic function $\mathcal{F}P$ from ordinary matrices to supermatrices is not unique. This results from the fact that symmetric supermatrices comprise two kinds of eigenvalues, i.e. bosonic and fermionic eigenvalues. Whereas ordinary symmetric matrices have only one kind of eigenvalues. In the supertraces, these two different kinds are differently weighted by a minus sign. To illustrate this problem, we also give a simple example.

The rotation invariance of $\mathcal{F}P$ enables us to choose a representation $\mathcal{F}P_0$ of $\mathcal{F}P$ acting on an arbitrary number of matrix invariants

$$\mathcal{F}P_0\left(\mathrm{tr}\,\widehat{K}^m | m \in \mathbb{N}\right) = \mathcal{F}P(\widehat{K})\,. \tag{7.25}$$

For this representation, a unique superfunction exists defined by

$$\Phi_0(\sigma) = \mathcal{F}P_0\left(\mathrm{Str}\,\sigma^m | m \in \mathbb{N}\right)\,, \tag{7.26}$$

where

$$\mathcal{F}P_0\left(\mathrm{Str}\,\widehat{B}^m | m \in \mathbb{N}\right) = \mathcal{F}P_0\left(\mathrm{tr}\,\widehat{K}^m | m \in \mathbb{N}\right) \tag{7.27}$$

with

$$\widehat{B} = \tilde{\gamma}^{-1}\widehat{V}\widehat{V}^\dagger\,. \tag{7.28}$$

However, the choice of the representation $\mathcal{F}P_0$ is not unique. The question arises whether it is a well defined object. It is clear that two representations $\mathcal{F}P_0$ and $\mathcal{F}P_1$ are equal on $\mathrm{Herm}\,(\beta, N)$ due to the Cayley–Hamilton theorem,

$$\mathcal{F}P_0(H) = \mathcal{F}P_1(H) \quad \text{, for all } H \in \mathrm{Herm}\,(\beta, N). \tag{7.29}$$

The Cayley–Hamilton theorem states that there is a polynomial which is zero for H. Thus, H^M with $M > N$ is a polynomial in $\{H^n\}_{1\leq n\leq N}$. Plugging an arbitrary symmetric supermatrix σ into the corresponding superfunctions Φ_0

and Φ_1 we realize that the choices do not yield the same superfunctions such that

$$\Phi_0(\sigma) \neq \Phi_1(\sigma) \tag{7.30}$$

holds for some σ.

For example with $N = 2$, $k = 1$ and $\beta = 2$, let the characteristic function $\mathcal{F}P(H) = \mathcal{F}P_0\left(\operatorname{tr} H^3\right)$. We get with help of the Cayley–Hamilton theorem

$$\mathcal{F}P_1\left(\operatorname{tr} H^2, \operatorname{tr} H\right) = \mathcal{F}P_0\left(2\operatorname{tr} H \operatorname{tr} H^2 - \operatorname{tr}^3 H\right) = \mathcal{F}P_0\left(\operatorname{tr} H^3\right) = \mathcal{F}P(H)\,. \tag{7.31}$$

We consider a $\mathrm{U}\,(1/1)$–symmetric supermatrix σ. This yields for the supersymmetric extension of Eq. (7.31)

$$\begin{aligned}\mathcal{F}P_0\left(2\mathrm{Str}\,\sigma \mathrm{Str}\,\sigma^2 - \mathrm{Str}^{\,3}\sigma\right) \;&\neq\; \mathcal{F}P_0\left(\mathrm{Str}\,\sigma^3\right) = \\ &= \;\mathcal{F}P_0\left(\frac{1}{4}\left[3\frac{\mathrm{Str}^{\,2}\sigma^2}{\mathrm{Str}\,\sigma} + \mathrm{Str}^{\,3}\sigma\right]\right)\,.\end{aligned} \tag{7.32}$$

One obtains the last equation with a theorem similar to the Cayley–Hamilton theorem. More specificly, there exists a unique polynomial equation of order two,

$$\sigma^2 - \frac{\mathrm{Str}\,\sigma^2}{\mathrm{Str}\,\sigma}\sigma - \frac{1}{4}\left(\mathrm{Str}^{\,2}\sigma - \frac{\mathrm{Str}^{\,2}\sigma^2}{\mathrm{Str}^{\,2}\sigma}\right) = 0\,, \tag{7.33}$$

for a $\mathrm{U}\,(1/1)$–symmetric supermatrix σ.

The resulting integral of the generating function $Z_k|_{\mathfrak{M}_N=\mathrm{Herm}\,(\beta,N)}$, see Sec. 8.2, is invariant under the choice of Φ_0. This is proven in Sec. 9.3. Such an ambiguity of the supersymmetric extension of the characteristic function was also investigated by the authors of Ref. [86]. They avoided the question of the definition of a Dirac distribution on superspace by the superbosonization formula. In the generalized Hubbard–Stratonovich transformation, we introduce for the supersymmetric extension from Eq. (7.25) to Eq. (7.26) a Dirac distribution depending on the representation of the superfunction.

7.6 Symmetries of the supermatrices

We find for a chosen representation $\mathcal{F}P_0$

$$Z_k(x^- + J) = (-\imath)^{k_2 N} \int\limits_{\mathfrak{C}^{k_2 N}} \Phi_0(\widehat{B}) \exp\left[-\imath \mathrm{Str}\,(x^- + J)\widehat{B}\right] d[\zeta]d[z]\,. \tag{7.34}$$

Here, we introduce $k_2 = \gamma_2 k$, $k_1 = \gamma_1 k$ and $\tilde{k} = \tilde{\gamma} k$. We will simplify the integral (7.34) to integrals over k_1 eigenvalues in the boson–boson block and over k_2 eigenvalues in the fermion–fermion block.

For every β, we have

$$\widehat{B}^\dagger = \widehat{B}\,, \tag{7.35}$$

i.e. $\widehat{B}$ is self-adjoint. The complex conjugation yields

$$\widehat{B}^* = \left\{ \begin{array}{ll} \widehat{Y}\widehat{B}\widehat{Y}^T & ,\ \beta\in\{1,4\} \\ \widehat{Y}\widehat{B}^*\widehat{Y}^T & ,\ \beta=2 \end{array}\right. \tag{7.36}$$

with the $(2k+2k)\times(2k+2k)$ supermatrices

$$\widehat{Y}\Big|_{\beta=1} = \left[\begin{array}{ccc} 0 & \mathbb{1}_k & 0 \\ \mathbb{1}_k & 0 & 0 \\ 0 & 0 & Y_{\rm s}\otimes\mathbb{1}_k \end{array}\right] \quad , \quad \widehat{Y}\Big|_{\beta=4} = \left[\begin{array}{ccc} Y_{\rm s}\otimes\mathbb{1}_k & 0 & 0 \\ 0 & 0 & \mathbb{1}_k \\ 0 & \mathbb{1}_k & 0 \end{array}\right] \tag{7.37}$$

and $\widehat{Y}|_{\beta=2} = {\rm diag}\,(1,0,1,0)\otimes\mathbb{1}_k$. We notice that for the unitary case $\widehat{B}$ is effectively a $(k+k)\times(k+k)$ supermatrix, i.e. half the dimension. With help of the properties (7.35) and (7.36) we construct the supermatrix sets

$$\widehat{\Sigma}_{\beta,k} = \left\{\sigma\in{\rm Mat}_2(2k/2k)\,\middle|\,\sigma^\dagger=\sigma,\ \sigma^* = \left\{\begin{array}{ll} \widehat{Y}\sigma\widehat{Y}^T & ,\ \beta\in\{1,4\} \\ \widehat{Y}\sigma^*\widehat{Y}^T & ,\ \beta=2 \end{array}\right\}\right\}. \tag{7.38}$$

A matrix in $\widehat{\Sigma}_0(\beta,k)$ fulfills the odd symmetry (7.36). We transform this symmetry with the unitary transformations

$$\widehat{U}|_{\beta=1} = \frac{1}{\sqrt{2}}\left[\begin{array}{ccc} \mathbb{1}_k & \mathbb{1}_k & 0 \\ -\imath\mathbb{1}_k & \imath\mathbb{1}_k & 0 \\ 0 & 0 & \sqrt{2}\,\mathbb{1}_{2k} \end{array}\right], \tag{7.39}$$

$$\widehat{U}|_{\beta=4} = \frac{1}{\sqrt{2}}\left[\begin{array}{ccc} \sqrt{2}\,\mathbb{1}_{2k} & 0 & 0 \\ 0 & \mathbb{1}_k & \mathbb{1}_k \\ 0 & -\imath\mathbb{1}_k & \imath\mathbb{1}_k \end{array}\right], \tag{7.40}$$

$\widehat{U}|_{\beta=2} = \mathbb{1}_{4k}$, according to the Dyson–index, arriving at the well–known symmetries of symmetric supermatrices $\Sigma_{\beta,k}$ [99], see Eq. (2.22). For $\beta\in\{1,4\}$ the set $\Sigma_{\beta,k}$ equals $\Sigma_{\beta,k/k}$. In the unitary case the non-zero part of $\Sigma_{2,k}$ consisting of a $(k+k)\times(k+k)$ supermatrix which is drawn from $\Sigma_{2,k/k}$. As in Sec. 2.2, the set $\Sigma^\psi_{\beta,k}$ is the Wick–rotated set of $\Sigma_{\beta,k}$ with entries in $\Lambda_0\oplus\Lambda_1$.

Chapter 8

The generalized Hubbard–Stratonovich transformation

Originally, Stratonovich introduced an inverse Fourier transformation to trace a two body interaction of particles back to an interaction between independent particles and an auxiliary field [117]. This transformation is nowadays called Hubbard–Stratonovich transformation and was popularized in the western hemisphere by Hubbard [118]. It was applied in random matrix theory very early, due to the Gaussian ensembles considered by Efetov [58], Verbaarschot et al. [59] and others. However, the original Hubbard–Stratonovich transformation does not apply to non-Gaussian ensembles. For such ensembles, there are two approaches referred to as superbosonization. The first approach is a generalization of the Hubbard–Stratonovich transformation for rotation invariant random matrix ensembles [61]. The basic idea is the introduction of a Dirac distribution in superspace by two Fourier transformations, extending earlier work in the context of scattering theory [79], universality considerations [10], field theory [80, 81] and quantum chromodynamics [82]. The connection of the generalized with the original Hubbard–Stratonovich transformation is the Fourier transformation, which explains the name of the former one. The second approach is the superbosonization formula developed in Refs. [83, 84].

In Sec. 8.1, we introduce the Dirac distribution described above. Thereby, we have to calculate the two Fourier transformations. The second one yields a generalization of the supersymmetric Ingham–Siegel integral [61] which is calculated in Sec. 8.2. To this end, we use an analog of the Sekiguchi differential operator for ordinary matrix Bessel functions. This further extends the generalized Hubbard–Stratonovich transformation to the orthogonal and the unitary symplectic symmetry class in a unifying way. We extend the generalized Hubbard–Stratonovich transformation to arbitrary and independent numbers of determinants in the numerator and denominator of the generating function Z in Sec. 8.3, since this approach works for such averages, too.

8.1 Transformation to supermatrices by a Dirac distribution

Following Refs. [79, 61, 82], $\Phi_0(\widehat{B})$ can be written as a convolution in the space of supermatrices $\Sigma_\psi^0(\beta,k)$ with a Dirac distribution. We have

$$\begin{aligned} Z_k(x^-+J) &= (-\imath)^{k_2N}\int\limits_{\mathfrak{C}_{k_2N}}\int\limits_{\Sigma_{\beta,k}^{\psi}}\Phi_0(\rho)\delta\left(\rho-\widehat{U}\widehat{B}\widehat{U}^\dagger\right)d[\rho] \\ &\times \exp\left[-\imath\mathrm{Str}\,(x^-+J)\widehat{B}\right]d[\zeta]d[z]\,, \end{aligned} \tag{8.1}$$

where the measure is defined as in theorems 3.3.1 and 3.3.2. We exchange the Dirac distribution by two Fourier transforms as in Refs. [61, 82]. Then, Eq. (8.1) becomes

$$\begin{aligned} Z_k(x^-+J) &= (-\imath)^{k_2N}\int\limits_{\mathfrak{C}_{k_2N}}\int\limits_{\Sigma_{\beta,k}^{-\psi}}\mathcal{F}\Phi_0(\sigma) \\ &\times \exp\left[\imath\mathrm{Str}\,\widehat{B}\left(\widehat{U}^\dagger\sigma\widehat{U}-x^--J\right)\right]d[\sigma]d[\zeta]d[z]\,, \end{aligned} \tag{8.2}$$

where the Fourier transform of Φ_0 is

$$\mathcal{F}\Phi_0(\sigma)=2^{2k(k-\tilde{\gamma})}\int\limits_{\Sigma_{\beta,k}^{\psi}}\Phi_0(\rho)\exp\left(-\imath\mathrm{Str}\,\rho\sigma\right)d[\rho]\,. \tag{8.3}$$

We write the supertrace in the exponent in Eq. (8.2) as a sum over expectation values

$$\mathrm{Str}\,\widehat{B}\left(\widehat{U}^\dagger\sigma\widehat{U}-x^--J\right)=\frac{1}{\tilde{\gamma}}\sum_{j=1}^{N}\mathrm{tr}\,\Psi_j^\dagger\left(\widehat{U}^\dagger\sigma\widehat{U}-x^--J\right)\Psi_j \tag{8.4}$$

with respect to the real, complex or quaternionic supervectors

$$\Psi_j^\dagger = \left\{ \begin{array}{ll} \left\{z_{jn}, z_{jn}^*, \zeta_{jn}, \zeta_{jn}^*\right\}_{1\leq n\leq k} & , \ \beta=1, \\ \left\{z_{jn}, 0, \zeta_{jn}, 0\right\}_{1\leq n\leq k} & , \ \beta=2, \\ \left\{\left[\begin{array}{c} z_{jn} \\ z_{j+N,n} \end{array}\right], \left[\begin{array}{c} -z_{j+N,n}^* \\ z_{jn}^* \end{array}\right], \left[\begin{array}{c} \zeta_{jn} \\ \zeta_{j+N,n} \end{array}\right], \left[\begin{array}{c} -\zeta_{j+N,n}^* \\ \zeta_{jn}^* \end{array}\right]\right\}_{1\leq n\leq k} & , \ \beta=4. \end{array}\right. \tag{8.5}$$

These supervectors are equivalent to those defined in Eq. (2.14). The integration over one of these supervectors yields

$$\int\limits_{\mathfrak{C}_{k_2}} \exp\left[\frac{\imath}{\tilde{\gamma}}\mathrm{tr}\,\Psi_j^\dagger\left(\widehat{U}^\dagger\sigma\widehat{U}-x^--J\right)\Psi_j\right]d[\Psi_j] = \imath^{k_2}\mathrm{Sdet}^{\,-1/\gamma_1}\mathfrak{p}\left(\sigma-x^--J\right)\,. \tag{8.6}$$

$\mathfrak{p}$ projects onto the non-zero matrix blocks of $\Sigma_{\beta,k}^{-\psi}$ which are only $(k+k)\times(k+k)$ supermatrices for $\beta=2$. $\mathfrak{p}$ is the identity for $\beta\in\{1,4\}$. The Eq. (8.6) is true because $\widehat{U}$ commutes with x^-+J. Then, Eq. (8.2) reads

$$Z_k(x^-+J) = \int\limits_{\Sigma_{\beta,k}^{-\psi}} \mathcal{F}\Phi_0(\sigma)\mathrm{Sdet}^{\,-N/\gamma_1}\mathfrak{p}\left(\sigma-x^--J\right)d[\sigma]\,. \tag{8.7}$$

Indeed, this result coincides with Ref. [61] for $\beta=2$ where the Fourier transform $\mathcal{F}\Phi_0(\sigma)$ was denoted by $Q(\sigma)$, cf. Sec. 6.2. Eq. (8.7) reduces for Gaussian ensembles with arbitrary β to expressions as in Refs. [68] and [59]. The integral is well defined because ε is greater than zero and the body of the eigenvalues of the boson–boson block is real. The representation (8.7) for the generating function can also be considered as a random matrix ensemble lying in superspace.

Equation (8.7) is one reason why we denote this integral transformation from the space over ordinary matrices to supermatrices as generalized Hubbard–Stratonovich transformation. If the probability density P is Gaussian then we can choose Φ_0 also as a Gaussian. Thus, this transformation above reduces to the ordinary Hubbard–Stratonovich transformation and the well-known result (8.7).

8.2 The supersymmetric Ingham–Siegel integral

We perform a Fourier transform in superspace for the convolution integral (8.7) and find

$$Z_k(x^-+J) = 2^{2k(k-\tilde{\gamma})}\int\limits_{\Sigma_{\beta,k}^{\psi}} \Phi_0(\rho)I_k^{(\beta,N)}(\rho)\exp\left[-\imath\mathrm{Str}\,\rho\left(x^-+J\right)\right]d[\rho]\,. \tag{8.8}$$

Here, we have to calculate the supersymmetric Ingham–Siegel integral

$$I_k^{(\beta,N)}(\rho) = \int\limits_{\Sigma_{\beta,k}^{-\psi}} \exp\left(-\imath \mathrm{Str}\,\rho\sigma^+\right) \mathrm{Sdet}^{\,-N/\gamma_1}\mathfrak{p}\sigma^+ d[\sigma] \tag{8.9}$$

with $\sigma^+ = \sigma + \imath\varepsilon \mathbb{1}_{4k}$.

Ingham [119] and Siegel [120] independently calculated a version of (8.9) for ordinary real symmetric matrices. The case of Hermitian matrices was discussed in Ref. [121]. Since we were unable to find the ordinary Ingham–Siegel integral also for the quaternionic case, we give the result here. It is related to Selbergs integral [122]. Let $R \in \mathrm{Herm}\,(\beta, m)$, $\varepsilon > 0$ and a real number $n \geq m$, then we have

$$\int\limits_{\mathrm{Herm}\,(\beta,m)} \exp\left(-\imath \mathrm{tr}\, RS^+\right) \det{}^{-n/\gamma_1} S^+ d[S] = \imath^{-\beta mn/2} G_{n-m,m}^{(\beta)} \det{}^{\lambda} R\; \Theta(R)\,, \tag{8.10}$$

where $S^+ = S + \imath\varepsilon \mathbb{1}_{\gamma_2 m}$, the exponent is

$$\lambda = \frac{n-m}{\gamma_1} - \frac{\gamma_1 - \gamma_2}{2} \tag{8.11}$$

and the constant is

$$G_{n-m,m}^{(\beta)} = \left(\frac{\gamma_2}{\pi}\right)^{\beta m(n-m+1)/2-m} \prod_{j=n-m+1}^{n} \frac{2\pi^{\beta j/2}}{\Gamma\left(\beta j/2\right)}\,. \tag{8.12}$$

$\Gamma(.)$ is the Euler gamma–function and $\Theta(.)$ is the Heavyside–distribution for matrices which is defined as

$$\Theta(R) = \left\{ \begin{array}{ll} 1 & , \ R \text{ is positive definite} \\ 0 & , \ \text{else} \end{array} \right. . \tag{8.13}$$

The ordinary Ingham–Siegel integral was recently used in the context of supersymmetry by Fyodorov [121]. The integral was extended to the superspace $\Sigma_{2,k}^{\pi/2}$ in Ref. [61]. In this article, we need a generalization to all $\Sigma_{\beta,k}^{-\psi}$, in particular $\beta \in \{1, 4\}$.

The integral (8.9) is invariant under the action of $\mathrm{U}^{(\beta)}(k_1/k_2)$. Thus, it is convenient to consider $I_k^{(\beta,N)}(r)$, where $r = \mathrm{diag}\,(r_{11}, \ldots, r_{\tilde{k}1}, r_{12}, \ldots, r_{\tilde{k}2})$ is the diagonal matrix of eigenvalues of ρ and contains nilpotent terms. The authors of Ref. [82] claimed in their proof of theorem 1 in chapter 6 that the diagonalization at this point of the calculation yields Efetov–Wegner terms. These terms do not appear in the ρ_2 integration because we do not change the integration variables, i.e. the integration measure $d[\rho]$ remains the same. For the unitary case, see Ref. [61]. We consider the eigenvalues of ρ as functions of the Cartesian variables. We may certainly differentiate a function with respect to the eigenvalues if we keep track of how these differential operators are defined in the Cartesian representation.

As worked out in App. B.4.1, the supersymmetric Ingham–Siegel integral (8.9) reads

$$\begin{aligned} I_k^{(\beta,N)}(\rho) &= C\det{}^{\kappa} r_1 \Theta(r_1) \det{}^k r_2 \exp\left(-e^{\imath\psi}\varepsilon \mathrm{tr}\, r_2\right) \\ &\times \left[D_{k_2 r_2}^{(4/\beta)}\left(\imath e^{\imath\psi}\gamma_1\varepsilon\right)\right]^N \frac{\delta(r_2)}{|\Delta_{k_2}(r_2)|^{4/\beta}}\,. \end{aligned} \tag{8.14}$$

The constant is

$$C = \left(-\frac{e^{-\imath\psi}}{\gamma_1}\right)^{k_2 N} \left(-\frac{\tilde{\gamma}}{2\pi}\right)^{k_1 k_2} \left(\frac{2\pi}{\gamma_1}\right)^{k_2} \left(\frac{\pi}{\gamma_1}\right)^{2k_2(k_2-1)/\beta} \frac{G_{N,k_1}^{(\beta)}}{\mathrm{FU}_{k_2}^{(4/\beta)}} \tag{8.15}$$

with $\mathrm{FU}_{k_2}^{(4/\beta)}$ defined as in Eq. (4.32). The exponent is given by

$$\kappa = \frac{N}{\gamma_1} + \frac{\gamma_2 - \gamma_1}{2} \tag{8.16}$$

and the differential operator

$$D_{k_2 r_2}^{(4/\beta)}(\delta) = \frac{1}{\Delta_{k_2}(r_2)} \det\left[r_{a2}^{N-b}\left(\frac{\partial}{\partial r_{a2}} + (k_2 - b)\frac{2}{\beta}\frac{1}{r_{a2}} + \imath\delta\right)\right]_{1\leq a,b\leq k_2} \tag{8.17}$$

with $\delta = \imath e^{\imath\psi}\gamma_1\varepsilon$ is the analog to the Sekiguchi differential operator [123]. We derive it in App. B.3.

The complexity of $D_{k_2 r_2}^{(4/\beta)}(\delta)$ makes Eq. (8.14) cumbersome, a better representation is desirable. To simplify Eq. (8.14), we need the following lemma which is shown in App. B.4.2.

Lemma 8.2.1
We consider two functions $F, f : \mathrm{Herm}\,(4/\beta, k_2) \to \mathbb{C}$ invariant under the action of $\mathrm{U}^{(4/\beta)}(k_2)$ and Schwartz functions of the matrix eigenvalues. Let F and f have the relation

$$F(\rho_2) = f(\rho_2) \det \rho_2^{N/\gamma_1 - k} \quad \text{for all } \rho_2 \in \mathrm{Herm}\,(4/\beta, k_2)\,. \tag{8.18}$$

Then, we have

$$\begin{aligned}
& \int\limits_{\mathbb{R}^{k_2}} \int\limits_{\mathrm{Herm}\,(4/\beta,k_2)} F(r_2)\det{}^k r_2 |\Delta_{k_2}(r_2)|^{4/\beta} \exp\left(\imath \mathrm{tr}\, r_2\sigma_2\right) \\
\times\ & \det{}^{N/\gamma_1}\left(e^{-\imath\psi}\sigma_2 + \imath\varepsilon 1\!\!1_{\tilde{k}}\right) d[\sigma_2]d[r_2] \\
=\ & w_1 f(0) \qquad (8.19) \\
=\ & \int\limits_{\mathbb{R}^{k_2}} F(r_2)|\Delta_{k_2}(r_2)|^{4/\beta} \left[\frac{w_2 \exp\left(\varepsilon e^{\imath\psi}\mathrm{tr}\, r_2\right)}{|\Delta_{k_2}(r_2)|^{4/\beta}} \prod_{j=1}^{k_2}\left(\frac{\partial}{\partial r_{j2}}\right)^{N-k_1}\delta(r_{j2})\right] d[r_2],
\end{aligned}$$

where the constants are

$$w_1 = \left(\frac{2\pi}{\gamma_1}\right)^{k_2}\left(\frac{\pi}{\gamma_1}\right)^{2k_2(k_2-1)/\beta} \frac{\left(\imath^N e^{-\imath\psi N}\right)^{k_2}}{\mathrm{FU}_{k_2}^{(4/\beta)}} \prod_{b=1}^{k_2}\prod_{a=1}^{N}\left(\frac{a}{\gamma_1}+\frac{b-1}{\gamma_2}\right) \tag{8.20}$$

$$\begin{aligned}
w_2 =\ & \frac{(-1)^{k_1k_2}}{\mathrm{FU}_{k_2}^{(4/\beta)}}\left(\frac{2\pi}{\gamma_1}\right)^{k_2}\left(\frac{\pi}{\gamma_1}\right)^{2k_2(k_2-1)/\beta}\left[\frac{(-\imath)^N e^{-\imath\psi N}}{(N-k_1)!\gamma_1^N}\right]^{k_2} \\
\times\ & \prod_{j=0}^{k_2-1}\frac{\Gamma\left(N+1+2j/\beta\right)}{\Gamma\left(1+2j/\beta\right)}.
\end{aligned} \tag{8.21}$$

This lemma yields for the supersymmetric Ingham–Siegel integral

$$I_k^{(\beta,N)}(\rho) = W\Theta(r_1)\frac{\det^\kappa r_1}{|\Delta_{k_2}(r_2)|^{4/\beta}}\prod_{j=1}^{k_2}\left(\frac{\partial}{\partial r_{j2}}\right)^{N-k_1}\delta(r_{j2})\,, \tag{8.22}$$

where the constant reads

$$\begin{aligned}
W =\ & \left(\frac{\tilde{\gamma}}{2\pi}\right)^{k_1k_2}\left(\frac{2\pi}{\gamma_1}\right)^{k_2}\left(\frac{\pi}{\gamma_1}\right)^{2k_2(k_2-1)/\beta}\left[\frac{\left(-e^{-\imath\psi}\right)^N}{(N-k_1)!\gamma_1^N}\right]^{k_2} \\
\times\ & \frac{G_{Nk_1}^{(\beta)}}{\mathrm{FU}_{k_2}^{(4/\beta)}}\prod_{j=0}^{k_2-1}\frac{\Gamma\left(N+1+2j/\beta\right)}{\Gamma\left(1+2j/\beta\right)}.
\end{aligned} \tag{8.23}$$

We further simplify this formula for $\beta = 1$ and $\beta = 2$. The powers of the Vandermonde determinant $\Delta_{k_2}^{4/\beta}(r_2)$ are polynomials of degree $k_2 \times 2(k_2 - 1)/\beta$. The single power of one eigenvalue derivative must be $2(k_2-1)/\beta$ if we substitute these terms in Eq. (8.17) by partial derivatives of the eigenvalues, for details see App. B.4.2. Hence, this power is a half-integer for $\beta = 4$. Also, $\Delta_{k_2}(r_2)$ has no symmetric term where all eigenvalues have the same power. Therefore, we cannotsimplify the quaternionic case in the same manner.

We use the identities

$$\prod_{j=1}^{n}\frac{\partial^{n-1}}{\partial x_j^{n-1}}\Delta_n^2(x) = (-1)^{n(n-1)/2}n!\,[(n-1)!]^n\ , \tag{8.24}$$

$$\prod_{j=1}^{n}\frac{\partial^{2(n-1)}}{\partial x_j^{2(n-1)}}\Delta_n^4(x) = n!\,[(2n-2)!]^n\prod_{j=0}^{n-1}(2j+1) \tag{8.25}$$

and find

$$\begin{aligned} I_k^{(1,N)}(\rho) &= 2^{-k(k-2)}\left[\frac{2\pi e^{-\imath\psi N}}{(N-2)!}\right]^k \\ &\times\ \Theta(r_1)\det r_1^{(N-1)/2}\prod_{j=1}^{k}\left(-\frac{\partial}{\partial r_{j2}}\right)^{N-2}\delta(r_{j2}) \end{aligned}\tag{8.26}$$

and

$$\begin{aligned} I_k^{(2,N)}(\rho) &= (-1)^{k(k+1)/2}2^{-k(k-1)}\left[\frac{2\pi e^{-\imath\psi N}}{(N-1)!}\right]^k \\ &\times\ \Theta(r_1)\det r_1^{N}\prod_{j=1}^{k}\left(-\frac{\partial}{\partial r_{j2}}\right)^{N-1}\delta(r_{j2})\ . \end{aligned}\tag{8.27}$$

For $\beta=4$, we summarize the constants and have

$$\begin{aligned} I_k^{(4,N)}(\rho) &= 2^{-k(k-2)}\left[\frac{2\pi e^{-\imath\psi N}}{(N-k)!}\right]^{2k}\Theta(r_1) \\ &\times\ \det r_1^{N+1/2}\frac{4^k k!}{\pi^k|\Delta_{2k}(r_2)|}\prod_{j=1}^{2k}\left(-\frac{\partial}{\partial r_{j2}}\right)^{N-k}\delta(r_{j2})\,. \end{aligned}\tag{8.28}$$

These distributions are true for superfunctions whose fermion–fermion block dependence is as in Eq. (8.18). Equations (8.26) and (8.27) can be extended to distributions on arbitrary Schwartz functions which is not the case for Eq. (8.28). The constants in Eqs. (8.26) and (8.27) must be the same due to the independence of the test–function.

Theorem 8.2.2
Equations (8.26) *and* (8.27) *are true for rotation invariant superfunctions* Φ_0 *which are Schwartz functions in the fermion–fermion block entries along the Wick–rotated real axis.*

We derive this theorem in App. B.4.3.

Indeed, Eq. (8.27) is the same as the formula for the supersymmetric Ingham–Siegel integral for $\beta = 2$ in Ref. [61]. Comparing both results, the different definitions of the measures have to be taken into account. We also see the similarity to the superbosonization formula [81, 80, 84, 83, 86, 82] for $\beta \in \{1, 2\}$, see also Sec. 10.1. One can replace the partial derivatives in Eq. (8.26) and (8.27) by contour integrals if the characteristic function Φ_0 is analytic. However for $\beta = 4$, more effort is needed. For our purposes with the generalized Hubbard–Stratonovich transformation, Eqs. (8.14) and (8.28) are sufficient for the quaternionic case. In the unitary case, the equivalence of Eq. (8.27) with the superbosonization formula was confirmed with help of Cauchy integrals by Basile and Akemann [82].

8.3 The generalized Hubbard–Stratonovich transformation for supersymmetric Wishart matrices

The generalized Hubbard–Stratonovich transformation is much more general as shown in the previous section. Instead of the $(2k + 2k) \times (2k + 2k)$ dimensional supermatrix $\widehat{B}$ in Eq. (7.34) with the definition (7.28), we now consider the $(\gamma_2 c + \gamma_1 d) \times (\gamma_2 c + \gamma_1 d)$ supersymmetric Wishart matrix

$$B = \tilde{\gamma}^{-1} V V^\dagger . \tag{8.29}$$

The $(\gamma_2 c+\gamma_1 d)\times(\gamma_2 a+\gamma_1 b)$ rectangular supermatrix V is given in Eq. (2.13). The supermatrix B can be written in the columns of V,

$$B = \frac{1}{\tilde{\gamma}} \left[\sum_{j=1}^{a} \Psi_{j1}^{(\mathrm{C})} \Psi_{j1}^{(\mathrm{C})\,\dagger} + \sum_{j=1}^{b} \Psi_{j2}^{(\mathrm{C})} \Psi_{j2}^{(\mathrm{C})\,\dagger} \right] . \tag{8.30}$$

The corresponding generating function (6.8) is an integral over a rotation invariant superfunction P on a superspace, which is sufficiently convergent in the Wick–rotated space $\Sigma_{\beta,a/b}^{-\psi}$ and analytic in its real independent variables,

$$Z_{cd}^{ab}(x^-) = \int\limits_{\Sigma_{\beta,a/b}^{-\psi}} P(\sigma) \mathrm{Sdet}^{\,-1/\tilde{\gamma}} \left(\sigma \otimes \widehat{\Pi}_{2\psi} - \mathbb{1}_{\gamma_2 a+\gamma_1 b} \otimes x^- \right) d[\sigma] \,, \tag{8.31}$$

where

$$x^- = \mathrm{diag}\left(x_{11} \otimes \mathbb{1}_{\gamma_2}, \ldots, x_{c1} \otimes \mathbb{1}_{\gamma_2}, x_{12} \otimes \mathbb{1}_{\gamma_1}, \ldots, x_{d2} \otimes \mathbb{1}_{\gamma_1}\right) - \imath\varepsilon \mathbb{1}_{\gamma_2 c+\gamma_1 d} \,. \tag{8.32}$$

The additional Wick–rotation $\widehat{\Pi}_{2\psi}$ guarantees the convergence of this integral. The analog to Eq. (7.34) is

$$Z_k(x^- + J) = \frac{1}{K_{ab}^{cb}} \int\limits_{\mathrm{Mat}_\beta^0(c\times a/d\times b)} \Phi_0(B_\psi) \exp\left[-\imath \mathrm{Str}\,(x^- + J)\widehat{\Pi}_{-2\psi} B_\psi\right] d[V]\,, \tag{8.33}$$

where the measure $d[V]$ and the constant K_{ab}^{cb} are the same as in Eqs. (2.36) and (2.35), respectively.

Due to corollary 3.2.5, we may restrict the discussion to $b = 0$ for all β or to $b = 1$ for $\beta = 4$. This is equivalent to the integral theorems 3.3.1, 3.3.2 and 3.3.3 applied on Eq. (8.31) because the integral is rotation invariant under $\mathrm{U}^{(\beta)}(a/b)$. Here, we consider the case $b = 0$. Thus, we omit the additional Wick–rotation $\widehat{\Pi}_{2\psi}$ in Eq. (8.31). In Sec. 10.3 we extend this result to the case $b = 1$ for $\beta = 4$.

The following theorem is proven in a way similar to Ref. [61], see also Secs. 8.1 and 8.2. The proof is given in App. B.5.

Theorem 8.3.1 (Gen. Hubbard–Stratonovich transformation)
Let F be a conveniently integrable and analytic superfunction on the set of $(\gamma_2 c + \gamma_1 d) \times (\gamma_2 c + \gamma_1 d)$ supermatrices and

$$\kappa = \frac{a - c + 1}{\gamma_1} + \frac{d - 1}{\gamma_2}. \tag{8.34}$$

With

$$a \geq c\ , \tag{8.35}$$

we find

$$\begin{aligned}
&\int\limits_{\mathrm{Mat}_\beta^0(c\times a/d)} F(B) \exp\left(-\varepsilon \mathrm{Str}\, B\right) d[V] \\
&= \widetilde{C}_{acd}^{(\beta)} \int\limits_{\Sigma_{\beta,c/d}^\psi} F\left(\hat\rho\right) \exp\left(-\varepsilon \mathrm{Str}\,\hat\rho\right) \det \rho_1^\kappa \Theta(\rho_1) \\
&\times \left(e^{-\imath\psi d} D_{dr_2}^{(4/\beta)}\right)^{a-c} \frac{\delta(r_2)}{|\Delta_d(e^{\imath\psi} r_2)|^{4/\beta}} e^{-\imath\psi c d} d[\rho] \\
&= \widetilde{C}_{acd}^{(\beta)} \int\limits_{\Sigma_{\beta,c/d}^0} \det \rho_1^\kappa \Theta(\rho_1) \frac{\delta(r_2)}{|\Delta_d(r_2)|^{4/\beta}} \\
&\times \left((-1)^d D_{dr_2}^{(4/\beta)}\right)^{a-c} F\left(\hat\rho\right) \exp\left(-\varepsilon \mathrm{Str}\,\hat\rho\right)\big|_{\psi=0}\, d[\rho]
\end{aligned} \tag{8.36}$$

with

$$\hat{\rho} = \left[\begin{array}{c|c} \rho_1 & e^{\imath\psi/2}\rho_\eta \\ \hline -e^{\imath\psi/2}\rho_\eta^\dagger & e^{\imath\psi}\left(\rho_2 - \rho_\eta^\dagger\rho_1^{-1}\rho_\eta\right) \end{array}\right]. \tag{8.37}$$

The variables r_2 are the eigenvalues of the supermatrix ρ_2. The measure $d[\rho]$ is equal to these in theorems 3.3.1 and 3.3.2, according to the Dyson index β. The differential operator in Eq. (8.36) *is the analog of the Sekiguchi differential operator [123], see Eq.* (8.17)*. The constant is*

$$\widetilde{C}_{acd}^{(\beta)} = 2^{-c}\left(2\pi\gamma_1\right)^{-ad}\left(\frac{2\pi}{\gamma_2}\right)^{cd}\tilde{\gamma}^{\beta ac/2}\frac{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a)\right)}{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a-c)\right)\mathrm{FU}_d^{(4/\beta)}}. \tag{8.38}$$

The volume of the rotation group $\mathrm{U}^{(\beta)}(n)$ *is given by*

$$\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(n)\right) = \prod_{j=1}^{n}\frac{2\pi^{\beta j/2}}{\Gamma\left(\beta j/2\right)}. \tag{8.39}$$

Since the diagonalization of ρ_2 yields $|\Delta_d(r_2)|^{4/\beta}$ in the measure, the ratio of the Dirac distribution with the Vandermonde determinant is for Schwartz functions on $\mathrm{Herm}\,(4/\beta, d)$ well–defined. Also, the action of $D_{dr_2}^{(4/\beta)}$ on such a Schwartz function integrated over the corresponding rotation group is finite at zero.

As in Sec. 8.2, the distribution in the fermion–fermion block in Eq. (8.36) takes for $\beta \in \{1,2\}$ the simpler form [61]

$$\begin{aligned} &\left(D_{dr_2}^{(4/\beta)}\right)^{a-c}\frac{\delta(r_2)}{|\Delta_d(r_2)|^{4/\beta}} \\ =\;& \mathrm{FU}_d^{(4/\beta)}\prod_{n=1}^{d}\frac{\Gamma\left(a-c+1+2(n-1)/\beta\right)}{(-\pi)^{2(n-1)/\beta}\Gamma\left(\gamma_1\kappa\right)}\prod_{n=1}^{d}\frac{\partial^{\gamma_1\kappa-1}}{\partial r_{n2}^{\gamma_1\kappa-1}}\delta(r_{2n}). \end{aligned} \tag{8.40}$$

This expression written as a contour integral is the superbosonization formula [82], see Sec. 10.2. We do not find such a simplification for β due to the same reason as in Sec. 8.2. The Vandermonde determinant $|\Delta_d(r_2)|$ as the Jacobian in the eigenvalue–angle coordinates is not polynomial and can, hence, not be absorbed by the derivatives.

Chapter 9

Discussion and results of the integrals in superspace

We again restrict ourselves to the case (6.8), i.e. $p = k_1 = \gamma_1 k$ and $q = k_2 = \gamma_2 k$. In Sec. 9.1, we present the generating function as a supersymmetric integral over eigenvalues and introduce the supersymmetric Bessel functions. In Sec. 9.2, we revisit the unitary case and point out certain properties of the generating function. Some of these properties, independence of the Wick–rotation and the choice of Φ_0, are also proven for the orthogonal and the unitary–symplectic case in Sec. 9.3. In Sec. 9.4, we calculate the k-point correlation functions.

9.1 Eigenvalue integral representation

The next step to calculate the generating function $Z_k(x^- + J)$ is the integration over the supergroup. The function $\Phi_0(\rho)I_k^{(\beta,N)}(\rho)$ is invariant under the action of $\mathrm{U}^{(\beta)}(k_1/k_2)$.

With help of the supermatrix Bessel function $\varphi^{(\beta)}_{k_1/k_2}(s,r)$, see Eq. (4.3), with the normalization (4.30), we find for the generating function

$$\begin{aligned} Z_k(x^- + J) &= 2^{2k(k-\tilde{\gamma})} e^{\imath\psi k_1} \int\limits_{\mathbb{R}^{k_1+k_2}} \Phi_0(r) I_k^{(\beta,N)}(r) \\ &\times \varphi^{(\beta)}_{k_1/k_2}(-\imath r, x^- + J) \left| B_k^{(\beta)}(r) \right| d[r_2]d[r_1] + \text{b.t.} \end{aligned} \tag{9.1}$$

The normalization of Z_k is guaranteed by the Efetov–Wegner terms "b.t.". One can see this when setting $(k-l)$ with $l < k$ of the source variables J_p to zero. Then we have

$$Z_k(x^- + J)\Big|_{J_l=\ldots=J_k=0} = Z_{l-1}(\tilde{x}^- + \widetilde{J}) \,, \tag{9.2}$$

$\tilde{x} = \mathrm{diag}\,(x_1, \ldots, x_{l-1})$, $\widetilde{J} = \mathrm{diag}\,(J_1, \ldots, J_{l-1})$, by the integration theorems in Refs. [58, 91, 92, 93, 68], see also theorems 3.3.1, 3.3.2 and 3.3.3. This agrees with the definition (6.8). In Eq. (6.8) the determinants cancel each other in the numerator and the denominator when $J_l = 0$.

9.2 The unitary case revisited

To make contact with the discussion in Ref. [61], we revisit the unitary case using the insight developed here.

For a further calculation we need the explicit structure of the supersymmetric matrix Bessel functions. However, the knowledge of these functions is limited. Only for certain β and k we know the exact structure. In particular for $\beta = 2$ the supermatrix Bessel function was first calculated in Ref. [94, 105] with help of the heat equation. In Sec. 4.2 we calculate it with help of partial integrations. For $p = q = k$ we find, see Eq. (4.27),

$$\varphi_{k/k}^{(2)}(-\imath r, x^- + J) = \frac{\imath^k \exp\left(-\varepsilon \mathrm{Str}\, r\right)}{2^{k^2}\pi^k (k!)^2} \tag{9.3}$$
$$\times \quad \frac{\det\left[\exp\left(-\imath r_{m1}(x_n - J_n)\right)\right]_{1\leq m,n\leq k} \det\left[\exp\left(\imath e^{\imath\psi} r_{m2}(x_n + J_n)\right)\right]_{1\leq m,n\leq k}}{\sqrt{B_k^{(2)}(r) B_k^{(2)}\left(x^- + J\right)}}$$

with $x \pm J = \mathrm{diag}\,(x_1 \pm J_1, \ldots, x_k \pm J_k)$ and the positive square root of the Berezinian

$$\sqrt{B_k^{(2)}(r)} = \frac{\Delta_k(s_1)\Delta_k(e^{\imath\psi}s_2)}{V_k(s)} = (-1)^{k(k-1)/2} \det\left[\frac{1}{r_{a1} - e^{\imath\psi} r_{b2}}\right]_{1\leq a,b\leq k}. \tag{9.4}$$

Due to the structure of $\varphi_{k/k}^{(2)}$ and $B_k^{(2)}$, we write the generating function for $\beta = 2$ as an integral over Φ_0 times a determinant [61]

$$Z_k(x^- + J) = (-1)^{k(k+1)/2} \det{}^{-1}\left[\frac{1}{x_a - x_b - J_a - J_b}\right]_{1\leq a,b\leq k} \int\limits_{\mathbb{R}^{2k}} \Phi_0(r)$$
$$\times \det\left[\mathfrak{F}_N(\tilde{r}_{mn}, \tilde{x}_{mn}) \Theta(r_{m1}) \exp\left(-\varepsilon \mathrm{Str}\, \tilde{r}_{mn}\right)\right]_{1\leq m,n\leq k} d[r_2] d[r_1] + \text{b.t.} \tag{9.5}$$

where $\tilde{r}_{mn} = \mathrm{diag}\,\left(r_{m1}, e^{\imath\psi} r_{n2}\right)$, $\tilde{x}_{mn} = \mathrm{diag}\,\left(x_m - J_m, x_n + J_n\right)$ and

$$\mathfrak{F}_N(\tilde{r}_{mn}, \tilde{x}_{mn}) = \frac{\imath r_{m1}^N \exp\left(-\imath \mathrm{Str}\, \tilde{r}_{mn}\tilde{x}_{mn}\right)}{(N-1)!(r_{m1} - e^{\imath\psi} r_{n2})} \left(-e^{-\imath\psi}\frac{\partial}{\partial r_{n2}}\right)^{N-1} \delta(r_{n2})\,. \tag{9.6}$$

Then, the modified k–point correlation function is

$$\begin{aligned}\widehat{R}_k(x^-) &= \text{b.t.} + \int\limits_{\mathbb{R}^{2k}} d[r_2]d[r_1]\Phi_0(r) \\ &\times \det\left[\mathfrak{F}_N(\tilde{r}_{mn}, x_{mn})\Theta(r_{m1})\exp\left(-\varepsilon \operatorname{Str}\tilde{r}_{mn}\right)\right]_{1\leq m,n\leq k} \end{aligned} \quad (9.7)$$

and the k–point correlation function is

$$R_k(x) = \int\limits_{\mathbb{R}^{2k}} \Phi_0(r)\det\left[\frac{\mathfrak{F}_N(\tilde{r}_{mn}, x_{mn})}{2\pi\imath}\right]_{1\leq m,n\leq k} d[r_2]d[r_1] + \text{b.t.} \quad (9.8)$$

We defined $x_{mn} = \operatorname{diag}(x_m, x_n)$. The boundary terms comprise the lower correlation functions. The k–point correlation function for $\beta = 2$ is a determinant of the fundamental function

$$R^{(\text{fund})}(x_m, x_n) = \int\limits_{\mathbb{R}^2} \Phi_0(r)\frac{\mathfrak{F}_N(r, x_{mn})}{2\pi\imath}dr_2dr_1 \quad (9.9)$$

if there is one characteristic function $\mathcal{F}P_0$ with a supersymmetric extension Φ_0 factorizing for diagonal supermatrices,

$$\Phi_0(r) = \operatorname{Sdet}\operatorname{diag}\left[\widehat{\Phi}_0(r_{11}), \ldots, \widehat{\Phi}_0(r_{k1}), \widehat{\Phi}_0\left(e^{\imath\psi}r_{12}\right), \ldots, \widehat{\Phi}_0\left(e^{\imath\psi}r_{k2}\right)\right], \quad (9.10)$$

with $\widehat{\Phi}_0 : \mathbb{C} \to \mathbb{C}$. For example, the shifted Gaussian ensemble in App. F of Ref. [61] or the Laguerre ensemble in subsection 16.2.4 of this thesis are of such a type.

In Eq. (9.9) we notice that this expression is independent of the generalized Wick–rotation. Every derivative of the fermionic eigenvalue r_2 contains the inverse Wick–rotation as a prefactor. Moreover, the Wick–rotation in the functions are only prefactors of r_2. Thus, an integration over the fermionic eigenvalues r_2 in Eq. (9.8) cancels the Wick–rotation by using the Dirac distribution. Also, this integration shows that every representation of the characteristic function gives the same result, see theorem 9.3.1 in the next section. However, the determinantal structure with the fundamental function in Eq. (9.9) depends on a special choice of Φ_0.

