Cohomology of graph hypersurfaces associated to certain Feynman graphs

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Contents

Introduction

There are interesting zeta and multi zeta values appearing in the calculation of Feynman integrals in physics. One hopes that there exist Tate mixed Hodge structures with periods given by Feynman integrals at least for some identifiable subset of graphs.

This paper is a natural continuation of the work started in [BEK]. For technical reasons we restrict our attention to primitively log divergent graphs. In [BEK], Sections 11,12 the series WS_n was work out in all details. Let $X_n \subset \mathbb{P}^{2n-1}$ be the graph hypersurface for the graph "wheel with n spokes" WS_n, it was proved that (as a Hodge structure)

$$H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash X) \cong \mathbb{Q}(-2)$$

and that the de Rham cohomology $H_{DR}^{2n-1}(\mathbb{P}^{2n-1} \backslash X)$ is generated by the integrand of graph period (1.62). In this paper we succeeded to do the same computation for the graph ZZ_5, the zigzag graph with Betti's number equals 5. We define a big series of graphs for which the minimal nontrivial weight piece of Hodge structure is of Tate type: $\mathrm{gr}\,_{min}^{W} H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash X) \cong \mathbb{Q}(-2)$ for graph hypersurfaces X in this situation. We study gluings of primitively log divergent graphs and compute $\mathrm{gr}\,_{min}^{W} H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash X) = \mathbb{Q}(-3)$ for the case $WS_3 \times WS_n$.

This paper is organized as follows. Section 1.1 contains some theorems on determinants, this is a key ingredient of our computation. The second section is a remainder of the construction of graph polynomials and Feynman integrals. The cohomological tools are presented in Section 1.3.

In Section 2.1 we compute the middle dimensional cohomology of the graph ZZ_5. We define *generalized zigzag graphs* GZZ and prove that they are primitively log divergent in Section 2.2. Then we present the main result that the minimal nontrivial weight pieces of mixed Hodge structures of such graphs are Tate. In Section 2.3 it is proved that the integrand (1.62) is nonzero in the de Rham cohomology $H_{DR}^{2n-2}(\mathbb{P}^{2n-1} \backslash X)$ for $GZZ(n,2)$ and generates it in the case of ZZ_5.

Section 3.1 contains the classification of primitively log divergent graphs with small number of edges. In the next section we compute the cohomology for the new found graph with 10 edges, the graph XX_5. This graph is obtained from two copies of the graph WS_3 by the operation of gluing. We study gluings of primitively log divergent graphs in Section 3.3 and try to compute the middle dimensional cohomology for the series $WS_3 \times WS_n$.

Chapter 1

Preliminaries

1.1 Determinants

Fix some commutative ring R with 1 and let $\mathcal{M} = (a_{ij})_{0\le i,j\le n}$ be an $(n+1)\times(n+1)$-matrix with entries in R. The numeration of rows and columns goes 0 through n. Let $\mathcal{M}(i_0,\dots,i_k;j_0,\dots,j_t)$ be the submatrix which we get from the matrix after removing rows i_0 to i_k and the columns j_0 to j_k. It is very convenient to denote the determinant of $\mathcal{M}$ just by M. We assume that the determinant of zero-dimensional matrix is 1. For example, $M(0,n;0,n) = 1$ for the matix in the definition above with $t = n = 1$.

Theorem 1.1.1
Let $n \ge 1$. For any $(n+1)\times(n+1)$-matrix $\mathcal{M}$ and any integers $0 \le i, j, k, t \le n$, satisfying $i \ne k$ and $j \ne t$, we have

$$M(i;j)M(k;t) - M(k;j)M(i;t) = M \cdot M(i,k;j,t). \tag{1.1}$$

Proof. First, we show that it is enough to prove the statement for the case $i = j = 0$ and $k = t = n$. Fix a matrix $\mathcal{M}$ and some i, j, k and t. Let $\widetilde{\mathcal{M}}$ be a matrix that we get from $\mathcal{M}$ after interchanging the pairs of rows $0 \leftrightarrow i$, $k \leftrightarrow n$ and columns $0 \leftrightarrow j$, $t \leftrightarrow n$. Notice that the operation of interchanging rows commutes with the operation of interchanging columns. Suppose that the statement of the theorem is true for $\widetilde{\mathcal{M}}$:

$$\widetilde{M}(0;0)\widetilde{M}(n;n) - \widetilde{M}(0;n)\widetilde{M}(n;0) = \widetilde{M}\widetilde{M}(0,n;0,n). \tag{1.2}$$

Both matrices $\mathcal{M}(k;t)$ and $\widetilde{\mathcal{M}}(n;n)$ have the same missing column and row, namely the t-th column and k-th row of $\mathcal{M}$, thus the matrices differ only by the order of rows and columns. This means that the determinants $M(k;t)$

and $\widetilde{M}(n;n)$ are the same up to sign. But each interchange of two rows or two columns changes the sign of a determinant, hence

$$M(k;t) = \widetilde{M}(n;n).$$

Similarly,

$$\begin{aligned} M(k;j) &= \widetilde{M}(n;0) && (1.3)\\ M(i;t) &= \widetilde{M}(0;n) && (1.4)\\ M(i;j) &= \widetilde{M}(0;0) && (1.5)\\ M(i,k;j,t) &= \widetilde{M}(0,n;0,n) && (1.6)\\ M &= \widetilde{M} && (1.7) \end{aligned}$$

Thus, we can assume that $i = j = 0$ and $k = t = n$.

We prove the statement by induction on the dimension of $\mathcal{M}$. Suppose that for all $r \times r$-matrices with $r \le n$ and all i, j, k and t the statement is true. The strategy is to present the polynomials of both sides of (1.2) as polynomials of the variables which are entries of the first and last row and column. For simplicity, we denote by $\mathcal{N}$ the matrix $\mathcal{M}(0,n;0,n)$ and define I to be the set $\{1,2,\dots,n-1\}$. We start with $M(n,n)$ and, using the Laplace expansion along the zero column and then along the zero row, we get

$$\begin{aligned} M(n,n) &= a_{00}M(0,n;0,n) + \sum_{i\in I}(-1)^i a_{i0}M(i,n;0,n)\\ &= a_{00}N + \sum_{i\in I}(-1)^i a_{i0}\sum_{j\in I}(-1)^{j-1}a_{0j}M(0,i,n;0,j,n)\\ &= a_{00}N + \sum_{i,j\in I}(-1)^{i+j-1}a_{i0}a_{0j}N(i;j). \end{aligned} \tag{1.8}$$

Hence the left hand side of (1.1) equals

$$\begin{aligned} LHS = \quad & \Big(a_{00}N + \sum_{i,j\in I}(-1)^{i+j-1}a_{i0}a_{0j}N(i;j)\Big)\\ & \cdot\Big(a_{nn}N + \sum_{k,t\in I}(-1)^{k+t-1}a_{kn}a_{nt}N(k;t)\Big) - \end{aligned}$$

$$
\begin{aligned}
& -\Big((-1)^{n-1}a_{n0}N + \sum_{i,t\in I}(-1)^{i+t+n}a_{i0}a_{nt}N(i;t)\Big) \\
& \cdot\Big((-1)^{n-1}a_{0n}N + \sum_{k,j\in I}(-1)^{k+j+n}a_{0j}a_{kn}N(k;j)\Big) \\
= \quad & a_{00}a_{nn}N^2 + a_{nn}N\sum_{i,j\in I}(-1)^{i+j-1}a_{i0}a_{0j}N(i;j) \\
& +a_{00}N\sum_{k,t\in I}(-1)^{k+t-1}a_{kn}a_{nt}N(k;t) \\
& +\sum_{i,j,k,t\in I}(-1)^{i+j+k+t}a_{i0}a_{0j}a_{kn}a_{nt}N(i;j)N(k;t) \\
& -a_{n0}a_{0n}N^2 - a_{0n}N\sum_{i,t\in I}(-1)^{i+t-1}a_{i0}a_{nt}N(i;t) \\
& -a_{n0}N\sum_{k,j\in I}(-1)^{k+j-1}a_{0j}a_{kn}N(k;j) \\
& -\sum_{i,j,k,t\in I}(-1)^{k+j+i+t}a_{i0}a_{nt}a_{0j}a_{kn}N(i;t)N(k;j).
\end{aligned} \tag{1.9}
$$

By the assumption, for $i \neq k$ and $j \neq t$ one has

$$N(i;j)N(k;t) - N(i;t)N(k;j) = NN(i,k;j,t). \tag{1.10}$$

Note that, in the case $i = k$ or $j = t$, the products $N(i;j)N(k;t)$ and $N(i;t)N(k;j)$ are the same, and then all summands of (1.9) which are multiples of $N(i;j)N(k;t)$ or $N(i;t)N(k;j)$ cancel. Thus we can rewrite the big sum (1.9) in the following way

$$
\begin{aligned}
LHS = \quad & (a_{00}a_{nn} - a_{0n}a_{n0})N^2 \\
& +a_{nn}N\sum_{i,j\in I}(-1)^{i+j-1}a_{i0}a_{0j}N(i;j) \\
& +a_{00}N\sum_{k,t\in I}(-1)^{k+t-1}a_{kn}a_{nt}N(k;t) \\
& +a_{0n}N\sum_{i,t\in I}(-1)^{i+t}a_{i0}a_{nt}N(i;t) \\
& +a_{n0}N\sum_{k,j\in I}(-1)^{k+j}a_{0j}a_{kn}N(k;j) \\
& +N\sum_{\substack{i\neq k\in I\\ j\neq t\in I}}(-1)^{i+j+k+t}a_{i0}a_{0j}a_{kn}a_{nt}N(i,k;j,t).
\end{aligned} \tag{1.11}
$$

The last sum is equal to NM. Indeed, define I' to be $I\cup\{n+1\}$. We expand M along the zero row and the zero column using a formula like (1.8) and then expand it further along the n-th row and the n-th column:

$$
\begin{aligned}
M = \quad & a_{00}M(0;0) + \sum_{i,j\in I'}(-1)^{i+j-1}a_{i0}a_{0j}M(0,i;0,j) \\
= \quad & a_{00}M(0;0) - a_{n0}a_{0n}M(0,n;0,n) \\
& +a_{n0}\sum_{j\in I}(-1)^{n+j-1}a_{0j}M(0,n;0,j) \\
& +a_{0n}\sum_{i\in I}(-1)^{i+n-1}a_{i0}M(0,i;0,n) \\
& +\sum_{i,j\in I}(-1)^{i+j-1}a_{i0}a_{0j}M(0,i;0,j) \\
= \quad & a_{00}\Big(a_{nn}M(0,n;0,n) + \sum_{k,t\in I}(-1)^{k+t-1}a_{kn}a_{nt}M(0,i,n;0,j,n)\Big) \\
& -a_{n0}a_{0n}M(0,n;0,n) + \sum_{j,k\in I}(-1)^{k+j}a_{n0}a_{0j}a_{kn}M(0,k,n;0,j,n) \\
& +\sum_{i,t\in I}(-1)^{i+t}a_{i0}a_{0n}a_{nt}M(0,i,n;0,t,n) \\
& +\sum_{i,j\in I}(-1)^{i+j-1}a_{i0}a_{0j}\Big(a_{nn}M(0,i,n;0,j,n) \\
& +\sum_{\substack{k,t\in I\\ i\neq k,j\neq t}}(-1)^{k+t-1}a_{kn}a_{nt}M(0,i,k,n;0,j,t,n)\Big)
\end{aligned}
\tag{1.12}
$$

Hence the sum (1.11) equals MN.

□

For a matrix $\mathcal{M}=(a_{ij})_{0\le i,j\le n}$ we define the minors

$$
\begin{aligned}
I_k^i := M(0,1,\dots,&i-1,i+k,i+k+1,\dots,n; \\
&0,1,\dots,i-1,i+k,i+k+1,\dots,n).
\end{aligned}
\tag{1.13}
$$

and

$$S_t := M(t,t+1,\dots,n;0,t+1,t+2,\dots,n), \tag{1.14}$$

where $1\le k\le n+1$ and $1\le t\le n$. We usually write I_n for I_n^0. For example, $I_{n+1}=M$, $I_n^1=M(0;0)$ and $I_n=M(n;n)$.

Corollary 1.1.2
For a symmetric matrix $\mathcal{M} = (a_{ij})_{0\le i,j\le n}$ one has the following equality

$$I_n I_n^1 - I_{n-1}^1 I_{n+1} = (S_n)^2. \tag{1.15}$$

Proof. Since $\mathcal{M}$ is symmetric, $M(0;n) = M(n;0) = S_n$ and the statement follows immediately from Theorem 1.1.1.

□

Take now an $(n+1)\times(n+1)$-matrix $\mathcal{M} = (a_{ij})$ with entries in R and suppose that the transpose of the last row equals the last column with elements, numerated by single lower indices.

$$\mathcal{M} = \begin{pmatrix} a_{00} & a_{01} & \vdots & a_{0\,n-2} & a_{0\,n-1} & a_0 \\ a_{10} & a_{11} & \vdots & a_{1\,n-2} & a_{1\,n-1} & a_1 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ a_{n-2\,0} & a_{n-2\,1} & \vdots & a_{n-2\,n-2} & a_{n-2\,n-1} & a_{n-2} \\ a_{n-1\,0} & a_{n-1\,1} & \vdots & a_{n-1\,n-2} & a_{n-1\,n-1} & a_{n-1} \\ a_0 & a_1 & \vdots & a_{n-2} & a_{n-1} & a_n \end{pmatrix}. \tag{1.16}$$

The determinant of $\mathcal{M}$ is thought of as an element in $R[a_0,\dots,a_n]$. It can be written as

$$M = I_{n+1} = a_n I_n - G_n, \tag{1.17}$$

defining G_n. Then G_n is computed as

$$G_n := \sum_{0\le i,j\le n-1} (-1)^{i+j} a_i a_j I_n(i;j). \tag{1.18}$$

The entries a_i play the role of variables while the other entries and minors are coefficients. The element $G_n \in R[a_0,\dots,a_n]$ is of degree 2 as a polynomial of the variables. We claim

Theorem 1.1.3
Let $I_{n-1} \not\equiv 0 \mod I_n$. Then

$$I_{n-1} G_n \equiv Li_n Li'_n \mod I_n \tag{1.19}$$

for some Li_n and Li'_n, linear as polynomials of the "variables".

Proof. By Theorem 1.1.1, we have

$$I_n(i;j)I_n(n-1;n-1) \equiv I_n(i;n-1)I_n(n-1;j) \mod I_n \tag{1.20}$$

for all $1 \leq i,j \leq n-2$. We multiply G_n by $I_{n-1} = I_n(n-1,n-1)$ and get

$$\begin{aligned} I_{n-1}G_n = \quad & a_{n-1}^2(I_{n-1})^2 + \sum_{0\leq i,j\leq n-2} (-1)^{i+j} a_i a_j I_{n-1} I_n(i;j) \\ & +a_{n-1} \sum_{0\leq i\leq n-2} (-1)^{i+n-1} a_i I_{n-1}\Big(I_n(i;n-1) + I_n(n-1;i)\Big) \\ \equiv \quad & \Big(a_{n-1}I_{n-1} + \sum_{0\leq i\leq n-2} (-1)^{i+n-1} a_i I_n(i;n-1)\Big) \\ & \cdot\Big(a_{n-1}I_{n-1} + \sum_{0\leq j\leq n-2} (-1)^{j+n-1} a_j I_n(n-1;j)\Big) \mod I_n. \end{aligned} \tag{1.21}$$

We set

$$Li_n = a_{n-1}I_{n-1} + \sum_{0\leq i\leq n-2} (-1)^{i+n-1} a_i I_n(i;n-1) \tag{1.22}$$

and

$$Li'_n = a_{n-1}I_{n-1} + \sum_{0\leq j\leq n-2} (-1)^{j+n-1} a_i I_n(n-1;j). \tag{1.23}$$

□

In the next chapters we deal only with symmetric matrices, thus we make a

Corollary 1.1.4
Let $\mathcal{M} = (a_{ij})$ be a symmetric $(n+1)\times(n+1)$-matrix with entries in a ring R (see (1.16)). If $I_{n-1} \not\equiv 0 \mod I_n$, the congruence

$$I_{n-1}G_n \equiv (Li_n)^2 \mod I_n \tag{1.24}$$

holds, where G_n and Li_n are given by (1.18) and (1.22) respectively.

One more fact about G_n will be used frequently in the next chapters.

Theorem 1.1.5
Let $\mathcal{M} = (a_{ij})$ be a symmetric $(n+1)\times(n+1)$-matrix with entries in a ring R and assume that the quotient ring $R/(I_n)$ is a domain. If $I_{n-1} \equiv 0 \mod I_n$, then

$$G_n \equiv \sum_{0\leq i\leq n-2} a_i^2 I_n(i,i) + 2\sum_{0\leq i<j\leq n-2} (-1)^{i+j} a_i a_j I_n(i;j) \mod I_n. \tag{1.25}$$

Proof. We prove that G_n forgets the "variable" a_{n-1}. Since $\mathcal{M}$ is symmetric, we can rewrite G_n (see (1.18)) as

$$\begin{aligned} G_n = \quad & a_{n-1}^2 I_{n-1} + \sum_{0\le i\le n-2} (-1)^{i+n-1} a_i a_{n-1}\Big(I_n(i,n-1)+I_n(n-1,i)\Big) \\ & + \sum_{0\le i\le j\le n-2} (-1)^{i+j} a_i a_j I_n(i;j) \\ = \quad & a_{n-1}^2 I_{n-1} + 2\sum_{0\le i\le n-2} (-1)^{i+n-1} a_i a_{n-1} I_n(i,n-1) \\ & + \sum_{0\le i\le n-2} a_i^2 I_n(i,i) + 2\sum_{0\le i<j\le n-2} (-1)^{i+j} a_i a_j I_n(i;j). \end{aligned} \tag{1.26}$$

Theorem 1.1.1 implies

$$(I_n(i,n-1))^2 \equiv I_n(i,i)I_n(n-1,n-1) \mod I_n \tag{1.27}$$

for all $0 \le i \le n-2$. Because $R/(I_n)$ is a domain and $I_{n-1} = I_n(n-1, n-1) \equiv 0 \mod I_n$, we get

$$I_n(i,n-1) \equiv 0 \mod I_n. \tag{1.28}$$

Hence, (1.26) implies the congruence

$$G_n \equiv \sum_{0\le i\le n-2} a_i^2 I_n(i,i) + 2\sum_{0\le i<j\le n-2} (-1)^{i+j} a_i a_j I_n(i;j) \mod I_n. \tag{1.29}$$

□

Remark 1.1.6
In our computations we will apply Corollary 1.1.4 and Theorem 1.1.3 only for the case where R in a polynomial ring over an algebraically closed field of characteristic zero. Moreover, entries of matrices will be only linear polynomials in R.

Take a symmetric $(n+1) \times (n+1)$-matrix $\mathcal{M} = (a_{ij})$ and numerate the variables in the last column and row by a_i's with single lower indices as in (1.16). We assume that the entries are in the ring $R = K[x_0, ..., x_m]$, for some field $K = \bar{K}$ of char 0, for some set of variables $X = \{x_0, \ldots, x_m\}$ and for some m (in the matrices associated to graphs we have always $m = 2n+1$). Assume that the a_i are in $X \cup \{0\}$ for $0 \le i \le n-2$ and all nonzero a_i for $0 \le i \le n$ are mutually different. Without loss of generality, we can assume $x_n = a_n$ and $x_{n-1} = a_{n-1}$.

Consider the projective space $\mathbb{P}^m = \mathbb{P}^m(x_0 : \ldots : x_m)$. The vanishing of polynomials in R, or determinants of submatrices of $\mathcal{M}$, or $\mathcal{M}$ itself define hypersurfaces in this projective space. Throughout the whole paper, for a finite set $f_1, f_2, \ldots$ of homogeneous polynomials we denote by $\mathcal{V}(f_1, f_2, ...)$ the corresponding reduced projective scheme. We consider the following situation. Let

$$V := \mathcal{V}(I_n, I_{n+1}) \in \mathbb{P}^m, \tag{1.30}$$

and define an open U in V by

$$\begin{cases} I_n = 0 \\ I_{n+1} = 0 \\ I_{n-1} \neq 0. \end{cases} \tag{1.31}$$

This is equivalent to say $U := V \backslash V \cap \mathcal{V}(I_{n-1})$; we usually write down such systems in order to see the way of stratifying the schemes further. We write

$$I_{n+1} = a_n I_n - G_n \tag{1.32}$$

and see that U is defined by the system

$$\begin{cases} I_n = 0 \\ G_n = 0 \\ I_{n-1} \neq 0. \end{cases} \tag{1.33}$$

These three polynomials are independent of a_n. Let $P_1 \in \mathbb{P}^m$ be the point where all variables but a_n vanish. Consider the natural projection $\pi_1 : \mathbb{P}^m \backslash Pt_1 \longrightarrow \mathbb{P}^{m-1}$ and denote by V_1 resp. U_1 the images of V resp. U under π_1, i.e.

$$V_1 = \mathcal{V}(I_n, G_n) \subset \mathbb{P}^{m-1} \tag{1.34}$$

and $U_1 \subset \mathbb{P}^{m-1}$ defined by the system (1.33). By the Corollary 1.1.4, on U and U_1 we have

$$I_{n-1} G_n = (Li_n)^2, \tag{1.35}$$

thus the vanishing of G_n is equivalent to the vanishing of Li_n. This means that U_1 is defined by

$$\begin{cases} I_n = 0 \\ Li_n = 0 \\ I_{n-1} \neq 0. \end{cases} \tag{1.36}$$

On U_1 we can express a_{n-1} from the equation

$$Li_n = a_{n-1} I_{n-1} + \sum_{1 \leq i \leq n-2} (-1)^{i+n-1} a_i I_n(i; n-1) = 0 \tag{1.37}$$

and get

$$a_{n-1} = \sum_{0 \le i \le n-2} (-1)^{i+n-1} \frac{a_i I_n(i; n-1)}{I_{n-1}}. \quad (1.38)$$

Of course, the right hand side of the last equality can depend on a_{n-1}. If this is not the case, we consider the point $P_2 \subset \mathbb{P}^{m-1}$ where all variables but a_{n-1} are zeros and claim

Theorem 1.1.7
Suppose that all entries a_{ij} of $\mathcal{M}$ are independent of a_{n-1}. The natural projection

$$\pi_2 : \mathbb{P}^{m-1} \backslash P_2 \longrightarrow \mathbb{P}^{m-2}$$

induces an isomorphism between U_1 and an open $U_2 \subset \mathbb{P}^{m-2}$ defined by

$$U_2 := \mathcal{V}(I_n) \backslash \mathcal{V}(I_n, I_{n-1}). \quad (1.39)$$

Proof. $\pi_2(U_1) = U_2$ and the expression (1.38) for a_{n-1} gives the map

$$\phi : U_2 \longrightarrow \mathbb{P}^{m-1} \quad (1.40)$$

inverse to π_2.

□

1.2 Graph polynomials

Let Γ be a finite graph with edges E and vertices V. We choose an orientation of edges. For a given vertex v and a given edge e we define $\text{sign}(e, v)$ to be -1 if e enters v and $+1$ if e exits v. Denote by $\mathbb{Z}[E]$ (resp. $\mathbb{Z}[V]$) the free $\mathbb{Z}$-module generated by the elements of E (resp. V). Consider the homology sequence

$$0 \longrightarrow H_1(\Gamma, \mathbb{Z}) \xrightarrow{\iota} \mathbb{Z}[E] \xrightarrow{\partial} \mathbb{Z}[V] \longrightarrow H_0(\Gamma, \mathbb{Z}) \longrightarrow 0, \tag{1.41}$$

where the $\mathbb{Z}$-linear map ∂ is defined by $\partial(e) = \sum_{v \in V} sign(v, e)$. The elements $e^\vee$ of a dual basis of $\mathbb{Z}[E]$ define linear forms $e^\vee \circ \iota$ on $H = H_1(\Gamma, \mathbb{Z})$. We view the squares of this functions $(e^\vee \circ \iota)^2 : H \to \mathbb{Z}$ as rank 1 quadratic forms. For a fixed basis of H we can associate a rank 1 symmetric matrix M_e to each such form.

Definition 1.2.1
We define the *graph polynomial* of Γ

$$\Psi_\Gamma := \det(\sum_{e \in E} A_e M_e) \tag{1.42}$$

in some variables A_e.

The polynomial Ψ is homogeneous of degree rank H. A change of the basis of H only changes Ψ_Γ by $+1$ or -1.

Definition 1.2.2
The *Betti number* of a graph Γ is defined to be

$$h_1(\Gamma) = \text{rank } H_1(\Gamma, \mathbb{Z}). \tag{1.43}$$

Recall that a tree $T \subset \Gamma$ is a *spanning tree* for the connected graph Γ if every vertex of Γ lies in T. We can extend this notion to a disconnected Γ by simply requiring $T \cap \Gamma_i$ be a spanning tree for each connected component $\Gamma_i \subset \Gamma$. The following proposition (see [BEK], Proposition 2.2) is often used as a definition of graph polynomial.

Proposition 1.2.3
With notation as above, we have

$$\Psi_\Gamma(A) = \sum_{T \; span \; tr.} \prod_{e \notin T} A_e. \tag{1.44}$$

Corollary 1.2.4
The coefficients of Ψ_Γ *are all either* 0 *or* +1.

For the graph Γ we build the table $Tab(\Gamma)$ with $h(\Gamma)$ rows and $|E(\Gamma)|$ columns. Each row corresponds to a loop of Γ, and these loops form a basis of $H_1(\Gamma, \mathbb{Z})$. For each such loop we choose some direction of loop tracing. The entry $Tab(\Gamma)_{ij}$ equals 1 if the edge e_j in the i's loop is in the tracing direction of the loop and equals -1 if this edge is in the opposite direction; if the edge e_j does not appear in the i's loop, then $Tab(\Gamma)_{ij} = 0$. We take $N := |E(\Gamma)|$ variables $T_1, \dots, T_N$ and build a matrix

$$M_\Gamma(T) = \sum_{k=1}^{N} T_k M^k \tag{1.45}$$

where M^k is a $h_1(\Gamma) \times h_1(\Gamma)$ matrix with entries

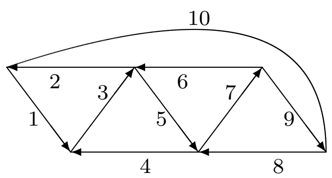
$$M^d_{ij} = Tab(\Gamma)_{id} \cdot Tab(\Gamma)_{jd}. \tag{1.46}$$

By definition, the graph polynomial for Γ is

$$\Psi_\Gamma(T) = \det M_\Gamma(T). \tag{1.47}$$

Consider the following example which will appear in section 2.1.

Example 1.2.5
Let Γ be the graph ZZ_5 (see the drawing). This graph has 10 edges and the Betti number equals 5.



	1	2	3	4	5	6	7	8	9	10
1	1	1	1	0	0	0	0	0	0	0
2	0	0	1	1	1	0	0	0	0	0
3	0	0	0	0	1	1	1	0	0	0
4	0	0	0	0	0	0	1	1	1	0
5	0	-1	0	0	0	-1	0	0	1	1

We choose the orientation and the numbering of edges as on the drawing to the left. Following the construction above, we build the table $Tab(ZZ_5)$ to the right and get the following matrix

$$M_{ZZ_5}(T) = \begin{pmatrix} T_1+T_2+T_3 & T_3 & 0 & 0 & -T_2 \\ T_3 & T_3+T_4+T_5 & T_5 & 0 & 0 \\ 0 & T_5 & T_5+T_6+T_7 & T_7 & -T_6 \\ 0 & 0 & T_7 & T_7+T_8+T_9 & T_9 \\ -T_2 & 0 & -T_6 & T_9 & T_2+T_6+T_9+T_{10} \end{pmatrix} \tag{1.48}$$

Obviously, using a different numbering of edges we get the same matrix and polynomial up to reindexing of T's. The different numeration of loops which we take will give us the same graph polynomial. Indeed, the interchange of the i's and the j's loop gives us the matrix $\tilde{M}_\Gamma(T)$ which can be gotten from $M_\Gamma(T)$ by interchanging of the i's and the j's row and then the i's and the j's column. The change of the orientation of one edge e_i gives us the same matrix M_Γ. If we change the direction of tracing of the i's loop, then all entries of the i row change signs and then all the entries of the i column, thus the polynomial remains the same. Finally, if we choose another basis of $H_1(\Gamma, \mathbb{Z})$ then Ψ_Γ remains the same by Proposition 1.2.3, but the matrix changes. We are interested not only on the graph polynomial itself, but also in the matrix M_Γ. We will always choose loops of small length to get as much zeros in M_Γ as possible.

Definition 1.2.6
The graph hypersurface $X_\Gamma \subset \mathbb{P}^{N-1}$ is the hypersurface cut out by $\Psi_\Gamma = 0$.

Throughout the whole paper we deal with such graph hypersurfaces. Sometimes it is convenient to make a linear change of coordinate in $\mathbb{P}^{N-1}$ to simplify the matrix. Clearly, this new matrix $\widetilde{M_\Gamma}$ will define a hypersurface isomorphic to X, which we denote again by X. For the graph in Example 1.2.5, we note that T_0, T_4, T_8 and T_{10} appear only in the diagonal of $M_{ZZ_5}(T)$. Changing the coordinates and redefining the variables by $A_0, \ldots, A_5$, B_0, B_1, B_3 and B_4, we get the matrix

$$M = M_{ZZ_5}(A, B) = \begin{pmatrix} B_0 & A_0 & 0 & 0 & A_5 \\ A_0 & B_1 & A_1 & 0 & 0 \\ 0 & A_1 & C_2 & A_2 & A_4 \\ 0 & 0 & A_2 & B_3 & A_3 \\ A_5 & 0 & A_4 & A_3 & B_4 \end{pmatrix}, \tag{1.49}$$

where $C_2 = A_1 + A_2 - A_4$. If we change the direction of tracing of the fifth loop for ZZ_5, then we come the the matrix of the same shape as (1.49) but with $C_2 = A_1 + A_2 + A_4$.

Remark 1.2.7
Proposition 2.3 shows that for a variable T_i the determinant of the matrix $M_\Gamma(T)$ is linear as a polynomial in T_i. The statement does not hold for the matrix $\tilde{M}_\Gamma$ which we get form $M_\Gamma(T)$ by linear change of variables.

Now we are going to define *Feynman quadrics* and explain where the graph polynomials are coming from. We follow [BEK], section 5. Let $K \subset \mathbb{R}$ be a real field. (We use $K = \mathbb{Q}$ for the applications to Feynman quadrics.) Take some homogeneous quadrics

$$Q_i : q_i(Z_1, \dots, Z_{2r}) = 0, \quad \text{for } 1 \le i \le r, \tag{1.50}$$

in the space $\mathbb{P}^{2r-1}(Z_1 : \dots : Z_{2r})$. The union $\cup_1^i Q_i$ of this quadrics has degree $2r$. Thus $\Gamma(\mathbb{P}^{2r-1}, \omega(\sum_1^r Q_i)) = K[\eta]$ where η is given by

$$\eta|_{Z_{2r-1}\neq 0} = \frac{dz_1 \wedge \dots \wedge dz_{2r-1}}{\tilde{q}_1 \dots \tilde{q}_r} \tag{1.51}$$

on the affine open $Z_{2r} \neq 0$ with coordinates $z_i = \frac{Z_i}{Z_{2r}}$ and $\tilde{q}_i = \frac{q_i}{Z_{2r}^2}$. We write

$$\eta = \frac{\Omega_{2r-1}}{q_1 \dots q_r}, \quad \text{where } \Omega_{2r-1} = \sum_{i=1}^{2r} (-1)^i Z_i dZ_1 \wedge \dots \widehat{dZ_i} \dots \wedge dZ_{2r}. \tag{1.52}$$

The transcendental quantity of interest is the *period*

$$P(Q) := \int_{\mathbb{P}^{2r-1}(\mathbb{R})} \eta = \int_{z_1,\dots,z_{2r-1}=-\infty}^{\infty} \frac{dz_1 \wedge \dots \wedge dz_{2r-1}}{\tilde{q}_1 \dots \tilde{q}_r}. \tag{1.53}$$

(when the integral is convergent).

Suppose now $r = 2n$ above and let $H \cong K^n$ be an n-dimensional vector space; we identify $\mathbb{P}^{4n-1} = \mathbb{P}(H^4)$. For a linear functional $l : H \to K$, l^2 gives a rank 1 quadratic form of H. A *Feynman quadric* is a rank 4 positive semi-definite form on $\mathbb{P}^{4n-1}$ of the form $q = q_l = (l^2, l^2, l^2, l^2)$. We are interested in the quadrics Q_i of this form. So, we suppose that the linear forms l_i are given, $1 \le i \le 2n$ and consider the period $P(Q)$ for $q_i = (q_{l_i}, q_{l_i}, q_{l_i}, q_{l_i})$.

For a linear form $l : H \to K$, we define $\lambda = \ker(l)$ and $\Lambda = \mathbb{P}(\lambda, \lambda, \lambda, \lambda) \subset \mathbb{P}^{4n-1}$. The Feynman quadric q_l is then a cone over the codimension 4 linear space Λ. We have $q_l = Z_1^2 + \dots + Z_4^2$ for a suitable choice of homogeneous coordinates $Z_1, \dots, Z_{2n}$.

Consider now a graph Γ with N edges and the Betti number $n = h_1(\Gamma)$. By (1.41), we have the configuration of N hyperplanes in the n-dimensional vector space $H = H_1(\Gamma)$. As above, we consider the Feynman quadrics $q_i = (l^2, l^2, l^2, l^2)$ on $\mathbb{P}^{4n-1}$, $1 \le i \le N$.

Definition 1.2.8
The graph Γ is said to be *convergent* (resp. *logarithmically divergent*) if $N > 2h_1(\Gamma)$ (resp. $N = 2h_1(\Gamma)$). The logarithmically divergent graph Γ is *primitively log divergent* if any connected proper subgraph $\Gamma' \subset \Gamma$ is convergent.

