Numerical concepts and error analysis for elliptic Neumann boundary control problems with pointwise state and control constraints

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Contents

Chapter 1

Introduction

In this work we study linear-quadratic optimal control problems governed by elliptic partial differential equations with inhomogeneous Neumann boundary conditions. The inhomogeneity in the boundary condition will be considered as the control variable. Furthermore, we investigate inequality constraints on the variables of the optimal control problem.

1.1 Motivation

This section is concerned with a brief overview on optimal control theory and applications. In optimal control theory, one is interested in governing the state of a system by using the controls under notice of minimizing a certain quantity of interest. In other words, we are considering the minimization of an *objective functional* $J(y, u)$ that depends on the *state* y and the *control* u, where the state and the control are coupled by a so-called *state equation*.

We will explain this concept by a simple example referred to as *The Rocket Car Problem* in the book of Macki and Strauss [53]. The car runs on a straight road and is equipped with rocket engines on each end in such a way, that it can be accelerated in both directions. The goal is to move the car as fast as possible from a given position A to an endpoint B. For simplicity, one assumes that the velocity in the points A and B, respectively, is zero and the mass of the car is one. This problem exhibits all essential elements of an optimal control problem. Here, the objective function is the running time. Furthermore, the state $y(t)$ is the position of the car at time t and the control variable $u(t)$ is the force induced by the rocket engines. The state equation $y''(t) = u(t)$ is obtained by Newton's law endowed with the particular initial and end conditions. For instance, based on the size of the rocket engines, there are constraints on the magnitude of $u(t)$, e.g., $|u(t)| \leq 1$. Thus, we end up with a control constrained optimal control problem. The control can be chosen arbitrarily within the given constraints. Then, the associated state is derived as the unique solution of the underlying differential equation. Now, the goal is to choose the control in a way such that the objective function becomes minimal. Such a control is then called *optimal*. We mention that, except the running time, one can also minimize other quantities, e.g., the least energy demand or the least fuel consumption.

In the previous example the state equation was an ordinary differentiable equation. However, there are a lot of processes in nature that can only be prescribed by partial differential equations. For instance heat transport, fluid mechanics, reaction diffusion problems, continuum mechanics, and other physical processes are modelled by partial differential equations.

A huge amount of applications of optimal control problems can be found within the ongoing DFG priority program 1253 *"Optimization with Partial Differential Equations"*. We will mention an application from the building and construction industry. The solidification of concrete begins at the surface of the structure such that the thermal extension of the material is restrained. This might cause cracks in the construction. Of course, if these cracks become too large the concrete structure is damaged and not utilizable. In simple cases the stress cracking can be prescribed, e.g., by requiring that the thermal stresses do not exceed a specified limit. This leads to state constraints or constraints on the temperature gradient.

An interesting medical application is currently investigated in the DFG research center MATHEON, see MATHEON project A1 *"Modelling, simulation, and optimal control of thermoregulation in the human vascular system"*. The so-called regional hyperthermia is used as a cancer therapy, where for instance radio frequency radiation is used to heat a tumor in order to make it more susceptible to other therapies. The heat transport within the vascular system is modelled by partial differential equations. The goal is the determination of optimal parameters for the radio antennas under notice of a desired temperature distribution. Thus, one ends up with optimal control problems governed by partial differential equations. Of course, the consideration of constraints on the temperature within the human body is reasonable. Thereby, we arrive at state constrained optimal control problems.

Most of the above mentioned processes are modelled by nonlinear partial differential equations. Often, optimal control problems with nonlinear partial differential equations are solved by various Newton's methods applied to the associated optimality system. These methods lead to the solution of linearized partial differential equations in every iteration. Hence, we will focus in this work on optimal control problems with linear elliptic partial differential equations.

1.2 Notations and function spaces

In the context of optimal control problems we will look for the weak solution of the underlying partial differential equation. For that purpose, we introduce briefly certain function spaces. For a more detailed introduction, we refer to [1] or [78].

1.2.1 The domain Ω

All functions involved in optimal control problems are defined on a domain Ω. We will specify certain restrictions on the domain. Let Ω be a bounded domain in $\mathbb{R}^d$. The boundary of Ω is denoted by $\Gamma := \partial\Omega$. A rather weak restriction on domains is the so-called inner cone condition, formulated in the following definition.

Definition 1.1 (Inner cone condition) *The domain $\Omega \subset \mathbb{R}^d$ satisfies the inner cone condition if there exists a finite cone C with a fixed radius $r > 0$ and a fixed aperture angle $\kappa > 0$ such that each $x \in \Omega$ is the vertex of a finite cone C_x that is contained in Ω and congruent to C.*

Often, the theory of partial differential equations requires domains with a sufficiently smooth boundary. The following definition can be found in [61] or [34].

Definition 1.2 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain. We say that its boundary Γ is of class $C^{k,1}$, $k \in \mathbb{N} \cup \{0\}$, if there are finitely many local cartesian coordinate systems $S_1, \ldots, S_M$, functions $h_1, \ldots, h_M$ and positive constants a, b such that*

(i) all functions h_i are k times continuously differentiable on the $(d-1)$-dimensional closed hypercube

$$\bar{Q}_{d-1} = \{y = (y_1, \ldots, y_{d-1}) \,:\, |y_i| \leq a,\, i = 1, \ldots, d-1\}.$$

Furthermore, the k-th derivative of the functions h_i is Lipschitz continuous.

(ii) for every $x \in \Gamma$ there exists a $i \in \{1, \ldots, d-1\}$ such that the point x has the representation $x = (y,\, h_i(y))$, $y \in Q_{d-1}$ in the cartesian coordinate system S_i.

(iii) for the local cartesian coordinate systems S_i

$$\begin{aligned}(y,\, y_d) \in \Omega \;&\Rightarrow\; y \in \bar{Q}_{d-1},\; h_i(y) < y_d < h_i(y) + b\\ (y,\, y_d) \notin \Omega \;&\Rightarrow\; y \in \bar{Q}_{d-1},\; h_i(y) - b < y_d < h_i(y)\end{aligned}$$

is valid.

In simple words, we say that Ω is of class $C^{k,1}$ if each point $x \in \Gamma$ has a neighbourhood $\mathcal{U}_x$ such that the intersection of $\mathcal{U}_x$ with the boundary Γ is the graph of a k times continuously differentiable function, where the k-th derivative is Lipschitz continuous. In this work we will consider bounded convex domains $\Omega \in \mathbb{R}^d$, $d = 2, 3$ with a polygonal or polyhedral boundary Γ. The following result, given in [36, Corollary 1.2.2.3], provides that such domains are Lipschitz.

Proposition 1.3 *Let Ω be a convex and polygonally or polyhedrally bounded domain in $\mathbb{R}^d$, $d = 2, 3$, then Ω has a Lipschitz boundary.*

1.2.2 L^p and Sobolev spaces

In this section we will introduce basic results with respect to Lebesgue integrable functions and Sobolev spaces which are required for understanding optimal control problems governed by partial differential equations.

We denote by $L^p(\Omega), 1 \leq p < \infty$ the space of real valued functions that are defined on the domain Ω and whose p-th powers are integrable with respect to the Lebesgue measure dx. The space $L^p(\Omega)$ endowed with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty$$

is a Banach space. For $p = 2$, $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u\,,\, v)_{L^2(\Omega)} = \int_\Omega u(x) v(x) \, dx.$$

We denote by $L^\infty(\Omega)$ the Banach space of real valued functions that are essentially bounded, where the norm is given by

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |u(x)|.$$

The dual space to $L^p(\Omega)$, $1 < p < \infty$, can be identified by another space of integrable functions. More precisely, the dual space is given by $L^q(\Omega)$, where $q := p/(p-1)$ is the conjugate exponent of p satisfying $1/p + 1/q = 1$. Moreover, the dual pairing is defined by

$$\langle u\,,\, v\rangle_{L^q(\Omega), L^p(\Omega)} = \int_\Omega u(x) v(x) \, dx.$$

A basic inequality to deal with Lebesgue integrable functions is namely the *Hölder inequality*

$$\int_\Omega |u(x) v(x)| \, dx \leq \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} \left(\int_\Omega |v(x)|^q \, dx \right)^{1/q} \tag{1.1}$$

for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$.

Forthcoming, let m be a nonnegative integer and let p be a real number with $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is the space of functions whose weak derivatives up to order m are functions of $L^p(\Omega)$. Equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p},$$

$W^{m,p}(\Omega)$ is a Banach space. Again, for the case $p = 2$ the space $H^m(\Omega) := W^{m,2}(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \le m} (D^\alpha u\,,\, D^\alpha v)_{L^2(\Omega)}.$$

An essential tool for dealing with Sobolev spaces is the following theorem. We refer to [1] for the proof and further results.

Theorem 1.4 (Sobolev embedding theorem) *Let Ω be a domain in $\mathbb{R}^d$ satisfying the inner cone condition. If $mp < d$, then the embedding*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \textit{for } q \le \frac{dp}{d - mp}$$

is continuous.
Suppose $mp > d$. Then we have the continuity of the embedding

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \textit{for } p \le q \le \infty.$$

If $mp = d$, then we have the continuous embedding

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \textit{for } p \le q < \infty.$$

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. If $mp > d$, then the embedding

$$W^{m,p}(\Omega) \hookrightarrow C^{m-\left[\frac{m}{p}\right]-1,\lambda}(\bar{\Omega})$$

is continuous, where

$$\lambda = \begin{cases} \left[\dfrac{m}{p}\right] + 1 - \dfrac{m}{p}, & \textit{if } \dfrac{m}{p} \textit{ is not an integer} \\ \textit{any positive number } < 1, & \textit{if } \dfrac{m}{p} \textit{ is an integer.} \end{cases}$$

For a comprehensive introduction into Sobolev spaces and continuous embeddings we refer again to [1].

Another difficulty is the definition of boundary values for functions of Sobolev spaces. To this end, we introduce the trace of functions belonging to some Sobolev space.

Theorem 1.5 (Trace theorem) *Let Ω be a bounded Lipschitz domain and $1 \le p \le \infty$. Then, there exists a linear and continuous mapping $\tau : W^{1,p}(\Omega) \to L^p(\Gamma)$, such that for all $y \in C(\bar{\Omega})$*

$$(\tau y)(x) = y(x), \quad x \in \Gamma \tag{1.2}$$

is valid. Moreover, we have the estimate:

$$\|\tau y\|_{L^p(\Gamma)} \le c\|y\|_{W^{1,p}(\Omega)}. \tag{1.3}$$

Particularly, we have for the case $p = 2$ the property $\tau : H^1(\Omega) \to L^2(\Gamma)$. The proof of the theorem can be found for instance in [34] or [79]. Consequently, for continuous functions y the image of τ coincides with the usual boundary values.

Definition 1.6 *The element τy is called trace of y on the boundary Γ and the mapping τ is called trace operator.*

We mention that often the denotation $y|_\Gamma$ is used instead of τy. An extension of the Theorem 1.5 to Sobolev-Slobodezki spaces can be found in [61]. Furthermore, we state the following result that is important for this work.

Theorem 1.7 *Let $\Omega \in \mathbb{R}^d$, $d = 2, 3$ be a bounded Lipschitz domain. Then the trace operator τ is continuous from $H^2(\Omega)$ to $H^1(\Gamma)$ and the estimate*

$$\|\tau y\|_{H^1(\Gamma)} \leq c\|y\|_{H^2(\Omega)}, \qquad \forall y \in H^2(\Omega) \tag{1.4}$$

is valid.

For the proof and a precise definition of $H^1(\Gamma)$, we refer to [61, Theorem II.4.11].

1.3 The optimal control problems

Now, let us introduce the optimal control problem, which we will investigate in the sequel. We consider the optimal control of a linear elliptic partial differential equation where the control acts on the boundary and a quadratic cost functional has to be minimized:

$$\left.\begin{aligned} \min \quad J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} &+ \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \\ -\Delta y + y = 0 &\qquad \text{in } \Omega \\ \partial_n y = u &\qquad \text{on } \Gamma \\ u_a \leq u(x) \leq u_b &\quad \text{a.e. on } \Gamma \\ y(x) \geq y_c(x) &\quad \text{a.e. in } \Omega'. \end{aligned}\right\} \tag{P}$$

The partial differential equation is denoted as state equation, where the functions y and u are called state and control, respectively. Moreover, $\partial_n y$ is the normal derivative with respect to the unit outward normal on Γ. Furthermore, the control has to satisfy control constraints and we require pointwise state constraints. This problem belongs to the class of linear-quadratic elliptic boundary control problems.

Let us specify the problem setting. We assume that the ingredients of the optimal control problem (P) satisfy the following:

- The domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is supposed to be a convex bounded polygonal or polyhedral domain with the boundary Γ. Moreover, $\Omega' \subset\subset \Omega$ is an inner subdomain, i.e., $\mathrm{dist}\{\Omega', \Gamma\} > 0$.
- The desired state y_d is a given function in $L^2(\Omega)$. The function y_c of the pointwise state constraints belongs to $C^{0,1}(\bar{\Omega})$.

- The control bounds u_a and u_b are real numbers satisfying $u_a < u_b$.
- The Tikhonov regularization parameter ν is a fixed positive real number.

It is well known that optimal control problems with pointwise state constraints imply some difficulties, namely that the associated Lagrange multipliers are in general only measures, see, for instance, Casas [17] for the elliptic case and for parabolic control problems, see Raymond [66]. Therefore, the underlying analysis and further numerical treatment of such problems is difficult.

In order to avoid these challenges, different regularization concepts are developed in the last years. The penalization of the state constraints by a logarithmic barrier function was used by Meyer, Prüfert and Tröltzsch [56], Prüfert, Tröltzsch and Weiser [65] and by Schiela and Weiser [72]. This approach leads to interior point methods. We will also refer to Ito and Kunisch [40] and to Hintermüller and Kunisch [38], where the authors made use of a quadratic penalization term, which is based on a Moreau-Yosida-type regularization.

A direct discretization of state constrained optimal control problems was discussed by Deckelnick and Hinze [29] and by Hinze and Meyer [52].

Another regularization concept was motivated by ideas from inverse problems. The so-called Lavrentiev regularization technique has a long tradition in that field of mathematics, see, for instance, [41, 49, 60]. In connection with optimal control problems, Lavrentiev regularization is a quite new technique. To the best knowledge of the author the first publication was by Meyer, Rösch and Tröltzsch [59]. In this concept, the pointwise state constraints are regularized by mixed control-state constraints. This is based on the knowledge that Lagrange multipliers associated with mixed control-state constraints are regular measurable functions. The existence of regular Lagrange multipliers was proven for different types of problems. For elliptic optimal control problems we refer to Tröltzsch [74] or Rösch and Tröltzsch [68, 69]. For parabolic problems we mention Bergonioux and Tröltzsch [12], Arada and Raymond [9] and Casas, Raymond and Zidani [23]. In the case of optimal control problems with distributed control, the Lavrentiev-type regularization is directly applicable since the control itself is used for regularization of the state constraints, see, for instance, [24, 26, 59, 63]. Furthermore, regularization error estimates for the distributed control case were developed by Cherednichenko and Rösch [26], Cherednichenko, Krumbiegel and Rösch [24] and Neitzel and Tröltzsch [62].

A direct extension of the Lavrentiev regularization concept to our original problem (P) is not possible, since the control is undefined in the domain, where the state constraints are given. A source representation was used to overcome this problem in a recent contribution of Tröltzsch and Yousept [75]. We will go a different way in this work: by introducing a new distributed control v, that is called *virtual control*, we make use of the Lavrentiev regularization concept. The original problem (P) is then replaced by the following family of regularized optimal control problems with mixed control-state constraints.

$$\left.\begin{aligned} \min \quad J_\varepsilon(y,u,v) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} \\ -\Delta y + y = \phi(\varepsilon)v \qquad \text{in } \Omega \\ \partial_n y = u \qquad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. on } \Gamma \\ y(x) \ge y_c(x) - \xi(\varepsilon)v(x) \quad \text{a.e. in } \Omega' \\ 0 \le v(x) \le v_b \quad \text{a.e. in } \Omega \end{aligned}\right\} \qquad (\mathrm{P}_1^\varepsilon)$$

with a regularization parameter $\varepsilon > 0$ and Ω, Ω', ν, y_d, y_c, u_a, and u_b are same defined as above. The functions ψ, ϕ, and ξ are real valued and positive. Furthermore, the constraint v_b is a positive real number. One part of this work is the estimation of the regularization error. An essential keypoint in the derivation of such error estimates is the presence of constraints on the virtual control v, which can cause numerical difficulties. It may happen, that the different constraints in the domain are active simultaneously. This leads to nonuniqueness of the adjoint variables, see [3, Remark 2.6]. For a certain type of optimal control problems this is even the generic situation, see [18]. The nonuniqueness of the dual variables is reflected by singular matrices in all numerical methods, which try to solve the optimality system (Karush-Kuhn-Tucker system) directly. Thus, it is desirable to avoid the additional constraints on the virtual control v.

Due to the previous arguments, we consider the same regularization concept for the original problem (P), but without control constraints on the virtual control. We introduce the regularized optimal control problems:

$$\left.\begin{aligned} \min \quad J_\varepsilon(y,u,v) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)}, \\ -\Delta y + y = \phi(\varepsilon)v \qquad \text{in } \Omega, \\ \partial_n y = u \qquad \text{on } \Gamma, \\ u_a \le u(x) \le u_b \quad \text{a.e. on } \Gamma, \\ y(x) \ge y_c(x) - \xi(\varepsilon)v(x) \quad \text{a.e. in } \Omega', \end{aligned}\right\} \qquad (\mathrm{P}_2^\varepsilon)$$

where the functions $\psi(\varepsilon)$, $\phi(\varepsilon)$, and $\xi(\varepsilon)$ are real valued and positive, and $\varepsilon > 0$ is a regularization parameter. The assumptions on the domain and the data functions are the same as for the original problem (P). Again, we aim to derive error estimates for the regularized problems.

In order to solve constrained optimal control problems governed by partial differential equations numerically, a reasonable discretization of the problem is necessary. We give a brief overview to earlier and recent results concerned with the application of finite element methods to PDE constrained optimal control problems. In the context of control constraints the FEM is well investigated. We start with the early contributions by Falk in [33] and Geveci in [35]. Further L^2-approximation results in the linear-quadratic elliptic case can be found in Rösch [67]. In addition to that, L^2-error estimates for semilinear elliptic control problems were derived by Arada, Casas, Tröltzsch [8]. Regarding different boundary control problems, we refer to Casas, Mateos [19], Casas, Mateos, Tröltzsch [20], Casas, Raymond [21] and Vexler [77]. In a very recent work of Mateos and Rösch [54] the dependence of the approximation order on the largest angle of polygonally bounded domains was investigated. For nonconvex domains specific mesh-grading techniques are developed providing the same approximation orders as in convex domains, see Apel, Rösch, Winkler [5,6] and Apel, Winkler [7]. Results on L^∞-estimates can be found in Arada, Casas, Tröltzsch [8] for semilinear problems. For the linear-quadratic case we refer to Meyer, Rösch [58] and Apel, Rösch, Sirch [4]. Contrary to the previous results, the variational discretization concept of Hinze in [39] proposes only the discretization of the state, and not the control. Connected with a suitable discretization scheme, this method delivers an optimal approximation order for the control. An optimal convergence order was also derived by means of superconvergence properties, see Meyer and Rösch [57].

In the case of state constrained optimal control problems the numerical analysis becomes more difficult. Nevertheless, there are several contributions regarding L^2-approximation error estimates for the distributed control case in the recent past, see, e.g., Cherednichenko and Rösch [25], Deckelnick and Hinze [30, 31], Hinze and Meyer [52] and Meyer [55]. The main purpose of this work is the development of an error estimate between a finite element approximation of problem (P_2^ε) and the solution of the original problem (P), i.e., the regularization and discretization error is considered simultaneously.

We mention that an overview concerning optimization methods for constrained optimal control problems with PDEs will be given in the respective chapter of this work.

1.4 Structure of the thesis

Chapter 2 is devoted to the analysis of the original problem (P). First, we clarify the existence and uniqueness of optimal solutions to problem (P) and the underlying state equation. We derive necessary and sufficient optimality conditions in a variational inequality form, where all inequality constraints of problem (P) are handled by an admissible set. On the other hand the formal Lagrange method, connected with several regularity results for solutions of PDEs, is used to obtain higher smoothness of the optimal solution.

Chapter 3 concentrates on the investigation of the regularized optimal control problem (P_1^ε). We focus on the derivation of an error estimate for the L^2-error between the optimal solution of problem (P) and the optimal solution of the regularized problem (P_1^ε). The proof of this estimate is predicated on the optimality conditions of both problems in variational inequality form and particular feasible controls. The construction of such feasible controls for problem (P) is based on a possible violation of the pure state constraints by the regularized solution. Furthermore, the regularization error estimate provides assumptions on the parameter functions of problem (P_1^ε) that delivers the convergence of the regularized solution to the original one. Finally, the theoretical results are illustrated by numerical tests.

Chapter 4 concerns the discussion of our second regularization concept, given by the problems (P_2^ε). Contrary to the previous chapter, we establish necessary and sufficient optimality conditions also by the use of the formal Lagrange method. We prove several regularity and boundedness results for the optimal solution of problem (P_2^ε). In view of a further discretization of the regularized problem, these results are important with respect to better approximation properties. The strategy for deriving a regularization error estimate is similar to Chapter 3. However, the absence of constraints on the virtual control makes the construction of feasible controls for the original problem (P) more difficult. In Chapter 3, the L^∞-bound of the virtual control is helpful for the estimation of the violation concerning the pure state constraints. Hence, we develop a more sophisticated technique: we prove Lipschitz-continuity of the respective violation function, where we benefit from the consideration of the state constraints in an inner subdomain Ω'. Based on this fact, we avoid the L^∞-estimate of the virtual control during the estimation of the violation with respect to the pure state constraints. The final regularization error estimate delivers slightly different assumptions on the parameter functions than the results of Chapter 3. In the end, we compare the virtual control approach, given by the problems (P_2^ε), with the Moreau-Yosida regularization concept introduced by Ito and Kunisch in [40].

In Chapter 5 we establish a finite element based approximation of the regularized optimal control problem (P_2^ε). The main purpose is the development of an a priori error estimate between the discretized and regularized problem and the original optimal control problem (P). Thereby, we consider the regularization and discretization error simultaneously and we propose a suitable coupling of the parameter functions and the mesh size. In order to achieve this goal, we investigate the optimality conditions of the respective discretized analogon to problem (P_2^ε). We establish regularity and boundedness results for the discretized solution similarly to the continuous case in Chapter 4. Again, the proof for the final error estimate is based on the optimality conditions in variational inequality form and appropriate feasible controls for the particular optimal control problems. In connection with the construction of such feasible controls, an interior maximum norm estimate for finite element approximations to solutions of the state equation of problem (P) is needed. We obtain an optimal approximation order for this estimates, where we benefit again from the consideration of the state constraints in the interior of the domain Ω.

In Chapter 6 we start with deriving error estimates for feasible and infeasible controls of optimal control problems like (P_2^ε). These controls can be interpreted as current iterates of an optimization algorithm. Based on this theory, we construct an error estimator, which is reliable as stopping criterion for iterative optimization methods. Forthcoming, we focus on the primal-dual active set strategy as a solution method for optimal control problems with mixed control-state constraints. We formulate this method in a function space setting and for a fully discretized scheme. Moreover, the previously mentioned error estimator is used as an alternative termination condition. In the end, we construct an analytical example in order to illustrate the theoretical results of the Chapters 4-6.

Chapter 2

The purely state constrained problem

In this section we will consider the optimal control problem (P) with respect to solvability and uniqueness of solutions. Moreover, we will establish optimality conditions. Furthermore, a key point will be the investigation of the adjoint equation. Throughout the whole work, we will use the constant c as a generic one.

2.1 Analysis of the state equation

Before we discuss the optimal control problem (P) itself, we recall some well known results on the state equation in (P) that is given by

$$\begin{aligned} -\Delta y + y &= 0 \quad \text{in } \Omega, \\ \partial_n y &= u \quad \text{on } \Gamma. \end{aligned} \tag{2.1}$$

It is reasonable to look for weak solutions of the state equation (2.1). Consequently, we replace the classical formulation by a variational formulation which is also-called weak formulation. First, we introduce the corresponding bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(y, z) := \int_\Omega \nabla y \cdot \nabla z + yz \, dx. \tag{2.2}$$

Next, we define by

$$F(z) := \int_\Gamma u\tau z \, ds, \quad z \in H^1(\Omega) \tag{2.3}$$

a linear and continuous functional on $H^1(\Omega)$, where $\tau : H^1(\Omega) \to L^2(\Gamma)$ is the usual trace operator. Hence, the weak formulation associated with (2.1) is given by

$$a(y, z) = F(z) \qquad \forall z \in H^1(\Omega). \tag{2.4}$$

The existence and uniqueness of weak solutions of linear elliptic partial differential equations is based on the Lax-Milgram lemma that we state here.

Lemma 2.1 (Lax-Migram) *Let V be a real Hilbert space and let $a : V \times V \to \mathbb{R}$ be a bilinear form satisfying the following conditions: There exist positive real constants α_0 and β_0 such that*

$$|a(y,v)| \leq \alpha_0 \|y\|_V \|v\|_V \qquad \forall y, v \in V \qquad \textit{(Continuity)} \tag{2.5}$$
$$a(y,y) \geq \beta_0 \|y\|_V^2 \qquad \forall y \in V \qquad \textit{(V-ellipticity).} \tag{2.6}$$

Then for every $F \in V^*$ the variational form

$$a(y,v) = F(v) \qquad \forall v \in V$$

admits a unique solution $y \in V$. Furthermore, there exists a constant c independent of F such that

$$\|y\|_V \leq c \|F\|_{V^*}.$$

The proof can be found in several books, where we mention e.g., [13], [14] and [27]. The following theorem covers the existence of unique solutions to the state equation (2.1) in $H^1(\Omega)$.

Theorem 2.2 *For every $u \in L^2(\Gamma)$ the state equation* (2.1) *admits a unique solution $y \in H^1(\Omega)$. Moreover, there exists a constant $c > 0$, depending only on the domain Ω, such that*

$$\|y\|_{H^1(\Omega)} \leq c \|u\|_{L^2(\Gamma)}$$

is satisfied.

Proof: The proof is standard and directly follows from applying the Lax-Milgram lemma 2.1 to the weak formulation (2.4). □

Next, we define the control-to-state mapping associated with the weak formulation (2.4). Based on the previous theorem and the Lax-Milgram lemma, we introduce a linear and continuous solution operator denoted by $G : H^1(\Omega)^* \to H^1(\Omega)$ that maps an arbitrary $F \in H^1(\Omega)^*$ to the unique solution $y \in H^1(\Omega)$ of (2.4). Due to

$$\langle \tau^* u \,,\, z \rangle_{H^1(\Omega)^*, H^1(\Omega)} := (u \,,\, \tau z)_{L^2(\Gamma)} = \int_\Gamma u \tau z \, ds, \tag{2.7}$$

where $\tau : H^1(\Omega) \to L^2(\Gamma)$ denotes again the trace operator, the control $u \in L^2(\Gamma)$ defines an element in $H^1(\Omega)^*$. Since the space $H^1(\Omega)$ is continuously embedded in $L^2(\Omega)$, we consider the solution of (2.4) as an element in $L^2(\Omega)$. Introducing the embedding operator $E_H : H^1(\Omega) \to L^2(\Omega)$ and using (2.7), the control-to-state mapping is given by

$$u \mapsto y, \quad y = S\tau^* u = E_H G \tau^* u \tag{2.8}$$

with the solution operator $S : H^1(\Omega)^* \to L^2(\Omega)$. The treatment of the operator S in this way has the advantage that the adjoint operator S^* maps $L^2(\Omega)$ to $H^1(\Omega)$.

As already mentioned, we require pointwise state constraints in an interior domain Ω' of Ω. In Section 2.3 we will derive optimality conditions for problem (P) that are based on the so-called Karush-Kuhn-Tucker theory. For that purpose, we have to guarantee the continuity of the state y in Ω'. This will be explained in detail in Section 2.3. Unfortunately, this cannot be obtained from the standard regularity result of Theorem 2.2, since $H^1(\Omega) \not\hookrightarrow C(\bar{\Omega})$ for $d \geq 2$. However, we benefit from the consideration of control constraints on the boundary. These constraints ensure that feasible controls for problem (P) belong to $L^\infty(\Gamma)$. Now, the following theorem guarantees $y \in C(\bar{\Omega})$, see [17].

Theorem 2.3 *Let Ω be a convex and bounded domain with polygonal or polyhedral boundary in $\mathbb{R}^d$, $d = 2, 3$. For all $g \in L^t(\Gamma)$, with $t > d - 1$ and for all $f \in L^p(\Omega)$, $p > d/2$, there exists a unique solution w of the boundary value problem*

$$\begin{aligned} -\Delta w + w &= f \quad \text{in } \Omega \\ \partial_n w &= g \quad \text{on } \Gamma \end{aligned} \tag{2.9}$$

belonging to $H^1(\Omega) \cap C(\bar{\Omega})$. Moreover, there exists a constant c independent of f and g such that

$$\|w\|_{H^1(\Omega)} + \|w\|_{C(\bar{\Omega})} \leq c(\|f\|_{L^p(\Omega)} + \|g\|_{L^t(\Gamma)}). \tag{2.10}$$

Thanks to this theorem, the state constraints of problem (P) are well-defined with respect to the $C(\Omega')$-topology.

2.2 Existence of solutions and multiplierfree optimality conditions

In this section we will show the existence and uniqueness of solutions for problem (P). Furthermore, we will derive necessary and sufficient optimality conditions. The proof of the existence of optimal solutions for linear-quadratic optimal control problems follows standard arguments. However, for convenience to the reader we will shortly sketch the proof for problem (P). We start with the following assumption of an inner point with respect to the state constraints.

Assumption 2.4 There exists a function $\hat{u} \in H^1(\Gamma)$ with $u_a \leq \hat{u}(x) \leq u_b$ a.e. on Γ and $\hat{y}(x) \geq y_c(x) + \gamma\ \forall x \in \Omega'$ with $\gamma > 0$, where $\hat{y} = S\tau^*\hat{u}$.

We mention that the existence theory of an optimal solution for problem (P) requires only one feasible point. However, the more stronger Slater condition in Assumption 2.4 is essential for the existence of a Lagrange multiplier associated with the pure state constraints. This will be discussed in detail in Section 2.3. Furthermore, this condition will be needed in several estimates.

Now, let us introduce the set of admissible controls for problem (P):

$$U_{ad} = \{u \in L^2(\Gamma) |\, u_a \leq u(x) \leq u_b \text{ a.e. on } \Gamma; (S\tau^*u)(x) \geq y_c(x) \text{ a.e. in } \Omega'\} \tag{2.11}$$

Due to Assumption 2.4, the admissible set is nonempty. Moreover, U_{ad} is convex and closed. Using the admissible set (2.11) and the control-to-state mapping (2.8), we formulate problem (P) as an optimization problem only with respect to the control u. The so-called reduced form is given by

$$\min_{u \in U_{ad}} f(u) := J(S\tau^*u, u) = \|S\tau^*u - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}. \tag{2.12}$$

Theorem 2.5 *Suppose that Assumption 2.4 is fulfilled. Then the optimization problem* (2.12) *admits a unique optimal solution* $\bar{u} \in U_{ad}$.

Proof: By means of Assumption 2.4, the admissible set U_{ad} is nonempty. Thus, we find a minimizing sequence $\{u_n\} \in U_{ad}$ of the objective functional in (2.12), i.e. $f(u_n) \to j$. Moreover, the admissible set is bounded. Hence, we can select a weakly converging subsequence $\{u_{n_k}\}$ from $\{u_n\}$ such that $u_{n_k} \rightharpoonup \bar{u}$. The convexity and closedness of the admissible set implies $\bar{u} \in U_{ad}$. It remains to show the optimality of $\bar{u}$. The weak continuity of the solution operator S ensures that

$$S\tau^* u_{n_k} \rightharpoonup S\tau^* \bar{u}.$$

Moreover, the convexity and continuity of the objective functional f itself implies that f is weakly lower semicontinuous. Thus, we obtain

$$j = \lim_{k\to\infty} f(u_{n_k}) \geq f(\bar{u}),$$

delivering the optimality of $\bar{u}$. Due to $\nu > 0$, the functional f is strictly convex such that the optimal solution $\bar{u}$ is unique. □

Based on this theorem and Theorem 2.2, the linear-quadratic optimal control problem (P) admits a unique optimal solution $(\bar{y}, \bar{u})$, where the optimal state is given by $\bar{y} = S\tau^*\bar{u}$.

Next, we derive first-order necessary optimality conditions for the unique optimal solution of problem (P). We consider again the reduced form (2.12). Since the solution operator in (2.8) is linear and continuous, it is Fréchet differentiable and the derivative is the operator itself. Then the chain rule immediately implies the Fréchet differentiability of the objective functional in (2.12). The derivative is stated in the next lemma. For more detailed explanations about differentiability of operators in the context of optimal control problems, we refer for instance to [73].

Lemma 2.6 *The functional* f *given in* (2.12) *is Fréchet differentiable from* $L^2(\Gamma)$ *to* $\mathbb{R}$. *Its derivative is given by*

$$f'(u)h = (\tau S^*(S\tau^* u - y_d) + \nu u\,,\, h)_{L^2(\Gamma)}, \tag{2.13}$$

where $S^* : L^2(\Omega) \to H^1(\Omega)$ *is the adjoint operator of* S.

The particular form of f' follows by straight forward computations, see e.g. [73]. Let us now state first-order necessary optimality conditions for problem (P).

Lemma 2.7 *Let* $(\bar{y}, \bar{u})$ *be the optimal solution of problem (P). The optimality condition is given by*

$$(\tau S^*(\bar{y} - y_d) + \nu\bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{ad}. \tag{2.14}$$

Proof: The proof follows standard arguments for optimization problems on convex sets. Let $u \in U_{ad}$ be arbitrary. The convexity of the admissible set U_{ad} ensures that the convex linear combination

$$u(t) = \bar{u} + t(u - \bar{u}), \quad t \in (0, 1]$$

belongs to U_{ad}. Due to the optimality of $\bar{u}$, we have $f(u(t)) \geq f(\bar{u})$. Furthermore, we find

$$\frac{1}{t}(f(\bar{u} + t(u - \bar{u})) - f(\bar{u})) \geq 0.$$

Going to the limit for $t \downarrow 0$, delivers the variational inequality

$$f'(\bar{u})(u - \bar{u}) \geq 0 \qquad \forall u \in U_{ad}$$

since $u \in U_{ad}$ was chosen arbitrarily. By the use of (2.13) and the control-to-state mapping (2.8), we obtain the assertion. □

Since the functional in problem (P) is strictly convex, the first-order necessary optimality condition, given in the previous lemma, is sufficient, too. In the next step, we want to find a representation for the adjoint operator S^*. First, we introduce the following auxiliary problem

$$\begin{aligned} -\Delta \hat{p} + \hat{p} &= w \quad \text{in } \Omega \\ \partial_n \hat{p} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.15}$$

Due to Theorem 2.3, the elliptic partial differential equation (2.15) admits a unique solution $\hat{p} \in H^1(\Omega)$ for every right hand side $w \in L^2(\Omega)$. Now, the following lemma delivers a representation for the adjoint operator S^*.

Lemma 2.8 *The adjoint operator $S^* : L^2(\Omega) \to H^1(\Omega)$ with respect to the state equation* (2.1) *is given by*

$$S^* w := \hat{p},$$

where $\hat{p} \in H^1(\Omega)$ is the weak solution of (2.15) *with respect to $w \in L^2(\Omega)$.*

Proof: Since $\hat{p} \in H^1(\Omega)$ is the weak solution of (2.15) with respect to $w \in L^2(\Omega)$, we find

$$a(z, \hat{p}) = \int_\Omega wz \, dx \qquad \forall z \in H^1(\Omega)$$

with the bilinear form $a(\cdot, \cdot)$ defined in (2.2). Moreover, the definition of the solution operator $S : H^1(\Omega)^* \to L^2(\Omega)$, $y = Sf$ for an arbitrary element $f \in H^1(\Omega)^*$ implies

$$a(y, z) = \langle f \, , \, z \rangle_{H^1(\Omega)^*, H^1(\Omega)} \qquad \forall z \in H^1(\Omega).$$

Choosing $y \in H^1(\Omega)$ as test function in the first weak formulation and $\hat{p} \in H^1(\Omega)$ in the second one, respectively, we derive

$$\int_\Omega wy \, dx = \langle f \, , \, \hat{p} \rangle_{H^1(\Omega)^*, H^1(\Omega)}.$$

Since $f \in H^1(\Omega)^*$ and $w \in L^2(\Omega)$ are arbitrary, we obtain the definition of the adjoint operator S^*:

$$(y \, , \, w)_{L^2(\Omega)} = (Sf \, , \, w)_{L^2(\Omega)} = \langle f \, , \, S^* w \rangle_{H^1(\Omega)^*, H^1(\Omega)} = \langle f \, , \, \hat{p} \rangle_{H^1(\Omega)^*, H^1(\Omega)}.$$

Due to Theorem 2.3, the mapping $w \mapsto \hat{p}$, $\hat{p} = S^* w$ is linear and continuous from $L^2(\Omega)$ to $H^1(\Omega)$ such that the adjoint operator is well-defined. □

Let us come back to the optimal control problem (P). Based on Lemma 2.8, we introduce the so-called adjoint state.

Definition 2.9 *The function* $\hat{p} = S^*(y - y_d) \in H^1(\Omega)$ *denotes the adjoint state associated with the state* $y = S\tau^* u$*, where* $\hat{p}$ *is the weak solution of* (2.15) *with respect to the right hand side* $w := y - y_d \in L^2(\Omega)$*.*

Now, the necessary and sufficient optimality condition can also be written as:

Corollary 2.10 *Let* $(\bar{y}, \bar{u})$ *be the optimal solution of problem (P) and* $\bar{p}$ *is the adjoint state associated with* $\bar{y}$*. Then* $(\bar{y}, \bar{u}, \bar{p})$ *satisfy the optimality system*

$$\bar{y} = S\tau^* \bar{u} \tag{2.16}$$
$$\bar{p} = S^*(\bar{y} - y_d) \tag{2.17}$$
$$(\tau\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{ad}. \tag{2.18}$$

We note that the optimality system (2.16)-(2.18) is an essential part of the later error analysis. However, the difficulties arising from the pure state constraints are not directly visible in this formulation. We will indicate the problems by the following consideration. Let us assume that the pure state constraints are active in an inner subdomain Ω_a of Ω', i.e.

$$y(x) = (S\tau^* u)(x) = y_c(x) \qquad \text{a.e. in } \Omega_a \subset \Omega'.$$

We can consider this as an equation determining the control u from the data function y_c. The linear and continuous control-to-state mapping is compact, since the range of the mapping is considered in $L^2(\Omega)$. Hence, the previous equation is ill-posed. Of course, ill-posedness can cause several difficulties, for instance a loss in the performance of numerical methods. In the next chapter we elaborate the details for state constrained optimal control problems by the use of the generalized Karush-Kuhn-Tucker theory, i.e., the introduction of Lagrange multipliers with respect to the pure state constraints. The associated theory was developed by Casas in [17] for a more general class of optimal control problems with pointwise state constraints.

Often in optimal control theory, the variational inequality and the adjoint state of the optimality system is used to obtain higher regularity of the optimal solution, particularly of the control. The variational inequality can be interpreted as the projection of the adjoint state on the admissible set. In the case of control constraints one obtains a simple pointwise projection formula such that a certain smoothness of the adjoint state is assigned to the control. Due to the definition of our admissible set U_{ad} for problem (P), a pointwise evaluation of the variational inequality (2.18) in connection with an increase of regularity of the control is not possible. This is a further reason for discussing the state constraints with the help of Lagrange multipliers.

2.3 Lagrange formulation and adjoint equation with multiplier

In this section we will derive optimality conditions for problem (P) applying the Karush-Kuhn-Tucker theory. This theory implies the existence of Lagrange multipliers with respect to the pure state constraints under certain so-called constraint qualifications. First, we consider a more general class of optimization problems, that covers the original problem (P). We follow the argumentation of Tröltzsch in [73, Chapter 6].

In the sequel, let U and Y be real Banach spaces and $C \subset U$ a nonempty convex set. Moreover, we introduce two Fréchet differentiable mappings $f : U \to \mathbb{R}$ and $G : U \to Y$. First, let us recall the definition of the convex cone.

Definition 2.11 *A convex set $K \subset Y$ is called convex cone, if $y \in K$ implies $\alpha y \in K$ for all positive $\alpha \in \mathbb{R}$.*

Definition 2.12 *Let $K \in Y$ be a convex cone. We write $y \geq_K 0$ if and only if $y \in K$. The cone defining this relation is called the positive cone in Y. Analogously, $y \leq_K 0$ is equivalent to $-y \in K$. Moreover, $y >_K 0$ implies $y \in \mathrm{int} K$.*

In the sequel, let K be a positive cone. The general optimization problem is given by

$$\left.\begin{array}{l} \min f(u) \\ \text{s. t. } G(u) \leq_K 0, \quad u \in C \end{array}\right\} \tag{PG}$$

Definition 2.13 *A function $\bar{u} \in C$ is called local solution of problem* (PG)*, if the constraint $G(\bar{u}) \leq_K 0$ is satisfied, and a constant $\delta > 0$ exists such that*

$$f(\bar{u}) \leq f(u)$$

for all $u \in C$ fulfilling $G(u) \leq_K 0$ and $\|\bar{u} - u\|_U \leq \delta$.

The rather general constraint in (PG) is to be eliminated by a Lagrange multiplier. We introduce the following definitions.

Definition 2.14 *The function $\mathcal{L} : U \times Y^* \to \mathbb{R}$, given by*

$$\mathcal{L}(u, \mu) := f(u) + \langle \mu \, , \, G(u) \rangle_{Y^*,Y}$$

is called Lagrange function.

Definition 2.15 *Let $\bar{u}$ be local solution of* (PG)*. Then, $\mu \in Y^*$ is called Lagrange multiplier, if it satisfies the following conditions:*

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u}, \mu)(u - \bar{u}) \geq 0 \quad \forall u \in C \tag{2.19}$$

$$\langle \mu \, , \, G(\bar{u}) \rangle_{Y^*,Y} = 0 \tag{2.20}$$

$$\langle \mu \, , \, y \rangle_{Y^*,Y} \geq 0 \quad \forall y \in Y. \tag{2.21}$$

With these definitions, we will state the well known Karush-Kuhn-Tucker Theorem. For a proof we refer for instance to Luenberger in [51] or, in a more general case to Zowe and Kurcyusz in [80].

Theorem 2.16 *Let $K \subset Y$ be a positive cone. We assume that $\bar{u}$ is a local solution of* (PG) *and that there is an $\tilde{u} \in C$ such that*

$$G(\bar{u}) + G'(\bar{u})\tilde{u} <_K 0 \tag{2.22}$$

is satisfied. Then, there exists a Lagrange multiplier $\mu \in Y^*$.

We mention that the so-called local Slater condition (2.22) is equivalent to the *constraint qualification* introduced by Zowe and Kurcyusz in [80]. In particular, it means that K contains inner points, cf. Penot [64].

We will now assign this general theory to our specific optimal control problem (P). We choose $U = L^2(\Gamma)$ and we recall the reduced objective functional $f : L^2(\Gamma) \to \mathbb{R}$ by

$$f(u) := J(S\tau^* u, u) = \|S\tau^* u - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}, \tag{2.23}$$

where the mapping $u \mapsto y$, $y = S\tau^* u$ is defined as in (2.8). Moreover, we define an admissible set

$$U^L_{ad} := \{u \in L^2(\Gamma) : \ u_a \le u \le u_b \ \text{ a.e. on } \Gamma\} \tag{2.24}$$

with respect to the control constraints. Thus, we set $C := U^L_{ad}$. In order to specify the mapping G, we have to choose the space Y and the corresponding convex cone $K \subset Y$. We mention that the range of the solution operator S is $L^2(\Omega)$, and the choice $Y = L^2(\Omega)$ would directly imply $\mu \in Y^* \cong L^2(\Omega)$. Defining $G : L^2(\Gamma) \to L^2(\Omega)$ by

$$G(u) = y_c - S\tau^* u,$$

the positive cone related to the state constraints is given by

$$K := \{y \in L^2(\Omega) : \ y \ge 0 \ \text{ a.e. in } \Omega'\}.$$

Unfortunately, K does not contain an inner point, i.e. int $K = \emptyset$. We will explain this fact by a simple example in the onedimensional case, i.e., we set $K := \{y \in L^2(0,1) : \ y \ge 0 \text{ a.e. in } (0,1)\}$. One could guess that the function $y(x) \equiv 1$ is an inner point of K. We consider the following sequence of functions

$$y_n(x) = \begin{cases} \ \ 1, & x \in [0, 1 - 1/n) \\ -1, & x \in [1 - 1/n, 1], \end{cases}$$

which does not belong to K. But, the sequence is converging to $y \equiv 1$ in the L^2-norm, i.e. $y \notin$ int K. Consequently, we cannot apply Theorem 2.16 since the assumption (2.22) is not satisfied. Therefore, we have to consider the state constraints of (P) in another space Y. One can easily see that the positive cone of nonnegative continuous functions contains inner points. Hence, it is possible to find functions $\tilde{u}$ such that the assumption (2.22) of Theorem 2.16 is fulfilled. Notice, that this assumption is equivalent to Assumption 2.4 for the linear-quadratic case. We remind that the pure state constraints are required in an inner subdomain $\Omega' \subset\subset \Omega$. Hence, we choose $Y = C(\Omega')$ and the corresponding positive cone is given by

$$K := \{y \in C(\Omega') : \ y(x) \ge 0 \ \forall x \in \Omega'\}.$$

The positive cone K is well defined, since the control constraints $u \in U_{ad}^L$ imply the continuity of the associated solution of the state equation (2.1), see Theorem 2.3. Finally, we have to specify the dual space of $C(\Omega')$. To this aim, we mention that every $\mu \in (C(\Omega'))^*$ can be identified with a regular Borel measure, which is also denoted by $\mu \in \mathcal{M}(\Omega')$, see for instance Alt [2]. The dual product is given by

$$\langle \mu \, , \, y \rangle_{(C(\Omega'))^*, C(\Omega')} = \int\limits_{\Omega'} y(x) d\mu, \qquad \forall y \in C(\Omega').$$

Now we can guess the difficulties of optimal control problems with pure state constraints. Applying now Theorem 2.16 and introducing an adjoint state, we end up with the following result. For a more detailed elaboration and a proof of the theorem, we refer to Casas [17].