9.3 Independence statement

For $\beta = 1$ and $\beta = 4$ we do not know the ordinary matrix Bessel function explicitly. Hence, we cannotgive such a compact expression as in the case $\beta = 2$. On the other hand, we can derive the independence of the Wick–rotation and of the Φ_0 choice of the generating function.

Theorem 9.3.1
The generating function Z_k is independent of the Wick–rotation and of the choice of the characteristic functions supersymmetric extension Φ_0 corresponding to a certain matrix ensemble $(P, \mathrm{Herm}\,(\beta, N))$.

Proof:
We split the proof in two parts. The first part regards the Wick–rotation and the second part yields the independence of the choice of Φ_0.

Due to the normalization of the supermatrix Bessel function (4.28), $\varphi^{(\beta)}_{k_1/k_2}(-\imath r, x^- + J)$ only depends on $e^{\imath\psi} r_2$. The same is true for Φ_0. Due to the property

$$D^{(4/\beta)}_{k_2 r_2}\left(\imath e^{\imath\psi}\gamma_1\varepsilon\right) = e^{\imath k_2\psi} D^{(4/\beta)}_{k_2, e^{\imath\psi} r_2}\left(\imath\gamma_1\varepsilon\right)\ , \tag{9.11}$$

the Ingham–Siegel integral in the form (8.14) times the phase $e^{\imath(k_1-k_2)\psi}$ only depends on $e^{\imath\psi} r_2$ and $e^{-\imath\psi}\partial/\partial r_2$. The additional phase comes from the ρ–integration. Thus, we see the independence of the Wick–rotation because of the same reason as in the $\beta = 2$ case.

Let Φ_0 and Φ_1 be two different supersymmetric extensions of the characteristic function $\mathcal{F}P$. Then these two superfunctions only depend on the invariants $\{\mathrm{Str}\,\sigma^{m_j}\}_{1\leq j\leq l_0}$ and $\{\mathrm{Str}\,\sigma^{n_j}\}_{1\leq j\leq l_1}$, $m_j, n_j, l_0, l_1 \in \mathbb{N}$. We consider Φ_0 and Φ_1 as functions of $\mathbb{C}^{l_0} \to \mathbb{C}$ and $\mathbb{C}^{l_1} \to \mathbb{C}$, respectively. Defining the function

$$\Delta\Phi(x_1, \ldots, x_M) = \Phi_0(x_{m_1}, \ldots, x_{m_{l_0}}) - \Phi_1(x_{n_1}, \ldots, x_{n_{l_1}}), \tag{9.12}$$

where $M = \max\{m_a, n_b\}$, we notice with the discussion in Sec. 7.5 that

$$\Delta\Phi(x_1, \ldots, x_M)|_{x_j = \mathrm{tr}\, H^j} = 0 \tag{9.13}$$

for every Hermitian matrix H. However, there could be a symmetric supermatrix σ with

$$\Delta\Phi(x_1, \ldots, x_M)|_{x_j = \mathrm{Str}\,\sigma^j} \neq 0. \tag{9.14}$$

With the differential operator

$$\mathfrak{D}_r = \left[D^{(4/\beta)}_{k_2 r_2}\left(\imath e^{\imath\psi}\gamma_1\varepsilon\right)\right]^{N-k_1} \frac{\varphi^{(\beta)}_{k_1/k_2}(-\imath r, x^- + J)}{V_k(r_1, e^{\imath\psi} r_2)}, \tag{9.15}$$

we consider the difference of the generating functions

$$\begin{aligned} \Delta Z_k(x^- + J) &= Z_k(x^- + J)|_{\Phi_0} - Z_k(x^- + J)|_{\Phi_1} \qquad (9.16) \\ &= \int_{\mathbb{R}^{k_1}} |\Delta_{k_2}(r_1)|^\beta \det{}^\kappa r_1 \Theta(r_1)\, \mathfrak{D}_r \Delta\Phi(x)|_{x_j = \mathrm{Str}\, r^j}\Big|_{r_2 = 0} d[r_1] \end{aligned}$$

Here, we omit the Efetov–Wegner terms. The differential operator is invariant under the action of the permutation group $S(k_2)$ on the fermionic block Herm $(4/\beta, k_2)$. Hence, we find

$$\begin{aligned}
\left.\left.\mathfrak{D}_r \Delta\Phi(x)\right|_{x_j=\mathrm{Str}\, r^j}\right|_{r_2=0} &= \left.\left.\sum_{\substack{a\in\{0,\ldots,N-k_1\}^M \\ |a|\leq k_2(N-k_1)}} d_a(r) \prod_{j=1}^{M} \frac{\partial^{a_j}}{\partial x_j^{a_j}} \Delta\Phi(x)\right|_{x_j=\mathrm{Str}\, r^j}\right|_{r_2=0} \\
&= \left.\sum_{\substack{a\in\{0,\ldots,N-k_1\}^M \\ |a|\leq k_2(N-k_1)}} d_a(r_1) \prod_{j=1}^{M} \frac{\partial^{a_j}}{\partial x_j^{a_j}} \Delta\Phi(x)\right|_{x_j=\mathrm{tr}\, r^j} \\
&= 0, \qquad (9.17)
\end{aligned}$$

where d_a are certain symmetric functions depending on the eigenvalues r. At $r_2 = 0$ these functions are well-defined since the supermatrix Bessel functions and the term $V_k^{-1}(r)$ are C^∞ at this point. Thus, we find that

$$\Delta Z_k(x^- + J) = 0. \qquad (9.18)$$

This means that the generating function is independent of the supersymmetric extension of the characteristic function. □

9.4 One–point and higher order correlation functions

We need an explicit expression or some properties of the supermatrix Bessel function to simplify the integral for the generating function. For $k = 1$ we know the supermatrix Bessel functions for all β. The simplest case is $\beta = 2$ where we take the formula (9.8) with $k = 1$ and obtain

$$R_1(x) = R^{(\mathrm{fund})}(x,x) = \int_{\mathbb{R}^2} \Phi_0(r) \frac{\mathfrak{F}_N\left(r, x\mathbb{1}_2\right)}{2\pi\imath} dr_2 dr_1 \,. \qquad (9.19)$$

Since the Efetov–Wegner term in the generating function is just unity there are no boundary terms in the level density. For $\beta \in \{1, 4\}$ we use the supermatrix Bessel function [89, 108], see also Eq. (4.18),

$$\begin{aligned}
\varphi_{2/1}^{(1)}(-\imath r, x^- + J) &= \frac{-2J}{\pi} \exp\left[-\imath \mathrm{Str}\, r(x^- + J)\right] \\
&\times \left[\imath \mathrm{Str}\, r + J\left(r_{11} - e^{\imath\psi} r_2\right)\left(r_{21} - e^{\imath\psi} r_2\right)\right] \,. \qquad (9.20)
\end{aligned}$$

We find

$$\widehat{R}_1(x^-) = -\imath \int\limits_{\mathbb{R}^3} \Phi_0(r) \det r_1^{(N-1)/2} \mathrm{Str}\, r \frac{|r_{11} - r_{21}|}{(r_{11} - e^{\imath\psi} r_2)^2 (r_{21} - e^{\imath\psi} r_2)^2}$$
$$\times \quad \exp\left(-\imath x^- \mathrm{Str}\, r\right) \Theta(r_1) \frac{1}{(N-2)!} \left(-e^{-\imath\psi} \frac{\partial}{\partial r_2}\right)^{N-2} \delta(r_2) d[r_1] dr_2 \tag{9.21}$$

for $\beta = 1$ and

$$\widehat{R}_1(x^-) = -4\imath \int\limits_{\mathbb{R}^3} \Phi_0(r) r_1^{2N+1} \mathrm{Str}\, r \frac{e^{\imath\psi} r_{12} - e^{\imath\psi} r_{22}}{(r_1 - e^{\imath\psi} r_{12})^2 (r_1 - e^{\imath\psi} r_{22})^2} \tag{9.22}$$
$$\times \exp\left(-\imath x^- \mathrm{Str}\, r\right) \Theta(r_1) \frac{\det e^{\imath\psi} r_2}{(2N+1)!} \left(4 e^{-2\imath\psi} D_{2,r_2}^{(1)}\right)^N \frac{\delta(r_{12})\delta(r_{22})}{e^{\imath\psi} r_{12} - e^{\imath\psi} r_{22}} d[r_2] dr_1$$

for $\beta = 4$. The differential operator has the explicit form

$$D_{2,r_2}^{(1)} = \frac{\partial^2}{\partial r_{12} \partial r_{22}} - \frac{1}{2} \frac{1}{r_{12} - r_{22}} \left(\frac{\partial}{\partial r_{12}} - \frac{\partial}{\partial r_{22}}\right). \tag{9.23}$$

For the level density we have

$$R_1(x) = -\frac{1}{2\pi} \int\limits_{\mathbb{R}^3} d[r_1] dr_2 \Phi_0(r) \det r_1^{(N-1)/2} \exp\left(-\imath x \mathrm{Str}\, r\right) \mathrm{Str}\, r \tag{9.24}$$
$$\times \quad \frac{|r_{11} - r_{21}|}{(r_{11} - e^{\imath\psi} r_2)^2 (r_{21} - e^{\imath\psi} r_2)^2} \frac{\Theta(r_1) + \Theta(-r_1)}{(N-2)!} \left(-e^{-\imath\psi} \frac{\partial}{\partial r_2}\right)^{N-2} \delta(r_2)$$

for $\beta = 1$ and

$$R_1(x) = -\frac{2}{\pi} \int\limits_{\mathbb{R}^3} d[r_2] dr_1 \Phi_0(r) r_1^{2N+1} \exp\left(-\imath x \mathrm{Str}\, r\right) \mathrm{Str}\, r \tag{9.25}$$
$$\times \quad \frac{e^{\imath\psi} r_{12} - e^{\imath\psi} r_{22}}{(r_1 - e^{\imath\psi} r_{12})^2 (r_1 - e^{\imath\psi} r_{22})^2} \frac{\det e^{\imath\psi} r_2}{(2N+1)!} \left(4 e^{-2\imath\psi} D_{2,r_2}^{(1)}\right)^N \frac{\delta(r_{12})\delta(r_{22})}{e^{\imath\psi} r_{12} - e^{\imath\psi} r_{22}}$$

for $\beta = 4$. The equations (9.22) to (9.25) comprise all level–densities for arbitrary matrix ensembles invariant under orthogonal and unitary–symplectic rotations. As probability densities which do not factorize are included, these results considerably extend those obtained by orthogonal polynomials.

For higher order correlation functions we use the definition (6.2) and the definition of the matrix Green function. With help of the quantities

$L = \mathrm{diag}\,(L_1, \ldots, L_k) \in \{\pm 1\}^k$ and $\widehat{L} = L \otimes 1\!\!1_{2\tilde{\gamma}}$, this yields

$$R_k(x) = 2^{2k(k-\tilde{\gamma})} \int\limits_{\mathbb{R}^{k_1+k_2}} \Phi_0(r) \lim_{\epsilon \searrow 0} \sum_{L \in \{\pm 1\}^k} \prod_{j=1}^{k} L_j \, \frac{I_k^{(\beta,N)}\left(\widehat{L}r\right) \exp\left(-\varepsilon \mathrm{Str}\, \widehat{L}r\right)}{\left(2\pi \imath e^{-\imath\psi\gamma_1}\right)^k}$$
$$\times \left(\prod_{j=1}^{k} -\frac{1}{2}\frac{\partial}{\partial J_j}\right) \varphi_{k_1 k_2}^{(\beta)}(-\imath r, x^{(0)} + J)\Bigg|_{J=0} \left|B_k^{(\beta)}(r_1, e^{\imath\psi} r_2)\right| d[r_2] d[r_1] + \mathrm{b.t.} \tag{9.26}$$

for analytic correlation functions. We extend this formula to all rotation invariant ensembles by the universality of the integral kernel. First, we make a limit of a uniformly convergent series of Schwartz functions analytic in the real components of its entries to every arbitrary Schwartz function describing a matrix ensemble. The Schwartz functions are dense in a weak sense in the sets of Lebesgue–integrable functions L^p and the tempered distributions. Thus, we integrate Eq. (9.26) with an arbitrary Schwartz function on $\mathbb{R}^k$ and take the limit of a series of Schwartz functions describing the ensembles to a tempered distribution which completes the extension.

Chapter 10

Comparison of the generalized Hubbard–Stratonovich transformation and the superbosonization formula

In Sec. 10.1, we present and further generalize the superbosonization formula, respectively. The theorem stating the equivalence of the generalized Hubbard–Stratonovich transformation and the superbosonization formula is given in Sec. 10.2 including a clarification of their mutual connection. In Sec. 10.3, we extend the theorems 8.3.1 and 10.1.1 to arbitrary matrix dimension.

10.1 The superbosonization formula

To compare the generalized Hubbard–Stratonovich transformation in its general form, see Sec. 8.3, with the superbosonization formula [83, 84], we have to extend the latter to arbitrary dimensions of V, see Eq. (2.13). Due to corollary 3.2.5 we consider the case $b = 0$ for all rotation classes as in Sec. 8.3. The case $b = 1$ for $\beta = 4$ will be discussed in Sec. 10.3.

We need for the following theorem the definition of the set

$$\begin{aligned} &\Sigma^{(\mathrm{c})}_{\beta,p/q} \qquad\qquad (10.1)\\ = &\left\{\sigma \in \mathrm{Mat}\,^0_\beta(p/q) \,\middle|\, \sigma = \begin{bmatrix} \sigma_1 & \sigma_{12}^\dagger \\ \sigma_{12} & \sigma_2 \end{bmatrix}, \sigma_1 \in \mathrm{Herm}\,(\beta, p), \sigma_2 \in \mathrm{CU}^{\,(4/\beta)}\,(q)\right\}\end{aligned}$$

where $\mathrm{CU}^{\,(\beta)}\,(q)$ is the set of the circular orthogonal (COE, $\beta = 1$), unitary

(CUE, $\beta = 2$) or unitary-symplectic (CSE, $\beta = 4$) ensembles,

$$\begin{aligned}&\mathrm{CU}^{(\beta)}(q) \qquad\qquad (10.2)\\ = &\left\{A \in \mathrm{Gl}(\gamma_2 q, \mathbb{C}) \left| \begin{array}{ll} A = A^T \in \mathrm{U}^{(2)}(q) & ,\ \beta = 1 \\ A \in \mathrm{U}^{(2)}(q) & ,\ \beta = 2 \\ A = (Y_s \otimes 1\!\!1_q) A^T (Y_s^T \otimes 1\!\!1_q) \in \mathrm{U}^{(2)}(2q) & ,\ \beta = 4 \end{array}\right.\right\}\end{aligned}$$

The index "c" indicates the relation to the circular ensembles. We notice that the set classes $\Sigma^0_{\beta,p/q}$ and $\Sigma^{(c)}_{\beta,p/q}$ differ in the fermion–fermion block. In Sec. 10.2, we show that this is the crucial difference between both methods. Due to the nilpotence of B's fermion–fermion block, we can change the set in this block for the Fourier transformation.

The proof of the superbosonization formula [83, 84] given below is based on the proofs of the superbosonization formula for arbitrary superfunctions on real supersymmetric Wishart matrices in Ref. [83] and for Gaussian functions on real, complex and quaternionic Wishart matrices in Ref. [124]. This theorem extends the superbosonization formula of Ref. [84] to averages of square roots of determinants over unitary-symplectically invariant ensembles, i.e. $\beta = 4$, $b = c = 0$ and d odd in Eq. (8.31). The proof of this theorem is given in App. B.6.

Theorem 10.1.1 (Superbosonization formula)
Let F and κ be the same as in theorem 8.3.1. If the inequality (8.35) *holds, we have*

$$\begin{aligned}&\int\limits_{\mathrm{Mat}^0_\beta(c\times a/d)} F(B)\exp\left(-\varepsilon \mathrm{Str}\, B\right) d[V] \\ = \ & C^{(\beta)}_{acd} \int\limits_{\Sigma^{(c)}_{\beta,p/q}} F(\rho)\exp\left(-\varepsilon \mathrm{Str}\,\rho\right) \mathrm{Sdet}\,\rho^{\kappa}\Theta(\rho_1) d[\rho]\,, \qquad (10.3)\end{aligned}$$

where the constant is

$$\begin{aligned} C^{(\beta)}_{acd} &= (-2\pi\gamma_1)^{-ad}\left(-\frac{2\pi}{\gamma_2}\right)^{cd} 2^{-c\tilde{\gamma}\beta ac/2}\frac{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a)\right)}{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a-c)\right)} \\ &\times \prod_{n=1}^{d}\frac{\Gamma\left(\gamma_1\kappa + 2(n-d)/\beta\right)}{\imath^{4(n-1)/\beta}\pi^{2(n-1)/\beta}}\,. \qquad (10.4)\end{aligned}$$

We define the measure $d[\widehat{V}]$ as in Eq. (2.36) *and the measure on the right hand side is $d[\rho] = d[\rho_1]d[\rho_2]d[\eta]$ where $d[\rho_1]$ and $d[\eta]$ is defined as in theorems 3.3.1* *and 3.3.2 and*

$$d[\rho_2] \;=\; \mathrm{FU}_d^{(4/\beta)}|\Delta_d(e^{\imath\varphi_j})|^{4/\beta}\prod_{n=1}^{d}\frac{de^{\imath\varphi_n}}{2\pi\imath}d\mu(U)\,. \tag{10.5}$$

Here, $\rho_2 = U\mathrm{diag}\,(e^{\imath\varphi_1},\ldots,e^{\imath\varphi_d})\,U^\dagger$*,* $U \in \mathrm{U}^{(4/\beta)}(d)$ *and* $d\mu(U)$ *is the normalized Haar–measure of* $\mathrm{U}^{(4/\beta)}(d)$*. The absolute value of the Vandermonde determinant* $\Delta_d(e^{\imath\varphi_j})$ *refers to a change of sign in every single difference* $(e^{\imath\varphi_n} - e^{\imath\varphi_m})$ *with "+" if* $\varphi_m < \varphi_n$ *and with "−" otherwise. Thus, it is not an absolute value in the complex plane. The constant* $\mathrm{FU}_d^{(4/\beta)}$ *is defined in Eq.* (4.32)*.*

The exponential term can also be shifted in the superfunction F. We need this additional term to regularize an intermediate step in the proof.

The inequality (8.35) is also crucial for this theorem. For example, let $\beta = 2$ and $F(\rho) = 1$. Then, the left hand side of Eq. (8.36) is not equal to zero. On the right hand side of Eq. (10.3), the dependence on the Grassmann variables only stems from the superdeterminant and we find

$$\int\limits_{\Lambda_{2cd}} \mathrm{Sdet}\,\rho^\kappa d[\eta] = \int\limits_{\Lambda_{2cd}} \frac{\det\left(\rho_1 + \eta\rho_2^{-1}\eta^\dagger\right)^\kappa}{\det\rho_2^\kappa}d[\eta] = 0 \tag{10.6}$$

for $\kappa < d$. The superdeterminant $\mathrm{Sdet}\,\rho$ is a polynomial of order $2c$ in the Grassmann variables $\{\eta_{nm},\eta_{nm}^*\}$ and the integral over the remaining variables is finite for $\kappa \geq 0$. Hence, it is easy to see that the right hand side of Eq. (10.3) is zero for $\kappa < d$. This inequality is equivalent to $a < c$.

This problem was also discussed in Ref. [86]. These authors gave a solution for the case that Eq. (8.35) is violated. This solution differs from our approaches in Secs. 10.2 and 11.3.

10.2 Equivalence of and connections between the two approaches

In Secs. 8.2 and 8.3, we have argued that both expressions in theorems 8.3.1 and 10.1.1 are equivalent for $\beta \in \{1,2\}$. Now we address all $\beta \in \{1,2,4\}$. The theorem below is proven in App. B.7. The proof treats all three cases in a unifying way. Properties of the ordinary matrix Bessel functions are used.

Theorem 10.2.1 (Equivalence of theorems 8.3.1 and 10.1.1)
The generalized Hubbard–Stratonovich transformation, 8.3.1, and the superbosonization formula, 10.1.1, are equivalent for superfunctions which are

Schwartz functions on the Wick–rotated real axis and analytic in the fermionic eigenvalues.

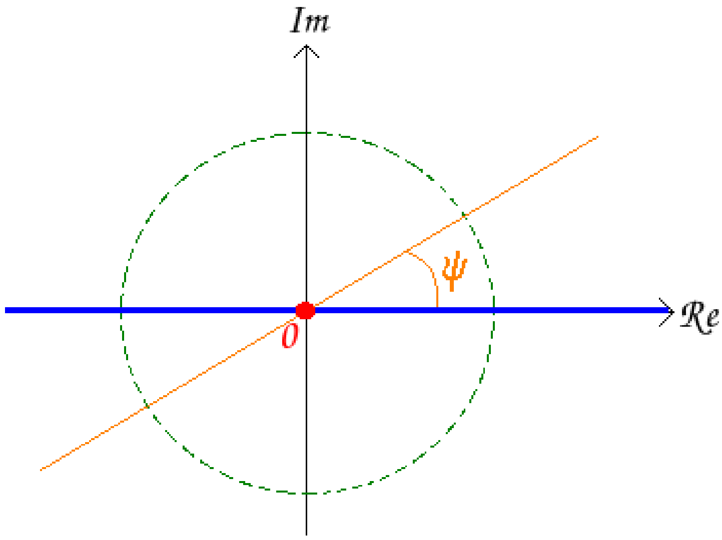

Figure 10.1: In the superbosonization formula, the integration of the fermionic eigenvalues is along the unit circle in the complex plane (dotted circle). The eigenvalue integrals in the generalized Hubbard–Stratonovich transformation are integrations over the real axis (bold line) or on the Wick–rotated real axis (thin line at angle ψ). The differential operator acts on the superfunction or on the Dirac distribution at zero (bold dot, 0), respectively. Taken from Ref. [3]

The compact integral in the fermion–fermion block of the superbosonization formula can be considered as a contour integral. In the proof of theorem 10.2.1, we find the integral identity

$$\begin{aligned}
&\int\limits_{[0,2\pi]^d} \widetilde{F}\left(e^{\imath\varphi_j}\right)\left|\Delta_d\left(e^{\imath\varphi_j}\right)\right|^{4/\beta}\prod_{n=1}^{d}\frac{e^{\imath(1-\gamma_1\kappa)\varphi_n}d\varphi_n}{2\pi} \\
&= \prod_{n=1}^{d}\frac{\imath^{4(n-1)/\beta}\Gamma(1+2n/\beta)}{\Gamma(2/\beta+1)\Gamma(\gamma_1\kappa-2(n-1)/\beta)}\left(D_{dr_2}^{(4/\beta)}\right)^{a-c}\widetilde{F}(r_2)\bigg|_{r_2=0}
\end{aligned} \tag{10.7}$$

for an analytic function $\widetilde{F}$ on $\mathbb{C}^d$ with permutation invariance. Hence, we can relate both constants (8.38) and (10.4),

$$\frac{\widetilde{C}_{acd}^{(\beta)}}{C_{acd}^{(\beta)}} = (-1)^{d(a-c)}\prod_{n=1}^{d}\frac{\imath^{4(n-1)/\beta}\Gamma(1+2n/\beta)}{\Gamma(2/\beta+1)\Gamma(\gamma_1\kappa-2(n-1)/\beta)}\,. \tag{10.8}$$

The integral identity (10.7) is reminiscent of the residue theorem. It is the analog of the connection between the contour integral and the differential operator in the cases $\beta \in \{1, 2\}$, see Fig. 10.1. Thus, the differential operator with the Dirac distribution in the generalized Hubbard–Stratonovich transformation restricts the non-compact integral in the fermion–fermion block to the point zero and its neighborhood. Therefore it is equivalent to a compact fermion–fermion block integral as appearing in the superbosonization formula.

10.3 The general case for arbitrary positive integers a, b, c, d and arbitrary Dyson–index $\beta \in \{1, 2, 4\}$

We consider an application of our results. The inequality (8.35) reads

$$N \geq \gamma_1 k = k_1 \tag{10.9}$$

for the calculation of the k–point correlation function (6.1) with help of the matrix Green function. For $\beta = 1$, a $N \times N$ real symmetric matrix has in the absence of degeneracies N different eigenvalues. However, we can only calculate k–point correlation functions with $k < N/2$. For $N \to \infty$, this restriction does not matter. But for exact finite N calculations, we have to modify the line of reasoning.

We construct the symmetry operator

$$\widehat{\mathfrak{S}}\left(\sigma\right) = \widehat{\mathfrak{S}}\left(\left[\begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{array}\right]\right) = \left[\begin{array}{cc} -\sigma_{22} & -\sigma_{21} \\ \sigma_{12} & \sigma_{11} \end{array}\right] \tag{10.10}$$

from $(m_1+m_2)\times(n_1+n_2)$ supermatrix to $(m_2+m_1)\times(n_2+n_1)$ supermatrix. This operator has the properties

$$\begin{aligned} \widehat{\mathfrak{S}}(\sigma^\dagger) &= \widehat{\mathfrak{S}}(\sigma)^\dagger \quad , & (10.11) \\ \widehat{\mathfrak{S}}(\sigma^*) &= \widehat{\mathfrak{S}}(\sigma)^*, & (10.12) \\ \widehat{\mathfrak{S}}^2(\sigma) &= -\sigma & (10.13) \end{aligned}$$

and

$$\widehat{\mathfrak{S}}\left(\left[\begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{array}\right]\left[\begin{array}{cc} \rho_{11} & \rho_{12} \\ 0 & 0 \end{array}\right]\right) = \widehat{\mathfrak{S}}\left(\left[\begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{array}\right]\right)\widehat{\mathfrak{S}}\left(\left[\begin{array}{cc} \rho_{11} & \rho_{12} \\ 0 & 0 \end{array}\right]\right). \tag{10.14}$$

Property (10.13) tells us that the eigenvalues of $\widehat{\mathfrak{S}}$ are $\pm\imath$ when $m_1 = m_2$ and $n_1 = n_2$.

Let a, b, c, d be arbitrary positive integers and $\beta \in \{1, 2, 4\}$. Then, Eq. (10.14) reads for a matrix product of a $(\gamma_2 c+\gamma_1 d)\times(0+\gamma_1 b)$ supermatrix with a $(0 + \gamma_1 b) \times (\gamma_2 c + \gamma_1 d)$ supermatrix

$$\left[\begin{array}{c} \zeta^\dagger \\ \tilde{z}^\dagger \end{array}\right]\left[\begin{array}{cc} \zeta & \tilde{z} \end{array}\right] = \widehat{\mathfrak{S}}\left(\left[\begin{array}{c} \tilde{z}^\dagger \\ -\zeta^\dagger \end{array}\right]\right)\widehat{\mathfrak{S}}\left(\left[\begin{array}{cc} \tilde{z} & \zeta \end{array}\right]\right) = \widehat{\mathfrak{S}}\left(\left[\begin{array}{c} \tilde{z}^\dagger \\ -\zeta^\dagger \end{array}\right]\left[\begin{array}{cc} \tilde{z} & \zeta \end{array}\right]\right). \tag{10.15}$$

With help of the operator $\widehat{\mathfrak{S}}$, we split the supersymmetric Wishart matrix B into two parts,

$$B = B_1 + \widehat{\mathfrak{S}}(B_2) \tag{10.16}$$

such that

$$B_1 = \tilde{\gamma}^{-1}\sum_{j=1}^{a}\Psi_{j1}^{(\mathrm{C})}\Psi_{j1}^{(\mathrm{C})\dagger} \qquad \text{and} \qquad B_2 = \tilde{\gamma}^{-1}\sum_{j=1}^{b}\widehat{\mathfrak{S}}\left(\Psi_{j2}^{(\mathrm{C})}\right)\widehat{\mathfrak{S}}\left(\Psi_{j2}^{(\mathrm{C})}\right)^\dagger . \tag{10.17}$$

The supervectors $\widehat{\mathfrak{S}}\left(\Psi_{j2}^{(\mathrm{C})}\right)$ are of the same form as $\Psi_{j1}^{(\mathrm{C})}$. Let σ be a quadratic supermatrix, i.e. $m_1 = n_1$ and $m_2 = n_2$. Then, we find the additional property

$$\mathrm{Sdet}\,\widehat{\mathfrak{S}}(\sigma) = (-1)^{m_2}\mathrm{Sdet}^{-1}\sigma\,. \tag{10.18}$$

Let $\widehat{\Sigma}^0_{\beta,p/q} = \widehat{\mathfrak{S}}\left(\Sigma^0_{\beta,p/q}\right)$, $\widehat{\Sigma}^{(\mathrm{c})}_{\beta,p/q} = \widehat{\mathfrak{S}}\left(\Sigma^{(\mathrm{c})}_{\beta,p/q}\right)$ and the Wick–rotated set $\widehat{\Sigma}^{\psi}_{\beta,p/q} = \widehat{\Pi}_\psi\widehat{\Sigma}^0_{\beta,p/q}\widehat{\Pi}_\psi$. Then, we construct the analog of the superbosonization formula and the generalized Hubbard–Stratonovich transformation.

Theorem 10.3.1 (Extension of both supersymmetry approaches)
Let F be the superfunction as in theorem 8.3.1 and

$$\kappa = \frac{a-c+1}{\gamma_1} - \frac{b-d+1}{\gamma_2}\,. \tag{10.19}$$

Also, let $e \in \mathbb{N}_0$ and

$$\tilde{a} = a + \gamma_1 e \qquad \text{and} \qquad \tilde{b} = b + \gamma_2 e \tag{10.20}$$

with

$$\tilde{a} \geq c \quad , \quad \tilde{b} \geq d. \tag{10.21}$$

We choose the Wick–rotation $e^{\imath\psi}$ such that all integrals are convergent. Then, we have

$$\begin{aligned}
&\int\limits_{\mathrm{Mat}_{\beta}^{\psi}(c\times a/d\times b)} F(B_\psi)\exp\left(-\varepsilon\mathrm{Str}\,\widehat{B}_\psi\right)d[V] \\
= & \left(-\frac{2}{\gamma_1}\right)^{\gamma_2 ec}\left(\frac{2}{\gamma_2}\right)^{\gamma_1 ed}\int\limits_{\mathrm{Mat}_{\beta}^{0}(c\times\tilde{a}/d\times\tilde{b})} F(\widetilde{B}_\psi)\exp\left(-\varepsilon\mathrm{Str}\,\widetilde{B}_\psi\right)d[\widetilde{V}] \\
= & C_{\mathrm{SF}}\int\limits_{\Sigma_{\beta,c/d}^{(c)}}\int\limits_{\widehat{\Sigma}_{4/\beta,dc}^{0(c)}} d[\rho^{(2)}]d[\rho^{(1)}]F(\rho^{(1)}+e^{\imath\psi}\rho^{(2)}) \\
\times & \exp\left[-\varepsilon\mathrm{Str}\,(\rho^{(1)}+e^{\imath\psi}\rho^{(2)})\right]\mathrm{Sdet}^{\,\kappa+\tilde{b}/\gamma_2}\rho^{(1)}\mathrm{Sdet}^{\,\kappa-\tilde{a}/\gamma_1}\rho^{(2)} \qquad (10.22) \\
= & C_{\mathrm{HS}}\int\limits_{\Sigma_{\beta,c/d}^{0}}\int\limits_{\widehat{\Sigma}_{4/\beta,cd}^{(0)}} d[\rho^{(2)}]d[\rho^{(1)}]\det{}^{\kappa+b/\gamma_2}\rho_1^{(1)}\det{}^{a/\gamma_1-\kappa}\rho_2^{(2)} \\
\times & \frac{\delta\left(r_2^{(1)}\right)}{\left|\Delta_d\left(r_2^{(1)}\right)\right|^{4/\beta}}\frac{\delta\left(r_1^{(2)}\right)}{\left|\Delta_c\left(r_1^{(2)}\right)\right|^{\beta}}\left(D_{dr_2^{(1)}}^{(4/\beta)}\right)^{\tilde{a}-c}\left(D_{cr_1^{(2)}}^{(\beta)}\right)^{\tilde{b}-d} \\
\times & F(\hat{\rho}^{(1)}+e^{\imath\psi}\hat{\rho}^{(2)})\exp\left[-\varepsilon\mathrm{Str}\,(\hat{\rho}^{(1)}+e^{\imath\psi}\hat{\rho}^{(2)})\right], \qquad (10.23)
\end{aligned}$$

where the constants are

$$C_{\mathrm{SF}} = (-1)^{c(b-d)}e^{\imath\psi(\tilde{a}d-\tilde{b}c)}\left(\frac{2}{\gamma_1}\right)^{\gamma_2 ec}\left(\frac{2}{\gamma_2}\right)^{\gamma_1 ed}C_{\tilde{a}cd}^{(\beta)}C_{\tilde{b}dc}^{(4/\beta)}, \qquad (10.24)$$

$$C_{\mathrm{HS}} = (-1)^{d(a-c)}e^{\imath\psi(\tilde{a}d-\tilde{b}c)}\left(-\frac{2}{\gamma_1}\right)^{\gamma_2 ec}\left(-\frac{2}{\gamma_2}\right)^{\gamma_1 ed}\widetilde{C}_{\tilde{a}cd}^{(\beta)}\widetilde{C}_{\tilde{b}dc}^{(4/\beta)}. \qquad (10.25)$$

Here, we define the supermatrix

$$\begin{aligned}
&\hat{\rho}^{(1)}+e^{\imath\psi}\hat{\rho}^{(2)} \qquad (10.26)\\
= & \left[\begin{array}{c|c} \rho_1^{(1)}+e^{\imath\psi}\left(\rho_1^{(2)}-\rho_{\tilde{\eta}}^{(2)}\rho_2^{(2)-1}\rho_{\tilde{\eta}}^{(2)\dagger}\right) & \rho_\eta^{(1)}+e^{\imath\psi}\rho_{\tilde{\eta}}^{(2)} \\ \hline -\rho_\eta^{(1)\dagger}-e^{\imath\psi}\rho_{\tilde{\eta}}^{(2)\dagger} & \rho_2^{(1)}-\rho_\eta^{(1)\dagger}\rho_1^{(1)-1}\rho_\eta^{(1)}+e^{\imath\psi}\rho_2^{(2)} \end{array}\right]
\end{aligned}$$

The measures $d[\rho^{(1)}]=d[\rho_1^{(1)}]d[\rho_2^{(1)}]d[\eta]$ and $d[\rho^{(2)}]=d[\rho_1^{(2)}]d[\rho_2^{(2)}]d[\tilde{\eta}]$ are the same as in theorems 8.3.1 and 10.1.1. The measure for $d[\rho^{(2)}]$ is similar to the measure $d[\rho^{(1)}]$ by the replacement $\beta\to 4/\beta$.

Since this theorem is a consequence of corollary 3.2.5 and theorems 8.3.1 and 10.1.1, the proof is quite simple.

Proof:
Let $e \in \mathbb{N}_0$ as in Eq. (10.21). Then, we use corollary 3.2.5 to extend the integral over V to an integral over $\widetilde{V}$. Hence, without loss of generality, we assume that the inequalities (10.21) are fulfilled for $e = 0$. We split the supersymmetric Wishart matrix B as in Eq. (10.16). Both Wishart matrices B_1 and B_2 fulfill the requirement (8.35) according to their dimensions. Thus, we singly apply both theorems 8.3.1 and 10.1.1 to B_1 and B_2. $\square$

Our approach of a violation of inequality (8.35) is quite different from the solution given in Ref. [86]. These authors introduce a matrix which projects the boson–boson block and the bosonic side of the off-diagonal blocks onto a space of the smaller dimension a. Then, they integrate over all of such orthogonal projectors. This integral becomes more difficult due to an additional measure on a curved, compact space. We use a second symmetric supermatrix. Hence, we have up to the dimensions of the supermatrices a symmetry between both supermatrices produced by $\widehat{\mathfrak{S}}$. There is no additional complication for the integration, since the measures of both supermatrices are of the same kind. Moreover, our approach extends the results to the case of $\beta = 4$ and odd b which is not considered in Ref. [86].

In Sec. 11.3, we will find a simpler solution for the case that inequality (8.35) is violated than the one in theorem 10.3.1. Instead of an integration over two supermatrices, we can modify the supersymmetry method that we have to integrate over one supermatrix, only.

Chapter 11

Explicit formulas for the superfunctions Φ and $\mathcal{F}\Phi = Q$

In Chap. 7 we have shown how one can obtain the supersymmetric extension Φ of the characteristic function $\mathcal{F}P$. However, this construction is based on the duality relation (7.20). This gives us only a procedure to find Φ but no explicit formula. This means that we have first to calculate the characteristic function $\mathcal{F}P$ and then to replace the traces by the supertraces after choosing a representation of $\mathcal{F}P$. This can be quite cumbersome for particular probability densities P, not to mention for arbitrary one.

Here, we give an alternative way to map from ordinary space to superspace. This map avoids the weakness of the common procedure namely to calculate the characteristic function. The crucial ingredients of our approach are the integration theorems 3.3.1, 3.3.2 and 3.3.3.

In Sec. 11.1, we consider the mapping from ordinary space to ordinary space. This is the case when the generating function (8.31) for $b = 0$ has only characteristic polynomials in the denominator. In Sec. 11.2, we discuss the case $b = 0$ and $c = d = 1$ for rotation invariant probability densities on Hermitian matrices. It is the simplest case to investigate our approach for characteristic polynomials in the numerator. We extend this mapping to arbitrary dimensions a, b, c and d in Sec. 11.3.

11.1 The case $c \leq a$, $b = d = 0$ and arbitrary Dyson index

We consider the integral

$$Z_{c0}^{a0}(x^-) = \int\limits_{\text{Herm}\,(\beta,a)} P(H) \prod_{j=1}^{c} \frac{1}{\det^{1/\gamma_1}\left(H - x_j^- \mathbb{1}_a\right)} d[H] \tag{11.1}$$

with the restriction $c \leq a$. Applying Eq. (2.34) backwards, we find

$$Z_{c0}^{a0}(x^-) = \frac{1}{\mathbf{K}_{a0}^{c0}} \int\limits_{\text{Herm}\,(\beta,a)} \int\limits_{\text{Mat}_\beta^0(c\times a/0)} P(H) \exp\left[\imath \text{tr}\, V^\dagger V H - \imath \text{tr}\, x^- V V^\dagger\right] d[V] d[H]\,. \tag{11.2}$$

We emphasize that V is here a purely ordinary real, complex or quaternionic rectangular matrix, according to the Dyson index β. No Grassmann variables appear in this integral.

With help of the characteristic function $\mathcal{F}P$, see Eq. (7.6), we have

$$Z_{c0}^{a0}(x^-) = \frac{1}{\mathbf{K}_{a0}^{c0}} \int\limits_{\text{Mat}_\beta^0(c\times a/0)} \mathcal{F}P(V^\dagger V) \exp\left[-\imath \text{tr}\, x^- V V^\dagger\right] d[V]\,. \tag{11.3}$$

The Wishart matrix $V^\dagger V$ is an element of Herm (β, a) whereas $VV^\dagger$ is in Herm (β, c). Since $c \leq a$ the set Herm (β, c) can be canonically embedded in the larger set Herm (β, a), i.e.

$$\begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \in \text{Herm}\,(\beta, a) \tag{11.4}$$

for all $K \in$ Herm (β, c). Defining the new Wishart matrix

$$\widetilde{V} = \begin{bmatrix} 0 \\ V \end{bmatrix} \in \text{Herm}\,(\beta, a)\,, \tag{11.5}$$

we use the duality relation (7.20) for ordinary matrices. Equation (11.3) becomes

$$Z_{c0}^{a0}(x^-) = \frac{1}{\mathbf{K}_{a0}^{c0}} \int\limits_{\text{Mat}_\beta^0(c\times a/0)} \mathcal{F}P(\widetilde{V}\widetilde{V}^\dagger) \exp\left[-\imath \text{tr}\, x^- \widetilde{V}\widetilde{V}^\dagger\right] d[V]\,. \tag{11.6}$$

We remark that no ambiguity of a representation of $\mathcal{F}P$, see Sec. 7.5, appears because all matrices are fully ordinary and contain no Grassmann variables.

The Fourier back transform gives a result similar to Eq. (11.2). To integrate over the Gaussian integrals, we split the matrix H into four blocks

$$H = \begin{bmatrix} H_0 & W^\dagger \\ W & \sigma \end{bmatrix}, \tag{11.7}$$

where $H_0 \in \mathrm{Herm}\,(\beta, a-c)$, $\sigma \in \mathrm{Herm}\,(\beta, c)$ and $W \in \mathrm{Mat}\,_\beta^0(c \times (a-c)/0)$. This yields

$$\begin{aligned} Z_{c0}^{a0}(x^-) &= \int\limits_{\mathrm{Herm}\,(\beta,a)} P\left(\begin{bmatrix} H_0 & W^\dagger \\ W & \sigma \end{bmatrix}\right) \det{}^{-a/\gamma_1}\left(\sigma - x^-\right) d[H_0]d[W]d[\sigma] \\ &\overset{!}{=} \int\limits_{\mathrm{Herm}\,(\beta,c)} \mathcal{F}\Phi_0(\sigma)\det{}^{-a/\gamma_1}\left(\sigma - x^-\right) d[\sigma]\,. \end{aligned} \tag{11.8}$$

The second equality is reminiscent of Eq. (8.7) where $\mathcal{F}\Phi_0(\sigma)$ is the Fourier transform of Φ_0. We recall that Φ_0 restricts the characteristic function $\mathcal{F}P$ on the set $\mathrm{Herm}\,(\beta, a)$ to the set $\mathrm{Herm}\,(\beta, c)$ by the duality relation (7.20) for V.

The comparison of the first line with the second one in Eq. (11.8) yields the explicit expression of $\mathcal{F}\Phi_0(\sigma)$,

$$\mathcal{F}\Phi_0(\sigma) = \int\limits_{\mathrm{Herm}\,(\beta,a-c)} \int\limits_{\mathrm{Mat}\,_\beta^0(c\times(a-c)/0)} P\left(\begin{bmatrix} H_0 & W^\dagger \\ W & \sigma \end{bmatrix}\right) d[W]d[H_0]\,. \tag{11.9}$$

We will see that this simple formula can be extended to characteristic polynomials in the numerator with help of supersymmetry.

11.2 The case $c = d = 1$, $b = 0$ and $\beta = 2$

To show the fundamental steps of the derivation and to keep the calculation as simple as possible, we consider the generating function

$$Z_{11}^{a0}(x^-) = \int\limits_{\mathrm{Herm}\,(2,a)} P(H)\frac{\det\left(H - x_2^-\mathbb{1}_a\right)}{\det\left(H - x_1^-\mathbb{1}_a\right)}d[H]\,. \tag{11.10}$$

Instead of rewriting the determinants as Gaussian integrals as in the common procedure, we first choose a representation P_0 of the probability distribution P which depends on the traces of H. We apply theorem 3.3.1 backwards for the parameters $p-a=q=1$ and the superfunction $f=P_0$. Choosing a Wick–rotation, the generating function becomes

$$Z_{11}^{a0}(x^-)=\imath\left(-2e^{\imath\psi}\right)^a\int\limits_{\Sigma_{2,a+1/1}^{-\psi}}P\left(\begin{bmatrix}H & W^\dagger\\ W & \sigma\end{bmatrix}\right)\frac{\det\left(H-x_2^-\mathbb{1}_a\right)}{\det\left(H-x_1^-\mathbb{1}_a\right)}d[H]d[W]d[\sigma]\,. \tag{11.11}$$

With this expression, we are able to do the same trick as in the previous section. We express the determinant as Gaussian integrals and perform a Fourier transformation. The difference of this Fourier transformation with the one in Sec. 11.1 is that we Fourier transform in the larger space $\Sigma_{2,a+1/1}^{\psi}$. We obtain

$$Z_{11}^{a0}(x^-)=\frac{\imath\left(-2e^{\imath\psi}\right)^a}{\mathbf{K}_{a0}^{11}}\int\limits_{\mathrm{Mat}\,_\beta^0(1\times a/1)}\mathcal{F}P_0(\widetilde{V}^\dagger\widetilde{V})\exp\left[-\imath\mathrm{tr}\,x^-\widetilde{V}\widetilde{V}^\dagger\right]d[V]\,, \tag{11.12}$$

where

$$\widetilde{V}=\begin{bmatrix}\overbrace{0}^{a} & \overbrace{0}^{1+1} \\ V & 0\end{bmatrix}\begin{matrix}\}a\\ \}1+1\end{matrix}\in\mathrm{Mat}\,_2^0(a+1/1) \tag{11.13}$$

with $V\in\mathrm{Mat}\,_\beta^0(1\times a/1)$.

Again we use the duality relation (7.20) for $\widetilde{V}$ and perform the Fourier transformation backwards. After integrating over the rectangular supermatrix we get

$$\begin{aligned}Z_{11}^{a0}(x^-) &= \imath\left(-2e^{\imath\psi}\right)^a\int\limits_{\Sigma_{2,a+1/1}^{-\psi}}P\left(\begin{bmatrix}H & W^\dagger\\ W & \sigma\end{bmatrix}\right)\mathrm{Sdet}\,^{-a/\gamma_1}\left(\sigma-x^-\right)d[H]d[W]d[\sigma]\\ &\overset{!}{=}\int\limits_{\Sigma_{2,1/1}^{-\psi}}\mathcal{F}\Phi_0(\sigma)\mathrm{Sdet}\,^{-a/\gamma_1}\left(\sigma-x^-\right)d[\sigma]\,.\end{aligned} \tag{11.14}$$

We identify both expressions in Eq. (11.14) and find

$$\mathcal{F}\Phi_0(\sigma)=\imath\left(-2e^{\imath\psi}\right)^a\int\limits_{\mathrm{Herm}\,(2,a)}\int\limits_{\mathrm{Mat}\,_2^{-\psi}(1\times a/1)}P\left(\begin{bmatrix}H & W^\dagger\\ W & \sigma\end{bmatrix}\right)d[W]d[H]\,. \tag{11.15}$$

Indeed, this is similar to Eq. (11.9). Moreover a new property of the probability distributions depending on $\mathrm{tr}\,H$ and $\mathrm{tr}\,H^2$ arises. Due to the integration theorems for vectors [91, 93], see also theorems 3.2.1, 3.2.2 and 3.2.3, we find

$$\mathcal{F}\Phi_0(\sigma) = \imath \int\limits_{\mathrm{Herm}\,(2,a)} P\left(\begin{bmatrix} H & 0 \\ 0 & \sigma \end{bmatrix}\right) d[H]\,. \tag{11.16}$$

This result agrees with the one in Ref. [125] for norm-dependent ensembles.