When Γ is logarithmically divergent, the form

$$\omega_\Gamma := \frac{d^{4n-1}x}{q_1 \cdots q_{2n}} \tag{1.54}$$

has poles only along $\bigcup Q_i$, and we define the period

$$P(\Gamma) := \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \omega_\Gamma \tag{1.55}$$

as in (1.53).

Proposition 1.2.9
Let Γ be a logarithmically divergent graph with $2n$ edges and $h_1(\Gamma)$. The period $P(\Gamma)$ converges if and only if Γ is primitively log divergent.

Proof. See [BEK], Proposition 5.2. □

Now we explain the Schwinger trick. Let $Q_i : q_i(Z_1, \ldots, Z_{4n}) = 0$, $1 \leq i \leq 2n$ be quadrics in $\mathbb{P}^{4n-1}$, and assume that the period integral (1.53) converges. Let M_i be the $4n \times 4n$ symmetric matrix corresponding to q_i, we define

$$\Phi(A_1, \ldots, A_{2n}) := \det(A_1 M_1 + \ldots + A_{2n} M_{2n}). \tag{1.56}$$

The Schwinger trick relates the period integral $P(Q)$ (see (1.53)) to an integral on $\mathbb{P}^{2n-1}$.

$$\int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}(Z)}{q_1 \cdots q_{2n}} = C \int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}(A)}{\sqrt{\Phi}}. \tag{1.57}$$

Here by $\sigma^{2n-1}(\mathbb{R}) \subset \mathbb{P}^{2n-1}(\mathbb{R})$ we denote the locus of all points $s = [s_1, \ldots, s_{2n}]$ such that the projective coordinates $s_i \geq 0$. C is an elementary constant, and Ω's are as in (1.52). More precisely, we have the following

Proposition 1.2.10
Assuming that the integral $P(Q)$ is convergent, we have

$$P(Q) := \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}(Z)}{q_1 \cdots q_{2n}} = \frac{c}{\pi^{2n}} \int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}(A)}{\sqrt{\Phi}}, \tag{1.58}$$

where $c \in \overline{\mathbb{Q}}^\times$, $[\mathbb{Q}(c) : \mathbb{Q}] \leq 2$. If $\Phi = \Xi$ for some $\Xi \in \mathbb{Q}[A_1, \ldots, A_{2n}]$, then $c \in \mathbb{Q}^\times$.

Proof. See [BEK], Proposition 6.2.

□

Corollary 1.2.11
Let Γ be a primitively log divergent graph with $2n$ edges and $q_1, \ldots, q_{2n}$ are the Feynman quadrics associated to Γ. The symmetric matrices M_i in this case are block diagonal

$$M = \begin{pmatrix} N_i & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 \\ 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & N_i \end{pmatrix}, \tag{1.59}$$

and we can write $\Phi = \Psi_\Gamma^4$, where $\Psi_\Gamma = \det(A_1 N_1 + \ldots + A_{2n} N_{2n})$ is a graph polynomial (1.42). Applying Schwinger trick, we get

$$P(Q) := \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}(Z)}{q_1 \ldots q_{2n}} = \frac{c}{\pi^{2n}} \int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}(A)}{\Psi_\Gamma^2} \tag{1.60}$$

for $c \in \mathbb{Q}^\times$.

In Section 7 of [BEK], the following construction was defined for a primitively log divergent graph Γ. Consider $\mathbb{P}^{2n-1}$ with homogeneous coordinates $A_1, \ldots, A_{2n}$ associated with edges of Γ. We refer to linear spaces $L \subset \mathbb{P}^{2n-1}$ defined by vanishing of the A_i as coordinate linear spaces. For such an L, we write $L(\mathbb{R}^{\geq 0})$ for the subset of real points with non-negative coordinates. We know that

$$X_\Gamma(\mathbb{C}) \cap \sigma^{2n-1}(\mathbb{R}) = \bigcup_{L \subset X_\Gamma} L(\mathbb{R}^{\geq 0}), \tag{1.61}$$

where the union goes over all coordinate linear spaces $L \subset X_\Gamma$ (see [BEK], Lemma 7.1).

Proposition 1.2.12
For Γ a primitively log divergent graph, define

$$\eta = \eta_\Gamma = \frac{\Omega_{2n-1}(A)}{\Psi_\Gamma^2}. \tag{1.62}$$

There exist a tower

$$\begin{aligned} &P = P_r \xrightarrow{\pi_r} P_{r-1} \xrightarrow{\pi_{r-1}} \ldots \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} \mathbb{P}^{2n-1}, \\ &\pi = \pi_r \circ \cdots \circ \pi_1, \end{aligned} \tag{1.63}$$

where P_i is obtained from P_{i-1} by blowing up the strict transform of a coordinate linear space $L_i \subset X_\Gamma$ and such that

(i) *$\pi^*\eta_\Gamma$ has no poles along the exceptional divisors associated to the blowups.*

(ii) *Let $B \subset P$ be a total transform of coordinate hyperplanes Δ^{2n-2} : $A_1A_2\ldots A_{2k} = 0$. Then B is a normal crossings divisor in P. No face (on-empty intersection of components) of B is contained in the strict transform of Y of X_Γ in P.*

(iii) *the strict transform of $\sigma^{2n-1}(\mathbb{R})$ does not meet Y.*

Proof. See [BEK], Proposition 7.3.

□

So we define a relative cohomology

$$H := H^{2n-1}(P\backslash Y, B\backslash B \cap Y). \tag{1.64}$$

The period of this relative cohomology (i.e. the integration along a homology of H with a de Rham cohomology of H) is exactly

$$\int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}(A)}{\Psi_\Gamma^2}, \tag{1.65}$$

that one appeared in (1.60). For more explanation see [BEK] and [Bl] section 7 and 8. We can consider the system of realizations of H (see, for example, [Hub]). There is a hope (see [BEK], 7.25) that for all primitively log divergent graph, or for an identifiable subset of them, the maximal weight piece of the Betti realization H_B is Tate,

$$\mathrm{gr}^{W}_{max} H_B = \mathbb{Q}(-p)^{\oplus r}. \tag{1.66}$$

One would like to that there should be a rank 1 sub-Hodge structure ι : $\mathbb{Q}(-p) \hookrightarrow \mathrm{gr}^{W}_{max} H_B$ such that the image of $\eta_\Gamma \in H_{DR}$ in $\mathrm{gr}^{W}_{max} H_{DR}$ spans $\iota(\mathbb{Q}(-p))_{DR}$.

Unfortunately, we cannot compute this even in very simple cases, but something can be done here. Note that by the construction the blow up above, we have natural inclusion $\mathbb{P}^{2n-1}\backslash X \hookrightarrow P\backslash Y$. This implies a morphism

$$H^{2n-1}(P\backslash Y) \xrightarrow{j} H^{2n-1}(\mathbb{P}^{2n-1}\backslash X). \tag{1.67}$$

Furthermore, the relative cohomology in (1.64) fits into an exact sequence

$$\longrightarrow H^{2n-2}(B\backslash B \cap Y) \longrightarrow H \longrightarrow H^{2n-1}(P\backslash Y) \longrightarrow \tag{1.68}$$

The idea (and the only thing we can do) is to compute $H^{2n-1}(\mathbb{P}^{2n-1}\backslash X)$. We hope that the map j in (1.67) is nonzero, otherwise our computations give no information about H. In the paper [BEK], Section 11, there was computed $H^{2n-1}(\mathbb{P}^{2n-1}\backslash X_n)$ for X_n a graph hypersurface of WS_n, $n \geq 3$ (for Betti or l-adic cohomology)

$$H^{2n-1}(\mathbb{P}^{2n-1}\backslash X_n) \cong \mathbb{Q}(-2n+3). \tag{1.69}$$

Moreover, motivated by discussion above about the weights of realizations of H, for the de Rham cohomology there was proved (see Section 12) that the class of

$$\eta_n := \frac{\Omega_{2r-1}}{\Psi_n^2} \in \Gamma(\mathbb{P}^{2n-1}, \omega(2X_n)) \tag{1.70}$$

lies in the second level of the Hodge filtration (and generates the whole cohomology because $H_{DR}^{2n-1}(\mathbb{P}^{2n-1}\backslash X_n))$ is one dimensional).

In the next chapters we compute $H^{2n-1}(\mathbb{P}^{2n-1}\backslash X)$ (or the maximal graduate piece of weight filtration) for new examples of primitively divergent graphs. For ZZ_5 it also succeeded to do the computation for $H_{DR}^{2n-1}(\mathbb{P}^{2n-1}\backslash X)$.

1.3 Cohomology

In this section we explain the cohomological tools we will use. We start with étale cohomology theory and at some place we proceed both with étale and Betti cases.

Definition 1.3.1
Let X be a separated scheme over some field K of char. 0. We and $\mathscr{F}$ be a torsion constructible sheaf of abelian groups on X. We can consider $\mathscr{F}$ as a contravariant functor $\mathscr{F} : Et/X \to Ab$. The category of sheaves for the étale topology on X is abelian with enough injectives. Now we can define étale cohomology groups $H^r(X_{et}, \mathscr{F})$ exactly as in the classical case using the derived functors of $\mathscr{F} \mapsto \mathscr{F}(X)$.

Suppose now that there exists an embedding $j : X \hookrightarrow \overline{X}$ into some complete scheme $\overline{X}$ as an open subscheme. Then the *cohomology with compact support* of X is defined to be

$$H^r_c(X, \mathscr{F}) = H^r(\overline{X}, j_!\mathscr{F}).$$

For any closed subscheme $Z \subset X$ in the assumptions above, one has the following exact sequence

$$\longrightarrow H^r_c(X - Z, \mathscr{F}) \longrightarrow H^r_c(X, \mathscr{F}) \longrightarrow H^r_c(Z, \mathscr{F}) \longrightarrow \tag{1.71}$$

induced by the exact sequence of abelian sheaves

$$0 \longrightarrow j'_!(\mathscr{F}|_{X-Z}) \longrightarrow \mathscr{F} \longrightarrow i_*(\mathscr{F}|_Z) \longrightarrow 0, \tag{1.72}$$

where $j' : X - Z \hookrightarrow X$ and $i : Z \hookrightarrow X$ denote the inclusions (see [Mi1], ch. 3, Remark 1.30).

Definition 1.3.2
A variety X is said to have *cohomological dimension* c if c is the least integer such that

$$H^r(X, \mathscr{F}) = 0 \tag{1.73}$$

for $r > c$ and all torsion sheaves $\mathscr{F}$ on X.

From now on we suppose K to be algebraically closed because in our computations the following theorem is used frequently.

Theorem 1.3.3
For a variety X over algebraically closed field K,

$$\mathrm{cd}(X) \leq 2\dim(X). \tag{1.74}$$

If X is affine, then

$$\mathrm{cd}(X) \leq \dim(X). \tag{1.75}$$

Proof. For the proof of the first statement see [Mi2], Theorem 15.1. The proof of the second statement is given in [SGA7], XIV, Theorem 3.1, this statement is usually called the *Artin vanishing*.

□

We return to the exact sequence (1.71); applying it for the constant sheaves $\mathscr{F}_n$ determined by $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$ for each n, and taking the inverse limit, we get the *localization* sequence:

$$\longrightarrow H_c^r(X - Z, \mathbb{Q}_l) \longrightarrow H^r(X, \mathbb{Q}_l) \longrightarrow H^r(Z, \mathbb{Q}_l) \longrightarrow \tag{1.76}$$

Because the operation of taking the inverse limit is an exact functor on modules of finite length, this sequence is exact. Theorem 1.3.3 has the following corollary.

Corollary 1.3.4
For an affine smooth X and a locally constant sheaf $\mathscr{F}$ (in particular, $\mathscr{F} = \mathbb{Q}_\ell$) on X

$$H_c^r(X, \mathscr{F}) = 0 \tag{1.77}$$

for $r < dim(X)$.

Proof. Since X is smooth and $\mathscr{F}$ is locally constant, we can apply the Poincaré duality ([Mi2], Theorem 24.1) to X. This implies the first statement. Considering the inverse system of constant sheaves determined by $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$, for each $n \geq 1$, and using the same argument as that one for the sequence (1.76), we get $H_c^r(X, \mathbb{Q}_l) = 0$ for $r < dim(X)$.

□

Consider the following situation: $X \subset \mathbb{P}^m$ is defined by the vanishing of one homogeneous polynomial $f \in K[x_0, ..., x_m]$, $m \geq 2$, we write $X = \mathcal{V}(f)$ in such situation. Applying (1.76) for the inclusion $X \hookrightarrow \mathbb{P}^m$, we get an exact sequence

$$\longrightarrow H_c^r(\mathbb{P}^m \backslash X, \mathbb{Q}_l) \longrightarrow H^r(\mathbb{P}^m, \mathbb{Q}_l) \longrightarrow H^r(X, \mathbb{Q}_l) \longrightarrow . \tag{1.78}$$

Note that $\mathbb{P}^m \backslash X$ is affine (and smooth) of dimension m, thus, by Artin's vanishing or Corollary 1.3.4,

$$H_c^r(\mathbb{P}^m \backslash X, \mathbb{Q}_l)) = 0 \tag{1.79}$$

for $0 \leq r \leq m - 1$. This implies

$$H^r(X, \mathbb{Q}_l) \cong H^r(\mathbb{P}^m, \mathbb{Q}_l) \tag{1.80}$$

for $0 \leq r \leq m-2$ and $H^{m-1}(\mathbb{P}^m, \mathbb{Q}_l) \hookrightarrow H^{m-1}(X, \mathbb{Q}_l)$. So, the first interesting cohomology of a hypersurface in $\mathbb{P}^m$ is in degree $m-1$, we call it sometimes a *middle dimensional* cohomology $H^{mid}(X)$. Now we formulate some statements both for étale and Betti cohomology and write $H^r(X)$ to unify the notation.

Definition 1.3.5
Define

$$H^r_{prim}(X) := coker(H^r(\mathbb{P}^m) \longrightarrow H^r(X)) \tag{1.81}$$

for all r.

We have no good reference for the following statement and we will prove in here.

Theorem 1.3.6
Let $X \subset \mathbb{P}^n$ be a variety over algebraically closed field of characteristic 0. Then the morphism

$$\phi_r : H^r(\mathbb{P}^n) \longrightarrow H^r(X) \tag{1.82}$$

is injective for $0 \leq r \leq 2\dim X$.

Proof. First consider the case of X being a hypersurface, so $\dim X = n-1$. Since $H(X) = H(X_{\text{red}})$, we can assume that X is reduced. For odd r the cohomology $H^r(\mathbb{P}^n)$ vanishes and there is nothing to prove. We start with top cohomology $H^{2n-2}(X)$, $r = 2n-2$. The singular locus Σ of the reduced hypersurface X is of dimension at most $n-2$. Define the complement $U := X - \Sigma$. Consider the localization sequence

$$\longrightarrow H^{2n-3}(\Sigma) \longrightarrow H_c^{2n-2}(U) \longrightarrow H^{2n-2}(X) \longrightarrow H^{2n-2}(\Sigma) \longrightarrow . \tag{1.83}$$

Both the leftmost and the rightmost terms vanish for dimensional reasons, and we get an isomorphism

$$H^{2n-2}(X) \cong H_c^{2n-2}(U). \tag{1.84}$$

Let X be a union of irreducible components $X = \bigcup_{i=1}^j X_i$. We resolve singularities and get some $\hat{X} = \coprod_{i=1}^j \hat{X}_i$ with the inclusion $U \hookrightarrow \hat{X}$. This gives us a localization sequence

$$\begin{aligned} \longrightarrow H^{2n-3}(\hat{X}\backslash U) \longrightarrow H_c^{2n-2}(U) \longrightarrow \\ H^{2n-2}(\hat{X}) \longrightarrow H^{2n-2}(\hat{X}\backslash U) \longrightarrow \end{aligned} \tag{1.85}$$

Again, the term to the left and the term to the right are zero for reason of dimension, and we get an isomorphism

$$H_c^{2n-2}(U) \cong H^{2n-2}(\hat{X}). \tag{1.86}$$

Each $\hat{X}_i$ is a smooth projective scheme of dimension $n-1$, and we can compute

$$H^{2n-2}(X) \cong H^{2n-2}(\hat{X}) = \bigoplus_{i=1}^{j} \mathbb{Q}(-n+1) \tag{1.87}$$

For a general line $\ell \in \mathbb{P}^n$ we have $\ell \cap X \subset U$. Now,

$$H^{2n-2}(\mathbb{P}^n) \cong H^{2n-2}_{\ell}(\mathbb{P}^n) \cong \mathbb{Q}(-n+1). \tag{1.88}$$

The intersection with X induces a map

$$H^{2n-2}_{\ell}(\mathbb{P}^n) \xrightarrow{\alpha} H^{2n-2}_{\ell\cap X}(X). \tag{1.89}$$

By excision, we have an isomorphism

$$\beta : H^{2n-2}_{\ell\cap X}(X) \cong H^{2n-2}_{\ell\cap X}(U) \cong H^{2n-2}_{\ell\cap X}(\hat{X}). \tag{1.90}$$

Note that $\ell \cap X$ is a union of $\deg X$ points (lying on U). We have a natural morphism

$$H^{2n-2}_{\ell\cap X}(\hat{X}) \xrightarrow{\gamma} H^{2n-2}(\hat{X}) \cong H^{2n-2}(\coprod_{i=1}^{j} \hat{X}_i). \tag{1.91}$$

Here γ maps the class of a point $p \in \ell \cap X$ to the class of this point in $H^{2n-2}(\hat{X}_i)$ when $p \in X_i$. All the $\hat{X}_i$ are smooth projective of dimension $n-1$. Thus $H^{2n-2}(\hat{X}_i)$ is one-dimensional and generated by the class of a point. Then the composition $\gamma\beta\alpha$ is a nonzero map. Since $H^{2n-2}(\mathbb{P}^n)$ is one-dimensional, this proves that ϕ_{2n-2} is injective.

Now we consider maps

$$H^{2i}(\mathbb{P}^n) \xrightarrow{\phi_{2i}} H^{2i}(X), \tag{1.92}$$

$i \leq n-1$, and take $n-1$ general hyperplanes $H_1, \ldots, H_{n-1} \subset \mathbb{P}^n$. The cohomology to the left is generated by the class $[D_i]$ with $D_i := H_1 \cap \ldots \cap H_i$. For injectivity it is enough to show that $\phi_{2i}([D_i]) \neq 0$. Using the cup-product on $H^*(X)$, we obtain

$$\phi_{2i}([D_i]) = \phi_2([D_1])^i \in H^{2i}(X). \tag{1.93}$$

We see that $D_{n-1} = H_1 \cap \ldots \cap H_{n-1}$ is a general line $\ell \in \mathbb{P}^n$ and it was proved above that $\phi_{2n-2}([\ell]) = \phi_2([D_1])^{n-1} \neq 0$ in $H^{2n-2}(X)$. Thus $\phi_{2i}([D_i]) \neq 0$ and ϕ_{2i} in injective for all $i \leq n-1$.

Suppose now that $X \subset \mathbb{P}^n$ is defined by m homogeneous polynomials, $X = \mathcal{V}(f_1, \ldots, f_m)$, and is of dimension d. We can play the same game for X_{red} to show that ϕ_i are injective for $i \leq 2d$. Indeed, take general hyperplanes H_i, $1 \leq i \leq d$ and define $D_d := H_1 \cap \ldots \cap H_d$. Then $D_d \cap X \subset X_{smooth}$. Denote by $\hat{X}$ the Hironaka resolution of singularities of X_{res} (exists since $K = \bar{K}$). By the same argument as above,

$$H^{2d}(X) \cong H^{2d}(\hat{X}) \cong \bigoplus_{i=1}^{j} \mathbb{Q}(-d). \tag{1.94}$$

Note that in $H^{2d}(\hat{X})$ only the summands which correspond to the resolutions of the irreducible components of maximal dimension ($= \dim(X)$) may survive, all other die for reason of dimension.

The intersection $D_d \cap X$ is a union of points. We explain the map ϕ_{2d} as above and conclude that $\phi_{2d}([D_d]) \neq 0$ in $H^{2d}(X)$. Now it follows that $\phi_{2i}([D_i]) \neq 0$ and ϕ_{2i} is injective for all $i \leq d$.

□

Let X be a proper scheme and $Y \subset X$ be a closed subscheme. By the theorem above, the localization sequence for $Y \subset X$ implies that the sequence

$$\longrightarrow H^i_c(X \backslash Y) \longrightarrow H^i_{prim}(X) \longrightarrow H^i_{prim}(Y) \longrightarrow \tag{1.95}$$

is exact in all terms up to $H^i_{prim}(Y)$ for $i = 2\dim Y$.

The Mayer-Vietoris sequence for the closed covering $X = X_1 \cup X_2$ yields the sequence

$$\longrightarrow H^i_{prim}(X) \longrightarrow H^i_{prim}(X_1) \oplus H^i_{prim}(X_2) \longrightarrow H^i_{prim}(X_1 \cap X_2) \longrightarrow \tag{1.96}$$

which is exact in terms up to $H^i_{prim}(X_1 \cap X_2)$ for $i = 2\dim X_1 \cap X_2$.

For our computations we need some vanishing theorems. First, Artin's vanishing holds in the analytic category.

Theorem 1.3.7
Let X be an affine variety defined over the field of complex numbers, and $\mathscr{F}$ be a constructible sheaf. Then $H^m(X_{an}, \mathscr{F}) = 0$ for $m > dim(X)$.

Proof. The direct analytic proof can be found in [Es].

□

The next two theorems are often referred to in the next chapters.

Theorem 1.3.8 (Vanishing Theorem A)
Let Y be a variety $\mathcal{V}(f_1, f_2, \ldots, f_k) \subset \mathbb{P}^N(a_0 : a_1 : \ldots : a_N)$ for some homogeneous polynomials $f_1, \ldots, f_k \in K[a_0, \ldots, a_N]$, and suppose that f_i are independent of the first t variables $a_0, \ldots, a_{t-1}$ for each i, $1 \leq i \leq k$. Then

1) $H^r_{prim}(Y) = 0$ *for* $r < N - k + t$.

2) $H^r(Y) = H^{r-2t}(Y')(-t)$ *for* $r \geq 2t$*, where* $Y' \subset \mathbb{P}^{N-t}$ *is defined by the same polynomials.*

Proof. Suppose first that $t = 0$. We prove that $H^r(\mathbb{P}^N \backslash Y) = 0$ for $r \geq N + k$ using induction on k. For $k = 1$ we have an affine $\mathbb{P}^N \backslash Y$ and the statement is exactly Atrin's vanishing. Assume that $k > 1$ and the statement holds for all Y defined by at most $s < k$ polynomials. Let $Y := \mathcal{V}(f_1, f_2, \ldots, f_k)$ and $U = \mathbb{P}^N \backslash Y$. Define the covering $U_1, U_2 \subset U$ by $U_1 := \mathbb{P}^N \backslash \mathcal{V}(f_1)$ and $U_2 := \mathbb{P}^N \backslash \mathcal{V}(f_2, \ldots, f_k)$. Note that the intersection

$$U_3 := U_1 \cap U_2 = \mathbb{P}^N \backslash (\mathcal{V}(f_1) \cup \mathcal{V}(f_2, \ldots, f_k)) = \mathbb{P}^N \backslash \mathcal{V}(f_1 f_2, \ldots, f_1 f_k) \quad (1.97)$$

is again the complement of a complete intersection defined by at most $k-1$ polynomials. We write a Mayer-Vietoris sequence

$$\longrightarrow H^{r-1}(U_3) \longrightarrow H^r(U) \longrightarrow H^r(U_1) \oplus H^r(U_2) \longrightarrow \quad (1.98)$$

By the assumption both the cohomology to the left and the summands to the right vanish for $r-1 \geq N+k-1$. Thus, the sequence implies $H^r(U) = 0$ for $r \geq N + k$. The induction hypothesis follows.

By duality, one has $H^r_c(\mathbb{P}^N \backslash Y) = 0$ for $r \leq N - k$. We have an exact sequence

$$\longrightarrow H^{r-1}_{prim}(\mathbb{P}^N) \longrightarrow H^{r-1}_{prim}(Y) \longrightarrow H^r_c(\mathbb{P}^N \backslash Y) \longrightarrow H^r_{prim}(\mathbb{P}^N) \longrightarrow \quad (1.99)$$

Since $H^i_{prim}(\mathbb{P}^N) = 0$ for all i, the sequence gives us an isomorphism $H^{r-1}_{prim}(Y) \cong H^r_c(\mathbb{P}^N \backslash Y)$. Thus $H^r_{prim}(Y) = 0$ for $r < N - k$.

Suppose now that $t \geq 1$. Define $\Delta := \mathcal{V}(a_t, \ldots, a_N) \cong \mathbb{P}^{t-1}$. Consider the natural projection $\pi : \mathbb{P}^N \backslash \Delta \longrightarrow \mathbb{P}^{N-t}$. Note that $\Delta \subset Y$ is a closed subscheme, thus one has an exact sequence

$$\longrightarrow H^r_c(Y \backslash \Delta) \longrightarrow H^r_{prim}(Y) \longrightarrow H^r_{prim}(\Delta) \longrightarrow . \quad (1.100)$$

The map π gives us an $\mathbb{A}^t$-fibration over $\pi(Y \backslash \Delta) = Y'$, by homotopy invariance

$$H^r_c(Y \backslash \Delta) \cong H^{r-2t}(Y')(-t). \quad (1.101)$$

Now, $H^r(Y \backslash \Delta) = 0$ for $r \leq 2n - 1$ and $H^r(\Delta) = 0$ for $r \geq 2n - 1$. The sequence above implies $H^r_{prim}(Y) \cong H^r_{prim}(\Delta) = 0$ for $r \leq 2n - 2$, $H^{2n-1}(Y) = 0$, and $H^r(Y) \cong H^r(Y \backslash \Delta) \cong H^{r-2t}(Y')(-t) = 0$ for $2t \leq r \leq N - k + t$. We applied here the case $t = 0$ for Y'. The statement follows.

□

Theorem 1.3.9 (Vanishing Theorem B)
For homogeneous polynomials $f_1, \dots, f_k, h \in K[a_0, \dots, a_N]$, $k \geq 0$, *define subscheme* $U \subset \mathbb{P}^N$ *by equations* $f_1 = \dots = f_k = 0$ *and inequality* $h \neq 0$, *i.e.*

$$U := \mathcal{V}(f_1, \dots, f_k) \backslash \mathcal{V}(f_1, \dots, f_k, h).$$

Suppose that all the polynomials are independent of the first t *variables* $a_0, \dots, a_{t-1}$, *and let* $U' \subset \mathbb{P}^{N-t}$ *be defined by the same polynomials but in* $\mathbb{P}^{N-t}(a_t : \dots : a_N)$. *Then the following equalities hold:*

1) $H^i_c(U) = 0$ *for* $i < N - k + t$.

2) $H^i_c(U) = H^{i-2t}_c(U')(-t)$.

Proof. Suppose first that $t = 0$. Let $Y := \mathcal{V}(f_1 \dots, f_n) \subset \mathbb{P}^N$, then $U \cong Y \backslash Y \cap \mathcal{V}(h)$. We have an exact sequence

$$\longrightarrow H^{r-1}_{prim}(Y \cap \mathcal{V}(h)) \longrightarrow H^r_c(U) \longrightarrow H^r_{prim}(Y) \longrightarrow \tag{1.102}$$

By Theorem A, both the cohomology to the right and the cohomology to the left vanish for $r - 1 < N - k - 1$. Thus $H^r_c(U) = 0$, $r < N - k$.

Let $t \geq 1$, consider the natural projection $\mathbb{P}^N \backslash \Delta \longrightarrow \mathbb{P}^{N-t}$, where $\Delta := \mathcal{V}(a_t, \dots, a_N)$. It maps U onto U' with fibres $\mathbb{A}^t$. Thus

$$H^r_c(U) \cong H^{r-2t}(U')(-t). \tag{1.103}$$

for all r. The case $t = 0$ applied to $U' \subset \mathbb{P}^{N-t}$ gives us $H^{r-2t}_c(U') = 0$ for $r - 2t < N - t - k$, thus $H^r_c(U) = 0$ for $r < N - k + t$.

□

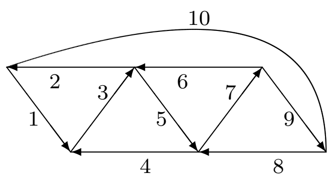
In the previous section we introduced the notion of graph hypersurface. By Corollary 1.2.4, a graph hypersurface X is always defined over $\mathbb{Z}$. In the étale case we work with $H^i(X \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$. We write $H^i(X)$ for this kipping in mind that we have $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acting on this $\mathbb{Q}_\ell$-vector spaces. This action will distinguish $\mathbb{Q}_\ell(-i)$ from $\mathbb{Q}_\ell(-j)$ for $i \neq j$. Finally, for Betti cohomology we write $H^i(X)$ for $H^i(X \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{Q})$.

Chapter 2

GZZ

2.1 ZZ_5

Here we are going to compute the middle dimensional cohomology of the graph hypersurface for the graph ZZ_5 in all details.
Recall that ZZ_5 is a primitively divergent graph and the smallest graph in the zigzag series (see, for example, [BrKr], sect. 1) which is not isomorphic to a WS_n graph for some n.



	1	2	3	4	5	6	7	8	9	10
1	1	1	1	0	0	0	0	0	0	0
2	0	0	1	1	1	0	0	0	0	0
3	0	0	0	0	1	1	1	0	0	0
4	0	0	0	0	0	0	1	1	1	0
5	0	-1	0	0	0	-1	0	0	1	1

In the way it was done in Example 1.2.5, we build a matrix $M_{ZZ_5}(T)$, and then, changing the variables, we come to the matrix

$$M := M_{ZZ_5}(A,B) = \begin{pmatrix} B_0 & A_0 & 0 & 0 & A_5 \\ A_0 & B_1 & A_1 & 0 & 0 \\ 0 & A_1 & C_2 & A_2 & A_4 \\ 0 & 0 & A_2 & B_3 & A_3 \\ A_5 & 0 & A_4 & A_3 & B_4 \end{pmatrix} \tag{2.1}$$

with variables $A_0, \ldots, A_5, B_0, B_1, B_3, B_4$ and

$$C_2 = A_1 + A_2 - A_4. \tag{2.2}$$

This dependent C_2 is a main problem for translating the technics of the case of WS_n (see [BEK], sect. 11) to the ZZ_5 case. Define the hypersurface associated to the graph ZZ_5:

$$X := \mathcal{V}(\det M) \subset \mathbb{P}^9. \tag{2.3}$$

We will work in $\mathbb{P}^9(A_0 : \ldots : A_5 : B_0 : B_1 : B_3 : B_4)$. Similarly to notation in Chapter 1, the projective scheme corresponding to the finite set of homogeneous polynomials $f_1, \ldots, f_k$ will be denoted by $\mathcal{V}(f_1, \ldots, f_k)$. Sometimes we will write $\mathcal{V}(f_1, \ldots, f_k)^{(N)}$ to indicate that we consider the variety in $\mathbb{P}^N$. We use the term *variety* for reduced (but not necessary irreducible) schemes. We define $H^*_{prim}(X) = \operatorname{coker}(H^*(\mathbb{P}^N) \longrightarrow H^*(X))$ for a subvariety $X \subset \mathbb{P}^N$.

Theorem 2.1.1
Let $X \subset \mathbb{P}^9$ be the hypersurface associated to ZZ_5, then

$$H^8_{prim}(ZZ_5) \cong \mathbb{Q}(-2). \tag{2.4}$$

Proof. The variety X is defined by the equation $I_5 = 0$. We write

$$I_5 = I_4 B_4 - G_4, \tag{2.5}$$

where

$$\begin{aligned} G_4 = G_4' + A_3^2 I_3 - 2A_3A_4A_2I_2 - 2A_3A_5S_3 = \\ G_4' + A_3^2 I_3 - 2A_3A_4A_2I_2 - 2A_3A_5A_2A_1A_0 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} G_4' = A_4^2 B_3 I_2 + 2A_4A_5B_3S_2 + A_5^2 I_3^1 = \\ A_4^2 B_3 I_2 + 2A_4A_5B_3A_1A_0 + A_5^2 I_3^1, \end{aligned} \tag{2.7}$$

see (1.13) and (1.14). By (2.5), I_5 is linear in the variable B_4. Consider

$$\hat{Y} := \mathcal{V}(I_5, I_4) \subset \mathbb{P}^9. \tag{2.8}$$

The variety $\hat{Y}$ is closed in X and one has the localization sequence

$$\to H^8_c(X \backslash \hat{Y}) \to H^8(X) \to H^8(\hat{Y}) \to H^9_c(X \backslash \hat{Y}) \to . \tag{2.9}$$

Let $P_1 = (0, ..., 0, 1) \subset \mathbb{P}^9$ be the point where all the variables but B_4 are zero. We project from P_1 and get an isomorphism

$$X \backslash \hat{Y} \cong \mathbb{P}^8 \backslash \mathcal{V}(I_4). \tag{2.10}$$

Note that I_4 is independent of A_5 and A_3, thus we can apply *Theorem B* to $\mathbb{P}^8 \backslash \mathcal{V}(I_4)$ ($N = 8$, $k = 0$, $t = 2$) and get $H^i(\mathbb{P}^8 \backslash \mathcal{V}(I_4)) = 0$ for $i < 10$.