Theorem 2.17 *Suppose that Assumption 2.4 is fullfilled. Moreover, let* $(\bar{y}, \bar{u})$ *be the optimal solution of problem* (P)*. Then a regular Borel measure* $\mu \in \mathcal{M}(\Omega')$*, which is extended by zero outside of* $\bar{\Omega}'$*, and an adjoint state* $p \in W^{1,s}(\Omega)$*,* $s < d/(d-1)$ *exists such that the following optimality system is satisfied:*

$$\begin{aligned} -\Delta \bar{y} + \bar{y} &= 0 & \qquad -\Delta p + p &= \bar{y} - y_d - \mu \\ \partial_n \bar{y} &= \bar{u} & \partial_n p &= 0 \end{aligned} \tag{2.25}$$

$$(\tau p + \nu \bar{u} \, , \, u - \bar{u})_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{ad}^L \tag{2.26}$$

$$\begin{aligned} &\int\limits_{\Omega'} (y_c - \bar{y}) d\mu = 0, \quad \bar{y}(x) \geq y_c(x) \quad a.e \ in \ \Omega' \\ &\int\limits_{\Omega'} \varphi d\mu \geq 0 \quad \forall \varphi \in C(\Omega'), \quad \varphi(x) \geq 0 \quad \forall x \in \Omega'. \end{aligned} \tag{2.27}$$

This result of Casas illustrates the difficulties of pure state constraints namely that the Lagrange multipliers associated with these constraints are in general no regular functions. Of course, the resulting lack of regularity in the adjoint state causes difficulties in the numerical analysis. But, another crucial problem in the case of boundary control problems becomes visible: due to the structure of the variational inequality (2.26), the adjoint state is uniquely determined only on the boundary. The remaining adjoint equation cannot guarantee the uniqueness of both of the dual variables p and μ, respectively. We refer to an example below. Of course, the nonuniqueness of the dual variables implies trouble to numerical optimization methods that attacks the full Karush-Kuhn-Tucker system directly.

Next, we illustrate the nonuniqueness of the dual variables by an example. We adapt the idea that is given in [3, Proposition 3.5]. Let $(\bar{y}, \bar{u})$ be the solution of problem (P). Furthermore, let $\hat{p}$ be an adjoint state and let $\hat{\mu}$ be a Lagrange multiplier such that the optimality system (2.25)-(2.27) is satisfied. Moreover, we assume that the Lagrange multiplier is a regular function. Notice, that one can easily construct examples, where the Lagrange multiplier associated with pure state constraints is a regular function, see for instance [45, Section 4.1.]. Furthermore, let $B_r(x_0)$ be an open ball centered at $x_0 \in \operatorname{int} \Omega'$ with radius $r > 0$ such that $\hat{\mu} \geq M > 0$ holds on $\overline{B_r(x_0)}$. Forthcoming, let $\tilde{p}$ be a sufficiently smooth function with $\tilde{p} \equiv 0$ a.e. in $\Omega \setminus \overline{B_r(x_0)}$. Hence, the function f defined by

$$f := -\Delta\tilde{p} + \tilde{p}$$

belongs to $L^\infty(\Omega)$ and $\operatorname{supp} f \subseteq B_r(x_0)$. Next, we set

$$\mu := \hat{\mu} + Cf \quad \text{and} \quad p := \hat{p} + C\tilde{p}$$

for a constant C. It easy to check that p and μ satisfy the optimality system (2.25)-(2.27) for all constants $0 < C < \frac{M}{\|f\|_{L^\infty(\Omega)}}$.

2.4 Higher regularity of the solution

In the previous section we derived the optimality system for problem (P). Furthermore, we asserted that the Lagrange multiplier with respect to the state constraints is in general only a Borel measure. Hence, the adjoint state is of low regularity. We will explain now the benefit arising from the consideration of state constraints in an inner subdomain Ω' instead of Ω. Although, Theorem 2.17 shows that the adjoint state is in general of low regularity, we will derive higher regularity of the adjoint state close to the boundary of Ω and on the boundary Γ itself. This is caused by the localization of the Lagrange multiplier in the inner subdomain Ω'. Thus, the smoothness of the optimal solution $(\bar{y}, \bar{u})$ for problem (P) will be improved.

First, let us recall a classical result for convex and polygonally or polyhedrally bounded domains $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, see e.g. [36].

Theorem 2.18 *Let Ω be a convex and polygonally or polyhedrally bounded domain in $\mathbb{R}^d$, $d = 2, 3$. Then, for every $(f, g) \in L^2(\Omega) \times H^{1/2}(\Gamma)$ the elliptic partial differential equation* (2.9) *admits a unique solution $w \in H^2(\Omega)$ and there exists a constant $c > 0$, depending only on the domain, such that*

$$\|w\|_{H^2(\Omega)} \leq c(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)})$$

is satisfied.

In view of the adjoint equation in Theorem 2.17, we consider the equation

$$\begin{aligned} -\Delta\, p + p &= \tilde{\mu} \quad \text{in } \Omega \\ \partial_n p &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2.28}$$

where $\tilde{\mu}$ is defined as follows

$$\tilde{\mu} = \begin{cases} \mu, & \text{in } \Omega' \\ 0, & \text{in } \Omega \setminus \Omega' \end{cases}$$

with given $\mu \in \mathcal{M}(\Omega')$, i.e.

$$\int_{\Omega} d\tilde{\mu} = \int_{\Omega'} d\mu.$$

According to Casas [17], there is a unique solution of (2.28) in $W^{1,s}(\Omega)$, $s < d/(d-1)$ that fulfills

$$\|p\|_{W^{1,s}(\Omega)} \leq c\,\|\mu\|_{\mathcal{M}(\Omega')}. \tag{2.29}$$

However, on a domain that is separated from Ω', p is more regular as stated in the following lemma.

Lemma 2.19 *Let Ω'', and Ω''' be subdomains of Ω that satisfy*

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega''' \subset\subset \Omega.$$

Furthermore, let $p \in W^{1,s}(\Omega)$ be the solution of (2.28). There is a constant $c > 0$ such that

$$\|p\|_{H^2(\Omega\setminus\Omega''')} \leq c\,\|\mu\|_{\mathcal{M}(\Omega')},$$

where c only depends on Ω', Ω'', Ω''', and Ω.

Proof: We start by defining

$$\varphi \in C^\infty(\bar{\Omega})\ ,\ \varphi|_{\Omega'} \equiv 0\ ,\ \varphi|_{\Omega\setminus\Omega''} \equiv 1.$$

Note that such a function exists since $\operatorname{dist}(\Omega', \partial\Omega'') > 0$ by assumption. Furthermore, the weak formulation of (2.28) is given by

$$\int_{\Omega} (\nabla p \cdot \nabla z + pz)\,dx = \int_{\Omega'} z d\mu \quad \forall z \in W^{1,s'}(\Omega),\ s' = s/(s-1).$$

Now, we will consider $p\varphi$ in the weak formulation. Integration by parts yields

$$\begin{aligned}
\int_{\Omega} (\nabla(p\varphi) \cdot \nabla z + (p\varphi)z)\,dx = & \int_{\Omega} \varphi\nabla p \cdot \nabla z + p\nabla\varphi \cdot \nabla z + p\varphi z\,dx \\
& = \int_{\Omega} (p\nabla\varphi \cdot \nabla z - z\nabla p \cdot \nabla\varphi)\,dx \\
& + \underbrace{\int_{\Omega} (\nabla p \cdot \nabla(\varphi z) + p\,\varphi\,z)\,dx}_{=\int_{\Omega'} \varphi\, z\, d\mu = 0} \\
& = -\int_{\Omega} (p\,z\,\Delta\varphi + 2\,z\nabla p \cdot \nabla\varphi)\,dx + \underbrace{\int_{\Gamma} z\,p\,\partial_n\varphi\,ds}_{=0},
\end{aligned}$$

where we used $\nabla\varphi|_\Gamma = 0$ which holds due to $\operatorname{dist}(\partial\Omega, \Omega'') > 0$ and $\varphi|_{\Omega\setminus\Omega''} \equiv 1$. Hence we obtain the following variational formulation for $w := p\varphi$

$$\int_\Omega (\nabla w \cdot \nabla z + wz)\, dx = -\int_\Omega (p\,\Delta\varphi + 2\nabla p \cdot \nabla\varphi) z\, dx \quad \forall\, z \in W^{1,s'}(\Omega). \tag{2.30}$$

Clearly, due to $p \in W^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$, see Sobolev embedding theorem 1.4, and $\varphi \in C^\infty(\bar\Omega)$, the right hand side in (2.30) defines an element of $H^1(\Omega)^*$. Applying the Lax-Milgram lemma, (2.30) admits a unique solution $w \in H^1(\Omega)$ giving in turn $p \in H^1(\Omega \setminus \Omega'')$ by the definition of φ. Next, we repeat the argument w.r.t. Ω''', i.e. we define a function ψ with

$$\psi \in C^\infty(\bar\Omega)\ ,\ \psi|_{\Omega''} \equiv 0\ ,\ \psi|_{\Omega\setminus\Omega'''} \equiv 1.$$

Then $\zeta := w\psi$ solves for all $z \in H^1(\Omega)$

$$\begin{aligned}\int_\Omega (\nabla\zeta \cdot \nabla z + \zeta z)\, dx &= \int_\Omega w\nabla\psi \cdot \nabla z - z\nabla w \cdot \nabla\psi\, dx + \int_\Omega (\nabla w \cdot \nabla(\psi z) + w\,\psi\, z)\, dx \\ &= -\int_\Omega (w\,\Delta\psi + 2\nabla w \cdot \nabla\psi) z\, dx + \int_\Omega (\nabla w \cdot \nabla(\psi z) + w\,\psi\, z)\, dx\end{aligned}$$

Using (2.30) for the second term, we obtain

$$\begin{aligned}\int_\Omega (\nabla\zeta \cdot \nabla z + \zeta z)\, dx = &-\int_\Omega (w\,\Delta\psi + 2\nabla w \cdot \nabla\psi) z\, dx \\ &- \int_\Omega (p\,\Delta\varphi + 2\nabla p \cdot \nabla\varphi)\psi z\, dx.\end{aligned}$$

Due to $p \in H^1(\Omega \setminus \Omega'')$, $w \in H^1(\Omega)$, $\psi|_{\Omega''} \equiv 0$, and $\varphi, \psi \in C^\infty(\bar\Omega)$ we have

$$w\,\Delta\psi + 2\nabla w \cdot \nabla\psi + (p\,\Delta\varphi + 2\nabla p \cdot \nabla\varphi)\psi \in L^2(\Omega)$$

and consequently $\zeta \in H^2(\Omega)$ by Theorem 2.18 implying in turn $p \in H^2(\Omega\setminus\Omega''')$. The estimate on $\|p\|_{H^2(\Omega\setminus\Omega''')}$ finally follows by straight forward estimation from (2.29) and the estimate in Theorem 2.18. □

Let us summarize the previous results in the following corollary.

Corollary 2.20 *Let the assumptions of Theorem 2.17 be fulfilled. Moreover, let $\mu \in \mathcal{M}(\Omega')$ be a regular Borel measure and $p \in W^{1,s}(\Omega)$, $s < d/(d-1)$ an adjoint state such that the optimality system (2.25)-(2.27) is satisfied. Then, we have $p \in H^1(\Gamma)$ and there is a positive constant c such that*

$$\|p\|_{H^1(\Gamma)} \le c(\|\bar y\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} + \|\mu\|_{\mathcal{M}(\Omega')}) \tag{2.31}$$

is valid.

In connection with the Trace theorem 1.7, the boundedness of the adjoint state with respect to the the regular functions $\bar{y}$ and y_d is implied by the standard regularity result of Grisvard, given in Theorem 2.18. Lemma 2.19 delivers the boundedness of the adjoint state associated with the measure part μ.

Considering now the variational equality (2.26) more precisely, one can increase the smoothness of the optimal control by the trace of the adjoint state. It is well known that the variational inequality is equivalent to the following pointwise projection on the admissible set U_{ad}^L

$$\bar{u} = P\left\{-\frac{\tau p}{\nu}\right\}, \tag{2.32}$$

where the projection operator P is defined by

$$P(f(x)) = \max\{u_a, \min\{f(x), u_b\}\}.$$

An important property of the pointwise projection operator is stated in the following lemma.

Lemma 2.21 *Let $f \in H^1(\Gamma)$ be a given function. Then, we have $P(f) \in H^1(\Gamma)$ and there exist positive constants C_1 and C_2 depending on the boundary and the bounds u_a, u_b such that*

$$\|P(f(x))\|_{H^1(\Gamma)} \leq C_1\|f\|_{H^1(\Gamma)} + C_2$$

is valid.

For a proof of we refer, for instance, to [43] or [48]. Thanks to Lemma 2.21 and Corollary 2.20, the optimal control $\bar{u}$ belongs to $H^1(\Gamma)$ and there exists a constant $C > 0$ such that

$$\|\bar{u}\|_{H^1(\Gamma)} \leq C. \tag{2.33}$$

is satisfied.

The higher regularity of the control at the boundary implies also higher smoothness of the state. According to Theorem 2.18, the weak solution $\bar{y}$ of the state equation (2.1) with respect to the right hand side $\bar{u} \in H^1(\Gamma)$ belongs to $H^2(\Omega)$ and the estimate

$$\|\bar{y}\|_{H^2(\Omega)} \leq c\|\bar{u}\|_{H^{1/2}(\Gamma)}$$

is satisfied for a positive constant depending only on the domain. The following result is one of the key points, where we benefit from the consideration of the state constraints in Ω' instead of Ω. It is devoted to the higher interior regularity of weak solutions of elliptic partial differential equations.

Theorem 2.22 *Let Ω be a convex and polygonally or polyhedrally bounded domain in $\mathbb{R}^d$, $d = 2, 3$. Suppose $w \in H^1(\Omega)$ is the weak solution of the elliptic partial differential equation*

$$\begin{aligned} -\Delta w + w &= f \quad \text{in } \Omega \\ \partial_n w &= g \quad \text{on } \Gamma \end{aligned}$$

for some $(f, g) \in L^2(\Omega) \times L^2(\Gamma)$*. If* $f \in H^m(\Omega)$ *for some nonnegative integer* m*, then* w *is an element of* $H^{m+2}(U)$ *for each subdomain* $U \subset\subset \Omega$ *and the estimate*

$$\|w\|_{H^{m+2}(U)} \leq c(\|f\|_{H^m(\Omega)} + \|w\|_{L^2(\Omega)})$$

is satisfied, where the positive constant c *is depending only on* Ω*,* U *and* m*.*

For a proof and more detailed explanations about interior regularity, we refer to [32, Chapter 6.3.1.]. As an immediate consequence, we deduce the following corollary.

Corollary 2.23 *Let* $y = S\tau^*u \in H^1(\Omega)$ *be the weak solution of* (2.1) *for a given* $u \in L^2(\Gamma)$*. Then* y *is an element of* $W^{2,\infty}(\Omega')$ *and there exists a constant* c*, depending on* Ω *and* Ω'*, such that*

$$\|y\|_{W^{2,\infty}(\Omega')} \leq c\|y\|_{L^2(\Omega)}. \tag{2.34}$$

Due to the fact that there is no source term in the classical formulation (2.1) of $y = S\tau^*u$, the previous result directly follows from Sobolev embeddings and Theorem 2.22. The $W^{2,\infty}$-regularity will be essential for interior maximum norm estimates of finite element approximations to solutions $y = S\tau^*u$ arising in Chapter 5. In the previous corollary, the L^2-norm of the weak solution $y = S\tau^*u$ of (2.1) appears. The next lemma provides an estimate of this norm, where the right hand side is considered in a weaker norm.

Lemma 2.24 *Let* $y \in H^1(\Omega)$ *be the solution of the weak formulation* (2.4) *for a given* $u \in L^2(\Gamma)$*. Then, there is a constant* $c > 0$*, independent of* u*, such that*

$$\|y\|_{L^2(\Omega)} \leq c\,\|u\|_{H^1(\Gamma)^*}.$$

Proof: We introduce a dual problem for a given function $f \in L^2(\Omega)$:

$$a(z, w) = \int_\Omega f z\, dx, \quad \forall z \in H^1(\Omega),$$

where the bilinear form $a(\cdot, \cdot)$ is defined in (2.2). According to Theorem 2.18, there is a unique solution $w \in H^2(\Omega)$ and the estimate

$$\|w\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)} \tag{2.35}$$

is satisfied. Furthermore, we have

$$a(y, z) = \int_\Gamma u\tau z\, ds, \quad \forall z \in H^1(\Omega)$$

since $y \in H^1(\Omega)$ is the solution of the weak formulation (2.4) for $u \in L^2(\Gamma)$. By means of the dual problem, the estimate (2.35) and the Trace theorem 1.7, we derive

$$\begin{aligned}
\|y\|_{L^2(\Omega)} &= \sup_{f\in L^2(\Omega)} \frac{|(f\,,\,y)_{L^2(\Omega)}|}{\|f\|_{L^2(\Omega)}} = \sup_{f\in L^2(\Omega)} \frac{|a(y,w)|}{\|f\|_{L^2(\Omega)}} \\
&= \sup_{f\in L^2(\Omega)} \frac{|(u\,,\,\tau w)_{L^2(\Omega)}|}{\|f\|_{L^2(\Omega)}} \\
&\leq \sup_{f\in L^2(\Omega)} \frac{\|u\|_{H^1(\Gamma)^*}\|\tau w\|_{H^1(\Gamma)}}{\|f\|_{L^2(\Omega)}} \\
&\leq \sup_{f\in L^2(\Omega)} \frac{c\|u\|_{H^1(\Gamma)^*}\|w\|_{H^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \\
&\leq \sup_{f\in L^2(\Omega)} \frac{c\|u\|_{H^1(\Gamma)^*}\|f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \\
&= c\|u\|_{H^1(\Gamma)^*},
\end{aligned}$$

which is the assertion. $\square$

Chapter 3

The virtual control approach with additional control constraints

In this chapter we will investigate the regularized problem (P_1^ε), where the pure state constraints of the original problem (P) are replaced by mixed control-state constraints with the help of a additional distributed control v_ε. Furthermore, we consider control constraints to the so-called virtual control. First, let us recall the problem:

$$\left.\begin{aligned} \min \quad J_\varepsilon(y_\varepsilon, u_\varepsilon, v_\varepsilon) := \frac{1}{2}\|y_\varepsilon - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_\varepsilon\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)} \\ -\Delta y_\varepsilon + y_\varepsilon = \phi(\varepsilon)v_\varepsilon \qquad \text{in } \Omega \\ \partial_n y_\varepsilon = u_\varepsilon \qquad \text{on } \Gamma \\ u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. on } \Gamma \\ y_\varepsilon(x) \ge y_c(x) - \xi(\varepsilon)v_\varepsilon(x) \quad \text{a.e. in } \Omega' \\ 0 \le v_\varepsilon(x) \le v_b \quad \text{a.e. in } \Omega. \end{aligned}\right\} \tag{P_1^ε}$$

The assumptions on the problem are stated in Chapter 1.3. Again, we mention that Lagrange multipliers associated with mixed control-state constraints are regular functions, see [68] and [74]. The main purpose is the development of an estimate between the solutions of the problem (P) and the regularized problem (P_1^ε). An essential keypoint for the derivation of such an estimate is the presence of additional control constraints on the new virtual control v_ε, see e.g. the proof of Lemma 3.6 below.

3.1 Analysis of the regularized problem (P_1^ε)

First, we verify the existence and uniqueness of solutions for problem (P_1^ε). Similarly to (2.8), we replace the state equation

$$\begin{aligned} -\Delta y_\varepsilon + y_\varepsilon &= \phi(\varepsilon)v_\varepsilon \quad \text{in } \Omega \\ \partial_n y_\varepsilon &= u_\varepsilon \qquad \text{on } \Gamma \end{aligned} \tag{3.1}$$

of (P_1^ε) by a control-to-state mapping. Using the bilinear form that was defined in (2.2), the weak formulation associated with (3.1) is given by

$$a(y_\varepsilon, z) = \int_\Gamma u_\varepsilon \tau z \, ds + \int_\Omega \phi(\varepsilon) v_\varepsilon z \, dx, \qquad \forall z \in H^1(\Omega).$$

For convenience, we use the same solution operator as in (2.8). Thus, we have to identify the right hand sides of the state equation as elements in $H^1(\Omega)^*$. According to (2.7), the boundary control $u_\varepsilon \in L^2(\Gamma)$ defines an element in the dual space of $H^1(\Omega)$. Furthermore, the distributed virtual control $v_\varepsilon \in L^2(\Omega)$ belongs to $H^1(\Omega)^*$ by

$$\langle E_H^* v \, , \, z \rangle_{H^1(\Omega)^*, H^1(\Omega)} := \int_\Omega v E_H z \, dx, \tag{3.2}$$

where $E_H : H^1(\Omega) \to L^2(\Omega)$ is the embedding operator from $H^1(\Omega)$ to $L^2(\Omega)$. Hence, the control-to-state mapping is given by

$$(u_\varepsilon, v_\varepsilon) \mapsto y_\varepsilon, \quad y_\varepsilon = S(\tau^* u_\varepsilon + \phi(\varepsilon) E_H^* v_\varepsilon), \tag{3.3}$$

again with the solution operator $S : H^1(\Omega)^* \to L^2(\Omega)$.
Now, let us introduce the admissible set of controls for problem (P_1^ε):

$$\begin{aligned} V_{ad}^{\varepsilon,1} =&\{(u,v) \in L^2(\Gamma) \times L^2(\Omega) | \, u_a \le u \le u_b \text{ a.e. on } \Gamma; \\ &0 \le v \le v_b \text{ a.e. in } \Omega; \, S(\tau^* u + \phi(\varepsilon) E_H^* v) \ge y_c - \xi(\varepsilon) v \text{ a.e. in } \Omega'\}. \end{aligned} \tag{3.4}$$

Due to Assumption 2.4, the admissible set $V_{ad}^{\varepsilon,1}$ contains at least the pair $(\hat{u}, 0)$ of controls, i.e. the set is nonempty. Moreover, the admissible set is convex, closed and bounded. With the help of (3.3) and (3.4), we state the reduced form of problem (P_1^ε):

$$\begin{aligned} \min_{(u_\varepsilon, v_\varepsilon) \in V_{ad}^{\varepsilon,1}} f_\varepsilon(u_\varepsilon, v_\varepsilon) := \frac{1}{2} \| S(\tau^* u_\varepsilon + \phi(\varepsilon) E_H^* v_\varepsilon) - y_d \|_{L^2(\Omega)}^2 + \frac{\nu}{2} \| u_\varepsilon \|_{L^2(\Gamma)}^2 \\ + \frac{\psi(\varepsilon)}{2} \| v_\varepsilon \|_{L^2(\Omega)}^2. \end{aligned} \tag{3.5}$$

Theorem 3.1 *Suppose that Assumption 2.4 is fulfilled. Then the optimization problem* (3.5) *admits a unique optimal solution* $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \in V_{ad}^{\varepsilon,1}$.

The proof is along the lines as in Theorem 2.5. Furthermore, the unique optimal state associated with the optimal controls $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ is given by

$$\bar{y}_\varepsilon = S(\tau^* \bar{u}_\varepsilon + \phi(\varepsilon) E_H^* \bar{v}_\varepsilon).$$

Hence, the linear-quadratic optimal control problem (P_1^ε) admits a unique optimal solution $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$. The objective functional in (3.5) is Fréchet differentiable. Due to $\psi(\varepsilon) > 0$, the functional is also strictly convex. The necessary and sufficient optimality condition for problem (P_1^ε) is formulated in the following lemma.

Lemma 3.2 *Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be the optimal solution of problem* $(\mathrm{P}_1^\varepsilon)$. *The necessary and sufficient optimality condition is given by*

$$(\tau S^*(\bar{y}_\varepsilon - y_d) + \nu \bar{u}_\varepsilon, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} + \\ (\phi(\varepsilon) E_H S^*(\bar{y}_\varepsilon - y_d) + \psi(\varepsilon)\bar{v}_\varepsilon, v - \bar{v}_\varepsilon)_{L^2(\Omega)} \geq 0 \quad \forall (u,v) \in V_{ad}^{\varepsilon,1}, \tag{3.6}$$

where $S^* : L^2(\Omega) \to H^1(\Omega)$ is the adjoint operator to S.

Since the solution operator S is the same as for problem (P), the representation of the adjoint operator is the same, too. With the help of Lemma 2.8 and Definition 2.9, the adjoint state associated with a state y_ε for problem $(\mathrm{P}_1^\varepsilon)$ is defined by $p_\varepsilon = S^*(y_\varepsilon - y_d)$, where p_ε is the weak solution of (2.15) with respect to the right hand side $w := y_\varepsilon - y_d$. Using the adjoint state, the necessary and sufficient optimality condition can be also written as:

Corollary 3.3 *Let $(\bar{y}_\varepsilon,\ \bar{u}_\varepsilon,\ \bar{v}_\varepsilon)$ be the optimal solution of problem $(\mathrm{P}_1^\varepsilon)$ and $\bar{p}_\varepsilon$ is the adjoint state associated with $\bar{y}_\varepsilon$. Then $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{p}_\varepsilon)$ satisfies the optimality system*

$$\bar{y}_\varepsilon = S(\tau^* \bar{u}_\varepsilon + \phi(\varepsilon) E_H^* \bar{v}_\varepsilon) \tag{3.7}$$

$$\bar{p}_\varepsilon = S^*(\bar{y}_\varepsilon - y_d) \tag{3.8}$$

$$(\tau \bar{p}_\varepsilon + \nu \bar{u}_\varepsilon, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\phi(\varepsilon) E_H \bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon\ ,\ v - \bar{v}_\varepsilon)_{L^2(\Omega)} \geq 0 \quad \forall (u,v) \in V_{ad}^{\varepsilon,1}. \tag{3.9}$$

We note that all inequality constraints of problem $(\mathrm{P}_1^\varepsilon)$ are handled in the convex admissible set $V_{ad}^{\varepsilon,1}$. Consequently, no Lagrange multiplier occurs in the adjoint equation (3.8) and in the variational inequality (3.9).

3.2 Convergence analysis

In this section we derive a regularization error estimate between the original solution of problem (P) and the optimal regularized solution of problem $(\mathrm{P}_1^\varepsilon)$. This estimate is based on the variational inequalities given in the optimality conditions of both problems. Clearly, we need appropriate feasible solutions for both problems (P) and $(\mathrm{P}_1^\varepsilon)$, respectively. These feasible controls should base on the optimal solution of the particular other problem. The next lemma provides a preliminary estimate depending on arbitrary feasible controls for the problems.

Lemma 3.4 *Let $(\bar{y},\ \bar{u})$ and $(\bar{y}_\varepsilon,\ \bar{u}_\varepsilon,\ \bar{v}_\varepsilon)$ be the optimal solution of* (P) *and* $(\mathrm{P}_1^\varepsilon)$, *respectively. For all $u_\delta \in U_{ad}$ and $(u_\sigma, 0) \in V_{ad}^{\varepsilon,1}$ there holds*

$$\nu \|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2} \|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \leq \\ (\tau \bar{p} + \nu \bar{u}\ ,\ u_\delta - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\tau \bar{p}_\varepsilon + \nu \bar{u}_\varepsilon\ ,\ u_\sigma - \bar{u})_{L^2(\Gamma)} \\ + c \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} \tag{3.10}$$

for a certain constant $c > 0$ independent of ε.

Proof: We start with the variational inequalities for (P) and (P_1^ε) given by (2.18) and (3.9), respectively. Adding both inequalities yields

$$(\tau\bar{p} + \nu\bar{u}, u_\delta - \bar{u})_{L^2(\Gamma)} + (\tau\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon, u_\sigma - \bar{u})_{L^2(\Gamma)} \\ + (\phi(\varepsilon)E_H\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon\,,\,-\bar{v}_\varepsilon)_{L^2(\Omega)} \geq 0$$

for all $u_\delta \in U_{ad}$ and $(u_\sigma, 0) \in V_{ad}^{\varepsilon,1}$. Next, we rewrite the previous inequality in the form

$$(\tau\bar{p} + \nu\bar{u}\,,\, u_\delta - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\tau\bar{p} + \nu\bar{u}\,,\, \bar{u}_\varepsilon - \bar{u})_{L^2(\Gamma)} \\ +(\tau\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon\,,\, u_\sigma - \bar{u})_{L^2(\Gamma)} + (\tau\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon\,,\, \bar{u} - \bar{u}_\varepsilon)_{L^2(\Gamma)} \\ +(\phi(\varepsilon)E_H\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon\,,\,-\bar{v}_\varepsilon)_{L^2(\Omega)} \geq 0$$

or in a more suitable way

$$(\tau\bar{p} + \nu\bar{u}\,,\, u_\delta - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\tau\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon\,,\, u_\sigma - \bar{u})_{L^2(\Gamma)} \\ +(\tau(\bar{p} - \bar{p}_\varepsilon)\,,\, \bar{u}_\varepsilon - \bar{u})_{L^2(\Gamma)} + \nu(\bar{u} - \bar{u}_\varepsilon\,,\, \bar{u}_\varepsilon - \bar{u})_{L^2(\Gamma)} \\ +(\phi(\varepsilon)E_H\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon\,,\,-\bar{v}_\varepsilon)_{L^2(\Omega)} \geq 0.$$

We proceed with considering the third term. By the use of the definitions of the respective states and adjoint states, given in Corollary 2.10 and Corollary 3.3, respectively, we obtain

$$\begin{aligned}(\tau(\bar{p} - \bar{p}_\varepsilon)\,,\, \bar{u}_\varepsilon - \bar{u})_{L^2(\Gamma)} &= (\tau S^*(\bar{y} - \bar{y}_\varepsilon)\,,\, \bar{u}_\varepsilon - \bar{u})_{L^2(\Gamma)} \\ &= (\bar{y} - \bar{y}_\varepsilon\,,\, S\tau^*(\bar{u}_\varepsilon - \bar{u}))_{L^2(\Omega)} \\ &= (\bar{y} - \bar{y}_\varepsilon\,,\, \bar{y}_\varepsilon - \bar{y})_{L^2(\Omega)} - (\bar{y} - \bar{y}_\varepsilon\,,\, SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega)} \\ &= -\|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} - (E_H(\bar{p} - \bar{p}_\varepsilon)\,,\, \phi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega)}.\end{aligned}$$

Summarizing the terms, we find

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} + \psi(\varepsilon)\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \leq \\ (\tau\bar{p} + \nu\bar{u}\,,\, u_\delta - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\tau\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon\,,\, u_\sigma - \bar{u})_{L^2(\Gamma)} \\ + \phi(\varepsilon)|(\bar{p}\,,\, \bar{v}_\varepsilon)_{L^2(\Omega)}|.$$

Finally, the last term is estimated by Young's inequality:

$$|(\bar{p}\,,\, \bar{v}_\varepsilon)_{L^2(\Omega)}| \leq \frac{\phi(\varepsilon)}{2\psi(\varepsilon)}\|\bar{p}\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2\phi(\varepsilon)}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)}.$$

Hence, we attained the assertion with the constant $c = 1/2\|\bar{p}\|^2_{L^2(\Omega)}$ that is independent of ε. □

The estimate (3.10) shows that the main goal for a final regularization error estimate is the determination of suitable feasible controls for the problems depending on the optimal solutions of the particular other problem.

3.2.1 Construction of feasible solutions

In this section we construct feasible solutions for the problem (P) and (P_1^ε), respectively. The following lemma shows the feasibility of the optimal control $\bar{u}$ of the original problem (P) for the regularized problem.

Lemma 3.5 *For every $\varepsilon > 0$ the control $(\bar{u}, 0)$ is feasible for (P_1^ε), i.e. $(\bar{u}, 0) \in V_{ad}^{\varepsilon,1}$.*

Proof: Since $\bar{u}$ is feasible for (P), the control constraints at the boundary and in the domain of problem (P_1^ε) are satisfied by $(\bar{u}, 0)$. Moreover, we find for all $\varepsilon > 0$

$$\xi(\varepsilon)0 + S(\tau^*\bar{u} + E_H^*\phi(\varepsilon)0) = \bar{y} \geq y_c \quad \text{a.e. in } \Omega'.$$

Hence, $(\bar{u}, 0)$ also fulfills the mixed control-state constraints of (P_1^ε). □

Unfortunately, the optimal regularized control $\bar{u}_\varepsilon$ of problem (P_1^ε) is in general infeasible for problem (P). To that end, we investigate the violation of the control $\bar{u}_\varepsilon$ with respect to the pure state constraints of problem (P). We define the violation function by

$$d[\bar{u}_\varepsilon, (P)] := (y_c - S\tau^*\bar{u}_\varepsilon)_+ = \max\{0, y_c - S\tau^*\bar{u}_\varepsilon\}. \tag{3.11}$$

The $L^\infty(\Omega')$-norm of this function is called maximal violation of $\bar{u}_\varepsilon$ with respect to problem (P).

Lemma 3.6 *The maximal violation $\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')}$ of $\bar{u}_\varepsilon$ w.r.t. problem (P) can be estimated by*

$$\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} \leq c(\xi(\varepsilon) + \phi(\varepsilon)\|\bar{v}_\varepsilon\|_{L^2(\Omega)}), \tag{3.12}$$

where $c > 0$ is a constant independent of ε.

Proof: The first step is done by using the definition (3.7) of the optimal state for problem (P_1^ε) and twice the triangle inequality:

$$\begin{aligned}
\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} &= \|(y_c - S\tau^*\bar{u}_\varepsilon)_+\|_{L^\infty(\Omega')} \\
&= \|(y_c - S(\tau^*\bar{u}_\varepsilon + E_H^*\phi(\varepsilon)\bar{v}_\varepsilon) + SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon)_+\|_{L^\infty(\Omega')} \\
&\leq \|(y_c - \bar{y}_\varepsilon)_+\|_{L^\infty(\Omega')} + \|SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon\|_{L^\infty(\Omega')}.
\end{aligned}$$

Since $\bar{y}_\varepsilon$ is the optimal state for problem (P_1^ε), the mixed control-state constraints are satisfied. Furthermore, the optimal virtual control $\bar{v}_\varepsilon$ fulfills the control constraints. Thus, we derive for the first term in the last inequality

$$\|(y_c - \bar{y}_\varepsilon)_+\|_{L^\infty(\Omega')} \leq \|(\xi(\varepsilon)\bar{v}_\varepsilon)_+\|_{L^\infty(\Omega')} \leq \xi(\varepsilon)\|\bar{v}_\varepsilon\|_{L^\infty(\Omega)} \leq \xi(\varepsilon)v_b.$$

Due to the definition of the solution operator S, the function $SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon$ is the weak solution of the partial differential equation (2.9) with respect to homogeneous Neumann boundary condition and the right hand side $\phi(\varepsilon)\bar{v}_\varepsilon \in L^2(\Omega)$. Hence, the estimate (2.10) of Theorem 2.3 delivers

$$\|SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon\|_{L^\infty(\Omega')} \leq c\phi(\varepsilon)\|\bar{v}_\varepsilon\|_{L^2(\Omega)}.$$

Summarizing all estimates, we obtain the assertion. □

In the next lemma, we construct a feasible solution u_δ for the problem (P) depending on the optimal regularized control $\bar{u}_\varepsilon$ and the inner point $\hat{u}$ of Assumption 2.4.

Lemma 3.7 *Let the Assumption 2.4 be satisfied. Then, for every $\varepsilon > 0$ there exists a $\delta_\varepsilon \in (0,1)$, such that $u_\delta := (1-\delta)\bar{u}_\varepsilon + \delta\hat{u}$ is feasible for (P) for all $\delta \in [\delta_\varepsilon, 1]$.*

Proof: One can easily see, that the convex linear combination

$$u_\delta := (1-\delta)\bar{u}_\varepsilon + \delta\hat{u} \tag{3.13}$$

fulfills the boundary constraints

$$u_a \le u_\delta \le u_b, \qquad \text{a.e. on } \Gamma,$$

since both of the controls $\bar{u}_\varepsilon$ and $\hat{u}$, respectively, satisfy this constraints. Consequently, we only have to check the state constraints. The associated state to u_δ is defined by

$$y_\delta = S\tau^* u_\delta.$$

By means of the violation function (3.11) and Assumption 2.4, we continue with

$$\begin{aligned} y_\delta = S\tau^* u_\delta &= (1-\delta)S\tau^*\bar{u}_\varepsilon + \delta S\tau^*\hat{u} \\ y_\delta - y_c &= (1-\delta)(S\tau^*\bar{u}_\varepsilon - y_c) + \delta(\hat{y} - y_c) \\ &\ge -(1-\delta)d[\bar{u}_\varepsilon, (P)] + \delta\gamma \\ &\ge -(1-\delta)\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} + \delta\gamma. \end{aligned}$$

One can easily see, that $\delta\gamma - (1-\delta)\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} \ge 0$ implies the feasibility of u_δ for problem (P). Hence, we set

$$\delta_\varepsilon := \frac{\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')}}{\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} + \gamma} \in (0,1) \qquad \forall \varepsilon > 0 \tag{3.14}$$

such that u_δ belongs to U_{ad} for all $\delta \in [\delta_\varepsilon, 1]$. □

3.2.2 Regularization error estimate

In the previous section we constructed feasible controls to both problems (P) and (P_1^ε), respectively. Next, we derive the main result of this chapter. The following theorem provides a preliminary error estimate of the optimal solution of problem (P) with respect to the optimal regularized one of problem (P_1^ε).

Theorem 3.8 *Let $(\bar{y}, \bar{u})$ and $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be the optimal solution of (P) and (P_1^ε), respectively. Then, there exists a positive constant c independent of ε, such that*

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \le c\left(\phi(\varepsilon)\|\bar{v}_\varepsilon\|_{L^2(\Omega)} + \xi(\varepsilon) + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right) \tag{3.15}$$

Proof: The basis for the proof is the estimate (3.10) given in Lemma 3.4. Thus, we start with choosing $u_\sigma := \bar{u}$ and u_δ, as defined by (3.13) in Lemma 3.7:

$$\begin{aligned}\nu\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y}-\bar{y}_\varepsilon\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \\ &\le \delta(\bar{p}+\nu\bar{u}\,,\,\hat{u}-\bar{u}_\varepsilon)_{L^2(\Gamma)} + c\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} \\ &\le \delta\|\bar{p}+\nu\bar{u}\|_{L^2(\Gamma)}\|\hat{u}-\bar{u}_\varepsilon\|_{L^2(\Gamma)} + c\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} \\ &\le \delta\|\bar{p}+\nu\bar{u}\|_{L^2(\Gamma)}|\Gamma||u_b-u_a| + c\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}.\end{aligned}$$

Let us mention that the term $\|\bar{p}+\nu\bar{u}\|_{L^2(\Gamma)}$ can also be limited by expressions containing only data of the problem. According to Lemma 3.7, we choose the specific parameter

$$\delta := \delta_\varepsilon = \frac{\|d[\bar{u}_\varepsilon,(P)]\|_{L^\infty(\Omega')}}{\|d[\bar{u}_\varepsilon,(P)]\|_{L^\infty(\Omega')}+\gamma}.$$

Moreover, we find

$$\delta_\varepsilon \le c\|d[\bar{u}_\varepsilon,(P)]\|_{L^\infty(\Omega')}$$

for all constants $c \ge \frac{1}{\gamma}$. Due to (3.12), we derive

$$\begin{aligned}\nu\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y}-\bar{y}_\varepsilon\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \le \\ c\left(\phi(\varepsilon)\|\bar{v}_\varepsilon\|_{L^2(\Omega)} + \xi(\varepsilon) + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right),\end{aligned}$$

which is the assertion. □

As one can see in the previous result, an estimate of the $L^2(\Omega)$-norm of the virtual control $\bar{v}_\varepsilon$ is necessary for the completion of the regularization error estimate. A simple estimate is given by the objective functional of problem (P_1^ε). Since $(\bar{y}_\varepsilon,\, \bar{u}_\varepsilon,\, \bar{v}_\varepsilon)$ is the optimal solution of problem (P_1^ε), we have $J_\varepsilon(\bar{y}_\varepsilon,\bar{u}_\varepsilon,\bar{v}_\varepsilon) < \infty$, and we accomplish

$$\begin{aligned}\frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \le J_\varepsilon(\bar{y}_\varepsilon,\bar{u}_\varepsilon,\bar{v}_\varepsilon) \le J_\varepsilon(\hat{y},\hat{u},0) = J(\hat{y},\hat{u}) \\ \|\bar{v}_\varepsilon\|_{L^2(\Omega)} \le \sqrt{\frac{2J(\hat{y},\hat{u})}{\psi(\varepsilon)}},\end{aligned} \tag{3.16}$$

where $(\hat{y},\hat{u})$ is the inner point defined in Assumption 2.4. However, this estimate is not yet optimal as the following corollary shows. This result is based on the preliminary error estimate of Theorem 3.8.

Corollary 3.9 *Let the assumptions of Theorem 3.8 be fulfilled. Then the estimate*

$$\|\bar{v}_\varepsilon\|_{L^2(\Omega)} \le c\frac{\sqrt{\xi(\varepsilon)\psi(\varepsilon)}+\phi(\varepsilon)}{\psi(\varepsilon)} \tag{3.17}$$

is satisfied for some constant $c > 0$ independent of ε.

Proof: Considering the error estimate (3.15), we have

$$\frac{\psi(\varepsilon)}{2}\|\bar v_\varepsilon\|^2_{L^2(\Omega)} \le c\left(\phi(\varepsilon)\|\bar v_\varepsilon\|_{L^2(\Omega)} + \xi(\varepsilon) + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right).$$

Moreover, this estimate implies

$$\|\bar v_\varepsilon\|^2_{L^2(\Omega)} \le \frac{c}{\psi(\varepsilon)}\max\left\{\phi(\varepsilon)\|\bar v_\varepsilon\|_{L^2(\Omega)}, \xi(\varepsilon) + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right\}.$$

Now, we consider the two cases where the maximum can be attained.
Case 1: First, we assume that the maximum in the right hand side of the previous inequality is determined by the first term. Hence, we obtain the following upper bound for the virtual control:

$$\|\bar v_\varepsilon\|_{L^2(\Omega)} \le c\frac{\phi(\varepsilon)}{\psi(\varepsilon)}.$$

Case 2: Considering the other case, we derive the following estimate:

$$\|\bar v_\varepsilon\|_{L^2(\Omega)} \le c\sqrt{\frac{\xi(\varepsilon)\psi(\varepsilon) + \phi(\varepsilon)^2}{\psi(\varepsilon)^2}} \le c\frac{\sqrt{\xi(\varepsilon)\psi(\varepsilon)} + \phi(\varepsilon)}{\psi(\varepsilon)}.$$

Summarizing both cases, we end up with

$$\|\bar v_\varepsilon\|_{L^2(\Omega)} \le c\max\left\{\frac{\phi(\varepsilon)}{\psi(\varepsilon)}, \frac{\sqrt{\xi(\varepsilon)\psi(\varepsilon)} + \phi(\varepsilon)}{\psi(\varepsilon)}\right\}.$$

One can easily see that the maximum is attained by the second term, which is the assertion. □

We proceed with the final error estimate of the optimal solution of problem (P) concerning the optimal regularized one of problem (P_1^ε).

Theorem 3.10 *Let $(\bar y, \bar u)$ and $(\bar y_\varepsilon, \bar u_\varepsilon, \bar v_\varepsilon)$ be the optimal solution of* (P) *and* (P_1^ε)*, respectively. Then, there exists a positive constant c, independent of ε, such that*

$$\nu\|\bar u - \bar u_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar y - \bar y_\varepsilon\|^2_{L^2(\Omega)} \le c\left(\xi(\varepsilon) + \frac{\phi(\varepsilon)\sqrt{\xi(\varepsilon)}}{\sqrt{\psi(\varepsilon)}} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right). \tag{3.18}$$

The result directly follows from Theorem 3.8 and Corollary 3.9. By means of Corollary 3.9, a final estimate of the maximal violation $\|d[\bar u_\varepsilon, (P)]\|_{L^\infty(\Omega')}$ is available, too.

Corollary 3.11 *The maximal violation $\|d[\bar u_\varepsilon, (P)]\|_{L^\infty(\Omega')}$ of $\bar u_\varepsilon$ w.r.t. problem* (P) *can be estimated by*

$$\|d[\bar u_\varepsilon, (P)]\|_{L^\infty(\Omega')} \le c\left(\xi(\varepsilon) + \frac{\phi(\varepsilon)\sqrt{\xi(\varepsilon)}}{\sqrt{\psi(\varepsilon)}} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right), \tag{3.19}$$

where $c > 0$ is a constant independent of ε.

The estimate of this corollary is a direct consequence of (3.12) and (3.17).

One can easily see in the regularization error estimate (3.18), that an appropriate choice of the parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$ should satisfy the following limit relations:

$$\lim_{\varepsilon\to 0}\xi(\varepsilon)=0, \qquad \lim_{\varepsilon\to 0}\frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}}=0. \tag{3.20}$$

Under these conditions, we can state the following remark.

Remark 3.12 Let the parameter functions fulfill (3.20). Then the maximal violation $\|d[\bar{u}_\varepsilon,(P)]\|_{L^\infty(\Omega')}$ tends to zero as $\varepsilon\to 0$. Moreover, the strong convergence

$$\bar{u}_\varepsilon \xrightarrow{L^2} \bar{u} \quad \text{as} \quad \varepsilon\to 0$$

holds.

Let us remind that the regularized control $\bar{u}_\varepsilon$ is in general infeasible for the original problem (P). In practical applications, this infeasibility might be a problem. The next corollary shows that the feasible control u_δ, which was constructed in Lemma 3.7, exhibits the same approximation properties as $\bar{u}_\varepsilon$.

Corollary 3.13 *Let* $(\bar{y},\bar{u})$ *be the optimal solution of problem* (P) *and let* u_δ *be the control introduced in Lemma 3.7 with* $\delta=\delta_\varepsilon$. *Furthermore, we assume that the parameter functions fulfill the conditions* (3.20). *Then, there exists a positive constant* $c>0$, *independent of* ε, *such that the estimate*

$$\nu\|\bar{u}-u_\delta\|^2_{L^2(\Gamma)}+\|\bar{y}-y_\delta\|^2_{L^2(\Omega)}\le c\left(\xi(\varepsilon)+\frac{\phi(\varepsilon)\sqrt{\xi(\varepsilon)}}{\sqrt{\psi(\varepsilon)}}+\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right) \tag{3.21}$$

is satisfied.