11.3 Extension to arbitrary dimensions

We choose the Wick–rotation in the interval $]\pi, 2\pi[$. For the general case of our approach we consider Eq. (8.31). Let $\tilde{a} = a + \gamma_1 e, \tilde{b} = b + \gamma_2 e \in \mathbb{N}$ be chosen such that the inequalities (10.21) are fulfilled. Then we enlarge the integral (8.31) to

$$\begin{aligned} Z_{cd}^{ab}(x^-) = C^{(\beta)} \times & \qquad\qquad (11.17) \\ \times \int\limits_{\Sigma_{\beta,\tilde{a}/\tilde{b}}^{-\psi}} P\left(\begin{bmatrix} \sigma_0 & W^\dagger \\ W & \sigma \end{bmatrix}\right) \mathrm{Sdet}^{\,-1/\tilde{\gamma}} & \left(\sigma_0 \otimes \widehat{\Pi}_{2\psi} - \mathbb{1}_{\gamma_2 a + \gamma_1 b} \otimes x^-\right) d[\sigma_0] d[W] d[\sigma]\,, \end{aligned}$$

where the constant is

$$C^{(\beta)} = \left(-e^{\imath\psi}\right)^{\gamma_2 e(a-b)} 2^{e(1-e)} \tilde{\gamma}^{e(a+b)} \begin{cases} \left(-2\imath e^{\imath\psi}\right)^e & , \beta = 1 \\ \imath^{e^2} & , \beta = 3 \\ \left(2\imath e^{-\imath\psi}\right)^e & , \beta = 4 \end{cases}. \tag{11.18}$$

As in the previous sections, we replace the superdeterminants by Gaussian integrals and Fourier transform P in the set $\Sigma_{\beta,\tilde{a}/\tilde{b}}^{\psi}$. After applying the duality relation (7.20) for the enlarged rectangular supermatrix

$$\widetilde{V} = \begin{bmatrix} \overbrace{\phantom{V_{11}}}^{a} & \overbrace{}^{e} & \overbrace{\phantom{V_{12}}}^{b} & \overbrace{}^{e} & \\ 0 & 0 & 0 & 0 & \}\tilde{a} - \gamma_2 c \\ V_{11} & 0 & V_{12} & 0 & \}\gamma_2 c \\ 0 & 0 & 0 & 0 & \}\tilde{b} - \gamma_1 d \\ V_{21} & 0 & V_{22} & 0 & \}\gamma_1 d \end{bmatrix} \in \mathrm{Mat}_{\beta}^{\psi}(\tilde{a}/\tilde{b}) \tag{11.19}$$

with

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \in \mathrm{Mat}_{\beta}^{\psi}(c \times a/d \times b)\,, \tag{11.20}$$

and $\mathrm{Mat}\,_\beta^\psi(c \times a/d \times b) = \widehat{\Pi}_\psi \mathrm{Mat}\,_\beta^0(c \times a/d \times b)\widehat{\Pi}_\psi$ we perform the inverse Fourier transformation and integrate over V. We find

$$\begin{aligned} Z_{cd}^{ab}(x^-) &= C^{(\beta)} \int\limits_{\Sigma_{\beta,\tilde{a}/\tilde{b}}^{-\psi}} P\left(\begin{bmatrix}\sigma_0 & W^\dagger \\ W & \sigma\end{bmatrix}\right) \mathrm{Sdet}\,^{(b-a)/\gamma_1}\left(\sigma\widehat{\Pi}_{2\psi} - x^-\right) d[\sigma_0]d[W]d[\sigma] \\ &\overset{!}{=} \int\limits_{\Sigma_{\beta,c/d}^{-\psi}} \mathcal{F}\Phi_0(\sigma)\mathrm{Sdet}\,^{(b-a)/\gamma_1}\left(\sigma\widehat{\Pi}_{2\psi} - x^-\right) d[\sigma]\,. \end{aligned} \tag{11.21}$$

We notice that the additional Wick–rotation $\widehat{\Pi}_{2\psi}$ is important to regularize the integral when $a > b$ which is not excluded.

Apart from the constant, Eq. (11.20) only depends on the difference of the dimensions b and a. This reflects the Cauchy–like theorems for supermatrices in Eq. (8.31), see Refs. [91, 93]. Also it shows another and simpler solution for the breaking of the inequality (8.35) as in Ref. [86] or in Sec. 10.3.

We compare both lines in Eq. (11.21) and have

$$\mathcal{F}\Phi_0(\sigma) = C^{(\beta)} \int\limits_{\Sigma_{\beta,\tilde{a}-c/\tilde{b}-d}^{-\psi}} \int\limits_{\mathrm{Mat}\,_\beta^{-\psi}(c\times(\tilde{a}-c)/d\times(\tilde{b}-d))} P\left(\begin{bmatrix}\sigma_0 & W^\dagger \\ W & \sigma\end{bmatrix}\right) d[\sigma_0]d[W]\,. \tag{11.22}$$

The probability density P is indeed directly connected with the supersymmetric version of the probability density $\mathcal{F}\Phi_0$. The superfunction Φ_0 is the Fourier transform of Eq. (11.22). Thus, we have found explicit integral equations for both superfunctions. These integrals might be easier to perform than the calculation of the characteristic function $\mathcal{F}P$ for large dimension $a = N$.

Chapter 12

Summary of part II

We extended the method of the generalized Hubbard–Stratonovich transformation in two steps. First, we showed that this approach also applies to arbitrary orthogonally and unitary–symplectically invariant random matrix ensembles. Due to a duality between ordinary and supersymmetric matrix spaces, the integrals for the k–point correlation functions are over superspaces. The integrals were reduced to eigenvalue integrals for all probability densities, including those which do not factorize. The results are in terms of the characteristic function. Thus, the characteristic function has to be calculated for the ensemble in question. Since the matrix Bessel functions of the ordinary orthogonal and unitary–symplectic group [88, 89, 90] and, hence, the supermatrix Bessel functions of $\mathrm{UOSp}\,(2k/2k)$ are not known explicitly beyond $k = 1$, we cannot further simplify our results. Up to the restriction $N \geq k_1$, formula (9.26) is exact for every k, N and rotation invariant ensemble. Thus, it can serve not only as starting point for universality considerations [10], but for all other studies.

In the second step, the generalized Hubbard–Stratonovich transformation was extended to arbitrary dimensional supersymmetric Wishart matrices which are not only related to averages over the matrix Green functions [9, 60, 61]. This gave us the opportunity to prove the equivalence of our approach and the superbosonization formula [83, 84] on a general level. Thereby, we generalized also the superbosonization formula. The superbosonization formula was proven in a new way and is now extended to odd dimensional supersymmetric Wishart matrices in the fermion–fermion block for the quaternionic case.

The generalized Hubbard–Stratonovich transformation and the superbosonization formula reduce in the absence of Grassmann variables to the ordinary integral identity for ordinary Wishart matrices [121, 84]. In the general case with the restriction $N \geq k_1$, both approaches differ in the fermion–fermion block integration. Due to the Dirac distribution and the differential operator, the integration over the non-compact domain in the generalized Hubbard–Stratonovich transformation is equal to a contour integral. This identity is reminiscent of the residue theorem. This contour integral is equivalent to the integration over the compact domain in the superbosonization formula. Hence, we found an integral identity between a compact integral and a differentiated Dirac distribution.

Furthermore, we derived an explicit functional dependence of the superfunctions Φ and $Q = \mathcal{F}\Phi$ which is indeed a new result. With formula (11.22) one is able to study the influence of deformation in the probability density P.

Also, we got two solutions for the problem when the restriction $N \geq k_1$ is violated, see theorem 10.3.1 and Sec. 11.3. Both approaches differ from the method presented in Ref. [86]. In Ref. [86] the authors introduce a projection matrix over which one has to integrate. However, this integration can be quite cumbersome because the integration domain is curved. We introduce in the first approach, see Sec. 10.3, an additional supermatrix which has the same symmetries as the original one. Thus, we integrate over a flat supermanifold. Nevertheless, we think the second solution is more appropriate since we have not introduced an additional matrix at all. The explicit dependence of the superfunction $Q = \mathcal{F}\Phi$ allows us to encode the difficulty of the case $N < k_1$ as an integral over a subblock of the original supermatrix, cf. Eq. (11.22). Moreover both approaches presented here are applicable on the artificial example $\beta = 4$ and odd b, as well, which has not been considered in Ref. [86].

Part III

An algebraical method to derive determinantal and Pfaffian structures

Chapter 13

Sketch of the idea: "Supersymmetry without supersymmetry"

In the orthogonal polynomial method as well as in the supersymmetry method every single ensemble has to be calculated in a particular way. Either one has to find the measure to construct the orthogonal polynomials or one has to identify the superspace corresponding to the ordinary integration domain. We consider three types of integrals related to mean values of ratios of characteristic polynomials. Determinantal structures stemming from supersymmetry, such as those found by Basor and Forrester [95], yield determinantal and Pfaffian structures of these integrals. Here, we establish the link to supersymmetry. To the best of our knowledge this connection has not been observed before. Our method is based on an algebraic manipulation of the characteristic polynomials and the Jacobian or the Berezinian resulting from changing integration variables. We neither use the Mehta–Mahoux theorem nor a mapping onto superspace.

In this chapter, we give an outline of our approach. In Sec. 13.1, we show with help of our method how the determinantal structure arises for a Hermitian matrix ensemble, while we derive the Pfaffian structure for an ensemble of Hermitian self-dual matrices in Sec. 13.2.

13.1 Determinantal structures

As a guideline for the reader, we present here the main ideas of our approach for one particular ensemble. We choose $\kappa = \operatorname{diag}(\kappa_{11}, \ldots, \kappa_{k1}, \kappa_{12}, \ldots, \kappa_{k2}) = \operatorname{diag}(\kappa_1, \kappa_2)$ in such a way that the integrals below are well defined. For many applications such as for Hermitian matrix ensembles one considers averages over ratios of characteristic polynomials

$$Z(\kappa) = \int P(H) \prod_{j=1}^{k} \frac{\det(H - \kappa_{j2} \mathbb{1}_N)}{\det(H - \kappa_{j1} \mathbb{1}_N)} d[H]\,, \tag{13.1}$$

cf. Eq. (6.8). The probability density P is rotation invariant and factorizes in the eigenvalues of the matrix H. We diagonalize H in its eigenvalues $E_1, \ldots, E_N$. The Jacobian is the second power of the Vandermonde determinant $\Delta_N(E)$. We expand one of the Vandermonde determinants and have up to a constant c

$$Z(\kappa) = c \int \prod_{a=1}^{N} \left[P(E_a) E_a^{a-1} \prod_{b=1}^{k} \frac{E_a - \kappa_{b2}}{E_a - \kappa_{b1}} \right] \Delta_N(E) d[E]\,. \tag{13.2}$$

We pursue an idea similar to the one by Basor and Forrester [95]. We supplement the factor

$$\Delta_N(E) \prod_{a=1}^{N} \prod_{b=1}^{k} \frac{E_a - \kappa_{b2}}{E_a - \kappa_{b1}} \tag{13.3}$$

by

$$\sqrt{\mathrm{Ber}\,^{(2)}_{k/k}(\kappa)} = \frac{\Delta_k(\kappa_1)\Delta_k(\kappa_2)}{\prod\limits_{a,b=1}^{k} (\kappa_{a1} - \kappa_{b2})}\,. \tag{13.4}$$

Both factors together are up to a sign

$$\sqrt{\mathrm{Ber}\,^{(2)}_{k/k+N}(\kappa_1; \kappa_2, E)} = \pm \frac{\Delta_k(\kappa_1)\Delta_{k+N}(\kappa_2, E)}{\prod\limits_{a,b=1}^{k} (\kappa_{a1} - \kappa_{b2}) \prod\limits_{a=1}^{k} \prod\limits_{b=1}^{N} (\kappa_{a1} - E_b)}\,. \tag{13.5}$$

The authors in Ref. [95] have shown that this function has for all $N \in \mathbb{N}_0$ a determinantal structure mixing terms of the Vandermonde determinant and the Cauchy determinant,

$$\sqrt{\mathrm{Ber}\,^{(2)}_{k/k+N}(\kappa_1; \kappa_2, E)} = \pm \det \left[\begin{array}{c|c} \dfrac{1}{\kappa_{a1} - \kappa_{b2}} & \dfrac{1}{\kappa_{a1} - E_b} \\ \hline \kappa_{b2}^{a-1} & E_b^{a-1} \end{array} \right]\,. \tag{13.6}$$

The insight crucial for our approach and not contained in Ref. [95] is the intimate connection of Eq. (13.6) to superspace. We will prove Eq. (13.6) in a new way and obtain also an interesting intermediate result not given in Ref. [95], see Chap. 14.

We now proceed with the evaluation of $Z(\kappa)$. We pull the eigenvalue integrals into the determinant and obtain

$$Z(\kappa) = \frac{c}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \det \left[\begin{array}{c|c} \dfrac{1}{\kappa_{a1} - \kappa_{b2}} & F_b(\kappa_{a1}) \\ \hline \kappa_{b2}^{a-1} & M_{ab} \end{array} \right] . \tag{13.7}$$

The symmetric matrix M_{ab} comprises the moments of the probability density P and the functions F_b are the Cauchy transform of those moments. At this point we have a choice for how to proceed further. For instance, we can reorder the monomials in the entries of the determinant to orthogonal polynomials with respect to P. Then, M_{ab} becomes diagonal and F_b are the Cauchy transforms of the orthogonal polynomials. Thus we arrive at the well known result, see Ref. [126]. On the other hand, we can choose an arbitrary set of linearly independent polynomials. Then, we use the important property of the determinant

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det[A - BD^{-1}C] \tag{13.8}$$

for arbitrary matrices A, B and C and an invertible quadratic matrix D. This finally yields

$$\begin{aligned} Z(\kappa) &= \frac{c}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \det \left[\frac{1}{\kappa_{a1} - \kappa_{b2}} - \sum_{m,n=1}^{N} F_m(\kappa_{a1}) M_{nm}^{-1} \kappa_{b2}^{n-1} \right] \\ &= \frac{c}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \det K(\kappa_{a1}, \kappa_{b2}) . \end{aligned} \tag{13.9}$$

We obtain the correct result [127] without the Dyson–Mehta–Mahoux integration theorem [33, 22, 128] for an arbitrary choice of polynomials. The orthogonal polynomials are not the tool to identify the determinantal structures. They are a result of the calculation.

In Sec. 16.1.2, we extend this sketch to a careful discussion for a large class of integrals. We will see that determinantal structures derived in many different fields of random matrix theory have a common origin.

13.2 Pfaffian structures

We consider averages of ratios of characteristic polynomials over the Hermitian self-dual matrices

$$Z(\kappa) = \int P(H) \prod_{j=1}^{k} \frac{\det(H - \kappa_{j2} \mathbb{1}_{2N})}{\det(H - \kappa_{j1} \mathbb{1}_{2N})} d[H] \,. \tag{13.10}$$

The probability density P is rotation invariant and factorizes in the eigenvalues of H, $E = \operatorname{diag}(E_1, \ldots, E_N) \otimes \mathbb{1}_2$. We choose $\kappa = \operatorname{diag}(\kappa_{11}, \ldots, \kappa_{k1}, \kappa_{12}, \ldots, \kappa_{k2}) = \operatorname{diag}(\kappa_1, \kappa_2)$ in such a way that the integrals are well defined. Changing to eigenvalue–angle coordinates yields

$$Z(\kappa) = c \int \prod_{a=1}^{N} \prod_{b=1}^{k} P(E_a) \frac{(E_a - \kappa_{b2})^2}{(E_a - \kappa_{b1})^2} \Delta_N^4(E) d[E] \tag{13.11}$$

with a normalization constant c. Introducing Dirac distributions, we extend the N eigenvalue integrals to $2N$ eigenvalue integrals and have

$$Z(\kappa) = c \int \prod_{j=1}^{N} g(E_j, E_{j+N}) \prod_{a=1}^{2N} \prod_{b=1}^{k} \frac{(E_a - \kappa_{b2})}{(E_a - \kappa_{b1})} \Delta_{2N}(E) d[E] \,, \tag{13.12}$$

where

$$g(E_j, E_{j+N}) = P(E_j) \frac{\delta(E_j - E_{j+N})}{E_j - E_{j+N}} \,. \tag{13.13}$$

In the next step we use the same method developed in the previous section. We extend the product of the characteristic polynomials times the Vandermonde determinant by $\sqrt{\operatorname{Ber}_{k/k}^{(2)}(\kappa)}$ which is a Cauchy determinant, see Eq. (9.4). This yields

$$\begin{aligned} \sqrt{\operatorname{Ber}_{k/k+2N}^{(2)}(\kappa_1; \kappa_2, E)} &= \pm \frac{\Delta_k(\kappa_1) \Delta_{k+2N}(\kappa_2, E)}{\prod\limits_{a,b=1}^{k} (\kappa_{a1} - \kappa_{b2}) \prod\limits_{a=1}^{k} \prod\limits_{b=1}^{N} (\kappa_{a1} - E_b)} \\ &= \pm \det \left[\begin{array}{c|c} \dfrac{1}{\kappa_{a1} - \kappa_{b2}} & \dfrac{1}{\kappa_{a1} - E_b} \\ \hline \kappa_{b2}^{a-1} & E_b^{a-1} \end{array} \right] . \end{aligned} \tag{13.14}$$

As in the previous section, the Berezinian $\operatorname{Ber}_{k/k+2N}^{(2)}$ plays a crucial role in this approach.

Integrating over all energies E_j with $j > N$ in Eq. (13.12), we obtain

$$\begin{aligned} Z(\kappa) &= \frac{c}{\sqrt{\mathrm{Ber}\,^{(2)}_{k/k}(\kappa)}} \qquad (13.15) \\ &\times \int \det \left[\begin{array}{c|c|c} \dfrac{1}{\kappa_{a1}-\kappa_{b2}} & \dfrac{1}{\kappa_{a1}-E_b} & \displaystyle\int \frac{g(E_b,E)}{\kappa_{a1}-E} dE \\ \hline \kappa_{b2}^{a-1} & E_b^{a-1} & \int g(E_b,E)E^{a-1}dE \end{array} \right] d[E]\,. \end{aligned}$$

With help of a modified version of de Bruijn's integral theorem [129], see App. C.3.2, we intergrate over the remaining variables and find the Pfaffian expression

$$Z(\kappa) = \frac{c}{\sqrt{\mathrm{Ber}\,^{(2)}_{k/k}(\kappa)}} \mathrm{Pf} \left[\begin{array}{c|c|c} 0 & \dfrac{1}{\kappa_{b1}-\kappa_{a2}} & \kappa_{a2}^{b-1} \\ \hline \dfrac{1}{\kappa_{b2}-\kappa_{a1}} & \mathbf{F}(\kappa_{a1},\kappa_{b1}) & G_b(\kappa_{a1}) \\ \hline -\kappa_{b2}^{a-1} & -G_a(\kappa_{b1}) & M_{ab} \end{array} \right] . \qquad (13.16)$$

We give a detailed definition of the functions $\mathbf{F}$, G_a and M_{ab} in Sec. 15.3. Here, we schematically explain what these functions are. The function $\mathbf{F}$ is almost the average over two-dimensional Hermitian self-dual matrices of two characteristic polynomials in the denominator. The functions G_a are Cauchy transforms of P's moments and M_{ab} is the anti-symmetric moment matrix of P generating the skew orthogonal polynomials of quaternion type [75].

Since the Pfaffian determinant is skew symmetric in the pairs of rows and columns, we can construct any linear independent set of polynomials in the last columns and rows in Eq. (13.16). For example, the skew orthogonal polynomials yield a block diagonal moment matrix M_{ab} which leads immediately to the well known result expressed in terms of skew orthogonal polynomials. Here, we leave the monomials as they are and use

$$\mathrm{Pf} \left[\begin{array}{cc} A & B \\ -B^T & C \end{array} \right] = \mathrm{Pf}\, C\, \mathrm{Pf}\, [A + BC^{-1}B^T] \qquad (13.17)$$

for arbitrary matrices A, B and an invertible, even-dimensional matrix C. The matrices A and C are anti-symmetric and they are even dimensional. Equation (13.17) is the analog to Eq. (13.8) for the determinant. As M_{ab} is even-dimensional, we finally arrive at the result

$$Z(\kappa) = \frac{c}{\sqrt{\mathrm{Ber}\,^{(2)}_{(k/k)}(\kappa)}} \mathrm{Pf} \left[\begin{array}{c|c} K_{11}(\kappa_{b2},\kappa_{a2}) & K_{12}(\kappa_{a2},\kappa_{b1}) \\ \hline -K_{12}(\kappa_{a1},\kappa_{b2}) & K_{22}(\kappa_{a1},\kappa_{b1}) \end{array} \right] , \qquad (13.18)$$

where the kernels are

$$K_{11}(\kappa_{b2},\kappa_{a2}) = \sum_{m,n=1}^{2N} \kappa_{a2}^{m-1} M_{mn}^{-1} \kappa_{b2}^{n-1}\,, \tag{13.19}$$

$$K_{12}(\kappa_{a2},\kappa_{b1}) = \frac{1}{\kappa_{b1}-\kappa_{a2}} + \sum_{m,n=1}^{2N} \kappa_{a2}^{m-1} M_{mn}^{-1} G_n(\kappa_{b1})\,, \tag{13.20}$$

$$K_{22}(\kappa_{a1},\kappa_{b1}) = \mathbf{F}(\kappa_{a1},\kappa_{b1}) + \sum_{m,n=1}^{2N} G_m(\kappa_{a1}) M_{mn}^{-1} G_n(\kappa_{b1})\,. \tag{13.21}$$

This is, indeed, the correct result which we found without making use of the Dyson–Mehta–Mahoux integration theorem [33, 22, 128]. Although we can employ an arbitrary choice of polynomial basis, we obtain the skew orthogonal polynomials generated by M_{ab}. Thus, the skew orthogonal polynomials are the result.

We show in Sec. 15.3 how Pfaffian structures for a wide class of matrix ensembles can be obtained in a unifying way. Our method is not only applicable for unitary-symplectic symmetry but also for ensembles of orthogonal rotation invariance. We will notice that there is no difference between both symmetries in the derivation. Hence, the Pfaffian structure of averages similar to Eq. (13.12) is elementary.

Chapter 14

Determinantal structure of Berezinians

In Sec. 14.1, we investigate the determinantal structure of the Berezinians resulting from Hermitian supermatrices. These Berezinians are crucial for the calculations in the ensuing sections. For the sake of completeness, we give the determinantal structure according to the supergroup $\mathrm{UOSp}\,(p/q)$ in Sec. 14.2.

14.1 Berezinians related to the supergroup $\mathrm{U}^{(2)}(p/q)$

As we have seen in Chap. 13, Berezinians resulting from diagonalization of supermatrices play a role of paramount importance for our method. Although we do not use any integral in superspace we find those Berezinians in the ratios of characteristic polynomials times the Vandermonde determinant. The crucial step is here to understand that those Berezinians have always a determinantal structure.

For the Vandermonde determinant the determinantal structure has been known for a long time [114], see also Eq. (3.28). Moreover, the square root of the Berezinian resulting from a diagonalization of the supersymmetric analog of a $\mathrm{U}^{(2)}(k/k)$–symmetric matrix is known [94], up to a sign, to be equal to the Cauchy determinant, see Eq. (9.4), due to Cauchy's lemma [130].

The next step is to generalize these structures to an arbitrary number of bosonic eigenvalues $\kappa_1 = \mathrm{diag}\,(\kappa_{11}, \ldots, \kappa_{p1})$ and of fermionic eigenvalues $\kappa_2 = \mathrm{diag}\,(\kappa_{12}, \ldots, \kappa_{q2})$. In App. C.1.1, we derive the determinantal structures of the Berezinians, see Ref. [105],

$$\sqrt{\mathrm{Ber}_{p/q}^{(2)}(\kappa)} = \frac{\prod\limits_{1\leq a<b\leq p}(\kappa_{a1}-\kappa_{b1})\prod\limits_{1\leq a<b\leq q}(\kappa_{a2}-\kappa_{b2})}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(\kappa_{a1}-\kappa_{b2})}, \tag{14.1}$$

for arbitrary p and q. Under the condition $p \leq q$, we obtain

$$\sqrt{\mathrm{Ber}_{p/q}^{(2)}(\kappa)} = (-1)^{q(q-1)/2+(q+1)p}\det\left[\begin{array}{c}\left\{\dfrac{\kappa_{a1}^{p-q}\kappa_{b2}^{q-p}}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq p\\ 1\leq b\leq q}}\\ \left\{\kappa_{b2}^{a-1}\right\}_{\substack{1\leq a\leq q-p\\ 1\leq b\leq q}}\end{array}\right]. \tag{14.2}$$

Since the left hand side is up to $(-1)^{pq}$ symmetric under exchanging the bosonic eigenvalues with the fermionic ones, the condition $p \leq q$ is not a restriction. This result is similar to Eq. (13.6). In Sec. 16.2 and App. C.6, we show that this intermediate result is useful for some calculations.

The left hand side of Eq. (14.2) is translation invariant $\kappa \to \kappa + \varepsilon \mathbb{1}_{p+q}$ with a constant ε. Thus, we may shift the expressions on the right hand side by ε,

$$\begin{aligned}\sqrt{\mathrm{Ber}_{p/q}^{(2)}(\kappa)} &= (-1)^{q(q-1)/2+(q+1)p}\\ &\times \det\left[\begin{array}{c}\left\{\dfrac{(\kappa_{a1}+\varepsilon)^{p-q}(\kappa_{b2}+\varepsilon)^{q-p}}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq p\\ 1\leq b\leq q}}\\ \left\{(\kappa_{b2}+\varepsilon)^{a-1}\right\}_{\substack{1\leq a\leq q-p\\ 1\leq b\leq q}}\end{array}\right].\end{aligned} \tag{14.3}$$

We expand the entries in the lower $(q-p)\times q$ block in ε. We notice that all rows together are a linearly independent set of polynomials from order zero to order $q-p-1$. As the determinant is skew symmetric, it yields

$$\begin{aligned}\sqrt{\mathrm{Ber}_{p/q}^{(2)}(\kappa)} &= (-1)^{q(q-1)/2+(q+1)p}\\ &\times \det\left[\begin{array}{c}\left\{\dfrac{(\kappa_{a1}+\varepsilon)^{p-q}(\kappa_{b2}+\varepsilon)^{q-p}}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq p\\ 1\leq b\leq q}}\\ \left\{\kappa_{b2}^{a-1}\right\}_{\substack{1\leq a\leq q-p\\ 1\leq b\leq q}}\end{array}\right].\end{aligned} \tag{14.4}$$

Since ε is arbitrary, we take the limit for ε to infinity and obtain the final result

$$\sqrt{\mathrm{Ber}_{p/q}^{(2)}(\kappa)} = (-1)^{q(q-1)/2+(q+1)p} \det \left[\begin{array}{c} \left\{ \dfrac{1}{\kappa_{a1} - \kappa_{b2}} \right\}_{\substack{1 \leq a \leq p \\ 1 \leq b \leq q}} \\ \left\{ \kappa_{b2}^{a-1} \right\}_{\substack{1 \leq a \leq q-p \\ 1 \leq b \leq q}} \end{array} \right] \tag{14.5}$$

which is identical to the one found by Basor and Forrester [95]. Indeed, Eq. (14.5) does not exhibit the nice symmetry between the bosonic and fermionic eigenvalues as in Eqs. (9.4) and (14.1).

14.2 Berezinians related to the supergroup UOSp (p/q)

As for the Vandermonde determinant itself, the determinantal structure for the forth power thereof is also well known [114],

$$\Delta_k^4(\kappa) = \det \left[\kappa_b^{a-1} \;\middle|\; (a-1)\kappa_b^{a-2} \right]_{\substack{1 \leq a \leq 2k \\ 1 \leq b \leq k}} . \tag{14.6}$$

In appendices C.1.2 and C.1.3, we also find a determinantal structure of the Berezinians

$$\mathrm{Ber}_{p/q}^{(1)}(\kappa) = \mathrm{Ber}_{q/p}^{(4)}(\tilde{\kappa}) \tag{14.7}$$

with $\tilde{\kappa} = \mathrm{diag}\,(\kappa_2, \kappa_1)$, see Eq. (3.45). Here, we have to distinguish between $p \leq 2q$ and $p \geq 2q$. For the first case, we obtain

$$\mathrm{Ber}_{p/q}^{(1)}(\kappa) = (-1)^p \det \left[\begin{array}{c|c} \left\{ \dfrac{\kappa_{a1}^{p-2q} \kappa_{b2}^{2q-p}}{\kappa_{a1} - \kappa_{b2}} \right. & \left. \dfrac{\partial}{\partial \kappa_{b2}} \dfrac{\kappa_{a1}^{p-2q} \kappa_{b2}^{2q-p}}{\kappa_{a1} - \kappa_{b2}} \right\}_{\substack{1 \leq a \leq p \\ 1 \leq b \leq q}} \\ \left\{ \kappa_{b2}^{a-1} \right. & \left. (a-1)\kappa_{b2}^{a-2} \right\}_{\substack{1 \leq a \leq 2q-p \\ 1 \leq b \leq q}} \end{array} \right] \tag{14.8}$$

and, for the second one, we get

$$\begin{aligned} \mathrm{Ber}_{p/q}^{(1)}(\kappa) &= (-1)^{p(p-1)/2+q} \\ &\times \det \left[\left\{ \frac{\kappa_{a1}^{p-2q} \kappa_{b2}^{2q-p}}{\kappa_{a1} - \kappa_{b2}} \;\middle|\; \frac{\kappa_{a1}^{p-2q} \kappa_{b2}^{2q-p}}{(\kappa_{a1} - \kappa_{b2})^2} \right\}_{\substack{1 \leq a \leq p \\ 1 \leq b \leq q}} \quad \left\{ \kappa_{a1}^{b-1} \right\}_{\substack{1 \leq a \leq p \\ 1 \leq b \leq p-2q}} \right] . \end{aligned} \tag{14.9}$$

We apply the same procedure as in Sec. 14.1 and shift all elements by ε. Taking the limit $\varepsilon \to \infty$, we find

$$\mathrm{Ber}_{p/q}^{(1)}(\kappa) = (-1)^p \det \left[\begin{array}{c|c} \left\{ \dfrac{1}{\kappa_{a1} - \kappa_{b2}} \right. & \left. \dfrac{1}{(\kappa_{a1} - \kappa_{b2})^2} \right\}_{\substack{1 \le a \le p \\ 1 \le b \le q}} \\ \left\{ \kappa_{b2}^{a-1} \right. & \left. (a-1)\kappa_{b2}^{a-2} \right\}_{\substack{1 \le a \le 2q-p \\ 1 \le b \le q}} \end{array} \right] \tag{14.10}$$

for $p \le 2q$ and

$$\begin{aligned} \mathrm{Ber}_{p/q}^{(1)}(\kappa) &= (-1)^{p(p-1)/2+q} \\ &\times \det \left[\begin{array}{c|c} \left\{ \dfrac{1}{\kappa_{a1} - \kappa_{b2}} \right. & \left. \dfrac{1}{(\kappa_{a1} - \kappa_{b2})^2} \right\}_{\substack{1 \le a \le p \\ 1 \le b \le q}} \quad \left\{ \kappa_{a1}^{b-1} \right\}_{\substack{1 \le a \le p \\ 1 \le b \le p-2q}} \end{array} \right] . \end{aligned} \tag{14.11}$$

for $p \ge 2q$. We notice that the determinantal structure of the ordinary Jacobians which are powers of Vandermonde determinants mixes with the structure of the Cauchy determinant, as for the $\mathrm{U}^{(2)}(p/q)$ case.

Chapter 15

Integrals with determinantal structures

We discuss three types of integrals which cover averages over ratios of characteristic polynomials for a large class of matrix ensembles. For the first two types we find determinantal structures. These types are integrals with a square root of a Berezinian (14.1) and with a Vandermonde determinant to the second power. They are studied in Secs. 15.1 and 15.2, respectively. The third type of integrals leads to Pfaffian determinants due to a pairwise coupling of integration variables. This is shown in Sec. 15.3.

15.1 Integrals of square root–Berezinian type

We consider the integral

$$\begin{aligned} Z_{k_1/k_2}^{(N_1/N_2)}(\kappa) &= \int\limits_{\mathbb{C}^{N_1+N_2}} \prod_{j=1}^{N_1} g_j(z_{j1}) \prod_{j=1}^{N_2} f_j(z_{j2}) \\ &\times \frac{\prod\limits_{a=1}^{N_1}\prod\limits_{b=1}^{k_1}(z_{a1}-\kappa_{b1})\prod\limits_{a=1}^{N_2}\prod\limits_{b=1}^{k_2}(z_{a2}-\kappa_{b2})}{\prod\limits_{a=1}^{N_1}\prod\limits_{b=1}^{k_2}(z_{a1}-\kappa_{b2})\prod\limits_{a=1}^{k_1}\prod\limits_{b=1}^{N_2}(\kappa_{a1}-z_{b2})}\sqrt{\mathrm{Ber}_{N_1/N_2}^{(2)}(z)}d[z]\,, \end{aligned} \tag{15.1}$$

where the integration variables z_j are complex variables. The functions g_j and f_j and the variables κ are chosen in such a way that the integral is convergent. The measure $d[z]$ is the product of the differentials of the real and imaginary parts. The applications which we will give are particular cases of this integral, although these applications correspond to essentially different ensembles. Thus, we show a fundamental relation which yields determinantal structures.

The crucial step is to extend the integral (15.1) by $\sqrt{\mathrm{Ber}^{(2)}_{k_1/k_2}(\kappa)}$ and to recognize that we obtain a new Berezinian

$$Z_{k_1/k_2}^{(N_1/N_2)}(\kappa) = \int\limits_{\mathbb{C}^{N_1+N_2}} \prod_{j=1}^{N_1} g_j(z_{j1}) \prod_{j=1}^{N_2} f_j(z_{j2}) \frac{\sqrt{\mathrm{Ber}^{(2)}_{N_1+k_1/N_2+k_2}(\tilde{z})}}{\sqrt{\mathrm{Ber}^{(2)}_{k_1/k_2}(\kappa)}} d[z] \,, \tag{15.2}$$

where the new bosonic eigenvalues are $\tilde{z}_1 = \mathrm{diag}\,(z_1, \kappa_1)$ and the new fermionic eigenvalues are $\tilde{z}_2 = \mathrm{diag}\,(z_2, \kappa_2)$. Now we use the determinantal structure of the square root Berezinian shown in Sec. 14.1.

Without loss of generality, we assume $N_2 \geq N_1$. In App. C.2, we show the details of this calculation and only give the results here. The simplest case is $k_1 = k_2 = k$. Then, the condition $(N_1 + k_1) \leq (N_2 + k_2)$ is automatically fulfilled. The integral (15.2) is then a quotient of two determinants times a constant

$$Z_{k/k}^{(N_1/N_2)}(\kappa) = \frac{(-1)^{(N_2+k)(N_2+k-1)/2} \det \mathbf{M}_{N_1/N_2}}{\sqrt{\mathrm{Ber}^{(2)}_{k/k}(\kappa)}} \det \left[K^{(N_1/N_2)}(\kappa_{a1}, \kappa_{b2})\right]_{1\leq a,b\leq k} \tag{15.3}$$

where we define

$$\mathbf{G}^{(N_1/N_2)}(\kappa_{b2}) = \begin{bmatrix} \left\{\kappa_{b2}^{a-1}\right\}_{1\leq a\leq N_2-N_1} \\ \left\{\int\limits_{\mathbb{C}} \dfrac{g_a(z)}{z-\kappa_{b2}} d[z]\right\}_{1\leq a\leq N_1} \end{bmatrix}, \tag{15.4}$$

$$\mathbf{F}^{(N_2)}(\kappa_{a1}) = \left[\int\limits_{\mathbb{C}} \frac{f_b(z)}{\kappa_{a1}-z} d[z]\right]_{1\leq b\leq N_2}, \tag{15.5}$$

$$\mathbf{M}_{N_1/N_2} = \begin{bmatrix} \left\{\int\limits_{\mathbb{C}} f_b(z) z^{a-1} d[z]\right\}_{\substack{1\leq a\leq N_2-N_1 \\ 1\leq b\leq N_2}} \\ \left\{\int\limits_{\mathbb{C}^2} \dfrac{g_a(z_1) f_b(z_2)}{z_1-z_2} d[z]\right\}_{\substack{1\leq a\leq N_1 \\ 1\leq b\leq N_2}} \end{bmatrix}, \tag{15.6}$$

$$K^{(N_1/N_2)}(\kappa_{a1}, \kappa_{b2}) = \frac{1}{\kappa_{a1}-\kappa_{a2}} - \mathbf{F}^{(N_2)}(\kappa_{a1}) \mathbf{M}_{N_1/N_2}^{-1} \mathbf{G}^{(N_1/N_2)}(\kappa_{b2}) \,. \tag{15.7}$$

Since the entries $K^{(N_1/N_2)}(\kappa_{a1}, \kappa_{b2})$ are independent of the dimension k, we identify

$$K^{(N_1/N_2)}(\kappa_{a1},\kappa_{b2}) = \frac{(-1)^{N_2(N_2+1)/2}}{\det \mathbf{M}_{N_1/N_2}} \frac{Z_{1/1}^{(N_1/N_2)}(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}} \tag{15.8}$$

which is the case $k = 1$. The normalization constant follows from $k = 0$ and is given by

$$C_{N_1/N_2} = Z_{0/0}^{(N_1/N_2)} = (-1)^{N_2(N_2-1)/2} \det \mathbf{M}_{N_1/N_2} \, . \tag{15.9}$$

This leads to the very compact result

$$Z_{k/k}^{(N_1/N_2)}(\kappa) = \frac{(-1)^{k(k-1)/2}}{C_{N_1/N_2}^{k-1}\sqrt{\mathrm{Ber}\,_{k/k}^{(2)}(\kappa)}} \det\left[\frac{Z_{1/1}^{(N_1/N_2)}(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}}\right]_{1\le a,b\le k} . \tag{15.10}$$

We recall that the functions g_j and f_j are arbitrary. This means that the fundamental structure of the ratios of the characteristic polynomials times the Berezinian fully generates the whole determinantal expression.

The cases $k_1 \le k_2$ and $k_1 \ge k_2$ cover all cases mentioned above. As in Ref. [131], we trace these cases back by introducing $|k_1-k_2|$ dummy variables. These variables enlarge the $(k_1+k_2)\times(k_1+k_2)$ eigenvalue matrix of κ to a $(k+k)\times(k+k)$ eigenvalue matrix, where $k = \max\{k_1,k_2\}$. Then, we use our result obtained above and remove these additional eigenvalues. The explicit calculations are given in appendices C.2.2 and C.2.3. We obtain

$$\begin{aligned} Z_{k_1/k_2}^{(N_1/N_2)}(\kappa) &= \frac{(-1)^{k_1(k_1-1)/2+(k_2-k_1)N_1}}{C_{N_1/N_2}^{k_2-1}\sqrt{\mathrm{Ber}\,_{k_1/k_2}^{(2)}(\kappa)}} \\ &\times \det\left[\begin{array}{c} \left\{\dfrac{Z_{1/1}^{(N_1/N_2)}(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\le a\le k_1\\ 1\le b\le k_2}} \\ \left\{\left(\kappa_0^2\dfrac{\partial}{\partial\kappa_0}\right)^{a-1} \left.\dfrac{\kappa_0^{N_2-N_1+1}Z_{1/1}^{(N_1/N_2)}(\kappa_0,\kappa_{b2})}{\kappa_0-\kappa_{b2}}\right|_{\kappa_0\to\infty}\right\}_{\substack{1\le a\le k_2-k_1\\ 1\le b\le k_2}} \end{array}\right] \end{aligned} \tag{15.11}$$

for $k_1 \le k_2$ and

$$\begin{aligned} Z_{k_1/k_2}^{(N_1/N_2)}(\kappa) &= \frac{(-1)^{(k_2+2k_1)(k_2-1)/2+(k_1-k_2)(N_2-N_1)}}{C_{N_1/N_2}^{k_1-1}\sqrt{\mathrm{Ber}\,_{k_1/k_2}^{(2)}(\kappa)}} \\ &\times \det\left[\begin{array}{c} \left\{\dfrac{Z_{1/1}^{(N_1/N_2)}(\kappa_{b1},\kappa_{a2})}{\kappa_{b1}-\kappa_{a2}}\right\}_{\substack{1\le a\le k_2\\ 1\le b\le k_1}} \\ \left\{\left(\kappa_0^2\dfrac{\partial}{\partial\kappa_0}\right)^{a-1} \left.\dfrac{\kappa_0^{N_1-N_2+1}Z_{1/1}^{(N_1/N_2)}(\kappa_{b1},\kappa_0)}{\kappa_{b1}-\kappa_0}\right|_{\kappa_0\to\infty}\right\}_{\substack{1\le a\le k_1-k_2\\ 1\le b\le k_1}} \end{array}\right] \end{aligned} \tag{15.12}$$

for $k_1 \geq k_2$. For $|k_1 - k_2| = 1$ the average in the last row is only over one characteristic polynomial, i.e. it is equal to $Z_{0/1}^{(N_1/N_2)}(\kappa_{b2})$ in Eq. (15.11) and $Z_{1/0}^{(N_1/N_2)}(\kappa_{b1})$ in Eq. (15.12). The limits in Eqs. (15.11) and (15.12) are well defined, as a comparison with Eq. (15.1) shows, and can be calculated by writing the derivative as a contour integral around $1/\kappa_0 = 0$. The limits are explicitly given as

$$\begin{aligned}
&\left(\kappa_0^2 \frac{\partial}{\partial \kappa_0}\right)^{a-1} \frac{\kappa_0^{N_2-N_1+1} Z_{1/1}^{(N_1/N_2)}(\kappa_0, \kappa_{b2})}{\kappa_0 - \kappa_{b2}}\Bigg|_{\kappa_0 \to \infty} \qquad (15.13)\\
&= \quad (-1)^{a-1+N_2}(a-1)! C_{N_1/N_2}\left[\kappa_{b2}^{a-1+N_2-N_1} - \mathbf{f}_a \mathbf{M}_{N_1/N_2}^{-1} \mathbf{G}^{(N_1/N_2)}(\kappa_{b2})\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left(\kappa_0^2 \frac{\partial}{\partial \kappa_0}\right)^{a-1} \frac{\kappa_0^{N_1-N_2+1} Z_{1/1}^{(N_1/N_2)}(\kappa_{b1}, \kappa_0)}{\kappa_{b1} - \kappa_0}\Bigg|_{\kappa_0 \to \infty} = (-1)^{a+N_2}(a-1)! C_{N_1/N_2}\\
&\times \left[\kappa_{b1}^{a-1+N_1-N_2}\Theta(a+N_1-N_2-1) - \mathbf{F}^{(N_2)}(\kappa_{b1}) \mathbf{M}_{N_1/N_2}^{-1} \mathbf{g}_a\right], \qquad (15.14)
\end{aligned}$$

where we define the matrices

$$\mathbf{f}_a = \left[\int_{\mathbb{C}} f_b(z) z^{a-1+N_2-N_1} d[z]\right]_{1 \leq b \leq N_2}, \qquad (15.15)$$

$$\mathbf{g}_a = \left[\begin{array}{c} \left\{-\delta_{N_2-N_1+1-a,b}\right\}_{1 \leq b \leq N_2-N_1} \\ \left\{\int_{\mathbb{C}} g_b(z) z^{a-1+N_1-N_2} d[z] \Theta(a+N_1-N_2-1)\right\}_{1 \leq b \leq N_1} \end{array}\right] \qquad (15.16)$$

The function Θ is the Heaviside distribution for discrete numbers which means it is the integrated Kronecker-δ and, hence, unity at zero.

15.2 Integrals of squared–Vandermonde type

Now, we investigate integrals of the type

$$\widetilde{Z}_{\frac{k_1/k_2}{l_1/l_2}}^{(N)}(\kappa, \lambda) = \int_{\mathbb{C}^N} \prod_{j=1}^{N} g(z_j) \frac{\prod\limits_{a=1}^{k_2}\prod\limits_{b=1}^{N}(\kappa_{a2} - z_b) \prod\limits_{a=1}^{l_2}\prod\limits_{b=1}^{N}(\lambda_{a2} - z_b^*)}{\prod\limits_{a=1}^{k_1}\prod\limits_{b=1}^{N}(\kappa_{a1} - z_b) \prod\limits_{a=1}^{l_1}\prod\limits_{b=1}^{N}(\lambda_{a1} - z_b^*)} |\Delta_N(z)|^2 d[z]\,. \qquad (15.17)$$

The function g is, as in Sec. 15.1, an arbitrary function with the only restriction that the integral above is convergent. Instead of one eigenvalue set as in the section above, we have now two eigenvalue sets $\kappa = \mathrm{diag}\,(\kappa_{11},\ldots,\kappa_{k_1 1},\kappa_{12},\ldots,\kappa_{k_2 2})$ and $\lambda = \mathrm{diag}\,(\lambda_{11},\ldots,\lambda_{l_1 1},\lambda_{12},\ldots,\lambda_{l_2 2})$. Because of these two sets, we have to extend the fraction by two square roots of Berezinians and find

$$\widetilde{Z}^{(N)}_{\substack{k_1/k_2\\ l_1/l_2}}(\kappa,\lambda) \;=\; \int\limits_{\mathbb{C}^N}\prod_{j=1}^{N} g(z_j)\frac{\sqrt{\mathrm{Ber}^{(2)}_{k_1/k_2+N}(\tilde{z})}\sqrt{\mathrm{Ber}^{(2)}_{l_1/l_2+N}(\hat{z})}}{\sqrt{\mathrm{Ber}^{(2)}_{k_1/k_2}(\kappa)}\sqrt{\mathrm{Ber}^{(2)}_{l_1/l_2}(\lambda)}}d[z]\,. \quad (15.18)$$

We introduce $\tilde{z} = \mathrm{diag}\,(\kappa_1,\kappa_2,z)$ and $\hat{z} = \mathrm{diag}\,(\lambda_1,\lambda_2,z^*)$.