The isomorphism (2.10) implies the vanishing $H^i(X\backslash\hat{Y}) = 0$ for $i < 10$. Substituting this into the sequence (2.9), we get an isomorphism

$$H^8(X) \cong H^8(\hat{Y}). \tag{2.11}$$

By (2.5), we can rewrite

$$\hat{Y} = \mathcal{V}(I_4, G_4)^{(9)}. \tag{2.12}$$

The superscript means that this is a subscheme in $\mathbb{P}^9$. We will use the same notation without superscript for the variety after forgetting B_4. Both the polynomials do not depend on B_4. Applying *theorem A* ($N = 9$, $k = 2$, $t = 1$), we get

$$H^8(\hat{Y}) \cong H^6(Y)(-1), \tag{2.13}$$

where $Y := \mathcal{V}(I_4, G_4) \subset \mathbb{P}^8(\text{no } B_4)$. Together with (2.11) and (2.12), this implies

$$H^8(X) \cong H^6(Y)(-1). \tag{2.14}$$

Define $\hat{V}, U \subset \mathbb{P}^8(\text{no } B_4)$ by

$$\widehat{V} = \mathcal{V}(I_4, I_3, G_4) \quad \text{and} \quad U := \mathcal{V}(I_4, G_4)\backslash\widehat{V}. \tag{2.15}$$

We can write an exact sequence

$$\longrightarrow H^6_c(U) \longrightarrow H^6(Y) \longrightarrow H^6(\widehat{V}) \longrightarrow H^7_c(U) \longrightarrow . \tag{2.16}$$

Lemma 2.1.2
One has $H^i_c(U) = 0$ for $i < 8$.

Proof. U is defined by the following system

$$\begin{cases} I_4 = 0 = G_4 \\ I_3 \neq 0. \end{cases} \tag{2.17}$$

We have studied such schemes in Section 1.1 (see 1.33). By Theorem 1.1.7, we have an isomorphism

$$U \cong\ U_2 := \mathcal{V}(I_4)\backslash\mathcal{V}(I_4, I_3) \subset \mathbb{P}^7(\text{no } B_4, A_3). \tag{2.18}$$

Using

$$I_4 = B_3I_3 - A_2^2I_2 \tag{2.19}$$

and projecting from the point $P_3 \in \mathbb{P}^7$ where all variables but B_3 are zero, we get

$$U_2 \cong \mathbb{P}^6\backslash I_3. \tag{2.20}$$

One has

$$I_3 = \begin{vmatrix} B_0 & A_0 & 0 \\ A_0 & B_1 & A_1 \\ 0 & A_1 & C_2 \end{vmatrix}, \quad C_2 = A_1 + A_2 - A_4 \tag{2.21}$$

After a linear change of coordinates, we can assume that I_3 does not depend on A_4 and A_5. The *Theorem B* ($N = 6$, $k = 0$, $T = 2$) implies

$$H^i_c(U) \cong H^i_c(\mathbb{P}^6 \backslash \mathcal{V}(I_3)) = 0 \quad \text{for } \ i < 8. \tag{2.22}$$

□

From the sequence (2.16) we now obtain an isomorphism

$$H^6(Y) \cong H^6(\widehat{V}). \tag{2.23}$$

Combining this with (2.14), one has

$$H^8(X) \cong H^6(\widehat{V})(-1). \tag{2.24}$$

Remember that $\widehat{V}$ lives in $\mathbb{P}^8$ (no B_4) and is defined by

$$\widehat{V} := \mathcal{V}(I_4, I_3, G_4). \tag{2.25}$$

Theorem 1.1.5 implies that G_4 is independent of A_3 on $\widehat{V}$. Define $G'_4 = G_4 \mid_{A_3=0}$ as in (2.7). One has

$$\widehat{V} = \mathcal{V}(I_4, I_3, G'_4) \subset \mathbb{P}^8(\text{no } B_4). \tag{2.26}$$

Applying *Theorem A* ($N = 8$, $k = 3$, $t = 1$) for the last variety, the defining equations of which are independent of A_3, we obtain

$$H^6(\widehat{V}) \cong H^4(V)(-1), \tag{2.27}$$

with

$$V = \mathcal{V}(I_4, I_3, G'_4) \subset \mathbb{P}^7(\text{no } B_4, A_4). \tag{2.28}$$

We combine this with (2.24) and get

$$H^8(X) \cong H^4(V)(-2). \tag{2.29}$$

The next step is to get rid of B_3. Define

$$G_3 := A_4^2 \begin{vmatrix} B_0 & A_0 \\ A_0 & B_1 \end{vmatrix} + 2A_4A_5 \begin{vmatrix} A_0 & 0 \\ B_1 & A_1 \end{vmatrix} + A_5^2 \begin{vmatrix} B_1 & A_1 \\ A_1 & C_2 \end{vmatrix}. \tag{2.30}$$

We have the following equality

$$G'_4 = A_4^2 B_3 \begin{vmatrix} B_0 & A_0 \\ A_0 & B_1 \end{vmatrix} + 2A_4A_5B_3 \begin{vmatrix} A_0 & 0 \\ B_1 & A_1 \end{vmatrix} + A_5^2 \left(\begin{vmatrix} B_1 & A_1 \\ A_1 & C_2 \end{vmatrix} B_3 - A_2^2 B_1 \right) = B_3G_3 - A_2^2A_5^2B_1. \quad (2.31)$$

So, one has

$$V = \mathcal{V}(I_4, I_3, G'_4) = \mathcal{V}(B_3I_3 - A_2^2I_2, I_3, B_3G_3 - A_2^2A_5^2B_1) = \mathcal{V}(I_3, A_2I_2, B_3G_3 - A_2^2A_5^2B_1). \quad (2.32)$$

Set

$$\widehat{W} := V \cap \mathcal{V}(G_3) = \mathcal{V}(I_3, A_2I_2, G_3, A_2A_5B_1) \subset \mathbb{P}^7 \quad (2.33)$$

with all the polynomials to the right independent of B_3. We have an exact sequence

$$\to H^3(\widehat{W}) \to H_c^4(V \backslash \widehat{W}) \to H^4_{prim}(V) \to H^4_{prim}(\widehat{W}) \to \quad (2.34)$$

We apply *theorem A* ($N = 7$, $k = 4$, $t = 1$) to $\widehat{W} \subset \mathbb{P}^7$(no B_4, A_3) and get

$$H^3(\widehat{W}) = 0 \quad \text{and} \quad H^4_{prim}(\widehat{W}) \cong H^2_{prim}(W)(-1) \quad (2.35)$$

with

$$W = \mathcal{V}(I_3, A_2I_2, G_3, A_2A_5B_1) \subset \mathbb{P}^6(\text{no } B_4, A_4, B_3). \quad (2.36)$$

Substituting this in (2.34), we get

$$0 \to H_c^4(V \backslash \widehat{W}) \to H^4_{prim}(V) \to H^2_{prim}(W)(-1) \to . \quad (2.37)$$

Now we show that $H^2_{prim}(W)$ also vanishes.
Consider the subvariety $W \cap \mathcal{V}(A_2) \subset W$ and an exact sequence

$$\to H^1(W \cap \mathcal{V}(A_2)) \to H_c^2(W \backslash W \cap \mathcal{V}(A_2)) \to H^2_{prim}(W) \to H^2_{prim}(W \cap \mathcal{V}(A_2)) \to . \quad (2.38)$$

We write

$$W \cap \mathcal{V}(A_2) = \mathcal{V}(I_3, A_2I_2, G_3, A_2A_5B_1, A_2) = \mathcal{V}(I_3, G_3, A_2) \subset \mathbb{P}^6, \quad (2.39)$$

Theorem A($N = 6$, $k = 3$, $t = 0$) implies

$$H^i_{prim}(W \cap \mathcal{V}(A_2)) = 0, \quad i \leq 2. \quad (2.40)$$

The sequence (2.38) gives us

$$H^2_{prim}(W) \cong H^2_c(W\backslash W \cap \mathcal{V}(A_2)). \tag{2.41}$$

The scheme $W\backslash W \cap \mathcal{V}(A_2)$ is defined by

$$\begin{cases} I_3 = A_2I_2 = 0 \\ G_3 = A_2A_5B_1 = 0 \\ A_2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} I_3 = I_2 = 0 \\ G_3 = A_5B_1 = 0 \\ A_2 \neq 0. \end{cases} \tag{2.42}$$

Now define $S, T \subset \mathbb{P}^6(\text{no } B_4,\ A_3,\ B_3)$ by

$$S = \mathcal{V}(I_3, I_2, G_3, A_5B_1) \tag{2.43}$$

and $T = S \cap \mathcal{V}(A_2)$. For these varieties we have an exact sequence

$$\to H^1(S) \to H^1(T) \to H^2_c(W\backslash W \cap \mathcal{V}(A_2)) \to H^2_{prim}(S) \to \tag{2.44}$$

Note that the polynomial

$$G_3 = A_4^2 \begin{vmatrix} B_0 & A_0 \\ A_0 & B_1 \end{vmatrix} + 2A_4A_5 \begin{vmatrix} A_0 & 0 \\ B_1 & A_1 \end{vmatrix} + A_5^2 \begin{vmatrix} B_1 & A_1 \\ A_1 & C_2 \end{vmatrix} \tag{2.45}$$

is of the same shape as G_n studied in sect. 1.1, and by the Theorem 1.1.5, G_3 loses first two summands on S. Thus,

$$\begin{aligned} S = \mathcal{V}(I_3, I_2, G_3, A_5B_1) = \mathcal{V}(I_3, I_2, A_5^2(B_1C_2 - A_1^2), A_5B_1) = \\ \mathcal{V}(C_2I_2 - A_1^2B_0, I_2, A_5A_1, A_5B_1) = \mathcal{V}(A_1B_0, I_2, A_5A_1, A_5B_1). \end{aligned} \tag{2.46}$$

We see that S is defined by the equations all independent of A_2 and A_4. *Theorem A* ($N = 6$, $k = 4$, $t = 2$) implies

$$H^i_{prim}(S) = 0, \quad \text{for } i < 4. \tag{2.47}$$

Let us look closely at T. By (2.46), we have

$$T = S \cap \mathcal{V}(A_2) = \mathcal{V}(A_1B_0, I_2, A_5A_1, A_5B_1, A_2) \subset \mathbb{P}^6(\text{no } B_4, A_4, B_3). \tag{2.48}$$

The defining polynomials do not depend on A_4, and *Theorem A* ($N = 6$, $k = 5$, $t = 1$) implies

$$H^i_{prim}(T) = 0, \quad \text{for } i < 2. \tag{2.49}$$

By (2.47) and (2.49), the sequence (2.44) implies the vanishing

$$H^2_c(W\backslash W\cap\mathcal{V}(A_2))=0. \tag{2.50}$$

Hence, (2.29), (2.37) and (2.41) yield an isomorphism

$$H^8_{prim}(X)\cong H^4_{prim}(V)(-2)\cong H^4_c(V\backslash\widehat{W})(-2). \tag{2.51}$$

The subscheme $V\backslash\widehat{W}\subset\mathbb{P}^7(\text{no } B_4,\ A_3)$ is defined by

$$\begin{cases} I_3=A_2I_2=0\\ B_3G_3-A_2^2A_5^2B_1=0\\ G_3\neq 0.\end{cases} \tag{2.52}$$

We solve the middle equation on B_3. Projecting from the point where all the coordinates but B_3 are zero, one gets an isomorphism

$$V\backslash\widehat{W}\cong Y\backslash Y\cap\mathcal{V}(G_3) \tag{2.53}$$

for $Y=\mathcal{V}(I_3,A_2I_2)\subset\mathbb{P}^6(\text{no } B_4,A_3,B_3)$. Consider the exact sequence

$$\to H^3(Y)\to H^3(Y\cap\mathcal{V}(G_3))\to H^4_c(Y\backslash Y\cap\mathcal{V}(G_3))\to H^4_{prim}(Y)\to . \tag{2.54}$$

The equations of Y do not depend on A_5. Applying *Theorem A* ($N=6$, $k=2$, $t=1$), we obtain

$$H^i_{prim}(Y)=0\quad\text{for }\ i<5. \tag{2.55}$$

Then the sequence (2.54) implies

$$H^4_c(Y\backslash Y\cap\mathcal{V}(G_3))\cong H^3(Y\cap\mathcal{V}(G_3)). \tag{2.56}$$

Comparing this with (2.51) and (2.53), one gets

$$H^8_{prim}(X)\cong H^4_c(V\backslash\widehat{W})(-2)\cong H^3(Y\cap\mathcal{V}(G_3))(-2). \tag{2.57}$$

Consider the subvariety

$$Y_1:=\mathcal{V}(I_3,I_2,G_3)\subset\mathcal{V}(I_3,A_2I_2,G_3)=Y\cap\mathcal{V}(G_3)\subset\mathbb{P}^6. \tag{2.58}$$

One has an exact sequence

$$\to H^2_{prim}(Y_1)\to H^3_c(Y\cap\mathcal{V}(G_3)\backslash Y_1)\to H^3(Y\cap\mathcal{V}(G_3))\to H^3(Y_1)\to . \tag{2.59}$$

Theorem 1.1.5 applied to G_3 on Y_1 allows us to rewrite $Y_1 \subset \mathbb{P}^6$(no B_4, A_3, B_3) as

$$Y_1 = \mathcal{V}(I_3, I_2, G_3) = \mathcal{V}(I_2, A_1B_0, A_5(B_1C_2 - A_1^2)). \tag{2.60}$$

After the change of the variables $C_2 := A_2$, the defining equations of Y_1 become independent of A_4. Applying *Theorem A*$(N = 6,\ k = 3,\ t = 1)$ to Y_1, we get

$$H^i(Y_1) = 0 \quad \text{for} \ \ i < 4. \tag{2.61}$$

The sequence (2.59) gives us

$$H^3(Y \cap \mathcal{V}(G_3)) \cong H_c^3(Y \cap \mathcal{V}(G_3)\backslash Y_1). \tag{2.62}$$

The scheme to the right $Y \cap \mathcal{V}(G_3)\backslash Y_1 \subset \mathbb{P}^6$(no B_4,A_3,B_3) is defined by

$$\begin{cases} I_3 = A_2I_2 = 0 \\ \qquad\quad G_3 = 0 \\ \qquad\quad I_2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} C_2I_2 - A_1^2B_0 = A_2 = 0 \\ A_4^2I_2 + 2A_4A_5A_0A_1 + A_5^2(B_1C_2 - A_1^2) = 0 \\ I_2 \neq 0. \end{cases} \tag{2.63}$$

By Corollary 1.1.4,

$$G_3I_2 = (A_4I_2 + A_0A_1A_5)^2 =: Li_3^2 \tag{2.64}$$

on $Y \cap \mathcal{V}(G_3)\backslash Y_1$, thus $G_3 = 0$ implies

$$A_4 = -\frac{A_0A_1A_5}{I_2}. \tag{2.65}$$

Furthermore, solving the first equation of (2.63) on C_2, we get

$$C_2 = A_1 - A_4 = \frac{A_1^2B_0}{I_2} \quad \Leftrightarrow \quad A_4 = \frac{A_1I_2 - A_1^2B_0}{I_2}. \tag{2.66}$$

By (2.65) and (2.66), it follows that

$$A_1I_2 - A_1^2B_0 = -A_0A_1A_5. \tag{2.67}$$

on $Y \cap \mathcal{V}(G_3)\backslash Y_1$.

We project from the point where the all the coordinates but A_4 are zero and forgetting A_2, which is zero on $Y \cap \mathcal{V}(G_3)\backslash Y_1$, we get an isomorphism

$$Y \cap \mathcal{V}(G_3)\backslash Y_1 \cong R\backslash Z, \tag{2.68}$$

where $R, Z \subset \mathbb{P}^4$ (no B_4, A_3, B_3, A_4, A_2) are defined by

$$R := \mathcal{V}(A_1I_2 + A_5A_0A_1 - A_1^2B_0) \tag{2.69}$$

and $Z := R \cap \mathcal{V}(I_2)$. By (2.57) and (2.62), one gets

$$\begin{aligned} H^8_{prim}(X) \cong & H^3(Y \cap \mathcal{V}(G_3))(-2) \cong \\ & H^3_c(Y \cap \mathcal{V}(G_3) \backslash Y_1)(-2) \cong H^3_c(R \backslash Z)(-2). \end{aligned} \tag{2.70}$$

The last step of the proof is the following.

Lemma 2.1.3
We have $H^3_c(R \backslash Z) \cong Q(0)$.

Proof. The variety R is defined by the equation

$$A_1 I_2 + A_5 A_0 A_1 - A_1^2 B_0 = A_1(I_2 + A_0 A_5 - A_1 B_0) = 0. \tag{2.71}$$

Consider the Mayer-Vietoris sequence for $R \backslash Z$:

$$\begin{aligned} \to H^2_c(\mathcal{V}(A_1) \backslash \mathcal{V}(A_1, I_2)) \oplus H^2_c(R_1 \backslash Z_1) \to H^2_c(\mathcal{V}(A_1) \cap R_1 \backslash Z_1) \to \\ H^3_c(R \backslash Z) \to H^3_c(\mathcal{V}(A_1) \backslash \mathcal{V}(A_1, I_2)) \oplus H^3_c(R_1 \backslash Z_1) \to \end{aligned} \tag{2.72}$$

with $Z_1 \subset R_1 \subset R \subset \mathbb{P}^4(A_0 : A_1 : A_5 : B_0 : B_1)$ defined by

$$R_1 = \mathcal{V}(I_2 + A_0 A_5 - A_1 B_0) \tag{2.73}$$

and $Z_1 = R_1 \cap \mathcal{V}(I_2)$. *Theorem B* ($N = 4$, $k = 1$, $t = 0$) implies

$$H^2_c(R_1 \backslash Z_1) = 0. \tag{2.74}$$

We prove that $H^3_c(R_1 \backslash Z_1)$ also vanishes. One has an exact sequence

$$\to H^2_{prim}(R_1) \to H^2_{prim}(Z_1) \to H^3_c(R_1 \backslash Z_1) \to H^3(R_1) \to . \tag{2.75}$$

The leftmost term vanishes because $R_1 \subset \mathbb{P}^4$ is a hypersurface. To compute $H^3(R_1)$, we write the following exact sequence

$$\begin{aligned} \longrightarrow H^2_{prim}(R_1 \cap \mathcal{V}(A_0)) \longrightarrow & H^3_c(R_1 \backslash R_1 \cap \mathcal{V}(A_0)) \longrightarrow \\ & H^3(R_1) \longrightarrow H^3(R_1 \cap \mathcal{V}(A_0)) \longrightarrow . \end{aligned} \tag{2.76}$$

Since

$$\begin{aligned} R_1 \cap \mathcal{V}(A_0) = \mathcal{V}(B_0 B_1 - A_0^2 + A_0 A_5 - A_1 B_0, A_0) = \\ \mathcal{V}(A_0, B_0(B_1 - A_1))^{(4)}, \end{aligned} \tag{2.77}$$

the defining polynomials are independent of A_5. Applying *Theorem A* ($N = 4$, $k = 2$, $t = 1$), we get $H^2_{prim}(R_1 \cap \mathcal{V}(A_0)) = 0$ and

$$H^3(R_1 \cap \mathcal{V}(A_0)) \cong H^1(\mathcal{V}(A_0, B_0(B_1 - A_1)))(-1) \tag{2.78}$$

with the variety to the right living in $\mathbb{P}^3(A_0 : A_1 : B_0 : B_1)$. But this variety is just the union of two lines intersected at one point and has trivial first cohomology group. Thus

$$H^3(R_1 \cap \mathcal{V}(A_0)) \cong H^3(\mathcal{V}(A_0, B_0(B_1 - A_1))^{(4)}) = 0, \tag{2.79}$$

and the sequence (2.76) implies an isomorphism

$$H^3(R_1) \cong H^3_c(R_1 \backslash R_1 \cap \mathcal{V}(A_0)). \tag{2.80}$$

The scheme $R_1 \backslash R_1 \cap \mathcal{V}(A_0) \subset \mathbb{P}^4(A_0 : A_1 : A_5 : B_0 : B_1)$ is defined by

$$\begin{cases} B_0B_1 - A_0^2 + A_0A_5 - A_1B_0 = 0 \\ \hfill A_0 \neq 0 \end{cases} \tag{2.81}$$

Projecting from the point where all the variables but A_5 are zero (and solving the first equation of the system on A_5), we get an isomorphism

$$R_1 \backslash R_1 \cap \mathcal{V}(A_0) \cong \mathbb{P}^3 \backslash \mathcal{V}(A_0) \cong \mathbb{A}^3. \tag{2.82}$$

Hence,

$$H^3_c(R_1 \backslash R_1 \cap \mathcal{V}(A_0)) = 0. \tag{2.83}$$

Together with (2.80), this simplifies (2.75) to

$$H^3_c(R_1 \backslash Z_1) \cong H^2_{prim}(Z_1). \tag{2.84}$$

Now,

$$Z_1 = R_1 \cap \mathcal{V}(I_2) = \mathcal{V}(B_0B_1 - A_0^2, A_0A_5 - A_1B_0) \subset \mathbb{P}^4. \tag{2.85}$$

Consider

$$Z_1 \cap \mathcal{V}(B_0) = \mathcal{V}(B_0, A_0^2, A_0A_5) = \mathcal{V}(A_0, B_0) \cong \mathbb{P}^2 \subset \mathbb{P}^4. \tag{2.86}$$

One has an exact sequence

$$\to H^1(\mathbb{P}^2) \to H^2_c(Z_1 \backslash Z_1 \cap \mathcal{V}(B_0)) \to H^2_{prim}(Z_1) \to H^2_{prim}(\mathbb{P}^2) \to . \tag{2.87}$$

Thus, we have an isomorphism

$$H^2_{prim}(Z_1) \cong H^2_c(Z_1 \backslash Z_1 \cap \mathcal{V}(B_0)). \tag{2.88}$$

The scheme $Z_1 \backslash Z_1 \cap \mathcal{V}(B_0)$ is defined by the following system:

$$\begin{cases} B_0B_1 - A_0^2 = 0 \\ A_0A_5 - A_1B_0 = 0 \\ B_0 \neq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} B_1 = \dfrac{A_0^2}{B_0} \\ A_1 = \dfrac{A_0A_5}{B_0} \\ B_0 \neq 0. \end{cases} \tag{2.89}$$

Set

$$\ell = (0 : A_1 : 0 : 0 : B_1) \subset \mathbb{P}^4(A_0 : A_1 : A_5 : B_0 : B_1). \tag{2.90}$$

The projection

$$\pi : \mathbb{P}^4 \backslash \ell \longrightarrow \mathbb{P}^2(A_0 : A_5 : B_0) \tag{2.91}$$

from the line ℓ gives an isomorphism

$$Z_1 \backslash Z_1 \cap \mathcal{V}(B_0) \cong \mathbb{P}^2 \backslash \mathcal{V}(B_0) \cong \mathbb{A}^2. \tag{2.92}$$

Thus, together with (2.88) and (2.84), we get

$$H_c^3(R_1 \backslash Z_1) \cong H^2_{prim}(Z_1) \cong H_c^2(Z_1 \backslash Z_1 \cap \mathcal{V}(B_0)) = 0. \tag{2.93}$$

Return now to the sequence (2.72). The defining polynomials of

$$\mathcal{V}(A_1) \backslash \mathcal{V}(A_1, I_2) \subset \mathbb{P}^4(A_0, A_1, A_5, B_0, B_1) \tag{2.94}$$

are independent of A_5, *Theorem B* ($N = 4$, $k = 1$, $t = 1$) implies

$$H_c^i(\mathcal{V}(A_1) \backslash \mathcal{V}(A_1, I_2)) = 0 \quad \text{for } \ i < 4. \tag{2.95}$$

By (2.93) and (2.74), the sequence (2.72) gives us

$$H_c^3(R \backslash Z) \cong H_c^2(\mathcal{V}(A_1) \cap R_1 \backslash Z_1). \tag{2.96}$$

The variety $\mathcal{V}(A_1) \cap R_1 \backslash Z_1$ is defined by

$$\begin{cases} I_2 + A_0A_5 - A_1B_0 = 0 \\ A_1 = 0 \\ I_2 \neq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} I_2 + A_0A_5 = 0 \\ A_1 = 0 \\ I_2 \neq 0 \end{cases} \tag{2.97}$$

Define $S, T \subset \mathbb{P}^3(A_0 : A_5 : B_0 : B_1)$ by

$$\begin{aligned} S &:= B_0B_1 - A_0^2 + A_0A_5, \\ T &:= S \cap \mathcal{V}(B_0B_1 - A_0^2). \end{aligned} \tag{2.98}$$

We get an isomorphism (forgetting A_1)

$$\mathcal{V}(A_1) \cap R_1 \backslash Z_1 \cong S \backslash T. \tag{2.99}$$

We have to compute

$$H_c^2(\mathcal{V}(A_1) \cap R_1 \backslash Z_1) \cong H_c^2(S \backslash T). \tag{2.100}$$

The exact sequence

$$\longrightarrow H^1(T) \longrightarrow H_c^2(S \backslash T) \longrightarrow H_{prim}^2(S) \longrightarrow \tag{2.101}$$

and stratification further gives us only

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow H_c^2(S \backslash T) \longrightarrow \mathbb{Q}(-1) \longrightarrow, \tag{2.102}$$

so we must compute $H_c^2(S \backslash T)$ directly.

The variety $S \subset \mathbb{P}^3(A_0 : A_5 : B_0 : B_1)$ is a quadric which is smooth. Up to a change of variables S is the image of Segre imbedding. More precisely, $S = Im(\gamma)$ for

$$\gamma : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 : (a : b), (c : d) \mapsto (ac : ac - bd : ad : bc). \tag{2.103}$$

Now, $T \subset S \subset \mathbb{P}^3$ is defined by

$$T := S \cap \mathcal{V}(B_0 B_1 - A_0^2) = \mathcal{V}(A_0 A_5, B_0 B_1 - A_0^2). \tag{2.104}$$

So T is a union of 3 components $T = \ell_1 \cup \ell_2 \cup \ell_3$, where ℓ_1 and ℓ_2 coincide with the lines $\gamma(\{\infty\} \times \mathbb{P}^1)$ and $\gamma(\mathbb{P}^1 \times \{\infty\})$ respectively, and ℓ_3 is a zero of a nontrivial section of $\mathcal{O}(1,1)$. Now,

$$S \backslash (\ell_1 \cup \ell_2) \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus (\mathbb{P}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1) = \mathbb{A}^2 \tag{2.105}$$

has affine coordinates b, d and then $\ell_3 \cap \mathbb{A}^2$ has defining ideal $1 - bd$, so is isomorphic to $\mathbb{G}_m$. Thus we get

$$S \backslash T \cong \mathbb{A}^2 \backslash \mathbb{G}_m. \tag{2.106}$$

Since $\mathbb{G}_m$ is closed in $\mathbb{A}^2$, we can consider an exact sequence

$$\longrightarrow H_c^1(\mathbb{A}^2) \longrightarrow H_c^1(\mathbb{G}_m) \longrightarrow H_c^2(S \backslash T) \longrightarrow H_c^2(\mathbb{A}^2) \longrightarrow . \tag{2.107}$$

Now it follows that

$$H_c^2(S \backslash T) \cong H_c^1(\mathbb{G}_m) \cong \mathbb{Q}(0). \tag{2.108}$$

By (2.96), (2.100) and (2.108), we get

$$\mathbb{Q}(0) \cong H_c^2(S \backslash T) \cong H_c^2(\mathcal{V}(A_1) \cap R_1 \backslash Z_1) \cong H_c^3(R \backslash Z). \tag{2.109}$$

□

The isomorphism (2.70) now yields the desired

$$H^8_{prim}(X) \cong H^3_c(R\backslash Z)(-2) \cong \mathbb{Q}(-2). \tag{2.110}$$

This concludes the proof.

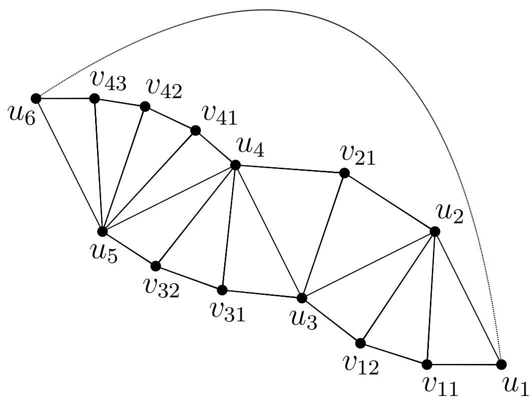
□

2.2 Generalized zigzag graphs

Definition 2.2.1
Fix some $t \geq 1$ and consider a set $V(\Gamma)$ of $t+2$ vertexes u_i, $1 \leq i \leq t+2$. Define $p(u_1, \ldots, u_{t+2})$ to be the set of $t+1$ edges (u_i, u_{i+1}), $1 \leq i \leq t+1$. Let $E(\Gamma) := p(u_1, \ldots, u_{t+2})$. Now choose some positive integers l_i for $1 \leq i \leq t$ with $l_1 \geq 2$ and $l_t \geq 2$. For each i, $1 \leq i \leq t$, we add $l_i - 1$ new vertexes v_{ij}, $1 \leq j \leq l_i - 1$, and l_i new edges $p(u_i, v_{i1}, \ldots, v_{i\,l_i-1}, u_{i+2})$, and $l_i - 1$ edges (v_{ij}, u_{i+1}), $1 \leq j \leq l_i - 1$. Finally, we add an edge (u_1, u_{t+2}). We call the constructed graph $\Gamma = (V(\Gamma), E(\Gamma))$ the *generalized zigzag* graph $GZZ(l_1, \ldots, l_t)$.

For $GZZ(l_1, \ldots, l_t)$ we define $n = 1+\sum_{i=1}^{t} l_i$. The graph $GZZ(l_1, \ldots, l_t)$ has $n+1$ vertexes, $2n$ edges and the Betti number equals n. Thus $GZZ(l_1, \ldots, l_t)$ is a logarithmically divergent graph.

Example 2.2.2
The graph $GZZ(3, 2, 3, 4)$ looks like



Example 2.2.3
The wheel with spokes graph WS_n is isomorphic to the generalized zigzag graph $GZZ(n-1)$, $n \geq 3$.

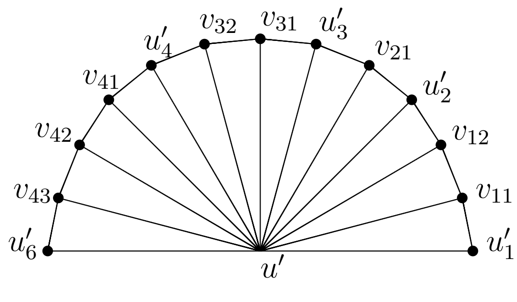
Example 2.2.4
The zigzag graph ZZ_n is isomorphic to the $GZZ(2, 1, \ldots, 1, 2)$ (with $n-5$ 1's in the middle) for $n \geq 5$.

Theorem 2.2.5
A generalized zigzag graph $\Gamma = GZZ(l_1, \ldots, l_t)$ is primitively log divergent.

Proof. We need to prove that for any subgraph $\Gamma' \subset \Gamma$ the inequality $|E(\Gamma')| > 2h_1(\Gamma')$ holds, which means that Γ' is not logarithmically divergent. We do not distinguish between a graph and it's set of edges. Because our graph Γ is planar, it partitions the plain into exactly $h_1 + 1$ pieces. This is a good way to compute h_1. We will call the loops of length 3 simple loops. We can order the simple loops from the the right bottom corner to the left top keeping in mind the drawing like in Example 2.2.2. Formally, let $\Delta_1 = p(u_1, u_2, v_{11}, u_1)$ and for each i we define the next simple loop Δ_{i+1} to be a simple loop which has a common edge with Δ_i but was not already labeled. Define $\Gamma_0 := \Gamma \backslash (u_1, u_{t+2})$. The main point of the proof is the following. The graph Γ_0 is a strip of Δ's, for each i, $1 \leq i \leq k$, we can cut this strip along (u_i, u_{i+1}), turn over one piece and glue along the same edge. Denote this operation by ϕ_i. This gives a map

$$\phi := \phi_t \circ \ldots \circ \phi_2 : \Gamma_0 \longrightarrow \hat{\Gamma}_0, \tag{2.111}$$

where $\hat{\Gamma}_0$ is isomorphic to WS_n without one boundary edge; this graph is topologically the same as a half of WS_n, we denote it by hWS_n. Note that the maps ϕ_i and ϕ are the isomorphisms between sets of edges of the graphs in the described way. On some vertexes this map is not single-valued. For the Example 2.2.2 we have the following $\hat{\Gamma}_0 = hWS_{13}$



The vertex u_i under described operations goes to u'_{i-1}, or u'_i, or u' depending on the edge that we take. We can label the simple loops of hWS_n from the right to the left by $\hat{\Delta}_1, \ldots, \hat{\Delta}_{n-1}$, these are the images of Δ's

$$\phi(\Delta_i) = \hat{\Delta}_i. \tag{2.112}$$

Each ϕ_i preserves loops; this means that a subgraph $\gamma \subset \phi_{i-1} \ldots \phi_2(\Gamma)$ is a loop if and only if $\phi_i(\gamma)$ is a loop of the same length. Thus this condition holds for ϕ. It follows that Γ_0 and hWS_n have the same Betti numbers. Moreover, for each subgraph $\Gamma''_0 \subset \Gamma_0$ we have

$$h_1(\Gamma''_0) = h_1(\phi(\Gamma''_0)). \tag{2.113}$$

To involve the "special" edge (u_1, u_{t+2}) into consideration, note that if the graph Γ_0'' is disconnected and we have no path $p'(u_1, \ldots, u_{t+1})$ with endpoints u_1 and u_{t+1}, then the adding of (u_1, u_{t+2}) doesn't change the Betti number; otherwise this increases the number by one.

$$h(\Gamma_0'' \cup (u_1, u_{t+1})) = \begin{cases} h(\Gamma_0''), & p'(u_1, \ldots, u_{t+2}) \not\subset \Gamma_0'', \\ h(\Gamma_0'') + 1, & \text{otherwise.} \end{cases} \tag{2.114}$$

This proves that we can extend the map ϕ to

$$\bar{\phi} : \Gamma \longrightarrow \hat{\Gamma} \tag{2.115}$$

which maps our graph to $\hat{\Gamma}$; this graph is nothing but $\hat{\Gamma}_0 \cong hWS_n$ compactified by adding the missing boundary edge and is isomorphic to WS_n. The map $\bar{\phi}$ satisfies the same condition as ϕ in (2.113). For the example of hWS_{13} above, we add the edge (u_1, u_6) on the drawing and get WS_{13}.