Proof: With the help of $a^2-b^2=(a+b)(a-b)$, we find

$$\begin{aligned}\nu\|\bar{u}-u_\delta\|^2_{L^2(\Gamma)} &\le \nu((2\bar{u}-\bar{u}_\varepsilon-u_\delta\,,\,\bar{u}_\varepsilon-u_\delta)_{L^2(\Gamma)}+\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Gamma)})\\ &\le \nu(\|2\bar{u}-\bar{u}_\varepsilon-u_\delta\|_{L^2(\Gamma)}\|\bar{u}_\varepsilon-u_\delta\|_{L^2(\Gamma)}+\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Gamma)})\end{aligned}$$

Due to control constraints on the boundary, the term $\|2\bar{u}-\bar{u}_\varepsilon-u_\delta\|_{L^2(\Gamma)}$ is bounded by a constant independent of ε. Using the definition (3.13) of u_δ, we continue with

$$\|\bar{u}_\varepsilon-u_\delta\|_{L^2(\Gamma)}\le\delta_\varepsilon\|\bar{u}_\varepsilon-\hat{u}\|_{L^2(\Gamma)}\le\delta|u_b-u_a||\Gamma|.$$

Thanks to (3.14) and (3.19), we find

$$\delta_\varepsilon\le c\|d[\bar{u}_\varepsilon,(\mathrm{P})]\|_{L^\infty(\Omega')}\le c\left(\xi(\varepsilon)+\frac{\phi(\varepsilon)\sqrt{\xi(\varepsilon)}}{\sqrt{\psi(\varepsilon)}}+\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right).$$

By means of the estimate (3.18) in Theorem 3.10, we arrive at

$$\nu\|\bar{u}-u_\delta\|^2_{L^2(\Gamma)}\le c\left(\xi(\varepsilon)+\frac{\phi(\varepsilon)\sqrt{\xi(\varepsilon)}}{\sqrt{\psi(\varepsilon)}}+\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right).$$

Finally, the continuity of the control-to-state mapping gives the estimate

$$\|\bar{y} - y_\delta\|_{L^2(\Omega)} = \|S\tau^*(\bar{u} - u_\delta)\|_{L^2(\Omega)} \leq c\|\bar{u} - u_\delta\|_{L^2(\Gamma)}.$$

Hence, the assertion is proven. $\square$

This result shows the convergence of the constructed feasible control u_δ to the optimal control $\bar{u}$ of problem (P) as $\varepsilon \to 0$.

Remark 3.14 We mention that the consideration of the state constraints in an inner subdomain $\Omega' \subset\subset \Omega$ was not necessary for constructing the feasible control u_δ and deriving the final error estimate (3.18). If one uses the regularization concept given by the problems (P_1^ε), one can also set $\Omega' = \Omega$, see [46]. However, in the other concept (P_2^ε) will benefit from the fact that Ω' is an inner subdomain of Ω. This will be discussed in detail in the next chapter.

3.3 Numerical example

In this section we want to illustrate the results of this chapter by considering a numerical example. The optimal control problem (P_1^ε) can be solved by several numerical methods, for instance by inner point methods (see e.g., [56], [72]) or active set strategies. For our purposes, we want to apply the primal-dual active set strategy (PDAS), see e.g. [10], [37], [47]. We note that a detailed explanation concerning the numerical implementation of the PDAS-method will be done in Chapter 6.2. Furthermore, we recall that one can attack the original problem also by direct discretization, see [29] and [52], where the authors considered state constrained elliptic control problems with distributed control. But, a naive implementation of this method in the case of boundary control problems is difficult: assume that the state constraints are active in an inner subdomain of Ω'. That means, the possible number of active state constraints might be of order $\mathcal{O}(N^2)$ after discretization. However, there are only $\mathcal{O}(N)$ control variables available to satisfy the constraints. In addition, we note that for the case of pure state constraints a convergence theory for the primal-dual active set strategy is not established.

We consider the following optimal control problem with pure state constraints

$$\left.\begin{array}{rl} \min \quad J(y,u) := & \dfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \\ -\Delta y + y = f & \quad \text{in } \Omega \\ \partial_n y = u & \quad \text{on } \Gamma \\ u_a \leq u(x) \leq u_b & \quad \text{a.e. on } \Gamma \\ y(x) \geq y_c(x) & \quad \text{a.e. in } \Omega', \end{array}\right\} \qquad \text{(PT)}$$

where we do not know the exact solution. Let $\Omega = (0,1)^2$ be the unit square and let $\Omega' = (0.25, 0.75)^2$ be an inner square of Ω. The functions f, $y_d \in L^2(\Omega)$ are chosen by

$$\begin{aligned} f(x_1, x_2) &= \sin(\pi x_1)\sin(\pi x_2) \\ y_d(x_1, x_2) &= 10(\sin(\pi x_1) + x_2). \end{aligned}$$

The constraints are defined as follows:

$$u_a = 0.1, \quad u_b = 1.9, \quad \text{and} \quad y_c(x) \equiv 4 \quad \text{a.e. in } \Omega'.$$

Furthermore, the Tikhonov parameter is chosen by $\nu = 1$. We note that the additional source term $f \in L^2(\Omega)$ in the state equation of (PT) does not influence the analysis of Chapter 3. With the help of a simple transformation, the problem (PT) can be rewritten to a problem of type (P).

As described in Chapter 1.3, the original problem is replaced by a family of regularized problems, where the Lavrentiev regularization is used by the introduction of a virtual control. Moreover, additional control constraints to the new control are considered.

$$\left.\begin{array}{c} \min \quad J(y,u,v) := \dfrac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u\|^2_{L^2(\Gamma)} + \dfrac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} \\ -\Delta y + y = \phi(\varepsilon)v + f \qquad \text{in } \Omega \\ \partial_n y = u \qquad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. on } \Gamma \\ y(x) \ge y_c(x) - \xi(\varepsilon)v(x) \quad \text{a.e. in } \Omega' \\ 0 \le v(x) \le v_b \quad \text{a.e. in } \Omega. \end{array}\right\} \qquad (\mathrm{PT}_1^\varepsilon)$$

In the following, the upper constraint concerning the virtual control is given by $v_b = 1000$. The regularized problems $(\mathrm{PT}_1^\varepsilon)$ were solved numerically by the primal-dual active set strategy. The method was implemented using Matlab. Using a regular and uniform triangulation of the domain Ω, all functions were discretized by piecewise linear finite element functions. The optimal solution of problem (PT)

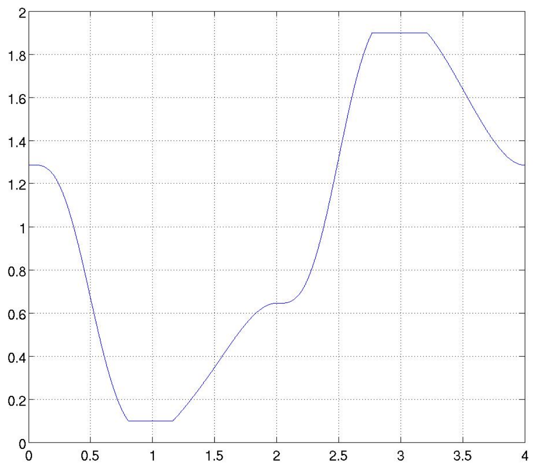

Figure 3.1: Control $\bar{u}$

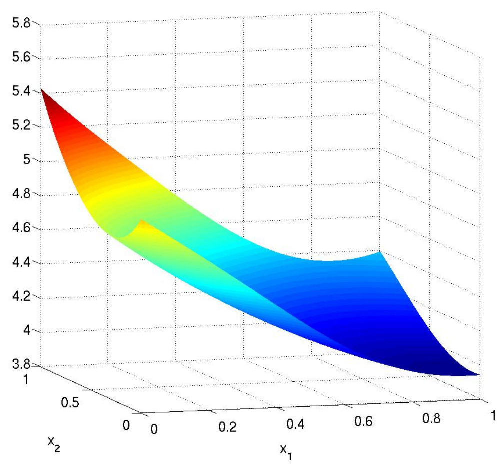

Figure 3.2: State $\bar{y}$

is unknown. Thus, the numerical solution of $(\mathrm{PT}_1^\varepsilon)$ for $\psi(\varepsilon) \equiv 1$, $\phi(\varepsilon) = \varepsilon^2$ and $\xi(\varepsilon) = \varepsilon^2$ with $\varepsilon = 1e-8$ on a very fine mesh with mesh size $h = 0.00125$ is used as a reference solution. For convenience, this solution is denoted by $(\bar{u}, \bar{y})$. The solution is shown in the Figures 3.1 and 3.2, where the control is presented on the boundary in counterclockwise direction. In all further computations, the numerical solutions of problem $(\mathrm{PT}_1^\varepsilon)$ are denoted by $(\cdot)_\varepsilon$. The Figures 3.3-3.6 show the numerical solutions u_ε on the whole boundary, the state y_ε, the virtual control v_ε and the adjoint state p_ε for the choice $\psi(\varepsilon) \equiv 1$, $\phi(\varepsilon) = \varepsilon$ and $\xi(\varepsilon) = \varepsilon^2$ of parameter functions and a rather moderate regularization parameter $\varepsilon = 0.05$. Particularly

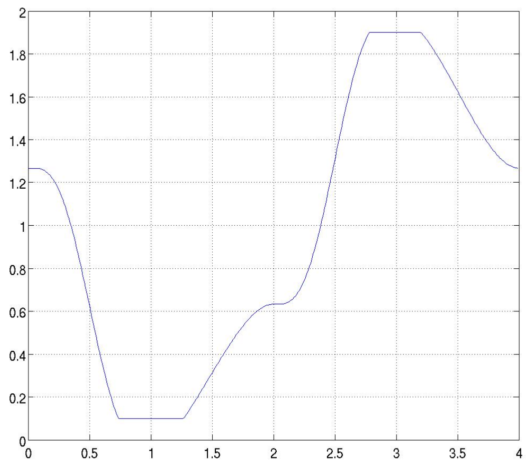

Figure 3.3: Control $\bar{u}_\varepsilon$

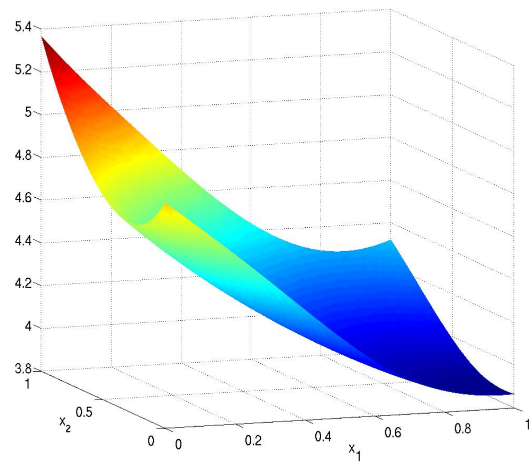

Figure 3.4: State $\bar{y}_\varepsilon$

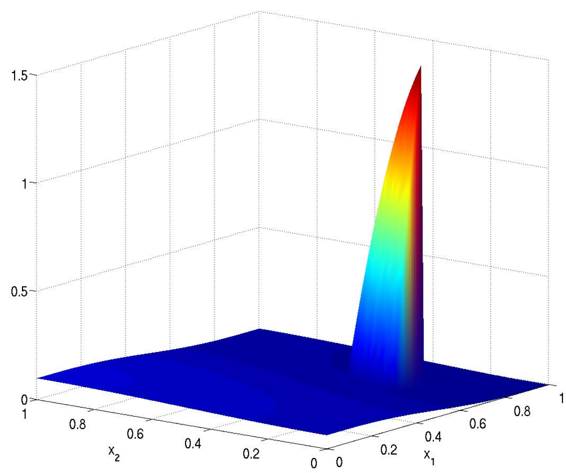

Figure 3.5: Virtual control $\bar{v}_\varepsilon$

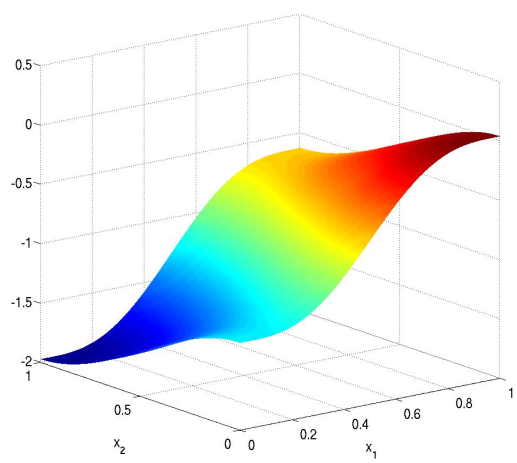

Figure 3.6: Adjoint state $\bar{p}_\varepsilon$

the shape of the virtual control in Figure 3.5 shows that the mixed control-state constraints are active in one corner of the inner subdomain $\Omega' = (0.25, 0.75)^2$.

Forthcoming, we investigate the behaviour of the error between the regularized solutions and the computed reference solution as ε becomes small. We consider different settings for the parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$. First, we illustrate the dependence of the error on the parameter function $\xi(\varepsilon)$. We set

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \varepsilon^{3/2}, \quad \xi(\varepsilon) = \varepsilon^{1/2}, \varepsilon^{3/4}, \varepsilon. \tag{3.22}$$

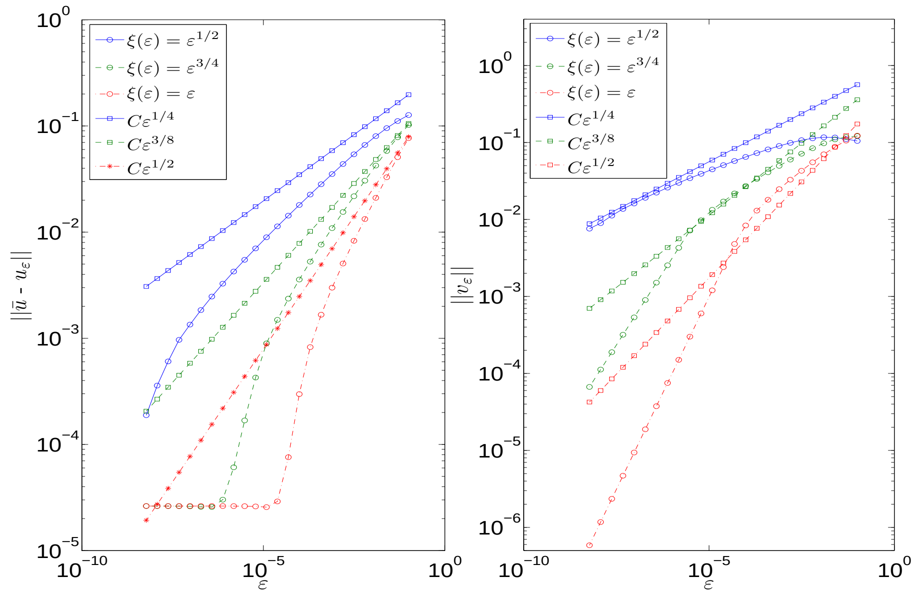

Figure 3.7: Error behaviour for different $\xi(\varepsilon)$

All further calculations are done for a meshsize $h = 0.0025$. The behaviour of the error for this choice is shown in Figure 3.7, where the left illustrates the error $\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$ and the right the $L^2(\Omega)$-norm of the virtual control v_ε. The curves illustrate the validity of the error estimates derived in Corollary 3.9 and Theorem 3.10. Furthermore, the descent rates of the errors are increasing if the exponent in the choice of $\xi(\varepsilon)$ is increasing. Since the numerical solutions are computed on a coarser mesh than the reference solution, a dominating influence of the discretization error is visible for smaller ε, particularly in the error of the control for the choices $\xi(\varepsilon) = \varepsilon^{3/4}$ and $\xi(\varepsilon) = \varepsilon$. However, we obtain better convergence rates than the theoretical ones displayed by the circled lines. For the setting $\psi(\varepsilon) \equiv 1$, $\phi(\varepsilon) = \varepsilon^{3/2}$ and $\xi(\varepsilon) = \varepsilon^{3/4}$ in (3.22) we determined an experimental order of convergence with respect to ε. This value is defined as follows: for a positive error functional $E(\varepsilon)$ with $\varepsilon > 0$ and two parameters $\varepsilon_1 \neq \varepsilon_2$ we set

$$r_E := \frac{\ln E(\varepsilon_1) - \ln E(\varepsilon_2)}{\ln \varepsilon_1 - \ln \varepsilon_2}. \tag{3.23}$$

Furthermore, we introduce the error functionals

$$E_u(\varepsilon) = \|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}, \qquad E_v(\varepsilon) = \|v_\varepsilon\|_{L^2(\Omega)}.$$

Table 3.1 shows the values of the regularization errors according to the control and the values of the $L^2(\Omega)-$norm of the virtual control. Moreover, the experimental order of convergence with respect to ε is presented. According to Corollary 3.9 and Theorem 3.10, we expect a convergence rate of $\mathcal{O}(\varepsilon^{3/8})$. As already visible in Figure 3.7, the experimental order is growing for $\varepsilon \to 0$ and it is better than the theoretical one. It seems that the parameter function $\phi(\varepsilon)$ becomes dominating as $\varepsilon \to 0$.

ε	$\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$	r_{E_u}	$\|v_\varepsilon\|_{L^2(\Omega)}$	r_{E_v}
$1.25e-2$	$4.2338e-2$	$-$	$9.7602e-2$	$-$
$6.25e-3$	$3.0491e-2$	0.47	$8.4309e-2$	0.21
$3.125e-3$	$2.179e-2$	0.48	$7.1471e-2$	0.24
$1.5625e-3$	$1.547e-2$	0.49	$5.9869e-2$	0.26
$7.8125e-4$	$1.0915e-2$	0.50	$4.9743e-2$	0.27
$3.9063e-4$	$7.6299e-3$	0.52	$4.1035e-2$	0.28
$1.9531e-4$	$5.2701e-3$	0.53	$3.3611e-2$	0.29
$9.7656e-5$	$3.5753e-3$	0.56	$2.7315e-2$	0.30
$4.8828e-5$	$2.3638e-3$	0.60	$2.1777e-2$	0.32
$2.4414e-5$	$1.4929e-3$	0.66	$1.7095e-2$	0.35
$1.2207e-5$	$8.9584e-4$	0.74	$1.3338e-2$	0.36

Table 3.1: Experimental convergence rates for $\psi(\varepsilon) \equiv 1$, $\phi(\varepsilon) = \varepsilon^{3/2}$, $\xi(\varepsilon) = \varepsilon^{3/4}$

Finally, we observe the dependence on $\phi(\varepsilon)$ by the following attitude

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \varepsilon^{1/4}, \varepsilon^{1/2}, \varepsilon, \quad \xi(\varepsilon) = \varepsilon^{3/2}. \tag{3.24}$$

The results are shown in Figure 3.8 and the different curves illustrate the error estimates of Corollary 3.9 and Theorem 3.10. As in the previous numerical test, the behaviour of the regularization error with respect to the boundary control is better than expected. However, for the first two choices in (3.24) the L^2-norm of v_ε decreases as predicted in Corollary 3.9. Notice that for the last choice the order of convergence is determined by the parameter function $\xi(\varepsilon)$, see (3.17). Although the rate is based on $\xi(\varepsilon)$, the influence of the parameter function $\phi(\varepsilon)$ seems to dominate the error behaviour for smaller regularization parameters ε. Table 3.2 shows the experimental order of convergence for $\psi(\varepsilon) \equiv 1$, $\phi(\varepsilon) = \varepsilon^{1/2}$ and $\xi(\varepsilon) = \varepsilon^{3/2}$. For both of the errors we expect a convergence rate $\mathcal{O}(\varepsilon^{1/2})$, see Corollary 3.9 and Theorem 3.10. Particularly, for the regularization error associated with the boundary control the experimental rate is better.

ε	$\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$	r_{E_u}	$\|v_\varepsilon\|_{L^2(\Omega)}$	r_{E_v}
$1e-1$	$8.7138e-2$	$-$	$3.6096e-1$	$-$
$5e-2$	$4.3488e-2$	1.00	$2.5992e-1$	0.47
$2.5e-2$	$2.1385e-2$	1.02	$1.8481e-1$	0.49
$1.25e-2$	$1.0383e-2$	1.04	$1.3066e-1$	0.50
$6.25e-3$	$4.9211e-3$	1.08	$9.2046e-2$	0.51
$3.125e-3$	$2.2055e-3$	1.16	$6.4657e-2$	0.51
$1.5625e-3$	$9.4131e-4$	1.23	$4.5341e-2$	0.51
$7.8125e-4$	$4.2694e-4$	1.14	$3.1848e-2$	0.50
$3.9063e-4$	$2.0834e-4$	1.03	$2.2482e-2$	0.50
$1.9531e-4$	$1.0501e-4$	0.98	$1.5891e-2$	0.50

Table 3.2: Experimental convergence rates for $\psi(\varepsilon) \equiv 1$, $\phi(\varepsilon) = \varepsilon^{1/2}$, $\xi(\varepsilon) = \varepsilon^{3/2}$

Let us summarize the numerical test. We observed the convergence of the optimal

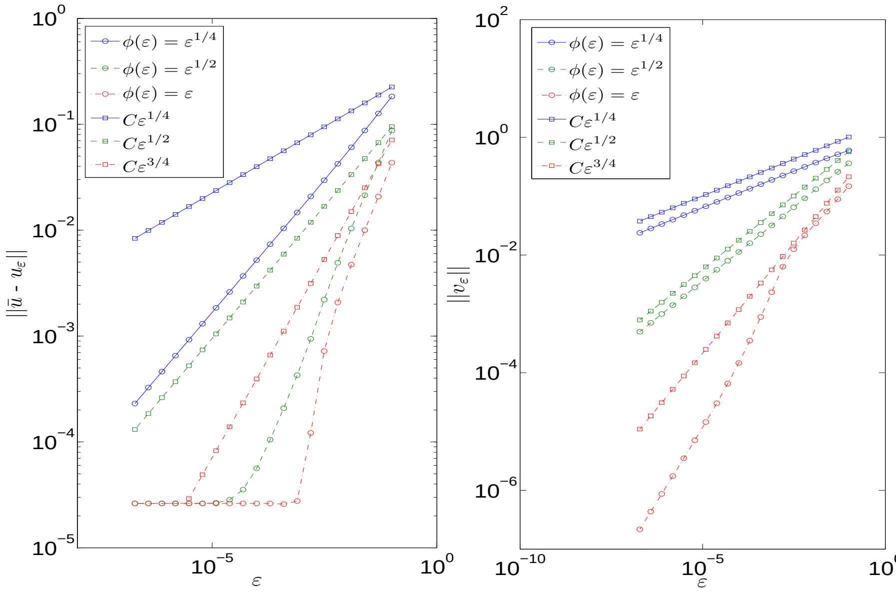

Figure 3.8: Error behaviour for different $\phi(\varepsilon)$

control of problem ($\mathrm{PT}_1^\varepsilon$) for different choices of parameter functions. Moreover, the behaviour of the virtual control was considered and the tests confirmed the estimates of Corollary 3.9. In all numerical tests the approximated convergence rates associated with the regularization error were better than the theory predicted. We note, that the presented error estimates are worst case scenarios. Moreover, in our numerical test the regularization error is only one error. Of course, also a discretization error occurs.

Chapter 4

The virtual control approach without control constraints

In this chapter we will analyze the second family of regularized problems (P_2^ε):

$$\left.\begin{aligned} \min \quad J_\varepsilon(y_\varepsilon,u_\varepsilon,v_\varepsilon) := \frac{1}{2}\|y_\varepsilon - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_\varepsilon\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)} \\ -\Delta y_\varepsilon + y_\varepsilon = \phi(\varepsilon)v_\varepsilon \qquad \text{in } \Omega \\ \partial_n y_\varepsilon = u_\varepsilon \qquad \text{on } \Gamma \\ u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. on } \Gamma \\ y_\varepsilon(x) \ge y_c(x) - \xi(\varepsilon)v_\varepsilon(x) \quad \text{a.e. in } \Omega' \end{aligned}\right\} \qquad (\mathrm{P}_2^\varepsilon)$$

The difference to the previous considered regularized problems (P_1^ε) is the absence of constraints to the virtual control v_ε. This fact ensures the uniqueness of the dual variables. Hence, efficient numerical methods are applicable for solving the regularized problem. On the other hand, the convergence analysis becomes more complicated, since the L^∞-bound of v_ε given by the constraints in (P_1^ε), was essential for the construction of feasible controls. This fact was particularly used in the proof of Lemma 3.6, where the maximal violation of the regularized control with respect to the pure state constraints of problem (P) was estimated. In this chapter, we prove the Lipschitz continuity of the respective violation function, where we benefit from the consideration of the state constraints in the interior of Ω. This allows us to avoid the L^∞-estimate of the virtual control.

4.1 Analysis of the regularized problem (P_2^ε)

First, we state also the existence and uniqueness of solutions for problem (P_2^ε). Similarly to problem (P_1^ε), the classical formulation of the corresponding state equation

$$\begin{aligned} -\Delta y_\varepsilon + y_\varepsilon &= \phi(\varepsilon)v_\varepsilon \quad \text{in } \Omega \\ \partial_n y_\varepsilon &= u_\varepsilon \qquad \text{on } \Gamma \end{aligned} \tag{4.1}$$

is replaced by a control-to-state mapping. Since the state equations are identical in both of the problems (P_1^ε) and (P_2^ε), respectively, we use again the mapping

$$y_\varepsilon = S(\tau^* u_\varepsilon + \phi(\varepsilon)E_H^* v_\varepsilon), \tag{4.2}$$

which was defined in (3.3). Due to the absence of control constraints to the virtual control, the admissible set for (P_2^ε) is now defined by

$$\begin{aligned} V_{ad}^{\varepsilon,2} = &\{(u,v) \in L^2(\Gamma) \times L^2(\Omega) |\, u_a \le u \le u_b \text{ a.e. on } \Gamma; \\ &S(\tau^* u + \phi(\varepsilon) E_H^* v) \ge y_c - \xi(\varepsilon) v \text{ a.e. in } \Omega'\}. \end{aligned} \tag{4.3}$$

If Assumption 2.4 is satisfied, the admissible set is nonempty. One can easily see, that $(\hat{u}, 0)$ is an element of $V_{ad}^{\varepsilon,2}$. Moreover, $V_{ad}^{\varepsilon,2}$ is convex and closed. The reduced form of (P_2^ε) is given by:

$$\begin{aligned} \min_{(u_\varepsilon,v_\varepsilon)\in V_{ad}^{\varepsilon,2}} f_\varepsilon(u_\varepsilon, v_\varepsilon) := \frac{1}{2}\|S(\tau^* u_\varepsilon + \phi(\varepsilon) E_H^* v_\varepsilon) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_\varepsilon\|^2_{L^2(\Gamma)} \\ + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.4}$$

Theorem 4.1 *Suppose that Assumption 2.4 is fulfilled. Then the optimization problem* (4.4) *admits a unique optimal solution* $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \in V_{ad}^{\varepsilon,2}$.

The proof is standard and quite similar to the proof of Theorem 2.5. Since the admissible set is nonempty and the objective functional is strictly convex and radially unbounded, the existence and uniqueness of a solution for (4.4) is obtained. Thus, the regularized optimal control problem (P_2^ε) admits a unique solution $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$, where $\bar{y}_\varepsilon = S(\tau^*\bar{u}_\varepsilon + \phi(\varepsilon) E_H^* \bar{v}_\varepsilon)$ is the unique optimal state associated with the controls $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$.

Except the admissible set, the necessary and sufficient optimality condition for problem (P_2^ε) is the same as for problem (P_1^ε), given by (3.6). For the sake of completeness, we formulate the optimality condition in the following lemma.

Lemma 4.2 *Let* $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ *be the optimal solution of problem* (P_2^ε). *The necessary and sufficient optimality condition is given by*

$$(\tau\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\phi(\varepsilon) E_H \bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon, v - \bar{v}_\varepsilon)_{L^2(\Omega)} \ge 0 \quad \forall (u,v) \in V_{ad}^{\varepsilon,2}, \tag{4.5}$$

where $\bar{p}_\varepsilon = S^*(\bar{y}_\varepsilon - y_d)$ *denotes the adjoint state.*

Again, the adjoint state $\bar{p}_\varepsilon$ is the weak solution of (2.15) with respect to the right hand side $\bar{y}_\varepsilon - y_d$, see Lemma 2.8 and Definition 2.9.

Unlike the first regularized problem (P_1^ε), we discuss the optimality conditions also for the classical approach with a Lagrange multiplier associated with the mixed control-state-constraints. We derive several boundedness results, and we deduce higher regularity of the optimal regularized control $\bar{u}_\varepsilon$, similarly to the original problem (P) in Section 2.4. Further on, only the control constraints are handled by an admissible set and we define

$$U_{ad}^L := \{u \in L^2(\Gamma) : \quad u_a \le u \le u_b \text{ a.e. on } \Gamma\}. \tag{4.6}$$

In the case of pointwise control-state-constraints the existence of regular Lagrange multipliers is proven, see e.g. [68] or [74]. Introducing a regular Lagrange multiplier associated with the mixed control-state constraints in problem (P_2^ε), the optimality system is formulated in the following theorem.

Theorem 4.3 *Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be the optimal solution of problem* $(\mathrm{P}_2^\varepsilon)$. *Then, a regular multiplier $\mu_\varepsilon \in L^2(\Omega')$ and an adjoint state $p_\varepsilon \in H^1(\Omega)$ exist such that the following optimality system is satisfied*

$$\begin{aligned} -\Delta \bar{y}_\varepsilon + \bar{y}_\varepsilon &= \phi(\varepsilon)\bar{v}_\varepsilon & \qquad -\Delta p_\varepsilon + p_\varepsilon &= \bar{y}_\varepsilon - y_d - \mu_\varepsilon \\ \partial_n \bar{y}_\varepsilon &= \bar{u}_\varepsilon & \partial_n p &= 0 \end{aligned} \tag{4.7}$$

$$(\tau p_\varepsilon + \nu \bar{u}_\varepsilon \, , \, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{ad}^L \tag{4.8}$$

$$\phi(\varepsilon) p_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\mu_\varepsilon = 0 \quad a.e. \ in \ \Omega \tag{4.9}$$

$$(\mu_\varepsilon \, , \, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega')} = 0, \quad \mu_\varepsilon \geq 0, \quad \bar{y}_\varepsilon \geq y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad a.e. \ in \ \Omega'. \tag{4.10}$$

Notice that the Lagrange multiplier is extended by zero outside of Ω'. We mention, that the existence of a regular Lagrange multiplier can be proven in a direct and rather trivial way. Transforming the problem $(\mathrm{P}_2^\varepsilon)$ in an equivalent and completely control constrained problem, one can use standard techniques to show the existence of a regular Lagrange multiplier. For a more detailed elaboration of the proof, we refer to [74, Ch. 4.1].

With the help of the optimality system (4.7)-(4.10), we improve the regularity of the optimal solution similarly to the original problem (P), see Section 2.4. Again, the variational inequality (4.8) is replaced by the equivalent pointwise projection

$$\bar{u}_\varepsilon = P\left\{-\frac{\tau p_\varepsilon}{\nu}\right\}$$

on the admissible set U_{ad}^L. Considering now the adjoint equation in (4.7) more precisely, we find $p_\varepsilon \in H^2(\Omega)$ for every $\varepsilon > 0$, see the classical result of Grisvard in Theorem 2.18. Due to the Theorem 1.7, the trace of the adjoint state belongs to $H^1(\Gamma)$. By means of Lemma 2.21, the optimal regularized control is an element of $H^1(\Gamma)$ and the estimate

$$\|\bar{u}_\varepsilon\|_{H^1(\Gamma)} \leq C_1 \|p_\varepsilon\|_{H^1(\Gamma)} + C_2$$

is valid for some positive constants C_1 and C_2. Thanks to the Trace theorem 1.7 and Theorem 2.18, we end up with the estimate:

$$\|\bar{u}_\varepsilon\|_{H^1(\Gamma)} \leq C_1 \|p_\varepsilon\|_{H^2(\Omega)} + C_2 \leq C_1(\|\bar{y}_\varepsilon\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} + \|\mu_\varepsilon\|_{L^2(\Omega)}) + C_2.$$

Unfortunately, this estimate is useless for passing to the limit, i.e. $\varepsilon \to 0$, since we can not expect an upper bound for the Lagrange multiplier μ_ε in $L^2(\Omega)$ that is independent of ε. Thus, we adapt the strategy of Section 2.4, and we use the localization of the regular multiplier in the inner subdomain Ω' of Ω. We consider the multiplier in a weaker norm, where a uniform bound independent of ε is available. The next lemma shows, that the multiplier μ_ε is uniformly bounded in $L^1(\Omega')$ for every $\varepsilon > 0$. The proof follows a strategy similarly to [52, Lemma 2.2].

Lemma 4.4 *Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be the optimal solution of problem* (P_2^ε)*. Furthermore, let p_ε be the adjoint state and μ_ε the Lagrange multiplier, such that the optimality system* (4.7)-(4.10) *is fulfilled. Then, the Lagrange multiplier μ_ε is uniformly bounded in $L^1(\Omega')$, i.e.*

$$\|\mu_\varepsilon\|_{L^1(\Omega')} \leq C \tag{4.11}$$

with a positive constant C independent of the regularization parameter ε.

Proof: First, we rewrite the equation (4.9) to a variational form

$$(\phi(\varepsilon)E_H p_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\mu_\varepsilon\,,\, v - \bar{v}_\varepsilon)_{L^2(\Omega)} = 0, \quad \forall v \in L^2(\Omega).$$

Adding the previous variational equation and (4.8) and using the representation of the adjoint state p_ε by the adjoint of the solution operator $S : H^1(\Omega)^* \to L^2(\Omega)$, we deduce

$$\begin{aligned}(\phi(\varepsilon)E_H S^*(\bar{y}_\varepsilon - y_d - \mu_\varepsilon) + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\mu_\varepsilon\,,\, v - \bar{v}_\varepsilon)_{L^2(\Omega)} +& \\ (\tau S^*(\bar{y}_\varepsilon - y_d - \mu_\varepsilon) + \nu\bar{u}_\varepsilon\,,\, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \geq 0 \qquad& \forall (u,v) \in U_{ad}^L \times L^2(\Omega).\end{aligned}$$

Sorting all terms where the multiplier arises and applying the adjoint operators, we arrive at

$$\begin{aligned}(\mu_\varepsilon\,,\, \xi(\varepsilon)(v - \bar{v}_\varepsilon) + SE_H^*\phi(\varepsilon)(v - \bar{v}_\varepsilon) + S\tau^*(u - \bar{u}_\varepsilon))_{L^2(\Omega)}& \\ \leq (\psi(\varepsilon)\bar{v}_\varepsilon + \phi(\varepsilon)E_H S^*(\bar{y}_\varepsilon - y_d)\,,\, v - \bar{v}_\varepsilon)_{L^2(\Omega)}& \\ + (\nu\bar{u}_\varepsilon + \tau S^*(\bar{y}_\varepsilon - y_d)\,,\, u - \bar{u}_\varepsilon)_{L^2(\Gamma)},&\end{aligned} \tag{4.12}$$

for all $(u,v) \in U_{ad}^L \times L^2(\Omega)$. Now, we choose the special test function $(\hat{u}, 0) \in U_{ad}^L \times L^2(\Omega)$, where $\hat{u}$ is the inner point with respect to the pure state constraints defined in Assumption 2.4. Using the control-to-state mapping (4.2), we find for the left side of the previous inequality

$$\begin{aligned}(\mu_\varepsilon\,,\, \xi(\varepsilon)(-\bar{v}_\varepsilon) + SE_H^*\phi(\varepsilon)(-\bar{v}_\varepsilon) + S\tau^*(\hat{u} - \bar{u}_\varepsilon))_{L^2(\Omega)}& \\ = (\mu_\varepsilon\,,\, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega)} + (\mu_\varepsilon\,,\, \hat{y} - y_c)_{L^2(\Omega)}& \\ = (\mu_\varepsilon\,,\, \hat{y} - y_c)_{L^2(\Omega)},&\end{aligned} \tag{4.13}$$

since the first term in the second line vanishes by (4.10). With the help of Assumption 2.4 and the positivity of the Lagrange multiplier, we derive the estimate

$$\gamma\|\mu_\varepsilon\|_{L^1(\Omega')} = \int\limits_{\Omega'} \gamma\mu_\varepsilon dx \leq (\mu_\varepsilon\,,\, \hat{y} - y_c)_{L^2(\Omega)}. \tag{4.14}$$

We note that the multiplier is zero in $\Omega \setminus \Omega'$. Summarizing (4.12) for $(\hat{u}, 0) \in U_{ad}^L \times L^2(\Omega)$, (4.13) and (4.14), we conclude

$$\begin{aligned}\gamma\|\mu_\varepsilon\|_{L^1(\Omega')} \leq (\mu_\varepsilon\,,\, \hat{y} - y_c)_{L^2(\Omega)} \leq (\psi(\varepsilon)\bar{v}_\varepsilon + \phi(\varepsilon)E_H S^*(\bar{y}_\varepsilon - y_d)\,,\, -\bar{v}_\varepsilon)_{L^2(\Omega)}& \\ + (\nu\bar{u}_\varepsilon + \tau S^*(\bar{y}_\varepsilon - y_d)\,,\, \hat{u} - \bar{u}_\varepsilon)_{L^2(\Gamma)}&\end{aligned}$$

We proceed with several estimations of the right side of the previous inequality such that

$$\begin{aligned}
\gamma\|\mu_\varepsilon\|_{L^1(\Omega')} \le & -\psi(\varepsilon)\|\bar v_\varepsilon\|^2_{L^2(\Omega)} + (\bar y_\varepsilon - y_d\,,\, -SE^*_H\phi(\varepsilon)\bar v_\varepsilon)_{L^2(\Omega)} \\
& + (\bar y_\varepsilon - y_d\,,\, S\tau^*(\hat u - \bar u_\varepsilon))_{L^2(\Omega)} + \nu(\bar u_\varepsilon\,,\, \hat u - \bar u_\varepsilon)_{L^2(\Gamma)} \\
\le & (\bar y_\varepsilon - y_d\,,\, \hat y - \bar y_\varepsilon)_{L^2(\Omega)} + \nu(\bar u_\varepsilon\,,\, \hat u)_{L^2(\Gamma)} - \nu\|\bar u_\varepsilon\|^2_{L^2(\Gamma)} \\
\le & (\bar y_\varepsilon - y_d\,,\, \hat y)_{L^2(\Omega)} + (y_d\,,\, \bar y_\varepsilon)_{L^2(\Omega)} + \nu(\bar u_\varepsilon\,,\, \hat u)_{L^2(\Gamma)} \\
\le & \|\bar y_\varepsilon - y_d\|_{L^2(\Omega)}\|\hat y\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\|\bar y_\varepsilon\|_{L^2(\Omega)} \\
& + \nu\|\bar u_\varepsilon\|_{L^2(\Gamma)}\|\hat u\|_{L^2(\Gamma)},
\end{aligned}$$

where we again used (4.2). The optimality of $(\bar u_\varepsilon, \bar y_\varepsilon)$ yields uniform boundedness with respect to ε of the remaining terms in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. Since $(\bar y_\varepsilon,\, \bar u_\varepsilon, \bar v_\varepsilon)$ is the optimal solution of problem $(\mathrm{P}_2^\varepsilon)$, we have $J_\varepsilon(\bar y_\varepsilon, \bar u_\varepsilon, \bar v_\varepsilon) < \infty$. Moreover, we find for the term $\|\bar y_\varepsilon - y_d\|_{L^2(\Omega)}$

$$\begin{aligned}
\frac{1}{2}\|\bar y_\varepsilon - y_d\|^2_{L^2(\Omega)} \le J_\varepsilon(\bar y_\varepsilon, \bar u_\varepsilon, \bar v_\varepsilon) \le J_\varepsilon(\hat y, \hat u, 0) = J(\hat y, \hat u) \\
\|\bar y_\varepsilon - y_d\|_{L^2(\Omega)} \le \sqrt{2J(\hat y, \hat u)},
\end{aligned}$$

where $(\hat y,\, \hat u)$ is the inner point defined in Assumption 2.4. The other terms can be estimated analogously. This completes the proof. $\square$

Corollary 4.5 *Let $(\bar y_\varepsilon, \bar u_\varepsilon, \bar v_\varepsilon)$ satisfy the optimality system (4.7)-(4.10) with the associated adjoint state p_ε and the Lagrange multiplier μ_ε. Then, there exist a constant $C > 0$, independent of ε, such that*

$$\|p_\varepsilon\|_{H^1(\Gamma)} \le C. \tag{4.15}$$

is satisfied.

Proof: The arguments are similar as in Corollary 2.20. The standard result of Grisvard, see Theorem 2.18, and the Trace theorem 1.7 provides the estimate for the adjoint state with respect to the regular part by $\bar y_\varepsilon$ and y_d. The boundedness of the part of the adjoint state associated with the Lagrange multiplier follows from Lemma 2.19 and Lemma 4.4. $\square$

Let us come back to the optimal control $\bar u_\varepsilon = P\{-\tau p_\varepsilon/\nu\}$, where P is again the pointwise projection on the admissible set U^L_{ad}. Due to Lemma 2.21 and the previous corollary, there exist a constant C that is independent of ε such that

$$\|\bar u_\varepsilon\|_{H^1(\Gamma)} \le C. \tag{4.16}$$

is valid. The uniform boundedness of the optimal control $\bar u_\varepsilon$ in $H^1(\Gamma)$ with respect to ε is essential in the next chapter, where approximation error estimates, caused by a finite element discretization of the boundary control, will be examined.

4.2 Regularization error estimate

In this section we derive a regularization error estimate between the original solution of problem (P) and the regularized one of problem $(\mathrm{P}_2^\varepsilon)$. Similarly to Section 3.2, the basis for the estimate are the variational inequalities in the optimality conditions of both problems and suitable feasible controls. For arbitrary feasible controls, one deduces the same basis error estimate like in Lemma 3.4. For the sake of completeness, the result is stated in the next lemma.

Lemma 4.6 *For all $u_\delta \in U_{ad}$ and $(u_\sigma, 0) \in V_{ad}^{\varepsilon,2}$ there holds*

$$\begin{aligned}\nu\|\bar u - \bar u_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar y - \bar y_\varepsilon\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar v_\varepsilon\|^2_{L^2(\Omega)} \le \\ (\tau\bar p + \nu\bar u\,,\, u_\delta - \bar u_\varepsilon)_{L^2(\Gamma)} + (\tau\bar p_\varepsilon + \nu\bar u_\varepsilon\,,\, u_\sigma - \bar u)_{L^2(\Gamma)} \\ + c\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\end{aligned} \tag{4.17}$$

for a certain constant $c > 0$ independent of ε.

The proof is done along the lines of Lemma 3.4. Again, the main purpose is the determination of suitable feasible controls for the problems that are predicated on the optimal solution of the particular other problem.

4.2.1 Auxiliary results and feasibility

Now, let us discuss the construction of feasible solutions. Obviously, the control $(\bar u, 0) \in L^2(\Gamma) \times L^2(\Omega)$ is an element of the admissible set $V_{ad}^{\varepsilon,2}$ of problem (P_2^ε) for every $\varepsilon > 0$. One can easily adapt the proof of Lemma 3.5. The construction of a feasible control for the original problem (P) is again based on the investigation of the violation of the control $\bar u_\varepsilon$ with respect to the pure state constraints. We recall the respective violation function:

$$d[\bar u_\varepsilon, (P)] := (y_c - S\tau^*\bar u_\varepsilon)_+ = \max\{0, y_c - S\tau^*\bar u_\varepsilon\}. \tag{4.18}$$

First, we state an auxiliary result, which is important for the estimation of the maximal violation of $\bar u_\varepsilon$ w.r.t problem (P).

Lemma 4.7 *Let $E \subset \mathbb{R}^d$ be a bounded domain satisfying the inner cone condition. Moreover, let the function f be in $C^{0,\alpha}(\bar E)$ for some $0 < \alpha \le 1$ with $\|f\|_{C^{0,\alpha}(\bar E)} \le \sigma$. Then, there exists a constant $c > 0$ such that the estimate*

$$\|f\|_{L^\infty(E)} \le c\sigma^{\frac{d}{2\alpha+d}}\|f\|^{\frac{2\alpha}{2\alpha+d}}_{L^2(E)} \tag{4.19}$$

is satisfied.

Proof: Let $\bar x \in \bar E$ be the point, where we obtain

$$M := |f(\bar x)| = \max_{x\in\bar E}\{|f(x)|\}\,.$$

Moreover, let $\mathcal{U}_\delta(\bar x)$ be a ball with center $\bar x$ and sufficiently small radius δ such that

$$|f(x) - f(\bar x)| \le \frac{M}{2}, \qquad \forall x \in \mathcal{U}_\delta(\bar x) \cap \bar E$$

is satisfied. The definition of Hölder-continuous functions and $\|f\|_{C^{0,\alpha}(E)} \leq \sigma$ yields

$$|f(x) - f(\bar{x})| \leq \sigma\delta^\alpha, \qquad \forall x \in \mathcal{U}_\delta(\bar{x}) \cap \bar{E}.$$

By choosing δ as follows

$$\delta := \left(\sigma^{-1}\frac{M}{2}\right)^{1/\alpha},$$

we ensure the validity of the first inequality. The domain E satisfies the inner cone condition, see Definition 1.1. Now, we define $\tilde{\delta} := \min\{r, \delta\}$, where $r > 0$ is the radius obtained from the fixed cone of Definition 1.1. We can guarantee

$$|f(x)| \geq \frac{M}{2}, \qquad \forall x \in \mathcal{U}_{\tilde{\delta}}(\bar{x}) \cap \bar{E}$$

Now, we will estimate the L^2-norm from below.

$$\begin{aligned}
\|f\|^2_{L^2(E)} &= \int\limits_E f^2 dx \geq \int\limits_{\mathcal{U}_{\tilde{\delta}}(\bar{x})\cap E} f^2 dx \geq \left(\frac{M}{2}\right)^2 \int\limits_{\mathcal{U}_{\tilde{\delta}}(\bar{x})\cap E} dx \\
&= c\left(\frac{M}{2}\right)^2 \tilde{\delta}^d = cM^2\left(\min\left\{r, \left(\sigma^{-1}\frac{M}{2}\right)^{1/\alpha}\right\}\right)^d \\
&= cM^2(\sigma^{-1}M)^{d/\alpha}\left(\min\left\{r\left(\frac{\sigma}{M}\right)^{1/\alpha}, 2^{-1/\alpha}\right\}\right)^d
\end{aligned}$$

Due to $\|f\|_{C^{0,\alpha}(\bar{E})} \leq \sigma$ and $M = \max_{x\in\bar{E}}\{|f(x)|\}$, we have $M \leq \sigma$. We proceed with the estimate as follows:

$$\begin{aligned}
\|f\|^2_{L^2(E)} &\geq cM^{\frac{2\alpha+d}{\alpha}}\sigma^{-\frac{d}{\alpha}}\left(\min\left\{r\left(\frac{\sigma}{M}\right)^{1/\alpha}, 2^{-1/\alpha}\right\}\right)^d \\
&\geq cM^{\frac{2\alpha+d}{\alpha}}\sigma^{-\frac{d}{\alpha}}\left(\min\left\{r, 2^{-1/\alpha}\right\}\right)^d \\
&= c\sigma^{-\frac{d}{\alpha}}\|f\|^{\frac{2\alpha+d}{\alpha}}_{L^\infty(E)}
\end{aligned}$$

This estimate implies the assertion. □

In the next lemma, we present an estimate of the maximal violation of $\bar{u}_\varepsilon$ w.r.t. (P).