To integrate Eq. (15.18), we first discuss the case $d = k_2 + N - k_1 = l_2 + N - l_1 \geq 0$. Under the integral we have two determinants with N rows depending on one z_a or one z_a^*. The other rows are independent of any z_a and z_a^*. Thus, we use an integration theorem similar to Andréief's [132] which we derive in App. C.3.1. In App. C.4 we carry out the calculation and find

$$\widetilde{Z}^{(N)}_{\substack{k_1/k_2\\ l_1/l_2}}(\kappa,\lambda) = \frac{(-1)^{(l_2+k_2)(l_1+k_1-1)/2}N!\det\widetilde{\mathbf{M}}_d}{\sqrt{\mathrm{Ber}^{(2)}_{k_1/k_2}(\kappa)}\sqrt{\mathrm{Ber}^{(2)}_{l_1/l_2}(\lambda)}}$$
$$\times\quad\det\left[\begin{array}{cc} \left\{\widetilde{K}^{(d)}_{11}(\lambda_{a2},\kappa_{b2})\right\}_{\substack{1\leq a\leq l_2\\ 1\leq b\leq k_2}} & \left\{\widetilde{K}^{(d)}_{12}(\lambda_{b1},\lambda_{a2})\right\}_{\substack{1\leq a\leq l_2\\ 1\leq b\leq l_1}} \\ \left\{\widetilde{K}^{(d)}_{21}(\kappa_{a1},\kappa_{b2})\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}} & \left\{\widetilde{K}^{(d)}_{22}(\kappa_{a1},\lambda_{b1})\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq l_1}} \end{array}\right], \quad (15.19)$$

where

$$\widetilde{Z}^{(1)}_{\substack{1/0\\ 1/0}}(\kappa_{a1},\lambda_{b1}) \;=\; \int\limits_{\mathbb{C}}\frac{g(z)}{(\kappa_{a1}-z)(\lambda_{b1}-z^*)}d^2z\,, \quad (15.20)$$

$$\widetilde{\mathbf{F}}_d(\kappa_{a1}) \;=\; \left[\int\limits_{\mathbb{C}}\frac{g(z)z^{*\,b-1}}{\kappa_{a1}-z}d^2z\right]_{1\leq b\leq d}, \quad (15.21)$$

$$\widetilde{\mathbf{F}}^{(*)}_d(\lambda_{b1}) \;=\; \left[\int\limits_{\mathbb{C}}\frac{g(z)z^{a-1}}{\lambda_{b1}-z^*}d^2z\right]_{1\leq a\leq d}, \quad (15.22)$$

$$\widetilde{\mathbf{M}}_d \;=\; \left[\int\limits_{\mathbb{C}} g(z)z^{a-1}z^{*\,b-1}d^2z\right]_{1\leq a,b\leq d}, \quad (15.23)$$

$$\mathbf{\Lambda}_d(\lambda_{a2}) \;=\; \left[\lambda_{a2}^{b-1}\right]_{1\leq b\leq d}\,, \quad (15.24)$$

$$
\begin{aligned}
\mathbf{K}_d(\kappa_{b2}) &= \left[\kappa_{b2}^{a-1}\right]_{1\le a\le d}\,, && (15.25)\\
\widetilde{K}_{11}^{(d)}(\kappa_{b2},\lambda_{a2}) &= -\mathbf{\Lambda}_d(\lambda_{a2})\widetilde{\mathbf{M}}_d^{-1}\mathbf{K}_d(\kappa_{b2})\,, && (15.26)\\
\widetilde{K}_{12}^{(d)}(\lambda_{b1},\lambda_{a2}) &= \frac{1}{\lambda_{b1}-\lambda_{a2}} - \mathbf{\Lambda}_d(\lambda_{a2})\widetilde{\mathbf{M}}_d^{-1}\widetilde{\mathbf{F}}_d^{(*)}(\lambda_{b1})\,, && (15.27)\\
\widetilde{K}_{21}^{(d)}(\kappa_{a1},\kappa_{b2}) &= \frac{1}{\kappa_{a1}-\kappa_{b2}} - \widetilde{\mathbf{F}}_d(\kappa_{a1})\widetilde{\mathbf{M}}_d^{-1}\mathbf{K}_d(\kappa_{b2}) && (15.28)\\
\widetilde{K}_{22}^{(d)}(\kappa_{a1},\lambda_{b1}) &= \widetilde{Z}^{(1)}_{\frac{1/0}{1/0}}(\kappa_{a1},\lambda_{b1}) - \widetilde{\mathbf{F}}_d(\kappa_{a1})\widetilde{\mathbf{M}}_d^{-1}\widetilde{\mathbf{F}}_d^{(*)}(\lambda_{b1})\,. && (15.29)
\end{aligned}
$$

With help of the particular cases $(k_1 = k_2 = 1, l_1 = l_2 = 0)$, $(k_1 = k_2 = 0, l_1 = l_2 = 1)$, $(k_1 = l_1 = 1, k_2 = l_2 = 0)$ and $(k_1 = l_1 = 0, k_2 = l_2 = 1)$, we identify

$$
\begin{aligned}
\widetilde{K}_{21}^{(N)}(\kappa_{a1},\kappa_{b2}) &= \frac{1}{N!\det\widetilde{\mathbf{M}}_N}\frac{\widetilde{Z}^{(N)}_{\frac{1/1}{0/0}}(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}}\,, && (15.30)\\
\widetilde{K}_{12}^{(N)}(\lambda_{b1},\lambda_{a2}) &= \frac{1}{N!\det\widetilde{\mathbf{M}}_N}\frac{\widetilde{Z}^{(N)}_{\frac{0/0}{1/1}}(\lambda_{b1},\lambda_{a2})}{\lambda_{b1}-\lambda_{a2}}\,, && (15.31)\\
\widetilde{K}_{22}^{(N-1)}(\kappa_{a1},\lambda_{b1}) &= \frac{1}{N!\det\widetilde{\mathbf{M}}_{N-1}}\widetilde{Z}^{(N)}_{\frac{1/0}{1/0}}(\kappa_{a1},\lambda_{b1})\,, && (15.32)\\
\widetilde{K}_{11}^{(N+1)}(\lambda_{a2},\kappa_{b2}) &= -\frac{1}{N!\det\widetilde{\mathbf{M}}_{N+1}}\widetilde{Z}^{(N)}_{\frac{0/1}{0/1}}(\lambda_{a2},\kappa_{b2})\,. && (15.33)
\end{aligned}
$$

The normalization constant is fixed by the case $(k_1 = k_2 = l_1 = l_2 = 0)$

$$
\widetilde{C}_N = \widetilde{Z}^{(N)}_{\frac{0/0}{0/0}} = N!\det\widetilde{\mathbf{M}}_N\,. \tag{15.34}
$$

Thus, we find

$$
\begin{aligned}
\widetilde{Z}^{(N)}_{\frac{k_1/k_2}{l_1/l_2}}(\kappa,\lambda) &= \frac{(-1)^{(l_2+k_2)(l_1+k_1-1)/2}N!}{\det^{l_2+k_1-1}\widetilde{\mathbf{M}}_d\sqrt{\mathrm{Ber}^{(2)}_{k_1/k_2}(\kappa)}\sqrt{\mathrm{Ber}^{(2)}_{l_1/l_2}(\lambda)}} \qquad (15.35)\\
\times\ \det &\left[\begin{array}{cc}
\left\{-\dfrac{\widetilde{Z}^{(d-1)}_{\frac{0/1}{0/1}}(\lambda_{a2},\kappa_{b2})}{(d-1)!}\right\}_{\substack{1\le a\le l_2\\1\le b\le k_2}} & \left\{\dfrac{\widetilde{Z}^{(d)}_{\frac{0/0}{1/1}}(\lambda_{b1},\lambda_{a2})}{d!(\lambda_{b1}-\lambda_{a2})}\right\}_{\substack{1\le a\le l_2\\1\le b\le l_1}}\\
\left\{\dfrac{\widetilde{Z}^{(d)}_{\frac{1/1}{0/0}}(\kappa_{a1},\kappa_{b2})}{d!(\kappa_{a1}-\kappa_{b2})}\right\}_{\substack{1\le a\le k_1\\1\le b\le k_2}} & \left\{\dfrac{\widetilde{Z}^{(d+1)}_{\frac{1/0}{1/0}}(\kappa_{a1},\lambda_{b1})}{(d+1)!}\right\}_{\substack{1\le a\le k_1\\1\le b\le l_1}}
\end{array}\right].
\end{aligned}
$$

Once more, we notice that the distribution $g(z)$ is quite arbitrary and, thus, a large class of ensembles is covered. The result (15.35) is equivalent to the one found by Bergere [133] with the method of biorthogonal polynomials.

We derive the integral (15.17) for the case that $d = k_2 + N - k_1 = l_2 + N - l_1 \geq 0$ is violated by the same method as used in Sec. 15.1. By extending the quotient of characteristic polynomials to the case discussed above, we apply the known result and take the limits with help of l'Hospital's rule. This procedure gives us expressions similar to Eqs. (15.11) and (15.12).

For the particular case that $d_\kappa = k_1 - k_2 - N$ and $d_\lambda = l_1 - l_2 - N$ are positive, we find a much simpler expression

$$\widetilde{Z}^{(N)}_{\frac{k_1/k_2}{l_1/l_2}}(\kappa,\lambda) = \frac{(-1)^{(l_1+k_1)(l_1+k_1-1)/2+N(k_2+l_2+1)}N!}{\sqrt{\mathrm{Ber}^{(2)}_{k_1/k_2}(\kappa)}\sqrt{\mathrm{Ber}^{(2)}_{l_1/l_2}(\lambda)}} \tag{15.36}$$
$$\times \quad \det\begin{bmatrix} 0 & 0 & \left\{\dfrac{1}{\lambda_{b1}-\lambda_{a2}}\right\}_{\substack{1\leq a\leq l_2\\1\leq b\leq l_1}} \\ 0 & 0 & \left\{\lambda_{b1}^{a-1}\right\}_{\substack{1\leq a\leq d_\lambda\\1\leq b\leq l_1}} \\ \left\{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq k_1\\1\leq b\leq k_2}} & \left\{\kappa_{a1}^{b-1}\right\}_{\substack{1\leq a\leq k_1\\1\leq b\leq d_\kappa}} & \left\{\widetilde{Z}^{(1)}_{\frac{1/0}{1/0}}(\kappa_{a1},\lambda_{b1})\right\}_{\substack{1\leq a\leq k_1\\1\leq b\leq l_1}} \end{bmatrix},$$

which is derived in App. C.4.2. We see that the whole integral is determined by $\widetilde{Z}^{(1)}_{\frac{1/0}{1/0}}(\kappa_{a1},\lambda_{b1})$ and some algebraic combinations of the "Cauchy terms" $1/(\lambda_{b1}-\lambda_{a2})$ and $1/(\kappa_{a1}-\kappa_{b2})$ and of the "Vandermonde terms" λ_{b1}^{a-1} and κ_{a1}^{b-1} if the number of characteristic polynomials in the denominator exceeds a critical value. Here, we emphasize that Eq. (15.36) has a simpler structure than Eqs. (15.11), (15.12) and (15.35) for $d_\kappa \neq d_\lambda$ since there are no limits to perform. Furthermore, the parameters d_λ and d_κ are independent as long they are larger than or equal to zero.

Eqs. (15.35) and (15.36) are similar to the results found by Uvarov [134, 135]. He studied the transformation behavior of the orthogonal polynomials when the probability distributions differ in a rational function. This shows the connection between his approach and ours.

15.3 Integrals of coupled square–root Vandermonde type

We consider the integral

$$Z_{(k_1/k_2)}^{(2N+1)}(\kappa) = \int\limits_{\mathbb{C}^{2N+1}} h(z_{2N+1}) \prod_{j=1}^{N} g(z_{2j-1}, z_{2j}) \frac{\prod\limits_{a=1}^{2N+1} \prod\limits_{b=1}^{k_2} (z_a - \kappa_{b2})}{\prod\limits_{a=1}^{2N+1} \prod\limits_{b=1}^{k_1} (\kappa_{b1} - z_a)} \Delta_{2N+1}(z) d[z] \,. \tag{15.37}$$

We choose the functions h and g and the external variables κ in such a way that the integral exists. With the two-dimensional Dirac distribution $h(z_{2N+1}) = \delta^2(z_{2N+1})$ in the complex plane and with the function $\tilde{g}(z_{2j-1}, z_{2j})$ $= z_{2j-1} z_{2j} g(z_{2j-1}, z_{2j})$, we regain another important integral

$$Z_{(k_1/k_2)}^{(2N)}(\kappa) = \int\limits_{\mathbb{C}^{2N}} \prod_{j=1}^{N} \tilde{g}(z_{2j-1}, z_{2j}) \frac{\prod\limits_{a=1}^{k_2} \prod\limits_{b=1}^{2N} (\kappa_{a2} - z_b)}{\prod\limits_{a=1}^{k_1} \prod\limits_{b=1}^{2N} (\kappa_{a1} - z_b)} \Delta_{2N}(z) d[z] \ , \tag{15.38}$$

to be calculated in the following. The integration variables z are coupled by the function g and, hence we have to expect a quite different result as in Sec. 15.1 for fully factorizing probability densities. This explains why we name this type of integral coupled square–root Vandermonde.

As in Sec. 15.1, we extend the integrand in Eq. (15.37) by $\sqrt{\mathrm{Ber}\,_{(k_1/k_2)}^{(2)}(\kappa)}$ and obtain

$$Z_{(k_1/k_2)}^{(2N+1)}(\kappa) = \int\limits_{\mathbb{C}^{2N+1}} h(z_{2N+1}) \prod_{j=1}^{N} g(z_{2j-1}, z_{2j}) \frac{\sqrt{\mathrm{Ber}\,_{(k_1/k_2+2N+1)}^{(2)}(\tilde{z})}}{\sqrt{\mathrm{Ber}\,_{(k_1/k_2)}^{(2)}(\kappa)}} d[z] \,, \tag{15.39}$$

where we define $\tilde{z} = \mathrm{diag}\,(\kappa_1; \kappa_2, z)$. We then use the determinantal structure of the square root Berezinian in the numerator for the integration. In App. C.5.1 we explicitly calculate (15.39) for odd $d = k_2 - k_1 + 2N + 1 \geq 0$

and find

$$
\begin{aligned}
Z_{(k_1/k_2)}^{(2N+1)}(\kappa) \;&=\; \frac{(-1)^{N+1}N!\operatorname{Pf}\mathbf{M}_{(d)}}{\sqrt{\operatorname{Ber}_{(k_1/k_2)}^{(2)}(\kappa)}} \qquad (15.40)\\
&\times\;\operatorname{Pf}\begin{bmatrix} \left\{K_{11}^{(d)}(\kappa_{a2},\kappa_{b2})\right\}_{1\le a,b\le k_2} & \left\{K_{12}^{(d)}(\kappa_{b1},\kappa_{a2})\right\}_{\substack{1\le a\le k_2\\ 1\le b\le k_1}} \\ \left\{-K_{12}^{(d)}(\kappa_{a1},\kappa_{b2})\right\}_{\substack{1\le a\le k_1\\ 1\le b\le k_2}} & \left\{K_{22}^{(d)}(\kappa_{a1},\kappa_{b1})\right\}_{1\le a,b\le k_1} \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{F}(\kappa_{a1},\kappa_{b1}) &= -(\kappa_{a1}-\kappa_{b1})Z_{(2/0)}^{(2)}(\kappa_{a1},\kappa_{b1}) \qquad (15.41)\\
&= -\int\limits_{\mathbb{C}^2}\frac{(\kappa_{a1}-\kappa_{b1})(z_1-z_2)g(z_1,z_2)}{(\kappa_{a1}-z_1)(\kappa_{a1}-z_2)(\kappa_{b1}-z_1)(\kappa_{b1}-z_2)}d[z]\,,\\
\mathbf{G}_{(d)}(\kappa_{a1}) &= \begin{bmatrix} \left\{\int\limits_{\mathbb{C}^2}\det\begin{bmatrix} \dfrac{g(z_1,z_2)}{\kappa_{a1}-z_1} & \dfrac{g(z_1,z_2)}{\kappa_{a1}-z_2} \\ z_1^{b-1} & z_2^{b-1}\end{bmatrix} d[z]\right\}_{1\le b\le d} \\ -\int\limits_{\mathbb{C}}\dfrac{h(z)}{\kappa_{a1}-z}dz \end{bmatrix}^T, \qquad (15.42)\\
\mathbf{K}_{(d)}(\kappa_{a2}) &= \begin{bmatrix} \left\{\kappa_{a2}^{b-1}\right\}_{1\le b\le d} & 0 \end{bmatrix}, \qquad (15.43)\\
K_{11}^{(d)}(\kappa_{a2},\kappa_{b2}) &= \mathbf{K}_{(d)}(\kappa_{a2})\mathbf{M}_{(d)}^{-1}\mathbf{K}_{(d)}^T(\kappa_{b2})\,, \qquad (15.44)\\
K_{12}^{(d)}(\kappa_{b1},\kappa_{a2}) &= \frac{1}{\kappa_{b1}-\kappa_{a2}}+\mathbf{K}_{(d)}(\kappa_{a2})\mathbf{M}_{(d)}^{-1}\mathbf{G}_{(d)}^T(\kappa_{b1})\,, \qquad (15.45)\\
K_{22}^{(d)}(\kappa_{a1},\kappa_{b1}) &= \mathbf{F}(\kappa_{a1},\kappa_{b1})+\mathbf{G}_{(d)}(\kappa_{a1})\mathbf{M}_{(d)}^{-1}\mathbf{G}_{(d)}^T(\kappa_{b1})\,. \qquad (15.46)
\end{aligned}
$$

Here, we use the moment matrix

$$
\mathbf{M}_{(d)}=\begin{bmatrix} \left\{\int\limits_{\mathbb{C}^2}\det\begin{bmatrix} g(z_1,z_2)z_1^{a-1} & z_1^{b-1} \\ g(z_1,z_2)z_2^{a-1} & z_2^{b-1}\end{bmatrix} d[z]\right\}_{1\le a,b\le d} & \left\{-\int\limits_{\mathbb{C}}h(z)z^{a-1}dz\right\}_{1\le a\le d} \\ \left\{\int\limits_{\mathbb{C}}h(z)z^{b-1}dz\right\}_{1\le b\le d} & 0 \end{bmatrix} \qquad (15.47)
$$

of our probability densities h and g. We recall that $\mathfrak{S}_M$ is the permutation group of M elements. The function "sign" equals "+1" for even permutations and "-1" for odd ones. We fix the sign of the Pfaffian for an arbitrary anti-symmetric $2N \times 2N$ matrix $\{D_{ab}\}$ by

$$\mathrm{Pf}\,[D_{ab}]_{1\leq a,b\leq N} = \frac{1}{2^N N!} \sum_{\omega\in\mathfrak{S}_{2N}} \mathrm{sign}\,(\omega) \prod_{j=1}^{N} D_{\omega(2j-1)\omega(2j)}\,. \tag{15.48}$$

We identify the integral kernels (15.44) to (15.46) with the particular cases ($k_1 = 0$, $k_2 = 2$), ($k_1 = 1$, $k_2 = 1$) and ($k_1 = 2$, $k_2 = 0$) of the integral (15.37),

$$K_{11}^{(2N+3)}(\kappa_{a2},\kappa_{b2}) = (-1)^{N+1}\frac{\kappa_{a2}-\kappa_{b2}}{N!\mathrm{Pf}\,\mathbf{M}_{(2N+3)}} Z_{(0/2)}^{(2N+1)}(\kappa_{a2},\kappa_{b2})\,, \tag{15.49}$$

$$K_{12}^{(2N+1)}(\kappa_{b1},\kappa_{a2}) = (-1)^{N+1}\frac{Z_{(1/1)}^{(2N+1)}(\kappa_{b1},\kappa_{a2})}{N!\mathrm{Pf}\,\mathbf{M}_{(2N+1)}(\kappa_{b1}-\kappa_{a2})}\,, \tag{15.50}$$

$$K_{22}^{(2N-1)}(\kappa_{a1},\kappa_{b1}) = (-1)^{N+1}\frac{\kappa_{a1}-\kappa_{b1}}{N!\mathrm{Pf}\,\mathbf{M}_{(2N-1)}} Z_{(2/0)}^{(2N+1)}(\kappa_{a1},\kappa_{b1})\,. \tag{15.51}$$

The normalization constant is defined by the case $k_1 = k_2 = 0$,

$$C_{(2N+1)} = Z_{(0/0)}^{(2N+1)} = (-1)^{N+1} N!\mathrm{Pf}\,\mathbf{M}_{(2N+1,1)}\,. \tag{15.52}$$

Hence, Eq. (15.40) reads

$$Z_{(k_1/k_2)}^{(2N+1)}(\kappa) = \frac{(-1)^{(k_2^2-k_1^2)/4} N!\,\left[(-1)^{N+1}\mathrm{Pf}\,\mathbf{M}_{(d)}\right]^{1-(k_1+k_2)/2}}{\sqrt{\mathrm{Ber}\,_{(k_1/k_2)}^{(2)}(\kappa)}} \tag{15.53}$$
$$\times\ \mathrm{Pf}\left[\begin{array}{c|c} \dfrac{(\kappa_{b2}-\kappa_{a2})Z_{(0/2)}^{(d-2)}(\kappa_{a2},\kappa_{b2})}{[(d-3)/2]!} & \dfrac{Z_{(1/1)}^{(d)}(\kappa_{b1},\kappa_{a2})}{[(d-1)/2]!(\kappa_{b1}-\kappa_{a2})} \\ \hline \dfrac{Z_{(1/1)}^{(d)}(\kappa_{a1},\kappa_{b2})}{[(d-1)/2]!(\kappa_{b2}-\kappa_{a1})} & \dfrac{(\kappa_{b1}-\kappa_{a1})Z_{(2/0)}^{(d+2)}(\kappa_{a1},\kappa_{b1})}{[(d+1)/2]!} \end{array}\right],$$

where the indices a and b run over all labels of κ. When d is odd, $k_1 + k_2$ is even. Thus, the Pfaffians are well defined.

For the case that $k_2 + k_1$ is odd, we extend the integral

$$Z_{(k_1/k_2)}^{(2N+1)}(\kappa) = -\lim_{\kappa_{02}\to\infty} \frac{Z_{(k_1/k_2+1)}^{(2N+1)}(\kappa)}{\kappa_{02}^{2N+1}} \tag{15.54}$$

by an additional parameter κ_{02}. This trick is similar to the one in Ref. [131], see also Sec. 15.1. Defining $\tilde{d} = k_2 - k_1 + 2N + 2 \geq 0$, we find

$$Z^{(2N+1)}_{(k_1/k_2)}(\kappa) = \frac{(-1)^{(k_2+k_1+1)/2} N!}{\sqrt{\mathrm{Ber}\,^{(2)}_{(k_1/k_2)}(\kappa)}} \tag{15.55}$$
$$\times \quad \mathrm{Pf}\left[\begin{array}{c|c|c} 0 & \dfrac{-Z^{(\tilde{d}-2)}_{(0/1)}(\kappa_{b2})}{[(\tilde{d}-3)/2]!} & \dfrac{-Z^{(\tilde{d})}_{(1/0)}(\kappa_{b1})}{[(\tilde{d}-1)/2]!} \\ \hline \dfrac{Z^{(\tilde{d}-2)}_{(0/1)}(\kappa_{a2})}{[(\tilde{d}-3)/2]!} & K^{(\tilde{d})}_{11}(\kappa_{a2}, \kappa_{b2}) & K^{(\tilde{d})}_{12}(\kappa_{b1}, \kappa_{a2}) \\ \hline \dfrac{Z^{(\tilde{d})}_{(1/0)}(\kappa_{a1})}{[(\tilde{d}-1)/2]!} & -K^{(\tilde{d})}_{12}(\kappa_{a1}, \kappa_{b2}) & K^{(\tilde{d})}_{22}(\kappa_{a1}, \kappa_{b1}) \end{array}\right],$$

where the indices a and b label all components of κ such that we have to take the Pfaffian determinant over a $(k_1+k_2+1) \times (k_1+k_2+1)$ matrix. We notice the appearance of averages over one characteristic polynomial.

The results above also hold for the integral (15.38). We simply have to choose h as a Dirac distribution. This relation is well known [75] for odd and even dimensional ensembles of real symmetric matrices or circular orthogonal matrices. Since the probability densities g and h are quite arbitrary, this result considerably extends the one found by Borodin and Strahov [136].

We are also interested in the case of $d = k_2 - k_1 + 2N + 1 \leq 0$. Employing the sketched derivation in App. C.5.2, we have

$$Z^{(2N+1)}_{(k_1/k_2)}(\kappa) = \frac{(-1)^N N!}{\sqrt{\mathrm{Ber}\,^{(2)}_{(k_1/k_2)}(\kappa)}} \tag{15.56}$$
$$\times \quad \mathrm{Pf}\left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \dfrac{1}{\kappa_{b1} - \kappa_{a2}} \\ \hline 0 & 0 & 0 & Z^{(1)}_{(1/0)}(\kappa_{b1}) \\ \hline 0 & 0 & 0 & \kappa_{b1}^{a-1} \\ \hline \dfrac{1}{\kappa_{b2} - \kappa_{a1}} & -Z^{(1)}_{(1/0)}(\kappa_{a1}) & -\kappa_{a1}^{b-1} & -(\kappa_{a1} - \kappa_{b1}) Z^{(2)}_{(2/0)}(\kappa_{a1}, \kappa_{b1}) \end{array}\right].$$

Again, the index a labels the rows and b labels the columns. The exponent of the Vandermonde terms κ_{b1}^{a-1} goes from zero to $d-1$ such that Eq. (15.56) is a $2(N+k_2+1)$ dimensional Pfaffian determinant. For the integral (15.38), we have to omit the column and the row with $Z^{(1)}_{(1/0)}$ and to replace d by $2N+k_2-k_1$. The matrix in the Pfaffian (15.56) is, indeed, even-dimensional. Thus, the expression is well defined.

Chapter 16

Applications to random matrix theory

All three types of integrals shown in the previous chapter cover a wide range of applications for rotation invariant random matrix ensembles with real, complex as well as quaternionic symmetry. This also includes real Ginibre ensembles [137, 138, 83] and Gaussian real chiral ensembles [139, 41] with two independent matrices. For these ensembles the Pfaffian structures were recently shown in a more or less complicate way. With our method we have found these structures in a unifying and straitforward way.

Ensembles with a squared–Vandermonde or a square root–Berezinian are shown in Secs. 16.1 and 16.2, respectively. As we have seen both types lead to determinantal structures. The Pfaffian structures are obtained for matrix ensembles with a coupled square root–Vandermonde. Particular examples and a list of matrix ensembles for this type of integrals are shown in Sec. 16.3.

16.1 Applications for integrals of squared–Vandermonde type

In subsection 16.1.1 we apply our method for integrals of squared–Vandermonde type to the example of Hermitian matrices. Since the method has a broad field of applications, we give a list of matrix ensembles in subsection 16.1.2.

16.1.1 Hermitian matrix ensemble

We first consider rotation invariant ensembles of the $N \times N$ Hermitian matrices $\mathrm{Herm}\,(2,N)$. In part II, we have seen that the averages

$$Z_{\tilde{k}_1/\tilde{k}_2}^{(N)}(\kappa) = \int\limits_{\mathrm{Herm}\,(2,N)} P(H) \frac{\prod\limits_{j=1}^{\tilde{k}_2} \det(H - \kappa_{j2}\mathbb{1}_N)}{\prod\limits_{j=1}^{\tilde{k}_1} \det(H - \kappa_{j1}\mathbb{1}_N)} d[H] \tag{16.1}$$

are of considerable interest. Here, all κ_{j1} have an imaginary part. Equation (16.1) yields the k–point correlation function by differentiation with respect to the source variables J in κ [9, 60, 61], see also Sec. 6.1. The average over characteristic polynomials in general and their relation to determinantal structures are considered as well, see Refs. [104, 140, 141, 142, 121, 87, 136].

Let $k_2 + l_2 = \tilde{k}_2$, $k_1 + l_1 = \tilde{k}_1$ and $d = k_2 + N - k_1 = l_2 + N - l_1 \geq 0$. We consider probability densities P which factorize in the eigenvalue representation of the matrix H. Due to the rotation invariance, we diagonalize $H = UEU^\dagger$ with a unitary matrix $U \in \mathrm{U}^{(2)}(N)$. The measure is

$$d[H] = \frac{1}{N!} \prod_{j=1}^{N} \frac{\pi^{j-1}}{(j-1)!} \Delta_N^2(E) d[E] d\mu(U), \tag{16.2}$$

where $d\mu(U)$ is the normalized Haar measure, see Chap. 4. Thus, we find from Eq. (15.17)

$$\widetilde{Z}^{(N)}_{\underline{k_1/k_2}\atop l_1/l_2}(\tilde{\kappa};\lambda) = (-1)^{(\tilde{k}_1+\tilde{k}_2)N} N! \prod_{j=1}^{N} \frac{(j-1)!}{\pi^{j-1}} Z^{(N)}_{\tilde{k}_1/\tilde{k}_2}(\kappa) \tag{16.3}$$

with

$$g(z_j) = P(E_j)\delta(y_j)\,. \tag{16.4}$$

We decompose z_j into real and imaginary part, $z_j = E_j + \imath y_j$, and define the two sets $\tilde{\kappa} = \mathrm{diag}\,(\kappa_{11}, \ldots, \kappa_{k_1 1}, \kappa_{12}, \ldots, \kappa_{k_2 2})$ and $\lambda = \mathrm{diag}\,(\kappa_{k_1+1,1}, \ldots, \kappa_{\tilde{k}_1 1}, \kappa_{k_2+1,2}, \ldots, \kappa_{\tilde{k}_2,2})$. Our result for this choice, indeed, coincides with the one found by Borodin and Strahov [136]. They as well splitted the number of characteristic polynomials in two sets and derived the determinantal structure by discrete approximation. They used similar algebraic manipulations but they did not consider the connection to supersymmetry. Hence our proof is truly a short-cut. As in Ref. [136], the splitting of $\tilde{k}_1$ and of $\tilde{k}_2$ in four positive integers is not unique. Thus, we find different determinantal expressions.

We remark that we have only used the structure of the square roots of the Berezinians and no other property of superspaces. However, we may identify the terms $1/(\kappa_{a1} - \kappa_{b2})$ in Eqs. (15.27) and (15.28) with the Efetov–Wegner terms [91, 58, 61] which only appear in superspace. When calculating Eq. (16.1) with the supersymmetry method, such terms occur by a change of coordinates in superspace from Cartesian coordinates to eigenvalue–angle coordinates [101].

The second term in Eqs. (15.27) and (15.28), also contained in Eqs. (15.26) and (15.29), are intimately connected to the well known sum over products of orthogonal polynomials. This is borne out in the presence of $\widetilde{\mathbf{M}}_N^{-1}$ which generates the bi-orthogonal polynomials [75]. Also, we might choose arbitrary polynomials in the square root of the Berezinian (14.5) instead of the powers κ_{b2}^{a-1}. If we take the orthogonal polynomials of the probability density P, then $\widetilde{\mathbf{M}}_N$ becomes diagonal and Eq. (15.26) is indeed the well known result. The k–point correlation function can be derived by the case $k_1 = k_2 = k$ and $l_1 = l_2 = 0$ with $\kappa = \mathrm{diag}\,(x_1 + L_1\imath\varepsilon - J_1, \ldots, x_k + L_k\imath\varepsilon - J_k, x_1 + L_1\imath\varepsilon + J_1, \ldots, x_k + L_k\imath\varepsilon + J_k)$, cf. Eq. (6.8). The Cauchy integrals (15.21) and (15.22) become integrals over Dirac distributions by summation over all terms with $L_j = \pm 1$ in the limit $\varepsilon \searrow 0$. Thus, we find the orthogonal polynomials, too. Due to the differentiation with respect to the J_j at zero the Efetov–Wegner terms vanish and the well known result [75] remains.

16.1.2 List of other matrix ensembles

As we have seen for the ensemble of Hermitian matrices, we find determinantal structures (15.35) and (15.36) with help of the general integral (15.17). Here, we collect a variety of different matrix ensembles. Those ensembles share not more than two features: (i) the probability density function factorizes in functions of the individual eigenvalues and (ii) the non-factorizing part in the integrand is the squared Vandermonde determinant. We emphasize that this list is not complete. One can certainly find other applications.

We introduce the decomposition into real and imaginary part, $z_j = x_j + \imath y_j$, and the polar coordinates $z_j = r_j e^{\imath\varphi_j}$. The probability density $g(z)$ in Eq. (15.17) for particular ensembles with unitary rotation symmetry is up to constants listed in table 16.1.

Since the unitary ensembles describe physical systems with broken time reversal symmetry, one is also interested in ensembles which have orthogonal and unitary-symplectic rotation symmetry. For most of such ensembles the average over ratios of characteristic polynomials cannot be transformed to one of the types of integrals discussed in Secs. 15.1 and 15.2. However for

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability density $g(z)$
Hermitian ensemble [141, 142, 108, 126, 87, 136]	$\widetilde{P}\left(\mathrm{tr}\, H^m, m \in \mathbb{N}\right)$ $H = H^\dagger$	H	$P(x)\delta(y)$
circular unitary ensemble (unitary group) [95, 143, 63] [144, 145, 65, 146, 66]	$\widetilde{P}\left(\mathrm{tr}\, U^m, m \in \mathbb{N}\right)$ $U^\dagger U = \mathbb{1}_N$	U and $U^\dagger$	$P\left(e^{\imath\varphi}\right)\delta(r-1)$
Hermitian chiral (complex Laguerre) ensemble [34, 147, 148, 149]	$\widetilde{P}\left(\mathrm{tr}\,(AA^\dagger)^m, m \in \mathbb{N}\right)$ A is a complex $N \times M$ matrix with $N \leq M$	$AA^\dagger$	$P(x)x^{M-N}\Theta(x)\delta(y)$
Gaussian elliptical ensemble [39, 150, 151, 152]; for $\tau = 1$ complex Ginibre ensemble	$\exp\left[-\frac{(\tau+1)}{2}\mathrm{tr}\, H^\dagger H\right]\times$ $\times\exp\left[-\frac{(\tau-1)}{2}\mathrm{Re}\ \mathrm{tr}\, H^2\right]$ H is a complex matrix; $\tau > 0$	H and $H^\dagger$	$\exp\left[-r^2\left(\sin^2\varphi + \tau\cos^2\varphi\right)\right]$
Gaussian complex chiral ensemble [153]	$\exp\left[-\mathrm{tr}\, A^\dagger A - \mathrm{tr}\, B^\dagger B\right]$ $C = \imath A + \mu B$ $D = \imath A^\dagger + \mu B^\dagger$ A and B are complex $N \times M$ matrices with $N \leq M$	CD and $D^\dagger C^\dagger$	$K_{M-N}\left(\frac{1+\mu^2}{2\mu^2}r\right)r^{M-N}\times$ $\times\exp\left(\frac{1-\mu^2}{2\mu^2}r\cos\varphi\right)$

Table 16.1: Particular cases of the probability densities $g(z)$ and their corresponding matrix ensembles of unitary rotation symmetry. The joint probability densities are equivalent to the single $g(z)$. K_{M-N} is the modified Bessel function.

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability density $g(z)$
real anti-symmetric matrices (Lie algebra of the orthogonal group)[75]	$\widetilde{P}\left(\operatorname{tr} H^m, m \in \mathbb{N}\right)$ $H = -H^T = H^*$ $N = 2L + \chi$ dimensional	H	$P(x)x^{\chi-1/2}\Theta(x)\delta(y)$
special orthogonal group [145, 65, 146]	$\widetilde{P}\left(\operatorname{tr} O^m, m \in \mathbb{N}\right)$ $O^T O = \mathbb{1}_{2L+\chi}$, $\det O = 1$	O	$\dfrac{P(x)}{\sqrt{1-x^2}}\delta(y)\,\lvert 1-x\rvert^{\chi}\times$ $\times\Theta(x-1)\Theta(1-x)$
anti-selfdual matrices (Lie algebra of the unitary–symplectic group)	$\widetilde{P}\left(\operatorname{tr} H^m, m \in \mathbb{N}\right)$ $H = \begin{bmatrix} 0 & \mathbb{1}_L \\ -\mathbb{1}_L & 0 \end{bmatrix} H^T \begin{bmatrix} 0 & \mathbb{1}_L \\ -\mathbb{1}_L & 0 \end{bmatrix}$	H	$P(x)x^{1/2}\Theta(x)\delta(y)$
unitary–symplectic group [145, 65, 146]	$\widetilde{P}\left(\operatorname{tr} S^m, m \in \mathbb{N}\right)$ $S^T \begin{bmatrix} 0 & \mathbb{1}_L \\ -\mathbb{1}_L & 0 \end{bmatrix} S = \begin{bmatrix} 0 & \mathbb{1}_L \\ -\mathbb{1}_L & 0 \end{bmatrix}$	S	$P(x)\sqrt{1-x^2}\delta(y)\times$ $\times\Theta(x-1)\Theta(1-x)$

Table 16.2: Particular cases of the probability densities $g(z)$ and their corresponding matrix ensembles of orthogonal and unitary–symplectic rotation symmetry. The joint probability densities are equivalent to the single $g(z)$. The variable χ is either zero or unity.

the special orthogonal group and the unitary-symplectic group and the Lie algebras thereof, they are integrals of the squared–Vandermonde type. They are listed in table 16.2.

We remark that the integrals which have to be performed are different for every single matrix ensemble and can be quite difficult to calculate. Nonetheless, all averages over ratios of characteristic polynomials have the determinantal structures (15.35) and (15.36). The entries of the matrix in the determinant are more or less averages of two characteristic polynomials only. Thus, we achieve a drastic reduction from averages over a large number of characteristic polynomial ratios to averages over two characteristic polynomials for a broad class of random matrix ensembles.

16.2 Applications for integrals of square root–Berezinian type

In subsection 16.2.1, we consider the Hermitian matrices again. We will show that the integral (16.1) can also be understood as an integral of square root–Berezinian type. In subsection 16.2.2, we shift this ensemble by an external field H_0 and transform the average in the usual way to an integral over a superspace. This integral is an integral of square root–Berezinian type.

As a particular example, we consider an intermediate ensemble from arbitrary unitarily invariant ensembles of Hermitian matrices to a rotation invariant ensemble of one of the symmetric spaces in subsection 16.2.3. We derive the k–point correlation function thereof. This generalizes known results [154, 155, 104, 140, 156, 157, 158, 125, 159]. For this example we use the supersymmetry method. Thereby, we demonstrate that our method works for calculations within superspace, too.

16.2.1 The Hermitian matrices revisited

The integral (16.1) in eigenvalue–angle coordinates is invariant under permutation of the eigenvalues of the matrix H. As in Sec. 13.1, we present one of the Vandermonde determinants as a product over powers of the eigenvalues,

$$Z_{\tilde{k}_1/\tilde{k}_2}^{(N)}(\kappa) = \prod_{j=1}^{N} \frac{(-\pi)^{j-1}}{(j-1)!} \int\limits_{\mathbb{R}^N} \prod_{j=1}^{N} P(E_j) E_j^{j-1} \frac{\prod\limits_{a=1}^{N}\prod\limits_{b=1}^{\tilde{k}_2}(E_a - \kappa_{b2})}{\prod\limits_{a=1}^{N}\prod\limits_{b=1}^{\tilde{k}_1}(E_a - \kappa_{b1})} \Delta_N(E) d[E] \,. \tag{16.5}$$

Using the same decomposition of $z_j = E_j + \imath y_j$ as in Sec. 16.1, we identify this integral with the integral (15.1) and find

$$Z^{(0/N)}_{\tilde{k}_1/\tilde{k}_2}(\kappa) = (-1)^{k_1 N} \prod_{j=1}^{N} \frac{(j-1)!}{(-\pi)^{j-1}} Z^{(N)}_{\tilde{k}_1/\tilde{k}_2}(\kappa) \tag{16.6}$$

with

$$f_j(z_j) = E_j^{j-1} P(E_j) \delta(y_j) \,. \tag{16.7}$$

The integral (16.1) is permutation invariant with respect to the bosonic and fermionic entries of κ. However, we do not see this symmetry in the expression found in subsection. 16.1.1 because we split κ into two parts. In the present section, we find a result which shows this symmetry from the beginning, cf. Eqs. (15.11) and (15.12).

16.2.2 The Hermitian matrix ensemble in an external field

Another calculation of integrals of the squared–Vandermonde type is not the only reason to consider integrals of the square root–Berezinian type. One of its powerful applications is the calculation of the k–point correlation functions of ensembles in the presence of an external field. We generalize the result for arbitrary unitarily invariant ensembles of Hermitian matrices in Ref. [61] to ensembles in an external field. Assuming $k \leq N$, we consider the integral

$$Z = \int\limits_{\mathrm{Herm}\,(2,N)} P(H) \prod_{j=1}^{k} \frac{\det(H + \alpha H_0 - \kappa_{j2} \mathbb{1}_N)}{\det(H + \alpha H_0 - \kappa_{j1} \mathbb{1}_N)} d[H] \,, \tag{16.8}$$

where P is an arbitrary rotation invariant ensemble and H_0 is an external field with a coupling constant α. For simplicity we set all imaginary parts of κ equal to $-\varepsilon$ which means $\kappa = \mathrm{diag}\,(x_1 - \imath\varepsilon - J_1, \ldots, x_k - \imath\varepsilon - J_k, x_1 - \imath\varepsilon + J_1, \ldots, x_k - \imath\varepsilon + J_k) = x^- + J$.

We use the generalized Hubbard–Stratonovich transformation [61], see also part II, to transform this integral to an integral over supermatrices. With help of this transformation we arrive at

$$\begin{aligned} Z &= 2^{2k(k-1)} \int\limits_{\Sigma^{\psi}_{2,k}} \int\limits_{\Sigma^{-\psi}_{2,k}} \Phi_0(\rho) \mathrm{Sdet}^{\,-1}(\sigma^+ \otimes \mathbb{1}_N + \alpha \mathbb{1}_{k+k} \otimes H_0) \\ & \quad \times \exp\left[-\imath \mathrm{Str}\, \rho(\sigma^+ + \kappa)\right] d[\sigma] d[\rho] \end{aligned} \tag{16.9}$$

with $\sigma^+ = \sigma + \imath\varepsilon 1\!\!1_{k+k}$. The superfunction Φ_0 is a rotation invariant supersymmetric extension of the characteristic function, see Secs. 7.1 and 7.5.

We diagonalize the matrices ρ and σ, $\rho = UrU^{-1}$ and $\sigma = VsV^{-1}$, and integrate over U and V which are in the supergroup $\mathrm{U}(k/k)$. Since we are interested in the k–point correlation function R_k of the shifted probability density $P(H - \alpha H_0)$, we omit the Efetov–Wegner terms occurring from this diagonalization because they yield lower order correlation functions. The supergroup integrals are supersymmetric versions of the Itzykson–Zuber integral [94, 105], see also Eq. (4.27). The integral (16.9) reads

$$\begin{aligned} Z = & \frac{1}{(2\pi\imath)^{2k}(k!)^4}\int\limits_{\mathbb{R}^{4k}} d[s]d[r]\frac{\Phi_0(r)\mathrm{Sdet}^{-1}(s^+\otimes 1\!\!1_N + \alpha 1\!\!1_{k+k}\otimes H_0)}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \\ & \times \sqrt{\mathrm{Ber}_{k/k}^{(2)}(s)}\det\left[\exp(-\imath r_{a1}s_{b1})\right]_{1\leq a,b\leq k}\det\left[\exp(\imath r_{a2}s_{b2})\right]_{1\leq a,b\leq k} \\ & \times \det\left[\exp(-\imath r_{a1}(x_b - J_b))\right]_{1\leq a,b\leq k}\det\left[\exp\left(\imath e^{\imath\psi}r_{a2}(x_b + J_b)\right)\right]_{1\leq a,b\leq k}. \end{aligned} \tag{16.10}$$

Using the permutation invariance within the bosonic and fermionic eigenvalues of r and s, we find

$$\begin{aligned} Z = & \frac{1}{(2\pi\imath)^{2k}}\int\limits_{\mathbb{R}^{2k}}\int\limits_{\mathbb{R}^{2k}} d[s]d[r]\frac{\Phi_0(r)\exp\left[-\imath\mathrm{Str}\, r\kappa\right]}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \\ & \times \prod_{a=1}^{k}\prod_{b=1}^{N}\frac{e^{-\imath\psi}s_{a2} + \imath\varepsilon + \alpha E_b^{(0)}}{s_{a1} + \imath\varepsilon + \alpha E_b^{(0)}}\exp\left[-\imath\mathrm{Str}\, rs^+\right]\sqrt{\mathrm{Ber}_{k/k}^{(2)}(s)}\,. \end{aligned} \tag{16.11}$$

The integration over s is exactly an integral of the square root–Berezinian type (15.1) with the parameters $N_1 = N_2 = k$, $k_1 = 0$ and $k_2 = N$. In App. C.6, we perform the Fourier transform and find

$$\begin{aligned} Z = & \frac{(-1)^{k(k-1)/2}}{(2\pi)^k}\int\limits_{\mathbb{R}_+^k}\int\limits_{\mathbb{R}^k} d[r]\frac{\Phi_0(r)\exp\left[-\imath\mathrm{Str}\, r\kappa\right]}{\Delta_N(\alpha E^{(0)})\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \\ & \times \det\left[\begin{array}{c|c} \dfrac{r_{a1}^N}{r_{a1} - e^{\imath\psi}r_{b2}}\left(-e^{-\imath\psi}\dfrac{\partial}{\partial r_{b2}}\right)^{N-1} & \imath\displaystyle\sum_{n=N}^{\infty}\frac{1}{n!}\left(\imath\alpha E_b^{(0)}r_{a1}\right)^n \\ \hline 2\pi\left(\dfrac{e^{-\imath\psi}}{\imath}\dfrac{\partial}{\partial r_{b2}}\right)^{a-1} & \left(-\alpha E_b^{(0)}\right)^{a-1} \end{array}\right]\delta(r_2), \end{aligned} \tag{16.12}$$

where the indices in the left upper block are $1 \leq a, b \leq k$ and in the right lower block they are $1 \leq a, b \leq N$. In the right upper and left lower block we have $(1 \leq a \leq k, 1 \leq b \leq N)$ and $(1 \leq a \leq N, 1 \leq b \leq 1)$, respectively.

We notice that the integration domain for the bosonic eigenvalues r_{a1} is the positive real axis whereas the integral for the fermionic eigenvalues is evaluated at zero.

Indeed, we obtain the known result [61], see also Sec. 9.2, for non–shifted arbitrary unitarily rotation invariant ensembles for $\alpha \to 0$. To show this, we put the $1/\alpha$ terms of the Vandermonde determinant $\Delta_N(\alpha E^{(0)})$ in the last N rows such that the lower right block is independent of α. The first N terms of the power series of exponential function in the upper right block are missing. Hence, an expansion in k columns yields that up to one term all other terms are at least of order α at the zero point. We find the limit

$$
\begin{aligned}
\lim_{\alpha\to 0} Z &= \frac{(-1)^{k(k-1)/2}}{(2\pi)^k} \int\limits_{\mathbb{R}_+^k} \int\limits_{\mathbb{R}^k} \frac{\Phi_0(r)\exp\left[-\imath \mathrm{Str}\, r\kappa\right]}{\sqrt{\mathrm{Ber}_{k/k}^{(2)}(\kappa)}} \\
&\times \det\left[\frac{r_{a1}^N}{r_{a1}-e^{\imath\psi}r_{b2}}\left(-e^{-\imath\psi}\frac{\partial}{\partial r_{b2}}\right)^{N-1}\right]_{1\le a,b\le k} \delta(r_2)d[r]\,, \quad (16.13)
\end{aligned}
$$

cf. Eq. (9.5).