It remains to prove that WS_n is primitively divergent. We label the spokes by a_i, and any other edge that has common vertexes with a_i and a_{i+1} (the indices modulo n) for some i is denoted by b_i. Take a subgraph $\Gamma' \subset WS_n$ and assume that $WS_n \backslash \Gamma'$ has p b-edges and q a-edges. Let Γ by an intermediate graph which we get from WS_n after removing this p a's. This graph Γ is a disjoint union of p graphs Γ_i isomorphic to hWS_{n_i} for some $n_i \leq 1$ and $1 \leq i \leq p$, assuming hWS_1 and hWS_2 to be an edge and a triangle respectively. It follows that

$$h_1(\Gamma'') = \bigoplus_{i=1}^{p} h_1(\Gamma_i'') \tag{2.116}$$

for any subgraph $\Gamma'' \subset \Gamma$ and $\Gamma_i'' = \Gamma_i \cap \Gamma''$, $1 \leq i \leq p$.

To get the initial subgraph $\Gamma' \subset \Gamma \subset WS_n$ we need to drop q b's. Assume that we drop q_i b's in Γ_i and get $\Gamma_i' \cong hWS_{n_i}'$ for $1 \leq i \leq p$. One can easily compute

$$h_1(hWS_{n_i}) - h_1(hWS_{n_i}') = \begin{cases} q_i - 1, & \text{if } q_i = n_i \\ q_i, & \text{otherwise.} \end{cases} \tag{2.117}$$

Assume that we have p_1 i's with the upper assumption, $p_1 \leq p$. Taking the sum over all i, we get

$$h_1(\Gamma) - h_1\Gamma' = q - p_1. \tag{2.118}$$

Now we recall that Γ is the WS_n without p a's. Each dropping of a-edge decreases the Betti number by one. Thus,

$$h_1(\Gamma) - p. \tag{2.119}$$

Together with (2.118), this gives us

$$h_1(\Gamma') - p - q + p_1. \tag{2.120}$$

Now we can compute

$$2h_1(\Gamma') - |E(\Gamma')| = 2(n-p-q+p_1) - (2n-p-q) = -p-q+2p_1 \leq 0 \tag{2.121}$$

since $p_1 \leq p$ by definition, and $p_1 \leq q$ since each of p_1 i's gives us some $q_i = n_i \neq 0$. The equality can hold only when $q_i = n_i = 1$, $p = p_1$, but this means that all edges are dropped. This concludes the proof.

□

Let $X \subset \mathbb{P}^{2n-1}$ be a graph hypersurface. We consider the middle dimensional Betti cohomology $H^{mid}(X) = H^{2n-2}(X)$. By Deligne's theory of MHS ([De2], [De3]), there is a $\mathbb{Q}$-mixed Hodge structure associated to $H^{mid}(X)$. We try to study the graded pieces of weight filtration W: $\mathrm{gr}\,^W_i(H^{2n-2}(X))$, $0 \leq i \leq 2n-2$.

Theorem 2.2.6
For the hypersurface X associated to a generalized zigzag graph $GZZ(l_1, \ldots, l_t)$, one has an inclusion

$$\mathrm{gr}\,^W_4(H^{mid}(X)) = W_4(H^{mid}(X)) = W_5(H^{mid}(X)) \cong \mathbb{Q}(-2). \tag{2.122}$$

Proof. Denote $GZZ(l_1, \ldots, l_t)$ by Γ. We consider the case when t is even and start with labeling of edges and choosing orientations. For simplicity, let $n_0 := 0$ and

$$n_i := \sum_{j=1}^{i} l_j, \quad \text{for } 1 \leq i \leq t. \tag{2.123}$$

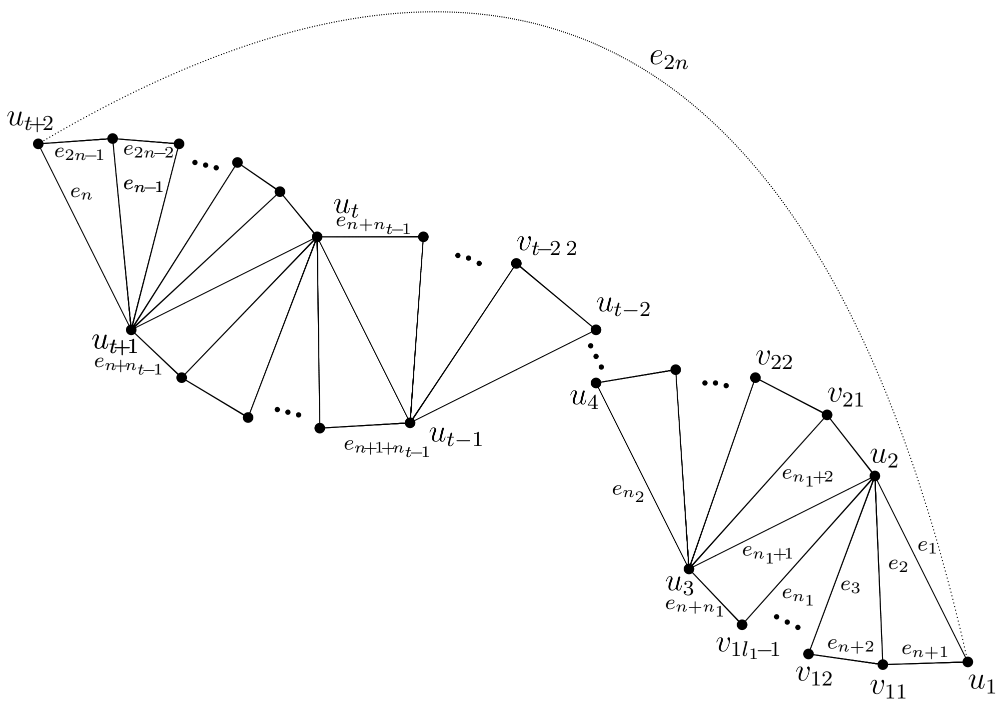
For each i, $1 \leq i \leq t$, define $e_{n_{i-1}+1} := (u_{i+1}, u_i)$ for odd i and $e_{n_{i-1}+1} := (u_i, u_{i+1})$ for even i,

$$e_{n_{i-1}+j} := \begin{cases} (u_{i+1}, v_{i\,j-1}) & \text{for } 2 \leq j \leq l_i, \; i \text{ odd}, \\ (v_{i\,j-1}, u_{i+1}) & \text{for } 2 \leq j \leq l_i, \; i \text{ even}. \end{cases} \tag{2.124}$$

Together with $e_{n_t+1} := (u_{t+2}, u_{t+1})$ for even t and $e_{n_t+1} := (u_{t+1}, u_{t+2})$ for odd t, these are the first $n_t + 1 =: n$ edges. Now, for each i, $1 \leq i \leq t$, define $e_{n+n_{i-1}+1} := (v_{i1}, u_i)$,

$$e_{n+n_{i-1}+j} := (v_{ij}, v_{ij-1}), \quad \text{for } 2 \leq j \leq l_i - 1, \tag{2.125}$$

and $e_{n+n_{i-1}+l_i} := (u_{i+2}, v_{i\,l_i-1})$. Roughly speaking, all edges are oriented from the left top corner to the right bottom and from the right top corner to the left bottom corner. Define $e_{2n} := (u_1, u_{t+2})$.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
3	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
6	0	0	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
7	0	0	0	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
8	0	0	0	0	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
12	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0
13	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	1	1

For building the table, we take the small loops from right bottom corner of the drawing to the left top corner, and the last loop to be chosen is the loop with the edge (u_1, u_{t+2}). Because of lack of space, we draw the table for the graph in Example 2.2.2.

Now we take $2n$ variables $T_1, \ldots, T_{2n}$ and build a matrix $M(T)$ as the sum of elementary matrices (see Section 2, Chapter 1). After a change of the coordinates similar to the case of ZZ_5, we get the matrix

$$M_{GZZ} = \begin{pmatrix} B_0 & A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{17} \\ A_0 & B_1 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & B_2 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & C_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{16} \\ 0 & 0 & 0 & A_3 & C_4 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & A_{15} \\ 0 & 0 & 0 & 0 & A_4 & B_5 & A_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_5 & B_6 & A_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_6 & B_7 & A_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_7 & C_8 & A_8 & 0 & 0 & A_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_8 & C_9 & A_9 & 0 & A_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_9 & C_{10} & A_{10} & A_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{10} & B_{12} & A_{11} \\ A_{17} & 0 & 0 & A_{16} & A_{15} & 0 & 0 & 0 & A_{14} & A_{13} & A_{12} & A_{11} & B_{12} \end{pmatrix}. \tag{2.126}$$

The A's appear in the last row in the zero column and in the columns $n_i + j - 1$ for all $i \not\equiv t \bmod 2$, $1 \leq i \leq t$, and all $1 \leq j \leq l_i$. In the same columns (but 0 and $n-2$) we have C's in the main diagonal. This C's are defined by

$$C_k := \begin{cases} A_v + A_{k-1} - A_k, & k_i, l_{i+1} > 1, i \neq 0, \\ A_v - A_{k-1} - A_k, & k_i + j, 1 \leq j \leq l_{i+1} - 2, \\ A_v - A_{k-1} + A_k, & k_{i+1} - 1, l_{i+1} > 1, i \neq t-1, \\ A_v + A_{k-1} + A_k, & k_i, l_{i+1} = 1, \end{cases} \tag{2.127}$$

where $i \not\equiv t \bmod 2$, and A_v is always in the last row in the same column as C_k. Formally, if $k_i + j - 1$, then

$$v = v(k) - 2 + \sum_{\substack{r=i+2 \\ r \not\equiv t \bmod 2}}^{t-1} l_r + l_{i+1} - j. \tag{2.128}$$

Sometimes we denote by A_m the entry in the left bottom corner of M_{GZZ}.

For the case of odd t we can derive the tables and the matrices from the even case. Indeed, consider some $\Gamma' = GZZ(l_1, \ldots, l_t)$ with even t and let Γ be the graph which we get from Γ' after forgetting edges of simple loops

$\Delta_1, \ldots, \Delta_{l_1}$ (see Theorem 2.2.5 for definition), we assume that (u_2, u_3) remains, and we take (u_{t+2}, u_2) instead of (u_{t+2}, u_1). So, $\Gamma = GZZ(l_2, \ldots, l_t)$. Constructing everything similar, the table for Γ is that for Γ' without first l_1 rows. The matrix of Γ looks similar to that of Γ' with the same assumptions on A's in the last row and on C's.

Consider the projective space $\mathbb{P}^{2n-1}$ with coordinates all the A_i's and B_j's appearing in the matrix and define

$$X := \mathcal{V}(\det(M_{GZZ})) = \mathcal{V}(I_n) \subset \mathbb{P}^{2n-1}, \tag{2.129}$$

where

$$M_{GZZ} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \ldots & C_{n-3} & A_{n-3} & A_{n-1} \\ \ldots & A_{n-3} & B_{n-2} & A_{n-2} \\ \ldots & A_{n-1} & A_{n-2} & B_{n-1} \end{pmatrix}. \tag{2.130}$$

Since $l_t > 1$, the entry $a_{n-3\,n-3}$ is really not independent, thus C_{n-3}.

Step 1. For the closed subscheme $\mathcal{V}(I_n, I_{n-1}) \subset X$ we have the localization sequence

$$\begin{aligned} &\to H_c^{2n-2}(X \backslash \mathcal{V}(I_n, I_{n-1})) \to H^{2n-2}(X) \to \\ &\qquad H^{2n-2}(\mathcal{V}(I_n, I_{n-1})) \to H_c^{2n-1}(X \backslash \mathcal{V}(I_n, I_{n-1})) \to . \end{aligned} \tag{2.131}$$

We can write

$$I_n = B_{n-1} I_{n-1} - G_{n-1}, \tag{2.132}$$

where G_n is independent of B_{n-1}. Projecting from the point where all the variables but B_{n-1} are zero, we get

$$X \backslash \mathcal{V}(I_n, I_{n-1}) \cong \mathbb{P}^{2n-2} \backslash \mathcal{V}(I_{n-1}). \tag{2.133}$$

Because I_{n-1} is independent of A_{n-2} and A_m, *Theorem B* ($N = 2n-2$, $k = 0$, $t = 2$) applied to the scheme on the right hand side of (2.133) implies

$$H_c^i(X \backslash \mathcal{V}(I_n, I_{n-1})) \cong H_c^i(\mathbb{P}^{2n-2} \backslash \mathcal{V}(I_{n-1})) = 0 \tag{2.134}$$

for $i < 2n$. The sequence (2.131) implies an isomorphism

$$H^{2n-2}(X) \cong H^{2n-2}(\mathcal{V}(I_n, I_{n-1})). \tag{2.135}$$

By (2.132), one has

$$\mathcal{V}(I_n, I_{n-1}) \cong \mathcal{V}(I_{n-1}, G_{n-1})^{(2n-1)}. \tag{2.136}$$

Both polynomials to the right are independent of are independent of B_{n-1}. *Theorem A* ($N = 2n-1$, $k = 2$, $t = 1$) and (2.135) imply

$$H^{2n-2}(X) \cong H^{2n-4}(\mathcal{V}(I_{n-1}, G_{n-1}))(-1). \tag{2.137}$$

The variety to the right lives in $\mathbb{P}^{2n-2}(\text{no } B_{n-1})$. Define the closed subscheme $\hat{V} \subset \mathcal{V}(I_{n-1}, G_{n-1})$ by

$$\hat{V} := \mathcal{V}(I_{n-1}, I_{n-2}, G_{n-1}) \subset \mathbb{P}^{2n-2}(\text{ no } B_{n-1}). \tag{2.138}$$

One has an exact sequence

$$\begin{aligned} \to H_c^{2n-4}(\mathcal{V}(I_{n-1}, G_{n-1})\backslash\hat{V}) \to H^{2n-4}(\mathcal{V}(I_{n-1}, G_{n-1})) \to \\ H^{2n-4}(\hat{V}) \to H_c^{2n-3}(\mathcal{V}(I_{n-1}, G_{n-1})\backslash\hat{V}) \to \end{aligned} \tag{2.139}$$

The polynomial I_{n-1} is independent of A_{n-2} and the coefficient of A_{n-2}^2 in G_{n-1} is I_{n-2}. By Theorem 1.1.7, we have

$$\mathcal{V}(I_{n-1}, G_{n-1})\backslash\hat{V} \cong \mathcal{V}(I_{n-1})\backslash\mathcal{V}(I_{n-1}, I_{n-2}) \subset \mathbb{P}^{2n-3}(\text{ no } B_{n-1}, A_{n-2}). \tag{2.140}$$

The polynomials I_{n-2} and I_{n-1} are independent of A_{n-1} and A_m. Applying *Theorem B* ($N = 2n-3$, $k = 1$, $t = 2$), we get

$$H_c^i(\mathcal{V}(I_{n-1}, G_{n-1})\backslash\hat{V}) \cong H_c^i(\mathcal{V}(I_{n-1})\backslash\mathcal{V}(I_{n-1}, I_{n-2})) = 0 \tag{2.141}$$

for $i \leq 2n-3$. The sequence (2.139) yields

$$H^{2n-4}(\mathcal{V}(I_{n-1}, G_{n-1})) \cong H^{2n-4}(\hat{V}). \tag{2.142}$$

By the Theorem 1.1.5, the polynomial G_{n-1} is independent of A_{n-2} on $\hat{V}$. Thus, $\hat{V}$ is defined by the vanishing of three polynomials that are independent of A_{n-2}. Applying the *Theorem A* ($N = 2n-2$, $k = 3$,$t = 1$), we get

$$H^{2n-4}(\hat{V}) \cong H^{2n-6}(V)(-1), \tag{2.143}$$

where the variety on the right hand side is defined by

$$V := \mathcal{V}(I_{n-1}, I_{n-2}, G'_{n-1}) \subset \mathbb{P}^{2n-3}(\text{ no } B_{n-1}, A_{n-1}) \tag{2.144}$$

and

$$G'_{n-1} := G_{n-1}|_{A_{n-1}=0}. \tag{2.145}$$

Combining (2.137), (2.142) and (2.143), we get

$$H^{2n-2}(X) \cong H^{2n-6}(V)(-2). \tag{2.146}$$

<u>*Step 2.*</u> Now we get rid of B_{n-2}. We can write

$$G'_{n-1} = B_{n-2}G_{n-2} - A_{n-3}^2 G_{n-3}, \tag{2.147}$$

where G_{n-2} and G_{n-3} are considered to be polynomials of variables $A_{n-1}, \dots, A_m$ and $A_n, \dots, A_m$ with "coefficients" from the matrices I_{n-2} and I_{n-3} respectively. The decomposition follows from the fact that each coefficient of G'_{n-1} is a factor of some I_{n-j-1}^j for $0 \le j \le n-2$, and the 3-diagonal matrix I_{n-j-1}^j has the right bottom entry B_{n-2}. Define the variety $\hat{T}_{n-2} \subset V$ by

$$\begin{aligned} &\hat{T}_{n-2} := V \cap \mathcal{V}(G_{n-2}) = \\ &\quad \mathcal{V}(A_{n-3}I_{n-3}, I_{n-2}, G_{n-2}, A_{n-3}G_{n-3})) \subset \mathbb{P}^{2n-3}(\text{ no } B_{n-1}, A_{n-2}). \end{aligned} \tag{2.148}$$

One has an exact sequence

$$H^{2n-7}(\hat{T}) \to H_c^{2n-6}(V \backslash \hat{T}) \to H_{prim}^{2n-6}(V) \to H_{prim}^{2n-6}(\hat{T}) \to . \tag{2.149}$$

Since the defining polynomials of $\hat{T}$ are independent of B_{n-2}, we apply *Theorem A* ($N = 2n-3$, $k = 4$, $t = 1$) and get

$$0 \to H_c^{2n-6}(V \backslash \hat{T}) \to H_{prim}^{2n-6}(V) \to H_{prim}^{2n-8}(T)(-1) \to \tag{2.150}$$

for $T \subset \mathbb{P}^{2n-4}$ (no B_{n-1}, A_{n-2} and B_{n-2}) defined by the same equations as $\hat{T}$. Applying the exact functors gr_i^W to the sequence above, we obtain

$$\mathrm{gr}_i^W H_{prim}^{2n-6}(V) \cong \mathrm{gr}_i^W H_c^{2n-6}(V \backslash \hat{T}), \quad i = 0, 1. \tag{2.151}$$

The subscheme $V \backslash \hat{T} \subset V$ is defined by the system

$$\begin{cases} I_{n-2} = A_{n-3}I_{n-3} = 0 \\ B_{n-2}G_{n-2} - A_{n-3}^2 G_{n-3} = 0 \\ G_{n-2} \neq 0. \end{cases} \tag{2.152}$$

Projecting from the point where all the variables but B_{n-2} are zero and solving the middle equation on B_{n-2}, we get an isomorphism

$$\begin{aligned} V \backslash \hat{T} &\cong \mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}) \backslash \mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2}) \\ &=: U_1 \subset \mathbb{P}^{2n-4}(\text{ no } B_{n-1}, A_{n-2}, B_{n-2}). \end{aligned} \tag{2.153}$$

One has an exact sequence

$$\begin{aligned} &H^{2n-7}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3})) \to H^{2n-7}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})) \to \\ &\quad H_c^{2n-6}(U_1) \to H_{prim}^{2n-6}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3})) \to . \end{aligned} \tag{2.154}$$

The variety $\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}) \subset \mathbb{P}^{2n-4}$ is defined by the polynomials that are independent of A_m. *Theorem A* ($N = 2n-4$, $k = 2$, $t = 1$) implies the vanishing of the rightmost and the leftmost terms, and the sequence simplifies to

$$H_c^{2n-6}(U_1) \cong H^{2n-7}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})). \tag{2.155}$$

Define $\hat{S}, U_2 \subset \mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2}) \subset \mathbb{P}^{2n-4}$ by

$$\hat{S} := \mathcal{V}(I_{n-2}, I_{n-3}, G_{n-2}) \tag{2.156}$$

and $U_2 := \mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})\backslash\hat{S}$. One has an exact sequence

$$\begin{aligned} &\longrightarrow H_{prim}^{2n-8}(\hat{S}) \longrightarrow H_c^{2n-7}(U_2) \longrightarrow \\ &\qquad H^{2n-7}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})) \longrightarrow H^{2n-7}(\hat{S}) \longrightarrow . \end{aligned} \tag{2.157}$$

The only appearance of A_{n-3} in the polynomials defining S is in G_{n-2}, namely in C_{n-3}. After a linear change of the variables we may assume that $C_{n-3} = A_{n-3}$ is independent. Furthermore, the same argument as for $\hat{V}$ at *step 1* (see (2.138) and (2.143)) gives us

$$H^{2n-7}(\hat{S}) \cong H^{2n-9}(S)(-1) \tag{2.158}$$

with

$$S := \mathcal{V}(I_{n-2}, I_{n-3}, G''_{n-2}) \subset \mathbb{P}^{2n-5}(\text{ no } B_{n-1}, A_{n-2}, B_{n-2}, A_{n-1}) \tag{2.159}$$

and $H_{prim}^{2n-6}(\hat{S}) = 0$. The sequence (2.157) simplifies to

$$\begin{aligned} 0 \longrightarrow H_c^{2n-7}(U_2) \longrightarrow H^{2n-7}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})) \longrightarrow \\ H^{2n-9}(S)(-1) \longrightarrow . \end{aligned} \tag{2.160}$$

Applying the functors gr_i^W for the sequence, by (2.151), (2.153), (2.155) and (2.160), we get

$$\operatorname{gr}_i^W H_{prim}^{2n-6}(V) \cong \operatorname{gr}_i^W H_c^{2n-7}(U_2) \quad \text{for } i = 0, 1. \tag{2.161}$$

Now, the scheme U_2 is defined by the system

$$\begin{cases} I_{n-2} = G_{n-2} = 0 \\ A_{n-3}I_{n-3} = 0 \\ I_{n-3} \neq 0 \end{cases} \Leftrightarrow \begin{cases} G_{n-2} = I_{n-2} = 0 \\ A_{n-3} = 0. \\ I_{n-3} \neq 0 \end{cases} \tag{2.162}$$

Eliminating A_{n-3}, which is zero on U_2, we get an isomorphism

$$U_2 \cong U_2' \tag{2.163}$$

with $U_2' \subset \mathbb{P}^{2n-5}$ (no B_{n-1}, A_{n-2}, B_{n-2}, A_{n-3}) defined by the system

$$\begin{cases} I_{n-2}' = 0 \\ G_{n-2}' = 0 \\ I_{n-3} \neq 0, \end{cases} \tag{2.164}$$

where primes mean that we set $A_{n-3} = 0$ in the polynomials, namely in C_{n-3}. Now we write

$$C_{n-3}' = A_{n-1} \pm A_{n-4}, \tag{2.165}$$

with "+" only when $a_{n-4n-4} = B_{n-4}$ in the matrix.

Such schemes U_2 were studied in the first chapter of Chapeter 1 (see (1.33) and (1.36)). It follows that U_2' is defined by the system

$$\begin{cases} C_{n-3}' I_{n-3} - A_{n-4}^2 I_{n-4} = 0 \\ Li_{n-2} = 0 \\ I_{n-3} \neq 0 \end{cases} \tag{2.166}$$

with

$$\begin{aligned} Li_{n-2} := A_{n-1} I_{n-3} + \sum_s (-1)^{s+n-1} A_{v(s)} I_{n-2}(s, n-3) = \\ A_{n-1} I_{n-3} + \sum_s (-1)^{s+n-1} A_{v(s)} I_s \prod_{k=s}^{n-4} A_k. \end{aligned} \tag{2.167}$$

The sum goes over all $s_i + j - 1 < n-3$, $i \not\equiv t \mod 2$, $1 \leq i \leq t$, $1 \leq j \leq l_i$, so over all $s < n-3$ such that $a_{ss} = C_s$. It is convenient to use the recurrence formula

$$Li_{s+1} = \begin{cases} A_{v(s)} I_s - A_{s-1} Li_s, & a_{s+1s+1} = C_{s+1}, \\ -A_{s-1} Li_s, & a_{s+1s+1} = B_{s+1}. \end{cases} \tag{2.168}$$

We can express A_{n-1} from the second equation of the system (2.166) and C_2 from the first one.

$$\begin{cases} A_{n-1} \pm A_{n-4} = C_{n-3}' = A_{n-4}^2 I_{n-4} / I_{n-3} \\ A_{n-1} = A_{n-4} Li_{n-3} / I_{n-3} \\ I_{n-3} \neq 0. \end{cases} \tag{2.169}$$

These two expressions for A_{n-1} must be equal on U_2'. We introduce the polynomials Ni_s defined by

$$A_{s-1}Ni_s = \pm A_{s-1}I_s + A_{s-1}^2 I_{s-1} - A_{s-1}Li_s. \tag{2.170}$$

Sometimes we can write Ni_s^- and Ni_s^+ to indicate the sign taken in Ni_s. The natural projection from the point where all the variables but A_{n-1} are zero induces an isomorphism

$$U_2' \cong U_3 := \mathcal{V}(A_{n-4}Ni_{n-3}) \backslash \mathcal{V}(A_{n-4}Ni_{n-3}, I_{n-3}) \tag{2.171}$$

with U_3 living in $\mathbb{P}^{2n-6}$(no B_{n-1}, A_{n-2}, B_{n-2}, A_{n-3}, A_{n-1}). By (2.161), (2.163) and (2.171),

$$\mathrm{gr}_i^W H_{prim}^{2n-6}(V) \cong \mathrm{gr}_i^W H_c^{2n-7}(U_3) \quad \text{for } i = 0, 1. \tag{2.172}$$

We have two possibilities : $a_{n-4\,n-4} = C_{n-4}$ or $a_{n-4\,n-4} = B_{n-4}$. When the letter holds, go to *Step 4* with $Ni_{n-3} = Ni_{n-3}^-$; do the next step with $Ni_{n-3} = Ni_{n-3}^+$ otherwise.

Step 3. Suppose that the entry a_{ss} of M_{GZZ} is C_s and $a_{s+1s+1} = C_{s+1}$. This means that $n_i \leq s \leq n_i + l_i - 2$ for some $i \not\equiv t$ mod 2. This corresponds to the case $s-4$ if we had come from *Step 2*. One has

$$C_s = A_v - A_s \pm A_{s-1} \tag{2.173}$$

with "+" only when $a_{s-1\,s-1} = B_s$. We work in $\mathbb{P}^N$(no DV_s) for

$$N = 2n - 1 - 2(n - 1 - s - 1) - 1 = 2s + 2, \tag{2.174}$$

and the Dropped Variables (DV_s) are all the variables in I_{n-1-s}^{s+1} but A_s. The thing to compute is $H_c^{2s+1}(U)$ for U defined by

$$U := \mathcal{V}(A_s Ni_{s+1}) \backslash \mathcal{V}(A_s Ni_{s+1}, I_{s+1}), \tag{2.175}$$

where

$$Ni_{s+1} = I_{s+1} + A_s I_s - Li_{s+1}. \tag{2.176}$$

Define $T, Y \subset \mathbb{P}^{2s+2}$(no DV_s) by

$$\begin{aligned} T &:= \mathcal{V}(A_s Ni_{s+1}), \\ Y &:= \mathcal{V}(A_s Ni_{s+1}, I_{s+1}). \end{aligned} \tag{2.177}$$

One has an exact sequence

$$\to H_{prim}^{2s}(T) \to H_{prim}^{2s}(Y) \to H_c^{2s+1}(U) \to H^{2s+1}(T) \to . \tag{2.178}$$

Using (2.173), we rewrite

$$\begin{aligned} Ni_{s+1} =& I_{s+1} + A_s I_s - Li_{s+1} = (A_v - A_s \pm A_{s-1}) I_s - \\ & A_{s-1}^2 I_{s-1} + A_s I_s - A_v I_s + A_{s-1} Li_s = \\ & - A_{s-1}(\pm I_s + A_{s-1} I_{s-1} - Li_s) = -A_{s-1} Ni_s. \end{aligned} \tag{2.179}$$

and see that Ni_{s+1} is actually independent of A_v and A_s. This allows us to apply *Theorem A* ($N = 2s+2$, $k = 1$, $t = 1$) to T and get

$$H^i_{prim}(T) = 0, \quad i < 2s+2. \tag{2.180}$$

Thus, the sequence (2.178) implies an isomorphism

$$H^{2s+1}_c(U) \cong H^{2s}_{prim}(Y). \tag{2.181}$$

Define the subvariety $\hat{Y}_1 \subset Y$ by

$$\hat{Y}_1 := Y \cap \mathcal{V}(I_s) = \mathcal{V}(A_s Ni_{s+1}, I_s, A_{s-1} I_{s-1}). \tag{2.182}$$

The polynomial Ni_{s-1} is independent of A_v by (2.179). Applying *Theorem A* ($N = 2s+2$, $k = 3$, $t = 1$) to $\hat{Y}_1$, we come to an exact sequence

$$0 \to H^{2s}_c(Y \backslash \hat{Y}_1) \to H^{2s}_{prim}(Y) \to H^{2s-2}_{prim}(Y_1)(-1) \to \tag{2.183}$$

with $Y_1 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_v)$ defined by the same polynomials. The scheme $Y \backslash \hat{Y}_1$ is defined by the system

$$\begin{cases} A_s A_{s-1} Ni_s = 0 \\ C_s I_s - A_{s-1}^2 I_{s-1} = 0 \\ I_s \neq 0. \end{cases} \tag{2.184}$$

By (2.173), we express A_v from the second equation. Projecting from the point where all the variables but A_v are zero, we get isomorphisms

$$Y \backslash \hat{Y}_1 \cong R \qquad \text{and} \qquad H^{2s}_c(Y \backslash \hat{Y}_1) \cong H^{2s}_c(R), \tag{2.185}$$

where $R \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_v)$ is given by the system

$$\begin{cases} A_s A_{s-1} Ni_s = 0 \\ I_s \neq 0. \end{cases} \tag{2.186}$$

Define $R_1, R_2 \subset R$ by

$$\begin{cases} A_{s-1} Ni_s = 0 \\ I_s \neq 0 \end{cases} \qquad \text{and} \qquad \begin{cases} A_s = 0 \\ I_s \neq 0. \end{cases} \tag{2.187}$$

One has the Mayer-Vietoris sequence

$$\begin{aligned}\longrightarrow H_c^{2s-1}(R_1)\oplus H_c^{2s-1}(R_2)\longrightarrow H_c^{2s-1}(R_3)\longrightarrow\\ H_c^{2s}(R)\longrightarrow H_c^{2s}(R_1)\oplus H_c^{2s}(R_2)\longrightarrow\end{aligned} \qquad (2.188)$$

with $R_3 := R_1 \cap R_2$. The defining polynomials of R_1 and R_2 are independent of A_s and A_m respectively. Applying *Theorem B* ($N = 2s+1$, $k = 1$, $t = 1$) to them, we get

$$H_c^i(R_1) = H_c^i(R_2) = 0 \qquad (2.189)$$

for $i < 2s + 1$. The sequence (2.188) implies an isomorphism

$$H_c^{2s}(R) \cong H_c^{2s-1}(R_3). \qquad (2.190)$$

Now, $R_3 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\, A_v)$ is defined by the system

$$\begin{cases} A_s = A_{s-1}Ni_s = 0 \\ I_s \neq 0. \end{cases} \qquad (2.191)$$

Projecting from the point where all the variables but A_s are zero, we get isomorphisms

$$R_3 \cong U' \qquad \text{and} \qquad H_c^{2s-1}(R_3) \cong H_c^{2s-1}(U') \qquad (2.192)$$

for $U' \subset \mathbb{P}^{2s}(\text{no } DV_s,\, A_v,\, A_s)$ defined by

$$U' = \mathcal{V}(A_{s-1}Ni_s)\backslash\mathcal{V}(A_{s-1}Ni_s, I_s). \qquad (2.193)$$

Collecting (2.181), (2.183), (2.185) (2.190) and (2.192) together, we obtain an exact sequence

$$0 \to H_c^{2s-1}(U') \to H_c^{2s+1}(U) \to H_{prim}^{2s-2}(Y_1)(-1) \to, \qquad (2.194)$$

where U is defined by (2.175). Applying gr_i^W, one gets

$$\mathrm{gr}_i^W H_c^{2n+1}(U) \cong \mathrm{gr}_i^W H_c^{2n+1}(U'), \quad i = 0, 1. \qquad (2.195)$$

If $s = 1$, go to *the Last Step.*

When we come to *Step 3* with some s, $n_i \leq s \leq n_i + l_i - 2$, $i \not\equiv t \bmod 2$, we must apply this step $s - n_i - 1$ times with $Ni_s = Ni_s^+$ and then one more time with $Ni_s = Ni_s^-$. After this, we are in a new situation.