Lemma 4.8 *Let $\bar{u}_\varepsilon$ be the optimal control of problem* (P_2^ε)*. The maximal violation $\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')}$ of $\bar{u}_\varepsilon$ w.r.t.* (P) *can be estimated by*

$$\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} \leq c(\xi(\varepsilon) + \phi(\varepsilon))^{\frac{2}{2+d}}\|\bar{v}_\varepsilon\|^{\frac{2}{2+d}}_{L^2(\Omega)}, \tag{4.20}$$

where c is a positive constant independent of ε.

Proof: According to Sobolev embeddings and (2.34) in Corollary 2.23, we obtain $S\tau^*\bar{u}_\varepsilon \in C^{0,1}(\bar{\Omega}')$ and

$$\|S\tau^*\bar{u}_\varepsilon\|_{C^{0,1}(\bar{\Omega}')} \leq c\|S\tau^*\bar{u}_\varepsilon\|_{W^{2,\infty}(\Omega')} \leq c\|\bar{u}_\varepsilon\|_{L^2(\Gamma)}.$$

Since $y_c \in C^{0,1}(\bar{\Omega}')$ and the max-function is continuous, the violation function $d[\bar{u}_\varepsilon, (P)]$ belongs also to the space $C^{0,1}(\bar{\Omega}')$. Furthermore, we find an upper bound, independent of ε, by:

$$\begin{aligned}\|d[\bar{u}_\varepsilon, (P)]\|_{C^{0,1}(\bar{\Omega}')} = \|(y_c - S\tau^*\bar{u}_\varepsilon)_+\|_{C^{0,1}(\bar{\Omega}')} &\leq \|y_c\|_{C^{0,1}(\bar{\Omega}')} + \|S\tau^*\bar{u}_\varepsilon\|_{C^{0,1}(\bar{\Omega}')} \\ &\leq \|y_c\|_{C^{0,1}(\bar{\Omega}')} + c\|\bar{u}_\varepsilon\|_{L^2(\Gamma)}.\end{aligned}$$

The optimality of $\bar{u}_\varepsilon$ yields the boundedness of the respective norm independent of ε. By means of Lemma 4.7 with $E = \Omega' \subset \mathbb{R}^d$, $d = 2, 3$, we derive

$$\begin{aligned}\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} &\leq c\|d[\bar{u}_\varepsilon, (P)]\|_{L^2(\Omega')}^{\frac{2}{2+d}} \\ &= c\|(y_c - S\tau^*\bar{u}_\varepsilon)_+\|_{L^2(\Omega')}^{\frac{2}{2+d}} \\ &= c\|(y_c - S(\tau^*\bar{u}_\varepsilon + E_H^*\phi(\varepsilon)\bar{v}_\varepsilon) + SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon)_+\|_{L^2(\Omega')}^{\frac{2}{2+d}} \\ &\leq c\|(y_c - \bar{y}_\varepsilon)_+ + (SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon)_+\|_{L^2(\Omega')}^{\frac{2}{2+d}}\end{aligned}$$

The feasibility of ($\bar{y}_\varepsilon$, $\bar{u}_\varepsilon$, $\bar{v}_\varepsilon$) for (P_2^ε) and the continuity of the solution operator yield

$$\begin{aligned}\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} &\leq c\left(\|(\xi(\varepsilon)\bar{v}_\varepsilon)_+\|_{L^2(\Omega')} + \|SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon\|_{L^2(\Omega')}\right)^{\frac{2}{2+d}} \\ &\leq c(\xi(\varepsilon) + \phi(\varepsilon))^{\frac{2}{2+d}}\|\bar{v}_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{2+d}}.\end{aligned}$$

This completes the proof. □

Next, we carry over the construction of a feasible solution u_δ for problem (P), presented in Lemma 3.7, to the current problem. Thus, we state the following lemma.

Lemma 4.9 *Let the Assumption 2.4 be satisfied. Then for every $\varepsilon > 0$ the control $u_\delta := (1-\delta)\bar{u}_\varepsilon + \delta\hat{u}$ is feasible for (P) for all $\delta \in [\delta_\varepsilon, 1]$, where δ_ε is given by*

$$\delta_\varepsilon := \frac{\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')}}{\|d[\bar{u}_\varepsilon, (P)]\|_{L^\infty(\Omega')} + \gamma}. \tag{4.21}$$

The proof is carried out along the lines of Lemma 3.7.

4.2.2 Error estimates

Now, we will derive the main result of this chapter. Thanks to the feasible controls, constructed in the previous section, we use the basis estimate of Lemma 4.6 for the following result.

Theorem 4.10 *Let ($\bar{y}$, $\bar{u}$) and ($\bar{y}_\varepsilon$, $\bar{u}_\varepsilon$, $\bar{v}_\varepsilon$) be the optimal solutions of* (P) *and* (P_2^ε)*, respectively. Then, there exists a positive constant c, independent of ε, such that*

$$\begin{aligned}\nu\|\bar{u} - \bar{u}_\varepsilon\|_{L^2(\Gamma)}^2 + \|\bar{y} - \bar{y}_\varepsilon\|_{L^2(\Omega)}^2 + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|_{L^2(\Omega)}^2 \leq \\ c\left((\xi(\varepsilon) + \phi(\varepsilon))^{\frac{2}{2+d}}\|\bar{v}_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{2+d}} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right)\end{aligned} \tag{4.22}$$

is valid.

Proof: Although, the proof is similar to the proof of Theorem 3.8, we will sketch at least the main steps. Choosing $u_\sigma := \bar{u}$ and u_δ as in Lemma 4.9, we derive by (4.17):

$$
\begin{aligned}
\nu\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y}-\bar{y}_\varepsilon\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \\
&\le \delta(\bar{p}+\nu\bar{u}\,,\,\hat{u}-\bar{u}_\varepsilon)_{L^2(\Gamma)} + c\frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} \\
&\le c\left(\delta + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right)
\end{aligned}
$$

According to Lemma 4.9, we choose the specific parameter $\delta := \delta_\varepsilon$ that is defined in (4.21). Moreover, we find

$$\delta_\varepsilon \le c\|d[\bar{u}_\varepsilon,(P)]\|_{L^\infty(\Omega')}$$

for all constants $c \ge 1/\gamma$. We proceed with the estimate (4.20) of the maximal violation such that the assertion is proven. □

Again, a $L^2(\Omega)$-estimate of the virtual control is needed for completion of the regularization error estimate. As already mentioned in Section 3.2.2, the $L^2(\Omega)$-estimate (3.16), which one obtains easily by the objective functional of the problem, is again not optimal. We continue with the following assumption on the parameter functions of problem (P_2^ε).

Assumption 4.11 For sufficiently small $\varepsilon > 0$ we assume that

$$\frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} < 1. \tag{4.23}$$

In view of (4.22), the term $\frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}}$ has to become small as $\varepsilon \to 0$ for convergence of the optimal regularized solution to the optimal solution of problem (P) such that the previous assumption is reasonable. Now, the result of Theorem 4.10 provides a better estimate of the virtual control.

Corollary 4.12 *Let the assumptions of Theorem 4.10 and Assumption 4.11 be fulfilled. Then, the estimate*

$$\|\bar{v}_\varepsilon\|_{L^2(\Omega)} \le c\frac{1}{\sqrt{\psi(\varepsilon)}}\left(\frac{\xi(\varepsilon)+\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}}\right)^{\frac{1}{d+1}} \tag{4.24}$$

is satisfied for sufficiently small $\varepsilon > 0$ and some positive constant $c > 0$ independent of ε.

Proof: Considering (4.22), we have the estimate:

$$\frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \le c\left((\xi(\varepsilon)+\phi(\varepsilon))^{\frac{2}{2+d}}\|\bar{v}_\varepsilon\|^{\frac{2}{2+d}}_{L^2(\Omega)} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}\right).$$

Moreover, this estimate implies

$$\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} \le \frac{2c}{\psi(\varepsilon)} \max\left\{ (\xi(\varepsilon)+\phi(\varepsilon))^{\frac{2}{2+d}} \|\bar{v}_\varepsilon\|^{\frac{2}{2+d}}_{L^2(\Omega)}, \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} \right\}.$$

Let us consider the two cases, where the maximum can be attained.
Case 1: First, we assume

$$\max\left\{ (\xi(\varepsilon)+\phi(\varepsilon))^{\frac{2}{2+d}} \|\bar{v}_\varepsilon\|^{\frac{2}{2+d}}_{L^2(\Omega)}, \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} \right\} = \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)}.$$

Consequently, we obtain the following upper bound:

$$\|\bar{v}_\varepsilon\|_{L^2(\Omega)} \le c\frac{\phi(\varepsilon)}{\psi(\varepsilon)}.$$

Case 2: Next, we assume that the maximum is attained by the first term. Hence, we derive the estimate:

$$\begin{aligned}
\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} &\le \frac{c}{\psi(\varepsilon)}(\xi(\varepsilon)+\phi(\varepsilon))^{\frac{2}{2+d}} \|\bar{v}_\varepsilon\|^{\frac{2}{2+d}}_{L^2(\Omega)} \\
\|\bar{v}_\varepsilon\|^{\frac{2(1+d)}{2+d}}_{L^2(\Omega)} &\le \frac{c}{\psi(\varepsilon)}(\xi(\varepsilon)+\phi(\varepsilon))^{\frac{2}{2+d}} \\
\|\bar{v}_\varepsilon\|_{L^2(\Omega)} &\le c\left((\psi(\varepsilon))^{-\frac{2+d}{2}}(\xi(\varepsilon)+\phi(\varepsilon)) \right)^{\frac{1}{1+d}}.
\end{aligned}$$

Summarizing both cases, we deduce the following upper bound

$$\begin{aligned}
\|\bar{v}_\varepsilon\|_{L^2(\Omega)} &\le c\max\left\{ \left((\psi(\varepsilon))^{-\frac{2+d}{2}}(\xi(\varepsilon)+\phi(\varepsilon)) \right)^{\frac{1}{1+d}}, \frac{\phi(\varepsilon)}{\psi(\varepsilon)} \right\} \\
&= \frac{c}{\sqrt{\psi(\varepsilon)}} \max\left\{ \sqrt{\psi(\varepsilon)}\left((\psi(\varepsilon))^{-\frac{2+d}{2}}(\xi(\varepsilon)+\phi(\varepsilon)) \right)^{\frac{1}{1+d}}, \frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right\} \\
&= \frac{c}{\sqrt{\psi(\varepsilon)}} \max\left\{ \sqrt{\psi(\varepsilon)}\left((\psi(\varepsilon))^{-\frac{1+d}{2}}\frac{\xi(\varepsilon)+\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right)^{\frac{1}{1+d}}, \frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right\} \\
&= \frac{c}{\sqrt{\psi(\varepsilon)}} \max\left\{ \left(\frac{\xi(\varepsilon)+\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right)^{\frac{1}{d+1}}, \frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right\}.
\end{aligned}$$

Due to (4.23) of Assumption 4.11, the maximum is attained by the first term inside the max-function, which is the assertion. □

The final regularization error estimate of the optimal solution of problem (P) with respect to the optimal regularized one of problem (P_2^ε) is a direct result of Theorem 4.10 and Corollary 4.12 and is stated in the following theorem.

Theorem 4.13 *Let $(\bar{y}, \bar{u})$ and $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be the optimal solution of* (P) *and* (P_2^ε)*, respectively. Moreover, let Assumption 4.11 be fulfilled. Then, there exists a positive constant c, independent of ε, such that*

$$\nu\|\bar{u}-\bar{u}_\varepsilon\|^2_{L^2(\Gamma)} + \|\bar{y}-\bar{y}_\varepsilon\|^2_{L^2(\Omega)} \le c\left(\frac{\xi(\varepsilon)+\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \right)^{\frac{2}{d+1}} \tag{4.25}$$

is satisfied.

One can easily see in the estimates (4.24) and (4.25) that an appropriate choice of parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$ should satisfy the following conditions:

$$\lim_{\varepsilon\to 0}\frac{\xi(\varepsilon)}{\sqrt{\psi(\varepsilon)}}=0,\qquad \lim_{\varepsilon\to 0}\frac{\phi(\varepsilon)}{\sqrt{\psi(\varepsilon)}}=0. \tag{4.26}$$

Remark 4.14 Comparing both families (P_1^ε) and (P_2^ε) of regularized problems, one notices immediately differences in the assumptions to the parameter functions, see (3.20) and (4.26). We consider the following specific choice of $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$:

$$\psi(\varepsilon)=\frac{1}{\varepsilon^2},\qquad \phi(\varepsilon)=\varepsilon,\qquad \xi(\varepsilon)\equiv 1.$$

Due to Theorem 4.13, the optimal solution of problem (P_2^ε) converges to the original solution as $\varepsilon\to 0$. However, the error estimate (3.18) for problem (P_1^ε) implies no convergence. We conclude, that the convergence of the solution of (P_2^ε) is obtained under weaker assumptions on the parameter functions than for problem (P_1^ε). We mention, that the difference occurs during the estimation of the maximal violation for both problems. If one applies the technique of Lemma 4.8 in the proof of Lemma 3.6, one will obtain the same results. Hence, the limit relations given in (4.26) are sufficient for the convergence of (P_1^ε) to (P), too.

4.3 Comparison to the Moreau-Yosida regularization

In this section we point out similarities of the regularization approach by virtual controls (P_2^ε) to the penalization technique, introduced by Ito and Kunisch in [40]. This technique is based on a Moreau-Yosida approximation of the Lagrange multiplier. Furthermore, we refer to Hintermüller and Kunisch, [38], where this approach is applied to a general class of constrained minimization problems, connected with so called Primal-dual path-following methods . The convergence of this regularization approach for nonlinear parabolic optimal control problems with control and state constraints was observed by Neitzel and Tröltzsch in [62].

Applying the Moreau-Yosida regularization concept to the original problem (P), the following regularized optimal control problems are developed:

$$\left.\begin{array}{rl}
\min & J_\gamma(y_\gamma,u_\gamma):=J(y_\gamma,u_\gamma)+\dfrac{\gamma}{2}\displaystyle\int_{\Omega'}((y_c-y_\gamma)_+)^2dx\\
 & -\Delta y_\gamma+y_\gamma=0 \qquad \text{in } \Omega\\
 & \partial_n y_\gamma=u_\gamma \qquad \text{on } \Gamma\\
 & u_a\le u_\gamma(x)\le u_b \quad \text{a.e. on } \Gamma,
\end{array}\right\} \tag{P_γ}$$

where $\gamma>0$ is a regularization parameter that is taken large. We end up in a purely control contrained optimal control problem, where the state constraints have been removed by penalization. Introducing a reduced formulation of problem (P_γ) by a control-to-state mapping for the state equation, the existence and uniqueness of a solution for the problem is proven as well as for the previous considered optimal control problems, see e.g. Chapter 2.2. The associated necessary and sufficient optimality system is determined by straight forward computation.

Theorem 4.15 *Let $(\bar{y}_\gamma, \bar{u}_\gamma)$ be the optimal solution of problem (P_γ). Then, there exists a unique adjoint state $p_\gamma \in H^1(\Omega)$ such that the following optimality system is satisfied*

$$\begin{aligned} -\Delta\bar{y}_\gamma + \bar{y}_\gamma &= 0 & \qquad -\Delta p_\gamma + p_\gamma &= \bar{y}_\gamma - y_d - \lambda_\gamma \\ \partial_n\bar{y}_\gamma &= \bar{u}_\gamma & \partial_n p_\gamma &= 0 \end{aligned} \tag{4.27}$$

$$(\tau p_\gamma + \nu\bar{u}_\gamma \, , \, u - \bar{u}_\gamma)_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{ad}^L \tag{4.28}$$

$$\lambda_\gamma = \gamma(y_c - \bar{y}_\gamma)_+ \in L^2(\Omega') \tag{4.29}$$

We mention that the admissible set U_{ad}^L for the controls is defined as in (4.6). Furthermore, the regular function λ_γ in (4.27) can be extended by zero on the whole domain Ω.

First, we consider both types of problems $(\mathrm{P}_2^\varepsilon)$ and (P_γ) without any notice on the optimality conditions. We observe the regularized problem $(\mathrm{P}_2^\varepsilon)$ for the specific case $\phi(\varepsilon) \equiv 0$. Consequently, the state equation is given by

$$\begin{aligned} -\Delta y_\varepsilon + y_\varepsilon &= 0 \quad \text{in } \Omega \\ \partial_n y_\varepsilon &= u_\varepsilon \quad \text{on } \Gamma. \end{aligned}$$

Hence, there is no longer a coupling of the boundary control u_ε and the distributed control v_ε by the state equation of problem $(\mathrm{P}_2^\varepsilon)$. In order to investigate the mixed control-state constraints pointwise, we split the inner domain Ω' into two disjoint subsets $\Omega' = \Omega_1 \cup \Omega_2$, where we define

$$\begin{aligned} \Omega_1 &:= \{x \in \Omega' : \, y_c(x) - y_\varepsilon(x) < 0\} \\ \Omega_2 &:= \{x \in \Omega' : \, y_c(x) - y_\varepsilon(x) \geq 0\}. \end{aligned}$$

First, we consider Ω_1. The mixed constraints are given by $y_c(x) - y_\varepsilon(x) \leq \xi(\varepsilon)v_\varepsilon(x)$ a.e. in Ω'. Due to the minimization of the L^2-norm of the virtual control v_ε in the objective of $(\mathrm{P}_2^\varepsilon)$, we derive

$$v_\varepsilon \equiv 0 \quad \text{a.e. in } \Omega_1.$$

Considering Ω_2, the inequality

$$\xi(\varepsilon)v_\varepsilon(x) \geq y_c(x) - y_\varepsilon(x) \geq 0$$

has to be satisfied. Again under notice of the minimization of the virtual control, we obtain

$$v_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - y_\varepsilon) \quad \text{a.e. in } \Omega_2.$$

Concluding, the mixed control-state constraints can be replaced by the equation

$$v_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - y_\varepsilon)_+.$$

Furthermore, one can easily see, that the virtual control v_ε vanishes in $\Omega \setminus \Omega'$. Thus, the optimal control problem (P_2^ε) can be rewritten to

$$\begin{aligned} \min \quad J(y_\varepsilon, u_\varepsilon) + \frac{\psi(\varepsilon)}{2\xi(\varepsilon)^2}\|(y_c - y_\varepsilon)_+\|^2_{L^2(\Omega')} \\ -\Delta y_\varepsilon + y_\varepsilon = 0 \qquad \text{in } \Omega \\ \partial_n y_\varepsilon = u_\varepsilon \qquad \text{on } \Gamma \\ u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. on } \Gamma. \end{aligned}$$

Consequently, we formulate the following result.

Corollary 4.16 *For the specific parameter function $\phi(\varepsilon) \equiv 0$, the problem ($\mathrm{P}_2^\varepsilon$) is equivalent to the problem (P_γ) arising by the Moreau-Yosida regularization, if the regularization parameter $\gamma > 0$ is defined by $\gamma := \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$.*

For the sake of completeness, we will also show the equivalence of the particular optimality systems of the problems. The optimality system for problem (P_γ) is given in Theorem 4.15. Due to Theorem 4.3, the optimality system for (P_2^ε) with the specific parameter function $\phi(\varepsilon) \equiv 0$ simplifies to

$$\begin{aligned} -\Delta \bar{y}_\varepsilon + \bar{y}_\varepsilon &= 0 & \qquad -\Delta p_\varepsilon + p_\varepsilon &= \bar{y}_\varepsilon - y_d - \mu_\varepsilon \\ \partial_n \bar{y}_\varepsilon &= \bar{u}_\varepsilon & \partial_n p &= 0 \end{aligned} \tag{4.30}$$

$$(\tau p_\varepsilon + \nu \bar{u}_\varepsilon \,,\, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \ge 0, \quad \forall u \in U_{ad}^L \tag{4.31}$$

$$\psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\mu_\varepsilon = 0 \quad \text{a.e. in } \Omega \tag{4.32}$$

$$(\mu_\varepsilon \,,\, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega')} = 0, \quad \mu_\varepsilon \ge 0, \quad \bar{y}_\varepsilon \ge y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad \text{a.e. in } \Omega'. \tag{4.33}$$

Since the multiplier μ_ε is a regular function, it is well known that the complementary slackness conditions in (4.33) are equivalent to

$$\mu_\varepsilon - \max\{0, \mu_\varepsilon + c(y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)\} = 0$$

for every $c > 0$. For the specific choice $c = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$, we obtain instead of (4.32) and (4.33)

$$\mu_\varepsilon = \max\{0, \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_c - \bar{y}_\varepsilon)\} = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_c - \bar{y}_\varepsilon)_+.$$

Thus the optimality systems of both problems are equivalent and we conclude with the following result.

Corollary 4.17 *Let ($\bar{y}_\gamma$, $\bar{u}_\gamma$) be the optimal solution of problem (P_γ). Moreover, let ($\bar{y}_\varepsilon$, $\bar{u}_\varepsilon$, $\bar{v}_\varepsilon$) be the optimal solution of (P_2^ε). For the specific choices $\phi(\varepsilon) \equiv 0$ in (P_2^ε) and $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$ in (P_γ), the optimal solution ($\bar{y}_\gamma$, $\bar{u}_\gamma$) coincides with ($\bar{y}_\varepsilon$, $\bar{u}_\varepsilon$). Furthermore, the optimal virtual control $\bar{v}_\varepsilon$ satisfies $\bar{v}_\varepsilon = 1/\xi(\varepsilon)(y_c - \bar{y}_\varepsilon)_+$.*

Remark 4.18 The previous arguments show that the error estimates of the previous section are also valid for the Moreau-Yosida regularization concept. In order to obtain the strong convergence of the regularized solutions of problem (P_2^ε) to the solution of problem (P), we required the limit relation $\lim_{\varepsilon\to 0} \frac{\xi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} = 0$ in 4.26. Due to $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$, this condition on the parameter functions implies $\gamma \to \infty$ for the penalization parameter in problem (P_γ).

Finally, let us mention some differences between the concepts. First, the problem size of the virtual control concept is slightly higher than for the penalty approach. The main difference is the differentiability of the cost functionals, namely the objective of problem (P_γ) is not twice differentiable. Hence, in the case of nonlinear optimal control problems the classical second-order analysis is not applicable.

Concluding, we mention again that both regularization approaches, the virtual control concept and the Moreau-Yosida regularization, respectively, benefit from the higher regularity of the solutions contrary to the original problem (P). Thus, efficient optimization methods are well defined and applicable.

Chapter 5

Finite element error analysis for the virtual control approach

This chapter is devoted to the discretization of the regularized problem (P_2^ε). The main goal is the derivation of an error estimate of the particular discretized version of (P_2^ε) to the original problem (P), where the regularization and discretization error are considered simultaneously. First, let us recall the regularized and continuous problem:

$$\left.\begin{array}{c} \min \quad J_\varepsilon(y_\varepsilon, u_\varepsilon, v_\varepsilon) := \dfrac{1}{2}\|y_\varepsilon - y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u_\varepsilon\|^2_{L^2(\Gamma)} + \dfrac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)} \\ y_\varepsilon = S(\tau^* u_\varepsilon + E_H^* \phi(\varepsilon) v_\varepsilon) \\ u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. on } \Gamma \\ y_\varepsilon(x) \ge y_c(x) - \xi(\varepsilon) v_\varepsilon(x) \quad \text{a.e. in } \Omega', \end{array}\right\} \qquad (\mathrm{P}_2^\varepsilon)$$

where the classical formulation of the state equation is replaced by the control-to-state mapping, see (4.2). We want to establish a finite element based approximation of the regularized optimal control problem (P_2^ε).

5.1 General assumptions and results

In this section, we will recall standard results concerning the finite element method. Furthermore, necessary results concerning the discretization of the regularized optimal control problem (P_2^ε) will be stated. We start with the introduction of a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$. The mesh $\mathcal{T}_h$ consists of open and pairwise disjoint cells (triangles, tetrahedra) such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} \bar{T}.$$

Note that the domain Ω is polygonally or polyhedrally bounded. The vertices of the elements of $\mathcal{T}_h$ are denoted by $x_1, \ldots, x_n$. With each element $T \in \mathcal{T}_h$, we associate two parameters $R(T)$ and $\rho(T)$, where $R(T)$ denotes the diameter of the element T and $\rho(T)$ is the diameter of the largest ball contained in T. The mesh size of $\mathcal{T}_h$ is defined by

$$h = \max_{T \in \mathcal{T}_h} R(T).$$

We suppose the following regularity assumption for $\mathcal{T}_h$:

Assumption 5.1 There exist two positive constants ρ and R such that

$$\frac{R(T)}{\rho(T)} \leq \rho, \quad \frac{h}{R(T)} \leq R$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

For a fixed mesh size $h > 0$, we denote by x_i^Γ, $i = 1, \ldots, n_e$, and e_j, $j = 1, \ldots, n_\Gamma$ the vertices and edges or faces at the boundary, respectively.

Associated with the previous triangulation, we define

$$V_h = \{v \in C(\bar{\Omega}) \,|\, v|_T \in \mathcal{P}_1 \,\forall T \in \mathcal{T}_h\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less than or equal one. Notice that $V_h \subset H^1(\Omega) \cap C(\bar{\Omega})$. Furthermore, the corresponding linear finite element ansatz functions are denoted by φ_i, $i = 1, \ldots, n$. Let us introduce the usual nodal interpolation operator and recall some approximation properties.

Definition 5.2 *For every $z \in W^{m,p}(\Omega)$, $mp > d$, the nodal interpolate $I_h z \in V_h$ is defined by*

$$(I_h z)(x) = \sum_{i=1}^{n} z(x_i)\varphi_i(x).$$

In the following lemma, we recall some standard approximation results for the nodal interpolation operator. For the proof and more detailed information, we refer e.g. to [14, Corollary 4.4.24 ff.].

Lemma 5.3 *Let I_h be the interpolation operator of Definition 5.2. Then, there exists a positive constant c, depending on Ω, ρ, m, d and p, such that*

$$\|z - I_h z\|_{W^{s,p}(\Omega)} \leq c h^{m-s} \|z\|_{W^{m,p}(\Omega)} \tag{5.1}$$

for all $z \in W^{m,p}(\Omega)$, $mp > d$ and $0 \leq s \leq m-1$. Moreover, we have

$$\|z - I_h z\|_{W^{s,\infty}(\Omega)} \leq c h^{2-s-d/p} \|z\|_{W^{2,p}(\Omega)} \tag{5.2}$$

for all $z \in W^{2,p}(\Omega)$, $p > d$ and $0 \leq s \leq 1$.

Forthcoming, we state the well known inverse estimates for functions belonging to V_h.

Lemma 5.4 *For every $z_h \in V_h$, there exists a positive constant c, depending on m, p, q and ρ, such that*

$$\|z_h\|_{W^{m,p}(\Omega)} \leq c h^{-m-(\frac{d}{q}-\frac{d}{p})} \|z_h\|_{L^q(\Omega)} \tag{5.3}$$

is satisfied for $1 \leq q \leq p \leq \infty$ and $m = 0, 1$. Furthermore, there exists a positive constant, depending on Ω, ρ and d, such that

$$\|z_h\|_{H^1(\Gamma)} \leq c h^{-1} \|z_h\|_{L^2(\Gamma)} \tag{5.4}$$

is satisfied.

A proof of the inverse estimate for domains in $\mathbb{R}^d$ can be found in standard text books, such as for instance Brenner and Scott [14, Theorem (4.5.11)] or Ciarlet [27, Theorem 17.2.]. By a local bi-Lipschitz transformation of the variables, the proof can be adapted to estimates on the boundary.

The following elliptic partial differential equation covers all state equations and adjoint equations occurring in our optimal control problems:

$$\begin{aligned} -\Delta w + w &= f \quad \text{in } \Omega \\ \partial_n w &= g \quad \text{on } \Gamma \end{aligned} \tag{5.5}$$

for some $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. The weak formulation associated with (5.5) is given by

$$\int_\Omega \nabla w \cdot \nabla z + wz\, dx = \int_\Omega fz\, dx + \int_\Gamma gz\, ds \qquad \forall z \in H^1(\Omega). \tag{5.6}$$

The corresponding finite element approximation $w_h \in V_h$ has to satisfy the following variational equation

$$\int_\Omega \nabla w_h \cdot \nabla z_h + w_h z_h\, dx = \int_\Omega f z_h\, dx + \int_\Gamma g z_h\, ds \qquad \forall z_h \in V_h. \tag{5.7}$$

The existence and uniqueness of solutions for (5.6) and (5.7) directly results from the Lax-Milgram Lemma 2.1. The next theorem provides standard finite element error estimates.

Theorem 5.5 *Let $w \in H^1(\Omega)$ be the solution of* (5.6) *and $w_h \in V_h$ the solution of* (5.7).

(i) For every $(f, g) \in L^2(\Omega) \times H^{1/2}(\Gamma)$, there exists a constant $c > 0$, independent of h, such that

$$\|w - w_h\|_{L^2(\Omega)} \le ch^2(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}) \tag{5.8}$$

$$\|w - w_h\|_{L^\infty(\Omega)} \le ch^{2-d/2}(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}). \tag{5.9}$$

(ii) If $(f, g) \in L^2(\Omega) \times L^2(\Gamma)$, then the error estimate

$$\|w - w_h\|_{L^2(\Omega)} \le ch^{3/2}(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}) \tag{5.10}$$

is satisfied for a constant c independent on h.

Proof: (i) If $(f, g)) \in L^2(\Omega) \times H^{1/2}(\Gamma)$, then the weak solution w of the boundary value problem (5.5) is in $H^2(\Omega)$, see Theorem 2.18. Then, the results are quite standard and we refer for a proof and more detailed information to, e.g. [13] or [14, Theorem (4.4.20)].
(ii) Due to $g \in L^2(\Gamma)$, we can only assure that w is a function of $H^{3/2}(\Omega)$, see [42]. Now, the estimate can be deduced by real interpolation, see Brenner and Scott [14,

Theorem (14.3.3)]. $\square$

Particularly in the previous chapter, L^∞-estimates in the inner domain Ω' occurred. This fact leads us to interior maximum norm estimates for finite element approximations to solutions of the partial differential equation (5.5). The next result was derived by Schatz and Wahlbin in [71], where the finite element approximation error in the L^∞-norm in an inner subdomain is estimated by the best approximation error plus the error in a weaker norm on a slightly larger domain.

Theorem 5.6 *Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Furthermore, let $w \in H^1(\Omega) \cap C(\bar{\Omega})$ be the solution of* (5.6) *and $w_h \in V_h$ the solution of* (5.7)*. Then, there exists a constant c and $0 < h_0 < 1$ such that*

$$\|w - w_h\|_{L^\infty(\Omega')} \le c(|\log h| \inf_{\chi_h \in V_h} \|w - \chi_h\|_{L^\infty(\Omega'')} + \|w - w_h\|_{L^2(\Omega)}) \tag{5.11}$$

is satisfied for $0 < h \le h_0$.

For a proof, we refer to [71, Theorem 5.1], where the result is given in a more general way.

Since we are discussing boundary control problems, we introduce also a finite element space for functions defined on the boundary of Ω. For the space of discrete functions on Γ we define

$$U_h = \{u \in C(\Gamma) \,|\, u|_{e_j} \in \mathcal{P}_1 \ \text{ for } \ j = 1, \dots, n_\Gamma\} \tag{5.12}$$

as the space of edgewise or facewise linear finite elements, respectively. Notice that the restriction of the finite element space V_h to the boundary Γ coincides with the space U_h by construction of the mesh.

Definition 5.7 *The basis functions of the space U_h are denoted by ψ_i, $i = 1, \dots, n_e$ and the following conditions are satisfied for every $i, j = 1, \dots, n_e$:*

$$\psi_i(x_j^\Gamma) = \delta_{ij}. \tag{5.13}$$

Due to Definition 5.7, the basis functions of U_h satisfy:

$$\psi_i(x) \ge 0 \text{ a.e. on } \Gamma, \quad i = 1, \dots, n_e \quad \sum_{i=1}^{n_e} \psi_i(x) = 1. \tag{5.14}$$

Remark 5.8 We define by

$$\omega_i := \text{supp } \psi_i \qquad i = 1, \dots, n_e$$

the patch ω_i that consists of the M_i adjacent elements of $\{e_j\}_{j=1}^{n_\Gamma}$ that share the vertex x_i^Γ. Assumption 5.1 implies the existence of a constant $M \in \mathbb{N}$, independent of h, such that $M_i \le M$ for all $i = 1, \dots, n_e$.

One can easily see that for two-dimensional convex polygonal domains every patch ω_i consists of two edges e_j.

Now, we define an interpolation operator for functions u in $L^2(\Gamma)$. Since we are considering control constraints on the boundary, this interpolation operator should satisfy the following assumption:

$$u_a \le u(x) \le u_b \text{ a.e. on } \Gamma \quad \Rightarrow \quad u_a \le (\Pi_h u)(x) \le u_b \text{ a.e. on } \Gamma. \tag{5.15}$$

First let us consider the usual L^2-projection on the space U_h, which we denote by $\Pi_h u$ for an arbitrary $u \in L^2(\Gamma)$. It is well known, that the projection $\Pi_h u \in U_h$ satisfies the variational equation

$$(\Pi_h u - u\,,\, v_h)_{L^2(\Gamma)} = 0 \quad \forall v_h \in U_h.$$

However, the following simple example shows that the condition (5.15) is in general violated.

Example 5.9 *For simplicity we set $\Gamma = (0,1)$, $u_a = 0$ and $u_b = 1$. For the discrete space we choose $U_h = \operatorname{span}\{\psi_1, \psi_2\} = \{x, 1-x\} \subset L^2(0,1)$. The function*

$$u(x) = \begin{cases} 0, & 0 \le x \le 0.5 \\ 1, & 0.5 < x \le 1 \end{cases}$$

satisfies $u_a \le u(x) \le u_b$ a.e. in $(0,1)$. The straight forward evaluation of the previous mentioned variational equation yields

$$(\Pi_h u)(x) = 1.25\psi_1 - 0.25\psi_2 = 1.5x - 0.25.$$

Thus, the condition (5.15) is in general not satisfied by the usual L^2-projection on the space U_h.

To this end, we consider the quasi-interpolation operator introduced by Carstensen in [15]. For an arbitrary $u \in L^1(\Gamma)$, the operator is constructed as follows:

$$\Pi_h u = \sum_{i=1}^{n_e} \pi_i(u)\psi_i, \tag{5.16}$$

where the coefficients $\pi_i(u) \in \mathbb{R}$ are defined by

$$\pi_i(u) = \frac{\int_{\omega_i} u\psi_i ds}{\int_{\omega_i} \psi_i ds}. \tag{5.17}$$

One can easily see that the property (5.15) is fulfilled by construction of the quasi-interpolation operator. Forthcoming, we state error estimates for $u - \Pi_h u$ in different norms. The following result provides a local error estimate on a patch ω_i.

Lemma 5.10 *For all $i = 1, \ldots, n_e$, there is a constant c, which is independent of h, such that*

$$\|u - \pi_i(u)\|_{L^2(\omega_i)} \le ch\|u\|_{H^1(\omega_i)} \quad \forall u \in H^1(\omega_i).$$

In [28] the result was proven for functions defined in the domain. The proof can be easily adapted to the boundary case and we skip the proof. In the next lemma we derive a global L^2-estimate for the error $u - \Pi_h u$.

Lemma 5.11 *There is a positive constant c, independent of h, such that*

$$\|u - \Pi_h u\|_{L^2(\Gamma)} \leq ch\|u\|_{H^1(\Gamma)} \quad \forall u \in H^1(\Gamma).$$

Proof: Due to $\sum_{i=1}^{n_e} \psi_i \equiv 1$, $\omega_i = \operatorname{supp} \psi_i$ and the definition of Π_h, we find for all $v \in L^2(\Gamma)$

$$\begin{aligned}
(u - \Pi_h u\,,\, v)_{L^2(\Gamma)} &= \left(u \sum_{i=1}^{n_e} \psi_i - \sum_{i=1}^{n_e} \pi_i(u)\psi_i, v \right)_{L^2(\Gamma)} \\
&= \sum_{i=1}^{n_e} \int_{\omega_i} (u - \pi_i(u))\, \psi_i\, v\, ds, \\
&\leq \sum_{i=1}^{n_e} \|u - \pi_i(u)\|_{L^2(\omega_i)} \|v\|_{L^2(\omega_i)}
\end{aligned}$$

Applying Lemma 5.10, we deduce

$$\begin{aligned}
(u - \Pi_h u\,,\, v)_{L^2(\Gamma)} &\leq c\, h \sum_{i=1}^{n_e} \|u\|_{H^1(\omega_i)}\, \|v\|_{L^2(\omega_i)} \\
&\leq c\, h \left(\sum_{i=1}^{n_e} \|u\|^2_{H^1(\omega_i)} \right)^{1/2} \left(\sum_{i=1}^{n_e} \|v\|^2_{L^2(\omega_i)} \right)^{1/2}.
\end{aligned}$$

This yields

$$|(u - \Pi_h u\,,\, v)_{L^2(\Gamma)}| \leq c\, h\, \|u\|_{H^1(\Gamma)}\, \|v\|_{L^2(\Gamma)}.$$

We note that $\sum_{i=1}^{n_e} \|u\|^2_{H^1(\omega_i)} \leq c\|u\|^2_{H^1(\Gamma)}$ for every $u \in H^1(\Gamma)$. This directly follows from Assumption 5.1, see also Remark 5.8. The specific choice $v = u - \Pi_h u$ completes the proof. □

Lemma 5.12 *For every $u \in H^1(\Gamma)$, there exists a constant c, independent of h, such that*

$$\|u - \Pi_h u\|_{H^1(\Gamma)^*} \leq ch^2\|u\|_{H^1(\Gamma)}$$

Proof: The beginning is similar to the proof of the previous lemma. For all $v \in H^1(\Gamma)$ one obtains

$$(u - \Pi_h u\,,\, v)_{L^2(\Gamma)} = \sum_{i=1}^{n_e} \int_{\omega_i} (u - \pi_i(u))\, \psi_i\, v\, ds.$$

Due to the definition (5.17) of $\pi_i(u)$, we have

$$\int_{\omega_i} (u - \pi_i(u))\psi_i ds = 0.$$

Thanks to Lemma 5.10, we derive

$$
\begin{aligned}
(u - \Pi_h u\,,\, v)_{L^2(\Gamma)} &= \sum_{i=1}^{n_e} \int_{\omega_i} (u - \pi_i(u))\, \psi_i\, (v - \pi_i(v))\, ds \\
&\le ch^2 \sum_{i=1}^{n_e} \|u\|_{H^1(\omega_i)} \|v\|_{H^1(\omega_i)} \\
&\le ch^2 \left(\sum_{i=1}^{n_e} \|u\|^2_{H^1(\omega_i)} \right)^{1/2} \left(\sum_{i=1}^{n_e} \|v\|^2_{H^1(\omega_i)} \right)^{1/2} \\
&\le ch^2 \|u\|_{H^1(\Gamma)} \|v\|_{H^1(\Gamma)}.
\end{aligned}
$$

Following the definition of $\|\cdot\|_{H^1(\Gamma)^*}$, we obtain

$$
\|u - \Pi_h u\|_{H^1(\Gamma)^*} = \sup_{v \in H^1(\Gamma)} \frac{|(u - \Pi_h u\,,\, v)_{L^2(\Gamma)}|}{\|v\|_{H^1(\Gamma)}} \le ch^2 \|u\|_{H^1(\Gamma)},
$$

which is the assertion. □

We mention that the proof of the previous two approximation error estimates follows the ideas of de los Reyes, Meyer and Vexler in [28]. With the results of this section at hand, we will discuss the discretization of the regularized optimal control problem (P_2^ε) and the associated convergence analysis in the following sections.

5.2 Discretization of the regularized problem (P_2^ε)

Now, we are going to establish a finite element approximation of the problem (P_2^ε). First, we discretize the state equation and we introduce a discrete control-to-state mapping, similar to (4.2). For each element $f \in H^1(\Omega)^*$, we denote by y_h the unique element of V_h that satisfies

$$
a(y_h, z_h) = \langle f\,,\, z_h \rangle_{H^1(\Omega)^*, H^1(\Omega)} \qquad \forall z_h \in V_h,
$$

where $a : V_h \times V_h \to \mathbb{R}$ is the bilinear form defined in (2.2). The existence and uniqueness of $y_h \in V_h$ directly follows from the Lax-Milgram Lemma 2.1. Then, the discrete solution operator $S_h : H^1(\Omega)^* \to L^2(\Omega)$ is defined as follows:

$$
f \mapsto y_h,\; y_h = S_h f \quad \Longleftrightarrow \quad a(y_h, z_h) = \langle f\,,\, z_h \rangle_{H^1(\Omega)^*, H^1(\Omega)} \quad \forall z_h \in V_h. \tag{5.18}
$$

Hence, the discrete solution for the state equation of problem (P_2^ε) is given by:

$$
y_h^\varepsilon = S_h(\tau^* u + \phi(\varepsilon) E_H^* v) \quad \text{for } (u, v) \in L^2(\Gamma) \times L^2(\Omega). \tag{5.19}
$$

We proceed with the discretization of the controls. We mention that the virtual control is also discretized by piecewise linear ansatz functions, i.e. we consider $v \in V_h$. For the space of discrete controls on the boundary we use U_h defined in (5.12). For simplicity, we do not consider a discretization of the desired state y_d as well as the bound y_c.

Based on the previous arguments, the regularized and discretized optimal control problem is given by

$$\left.\begin{array}{lll}\min & J_\varepsilon(y_h^\varepsilon, u_h^\varepsilon, v_h^\varepsilon) = \frac{1}{2}\|y_h^\varepsilon - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|u_h^\varepsilon\|_{L^2(\Gamma)}^2 + \frac{\psi(\varepsilon)}{2}\|v_h^\varepsilon\|_{L^2(\Omega)}^2 \\ \text{s.t.} & y_h^\varepsilon = S_h(\tau^* u_h^\varepsilon + \phi(\varepsilon) E_H^* v_h^\varepsilon) \quad \text{for} \quad (u_h^\varepsilon, v_h^\varepsilon) \in V_{ad,h}^{\varepsilon,2},\end{array}\right\} \qquad (\mathrm{P}_{2,h}^\varepsilon)$$

where the discrete admissible set is defined by

$$\begin{aligned} V_{ad,h}^{\varepsilon,2} := \{(u_h, v_h) \in U_h \times V_h \,|\, & u_a \le u_h(x) \le u_b \text{ a.e. on } \Gamma, \\ & S_h(\tau^* u_h + \phi(\varepsilon) E_H^* v_h)(x) \ge y_c(x) - \xi(\varepsilon) v_h(x) \text{ a.e. in } \Omega'\}. \end{aligned}$$

The admissible set is convex and closed. Furthermore, the next lemma shows that the admissible set is nonempty for sufficiently small mesh sizes.

Lemma 5.13 *Let $\hat{u}$ be the inner point with respect to the state constraints defined in Assumption 2.4, i.e.*

$$u_a \le \hat{u}(x) \le u_b \quad \textit{a.e. on } \Gamma \quad \textit{and} \quad (S\tau^*\hat{u})(x) \ge y_c(x) + \gamma \;\; \forall x \in \Omega'$$

with $\gamma > 0$. Then, there is an $h_0 > 0$ such that

$$\hat{y}_h(x) = (S_h \tau^* \Pi_h \hat{u})(x) \ge y_c(x) + \gamma_0, \qquad \textit{a.e. in } \Omega'$$

is valid for all $0 < h \le h_0$ with a constant γ_0 independent of h.

Proof: Since $\hat{u}$ satisfies the control constraints and the quasi-interpolation operator Π_h by (5.16) preserves this property, we obtain $u_a \le \Pi_h \hat{u} \le u_b$. With the help of Assumption 2.4, we proceed with

$$\begin{aligned} (S_h\tau^*\Pi_h\hat{u})(x) &= (S\tau^*\hat{u})(x) + (S\tau^*(\Pi_h\hat{u} - \hat{u}))(x) + ((S_h - S)\tau^*\Pi_h\hat{u})(x) \\ &\ge y_c(x) + \gamma - \|S\tau^*(\Pi_h\hat{u} - \hat{u})\|_{L^\infty(\Omega')} - \|(S_h - S)\tau^*\Pi_h\hat{u}\|_{L^\infty(\Omega')} \end{aligned}$$

The first L^∞-error in the subdomain Ω' is estimated by Corollary 2.23 and Lemma 2.24, which gives

$$\|S\tau^*(\Pi_h\hat{u} - \hat{u})\|_{L^\infty(\Omega')} \le c\|S\tau^*(\Pi_h\hat{u} - \hat{u})\|_{L^2(\Omega)} \le c\|\Pi_h\hat{u} - \hat{u}\|_{H^1(\Gamma)^*}.$$

Moreover, Lemma 5.12 and $\hat{u} \in H^1(\Gamma)$ yields

$$\|S\tau^*(\Pi_h\hat{u} - \hat{u})\|_{L^\infty(\Omega')} \le ch^2\|\hat{u}\|_{H^1(\Gamma)}.$$

Since the quasi-interpolation $\Pi_h\hat{u}$ belongs to $H^{1/2}(\Gamma)$ by construction, we obtain by (5.9)

$$\|(S_h - S)\tau^*\Pi_h\hat{u}\|_{L^\infty(\Omega')} \le ch^{2-d/2}\|\Pi_h\hat{u}\|_{H^{1/2}(\Gamma)}.$$

Analogue to [15, Theorem 3.1], the stability of the quasi-interpolation in $H^{1/2}(\Gamma)$ is given. Concluding, we end up with

$$\hat{y}_h(x) = (S_h\tau^*\Pi_h\hat{u})(x) \ge y_c + \gamma - c(h^2\|\hat{u}\|_{H^1(\Gamma)} + h^{2-d/2}\|\hat{u}\|_{H^{1/2}(\Gamma)}).$$

Hence, if h_0 is chosen sufficiently small, we obtain

$$\gamma_0 := \gamma - c(h_0^2 \|\hat{u}\|_{H^1(\Gamma)} + h_0^{2-d/2} \|\hat{u}\|_{H^{1/2}(\Gamma)}) > 0.$$

$\square$

The previous result implies $(\Pi_h \hat{u}, 0) \in V_{ad,h}^{\varepsilon,2}$ for sufficiently small mesh sizes h. Moreover, $(\Pi_h \hat{u}, 0)$ is an inner point with respect to the mixed control-state constraints of problem $(\mathrm{P}_{2,h}^\varepsilon)$. Consequently, one obtains by standard arguments that the discretized and regularized optimal control problem $(\mathrm{P}_{2,h}^\varepsilon)$ admits a unique solution $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$. By straight forward computation, we formulate the necessary and sufficient optimality condition in the following lemma.