By differentiating the source variables in Eq. (16.12) and setting them to zero, we obtain the modified k–point correlation function

$$
\begin{aligned}
&\widehat{R}_k(x^-) = \prod_{j=1}^{k}\left(\frac{1}{2}\frac{\partial}{\partial J_j}\right) Z\Bigg|_{J=0} \\
&= \frac{1}{(-2\pi)^k} \int\limits_{\mathbb{R}_+^k} \int\limits_{\mathbb{R}^k} d[r] \frac{\Phi_0(r)\exp\left[-\imath \mathrm{Str}\, rx^-\right]}{\Delta_N(\alpha E^{(0)})} \qquad (16.14) \\
&\times \det\left[\begin{array}{c|c} \dfrac{r_{a1}^N}{r_{a1}-e^{\imath\psi}r_{b2}}\left(-e^{-\imath\psi}\dfrac{\partial}{\partial r_{b2}}\right)^{N-1} & \imath\displaystyle\sum_{n=N}^{\infty}\frac{1}{n!}\left(\imath\alpha E_b^{(0)} r_{a1}\right)^n \\ \hline 2\pi\left(\dfrac{e^{-\imath\psi}}{\imath}\dfrac{\partial}{\partial r_{b2}}\right)^{a-1} & \left(-\alpha E_b^{(0)}\right)^{a-1} \end{array}\right]\delta(r_2).
\end{aligned}
$$

As discussed in Sec. 6, this correlation function is related to the k–point correlation function R_k over the flat Fourier transformation in x. Hence, we obtain

$$R_k(x) = \frac{\imath^k}{(2\pi)^{2k}} \int\limits_{\mathbb{R}^{2k}} d[r] \frac{\Phi_0(r) \exp[-\imath \mathrm{Str}\, rx]}{\Delta_N(\alpha E^{(0)})} \tag{16.15}$$

$$\times \quad \det \left[\begin{array}{c|c} \dfrac{r_{a1}^N}{r_{a1} - e^{\imath\psi} r_{b2}} \left(-e^{-\imath\psi} \dfrac{\partial}{\partial r_{b2}} \right)^{N-1} & \displaystyle\sum_{n=N}^{\infty} \frac{\imath}{n!} \left(\imath \alpha E_b^{(0)} r_{a1} \right)^n \\ \hline 2\pi \left(\dfrac{e^{-\imath\psi}}{\imath} \dfrac{\partial}{\partial r_{b2}} \right)^{a-1} & \left(-\alpha E_b^{(0)} \right)^{a-1} \end{array} \right] \delta(r_2).$$

In both Eqs. (16.14) and (16.15), the indices in the blocks of the determinants run over the same values as in Eq. (16.12). We emphasize that this result is exact for any rotation invariant probability density as long as this integral above is existent. It generalizes known results [104, 125] for norm-dependent ensembles.

16.2.3 Determinantal and Pfaffian structures for intermediate ensemble

In Eq. (16.15), we easily see that for factorizing characteristic function (7.6) and, thus, for factorizing superfunction (9.10) the k–point correlation function is a ratio of a $(k+N) \times (k+N)$ determinant and a $N \times N$ determinant

$$\begin{aligned} R_k(x) &= \frac{\imath^k}{(2\pi)^{2k} \Delta_N(\alpha E^{(0)})} \\ &\times \det \left[\begin{array}{cc} \left\{ \widetilde{R}_1(x_a, x_b) \right\}_{1 \le a,b \le k} & \left\{ \widetilde{R}_2(\alpha E_b^{(0)}, x_a) \right\}_{\substack{1 \le a \le k \\ 1 \le b \le N}} \\ \left\{ \widetilde{R}_{a3}(x_b) \right\}_{\substack{1 \le a \le N \\ 1 \le b \le k}} & \left\{ \left(-\alpha E_b^{(0)} \right)^{a-1} \right\}_{1 \le a,b \le N} \end{array} \right] . \end{aligned} \tag{16.16}$$

Here, the entries are

$$\begin{aligned} \widetilde{R}_1(x_a, x_b) &= \int\limits_{\mathbb{R}^2} \exp\left[-\imath (r_1 x_a - e^{\imath\psi} r_2 x_b) \right] \frac{\Phi(r) r_1^N}{r_1 - e^{\imath\psi} r_2} \\ &\times \left(-e^{-\imath\psi} \frac{\partial}{\partial r_2} \right)^{N-1} \delta(r_2) d[r] \,, \end{aligned} \tag{16.17}$$

$$\widetilde{R}_2(\alpha E_b^{(0)}, x_a) = \imath \int\limits_{\mathbb{R}} \Phi(r_1) \exp[-\imath r_1 x_a] \sum_{n=N}^{\infty} \frac{1}{n!} \left(\imath \alpha E_b^{(0)} r_1 \right)^n dr_1 \,, \tag{16.18}$$

$$\widetilde{R}_{a3}(x_b) = 2\pi \int\limits_{\mathbb{R}} \frac{1}{\Phi\left(e^{\imath\psi} r_2\right)} \exp\left[\imath e^{\imath\psi} r_2 x_b\right] \left(\frac{e^{-\imath\psi}}{\imath} \frac{\partial}{\partial r_2}\right)^{a-1} \delta(r_2) dr_2 \,. \tag{16.19}$$

By splitting off the lower right block from the determinant as in Eq. (13.8), we see that R_k is a $k \times k$ determinant in x which was also shown in Ref. [104]. However, the representation (16.16) is much better suited for further calculations than for the $k \times k$ determinant representation.

We can transform the characteristic function Φ_0 in Eqs. (16.17), (16.18) and (16.19) to probability densities in an ordinary space by the inverse procedure performed in subsection 16.2.2. We emphasize that these correlation functions can also be expressed in terms of mean values over ratios of characteristic polynomials. To illustrate this we explicitly work out Eqs. (16.17), (16.18) and (16.19) for Laguerre ensembles in subsection 16.2.4. For Gaussian ensembles we obtain the correct result [125].

Instead of taking H_0 as a constant external field, one can take it from a random matrix ensemble, too. For the Gaussian case this was discussed in Refs. [154, 155, 104, 140]. Here, we investigate H_0 as a real symmetric, a Hermitian and a Hermitian selfdual matrix with factorizing probability density $\widetilde{P}$. We remark that H_0 can also be drawn from a Wishart ensemble since it can be mapped to one of the symmetric ensembles.

Assuming that the characteristic function of P factorizes as well, the k–point correlation function is

$$R_k^{(2)}(x) = \frac{(-1)^{N(N-1)/2} \imath^k}{(2\pi)^{2k}} \det\left[K^{(2)}(x_a, x_b)\right]_{1 \leq a,b \leq k} \tag{16.20}$$

for a second ensemble of the Hermitian matrices. We define the kernel

$$\begin{aligned} K^{(2)}(x_a, x_b) &= \widetilde{R}_1(x_a, x_b) \qquad\qquad (16.21) \\ &- \sum_{m,n=1}^{N} \int\limits_{\mathbb{R}} \widetilde{P}(E) \widetilde{R}_2(\alpha E, x_a) \left(-E\right)^{m-1} dE \left(M^{(2)\,-1}\right)_{mn} \frac{\widetilde{R}_{n3}(x_b)}{\alpha^{n-1}} \end{aligned}$$

and the moment matrix

$$M_{mn}^{(2)} = \int\limits_{\mathbb{R}} \widetilde{P}(E)(-E)^{m+n-2} dE \tag{16.22}$$

for the probability density $\widetilde{P}$.

For quaternionic H_0, $N = 2Q$, we apply a generalization of de Bruijn's integral theorem [129] which we derive in App. C.3.2. This yields the Pfaffian structure

$$R_k^{(4)}(x) = \frac{\imath^k}{(2\pi)^{2k}} \mathrm{Pf} \left[\begin{array}{cc} K_1^{(4)}(x_a, x_b) & K_2^{(4)}(x_a, x_b) \\ -K_2^{(4)}(x_b, x_a) & K_3^{(4)}(x_a, x_b) \end{array} \right]_{1 \leq a,b \leq k} , \qquad (16.23)$$

where the sign of the Pfaffian determinant is defined as in Eq. (15.48). The kernels in the Pfaffian are given by

$$K_1^{(4)}(x_a, x_b) = \sum_{m,n=1}^{2Q} \frac{\widetilde{R}_{m3}(x_a)}{\alpha^{m-1}} \left(M^{(4)\,-1} \right)_{mn} \frac{\widetilde{R}_{n3}(x_b)}{\alpha^{n-1}} \,, \qquad (16.24)$$

$$K_2^{(4)}(x_a, x_b) = \widetilde{R}_1(x_b, x_a) + \sum_{m,n=1}^{2Q} \frac{\widetilde{R}_{m3}(x_a)}{\alpha^{m-1}} \left(M^{(4)\,-1} \right)_{mn} k_n^{(4)}(x_b) \,, \qquad (16.25)$$

$$K_3^{(4)}(x_a, x_b) = k^{(4)}(x_a, x_b) + \sum_{m,n=1}^{2Q} k_m^{(4)}(x_a) \left(M^{(4)\,-1} \right)_{mn} k_n^{(4)}(x_b) \,. \qquad (16.26)$$

The functions appearing in these definitions are

$$k^{(4)}(x_a, x_b) = \int\limits_{\mathbb{R}} \widetilde{P}(E) \det \left[\begin{array}{cc} \widetilde{R}_2(\alpha E, x_b) & \widetilde{R}_2(\alpha E, x_a) \\ \dfrac{\partial \widetilde{R}_2}{\partial E}(\alpha E, x_b) & \dfrac{\partial \widetilde{R}_2}{\partial E}(\alpha E, x_a) \end{array} \right] dE \,, \qquad (16.27)$$

$$k_n^{(4)}(x_b) = (-1)^n \int\limits_{\mathbb{R}} \widetilde{P}(E) \det \left[\begin{array}{cc} \widetilde{R}_2(\alpha E, x_b) & E^{n-1} \\ \dfrac{\partial \widetilde{R}_2}{\partial E}(\alpha E, x_b) & (n-1)E^{n-2} \end{array} \right] dE . \qquad (16.28)$$

In Eqs. (16.24), (16.25) and (16.26) the inverse of the skew–symmetric moment matrix

$$M_{ab}^{(4)} = (b-a) \int\limits_{\mathbb{R}} \widetilde{P}(E)(-E)^{a+b-3} dE \qquad (16.29)$$

arises which generates the skew orthogonal polynomials of quaternion type.

If H_0 stems from an ensemble of $(2Q+\chi) \times (2Q+\chi)$ real symmetric matrices, $\chi \in \{0, 1\}$, we obtain another Pfaffian

$$R_k^{(1)}(x) = \frac{(-1)^{N(N-1)/2} \imath^k}{(2\pi)^{2k}} \mathrm{Pf} \left[\begin{array}{cc} K_1^{(1)}(x_a, x_b) & K_2^{(1)}(x_a, x_b) \\ -K_2^{(1)}(x_b, x_a) & K_3^{(1)}(x_a, x_b) \end{array} \right]_{1 \leq a,b \leq k} . \qquad (16.30)$$

The entries are

$$K_1^{(1)}(x_a, x_b) = \sum_{m,n=1}^{2Q} \frac{\widetilde{R}_{m3}(x_a)}{\alpha^{m-1}} \left(M^{(1)\,-1}\right)_{mn} \frac{\widetilde{R}_{n3}(x_b)}{\alpha^{n-1}}\,, \tag{16.31}$$

$$K_2^{(1)}(x_a, x_b) = \widetilde{R}_1(x_b, x_a) + \sum_{m,n=1}^{2Q} \frac{\widetilde{R}_{m3}(x_a)}{\alpha^{m-1}} \left(M^{(1)\,-1}\right)_{mn} k_n^{(1)}(x_b)\,, \tag{16.32}$$

$$K_3^{(1)}(x_a, x_b) = k^{(1)}(x_a, x_b) + \sum_{m,n=1}^{2Q} k_m^{(1)}(x_a) \left(M^{(1)\,-1}\right)_{mn} k_n^{(1)}(x_b) \tag{16.33}$$

with the moment matrix

$$M_{mn}^{(1)} = \begin{cases} \int\limits_{-\infty<E_1<E_2<\infty} d[E]\widetilde{P}(E) & , 1 \leq m, n \leq 2Q + \chi \\ \quad \times \det \begin{bmatrix} (-E_1)^{b-1} & (-E_1)^{a-1} \\ (-E_2)^{b-1} & (-E_2)^{a-1} \end{bmatrix} & \\ -\int\limits_{\mathbb{R}} \widetilde{P}(E)(-E)^{m-1} dE & , \begin{cases} 1 \leq m \leq 2Q \\ n = 2Q + 2 \\ \chi = 1 \end{cases} \\ \int\limits_{\mathbb{R}} \widetilde{P}(E)(-E)^{n-1} dE & , \begin{cases} 1 \leq n \leq 2Q \\ m = 2Q + 2 \\ \chi = 1 \end{cases} \\ 0 & , \begin{cases} m = n = 2Q + 2 \\ \chi = 1 \end{cases} \end{cases} . \tag{16.34}$$

Here, the functions in Eqs. (16.32) and (16.33) are

$$k^{(1)}(x_a, x_b) = \int\limits_{-\infty<E_1<E_2<\infty} \widetilde{P}(E) \det \begin{bmatrix} \widetilde{R}_2(\alpha E_2, x_a) & \widetilde{R}_2(\alpha E_2, x_b) \\ \widetilde{R}_2(\alpha E_1, x_a) & \widetilde{R}_2(\alpha E_1, x_b) \end{bmatrix} d[E] \tag{16.35}$$

and

$$\begin{aligned} &\left[k_n^{(1)}(x_b)\right]_{1 \leq n \leq 2(Q+\chi)} \\ &= \begin{bmatrix} \left\{ \int\limits_{-\infty<E_1<E_2<\infty} \widetilde{P}(E) \det \begin{bmatrix} \widetilde{R}_2(\alpha E_2, x_b) & (-E_2)^{n-1} \\ \widetilde{R}_2(\alpha E_1, x_b) & (-E_1)^{n-1} \end{bmatrix} d[E] \right\}_{1 \leq n \leq 2Q+\chi} \\ -\int\limits_{\mathbb{R}} \widetilde{P}(E) \widetilde{R}_2(\alpha E, x_b) dE \end{bmatrix} . \end{aligned} \tag{16.36}$$

As in the other cases, the matrix $\{M_{mn}\}$ generates the skew orthogonal polynomials of real type with respect to $\widetilde{P}$. Pandey and Mehta [154] constructed these polynomials for the Gaussian measure. They also found a Pfaffian structure for the interpolation between GUE and GOE. In Ref. [104], one can implicitly recognize the determinantal and Pfaffian structure in the interpolation from an arbitrary Gaussian symmetric ensemble to GUE. Our results (16.20), (16.23) and (16.30) extend these determinantal and Pfaffian structures to intermediate ensembles between an arbitrary symmetric ensemble factorizing in the probability density and an arbitrary unitarily invariant ensemble factorizing in the characteristic function.

Moreover, we can omit the factorization of the unitarily rotation invariant ensemble and find an integral representation in the superspace for an interpolation of an arbitrary unitarily invariant ensemble to the other classes of rotation invariance. For this purpose we integrate over H_0 in Eq. (16.15) and find for the integral kernel a determinant or a Pfaffian determinant, depending on whether H_0 is Hermitian, Hermitian self-dual or real symmetric.

16.2.4 Laguerre ensembles in an external field

We consider the Laguerre ensemble

$$P_\nu(H)=\prod_{j=1}^{N}\left[\left(\frac{c}{\pi}\right)^{j-1}\frac{c^{\nu+1}}{\Gamma(\nu+j)}\right]\exp\left(-c\operatorname{tr}H\right)\det{}^{\nu}H\Theta(H)\,, \tag{16.37}$$

where $\nu,c\in\mathbb{R}^+$ are some constants and Θ is defined in Eq. (8.13). The characteristic function is

$$\mathcal{F}P_\nu(H)=c^{(N+\nu)N}\det{}^{-N-\nu}(c\mathbb{1}_N-\imath H) \tag{16.38}$$

and the supersymmetric extension is, hence,

$$\Phi_\nu(\rho)=c^{(N+\nu)(k_1-k_2)}\mathrm{Sdet}\,^{-N-\nu}(c\mathbb{1}_{k_1+k_2}-\imath\rho)\,. \tag{16.39}$$

We notice that Φ factorizes, i.e. it fulfills the condition (9.10). Thus we can apply the calculations in subsection 16.2.3.

The function $\widetilde{R}_1(x_a,x_b)$, see Eq. (16.17), is up to the Efetov–Wegner term, which is the normalization in this case, the same as the generating function (16.8) for $k=1$ and $\alpha=0$,

$$\widetilde{R}_1(x_a,x_b)\sim\frac{1}{x_a-x_b} \tag{16.40}$$
$$\times\lim_{\varepsilon\searrow 0}\int\limits_{\mathrm{Herm}\,(2,N)}P_\nu(H)\left[\frac{\det(H-x_b\mathbb{1}_N)}{\det(H-(x_a-\imath\varepsilon)\mathbb{1}_N)}-\frac{\det(H-x_b\mathbb{1}_N)}{\det(H-(x_a+\imath\varepsilon)\mathbb{1}_N)}\right]d[H]\,.$$

Let $\pi_N^{(\nu)}$ the orthogonal polynomials of order N with respect to the probability density P_ν, i.e. $\pi_N^{(\nu)}(x) = x^N + \ldots$ are the associated Laguerre polynomials. Then, we find

$$\begin{aligned} \widetilde{R}_1(x_a, x_b) &= \frac{\pi(-\imath)^{N-1}2}{(N+\nu-1)!} \qquad (16.41) \\ \times \quad & \frac{\pi_N^{(\nu)}(cx_a)\pi_{N-1}^{(\nu)}(cx_b) - \pi_{N-1}^{(\nu)}(cx_a)\pi_N^{(\nu)}(cx_b)}{x_a - x_b}(cx_a)^\nu \exp(-cx_a)\Theta(x_a)\,, \end{aligned}$$

which is indeed the determinantal kernel for the case $\alpha = 0$.

For calculating the second function $\widetilde{R}_2(\alpha E_b^{(0)}, x_a)$, see Eq. (16.18), we consider the integral

$$\mathcal{I}_n(\alpha E_b^{(0)}, x_a) = \imath \int\limits_{\mathbb{R}} \Phi_\nu(r_1) \exp\left[-\imath r_1 x_a\right] \frac{\left(\imath\alpha E_b^{(0)} r_1\right)^n}{n!} dr_1 \,. \qquad (16.42)$$

It has a structure similar to Eq. (16.40),

$$\begin{aligned} \mathcal{I}_n(\alpha E_b^{(0)}, x_a) &\sim \left(\alpha E_b^{(0)}\right)^n \lim_{\varepsilon \searrow 0} \int\limits_{\mathrm{Herm}\,(2,n+1)} d[H] P_{N+\nu-1-n}(H) \\ \times \quad & \left[\frac{1}{\det(H - (x_a - \imath\varepsilon)\mathbb{1}_{n+1})} - \frac{1}{\det(H - (x_a + \imath\varepsilon)\mathbb{1}_{n+1})}\right] . \qquad (16.43) \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathcal{I}_n(\alpha E_b^{(0)}, x_a) &= \frac{2\pi\imath c}{(N+\nu-1)!n!} \qquad (16.44) \\ \times \quad & \left(\alpha E_b^{(0)} c\right)^n \pi_{n+1}^{(N+\nu-1-n)}(cx_a)(cx_a)^{N+\nu-1-n} \exp(-cx_a)\Theta(x_a)\,, \end{aligned}$$

which also follows by directly integrating Eq. (16.42). We combine this result with the definition (16.18) and find

$$\begin{aligned} \widetilde{R}_2(\alpha E_b^{(0)}, x_a) &= \frac{2\pi\imath c}{(N+\nu-1)!} \left[\pi_1^{(N+\nu-1)}(c[x_a - \alpha E_b^{(0)}])(c[x_a - \alpha E_b^{(0)}])^{N+\nu-1}\right. \\ \times \quad & \exp(-c[x_a - \alpha E_b^{(0)}])\Theta([x_a - \alpha E_b^{(0)}]) - \sum_{n=0}^{N} \frac{1}{n!}\left(\alpha E_b^{(0)} c\right)^n \\ \times \quad & \left.\pi_{n+1}^{(N+\nu-1-n)}(cx_a)(cx_a)^{N+\nu-1-n} \exp(-cx_a)\Theta(x_a)\right] . \qquad (16.45) \end{aligned}$$

We notice that the first term of $\widetilde{R}_2(\alpha E_b^{(0)}, x_a)$ vanishes if x_a is smaller than $\alpha E_b^{(0)}$.

Also for the function $\widetilde{R}_{a3}(x_b)$, see Eq. (16.19), we find an expression of a form similar to Eq. (16.40) and Eq. (16.43),

$$\widetilde{R}_{a3}(x_b) \sim \int\limits_{\mathrm{Herm}\,(a-1)} P_{N+\nu+1-a}(H)\det(H - x_b \mathbb{1}_{a-1}) d[H] \,. \tag{16.46}$$

We easily see that this is

$$\widetilde{R}_{a3}(x_b) = 2\pi c^{1-a} \pi_{a-1}^{(N+\nu+1-a)}(cx_b) \,. \tag{16.47}$$

This result can also be obtained by performing the integration of Eq. (16.19) directly.

We emphasize that our result of the Laguerre ensemble in the presence of an external source is different from those in Refs. [160, 161] since the coupling is different.

16.3 Applications for integrals of coupled square–root Vandermonde type

In subsection. 16.3.1, we apply the general results to two ensembles of real symmetric matrices and Hermitian self-dual matrices. We give an overview of applications for ensembles which are rotation invariant under the orthogonal and unitary-symplectic group in subsection. 16.3.2.

16.3.1 Rotation invariant ensembles of real symmetric matrices and Hermitian self-dual matrices

We consider mean values of characteristic polynomials for a rotation invariant probability density P over the real symmetric matrices $\mathrm{Herm}\,(1, N)$ or the Hermitian self-adjoint matrices $\mathrm{Herm}\,(4, N)$,

$$Z_{(k_1/k_2)}^{(N,\beta)}(\kappa) = \int\limits_{\mathrm{Herm}\,(\beta,N)} P(H) \frac{\prod\limits_{j=1}^{k_2} \det(H - \kappa_{j2} \mathbb{1}_{\gamma N})}{\prod\limits_{j=1}^{k_1} \det(H - \kappa_{j1} \mathbb{1}_{\gamma N})} d[H] \,. \tag{16.48}$$

The constant γ equals one for the real case and two for the quaternionic case. For the quaternionic case, the diagonalization of H leads to the identification

$$\tilde{g}(z_1, z_2) = P(E_1)\delta(y_1)\delta(y_2) \frac{\delta(E_1 - E_2)}{E_1 - E_2} \,, \tag{16.49}$$

c.f. Eq. (15.38), and

$$Z_{(k_1/k_2)}^{(2N)}(\kappa) = (-1)^{N(N-1)/2}\frac{1}{N!}\prod_{j=1}^{N}\frac{\pi^{2(j-1)}}{\Gamma(2j)}Z_{(k_1/k_2)}^{(N,4)}(\kappa)\,. \tag{16.50}$$

Let $N = 2Q + \chi$ with $\chi \in \{0, 1\}$. The diagonalization in the real case leads to a product of Heavyside distributions $\Theta(E_{j+1} - E_j)$, $j \in \{1, \ldots, N-1\}$, which is equivalent to the ordering of the eigenvalues $E_1 \leq E_2 \leq \ldots \leq E_N$. Let $z_j = E_j + \imath y_j$. We split the product of Heavyside distributions in two products

$$\prod_{j=1}^{N-1}\Theta(E_{j+1} - E_j) = \prod_{j=1}^{Q+\chi-1}\Theta(E_{2j+1} - E_{2j})\prod_{j=1}^{Q}\Theta(E_{2j} - E_{2j-1})\,. \tag{16.51}$$

Putting the second product of Eq. (16.51) to the probability density, we define the probability densities

$$g(z_1, z_2) = \tilde{g}(z_1, z_2) = P(E_1)P(E_2)\delta(y_1)\delta(y_2)\Theta(E_2 - E_1) \tag{16.52}$$

and

$$h(z) = P(E)\delta(y)\,, \tag{16.53}$$

according to even and odd N. Due to the integration method over alternate variables [114], the identification is

$$Z_{(k_1/k_2)}^{(2L+\chi)}(\kappa) = (-1)^{\chi k_1}\frac{1}{Q!}\prod_{j=1}^{2Q+\chi}\frac{\pi^{(j-1)/2}}{\Gamma(j/2)}Z_{(k_1/k_2)}^{(2Q+\chi,1)}(\kappa)\,. \tag{16.54}$$

The Pfaffian structure of the results (15.53) for both examples are well known [136]. Let $k_2 - k_1$ be even and $d = k_2 - k_1 + \gamma N \geq 0$. The moment matrices

$$M_{(d)}^{(1)} = \left[\int\limits_{-\infty\leq E_1\leq E_2\leq\infty} P(E_1)P(E_2)(E_1^{a-1}E_2^{b-1} - E_1^{b-1}E_2^{a-1})dE_1dE_2\right]_{1\leq a,b\leq d} \tag{16.55}$$

for the real case with even d,

$$\widetilde{M}_{(d)}^{(1)} = \left[\begin{array}{cc} M_{(d)}^{(1)} & \left\{-\int\limits_{\mathbb{R}} P(E)E^{a-1}\right\}_{1\leq a\leq d} \\ \left\{\int\limits_{\mathbb{R}} P(E)E^{b-1}\right\}_{1\leq b\leq d} & 0 \end{array}\right] \tag{16.56}$$

for the real case with odd d and

$$M_{(d)}^{(4)} = \left[(a-b)\int\limits_{\mathbb{R}} P(E)E^{a+b-3}dE\right]_{1\leq a,b\leq d} \tag{16.57}$$

for the quaternionic case generate the skew orthogonal polynomials, corresponding to the symmetry. Considering the structure of the Berezinian, this shows an intimate connection between the method of orthogonal polynomials and the supersymmetry method.

A new result is the Pfaffian structure of the sparsely occupied matrix (15.56) if $d \leq 0$. The row and the column with $Z_{(1/0)}^{(1)}$ only appears for odd dimensional, real symmetric matrices. This factor is the Cauchy–transform of the probability density itself. The function $\mathbf{F}$ is almost the mean value of the two characteristic polynomials in the denominator which has to be calculated, too. However, the N eigenvalue integrals are drastically reduced to one or two dimensional integrals. Even with help of the supersymmetry method one could not reduce the number of integrals in such an impressive way.

16.3.2 A list of other matrix ensembles

We average ratios of characteristic polynomials similar to the type (16.48) where the integration domains are matrix sets different from the symmetric spaces. Those matrix sets have to be rotation invariant either under the orthogonal group or under the unitary symplectic group. For both symmetries we give a list of ensembles to which the integrals (15.37) or (15.38) are applicable. We use a decomposition in real and imaginary part, $z_j = x_j + \imath y_j$, and eigenvalue–angle coordinates $z_j = r_j e^{\imath\varphi_j}$. Then, the probability densities in Eqs. (15.37) and (15.38) are equivalent to the probability densities in Eq. (16.48) after suitable changes of variables. The ensembles with orthogonal symmetry are given in table 16.3 and those with unitary-symplectic symmetry are listed in table 16.4. Since two matrix models over the real numbers are currently investigated, we show two examples in table 16.5 for which our approach works as well.

The two-dimensional complex Dirac distribution used in table 16.3 is defined by

$$\delta^2(z_1 - z_2^*) = \delta(x_2 - x_1)\delta(y_2 + y_1) = \frac{1}{r_1}\delta(r_1 - r_2)\delta(\varphi_1 + \varphi_2)\,. \tag{16.58}$$

We use the short hand notation

$$\eta_\pm = \frac{1 \pm \mu^2}{4\mu^2}\,, \tag{16.59}$$

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability densities $g(z_1,z_2)$ and $\tilde{g}(z_1,z_2)$	probability density $h(z)$
real symmetric matrices [162, 87, 136]	$\widetilde{P}\left(\operatorname{tr} H^m, m\in\mathbb{N}\right)$ $H=H^T=H^*$	H	$P(x_1)P(x_2)$ $\times\delta(y_1)\delta(y_2)$ $\times\Theta(x_2-x_1)$	$P(x)\delta(y)$
circular orthogonal ensemble [63]	$\widetilde{P}\left(\operatorname{tr} U^m, m\in\mathbb{N}\right)$ $U^\dagger U=\mathbb{1}_N$ and $U^T=U$	U and $U^\dagger$	$P(e^{\imath\varphi_1})P(e^{\imath\varphi_2})$ $\times\delta(r_1-1)\delta(r_2-1)$ $\times\Theta(\varphi_2-\varphi_1)$	$P(e^{\imath\varphi})\delta(r-1)$
real symmetric chiral (real Laguerre) ensemble [163, 164, 35, 147]	$\widetilde{P}\left(\operatorname{tr}(AA^T)^m, m\in\mathbb{N}\right)$ A is a real $N\times M$ matrix with $\nu=M-N\geq 0$	AA^T	$P(x_1)P(x_2)$ $\times(x_1x_2)^{(\nu-1)/2}$ $\times\delta(y_1)\delta(y_2)$ $\times\Theta(x_2-x_1)$	$P(x)\delta(y)x^{(\nu-1)/2}$
Gaussian real elliptical ensemble; for $\tau=1$ real Ginibre ensemble [137, 138, 83] [165, 166, 167] [168, 169, 170]	$\exp\left[-\dfrac{(\tau+1)}{2}\operatorname{tr} H^TH\right]$ $\times\exp\left[-\dfrac{(\tau-1)}{2}\operatorname{tr} H^2\right]$ $H=H^*$; $\tau>0$	H	$\prod\limits_{j\in\{1,2\}}\exp\left[-\tau x_j^2\right]$ $\times\sqrt{\operatorname{erfc}(\sqrt{2(1+\tau)}y_j)}$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2-x_1)$ $+2\imath\delta^2(z_1-z_2^*)\Theta(y_1)]$	$\exp(-\tau x^2)\delta(y)$

Table 16.3: Particular cases of the probability densities $g(z_1,z_2)$ and $h(z)$ and their corresponding matrix ensembles of orthogonal rotation symmetry. The joint probability density is equivalent to $g(z_1,z_2)$ and $h(z)$. The density $h(z)$ only appears for odd-dimensional matrices.

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability density $\tilde{g}(z_1, z_2)$
Hermitian, self-dual matrices [87, 136]	$\widetilde{P}\,(\mathrm{tr}\, H^m, m \in \mathbb{N})$ $H = H^\dagger$	H	$P(x_1)\delta(y_1)\delta(y_2)\dfrac{\delta(x_2 - x_1)}{x_1 - x_2}$
circular unitary-symplectic ensemble [63]	$\widetilde{P}\,(\mathrm{tr}\, U^m, m \in \mathbb{N})$ $U^\dagger U = \mathbb{1}_N$	U and $U^\dagger$	$P\left(e^{\imath\varphi_1}\right)\delta(r_1 - 1)$ $\times\delta(r_2 - 1)\dfrac{\delta(\varphi_2 - \varphi_1)}{\sin(\varphi_1 - \varphi_2)}$
Hermitian self-dual chiral (quaternionic Laguerre) ensemble [163, 164, 35, 147]	$\widetilde{P}\left(\mathrm{tr}\,(AA^\dagger)^m, m \in \mathbb{N}\right)$ A is a quaternionic $N \times M$ matrix with $N \leq M$	$AA^\dagger$	$P(x_1)x_1^{M-N+1}$ $\times\delta(y_1)\delta(y_2)\dfrac{\delta(x_2 - x_1)}{x_1 - x_2}$
Gaussian quaternionic elliptical ensemble; $\tau = 1$ for quaternionic Ginibre ensemble [171, 172]	$\exp\left[-\dfrac{(\tau+1)}{2}\mathrm{tr}\, H^T H\right] \times$ $\times \exp\left[-\dfrac{(\tau-1)}{2}\mathrm{tr}\, H^2\right]$ H is a quaternionic matrix	H	$\exp\left[-2r_1^2(\sin^2\varphi_1\right.$ $\left.+\tau\cos^2\varphi_1)\right] r_1 \sin(2\varphi_1)$ $\times\delta(r_1 - r_2)\delta(\varphi_1 + \varphi_2)$
Gaussian quaternionic chiral ensemble [173]	$\exp\left[-\mathrm{tr}\, A^\dagger A - \mathrm{tr}\, B^\dagger B\right]$ $C = \imath A + \mu B$ $D = \imath A^\dagger + \mu B^\dagger$ A and B are quaternionic $N \times M$ matrices with $\nu = M - N \geq 0$	CD	$K_{2\nu}\left(2\eta_+ r_1\right) r_1^{2\nu}$ $\times\exp\left[2\eta_- r_1 \cos\varphi_1\right]$ $\times r_1 \sin\varphi_1$ $\times\delta(r_1 - r_2)\delta(\varphi_1 + \varphi_2)$

Table 16.4: Particular cases of the probability densities $\tilde{g}(z_1, z_2)$ and their corresponding matrix ensembles unitary–symplectic rotation symmetry. The joint probability density is equivalent to $\tilde{g}(z_1, z_2)$. All matrices have quaternion structure and, thus, they are even-dimensional.

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability densities $g(z_1,z_2)$ and $\tilde{g}(z_1,z_2)$	probability density $h(z)$
Gaussian real chiral ensemble [139, 41]	$\exp\left[-\mathrm{tr}\,(A^TA+B^TB)\right]$ $C=A+\mu B$ $D=-A^T+\mu B^T$ A and B are real $N\times M$ matrices with $\nu=M-N\geq 0$	CD	$\prod\limits_{j\in\{1,2\}}\exp\left[-2\eta_- z_j\right]$ $\times\lvert z_j\rvert^{\nu}\sqrt{f(2\eta_+ z_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2-x_1)$ $+2\imath\delta^2(z_1-z_2^*)\Theta(y_1)]$	$\exp\left[-2\eta_- x\right]$ $\times K_{\nu/2}(2\eta_+ x)$ $\times x^{\nu/2}\delta(y)$
generalized eigenvalues for a pair of real matices [174]	$\exp\left[-\mathrm{tr}\,(A^TA+B^TB)\right]$ A and B are real $N\times N$ matrices	AB^{-1}	$\tau(z_1)\tau(z_2)$ $\times[\delta(r_1-1)\delta(r_2-1)$ $\times\Theta(\phi_2-\phi_1)$ $+\dfrac{2\imath}{\lvert z_1\rvert^2}\Theta(1-\lvert z_1\rvert)$ $\times\delta^2\left(z_1-\dfrac{1}{z_2^*}\right)]$	$\tau(\phi)\delta(r-1)$

Table 16.5: Particular cases of the probability densities $g(z_1, z_2)$ and $h(z)$ and their corresponding two matrix models of orthogonal rotation symmetry. The joint probability density is equivalent to $g(z_1, z_2)$ and $h(z)$. The density $h(z)$ only appears for odd-dimensional matrices.

c.f. Ref. [41]. The functions erfc and K_ν are the complementary error–function and the K–Bessel function of order ν, respectively. The function f is calculated in Ref. [41] and given by

$$f(x+\imath y) = 2\int\limits_0^\infty \exp\left[-2t(x^2-y^2)-\frac{1}{4t}\right] K_{\nu/2}(2t(x^2+y^2))\mathrm{erfc}(2\sqrt{t}|y|)\frac{dt}{t}\,. \tag{16.60}$$

For the two matrix model with the combination AB^{-1} we need the function

$$\tau(z) = \left(\frac{1}{x}\right)^{N-1/2}\left[\frac{1}{\sqrt{\pi}}\int\limits_{(|z|^{-1}-|z|)/2}^{\infty}\frac{1}{(1+t^2)^{N/2+1}}dt\right]^{1/2}\,. \tag{16.61}$$

which is derived in Ref. [174].

We notice that the Pfaffian structure appearing for all those ensembles is fundamental. Particularly, the obvious difference between ensembles with orthogonal symmetry and those with unitary-symplectic symmetry vanishes in our derivation. The Pfaffian structure is exclusively due to the starting points (15.37) and (15.38). Furthermore, we expect that the list of those ensembles given here is not complete and can certainly be extended.

Chapter 17

Summary of part III

We presented a new method to calculate mean values for ratios of characteristic polynomials in a wide class of matrix ensembles with unitary, orthogonal and unitary symplectic symmetry and factorizing probability density, cf. tabels 16.1-16.5. Our approach is based on determinantal structures of Berezinians with arbitrary dimensions resulting from diagonalization of Hermitian supermarices. Although we did not map ordinary matrix ensembles into superspace, we managed to reconstruct those Berezinians in the product of the characteristic polynomials with powers of the Vandermonde determinant. Using these determinantal structures, we obtained determinants and Pfaffian determinants whose entries are given in terms of the inverse of the moment matrix for the particular ensemble. These matrices are connected to the orthogonal and skew orthogonal polynomials and show that the known results from the orthogonal polynomial method are obtained.

Results obtained with our approach coincide with known results for specific matrix ensembles [175, 142, 152, 133, 75, 87, 173, 136]. In particular, we re-derived the results of Borodin and Strahov [136] for the ensembles of the symmetric spaces in a more direct way.

Remarkably, the Pfaffian structure appearing for ensembles with real as well as with quaternionic structures emerges from the same type of integral. Thus, there is no difference between both symmetries when calculating the eigenvalue statistics.

For the case of a large number of characteristic polynomials in the denominator, the kernels in the determinants as well as in the Pfaffian determinants reduce to one and two dimensional integrals. These integrals are the mean value of one or two characteristic polynomials in the denominator over one or two dimensional matrices, respectively. Thus in this case, we have drastically reduced the number of integrals, even below the number that would result when mapping onto superspace [59, 61], cf. part II.

In this method, determinantal and Pfaffian structures stem from purely algebraic manipulations. This is the reason why our results are so general. No integration has to be performed. The determinantal and Pfaffian structures are already contained in the initial integrand.

Furthermore, we showed that the determinantal structure is stable when an external field is coupled to the random matrix. With help of the supersymmetry method, we derived this for arbitrary unitarily invariant Hermitian matrix ensembles in an external field. Our formula for the k–point correlation function is a generalization of recent results over arbitrary Hermitian matrix ensembles [61] and over norm–dependent ensembles in an external field [125]. Moreover, we considered an external field drawn from another symmetric ensemble. We calculated the k–point correlation function for an interpolation between an arbitrary Hermitian ensemble factorizing in the characteristic function and an arbitrary symmetric ensemble factorizing in the probability density. We found determinantal and Pfaffian structures, too. For Gaussian ensembles, this coincides with known results [154, 104]. We gave explicit results for the Laguerre ensembles coupled to an external field in a way which is different from couplings investigated in Ref. [160, 161].

Chapter 18

Conclusions and outlook

I extended the generalized Hubbard–Stratonovich transformation from rotation invariant Hermitian matrix ensembles [61] to matrix ensembles over the real symmetric and Hermitian self-dual matrices. In combination with the approaches developed in Chap. 4, I hope to generalize the results of the k–point correlation functions for unitary ensembles in the presence of an external field [154, 155, 104, 140, 125] to ensembles with orthogonal and unitary symplectic symmetry, see also subsections 16.2.2 and 16.2.3. As we have seen for the unitary case, one has to perform supergroup integrals. In particular, one has to calculate the supermatrix Bessel functions. These integrals are one of the main problems to solve for the orthogonal and unitary symplectic case. Therefore, it is helpful to know that they have the simple relation (4.35) to the ordinary matrix Bessel functions.

One open problem in the supersymmetry method is that there is still no solution to reproduce the Pfaffian structures for factorizing probability densities in the real and quaternionic case. Thus, it is important to understand how these structures appear in a natural way, as I did it in part III. To solve this problem one has also to understand what the relation between the original probability density P and the probability density in superspace $Q = \mathcal{F}\Phi$ is. The mapping from ordinary to superspace may destroy the factorization property. Therefore, it is useful to know how this happens. This can be studied with help of the relation between P and $Q = \mathcal{F}\Phi$ derived in Chap. 11.

The determinantal and Pfaffian structures derived in part III are not only of importance for computations since it drastically reduces the number of characteristic polynomials in the integrand. It has also physical meaning. These structures tell us that all eigenvalue correlations are completely determined by generating functions with one or two characteristic polynomials in the integrand. This property of factorizing probability densities carries over to the large N limit. Hence, such matrix ensembles cannot model more complicate spectra than those which are determined by the generating functions with only two characteristic polynomials in the integrand.

One important aspect to use the supersymmetry method in random matrix theory is the analysis of the large N limit [10]. Recently, the universality on the scale of the mean level spacing was proven for the generating functions for rotation invariant ensembles of Hermitian matrices by Mandt and Zirnbauer [11]. However, they have considered factorizing probability densities only. Since the supersymmetry method also works for non-factorizing probability densities, one can study the situation when universality breaks down. The explicit functional equation (11.22) seems to be promising to solve this problem.

To find universality proofs for other matrix ensembles, one has to generalize the supersymmetry method shown here. Currently, we have made such an attempt for an ordinary Wishart ensemble with broken rotation invariance on one side [176]. For the unitary group the supersymmetry method already exists [143] due to the color-flavor transformation. Of paramount importance seems to be the generalization of the supersymmetry method to matrix ensembles beyond the Cartan classification [36, 177]. For the most of these matrix ensembles the joint probability density is not derived yet. Hence, it is not guaranteed that their joint probability density belongs to one of the three types of integrals discussed in Chap. 15. In particular, this is a problem for the orthogonal polynomial method for which the factorization property is at the heart of its approach.

As it was mentioned in the previous chapter, the approach of part III has consequences reaching beyond the supersymmetry method. Recently, we could show with help of this approach that the orthogonality of the orthogonal and skew orthogonal polynomials is also a purely algebraical property [178]. It arises from the determinantal and Pfaffian structure of the mean values over ratios of characteristic polynomials. This allows to construct a compact formula for the orthogonal and skew orthogonal polynomials of arbitrary factorizing probability densities, cf. Eqs. (15.17), (15.37) and (15.38).

Appendix A

Derivations for Part I

In App. A.1, we prove theorem 3.1.1. A proof of lemma 3.1.2 is given in App. A.2. The Cauchy–like integration theorems 3.2.1, 3.3.1 and 3.3.2 are proven in Apps. A.3, A.4 and A.5, respectively.

A.1 Proof of theorem 3.1.1

One can project f onto all $f_{j_1 j_2}$ with a projector $P_{j_1 j_2}$ in the following way. The operator identity

$$1 = \frac{\partial}{\partial \eta_n}\eta_n + \eta_n \frac{\partial}{\partial \eta_n} \tag{A.1}$$

holds for every Grassmann generator η_n and also for its complex conjugate η_n^*. The first term on the right hand side projects onto the function $f(x, \eta_1, \ldots, \eta_{n-1}, 0, \eta_{n+1}, \ldots)$ and the second term projects, up to a sign, onto $\eta_n df/dz(x, \eta_1, \ldots, \eta_{n-1}, z, \eta_{n+1}, \ldots)|_{z=0}$. Thus, the generalization of Eq. (A.1) to all generators is

$$\begin{aligned} 1 &= \prod_{n=1}^{L}\left(\frac{\partial}{\partial \eta_n}\eta_n + \eta_n \frac{\partial}{\partial \eta_n}\right)\left(\frac{\partial}{\partial \eta_n^*}\eta_n^* + \eta_n^* \frac{\partial}{\partial \eta_n^*}\right) \\ &= \sum_{j_1, j_2 \in \mathbb{I}} (-1)^{J(j_1, j_2)} \left(\prod_{n=1}^{L} (\eta_n^*)^{j_{1n}} \eta_n^{j_{2n}}\right)\left(\prod_{n=1}^{L} \frac{\partial^2}{\partial \eta_n \partial \eta_n^*}\right)\left(\prod_{n=1}^{L} (\eta_n^*)^{1-j_{1n}} \eta_n^{1-j_{2n}}\right), \end{aligned} \tag{A.2}$$

where

$$J(j_1, j_2) = \sum_{n=1}^{L} (1 - j_{1n}) j_{2n} + \sum_{1 \le n < m \le L} (j_{1n} + j_{2n})(j_{1m} + j_{2m}) \, . \tag{A.3}$$

The action of this operator onto the left hand side of Eq. (2.29) gives an expression for f which can be compared term by term with the right hand side of Eq. (2.29). This yields

$$f_{j_1 j_2}(x) = (-1)^{J(j_1,j_2)} \left(\prod_{n=1}^{L} \frac{\partial^2}{\partial\eta_n \partial\eta_n^*}\right) \left(\prod_{n=1}^{L} (\eta_n^*)^{1-j_{1n}} \eta_n^{1-j_{2n}}\right) f(x,\eta)\,. \quad \text{(A.4)}$$

We define the projector onto the body $f_{0,0}$ of such a superfunction

$$P_{0,0} = \left(\prod_{n=1}^{L} \frac{\partial^2}{\partial\eta_n \partial\eta_n^*}\right) \left(\prod_{n=1}^{L} \eta_n^* \eta_n\right)\,. \quad \text{(A.5)}$$

An application of $P_{0,0}$ on $f_{j_1 j_2}(x)$ is the identity because $f_{j_1 j_2}$ does not depend on any Grassmann variable. However, the action of $P_{1,\ldots,1}$ onto $f_{j_1 j_2}$ is zero. When we analyze the projector $P_{1,\ldots,1}$, i.e. the first part on the left hand side of Eq. (A.5), we must take into account the commutating variables. We obtain

$$\begin{aligned} P_{1,\ldots,1} = \prod_{n=1}^{L} \frac{\partial^2}{\partial\eta_n \partial\eta_n^*} &= \frac{(-2)^{-L}}{L!} \prod_{m=1}^{L} h_m \left(2\sum_{n=1}^{L} \frac{1}{h_n} \frac{\partial^2}{\partial\eta_n^* \partial\eta_n}\right)^L \\ &= \frac{2^{-L}}{L!} \prod_{m=1}^{L} h_m \left(\Delta_C - \Delta_S\right)^L \\ &= \frac{2^{-L}}{L!} \prod_{m=1}^{L} h_m \sum_{n=0}^{L} \binom{L}{n} \Delta_C^{L-n} \left(-\Delta_S\right)^n\,. \end{aligned} \quad \text{(A.6)}$$

We plug Eq. (A.6) into Eq. (A.4) and act with the operator $P_{0,0}$ from the left. We use that $P_{0,0}$ commutes with Δ_C and get

$$\begin{aligned} f_{j_1 j_2}(x) &= \frac{(-1)^{J(j_1,j_2)} 2^{-L}}{L!} \prod_{m=1}^{L} h_m \\ &\times \sum_{n=0}^{L} \binom{L}{n} \Delta_C^{L-n} P_{0,0} \left(-\Delta_S\right)^n \left(\prod_{n=1}^{L} (\eta_n^*)^{1-j_{1n}} \eta_n^{1-j_{2n}}\right) f(x,\eta)\,. \end{aligned} \quad \text{(A.7)}$$

We now choose the metric g such that the Laplacian Δ_S can be written in radial and angular coordinates. Then it splits into a sum of two differential operators $\Delta_{S,r} + \Delta_{S,\varphi}$. The radial part $\Delta_{S,r}$ only contains radial coordinates and partial derivatives thereof. The angular part $\Delta_{S,\varphi}$ only depends on partial derivatives with respect to the angular coordinates. The action of $\Delta_{S,\varphi}$ onto an invariant function is zero. The radial part fulfills

$$P_{0,0} \Delta_{S,r}(r(x,\eta)) = \Delta_{S,r}(r(x,0)) P_{0,0} = \Delta_{S,r}(r(x)) P_{0,0}\,. \quad \text{(A.8)}$$

We summarize these results and apply them onto the integral over an invariant function.

$$
\begin{aligned}
\int\limits_{\Lambda_{2L}} f(x,\eta)d[\eta] =& \frac{1}{(2\pi)^L} f_{1,\ldots,1}(x) \\
\overset{(A.7)}{=}& \frac{1}{L!(4\pi)^L} \prod_{m=1}^{L} h_m \sum_{n=0}^{L} \binom{L}{n} \Delta_C^{L-n} P_{0,0} \left[-\Delta_S(r(x,\eta))\right]^n f(x,\eta) \\
=& \frac{1}{L!(4\pi)^L} \prod_{m=1}^{L} h_m \sum_{n=0}^{L} \binom{L}{n} \Delta_C^{L-n} P_{0,0} \left[-\Delta_{S,r}(r(x,\eta))\right]^n f(r(x,\eta)) \\
\overset{(A.8)}{=}& \frac{1}{L!(4\pi)^L} \prod_{m=1}^{L} h_m \sum_{n=0}^{L} \binom{L}{n} \Delta_C^{L-n} \left[-\Delta_{S,r}(r(x))\right]^n f(r(x)) \\
=& \left. D_{C,S}(r) f(r)\right|_{r=r(x)} .
\end{aligned}
\tag{A.9}
$$

This is the proposed result.