Step 4. Suppose that the entry a_{ss} of M_{GZZ} is B_s and $a_{s+1\,s+1} = C_{s+1}$. This means that $s = n_i - 1$ for some $i \not\equiv t \bmod 2$. Denote by DV_s the dropped variables that are all the variables appearing in I^{s+1}_{n-1-s}, but A_s. Again, we have to compute $H_c^{2s+1}(U)$ for $U \subset \mathbb{P}^{2s+2}$ defined by

$$U := \mathcal{V}(A_s Ni_{s+1}) \backslash \mathcal{V}(A_s Ni_{s+1}, I_{s+1}), \tag{2.196}$$

where

$$Ni_{s+1} = -I_{s+1} + A_s I_s - Li_{s+1}. \tag{2.197}$$

Define closed subschemes $U_1, U_2 \subset U$ by

$$\begin{aligned} U_1 &:= \mathcal{V}(A_s) \backslash \mathcal{V}(A_s, I_{s+1}), \\ U_2 &:= \mathcal{V}(Ni_{s+1}) \backslash \mathcal{V}(Ni_{s+1}, I_{s+1}). \end{aligned} \tag{2.198}$$

This covering gives us an exact sequence

$$\begin{aligned} \longrightarrow H_c^{2s}(U_1) \oplus H_c^{2s}(U_1) \longrightarrow H_c^{2s}(U_3) \longrightarrow \\ H_c^{2s+1}(U) \longrightarrow H_c^{2s+1}(U_1) \oplus H_c^{2s+1}(U_1) \longrightarrow, \end{aligned} \tag{2.199}$$

where $U_3 := U_1 \cap U_2$. The polynomials in the definition of U_1 do not depend on A_m. *Theorem B* ($N = 2s+2$, $k = 1$, $t = 1$) implies

$$H_c^i(U_1) = 0 \quad \text{for } i < 2s+2. \tag{2.200}$$

We rewrite

$$\begin{aligned} Ni_{s+1} = & -I_{s+1} + A_s I_s - Li_{s+1} = -B_s I_s + A_{s-1}^2 I_{s-1} + A_s I_s + \\ & A_{s-1} Li_s = (A_s - B_s) I_s + A_{s-1}^2 I_{s-1} + A_{s-1} Li_s \end{aligned} \tag{2.201}$$

and see that Ni_{s+1} depends neither on B_s nor on A_s but on the difference $A_s - B_s$. After the change of variables $B_s := A_s - B_s$, the polynomial Ni_{s+1} becomes independent of A_s. Applying *Theorem B* ($N = 2s+2$, $k = 1$, $t = 1$), we get

$$H_c^i(U_2) = 0 \quad \text{for } i < 2s+2. \tag{2.202}$$

Together with (2.200), the sequence (2.199) gives an isomorphism

$$H_c^{2s+1}(U) \cong H_c^{2s}(U_3). \tag{2.203}$$

Now, $U_3 \subset \mathbb{P}^{2s+2}$(no DV_s) is given by the system

$$\begin{cases} A_s = 0 \\ Ni_{s+1} = 0 \\ I_{s+1} \neq 0 \end{cases} \Leftrightarrow \begin{cases} A_s = 0 \\ I_{s+1} + Li_{s+1} = 0 \\ I_{s+1} \neq 0. \end{cases} \tag{2.204}$$

We eliminate the variable A_s and consider the open $U_4 \subset \mathbb{P}^{2s+1}(\text{no } DV_s, A_s)$ defined by the last two conditions, then

$$H_c^{2s}(U_3) \cong H_c^{2s}(U_4). \tag{2.205}$$

Define $T, Y \subset \mathbb{P}^{2s+1}(\text{no } DV_s, A_s)$ by

$$\begin{aligned} T &:= \mathcal{V}(I_{s+1} + Li_{s+1}), \\ Y &:= \mathcal{V}(I_{s+1} + Li_{s+1}, I_{s+1}). \end{aligned} \tag{2.206}$$

We can write an exact sequence

$$\longrightarrow H^{2s-1}(T) \longrightarrow H^{2s-1}(Y) \longrightarrow H_c^{2s}(U_4) \longrightarrow H_{prim}^{2s}(T) \longrightarrow . \tag{2.207}$$

Theorem A ($N = 2s+1$, $k = 1$, $t = 0$) gives us the vanishing of the term to the left. Similar to (2.201), one has

$$I_{s+1} + Li_{s+1} = B_s I_s - A_{s-1}^2 I_{s-1} - A_{s-1} Li_s. \tag{2.208}$$

Let $\hat{T}_1$ be a subvariety of $T \subset \mathbb{P}^{2s+1}(\text{no } DV_s, A_s)$ defined by

$$\hat{T}_1 := T \cap \mathcal{V}(I_s) = \mathcal{V}(I_s, A_{s-1}^2 I_{s-1} + A_{s-1} Li_s). \tag{2.209}$$

We write an exact sequence

$$\longrightarrow H_c^{2s}(T \backslash \hat{T}_1) \longrightarrow H_{prim}^{2s}(T) \longrightarrow H_{prim}^{2s}(\hat{T}_1) \longrightarrow . \tag{2.210}$$

The defining polynomials of $\hat{T}_1$ are independent of B_s. We apply *Theorem A* ($N = 2s+1$, $k = 2$, $t = 1$) and get

$$H_{prim}^{2s}(\hat{T}_1) \cong H_{prim}^{2s-2}(T_1)(-1). \tag{2.211}$$

On $T \backslash \hat{T}_1$ we can express B_s (see (2.208)) and get an isomorphism

$$T \backslash \hat{T}_1 \cong \mathbb{P}^{2s} \backslash \mathcal{V}(I_s) \tag{2.212}$$

with $\mathbb{P}^{2s}(\text{no } DV_s, A_s, B_s)$. The polynomial I_s does not depend on A_m, and *Theorem B* ($N = 2s$, $k = 0$, $t = 1$) yields

$$H_c^i(\mathbb{P}^{2s} \backslash \mathcal{V}(I_s)) = 0 \quad \text{for } i < 2s+1. \tag{2.213}$$

By (2.211) and (2.213), the sequence (2.210) simplifies to

$$0 \longrightarrow H_{prim}^{2s}(T) \longrightarrow H_{prim}^{2s-2}(\hat{T}_1)(-1) \longrightarrow . \tag{2.214}$$

Applying gr_i^W, we get

$$\mathrm{gr}_0^W H^{2s}(T) = \mathrm{gr}_1^W H^{2s}(T) = 0. \tag{2.215}$$

We return to the variety Y which is defined by

$$\begin{aligned} Y := \mathcal{V}(I_{s+1} + Li_{s+1}, I_{s+1}) = \mathcal{V}(Li_{s+1}, I_{s+1}) = \\ \mathcal{V}(A_{s-1}Li_s, B_sI_s - A_{s-1}^2 I_{s-1}). \end{aligned} \tag{2.216}$$

One can write an exact sequence

$$\to H^{2s-2}_{prim}(\hat{Y}_1) \to H_c^{2s-1}(Y \backslash \hat{Y}_1) \to H^{2s-1}(Y) \to H^{2s-1}(\hat{Y}_1) \to, \tag{2.217}$$

where $\hat{Y}_1$ is the subvariety of $Y \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$ defined by

$$\hat{Y}_1 := Y \cap \mathcal{V}(I_s) = \mathcal{V}(I_s, A_{s-1}Li_{s-1}, A_{s-1}I_{s-1}). \tag{2.218}$$

The last three polynomials are independent of B_s; applying *Theorem A* ($N = 2s+1$, $k = 3$, $t = 1$), we get

$$H^{2s-1}(\hat{Y}_1) \cong H^{2s-3}(Y_1)(-1) \tag{2.219}$$

and $H^{2s-2}(\hat{Y}_1) = 0$. This implies that the sequence (2.217) simplifies to

$$0 \to H_c^{2s-1}(Y \backslash \hat{Y}_1) \to H^{2s-1}(Y) \to H^{2s-3}(Y_1)(-1) \to, \tag{2.220}$$

Now, $Y \backslash \hat{Y}_1 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$ is defined by the system

$$\begin{cases} A_{s-1}Li_s = 0 \\ B_sI_s - A_{s-1}^2 I_{s-1} = 0 \\ I_s \neq 0. \end{cases} \tag{2.221}$$

We can express B_s from the second equation and, projecting from the point where all variables but B_s are zero, we get an isomorphism

$$Y \backslash \hat{Y}_1 \cong U', \tag{2.222}$$

where $U' \subset \mathbb{P}^{2s}(\text{no } DV_s,\ A_s,\ B_s)$ is defined by the system

$$\begin{cases} A_{s-1}Li_s = 0 \\ I_s \neq 0. \end{cases} \tag{2.223}$$

Finally, combining (2.203), (2.205), (2.207),(2.215), (2.220) and (2.222), we get

$$\mathrm{gr}_i^W H_c^{2s+1}(U) \cong \mathrm{gr}_i^W H^{2s-1}(Y) \cong \mathrm{gr}_i^W H_c^{2s-1}(U'), \quad i = 0, 1, \tag{2.224}$$

for U and U' defined in (2.196) and (2.223) respectively.

Now, if $s = 1$, go to *the Last Step*. If $a_{s-1\,s-1} = C_{s-1}$, we go to *Step 6*. Otherwise do the next step.

Step 5. Consider an entry $a_{ss} = B_s$ of M_{GZZ} such that $a_{s+1s+1} = B_{s+1}$. With other words, s satisfies the condition $n_i \leq s \leq n_i + l_{i+1} - 2$ for some $i \equiv t \bmod 2$. Let $U \subset \mathbb{P}^{2s+2}(\text{no } DV_s)$ be defined by

$$U := \mathcal{V}(A_s Li_{s+1}) \backslash \mathcal{V}(A_s Li_{s+1}, I_{s+1}), \tag{2.225}$$

and denote by DV_s all the variables appearing in I^{s+1}_{n-1-s}. As usual, we try to compute $H^{2s+1}_c(U)$. Define $U_1, U_2 \subset U$ by

$$\begin{aligned} U_1 &:= \mathcal{V}(A_s) \backslash \mathcal{V}(A_s, I_{s+1}), \\ U_2 &:= \mathcal{V}(Li_{s+1}) \backslash \mathcal{V}(Li_{s+1}, I_{s+1}). \end{aligned} \tag{2.226}$$

One can write an exact sequence

$$\begin{aligned} \longrightarrow H^{2s}_c(U_1) \oplus H^{2s}_c(U_2) \longrightarrow H^{2s}_c(U_3) \longrightarrow \\ H^{2s+1}_c(U) \longrightarrow H^{2s+1}_c(U_1) \oplus H^{2s+1}_c(U_2) \longrightarrow \end{aligned} \tag{2.227}$$

where $U_3 := U_1 \cap U_2$. The defining polynomials of U_1 do not depend on A_m, thus *Theorem B* ($N = 2s+2$, $k = 1$, $t = 1$) implies

$$H^i_c(U_1) = 0 \quad \text{for } i < 2s+2. \tag{2.228}$$

Since

$$Li_{s+1} = -A_{s-1} Li_s \tag{2.229}$$

and I_{s+1} are independent of A_s, we apply *Theorem B* ($N = 2s+2$, $k = 1$, $t = 1$) to U_2 and get

$$H^i_c(U_2) = 0 \quad \text{for } i \leq 2s+1. \tag{2.230}$$

Thus, (2.227) gives us the isomorphism

$$H^{2s+1}_c(U) \cong H^{2s}_c(U_3). \tag{2.231}$$

We can eliminate A_s, which is zero along U_3, and get an isomorphism

$$U_3 \cong U_4 := \mathcal{V}(Li_{s+1}) \backslash \mathcal{V}(Li_{s+1}, I_{s+1}) \tag{2.232}$$

with $U_4 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\, A_s)$. Defining $T, Y \subset \mathbb{P}^{2s+1}$ by

$$\begin{aligned} T &:= \mathcal{V}(Li_{s+1}), \\ Y &:= \mathcal{V}(Li_{s+1}, I_{s+1}), \end{aligned} \tag{2.233}$$

we get an exact sequence

$$\to H^{2s-1}(T) \to H^{2s-1}(Y) \to H_c^{2s}(U_4) \to H_{prim}^{2s}(T) \to . \tag{2.234}$$

By (2.229), Li_s is independent of B_s, thus *Theorem A* ($N = 2s+1$, $k = 1$, $t = 1$) yields

$$H_{prim}^{i}(T) = 0 \quad \text{for} \ \ i < 2s+1. \tag{2.235}$$

The sequence (2.234) implies an isomorphism

$$H_c^{2s}(U_4) \cong H^{2s-1}(Y). \tag{2.236}$$

Now, let

$$\hat{Y}_1 := Y \cap \mathcal{V}(I_s) = \mathcal{V}(Li_s, I_{s+1}, I_s) \tag{2.237}$$

be a subvariety of $Y \subset \mathbb{P}^{2s+1}$(no DV_s, A_s). One has an exact sequence

$$\to H_{prim}^{2s-2}(\hat{Y}_1) \to H_c^{2s-1}(Y \backslash \hat{Y}_1) \to H^{2s-1}(Y) \to H^{2s-1}(\hat{Y}_1) \to . \tag{2.238}$$

Since

$$\hat{Y}_1 = \mathcal{V}(Li_{s+1}, B_s I_s - A_{s-1}^2 I_{s-1}, I_s) = \mathcal{V}(A_{s-1}Li_s, I_s, A_{s-1}I_{s-1}), \tag{2.239}$$

the defining polynomials forget B_s; by *Theorem A* ($N = 2s+1$, $k = 3$, $t = 1$), the sequence (2.238) simplifies to

$$0 \to H_c^{2s-1}(Y \backslash \hat{Y}_1) \to H^{2s-1}(Y) \to H^{2s-3}(Y_1)(-1) \to, \tag{2.240}$$

where $Y_1 \subset \mathbb{P}^{2s}$(no DV_s, A_s, B_s) is defined by

$$Y_1 := \mathcal{V}(A_{s-1}Li_{s-1}, I_s, A_{s-1}I_{s-1}). \tag{2.241}$$

The open subscheme $Y \backslash \hat{Y}_1 \subset Y$ is given by the system

$$\begin{cases} A_{s-1}Li_{s-1} = 0 \\ B_s I_s - A_{s-1}^2 I_{s-1} = 0 \\ I_s \neq 0. \end{cases} \tag{2.242}$$

Expressing B_s from the second equation and projecting from the point where all the variables but B_s are zero, we get an isomorphism

$$Y \backslash \hat{Y}_1 \cong U', \tag{2.243}$$

where $U' \subset \mathbb{P}^{2s}$(no DV_s, A_s, B_s) defined by

$$U' := \mathcal{V}(A_{s-1}Li_s) \backslash \mathcal{V}(A_{s-1}Li_s, I_s). \tag{2.244}$$

Collecting (2.231),(2.232),(2.240) and (2.243) together, we get an exact sequence

$$0 \longrightarrow H_c^{2s-1}(U') \longrightarrow H_c^{2s+1}(U) \longrightarrow H^{2s-3}(Y_1)(-1) \longrightarrow \tag{2.245}$$

for U and U' defined by (2.225) and (2.244) respectively. Consequently, we obtain

$$\operatorname{gr}_i^W H_c^{2n+1}(U) \cong \operatorname{gr}_i^W H_c^{2n-1}(U'), \quad i = 0, 1. \tag{2.246}$$

If $s = 1$, go to *the Last Step.*

After repeating a suitable number of times *Step 5*, we come to the following situation.

Step 6. Suppose that the entry a_{ss} of the matrix M_{GZZ} is C_s and $a_{s+1s+1} = B_{s+1}$. This happens when $s_i - 1$ for some $i \equiv t$ mod 2. For C_s we have

$$C_s = A_v + A_s \pm A_{s-1} \tag{2.247}$$

with "+" only when $l_i = 1$. Let $U \subset \mathbb{P}^{2s+2}$(no DV_s) be defined by

$$U := \mathcal{V}(A_s Li_{s+1}) \backslash \mathcal{V}(A_s Li_{s+1}, I_{s+1}), \tag{2.248}$$

and denote by DV_s all the variables appearing in I_{n-1-s}^{s+1}. As in the previous case, we define $U_1, U_2 \subset U$ to be

$$\begin{aligned} U_1 :&= \mathcal{V}(A_s) \backslash \mathcal{V}(A_s, I_{s+1}), \\ U_2 :&= \mathcal{V}(Li_{s+1}) \backslash \mathcal{V}(Li_{s+1}, I_{s+1}). \end{aligned} \tag{2.249}$$

and write an exact sequence

$$\begin{aligned} \longrightarrow H_c^{2s}(U_1) \oplus H_c^{2s}(U_2) \longrightarrow H_c^{2s}(U_3) \longrightarrow \\ H_c^{2s+1}(U) \longrightarrow H_c^{2s+1}(U_1) \oplus H_c^{2s+1}(U_2) \longrightarrow \end{aligned} \tag{2.250}$$

where $U_3 := U_1 \cap U_2$. For this step we have

$$\begin{aligned} Li_{s+1} &= A_v I_s - A_{s-1} Li_s, \\ I_{s+1} &= C_s I_s - A_{s-1}^2 I_{s-1}. \end{aligned} \tag{2.251}$$

Noting that the polynomials defining U_1 and U_2 are independent of A_m and A_s respectively, we apply *Theorem B* ($N = 2s + 2$, $k = 1$, $t = 1$) and get

$$H_c^i(U_1) = H_c^i(U_2) = 0 \quad \text{for } i < 2s + 2. \tag{2.252}$$

The sequence (2.250) implies an isomorphism

$$H_c^{2s+1}(U) \cong H_c^{2s}(U_3). \tag{2.253}$$

Eliminating A_s, which is zero on U_3, we get an isomorphism

$$U_3 \cong U_4 := \mathcal{V}(Li_{s+1}) \backslash \mathcal{V}(Li_{s+1}, I_{s+1}) \tag{2.254}$$

with $U_4 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\, A_s)$. Thus,

$$H_c^{2s}(U_3) \cong H_c^{2s}(U_4). \tag{2.255}$$

Denoting by I'_{s+1} the polynomial I_{s+1} after setting $A_s = 0$, we define $T, Y \subset \mathbb{P}^{2s+1}$ (no DV_s, A_s) by

$$\begin{aligned} T :&= \mathcal{V}(Li_{s+1}), \\ Y :&= \mathcal{V}(Li_{s+1}, I'_{s+1}). \end{aligned} \tag{2.256}$$

One gets an exact sequence

$$\to H^{2s-1}(T) \to H^{2s-1}(Y) \to H_c^{2s}(U_4) \to H_{prim}^{2s}(T) \to . \tag{2.257}$$

Theorem A ($N = 2s+1$, $k = 1$, $t = 0$) implies the vanishing of the term to the left. Motivated by (2.251), we define

$$\hat{T}_1 := T \cap \mathcal{V}(I_s) = \mathcal{V}(I_s, A_{s-1}Li_s) \subset \mathbb{P}^{2s+1}(\text{no}\, DV_s,\, A_s). \tag{2.258}$$

One can write an exact sequence

$$\longrightarrow H_c^{2s}(T \backslash \hat{T}_1) \longrightarrow H_{prim}^{2s}(T) \longrightarrow H_{prim}^{2s}(\hat{T}_1) \longrightarrow \tag{2.259}$$

On $T \backslash \hat{T}_1$ we can express A_v from the equation $Li_s = 0$. Projecting from the point where all the variables but A_v are zero, we get an isomorphism

$$T \backslash \hat{T}_1 \cong \mathbb{P}^{2s} \backslash \mathcal{V}(I_{s-1}). \tag{2.260}$$

The polynomial I_{s-1} does not depend on A_m. Applying *Theorem B* ($N = 2s$, $k = 0$, $t = 1$), one gets

$$H_c^{2s}(T \backslash \hat{T}_1) = 0. \tag{2.261}$$

The polynomials defining $\hat{T}_1$ are independent of A_v. Applying *Theorem A* ($N = 2s+1$, $k = 2$, $t = 1$) to $\hat{T}_1$, one gets

$$H_{prim}^{2s}(T) \cong H_{prim}^{2s-2}(T_1)(-1), \tag{2.262}$$

where $T_1 \subset \mathbb{P}^{2s}(\text{no } DV_s,\, A_s,\, A_v)$ is defined by the same equations as $\hat{T}_1$. By (2.261) and (2.262), the sequence (2.259) gives us

$$\operatorname{gr}_0^W H_{prim}^{2s}(T) = \operatorname{gr}_1^W H_{prim}^{2s}(T) = 0. \tag{2.263}$$

Now define $\hat{Y}_1 \subset Y \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\, A_s)$ by

$$\hat{Y}_1 := \mathcal{V}(Li_{s+1}, I'_{s+1}, I_s) = \mathcal{V}(A_{s-1}Li_s, A_{s-1}I_{s-1}, I_s). \tag{2.264}$$

One can write a exact sequence

$$\to H^{2s-2}(Y)_{prim} \to H_c^{2s-1}(Y\backslash\hat{Y}_1) \to H^{2s-1}(Y) \to H^{2s-1}(\hat{Y}_1) \to \tag{2.265}$$

The polynomials defining $\hat{Y}_1$ do not depend on A_v. After application of *Theorem A* ($N = 2s$, $k = 3$, $t = 1$), the sequence (2.265) simplifies to

$$0 \to H_c^{2s-1}(Y\backslash\hat{Y}_1) \to H^{2s-1}(Y) \to H^{2s-3}(\hat{Y}_1)(-1) \to, \tag{2.266}$$

where $Y_1 \subset \mathbb{P}^{2s}(\text{no } DV_s,\, A_s,\, A_v)$ is defined by the same equations. It follows that

$$\operatorname{gr}_i^W H^{2s-1}(Y) \cong \operatorname{gr}_i^W H_c^{2s-1}(Y\backslash\hat{Y}_1), \quad i = 0, 1. \tag{2.267}$$

The open subscheme $Y\backslash\hat{Y}_1 \subset Y$ is defined by the system

$$\begin{cases} Li_{s+1} = 0 \\ I'_{s+1} = 0 \\ I_s \neq 0 \end{cases} \Leftrightarrow \begin{cases} A_v I_s - A_{s-1}Li_s = 0 \\ (A_v \pm A_{s-1})I_s - A_{s-1}^2 I_{s-1} = 0 \\ I_s \neq 0. \end{cases} \tag{2.268}$$

We can express A_v from the first and second equation and this expressions must be equal. So we define Ni_s by

$$Ni_s := \pm I_s + A_{s-1}I_{s-1} - Li_s \tag{2.269}$$

with "$-$" only when $l_i = 1$. The expression for A_v and the natural projection from the point where all the variables but A_v are zero yield an isomorphism

$$Y\backslash\hat{Y}_1 \cong U', \tag{2.270}$$

where $U' \subset \mathbb{P}^{2s}(\text{no } DV_s,\, A_s,\, A_v)$ is defined by

$$U' := \mathcal{V}(A_{s-1}Ni_s)\backslash\mathcal{V}(A_{s-1}Ni_s, I_s). \tag{2.271}$$

By (2.253), (2.255), and (2.257), one has an exact sequence

$$0 \longrightarrow H^{2s-1}(Y) \longrightarrow H_c^{2s+1}(U) \longrightarrow H^{2s}_{prim}(T) \longrightarrow . \tag{2.272}$$

Hence, (2.263), (2.267) and (2.270) imply

$$\operatorname{gr}_i^W H_c^{2s+1}(U) \cong \operatorname{gr}_i^W H_c^{2s-1}(U'), \quad i = 0, 1, \tag{2.273}$$

$$(2.274)$$

with U and U' defined by (2.248) and (2.271) respectively.

If $s = 1$, go to *the Last Step.* If $a_{s-1\,s-1} = B_{s-1}$, return to *Step 4* with $Ni_s = Ni_s^-$; return to *Step 3* with $Ni_s = Ni_s^+$ otherwise.

the Last Step. Recall that $l_1 > 1$. In the case $t \equiv 0 \bmod 2$ we have come from *Step 4* or *Step 5.* The matrix looks like

$$M_{GZZ} = \begin{pmatrix} B_0 & A_0 & \vdots & A_m \\ A_0 & B_1 & \vdots & 0 \\ \dots & \dots & \ddots & \vdots \\ A_m & 0 & \dots & \ddots \end{pmatrix}. \tag{2.275}$$

We are interested $H^1(U)$, where $U \subset \mathbb{P}^2(A_0 : A_m : B_0)$ is defined by

$$U := \mathcal{V}(A_0 Li_1) \backslash \mathcal{V}(A_0 Li_1, I_1) = \mathcal{V}(A_0 A_m) \backslash \mathcal{V}(A_0 A_m, B_0). \tag{2.276}$$

The exact sequence

$$\begin{aligned} \longrightarrow H^0_{prim}(\mathcal{V}(A_0 A_m)) \longrightarrow H^0_{prim}(\mathcal{V}(A_0 A_m, B_0)) \longrightarrow \\ H^1_c(U) \longrightarrow H^1(\mathcal{V}(A_0 A_m)) \longrightarrow \end{aligned} \tag{2.277}$$

implies

$$H^1_c(U) \cong \mathbb{Q}(0). \tag{2.278}$$

In the opposite case, when $t \not\equiv 0 \bmod 2$, the matrix looks like

$$M_{GZZ} = \begin{pmatrix} B_0 & A_0 & \vdots & A_m \\ A_0 & C_1 & \vdots & A_{m-1} \\ \dots & \dots & \ddots & \vdots \\ A_m & A_{m-1} & \dots & \ddots \end{pmatrix}, \tag{2.279}$$

and we had come from *Step 3* or *Step 6.* We deal with $U \subset \mathbb{P}^2(A_0 : A_m : B_0)$ defined by

$$\begin{aligned} U := \mathcal{V}(A_0 Ni_1) \backslash \mathcal{V}(A_0 Ni_1, I_1) = \\ \mathcal{V}(A_0(B_0 + A_0 - A_m)) \backslash \mathcal{V}(A_0(B_0 + A_0 - A_m), B_0). \end{aligned} \tag{2.280}$$

Changing the variables $A_m := B_0 + A_0 - A_m$, we come to the situation above, and we again obtain

$$H^1_c(U) \cong \mathbb{Q}(0). \tag{2.281}$$

We have constructed a sequence of schemes $U = U^0$, U^1, $\ldots$, $U^{n-4} = U_3$ (see (2.171)) such that

$$\mathrm{gr}_i^W H_c^{2s+1}(U^s) \cong \mathrm{gr}_i^W H_c^{2s-1}(U^{s-1}), \quad i = 0, 1, \tag{2.282}$$

for $0 \leq s \leq n-3$, $U^s \subset \mathbb{P}^{2s+2}$. By (2.172), we obtain

$$\begin{gathered} \mathrm{gr}_i^W H_{prim}^{2n-6}(V) \cong \mathrm{gr}_i^W H_c^{2n-7}(U_3) \cong \\ \mathrm{gr}_i^W H_c^{2n-7}(U^{n-4}) \cong \ldots \cong \mathrm{gr}_i^W H_c^1(U^0), \quad i = 0, 1. \end{gathered} \tag{2.283}$$

Hence,

$$\mathrm{gr}_0^W H_{prim}^{2n-6}(V) \cong \mathbb{Q}(0) \quad \text{and} \quad \mathrm{gr}_1^W H_{prim}^{2n-6}(V) = 0. \tag{2.284}$$

Using the isomorphism

$$H^{2n-2}(X) \cong H^{2n-6}(V)(-2) \tag{2.285}$$

(see (e34)), we finally get

$$\begin{gathered} \mathrm{gr}_4^W H_{prim}^{2n-2}(X) \cong W_4 H_{prim}^{2n-2}(X) \cong \mathbb{Q}(-2), \\ \mathrm{gr}_5^W H_{prim}^{2n-2}(X) = 0. \end{gathered} \tag{2.286}$$

□

2.3 De Rham class for GZZ(n,2)

Fix some $n \geq 2$ and define $\Gamma = \Gamma_n := GZZ(n,2)$. This graph has $2n+6$ edges and $h_1(\Gamma) = 2(n+3)$. Let $X_n \subset \mathbb{P}^{2n+5}$ be the graph hypersurface associated to Γ_n. By the results of the previous section, one has an inclusion

$$\mathbb{Q}(-2) \hookrightarrow H^{2n+4}_{prim}(X_n) \cong H^{2n+5}_c(\mathbb{P}^{2n+5} \backslash X). \tag{2.287}$$

Hence, we get $\dim H^{2n+5}_{DR}(\mathbb{P}^{2n+5} \backslash X_n) \neq 0$. We do not know that this cohomology group is one-dimensional in general. Nevertheless, according to Example 2.2.4, $\Gamma_2 := GZZ(2,2) \cong ZZ_5$, thus $H^{2n+5}_{DR}(\mathbb{P}^{2n+5} \backslash X_n) \cong K$ for $n = 2$. In this chapter we consider

$$\eta = \eta_\Gamma = \frac{\Omega_{2n+5}}{\Psi^2_{\Gamma_n}} \in \Gamma(\mathbb{P}^{2n+5}, \omega(2X_n)) \tag{2.288}$$

(see (1.62)) and show that $[\eta_n] \neq 0$ in $H^{2n+5}(\mathbb{P}^{2n+5} \backslash X_n)$. We strongly follow Section 12, [BEK], where the computations for WS_n were done.

Lemma 2.3.1
Let $U = \operatorname{Spec} R$ be a smooth, affine variety and $0 \neq f, g \in R$. Define $Z := \mathcal{V}(f,g) \subset U$. We have a map of complexes

$$\left(\Omega^*_{R[1/f]}/\Omega^*_R\right) \oplus \left(\Omega^*_{R[1/g]}/\Omega^*_R\right) \xrightarrow{\gamma} \left(\Omega^*_{R[1/fg]}/\Omega^*_R\right) \tag{2.289}$$

Then the de Rham cohomology with supports $H^*_{Z,DR}(U)$ can by computed by the cone of γ shifted by -2.

Proof. We write the localization sequence for $\mathcal{V}(f) \subset U$

$$\to H^i_{\mathcal{V}(f),DR}(U) \to H^i_{DR}(U) \to H^i_{DR}(U \backslash \mathcal{V}(f)) \to H^{i+1}_{\mathcal{V}(f),DR}(U) \to . \tag{2.290}$$

Since U and $U \backslash \mathcal{V}(f)$ are affine, the de Rham cohomology is the cohomology of complexes of differential forms Ω^*_R and $\Omega^*_{R[1/f]}$. This implies

$$H^*_{\mathcal{V}(f),DR}(U) = H^*(\Omega^*_{R[1/f]}/\Omega^*_R[-1]). \tag{2.291}$$

We replace f by g resp. fg to get similar equalities for $\mathcal{V}(g)$ and $\mathcal{V}(fg)$. Consider the Mayer-Vietoris sequence for $\mathcal{V}(f), \mathcal{V}(g) \subset U$:

$$\begin{aligned} \longrightarrow H^*_{Z,DR}(U) \longrightarrow H^*_{\mathcal{V}(f)}(U) \oplus H^*_{\mathcal{V}(f)}(U) \longrightarrow \\ H^*_{\mathcal{V}(f)\cup\mathcal{V}(g)}(U) \longrightarrow H^{*+1}_{Z,DR}(U) \longrightarrow . \end{aligned} \tag{2.292}$$

Now the five-lemma yields that the natural map $H^*_{Z,DR}(U) \to H^*(C^*[-2])$ for $C^* := Cone(\gamma)$ becomes an isomorphism.

□

Remark 2.3.2
The direct computation shows that C^* is quasi-isomorphic to the cone of

$$\left(\Omega^*_{R[1/f]}/\Omega^*_R\right) \xrightarrow{\Delta} \left(\Omega^*_{R[1/fg]}/\Omega^*_{R[1/g]}\right). \tag{2.293}$$

For the application, we use $U := \mathbb{P}^{2n+5}\backslash X_n$. Recall that the matrix of Γ_n looks like

$$M_{GZZ(n,2)} = \begin{pmatrix} B_0 & A_0 & 0 & \vdots & 0 & 0 & 0 & A_{n+3} \\ A_0 & B_1 & A_1 & \vdots & 0 & 0 & 0 & 0 \\ 0 & A_1 & B_6 & \vdots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & B_{n-1} & A_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \vdots & A_{n-1} & C_n & A_n & A_{n+2} \\ 0 & 0 & 0 & \vdots & 0 & A_n & B_{n+1} & A_{n+1} \\ A_{n+3} & 0 & 0 & \vdots & 0 & A_{n+2} & A_{n+1} & B_{n+2} \end{pmatrix}. \tag{2.294}$$

Define $a_i := \frac{A_i}{A_{n+3}}$, $b_i := \frac{B_i}{A_{n+3}}$ and $c_n := \frac{C_n}{A_{n+3}} = a_{n+2} + a_{n-1} - a_n$. (We will see that the forms we work with have no poles along $A_{n+3} = 0$.) Let

$$i_j = \frac{I_j}{A^j_{n+3}}, \quad g_{n+2} = \frac{G_{n+2}}{A^{n+3}_{n+3}}. \tag{2.295}$$

Set $f = i_{n+2}$, $g = i_{n+1}$. The equation $b_{n+2}i_{n+2} - g_{n+2} = 0$ defines X_n on $A_{n+3} \neq 0$, then $b_{n+2}i_{n+2} - g_{n+2}$ is invertible on $U = \mathbb{P}^{2n+5}\backslash X$. Thus g_{n+2} is invertible on $\mathcal{V}(f)$. The element

$$\begin{aligned} \beta = &-db_0 \wedge \ldots \wedge db_{n-1} \wedge db_{n+1} \wedge \\ &\wedge da_0 \wedge \ldots \wedge da_{n+2} \frac{1}{g_{n+2}i_{n+2}} \left(\frac{g_{n+2}}{b_{n+2}i_{n+2} - g_{n+2}}\right) \end{aligned} \tag{2.296}$$

is defined in $\Omega^{2n+4}_{R[1/f]}/\Omega^{2n+4}_R$. It satisfies

$$d\beta = \eta = \frac{db_0 \wedge \ldots \wedge db_{n-1} \wedge db_{n+1} \wedge db_{n+2} \wedge da_0 \wedge \ldots \wedge da_{n+2}}{(b_{n+2}i_{n+2} - g_{n+2})^2}. \tag{2.297}$$

By Corollary 1.1.4, $I_{n+1}G_{n+2} \equiv (Li_{n+2})^2 \mod I_{n+2}$, thus

$$\begin{aligned} i_{n+1}g_{n+2} \equiv &(a_{n+1}i_{n+1} - a_{n+2}a_ni_n + \\ &(-1)^{n-1}a_na_{n-1}\ldots a_1a_0)^2 \mod i_{n+2}. \end{aligned} \tag{2.298}$$

We also use

$$i_k = b_{k-1}i_{k-1} - a_{k-2}^2 i_{k-2} \tag{2.299}$$

for $k+2$ or $k < n+1$. We now compute in $\Omega^*_{R[1/fg]}/\Omega^*_{R[1/g]}$ and get

$$\begin{aligned}\beta = \frac{di_{n+2}}{i_{n+2}} \wedge \frac{da_{n+2}}{g_{n+2}i_{n+1}} \wedge db_0 \wedge \ldots \wedge db_{n-1} \wedge \\ \wedge da_0 \wedge \ldots \wedge da_{n+1} \cdot \Big(1 - \frac{b_{n+2}i_{n+2}}{b_{n+2}i_{n+2} - g_{n+2}}\Big) = \\ -d\Big(\frac{1}{a_{n+1}i_{n+1} - a_{n+2}a_n i_n + (-1)^{n-1}a_n \ldots a_0} \cdot \frac{di_{n+2}}{i_{n+2}} \wedge \nu\Big),\end{aligned} \tag{2.300}$$

where

$$\nu := \frac{da_{n+2}}{i_{n+1}} \wedge db_0 \wedge \ldots \wedge db_{n-1} \wedge da_0 \wedge \ldots \wedge da_n. \tag{2.301}$$

Using the equality

$$i_{n+1} = c_n i_n - a_{n-1}^2 i_{n-1} = (a_{n+2} + a_{n-1} - a_n)i_n - a_{n-1}^2 i_{n-1} \tag{2.302}$$

and (2.299), we get

$$\begin{aligned}\nu = \frac{di_{n+1}}{i_{n+1}} \wedge \frac{db_{n-1}}{i_n} \wedge db_{n-2} \wedge \ldots \wedge db_0 \wedge da_0 \wedge \ldots \wedge da_n. \\ \frac{di_{n+1}}{i_{n+1}} \wedge \frac{di_n}{i_n} \wedge \ldots \wedge \frac{di_2}{i_2} \wedge \frac{db_0}{b_0} \wedge da_0 \wedge \ldots \wedge da_n.\end{aligned} \tag{2.303}$$

By (2.300) one has $\beta = d\theta$ with

$$\theta := -\frac{1}{a_{n+1}i_{n+1} - a_{n+2}a_n i_n + (-1)^{n-1}a_n \ldots a_0} \cdot \frac{di_{n+2}}{i_{n+2}} \wedge \nu. \tag{2.304}$$

Both β and θ have no poles along $A_{n+3} = 0$. Thus the pair

$$(\beta, \theta) \in H^{2n+5}_{Z,DR}(U) \tag{2.305}$$

(see Remark 2.3.2) represents a class mapping to $\eta_n \in H^{2n+5}_{DR}(\mathbb{P}^{2n+5} \backslash X_n)$, where Z is defined by

$$Z := \mathcal{V}(I_{n+2}, I_{n+1}). \tag{2.306}$$

Lemma 2.3.3
The natural map

$$H^{2n+5}_Z(\mathbb{P}^{2n+5} \backslash X_n) \longrightarrow H^{2n+5}(\mathbb{P}^{2n+5} \backslash X_n) \tag{2.307}$$

is injective.