Lemma 5.14 *Let $(\bar{y}_h^\varepsilon,\ \bar{u}_h^\varepsilon,\ \bar{v}_h^\varepsilon)$ be the optimal solution of problem $(\mathrm{P}_{2,h}^\varepsilon)$. The optimality condition is given by*

$$(\tau \bar{p}_h^\varepsilon + \nu \bar{u}_h^\varepsilon, u - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} + (\phi(\varepsilon) E_H \bar{p}_h^\varepsilon + \psi(\varepsilon) \bar{v}_h^\varepsilon, v - \bar{v}_h^\varepsilon)_{L^2(\Omega)} \geq 0 \quad \forall (u,v) \in V_{ad,h}^{\varepsilon,2}, \tag{5.20}$$

where $\bar{p}_h^\varepsilon = S_h^*(\bar{y}_h^\varepsilon - y_d)$ denotes the discrete adjoint state with respect to $\bar{y}_h^\varepsilon$ and S_h^* is adjoint operator of S_h.

Remark 5.15 The discrete adjoint state $\bar{p}_h^\varepsilon$ is defined by the following equivalence:

$$\bar{p}_h^\varepsilon = S_h^*(\bar{y}_h^\varepsilon - y_d) \quad \Longleftrightarrow \quad a(z_h, \bar{p}_h^\varepsilon) = (\bar{y}_h^\varepsilon - y_d\,,\, z_h)_{L^2(\Omega)} \quad \forall z_h \in V_h.$$

This equivalence is obtained by adapting the arguments of Lemma 2.8 to the discrete case.

Later, we need uniform boundedness of the discrete control $\bar{u}_h^\varepsilon$ in $H^1(\Gamma)$ with respect to ε and h. To this end, we consider the optimality conditions for problem $(\mathrm{P}_{2,h}^\varepsilon)$ using the classical approach with a Lagrange multiplier μ_h^ε associated with the mixed constraints. The control constraints are handled by the admissible set

$$U_{ad,h}^L := \{u_h \in U_h : \quad u_a \leq u_h(x) \leq u_b \text{ a.e. on } \Gamma\}. \tag{5.21}$$

In contrast to Lemma 5.14, the discrete optimal adjoint state is denoted by p_h^ε since the Lagrange multiplier μ_h^ε arises in the right-hand side. Introducing a Lagrange multiplier with respect to the control-state-constraints in $(\mathrm{P}_{2,h}^\varepsilon)$, we obtain the following optimality system

$$\bar{y}_h^\varepsilon = S_h(\tau^* \bar{u}_h^\varepsilon + \phi(\varepsilon) E_H^* \bar{v}_h^\varepsilon) \tag{5.22}$$

$$p_h^\varepsilon = S_h^*(\bar{y}_h^\varepsilon - y_d - \mu_h^\varepsilon) \tag{5.23}$$

$$(\tau p_h^\varepsilon + \nu \bar{u}_h^\varepsilon\,,\, u - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{ad,h}^L \tag{5.24}$$

$$\phi(\varepsilon) p_h^\varepsilon + \psi(\varepsilon) \bar{v}_h^\varepsilon - \xi(\varepsilon) \mu_h^\varepsilon = 0 \quad \text{a.e. in } \Omega \tag{5.25}$$

$$\begin{aligned} &(\mu_h^\varepsilon\,,\, y_c - \bar{y}_h^\varepsilon - \xi(\varepsilon) \bar{v}_h^\varepsilon)_{L^2(\Omega')} = 0, \\ &\mu_h^\varepsilon \geq 0, \quad \bar{y}_h^\varepsilon \geq y_c - \xi(\varepsilon) \bar{v}_h^\varepsilon \quad \text{a.e. in } \Omega'. \end{aligned} \tag{5.26}$$

Analogue to (4.11), the next lemma ensures the uniform boundedness of the discrete multiplier in $L^1(\Omega')$ with respect to ε and sufficiently small mesh sizes h.

Lemma 5.16 *Let $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ be the optimal solution of problem $(\mathrm{P}_{2,h}^\varepsilon)$. Furthermore, let p_h^ε be the adjoint state and μ_h^ε the Lagrange multiplier such that the optimality system (5.22)-(5.26) is fulfilled. Then, there exists a mesh size $h_0 > 0$ such that the Lagrange multiplier μ_h^ε is uniformly bounded in $L^1(\Omega')$ for all mesh sizes $0 < h \le h_0$, i.e.*

$$\|\mu_h^\varepsilon\|_{L^1(\Omega')} \le C, \tag{5.27}$$

where the constant $C > 0$ is independent of the regularization parameter ε and the mesh size h.

Proof: Since the proof is rather similar to the proof of Lemma 4.4, we will sketch only the main steps. We add a variational formulation of (5.25) and (5.24) for the specific test function $(\Pi_h \hat{u}, 0) \in U_{ad,h}^L \times V_h$:

$$\begin{aligned}(\phi(\varepsilon) E_H S_h^*(\bar{y}_h^\varepsilon - y_d - \mu_h^\varepsilon) + \psi(\varepsilon)\bar{v}_h^\varepsilon - \xi(\varepsilon)\mu_h^\varepsilon\,,\, -\bar{v}_h^\varepsilon)_{L^2(\Omega)} +& \\ (\tau S_h^*(\bar{y}_h^\varepsilon - y_d - \mu_h^\varepsilon) + \nu\bar{u}_h^\varepsilon, \Pi_h\hat{u} - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} &\ge 0.\end{aligned}$$

Sorting all terms, where the multiplier occurs, and applying the adjoint solution operator, we arrive at

$$\begin{aligned}(\mu_h^\varepsilon\,,\, \xi(\varepsilon)(-\bar{v}_h^\varepsilon) + S_h E_2^* \phi(\varepsilon)(-\bar{v}_h^\varepsilon) + S_h\tau^*(\Pi_h\hat{u} - \bar{u}_h^\varepsilon))_{L^2(\Omega)} \le& \\ (\psi(\varepsilon)\bar{v}_h^\varepsilon + \phi(\varepsilon)E_2 S_h^*(\bar{y}_h^\varepsilon - y_d)\,,\, -\bar{v}_h^\varepsilon)_{L^2(\Omega)}& \\ + (\nu\bar{u}_h^\varepsilon + \tau S_h^*(\bar{y}_h^\varepsilon - y_d)\,,\, \Pi_h\hat{u} - \bar{u}_h^\varepsilon)_{L^2(\Gamma)}&.\end{aligned} \tag{5.28}$$

By means of the discrete control-to-state mapping (5.19) and the complementary slackness condition (5.26), we find for the left side of the previous inequality:

$$\begin{aligned}&(\mu_h^\varepsilon\,,\, \xi(\varepsilon)(-\bar{v}_h^\varepsilon) + S_h E_H^* \phi(\varepsilon)(-\bar{v}_h^\varepsilon) + S_h\tau^*(\Pi_h\hat{u} - \bar{u}_h^\varepsilon))_{L^2(\Omega)} \\ &\quad = (\mu_h^\varepsilon\,,\, y_c - \bar{y}_h^\varepsilon - \xi(\varepsilon)\bar{v}_h^\varepsilon)_{L^2(\Omega)} + (\mu_h^\varepsilon\,,\, S_h\tau^*\Pi_h\hat{u} - y_c)_{L^2(\Omega)} \\ &\quad = (\mu_h^\varepsilon\,,\, S_h\tau^*\Pi_h\hat{u} - y_c)_{L^2(\Omega)}.\end{aligned} \tag{5.29}$$

Thanks to the positivity of the discrete multiplier and Lemma 5.13, we infer

$$\gamma_0\|\mu_h^\varepsilon\|_{L^1(\Omega')} \le (\mu_h^\varepsilon\,,\, S_h\tau^*\Pi_h\hat{u} - y_c)_{L^2(\Omega)} \tag{5.30}$$

for sufficiently small mesh sizes $h \le h_0$. Summarizing (5.28), (5.29) and (5.30), and applying similar simplifications like in the proof of Lemma 4.4, we derive the estimate

$$\begin{aligned}\gamma_0\|\mu_h^\varepsilon\|_{L^1(\Omega')} &\le (\psi(\varepsilon)\bar{v}_h^\varepsilon + \phi(\varepsilon)E_H S_h^*(\bar{y}_h^\varepsilon - y_d)\,,\, -\bar{v}_h^\varepsilon)_{L^2(\Omega)} \\ &\quad + (\nu\bar{u}_h^\varepsilon + \tau S_h^*(\bar{y}_h^\varepsilon - y_d)\,,\, \Pi_h\hat{u} - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} \\ &\le \|y_d\|_{L^2(\Omega)}\|\bar{y}_h^\varepsilon\|_{L^2(\Omega)} + \nu\|\bar{u}_h^\varepsilon\|_{L^2(\Gamma)}\|\Pi_h\hat{u}\|_{L^2(\Gamma)} \\ &\quad + \|\bar{y}_h^\varepsilon - y_d\|_{L^2(\Omega)}\|S_h\tau^*\Pi_h\hat{u}\|_{L^2(\Omega)},\end{aligned}$$

which implies the assertion since the optimality of $(\bar{u}_h^\varepsilon, \bar{y}_h^\varepsilon)$ yields uniform boundedness in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. □

5.3 Auxiliary results

In this section we will provide results that are necessary for the derivation of an error estimate between the problems ($\mathrm{P}^{\varepsilon}_{2,h}$) and the original problem (P), e.g. interior maximum norm estimates for finite element approximations to solutions $y = S\tau^* u$. Among other results, we show the uniform boundedness of the optimal discrete control $\bar{u}^{\varepsilon}_h$ in $H^1(\Gamma)$ w.r.t. h and ε. This is important for applying the interpolation error estimates of Lemma 5.11 and Lemma 5.12, respectively.

5.3.1 Approximation error estimates

During the discussion of feasibility of controls for the problems (P) or ($\mathrm{P}^{\varepsilon}_{2,h}$), respectively, the error $\|(S - S_h)\tau^* u\|_{L^\infty(\Omega')}$ will often arise. Thus, we state the following lemma.

Lemma 5.17 *Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. For every $u \in L^2(\Gamma)$, there exists a positive constant c, independent of h, and a mesh size $0 < h_0 < 1$, such that*

$$\|(S - S_h)\tau^* u\|_{L^\infty(\Omega')} \leq ch^{3/2}\|u\|_{L^2(\Gamma)} \tag{5.31}$$

is valid for all $h \leq h_0$. For every $u \in H^{1/2}(\Gamma)$, there exists a positive constant c, independent of h, and a mesh size $0 < h_0 < 1$ such that

$$\|(S - S_h)\tau^* u\|_{L^\infty(\Omega')} \leq ch^2|\log h|\|u\|_{H^{1/2}(\Gamma)} \tag{5.32}$$

is satisfied for all $h \leq h_0$.

Proof: Throughout the proof, we use abbreviations $y = S\tau^* u$ and $y_h = S_h\tau^* u$. By the definition (2.8) of the solution operator, y solves the variational formulation (5.6) for $f \equiv 0$ and $g := u$. Moreover, y_h denotes the associated discrete counterpart that solves (5.7). Due to Theorem 5.6, there is a mesh size $h_0 < 1$ such that we obtain

$$\|y - y_h\|_{L^\infty(\Omega')} \leq c(|\log h|\|y - I_h y\|_{L^\infty(\Omega'')} + \|y - y_h\|_{L^2(\Omega)}),$$

for $0 < h \leq h_0$. Moreover, I_h denotes the usual nodal interpolation operator. Due to Corollary 2.23, the weak solution $y = S\tau^* u$ belongs to $W^{2,\infty}(\Omega'')$. Thanks to the interpolation error estimate (5.2) and the estimate given in Corollary 2.23, we arrive at

$$\|y - I_h y\|_{L^\infty(\Omega'')} \leq ch^2\|y\|_{W^{2,\infty}(\Omega'')} \leq ch^2\|u\|_{L^2(\Gamma)}.$$

The second error is estimated the standard finite element approximation error estimates (5.10) and (5.8), respectively. This completes the proof. □

Lemma 5.18 *Let μ^{ε}_h be the Lagrange multiplier associated with the mixed control-state constraints in the optimality system (5.22)-(5.26). Then, there is a positive constant c, independent of h and ε, and a mesh size $h_0 < 1$ such that*

$$\|(S^* - S^*_h)\mu^{\varepsilon}_h\|_{L^2(\Gamma)} \leq ch^{3/2}$$

is valid for all $0 < h \leq h_0$.

Proof: Throughout the proof, we use abbreviations $p = S^*\mu_h^\varepsilon$ and $p_h = S_h^*\mu_h^\varepsilon$. By the definitions of the solution operators and $\text{supp}\{\mu_h^\varepsilon\} = \Omega'$, we have

$$p = S^*\mu_h^\varepsilon \quad \Longleftrightarrow \quad a(z,p) = \int\limits_{\Omega'} \mu_h^\varepsilon z dx \quad \forall z \in H^1(\Omega)$$

$$p_h = S_h^*\mu_h^\varepsilon \quad \Longleftrightarrow \quad a(z_h,p_h) = \int\limits_{\Omega'} \mu_h^\varepsilon z_h dx \quad \forall z \in V_h$$

with the bilinear form $a(\cdot,\cdot)$ defined in (2.2). Modifying the idea of Casas in [16, Theorem 3], we introduce first the dual problems:

$$a(w,z) = \int\limits_{\Gamma} f z\, ds \quad \forall z \in H^1(\Omega), \qquad a(w_h,z_h) = \int\limits_{\Gamma} f z_h\, ds \quad \forall z_h \in V_h$$

With the help of these dual problems and Galerkin orthogonality, we continue with

$$\begin{aligned}
\|(p-p_h)\|_{L^2(\Gamma)} &= \sup_{f\in L^2(\Gamma)} \frac{\left|\int\limits_{\Gamma}(p-p_h)f\, ds\right|}{\|f\|_{L^2(\Gamma)}} \\
&= \sup_{f\in L^2(\Gamma)} \frac{|a(w,p-p_h)|}{\|f\|_{L^2(\Gamma)}} \\
&= \sup_{f\in L^2(\Gamma)} \frac{|a(w-w_h,p)|}{\|f\|_{L^2(\Gamma)}} \\
&= \sup_{f\in L^2(\Gamma)} \frac{\left|\int\limits_{\Omega'}\mu_h^\varepsilon(w-w_h)dx\right|}{\|f\|_{L^2(\Gamma)}} \\
&\le \sup_{f\in L^2(\Gamma)} \frac{\|w-w_h\|_{L^\infty(\Omega')}\|\mu_h^\varepsilon\|_{L^1(\Omega')}}{\|f\|_{L^2(\Gamma)}}.
\end{aligned}$$

According to the definition of the solution operator S, we find $w = S\tau^* f$ and $w_h = S_h\tau^* f$. Due to Lemma 5.17, we derive for the error in the dual problem

$$\|w - w_h\|_{L^\infty(\Omega')} \le ch^{3/2}\|f\|_{L^2(\Gamma)}$$

for $0 < h \le h_0$, where $h_0 < 1$ is sufficiently small. Thus, the assertion is proven since the multiplier μ_h^ε is uniformly bounded in $L^1(\Omega')$ w.r.t. ε and h, see Lemma 5.16. □

5.3.2 Boundedness of the discrete variables

We start with the discussion of the uniform boundedness of the discrete adjoint state p_h^ε with respect to ε and h.

Lemma 5.19 *Let $p_h^\varepsilon \in V_h$ be the associated discrete adjoint state in the optimality system* (5.22)-(5.26)*. Then, there is positive constant C, independent of ε and h, such that*

$$\|p_h^\varepsilon\|_{H^1(\Gamma)} \le C. \tag{5.33}$$

Proof: First, we introduce the abbreviation $p^\varepsilon = S^*(\bar{y}_h^\varepsilon - y_d - \mu_h^\varepsilon)$. Note, that the discrete multiplier is located in $\Omega' \subset\subset \Omega$. Due to Theorem 2.18, Lemma 2.19 and the Trace Theorem 1.5 we obtain

$$\|p^\varepsilon\|_{H^1(\Gamma)} \le c(\|\bar{y}_h^\varepsilon - y_d\|_{L^2(\Omega)} + \|\mu_h^\varepsilon\|_{\mathcal{M}(\Omega')}).$$

A similar result is proven in Corollary 4.5. Due to the optimality of $\bar{y}_h^\varepsilon$, the term $\|\bar{y}_h^\varepsilon - y_d\|_{L^2(\Omega)}$ is bounded by the cost functional of problem $(\mathrm{P}_{2,h}^\varepsilon)$. Furthermore, Lemma 5.16 provides an upper bound for $\|\mu_h^\varepsilon\|_{\mathcal{M}(\Omega')}$ such that p^ε is uniformly bounded in $H^1(\Gamma)$ w.r.t. h and ε. Forthcoming, we apply the triangle inequality

$$\|p_h^\varepsilon\|_{H^1(\Gamma)} \le \|p_h^\varepsilon - \Pi_h p^\varepsilon\|_{H^1(\Gamma)} + \|\Pi_h p^\varepsilon\|_{H^1(\Gamma)}.$$

The stability of Π_h in the H_0^1-seminorm is shown in [15, Theorem 3.1] for functions defined in the domain. The arguments are based on the fact that $\sum_i \nabla\psi_i(x) = 0$ and can easily be adapted to the boundary case. Together with the boundedness of p^ε in $H^1(\Gamma)$, the second term is bounded by a constant independent of h and ε. With the help of the inverse inequality (5.4) and the triangle inequality, the first term is estimated by

$$\begin{aligned}
\|p_h^\varepsilon - \Pi_h p^\varepsilon\|_{H^1(\Gamma)} &\le ch^{-1}\|p_h^\varepsilon - \Pi_h p^\varepsilon\|_{L^2(\Gamma)} \\
&\le ch^{-1}(\|p_h^\varepsilon - p^\varepsilon\|_{L^2(\Gamma)} + \|p^\varepsilon - \Pi_h p^\varepsilon\|_{L^2(\Gamma)}) \\
&\le ch^{-1}\big(\|(S^* - S_h^*)(\bar{y}_h^\varepsilon - y_d)\|_{L^2(\Gamma)} + \|(S^* - S_h^*)\mu_h^\varepsilon\|_{L^2(\Gamma)} \\
&\qquad + \|p^\varepsilon - \Pi_h p^\varepsilon\|_{L^2(\Gamma)}\big).
\end{aligned}$$

Using a standard finite element error estimate and Lemma 5.18, we obtain for the first two errors

$$\|(S^* - S_h^*)(\bar{y}_h^\varepsilon - y_d)\|_{L^2(\Gamma)} + \|(S^* - S_h^*)\mu_h^\varepsilon\|_{L^2(\Gamma)} \le \\
ch^{3/2}(\|\bar{y}_h^\varepsilon - y_d\|_{L^2(\Omega)} + \|\mu_h^\varepsilon\|_{L^1(\Omega')}).$$

The interpolation error estimate in Lemma 5.11, applied to the last term, completes the proof since the remaining terms are uniformly bounded with respect to h and ε in the particular norms. $\square$

The next results are devoted to the uniform boundedness of the discrete optimal control $\bar{u}_h^\varepsilon$ in $H^1(\Gamma)$ w.r.t. ε and h. To this end, we investigate the projections on the convex sets U_{ad}^L and $U_{ad,h}^L$, respectively. For a given $\bar{w} \in L^2(\Gamma)$, $P(\bar{w}) \in U_{ad}^L$ denotes the solution of the least squares problem

$$\min_{w \in U_{ad}^L} \frac{1}{2}\|w - \bar{w}\|_{L^2(\Gamma)}^2.$$

It is well known, that $P(\bar{w}) \in U_{ad,h}^L$ is equivalent to the solution of the following variational inequality

$$(P(\bar{w}) - \bar{w}\,,\, w - P(\bar{w}))_{L^2(\Gamma)} \ge 0 \qquad \forall w \in U_{ad}^L. \tag{5.34}$$

The discrete counterpart $P_h(\bar{w})$ for a given $\bar{w} \in L^2(\Gamma)$ denotes the solution of

$$\min_{w_h \in U_{h,ad}^L} \frac{1}{2}\|w_h - \bar{w}\|_{L^2(\Gamma)}^2,$$

which is equivalent to

$$(P_h(\bar{w}) - \bar{w}\,,\, w_h - P_h(\bar{w}))_{L^2(\Gamma)} \geq 0 \qquad \forall w_h \in U^L_{ad,h}. \tag{5.35}$$

Lemma 5.20 *Let $\bar{w} \in H^1(\Gamma)$ be given. Furthermore, let $P(\bar{w})$ be the solution of (5.34) and let $P_h(\bar{w})$ be the solution of (5.35), respectively. Then, there exists a positive constant c depending on $\|\bar{w}\|_{H^1(\Gamma)}$, the boundary Γ and u_a, u_b such that*

$$\|P_h(\bar{w}) - P(\bar{w})\|_{L^2(\Gamma)} \leq ch \tag{5.36}$$

is valid.

Proof: The starting point of this proof is the variational inequality (5.34) and (5.35), respectively. Clearly, $P_h(\bar{w})$ is feasible for (5.34). Since $P(\bar{w})$ is the solution of the variational inequality (5.34) and the operator quasi-interpolation operator Π_h, defined in (5.16), preserves the validity of the inequality constraints of U^L_{ad}, we have $\Pi_h(P(\bar{w})) \in U^L_{ad,h}$. Thus, we are allowed to choose $\Pi_h(P(\bar{w}))$ as a feasible function in (5.35). Adding both inequalities, yields

$$\begin{aligned}
0 &\leq (P(\bar{w}) - \bar{w}\,,\, P_h(\bar{w}) - P(\bar{w}))_{L^2(\Gamma)} + (P_h(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P_h(\bar{w}))_{L^2(\Gamma)} \\
&= (P(\bar{w}) - P_h(\bar{w})\,,\, P_h(\bar{w}) - P(\bar{w}))_{L^2(\Gamma)} + (P_h(\bar{w}) - \bar{w}\,,\, P_h(\bar{w}) - P(\bar{w}))_{L^2(\Gamma)} \\
&\quad + (P_h(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P_h(\bar{w}))_{L^2(\Gamma)} \\
&= -\|P_h(\bar{w}) - P(\bar{w})\|^2_{L^2(\Gamma)} + (P_h(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P(\bar{w}))_{L^2(\Gamma)}
\end{aligned}$$

We continue with

$$\begin{aligned}
\|P_h(\bar{w}) - P(\bar{w})\|^2_{L^2(\Gamma)} &\leq (P_h(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P(\bar{w}))_{L^2(\Gamma)} \\
&= (P_h(\bar{w}) - P(\bar{w})\,,\, \Pi_h(P(\bar{w})) - P(\bar{w}))_{L^2(\Gamma)} \\
&\quad + (P(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P(\bar{w}))_{L^2(\Gamma)}
\end{aligned}$$

Applying Young's inequality to the first term, we obtain

$$\begin{aligned}
\frac{1}{2}\|P_h(\bar{w}) - P(\bar{w})\|^2_{L^2(\Gamma)} &\leq \frac{1}{2}\|\Pi_h(P(\bar{w})) - P(\bar{w})\|^2_{L^2(\Gamma)} \\
&\quad + \|\Pi_h(P(\bar{w})) - P(\bar{w})\|_{(H^1(\Gamma))^*}\|P(\bar{w}) - \bar{w}\|_{H^1(\Gamma)}.
\end{aligned}$$

With the help of the approximation error estimates (5.11) and (5.12), we derive

$$\frac{1}{2}\|P_h(\bar{w}) - P(\bar{w})\|^2_{L^2(\Gamma)} \leq ch^2\|P(\bar{w})\|^2_{H^1(\Gamma)} + ch^2\|P(\bar{w}) - \bar{w}\|_{H^1(\Gamma)}\|P(\bar{w})\|_{H^1(\Gamma)}.$$

Due to $\bar{w} \in H^1(\Gamma)$ and the estimate of Lemma 2.21, we derive

$$\|P_h(\bar{w}) - P(\bar{w})\|_{L^2(\Gamma)} \leq ch$$

with a positive constant c, that depends linearly on $\|\bar{w}\|_{H^1(\Gamma)}$, the boundary Γ and u_a, u_b. □

An immediate consequence of the previous result is the uniform boundedness of the discrete optimal control $\bar{u}^\varepsilon_h$ in $H^1(\Gamma)$ w.r.t. ε and h.

Lemma 5.21 *Let $\bar{u}_h^\varepsilon \in U_{ad,h}^L$ be the discrete optimal control determined by the optimality system* (5.22)-(5.26)*. Then, there exists a positive constant C, independent of h and ε, such that*

$$\|\bar{u}_h^\varepsilon\|_{H^1(\Gamma)} \leq C$$

is satisfied.

Proof: It is well known that the variational inequality (5.24) can be interpreted as the projection of $-p_\varepsilon^h/\nu$ on the convex set $U_{ad,h}^L$, e.g.

$$\bar{u}_h^\varepsilon = P_h(-p_\varepsilon^h/\nu).$$

According to the variational inequality (5.34), we introduce the projection of $-p_\varepsilon^h/\nu$ on the convex set U_{ad}^L, i.e. $P(-p_\varepsilon^h/\nu)$. By the triangle inequality, we find

$$\|\bar{u}_h^\varepsilon\|_{H^1(\Gamma)} \leq \|P_h(-p_\varepsilon^h/\nu) - \Pi_h(P(-p_\varepsilon^h/\nu))\|_{H^1(\Gamma)} + \|\Pi_h(P(-p_\varepsilon^h/\nu))\|_{H^1(\Gamma)} \quad (5.37)$$

with the quasi-interpolation operator Π_h defined in (5.16). Thanks to Lemma 2.21 and Lemma 5.19, we infer

$$\|P(-p_\varepsilon^h/\nu)\|_{H^1(\Gamma)} \leq C \quad \text{and} \quad \|\Pi_h(P(-p_\varepsilon^h/\nu))\|_{H^1(\Gamma)} \leq C$$

with certain positive constants C independent of ε and h. For the boundedness of the quasi-interpolation operator, we refer again to [15, Theorem 3.1], where the stability is shown for functions defined in the domain. Using a standard inverse estimate for the first term in (5.37), we continue with

$$\begin{aligned} \|P_h(p_\varepsilon^h) - \Pi_h(P(p_\varepsilon^h))\|_{H^1(\Gamma)} &\leq ch^{-1}\|P_h(p_\varepsilon^h) - \Pi_h(P(p_\varepsilon^h))\|_{L^2(\Gamma)} \\ &\leq ch^{-1}\big(\|P_h(p_\varepsilon^h) - P(p_\varepsilon^h)\|_{L^2(\Gamma)} \\ &\quad +\|\Pi_h(P(p_\varepsilon^h)) - P(p_\varepsilon^h)\|_{L^2(\Gamma)}\big) \end{aligned}$$

Thanks to (5.36), the first term can be estimated by

$$\|P_h(p_\varepsilon^h) - P(p_\varepsilon^h)\|_{L^2(\Gamma)} \leq ch,$$

where the constant c depends on $\|p_\varepsilon^h\|_{H^1(\Gamma)}$ which is uniformly bounded w.r.t. h and ε, see Lemma 5.19. The second error is estimated by (5.11) such that

$$\|\Pi_h(P(p_\varepsilon^h)) - P(p_\varepsilon^h)\|_{L^2(\Gamma)} \leq ch\|P(p_\varepsilon^h)\|_{H^1(\Gamma)}.$$

Finally, Lemma 2.21 and again Lemma 5.19 yield a bound for $\|P(p_\varepsilon^h)\|_{H^1(\Gamma)}$ independent of the meshsize h and ε. Consequently, we derive

$$\|P_h(p_\varepsilon^h) - \Pi_h(P(p_\varepsilon^h))\|_{H^1(\Gamma)} \leq C$$

with some positive constant C independent of ε and h. In conclusion, we obtain the uniform boundedness of $\bar{u}_h^\varepsilon$ in $H^1(\Gamma)$ w.r.t. ε and h. □

We mention, that, for the case $\Omega \subset \mathbb{R}^2$ and Ω also convex polygonally bounded, a different proof for the stability of $\bar{u}_h^\varepsilon$ in $H^1(\Gamma)$ can be found in a work of Casas and Raymond, see [22].

Finally, we derive an a priori bound for the discrete state $\tilde{y}_h := S_h\tau^*\bar{u}_h^\varepsilon$ in the space of Lipschitz continuous functions $C^{0,1}(\bar{\Omega}')$, which is needed in subsequent estimates. We start with the following inverse estimate:

Lemma 5.22 *Let $z_h \in V_h$. Then, there exists a positive constant c, independent of the mesh size h, such that the estimate*

$$\|z_h\|_{C^{0,1}(\bar{\Omega})} \leq ch^{-1}\|z_h\|_{L^\infty(\Omega)} \tag{5.38}$$

is valid.

The proof of this inverse estimate is quite standard and we skip it. We proceed with the a priori bound for the state $\tilde{y}_h = S_h\tau^*\bar{u}_h^\varepsilon$ in the space of Lipschitz continuous functions.

Lemma 5.23 *Let $\tilde{y}_h = S_h\tau^*\bar{u}_h^\varepsilon$, where $\bar{u}_h^\varepsilon$ is the optimal control of problem $(\mathrm{P}_{2,h}^\varepsilon)$. Then, there exists a positive constant C, independent of h and ε, such that*

$$\|\tilde{y}_h\|_{C^{0,1}(\bar{\Omega}')} \leq C. \tag{5.39}$$

Proof: We introduce the continuous counterpart $\tilde{y} := S\tau^*\bar{u}_h^\varepsilon$ to $\tilde{y}_h$. Due to Corollary 2.23, we have $\tilde{y} \in W^{2,\infty}(\Omega')$ and the estimate

$$\|\tilde{y}\|_{W^{2,\infty}(\Omega')} \leq c\|\tilde{y}\|_{L^2(\Omega)} \leq c\|\bar{u}_h^\varepsilon\|_{L^2(\Gamma)} \tag{5.40}$$

is valid. Moreover, we obtain

$$\|\tilde{y}\|_{C^{0,1}(\bar{\Omega}')} \leq c\|\tilde{y}\|_{W^{2,\infty}(\Omega')}$$

since $C^{0,1}(\bar{\Omega}')$ is embedded in $W^{2,\infty}(\Omega')$. By the use of the nodal interpolation $I_h\tilde{y}$, it turns out that

$$\begin{aligned}
\|\tilde{y}_h\|_{C^{0,1}(\bar{\Omega}')} &= \|\tilde{y}_h - I_h\tilde{y} + I_h\tilde{y}\|_{C^{0,1}(\bar{\Omega}')} \\
&\leq \|\tilde{y}_h - I_h\tilde{y}\|_{C^{0,1}(\bar{\Omega}')} + \|\tilde{y}\|_{C^{0,1}(\bar{\Omega}')} \\
&\leq \|\tilde{y}_h - I_h\tilde{y}\|_{C^{0,1}(\bar{\Omega}')} + c\|\bar{u}_h^\varepsilon\|_{L^2(\Gamma)}.
\end{aligned}$$

The inverse estimate of Lemma 5.22 and the triangle inequality yield

$$\begin{aligned}
\|\tilde{y}_h - I_h\tilde{y}\|_{C^{0,1}(\bar{\Omega}')} &\leq Ch^{-1}\|\tilde{y}_h - I_h\tilde{y}\|_{L^\infty(\Omega')} \\
&\leq ch^{-1}(\|\tilde{y}_h - \tilde{y}\|_{L^\infty(\Omega')} + \|\tilde{y} - I_h\tilde{y}\|_{L^\infty(\Omega')}).
\end{aligned}$$

For first error it is sufficient to use (5.31) such that

$$\|\tilde{y}_h - \tilde{y}\|_{L^\infty(\Omega')} \leq ch^{3/2}\|\bar{u}_h^\varepsilon\|_{L^2(\Gamma)}.$$

Thanks to the interpolation error estimate (5.2) and (5.40), we derive

$$\|\tilde{y} - I_h\tilde{y}\|_{L^\infty(\Omega')} \leq ch^2\|\tilde{y}\|_{W^{2,\infty}(\Omega')} \leq ch^2\|\bar{u}_h^\varepsilon\|_{L^2(\Gamma)}.$$

Due to the optimality of $\bar{u}_h^\varepsilon$, the remaining term is bounded by the objective functional of $(\mathrm{P}_{2,h}^\varepsilon)$ and the assertion is proven. $\square$

5.4 Convergence analysis for the discretized and regularized problem

In this section we establish an error estimate between the solution of the original problem (P) and solution of the discretized and regularized problem ($\mathrm{P}^{\varepsilon}_{2,h}$). Again, the strategy is based on the so-called two-way feasibility, i.e. the determination of suitable feasible controls for the particular other problem. We start with a basic estimate that directly results from the optimality conditions of problem (P) and ($\mathrm{P}^{\varepsilon}_{2,h}$), respectively.

Theorem 5.24 *Let $(\bar y, \bar u)$ and $(\bar y^{\varepsilon}_h, \bar u^{\varepsilon}_h, \bar v^{\varepsilon}_h)$ be the optimal solutions of* (P) *and* ($\mathrm{P}^{\varepsilon}_{2,h}$)*, respectively. For all $u^{\delta} \in U_{ad}$ and $(u^{\sigma}_h, 0) \in V^{\varepsilon,2}_{ad,h}$, there holds*

$$
\begin{aligned}
\frac{\nu}{2}\,\|\bar u - \bar u^{\varepsilon}_h\|^2_{L^2(\Gamma)} + \frac{1}{2}\|\bar y - \bar y^{\varepsilon}_h\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar v^{\varepsilon}_h\|^2_{L^2(\Omega)}& \\
\le c\Big((\tau\bar p + \nu\bar u\,,\, u^{\delta} - \bar u^{\varepsilon}_h)_{L^2(\Gamma)} + (\tau\bar p^{\varepsilon}_h + \nu\bar u^{\varepsilon}_h\,,\, u^{\sigma}_h - \bar u)_{L^2(\Gamma)}& \\
+\frac{(\phi(\varepsilon))^2}{\psi(\varepsilon)} + h^3\Big)&
\end{aligned}
\tag{5.41}
$$

for a certain constant $c > 0$ independent of h and ε.

Before proving this result, we will state two auxiliary results.

Lemma 5.25 *Let the assumptions of Theorem 5.24 be fulfilled. Then, we find*

$$
\begin{aligned}
(\bar y - \bar y^{\varepsilon}_h\,,\, S_h\tau^*(\bar u^{\varepsilon}_h - \bar u))_{L^2(\Omega)} = -\|\bar y - \bar y^{\varepsilon}_h\|^2 + (\bar y - \bar y^{\varepsilon}_h\,,\,(S - S_h)\tau^*\bar u)_{L^2(\Omega)} +& \\
(E_H S^*_h(\bar y - \bar y^{\varepsilon}_h)\,,\, -\phi(\varepsilon)\bar v^{\varepsilon}_h)_{L^2(\Omega)}.&
\end{aligned}
\tag{5.42}
$$

Proof: Due to $\bar y = S\tau^*\bar u$ and $\bar y^{\varepsilon}_h = S_h(\tau^*\bar u^{\varepsilon}_h + \phi(\varepsilon)E^*_H\bar v^{\varepsilon}_h)$, we derive

$$
\begin{aligned}
(\bar y - \bar y^{\varepsilon}_h\,,\, S_h\tau^*(\bar u^{\varepsilon}_h - \bar u))_{L^2(\Omega)} &= (\bar y - \bar y^{\varepsilon}_h\,,\, S_h(\tau^*\bar u^{\varepsilon}_h + E^*_H\phi(\varepsilon)\bar v^{\varepsilon}_h) - S_h\tau^*\bar u)_{L^2(\Omega)} - \\
&\quad (\bar y - \bar y^{\varepsilon}_h\,,\, S_h E^*_H\phi(\varepsilon)\bar v^{\varepsilon}_h)_{L^2(\Omega)} \\
&= (\bar y - \bar y^{\varepsilon}_h\,,\, \bar y^{\varepsilon}_h - \bar y)_{L^2(\Omega)} + (\bar y - \bar y^{\varepsilon}_h\,,\, \bar y - S_h\tau^*\bar u)_{L^2(\Omega)} + \\
&\quad (E_H S^*_h(\bar y - \bar y^{\varepsilon}_h)\,,\, -\phi(\varepsilon)\bar v^{\varepsilon}_h)_{L^2(\Omega)} \\
&= -\|\bar y - \bar y^{\varepsilon}_h\|^2 + (\bar y - \bar y^{\varepsilon}_h\,,\,(S - S_h)\tau^*\bar u)_{L^2(\Omega)} + \\
&\quad (E_H S^*_h(\bar y - \bar y^{\varepsilon}_h)\,,\, -\phi(\varepsilon)\bar v^{\varepsilon}_h)_{L^2(\Omega)}.
\end{aligned}
$$

□

Lemma 5.26 *Let the assumptions of Theorem 5.24 be fulfilled. Then, we obtain*

$$
\begin{aligned}
(\tau(\bar p - \bar p^{\varepsilon}_h)\,,\, \bar u^{\varepsilon}_h - \bar u)_{L^2(\Gamma)} &= -\|\bar y - \bar y^{\varepsilon}_h\|^2 + (\tau(S^* - S^*_h)(\bar y - y_d)\,,\, \bar u^{\varepsilon}_h - \bar u)_{L^2(\Gamma)} + \\
&\quad (\bar y - \bar y^{\varepsilon}_h\,,\,(S - S_h)\tau^*\bar u)_{L^2(\Omega)} + \\
&\quad (E_H S^*_h(\bar y - \bar y^{\varepsilon}_h)\,,\, -\phi(\varepsilon)\bar v^{\varepsilon}_h)_{L^2(\Omega)},
\end{aligned}
$$

where $\bar p$ and $\bar p^{\varepsilon}_h$ are the associated adjoint states.

Proof: Due to the definitions

$$\bar{p} = S^*(S\tau^*\bar{u} - y_d)$$
$$\bar{p}_h^\varepsilon = S_h^*(S_h(\tau^*\bar{u}_h^\varepsilon + E_H^*\phi(\varepsilon)\bar{v}_h^\varepsilon) - y_d)$$

of the adjoint states, we continue with

$$(\tau(\bar{p} - \bar{p}_h^\varepsilon)\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)} = -(\tau(S^* - S_h^*)y_d\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)} + \\ (\tau S^* S\tau^*\bar{u} - \tau S_h^*(S_h(\tau^*\bar{u}_h^\varepsilon + \phi(\varepsilon)E_H^*\bar{v}_h^\varepsilon))\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)}.$$

The last term in the right hand side can be rewritten as:

$$\begin{aligned}
&(S\tau^*\bar{u}\,,\, S\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} - (S_h(\tau^*\bar{u}_h^\varepsilon + \phi(\varepsilon)E_H^*\bar{v}_h^\varepsilon)\,,\, S_h\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} \\
&\quad = (S\tau^*\bar{u}\,,\, S\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} - (S\tau^*\bar{u}\,,\, S_h\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} + \\
&\qquad (S\tau^*\bar{u}\,,\, S_h\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} - (S_h(\tau^*\bar{u}_h^\varepsilon + \phi(\varepsilon)E_H^*\bar{v}_h^\varepsilon)\,,\, S_h\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} \\
&\quad = (S\tau^*\bar{u}\,,\, (S - S_h)\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} + \\
&\qquad (S\tau^*\bar{u} - S_h(\tau^*\bar{u}_h^\varepsilon + E_H^*\phi(\varepsilon)\bar{v}_h^\varepsilon)\,,\, S_h\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} \\
&\quad = (\bar{y}\,,\, (S - S_h)\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} + (\bar{y} - \bar{y}_h^\varepsilon\,,\, S_h\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)}.
\end{aligned}$$

By the use of Lemma 5.25 for the last term, we conclude

$$\begin{aligned}
(\tau(\bar{p} - \bar{p}_h^\varepsilon)\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)} = {}& -(\tau(S^* - S_h^*)y_d\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)} + \\
& (\bar{y}\,,\, (S - S_h)\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} - \|\bar{y} - \bar{y}_h^\varepsilon\|^2 \\
& (\bar{y} - \bar{y}_h^\varepsilon\,,\, (S - S_h)\tau^*\bar{u})_{L^2(\Omega)} + \\
& (E_H S_h^*(\bar{y} - \bar{y}_h^\varepsilon)\,,\, -\phi(\varepsilon)\bar{v}_h^\varepsilon)_{L^2(\Omega)}.
\end{aligned}$$

The first two terms in the previous right hand side simplify to

$$(\bar{y}\,,\, (S - S_h)\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} - (\tau(S^* - S_h^*)y_d\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)} = \\ (\tau(S^* - S_h^*)(\bar{y} - y_d)\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)}.$$

This completes the proof. □

Now, we prove the result of Theorem 5.24.

Proof of Theorem 5.24: We start with the variational inequalities of (P) and ($P_{2,h}^\varepsilon$) for $u := u^\delta \in U_{ad}$ and $(u, v) := (u_h^\sigma, 0) \in V_{ad,h}^{\varepsilon,2}$, see (2.18) and (5.20), respectively. Adding both inequalities and appropriate modifications similarly as in the beginning of the proof for Lemma 3.4, yields

$$\nu\,\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)}^2 + \psi(\varepsilon)\|\bar{v}_h^\varepsilon\|^2 \le (\tau\bar{p} + \nu\bar{u}\,,\, u^\delta - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} + (\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\,,\, u_h^\sigma - \bar{u})_{L^2(\Gamma)} \\ + (\tau(\bar{p} - \bar{p}_h^\varepsilon)\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)} + (\phi(\varepsilon)E_H\bar{p}_h^\varepsilon\,,\, -\bar{v}_h^\varepsilon)_{L^2(\Omega)}$$

for all $u^\delta \in U_{ad}$ and $(u_h^\sigma, 0) \in V_{ad,h}^{\varepsilon,2}$. It remains to consider the last two terms of the previous inequality. Thanks to Lemma 5.26, we obtain:

$$\begin{aligned}
\nu\,\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)}^2 &+ \|\bar{y} - \bar{y}_h^\varepsilon\|_{L^2(\Omega)}^2 + \psi(\varepsilon)\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^2 \\
\le\ & (\tau\bar{p} + \nu\bar{u}\,,\, u^\delta - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} + (\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\,,\, u_h^\sigma - \bar{u})_{L^2(\Gamma)} \\
& + (\tau(S^* - S_h^*)(\bar{y} - y_d)\,,\, \bar{u}_h^\varepsilon - \bar{u})_{L^2(\Gamma)} + (\bar{y} - \bar{y}_h^\varepsilon\,,\, (S - S_h)\tau^*\bar{u})_{L^2(\Omega)} \\
& + (E_H S_h^*(\bar{y} - \bar{y}_h^\varepsilon)\,,\, -\phi(\varepsilon)\bar{v}_h^\varepsilon)_{L^2(\Omega)} + (\phi(\varepsilon)E_H\bar{p}_h^\varepsilon\,,\, -\bar{v}_h^\varepsilon)_{L^2(\Omega)}.
\end{aligned} \tag{5.43}$$

Using $\bar{p}_h^\varepsilon = S_h^*(\bar{y}_h^\varepsilon - y_d)$, the last two terms simplify to

$$(E_H S_h^*(\bar{y} - \bar{y}_h^\varepsilon) + E_H \bar{p}_h^\varepsilon\,,\, -\phi(\varepsilon)\bar{v}_h^\varepsilon)_{L^2(\Omega)} = (E_H S_h^*(\bar{y} - y_d)\,,\, -\phi(\varepsilon)\bar{v}_h^\varepsilon)_{L^2(\Omega)}.$$

Moreover, we apply Young's inequality such that

$$(E_H S_h^*(\bar{y} - y_d)\,,\, -\phi(\varepsilon)\bar{v}_h^\varepsilon)_{L^2(\Omega)} \le \frac{\phi(\varepsilon)^2}{2\psi(\varepsilon)}\|E_H S_h^*(\bar{y} - y_d)\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar{v}_h^\varepsilon\|^2_{L^2(\Omega)}.$$

Again using Young's inequality to the third and the fourth term in (5.43), we derive

$$\begin{aligned}
\frac{\nu}{2}\,\|\bar{u} - \bar{u}_h^\varepsilon\|^2_{L^2(\Gamma)} &+ \frac{1}{2}\|\bar{y} - \bar{y}_h^\varepsilon\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar{v}_h^\varepsilon\|^2_{L^2(\Omega)} \\
&\le\ (\tau\bar{p} + \nu\bar{u}\,,\, u^\delta - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} + (\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\,,\, u_h^\sigma - \bar{u})_{L^2(\Gamma)} \\
&\quad + \frac{1}{2\nu}\|\tau(S^* - S_h^*)(\bar{y} - y_d)\|^2_{L^2(\Gamma)} + \frac{1}{2}\|(S - S_h)\tau^*\bar{u}\|^2_{L^2(\Omega)} \\
&\quad + \frac{\phi(\varepsilon)^2}{2\psi(\varepsilon)}\|E_H S_h^*(\bar{y} - y_d)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Standard finite element error estimates yield for the first two errors

$$\frac{1}{2\nu}\|\tau(S^* - S_h^*)(\bar{y} - y_d)\|^2_{L^2(\Gamma)} + \frac{1}{2}\|(S - S_h)\tau^*\bar{u}\|^2_{L^2(\Omega)} \le ch^3(\|\bar{y} - y_d\|_{L^2(\Omega)} + \|\bar{u}\|_{L^2(\Gamma)}).$$

Due to optimality, the remaining norms are bounded by the objective of the original problem (P). Finally, the embedding operator E_H and the adjoint of the discrete solution operator are linear and continuous such that the term $\|E_H S_h^*(\bar{y} - y_d)\|_{L^2(\Omega)}$ is bounded by constant independent of h and ε. □

5.4.1 Construction of feasible controls

Similarly to the previous chapters, we construct feasible control that are based on the optimal solution of the particular other problem. In the Chapters 3 and 4, the optimal control $\bar{u}$ of the original problem (P) was feasible for the regularized problem, see Lemma e.g. 3.5. Unfortunately, in the case of discretization of the controls this is in general not satisfied since $\bar{u} \notin U_h$. In order to construct a feasible control for $(\mathrm{P}^\varepsilon_{2,h})$, we will consider the violation of the control $(\Pi_h\bar{u}, \bar{v} \equiv 0)$ with respect to the mixed control-state constraints of the discretized problem $(\mathrm{P}^\varepsilon_{2,h})$, where Π_h is the quasi-interpolation operator given in (5.16). We define the violation function by

$$\begin{aligned}
d[(\Pi_h\bar{u}, 0), (\mathrm{P}^\varepsilon_{2,h})] &:= (y_c - S_h\tau^*\Pi_h\bar{u} - S_h E_H^*\phi(\varepsilon)0 - \xi(\varepsilon)0)_+ \\
&= \max\{0, y_c - S_h\tau^*\Pi_h\bar{u}\}.
\end{aligned} \tag{5.44}$$

The $L^\infty(\Omega')$-norm of (5.44) is called maximal violation of $(\Pi_h\bar{u}, 0)$ with respect to $(\mathrm{P}^\varepsilon_{2,h})$.