A.2 Proof of lemma 3.1.2

Consider two non-commuting finite dimensional, real matrices A and B, then

$$
\sum_{n=0}^{L} \binom{L}{n} A^{L-n}(B-A)^n = \frac{d^L}{dt^L} \left. \left(e^{At}e^{(B-A)t}\right)\right|_{t=0} . \tag{A.10}
$$

Using $\phi(s) = e^{Ats}e^{(B-A)ts}$ we obtain

$$
\frac{d}{ds}\phi(s) = \left([A,\phi(s)]_- + \phi(s)B\right)t \quad \text{and} \quad \phi(0) = 1 . \tag{A.11}
$$

Consequently, we find

$$
\phi(s) = e^{st\ \mathrm{IAd}[A,B]}(\mathbf{1}) \tag{A.12}
$$

and arrive at

$$
\sum_{n=0}^{L} \binom{L}{n} A^{L-n}(B-A)^n = \mathrm{IAd}[A,B]^L(\mathbf{1}) . \tag{A.13}
$$

Since this formula is a finite polynomial in the operators A and B, Eq. (A.13) holds for any linear operator. This means that we can perform this rearrangement for the Laplacians in the operator $D_{C,S}(r)$.

The operator $(\Delta_C-\Delta_{S,r})=P_{0,0}(\Delta_C-\Delta_S+\Delta_{S,\phi})$ is a differential operator of order one because the Grassmann variables can be viewed as a perturbative term in the pure commutative part $(\Lambda^0(p,2L))^p$. The flat operator $(\Delta_S-\Delta_C)$ only contains second derivatives with respect to the Grassmann variables. The derivative of a Grassmann variable reads in radial–angle coordinates

$$\frac{\partial}{\partial\eta_n}=\sum_{m=1}^{\dim(\mathcal{N})}\frac{\partial r_m}{\partial\eta_n}\frac{\partial}{\partial r_m}+\sum_{m=1}^{p+2L-\dim(\mathcal{N})}\frac{\partial\phi_m}{\partial\eta_n}\frac{\partial}{\partial\phi_m}\,. \tag{A.14}$$

$\partial r_m/\partial\eta_n$ is anticommuting and satisfies $P_{0,0}\partial r_m/\partial\eta_n=0$. Thus, $D_{C,S}(r)$ is a differential operator of order L.

A.3 Proof of theorem 3.2.1

The differential operator in Eq. (3.9) can be written as

$$D_r^{(1,L)}=\left(\frac{1}{2\pi}\right)^L\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^L\,. \tag{A.15}$$

This follows from the commutation relation

$$\left[\frac{\partial^2}{\partial r^2},\frac{1}{r}\frac{\partial}{\partial r}\right]_-=-2\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 \tag{A.16}$$

and from Eq. (3.5). One can also expand $f\left(\sqrt{r^2+2\sum\limits_{n=1}^{L}\eta_n^*\eta_n}\right)$ in a Taylor expansion in r^2 and take into account only the highest term of the power series in the Grassmann variables. For $p=2L$ one finds

$$\begin{aligned}\int\limits_{\mathrm{Mat}_1^0(2L\times1/L\times0)}f(x,\eta)d[x,\eta]=&\int\limits_{\mathbb{R}^{2L}}\left(\frac{1}{2\pi}\right)^L\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^L f(x)d[x]\\=&\frac{2}{2^L(L-1)!}\int\limits_{\mathbb{R}^+}r^{2L-1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^L f(r)dr\\=&\frac{1}{(L-1)!}\int\limits_{\mathbb{R}^+}\left(r^2\right)^{L-1}\left(\frac{\partial}{\partial(r^2)}\right)^L f\left(\sqrt{r^2}\right)dr^2\\=&(-1)^L f(0)\,.\end{aligned} \tag{A.17}$$

This proves the third case in Eq. (3.12). To prove the first case in Eq. (3.12), we expand the superfunction f in a power series in p pairs of Grassmann variables. We integrate over all real variables and over these pairs and apply the third case of Eq. (3.12). We can then expand the rest of the Grassmann variables since we know that f only depends on the length of the remaining supervector. This proves the first case in Eq. (3.12).

The second case in Eq. (3.12) can be treated similarly, however now there is an additional real variable which cannot be integrated. We consider $p = 1$ and $L = 2$. Then, we have to integrate

$$\int\limits_{\mathbb{R}} \frac{1}{r}\frac{\partial}{\partial r} f(r) dr\,. \tag{A.18}$$

We need an r in the numerator for cancellation with the singular term r^{-1}. Such a contribution is guaranteed if we have to every pair of Grassmann variables a pair of real variables.

The same reasoning applies for the fourth equation in Eq. (3.12). Here we do not find for every pair of real variables a pair of Grassmann variables. Thus, we can integrate over $2L$ real variables using the third equation of Eq. (3.12) and we are left with an integral over $(p - 2L)$ real variables.

A.4 Proof of theorem 3.3.1

First we prove the case $k_1 = k_2 = 1$. We define the complex number $z = s_1 - e^{\imath\psi} s_2$. Setting $f(s_1, s_2) = \tilde{f}(z, z^*)$ we get

$$\int\limits_{\Sigma^{\psi}_{2,1/1}} f(\sigma)d[\eta]d[\sigma_2]d[\sigma_1] = \frac{1}{2\pi}\int\limits_{\mathbb{C}} \frac{1}{z}\frac{\partial}{\partial z^*}\tilde{f}(z, z^*)dzdz^* = -\imath\tilde{f}(0) = -\imath f(0)\,. \tag{A.19}$$

For arbitrary $k = k_1 = k_2$ we rearrange and split the matrix σ in the following way

$$\sigma = \begin{bmatrix} \sigma_s & v \\ v^\dagger & \tilde{\sigma} \end{bmatrix}\,, \tag{A.20}$$

where σ_s is a $\mathrm{U}^{(2)}(1/1)$–symmetric supermatrix and $\tilde{\sigma}$ a Hermitian $\mathrm{U}^{(2)}(k-1/k-1)$–symmetric supermatrix. Moreover, we defined $v = (v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1})$ with the complex $(1/1)$–supervectors $v_j = \begin{bmatrix} z_j \\ e^{\imath\psi/2}\eta_j \end{bmatrix}$ and $w_j = \begin{bmatrix} e^{\imath\psi/2}\tilde{\eta}_j^* \\ e^{\imath\psi}\tilde{z}_j \end{bmatrix}$. The supervectors have the same structure as those of theorem

3.2.2. We integrate first over all variables except σ_s. The resulting function on the set of $\mathrm{U}^{(2)}(1/1)$–symmetric supermatrices

$$F_1(\sigma_s) = \int\limits_{\Sigma^{\psi}_{2,k-1/k-1}} \int\limits_{\mathrm{Mat}^{\,0}_{2}(1\times(k-1)/1\times(k-1))} f(\sigma) d[v] d[\tilde{\sigma}] \qquad \text{(A.21)}$$

is invariant under the action of $\mathrm{U}^{(2)}(1/1)$. Therefore, we can use Eq. (A.19) and have to calculate

$$\int\limits_{\Sigma^{\psi}_{2,k-1/k-1}} \int\limits_{\mathrm{Mat}^{\,0}_{2}(1\times(k-1)/1\times(k-1))} f\left(\begin{bmatrix} 0 & v \\ v^\dagger & \tilde{\sigma} \end{bmatrix}\right) d[v] d[\tilde{\sigma}] \,. \qquad \text{(A.22)}$$

We integrate over the remaining variables except over one pair of the (1/1)–supervectors $(u, u^\dagger) = (v_j, v_j^\dagger)$ or $(u, u^\dagger) = (w_j, w_j^\dagger)$. The function

$$F_2(u, u^\dagger) = \int\limits_{\Sigma^{\psi}_{2,k-1/k-1}} \int\limits_{\mathrm{Mat}^{\,0}_{2}(1\times(k-1)/1\times(k-2))} f\left(\begin{bmatrix} 0 & v \\ v^\dagger & \tilde{\sigma} \end{bmatrix}\right) d[v_{\neq u}] d[\tilde{\sigma}] \qquad \text{(A.23)}$$

fulfills the requirements of theorem 3.2.2. We perform the integration over all pairs $(v_j, v_j^\dagger)$ and $(w_j, w_j^\dagger)$ accordingly. We are left with an integration over $f(\tilde{\sigma})$. Since $\tilde{\sigma}$ has the same symmetry as σ with lower matrix dimension, we can proceed by induction. This proves the second case of Eq. (3.37). In order to prove the first and the third case of Eq. (3.37), we define the $\mathrm{U}^{(2)}(k/k)$–symmetric supermatrix σ_k, where $k = \min(k_1, k_2)$, and the $\Delta k \times \Delta k$–Hermitian matrix $\sigma_{\Delta k}$ in the boson–boson ($k = k_2$) or in the fermion–fermion ($k = k_1$) block, where $\Delta k = |k_1 - k_2|$. We define the function

$$F_3(\sigma_k) = \int\limits_{\mathrm{Herm}\,(2,\Delta k)} \int\limits_{\mathrm{Mat}^{\,0}_{2}(k\times\Delta k/0\times\Delta k)} f\left(\begin{bmatrix} \sigma_k & v \\ v^\dagger & \sigma_{\Delta k} \end{bmatrix}\right) d[v] d[\sigma_{\Delta k}] \qquad \text{(A.24)}$$

and apply the second case of Eq. (3.37) on F_3. The off-diagonal block matrix v consists of complex (k/k)–supervectors. We iteratively perform the integrations over these supervectors using theorem 3.2.2. This completes the proof.

A.5 Proof of theorem 3.3.2

As in theorem 3.3.1, we first prove the simplest nontrivial case $k_1 = 2k_2 = 2$. We use Eq. (3.54) and obtain after an integration over all angular coordinates

$$\begin{aligned}\int\limits_{\Sigma^{\psi}_{1,2/1}} f(\sigma)d[\eta]d[\sigma_2]d[\sigma_1] =& \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}} ds_2 d[s_1]\frac{e^{2\imath\psi}}{2\pi}\frac{|s_{11}-s_{21}|}{s_{11}-s_{21}} \\ \times& \left(2\frac{\partial}{\partial s_{11}} + e^{-\imath\psi}\frac{\partial}{\partial s_2}\right)\frac{1}{s_{21}-e^{\imath\psi}s_2}\left(2\frac{\partial}{\partial s_{21}} + e^{-\imath\psi}\frac{\partial}{\partial s_2}\right) f(s_{11},s_{21},s_2)\ .\end{aligned} \tag{A.25}$$

The equation is valid because the single terms in the integrand, see Eq. (3.54), are symmetric under interchange of the two bosonic eigenvalues. A change of variables $r = \frac{1}{2}(s_{11}-s_{21})$ and $z = \frac{1}{2}(s_{11}+s_{21})-e^{\imath\psi}s_2$, such that $f(s_{11}, s_{21}, s_2) = \tilde{f}(r, z, z^*)$, leads to

$$\begin{aligned}&\int\limits_{\Sigma^{\psi}_{1,2/1}} f(\sigma)d[\eta]d[\sigma_2]d[\sigma_1] \\ =& C_\psi \int\limits_{\mathbb{C}}\int\limits_{\mathbb{R}^+}\left(\left(1-e^{-2\imath\psi}\right)\frac{\partial}{\partial z^*} + \frac{\partial}{\partial r}\right)\frac{1}{z-r}\left(\left(1-e^{-2\imath\psi}\right)\frac{\partial}{\partial z^*} - \frac{\partial}{\partial r}\right) \\ &\times \tilde{f}(r,z,z^*)drdzdz^* \\ \overset{(2)}{=}& - C_\psi\int\limits_{\mathbb{C}}\frac{1}{z}\left(\left(1-e^{-2\imath\psi}\right)\frac{\partial}{\partial z^*} - \frac{\partial}{\partial r}\right)\tilde{f}(r,z,z^*)\bigg|_{r=0} dzdz^* \\ \overset{(3)}{=}& -\frac{e^{\imath\psi}}{\pi}\int\limits_{\mathbb{C}}\frac{1}{z}\frac{\partial}{\partial z^*}\tilde{f}(0,z,z^*)dzdz^* = 2\imath e^{\imath\psi}\tilde{f}(0) = 2\imath e^{\imath\psi}f(0)\end{aligned} \tag{A.26}$$

where $C_\psi = e^{2\imath\psi}/2\pi\imath\sin(\psi)$. The second equality (2) holds because the integral over the complex plane with the derivative with respect to z^* is up to a constant equal to

$$\begin{aligned}&\int\limits_{\mathbb{R}^+}\left(\left(1-e^{-2\imath\psi}\right)\frac{\partial}{\partial z^*} - \frac{\partial}{\partial r}\right)\tilde{f}(r,z,z^*)\bigg|_{z=z^*=r} dr \\ =& \int\limits_{\mathbb{R}^+}\left(2\frac{\partial}{\partial s_{21}} + e^{-\imath\psi}\frac{\partial}{\partial s_2}\right) f(s_{11},s_{21},s_2)\bigg|_{s_{21}=s_2=0} ds_{11} \overset{(3.55)}{=} 0\ .\end{aligned} \tag{A.27}$$

The third equality (3) holds because of the permutation symmetry in the two bosonic eigenvalues, s_{11} and s_{21}, and accordingly we get $\tilde{f}(r, z, z^*) = \tilde{f}(-r, z, z^*)$ and $\partial\tilde{f}/\partial r(r, z, z^*)|_{r=0} = 0$.

For arbitrary $k = k_1 = k_2/2$ we proceed as in the proof of theorem 3.3.1. We split off a $\mathrm{U}^{(1)}(2/1)$–symmetric supermatrix and integrate over the remaining variables such that the resulting function on the set of $\mathrm{U}^{(1)}(2/1)$–symmetric supermatrices is invariant. We apply the simplest case of the theorem above. We iteratively perform the integrals over the $2(k-1)$ real $(2/2)$–supervectors and $(k-1)$ quaternionic $(2/2)$–supervectors in the off-diagonal matrix block in the same manner as in the unitary case with help of the theorems 3.2.1 and 3.2.3. Finally, we carry out iteratively the integral over a $\mathrm{U}^{(1)}(2k-2/k-1)$–symmetric supermatrix. This proves the second equation in Eq. (3.56).

For arbitrary k_1 and k_2, one can split off the largest $\mathrm{U}^{(1)}(2k/k)$–symmetric supermatrix where k is the minimum of k_2 and $(k_1 - k_1 \mathrm{mod} 2)/2$. We use the second case of Eq. (3.56) to treat this block. The integrations over the $(2k/2k)$–supervectors, which are $(k_1 - 2k_2)$ real supervectors for $k_1 > 2k_2$ and $(2k_2 - k_1 + k_1 \mathrm{mod} 2)/2$ quaternionic supervectors for $k_1 < 2k_2$ plus an additional real supervector depending on whether k_1 is even or odd, can be iteratively calculated with the help of the theorems of Sec. 3.2. This proves the first, the third and the fourth case of Eq. (3.56).

Appendix B

Derivations for Part II

In App. B.1, we show that not only the trace but also the supertrace fulfills circularity in rectangular matrices. We prove proposition 7.4.1 in App. B.2. A differential operator analogous to the Sekiguchi differential operator is derived in App. B.3 and is used in the calculation of the supersymmetric Ingham–Siegel integral in App. B.4. The theorems 8.3.1, 10.1.1 and 10.2.1 are proven in the appendices B.5, B.6 and B.7, respectively.

B.1 Circularity of the supertrace for rectangular supermatrices

The circularity for rectangular matrices of pure commuting entries or anticommuting entries was derived by Berezin [96]. Since we have not found the general theorem for arbitrary rectangular supermatrices, we give the trivial statement.

Corollary B.1.1
Let the matrices V_1 and V_2 be the same as in Eq. (7.22)*. Then, we have*

$$\mathrm{Str}\, V_1 V_2 = \mathrm{Str}\, V_2 V_1 \,. \tag{B.1}$$

Proof:
We recall the circularity of the trace for rectangular matrices of commuting elements $\mathrm{tr}\, A_1 A_2 = \mathrm{tr}\, A_2 A_1$ and its anticommuting analogue $\mathrm{tr}\, B_1 B_2 = -\mathrm{tr}\, B_2 B_1$ which was proven by Berezin [96]. We make the simple calculation

$$\begin{aligned} \mathrm{Str}\, V_1 V_2 &= \mathrm{tr}\, A_1 A_2 + \mathrm{tr}\, B_1 C_2 - \mathrm{tr}\, C_1 B_2 - \mathrm{tr}\, D_1 D_2 \\ &= \mathrm{tr}\, A_2 A_1 - \mathrm{tr}\, C_2 B_1 + \mathrm{tr}\, B_2 C_1 - \mathrm{tr}\, D_2 D_1 \\ &= \mathrm{Str}\, V_2 V_1 \,. \end{aligned} \tag{B.2}$$

$\square$

For our purposes we must prove

$$\mathrm{tr}\,(\widehat{V}^\dagger\widehat{V})^m = \mathrm{Str}\,(\widehat{V}\widehat{V}^\dagger)^m\,. \tag{B.3}$$

We define $V_1 = \widehat{V}^\dagger$ and $V_2 = (\widehat{V}\widehat{V}^\dagger)^{m-1}\widehat{V}$ and get $a = 2k$, $b = 2k$, $c = \gamma_2 N$ and $d = 0$. Applying corollary B.1.1 and reminding that $\mathrm{tr}\,A = \mathrm{Str}\,A$ for a matrix of commuting elements and identification with the boson–boson block, we have the desired result (B.3).

B.2 Proof of proposition 7.4.1

Let λ be the wanted eigenvalue and be a commuting variable of the Grassmann algebra constructed from the $\{\tau_q^{(p)}, \tau_q^{(p)*}\}_{p,q}$. Then, we split this eigenvalue in its body $\lambda^{(0)} \in \Lambda_0$ and its soul $\lambda^{(1)}$, i.e. $\lambda = \lambda^{(0)} + \lambda^{(1)}$ with $\lambda^{(1)}$ nilpotent. Let v be the $\gamma_2 N$–dimensional eigenvector of H such that

$$Hv = \lambda v \quad \text{and} \quad v^\dagger v = 1\,. \tag{B.4}$$

In this equation, we recognize in the lowest order of Grassmann variables that $\lambda^{(0)}$ is an eigenvalue of $H^{(0)}$. Then, let $\lambda^{(0)}$ be an eigenvalue of the highest degeneracy δ of $H^{(0)}$, i.e. $\delta = \dim\ker(H^{(0)} - \lambda^{(0)}\mathbb{1}_N)$. Without loss of generality, we assume that $H^{(0)}$ is diagonal and the eigenvalue $\lambda^{(0)}$ only appears in the upper left $\delta \times \delta$ matrix block,

$$H^{(0)} = \begin{bmatrix} \lambda^{(0)}\mathbb{1}_\delta & 0 \\ 0 & \widetilde{H}^{(0)} \end{bmatrix}. \tag{B.5}$$

We also split the vectors in δ and $N - \delta$ dimensional vectors

$$v^{(0)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \tau_q = \begin{bmatrix} \tau_{q1} \\ \tau_{q2} \end{bmatrix}. \tag{B.6}$$

Thus, we find the two equations from Eq. (B.4)

$$\begin{aligned} T_{11}v_1 - \lambda^{(1)}v_1 + T_{12}v_2 &= 0\,, & \text{(B.7)} \\ T_{21}v_1 + \left[\widetilde{H}^{(0)} - \lambda\mathbb{1}_{N-\delta} + T_{22}\right] v_2 &= 0 & \text{(B.8)} \end{aligned}$$

where $T_{nm} = \sum\limits_{q=1}^{\widetilde{N}} l_q \left[\tau_{qn}\tau_{qm}^\dagger + \widetilde{Y}\left(\tau_{qn}^*\tau_{qm}^T\right)\right]$. Equation (B.8) yields

$$v_2 = -\left[\widetilde{H}^{(0)} - \lambda\mathbb{1}_{N-\delta} + T_{22}\right]^{-1} T_{21}v_1\,. \tag{B.9}$$

Hence, the body of v_2 is zero and we have for Eq. (B.7)

$$T_{11}v_1 - \lambda^{(1)}v_1 - T_{12}\left[\widetilde{H}^{(0)} - \lambda\mathbb{1}_{N-\delta} + T_{22}\right]^{-1} T_{21}v_1 = 0\,. \tag{B.10}$$

If the degeneracy is $\delta > \gamma_2$, we consider a δ–dimensional real vector $w \neq 0$ such that $w^\dagger v_1 = 0$. Then, we get for the lowest order in the Grassmann variables of Eq. (B.10) times $w^\dagger$

$$w^\dagger T_{11} v_1^{(0)} = 0\,, \tag{B.11}$$

where $v_1^{(0)}$ is the body of v_1. The entries of $w^\dagger T_{11}$ are linearly independent. Thus, the body of v_1 is also zero. This violates the second property of Eq. (B.4).

Let the degeneracy $\delta = \gamma_2$. Then, v_1 is γ_2-dimensional and is normalizable. For $\beta = 4$, we have the quaternionic case and the matrix before v_1 in Eq. (B.10) is a diagonal quaternion. Hence, it must be true

$$\lambda^{(1)}\mathbb{1}_{\gamma_2} = T_{11} - T_{12}\left[\widetilde{H}^{(0)} - \lambda\mathbb{1}_{N-\delta} + T_{22}\right]^{-1} T_{21}\,. \tag{B.12}$$

Considering the second order term in the Grassmann variables of Eq. (B.12), λ's second order term is T_{11} for $\beta \in \{1, 2\}$ and $\operatorname{tr} T_{11}/2$ for $\beta = 4$. Eq. (B.12) is unique solvable by recursive calculation. We plug the right hand side of Eq. (B.12) into the $\lambda^{(1)}$ on the same side and repeat this procedure. Hence, we define the operator

$$O(\mu) = \frac{1}{\gamma_2}\operatorname{tr}\left\{T_{11} - T_{12}\left[\widetilde{H}^{(0)} - (\lambda^{(0)} + \mu)\mathbb{1}_{N-\delta} + T_{22}\right]^{-1} T_{21}\right\} \tag{B.13}$$

with the recursion

$$O^{n+1}(\mu) = O\left[O^n(\mu)\right]\,. \tag{B.14}$$

Then, $\lambda^{(1)} = O^n(\lambda^{(1)})$ is true for arbitrary $n \in \mathbb{N}$. The recursion is finished for $n_0 \in \mathbb{N}$ if $\lambda^{(1)} = O^{n_0}(\lambda^{(1)}) = O^{n_0}(0)$. Due to the Grassmann variables, this recursion procedure eventually terminates after the $(\gamma_2 N\widetilde{N}/2)$'th time. Thus, the eigenvalue λ depends on Grassmann variables and is not a real number.

B.3 A matrix Bessel version of the Sekiguchi differential operator

We derive a version for the Sekiguchi differential operator for the ordinary matrix Bessel functions $\varphi_N^{(\beta)}(y, x)$ on the connection between the Jack–polynomials and the ordinary matrix Bessel functions.

The Sekiguchi differential operator is defined as in Ref. [123]

$$D_{Nz}(u,\beta)=\Delta_N^{-1}(z)\det\left[z_a^{N-b}\left(z_a\frac{\partial}{\partial z_a}+(N-b)\frac{\beta}{2}+u\right)\right]_{1\leq a,b\leq N} \tag{B.15}$$
$$=\Delta_N^{-1}(z)\det\left[\frac{\beta}{2}\left(z_a\frac{\partial}{\partial z_a}+u\right)z_a^{N-b}+\left(1-\frac{\beta}{2}\right)z_a^{N-b}\left(z_a\frac{\partial}{\partial z_a}+u\right)\right]_{1\leq a,b\leq N}.$$

Here, u is a boost and the expansion parameter to generate the elementary polynomials in the Cherednik operators, for more explicit information see Ref. [179]. Let $J_N^{(\beta)}(n,z)$ be the Jack–polynomial with the partition $n_1\geq\ldots\geq n_N$ and the standard parameter $\alpha=\frac{2}{\beta}$ in Macdonald's [180] notation. The Jack–polynomials are eigenfunctions with respect to $D_{Nz}(u,\beta)$

$$D_{Nz}(u,\beta)J_N^{(\beta)}(n,z)=\prod_{a=1}^{N}\left[n_a+(N-a)\frac{\beta}{2}+u\right]J_N^{(\beta)}(n,z)\,. \tag{B.16}$$

The aim is to find a similar differential operator for the ordinary matrix Bessel function $\varphi_N^{(\beta)}(y,x)$ such that

$$\begin{aligned}D_{Nx}^{(\beta)}(\mathbf{b})\varphi_N^{(\beta)}\left(\frac{y}{\gamma_2},x\right) &= \prod_{a=1}^{N}\imath\left(y_a+\mathbf{b}\right)\varphi_N^{(\beta)}\left(\frac{y}{\gamma_2},x\right)\\ &= \det{}^{1/\gamma_2}\imath(y+\mathbf{b}\mathbb{1}_{\gamma_2 N})\varphi_N^{(\beta)}\left(\frac{y}{\gamma_2},x\right).\end{aligned} \tag{B.17}$$

Proposition B.3.1
The differential operator which fulfils Eq. (B.17) *is*

$$D_{Nx}^{(\beta)}(\boldsymbol{b})=\Delta_N^{-1}(x)\det\left[x_a^{N-b}\left(\frac{\partial}{\partial x_a}+(N-b)\frac{\beta}{2}\frac{1}{x_a}+\imath\boldsymbol{b}\right)\right]_{1\leq a,b\leq N}. \tag{B.18}$$

Proof:
Kohler [181] has presented a connection between the Jack–polynomials and the matrix Bessel functions. Let

$$z_a=\exp\left[\imath\frac{2\pi}{L}x_a\right]\quad\text{and}\quad n_a=\frac{L}{2\pi}y_a-\left(\frac{N+1}{2}-a\right)\frac{\beta}{2} \tag{B.19}$$

then it is true

$$\varphi_N^{(\beta)}\left(\frac{y}{\gamma_2},x\right)=\lim_{L\to\infty}\left(\frac{\Delta_N(z)}{\Delta_N(x)\Delta_N(y)}\right)^{\beta/2}\prod_{a=1}^{N}z_a^{-\beta(N-1)/4}J_N^{(\beta)}(n,z)\,. \tag{B.20}$$

We expand the determinant in Eq. (B.15) and have

$$D_{Nz}(u,\beta) = \Delta_N^{-1}(z) \tag{B.21}$$
$$\times \sum_{m\in\{0,1\}^N} \prod_{a=1}^N \left[\frac{\beta}{2}\left(z_a\frac{\partial}{\partial z_a}+u\right)\right]^{m_a} \Delta_N(z) \prod_{a=1}^N \left[\left(1-\frac{\beta}{2}\right)\left(z_a\frac{\partial}{\partial z_a}+u\right)\right]^{1-m_a}.$$

Using the substitution (B.19) and

$$\widetilde{\Delta}_N(x) = \prod_{1\leq a<b\leq N} 2\imath\sin\left(\frac{\pi}{L}(x_a-x_b)\right)\exp\left(\imath\pi\frac{x_a+x_b}{L}\right), \tag{B.22}$$

we consider the limit

$$\begin{aligned}
&\lim_{L\to\infty}\left(\frac{2\pi\imath}{L}\right)^N \Delta_N(x)D_{Nz}(u,\beta)\\
=&\lim_{L\to\infty}\frac{\Delta_N(x)}{\widetilde{\Delta}_N(x)}\sum_{m\in\{0,1\}^N}\prod_{a=1}^N\left[\frac{\beta}{2}\left(\frac{\partial}{\partial x_a}+\imath\frac{2\pi u}{L}\right)\right]^{m_a}\widetilde{\Delta}_N(x)\\
\times&\prod_{j=1}^N\left[\left(1-\frac{\beta}{2}\right)\left(\frac{\partial}{\partial x_a}+\imath\frac{2\pi u}{L}\right)\right]^{1-m_a}\\
=&\sum_{m\in\{0,1\}^N}\prod_{a=1}^N\left[\frac{\beta}{2}\left(\frac{\partial}{\partial x_a}+\imath\mathbf{b}\right)\right]^{m_a}\Delta_N(x)\left[\left(1-\frac{\beta}{2}\right)\left(\frac{\partial}{\partial x_a}+\imath\mathbf{b}\right)\right]^{1-m_a}\\
=&\det\left[\frac{\beta}{2}\left(\frac{\partial}{\partial x_a}+\imath\mathbf{b}\right)x_a^{N-b}+\left(1-\frac{\beta}{2}\right)x_a^{N-b}\left(\frac{\partial}{\partial x_a}+\imath B\right)\right]_{1\leq a,b\leq N}\\
=&\det\left[x_a^{N-b}\left(\frac{\partial}{\partial x_a}+(N-b)\frac{\beta}{2}\frac{1}{x_a}+\imath\mathbf{b}\right)\right]_{1\leq a,b\leq N}.
\end{aligned}\tag{B.23}$$

Here, we define the boost $\mathbf{b}=\lim\limits_{L\to\infty}2\pi u/L$. The eigenvalue in Eq. (B.16) is in the limit

$$\begin{aligned}
\lim_{L\to\infty}\left(\frac{2\pi\imath}{L}\right)^N\prod_{a=1}^N\left[n_a+(N-a)\frac{\beta}{2}+u\right] &= \prod_{a=1}^N\imath\,(y_a+\mathbf{b})\\
&= \det{}^{1/\gamma_2}\imath(y+\mathbf{b}\mathbb{1}_{\gamma_2N})\,.
\end{aligned}\tag{B.24}$$

We assume that Eq. (B.20) is a uniformly convergent limit. Thus, we combine Eqs. (B.20), (B.23) and (B.24) with the eigenvalue equation (B.16) and find formula (B.15). □

Indeed, $D_{Nx}^{(\beta)}(\mathbf{b})$ is for the unitary case, $\beta=2$,

$$D_{Nx}^{(2)}(\mathbf{b}) = \Delta_N^{-1}(x)\prod_{a=1}^N\left(\frac{\partial}{\partial x_a}+\imath\mathbf{b}\right)\Delta_N(x)\,. \tag{B.25}$$

B.4

In App. B.4.1, we compute the Ingham–Siegel integral. We derive the results of lemma 8.2.1 and theorem 8.2.2 in the appendices B.4.2 and B.4.3, respectively.

B.4.1 Decomposition of the boson–boson and fermion–fermion block integration

We split σ in its boson–fermion block structure

$$\mathfrak{p}\sigma = \left[\begin{array}{cc} \sigma_1 & e^{-\imath\psi/2}\sigma_\eta^\dagger \\ e^{-\imath\psi/2}\sigma_\eta & e^{-\imath\psi}\sigma_2 \end{array}\right] . \tag{B.26}$$

The following calculation must be understood in a weak sense. We first integrate over a conveniently integrable function and, then, perform the integral transformations. Hence, we understand $I_k^{(\beta,N)}$ as a distribution where we must fix the underlying set of test–functions. For our purposes, we need Schwartz functions analytic in the real independent variables with respect to the Wick–rotation.

Since the superdeterminant of $\mathfrak{p}\left(\sigma + \imath\varepsilon\mathbb{1}_{4k}\right)$ is

$$\mathrm{Sdet}\,\mathfrak{p}\sigma^+ = \frac{\det\left(\sigma_1 + \imath\varepsilon\mathbb{1}_{\tilde{k}}\right)}{\det\left[e^{-\imath\psi}\sigma_2 + \imath\varepsilon\mathbb{1}_{\tilde{k}} - e^{-\imath\psi}\sigma_\eta\left(\sigma_1 + \imath\varepsilon\mathbb{1}_{\tilde{k}}\right)^{-1}\sigma_\eta^\dagger\right]} \tag{B.27}$$

we shift σ_2 by analytic continuation to $\sigma_2 + \sigma_\eta\left(\sigma_1 + \imath\varepsilon\mathbb{1}_{\tilde{k}}\right)^{-1}\sigma_\eta^\dagger$ and obtain

$$\begin{aligned} I_k^{(\beta,N)}(\rho) &= \int\limits_{\Sigma_{\beta,k}^{-\psi}} \exp\left(-\imath\mathrm{tr}\,r_1\sigma_1 + \imath\mathrm{tr}\,r_2\left[\sigma_2 + \sigma_\eta\left(\sigma_1 + \imath\varepsilon\mathbb{1}_{\tilde{k}}\right)^{-1}\sigma_\eta^\dagger\right]\right) \\ &\times \exp\left(\varepsilon\mathrm{Str}\,r\right)\left[\frac{\det\left(e^{-\imath\psi}\sigma_2 + \imath\varepsilon\mathbb{1}_{\tilde{k}}\right)}{\det\left(\sigma_1 + \imath\varepsilon\mathbb{1}_{\tilde{k}}\right)}\right]^{N/\gamma_1} d[\sigma] . \end{aligned} \tag{B.28}$$

An integration over the Grassmann variables yields

$$\begin{aligned} I_k^{(\beta,N)}(\rho) &= \left(\frac{-\imath\tilde{\gamma}}{2\pi}\right)^{k_1k_2}\exp\left(\varepsilon\mathrm{Str}\,r\right)\det{}^k r_2 \\ &\times \int\limits_{\mathrm{Herm}\,(\beta,k_1)} \exp\left(-\imath\mathrm{tr}\,r_1\sigma_1\right)\det\left(\sigma_1 + \imath\varepsilon\mathbb{1}_{\tilde{k}}\right)^{-N/\gamma_1-k} d[\sigma_1] \\ &\times \int\limits_{\mathrm{Herm}\,(4/\beta,k_2)} \exp\left(\imath\mathrm{tr}\,r_2\sigma_2\right)\det\left(e^{-\imath\psi}\sigma_2 + \imath\varepsilon\mathbb{1}_{\tilde{k}}\right)^{N/\gamma_1} d[\sigma_2] . \end{aligned} \tag{B.29}$$

With help of Eq. (8.10) we have

$$I_k^{(\beta,N)}(\rho) = \imath^{-k_2 N} G_{Nk_1}^{(\beta)} \left(-\frac{\tilde{\gamma}}{2\pi}\right)^{k_1 k_2} \det{}^\kappa r_1 \Theta(r_1) \exp\left(-e^{\imath\psi}\varepsilon \mathrm{tr}\, r_2\right)$$
$$\times \quad \det{}^k r_2 \int\limits_{\mathrm{Herm}\,(4/\beta,k_2)} \exp\left(\imath \mathrm{tr}\, r_2\sigma_2\right) \det{}^{N/\gamma_1}\left(e^{-\imath\psi}\sigma_2 + \imath\varepsilon \mathbb{1}_{\tilde{k}}\right) d[\sigma_2]\,. \tag{B.30}$$

The remaining integral over the fermion–fermion block σ_2,

$$\mathfrak{I}(r_2) = \exp\left(-e^{\imath\psi}\varepsilon \mathrm{tr}\, r_2\right) \int\limits_{\mathrm{Herm}\,(4/\beta,k_2)} \exp\left(\imath \mathrm{tr}\, r_2\sigma_2\right) \det{}^{N/\gamma_1}\left(\sigma_2 + \imath e^{\imath\psi}\varepsilon \mathbb{1}_{\tilde{k}}\right) d[\sigma_2]\,, \tag{B.31}$$

is up to a constant a differential operator with respect to r_2 times the Dirac–distribution of r_2 because the determinant term is for $\beta \in \{1,2\}$ a polynomial in σ_2 and for $\beta = 4$ we use Kramer's–degeneracy. We give several representations of this distribution.

We first start with an eigenvalue–angle decomposition of $\sigma_2 = U s_2 U^\dagger$ where s_2 is diagonal and $U \in \mathrm{U}^{(4/\beta)}(k_2)$. Integrating over the group $\mathrm{U}^{(4/\beta)}(k_2)$, Eq. (B.31) becomes

$$\begin{aligned} \mathfrak{I}(r_2) &= \mathrm{FU}_{k_2}^{(4/\beta)} \exp\left(-e^{\imath\psi}\varepsilon \mathrm{tr}\, r_2\right) \\ &\times \int\limits_{\mathbb{R}^{k_2}} \varphi_{k_2}^{(4/\beta)}(r_2, s_2) \det{}^{N/\gamma_1}\left(s_2 + \imath e^{\imath\psi}\varepsilon \mathbb{1}_{\tilde{k}}\right) |\Delta_{k_2}(s_2)|^{4/\beta} d[s_2]. \end{aligned} \tag{B.32}$$

The definition of the ordinary matrix Bessel function is given in Eq. (4.4). The constant $\mathrm{FU}_n^{(\beta)}$ is defined in Eq. (4.32). It agrees with the one in Ref. [182] denoted by Vol_B.

We plug the differential operator (B.18) of App. B.3 into Eq. (B.32) and have

$$\begin{aligned} \mathfrak{I}(r_2) &= \mathrm{FU}_{k_2}^{(4/\beta)} \exp\left(-e^{\imath\psi}\varepsilon \mathrm{tr}\, r_2\right) (\imath\gamma_1)^{-k_2 N} \\ &\times \left[D_{k_2 r_2}^{(4/\beta)}\left(\imath e^{\imath\psi}\gamma_1\varepsilon\right)\right]^N \int\limits_{\mathbb{R}^{k_2}} \phi_{k_2}^{(4/\beta)}(r_2, s_2) |\Delta_{k_2}(s_2)|^{4/\beta} d[s_2]\,. \end{aligned} \tag{B.33}$$

The integration over the eigenvalues leads to the Dirac distribution

$$\begin{aligned} \mathfrak{I}(r_2) &= \left(\frac{2\pi}{\gamma_1}\right)^{k_2} \left(\frac{\pi}{\gamma_1}\right)^{2k_2(k_2-1)/\beta} \frac{\exp\left(-e^{\imath\psi}\varepsilon \mathrm{tr}\, r_2\right)}{\mathrm{FU}_{k_2}^{(4/\beta)}} (\imath\gamma_1)^{-k_2} \\ &\times \left[D_{k_2 r_2}^{(4/\beta)}\left(\imath e^{\imath\psi}\gamma_1\varepsilon\right)\right]^N \frac{\delta(r_2)}{|\Delta_{k_2}(r_2)|^{4/\beta}} \end{aligned} \tag{B.34}$$

and we find the representation for the supersymmetric Ingham–Siegel integral (8.14).

B.4.2 Proof of lemma 8.2.1

The boost $\imath e^{\imath\psi}\varepsilon$ in the determinant can simply be shifted away because of

$$\begin{aligned} D_{k_2r_2}^{(4/\beta)}\left(\imath e^{\imath\psi}\gamma_1\varepsilon\right)\exp\left(\varepsilon e^{\imath\psi}\operatorname{tr} r_2\right) &= \exp\left(\varepsilon e^{\imath\psi}\operatorname{tr} r_2\right) D_{k_2r_2}^{(4/\beta)}(0) \\ &= \exp\left(\varepsilon e^{\imath\psi}\operatorname{tr} r_2\right) D_{k_2r_2}^{(4/\beta)} \end{aligned} \tag{B.35}$$

and Eq. (B.34). Let $\mathfrak{S}$ the set of $\mathrm{U}^{(4/\beta)}(k_2)$–invariant Schwartz functions on $\mathrm{Herm}\,(4/\beta,k_2)\to\mathbb{C}$. The ordinary matrix Bessel functions, see Eq. (4.4), are complete and orthogonal in $\mathfrak{S}$ with the sesquilinear scalar product

$$\langle f_1|f_2\rangle=\int\limits_{\mathbb{R}^{k_2}} f_1^*(x)f_2(x)|\Delta_{k_2}(x)|^{4/\beta}d[x]\,. \tag{B.36}$$

The completeness and the orthogonality are

$$\begin{aligned} \langle\phi_{k_2}^{(4/\beta)}(x)|\phi_{k_2}^{(4/\beta)}(x')\rangle &= \int\limits_{\mathbb{R}^{k_2}} |\phi_{k_2}^{(4/\beta)}(y)\rangle\langle\phi_{k_2}^{(4/\beta)}(y)|\ |\Delta_{k_2}(y)|^{4/\beta}d[y] \\ &= \int\limits_{\mathbb{R}^{k_2}} \phi_{k_2}^{(4/\beta)}(y,x)\phi_{k_2}^{(4/\beta)*}(y,x')|\Delta_{k_2}(y)|^{4/\beta}d[y] \\ &= C_k^{(\beta)}\frac{1}{k_2!}\sum_{p\in\mathfrak{S}_{k_2}}\frac{\prod\limits_{j=1}^{k_2}\delta(x_j-x'_{p(j)})}{|\Delta_{k_2}(x)|^{2/\beta}|\Delta_{k_2}(x')|^{2/\beta}}\,. \end{aligned} \tag{B.37}$$

We defined the constant

$$C_k^{(\beta)}=\left(\frac{2\pi}{\gamma_1}\right)^{k_2}\left(\frac{\pi}{\gamma_1}\right)^{2k_2(k_2-1)/\beta}\left(\mathrm{FU}_{k_2}^{(4/\beta)}\right)^{-2}\,. \tag{B.38}$$

Thus, we write $D_{k_2r_2}^{(4/\beta)}$ in the Bessel function basis

$$\begin{aligned} D_{k_2}^{(4/\beta)} &= C_k^{(\beta)\ -2}\int\limits_{\mathbb{R}^{k_2}} |\phi_{k_2}^{(4/\beta)}(y)\rangle\langle\phi_{k_2}^{(4/\beta)}(y)|\ |\Delta_{k_2}(y)|^{4/\beta}d[y] \\ &\times\ D_{k_2x}^{(4/\beta)}\int\limits_{\mathbb{R}^{k_2}} |\phi_{k_2}^{(4/\beta)}(y')\rangle\langle\phi_{k_2}^{(4/\beta)}(y')|\ |\Delta_{k_2}(y')|^{4/\beta}d[y'] \qquad \text{(B.39)} \\ &= C_k^{(\beta)\ -1}\int\limits_{\mathbb{R}^{k_2}} \det(i\gamma_1 y)^{1/\gamma_1}\phi_{k_2}^{(4/\beta)}(y,x)\phi_{k_2}^{(4/\beta)*}(y,x')|\Delta_{k_2}(y)|^{4/\beta}d[y] \end{aligned}$$

with the action on a function $f \in \mathfrak{S}$

$$\begin{aligned} D_{k_2}^{(4/\beta)}|f\rangle &= C_k^{(\beta)\ -1} \int\limits_{\mathbb{R}^{k_2}} \int\limits_{\mathbb{R}^{k_2}} \det(i\gamma_1 y)^{1/\gamma_1} \phi_{k_2}^{(4/\beta)}(y,x) \phi_{k_2}^{(4/\beta)*}(y,x') f(x') \\ &\times |\Delta_{k_2}(x')|^{4/\beta} |\Delta_{k_2}(y)|^{4/\beta} d[x'] d[y] \,. \end{aligned} \tag{B.40}$$

Due to this representation, the Sekiguchi differential operator analog, $\imath^{k_2} D_{k_2}^{(4/\beta)}$, is symmetric with respect to the scalar product (B.36)

$$\langle f_1 | \imath^{k_2} D_{k_2}^{(4/\beta)} | f_2 \rangle = \langle \imath^{k_2} D_{k_2}^{(4/\beta)} f_1 | f_2 \rangle \,. \tag{B.41}$$

Let L be a complex number. Then, we easily see with help of Eq. (B.18)

$$D_{k_2 x}^{(4/\beta)} \det x^{L/\gamma_1} = \prod_{b=1}^{k_2} \left(L + \frac{2}{\beta} b - \frac{2}{\beta} \right) \det x^{(L-1)/\gamma_1} \,. \tag{B.42}$$

Since the property (B.41), we obtain for a function $f \in \mathfrak{S}$

$$\begin{aligned} & \int\limits_{\mathbb{R}^{k_2}} \det x^{L/\gamma_1} |\Delta_{k_2}(x)|^{4/\beta} D_{k_2 x}^{(4/\beta)} f(x) d[x] \\ =\ & (-1)^{k_2} \int\limits_{\mathbb{R}^{k_2}} f(x) |\Delta_{k_2}(x)|^{4/\beta} D_{k_2 x}^{(4/\beta)} \det x^{L/\gamma_1} d[x] \qquad \text{(B.43)} \\ =\ & (-1)^{k_2} \prod_{b=1}^{k_2} \left(L + \frac{2}{\beta} b - \frac{2}{\beta} \right) \int\limits_{\mathbb{R}^{k_2}} f(x) |\Delta_{k_2}(x)|^{4/\beta} \det x^{(L-1)/\gamma_1} d[x] \,. \end{aligned}$$

The boundary terms of the partial integration do not appear because f is a Schwartz function and $D_{k_2 x}^{(4/\beta)}$ has the representation (B.39).