Proof. The proof goes almost word for word as the proof of Lemma 12.3. in [BEK] and works for Betti's, de Rham or étale cohomology. Define $Y = \mathcal{V}(I_{n+2})$. We will show that the desired map is a composition of two injective maps

$$H_Z^{2n+5}(\mathbb{P}^{2n+5}\backslash X) \xrightarrow{u} H_Y^{2n+5}(\mathbb{P}^{2n+5}\backslash X) \xrightarrow{v} H^{2n+5}(\mathbb{P}^{2n+5}\backslash X). \tag{2.308}$$

One has the localization sequence

$$\to H^{2n+4}(\mathbb{P}^{2n+5}\backslash(X \cup Y)) \to H_Y^{2n+5}(\mathbb{P}^{2n+5}\backslash X_n) \to H^{2n+5}(\mathbb{P}^{2n+5}\backslash X_n) \to \tag{2.309}$$

Recall that I_{2n+2} is independent of B_{n+2}, A_{n+1} and A_{n+3}. Define $Y_0 := \mathcal{V}(I_4) \subset \mathbb{P}^{2n+4}$(no B_{n+2}) and $Y_1 := \mathcal{V}(I_4) \subset \mathbb{P}^{2n+2}$(no B_{n+2}, A_{n+1}, A_{n+3}). Consider the projections

$$\mathbb{P}^{2n+5}\backslash(X_n \cup Y) \xrightarrow{\pi_1} \mathbb{P}^{2n+4}\backslash Y_0 \xrightarrow{\pi_2} \mathbb{P}^{2n+2}\backslash Y_1, \tag{2.310}$$

where π_1 forgets the variable B_{n+2} and π_2 forgets A_{n+1} and A_{n+3}. Since the map $X\backslash Y \to \mathbb{P}^{2n+4}\backslash Y_0$ induced by projection is an isomorphism, it follows that π_1 is an $\mathbb{G}_m$-bundle. The map π_2 is an $\mathbb{A}^2$-bundle. One gets

$$H^{2n+4}(\mathbb{P}^{2n+5}\backslash(X \cup Y)) \cong H^{2n+4}(\mathbb{P}^{2n+2}\backslash Y_1) \oplus H^{2n+3}(\mathbb{P}^{2n+2}\backslash Y_1)(-1) = 0 \tag{2.311}$$

by homotopy invariance and Artin's vanishing. This proves that v is injective.

Consider the two open subschemes

$$\mathbb{P}^{2n+5}\backslash(X_n \cup Y) \subset \mathbb{P}^{2n+5}\backslash(X_n \cup Z) \subset \mathbb{P}^{2n+5}\backslash X_n, \tag{2.312}$$

then [Mi1],ch.3, Remark 1.26 gives us the following exact sequence

$$\begin{aligned} &\longrightarrow H_{Y-Z}^{2n+4}(\mathbb{P}^{2n+5}\backslash(X \cup Z)) \longrightarrow \\ &\qquad H_Z^{2n+4}(\mathbb{P}^{2n+5}\backslash X) \xrightarrow{u} H_Y^{2n+4}(\mathbb{P}^{2n+5}\backslash X) \longrightarrow . \end{aligned} \tag{2.313}$$

Because $I_{n+2} = B_{n+1}I_{n+1} - A_n^2 I_n$, the singular locus of Y is contained in Z. Thus, for the smooth subscheme $Y\backslash Z$ in $\mathbb{P}^{2n+5}\backslash(X \cup Z)$ of codimension 1 we may apply Gysin isomorphism (see [Mi2], Corollary 16.2)

$$H_{Y-Z}^{2n+4}(\mathbb{P}^{2n+5}\backslash(X \cup Z)) \cong H^{2n+2}(Y\backslash((X \cap Y) \cup Z))(-1). \tag{2.314}$$

The injectivity of u will follow from the vanishing of the cohomology to the right. Denote by π_3 the projection from the point where all the variables but B_{n+2} are zero and define

$$T := \pi_3(Y\backslash((X \cap Y) \cup Z)) \subset \mathbb{P}^{2n+4}. \tag{2.315}$$

Since $X \cap Y = \mathcal{V}(I_{n+2}, G_{n+2})$, the defining polynomials are independent of B_{n+2}, thus π_3 induces an $\mathbb{A}^1$-fibration and

$$H^{2n+2}(Y\backslash((X \cap Y) \cup Z)) \cong H^{2n+2}(T). \tag{2.316}$$

Now define the projection π_4 obtained by dropping A_{n+1} and A_{n+3}

$$\pi_4 : T \to Y_1\backslash Z_1 \subset \mathbb{P}^{2n+2}. \tag{2.317}$$

The map $\pi_3(Y\backslash Z) \to Y_1\backslash Z_1$ is an $\mathbb{A}^2$-fibration while $\pi_3(Y\backslash(X \cap Y)) \to Y_1\backslash Z_1$ is an $\mathbb{A}^1$-fibration. It follows that the fibers of π_4 are $\mathbb{A}^1 \times \mathbb{G}_m$. One gets

$$H^{2n+2}(T) \cong H^{2n+2}(Y_1\backslash Z_1) \oplus H^{2n+1}(Y_1\backslash Z_1)(-1). \tag{2.318}$$

We know that

$$Y_1\backslash Z_1 = \mathcal{V}(I_{n+2})\backslash\mathcal{V}(I_{n+2}, I_{n+1}) \cong \mathbb{P}^{2n+1}\backslash\mathcal{V}(I_{n+1}). \tag{2.319}$$

We may change the variables $C_n := A_{n+2}$, then I_{n+1} forgets A_2 and thus the scheme to the right becomes a cone over $\mathbb{P}^{2n}\backslash\mathcal{V}(I_{n+1})$. Applying Artin's vanishing, we get

$$H^i(Y_1\backslash Z_1) = 0 \tag{2.320}$$

for $i > 2n$. The equalities (2.314), (2.316) and (2.318) imply the injectivity of u.

□

Theorem 2.3.4
Let X_n be the graph hypersurface for $\Gamma_n = GZZ(n, 2)$ and let $[\eta_n] \in H^{2n+5}_{DR}(\mathbb{P}^{2n+5}\backslash X)$ be the de Rham class of (2.288). Then $[\eta_n] \neq 0$.

Proof. The proof is almost the same as that of Theorem 12.4, [BEK]. We have lifted the class $[\eta_n]$ to a class $(\beta, \eta) \in H^{2n+5}(\mathbb{P}^{2n+5}\backslash X)$, see (2.305). By Lemma 2.3.3, it is enough to show that $(\beta, \eta) \neq 0$. We localize an the generic point of Z and compute further in the function field of Z. Consider the long denominator of β in (2.300):

$$D := a_{n+1}i_{n+1} - a_{n+2}a_n i_n + (-1)^{n-1}a_n \dots a_0. \tag{2.321}$$

On $\mathcal{V}(i_{n+2}, i_{n+1})$ we have

$$\left\{ \begin{aligned} b_{n+1}i_{n+1} - a_n^2 i_n &= 0 \\ i_{n+1} &= 0 \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} a_n i_n &= 0 \\ i_{n+1} &= 0, \end{aligned} \right. \tag{2.322}$$

thus both the left and the middle summand of D vanish. Now it follows that as the class in the function field of Z, the class (β, η) is represented by

$$\pm d\log(i_n) \wedge \ldots \wedge d\log(i_1) \wedge d\log(a_0) \wedge \ldots \wedge d\log(a_n) \tag{2.323}$$

This is a nonzero multiple of

$$d\log(b_{n-1}) \wedge \ldots \wedge d\log(b_0) \wedge d\log(a_0) \wedge \ldots \wedge d\log(a_n), \tag{2.324}$$

so in non-zero as a form. The Deligne theory of MHS yields that the vector space of logarithmic forms injects into de Rham cohomology of the open on which those forms are smooth (see (3.1.5.2) in [De2]). Thus the form above is nonzero.

□

Corollary 2.3.5
Let X be the graph hypersurface for $\Gamma = ZZ_5$. Then for the class of η defined in (2.288) one has

$$K[\eta] = H^9_{DR}(\mathbb{P}^9 \backslash X). \tag{2.325}$$

Chapter 3

Gluings

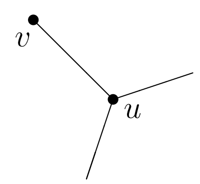
3.1 Classification

Definition 3.1.1
The degree $\deg(v)$ of a vertex v (of an undirected graph) is defined to be the number of edges entering this vertex.

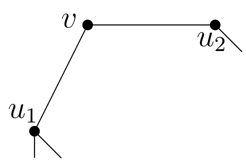
Lemma 3.1.2
For any vertex v of a primitively divergent graph Γ the following inequality holds

$$\deg(v) \geq 3. \tag{3.1}$$

Proof. Suppose that $\deg(v) = 1$ for a vertex $v \in V(\Gamma)$, so we have an edge $uv \in E(\Gamma)$ for some vertex $u \in V(\Gamma)$. We delete the edge uv — the only one edge connecting v with the other vertexes of Γ — together with the vertex v and define $\Gamma' = \Gamma \backslash \{uv\}$. This graph has the same Betti number but smaller number of edges. This is a contradiction with the assumption that Γ is primitively divergent.



deg(v) = 1.



deg(v) = 2.

In the case $\deg(v) = 2$ for some vertex $v \in V(\Gamma)$, we denote by u_1 and u_2 the two vertexes which are adjacent to v. Let $\Gamma' = \Gamma \backslash \{u_1 v, u_2 v\}$. Note that $|E(\Gamma')| = |E(\Gamma)| - 2$. We know that $h_1(\Gamma)$ is independent of the choice of a basis of $H_1(\Gamma)$, thus we can take such a basis that u_1v and u_2v will only appear in one basis element, and we get

$$h_1(\Gamma') = h_1(\Gamma) - 1. \tag{3.2}$$

Since we found an divergent subgraph of Γ, Γ is not primitively divergent. Hence, $d(v) \geq 3$ for any $v \in V(\Gamma)$.

□

We classify primitively divergent graphs with small number of edges.

Theorem 3.1.3
Let Γ be a primitively divergent graph with $E(\Gamma) = 2n$ and $n \leq 6$. Then for Γ we have one of the following possibilities

- *n=3, then $\Gamma \cong WS_3$.*
- *n=4, then $\Gamma \cong WS_4$.*
- *n=5, then Γ is isomorphic to the one of the following graphs WS_5, ZZ_5, XX_5 or ST_5.*

Proof. Set $m := |V(\Gamma)|$. Denote by α_i the degree of the vertex $v_i \in V$ for $1 \leq i \leq m$. We can compute the number of edges of Γ by taking the sum of all α's, and each edge will be counted twice. Thus, one has

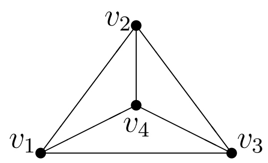
$$\sum_{i=1}^{m} \alpha_i = 4n. \tag{3.3}$$

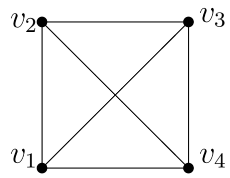
By Lemma 3.1.2, $\alpha_i \geq 3$ for every i. It follows that

$$3m \leq 4n. \tag{3.4}$$

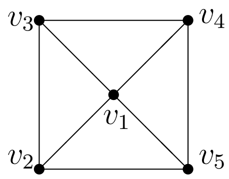
Since we do not allow multiple edges and self-loops, we may assume that $n \geq 3$.

n = 3. The inequality (3.4) gives us $m \leq 4$. Define K_r to be a complete graph with r vertexes Because K_3 has only 3 edges, $m \neq 3$. Hence, $m = 4$, the inequality (3.4) becomes equality, and $\alpha_i = 3$ for all i. A graph with 4 vertexes can have an most $\frac{4 \cdot 3}{2} = 6$ edges, thus Γ is isomorphic to K_4. Note that graph WS_3 is isomorphic to K_4.





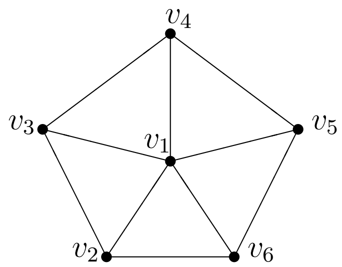
<u>$n = 4$</u>. By the inequality (3.4) we have $m \leq 5$. Because K_4 has only 6 edges, we get $m = 5$. Moreover, by (3.3), the only one possibility for $\alpha = (\alpha_i)$ with $\alpha_1 \geq \ldots \geq \alpha_5$ is $(4, 3, 3, 3, 3)$. This means that up to a graph isomorphism, we have the following situation. The vertex v_1 is connected by an edge to each other four vertexes, and this 4 vertexes are lying on a loop of length 4. Indeed, the vertex v_3 is adjacent to v_1 and to two more vertexes, say v_2 and v_4. If v_4 is adjacent to v_2, then v_5 must be connected with itself, this is not allowed. Thus, v_4 is adjacent to v_5 and the remaining edge is $v_5 v_2$.

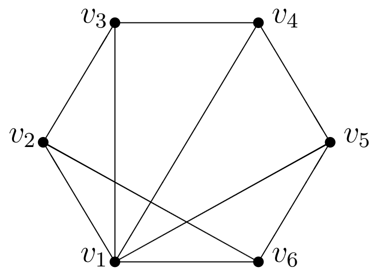


We get an isomorphism $\Gamma \cong WS_4$.

<u>$n = 5$</u>. By the same argument as above, we get $m = 6$. Take an order $\alpha_1 \geq \ldots \geq \alpha_6$, it follows that we have 2 possibilities: $(5, 3, 3, 3, 3, 3)$ and $(4, 4, 3, 3, 3, 3)$.

<u>Case A:</u> For the first case $\alpha = (5, 3, 3, 3, 3, 3)$ we have again one vertex, v_1, adjacent to all other other ones, and this 5 vertexes $v_2, \ldots, v_6$ build a loop of length 5.





Recall that the adjacency matrix for a undirected graph with m loops is a symmetric $m \times m$ matrix $Ad = (a_{i,j})$ such that $a_{ij} = 1$ when v_i is connected to v_j and $a_{ij} = 0$ otherwise. The degree of a vertex v_i can be computed as

$$\deg(v_i) = \sum_{j=1}^{m} a_{ij}. \tag{3.5}$$

For Case A the adjacency matrix is the following.

$$Ad_A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Case B: Suppose now that $\alpha = (4, 4, 3, 3, 3, 3)$. There are two different situations, depending on wether the vertexes of degree 4 are connected to each other or not.

Case B.1: Consider the case where the two vertexes of degree 4 (namely v_1 and v_2) are adjacent. Without loss of generality, we may assume that the one vertex that is not adjacent to v_1 is v_6. We get the following adjacency matrix.

$$Ad_{B.1} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & * & * & * & * \\ 1 & * & 0 & * & * & * \\ 1 & * & * & 0 & * & * \\ 1 & * & * & * & 0 & * \\ 0 & * & * & * & * & 0 \end{pmatrix}.$$

Again, we have to distinguish two cases.

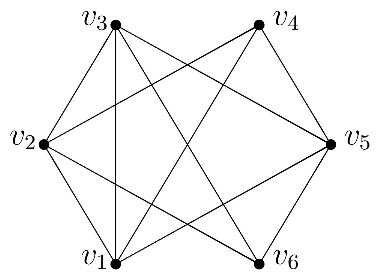
Case B.1.1: Suppose that $v_2v_6 \in E(\Gamma)$. The vertex v_2 is adjacent to v_1 and v_6, and $\deg(v_2) = 4$. Without loss of generality, we may assume that $v_2v_3, v_2v_4 \in E(\Gamma)$ and $v_2v_5 \notin E(\Gamma)$.

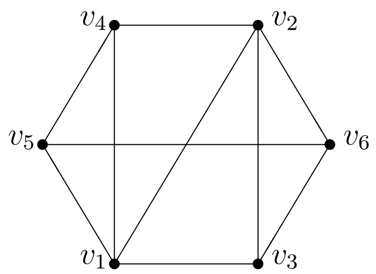
$$Ad_{B.1.1} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & * & * & * \\ 1 & 1 & * & 0 & * & * \\ 1 & 0 & * & * & 0 & * \\ 0 & 1 & * & * & * & 0 \end{pmatrix}.$$

Assume for a moment that v_3 is adjacent to v_4. Because $\deg(v_3) = \deg(v_4) = 3$ and both v_3 and v_4 are adjacent to v_1 and v_2, we conclude that v_6v_3, $v_6v_4 \notin E(\Gamma)$. Then v_6 is only adjacent to v_1 and, may be, to v_5; this contradicts $\deg(v_6) = 3$. Hence $v_3v_4 \notin E(\Gamma)$ and v_3 is adjacent to v_5 or v_6. Note that if we interchange vertexes $v_1 \leftrightarrow v_2$ and $v_5 \leftrightarrow v_6$, we get the same adjacency matrix and the graph isomorphic to Γ. Thus, one can assume that $v_3v_6 \in E(\Gamma)$ and, consequently, $v_3v_5 \notin E(\Gamma)$. Since $\deg(v_5) = 3$, it follows that v_5 is adjacent to v_4 and v_6. We obtain

$$Ad_{B.1.1} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

We can redraw this graph in a suitable way and see that it is isomorphic to ZZ_5.

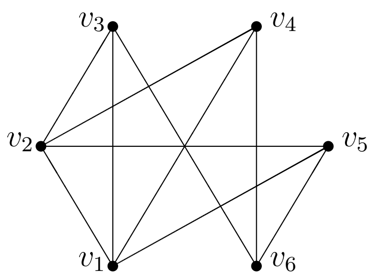


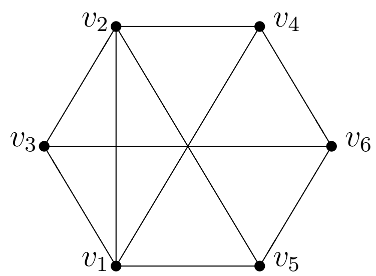


Case B.1.2: Now we suppose that the vertex v_2 is not adjacent to v_6. Together with $\deg(v_6) = 3$ this implies that v_6 is connected to v_5, v_4 and v_3. Since $\deg(v_3) = \deg(v_4) = \deg(v_5) = 3$, all this three vertexes are mutually not connected. We finally get the following adjacency matrix

$$Ad_{B.1.2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

We will refer to this graph as ST_5, which means "strange". For this graph we cannot say anything important about the graph hypersurface on the cohomological level.

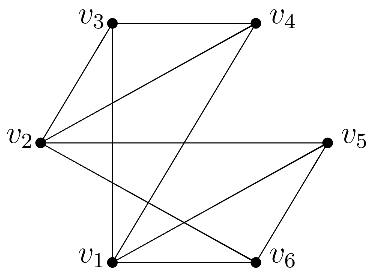


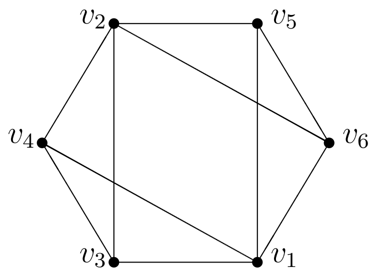


Case B.2: We consider the case when the two vertexes of degree four are not adjacent. Since $\deg(v_3) = 3$, the vertex v_3 must be adjacent to only one of the vertexes v_4, v_5 or v_6. Without loss of generality, we may assume that $v_3v_4 \in E(\Gamma)$. Now, v_6 cannot be adjacent to v_4, thus $v_5v_6 \in E(\Gamma)$. We obtain

$$Ad_{B.2} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We redraw the graph in a suitable way and call the right drawing XX_5.





□

In the next sections we study the graph hypersurface for XX_5 and the gluings — the graph obtained by the construction of gluing motivated by the shape of the graph XX_5.

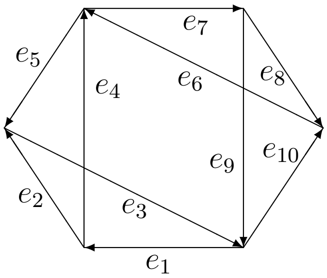
3.2 XX_5

In this section we compute the middle dimensional cohomology of the graph hypersurface for the graph XX_5 (see Theorem 3.1.3) in the similar way as it was done for ZZ_5 in Ch.2, Sect.1. We again consider $H^*(\cdot)$ to be étale or Betti's cohomology.

Theorem 3.2.1
Let X be the graph hypersurface for XX_5. Then

$$H^8_{prim}(X) \cong \mathbb{Q}(-3). \tag{3.6}$$

Proof. According to the classification, XX_5 is a primitively log divergent graph with 10 edges and $h_1(XX_5) = 5$. We orient and number edges in the following way:



	1	2	3	4	5	6	7	8	9	10
1	1	1	1	0	0	0	0	0	0	0
2	0	-1	0	1	1	0	0	0	0	0
3	0	0	1	0	1	1	0	0	0	1
4	0	0	0	0	0	1	1	1	0	0
5	0	0	0	0	0	0	0	-1	1	1

By the construction, we get a matrix in variables T_i, and changing coordinates, we obtain the matrix

$$M_{XX_5}(A,B) := \begin{pmatrix} B_0 & A_0 & A_4 & 0 & 0 \\ A_0 & B_1 & A_1 & 0 & 0 \\ A_4 & A_1 & C_2 & A_2 & A_5 \\ 0 & 0 & A_2 & B_3 & A_3 \\ 0 & 0 & A_5 & A_3 & B_4 \end{pmatrix} \tag{3.7}$$

with variables $A_0, \ldots, A_5, B_0, B_1, B_3, B_4$ and $C_2 := A_1 + A_2 + A_4 + A_5$. Define

$$X := \mathcal{V}(I_5) \subset \mathbb{P}^9(A_0 : \ldots : A_5 : B_0 : B_1 : B_3 : B_4), \tag{3.8}$$

and we compute $H^{mid}_{prim}(X) = H^8_{prim}(X)$ here. The determinant I_5 can be written in the following way:

$$I_5 = I_4 B_4 - G_4 \tag{3.9}$$

with

$$G_4 = A_3^2 I_3 + A_5^2 I_2 B_3 - 2A_3 A_5 I_2 A_2. \tag{3.10}$$

Consider the subvariety $\mathcal{V}(I_5, I_4)^{(9)} \subset X$ and define $U := X \backslash \mathcal{V}(I_5, I_4)^{(9)}$. One has an exact sequence

$$\to H^8_c(U) \to H^8_{prim}(X) \to H^8_{prim}(\mathcal{V}(I_4, G_4)^{(9)}) \to H^9_c(U) \to . \tag{3.11}$$

Lemma 3.2.2
One has $H^i_c(U)$ for $i < 10$.

Proof. The scheme U is defined by the system

$$\begin{cases} I_5 = B_4 I_4 - G_4 = 0 \\ \qquad\qquad I_4 \neq 0. \end{cases} \tag{3.12}$$

We solve the first equation on B_4; projecting from the point where all the variables but B_4 are zero, we get an isomorphism

$$U \cong \mathbb{P}^8 \backslash \mathcal{V}(I_4) \tag{3.13}$$

with the scheme to the right in $\mathbb{P}^8$(no B_4). The polynomial I_4 is independent of A_3. Applying *Theorem B* ($N = 8$, $k = 0$, $t = 1$) to $\mathbb{P}^8 \backslash \mathcal{V}(I_4)$, we get

$$H^i(U) = H^i_c(\mathbb{P}^8 \backslash \mathcal{V}(I_4)) = 0 \quad \text{for} \quad i < 9 \tag{3.14}$$

and

$$H^9_c(U) \cong H^9_c(\mathbb{P}^8 \backslash \mathcal{V}(I_4)) \cong H^7_c(\mathbb{P}^7 \backslash \mathcal{V}(I_4))(-1), \tag{3.15}$$

where the scheme to the right lives in $\mathbb{P}^7$(no B_4, A_3). One has an exact sequence

$$\to H^6_{prim}(\mathbb{P}^7) \to H^6_{prim}(\mathcal{V}(I_4)) \to H^7_c(\mathbb{P}^7 \backslash \mathcal{V}(I_4)) \to H^7(\mathbb{P}^7) \to . \tag{3.16}$$

Since the outermost terms vanish, we get an isomorphism

$$H^7_c(\mathbb{P}^7 \backslash \mathcal{V}(I_4)) \cong H^6_{prim}(\mathcal{V}(I_4)). \tag{3.17}$$

The only one appearance of A_5 in I_4 is in the sum C_2. We make the linear change of coordinates $C_2 := A_5$ and think of C_2 as independent variable. Denote by I'_4 and I'_3 the images of the polynomials I_4 and I_3 under this transformation. One has an isomorphism

$$H^6_{prim}(\mathcal{V}(I_4)) \cong H^6_{prim}(\mathcal{V}(I'_4)). \tag{3.18}$$

Together with (3.15) and (3.17), we get

$$H^9_c(U) \cong H^6_{prim}(\mathcal{V}(I'_4))(-1), \tag{3.19}$$

where $\mathcal{V}(I_4') \subset \mathbb{P}^7$(no B_4, A_3). We can write

$$I_4' = B_3 I_3' - A_2^2 I_2. \tag{3.20}$$

Define $\hat{T} := \mathcal{V}(I_4', I_3') \subset \mathcal{V}(I_4') \subset \mathbb{P}^7$ and $S := \mathcal{V}(I_4')\backslash\hat{T}$. One has an exact sequence

$$\longrightarrow H_c^6(S) \longrightarrow H_{prim}^6(\mathcal{V}(I_4')) \longrightarrow H_{prim}^6(\hat{T}) \longrightarrow . \tag{3.21}$$

On $\mathcal{V}(I_4')\backslash\hat{T}$ we solve (3.20) on B_3; projecting from the point where all the variables but B_3 are zero, we get an isomorphism

$$S \cong \mathbb{P}^6\backslash\mathcal{V}(I_3'). \tag{3.22}$$

The polynomial I_3' is independent of A_2, *Theorem B* ($N = 6$, $k = 0$, $t = 1$) implies

$$H_c^6(S) = 0. \tag{3.23}$$

The variety

$$\hat{T} = \mathcal{V}(I_4', I_3') = \mathcal{V}(I_3', A_2 I_2) \subset \mathbb{P}^7 \tag{3.24}$$

is defined by the polynomials both independent of B_3. By *Theorem A* ($N = 7$, $k = 2$, $t = 1$), we obtain

$$H_{prim}^6(\hat{T}) \cong H_{prim}^4(T)(-1), \tag{3.25}$$

where $T := \mathcal{V}(I_3', A_2 I_2) \subset \mathbb{P}^6$(no B_4, A_3, B_3). The sequence (3.21) simplifies to

$$0 \longrightarrow H_{prim}^6(\mathcal{V}(I_4')) \longrightarrow H_{prim}^4(T)(-1) \rightarrow . \tag{3.26}$$

Define $T_1 := \mathcal{V}(I_3', I_2) \subset T$ and $T_c := T\backslash T_1$. We have an exact sequence

$$\longrightarrow H^3(T_1) \longrightarrow H_c^4(T_c) \longrightarrow H_{prim}^4(T) \longrightarrow H_{prim}^4(T_1) \longrightarrow . \tag{3.27}$$

Note that $T_1 = \mathcal{V}(I_3', I_2) = \mathcal{V}(I_2, G_2) \subset \mathbb{P}^6$ with $G_2 := I_3' - C_2 I_2$, the defining polynomials are independent of C_2 and A_2. Thus, *Theorem A* ($N = 6$, $k = 2$, $t = 2$) gives us $H_{prim}^i(T_1) = 0$ for $i < 6$. The sequence (3.27) implies an isomorphism

$$H_{prim}^4(T) \cong H_c^4(T_c). \tag{3.28}$$

The scheme $T_c \subset \mathbb{P}^6$ is defined by the sequence

$$\begin{cases} I_3' = 0 \\ A_2 I_2 = 0 \\ I_2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} C_2 I_2 - G_2 = 0 \\ A_2 = 0 \\ I_2 \neq 0. \end{cases} \tag{3.29}$$

We solve the first equation on C_2. Applying *Theorem B* ($N = 4$, $k = 0$, $t = 2$) to $\mathbb{P}^4 \backslash \mathcal{V}(I_2)$, we get

$$H_c^4(T_c) = H_c^4(\mathbb{P}^4 \backslash \mathcal{V}(I_2)) = 0. \tag{3.30}$$

By (3.19), (3.26) and (3.28), we finally obtain

$$H^4_{prim}(\mathcal{V}(T))(-2) = 0, \quad H_c^9(U) \cong H^6_{prim}(\mathcal{V}(I_4'))(-1) = 0. \tag{3.31}$$

□

The lemma gives us the vanishing of the outmost terms of the sequence (3.11), thus one gets an isomorphism

$$H^8_{prim}(X) \cong H^8_{prim}(\mathcal{V}(G_4, I_4)^{(9)}). \tag{3.32}$$

The variety to the right is defined by the equations independent of B_4. *Theorem A* implies

$$H^8_{prim}(X) \cong H^6_{prim}(\mathcal{V}(G_4, I_4))(-1), \tag{3.33}$$

where the variety to the right lives in $\mathbb{P}^8$(no B_4). The polynomial G_4 is defined by

$$G_4 := A_3^2 \begin{vmatrix} B_0 & A_0 & A_4 \\ A_0 & B_1 & A_1 \\ A_4 & A_1 & C_2 \end{vmatrix} + A_5^2 \begin{vmatrix} B_0 & A_0 & 0 \\ A_0 & B_1 & 0 \\ 0 & 0 & B_3 \end{vmatrix} - 2A_3A_5 \begin{vmatrix} B_0 & A_0 & A_4 \\ A_0 & B_1 & A_1 \\ 0 & 0 & A_2 \end{vmatrix} =$$
$$= A_3^2 I_3 + A_5^2 I_2 B_3 - 2A_3A_5I_2A_2. \tag{3.34}$$

Define $\hat{V}, U_1 \subset \mathcal{V}(G_4, I_4) \subset \mathbb{P}^8(\text{no } B_4)$ by

$$\hat{V} := \mathcal{V}(G_4, I_4, I_3) \tag{3.35}$$

and $U_1 := \mathcal{V}(G_4, I_4) \backslash \hat{V}$. One can write an exact sequence

$$\longrightarrow H_c^6(U_1) \longrightarrow H^6_{prim}(\mathcal{V}(I_4, G_4)) \longrightarrow H^6_{prim}(\hat{V}) \longrightarrow H_c^7(U_1) \longrightarrow \tag{3.36}$$

The scheme U_1 is defined by the system

$$\begin{cases} G_4 = 0 \\ I_4 = 0 \\ I_3 \neq 0. \end{cases} \tag{3.37}$$

Such U's were studied in Chapter 1, Section 1. By Theorem 1.1.7, it follows that

$$U_1 \cong U_2 := \mathcal{V}(I_4)\backslash\mathcal{V}(I_4, I_3) \subset \mathbb{P}^7(\text{no } B_4, A_3) \tag{3.38}$$