Lemma 5.27 *There is a sufficiently small mesh size $0 < h_0 < 1$ such that the maximal violation $\|d[(\Pi_h\bar{u}, 0), (\mathrm{P}^\varepsilon_{2,h})]\|_{L^\infty(\Omega')}$ of $(\Pi_h\bar{u}, 0)$ w.r.t. $(\mathrm{P}^\varepsilon_{2,h})$ can be estimated by*

$$\|d[(\Pi_h\bar{u}, 0), (\mathrm{P}^\varepsilon_{2,h})]\|_{L^\infty(\Omega')} \le ch^2|\log h| \tag{5.45}$$

for all $0 < h \le h_0$, where the constant $c > 0$ is independent of h and ε.

Proof: Using the triangle inequality and $\bar{y} = S\tau^*\bar{u}$, we find

$$
\begin{aligned}
\|d[(\Pi_h\bar{u}, 0), (P_\varepsilon^h)]\|_{L^\infty(\Omega')} &= \|(y_c - S_h\tau^*\Pi_h\bar{u})_+\|_{L^\infty(\Omega')} \\
&= \|(y_c - S\tau^*\bar{u} + S\tau^*(\bar{u} - \Pi_h\bar{u}) + (S - S_h)\tau^*\Pi_h\bar{u})_+\|_{L^\infty(\Omega')} \\
&\le \|(y_c - \bar{y})_+\|_{L^\infty(\Omega')} + \|S\tau^*(\bar{u} - \Pi_h\bar{u})\|_{L^\infty(\Omega')} + \\
&\quad\ \|(S - S_h)\tau^*\Pi_h\bar{u}\|_{L^\infty(\Omega')}.
\end{aligned}
$$

Due to the feasibility of $\bar{y}$ for problem (P), the first term vanishes. In Section 2.4 we have shown that the optimal control $\bar{u}$ belongs to $H^1(\Gamma)$, see (2.33). Thanks to Corollary 2.23, Lemma 2.24 and Lemma 5.12, we find for the second term

$$
\|S\tau^*(\bar{u} - \Pi_h\bar{u})\|_{L^\infty(\Omega')} \le c\|S\tau^*(\bar{u} - \Pi_h\bar{u})\|_{L^2(\Omega)} \le c\|\bar{u} - \Pi_h\bar{u}\|_{H^1(\Gamma)^*} \le ch^2\|\bar{u}\|_{H^1(\Gamma)}.
$$

The last term $\|(S - S_h)\tau^*\Pi_h\bar{u}\|_{L^\infty(\Omega')}$ is estimated by (5.32) such that

$$
\|(S - S_h)\tau^*\Pi_h\bar{u}\|_{L^\infty(\Omega')} \le ch^2|\log h|\|\Pi_h\bar{u}\|_{H^{1/2}(\Gamma)}
$$

for all $0 < h \le h_0$, where $0 < h_0 < 1$ is sufficiently small. The stability of the quasi-interpolation operator and the boundedness of $\bar{u}$ in $H^{1/2}(\Gamma)$ by (2.33) yield the assertion. □

Now, we construct a feasible solution u_h^σ for $(\mathrm{P}_{2,h}^\varepsilon)$ depending on the inner point $\hat{u}$ of Assumption 2.4 and the optimal solution $\bar{u}$ of problem (P).

Lemma 5.28 *Let Assumption 2.4 be satisfied. Then, for all sufficiently small mesh sizes h, there exists a $\sigma_h \in (0, 1)$ such that $(u_h^\sigma, 0)$ is feasible for $(\mathrm{P}_{2,h}^\varepsilon)$ for all $\sigma \in [\sigma_h, 1]$, where u_h^σ is defined by*

$$
u_h^\sigma := (1 - \sigma)\Pi_h\bar{u} + \sigma\Pi_h\hat{u}.
$$

Proof: Since the control constraints on the boundary are identical for both problems (P) and $(\mathrm{P}_{2,h}^\varepsilon)$ and the operator Π_h defined in (5.16) preserves the validity of this constraints, the convex linear combination $u_h^\sigma = (1 - \sigma)\Pi_h\bar{u} + \sigma\Pi_h\hat{u}$ satisfy the box constraints. Now, it suffices to check

$$
y_h^\sigma = S_h\tau^*u_h^\sigma \ge y_c \quad \text{a.e. in } \Omega'.
$$

Using the violation function (5.44) and Lemma 5.13, we continue with

$$
\begin{aligned}
y_h^\sigma - y_c &= (1 - \sigma)(S_h\tau^*\Pi_h\bar{u} - y_c) + \sigma(S_h\tau^*\Pi_h\hat{u} - y_c) \\
&\ge -(1 - \sigma)d[(\Pi_h\bar{u}, 0), (\mathrm{P}_{2,h}^\varepsilon)] + \sigma\gamma_0 \\
&\ge -(1 - \sigma)\|d[(\Pi_h\bar{u}, 0), (\mathrm{P}_{2,h}^\varepsilon)]\|_{L^\infty(\Omega')} + \sigma\gamma_0
\end{aligned}
$$

for sufficiently small mesh sizes h. Consequently, the choice

$$
\sigma_h := \frac{\|d[(\Pi_h\bar{u}, 0), (\mathrm{P}_{2,h}^\varepsilon)]\|_{L^\infty(\Omega')}}{\|d[(\Pi_h\bar{u}, 0), (\mathrm{P}_{2,h}^\varepsilon)]\|_{L^\infty(\Omega')} + \gamma_0} \in (0, 1) \tag{5.46}
$$

implies the assertion. □

Next, we consider the other direction. We introduce the violation function

$$
d[\bar{u}_h^\varepsilon, (P)] := (y_c - S\tau^*\bar{u}_h^\varepsilon)_+ \tag{5.47}
$$

with respect to the pure state constraints of the original problem (P). Of course, this is similarly to (3.11), where the continuous regularized control was considered.

Lemma 5.29 *The maximal violation $\|d[\bar{u}_h^\varepsilon,(P)]\|_{L^\infty(\Omega')}$ of $\bar{u}_h^\varepsilon$ w.r.t. problem (P) can be estimated by*

$$\|d[\bar{u}_h^\varepsilon,(P)]\|_{L^\infty(\Omega')} \le c\left((\xi(\varepsilon)+\phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^{2/(2+d)} + h^2|\log h| \right) \tag{5.48}$$

for mesh sizes $0<h\le h_0$, where $h_0<1$ is chosen sufficiently small and the constant $c>0$ is independent of ε and h.

Proof: The first step is done by the use of the triangle inequality:

$$\begin{aligned}\|d[\bar{u}_h^\varepsilon,(P)]\|_{L^\infty(\Omega')} &= \|(y_c - S\tau^*\bar{u}_h^\varepsilon)_+\|_{L^\infty(\Omega')}\\ &\le \|(y_c - S_h\tau^*\bar{u}_h^\varepsilon)_+\|_{L^\infty(\Omega')} + \|(S_h - S)\tau^*\bar{u}_h^\varepsilon\|_{L^\infty(\Omega')}.\end{aligned} \tag{5.49}$$

We recall the notation $\tilde{y}_h = S_h\tau^*\bar{u}_h^\varepsilon$ of Lemma 5.23. Since we required $y_c \in C^{0,1}(\bar{\Omega})$, we derive

$$\|(y_c-\tilde{y}_h)_+\|_{C^{0,1}(\bar{\Omega}')} \le \|y_c\|_{C^{0,1}(\bar{\Omega}')} + \|\tilde{y}_h\|_{C^{0,1}(\bar{\Omega}')}.$$

Due to Lemma 5.23, the function $(y_c-\tilde{y}_h)_+$ is uniformly bounded in $C^{0,1}(\bar{\Omega}')$ with respect to ε and h. We proceed with the use of Lemma 4.7 such that

$$\begin{aligned}\|(y_c-\tilde{y}_h)_+\|_{L^\infty(\Omega')} &\le c\|(y_c-\tilde{y}_h)_+\|_{L^2(\Omega')}^{2/(2+d)}\\ &\le c\left(\|(y_c - S_h\tau^*\bar{u}_h^\varepsilon - S_hE_2^*\phi(\varepsilon)\bar{v}_h^\varepsilon)_+\|_{L^2(\Omega')} + \|(S_hE_H^*\phi(\varepsilon)\bar{v}_h^\varepsilon)_+\|_{L^2(\Omega')}\right)^{2/(2+d)}\\ &= c\left(\|(y_c-\bar{y}_h^\varepsilon)_+\|_{L^2(\Omega')} + \|(S_hE_H^*\phi(\varepsilon)\bar{v}_h^\varepsilon)_+\|_{L^2(\Omega')}\right)^{2/(2+d)}.\end{aligned}$$

The optimality of $(\bar{y}_h^\varepsilon,\,\bar{u}_h^\varepsilon,\,\bar{v}_h^\varepsilon)$ for $(\mathrm{P}_{2,h}^\varepsilon)$ and the continuity of the discrete solution operator S_h yield

$$\begin{aligned}\|(y_c-\tilde{y}_h)_+\|_{L^\infty(\Omega')} &\le c\left(\|(\xi(\varepsilon)\bar{v}_h^\varepsilon)_+\|_{L^2(\Omega')} + \|(S_hE_H^*\phi(\varepsilon)\bar{v}_h^\varepsilon)_+\|_{L^2(\Omega')}\right)^{2/(2+d)}\\ &\le c\,(\xi(\varepsilon)+\phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^{2/(2+d)}.\end{aligned}$$

Due to Lemma 5.21, the control $\bar{u}_h^\varepsilon$ belongs to $H^1(\Gamma)$ and is uniformly bounded with respect to ε and h. Thus, we use the estimate (5.32) for the second term in (5.49) such that

$$\|(S_h-S)\tau^*\bar{u}_h^\varepsilon\|_{L^\infty(\Omega')} \le ch^2|\log h|$$

with a positive constant c independent of ε and h. This completes the proof. □
The following lemma provides feasible controls for the original problem based on the optimal discretized and regularized control $\bar{u}_h^\varepsilon$.

Lemma 5.30 *Let the Assumption 2.4 be satisfied. Then, for every $\varepsilon>0$ the control $u^\delta := (1-\delta)\bar{u}_h^\varepsilon + \delta\hat{u}$ is feasible for (P) for all $\delta\in[\delta_\varepsilon,1]$, where δ_ε is given by*

$$\delta_\varepsilon := \frac{\|d[\bar{u}_h^\varepsilon,(P)]\|_{L^\infty(\Omega')}}{\|d[\bar{u}_h^\varepsilon,(P)]\|_{L^\infty(\Omega')}+\gamma}. \tag{5.50}$$

The proof is done along the lines of Lemma 3.7.

5.4.2 Discretization and regularization error estimate

Now, we are in the position to derive an error estimate between the original problem (P) and the discretized and regularized one ($\mathrm{P}^{\varepsilon}_{2,h}$) since feasible controls for both of the problems were constructed in the previous section.

Theorem 5.31 *Let $(\bar{y}, \bar{u})$ and $(\bar{y}^{\varepsilon}_h, \bar{u}^{\varepsilon}_h, \bar{v}^{\varepsilon}_h)$ be the optimal solution of (P) and ($\mathrm{P}^{\varepsilon}_{2,h}$), respectively. Then, there exists a positive constant c, independent of the mesh size h and ε, and a mesh size $h_0 < 1$ such that*

$$\begin{aligned} \frac{\nu}{2}\|\bar{u} - \bar{u}^{\varepsilon}_h\|^2_{L^2(\Gamma)} + \frac{1}{2}\|\bar{y} - \bar{y}^{\varepsilon}_h\|^2_{L^2(\Omega)} + \frac{\psi(\varepsilon)}{2}\|\bar{v}^{\varepsilon}_h\|^2_{L^2(\Omega)} & \\ c\left((\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}^{\varepsilon}_h\|^{2/(2+d)}_{L^2(\Omega)} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} + h^2|\log h| \right) & \end{aligned} \tag{5.51}$$

is satisfied for all mesh sizes $0 < h \le h_0$ and regularization parameters $\varepsilon > 0$.

Proof: The basis of the proof is the estimate (5.41) given in Lemma 5.24. We start with choosing $u_\delta \in U_{ad}$ as defined in Lemma 5.30 connected with the specific parameter $\delta := \delta_\varepsilon$ given in (5.50). Moreover, for constants $c \ge 1/\gamma$ we obtain

$$\delta_\varepsilon \le c\|d[\bar{u}^{\varepsilon}_h, (P)]\|_{L^\infty(\Omega')}.$$

Due to the estimate (5.48) of Lemma 5.29, there exists a mesh size $h_0 < 1$ such that we derive

$$\begin{aligned} (\tau\bar{p} + \nu\bar{u}\,,\, u^{\delta} - \bar{u}^{\varepsilon}_h)_{L^2(\Gamma)} &\le \delta\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}\|\hat{u} - \bar{u}^{\varepsilon}_h\|_{L^2(\Gamma)} \\ &\le c\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}|\Gamma||u_b - u_a|\|d[\bar{u}^{\varepsilon}_h, (P)]\|_{L^\infty(\Omega')} \\ &\le c((\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}^{\varepsilon}_h\|^{2/(2+d)}_{L^2(\Omega)} + h^2|\log h|) \end{aligned}$$

for all $h \le h_0$. The optimality of $\bar{u}$ and $\bar{p}$ yield the boundedness of the term $\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}$. We proceed with the choice $(u^{\sigma}_h, 0) \in V^{\varepsilon,2}_{ad,h}$ given by Lemma 5.28 for $\sigma := \sigma_h$ defined in (5.46). Similarly to the estimate of δ_ε above, we obtain with (5.45)

$$\sigma_h \le c\|d[(\Pi_h\bar{u}, 0), (\mathrm{P}^{\varepsilon}_{2,h})]\|_{L^\infty(\Omega')} \le ch^2|\log h|$$

for all mesh sizes $0 < h \le h_0$, where $h_0 < 1$ is chosen sufficiently small. We continue with

$$\begin{aligned} (\tau\bar{p}^{\varepsilon}_h + \nu\bar{u}^{\varepsilon}_h\,,\, u^{\sigma}_h - \bar{u})_{L^2(\Gamma)} =& \,\sigma_h(\tau\bar{p}^{\varepsilon}_h + \nu\bar{u}^{\varepsilon}_h\,,\, \Pi_h\hat{u} - \Pi_h\bar{u})_{L^2(\Gamma)} + \\ & (\tau\bar{p}^{\varepsilon}_h + \nu\bar{u}^{\varepsilon}_h\,,\, \Pi_h\bar{u} - \bar{u})_{L^2(\Gamma)} \\ \le& \, ch^2|\log h|\|\tau\bar{p}^{\varepsilon}_h + \nu\bar{u}^{\varepsilon}_h\|_{L^2(\Gamma)}\|\Pi_h(\hat{u} - \bar{u})\|_{L^2(\Gamma)} + \\ & \|\tau\bar{p}^{\varepsilon}_h + \nu\bar{u}^{\varepsilon}_h\|_{H^1(\Gamma)}\|\Pi_h\bar{u} - \bar{u}\|_{H^1(\Gamma)^*}. \end{aligned}$$

Thanks to Lemma 5.19 and Lemma 5.21, $\bar{u}^{\varepsilon}_h$ and $\bar{p}^{\varepsilon}_h$ belong to $H^1(\Gamma)$. Furthermore, the functions are bounded by constants independent of h and ε. Due to (2.33), $\bar{u}$ is also an element in $H^1(\Gamma)$ and the estimate of Lemma 5.12 yields at last

$$(\tau\bar{p}^{\varepsilon}_h + \nu\bar{u}^{\varepsilon}_h\,,\, u^{\sigma}_h - \bar{u})_{L^2(\Gamma)} \le ch^2|\log h|.$$

Summarizing all, we obtain the assertion. □

Similarly to the undiscretized case, considered in Chapter 4, an estimate for the virtual control is necessary to complete the error estimate. At the same time, we will discuss the coupling between the mesh size and the parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$. We formulate the following assumption.

Assumption 5.32 The parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$ are chosen in way such that

$$\frac{\phi(\varepsilon)+\xi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \sim (h|\log h|^{1/2})^{1+d}. \tag{5.52}$$

We continue with an estimate of the L^2-norm of the discrete virtual control. Like in the previous chapters, the proof is based on the preliminary estimate given in Theorem 5.31.

Corollary 5.33 *Let the assumptions of Theorem 5.31 be fulfilled. Furthermore, let Assumption 5.32 be satisfied. Then, there exist a positive constant c, independent of h and ε, and a mesh size $h_0 < 1$ such that*

$$\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)} \le c\frac{h|\log h|^{1/2}}{\sqrt{\psi(\varepsilon)}} \tag{5.53}$$

is fulfilled for all mesh sizes $0 < h \le h_0$.

Proof: Considering (5.51), it turns out that the estimate

$$\frac{\psi(\varepsilon)}{2}\|\bar{v}_h^\varepsilon\|^2_{L^2(\Omega)} \le c\left((\xi(\varepsilon)+\phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}_h^\varepsilon\|^{2/(2+d)}_{L^2(\Omega)} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} + h^2|\log h|\right)$$

is valid for all mesh sizes $0 < h \le h_0$, where is $h_0 < 1$ is chosen sufficiently small. Due to Assumption 5.32, we infer

$$\frac{\psi(\varepsilon)}{2}\|\bar{v}_h^\varepsilon\|^2_{L^2(\Omega)} \le c\left(\left(h|\log h|^{1/2}\right)^{\frac{2(1+d)}{2+d}}(\sqrt{\psi(\varepsilon)}\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)})^{\frac{2}{2+d}} + h^2|\log h|\right).$$

Notice, that the term $\phi(\varepsilon)^2/\psi(\varepsilon)$ is neglected since Assumption 5.32 yields a higher order of convergence with respect to h. Forthcoming, the previous estimate implies

$$\|\bar{v}_h^\varepsilon\|^2_{L^2(\Omega)} \le \frac{c}{\psi(\varepsilon)}\max\left\{\left(h|\log h|^{1/2}\right)^{\frac{2(1+d)}{2+d}}(\sqrt{\psi(\varepsilon)}\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)})^{\frac{2}{2+d}}, h^2|\log h|\right\}.$$

We continue with considering the two cases, where the maximum can be attained.
Case1: We start with assuming that the maximum is given by $h^2|\log h|$. It turns out that the estimate

$$\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)} \le c\frac{h|\log h|^{1/2}}{\sqrt{\psi(\varepsilon)}}.$$

is valid.
Case2: Next, we assume that the maximum is attained by the first term. Consequently, we find

$$\begin{aligned}
\|\bar{v}_h^\varepsilon\|^{\frac{2(1+d)}{2+d}}_{L^2(\Omega)} &\le c(h|\log h|^{1/2})^{\frac{2(1+d)}{2+d}}(\psi(\varepsilon))^{-\frac{(1+d)}{2+d}} \\
\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)} &\le c\frac{h|\log h|^{1/2}}{\sqrt{\psi(\varepsilon)}}.
\end{aligned}$$

Summarizing, both cases imply the same order of convergence with respect to the mesh size h for $\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}$. Thus, the assertion is proven. □

Finally, we state the final error estimate of the optimal solution of problem (P) to the optimal discretized and regularized solution of problem $(\mathrm{P}_{2,h}^\varepsilon)$.

Theorem 5.34 *Let $(\bar{y}, \bar{u})$ and $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ be the optimal solution of (P) and $(\mathrm{P}_{2,h}^\varepsilon)$, respectively. Moreover, let Assumption 5.32 be satisfied. Then, there exist a positive constant c, independent of h, such that*

$$\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h^\varepsilon\|_{L^2(\Omega)} \le ch|\log h|^{1/2} \tag{5.54}$$

is fulfilled for all mesh sizes $0 < h \le h_0$, where $h_0 < 1$ is chosen sufficiently small.

Proof: This result is an immediate consequence of Theorem 5.31 and the estimate (5.53). According to (5.51), there exists a mesh size $h_0 < 1$ such that

$$\frac{\nu}{2}\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)}^2 + \frac{1}{2}\|\bar{y} - \bar{y}_h^\varepsilon\|_{L^2(\Omega)}^2 \le c\left((\xi(\varepsilon) + \phi(\varepsilon))^{\frac{2}{2+d}} \|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^{\frac{2}{2+d}} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} + h^2|\log h| \right)$$

is satisfied for all mesh sizes $0 < h \le h_0$ and regularization parameters $\varepsilon > 0$. Due to Assumption 5.32, the term $\phi(\varepsilon)^2/\psi(\varepsilon)$ is of higher order with respect to h and we neglect it. Thanks to the coupling (5.52) and the estimate (5.53) of the discrete virtual control, we derive the estimate

$$\begin{aligned}
\frac{\nu}{2}\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)}^2 &+ \frac{1}{2}\|\bar{y} - \bar{y}_h^\varepsilon\|_{L^2(\Omega)}^2 \\
&\le c\left((\xi(\varepsilon) + \phi(\varepsilon))^{\frac{2}{2+d}} \left(\frac{h|\log h|^{1/2}}{\sqrt{\psi(\varepsilon)}}\right)^{\frac{2}{2+d}} + h^2|\log h| \right) \\
&\le c\left(\left((h|\log h|^{1/2})^{1+d}\right)^{\frac{2}{2+d}} \left(h|\log h|^{1/2}\right)^{\frac{2}{2+d}} + h^2|\log h| \right) \\
&\le ch^2|\log h|.
\end{aligned}$$

This completes the proof. □

Remark 5.35 Particularly the last estimate in the previous proof shows that the coupling between the mesh size and the parameter functions seems to be optimal for the estimate given by (5.51). One can easily see, that the choice of a higher exponent than $1+d$ in the coupling (5.52) does not improve the order of convergence with respect to h. Moreover, this will cause a faster decreasing of the regularization effect if h tends to zero. On the other hand, a less exponent in (5.52) yields a lower order of convergence for the error between the problems (P) and $(\mathrm{P}_{2,h}^\varepsilon)$.

5.5 Finite element discretization of the unregularized problem - Error estimates

Finally, we consider the discretized and regularized problem $(\mathrm{P}_{2,h}^\varepsilon)$ for a specific choice of the parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$. Let us assume the following setting of parameter functions:

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) \equiv 0, \quad \xi(\varepsilon) \equiv 0.$$

Of course, this yields no occurrence of the virtual control in the discretized state equation and in the inequality constraints. Under notice of the minimization of the objective functional in $(\mathrm{P}^{\varepsilon}_{2,h})$, we obtain $\bar{v}^{\varepsilon}_h \equiv 0$ for every $\varepsilon > 0$. Hence, for every $\varepsilon > 0$ the problem $(\mathrm{P}^{\varepsilon}_{2,h})$ is equivalent to:

$$\left.\begin{array}{rc} \min\limits_{(y_h,u_h)\in V_h\times U_h} & J(y_h,u_h) := \dfrac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u_h\|^2_{L^2(\Gamma)} \\ & y_h = S_h(\tau^* u_h) \\ & u_a \le u_h(x) \le u_b \quad \text{a.e. on } \Gamma \\ & y_h(x) \ge y_c(x) \quad \text{a.e. in } \Omega'. \end{array}\right\} \qquad (\mathrm{P}_h)$$

This problem is a purely discretized analogon to the original optimal control problem (P). Due to Lemma 5.13, the admissible set

$$U_{ad,h} = \{u_h \in U_h : \, u_a \le u_h \le u_b \text{ a.e. on } \Gamma, \, S_h(\tau^* u_h)(x) \ge y_c(x) \text{ a.e. in } \Omega'\}$$

of problem (P_h) is nonempty. Consequently, the existence and uniqueness of an optimal solution, denoted by $(\bar{y}_h, \, \bar{u}_h)$, for the discretized problem (P_h) is obtained by standard arguments. The associated necessary and sufficient optimality condition is formulated in the following lemma.

Lemma 5.36 *Let $(\bar{y}_h, \, \bar{u}_h)$ be the optimal solution of problem (P_h). The optimality condition is given by*

$$(\tau\bar{p}_h + \nu\bar{u}_h \, , \, u - \bar{u}_h)_{L^2(\Gamma)} \ge 0 \quad \forall u \in U_{ad,h}, \tag{5.55}$$

where $\bar{p}_h = S^*_h(\bar{y}_h - y_d)$ denotes the discrete adjoint state.

We are interested in an error estimate of the solution of problem (P_h) to the solution of problem (P). A key point in the convergence analysis of the present chapter was the boundedness of the discrete Lagrange multiplier associated with the mixed control-state constraints, see Lemma 5.16. Based on this result, we proved the uniform boundedness of the discretized and regularized optimal control $\bar{u}^{\varepsilon}_h$ in $H^1(\Gamma)$ with respect to h and ε. First, let us introduce the discrete counterpart of the Karush-Kuhn-Tucker system of problem (P) given in Theorem 2.17:

$$a(\bar{y}_h, z_h) = \int_\Gamma \bar{u}_h z_h \, ds \quad \forall z_h \in V_h \tag{5.56}$$

$$a(z_h, p_h) = \int_\Omega (\bar{y}_h - y_d) z_h \, dx - \int_{\Omega'} z_h d\mu_h \quad \forall z_h \in V_h \tag{5.57}$$

$$(\tau p_h + \nu\bar{u}_h \, , \, u - \bar{u}_h)_{L^2(\Gamma)} \ge 0, \quad \forall u \in U^L_{ad,h} \tag{5.58}$$

$$\left.\begin{array}{l} \displaystyle\int_{\Omega'} \bar{y}_h - y_c d\mu_h = 0, \quad \bar{y}_h(x) \ge y_c(x) \quad \text{a.e in } \Omega' \\ \displaystyle\int_{\Omega'} \varphi d\mu_h \ge 0 \quad \forall \varphi \in C(\Omega'), \quad \varphi(x) \ge 0 \quad \forall x \in \Omega'. \end{array}\right. \tag{5.59}$$

with the bilinear form $a(\cdot,\cdot)$ defined in (2.2). Due to the presence of a measure, the state and the adjoint equation are considered by the associated discrete weak formulations instead of solution operators. In the proof of Lemma 5.16, we used that the discrete Lagrange multiplier associated with the mixed control-state constraints is a regular function. Of course, this is general not the case for problem (P_h) since μ_h denotes an approximation of the original Lagrange multiplier μ of problem (P) which is in general only a measure. The next lemma provides a uniform bound of μ_h in the space of regular Borel measures.

Lemma 5.37 *Let $(\bar{y}_h, \bar{u}_h)$ be the optimal solution of problem (P_h). Furthermore, let p_h be an adjoint state and μ_h a Lagrange multiplier such that the optimality system (5.56)-(5.59) is fulfilled. Then, there exist a constant $C > 0$, independent of h, such that*

$$\|\mu_h\|_{\mathcal{M}(\Omega')} \leq C.$$

Proof: We start by considering the variational inequality (5.58) for the specific function $\Pi_h \hat{u} \in U^L_{ad,h}$, where $\hat{u}$ is the inner point of Assumption 2.4 and Π_h denotes the quasi-interpolation operator defined in (5.16):

$$(\tau^* p_h\,,\, \Pi_h\hat{u} - \bar{u}_h)_{L^2(\Gamma)} + \nu(\bar{u}_h\,,\, \Pi_h\hat{u} - \bar{u}_h)_{L^2(\Gamma)} \geq 0. \tag{5.60}$$

Due to the definition (5.18) of the discrete control-to-state mapping and (2.7), we find for $\hat{y}_h = S_h\tau^*\Pi_h\hat{u}$

$$\hat{y}_h = S_h\tau^*\Pi_h\hat{u} \quad\Leftrightarrow\quad a(\hat{y}_h, z_h) = \int_\Gamma \Pi_h\hat{u}\tau z_h\,ds \quad \forall z_h \in V_h.$$

By means of this and the weak formulations (5.56) and (5.57), we reformulate the first term in (5.60) to

$$\begin{aligned}(\tau^* p_h\,,\, \Pi_h\hat{u} - \bar{u}_h)_{L^2(\Gamma)} &= a(\hat{y}_h - \bar{y}_h, p_h)\\ &= (\bar{y}_h - y_d\,,\, \hat{y}_h - \bar{y}_h)_{L^2(\Omega)} - \int_{\Omega'} (\hat{y}_h - \bar{y}_h)d\mu_h.\end{aligned} \tag{5.61}$$

Summarizing (5.60) and (5.61), we obtain

$$\begin{aligned}\int_{\Omega'} (\hat{y}_h - \bar{y}_h)d\mu_h &\leq \nu(\bar{u}_h\,,\, \Pi_h\hat{u} - \bar{u}_h)_{L^2(\Gamma)} + (\bar{y}_h - y_d\,,\, \hat{y}_h - \bar{y}_h)_{L^2(\Omega)}\\ &\leq \nu\|\bar{u}_h\|_{L^2(\Gamma)}\|\Pi_h\hat{u}\|_{L^2(\Gamma)} + \|\bar{y}_h\|_{L^2(\Omega)}\|y_d\|_{L^2(\Omega)}\\ &\quad + \|\bar{y}_h - y_d\|_{L^2(\Omega)}\|\hat{y}_h\|_{L^2(\Omega)}.\end{aligned}$$

Due to optimality of $(\bar{y}_h, \bar{u}_h)$ and Assumption 2.4, the remaining terms are bounded by a constant independent of h. Finally, the complementary slackness conditions (5.59) and Lemma 5.13 imply for the left side of the previous inequality:

$$\begin{aligned}\int_{\Omega'} (\hat{y}_h - \bar{y}_h)d\mu_h &= \int_{\Omega'} (\hat{y}_h - y_c)d\mu_h + \int_{\Omega'} (y_c - \bar{y}_h)d\mu_h\\ &\geq \gamma_0 \int_{\Omega'} d\mu_h\\ &\geq \gamma_0\|\mu_h\|_{\mathcal{M}(\Omega')}.\end{aligned}$$

Hence, the assertion is proven. □

We remind that the purely discretized problem (P_h) is equivalent to problem $(\mathrm{P}^{\varepsilon}_{2,h})$ for the specific choice $\psi(\varepsilon) = 1$, $\phi(\varepsilon) \equiv \xi(\varepsilon) \equiv 0$ of parameter functions. Thus, the convergence analysis of the previous sections can be easily adapted to problem (P_h) since the boundedness of the discrete Lagrange multiplier is guaranteed by the previous lemma. Concluding, we state the following finite element error estimate of problem (P_h) concerning problem (P).

Theorem 5.38 *Let $(\bar{y}, \bar{u})$ and $(\bar{y}_h, \bar{u}_h)$ be the optimal solutions of* (P) *and* (P_h), *respectively. Then, there exist a positive constant $c > 0$, independent of h, and a sufficiently mesh size $h_0 < 1$ such that*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch|\log h|^{1/2} \tag{5.62}$$

is fulfilled for all mesh sizes $0 < h \leq h_0$.

In view of the previous result, it seems that a regularization is not necessary. But, as already mentioned, for optimal control problems with pure state constraints efficient optimization methods are in general not well defined such that regularization concepts are reasonable. A brief overview concerning optimization methods for constrained optimal control problem governed with PDEs is given in the beginning of the next chapter.

Chapter 6

Numerical verification and optimization algorithms

In this chapter we will illustrate the convergence results of the previous chapters by different numerical examples. Hence, we are interested in efficient optimization methods for solving linear-quadratic optimal control problems with mixed constraints like (P_2^ε) or ($P_{2,h}^\varepsilon$), respectively. One class of algorithms are the interior point methods, where the inequality constraints are replaced by appropriate penalty terms in the objective functional, see e.g. [50, 56, 65, 72].

We will focus on active set strategies, especially the *primal-dual active set strategy* as for instance proposed by Bergonioux, Ito, and Kunisch in [10] for control constrained optimal control problems. However, for problems with pure state constraints it is not possible to formulate this method in function spaces since the Lagrange multipliers are only measures. Nevertheless, we mention a direct application of this method to a discretized version of a linear-quadratic optimal control problem with pure state constraints and distributed control by Bergonioux and Kunisch [11]. We remind that regularized problems like (P_2^ε) provide regular multipliers. Hence, the active and inactive sets can be defined also on the continuous level (cf. Section 6.2). Furthermore, several convergence results concerning the active set strategy in function spaces are known, see e.g. Hintermüller, Ito and Kunisch [37]. These authors showed that the active set strategy can be interpreted as a semi-smooth Newton method converging locally superlinear. For more detailed information about semi-smooth Newton methods we refer, for instance, to [40, 70, 76]. In Rösch, Kunisch [47], the authors presented a global convergence result underlying that the Tikhonov parameter in the objective functional satisfies a certain inequality. Furthermore, also the Moreau-Yosida regularization (see Section 4.3) allows to formulate the active set strategy in function spaces, see [38].

In the work [44] we derived error estimates for feasible and infeasible solutions of control constrained optimal control problems. Based on this theory we constructed an error estimator, which is reliable as a stopping criterion for iterative solvers for control constrained optimal control problems, e.g., the primal-dual active set strategy or the projected gradient method. The next section is devoted to the extension of this idea to problems like (P_2^ε), i.e. optimal control problems with boundary control and mixed control state constraints. In Section 6.2 we will describe the primal-dual active set strategy to solve regularized optimal control problems of type (P_2^ε). Finally, Section 6.3 is devoted to the numerical illustration of the results derived in Chapters 4-6.

6.1 Error estimates for feasible and infeasible solutions

6.1.1 Conversion to a control constrained problem

First, let us recall again the regularized optimal control problem (P_2^ε):

$$\left.\begin{array}{c} \min \quad J(y_\varepsilon, u_\varepsilon, v_\varepsilon) := \dfrac{1}{2}\|y_\varepsilon - y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u_\varepsilon\|^2_{L^2(\Gamma)} + \dfrac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)} \\ -\Delta y_\varepsilon + y_\varepsilon = \phi(\varepsilon) v_\varepsilon \qquad \text{in } \Omega \\ \partial_n y_\varepsilon = u_\varepsilon \qquad \text{on } \Gamma \\ u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. on } \Gamma \\ y_\varepsilon(x) \ge y_c(x) - \xi(\varepsilon) v_\varepsilon(x) \quad \text{a.e. in } \Omega'. \end{array}\right\} \qquad (\mathrm{P}_2^\varepsilon)$$

The existence and uniqueness of an optimal solution ($\bar{y}_\varepsilon$, $\bar{u}_\varepsilon$, $\bar{v}_\varepsilon$) was already clarified in Section 4.1, see Theorem 4.1. Furthermore, by introduction of an adjoint state p_ε and a Lagrange multiplier μ_ε associated with the mixed constraints we derived the following optimality system:

$$\begin{array}{cc} -\Delta \bar{y}_\varepsilon + \bar{y}_\varepsilon = \phi(\varepsilon)\bar{v}_\varepsilon & -\Delta p_\varepsilon + p_\varepsilon = \bar{y}_\varepsilon - y_d - \mu_\varepsilon \\ \partial_n \bar{y}_\varepsilon = \bar{u}_\varepsilon & \partial_n p = 0 \end{array} \tag{6.1}$$

$$(\tau p_\varepsilon + \nu \bar{u}_\varepsilon \,,\, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \ge 0, \quad \forall u \in U_{ad} \tag{6.2}$$

$$\phi(\varepsilon) p_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\mu_\varepsilon = 0 \quad \text{a.e. in } \Omega \tag{6.3}$$

$$(\mu_\varepsilon \,,\, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega')} = 0, \quad \mu_\varepsilon \ge 0, \quad \bar{y}_\varepsilon \ge y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad \text{a.e. in } \Omega', \tag{6.4}$$

where the admissible set of boundary controls U_{ad} is defined by

$$U_{ad} = \{u \in L^2(\Gamma) : \, u_a \le u \le u_b \ \text{ a.e. on } \Gamma\},$$

see Theorem 4.3.

In the work [44] the simple structure of the optimality conditions for control constrained optimal control problems was a key point for the construction of an error estimator. Thus, we transform (P_2^ε) into a control constrained problem by substituting

$$w_\varepsilon(x) := y_\varepsilon(x) + \xi(\varepsilon) v_\varepsilon(x). \tag{6.5}$$

With $v_\varepsilon = \frac{1}{\xi(\varepsilon)}(w_\varepsilon - y_\varepsilon)$ we convert problem (P_2^ε) into a purely control constrained optimal control problem with respect to the control variables (u_ε, w_ε):

$$\left.\begin{aligned} \min J(y_\varepsilon, u_\varepsilon, w_\varepsilon) := \frac{1}{2}\|y_\varepsilon - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_\varepsilon\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2\xi(\varepsilon)^2}\|w_\varepsilon - y_\varepsilon\|^2_{L^2(\Omega)} \\ -\Delta y_\varepsilon + (1 + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)})y_\varepsilon = \frac{\phi(\varepsilon)}{\xi(\varepsilon)}w_\varepsilon \qquad \text{in } \Omega \\ \partial_n y_\varepsilon = u_\varepsilon \qquad \text{on } \Gamma \\ u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. on } \Gamma \\ w_\varepsilon(x) \ge y_c(x) \quad \text{a.e. in } \Omega'. \end{aligned}\right\} \quad (Q_w^\varepsilon)$$

According to the Lax-Milgram Lemma 2.1, the state equation admits a unique solution in $H^1(\Omega)$ that depends continuously on w and u. Based on the bilinear form

$$a^w(y,z) := \int_\Omega \nabla y \cdot \nabla z + \left(1 + \frac{\phi(\varepsilon)}{\xi(\varepsilon)}\right) yz\,dx \tag{6.6}$$

we define the solution operator $S^w : H^1(\Omega)^* \to L^2(\Omega)$ by

$$f \mapsto y, \quad y = S^w f \quad \Longleftrightarrow \quad a^w(y,z) = \langle f\,,\, z\rangle_{H^1(\Omega)^*,H^1(\Omega)} \quad \forall z \in H^1(\Omega)$$

for each element $f \in H^1(\Omega)^*$. Similar to (3.3), we introduce the control-to-state mapping by

$$(u,w) \mapsto y, \quad y_\varepsilon = S^w(\tau^* u_\varepsilon + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H^* w_\varepsilon). \tag{6.7}$$

The operators $\tau^* : L^2(\Gamma) \to H^1(\Omega)^*$ and $E_H^* : L^2(\Omega) \to H^1(\Omega)^*$ are defined by (2.7) and (3.2), respectively.

In order to derive optimality conditions, we define the admissible set with respect to boundary control u and the distributed control w, respectively, as follows

$$\begin{aligned} U_{ad} &= \{u \in L^2(\Gamma) : \, u_a \le u \le u_b \text{ a.e. on } \Gamma\} \\ W_{ad} &= \{w \in L^2(\Omega) : \, w \ge y_c \text{ a.e. in } \Omega'\}. \end{aligned}$$

These sets are nonempty, convex and closed. With the help of the control-to-state mapping (6.7), we formulate the problem (Q_w^ε) in the reduced form:

$$\begin{aligned} \min_{(u_\varepsilon, w_\varepsilon) \in U_{ad} \times W_{ad}} f(u_\varepsilon, w_\varepsilon) := \frac{1}{2}\|S(\tau^* u_\varepsilon + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H^* w_\varepsilon) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_\varepsilon\|^2_{L^2(\Gamma)} \\ + \frac{\psi(\varepsilon)}{2\xi(\varepsilon)^2}\|w_\varepsilon - S(\tau^* u_\varepsilon + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H^* w_\varepsilon)\|^2_{L^2(\Omega)}. \end{aligned} \tag{6.8}$$

The existence of a unique solution $(\bar y_\varepsilon, \bar u_\varepsilon, \bar w_\varepsilon)$ can be obtained by standard arguments, see Section 2.2. Furthermore, straight forward computation yields the following necessary and sufficient optimality condition.

Lemma 6.1 *Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{w}_\varepsilon)$ be the optimal solution of problem $(\mathrm{Q}_w^\varepsilon)$. The necessary and sufficient optimality condition is given by*

$$
\begin{aligned}
(\tau p_\varepsilon^w + &\nu \bar{u}_\varepsilon, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \\
&+ \left(\tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H p_\varepsilon^w + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - \bar{y}_\varepsilon), w - \bar{w}_\varepsilon\right)_{L^2(\Omega)} \geq 0 \quad \forall (u,w) \in U_{ad} \times W_{ad},
\end{aligned}
\tag{6.9}
$$

where $p_\varepsilon^w = (S^w)^*((1 + \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2})\bar{y}_\varepsilon - y_d - \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}\bar{w}_\varepsilon)$ denotes the associated adjoint state.

We note that the both parts in the variational inequality (6.9) can be treated separately since the controls are not coupled. Following the ideas of Lemma 2.8, we find a representation for the adjoint operator $(S^w)^*$. Consequently, we obtain the following optimality system for problem $(\mathrm{Q}_w^\varepsilon)$:

$$
\begin{aligned}
-\Delta \bar{y}_\varepsilon + (1 + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)})\bar{y}_\varepsilon &= \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)}\bar{w}_\varepsilon \\
\partial_n \bar{y}_\varepsilon &= \bar{u}_\varepsilon \\
-\Delta p_\varepsilon^w + (1 + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)})p_\varepsilon^w &= (1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})\bar{y}_\varepsilon - y_d - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}\bar{w}_\varepsilon \\
\partial_n p_\varepsilon^w &= 0
\end{aligned}
\tag{6.10}
$$

$$
(\tau p_\varepsilon^w + \nu \bar{u}_\varepsilon\,,\, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{ad} \tag{6.11}
$$

$$
(\tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H p_\varepsilon^w + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - \bar{y}_\varepsilon)\,,\, w - \bar{w}_\varepsilon)_{L^2(\Omega)} \geq 0 \quad \forall w \in W_{ad} \tag{6.12}
$$

Next, we show the equivalence of the optimality systems (6.1)-(6.4) for $(\mathrm{P}_2^\varepsilon)$ and the previous one for $(\mathrm{Q}_w^\varepsilon)$, respectively. To this end, we introduce a Lagrange multiplier with respect to the control constraint of the distributed control w_ε in problem $(\mathrm{Q}_w^\varepsilon)$. For the case of control constrained optimal control problems, the associated Lagrange multipliers can be obtained by pointwise construction, see e.g. [73, Section 6.1.1]. Thus, we define

$$
\mu_\varepsilon^w := \frac{\phi(\varepsilon)}{\xi(\varepsilon)} p_\varepsilon^w + \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - \bar{y}_\varepsilon). \tag{6.13}
$$

A pointwise discussion of the variational inequality (6.12) yields the positivity of the multiplier μ_ε^w in Ω' and the complementary slackness conditions. Hence, we end up with

$$
\begin{aligned}
-\Delta \bar{y}_\varepsilon + (1 + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)})\bar{y}_\varepsilon = \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)}\bar{w}_\varepsilon \quad & -\Delta p_\varepsilon^w + (1 + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)})p_\varepsilon^w = (1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})\bar{y}_\varepsilon - y_d - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}\bar{w}_\varepsilon \\
\partial_n \bar{y}_\varepsilon = \bar{u}_\varepsilon \quad & \partial_n p_\varepsilon^w = 0
\end{aligned}
$$

$$
\begin{gathered}
(\tau p_\varepsilon^w + \nu \bar{u}_\varepsilon\,,\, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{ad} \\
\frac{\phi(\varepsilon)}{\xi(\varepsilon)} p_\varepsilon^w + \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - \bar{y}_\varepsilon) - \mu_\varepsilon^w = 0 \quad \text{a.e. in } \Omega \\
(\mu_\varepsilon^w\,,\, y_c - \bar{w}_\varepsilon)_{L^2(\Omega')} = 0 \quad \mu_\varepsilon^w \geq 0 \text{ a.e. in } \Omega', \quad \bar{w}_\varepsilon \geq y_c \text{ a.e. in } \Omega'
\end{gathered}
$$

By setting $p_\varepsilon := p_\varepsilon^w$, $\mu_\varepsilon := \mu_\varepsilon^w$, the transformation $\bar{w}_\varepsilon = \bar{y}_\varepsilon + \xi(\varepsilon)\bar{v}_\varepsilon$ and the definition (6.13) delivers the optimality system (6.1)-(6.4) of problem $(\mathrm{P}_2^\varepsilon)$ such that the equivalence is shown.

6.1.2 Error estimates

In this section, we derive error estimates for arbitrary feasible and infeasible controls of problem (P_2^ε) with respect to the optimal control of the problem (P_2^ε). These feasible and infeasible controls can be interpreted as current iterates of an optimization algorithm solving the problem (P_2^ε). Furthermore, we define error estimators in an explicit manner.

Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be the optimal solution of problem (P_2^ε). Consequently, $(\bar{u}_\varepsilon, \bar{w}_\varepsilon)$ with $\bar{w}_\varepsilon = \bar{y}_\varepsilon + \xi(\varepsilon)\bar{v}_\varepsilon$ are the optimal controls of problem (Q_w^ε). Furthermore, let (u_N, w_N) be a current feasible control of problem (Q_w^ε), i.e. $(u_N, w_N) \in U_{ad} \times W_{ad}$. Moreover, $y_N = S^w(\tau^* u_N + \frac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H^* w_N)$ and $p_N = (S^w)^*((1 + \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}) y_N - y_d - \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2} w_N)$ are the associated state and adjoint state, respectively.

Theorem 6.2 *Let $\delta_u \in L^2(\Gamma)$ and $\delta_w \in L^2(\Omega)$ be functions, such that*

$$(\tau p_N + \nu u_N + \delta_u \, , \, u - u_N)_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{ad} \tag{6.14}$$

$$(\tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H p_N + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(w_N - y_N) + \delta_w \, , \, w - w_N)_{L^2(\Omega)} \geq 0 \quad \forall w \in W_{ad} \tag{6.15}$$

holds true. Then, the error of the optimal control $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ to the control (u_N, v_N) with $v_N = \frac{1}{\xi(\varepsilon)}(w_N - y_N)$ can be estimated by

$$\nu \|\bar{u}_\varepsilon - u_N\|^2_{L^2(\Gamma)} + \psi(\varepsilon)\|\bar{v}_\varepsilon - v_N\|^2_{L^2(\Omega)} \leq \frac{1}{\nu}\|\delta_u\|^2_{L^2(\Gamma)} + \left(1 + \frac{\xi(\varepsilon)^2}{\psi(\varepsilon)}\right)\|\delta_w\|^2_{L^2(\Omega)}. \tag{6.16}$$

Proof: Since $\bar{u}_\varepsilon$, $u_N \in U_{ad}$, we set $u := \bar{u}_\varepsilon$ in (6.14) and $u := u_N$ in the variational inequality (6.11) of problem (Q_w^ε). Adding the inequalities yields

$$(\tau(p_\varepsilon^w - p_N) + \nu(\bar{u}_\varepsilon - u_N) - \delta_u \, , \, u_N - \bar{u}_\varepsilon)_{L^2(\Gamma)} \geq 0$$

or

$$-\nu\|\bar{u}_\varepsilon - u_N\|^2_{L^2(\Gamma)} + (\tau(p_\varepsilon^w - p_N) \, , \, u_N - \bar{u}_\varepsilon)_{L^2(\Gamma)} + (\delta_u \, , \, \bar{u}_\varepsilon - u_N)_{L^2(\Gamma)} \geq 0. \tag{6.17}$$

Analogously, we choose $w := \bar{w}_\varepsilon$ in (6.15) and $w := w_N$ in (6.12) as feasible test functions, respectively. By adding the inequalities, we obtain

$$\begin{aligned}
&(\tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H(p_\varepsilon^w - p_N) \, , \, w_N - \bar{w}_\varepsilon)_{L^2(\Omega)} \\
&\qquad + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - w_N + y_N - \bar{y}_\varepsilon \, , \, w_N - \bar{w}_\varepsilon)_{L^2(\Omega)} + (\delta_w \, , \, \bar{w}_\varepsilon - w_N)_{L^2(\Omega)} \geq 0
\end{aligned} \tag{6.18}$$

Before we will add (6.17) and (6.18), we rewrite the following term:

$$\begin{aligned}
&(\tau(p_\varepsilon^w - p_N) \, , \, u_N - \bar{u}_\varepsilon)_{L^2(\Gamma)} \\
&\quad = (\tau(S^w)^*((1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})(\bar{y}_\varepsilon - y_N) - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - w_N) \, , \, u_N - \bar{u}_\varepsilon)_{L^2(\Gamma)} \\
&\quad = ((1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})(\bar{y}_\varepsilon - y_N) - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - w_N) \, , \, S^w \tau^*(u_N - \bar{u}_\varepsilon))_{L^2(\Omega)} \\
&\quad = ((1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})(\bar{y}_\varepsilon - y_N) - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - w_N) \, , \, y_N - \bar{y}_\varepsilon)_{L^2(\Omega)} - \\
&\qquad ((1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})(\bar{y}_\varepsilon - y_N) - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - w_N) \, , \, \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} S^w E_H^*(w_N - \bar{w}_\varepsilon))_{L^2(\Omega)} \\
&\quad = ((1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})(\bar{y}_\varepsilon - y_N) - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_\varepsilon - w_N) \, , \, y_N - \bar{y}_\varepsilon)_{L^2(\Omega)} - \\
&\qquad (\tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H(p_\varepsilon^w - p_N) \, , \, w_N - \bar{w}_\varepsilon)_{L^2(\Omega)}.
\end{aligned}$$

Now, we add (6.17) and (6.18) such that we derive

$$-\nu\|\bar u_\varepsilon - u_N\|^2_{L^2(\Gamma)} + ((1+\tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})(\bar y_\varepsilon - y_N) - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar w_\varepsilon - w_N)\,,\, y_N - \bar y_\varepsilon)_{L^2(\Omega)}$$
$$+\tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar w_\varepsilon - w_N + y_N - \bar y_\varepsilon\,,\, w_N - \bar w_\varepsilon)_{L^2(\Omega)}$$
$$+(\delta_u\,,\, \bar u_\varepsilon - u_N)_{L^2(\Gamma)} + (\delta_w\,,\, \bar w_\varepsilon - w_N)_{L^2(\Omega)} \geq 0.$$

Summing up the second and the third term, and using the substition $w = y + \xi(\varepsilon)v$ implies

$$\begin{aligned}
&-\left(1+\tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}\right)\|\bar y_\varepsilon - y_N\|^2_{L^2(\Omega)} + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar w_\varepsilon - w_N + 2(y_N - \bar y_\varepsilon)\,,\, w_N - \bar w_\varepsilon)_{L^2(\Omega)}\\
&\quad= -\left(1+\tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}\right)\|\bar y_\varepsilon - y_N\|^2_{L^2(\Omega)} + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_N - \bar y_\varepsilon + \bar v_\varepsilon - v_N\,,\, y_N - \bar y_\varepsilon + v_N - \bar v_\varepsilon)_{L^2(\Omega)}\\
&\quad= -\|\bar y_\varepsilon - y_N\|^2_{L^2(\Omega)} - \psi(\varepsilon)\|\bar v_\varepsilon - v_N\|^2_{L^2(\Omega)}.
\end{aligned}$$

Concluding, we obtain the following inequality

$$\nu\|\bar u_\varepsilon - u_N\|^2_{L^2(\Gamma)} + \|\bar y_\varepsilon - y_N\|^2_{L^2(\Omega)} + \psi(\varepsilon)\|\bar v_\varepsilon - v_N\|^2_{L^2(\Omega)} \leq$$
$$(\delta_u\,,\, \bar u_\varepsilon - u_N)_{L^2(\Gamma)} + (\delta_w\,,\, \bar y_\varepsilon - y_N)_{L^2(\Omega)} + \xi(\varepsilon)(\delta_w\,,\, \bar v_\varepsilon - v_N)_{L^2(\Omega)}.$$

Finally, we apply Young's inequality to the remaining terms on the right side. This yields

$$\frac{\nu}{2}\|\bar u_\varepsilon - u_N\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|\bar v_\varepsilon - v_N\|^2_{L^2(\Omega)} \leq \frac{1}{2\nu}\|\delta_u\|^2_{L^2(\Gamma)} + \left(\frac{1}{4} + \frac{\xi(\varepsilon)^2}{2\psi(\varepsilon)}\right)\|\delta_w\|^2_{L^2(\Omega)}.$$

Thus, the assertion (6.16) is proven. □

If the feasible control (u_N, v_N) is interpreted as a current iterate of an optimization algorithm solving $(\mathrm{P}_2^\varepsilon)$, then the previous error estimate provides the possibility to construct a reliable estimator for a stopping criterion. We will consider the functions $\delta_u \in L^2(\Gamma)$ and $\delta_w \in L^2(\Omega)$ in detail. In order to construct appropriate functions δ_u and δ_w, respectively, we analyze the variational inequalities (6.14) and (6.15) pointwise. Let us start with the construction of δ_u. First, we assume $u_N \in (u_a, u_b)$. Due to the validity of variational inequality (6.14), we obtain

$$\delta_u = -\tau p_N - \nu u_N.$$

The case $u_N = u_a$ yields

$$\tau p_N + \nu u_N + \delta_u \geq 0.$$

Choosing the function δ_u as small as possible, we find

$$\delta_u = \max\{0, -\tau p_N - \nu u_N\}.$$

Of course, the last case $u_N = u_b$ can be treated analogously. Motivated by these arguments, we define the function δ_u by

$$\delta_u(x) := \begin{cases} \max\{0, -(\tau p_N)(x) - \nu u_N(x)\}, & \text{if } u_N(x) = a\\ \min\{0, -(\tau p_N)(x) - \nu u_N(x)\}, & \text{if } u_N(x) = b\\ -(\tau p_N)(x) - \nu u_N(x), & \text{if } u_N(x) \in (a,b). \end{cases} \tag{6.19}$$

One can easily see, that this choice of δ_u satisfies the variational inequality (6.14) in Theorem 6.2. A similar elaboration of the variational inequality (6.15) implies the following definition for the function δ_w

$$\delta_w(x) := \begin{cases} \max\{0, -\frac{\phi(\varepsilon)}{\xi(\varepsilon)} p_N(x) - \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(w_N(x) - y_N(x))\}, & \text{if } w_N(x) = y_c \\ -\frac{\phi(\varepsilon)}{\xi(\varepsilon)} p_N(x) - \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(w_N(x) - y_N(x)), & \text{else .} \end{cases} \tag{6.20}$$

Since the optimality systems of the problems (P_2^ε) and (Q_w^ε) are equivalent, as shown in Section 6.1.1, we substitute $w_N = y_N + \xi(\varepsilon)v_N$ in (6.20). Hence, the function δ_w, depending on the variables of problem (P_2^ε), is defined by

$$\delta_w(x) := \begin{cases} \max\{0, -\frac{\phi(\varepsilon)}{\xi(\varepsilon)} p_N(x) - \frac{\psi(\varepsilon)}{\xi(\varepsilon)} v_N(x)\}, & \text{if } y_N(x) + \xi(\varepsilon)v_N(x) = y_c(x) \\ -\frac{\phi(\varepsilon)}{\xi(\varepsilon)} p_N(x) - \frac{\psi(\varepsilon)}{\xi(\varepsilon)} v_N(x), & \text{else .} \end{cases} \tag{6.21}$$

It is easy to verify, that the constructed function δ_w satisfies the variational inequality (6.15). Moreover, we note that the functions δ_u and δ_w are equal to zero, if the feasible control (u_N, v_N) coincides with the optimal control $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ of problem (P_2^ε).

In the further discussion, we will not restrict to feasible controls, i.e. we consider controls $(\tilde{u}_N, \tilde{w}_N) \in L^2(\Gamma) \times L^2(\Omega)$ that are infeasible for the transformed problem (Q_w^ε). The corresponding state and adjoint state are denoted by $\tilde{y}_N$ and $\tilde{p}_N$, respectively. Again, let $(u_N, w_N) \in U_{ad} \times W_{ad}$ be a pair of feasible controls with the associated state y_N and the adjoint state p_N. The next corollary provides an error estimate of the infeasible controls to the optimal controls $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ of problem (P_2^ε).

Corollary 6.3 *Let $\delta_u \in L^2(\Gamma)$ and $\delta_w \in L^2(\Omega)$ be functions satisfying the variational inequalities* (6.14) *and* (6.15)*, respectively. Then, the error estimate*

$$\begin{aligned} \nu\|\bar{u}_\varepsilon - \tilde{u}_N\|^2_{L^2(\Gamma)} + \psi(\varepsilon)\|\bar{v}_\varepsilon - \tilde{v}_N\|^2_{L^2(\Omega)} \le 2 \Bigg(\frac{1}{\nu}\|\delta_u\|^2_{L^2(\Gamma)} + \left(1 + \frac{\xi(\varepsilon)^2}{\psi(\varepsilon)}\right) \|\delta_w\|^2_{L^2(\Omega)} \\ + \nu\|u_N - \tilde{u}_N\|^2_{L^2(\Gamma)} + \psi(\varepsilon)\|v_N - \tilde{v}_N\|^2_{L^2(\Omega)} \Bigg) \end{aligned} \tag{6.22}$$

is valid.

The result directly follows from Theorem 6.2 and the triangle inequality.

Remark 6.4 In particular the previous error estimate is applicable to optimization methods, where the iterates are in general infeasible. For instance, this scenario occurs for the primal-dual active set strategy. Then, it is reasonable to choose the necessary feasible control (u_N, w_N) by the pointwise projection of the infeasible control $(\tilde{u}_N, \tilde{w}_N)$ on the admissible sets $U_{ad} \times W_{ad}$, e.g.:

$$u_N := P_{[u_a, u_b]}\{\tilde{u}_N\} \quad \text{and} \quad w_N := P_{[y_c, \infty)}\{\tilde{w}_N\}.$$

We will point out this more detailed in Section 6.2.

6.1.3 Application to a fully discretized scheme

In this section we establish error estimates, similarly as in the previous section, for a fully discretized scheme. We mention, that all further considerations can be also done for the semidiscrete approach, proposed by Hinze in [39], where the controls are not discretized. We use the same discrete framework as introduced in the beginning of Section 5.1, i.e. we consider a mesh $\mathcal{T}_h$ of Ω consisting of open and pairwise disjoint triangles or tetrahedra, where the vertices are denoted by x_i, $i = 1, \ldots, n$. Let n_e be the number of vertices on the boundary. We suppose that $\mathcal{T}_h$ satisfies the Assumption 5.1 of a quasi-uniform mesh. Again, $V_h \subset H^1(\Omega) \cap C(\bar{\Omega})$ denotes the space of linear finite elements with

$$V_h = \operatorname{span}\{\varphi_i,\, i = 1, \ldots, n\}.$$

The space U_h is the space of continuous piecewise linear functions on the intervals or triangles of Γ and we set

$$U_h = \operatorname{span}\{\psi_j,\, j = 1, \ldots, n_e\}.$$

Note, that the restriction of the basis functions φ_i of V_h with nonempty support on the boundary to Γ coincides with basis functions ψ_j of U_h.

We start with the definition of a discrete solution operator $S_h^w : H^1(\Omega)^* \to L^2(\Omega)$ by

$$f \mapsto y_h,\, y_h = S_h^w f \quad \Longleftrightarrow \quad a^w(y_h, z_h) = \langle f\,,\, z_h\rangle_{H^1(\Omega)^*, H^1(\Omega)} \quad \forall z_h \in V_h \tag{6.23}$$

for each element $f \in H^1(\Omega)^*$, where $a^w : V_h \times V_h \to \mathbb{R}$ is the bilinear form defined in (6.6). The Lax-Milgram Lemma 2.1 ensures the existence of a unique element $y_h = S_h^w f \in V_h$ for every $f \in H^1(\Omega)^*$. Hence, the discrete solution of the state equation of problem $(\mathrm{Q}_w^\varepsilon)$ is given by:

$$y_h = S_h^w(\tau^* u + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H^* w) \quad \text{for } (u, w) \in L^2(\Gamma) \times L^2(\Omega).$$

By means of the following discrete admissible sets

$$U_{ad,h} = U_{ad} \cap U_h$$
$$W_{ad,h} = W_{ad} \cap V_h.$$

for the controls, the discretized analogon to problem $(\mathrm{Q}_w^\varepsilon)$ is given by:

$$\left.\begin{array}{l} \min J(y_h^\varepsilon, u_h^\varepsilon, w_h^\varepsilon) = \dfrac{1}{2}\|y_h^\varepsilon - y_d\|^2_{L^2(\Omega)} + \dfrac{\nu}{2}\|u_h^\varepsilon\|^2_{L^2(\Gamma)} + \dfrac{\psi(\varepsilon)}{2\xi(\varepsilon)^2}\|w_h^\varepsilon - y_h^\varepsilon\|^2_{L^2(\Omega)} \\ \text{s.t.} \qquad y_h^\varepsilon = S_h^w(\tau^* u_h^\varepsilon + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H^* w_h^\varepsilon) \quad \text{for} \quad (u_h^\varepsilon, w_h^\varepsilon) \in U_{ad,h} \times W_{ad,h}. \end{array}\right\} \tag{$\mathrm{Q}_{w,h}^\varepsilon$}$$

The following optimality system for problem $(\mathrm{Q}_{w,h}^\varepsilon)$ is obtained by standard arguments:

$$\bar{y}_h^\varepsilon = S_h^w(\tau^* \bar{u}_h^\varepsilon + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H^* \bar{w}_h^\varepsilon) \tag{6.24}$$

$$p_h^{w,\varepsilon} = (S_h^w)^*((1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})\bar{y}_h^\varepsilon - y_d - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}\bar{w}_h^\varepsilon) \tag{6.25}$$

$$(\tau p_h^{w,\varepsilon} + \nu \bar{u}_h^\varepsilon\,, u - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{ad,h} \tag{6.26}$$

$$(\tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H p_h^{w,\varepsilon} + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(\bar{w}_h^\varepsilon - \bar{y}_h^\varepsilon)\,,\, w - \bar{w}_h^\varepsilon)_{L^2(\Omega)} \geq 0 \quad \forall w \in W_{ad,h} \tag{6.27}$$

with the unique discrete optimal solution $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{w}_h^\varepsilon)$ and the associated discrete adjoint state $p_h^{w,\varepsilon}$. Consequently, $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ with $\bar{v}_h^\varepsilon = \frac{1}{\xi(\varepsilon)}(\bar{w}_h^\varepsilon - \bar{y}_h^\varepsilon)$ is the optimal solution of the discretized analogon to problem $(\mathrm{P}_2^\varepsilon)$.

In order to adapt the strategy of Theorem 6.2, we assume that

$$(u_N, w_N) \in U_{ad,h} \times W_{ad,h}$$

is a current feasible control for problem $(\mathrm{Q}_{w,h}^\varepsilon)$. Furthermore,

$$y_N = S_h^w(\tau^* u_N + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H^* w_N)$$

and

$$p_N = (S_h^w)^*((1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}) y_N - y_d - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2} w_N)$$

are the associated discrete state and adjoint state, respectively.

Corollary 6.5 *Let $\delta_u \in U_h$ and $\delta_w \in V_h$ be functions, such that*

$$(\tau p_N + \nu u_N + \delta_u \,,\, u - u_N)_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_{ad,h} \tag{6.28}$$

$$(\tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E_H p_N + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(w_N - y_N) + \delta_w \,,\, w - w_N)_{L^2(\Omega)} \geq 0 \quad \forall w \in W_{ad,h} \tag{6.29}$$

holds true. Then, the error of the discrete optimal control $(\bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ to the control (u_N, v_N) with $v_N = \frac{1}{\xi(\varepsilon)}(w_N - y_N)$ can be estimated by

$$\nu\|\bar{u}_h^\varepsilon - u_N\|^2_{L^2(\Gamma)} + \psi(\varepsilon)\|\bar{v}_h^\varepsilon - v_N\|^2_{L^2(\Omega)} \leq \frac{1}{\nu}\|\delta_u\|^2_{L^2(\Gamma)} + \left(1 + \frac{\xi(\varepsilon)^2}{\psi(\varepsilon)}\right)\|\delta_w\|^2_{L^2(\Omega)}. \tag{6.30}$$

The proof is done along the lines to Theorem 6.2 with the discrete solution operators S_h^w and $(S_h^w)^*$.

Also in the fully discretized scheme we are not restricted to feasible controls. To this end, we consider discrete controls $(\tilde{u}_N, \tilde{w}_N) \in U_h \times V_h$ that are infeasible for the transformed problem $(\mathrm{Q}_{w,h}^\varepsilon)$. The corresponding discrete state is denoted by $\tilde{y}_N$. Again, let $(u_N, w_N) \in U_{ad,h} \times W_{ad,h}$ be a pair of feasible controls with the associated state y_N and adjoint state p_N, respectively. The next result is a direct consequence of Corollary 6.5 and the triangle inequality.

Corollary 6.6 *Let $\delta_u \in U_h$ and $\delta_w \in V_h$ be functions satisfying the variational inequalities* (6.28) *and* (6.29)*, respectively. Then the error estimate*

$$\begin{aligned}\nu\|\bar{u}_h^\varepsilon - \tilde{u}_N\|^2_{L^2(\Gamma)} + \psi(\varepsilon)\|\bar{v}_h^\varepsilon - \tilde{v}_N\|^2_{L^2(\Omega)} \leq 2\Bigg(&\frac{1}{\nu}\|\delta_u\|^2_{L^2(\Gamma)} + \left(1 + \frac{\xi(\varepsilon)^2}{\psi(\varepsilon)}\right)\|\delta_w\|^2_{L^2(\Omega)}\\ &+\nu\|u_N - \tilde{u}_N\|^2_{L^2(\Gamma)} + \psi(\varepsilon)\|v_N - \tilde{v}_N\|^2_{L^2(\Omega)}\Bigg)\end{aligned} \tag{6.31}$$

is valid with $\tilde{v}_N = \frac{1}{\xi(\varepsilon)}(\tilde{w}_N - \tilde{y}_N)$ and $v_N = \frac{1}{\xi(\varepsilon)}(w_N - y_N)$, respectively.

For the purpose of constructing appropriate functions $\delta_u \in U_h$ and $\delta_w \in V_h$, we will sketch the main steps by means of the variational inequality (6.28). We choose

$$u = u_N + \alpha_j \psi_j, \qquad j = 1, \ldots, n_e \tag{6.32}$$

as a feasible test function for (6.28), where ψ_j is a basis function of the finite element space U_h. Thus, we obtain

$$(\tau p_N + \nu u_N + \delta_u \, , \, \alpha_j \psi_j)_{L^2(\Gamma)} \geq 0, \quad j = 1, \ldots, n_e. \tag{6.33}$$

Since the feasible control u_N is an element of the discrete space U_h, we write the function u in the form

$$u = \sum_{i=1}^{n_e} \mathbf{u}_{N,i} \psi_i + \alpha_j \psi_j, \qquad j = 1, \ldots, n_e. \tag{6.34}$$

In all further considerations, bold letters correspond to the vectors containing the particular values of the functions in the nodes of the triangulation. Under notice of feasibility of the test function u, we consider the different cases for the coefficients $\mathbf{u}_{N,j}$:

(i) First, we assume $u_a < \mathbf{u}_{N,j} < u_b$. In order to choose a feasible test function (6.34), the coefficient α_j can accept both signs. This yields for (6.33):

$$(\tau p_N + \nu u_N + \delta_u \, , \, \psi_j)_{L^2(\Gamma)} = 0 \qquad \text{if } \mathbf{u}_{N,j} \in (u_a, u_b).$$

By means of the basis representations of $p_N \in V_h$, $u_N \in U_h$ and $\delta_u \in U_h$, respectively, we obtain

$$\sum_{i=1}^{n} \mathbf{p}_{N,i} (\tau \varphi_i \, , \, \psi_j)_{L^2(\Gamma)} + \sum_{i=1}^{n_e} (\nu \mathbf{u}_{N,i} + \boldsymbol{\delta}_{u,i})(\psi_i \, , \, \psi_j)_{L^2(\Gamma)} = 0 \qquad \text{if } \mathbf{u}_{N,j} \in (u_a, u_b).$$

Introducing the mass matrices

$$\mathbb{M}_\Omega^\Gamma = \big((\psi_i \, , \, \tau \varphi_j)_{L^2(\Gamma)} \big)_{i,j}^{n_e, n} \in \mathbb{R}^{n_e \times n} \tag{6.35}$$

and

$$\mathbb{M}^\Gamma = \big((\psi_i \, , \, \psi_j)_{L^2(\Gamma)} \big)_{i,j}^{n_e, n_e} \in \mathbb{R}^{n_e \times n_e}, \tag{6.36}$$

we end up with

$$(\mathbb{M}^\Gamma \boldsymbol{\delta}_u)_j = - \big((\mathbb{M}_\Omega^\Gamma)^T \mathbf{p}_N + \nu \mathbb{M}^\Gamma \mathbf{u}_N) \big)_j \qquad \text{if } \mathbf{u}_{N,j} \in (u_a, u_b).$$

(ii) Under the assumption $\mathbf{u}_{N,j} = u_a$ and the feasibility of the test function (6.34), the coefficient α_j can only be chosen positive. This yields

$$\sum_{i=1}^{n} \mathbf{p}_{N,i} (\tau \varphi_i \, , \, \psi_j)_{L^2(\Gamma)} + \sum_{i=1}^{n_e} (\nu \mathbf{u}_{N,i} + \boldsymbol{\delta}_{u,i})(\psi_i \, , \, \psi_j)_{L^2(\Gamma)} \geq 0 \qquad \text{if } \mathbf{u}_{N,j} = u_a.$$

Similarly to the definition (6.19) for the continuous case, we choose the coefficients $\boldsymbol{\delta}_{u,j}$ as small as possible. Hence, we define

$$(\mathbb{M}^\Gamma \boldsymbol{\delta}_u)_j = \max \Big\{ 0, - \big((\mathbb{M}_\Omega^\Gamma)^T \mathbf{p}_N + \nu \mathbb{M}^\Gamma \mathbf{u}_N) \big)_j \Big\} \qquad \text{if } \mathbf{u}_{N,j} = u_a.$$

(iii) The case $\mathbf{u}_{N,j} = u_b$ can be treated analogously such that we define

$$(\mathbb{M}^\Gamma \boldsymbol{\delta}_u)_j = \min\left\{0, -\left((\mathbb{M}^\Gamma_\Omega)^T \mathbf{p}_N + \nu \mathbb{M}^\Gamma \mathbf{u}_N)\right)_j\right\} \qquad \text{if } \mathbf{u}_{N,j} = u_b.$$

A similar consideration is done for the variational inequality (6.29) and $\delta_w \in V_h$. We derive for $j = 1, \dots, n$:

$$(\mathbb{M}^\Omega \boldsymbol{\delta}_w)_j = \begin{cases} \max\left\{0, -\left(\mathbb{M}^\Omega (\frac{\phi(\varepsilon)}{\xi(\varepsilon)}\mathbf{p}_N + \frac{\psi(\varepsilon)}{\xi(\varepsilon)}\mathbf{v}_N)\right)_j\right\}, & \text{if } \mathbf{y}_{N,j} + \xi(\varepsilon)\mathbf{v}_{N,j} = \mathbf{y}_{c,j} \\ -\left(\mathbb{M}^\Omega (\frac{\phi(\varepsilon)}{\xi(\varepsilon)}\mathbf{p}_N + \frac{\psi(\varepsilon)}{\xi(\varepsilon)}\mathbf{v}_N)\right)_j, & \text{else} \end{cases}$$

with the mass matrix

$$\mathbb{M}^\Omega = \left((\varphi_i\,,\,\varphi_j)_{L^2(\Omega)}\right)_{i,j}^n \in \mathbb{R}^{n\times n}. \tag{6.37}$$

Summarizing, the quantity $\delta_u \in U_h$ is determined by solving the linear system of equations

$$\mathbb{M}^\Gamma \boldsymbol{\delta}_u = \mathbf{b}_u, \tag{6.38}$$

where the right hand side is given by

$$\mathbf{b}_{u,j} = \begin{cases} \max\left\{0, -\left((\mathbb{M}^\Gamma_\Omega)^T \mathbf{p}_N + \nu \mathbb{M}^\Gamma \mathbf{u}_N)\right)_j\right\}, \text{ if } \mathbf{u}_{N,j} = u_a \\ \min\left\{0, -\left((\mathbb{M}^\Gamma_\Omega)^T \mathbf{p}_N + \nu \mathbb{M}^\Gamma \mathbf{u}_N)\right)_j\right\}, \text{ if } \mathbf{u}_{N,j} = u_b \\ -\left((\mathbb{M}^\Gamma_\Omega)^T \mathbf{p}_N + \nu \mathbb{M}^\Gamma \mathbf{u}_N)\right)_j, \text{ else} \end{cases} \tag{6.39}$$

Analogously, the estimator $\delta_w \in V_h$ is constructed by solving the linear system of equations

$$\mathbb{M}^\Omega \boldsymbol{\delta}_w = \mathbf{b}_w \tag{6.40}$$

with the right hand side

$$\mathbf{b}_{w,j} = \begin{cases} \max\left\{0, -\left(\mathbb{M}^\Omega (\frac{\phi(\varepsilon)}{\xi(\varepsilon)}\mathbf{p}_N + \frac{\psi(\varepsilon)}{\xi(\varepsilon)}\mathbf{v}_N)\right)_j\right\}, & \text{if } \mathbf{y}_{N,j} + \xi(\varepsilon)\mathbf{v}_{N,j} = \mathbf{y}_{c,j} \\ -\left(\mathbb{M}^\Omega (\frac{\phi(\varepsilon)}{\xi(\varepsilon)}\mathbf{p}_N + \frac{\psi(\varepsilon)}{\xi(\varepsilon)}\mathbf{v}_N)\right)_j, & \text{else.} \end{cases} \tag{6.41}$$

We mention that the determination of these estimators is of low cost compared to one iterate of an optimization method solving problems like (P_2^ε) since the system matrices are symmetric positive definite and sparse.

6.2 The primal-dual active set strategy (PDAS)

This section is concerned to the description of the primal-dual active set strategy to solve problems like (P_2^ε). We introduce this method both for the continuous case and for the fully discretized case.

6.2.1 PDAS in function space setting

In order to realize the primal-dual active set strategy, we have two possibilities. One can transform the problem (P_2^ε) to a purely control constraint problem by introducing the new control

$$w_\varepsilon := \bar{y}_\varepsilon + \xi(\varepsilon)\bar{v}_\varepsilon,$$

see Section 6.1.1. Thus, one can apply the standard algorithm that is prescribed, for instance, in [10]. However, this transformation can lead to singular perturbed problems for certain choices of parameter functions and $\varepsilon \downarrow 0$. Consequently, one has to deal with the specific difficulties of these problems.

The second strategy is focused on directly solving the optimality system (6.1)-(6.4), where a Lagrange multiplier corresponding to the mixed control-state constraints was introduced. Deriving this method, we need the pointwise formulation of the complementary slackness condition (6.4) that is given by:

$$\int_{\Omega'} \mu_\varepsilon(x)(y_c(x) - \xi(\varepsilon)\bar{v}_\varepsilon(x) - \bar{y}_\varepsilon(x))dx = 0.$$

Due to $\mu_\varepsilon(x) \geq 0$ and $y_c(x) \leq \bar{y}_\varepsilon(x) + \xi(\varepsilon)\bar{v}_\varepsilon(x)$, this implies

$$\mu_\varepsilon(x)(y_c(x) - \bar{y}_\varepsilon(x) - \xi(\varepsilon)\bar{v}_\varepsilon(x)) = 0, \quad \text{a.e. in } \Omega'.$$

Given the optimal solution $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ for (P_2^ε), we will define the active and inactive sets. First, we discuss the mixed control-state constraints. The active and inactive sets are defined up to sets of measure zero as follows:

$$\begin{aligned} \mathcal{A}^\Omega &:= \{x \in \Omega' \,|\, \xi(\varepsilon)\bar{v}_\varepsilon(x) + \bar{y}_\varepsilon(x) - \mu_\varepsilon(x) < y_c(x)\} \\ \mathcal{I}^\Omega &:= \Omega \setminus \mathcal{A}^\Omega. \end{aligned}$$

Thus, the inequalities in (6.4) can be replaced by associated equalities on the sets $\mathcal{A}^\Omega$ and $\mathcal{I}^\Omega$:

$$\begin{aligned} \xi(\varepsilon)\bar{v}_\varepsilon(x) + \bar{y}_\varepsilon(x) = y_c(x), &\ \text{a.e. on } \mathcal{A}^\Omega \\ \mu_\varepsilon(x) = 0, \qquad &\ \text{a.e. on } \mathcal{I}^\Omega. \end{aligned}$$

We proceed with the control constraints acting at the boundary Γ. The active and inactive sets can be defined by

$$\begin{aligned} \mathcal{A}_-^\Gamma &:= \{x \in \Gamma \,|\, \bar{u}_\varepsilon(x) = u_a\} \\ \mathcal{A}_+^\Gamma &:= \{x \in \Gamma \,|\, \bar{u}_\varepsilon(x) = u_b\} \\ \mathcal{I}^- &:= \Gamma \setminus \{\mathcal{A}_-^\Gamma \cup \mathcal{A}_+^\Gamma\}. \end{aligned}$$

Hence, the variational inequality (6.2) in the optimality system can be replaced by the following explicit expression:

$$\bar{u}_\varepsilon(x) = \begin{cases} u_a \quad , & x \in \mathcal{A}_-^\Gamma \\ u_b \quad , & x \in \mathcal{A}_+^\Gamma \\ -\dfrac{\bar{p}_\varepsilon(x)|_\Gamma}{\nu}, & x \in \mathcal{I}^\Gamma. \end{cases}$$

With the previous considerations at hand, we transform the optimality system (6.1)-(6.4) into:

$$\left.\begin{array}{c}
-\Delta\bar{y}_\varepsilon + \bar{y}_\varepsilon = \phi(\varepsilon)\bar{v}_\varepsilon \qquad -\Delta p_\varepsilon + p_\varepsilon = \bar{y}_\varepsilon - y_d - \mu_\varepsilon \\
\partial_n\bar{y}_\varepsilon = \bar{u}_\varepsilon \qquad\qquad\qquad \partial_n p_\varepsilon = 0 \\
\bar{u}_\varepsilon(x) = \begin{cases} u_a \quad , & x \in \mathcal{A}_-^\Gamma \\ u_b \quad , & x \in \mathcal{A}_+^\Gamma \\ -\dfrac{p_\varepsilon(x)|_\Gamma}{\nu}, & x \in \mathcal{I}^\Gamma. \end{cases} \\
\phi(\varepsilon)p_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\mu_\varepsilon = 0 \quad \text{a.e. in } \Omega \\
\xi(\varepsilon)\bar{v}_\varepsilon(x) + \bar{y}_\varepsilon(x) = y_c(x), \quad \text{a.e. on } \mathcal{A}^\Omega \\
\mu_\varepsilon(x) = 0, \quad \text{a.e. on } \mathcal{I}^\Omega.
\end{array}\right\} \tag{6.42}$$

The primal-dual active set strategy proceeds as follows.

Algorithm 6.7

(1) Define initial sets $\mathcal{A}_-^{\Gamma,0}$, $\mathcal{A}_+^{\Gamma,0}$ *and* $\mathcal{A}^{\Omega,0}$. *Set* $\mathcal{I}^{\Gamma,0} = \Gamma \setminus \{\mathcal{A}_-^{\Gamma,0} \cup \mathcal{A}_+^{\Gamma,0}\}$, $\mathcal{I}^{\Omega,0} = \Omega \setminus \mathcal{A}^{\Omega,0}$ *and* $N = 0$.

(2) Determine the solution $(y_\varepsilon^N, u_\varepsilon^N, v_\varepsilon^N, p_\varepsilon^N, \mu_\varepsilon^N)$ *of the optimality system* (6.42) *on the current active and inactive sets.*

(3) Determine the new active and inactive sets by

$$\begin{aligned}
\mathcal{A}_-^{\Gamma,N+1} &= \left\{x \in \Gamma :\, u_\varepsilon^N(x) - p_\varepsilon^N(x)|_\Gamma - \nu u_\varepsilon^N(x) < u_a\right\} \\
\mathcal{A}_+^{\Gamma,N+1} &= \left\{x \in \Gamma :\, u_\varepsilon^N(x) - p_\varepsilon^N(x)|_\Gamma - \nu u_\varepsilon^N(x) > u_b\right\} \\
\mathcal{I}^{\Gamma,N+1} &= \Gamma \setminus \left\{\mathcal{A}_-^{\Gamma,N+1} \cup \mathcal{A}_+^{\Gamma,N+1}\right\} \\
\mathcal{A}^{\Omega,N+1} &= \{x \in \Omega' :\, \xi(\varepsilon)v_\varepsilon^N(x) + y_\varepsilon^N(x) - \mu_\varepsilon^N(x) < y_c(x)\} \\
\mathcal{I}^{\Omega,N+1} &= \Omega \setminus \mathcal{A}^{\Omega,N+1}
\end{aligned}$$

(4) IF $\mathcal{A}_-^{\Gamma,N+1} = \mathcal{A}_-^{\Gamma,N}$, $\mathcal{A}_+^{\Gamma,N+1} = \mathcal{A}_+^{\Gamma,N}$ *and* $\mathcal{A}^{\Omega,N+1} = \mathcal{A}^{\Omega,N}$, *then STOP ELSE: Set* $N := N + 1$ *and goto (2).*

The justification of the termination condition in step (4) is given by the following well known theorem.

Theorem 6.8 *If* $\mathcal{A}_-^{\Gamma,N+1} = \mathcal{A}_-^{\Gamma,N}$, $\mathcal{A}_+^{\Gamma,N+1} = \mathcal{A}_+^{\Gamma,N}$ *and* $\mathcal{A}^{\Omega,N+1} = \mathcal{A}^{\Omega,N}$ *for some* $N \in \mathbb{N}$ *then the current solution* $(y_\varepsilon^N, u_\varepsilon^N, v_\varepsilon^N, p_\varepsilon^N, \mu^N)$ *of* (6.42) *satisfies the optimality system* (6.1)-(6.4).

For a proof and more details concerning the termination condition and the convergence of the algorithm, we refer to [10, 37, 47, 70].

6.2.2 PDAS for the fully discretized problem

In this section we apply the primal-dual active set strategy to the optimality system (5.22)-(5.26) of problem ($\mathrm{P}^{\varepsilon}_{2,h}$). We start with the consideration of the discrete state equation. Due to the definition (5.18) of the discrete solution operator, the discrete state equation (5.22) is equivalent to the weak formulation

$$a(\bar{y}^{\varepsilon}_h, z_h) = \int\limits_{\Gamma} \bar{u}^{\varepsilon}_h z_h \, ds + \int\limits_{\Omega} \phi(\varepsilon)\bar{v}^{\varepsilon}_h z_h \, dx \quad \forall z_h \in V_h.$$

Within our discrete framework, introduced in the beginning of Section 5.1, we seek for discrete solutions $\bar{y}^{\varepsilon}_h, \bar{v}^{\varepsilon}_h \in V_h$ and $\bar{u}^{\varepsilon}_h \in U_h$ of the form

$$\bar{y}^{\varepsilon}_h(x) = \sum_{i=1}^{n} \mathbf{y}^{\varepsilon}_i \varphi_i(x), \quad \bar{v}^{\varepsilon}_h(x) = \sum_{i=1}^{n} \mathbf{v}^{\varepsilon}_i \varphi_i(x) \quad \text{and} \quad \bar{u}^{\varepsilon}_h(x) = \sum_{i=1}^{n_e} \mathbf{u}^{\varepsilon}_i \psi_i(x), \tag{6.43}$$

where the bold letters again correspond to the vectors containing the values of the functions in the nodes of the triangulation. Inserting (6.43) in the weak formulation and using $z_h = \varphi_j$, $j = 1, \ldots, n$ as test functions yields

$$(\mathbb{K} + \mathbb{M}^{\Omega})\mathbf{y}^{\varepsilon} = \mathbb{M}^{\Gamma}_{\Omega}\mathbf{u}^{\varepsilon} + \phi(\varepsilon)\mathbb{M}^{\Omega}\mathbf{v}^{\varepsilon}$$

with

$$\mathbb{K}_{ij} := \int\limits_{\Omega} \nabla\varphi_i \cdot \nabla\varphi_j \, dx, \quad i, j = 1, \ldots, n,$$

and the mass matrices $\mathbb{M}^{\Omega}$ and $\mathbb{M}^{\Gamma}_{\Omega}$ defined in (6.37) and (6.35), respectively. Analogously, we find for the discrete adjoint equation (5.23)

$$(\mathbb{K} + \mathbb{M}^{\Omega})\mathbf{p}^{\varepsilon} = \mathbb{M}^{\Omega}(\mathbf{y}^{\varepsilon} - \mathbf{y}_d - \boldsymbol{\mu}^{\varepsilon})$$

with $\mathbf{y}_{d,i} := y_d(x_i)$, $i = 1, \ldots, n$. We proceed with the evaluation of the gradient equation (5.25) at the nodes of triangulation such that we obtain

$$\phi(\varepsilon)\mathbf{p}^{\varepsilon}_i + \psi(\varepsilon)\mathbf{v}^{\varepsilon}_i - \xi(\varepsilon)\boldsymbol{\mu}^{\varepsilon}_i = 0, \quad i = 1, \ldots, n.$$

Next, we introduce the following index set

$$\mathcal{J} := \{i \in \{1, \ldots, n\} : x_i \in \Omega'\}$$

containing the indices of the nodes that belong to the inner subdomain Ω'. With that at hand, we define the discrete counterparts of the active and inactive sets associated with the mixed constraints by

$$\begin{aligned} \mathcal{A}^{\Omega}_h &:= \{i \in \mathcal{J} \,|\, \xi(\varepsilon)\mathbf{v}^{\varepsilon}_i + \mathbf{y}^{\varepsilon}_i - \boldsymbol{\mu}^{\varepsilon}_i < \mathbf{y}_{c,i}\} \\ \mathcal{I}^{\Omega}_h &:= \{1, \ldots, N\} \setminus \mathcal{A}^{\Omega}_h. \end{aligned} \tag{6.44}$$

Similarly to the continuous case, the inequalities in (5.26) can be replaced by equations using the previous sets and a pointwise evaluation. For that purpose, we define diagonal matrices $\mathbb{E}^{\Omega} \in \mathbb{R}^{n \times n}$ by

$$\mathbb{E}^{\Omega}_{ij} := \begin{cases} 1, & \text{if } i = j \text{ and } i \in \mathcal{A}^{\Omega}_h \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the discrete version of $\xi(\varepsilon)\bar{v}_\varepsilon + \bar{y}_\varepsilon = y_c$ a.e. in $\mathcal{A}^\Omega$ is given by

$$\mathbb{E}^\Omega(\xi(\varepsilon)\mathbf{v}^\varepsilon + \mathbf{y}^\varepsilon) = \mathbb{E}^\Omega \mathbf{y}_c. \tag{6.45}$$

Moreover, we find for the discrete Lagrange multiplier

$$(\mathbb{I}^\Omega - \mathbb{E}^\Omega)\boldsymbol{\mu}^\varepsilon = 0, \tag{6.46}$$

where $\mathbb{I}^\Omega$ denotes the $n \times n$ identity matrix. Adding (6.45) and (6.46) yields

$$\mathbb{E}^\Omega(\xi(\varepsilon)\mathbf{v}^\varepsilon + \mathbf{y}^\varepsilon) + (\mathbb{I} - \mathbb{E}^\Omega)\boldsymbol{\mu}^\varepsilon = \mathbb{E}^\Omega \mathbf{y}_c. \tag{6.47}$$

Finally, we consider the variational inequality (5.24). Unfortunately, a pointwise evaluation, like in the continuous case, is not possible since the control is discretized by piecewise linear ansatz functions. However, we adapted the strategy of Chapter 6.1.3 p.92 cf., where a similar variational inequality was considered. We continue with introducing index sets representing the discrete counterparts of the active and inactive sets on the boundary:

$$\begin{aligned}
\mathcal{A}^\Gamma_{-,h} &:= \{i \in \{1,\dots,n_e\} \,|\, \mathbf{u}^\varepsilon_i - \big((\mathbb{M}^\Gamma_\Omega)^T \mathbf{p}^\varepsilon + \nu \mathbb{M}^\Gamma \mathbf{u}^\varepsilon\big)_i < u_a\} \\
\mathcal{A}^\Gamma_{+,h} &:= \{i \in \{1,\dots,n_e\} \,|\, \mathbf{u}^\varepsilon_i - \big((\mathbb{M}^\Gamma_\Omega)^T \mathbf{p}^\varepsilon + \nu \mathbb{M}^\Gamma \mathbf{u}^\varepsilon\big)_i > u_b\} \\
\mathcal{I}^\Gamma_h &:= \{1,\dots,n_e\} \setminus \{\mathcal{A}^\Gamma_{-,h} \cup \mathcal{A}^\Gamma_{+,h}\}.
\end{aligned}$$

The matrix $\mathbb{M}^\Gamma$ was defined in (6.36). One can easily check that the variational inequality can be replaced by the following equations on these sets

$$\begin{aligned}
\mathbf{u}^\varepsilon_i &= u_a, \quad \text{if } i \in \mathcal{A}^\Gamma_{-,h} \\
\mathbf{u}^\varepsilon_i &= u_b, \quad \text{if } i \in \mathcal{A}^\Gamma_{+,h} \\
\big((\mathbb{M}^\Gamma_\Omega)^T \mathbf{p}^\varepsilon + \nu \mathbb{M}^\Gamma \mathbf{u}^\varepsilon\big)_i &= 0, \quad \text{if } i \in \mathcal{I}^\Gamma_h.
\end{aligned} \tag{6.48}$$

Forthcoming, we define diagonal matrices $\mathbb{E}^\Gamma_-$, $\mathbb{E}^\Gamma_+ \in \mathbb{R}^{n_e \times n_e}$ associated with the active sets by

$$\mathbb{E}^\Gamma_{-,ij} := \begin{cases} 1, & \text{if } i = j \text{ and } i \in \mathcal{A}^\Gamma_{-,h} \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \mathbb{E}^\Gamma_{+,ij} := \begin{cases} 1, & \text{if } i = j \text{ and } i \in \mathcal{A}^\Gamma_{+,h} \\ 0, & \text{otherwise.} \end{cases}$$

With that at hand, we find for (6.48)

$$\underbrace{(\mathbb{I}^\Gamma - \mathbb{E}^\Gamma_- - \mathbb{E}^\Gamma_+)}_{=:\mathbb{A}}\big((\mathbb{M}^\Gamma_\Omega)^T \mathbf{p}^\varepsilon + \nu \mathbb{M}^\Gamma \mathbf{u}^\varepsilon\big) + (\mathbb{E}^\Gamma_- + \mathbb{E}^\Gamma_+)\mathbf{u}^\varepsilon = \mathbb{E}^\Gamma_- \mathbf{u}_a + \mathbb{E}^\Gamma_+ \mathbf{u}_b, \tag{6.49}$$

where $\mathbf{u}_a$ and $\mathbf{u}_b$ are vectors of length n_e containing the real numbers u_a and u_b, respectively. Finally, we deduce the following linear system of equations as a discrete analogon to (6.42):

$$\mathbb{KKT}\mathbf{x} = \mathbf{b} \tag{6.50}$$

with

$$\mathbb{KKT} := \begin{pmatrix}
\mathbb{M}^\Gamma_\Omega & -(\mathbb{K} + \mathbb{M}^\Omega) & \phi(\varepsilon)\mathbb{M}^\Omega & 0 & 0 \\
0 & \mathbb{M}^\Omega & 0 & -(\mathbb{K} + \mathbb{M}^\Omega) & -\mathbb{M}^\Omega \\
0 & 0 & \psi(\varepsilon)\mathbb{I}^\Omega & \phi(\varepsilon)\mathbb{I}^\Omega & -\xi(\varepsilon)\mathbb{I}^\Omega \\
\nu \mathbb{A}\mathbb{M}^\Gamma + \mathbb{E}^\Gamma_- + \mathbb{E}^\Gamma_+ & 0 & 0 & \mathbb{A}(\mathbb{M}^\Gamma_\Omega)^T & 0 \\
0 & \mathbb{E}^\Omega & \xi(\varepsilon)\mathbb{E}^\Omega & 0 & \mathbb{I} - \mathbb{E}^\Omega
\end{pmatrix} \tag{6.51}$$

and

$$\mathbf{x} := (\mathbf{u}^\varepsilon,\, \mathbf{y}^\varepsilon,\, \mathbf{v}^\varepsilon,\, \mathbf{p}^\varepsilon,\, \boldsymbol{\mu}^\varepsilon)^T, \quad \mathbf{b} := (0,\, -\mathbb{M}^\Omega \mathbf{y}_d,\, 0,\, \mathbb{E}^\Gamma_- \mathbf{u}_a + \mathbb{E}^\Gamma_+ \mathbf{u}_b,\, \mathbb{E}^\Omega \mathbf{y}_c)^T.$$

Next, we formulate the primal-dual active set algorithm for this fully discretized scheme. Furthermore, we use the error estimates and the related quantities δ_u and δ_w, derived in Section 6.1.3, as an alternative stopping criterion. To this end, we denote by $(\tilde{y}^\varepsilon_N,\, \tilde{u}^\varepsilon_N,\, \tilde{v}^\varepsilon_N,\, \tilde{p}^\varepsilon_N,\, \tilde{\mu}^\varepsilon_N)$ the current iterate of the PDAS-algorithm. We emphasize that the current control $(\tilde{u}^\varepsilon_N,\, \tilde{v}^\varepsilon_N)$ is in general infeasible. However, for the construction of the estimators δ_u and δ_w, respectively, we need appropriate feasible controls. We start with introducing the new control $\tilde{w}^\varepsilon_N := \tilde{y}^\varepsilon_N + \xi(\varepsilon)\tilde{v}^\varepsilon_N$. Based on $(\tilde{u}^\varepsilon_N,\, \tilde{w}^\varepsilon_N)$, we define

$$u^\varepsilon_N = \sum_{i=1}^{n_e} \mathbf{u}^\varepsilon_{N,i}\psi_i \quad \text{with} \quad \mathbf{u}^\varepsilon_{N,i} = \max\{u_a, \min\{u_b, \tilde{\mathbf{u}}_{N,i}\}\} \tag{6.52}$$

and

$$w^\varepsilon_N := \sum_{i=1}^{n} \mathbf{w}^\varepsilon_{N,i}\varphi_i \quad \text{with} \quad \mathbf{w}^\varepsilon_{N,i} = \max\{\mathbf{y}_{c,i}, \tilde{\mathbf{w}}_{N,i}\}. \tag{6.53}$$

This control $(u^\varepsilon_N,\, w^\varepsilon_N)$ is feasible for the discretized converted problem $(\mathrm{Q}^\varepsilon_{w,h})$, where the admissible set $W_{ad,h}$ associated with the discrete distributed controls is defined by

$$W_{ad,h} := \{w \in V_h \,|\, w(x_i) \geq y_c(x_i),\, i \in \mathcal{J}\},$$

i.e. the inequality constraints are only required in the nodes of the triangulation. Due to (6.45), the previous definition of the admissible set is reasonable. The associated state y^ε_N to the control $(u^\varepsilon_N,\, w^\varepsilon_N)$ is given by

$$y^\varepsilon_N = S^w_h(\tau^* u^\varepsilon_N + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)} E^*_H w^\varepsilon_N).$$

The definition (6.23) of the discrete control-to-state operator yields

$$(\mathbb{K} + (1 + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)})\mathbb{M}^\Omega)\mathbf{y}^\varepsilon_N = \mathbb{M}^\Gamma_\Omega \mathbf{u}^\varepsilon_N + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)}\mathbb{M}^\Omega \mathbf{w}^\varepsilon_N. \tag{6.54}$$

Analogously, we find for the adjoint state $p^\varepsilon_N = (S^w_h)^*((1 + \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2})y^\varepsilon_N - y_d - \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2} w^\varepsilon_N)$

$$(\mathbb{K} + (1 + \tfrac{\phi(\varepsilon)}{\xi(\varepsilon)})\mathbb{M}^\Omega)\mathbf{y}^\varepsilon_N = \mathbb{M}^\Omega \left((1 + \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2})\mathbf{y}^\varepsilon_N - \mathbf{y}_d - \tfrac{\psi(\varepsilon)}{\xi(\varepsilon)^2}\mathbf{w}^\varepsilon_N\right). \tag{6.55}$$

We consider the feasible control $(u^\varepsilon_N,\, v^\varepsilon_N)$ with $v^\varepsilon_N = \frac{1}{\xi(\varepsilon)}(w^\varepsilon_N - y^\varepsilon_N)$ as the current iterative solution of the optimality system (5.22)-(5.26). Hence, we use the error estimate of Corollary 6.5 and the corresponding estimators δ_u and δ_w as a reliable stopping criterion for the primal-dual active set algorithm.