Let F and f be the functions of lemma 8.2.1. Then, we calculate

$$\begin{aligned} & \int\limits_{\mathbb{R}^{k_2}} \int\limits_{\mathrm{Herm}\,(4/\beta,k_2)} F(r_2) {\det}^k r_2 |\Delta_{k_2}(r_2)|^{4/\beta} \exp\left(\imath \mathrm{tr}\, r_2 \sigma_2\right) \\ \times\ & {\det}^{N/\gamma_1} \left(e^{-\imath\psi} \sigma_2 + \imath\varepsilon \mathbb{1}_{\tilde{k}} \right) d[\sigma_2] d[r_2] \\ =\ & \int\limits_{\mathbb{R}^{k_2}} \int\limits_{\mathrm{Herm}\,(4/\beta,k_2)} f(r_2) {\det}^{N/\gamma_1} r_2 |\Delta_{k_2}(r_2)|^{4/\beta} \exp\left(\imath \mathrm{tr}\, r_2 \sigma_2\right) \\ \times\ & {\det}^{N/\gamma_1} \left(e^{-\imath\psi} \sigma_2 + \imath\varepsilon \mathbb{1}_{\tilde{k}} \right) d[\sigma_2] d[r_2] \\ =\ & \left(\frac{-\imath e^{-\imath\psi}}{\gamma_1} \right)^{k_2 N} \mathrm{FU}_{k_2}^{(4/\beta)} \int\limits_{\mathbb{R}^{k_2}} \int\limits_{\mathbb{R}^{k_2}} f(r_2) \exp\left(\varepsilon e^{\imath\psi} \mathrm{tr}\, r_2 \right) |\Delta_{k_2}(r_2)|^{4/\beta} \\ \times\ & {\det}^{N/\gamma_1} s_2 |\Delta_{k_2}(s_2)|^{4/\beta} \left(D_{k_2 s_2}^{(4/\beta)} \right)^N \phi_{k_2}^{(4/\beta)}(r_2, s_2) d[s_2] d[r_2] \end{aligned}$$

$$\begin{aligned} &= (\imath e^{-\imath\psi})^{k_2N}\mathrm{FU}_{k_2}^{(4/\beta)}\prod_{a=1}^{N}\prod_{b=1}^{k_2}\left(\frac{a}{\gamma_1}+\frac{b-1}{\gamma_2}\right)\int\limits_{\mathbb{R}^{k_2}}\int\limits_{\mathbb{R}^{k_2}} f(r_2) \qquad \text{(B.44)}\\ &\times\ \exp\left(\varepsilon e^{\imath\psi}\mathrm{tr}\, r_2\right)|\Delta_{k_2}(r_2)|^{4/\beta}|\Delta_{k_2}(s_2)|^{4/\beta}\phi_{k_2}^{(4/\beta)}(r_2,s_2)d[s_2]d[r_2]\\ &= \left(\frac{2\pi}{\gamma_1}\right)^{k_2}\left(\frac{\pi}{\gamma_1}\right)^{2k_2(k_2-1)/\beta}\frac{\left(\imath e^{-\imath\psi}\right)^{k_2N}}{\mathrm{FU}_{k_2}^{(4/\beta)}\gamma_1^{k_2N}}\prod_{j=0}^{k_2-1}\frac{\Gamma\left(N+1+2j/\beta\right)}{\Gamma\left(1+2j/\beta\right)}f(0)\,. \end{aligned}$$

The second equality in Eq. (8.19) is true because of

$$f(0)=\prod_{j=1}^{k_2}\frac{1}{(N-k_1)!}\left(\frac{\partial}{\partial r_{j2}}\right)^{N-k_1}\left[f(r_2)\exp\left(\varepsilon e^{\imath\psi}\mathrm{tr}\, r_2\right)\det r_2^{N/\gamma_1-k}\right]\Bigg|_{r_2=0}. \tag{B.45}$$

The function in the bracket is F times the exponential term $\exp\left(\varepsilon e^{\imath\psi}\mathrm{tr}\, r_2\right)$.

B.4.3 Proof of theorem 8.2.2

We have to show

$$\begin{aligned} &\int\limits_{\mathrm{Herm}\,(4/\beta,k_2)}\int\limits_{\mathrm{Herm}\,(4/\beta,k_2)} F(\rho_2)\mathrm{det}^k\rho_2\exp\left(\imath\mathrm{tr}\,\rho_2\sigma_2\right)\mathrm{det}^{N/\gamma_1}\sigma_2 d[\sigma_2]d[\rho_2]\\ &\sim \int\limits_{\mathbb{R}^{k_2}} F(r_2)\prod_{j=1}^{k}\left(-\frac{\partial}{\partial r_{j2}}\right)^{N-2/\beta}\delta(r_{j2})d[r_2] \qquad \text{(B.46)} \end{aligned}$$

for every rotation invariant Schwartz function $F:\mathrm{Herm}\,(4/\beta,k_2)\to\mathbb{C}$ and $\beta\in\{1,2\}$. Due to

$$\begin{aligned} &\int\limits_{\mathrm{Herm}\,(4/\beta,k_2)}\exp\left(\imath\mathrm{tr}\, r_2\sigma_2\right)\mathrm{det}\sigma_2^{N/\gamma_1}d[\sigma_2]\\ &\sim \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{4(k_2-1)/\beta}} y^N\exp\left[\imath r_{k_22}\mathrm{tr}\left(y\mathbb{1}_{\tilde\gamma}+v^\dagger v\right)\right]d[v]dy\\ &\times \int\limits_{\mathrm{Herm}\,(4/\beta,k_2-1)}\exp\left(\imath\mathrm{tr}\,\tilde r_2\tilde\sigma_2\right)\mathrm{det}\tilde\sigma_2^{(N+2/\beta)/\gamma_1}d[\tilde\sigma_2] \qquad \text{(B.47)} \end{aligned}$$

with the decompositions $r_2=\mathrm{diag}\,(\tilde r_2, r_{k_22}\mathbb{1}_{\tilde\gamma})$ and

$$\sigma_2=\begin{bmatrix}\tilde\sigma_2 & v\\ v^\dagger & y\mathbb{1}_{\tilde\gamma}\end{bmatrix}, \tag{B.48}$$

we make a complete induction. Thus, we reduce the derivation to

$$\begin{aligned}
&\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{4(k_2-1)/\beta}} f(x)x^{k_1}y^N \exp\left[\imath x \mathrm{tr}\,(y+v^\dagger v)\right] d[v]dydx \\
\sim\ &\int\limits_{\mathbb{R}} f(x)\frac{\partial^{N-2/\beta}}{\partial x^{N-2/\beta}}\delta(x)d[x]
\end{aligned} \tag{B.49}$$

where $f:\mathbb{R}\to\mathbb{C}$ is a Schwartz function. The function

$$\tilde{f}(y)=\int\limits_{\mathbb{R}} f(x)x^{k_1}\exp\left(\imath xy\right)dx \tag{B.50}$$

is also a Schwartz function. Hence, we compute

$$\begin{aligned}
&\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{4(k_2-1)/\beta}} f(x)x^{k_1}y^N \exp\left[\imath x \mathrm{tr}\,(y+v^\dagger v)\right] d[v]dydx \\
=\ &\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{4(k_2-1)/\beta}} \tilde{f}\left[\mathrm{tr}\,(y+v^\dagger v)\right] y^N d[v]dy \\
=\ &\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^{4(k_2-1)/\beta}} y^{N-2(k_2-1)/\beta}\left(-\frac{\partial}{\partial y}\right)^{2(k_2-1)/\beta} \tilde{f}\left(\mathrm{tr}\,(y+v^\dagger v)\right) d[v]dy \\
\sim\ &\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^+} \tilde{v}^{2(k_2-1)/\beta-1}\left(-\frac{\partial}{\partial \tilde{v}}\right)^{2(k_2-1)/\beta} \tilde{f}\Big(\mathrm{tr}\,(y+\tilde{v})\Big) y^{N-2(k_2-1)/\beta} d\tilde{v}dy \\
\sim\ &\int\limits_{\mathbb{R}} \tilde{f}\left(\mathrm{tr}\,y\right) y^{N-2(k_2-1)/\beta}dy \\
\sim\ &\int\limits_{\mathbb{R}} f(x)x^{k_1}\left(-\frac{\partial}{\partial x}\right)^{N-2(k_2-1)/\beta} \delta(x)dx \\
\sim\ &\int\limits_{\mathbb{R}} f(x)\frac{\partial^{N-2/\beta}}{\partial x^{N-2/\beta}}\delta(x)d[x]\ ,
\end{aligned} \tag{B.51}$$

which is for $\beta\in\{1,2\}$ well–defined.

B.5 Proof of theorem 8.3.1

We choose a Wick–rotation $e^{\imath\psi}$ such that all calculations below are well defined. Then, we perform a Fourier transformation

$$\begin{aligned} &\int\limits_{\mathrm{Mat}^{0}_{\beta}(c\times a/d)} F(B)\exp\left(-\varepsilon \mathrm{Str}\, B\right) d[V] \\ =\ & \widetilde{C}_1 \int\limits_{\Sigma^{-\psi}_{\beta,c/d}} \int\limits_{\mathrm{Mat}^{0}_{\beta}(c\times a/d)} \mathcal{F}F(\sigma)\exp\left(\imath \mathrm{Str}\, B\sigma^{+}\right) d[V]d[\sigma]\,, \end{aligned} \tag{B.52}$$

where $\sigma^{+} = \sigma + \imath\varepsilon \mathbb{1}_{\gamma_2 c+\gamma_1 d}$,

$$\mathcal{F}F(\sigma) = \int\limits_{\Sigma^{\psi}_{\beta,c/d}} F(\rho)\exp\left(-\imath \mathrm{Str}\,\rho\sigma\right) d[\rho]\,, \tag{B.53}$$

and the constant is

$$\widetilde{C}_1 = \left(\frac{2\pi}{\tilde{\gamma}}\right)^{2cd} \left(\frac{\gamma_2}{2\pi}\right)^{c} \left(\frac{\gamma_2}{\pi}\right)^{\beta c(c-1)/2} \left(\frac{\gamma_1}{2\pi}\right)^{d} \left(\frac{\gamma_1}{\pi}\right)^{2d(d-1)/\beta}\,. \tag{B.54}$$

The integration over V yields

$$\int\limits_{\mathrm{Mat}^{0}_{\beta}(c\times a/d)} F(B)\exp\left(-\varepsilon \mathrm{Str}\, B\right) d[V] = \widetilde{C}_2 \int\limits_{\Sigma^{-\psi}_{\beta,c/d}} \mathcal{F}F(\sigma)\mathrm{Sdet}^{\,-a/\gamma_1}\sigma^{+} d[\sigma] \tag{B.55}$$

with

$$\widetilde{C}_2 = \left(\frac{2\pi}{\tilde{\gamma}}\right)^{2cd} \left(\frac{\gamma_2}{2\pi}\right)^{c} \left(\frac{\gamma_2}{\pi}\right)^{\beta c(c-1)/2} \left(\frac{\gamma_1}{2\pi}\right)^{d} \left(\frac{\gamma_1}{\pi}\right)^{2d(d-1)/\beta} \left(\frac{\imath}{2\pi}\right)^{ad} (\gamma_1\pi\imath)^{\beta ac/2}. \tag{B.56}$$

We transform this result back by a Fourier transformation

$$\int\limits_{\mathrm{Mat}^{0}_{\beta}(c\times a/d)} F(B)\exp\left(-\varepsilon \mathrm{Str}\, B\right) d[V] = \widetilde{C}_2 \int\limits_{\Sigma^{\psi}_{\beta,c/d}} F(\rho) I^{(\beta,a)}_{cd}(\rho)\exp\left(-\varepsilon \mathrm{Str}\,\rho\right) d[\rho], \tag{B.57}$$

where we have to calculate the supersymmetric Ingham–Siegel integral

$$I^{(\beta,a)}_{cd}(\rho) = \int\limits_{\Sigma^{-\psi}_{\beta,c/d}} \exp\left(-\imath \mathrm{Str}\,\rho\sigma^{+}\right) \mathrm{Sdet}^{\,-a/\gamma_1}\sigma^{+} d[\sigma]\,. \tag{B.58}$$

This distribution is rotation invariant under $\mathrm{U}^{(\beta)}(c/d)$. The ordinary version, $d=0$, of Eq. (B.58) is Eq. (8.10).

After performing four shifts

$$\begin{aligned}
\sigma_1 &\rightarrow \sigma_1 - \sigma_{\tilde{\eta}}\left(\sigma_2 + \imath e^{\imath\psi}\varepsilon \mathbb{1}_{\gamma_1 d}\right)^{-1}\sigma_{\tilde{\eta}}^{\dagger}, && \text{(B.59)}\\
\sigma_{\tilde{\eta}} &\rightarrow \sigma_{\tilde{\eta}} - \rho_1^{-1}\rho_{\eta}\left(\sigma_2 + \imath e^{\imath\psi}\varepsilon \mathbb{1}_{\gamma_1 d}\right), && \text{(B.60)}\\
\sigma_{\tilde{\eta}}^{\dagger} &\rightarrow \sigma_{\tilde{\eta}}^{\dagger} - \left(\sigma_2 + \imath e^{\imath\psi}\varepsilon \mathbb{1}_{\gamma_1 d}\right)\rho_{\eta}^{\dagger}\rho_1^{-1}, && \text{(B.61)}\\
\rho_2 &\rightarrow \rho_2 - \rho_{\eta}^{\dagger}\rho_1^{-1}\rho_{\eta}, && \text{(B.62)}
\end{aligned}$$

and defining

$$\hat{\rho} = \left[\begin{array}{c|c} \rho_1 & e^{\imath\psi/2}\rho_{\eta} \\ \hline -e^{\imath\psi/2}\rho_{\eta}^{\dagger} & e^{\imath\psi}\left(\rho_2 - \rho_{\eta}^{\dagger}\rho_1^{-1}\rho_{\eta}\right) \end{array}\right], \tag{B.63}$$

we find

$$\int\limits_{\mathrm{Mat}_{\beta}^{0}(c\times a/d)} F(B)\exp\left(-\varepsilon \mathrm{Str}\, B\right) d[V] = \widetilde{C}_2 \int\limits_{\Sigma_{\beta,c/d}^{\psi}} F\left(\hat{\rho}\right)\widetilde{I}(\rho)\exp\left(-\varepsilon \mathrm{Str}\,\hat{\rho}\right) d[\rho], \tag{B.64}$$

where

$$\begin{aligned}
\widetilde{I}(\rho) &= \int\limits_{\Sigma_{\beta,c/d}^{-\psi}} d[\sigma]\left(\frac{\det(e^{-\imath\psi}\sigma_2 + \imath\varepsilon \mathbb{1}_{\gamma_1 d})}{\det(\sigma_1 + \imath\varepsilon \mathbb{1}_{\gamma_2 c})}\right)^{a/\gamma_1} && \text{(B.65)}\\
&\times \exp\left[\varepsilon \mathrm{Str}\,\rho - \imath\left(\mathrm{tr}\,\rho_1\sigma_1 - \mathrm{tr}\,\rho_2\sigma_2 + \mathrm{tr}\,\sigma_{\tilde{\eta}}^{\dagger}\rho_1\sigma_{\tilde{\eta}}(\sigma_2 + \imath e^{\imath\psi}\varepsilon \mathbb{1}_{\gamma_1 d})^{-1}\right)\right].
\end{aligned}$$

We integrate over $d[\tilde{\eta}]$ and apply Eq. (8.10) for the $d[\sigma_1]$–integration. Then, Eq. (B.65) reads

$$\begin{aligned}
\widetilde{I}(\rho) &= \widetilde{C}_3 e^{-\imath\psi cd}\det\rho_1^{\kappa}\Theta(\rho_1) && \text{(B.66)}\\
&\times \int\limits_{\mathrm{Herm}\,(4/\beta,d)} \exp\left(-\imath \mathrm{tr}\,\rho_2(\sigma_2 + \imath e^{\imath\psi}\varepsilon \mathbb{1}_{\gamma_1 d})\right)\det(e^{-\imath\psi}\sigma_2 + \imath\varepsilon)^{(a-c)/\gamma_1} d[\sigma_2]
\end{aligned}$$

with the constant

$$\widetilde{C}_3 = \imath^{-\beta ac/2}\left(\frac{\tilde{\gamma}}{2\pi\imath}\right)^{cd} G_{a-c,c}^{(\beta)}, \tag{B.67}$$

see Eq. (8.12). The exponent κ is the same as in Eq. (8.34). As in App. B.4, we decompose σ_2 in angles and eigenvalues and integrate over the angles. Thus, we get the ordinary matrix Bessel function $\varphi_d^{(4/\beta)}(r_2, s_2)$, see Eq. (4.4), in Eq. (B.66) which are only for certain β and d explicitly known. However, the analog of the Sekiguchi differential operator for the ordinary matrix Bessel functions $D_{dr_2}^{(4/\beta)}$, see Eq. (8.17), fulfills the eigenvalue equation (B.17). Since the determinant of σ_2 stands in the numerator, we shift $\sigma_2 \to \sigma_2 - \imath e^{\imath\psi}\varepsilon \mathbb{1}_{\gamma_1 d}$ and replace the determinants in Eq. (B.66) by $D_{dr_2}^{(4/\beta)}$. After an integration over σ_2, we have

$$\widetilde{I}(\rho) = \widetilde{C}_4 e^{-\imath\psi cd} \det \rho_1^\kappa \Theta(\rho_1) \left(e^{-\imath\psi d} D_{dr_2}^{(4/\beta)}\right)^{a-c} \frac{\delta(r_2)}{|\Delta_d(e^{\imath\psi} r_2)|^{4/\beta}}. \tag{B.68}$$

The constant is

$$\widetilde{C}_4 = \imath^{-\beta ac/2} \left(\frac{\tilde{\gamma}}{2\pi\imath}\right)^{cd} G_{a-c,c}^{(\beta)} (\imath\gamma_1)^{(c-a)d} \left(\frac{\pi}{\gamma_1}\right)^{2d(d-1)/\beta} \left(\frac{2\pi}{\gamma_1}\right)^{d} \frac{1}{\mathrm{FU}_d^{(4/\beta)}}. \tag{B.69}$$

Summarizing the constants (B.56) and (B.69), we get

$$\widetilde{C}_{acd}^{(\beta)} = \widetilde{C}_2 \widetilde{C}_4 = 2^{-c} \left(2\pi\gamma_1\right)^{-ad} \left(\frac{2\pi}{\gamma_2}\right)^{cd} \tilde{\gamma}^{\beta ac/2} \frac{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a)\right)}{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a-c)\right) \mathrm{FU}_d^{(4/\beta)}}. \tag{B.70}$$

Due to the Dirac distribution, we shift $D_{dr_2}^{(4/\beta)}$ from the Dirac distribution to the superfunction and remove the Wick–rotation. Hence, we find the result of the theorem.

B.6 Proof of theorem 10.1.1

First, we consider two particular cases. Let $d = 0$ and $a \geq c$ be an arbitrary positive integer. Then, we find

$$B \in \Sigma_{\beta,c/0}^{0} = \Sigma_{\beta,c/0}^{(\mathrm{c})} \subset \mathrm{Herm}\,(\beta, c)\,. \tag{B.71}$$

We introduce a Fourier transformation

$$\int\limits_{\mathbb{R}^{\beta ac}} F(B) \exp\left(-\varepsilon \mathrm{tr}\, B\right) d[V] \tag{B.72}$$

$$= \left(\frac{\gamma_2}{2\pi}\right)^{c} \left(\frac{\gamma_2}{\pi}\right)^{\beta c(c-1)/2} \int\limits_{\mathrm{Herm}\,(\beta,c)} \int\limits_{\mathbb{R}^{\beta ac}} \mathcal{F}F(\sigma_1) \exp\left(\imath \mathrm{tr}\, B\sigma_1^+\right) d[V] d[\sigma_1]\,,$$

where the measure $d[\sigma_1]$ is defined as in theorems 3.3.1 and 3.3.2 and $\sigma_1^+ = \sigma_1 + \imath\varepsilon\mathbb{1}_{\gamma_2 c}$. The Fourier transform is

$$\mathcal{F}F(\sigma_1) = \int\limits_{\mathrm{Herm}\,(\beta,c)} F(\rho_1) \exp\left(-\imath \mathrm{tr}\, \rho_1\sigma_1\right) d[\rho_1]. \tag{B.73}$$

The integration over the supervectors, which are in this particular case ordinary vectors, yields

$$\int\limits_{\mathbb{R}^{\beta ac}} \exp\left(\imath \mathrm{tr}\, B\sigma_1^+\right) d[V] = \det\left(\frac{\sigma_1^+}{\imath\gamma_1\pi}\right)^{-a/\gamma_1}, \tag{B.74}$$

see Eqs. (2.34) and (8.29). The Fourier transform of this determinant is an Ingham–Siegel integral [119, 120], see Eq. (8.10). Thus, we find for Eq. (B.72)

$$\int\limits_{\mathbb{R}^{\beta ac}} F(B)\exp\left(-\varepsilon \mathrm{tr}\, B\right) d[V] = C_{ac0}^{(\beta)} \int\limits_{\Sigma_{\beta,c/0}^{(\mathrm{c})}} F(\rho)\exp\left(-\varepsilon \mathrm{tr}\,\rho_1\right)\det\rho_1^{\kappa} d[\rho_1] \tag{B.75}$$

with the exponent

$$\kappa = \frac{a-c+1}{\gamma_1} - \frac{1}{\gamma_2}, \tag{B.76}$$

which verifies this theorem. The product in the constant

$$C_{ac0}^{(\beta)} = 2^{-c}\tilde{\gamma}^{\beta ac/2}\frac{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a)\right)}{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a-c)\right)} \tag{B.77}$$

is a ratio of group volumes.

In the next case, we consider $c = 0$ and arbitrary d. We see that

$$B \in \Sigma_{\beta,0/d} \tag{B.78}$$

is true. We integrate over

$$\int\limits_{\Lambda_{2ad}} F(B)\exp\left(\varepsilon \mathrm{tr}\, B\right) d[V], \tag{B.79}$$

where the function F is analytic. As in Ref. [83], we expand $F(B)\exp\left(\varepsilon \mathrm{tr}\, B\right)$ in the entries of B and, then, integrate over every single term of this expansion. Every term is a product of B's entries and can be generated by differentiation of $(\mathrm{tr}\, AB)^n$ with respect to $A \in \Sigma_{\beta,0/d}^0$ for certain $n \in \mathbb{N}$. Thus, it is sufficient to proof the integral theorem

$$\int\limits_{\Lambda_{2ad}} (\mathrm{tr}\, AB)^n\, d[V] = C_{a0d}^{(\beta)} \int\limits_{\Sigma_{\beta,0/d}^{(\mathrm{c})}} (\mathrm{tr}\, A\rho_2)^n \det\rho_2^{-\kappa} d[\rho_2]\,. \tag{B.80}$$

Since $\Sigma^0_{\beta,0/d}$ is generated by $\Sigma^{(\mathrm{c})}_{\beta,0/d}$ by analytic continuation in the eigenvalues, it is convenient that $A \in \Sigma^{(\mathrm{c})}_{\beta,0/d}$. Then, $A^{-1/2}$ is well-defined and $A^{-1/2}\rho_2 A^{-1/2} \in \Sigma^{(\mathrm{c})}_{\beta,0/d}$. We transform in Eq. (B.80)

$$V \to A^{-1/2}V \quad , \quad V^\dagger \to V^\dagger A^{-1/2} \quad \text{and} \quad \rho_2 \to A^{-1/2}\rho_2 A^{-1/2} . \tag{B.81}$$

The measures turns under this change into

$$d[V] \;\; \longrightarrow \;\; \det A^{a/\gamma_1} d[V] \quad \text{and} \tag{B.82}$$

$$d[\rho_2] \;\; \longrightarrow \;\; \det A^{-\kappa+a/\gamma_1} d[\rho_2] \, , \tag{B.83}$$

where the exponent is

$$\kappa = \frac{a+1}{\gamma_1} + \frac{d-1}{\gamma_2} . \tag{B.84}$$

Hence, we have to calculate the remaining constant defined by

$$\int\limits_{\Lambda_{2ad}} (\mathrm{tr}\, B)^n \, d[V] = C^{(\beta)}_{a0d} \int\limits_{\Sigma^{(\mathrm{c})}_{\beta,0/d}} (\mathrm{tr}\, \rho_2)^n \det \rho_2^{-\kappa} d[\rho_2]. \tag{B.85}$$

This equation holds for arbitrary n. Then, this must also be valid for $F(B) = \varepsilon = 1$ in Eq. (B.79). The right hand side of Eq. (B.79) is

$$\int\limits_{\Lambda_{2ad}} \exp\left(\mathrm{tr}\, B\right) d[V] = (-2\pi)^{-ad} \;, \tag{B.86}$$

see Eqs. (2.34) and (8.29). On the left hand side, we first integrate over the group $\mathrm{U}^{(4/\beta)}(d)$ and get

$$\int\limits_{\Sigma^{(\mathrm{c})}_{\beta,0/d}} \exp\left(\mathrm{tr}\, \rho_2\right) \det \rho_2^{-\kappa} d[\rho_2] \tag{B.87}$$

$$= \;\; \mathrm{FU}^{(4/\beta)}_d \int\limits_{[0,2\pi]^d} |\Delta_d(e^{\imath\varphi_j})|^{4/\beta} \prod_{n=1}^{d} \exp\left(\gamma_1 e^{\imath\varphi_n}\right) e^{-\imath\varphi_n(\gamma_1\kappa-1)} \frac{d\varphi_n}{2\pi} .$$

We derive this integral with help of Selberg's integral formula [75]. We assume that $\tilde\beta = 4/\beta$ and $\gamma_1\kappa$ are arbitrary positive integers and $\tilde\beta$ is even. Then, we omit the absolute value and Eq. (B.87) becomes

$$\int\limits_{\Sigma^{(\mathrm{c})}_{\beta,0d}} \exp\left(\mathrm{tr}\, \rho_2\right) \det \rho_2^{-\kappa} d[\rho_2] = \mathrm{FU}^{(\beta)}_d \Delta^{\tilde\beta}_d \left(\frac{1}{\gamma_1}\frac{\partial}{\partial\lambda_j}\right) \prod_{n=1}^{d} \frac{(\gamma_1\lambda_n)^{\gamma_1\kappa-1}}{\Gamma\left(\gamma_1\kappa\right)}\Bigg|_{\lambda=1} . \tag{B.88}$$

We consider another integral which is the Laguerre version of Selberg's integral [75]

$$\begin{aligned}
&\int\limits_{\mathbb{R}_+^d} \Delta_d^{\tilde{\beta}}(x) \prod_{n=1}^{d} \exp\left(-\gamma_1 x_n\right) x_n^{\xi} dx_n \\
&= \Delta_d^{\tilde{\beta}}\left(-\frac{1}{\gamma_1}\frac{\partial}{\partial\lambda_j}\right) \prod_{n=1}^{d} \Gamma(\xi+1)\left(\gamma_1\lambda_n\right)^{-\xi-1}\Bigg|_{\lambda=1} \\
&= \prod_{n=1}^{d} \frac{\Gamma\left(1+n\tilde{\beta}/2\right)\Gamma\left(\xi+1+(n-1)\tilde{\beta}/2\right)}{\gamma_1^{\xi+1+\tilde{\beta}(d-1)/2}\Gamma\left(1+\tilde{\beta}/2\right)}, \qquad \text{(B.89)}
\end{aligned}$$

where ξ is an arbitrary positive integer. Since $\tilde{\beta}$ is even, the minus sign in the Vandermonde determinant vanishes. The equations (B.88) and (B.89) are up to the Gamma–functions polynomials in κ and ξ. We remind that (B.89) is true for every complex ξ. Let $\operatorname{Re}\xi > 0$, we have

$$\left|\Delta_d^{\tilde{\beta}}\left(-\frac{1}{\gamma_1}\frac{\partial}{\partial\lambda_j}\right) \prod_{n=1}^{d} \frac{\left(\gamma_1\lambda_n\right)^{-\xi-1}}{\Gamma\left(\xi+1+(n-1)\tilde{\beta}/2\right)}\Bigg|_{\lambda=1}\right| \le \text{const.} < \infty \qquad \text{(B.90)}$$

and

$$\left|\gamma_1^{-d(\xi+1+\tilde{\beta}(d-1)/2)} \prod_{n=1}^{d} \frac{\Gamma\left(1+n\tilde{\beta}/2\right)}{\Gamma(\xi+1)\Gamma\left(1+\tilde{\beta}/2\right)}\right| \le \text{const.} < \infty\,. \qquad \text{(B.91)}$$

The functions are bounded and regular for $\operatorname{Re}\xi > 0$ and we can apply Carlson's theorem [75]. We identify $\xi = -\gamma_1\kappa$ and find

$$\begin{aligned}
&\int\limits_{\Sigma_{\beta,0/d}^{(c)}} \exp\left(\operatorname{tr}\rho_2\right)\det\rho_2^{-\kappa} d[\rho_2] \qquad \text{(B.92)} \\
&= \gamma_1^{ad}\mathrm{FU}_d^{(4/\beta)} \prod_{n=1}^{d} \frac{\Gamma\left(1+n\tilde{\beta}/2\right)\Gamma\left(1-\gamma_1\kappa+(n-1)\tilde{\beta}/2\right)}{\Gamma\left(1+\tilde{\beta}/2\right)\Gamma\left(\gamma_1\kappa\right)\Gamma\left(1-\gamma_1\kappa\right)}.
\end{aligned}$$

Due to Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, this equation simplifies to

$$\int\limits_{\Sigma_{\beta,0/d}^{(c)}} \exp\left(\operatorname{tr}\rho_2\right)\det\rho_2^{-\kappa} d[\rho_2] = \mathrm{FU}_d^{(4/\beta)} \prod_{n=1}^{d} \frac{\imath^{4(n-1)/\beta}\gamma_1^{a}\Gamma\left(1+2n/\beta\right)}{\Gamma\left(1+2/\beta\right)\Gamma\left(\gamma_1\kappa-2(n-1)/\beta\right)} \qquad \text{(B.93)}$$

or equivalent

$$\begin{aligned}
&2^{\tilde\beta d(d-1)/2}\int\limits_{[0,2\pi]^d}\prod_{1\le n<m\le d}\left|\sin\left(\frac{\varphi_n-\varphi_m}{2}\right)\right|^{\tilde\beta}\prod_{n=1}^{d}\exp\left(\gamma_1 e^{\imath\varphi_n}\right)e^{-\imath\varphi_n a}\frac{d\varphi_n}{2\pi}\\
&=\ \gamma_1^{ad}\prod_{n=1}^{d}\frac{\Gamma\left(1+n\tilde\beta/2\right)}{\Gamma\left(1+\tilde\beta/2\right)\Gamma\left(a+1+(n-1)\tilde\beta/2\right)}\,. \qquad \text{(B.94)}
\end{aligned}$$

Since a is a positive integer for all positive and even $\tilde\beta$, the equations above are true for all such $\tilde\beta$. For constant natural numbers a, d, γ_1 and complex $\tilde\beta$ with $\operatorname{Re}\tilde\beta>0$, the inequalities

$$\begin{aligned}
&\left|\int\limits_{[0,2\pi]^d}\prod_{1\le n<m\le d}\left|\sin\left(\frac{\varphi_n-\varphi_m}{2}\right)\right|^{\tilde\beta}\prod_{n=1}^{d}\exp\left(\gamma_1 e^{\imath\varphi_n}\right)e^{-\imath\varphi_n a}\frac{d\varphi_n}{2\pi}\right|\\
&\le\ \int\limits_{[0,2\pi]^d}\prod_{1\le n<m\le d}\left|\sin\left(\frac{\varphi_n-\varphi_m}{2}\right)\right|^{\operatorname{Re}\tilde\beta}\prod_{n=1}^{d}\exp\left(\gamma_1\cos\varphi_n\right)\frac{d\varphi_n}{2\pi}<\infty \qquad \text{(B.95)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|2^{-\tilde\beta d(d-1)/2}\gamma_1^{ad}\prod_{n=1}^{d}\frac{\Gamma\left(1+n\tilde\beta/2\right)}{\Gamma\left(1+\tilde\beta/2\right)\Gamma\left(a+1+(n-1)\tilde\beta/2\right)}\right|\\
&\le\ \text{const.}\ 2^{-\operatorname{Re}\tilde\beta d(d-1)/2}<\infty \qquad \text{(B.96)}
\end{aligned}$$

are valid and allow us with Carlson's theorem to extend Eq. (B.94) to every complex $\tilde\beta$, in particular to $\tilde\beta=1$. Thus, we find for the constant in Eq. (B.85)

$$C_{a0d}=(-2\pi\gamma_1)^{-ad}\left[\prod_{n=1}^{d}\frac{\imath^{4(n-1)/\beta}\pi^{2(n-1)/\beta}}{\Gamma(a+1+2(n-1)/\beta)}\right]^{-1}. \qquad \text{(B.97)}$$

Now, we consider arbitrary d and $a\ge c$ and split

$$\Psi_{j1}^{(\mathrm{C})}=\left[\begin{array}{c}\mathbf{x}_j\\ \chi_j\end{array}\right] \qquad \text{(B.98)}$$

and

$$B=\frac{1}{\tilde\gamma}\sum_{j=1}^{a}\Psi_{j1}^{(\mathrm{C})}\Psi_{j1}^{(\mathrm{C})\dagger}=\left[\begin{array}{c|c}\displaystyle\sum_{j=1}^{a}\frac{\mathbf{x}_j\mathbf{x}_j^\dagger}{\tilde\gamma} & \displaystyle\sum_{j=1}^{a}\frac{\mathbf{x}_j\chi_j^\dagger}{\tilde\gamma}\\ \hline \displaystyle\sum_{j=1}^{a}\frac{\chi_j\mathbf{x}_j^\dagger}{\tilde\gamma} & \displaystyle\sum_{j=1}^{a}\frac{\chi_j\chi_j^\dagger}{\tilde\gamma}\end{array}\right]=\left[\begin{array}{cc}B_{11} & B_{12}\\ B_{21} & B_{22}\end{array}\right] \qquad \text{(B.99)}$$

such that $\mathbf{x}_j$ contains all commuting variables of $\Psi_{j1}^{(\mathrm{C})}$ and χ_j depends on all Grassmann variables. Then, we replace the sub-matrices B_{12}, B_{21} and B_{22} by Dirac distributions

$$\begin{aligned}
&\int\limits_{\mathrm{Mat}\,^0_\beta(c\times a/d)} F(B)\exp\left(-\varepsilon\mathrm{Str}\, B\right) d[V] \qquad \text{(B.100)}\\
= \;& C_1 \int\limits_{\mathrm{Herm}\,(4/\beta,d)^2}\ \int\limits_{\mathrm{Mat}\,^0_\beta(c\times a/d)}\ \int\limits_{(\Lambda_{2cd})^2} d[\eta]d[\tilde{\eta}]d[V]d[\tilde{\rho}_2]d[\sigma_2] F\left(\begin{bmatrix} B_{11} & \rho_\eta \\ -\rho_\eta^\dagger & \tilde{\rho}_2\end{bmatrix}\right)\\
&\times\exp\left[-\varepsilon\mathrm{Str}\, B - \imath\left(\mathrm{tr}\,(\rho_\eta^\dagger + B_{21})\sigma_{\tilde{\eta}} + \mathrm{tr}\,\sigma_{\tilde{\eta}}^\dagger(\rho_\eta - B_{12}) - \mathrm{tr}\,(\tilde{\rho}_2 - B_{22})\sigma_2\right)\right],
\end{aligned}$$

where

$$C_1 = \left(\frac{2\pi}{\tilde{\gamma}}\right)^{2cd}\left(\frac{\gamma_1}{\pi}\right)^{2d(d-1)/\beta}\left(\frac{\gamma_1}{2\pi}\right)^{d}. \qquad \text{(B.101)}$$

The matrices ρ_η and $\sigma_{\tilde{\eta}}$ are rectangular matrices depending on Grassmann variables. Shifting $\chi_j \to \chi_j + \left(\sigma_2^+\right)^{-1}\sigma_{\tilde{\eta}}^\dagger\mathbf{x}_j$ and $\chi_j^\dagger \to \chi_j^\dagger - \mathbf{x}_j^\dagger\sigma_{\tilde{\eta}}\left(\sigma_2^+\right)^{-1}$, we get

$$\begin{aligned}
&\int\limits_{\mathrm{Mat}\,^0_\beta(c\times a/d)} F(B)\exp\left(-\varepsilon\mathrm{Str}\, B\right) d[V] \qquad \text{(B.102)}\\
= \;& C_1 \int\limits_{\mathrm{Herm}\,(4/\beta,d)^2}\ \int\limits_{\mathrm{Mat}\,^0_\beta(c\times a/d)}\ \int\limits_{(\Lambda_{2cd})^2} d[\eta]d[\tilde{\eta}]d[\widehat{V}]d[\tilde{\rho}_2]d[\sigma_2] F\left(\begin{bmatrix} B_{11} & \rho_\eta \\ -\rho_\eta^\dagger & \tilde{\rho}_2\end{bmatrix}\right)\\
&\times\exp\left[-\varepsilon\mathrm{Str}\, B - \imath\mathrm{tr}\,\left(\sigma_{\tilde{\eta}}^\dagger B_{11}\sigma_{\tilde{\eta}}\left(\sigma_2^+\right)^{-1} + \rho_\eta^\dagger\sigma_{\tilde{\eta}} + \sigma_{\tilde{\eta}}^\dagger\rho_\eta - (\tilde{\rho}_2 - B_{22})\sigma_2\right)\right].
\end{aligned}$$

This integral only depends on B_{11} and B_{22}. Thus, we apply the first case of this proof and replace B_{11}. We find

$$\begin{aligned}
&\int\limits_{\mathrm{Mat}\,^0_\beta(c\times a/d)} F(B)\exp\left(-\varepsilon\mathrm{Str}\, B\right) d[V] \qquad \text{(B.103)}\\
= \;& C_{ac0}^{(\beta)} C_1 \int\limits_{\mathrm{Herm}\,(4/\beta,d)^2}\ \int\limits_{\Sigma_{\beta,c/0}^{(\mathrm{c})}}\ \int\limits_{(\Lambda_{2cd})^2}\ \int\limits_{\Lambda_{2ac}} d[\chi]d[\eta]d[\tilde{\eta}]d[\rho_1]d[\tilde{\rho}_2]d[\sigma_2]\\
&\times F\left(\begin{bmatrix} \rho_1 & \rho_\eta \\ -\rho_\eta^\dagger & \tilde{\rho}_2\end{bmatrix}\right)\det\rho_1^{\tilde{\kappa}}\exp\left[\varepsilon(\mathrm{tr}\, B_{22} - \mathrm{tr}\,\rho_1)\right]\\
&\times\exp\left[\imath\mathrm{tr}\,\left(\sigma_{\tilde{\eta}}^\dagger\rho_1\sigma_{\tilde{\eta}}\left(\sigma_2^+\right)^{-1} - \rho_\eta^\dagger\sigma_{\tilde{\eta}} - \sigma_{\tilde{\eta}}^\dagger\rho_\eta + (\tilde{\rho}_2 - B_{22})\sigma_2\right)\right]
\end{aligned}$$

with the exponent

$$\tilde{\kappa} = \frac{a-c+1}{\gamma_1} - \frac{1}{\gamma_2}. \tag{B.104}$$

After another shifting $\sigma_{\tilde{\eta}} \to \sigma_{\tilde{\eta}} - \rho_1^{-1}\rho_\eta\sigma_2^+$ and $\sigma_{\tilde{\eta}}^\dagger \to \sigma_{\tilde{\eta}}^\dagger - \sigma_2^+\rho_\eta^\dagger\rho_1^{-1}$, we integrate over $d[\tilde{\eta}]$ and B_{22} and have

$$\begin{aligned}
&\int\limits_{\mathrm{Mat}_\beta^0(c\times a/d)} F(B)\exp\left(-\varepsilon\mathrm{Str}\,B\right)d[V] \qquad\qquad (\mathrm{B}.105)\\
&= C_{ac0}^{(\beta)}C_2 \int\limits_{\Sigma_{\beta,c/0}^{(\mathrm{c})}} \int\limits_{\mathrm{Herm}\,(4/\beta,d)^2} \int\limits_{\Lambda_{2cd}} d[\eta]d[\rho_1]d[\tilde{\rho}_2]d[\sigma_2]F\left(\left[\begin{array}{cc}\rho_1 & \rho_\eta\\ -\rho_\eta^\dagger & \tilde{\rho}_2\end{array}\right]\right)\\
&\times\ \det\rho_1^\kappa\det\left(\sigma_2^+\right)^{(a-c)/\gamma_1}\exp\left[-\varepsilon\mathrm{tr}\,\rho_1 + \imath\mathrm{tr}\,\left(\rho_\eta^\dagger\rho_1^{-1}\rho_\eta\sigma_2^+ + \tilde{\rho}_2\sigma_2\right)\right],
\end{aligned}$$

where the exponent is

$$\kappa = \frac{a-c+1}{\gamma_1} + \frac{d-1}{\gamma_2} \tag{B.106}$$

and the new constant is

$$C_2 = \left(\frac{\imath}{2\pi}\right)^{ad}\left(\frac{2\pi}{\tilde{\gamma}\imath}\right)^{cd}\left(\frac{\gamma_1}{\pi}\right)^{2d(d-1)/\beta}\left(\frac{\gamma_1}{2\pi}\right)^{d}. \tag{B.107}$$

We express the determinant in σ_2^+ as in Sec. 7.1 as Gaussian integrals and define a new $(\gamma_2(a-c)+0)\times(0+\gamma_1 d)$ rectangular supermatrix V_{new} and its corresponding $(0+\gamma_1 d)\times(0+\gamma_1 d)$ supermatrix $B_{\mathrm{new}} = \tilde{\gamma}^{-1}V_{\mathrm{new}}V_{\mathrm{new}}^\dagger$. Integrating σ_2 and ρ_2, Eq. (B.105) becomes

$$\begin{aligned}
&\int\limits_{\mathrm{Mat}_\beta^0(c\times a/d)} F(B)\exp\left(-\varepsilon\mathrm{Str}\,B\right)d[V] \qquad\qquad (\mathrm{B}.108)\\
&= \tilde{\gamma}^{-cd}C_{ac0}^{(\beta)} \int\limits_{\Sigma_{\beta,c/0}^{(\mathrm{c})}} \int\limits_{\Lambda_{2(a-c)d}} F\left(\left[\begin{array}{c|c}\rho_1 & \rho_\eta\\ \hline -\rho_\eta^\dagger & B_{\mathrm{new}} - \rho_\eta^\dagger\rho_1^{-1}\rho_\eta\end{array}\right]\right)\\
&\times\ \exp\left(-\varepsilon\mathrm{tr}\,\rho_1 + \varepsilon\mathrm{tr}\left(B_{\mathrm{new}} - \eta^\dagger\rho_1^{-1}\eta\right)\right)\det\rho_1^\kappa d[V_{\mathrm{new}}]d[\eta]d[\rho_1].
\end{aligned}$$

Now, we apply the second case in this proof and shift $\rho_2 \to \rho_2 + \rho_\eta^\dagger\rho_1^{-1}\rho_\eta$ by analytic continuation. We get the final result

$$\int\limits_{\mathrm{Mat}_\beta^0(c\times a/d)} F(B)\exp\left(-\varepsilon\mathrm{Str}\,B\right)d[V] = C_{acd}^{(\beta)} \int\limits_{\Sigma_{\beta,c/d}^{(\mathrm{c})}} F\left(\rho\right)\exp\left(-\varepsilon\mathrm{Str}\,\rho\right)\mathrm{Sdet}\,\rho^\kappa d[\rho] \tag{B.109}$$

with

$$
\begin{aligned}
& C_{acd}^{(\beta)} = \tilde{\gamma}^{-cd} C_{ac0}^{(\beta)} C_{a-c,0d}^{(\beta)} && \text{(B.110)}\\
= \;& (-2\pi\gamma_1)^{-ad}\left(-\frac{2\pi}{\gamma_2}\right)^{cd} 2^{-c}\tilde{\gamma}^{\beta ac/2}\\
\times \;& \frac{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a)\right)}{\mathrm{Vol}\left(\mathrm{U}^{(\beta)}(a-c)\right)} \prod_{n=1}^{d} \frac{\Gamma\left(\gamma_1\kappa + 2(n-d)/\beta\right)}{\imath^{4(n-1)/\beta}\pi^{2(n-1)/\beta}}\\
= \;& \imath^{-2d(d-1)/\beta}\frac{(2\pi)^d\tilde{\gamma}^{\beta ac/2-cd}}{(-2)^{(c-a)d}2^c}\begin{cases} \dfrac{2^{d^2}\mathrm{Vol}\left(\mathrm{U}^{(1)}(a)\right)}{\mathrm{Vol}\left(\mathrm{U}^{(1)}(a-c+2d)\right)} & ,\ \beta=1\\ \dfrac{\mathrm{Vol}\left(\mathrm{U}^{(2)}(a)\right)}{\mathrm{Vol}\left(\mathrm{U}^{(2)}(a-c+d)\right)} & ,\ \beta=2\\ \dfrac{2^{-(2a+1-c)c}\mathrm{Vol}\left(\mathrm{U}^{(1)}(2a+1)\right)}{\mathrm{Vol}\left(\mathrm{U}^{(1)}(2(a-c)+d+1)\right)} & ,\ \beta=4 \end{cases} .
\end{aligned}
$$

This constant considerably simplifies for the case $d = \beta c/2 = \gamma_2 k$ considered in Sec. 7.