Using the equality

$$I_4 = B_3 I_3 - A_2^2 I_2 \tag{3.39}$$

and projecting from the point where all the coordinates but B_3 are zero, we obtain an isomorphism

$$U_2 \cong \mathbb{P}^6\backslash\mathcal{V}(I_3). \tag{3.40}$$

We change the coordinates $C_2 := A_5$ and denote by I_3' the image of I_3 under this transformation. The polynomial I_3' is independent of A_2. *Theorem B* ($N = 6$, $k = 0$, $t = 1$) yields

$$\begin{aligned} &H^i(\mathbb{P}^6\backslash\mathcal{V}(I_3')) = 0 \quad \text{for } \ i < 7, \\ &H^7(\mathbb{P}^6\backslash\mathcal{V}(I_3')) \cong H^5(\mathbb{P}^5\backslash\mathcal{V}(I_3'))(-1) \end{aligned} \tag{3.41}$$

with the scheme to the right in $\mathbb{P}^5(\text{no } B_4,\, A_3,\, B_3,\, A_2)$. Note that $\mathcal{V}(I_3') \subset \mathbb{P}^5$ is exactly the graph hypersurface for WS_3. By Theorem 3.2.3, and using the exact sequence

$$\longrightarrow H^4_{prim}(\mathbb{P}^5) \longrightarrow H^4_{prim}(\mathcal{V}(I_3')) \longrightarrow H^5_c(\mathbb{P}^5\backslash\mathcal{V}(I_3')) \longrightarrow H^5(\mathbb{P}^5) \longrightarrow, \tag{3.42}$$

we get

$$H^5_c(\mathbb{P}^5\backslash\mathcal{V}(I_3')) \cong H^4_{prim}(\mathcal{V}(I_3')) \cong \mathbb{Q}(-2). \tag{3.43}$$

Collecting together (3.38), (3.40) and (3.41), we obtain

$$\begin{aligned} &H^i_c(U_1) = 0 \quad \text{for } \ i < 7, \\ &H^7_c(\mathcal{V}(G_4, I_4)\backslash\hat{V}) \cong \mathbb{Q}(-3). \end{aligned} \tag{3.44}$$

The sequence (3.36) simplifies to

$$0 \to H^6_{prim}(\mathcal{V}(I_4, G_4)) \to H^6_{prim}(\hat{V}) \to \mathbb{Q}(-3) \to \tag{3.45}$$

Now consider $\hat{V} = \mathcal{V}(G_4, I_4, I_3) \subset \mathbb{P}^8(\text{no } B_4)$. By Theorem 1.1.5, the polynomial G_4 (see (3.34)) is independent of A_3 on $\hat{V}$. Thus, we can write

$$\hat{V} = \mathcal{V}(A_5 I_2 B_3, I_4, I_3)^{(8)}. \tag{3.46}$$

Now, all the three defining polynomials of $\hat{V}$ are independent of A_3. We apply *Theorem A* ($N = 8$, $k = 3$, $t = 1$) and obtain

$$H^6_{prim}(\hat{V}) \cong H^4_{prim}(V)(-1), \tag{3.47}$$

where

$$V := \mathcal{V}(A_5 I_2 B_3, I_4, I_3) = \mathcal{V}(A_5 I_2 B_3, I_3, A_2 I_2) \subset \mathbb{P}^7(\text{no } B_3, A_3). \tag{3.48}$$

Define $V_1 \subset V \subset \mathbb{P}^7$ by

$$V_1 := V \cap \mathcal{V}(I_2) = \mathcal{V}(I_3, I_2). \tag{3.49}$$

One has an exact sequence

$$\longrightarrow H^3(V_1) \longrightarrow H_c^4(V \backslash V_1) \longrightarrow H^4_{prim}(V) \longrightarrow H^4_{prim}(V_1) \longrightarrow \tag{3.50}$$

We see that the defining polynomials of V_1 are independent of A_2 and B_3. *Theorem A* ($N = 7$, $k = 2$, $t = 2$) yields

$$H^i(V_1) = 0 \quad \text{for } \ i < 7. \tag{3.51}$$

Then the sequence above implies an isomorphism

$$H^4_{prim}(V) \cong H_c^4(V \backslash V_1). \tag{3.52}$$

The scheme $V \backslash V_1 \subset \mathbb{P}^7$ (no B_4 or A_3) is defined by the system

$$\begin{cases} A_5 B_3 I_2 = 0 \\ I_3 = 0 \\ A_2 I_2 = 0 \\ I_2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} A_5 B_3 = 0 \\ I_3 = 0 \\ A_2 = 0 \\ I_2 \neq 0. \end{cases} \tag{3.53}$$

The variable B_3 appears only in the first equation. Set $V_{02} := (V \backslash V_1) \cap \mathcal{V}(A_5)$ and $V_{03} := (V \backslash V_1) \cap \mathcal{V}(B_3)$. We write the Meyer-Vietoris sequence for $V \backslash V_1$:

$$\begin{aligned} \longrightarrow H_c^3(V_{02}) \oplus H_c^3(V_{03}) \longrightarrow H_c^3(V_{02} \cap V_{03}) \longrightarrow \\ H_c^4(V \backslash V_1) \longrightarrow H_c^4(V_{02}) \oplus H_c^4(V_{03}) \longrightarrow \end{aligned} \tag{3.54}$$

The defining polynomials of V_{02} are all independent of B_3. *Theorem B* ($N = 7$, $k = 3$, $t = 1$) gives us

$$H_c^i(V_{02}) = 0 \quad \text{for } \ i < 5. \tag{3.55}$$

Now, the variety V_{03} is defined by the system

$$\begin{cases} B_3 = A_2 = 0 \\ I_3 = C_2 I_2 - G_2 = 0 \\ I_2 \neq 0. \end{cases} \tag{3.56}$$

The only one appearance of A_5 in the equations is in C_2, so we change the variables $C_2 := A_5$. Thus V_{03} is isomorphic with the variety V'_{03} defined by the same equations but with C_2 independent. We solve the second equation on C_2; projecting from the point where all the variables but C_2 are zero, we get

$$V_{03} \cong V'_{03} \cong \mathbb{P}^4 \backslash \mathcal{V}(I_2), \tag{3.57}$$

where the scheme to the right lives in $\mathbb{P}^4(B_0 : B_1 : A_0 : A_1 : A_4)$. Since I_2 in independent of A_1 and A_4, *Theorem B* ($N = 4$, $k = 0$, $t = 2$) implies

$$H^i_c(V_{03}) \cong H^i_c(\mathbb{P}^4 \backslash I_2) = 0 \quad \text{for } \ i < 6. \tag{3.58}$$

By (3.55) and (3.58), the sequence (3.54) yields an isomorphism

$$H^4_c(V \backslash V_1) \cong H^3_c(V_{02} \cap V_{03}). \tag{3.59}$$

On $V_{02} \cap V_{03}$ all the variables A_5,A_2 and B_3 are zero, thus we obtain

$$V_{02} \cap V_{03} \cong \mathcal{V}(I_3) \backslash \mathcal{V}(I_3, I_2), \tag{3.60}$$

where the scheme to the right lives in $\mathbb{P}^4(B_0 : B_1 : A_0 : A_1 : A_4)$. Together with (3.52) and (3.60), one gets an isomorphism

$$H^4_{prim}(V) \cong H^3_c(\mathcal{V}(I_3) \backslash \mathcal{V}(I_3, I_2)). \tag{3.61}$$

For the scheme to the right, we have an exact sequence

$$\begin{aligned} \to H^2_{prim}(\mathcal{V}(I_3)) \to H^2_{prim}(R) \to H^3_c(\mathcal{V}(I_3) \backslash \mathcal{V}(I_3, I_2)) \to \\ \to H^3(\mathcal{V}(I_3)) \to H^3(R) \to, \end{aligned} \tag{3.62}$$

where

$$R := \mathcal{V}(I_3, I_2) = \mathcal{V}(A_1^2 B_0 + A_4^2 B_1 - 2A_1 A_4 A_0, I_2) = \mathcal{V}(G_2, I_2). \tag{3.63}$$

Consider $R_1 = R \cap \mathcal{V}(B_0) = \mathcal{V}(B_0, A_0, A_4 B_1) \subset \mathbb{P}^4$ and an exact sequence

$$\begin{aligned} \longrightarrow H^1(R_1) \longrightarrow H^2_c(R \backslash R_1) \to H^2_{prim}(R) \longrightarrow \\ H^2_{prim}(R_1) \longrightarrow H^3_c(R \backslash R_1) \longrightarrow H^3(R) \longrightarrow H^3(R_1) \longrightarrow . \end{aligned} \tag{3.64}$$

The variety $R_1 = \mathcal{V}(B_0, A_0, A_4 B_1) \subset \mathbb{P}^4$ and isomorphic to the union of two lines intersected at one point. The sequence above simplifies to

$$\begin{aligned} 0 \longrightarrow H^2_c(R \backslash R_1) \longrightarrow H^2_{prim}(R) \longrightarrow \\ \mathbb{Q}(-1) \longrightarrow H^3_c(R \backslash R_1) \longrightarrow H^3(R) \longrightarrow 0 \end{aligned} \tag{3.65}$$

Now the scheme $R\backslash R_1$ is defined by

$$\begin{cases} G_2 = 0 \\ I_2 = 0 \\ B_0 \neq 0 \end{cases} \Leftrightarrow \begin{cases} A_1^2 B_0 + A_4^2 B_1 - 2A_1 A_4 A_0 = 0 \\ B_0 B_1 - A_0^2 = 0 \\ B_0 \neq 0. \end{cases} \tag{3.66}$$

We apply Theorem 1.1.5 to $R\backslash R_1$, and projecting further from the point where all the variables but B_1 are zero, we obtain

$$R\backslash R_1 \cong \mathcal{V}(I_2)\backslash\mathcal{V}(I_2, B_0) \cong \mathbb{P}^2\backslash\mathcal{V}(B_0) \cong \mathbb{A}^2. \tag{3.67}$$

Hence, $H_c^2(R\backslash R_1) = H_c^3(R\backslash R_1) = 0$. Substituting this into the sequence (3.65), one gets

$$H^2_{prim}(R) \cong Q(-1) \quad \text{and} \quad H^3(R) = 0. \tag{3.68}$$

Since $\mathcal{V}(I_3) \subset \mathbb{P}^4$ is a hypersurface, $H^2_{prim}(\mathcal{V}(I_3)) = 0$; by (3.68), the sequence (3.62) simplifies to

$$0 \longrightarrow \mathbb{Q}(-1) \longrightarrow H_c^3(\mathcal{V}(I_3)\backslash\mathcal{V}(I_3, I_2)) \longrightarrow H^3(\mathcal{V}(I_3)) \longrightarrow 0 \tag{3.69}$$

We need to compute $H^3(\mathcal{V}(I_3))$. Define $\hat{Y}, S_1 \subset \mathcal{V}(I_3) \subset \mathbb{P}^4(A_0 : A_1 : A_4 : B_0 : B_1)$ by

$$\hat{Y} := \mathcal{V}(I_3, I_2^1) \tag{3.70}$$

and $S_1 := \mathcal{V}(I_3)\backslash\hat{Y}$. One has an exact sequence

$$\longrightarrow H_c^3(S_1) \longrightarrow H^3(\mathcal{V}(I_3)) \longrightarrow H^3(\hat{Y}) \longrightarrow H_c^4(S_1) \longrightarrow . \tag{3.71}$$

The scheme S_1 is defined by

$$\begin{cases} B_0 I_2^1 - G_2' = 0 \\ I_2^1 \neq 0. \end{cases} \tag{3.72}$$

We solve the first equation on B_0, and projecting from the point where all the variables but B_0 are zero, we get

$$S_1 \cong \mathbb{P}^3\backslash\mathcal{V}(I_2^1). \tag{3.73}$$

The polynomial I_2^1 is independent of A_0. *Theorem B* ($N = 3$, $k = 0$, $t = 1$) applied to the variety to the right implies

$$\begin{aligned} &H_c^i(S_1) = 0 \quad \text{for} \quad i < 4, \\ &H_c^4(S_1) \cong H_c^2(\mathbb{P}^2\backslash\mathcal{V}(I_2^1))(-1). \end{aligned} \tag{3.74}$$

Since the variety $\mathcal{V}(I_2^1) \cong \mathcal{V}(B_1(A_1 + A_4) - A_1^2) \subset \mathbb{P}^2(A_1 : A_4 : B_1)$ is isomorphic to a line, we get

$$H_c^4(S_1) \cong H_c^2(\mathbb{A}^2)(-1) = 0. \tag{3.75}$$

Substituting this into the sequence (3.71), we get an isomorphism

$$H^3(\mathcal{V}(I_3)) \cong H^3(\hat{Y}). \tag{3.76}$$

Now, $\hat{Y} := \mathcal{V}(I_3, I_2^1) \cong \mathcal{V}(G_2', I_2^1)^{(4)}$ with

$$G_2' := I_3 - B_0 I_2^1 = A_0^2 C_2 + A_4^2 B_1 - 2A_0 A_4 A_1. \tag{3.77}$$

The defining polynomials of $\hat{Y} \subset \mathbb{P}^4$ are independent of B_0, we apply *Theorem A* and get

$$H^3(\hat{Y}) \cong H^1(Y)(-1), \tag{3.78}$$

where $Y := \mathcal{V}(G_2', I_2^1) \subset \mathbb{P}^3(A_0 : A_1 : A_4 : B_1)$. Define $Y_1, Y_c \subset Y \subset \mathbb{P}^3$ by

$$Y_1 = Y \cap \mathcal{V}(B_1) \tag{3.79}$$

and $Y_c := Y \backslash Y_1$. One has an exact sequence

$$\longrightarrow H^0_{prim}(Y) \longrightarrow H^0_{prim}(Y_1) \longrightarrow H^1_c(Y_c) \longrightarrow H^1(Y) \longrightarrow H^1(Y_1) \longrightarrow \tag{3.80}$$

The variety Y_1 is defined by the system of equations

$$\begin{cases} A_0^2 C_2 + A_4^2 B_1 - 2A_0 A_4 A_1 = 0 \\ B_1 C_2 - A_1^2 = 0 \\ B_1 = 0, \end{cases} \tag{3.81}$$

where $C_2 = A_1 + A_4$. It is easy to see that

$$Y_1 = \mathcal{V}(B_1, A_1, A_0 A_4) \subset \mathbb{P}^3, \tag{3.82}$$

so Y_1 is isomorphic to a union of two points. By *Theorem A* ($N = 3$, $k = 2$, $t = 0$), $H^0_{prim}(Y) = 0$, thus the sequence (3.80) simplifies to

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow H^1_c(Y_c) \longrightarrow H^1(Y) \longrightarrow 0. \tag{3.83}$$

The scheme Y_c is defined by the following system:

$$\begin{cases} A_0^2 C_2 + A_4^2 B_1 - 2A_0 A_4 A_1 = 0 \\ B_1 C_2 - A_1^2 = 0 \\ B_1 \neq 0. \end{cases} \tag{3.84}$$

By Corollary 1.1.4, one has

$$G_2' B_1 \equiv (Li_2)^2 \mod I_2, \tag{3.85}$$

where $Li_2 := A_4 B_1 - A_0 A_1$. Hence, the first equation of the system implies

$$A_4 B_1 - A_0 A_1 = 0 \quad \Leftrightarrow \quad A_4 = \frac{A_0 A_1}{B_1} \tag{3.86}$$

while the second equation gives us

$$C_2 = A_1 + A_4 = \frac{A_1^2}{B_1} \quad \Leftrightarrow \quad A_4 = \frac{A_1(A_1 - B_1)}{B_1}. \tag{3.87}$$

Thus,

$$A_1(B_1 + A_0 - A_1) = 0 \tag{3.88}$$

on Y_c. Projecting from the point where all the variables but A_4 are zero, we get an isomorphism

$$Y_c \cong \mathcal{V}(A_1(B_1 + A_0 - A_1)) \backslash \mathcal{V}(A_1(B_1 + A_0 - A_1), B_1) \subset \mathbb{P}^2. \tag{3.89}$$

For Y_c we can now write an exact sequence

$$\begin{aligned} \to H^0_{prim}(\mathcal{V}(A_1(B_1 - A_1 - A_0))) \to H^0_{prim}(\mathcal{V}(A_1(B_1 - A_1 - A_0), B_1)) \\ \to H^1_c(Y_c) \to H^1(\mathcal{V}(A_1(B_1 - A_1 - A_0))) \to . \end{aligned} \tag{3.90}$$

Changing the variables $B_1 := B_1 - A_1 - A_0$, we see that the variety $\mathcal{V}(A_1(B_1 - A_1 - A_0))$ is isomorphic to a union of two lines intersected at one point. Similarly, $\mathcal{V}(A_1(B_1 - A_1 - A_0), B_1)$ is isomorphic to a union of two points. Hence, the sequence (3.90) implies

$$H^1_c(Y_c) \cong \mathbb{Q}(0). \tag{3.91}$$

We return to the sequence (3.83) and obtain the vanishing

$$H^1(Y) = 0. \tag{3.92}$$

By (3.76), (3.78) and (3.92), the sequence (3.69) gives us an isomorphism

$$H^3_c(\mathcal{V}(I_3) \backslash \mathcal{V}(I_3, I_2)) \cong \mathbb{Q}(-1). \tag{3.93}$$

Using (3.45), (3.47) and (3.61) we obtain an exact sequence

$$0 \longrightarrow H^6_{prim}(\mathcal{V}(I_4, G_4)) \longrightarrow \mathbb{Q}(-2) \xrightarrow{j} \mathbb{Q}(-3) \longrightarrow \tag{3.94}$$

The map j must be zero both for Hodge structures and for $\mathbb{Q}_\ell$-modules with $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ action. Hence, the sequence yields

$$H^6_{prim}(\mathcal{V}(I_4, G_4)) \cong \mathbb{Q}(-2). \tag{3.95}$$

By (3.33), we finally get

$$H^8_{prim}(X) \cong H^6_{prim}(\mathcal{V}(I_4, G_4))(-1) \cong \mathbb{Q}(-3). \tag{3.96}$$

□

Now we want to recall the classical case of WS_n (see [BEK]) and prove two lemmas used in the next section.

Theorem 3.2.3
Let $X \subset \mathbb{P}^{2n-1}$ be the graph hypersurface associated to WS_n, $n \geq 3$. Then

$$H^{2n-2}_{prim}(X) \cong \mathbb{Q}(-2). \tag{3.97}$$

Proof. We only recall several steps of the proof. The proof itself can be found in [BEK].

In this case the matrix is a three-diagonal matrix 3diag$(B_0, \dots, B_{n-1}; A_0, \dots, A_{n-2})$ plus some extra term A_{n-1} at the corners.

$$M_{WS_n}(A,B) := \begin{pmatrix} B_0 & A_0 & \vdots & 0 & 0 & A_{n-1} \\ A_0 & B_1 & \vdots & 0 & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & \vdots & B_{n-3} & A_{n-3} & 0 \\ 0 & 0 & \vdots & A_{n-3} & B_{n-2} & A_{n-2} \\ A_{n-1} & 0 & \vdots & 0 & A_{n-2} & B_{n-1} \end{pmatrix} \tag{3.98}$$

We deal with the hypersurface $X := \mathcal{V}(I_n) \subset \mathbb{P}^{2n-1}(A,B)$. Projecting from the point where all the variables but B_{n-1} vanish, we get the following isomorphism

$$H^{2n-2}(X) \cong H^{2n-2}(\mathcal{V}(I_n, I_{n-1})) \cong H^{2n-4}(\mathcal{V}(I_{n-1}, G_{n-1}))(-1), \tag{3.99}$$

where G_{n-1} is such that $I_n = B_{n-1}I_{n-1} - G_{n-1}$. The variety to the right lives in $\mathbb{P}^{2n-2}$(no B_{n-1}). Next, one can prove that

$$\begin{aligned} H^{2n-4}(\mathcal{V}(I_{n-1}, G_{n-1})) &\cong H^{2n-4}(\mathcal{V}(I_{n-1}, G_{n-1}, I_{n-2})) \\ &\cong H^{2n-6}(\mathcal{V}(I_{n-1}, I_{n-2}, A_{n-1}I^1_{n-2}))(-1), \end{aligned} \tag{3.100}$$

where the variety on the right hand side lives in $\mathbb{P}^{2n-3}$(no B_{n-1}, A_{n-2}). The next step of the proof is an isomorphism

$$H^{2n-6}_{prim}(\mathcal{V}(I_{n-1}, I_{n-2}, A_{n-1}I^1_{n-2})) \cong H^{2n-7}(\mathcal{V}(I_{n-1}, I_{n-2}, A_{n-1}, I^1_{n-2})) \tag{3.101}$$

The last thing to prove is

$$H^{2n-7}(Z) = \mathbb{Q}(0), \tag{3.102}$$

where $Z := \mathcal{V}(I_{n-1}, I_{n-2}, I^1_{n-2}) \subset \mathbb{P}^{2n-4}$(no B_{n-1}, A_{n-2}, A_{n-1}). This is exactly the statement of Theorem 11.9 in [BEK]. In the next section we will slightly modify this part of the proof, and apply it in the computation of $WS_n \times WS_3$. Now, (3.102) implies

$$H^{2n-4}_{prim}(\mathcal{V}(I_{n-1}, G_{n-1})) \cong \mathbb{Q}(-1) \tag{3.103}$$

and

$$H^{2n-2}_{prim}(\mathcal{V}(I_n)) \cong \mathbb{Q}(-2). \tag{3.104}$$

□

The other statement is used several times.

Lemma 3.2.4
Let M be the three-diagonal matrix $\text{3diag}(B_0, \dots, B_{n-1}; A_0, \dots, A_{n-2})$, $n \geq 2$. *Then*

$$H^i_{prim}(\mathcal{V}(I_n)) = 0 \tag{3.105}$$

for $i \leq 2n-3$.

Proof. The matrix M looks like

$$M := \begin{pmatrix} B_0 & A_0 & \vdots & 0 & 0 & 0 \\ A_0 & B_1 & \vdots & 0 & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & \vdots & B_{n-3} & A_{n-3} & 0 \\ 0 & 0 & \vdots & A_{n-3} & B_{n-2} & A_{n-2} \\ 0 & 0 & \vdots & 0 & A_{n-2} & B_{n-1} \end{pmatrix} \tag{3.106}$$

The variety $\mathcal{V}(I_n)$ lives in $\mathbb{P}^{2n-2}(A, B)$. For $i \leq 2n-4$ the vanishing holds for dimensional reasons (by *Theorem A*). The statement means that the interesting cohomology of $\mathcal{V}(I_n)$ is not the middle dimensional one,

$H^{mid}(\mathcal{V}(I_n)) = H^{2n-3}(\mathcal{V}(I_n))$, but the cohomology of the degree one above. Actually, we will prove that

$$H^i_{prim}(\mathcal{V}(I_n)) = H^i_{prim}(\mathcal{V}(I_n, I_{n-1})) = 0 \tag{3.107}$$

for $i \leq 2n-3$ using induction on n. For $n = 2$, the variety $\mathcal{V}(B_0B_1 - A_0^2)$ is isomorphic to $\mathbb{P}^1$ and $\mathcal{V}(I_2, B_0) = \mathcal{V}(B_0, A_0)$ is a point, thus

$$H^1(\mathcal{V}(I_2)) = H^1(\mathcal{V}(I_2, I_1)) = 0. \tag{3.108}$$

Suppose now that for all three-diagonal matrices of dimension smaller then $n \times n$ the statement holds. Consider the exact sequence

$$\begin{aligned} &\longrightarrow H_c^{2n-3}(\mathcal{V}(I_n)\backslash\mathcal{V}(I_n, I_{n-1})) \longrightarrow H^{2n-3}(\mathcal{V}(I_n)) \longrightarrow \\ &\quad H^{2n-3}(\mathcal{V}(I_n, I_{n-1})) \longrightarrow H_c^{2n-2}(\mathcal{V}(I_n)\backslash\mathcal{V}(I_n, I_{n-1})) \longrightarrow . \end{aligned} \tag{3.109}$$

Using the formula

$$I_n = B_{n-1}I_{n-1} - A_{n-2}^2 I_{n-2}, \tag{3.110}$$

we can solve the equation $I_n = 0$ on B_{n-1}; projection from the point where all the variables but B_{n-1} are zero gives us an isomorphism

$$\mathcal{V}(I_n)\backslash\mathcal{V}(I_n, I_{n-1}) \cong \mathbb{P}^{2n-3}\backslash\mathcal{V}(I_{n-1}). \tag{3.111}$$

Now, I_{n-1} is independent of A_{n-2}. We project from the point where all the variables but A_{n-2} are zero, and get

$$H^*(\mathbb{P}^{2n-3}\backslash\mathcal{V}(I_{n-1})) \cong H^{*-2}(\mathbb{P}^{2n-4}\backslash\mathcal{V}(I_{n-1}))(-1). \tag{3.112}$$

The exact sequence

$$\begin{aligned} &\longrightarrow H^{*-1}_{prim}(\mathbb{P}^{2n-4}) \longrightarrow H^{*-1}_{prim}(\mathcal{V}(I_{n-1})) \longrightarrow \\ &\qquad H^*(\mathbb{P}^{2n-4}\backslash\mathcal{V}(I_{n-1})) \longrightarrow H^*_{prim}(\mathbb{P}^{2n-4}) \longrightarrow \end{aligned} \tag{3.113}$$

gives us an isomorphism

$$H^*(\mathbb{P}^{2n-4}\backslash\mathcal{V}(I_{n-1})) \cong H^{*-1}_{prim}(\mathcal{V}(I_{n-1})). \tag{3.114}$$

By the induction hypothesis, for $\mathcal{V}(I_{n-1}) \subset \mathbb{P}^{2n-4}$(no B_{n-1}, A_{n-2}) we have

$$H^i_{prim}(\mathcal{V}(I_{n-1})) = 0 \quad i \leq 2n-5. \tag{3.115}$$

By (3.111), (3.112) and (3.114), we get

$$H^i_c(\mathcal{V}(I_n)\backslash\mathcal{V}(I_n, I_{n-1})) \cong H^i_c(\mathbb{P}^{2n-3}\backslash\mathcal{V}(I_{n-1})) = 0 \tag{3.116}$$

for $i \leq (2n-5)+2+1 = 2n-2$. Thus, the sequence (3.109) implies an isomorphism

$$H^{2n-3}(\mathcal{V}(I_n)) \cong H^{2n-3}(\mathcal{V}(I_n, I_{n-1})). \tag{3.117}$$

We write $\mathcal{V}(I_n, I_{n-1}) = \mathcal{V}(I_{n-1}, A_{n-2}I_{n-2})^{(2n-2)}$, and projecting from the point where all variables but B_{n-1} are zero, we get

$$H^{2n-3}(\mathcal{V}(I_n, I_{n-1})) \cong H^{2n-5}(\mathcal{V}(I_{n-1}, A_{n-2}I_{n-2}))(-1) \tag{3.118}$$

with $\mathcal{V}(I_{n-1}, A_{n-2}I_{n-2}) \subset \mathbb{P}^{2n-3}(\text{no } B_{n-1})$. One has an exact sequence

$$\begin{aligned} H^{2n-6}_{prim}(\mathcal{V}(I_{n-1}, A_{n-2}, I_{n-2})) &\longrightarrow H^{2n-5}(\mathcal{V}(I_{n-1}, A_{n-2}I_{n-2})) \longrightarrow \\ &H^{2n-5}(\mathcal{V}(I_{n-1}, A_{n-2})) \oplus H^{2n-5}(\mathcal{V}(I_{n-1}, I_{n-2})^{(2n-3)}) \longrightarrow . \end{aligned} \tag{3.119}$$

By the induction assumption, both the term to the left and the sum to the right vanish, and we get

$$\begin{aligned} H^{2n-3}(\mathcal{V}(I_n)) \cong H^{2n-3}(\mathcal{V}(I_n, I_{n-1})) &\cong \\ H^{2n-5}(\mathcal{V}(I_{n-1}, A_{n-2}I_{n-2}))(-1) &= 0. \end{aligned} \tag{3.120}$$

□

Another lemma with the similar statement will be used in the next section.

Lemma 3.2.5
Let M be the three-diagonal matrix $\text{3diag}(B_0, \dots, B_{n-2}, C_{n-1}; A_0, \dots, A_{n-2})$, $n \geq 3$ *with* $C_{n-1} = A_{n-2}$. *Then*

$$H^i_{prim}(\mathcal{V}(I_n)) = 0 \tag{3.121}$$

for $i \leq 2n-4$.

Proof. We work with the matrix

$$M := \begin{pmatrix} B_0 & A_0 & \vdots & 0 & 0 & 0 \\ A_0 & B_1 & \vdots & 0 & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & \vdots & B_{n-3} & A_{n-3} & 0 \\ 0 & 0 & \vdots & A_{n-3} & B_{n-2} & A_{n-2} \\ 0 & 0 & \vdots & 0 & A_{n-2} & C_{n-1} \end{pmatrix} \tag{3.122}$$

The variety $\mathcal{V}(I_n(M))$ is taken in $\mathbb{P}^{2n-3}(A, B)$. *Theorem A* ($N = 2n-3$, $k = 1$, $t = 0$) implies $H^i(\mathcal{V}(I_n)) = 0$ for $i < 2n-4$. Consider $H^{2n-4}(\mathcal{V}(I_n))$. We can write

$$I_n = C_{n-1}I_{n-1} - A_{n-2}^2 I_{n-2} = A_{n-2}(I_{n-1} - A_{n-2}I_{n-2}). \tag{3.123}$$

Define $S, T \subset \mathcal{V}(I_n)$ by

$$T := \mathcal{V}(I_{n-1} - A_{n-2}I_{n-2}) \tag{3.124}$$

and $S := \mathcal{V}(A_{n-2})^{(2n-3)}$. Then $\mathcal{V}(I_n) = T \cup S$, and one has the Mayer-Vietoris sequence

$$\begin{aligned} \longrightarrow H^{2n-5}_{prim}(S \cap T) \longrightarrow H^{2n-4}_{prim}(\mathcal{V}(I_n)) \longrightarrow \\ H^{2n-4}_{prim}(S) \oplus H^{2n-4}_{prim}(T) \longrightarrow . \end{aligned} \tag{3.125}$$

Since $S \cap T = \mathcal{V}(A_{n-2}, I_{n-1})$, Lemma 3.2.4 implies $H^{2n-5}_{prim}(S \cap T) = 0$. The variety $S = \mathcal{V}(A_{n-2}) \subset \mathbb{P}^{2n-3}$ is isomorphic to $\mathbb{P}^{2n-4}$, thus $H^{2n-4}_{prim}(S) = 0$. For T, we can write

$$I_{n-1} - A_{n-2}I_{n-2} = (B_{n-2} - A_{n-2})I_{n-2} - A_{n-3}^2 I_{n-3}. \tag{3.126}$$

We see that this polynomial does not depend on A_{n-2} or B_{n-2} but only on $B_{n-2} - A_{n-2}$. We can change the variables $B_{n-2} := B_{n-2} - A_{n-2}$, then *Theorem A*($N = 2n-3$, $k = 1$, $t = 1$) implies $H^{2n-4}(T) = 0$. Finally, the sequence (3.125) implies

$$H^{2n-4}_{prim}(\mathcal{V}(I_n)) = 0. \tag{3.127}$$

□

3.3 Gluings of WS's

In general, it is not easy to verify whether the graph is primitively log divergent or not. Nevertheless, we can construct new primitively log divergent graphs from the existing one's by the operation of gluing.

Definition 3.3.1
Let Γ and Γ' be two graphs, choose two edges $(u,v) \in E(\Gamma)$ and $(u',v') \in E(\Gamma')$. We define the graph $\Gamma \times \Gamma'$ as follows. We drop the edges (u,v) and (u',v'), and identify vertices u with u' and v with v'. We say also that $\Gamma \times \Gamma'$ is the *gluing* of Γ and Γ' along edges (u,v) and (u',v').

Example 3.3.2
The graph XX considered in the previous section is isomorphic to $WS_3 \times WS_3$.

Theorem 3.3.3
The gluing $\Gamma \times \Gamma'$ of two primitively log divergent graphs Γ and Γ' (along edges (u,v) and (u',v')) is again primitively log divergent.

Proof. Suppose that Γ and Γ' have $2n$ and $2m$ edges respectively, then $h_1(\Gamma)$ and $h_1(\Gamma') = m$. We can chose a basis $\{\gamma_1, \ldots, \gamma_n\}$ of $H_1(\Gamma, \mathbb{Z})$ such that the edge (u,v) only appears in γ_n. Indeed, we take any basis $\{\gamma_1, \ldots, \gamma_{n-1}\}$ of $H_1(\Gamma \backslash \{(u,v)\}, \mathbb{Z})$ and define γ_n to be any loop containing (u,v), then $\{\gamma_1, \ldots, \gamma_n\}$ form a basis of $H_1(\Gamma, \mathbb{Z})$. Similarly, we choose a basis $\delta_1, \ldots, \delta_m$ such that the only appearance of (u',v') is in δ_m. It follows that the loops $\{\gamma_1, \ldots, \gamma_{n-1}, \delta_1, \ldots, \delta_{m-1}, \gamma_n \times \delta_m\}$ form a basis of $H_1(\Gamma \times \Gamma', \mathbb{Z})$. Thus, $|E(\Gamma \times \Gamma')| = 2n + 2m - 2 = 2h_1(\Gamma \times \Gamma')$ and $\Gamma \times \Gamma'$ is logarithmically divergent.

To prove that $\Gamma \times \Gamma'$ is primitively log divergent, we consider a proper subgraph $\Gamma_0 \subset \Gamma \times \Gamma'$ and define Γ_1 (respectively Γ_2) to be the graph $\Gamma_0 \cap \Gamma \cup \{(u,v)\}$ (respectively $\Gamma_0 \cap \Gamma' \cup \{(u',v')\}$). Because the graphs Γ and Γ' is primitively log divergent, for the subgraphs $\Gamma_1 \subset \Gamma$ and $\Gamma_2 \subset \Gamma'$ the inequalities

$$|E(\Gamma_1)| \leq 2h_1(\Gamma_1) \quad \text{and} \quad |E(\Gamma_2)| \leq 2h_1(\Gamma_2) \tag{3.128}$$

hold, and the inequalities become strict if subgraphs are proper. Since Γ_0 is the proper subgraph, at least one of the subgraphs Γ_1, Γ_2 is proper. Thus we get

$$|E(\Gamma_1)| + |E(\Gamma_2)| < 2(h_1(\Gamma_1) + h_1(\Gamma_2)) \tag{3.129}$$

The number of edges of Γ_0 equals

$$|E(\Gamma_0)| = |E(\Gamma_1)| + |E(\Gamma_2)| - 2, \tag{3.130}$$

and one has an inequality

$$h_1(\Gamma_1) + h_1(\Gamma_2) - 1 \leq h_1(\Gamma_0) \qquad (3.131)$$

which becomes an equality if the operation of adding (u, v) to $\Gamma_0 \cap \Gamma$ (or that of (u', v') to $\Gamma_0 \cap \Gamma'$) increases the Betti number. The inequality 3.129 implies

$$|E(\Gamma_0)| < 2h_1(\Gamma_0). \qquad (3.132)$$

Thus, every subgraph of $\Gamma \times \Gamma'$ is convergent and $\Gamma \times \Gamma'$ is primitively log divergent.