Algorithm 6.9

(1) *Define initial active sets* $\mathcal{A}^{\Gamma,0}_{-,h}$, $\mathcal{A}^{\Gamma,0}_{+,h}$, $\mathcal{A}^{\Omega,0}_{h}$ *and choose an error tolerance* $\rho > 0$. *Set*

$$\mathcal{I}^{\Gamma,0}_h = \{1, \ldots, n_e\} \setminus \{\mathcal{A}^{\Gamma,0}_{-,h} \cup \mathcal{A}^{\Gamma,0}_{+,h}\},$$
$$\mathcal{I}^{\Omega,0}_h = \{1, \ldots, n\} \setminus \mathcal{A}^{\Omega,0}_h$$

and $N = 0$.

(2) Assemble the matrix (6.51) *with respect to the current active and inactive sets. Determine* $(\tilde{\mathbf{u}}_N^\varepsilon, \tilde{\mathbf{y}}_N^\varepsilon, \tilde{\mathbf{v}}_N^\varepsilon, \tilde{\mathbf{p}}_N^\varepsilon, \tilde{\boldsymbol{\mu}}_N^\varepsilon)$ *as the solution of* (6.50).

(3) Set $\tilde{w}_N^\varepsilon := \tilde{y}_N^\varepsilon + \xi(\varepsilon)\tilde{v}_N^\varepsilon$. *Construct the feasible control* $(u_N^\varepsilon, w_N^\varepsilon)$ *by* (6.52) *and* (6.53), *respectively.*
Determine $\mathbf{y}_N^\varepsilon$ *and* $\mathbf{p}_N^\varepsilon$ *as the solutions of* (6.54) *and* (6.55).
Set $\mathbf{v}_N^\varepsilon = \frac{1}{\xi(\varepsilon)}(\mathbf{w}_N^\varepsilon - \mathbf{y}_N^\varepsilon)$.

(4) Determine the quantities $\boldsymbol{\delta}_u$ *and* $\boldsymbol{\delta}_w$ *according to* (6.38) *and* (6.40), *respectively.*

(5) IF

$$C_1(\nu,\varepsilon)\|\delta_u\|_{L^2(\Gamma)} + C_2(\nu,\varepsilon)\|\delta_w\|_{L^2(\Omega)} \le \rho$$

with $C_1(\nu,\varepsilon) = \frac{\sqrt{2}}{\sqrt{\nu}\min\{\sqrt{\nu},\sqrt{\psi(\varepsilon)}\}}$ *and* $C_2(\nu,\varepsilon) = \frac{\sqrt{2(1+\frac{\xi(\varepsilon)^2}{\psi(\varepsilon)})}}{\min\{\sqrt{\nu},\sqrt{\psi(\varepsilon)}\}}$ *is satisfied, then STOP,*
ELSE: Determine the new active and inactive sets by

$$\begin{aligned}
\mathcal{A}_{-,h}^{\Gamma,N+1} &= \left\{i \in \{1,\dots,n_e\} \mid \tilde{\mathbf{u}}_{N,i}^\varepsilon - \left((\mathbb{M}_\Gamma^\Omega)^T\tilde{\mathbf{p}}_N^\varepsilon + \nu\mathbb{M}^\Gamma\tilde{\mathbf{u}}_N^\varepsilon\right)_i < u_a\right\}\\
\mathcal{A}_{+,h}^{\Gamma,N+1} &= \left\{i \in \{1,\dots,n_e\} \mid \tilde{\mathbf{u}}_{N,i}^\varepsilon - \left((\mathbb{M}_\Gamma^\Omega)^T\tilde{\mathbf{p}}_N^\varepsilon + \nu\mathbb{M}^\Gamma\tilde{\mathbf{u}}_N^\varepsilon\right)_i > u_b\right\}\\
\mathcal{I}_h^{\Gamma,N+1} &= \{1,\dots,n_e\} \setminus \left\{\mathcal{A}_{-,h}^{\Gamma,N+1} \cup \mathcal{A}_{+,h}^{\Gamma,N+1}\right\}\\
\mathcal{A}_h^{\Omega,N+1} &= \{i \in \mathcal{J} \mid \xi(\varepsilon)\tilde{\mathbf{v}}_{N,i}^\varepsilon + \tilde{\mathbf{y}}_{N,i}^\varepsilon - \tilde{\boldsymbol{\mu}}_{N,i}^\varepsilon < \mathbf{y}_{c,i}\}\\
\mathcal{I}_h^{\Omega,N+1} &= \Omega \setminus \mathcal{A}_h^{\Omega,N+1}.
\end{aligned}$$

Set $N := N+1$ *and GOTO (2).*

Remark 6.10 In the previous algorithm, we used the constructed feasible control $(u_N^\varepsilon, v_N^\varepsilon)$ as the current iterative solution of the discrete optimality system (5.22)-(5.26). If we are more interested in the common iterate $(\tilde{u}_N^\varepsilon, \tilde{v}_N^\varepsilon)$ of the PDAS-algorithm, calculated in step (2), the situation with respect to the termination condition is slightly different. The estimators δ_u and δ_w are still determined with respect to a feasible control according to (6.38) and (6.40), respectively. However, as an reliable stopping criterion for $\|\bar{u}_h^\varepsilon - \tilde{u}_N^\varepsilon\|_{L^2(\Gamma)}$ and $\|\bar{v}_h^\varepsilon - \tilde{v}_N^\varepsilon\|_{L^2(\Omega)}$ we have to apply Corollary 6.6. Thus, the termination condition in step (5) of the previous algorithm has to be adapted respective the error estimate (6.31).

6.3 Numerical example

This section is devoted to the illustration of the theoretical results of the previous chapters. To this end, we will first construct an analytical solution for an optimal control problem similarly to our original problem (P). Forthcoming, we will compare the results concerning the regularization error estimate derived in Section 4.2.2 by several numerical tests. We will proceed with taking also the discretization error into account, i.e. we will illustrate the regularization and discretization error estimate of Section 5.4.2. Finally, we will observe the error arising during the optimization by the primal-dual active set strategy. Moreover, we will study the quality of the error estimators, constructed in Section 6.1.3, as a stopping criterion.

6.3.1 Construction of an analytical solution

In order to illustrate the theoretical results of Chapter 4 and Chapter 5, respectively, we consider the following optimal control problem with pure state constraints and control constraints acting on the boundary

$$\left.\begin{aligned} \min \quad J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u-u_d\|^2_{L^2(\Gamma)} \\ -\Delta y + y = f \qquad \text{in } \Omega \\ \partial_n y = u + g \qquad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. on } \Gamma \\ y(x) \ge y_c(x) \quad \text{a.e. in } \Omega', \end{aligned}\right\} \tag{PT}$$

with given functions y_d, $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ and $u_d \in H^1(\Gamma)$. Let $\Omega = (0,1)^2$ be the unit square and let $\Omega' = (0.25, 0.75)^2$ be an inner square of Ω. We construct an analytical solution ($\bar{y}$, $\bar{u}$, p, μ), such that the following optimality system of problem (PT) is satisfied

$$\begin{aligned} -\Delta\bar{y} + \bar{y} &= f & \qquad -\Delta p + p &= \bar{y} - y_d - \mu \\ \partial_n \bar{y} &= \bar{u} + g & \partial_n p &= 0 \end{aligned}$$

$$\begin{aligned} &(\tau p + \nu(\bar{u} - u_d)\,,\, u - \bar{u})_{L^2(\Gamma)} \ge 0, \quad \forall u \in U^L_{ad} \\ &\int_{\Omega'} (y_c - \bar{y})d\mu = 0, \quad \bar{y}(x) \ge y_c(x) \quad \text{a.e in } \Omega' \\ &\int_{\Omega'} \varphi d\mu \ge 0 \quad \forall \varphi \in C(\Omega'), \quad \varphi(x) \ge 0 \quad \forall x \in \Omega'. \end{aligned}$$

We are interested in an example, where the Lagrange multiplier μ associated with the state constraints is only a measure along a curve in the domain Ω'. First, we choose

$$\bar{y}(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2), \quad x \in \Omega$$

as the optimal state. Hence, the state equation implies

$$f(x_1,x_2) = (2\pi^2+1)\sin(\pi x_1)\sin(\pi x_2), \quad x \in \Omega.$$

In order to fulfill the boundary condition, we define first

$$\tilde{u}(x_1,x_2) = \partial_n\bar{y} = -\pi(\sin(\pi x_1) + \sin(\pi x_2)), \quad x \in \Gamma.$$

The optimal control is given by the pointwise projection on $[u_a, u_b]$:

$$\bar{u}(x_1,x_2) = P_{[u_a,u_b]}\left(\tilde{u}(x_1,x_2)\right), \quad x \in \Gamma.$$

With the help of $g = \tilde{u} - \bar{u}$, the boundary condition of the state equation is satisfied. The lower state constraint y_c is chosen by

$$y_c(x) = \begin{cases} C & , \bar{y}(x) > C \\ 2\bar{y}(x) - C & , \bar{y}(x) \le C, \end{cases}$$

with the constant $C = 0.9$. Due to this choice, the constraint is only active along the curve $\bar{y} = C$. This implies that the associated Lagrange multiplier is a line-measure concentrated on the curve $\bar{y} = C$. In order to achieve this property, the adjoint state is defined with a kink along this curve:

$$p(x_1, x_2) = \begin{cases} \nu \cos(\pi x_1) \cos(\pi x_2) + C_1(\bar{y}(x_1, x_2) - C), & \bar{y}(x_1, x_2) \ge C \\ \nu \cos(\pi x_1) \cos(\pi x_2), & \bar{y}(x_1, x_2) < C \end{cases}$$

with a constant $C_1 = 0.1$. It is easy to verify that $\bar{p}$ satisfies the homogenous Neumann boundary condition. Due to the kink, the gradient of the adjoint state p exhibits a discontinuity along the curve $\bar{y} = C$. Hence, the Lagrange multiplier is a measure concentrated on this curve. We proceed with introducing the regular parts of the domain Ω by

$$\Omega_1 = \{x \in \Omega \,|\, \bar{y}(x) < C\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega \,|\, \bar{y}(x) > C\}.$$

Since the adjoint state is infinitely many times differentiable in Ω_1 and Ω_2, respectively, we can evaluate the desired state y_d separately in Ω_1 and Ω_2 by

$$y_d = \triangle p - p + \bar{y}.$$

It remains to define u_d, which is given by

$$u_d(x) = \frac{1}{\nu} \bar{p}(x)|_\Gamma + \tilde{u}(x).$$

One can easily see that the variational inequality of the optimality system is satisfied. In all further computations the Tikhonov parameter is chosen by $\nu = 1$. Moreover, the boundary constraints are defined as follows:

$$u_a = -3, \qquad u_b = -0.4.$$

We note that the additional functions $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ and $u_d \in H^1(\Gamma)$ in the problem (PT) do not influence the theory of the previous chapters. By means of appropriate transformations, the problem (PT) can be converted to a problem of type (P).

6.3.2 Investigation of the regularization error

In this section we confirm the regularization error estimates derived in Section 4.2.2. For this purpose, the original problem (PT) is replaced by the following regularized optimal control problem

$$\left.\begin{aligned} \min \quad J(y,u,v) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u - u_d\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} \\ -\Delta y + y = \phi(\varepsilon) v + f \qquad \text{in } \Omega \\ \partial_n y = u + g \qquad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. on } \Gamma \\ y(x) \ge y_c(x) - \xi(\varepsilon) v(x) \quad \text{a.e. in } \Omega'. \end{aligned}\right\} \qquad (\mathrm{PT}_\varepsilon)$$

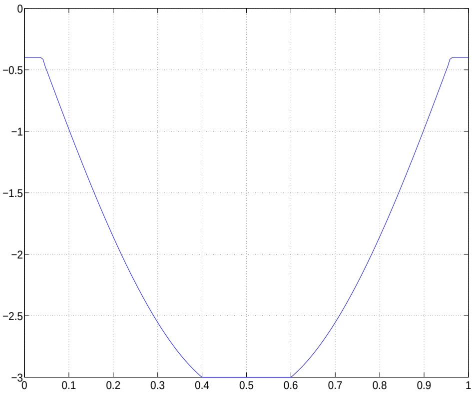

Figure 6.1: Control u_ε

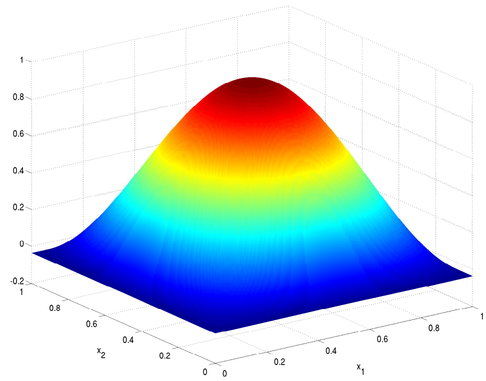

Figure 6.2: State y_ε

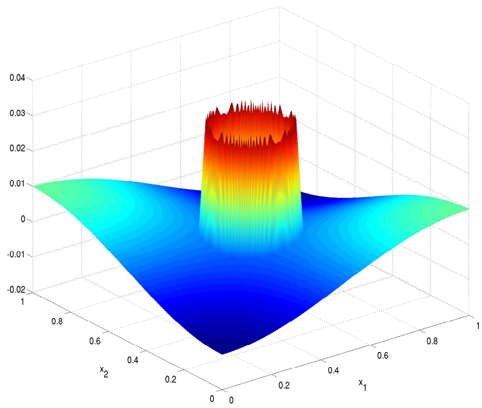

Figure 6.3: Virtual control v_ε

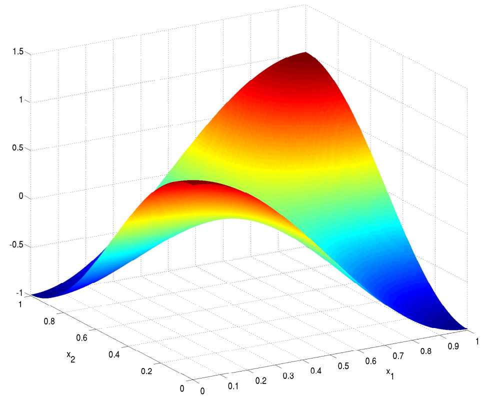

Figure 6.4: Adjoint state p_ε

The regularized problems (PT_ε) were solved by the primal-dual active set method, see Section 6.2. We used a regular and uniform triangulation of the domain Ω. All functions were discretized by piecewise linear finite element functions. The numerical solutions of the regularized problems (PT_ε) are denoted by the subscript ε. Moreover, the optimal control and the optimal state of the unregularized problem (PT) are $\bar{u}$ and $\bar{y}$, respectively. For the first numerical calculation, we used the mesh size $h = 0.005$. The Figures 6.1-6.5 present the numerical solution of (PT_ε) for $\varepsilon = 0.05$ connected with the following choice of parameter functions

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \varepsilon, \quad \xi(\varepsilon) = \varepsilon.$$

Notice that the control in Figure 6.1 is shown only on one part of the boundary. As one can see, the Lagrange multiplier and the virtual control exhibit some irregularities, especially in the active regions around the curve $\bar{y} = C$.

We proceed with investigating the behaviour of the error between the regularized solution and the optimal solution of problem (PT) for $\varepsilon \downarrow 0$. For that purpose, we consider different choices of the parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$. The $L^2(\Gamma)$-error of the numerical control u_ε to the original optimal control is evaluated by the use of the Boole rule. First, we illustrate the dependence of the regularization error

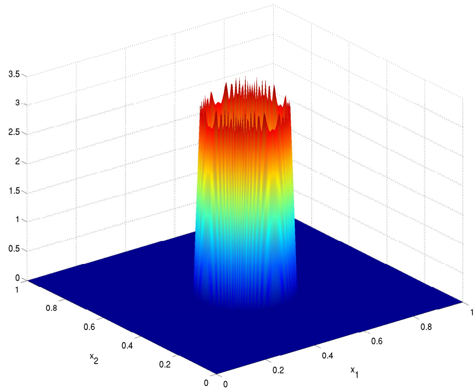

Figure 6.5: Lagrange multiplier μ_ε

on $\xi(\varepsilon)$. We set

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \varepsilon, \quad \xi(\varepsilon) = \varepsilon^{1/2}, \, \varepsilon. \tag{6.56}$$

In order to display the dependence of the error on the regularization parameter, we choose the mesh size $h = 0.0025$ such that the discretization error is sufficiently small. The behaviour of the error for this choice is shown in Figure 6.6, where the left shows the error $\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$ and the right the $L^2(\Omega)$-norm of the virtual control v_ε. The curves illustrate the validity of the error estimate given in Corollary

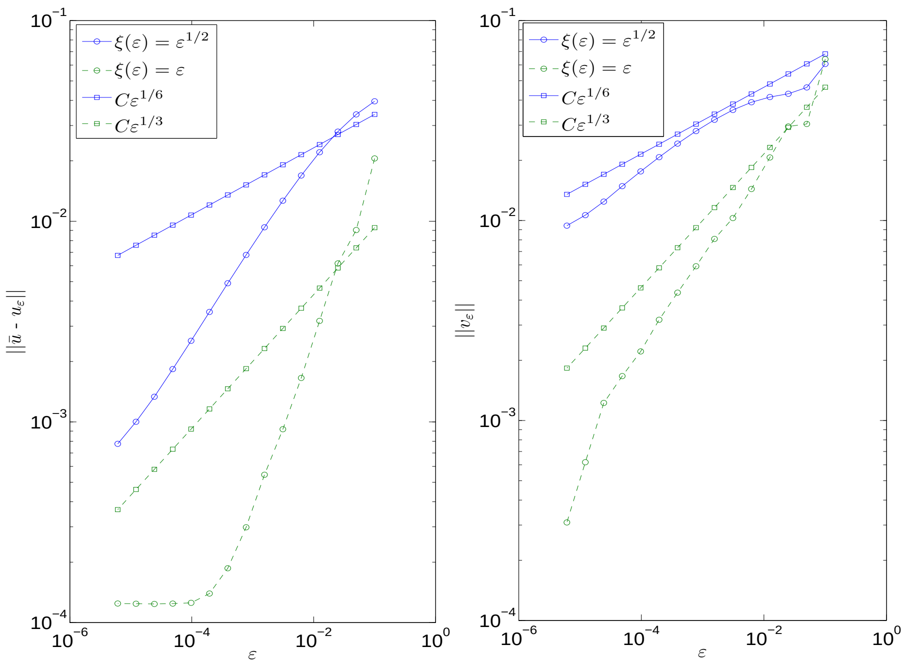

Figure 6.6: Error behaviour for different $\xi(\varepsilon)$

4.12 and Theorem 4.13, respectively. Furthermore, the descent rate of the error is increasing if the exponent of the regularization parameter in the choice of $\xi(\varepsilon)$ increases. Particularly in the dashed green curve for $\xi(\varepsilon) = \varepsilon$, one can see that the discretization error dominates as ε becomes smaller. Nevertheless, for all choices we obtain a better convergence rate than we expected by the theory. For the setting $\psi(\varepsilon) \equiv 1$, $\phi(\varepsilon) = \varepsilon$, $\xi(\varepsilon) = \varepsilon^{1/2}$ we evaluated the experimental order of convergence with respect to ε. The value associated with the boundary control is defined by

$$r_{E_u} := \frac{\ln E_u(\varepsilon_1) - \ln E_u(\varepsilon_2)}{\ln \varepsilon_1 - \ln \varepsilon_2} \quad \text{for} \quad \varepsilon_1 \neq \varepsilon_2$$

with the error functional $E_u(\varepsilon) = \|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$. The experimental convergence rate concerning the virtual control is defined analogously. Table 6.1 shows the values of

ε	$\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$	r_{E_u}	$\|v_\varepsilon\|_{L^2(\Omega)}$	r_{E_v}
$2.5e-2$	$2.7958e-2$	–	$4.3028e-2$	–
$1.25e-2$	$2.2065e-2$	0.34	$4.1452e-2$	0.05
$6.25e-3$	$1.6897e-2$	0.38	$3.9044e-2$	0.08
$3.125e-3$	$1.2653e-2$	0.41	$3.5725e-2$	0.13
$1.5625e-3$	$9.3256e-3$	0.44	$3.1918e-2$	0.16
$7.8125e-4$	$6.7903e-3$	0.46	$2.8001e-2$	0.19
$3.9063e-4$	$4.9075e-3$	0.47	$2.4237e-2$	0.21
$1.9531e-4$	$3.5301e-3$	0.47	$2.0746e-2$	0.22
$9.7656e-5$	$2.5369e-3$	0.48	$1.7602e-2$	0.24
$4.8828e-5$	$1.8321e-3$	0.47	$1.4856e-2$	0.24
$2.4414e-5$	$1.3342e-3$	0.46	$1.2439e-2$	0.25

Table 6.1: Experimental convergence rates for $\psi(\varepsilon) \equiv 1$, $\phi(\varepsilon) = \varepsilon$, $\xi(\varepsilon) = \varepsilon^{1/2}$

the regularization error according to the control and the values of the $L^2(\Omega)$-norm of the virtual control. Moreover, the experimental order of convergence with respect to ε is presented. According to Corollary 4.12 and Theorem 4.13 and the considered parameter functions, we expected for both errors a convergence rate of $\mathcal{O}(\varepsilon^{1/6})$. The experimental rates are better than the theoretical ones, where the rate concerning the L^2-norm of the virtual control differs not so strongly from the expected one.

Next, we study the dependence on $\psi(\varepsilon)$ by the following attitude

$$\phi(\varepsilon) \equiv 1, \quad \xi(\varepsilon) \equiv 1, \quad \psi(\varepsilon) = \varepsilon^{-1},\, \varepsilon^{-3/2}.$$

The results are shown in Figure 6.7. The behaviour of the different curves illustrates the validity of the error estimates given in Corollary 4.12 and Theorem 4.13. Again, we obtain better convergence rates than expected, particularly for error $\|\bar{u}-u_\varepsilon\|_{L^2(\Gamma)}$. In the curves corresponding to the norm of the virtual control the difference is quite small. Table 6.2 presents the errors and the associated experimental orders of convergence for the setting $\phi(\varepsilon) \equiv 1$, $\xi(\varepsilon) \equiv 1$, $\psi(\varepsilon) = \varepsilon^{-1}$. According to the results of Corollary 4.12 and Theorem 4.13, we expect convergence rates $\mathcal{O}(\varepsilon^{2/3})$ for $\|v_\varepsilon\|_{L^2(\Omega)}$ and $\mathcal{O}(\varepsilon^{1/6})$ for the error $\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$, respectively. Like in the first numerical test concerning the dependence on $\xi(\varepsilon)$, the experimental rates are better than the expected ones, where the discrepancy of the values associated with the virtual control is considerably smaller in the current test.

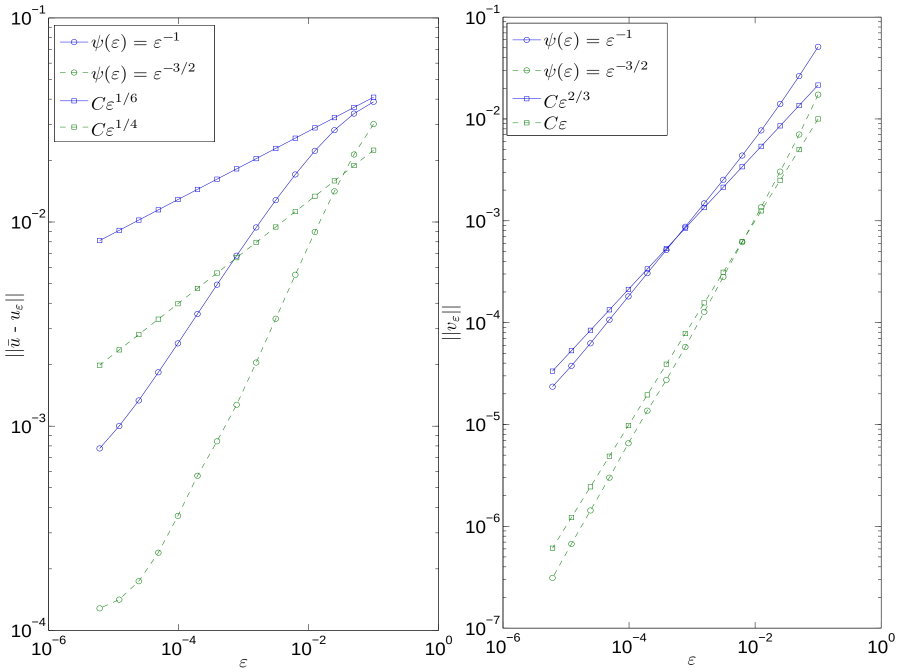

Figure 6.7: Error behaviour for different $\psi(\varepsilon)$

Let us briefly summarize the numerical results of this section. We observed the expected convergence of the optimal control of the regularized problem (PT_ε) to the optimal control of the original problem (PT). However, the approximation rates were better than the expected ones in all numerical tests. We note again, that the error estimate in Theorem 4.13 is the result concerning a worst case scenario.

6.3.3 Investigation of the regularization and discretization error

In this section we study the validity of the regularization and discretization error estimates derived in Section 5.4.2. We compare the solution of the original problem (PT) with the solution of a discretized version of the regularized problem (PT_ε). We will use the discrete framework introduced in the beginning of Chapter 5.1. The resulting discretized analogon to problem (PT_ε) is denoted by ($\mathrm{PT}_{\varepsilon,h}$). We mention that the theory of Chapter 5 can be easily adapted to problems of type (PT_ε). Furthermore, one can follow the lines of Section 6.2.2 in order to apply the primal-dual active set strategy to problem ($\mathrm{PT}_{\varepsilon,h}$). Again, the optimal control and the optimal state of the unregularized and continuous problem (PT) are $\bar{u}$ and $\bar{y}$, respectively. The numerical solutions of problem ($\mathrm{PT}_{\varepsilon,h}$) are denoted by $(\cdot)_h^\varepsilon$.
We observe the regularization and discretization error for the following choice of parameter functions:

$$\phi(\varepsilon) \equiv 1, \quad \xi(\varepsilon) \equiv 1, \quad \psi(\varepsilon) = \varepsilon^{-2}.$$

Due to Assumption 5.32, we have to couple the regularization parameter ε and the mesh size h as follows:

$$\varepsilon \sim h^3 |\log h|^{3/2}.$$

ε	$\|\bar{u} - u_\varepsilon\|_{L^2(\Gamma)}$	r_{E_u}	$\|v_\varepsilon\|_{L^2(\Omega)}$	r_{E_v}
$2.5e-2$	$2.8156e-2$	$-$	$1.4043e-2$	$-$
$1.25e-2$	$2.2317e-2$	0.33	$7.7205e-3$	0.86
$6.25e-3$	$1.7101e-2$	0.38	$4.3720e-3$	0.82
$3.125e-3$	$1.2791e-2$	0.42	$2.5288e-3$	0.79
$1.5625e-3$	$9.4093e-3$	0.44	$1.4816e-3$	0.77
$7.8125e-4$	$6.8381e-3$	0.46	$8.7396e-4$	0.76
$3.9063e-4$	$4.9336e-3$	0.47	$5.1706e-4$	0.76
$1.9531e-4$	$3.5441e-3$	0.48	$3.0586e-4$	0.76
$9.7656e-5$	$2.5442e-3$	0.48	$1.8065e-4$	0.76
$4.8828e-5$	$1.8358e-3$	0.47	$1.0664e-4$	0.76
$2.4414e-5$	$1.3360e-3$	0.46	$6.2663e-5$	0.77

Table 6.2: Experimental convergence rates for $\phi(\varepsilon) = \xi(\varepsilon) = 1$, $\psi(\varepsilon) = 1/\varepsilon$

Based on this setting, the results of Theorem 5.34 and Corollary 5.33 imply the error behaviour

$$\|\bar{u} - u_h^\varepsilon\|_{L^2(\Gamma)} = \mathcal{O}(h|\log h|^{1/2}) \quad \text{and} \quad \|v_h^\varepsilon\|_{L^2(\Omega)} = \mathcal{O}(h^4|\log h|^2).$$

The numerical results are presented in Figure 6.8. Note that we omitted the

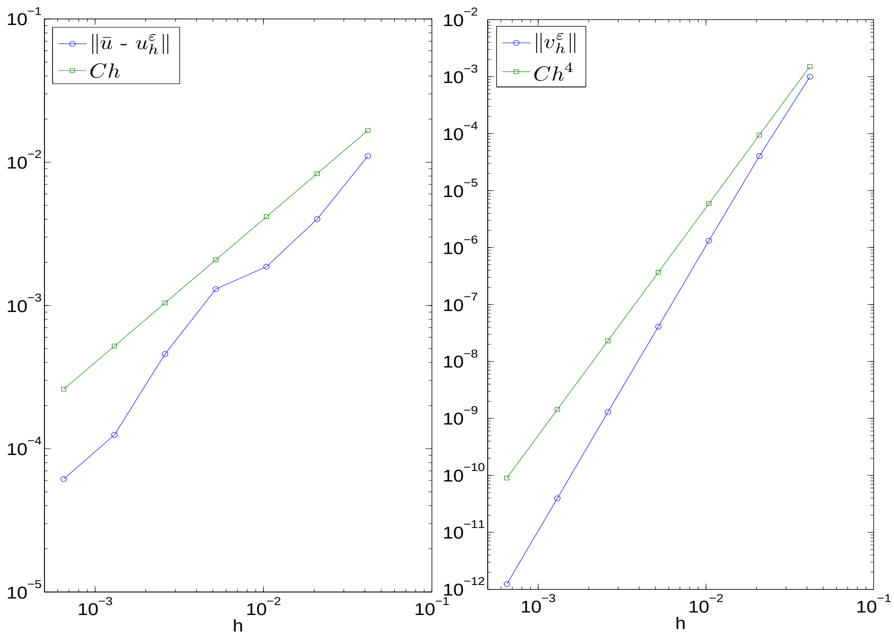

Figure 6.8: $\|\bar{u} - u_h^\varepsilon\|_{L^2(\Gamma)}$ and $\|v_h^\varepsilon\|_{L^2(\Omega)}$

logarithmic parts in the particular reference curves. As one can see, the errors $\|\bar{u} - u_h^\varepsilon\|_{L^2(\Gamma)}$ and $\|v_h^\varepsilon\|_{L^2(\Omega)}$ validate the results predicted in Theorem 5.34 and Corollary 5.33, whereas $\|v_h^\varepsilon\|_{L^2(\Omega)}$ shows a better convergence behaviour. Furthermore, the experimental orders of convergence with respect to h are presented in Table 6.3. One can see the small gap between the theory and the numerical results, especially in the experimental rate concerning the virtual control.

h	ε	$\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)}$	r_u	$\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}$	r_v
2^{-4}	$2.8935e-2$	$1.1043e-2$	1.25	$9.9821e-4$	4.93
2^{-5}	$3.6169e-3$	$4.0084e-3$	1.21	$4.0178e-5$	4.99
2^{-6}	$4.5211e-4$	$1.8653e-3$	1.23	$1.3128e-6$	5.00
2^{-7}	$5.6514e-5$	$1.3016e-3$	1.47	$4.0859e-8$	5.00
2^{-8}	$7.0643e-6$	$4.5900e-4$	1.45	$1.2913e-9$	5.01
2^{-9}	$8.8303e-7$	$1.2495e-4$	1.02	$3.9496e-11$	4.99
2^{-10}	$1.1038e-7$	$6.1408e-5$	–	$1.2387e-12$	–

Table 6.3: Errors and experimental order of convergence

6.3.4 Quality of the iteration error estimators

In this section we investigate the validity of the error estimate given in Corollary 6.5 by our numerical example, where the feasible controls are constructed by the current iterates of the primal-dual active set strategy. Moreover, we study the quality of the associated error estimators δ_u and δ_w, introduced in Section 6.1.3, as an alternative stopping criterion for the algorithm.

In the sequel, we consider the regularized and discretized problem $(\mathrm{PT}_{\varepsilon,h})$ for a fixed regularization parameter ε and a fixed mesh size h, respectively. The optimal solution $(\bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ of problem $(\mathrm{PT}_{\varepsilon,h})$ is determined by the primal-dual active set strategy using the common termination condition, i.e. if there is no change in the active and inactive sets, then the iterate is optimal, see [10]. We denote by $(\tilde{u}_N, \tilde{v}_N)$ the current iterate, computed in step (2) of Algorithm 6.9. Then, step (3) of Algorithm 6.9 delivers an associated feasible control (u_N, v_N). The quantities δ_u and δ_w, respectively, are constructed by (6.38) and (6.40). Note, that the right hand side for (6.38) has to be modified since the occurrence of the function u_d changes the variational inequality (6.28). We remind that the Tikhonov parameter is $\nu = 1$. For the numerical tests we fix the mesh size $h = 0.005$ and we choose the following parameter functions

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \varepsilon, \quad \xi(\varepsilon) = \varepsilon.$$

Due to this setting, the error estimate (6.30) can be simplified to

$$\|\bar{u}_h^\varepsilon - u_N\|_{L^2(\Gamma)} + \|\bar{v}_h^\varepsilon - v_N\|_{L^2(\Omega)} \leq \sqrt{2}(\|\delta_u\|_{L^2(\Gamma)} + \sqrt{1+\varepsilon^2}\|\delta_w\|_{L^2(\Omega)}). \tag{6.57}$$

The results for $\varepsilon = 0.01$ are presented in Table 6.4 and illustrate the validity of the previous error estimate. Furthermore, it is visible that the error estimators δ_u and δ_w, respectively, cannot be used separately. One can easily see that the individual quantity δ_u guarantees no reliable error estimate for the error $\|\bar{u}_h^\varepsilon - u_N\|_{L^2(\Gamma)}$. Except $N = 4$, the values of $\|\delta_u\|_{L^2(\Gamma)}$ imply that the active and inactive sets associated with the boundary control constraints are not changing. However, this is not the case for the mixed control-state constraints in Ω', and of course this iteration error influences the error at the boundary, too. Consequently, the error estimators δ_u and δ_w, respectively, ensure only together a reliable estimate of the iteration error. The quantities δ_u and δ_w are useful as an alternative stopping parameter and of low cost, compared to the effort for solving the whole Karush-Kuhn-Tucker system

N	$\|\bar{u}_h^\varepsilon - u_N\|_{L^2(\Gamma)}$	$\sqrt{2}\|\delta_u\|_{L^2(\Gamma)}$	$\|\bar{v}_h^\varepsilon - v_N\|_{L^2(\Omega)}$	$\sqrt{2(1+\varepsilon^2)}\|\delta_w\|_{L^2(\Omega)}$
1	$3.4198e-2$	$1.7039e+2$	$2.2818e+0$	$1.7739e+2$
2	$2.5424e-2$	$1.0035e-8$	$9.3675e-1$	$9.4092e+1$
3	$1.0848e-2$	$1.0959e-8$	$3.0877e-1$	$3.1441e+1$
4	$3.7372e-3$	$4.8961e-4$	$8.8023e-2$	$9.4249e+0$
5	$7.7522e-4$	$1.1412e-8$	$1.8401e-2$	$2.1961e+0$
6	$6.6326e-5$	$1.1697e-8$	$2.2044e-3$	$3.1091e-1$
7	$9.5666e-7$	$1.1296e-8$	$6.4341e-5$	$9.2847e-3$
8	0	$1.1468e-8$	$9.4792e-11$	$6.6913e-9$

Table 6.4: Iteration errors of the primal-dual active set strategy $\varepsilon = 0.01$

in one iteration of the primal-dual active set method. Due to the knowledge of the optimal solution of problem (PT), we evaluated the following discretization and regularization error

$$\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)} \approx 3.1947e-3 \quad \text{and} \quad \|\bar{v}_h^\varepsilon\|_{L^2(\Omega)} \approx 1.8426e-2$$

numerically. If the threshold value ρ for the quantities δ_u and δ_w, respectively, in Algorithm 6.9 is chosen in the order of magnitude of the discretization and regularization error, then one could stop at iterate $N = 7$. The numerical results indicate also that the estimate, given in (6.57), is not very sharp. Let us mention that the respective estimates for only control constrained optimal control problems are significantly better such that the the error estimator is very efficient, see [44, Section 7 ff.].

We conclude that one cannot save many iterations of the primal-dual active set algorithm if a reasonable threshold value ρ for the stopping criterion is chosen. This fact becomes even more noticeable in the results of Table 6.5, where we used a smaller regularization parameter ε in the numerical computations.

N	$\|\bar{u}_h^\varepsilon - u_N\|_{L^2(\Gamma)}$	$\|\delta_u\|_{L^2(\Gamma)}$	$\|\bar{v}_h^\varepsilon - v_N\|_{L^2(\Omega)}$	$\|\delta_w\|_{L^2(\Omega)}$
1	$3.6883e-2$	$1.7041e+4$	$2.2841e+1$	$1.7739e+4$
2	$3.4231e-2$	$3.3728-3$	$9.4175e+0$	$9.4217e+3$
3	$2.4999e-2$	$2.6365e+2$	$3.1514e+0$	$3.1862e+3$
4	$1.0914e-2$	$1.0981e-6$	$9.5113e-1$	$9.7623e+2$
5	$3.1498e-3$	$7.9424-4$	$2.4127e-1$	$2.5298e+2$
6	$5.3057e-3$	$5.5765e+2$	$9.5097e-2$	$8.2280e+1$
7	$1.2762e-3$	$1.1561e-6$	$3.3578e-2$	$3.5160e+1$
8	$5.5136e-4$	$1.3551e+2$	$1.3992e-2$	$1.2649e+1$
9	$1.7447e-5$	$7.4736e+0$	$5.6788e-3$	$4.2705e+0$
10	0	$1.1599e-6$	$9.6563e-10$	$6.8396e-7$

Table 6.5: Iteration errors of the primal-dual active set strategy $\varepsilon = 0.001$

Chapter 7

Conclusions and perspectives

In this thesis we considered a linear-quadratic Neumann boundary control problem with pointwise state and control constraints. We pointed out the well-known difficulty that Lagrange multipliers associated with pure state constraints are in general only regular Borel measures. Usually, this causes a lack of regularity in the optimal solution of such problems. Due to the consideration of the pure state constraints in an inner subdomain, the Lagrange multiplier is localized there. This fact allowed us to derive higher regularity of the adjoint state close to the boundary of the domain. Consequently, this implied higher smoothness of the optimal control on the boundary. However, due to the nonuniqueness of the dual variables in the case of boundary control problems, a regularization of such problems remains reasonable.

We developed the so called virtual control concept, where a new distributed control was used to apply the Lavrentiev regularization approach. First, additional control constraints to the virtual control were considered. This fact was helpful for the derivation of a regularization error estimate. But, the presence of these constraints can cause numerical difficulties with respect to the applicability of efficient optimization methods using adjoint variables.

Due to these arguments, we omitted the constraints to the virtual control in a second approach. A different argumentation and more sophisticated techniques permitted us to establish a slightly different regularization error estimate than before, where the consideration of the state constraints in an inner subdomain was beneficial. Furthermore, we compared the virtual control approach without control constraints to the Moreau-Yosida regularization concept. We pointed out similarities for specific settings of the virtual control approach.

The second part of this work was devoted to the discretization of the virtual control concept without control constraints. We focused on a linear finite element discretization in the domain and on the boundary, respectively. By the use of an appropriate coupling of the regularization parameter and the mesh size, we derived a discretization error estimate to the optimal solution of the continuous and unregularized original problem. We remind that the regularization and the discretization error was considered simultaneously.

In the last part of this work, we established error estimates of arbitrary feasible and infeasible controls of the regularized problem with respect to the optimal control of the problem. Since we interpreted these controls as current iterates of an numerical algorithm, the error estimate provides information on the accuracy of the iterate. Based on this theory, we constructed error estimators, which are reliable as stopping criterion for iterative optimization methods. We presented the primal-dual active set algorithm as a possible numerical method for solving optimal control problems with mixed control-state constraints. Finally, we illustrated the theoretical results of this work by several numerical tests.

Let us briefly comment on possible extensions of the concepts presented in this work. First, we considered a rather simple elliptic partial differential equation such that the investigation of more general elliptic partial differential equations is desirable. In order to ensure similar regularity results for the corresponding state, see e.g., Theorem 2.18 or Corollary 2.23, one has to discuss necessary assumptions on the coefficients occurring in the partial differential equation.

In connection with specific mesh-grading techniques one could also omit the restriction on convex polygonally or polyhedrally bounded domains. Particularly for the three-dimensional case, a discretization based on anisotropic finite elements might be of interest.

Furthermore, the extension of the virtual control concepts to optimal control problems governed by semi-linear elliptic partial differential equations is conceivable. Of course, due to the non-linearity, a global discussion of optimal solutions is no longer possible, e.g., a linearized Slater condition, depending on a local solution of the problem, instead of Assumption 2.4 has to be established. Moreover, a second-order analysis is needed to guarantee local convexity and optimality. A further challenge is to carry over these properties to the problems arising by the virtual control concept.

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