B.7 Proof of theorem 10.2.1

We define the function

$$
\begin{aligned}
\widetilde{F}(r_2) \;=\; & \int\limits_{\mathrm{U}^{4/\beta}(d)} \int\limits_{\mathrm{Herm}\,(\beta,c)} \int\limits_{\Lambda_{2cd}} F\left(\left[\begin{array}{c|c} \rho_1 & \rho_\eta \\ \hline -\rho_\eta^\dagger & Ur_2U^\dagger - \rho_\eta^\dagger\rho_1^{-1}\rho_\eta \end{array}\right]\right)\\
& \times\ \exp\left[-\varepsilon(\mathrm{tr}\,\rho_1 - \mathrm{tr}\,(r_2 - \rho_\eta^\dagger\rho_1^{-1}\rho_\eta)\right] \det{}^\kappa\rho_1 d[\eta]d[\rho_1]d\mu(U)\,. \quad \text{(B.111)}
\end{aligned}
$$

Then, we have to prove

$$
\begin{aligned}
& C_{acd}^{(\beta)} \int\limits_{[0,2\pi]^d} \widetilde{F}\left(e^{\imath\varphi_j}\right)\left|\Delta_d\left(e^{\imath\varphi_j}\right)\right|^{4/\beta}\prod_{n=1}^{d}\frac{e^{\imath(1-\kappa)\varphi_n}d\varphi_n}{2\pi}\\
= \;& \widetilde{C}_{acd}^{(\beta)}\left((-1)^d D_{dr_2}^{(4/\beta)}\right)^{a-c}\widetilde{F}(r_2)\Big|_{r_2=0}\,. \qquad \text{(B.112)}
\end{aligned}
$$

Since $\widetilde{F}$ is permutation invariant and a Schwartz function, we express $\widetilde{F}$ as an integral over ordinary matrix Bessel functions,

$$\widetilde{F}(r_2) = \int\limits_{\mathbb{R}^d} g(q)\varphi_d^{(4/\beta)}(r_2,q)|\Delta_d(q)|^{4/\beta}dq, \tag{B.113}$$

where g is a Schwartz function. The integral and the differential operator in Eq. (B.112) commute with the integral in Eq. (B.113). Thus, we only need to prove

$$\begin{aligned} & C_{acd}^{(\beta)} \int\limits_{[0,2\pi]^d} \varphi_d^{(4/\beta)}\left(e^{\imath\varphi_j},q\right)\left|\Delta_d\left(e^{\imath\varphi_j}\right)\right|^{4/\beta} \prod_{n=1}^{d} \frac{e^{\imath(1-\gamma_1\kappa)\varphi_n}d\varphi_n}{2\pi} \\ = \; & \widetilde{C}_{acd}^{(\beta)} \left((-1)^d D_{dr_2}^{(4/\beta)}\right)^{a-c} \varphi_d^{(4/\beta)}(r_2,q)\bigg|_{r_2=0} \end{aligned} \tag{B.114}$$

for all $q \in \mathcal{S}_1^d$ where $\mathcal{S}_1$ is the unit–circle in the complex plane. The right hand side of this equation is with help of Eq. (B.17)

$$\left(D_{dr_2}^{(4/\beta)}\right)^{a-c} \varphi_d^{(4/\beta)}(r_2,q)\bigg|_{r_2=0} = (-\imath\gamma_1)^{d(a-c)}\det q^{(a-c)/\gamma_1}. \tag{B.115}$$

The components of q are complex phase factors. The integral representation of the ordinary matrix Bessel functions (4.4) and the $d[\varphi]$–integral in Eq. (B.114) form the integral over the circular ensembles CU$^{(4/\beta)}(d)$. Thus, q can be absorbed by $e^{\imath\varphi_j}$ and we find

$$\begin{aligned} & \int\limits_{[0,2\pi]^d} \varphi_d^{(4/\beta)}\left(e^{\imath\varphi_j},q\right)\left|\Delta_d\left(e^{\imath\varphi_j}\right)\right|^{4/\beta} \prod_{n=1}^{d} \frac{e^{\imath(1-\gamma_1\kappa)\varphi_n}d\varphi_n}{2\pi} \\ = \; & \det q^{(a-c)/\gamma_1} \int\limits_{[0,2\pi]^d} \varphi_d^{(4/\beta)}\left(e^{\imath\varphi_j},1\right)\left|\Delta_d\left(e^{\imath\varphi_j}\right)\right|^{4/\beta} \prod_{n=1}^{d} \frac{e^{\imath(1-\gamma_1\kappa)\varphi_n}d\varphi_n}{2\pi}. \end{aligned} \tag{B.116}$$

The ordinary matrix Bessel function is at $q=1$ the exponential function

$$\varphi_d^{(4/\beta)}\left(e^{\imath\varphi_j},1\right) = \exp\left(\imath\gamma_1\sum_{n=1}^{d} e^{\imath\varphi_n}\right). \tag{B.117}$$

With Eq. (B.94), the integral on the left hand side in Eq. (B.116) yields with this exponential function

$$\begin{aligned}
&\int\limits_{[0,2\pi]^d} |\Delta_d\left(e^{\imath\varphi_j}\right)|^{4/\beta} \prod_{n=1}^{d} \frac{e^{\imath(1-\gamma_1\kappa)\varphi_n}\exp\left(\imath\gamma_1 e^{\imath\varphi_n}\right)d\varphi_n}{2\pi} \\
= \quad & (\imath\gamma_1)^{(a-c)d} \prod_{n=1}^{d} \frac{\imath^{4(n-1)/\beta}\Gamma\left(1+2n/\beta\right)}{\Gamma\left(1+2/\beta\right)\Gamma\left(a-c+1+2(n-1)/\beta\right)} \\
= \quad & \frac{(\imath\gamma_1)^{(a-c)d}}{\mathrm{FU}_d^{(4/\beta)}} \prod_{n=1}^{d} \frac{\imath^{4(n-1)/\beta}\pi^{2(n-1)/\beta}}{\Gamma\left(a-c+1+2(n-1)/\beta\right)}. \qquad \text{(B.118)}
\end{aligned}$$

Hence, the ratio of the constants on both sides in Eq. (B.112) times the constant in Eq. (B.115) equals to this expression.

Appendix C

Derivations for Part III

C.1 Derivation of the Berezinians

In App. C.1.1 we derive the determinantal structure for $\beta = 2$. The derivations for the other two cases $p \leq 2q$ and $p \geq 2q$ for $\beta \in \{1, 4\}$ are given in appendices C.1.2 and C.1.3, respectively.

C.1.1 The case $\beta = 2$

Let $p \leq q$. The trick is to extend the ratio of products by additional variables $\mathrm{diag}\,(\kappa_{p+1,1}, \ldots, \kappa_{q,1})$. Then, we apply Eq. (9.4). Since these additional variables are artificially introduced and, hence, arbitrary, we perform the limit $\mathrm{diag}\,(\kappa_{p+1,1}, \ldots, \kappa_{q,1}) \to 0$. We find

$$\begin{aligned}
&\frac{\prod\limits_{1\leq a<b\leq p} (\kappa_{a1} - \kappa_{b1}) \prod\limits_{1\leq a<b\leq q} (\kappa_{a2} - \kappa_{b2})}{\prod\limits_{a=1}^{p} \prod\limits_{b=1}^{q} (\kappa_{a1} - \kappa_{b2})} \\
= \quad & \frac{\prod\limits_{a=p+1}^{q} \prod\limits_{b=1}^{q} (\kappa_{a1} - \kappa_{b2})}{\prod\limits_{p+1\leq a<b\leq q} (\kappa_{a1} - \kappa_{b1}) \prod\limits_{a=1}^{p} \prod\limits_{b=p+1}^{q} (\kappa_{a1} - \kappa_{b1})} \\
\times \quad & \frac{\prod\limits_{1\leq a<b\leq q} (\kappa_{a1} - \kappa_{b1}) (\kappa_{a2} - \kappa_{b2})}{\prod\limits_{1\leq a,b\leq q} (\kappa_{a1} - \kappa_{b2})}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{q(q+1)/2+pq}\prod\limits_{a=1}^{q}\prod\limits_{b=p+1}^{q}(\kappa_{a2}-\kappa_{b1})}{\prod\limits_{p+1\leq a<b\leq q}(\kappa_{a1}-\kappa_{b1})\prod\limits_{a=1}^{p}\prod\limits_{b=p+1}^{q}(\kappa_{a1}-\kappa_{b1})} \\
&\times\ \det\left[\frac{1}{\kappa_{a1}-\kappa_{b2}}\right]_{1\leq a,b\leq q}\Bigg|_{\kappa_{p+1,1}=\ldots=\kappa_{q,1}=0} \\
&= (-1)^{q(q+1)/2+pq}\frac{\det\left[\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right]_{1\leq a,b\leq q}}{\prod\limits_{p+1\leq a<b\leq q}(\kappa_{a1}-\kappa_{b1})}\Bigg|_{\kappa_{p+1,1}=\ldots=\kappa_{q,1}=0} \mathrm{Sdet}^{\,p-q}\kappa\,. \qquad \text{(C.1)}
\end{aligned}
$$

This yields the result (14.2) with help of l'Hospital's rule.

C.1.2 The case $\beta \in \{1,4\}$ with $p \leq 2q$

Let $p \leq 2q$. This calculation is similar to the one in App. C.1.1. The only difference is that we have to take Kramers' degeneracy in the fermionic eigenvalues $\mathrm{diag}\,(\kappa_{12},\ldots,\kappa_{q2})$ into account. We first use non-degenerate entries to reduce the problem to the one for $\beta = 2$. Finally, we restore Kramers' degeneracy. We find

$$
\begin{aligned}
&\frac{\prod\limits_{1\leq a<b\leq p}(\kappa_{a1}-\kappa_{b1})\prod\limits_{1\leq a<b\leq q}(\kappa_{a2}-\kappa_{b2})^4}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(\kappa_{a1}-\kappa_{b2})^2} \\
&= (-1)^{q(q-1)/2}\frac{\prod\limits_{1\leq a<b\leq p}(\kappa_{a1}-\kappa_{b1})\prod\limits_{1\leq a<b\leq 2q}(\kappa_{a2}-\kappa_{b2})}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{2q}(\kappa_{a1}-\kappa_{b2})\prod\limits_{j=1}^{q}(\kappa_{j2}-\kappa_{j+q,2})}\Bigg|_{\kappa_{j2}=\kappa_{j+q,2}} \\
&= (-1)^{q(q+1)/2+p}\prod_{j=1}^{q}\frac{1}{(\kappa_{j2}-\kappa_{j+q,2})}\det\left[\begin{array}{c}\left\{\dfrac{\kappa_{a1}^{p-2q}\kappa_{b2}^{2q-p}}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq p\\1\leq b\leq 2q}}\\ \left\{\kappa_{b2}^{a-1}\right\}_{\substack{1\leq a\leq 2q-p\\1\leq b\leq 2q}}\end{array}\right]\Bigg|_{\kappa_{j2}=\kappa_{j+q,2}}. \qquad \text{(C.2)}
\end{aligned}
$$

This directly leads to Eq. (14.8).

C.1.3 The case $\beta \in \{1,4\}$ with $p \geq 2q$

Let $p \geq 2q$. Some modifications of the line of arguing in C.1.2 are necessary, we find

$$
\begin{aligned}
& \frac{\prod\limits_{1\leq a<b\leq p}(\kappa_{a1}-\kappa_{b1})\prod\limits_{1\leq a<b\leq q}(\kappa_{a2}-\kappa_{b2})^4}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(\kappa_{a1}-\kappa_{b2})^2} \\
= \;& (-1)^{q(q-1)/2}\left.\frac{\prod\limits_{1\leq a<b\leq p}(\kappa_{a1}-\kappa_{b1})\prod\limits_{1\leq a<b\leq 2q}(\kappa_{a2}-\kappa_{b2})}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{2q}(\kappa_{a1}-\kappa_{b2})\prod\limits_{j=1}^{q}(\kappa_{j2}-\kappa_{j+q,2})}\right|_{\kappa_{j2}=\kappa_{j+q,2}} \qquad (C.3)\\
= \;& (-1)^{[p(p-1)+q(q-1)]/2} \\
\times \;& \prod_{j=1}^{q}\frac{(\kappa_{j2}\kappa_{j+q,2})^{2q-p}}{(\kappa_{j2}-\kappa_{j+q,2})}\det\left.\left[\begin{array}{cc}\left\{\dfrac{\kappa_{a1}^{p-2q}}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq p\\ 1\leq b\leq 2q}} & \left\{\kappa_{a2}^{b-1}\right\}_{\substack{1\leq a\leq p\\ 1\leq b\leq p-2q}}\end{array}\right]\right|_{\kappa_{j2}=\kappa_{j+q,2}},
\end{aligned}
$$

which implies formula (14.9).

C.2 Calculating integrals of square root–Berezinian type

We derive Eqs. (15.3), (15.11) and (15.12) in appendices C.2.1, C.2.2 and C.2.3, respectively.

C.2.1 The case $k_1 = k_2 = k$

We calculate

$$
\begin{aligned}
& \int\limits_{\mathbb{C}^{N_1+N_2}}\prod_{j=1}^{N_1}g_j(z_{j1})\prod_{j=1}^{N_2}f_j(z_{j2})\sqrt{\mathrm{Ber}\,^{(2)}_{N_1+k/N_2+k}(\tilde{z})}d[z] \qquad (C.4)\\
= \;& (-1)^{(N_2+k)(N_2+k-1)/2}\int\limits_{\mathbb{C}^{N_1+N_2}}\prod_{j=1}^{N_1}g_j(z_{j1})\prod_{j=1}^{N_2}f_j(z_{j2})
\end{aligned}
$$

$$\times \quad \det\left[\begin{array}{cc} \left\{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{1\leq a,b\leq k} & \left\{\dfrac{1}{\kappa_{a1}-z_{b2}}\right\}_{\substack{1\leq a\leq k\\ 1\leq b\leq N_2}} \\ \left\{\kappa_{b2}^{a-1}\right\}_{\substack{1\leq a\leq N_2-N_1\\ 1\leq b\leq k}} & \left\{z_{b2}^{a-1}\right\}_{\substack{1\leq a\leq N_2-N_1\\ 1\leq b\leq N_2}} \\ \left\{\dfrac{1}{z_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq N_1\\ 1\leq b\leq k}} & \left\{\dfrac{1}{z_{a1}-z_{b2}}\right\}_{\substack{1\leq a\leq N_1\\ 1\leq b\leq N_2}} \end{array}\right] d[z]\,.$$

We use the definitions (15.4) to (15.6). The integral (C.4), then, reads

$$\begin{aligned} &\int\limits_{\mathbb{C}^{N_1+N_2}} \prod_{j=1}^{N_1} g_j(z_{j1}) \prod_{j=1}^{N_2} f_j(z_{j2}) \sqrt{\mathrm{Ber}\,^{(2)}_{N_1+k/N_2+k}(\tilde{z})}\, d[z] \qquad \text{(C.5)} \\ = \quad & (-1)^{(N_2+k)(N_2+k-1)/2} \det\left[\begin{array}{cc} \left\{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{1\leq a,b\leq k} & \left\{\mathbf{F}^{(N_2)}(\kappa_{a1})\right\}_{1\leq a\leq k} \\ \left\{\mathbf{G}^{(N_1/N_2)}(\kappa_{b2})\right\}_{1\leq b\leq k} & \mathbf{M}_{N_1/N_2} \end{array}\right]. \end{aligned}$$

The next step is to extract the matrix $\mathbf{M}_{N_1/N_2}$ from the determinant with help of Eq. (13.8). This yields

$$\begin{aligned} &\int\limits_{\mathbb{C}^{N_1+N_2}} \prod_{j=1}^{N_1} g_j(z_{j1}) \prod_{j=1}^{N_2} f_j(z_{j2}) \sqrt{\mathrm{Ber}\,^{(2)}_{N_1+k/N_2+k}(\tilde{z})}\, d[z] \qquad \text{(C.6)} \\ = \quad & (-1)^{(N_2+k)(N_2+k-1)/2} \det \mathbf{M}_{N_1/N_2} \det\left[K^{(N_1/N_2)}(\kappa_{a1},\kappa_{b2})\right]_{1\leq a,b\leq k} \end{aligned}$$

with $K^{(N_1/N_2)}$ as in definition (15.7).

C.2.2 The case $k_1 \leq k_2 = k$

Let $k_1 \leq k_2$. Then, we have

$$Z_{k_1/k_2}^{(N_1/N_2)}(\kappa) = (-1)^{(k_2-k_1)N_1} \lim_{\kappa_{k_1+1,1},\ldots,\kappa_{k_2,1}\to\infty} \prod_{j=k_1+1}^{k_2} \kappa_{j1}^{N_2-N_1} Z_{k_2/k_2}^{(N_1/N_2)}(\kappa) \qquad \text{(C.7)}$$

With help of Eq. (15.10), we obtain

$$\begin{aligned} Z_{k_1/k_2}^{(N_1/N_2)}(\kappa) = {}& \frac{(-1)^{k_1(k_1-1)/2+(k_2-k_1)N_1}}{C_{N_1/N_2}^{k_2-1}\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2}(\kappa)}} \qquad \text{(C.8)} \\ \times {}& \lim_{\kappa_{k_1+1,1},\ldots,\kappa_{k_2,1}\to\infty} \prod_{j=k_1+1}^{k_2} \kappa_{j1}^{N_2-N_1+k_2-k_1} \frac{\det\left[\dfrac{Z_{1/1}^{(N_1/N_2)}(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}}\right]_{1\leq a,b\leq k_2}}{\det\left[\kappa_{b1}^{a-1}\right]_{\substack{1\leq a\leq k_2-k_1\\ k_1+1\leq b\leq k_2}}} \end{aligned}$$

The limit expression is a function in $1/\kappa_{a1}$ which is differentiable at $1/\kappa_{a1} = 0$. Hence, using l'Hospital's rule we find Eq. (15.11)

C.2.3 The case $k = k_1 \geq k_2$

For $k_1 \geq k_2$, we have to proceed in a similar way. We extend the number of the fermionic eigenvalues κ_2 and find

$$\begin{aligned}
& Z_{k_1/k_2}^{(N_1/N_2)}(\kappa) \\
= \ & (-1)^{(k_1-k_2)(N_2-N_1)} \lim_{\kappa_{k_2+1,2},\ldots,\kappa_{k_1,2}\to\infty} \prod_{j=k_2+1}^{k_1} \kappa_{j2}^{N_1-N_2} Z_{k_1/k_1}^{(N_1/N_2)}(\kappa) \\
= \ & \frac{(-1)^{(k_2+2k_1)(k_2-1)/2+(k_1-k_2)(N_2-N_1)}}{C_{N_1/N_2}^{k_1-1}\sqrt{\mathrm{Ber}_{k_1/k_2}^{(2)}(\kappa)}} \qquad \qquad (\mathrm{C.9}) \\
\times \ & \lim_{\kappa_{k_1+1,1},\ldots,\kappa_{k_2,1}\to\infty} \prod_{j=k_2+1}^{k_1} \kappa_{j2}^{N_1-N_2+k_1-k_2} \frac{\det\left[\dfrac{Z_{1/1}^{(N_1/N_2)}(\kappa_{b1},\kappa_{a2})}{\kappa_{b1}-\kappa_{a2}}\right]_{1\leq a,b\leq k_1}}{\det\left[\kappa_{b2}^{a-1}\right]_{\substack{1\leq a\leq k_1-k_2 \\ k_2+1\leq b\leq k_1}}} .
\end{aligned}$$

This directly gives the result (15.12).

C.3 Extension of integration theorems for determinantal kernels

In App. C.3.1, we generalize Andréief's integral theorem which leads to determinantal structures. De Bruijn's integral theorem yields Pfaffian determinants and is extended in App. C.3.2.

C.3.1 Extension of Andréief's integral theorem

We consider the integral

$$\mathcal{I} = \int_{\mathbb{C}^N} \det\left[\begin{array}{c} \{r_{ab}\}_{\substack{1\leq a\leq k \\ 1\leq b\leq N+k}} \\ \{R_b(z_a,z_a^*)\}_{\substack{1\leq a\leq N \\ 1\leq b\leq N+k}} \end{array}\right] \det\left[\begin{array}{c} \{s_{ab}\}_{\substack{1\leq a\leq l \\ 1\leq b\leq N+l}} \\ \{S_b(z_a,z_a^*)\}_{\substack{1\leq a\leq N \\ 1\leq b\leq N+l}} \end{array}\right] d[z] . \tag{C.10}$$

The functions R_a and S_a are such that the integrals are convergent. Apart from this property they are arbitrary. We expand the first determinant in the first k rows and the second determinant in the first l rows and obtain

$$\begin{aligned}\mathcal{I} &= \frac{1}{k!(N-k)!l!(N-l)!} \\ &\times \sum_{\substack{\rho\in\mathfrak{S}_{N+k}\\ \sigma\in\mathfrak{S}_{N+l}}} \mathrm{sign}\,(\rho)\mathrm{sign}\,(\sigma)\det[r_{a\rho(b)}]_{1\le a,b\le k}\det[s_{a\sigma(b)}]_{1\le a,b\le l} \\ &\times \int_{\mathbb{C}^N} \det[R_{\rho(b)}(z_a,z_a^*)]_{\substack{1\le a\le N\\ k+1\le b\le N+k}} \det[S_{\sigma(b)}(z_a,z_a^*)]_{\substack{1\le a\le N\\ l+1\le b\le N+l}} d[z]\,.\end{aligned} \tag{C.11}$$

We apply Andréief's integration theorem for determinants [132] and obtain

$$\begin{aligned}\mathcal{I} &= \frac{N!}{k!(N-k)!l!(N-l)!}\sum_{\substack{\rho\in\mathfrak{S}_{N+k}\\ \sigma\in\mathfrak{S}_{N+l}}} \mathrm{sign}\,(\rho)\mathrm{sign}\,(\sigma)\det[r_{a\rho(b)}]_{1\le a,b\le k} \\ &\times \det[s_{a\sigma(b)}]_{1\le a,b\le l}\det\left[\int_{\mathbb{C}} R_{\rho(a)}(z,z^*)S_{\sigma(b)}(z,z^*)d^2z\right]_{\substack{k+1\le a\le N+k\\ l+1\le b\le N+l}}.\end{aligned} \tag{C.12}$$

This expression is an expansion of a $(N+k+l)\times(N+k+l)$ determinant in the first k columns and the first l rows. We find the final result

$$\mathcal{I} = (-1)^{kl}N!\det\left[\begin{array}{cc} 0 & \{s_{ab}\}_{\substack{1\le a\le l\\ 1\le b\le N+l}} \\ \{r_{ba}\}_{\substack{1\le a\le N+k\\ 1\le b\le k}} & \left\{\int_{\mathbb{C}} R_a(z,z^*)S_b(z,z^*)d^2z\right\}_{\substack{1\le a\le N+k\\ 1\le b\le N+l}}\end{array}\right]. \tag{C.13}$$

For $k=l=0$, we, indeed, obtain the original integral theorem by Andréief.

C.3.2 Extension of de Bruijn's integral theorem

Consider the integral

$$\mathcal{J} = \int_{\mathbb{C}^N}\det\left[\{A_{ab}\}_{\substack{1\le a\le 2N+l\\ 1\le b\le l}}\{B_a(z_b,z_b^*)\}_{\substack{1\le a\le 2N+l\\ 1\le b\le N}}\{C_a(z_b,z_b^*)\}_{\substack{1\le a\le 2N+l\\ 1\le b\le N}}\right]d[z]\,. \tag{C.14}$$

As in App. C.3.1, we expand the determinant in the first l columns and obtain

$$\mathcal{J} = \frac{1}{(2N)!} \sum_{\sigma \in \mathfrak{S}_{2N+l}} \operatorname{sign}(\sigma) \prod_{j=1}^{l} A_{\sigma(j)j} \times \int\limits_{\mathbb{C}^N} \det \left[\begin{array}{cc} \{B_{\sigma(a)}(z_b, z_b^*)\}_{\substack{l+1 \leq a \leq 2N+l \\ 1 \leq b \leq N}} & \{C_{\sigma(a)}(z_b, z_b^*)\}_{\substack{l+1 \leq a \leq 2N+l \\ 1 \leq b \leq N}} \end{array} \right] d[z] \,. \tag{C.15}$$

We define the quantity

$$D_{ab} = \int\limits_{\mathbb{C}} \left[B_a(z, z^*) C_b(z, z^*) - B_b(z, z^*) C_a(z, z^*) \right] d[z] \,. \tag{C.16}$$

Then, we apply the original version of de Bruijn's integral theorem [129] and find

$$\mathcal{J} = \frac{(-1)^{N(N-1)/2} N!}{(2N)!} \sum_{\sigma \in \mathfrak{S}_{2N+l}} \operatorname{sign}(\sigma) \prod_{j=1}^{l} A_{\sigma(j)j} \operatorname{Pf} \left[D_{\sigma(a)\sigma(b)} \right]_{l+1 \leq a,b \leq 2N+l} \,. \tag{C.17}$$

Summarizing all terms, the integral $\mathcal{J}$ is up to a constant

$$\mathcal{J} \sim \operatorname{Pf} \left[\begin{array}{cc} 0 & \{A_{ba}\}_{\substack{1 \leq a \leq l \\ 1 \leq b \leq 2N+l}} \\ \{-A_{ab}\}_{\substack{1 \leq a \leq 2N+l \\ 1 \leq b \leq l}} & \{D_{ab}\}_{1 \leq a,b \leq 2N+l} \end{array} \right] . \tag{C.18}$$

We fix the constant by the particular choice

$$[A_{ab}]_{\substack{1 \leq a \leq 2N+l \\ 1 \leq b \leq l}} = \left[\begin{array}{c} \mathbb{1}_l \\ 0 \end{array} \right] \tag{C.19}$$

which yields

$$\mathcal{J} = (-1)^{N(N-1)/2 + l(l-1)/2} N! \operatorname{Pf} \left[\begin{array}{cc} 0 & \{A_{ba}\}_{\substack{1 \leq a \leq l \\ 1 \leq b \leq 2N+l}} \\ \{-A_{ab}\}_{\substack{1 \leq a \leq 2N+l \\ 1 \leq b \leq l}} & \{D_{ab}\}_{1 \leq a,b \leq 2N+l} \end{array} \right] . \tag{C.20}$$

For $l = 0$, this is indeed de Bruijn's integral theorem.

C.4 Calculating integrals of squared–Vandermonde type

We derive the cases $(k_1-k_2)=(l_1-l_2)\leq N$ and $(k_2-k_1),(l_2-l_1)\leq N$ in appendices C.4.1 and C.4.2, respectively.

C.4.1 The case $(k_1-k_2)=(l_1-l_2)\leq N$

With help of Eq. (14.5), we rewrite the integrand (15.18) as a product of two determinants

$$\begin{aligned}
&\int\limits_{\mathbb{C}^N}\prod_{j=1}^{N} g(z_j)\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2+N}(\tilde{z})}\sqrt{\mathrm{Ber}\,^{(2)}_{l_1/l_2+N}(\hat{z})}d[z] \\
= \ & (-1)^{(l_1-k_1)(l_1+k_1-1)/2}\int\limits_{\mathbb{C}^N} d[z]\prod_{j=1}^{N} g(z_j) \\
\times \ & \det\begin{bmatrix} \left\{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}} & \left\{\dfrac{1}{\kappa_{a1}-z_b}\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq N}} \\ \left\{\kappa_{b2}^{a-1}\right\}_{\substack{1\leq a\leq d\\ 1\leq b\leq k_2}} & \left\{z_b^{a-1}\right\}_{\substack{1\leq a\leq d\\ 1\leq b\leq N}} \end{bmatrix} \\
\times \ & \det\begin{bmatrix} \left\{\dfrac{1}{\lambda_{a1}-\lambda_{b2}}\right\}_{\substack{1\leq a\leq l_1\\ 1\leq b\leq l_2}} & \left\{\dfrac{1}{\lambda_{a1}-z_b^*}\right\}_{\substack{1\leq a\leq l_1\\ 1\leq b\leq N}} \\ \left\{\lambda_{b2}^{a-1}\right\}_{\substack{1\leq a\leq d\\ 1\leq b\leq l_2}} & \left\{z_b^{*\,a-1}\right\}_{\substack{1\leq a\leq d\\ 1\leq b\leq N}} \end{bmatrix} .
\end{aligned} \tag{C.21}$$

Using the definitions (15.20)-(15.25), we apply the theorem of App. C.3.1 and find

$$\begin{aligned}
&\int\limits_{\mathbb{C}^N}\prod_{j=1}^{N} g(z_j)\sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2+N}(\tilde{z})}\sqrt{\mathrm{Ber}\,^{(2)}_{l_1/l_2+N}(\hat{z})}d[z] \\
= \ & (-1)^{(l_2+k_2)(l_1+k_1-1)/2}N!
\end{aligned} \tag{C.22}$$

$$\times\det\begin{bmatrix} 0 & \left\{\dfrac{1}{\lambda_{b1}-\lambda_{a2}}\right\}_{\substack{1\leq a\leq l_2\\ 1\leq b\leq l_1}} & \left\{\mathbf{\Lambda}_d(\lambda_{a2})\right\}_{1\leq a\leq l_2} \\ \left\{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq k_2}} & \left\{\widetilde{Z}^{(1)}_{\frac{1/0}{1/0}}(\kappa_{a1},\lambda_{b1})\right\}_{\substack{1\leq a\leq k_1\\ 1\leq b\leq l_1}} & \left\{\widetilde{\mathbf{F}}_d(\kappa_{a1})\right\}_{1\leq a\leq k_1} \\ \left\{\mathbf{K}_d(\kappa_{b2})\right\}_{1\leq b\leq k_2} & \left\{\widetilde{\mathbf{F}}_d^{(*)}(\lambda_{b1})\right\}_{1\leq b\leq l_1} & \widetilde{\mathbf{M}}_d \end{bmatrix}.$$

The last step is the same as in App. C.2.1. We separate the matrix $\widetilde{\mathbf{M}}_d$ from the determinant by inverting it. This yields Eq. (15.19).

C.4.2 The case $(k_1 - k_2), (l_1 - l_2) \geq N$

We consider the integral

$$\begin{aligned}
&\int\limits_{\mathbb{C}^N} \prod_{j=1}^{N} g(z_j) \sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2+N}(\tilde{z})} \sqrt{\mathrm{Ber}\,^{(2)}_{l_1/l_2+N}(\hat{z})} d[z] \\
= \quad & (-1)^{l_1(l_1-1)/2+k_1(k_1-1)/2} \int\limits_{\mathbb{C}^N} d[z] \prod_{j=1}^{N} g(z_j) \qquad\qquad \text{(C.23)} \\
\times \quad & \det \begin{bmatrix} \left\{ \dfrac{1}{\kappa_{b1} - \kappa_{a2}} \right\}_{\substack{1 \leq a \leq k_2 \\ 1 \leq b \leq k_1}} \\ \left\{ \dfrac{1}{\kappa_{b1} - z_a} \right\}_{\substack{1 \leq a \leq N \\ 1 \leq b \leq k_1}} \\ \left\{ \kappa_{b1}^{a-1} \right\}_{\substack{1 \leq a \leq d_\kappa \\ 1 \leq b \leq k_1}} \end{bmatrix} \det \begin{bmatrix} \left\{ \dfrac{1}{\lambda_{b1} - \lambda_{a2}} \right\}_{\substack{1 \leq a \leq l_2 \\ 1 \leq b \leq l_1}} \\ \left\{ \dfrac{1}{\lambda_{b1} - z_a^*} \right\}_{\substack{1 \leq a \leq N \\ 1 \leq b \leq l_1}} \\ \left\{ \lambda_{b1}^{a-1} \right\}_{\substack{1 \leq a \leq d_\lambda \\ 1 \leq b \leq l_1}} \end{bmatrix} .
\end{aligned}$$

Using the result of App. C.3.1 we get

$$\begin{aligned}
&\int\limits_{\mathbb{C}^N} \prod_{j=1}^{N} g(z_j) \sqrt{\mathrm{Ber}\,^{(2)}_{k_1/k_2+N}(\tilde{z})} \sqrt{\mathrm{Ber}\,^{(2)}_{l_1/l_2+N}(\hat{z})} d[z] \\
= \quad & (-1)^{(l_1+k_1)(l_1+k_1-1)/2+N(k_2+l_2+1)} N! \qquad\qquad \text{(C.24)} \\
\times \quad & \det \begin{bmatrix} 0 & 0 & \left\{ \dfrac{1}{\lambda_{b1} - \lambda_{a2}} \right\}_{\substack{1 \leq a \leq l_2 \\ 1 \leq b \leq l_1}} \\ 0 & 0 & \left\{ \lambda_{b1}^{a-1} \right\}_{\substack{1 \leq a \leq d_\lambda \\ 1 \leq b \leq l_1}} \\ \left\{ \dfrac{1}{\kappa_{a1} - \kappa_{b2}} \right\}_{\substack{1 \leq a \leq k_1 \\ 1 \leq b \leq k_2}} & \left\{ \kappa_{a1}^{b-1} \right\}_{\substack{1 \leq a \leq k_1 \\ 1 \leq b \leq d_\kappa}} & \left\{ \widetilde{Z}^{(1)}_{\substack{1/0 \\ 1/0}} (\kappa_{a1}, \lambda_{b1}) \right\}_{\substack{1 \leq a \leq k_1 \\ 1 \leq b \leq l_1}} \end{bmatrix} ,
\end{aligned}$$

which is the desired formula.

C.5 Calculating integrals of coupled square–root Vandermonde type

In App. C.5.1, we carry out the integrals in Eq. (15.39) for the case $k_2 + 2N + 1 \geq k_1$. We derive the other case $k_2 + 2N + 1 \leq k_1$ in App. C.5.2.

C.5.1 The case $k_2 + 2N + 1 \geq k_1$

Let $d = k_2 - k_1 + 2N + 1 \geq 0$ be odd. We are interested in the integral

$$\begin{aligned}
& \int\limits_{\mathbb{C}^{2N+1}} h(z_{2N+1}) \prod_{j=1}^{N} g(z_{2j-1}, z_{2j}) \sqrt{\mathrm{Ber}\,^{(2)}_{(k_1/k_2+2N+1)}(\tilde{z})}\, d[z] \qquad \text{(C.25)} \\
= \ & (-1)^{k_2(k_2-1)/2+N} \int\limits_{\mathbb{C}^{2N+1}} h(z_{2N+1}) d[z] \prod_{j=1}^{N} g(z_{2j-1}, z_{2j}) \\
\times \ & \det \left[\begin{array}{cc} \left\{ \dfrac{1}{\kappa_{a1} - \kappa_{b2}} \right\}_{\substack{1\leq a\leq k_1 \\ 1\leq b\leq k_2}} & \left\{ \dfrac{1}{\kappa_{a1} - z_b} \right\}_{\substack{1\leq a\leq k_1 \\ 1\leq b\leq 2N+1}} \\ \left\{ \kappa_{b2}^{a-1} \right\}_{\substack{1\leq a\leq d \\ 1\leq b\leq k_2}} & \left\{ z_b^{a-1} \right\}_{\substack{1\leq a\leq d \\ 1\leq b\leq 2N+1}} \end{array} \right].
\end{aligned}$$

The first step is the integration over all variables z_j with an odd index j. Thus, we have

$$\begin{aligned}
& \int\limits_{\mathbb{C}^{2N+1}} h(z_{2N+1}) \prod_{j=1}^{N} g(z_{2j-1}, z_{2j}) \sqrt{\mathrm{Ber}\,^{(2)}_{(k_1/2N+1+k_2)}(\tilde{z})}\, d[z] \qquad \text{(C.26)} \\
= \ & (-1)^{k_2(k_2-1)/2+N} \int\limits_{\mathbb{C}^{N}} d[z] \\
\times \ & \det \left[\begin{array}{cc} \left\{ \dfrac{1}{\kappa_{b1} - \kappa_{a2}} \right\}_{\substack{1\leq a\leq k_2 \\ 1\leq b\leq k_1}} & \left\{ \kappa_{a2}^{b-1} \right\}_{\substack{1\leq a\leq k_2 \\ 1\leq b\leq d}} \\ \left\{ \displaystyle\int_{\mathbb{C}} \dfrac{h(z)}{\kappa_{b1} - z} dz \right\}_{1\leq b\leq k_1} & \left\{ \displaystyle\int_{\mathbb{C}} h(z) z^{b-1} dz \right\}_{1\leq b\leq d} \\ \left\{ \begin{array}{c} \displaystyle\int_{\mathbb{C}} \dfrac{g(z, z_a)}{\kappa_{b1} - z} dz \\ \dfrac{1}{\kappa_{b1} - z_a} \end{array} \right\}_{\substack{1\leq a\leq N \\ 1\leq b\leq k_1}} & \left\{ \begin{array}{c} \displaystyle\int_{\mathbb{C}} g(z, z_a) z^{b-1} dz \\ z_a^{b-1} \end{array} \right\}_{\substack{1\leq a\leq N \\ 1\leq b\leq d}} \end{array} \right].
\end{aligned}$$

We perform the last integrals with help of a modified de Bruijn's integral theorem [129], see App. C.3.2, and find

$$\int\limits_{\mathbb{C}^{2N+1}} h(z_{2N+1})\prod_{j=1}^{N} g(z_{2j-1},z_{2j})\sqrt{\mathrm{Ber}\,^{(2)}_{(k_1/2N+1+k_2)}(\tilde{z})}d[z] = (-1)^{N+1}N!$$
$$\times \quad \mathrm{Pf}\begin{bmatrix} 0 & \left\{\dfrac{1}{\kappa_{b1}-\kappa_{a2}}\right\}_{\substack{1\le a\le k_2\\ 1\le b\le k_1}} & \left\{\mathbf{K}_{(d)}(\kappa_{a2})\right\}_{1\le a\le k_2} \\ \left\{-\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\le a\le k_1\\ 1\le b\le k_2}} & \{\mathbf{F}(\kappa_{a1},\kappa_{b1})\}_{1\le a,b\le k_1} & \left\{\mathbf{G}_{(d)}(\kappa_{a1})\right\}_{1\le a\le k_1} \\ \left\{-\mathbf{K}^T_{(d)}(\kappa_{b2})\right\}_{1\le b\le k_2} & \left\{-\mathbf{G}^T_{(d)}(\kappa_{b1})\right\}_{1\le b\le k_1} & \mathbf{M}_{(d)} \end{bmatrix} \tag{C.27}$$

with the matrices defined in Eqs. (15.41)-(15.47). Finally, we extract the matrix $\mathbf{M}_{(d)}$ from the Pfaffian by inversion, see Eq. (13.17), and arrive at Eq. (15.40).

C.5.2 The case $k_2 + 2N + 1 \le k_1$

Let $d = k_2 - k_1 + 2N + 1 \le 0$ be an arbitrary integer. Then, we calculate

$$\begin{aligned} &\int\limits_{\mathbb{C}^{2N+1}} h(z_{2N+1})\prod_{j=1}^{N} g(z_{2j-1},z_{2j})\sqrt{\mathrm{Ber}\,^{(2)}_{(k_1/k_2+2N+1)}(\tilde{z})}d[z] \\ = \quad & (-1)^{k_1(k_1-1)/2+k_1-k_2-1}\int\limits_{\mathbb{C}^{2N+1}} d[z]h(z_{2N+1})\prod_{j=1}^{N} g(z_{2j-1},z_{2j}) \qquad \text{(C.28)} \\ \times \quad & \det\begin{bmatrix} \left\{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}\right\}_{\substack{1\le a\le k_1\\ 1\le b\le k_2}} & \left\{\kappa_{a1}^{b-1}\right\}_{\substack{1\le a\le k_1\\ 1\le b\le -d}} & \left\{\dfrac{1}{\kappa_{a1}-z_b}\right\}_{\substack{1\le a\le k_1\\ 1\le b\le 2N+1}} \end{bmatrix}. \end{aligned}$$

As in App. C.5.1, we integrate first over all variables with an odd index. This yields

$$\begin{aligned} &\int\limits_{\mathbb{C}^{2N+1}} h(z_{2N+1})\prod_{j=1}^{N} g(z_{2j-1},z_{2j})\sqrt{\mathrm{Ber}\,^{(2)}_{(k_1/k_2+2N+1)}(\tilde{z})}d[z] \\ = \quad & (-1)^{k_1(k_1-1)/2+k_1-k_2-1}\int\limits_{\mathbb{C}^{N}} d[z] \end{aligned}$$

$$
\times\ \det\left[\begin{array}{c}
\left\{\dfrac{1}{\kappa_{b1}-\kappa_{a2}}\right\}_{\substack{1\leq a\leq k_2\\ 1\leq b\leq k_1}} \\
\left\{\kappa_{b1}^{a-1}\right\}_{\substack{1\leq a\leq -d\\ 1\leq b\leq k_1}} \\
\left\{\displaystyle\int\limits_{\mathbb{C}}\dfrac{h(\tilde{z})}{\kappa_{b1}-\tilde{z}}d[\tilde{z}]\right\}_{1\leq b\leq k_1} \\
\left\{\begin{array}{c}\displaystyle\int\limits_{\mathbb{C}}\dfrac{g(\tilde{z},z_a)}{\kappa_{b1}-\tilde{z}}d[\tilde{z}] \\ \dfrac{1}{\kappa_{b1}-z_a}\end{array}\right\}_{\substack{1\leq a\leq N\\ 1\leq b\leq k_1}}
\end{array}\right]. \tag{C.29}
$$

Again, we use the modified version of de Bruijn's integral theorem and obtain Eq. (15.56) up to the Berezinian in the denominator.

C.6 Calculation of the flat Fourier transform in Eq. (16.11)

We consider the flat Fourier transform

$$
J=\int\limits_{\mathbb{R}^{2k}}\prod_{a=1}^{k}\prod_{b=1}^{N}\frac{e^{-\imath\psi}s_{a2}+\imath\varepsilon+\alpha E_b^{(0)}}{s_{a1}+\imath\varepsilon+\alpha E_b^{(0)}}\exp\left[-\imath\mathrm{Str}\,rs^+\right]\sqrt{\mathrm{Ber}_{k/k}^{(2)}(s)}d[s]\,. \tag{C.30}
$$

By extending this integral with a Vandermonde determinant of $-\alpha E^{(0)}$ and using Eq. (14.4), we find the determinant

$$
J=\frac{(-1)^{k(k-1)/2}}{\Delta_N(\alpha E^{(0)})}\det\left[\begin{array}{cc}
\left\{J_1(\tilde{r}_{ab})\right\}_{1\leq a,b\leq k} & \left\{J_2(r_{a1},\alpha E_b^{(0)})\right\}_{\substack{1\leq a\leq k\\ 1\leq b\leq N}} \\
\left\{J_{3,a}(r_{b2})\right\}_{\substack{1\leq a\leq N\\ 1\leq b\leq k}} & \left\{\left(-\alpha E_b^{(0)}\right)^{a-1}\right\}_{1\leq a,b\leq N}
\end{array}\right], \tag{C.31}
$$

where $s_1^+=s_1+\imath\varepsilon$, $s_2^+=s_2+\imath e^{\imath\psi}\varepsilon$ and $\tilde{r}_{ab}=\mathrm{diag}\,(r_{a1},e^{\imath\psi}r_{b2})$. Hence, we have to calculate three types of integrals. The integrals in the off-diagonal blocks are the simpler ones. We have

$$
\begin{aligned}
J_2(r_{a1},\alpha E_b^{(0)}) &= \int\limits_{\mathbb{R}} \frac{\exp\left[-\imath r_{a1}s_1^+\right]}{(s_1^+ + \alpha E_b^{(0)})}\left(\frac{-\alpha E_b^{(0)}}{s_1^+}\right)^N ds_1 \\
= & \ \frac{(-\imath)^{N+1}\left(-\alpha E_b^{(0)}\right)^N}{(N-1)!}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} d[t,s]t_2^{N-1} \\
\times & \ \exp\left[-\imath r_{a1}s_1^+ + \imath(s_1^+ + \alpha E_b^{(0)})t_1 + \imath s_1^+ t_2\right] \\
= & \ \frac{(-\imath)^{N+1}2\pi\left(-\alpha E_b^{(0)}\right)^N}{(N-1)!} \\
\times & \int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^+} \delta(t_1+t_2-r_{a1})t_2^{N-1}\exp\left[\imath\alpha E_b^{(0)}t_1\right]dt_1dt_2 \\
= & \ \frac{(-\imath)^{N+1}2\pi\left(-\alpha E_b^{(0)}\right)^N}{(N-1)!}\Theta(r_{a1})\int\limits_0^{r_{a1}} t_2^{N-1}\exp\left[-\imath\alpha E_b^{(0)}(t_2-r_{a1})\right]dt_2 \\
= & \ -2\pi\imath\Theta(r_{a1})\exp\left[\imath\alpha E_b^{(0)}r_{a1}\right] + 2\pi\imath\Theta(r_{a1})\sum_{n=0}^{N-1}\frac{1}{n!}\left(\imath\alpha E_b^{(0)}r_{a1}\right)^n \\
= & \ -2\pi\imath\Theta(r_{a1})\sum_{n=N}^{\infty}\frac{1}{n!}\left(\imath\alpha E_b^{(0)}r_{a1}\right)^n
\end{aligned}
\tag{C.32}
$$

and

$$
J_{3,a}(r_{b2}) = \int\limits_{\mathbb{R}} \left(e^{-\imath\psi}s_2^+\right)^{a-1}\exp\left[\imath r_{b2}s_2^+\right]ds_2 = 2\pi\left(\frac{e^{-\imath\psi}}{\imath}\frac{\partial}{\partial r_{b2}}\right)^{a-1}\delta(r_{b2})
\tag{C.33}
$$

The integrand of the integral

$$
J_1(\tilde r_{ab}) = \int\limits_{\mathbb{R}^2}\frac{\exp\left[-\imath \mathrm{Str}\,\tilde r_{ab}s^+\right]}{s_1 - e^{-\imath\psi}s_2}\left(\frac{s_2^+}{s_1^+}\right)^N d[s]
\tag{C.34}
$$

has to be interpreted as a distribution. It is up to an Efetov–Wegner term

$$
J_1(\tilde r_{ab}) = 2\pi e^{\imath\psi}\int\limits_{\Sigma_{-\psi}(1)}\exp\left[-\imath\mathrm{Str}\,\tilde r_{ab}\sigma^+\right]\mathrm{Sdet}^{-N}\sigma^+ d[\sigma] + 2\pi\imath\,.
\tag{C.35}
$$

This is the supersymmetric Ingham-Siegel integral [61], cf. Eq. (8.27). We employ the result of Refs. [61], see also Sec. 8.2, and obtain

$$J_1(\tilde{r}_{ab}) = -2\pi \frac{r_{a1}^N \Theta(r_{a1})}{r_{a1} - e^{\imath\psi} r_{b2}} \left(-e^{-\imath\psi} \frac{\partial}{\partial r_{b2}} \right)^{N-1} \delta(r_{b2}) \, . \tag{C.36}$$

Thus, we get for the integral (C.30)

$$J = \frac{(-2\pi)^k (-1)^{k(k-1)/2} \Theta(r_1) \delta(r_2)}{\Delta_N(\alpha E^{(0)})} \tag{C.37}$$

$$\times \quad \det \left[\begin{array}{c|c} \dfrac{r_{a1}^N}{r_{a1} - e^{\imath\psi} r_{b2}} \left(-e^{-\imath\psi} \dfrac{\partial}{\partial r_{b2}} \right)^{N-1} & \displaystyle\sum_{n=N}^{\infty} \frac{\imath}{n!} \left(\imath\alpha E_b^{(0)} r_{a1} \right)^n \\ \hline 2\pi \left(\dfrac{e^{-\imath\psi}}{\imath} \dfrac{\partial}{\partial r_{b2}} \right)^{a-1} & \left(-\alpha E_b^{(0)} \right)^{a-1} \end{array} \right] ,$$

where $\Theta(r_1)$ indicates that every bosonic eigenvalue r_{a1} has to be positive definite. The distribution $\delta(r_2)$ is the product of all Dirac distributions $\delta(r_{b2})$. The indices a and b take the same values as in Eq. (16.12).

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Bibliography

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