□

Corollary 3.3.4
Every gluing Γ of finitely many GZZ graphs (along any pair of edges) is primitively log divergent.

Proof. This follows from the fact that a GZZ graph is primitively log divergent, see Theorem 2.2.5.

□

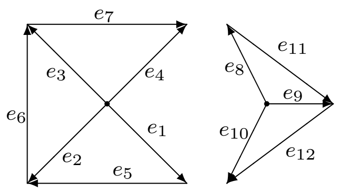
Throughout this section we only deal with gluings of WS's graphs. Our goal here is to analyse the middle dimensional (Betti) cohomology of hypersurfaces associated to graphs $WS_n \times WS_3$ for $n \geq 4$. The gluing for $WS_n \times WS_3$ goes along some two b-edges (not spokes).

Theorem 3.3.5
Let X be the graph hypersurface for the graph $WS_n \times WS_3$, $n \geq 4$. For the middle dimensional cohomology $H^{mid}(X)$, one has

$$gr_6^W(H^{mid}_{prim}(X)) = \mathbb{Q}(-3) \quad \textit{and} \quad gr_8^W(H^{mid}_{prim}(X)) = \mathbb{Q}(-4)^{\oplus d}, \qquad (3.133)$$

where $d = 0$, 1 or 2, and all other $\mathrm{gr}_i^W = 0$.

Proof. Fix $n \geq 4$ and consider the graph WS_n. We orient the spokes (a-edges) (v_0, v_i) as exiting the center v_0 and label them with e_1 through e_n. The boundary edges (v_i, v_{i+1}) (modulo n) are denoted by e_{n+i} and are oriented exiting v_i. Now we rename the last edge $e_{2n} =: e$, play the same game with the graph WS_3, shifting the numeration of edges by $2n - 1$, and glue WS_n with WS_3 along e and e_{2n+5}. Denote the resulting graph by Γ. To show the way of constructing the tables and the matrices associated to this gluing, we restrict to the case $WS_4 \times WS_3$.



	1	2	3	4	5	6	7	8	9	10	11	12
1	1	−1	0	0	1	0	0	0	0	0	0	0
2	0	1	−1	0	0	1	0	0	0	0	0	0
3	0	0	1	−1	0	0	1	0	0	0	0	0
4	1	0	0	−1	0	0	0	1	0	−1	0	0
5	0	0	0	0	0	0	0	1	−1	0	1	0
6	0	0	0	0	0	0	0	0	1	−1	0	1

The matrix M_Γ has two "blocks" coming from the matrices of WS_n and WS_3 intersected by one element which becomes dependent.

$$M_\Gamma(A,B) = \left(\begin{array}{ccccccccc} B_0 & A_0 & 0 & ... & 0 & 0 & A_{n+1} & 0 & 0 \\ A_0 & B_1 & A_1 & ... & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & B_2 & ... & 0 & 0 & 0 & 0 & 0 \\ \multicolumn{9}{c}{\dotfill} \\ 0 & 0 & 0 & ... & B_{n-3} & A_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & A_{n-3} & B_{n-2} & A_{n-2} & 0 & 0 \\ A_{n+1} & 0 & 0 & ... & 0 & A_{n-2} & C_{n-1} & A_{n-1} & A_{n+2} \\ 0 & 0 & 0 & ... & 0 & 0 & A_{n-1} & B_n & A_n \\ 0 & 0 & 0 & ... & 0 & 0 & A_{n+2} & A_n & B_{n+1} \end{array}\right) \quad (3.134)$$

We deal with polynomials in variables $A = \{A_0, A_1, \ldots, A_{n+2}\}$ and $B = \{B_0, \ldots, B_{n-2}, B_n, B_{n+1}\}$, the element C_{n-1} is equal to the sum

$$C_{n-1} := A_{n+1} + A_{n-2} + A_{n-1} + A_{n+2}. \quad (3.135)$$

The hypersurface $X \subset \mathbb{P}^{2n+3}(A,B)$ is defined by the vanishing of $I_{n+2}(M) = \det M_\Gamma$. The middle dimensional cohomology to compute is

$$H^{mid}(X) = H^{2n+2}(X). \quad (3.136)$$

One has

$$I_{n+2} = B_{n+1} I_{n+1} - G_{n+1}. \quad (3.137)$$

We have an exact sequence

$$\longrightarrow H_c^{2n+2}(U) \longrightarrow H^{2n+2}(X) \longrightarrow H^{2n+2}(\mathcal{V}(I_{n+2}, I_{n+1})) \longrightarrow H_c^{2n+3}(U) \longrightarrow, \quad (3.138)$$

where $U := X \backslash \mathcal{V}(I_{n+2}, I_{n+1}) \subset \mathbb{P}^{2n+3}(A,B)$.

Lemma 3.3.6
One has $H_c^i(U) = 0$ for $i \leq 2n+3$.

Proof. Denote by P_1 the point where all the variables but B_{n+1} are zero. The natural projection from the point P_1, $\pi_1 : \mathbb{P}^{2n+3}\backslash P_1 \to \mathbb{P}^{2n+2}(\text{no } B_{n+1})$, induces an isomorphism

$$U \cong \mathbb{P}^{2n+2}\backslash \mathcal{V}(I_{n+1}). \tag{3.139}$$

Note that I_{n+1} is independent of A_n. Thus, *Theorem B*$(N = 2n+2,\ k = 0,\ t = 1)$ already gives us $H^i_c(U) = 0$ for $i \leq 2n+2$. For $i = 2n+3$ we need to stratify further. Let $P_2 \in \mathbb{P}^{2n+2}$ be the point where all the variables but A_n are zero. The natural projection $\pi_1 : \mathbb{P}^{2n+2}\backslash P_2 \to \mathbb{P}^{2n+1}(\text{no } B_{n+1})$ gives us an $\mathbb{A}^1$-fibration over $\mathbb{P}^{2n+1}\backslash \mathcal{V}(I_{n+1})$, thus

$$H^{2n+3}_c(U) \cong H^{2n+3}_c(\mathbb{P}^{2n+2}\backslash \mathcal{V}(I_{n+1})) \cong H^{2n+1}_c(\mathbb{P}^{2n+1}\backslash \mathcal{V}(I_{n+1}))(-1). \tag{3.140}$$

Next, we make a change of variables: $C_{n-1} := A_{n+2}$ and denote by I'_i the image of I_i under this transformation, $i \leq n+1$. We get an isomorphism

$$\mathbb{P}^{2n+1}\backslash \mathcal{V}(I_{n+1}) \cong \mathbb{P}^{2n+1}\backslash \mathcal{V}(I'_{n+1}). \tag{3.141}$$

For the right hand side scheme, we have an exact sequence

$$\longrightarrow H^{2n}_{prim}(\mathbb{P}^{2n+1}) \longrightarrow H^{2n}_{prim}(\mathcal{V}(I'_{n+1})) \longrightarrow H^{2n+1}_c(\mathbb{P}^{2n+1}\backslash \mathcal{V}(I'_{n+1})) \longrightarrow H^{2n+1}(\mathbb{P}^{2n+1}) \longrightarrow . \tag{3.142}$$

It follows that

$$H^{2n+1}_c(\mathbb{P}^{2n+1}\backslash \mathcal{V}(I'_{n+1})) \cong H^{2n}_{prim}(\mathcal{V}(I'_{n+1})). \tag{3.143}$$

Define $\hat{T}, T_0 \subset \mathbb{P}^{2n+1}$ (no B_{n+1}, A_n) by $\hat{T} := \mathcal{V}(I'_{n+1}, I'_n)$ and $T_0 := \mathcal{V}(I'_{n+1})\backslash \hat{T}$. We need to analyse the exact sequence

$$\longrightarrow H^{2n}_c(T_0) \longrightarrow H^{2n}_{prim}(\mathcal{V}(I'_{n+1})) \longrightarrow H^{2n}_{prim}(\hat{T}) \longrightarrow \tag{3.144}$$

The open scheme T_0 is defined by the system

$$\begin{cases} I'_{n+1} = B_n I'_n - A^2_{n-1} I_{n-1} = 0 \\ I'_n \neq 0, \end{cases} \tag{3.145}$$

The projection from the point where all the variables but B_n vanish induces an isomorphism

$$T_0 \cong \mathbb{P}^{2n}\backslash \mathcal{V}(I'_n). \tag{3.146}$$

Since I'_n is independent of A_{n-1}, *Theorem B*$(N = 2n,\ k = 0,\ t = 1)$, applied to $\mathbb{P}^{2n}\backslash \mathcal{V}(I'_n)$, implies

$$H^{2n}_c(T_0) = 0. \tag{3.147}$$

Now,

$$\hat{T} = \mathcal{V}(I'_{n+1}, I'_n) = \mathcal{V}(I'_n, A_{n-1}I_{n-1}) \subset \mathbb{P}^{2n+1}. \tag{3.148}$$

Both polynomials I'_n and $A_{n-1}I_{n-1}$ are independent of B_n, and projecting from the point where all variables but B_n are zero, we get

$$H^{2n}_{prim}(\hat{T}) \cong H^{2n-2}_{prim}(T)(-1) \tag{3.149}$$

with

$$T := \mathcal{V}(I'_n, A_{n-1}I_{n-1}) \subset \mathbb{P}^{2n}(\text{no } B_{n+1}, A_n, B_n). \tag{3.150}$$

Consider $T_1 \subset T$ defined by

$$T_1 := T \cap \mathcal{V}(I_{n-1}), \tag{3.151}$$

and set $T_{00} = T \backslash T_1$. One has an exact sequence

$$\longrightarrow H^{2n-2}_c(T_{00}) \longrightarrow H^{2n-2}_{prim}(T) \longrightarrow H^{2n-2}_{prim}(T_1) \longrightarrow \tag{3.152}$$

The variety

$$T_1 = \mathcal{V}(I'_n, I_{n-1}) \tag{3.153}$$

is defined by two polynomials both independent of A_{n-1}. We apply *Theorem A* ($N = 2n$, $k = 2$, $t = 2$) to T_1 and get

$$H^{2n-2}_{prim}(T_1) = 0. \tag{3.154}$$

The open scheme T_{00} is defined by the system

$$\begin{cases} I'_n = 0 \\ A_{n-1}I_{n-1} = 0 \\ I_{n-1} \neq 0 \end{cases} \Leftrightarrow \begin{cases} I'_n = 0 \\ A_{n-1} = 0. \\ I_{n-1} \neq 0 \end{cases} \tag{3.155}$$

Write

$$I'_n = C_{n-1}I_{n-1} - G_{n-1}, \tag{3.156}$$

where G_{n-1} is independent of $C_{n-1} = A_{n+2}$. We can express C_{n-1} from the system, and the projection from the point where all the variables but C_{n-1} are zero induces an isomorphism

$$T_{00} \cong \mathbb{P}^{2n-1} \backslash \mathcal{V}(I_{n-1}). \tag{3.157}$$

The polynomial I_{n-1} is independent of A_{n-2} and A_{n+1}. We apply *Theorem B* ($N = 2n-1$, $k = 0$, $t = 2$) and get

$$H^{2n-2}_c(T_{00}) = 0. \tag{3.158}$$

By (3.154) and (3.158), the exact sequence (3.152) simplifies to

$$H^{2n+2}_{prim}(T) = 0. \tag{3.159}$$

The sequence (3.144) together with (3.149) gives us

$$H^{2n}_{prim}(\mathcal{V}(I'_{n+1})) = 0. \tag{3.160}$$

The vanishing of $H^{2n+3}(U)$ now follows from (3.140) and (3.143) .

□

We return to the sequence (3.138). Lemma 3.3.6 yields an isomorphism

$$H^{2n+2}(X) \cong H^{2n+2}(\mathcal{V}(I_{n+2}, I_{n+1})). \tag{3.161}$$

One has

$$I_{n+2} = B_{n+1}I_{n+2} - G_{n+1}, \tag{3.162}$$

thus

$$H^{2n+2}(X) \cong H^{2n+2}(\mathcal{V}(I_{n+1}, G_{n+1})^{(2n+3)}). \tag{3.163}$$

Both I_{n+1} and G_{n+1} are independent of B_{n+1}, we can project from the point P_1 (see Lemma 3.3.6) and get

$$H^{2n+2}(X) \cong H^{2n}(\mathcal{V}(I_{n+1}, G_{n+1}))(-1) \tag{3.164}$$

with the variety on the right hand side living in $\mathbb{P}^{2n+2}$(no B_{n+1}). Now define

$$\hat{V} := \mathcal{V}(I_{n+1}, G_{n+1}, I_n) \subset \mathcal{V}(I_{n+1}, G_{n+1}) \subset \mathbb{P}^{2n+2}. \tag{3.165}$$

We write an exact sequence

$$\begin{aligned} \longrightarrow H^{2n}_c(U_1) \longrightarrow H^{2n}(\mathcal{V}(I_{n+1}, G_{n+1})) \longrightarrow \\ H^{2n}(\hat{V}) \longrightarrow H^{2n+1}_c(U_1) \longrightarrow, \end{aligned} \tag{3.166}$$

where $U_1 := \mathcal{V}(I_{n+1}, G_{n+1})\backslash\hat{V}$. This U_1 can be defined by the system

$$\begin{cases} I_{n+1} = G_{n+1} = 0 \\ I_n \neq 0, \end{cases} \tag{3.167}$$

where

$$G_{n+1} := A_n^2 I_n + A_{n+2}^2 B_n I_{n-1} - 2A_n A_{n+1} I_{n+1}(n; n+1). \tag{3.168}$$

Such U_1 were studied in section 1 of chapter 1 (see (1.36)), and Theorem 1.1.7 claims that

$$U_1 \cong U_2 := \mathcal{V}(I_{n+1})\backslash\mathcal{V}(I_{n+1}, I_n) \tag{3.169}$$

with $U_2 \subset \mathbb{P}^{2n+1}$(no B_{n+1}, A_n). We use the equality

$$I_{n+1} = B_n I_n - A_{n-1}^2 I_{n-1}, \tag{3.170}$$

and we project further from the point P_3 where all the coordinates but B_n vanish. One gets

$$U_2 \cong \mathbb{P}^{2n} \backslash \mathcal{V}(I_n). \tag{3.171}$$

Now, the only appearance of A_{n+2} in I_n is inside the sum C_{n-1}. We again change the variables as in the lemma above and come to the polynomial I'_n which has $C_{n-1} := A_{n+2}$ and does not depend on A_{n-1}. *Theorem B* ($N = 2n$, $k = 0$, $t = 1$) implies that $H^i(\mathbb{P}^{2n} \backslash \mathcal{V}(I'_n)) = 0$ for $i \leq 2n$ and

$$H_c^{2n+1}(\mathbb{P}^{2n} \backslash \mathcal{V}(I'_n)) \cong H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash \mathcal{V}(I'_n))(-1). \tag{3.172}$$

Note that $\mathcal{V}(I'_n)$ is exactly the graph hypersurface for WS_n. The exact sequence

$$\begin{aligned} &\longrightarrow H_{prim}^{2n-2}(\mathbb{P}^{2n-1}) \longrightarrow H_{prim}^{2n-2}(\mathcal{V}(I'_n)) \longrightarrow \\ &\qquad H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash \mathcal{V}(I'_n)) \longrightarrow H^{2n-1}(\mathbb{P}^{2n-1}) \longrightarrow \end{aligned} \tag{3.173}$$

implies

$$H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash \mathcal{V}(I'_n)) \cong H_{prim}^{2n-2}(\mathcal{V}(I'_n)) \cong \mathbb{Q}(-2). \tag{3.174}$$

Collect (3.169), (3.171),(3.172) and (3.174) together; the sequence (3.166) simplifies to

$$0 \longrightarrow H^{2n}(\mathcal{V}(I_{n+1}, G_{n+1})) \longrightarrow H^{2n}(\hat{V}) \longrightarrow \mathbb{Q}(-3) \longrightarrow . \tag{3.175}$$

We can simplify the polynomial G_{n+1} on $\hat{V} \subset \mathbb{P}^{2n+2}$(no B_{n+1}). Indeed, Theorem 1.1.5 in the first chapter asserts that when the coefficient I_n of A_n^2 vanishes (see (3.168)), then the rightmost summand in (3.168) vanishes as well. Thus we can rewrite $\hat{V} = \mathcal{V}(I_n, I_{n+1}, A_{n+2}B_nI_{n-1})$. The defining equations of $\hat{V}$ are independent of A_n. We can project from the point P_2 where all the variables but A_n vanish and get

$$H^{2n}(\hat{V}) \cong H^{2n-2}(V)(-1), \tag{3.176}$$

where

$$V := \mathcal{V}(I_n, I_{n+1}, A_{n+2}B_nI_{n-1}) \subset \mathbb{P}^{2n+1}(\text{no } B_{n+1}, A_n). \tag{3.177}$$

By (3.164), (3.175) and (3.176), one has the following exact sequence

$$0 \longrightarrow H^{2n+2}(X) \longrightarrow H^{2n-2}(V)(-2) \longrightarrow \mathbb{Q}(-4) \longrightarrow, \tag{3.178}$$

One can avoid polarization and rewrite the exact sequence

$$0 \longrightarrow H^{2n+2}_{prim}(X) \longrightarrow H^{2n-2}_{prim}(V)(-2) \longrightarrow \mathbb{Q}(-4) \longrightarrow . \tag{3.179}$$

Now we attack V. Using the equality

$$I_{n+1} = B_n I_n - A^2_{n-1} I_{n-1}, \tag{3.180}$$

we can write

$$V = \mathcal{V}(I_n, A_{n+2}B_n I_{n-1}, A_{n-1}I_{n-1}). \tag{3.181}$$

Define the subvarieties $V_1, V_2 \subset V \subset \mathbb{P}^{2n+1}$(no B_{n+1},A_n) by

$$\begin{aligned} V_1 &:= \mathcal{V}(I_n, B_n, A_{n-1}I_{n-1}) \\ V_2 &:= \mathcal{V}(I_n, A_{n+2}I_{n-1}, A_{n-1}I_{n-1}). \end{aligned} \tag{3.182}$$

One has an exact sequence

$$\begin{aligned} \longrightarrow H^{2n-3}(V_1) \oplus H^{2n-3}(V_2) \longrightarrow H^{2n-3}(V_3) \longrightarrow \\ H^{2n-2}_{prim}(V) \longrightarrow H^{2n-2}_{prim}(V_1) \oplus H^{2n-2}_{prim}(V_2) \longrightarrow \end{aligned} \tag{3.183}$$

with

$$V_3 := V_1 \cap V_2 = \mathcal{V}(I_n, B_n, A_{n+2}I_{n-1}, A_{n-1}I_{n-1}). \tag{3.184}$$

Note that the defining polynomials of V_2 are independent of B_n. *Theorem A* ($N = 2n+1$, $k = 3$, $t = 1$) implies

$$H^i_{prim}(V_2) = 0 \quad \text{for } \ i \leq 2n-2. \tag{3.185}$$

Theorem A ($N = 2n+1$, $k = 3$, $t = 0$) also gives us

$$H^i_{prim}(V_1) = 0 \quad \text{for } \ i \leq 2n-3. \tag{3.186}$$

We show that $H^{2n-2}_{prim}(V_1)$ vanishes as well. Define

$$V_{11} := \mathcal{V}(I_n, B_n, I_{n-1}) \subset V_1 \subset \mathbb{P}^{2n+1}. \tag{3.187}$$

and denote by V_{10} the complement $V_1 \backslash V_{11}$. One has an exact sequence

$$\longrightarrow H^{2n-2}_c(V_{10}) \longrightarrow H^{2n-2}_{prim}(V_1) \longrightarrow H^{2n-2}_{prim}(V_{11}) \longrightarrow \tag{3.188}$$

The equations of V_{11} do not depend on A_{n-1} or A_{n+2} but only on the sum $A_{n-1} + A_{n+2}$ in C_{n-1}. After the change of variables $C_{n-1} := A_{n+2}$, we can apply *Theorem A* ($N = 2n+1$, $k = 3$, $t = 1$) and get

$$H^{2n-2}_{prim}(V_{11}) = 0. \tag{3.189}$$

The open subscheme V_{10} is defined by the system

$$\begin{cases} B_n = I_n = 0 \\ A_{n-1}I_{n-1} = 0 \\ I_{n-1} \neq 0 \end{cases} \Leftrightarrow \begin{cases} C_{n-1}I_{n-1} - G_{n-1} = 0 \\ B_n = A_{n-1} = 0. \\ I_{n-1} \neq 0 \end{cases} \tag{3.190}$$

Using the same change of the variables and expressing $C_{n-1} := A_{n+2}$ from the system, we obtain that the projection from the point P_4 where all the variables but A_{n+2} vanish, induces an isomorphism

$$V_{10} \cong \mathbb{P}^{2n-2} \backslash \mathcal{V}(I_{n-1}). \tag{3.191}$$

We have forgotten variables B_n, A_{n-1} identifying $\mathbb{P}^{2n-2}$(no B_{n+1}, A_n, B_n, A_{n+2}, A_{n-1}) with $\mathcal{V}(B_n, A_{n-1}) \subset \mathbb{P}^{2n}$(no B_{n+1}, A_n, A_{n+2}). Because the polynomial I_{n-1} is independent of A_{n-2} and A_{n+1}, *Theorem B* ($N = 2n-2$, $k = 0$, $t = 2$) implies

$$H^i_c(V_{10}) = 0 \quad \text{for } i \leq 2n-1. \tag{3.192}$$

By (3.188) and (3.189), one gets

$$H^{2n-2}_{prim}(V_1) = 0. \tag{3.193}$$

We return to the sequence (3.183). By (3.185), (3.186) and (3.193), we get an isomorphism

$$H^{2n-2}_{prim}(V) \cong H^{2n-3}(V_3), \tag{3.194}$$

where

$$V_3 := \mathcal{V}(I_n, B_n, A_{n+2}I_{n-1}, A_{n-1}I_{n-1}). \tag{3.195}$$

Consider $V_{31} \subset V_3 \subset \mathbb{P}^{2n+1}$(no B_{n+1}, A_n) defined by

$$V_{31} := V_3 \cap \mathcal{V}(I_{n-1}) = \mathcal{V}(I_n, B_n, I_{n-1}). \tag{3.196}$$

One has an exact sequence

$$\begin{aligned} \longrightarrow H^{2n-4}_{prim}(V_{31}) \longrightarrow H^{2n-3}_c(V_3 \backslash V_{31}) \longrightarrow \\ H^{2n-3}(V_3) \longrightarrow H^{2n-3}(V_{31}) \longrightarrow . \end{aligned} \tag{3.197}$$

Theorem A ($N = 2n+1$, $k = 3$, $t = 0$) implies

$$H^i_{prim}(V_{31}) = 0 \quad \text{for } i \leq 2n-3. \tag{3.198}$$

Thus, the sequence (3.197) yields an isomorphism

$$H^{2n-3}(V_3) \cong H^{2n-3}_c(V_3 \backslash V_{31}). \tag{3.199}$$

The subscheme $V_3 \backslash V_{31}$ is defined by the system

$$\begin{cases} A_{n-1}I_{n-1} = I_n = 0 \\ B_n = A_{n+2}I_{n-1} = 0 \\ I_{n-1} \neq 0 \end{cases} \Leftrightarrow \begin{cases} A_{n-1} = I_n = 0 \\ B_n = A_{n+2} = 0. \\ I_{n-1} \neq 0 \end{cases} \tag{3.200}$$

Now we consider $V_3 \backslash V_{31}$ as being in $\mathbb{P}^{2n-2}$(no DV_5) and define $Y, S \subset \mathbb{P}^{2n-2}$ by

$$\begin{aligned} Y &= \mathcal{V}(I_n), \\ S &= \mathcal{V}(I_n, I_{n-1}), \end{aligned} \tag{3.201}$$

where by DV_5 the set of the dropped variables $\{B_{n+1}, A_n, B_n, A_{n+2}, A_{n-1}\}$ is denoted. This gives us an exact sequence

$$0 \longrightarrow H^{2n-4}_{prim}(S) \longrightarrow H^{2n-3}(V_3 \backslash V_{31}) \longrightarrow H^{2n-3}(Y) \longrightarrow \tag{3.202}$$

After rewriting

$$S = \mathcal{V}(I_{n-1}, C_{n-1}I_{n-1} - G_{n-1}) = \mathcal{V}(I_{n-1}, G_{n-1}), \tag{3.203}$$

we note that S is exactly the variety which appears in the first reduction step of the case of WS_n (see Theorem 3.2.3), and we know that

$$H^{2n-4}(S) \cong \mathbb{Q}(-1). \tag{3.204}$$

The computation of $H^{2n-3}(Y)$ is less easy. The polynomial I_n is similar to the polynomial associated to WS_n with the only difference that C_{n-1} is not independent and is equal $A_{n+1} + A_{n-2}$. We start from the upper left corner of the matrix and write

$$I_n = B_0 I^1_{n-1} - \widetilde{G}_{n-1}, \tag{3.205}$$

where

$$\widetilde{G}_{n-1} = A_0^2 I^2_{n-2} + A^2_{n+1} I^1_{n-2} + (-1)^{n-1} A_0 A_{n+1} S_{n-2}. \tag{3.206}$$

Consider $\hat{Y}_1 \subset Y \subset \mathbb{P}^{2n-2}$(no DV_5) defined by

$$\hat{Y}_1 := \mathcal{V}(I_n, I^1_{n-1}). \tag{3.207}$$

One has

$$\rightarrow H^{2n-3}_c(Y \backslash \hat{Y}_1) \rightarrow H^{2n-3}(Y) \rightarrow H^{2n-3}(\hat{Y}_1) \rightarrow H^{2n-2}_c(Y \backslash \hat{Y}_1) \rightarrow . \tag{3.208}$$

Using the projection from the point $P_4 \subset \mathbb{P}^{2n-2}$ where all the variables but B_0 vanish, we get an isomorphism

$$Y\backslash\hat{Y}_1 \cong \mathbb{P}^{2n-3}\backslash\mathcal{V}(I^1_{n-1}). \tag{3.209}$$

Because I^1_{n-1} is independent of A_0, *Theorem B*$(N = 2n-3,\ k = 0,\ t = 1)$ implies

$$\begin{aligned} H_c^{2n-3}(Y\backslash\hat{Y}_1) = 0 \quad \text{and} \\ H_c^{2n-2}(Y\backslash\hat{Y}_1) \cong H_c^{2n-4}(\mathbb{P}^{2n-4}\backslash\mathcal{V}(I^1_{n-1})(-1). \end{aligned} \tag{3.210}$$

One has an exact sequence

$$\begin{aligned} \longrightarrow H^{2n-5}(\mathbb{P}^{2n-4}) \longrightarrow H^{2n-5}(\mathcal{V}(I^1_{n-1})) \longrightarrow \\ H_c^{2n-4}(\mathbb{P}^{2n-4}\backslash\mathcal{V}(I^1_{n-1})) \longrightarrow H^{2n-4}_{prim}(\mathbb{P}^{2n-4}) \longrightarrow \end{aligned} \tag{3.211}$$

and gets

$$H_c^{2n-2}(Y\backslash\hat{Y}_1) \cong H^{2n-5}(\mathcal{V}(I^1_{n-1})). \tag{3.212}$$

We can make a change of variables $C_{n-1} := A_{n+1}$ and, for the corresponding I'^1_{n-1} (by Lemma 3.2.4), we obtain

$$H^{2n-5}(\mathcal{V}(I'^1_{n-1})) \cong H^{mid}(\mathcal{V}(I'^1_{n-1})) = 0. \tag{3.213}$$

By (3.210) and (3.212), the sequence (3.208) simplifies to an isomorphism

$$H^{2n-3}(Y) \cong H^{2n-3}(\hat{Y}_1). \tag{3.214}$$

Using the equality (3.205), we can write

$$Y_1 := \mathcal{V}(I_n, I_{n-1}) = \mathcal{V}(I_{n-1}, \widetilde{G}_{n-1})^{(2n+2)}. \tag{3.215}$$

The polynomials to the right are independent of B_0. We project from the point where all the variables but B_0 vanish and come to $Y_1 \subset \mathbb{P}^{2n-3}$(no DV_5, B_0) defined by

$$Y_1 := \mathcal{V}(I_{n-1}, G_{n-1}), \tag{3.216}$$

and together with (3.214) this implies

$$H^{2n-3}(Y) \cong H^{2n-5}(Y_1)(-1). \tag{3.217}$$

Define

$$\hat{Y}_2 := Y_1 \cap \mathcal{V}(I^2_{n-2}) = \mathcal{V}(I_{n-1}, \widetilde{G}_{n-1}, I^2_{n-2}) \tag{3.218}$$

and consider an exact sequence

$$\to H_c^{2n-5}(Y_1\backslash\hat{Y}_2) \to H^{2n-5}(Y_1) \to H^{2n-5}(\hat{Y}_2) \to H_c^{2n-4}(Y_1\backslash\hat{Y}_2) \to . \tag{3.219}$$

The open subscheme $Y_1\backslash\hat{Y}_2 \subset Y_1 \subset \mathbb{P}^{2n-3}$(no DV_5, B_0) is defined by the system

$$\begin{cases} I^1_{n-1} = 0 \\ \widetilde{G}_{n-1} = 0 \\ I^2_{n-2} \neq 0 \end{cases} \tag{3.220}$$

with $\widetilde{G}_{n-1}$ as in (3.206). This G_{n-1} has the same shape as that one we have studied at Section 1 of Chapter 1 with the only difference that the "variables" are chosen from the zero row and column. We apply Theorem 1.1.7 and get an isomorphism

$$Y_1\backslash\hat{Y}_2 \cong U_3 := \mathcal{V}(I^1_{n-1})\backslash\mathcal{V}(I^1_{n-1}, I^2_{n-2}). \tag{3.221}$$

with $U_3 \subset \mathbb{P}^{2n-4}$(no DV_5, B_0, A_0). Using the equation

$$I^1_{n-1} = B_1 I^2_{n-2} - A_1^2 I^3_{n-3} \tag{3.222}$$

and projection from the point where all the variables but B_1 are zero, we get an isomorphism

$$U_3 \cong \mathbb{P}^{2n-5}\backslash\mathcal{V}(I^2_{n-2}) \cong \mathbb{P}^{2n-5}\backslash\mathcal{V}(I'^2_{n-2}), \tag{3.223}$$

where I'^2_{n-2} is the image of I^2_{n-2} under the change of the variables $C_{n-1} := A_{n+1}$. Note that I'^2_{n-2} is independent of A_1, thus *Theorem B*$(N = 2n-5$, $k = 0$, $t = 1)$ implies

$$H^i_c(Y_1\backslash\hat{Y}_2) \cong H^i_c(U_3) \cong H^i_c(\mathbb{P}^{2n-5}\backslash\mathcal{V}(I'^2_{n-2})) = 0, \;\; i \leq 2n-5. \tag{3.224}$$

and

$$H^{2n-4}_c(\mathbb{P}^{2n-5}\backslash\mathcal{V}(I'^2_{n-2})) \cong H^{2n-6}_c(\mathbb{P}^{2n-6}\backslash\mathcal{V}(I'^2_{n-2}))(-1) \tag{3.225}$$

The localization sequence for $\mathcal{V}(I'^2_{n-2}) \subset \mathbb{P}^{2n-6}$(no DV_5, B_0, A_0, B_1, A_1) gives us

$$H^{2n-6}_c(\mathbb{P}^{2n-5}\backslash\mathcal{V}(I'^2_{n-2})) \cong H^{2n-7}(\mathcal{V}(I'^2_{n-2})). \tag{3.226}$$

We know (see Theorem 3.2.4) that

$$H^{mid}_{prim}(\mathcal{V}(I'^2_{n-2})) = 0. \tag{3.227}$$

Thus,

$$H^{2n-4}_c(Y_1\backslash\hat{Y}_2) \cong H^{2n-4}_c(U_3) = H^{2n-6}_c(\mathbb{P}^{2n-6}\backslash\mathcal{V}(I'^2_{n-2}))(-1) = 0. \tag{3.228}$$

Using (3.224), (3.228) and the sequence (3.219), we get an isomorphism

$$H^{2n-5}(Y_1) \cong H^{2n-5}(\hat{Y}_2). \tag{3.229}$$

Now recall that in the case when $I^1_{n-1} = I^2_{n-2} = 0$, the polynomial $\widetilde{G}_{n-1}$ does not depend on A_0 (see Theorem 1.1.5). Thus, by (3.206),

$$\hat{Y}_2 := \mathcal{V}(I^1_{n-1}, \widetilde{G}_{n-1}, I^2_{n-2}) \cong \mathcal{V}(I^1_{n-1}, A_{n+1}I^1_{n-2}, I^2_{n-2})^{(2n-3)} \tag{3.230}$$

We project from the point where all the variables but A_0 vanish, and get

$$H^{2n-5}(Y_1) \cong H^{2n-5}(\hat{Y}_2) \cong H^{2n-7}(Y_2)(-1), \tag{3.231}$$

where $Y_2 \subset \mathbb{P}^{2n-4}(\text{no } DV_5,\ B_0,\ A_0)$ is defined by

$$Y_2 := \mathcal{V}(I^1_{n-1}, A_{n+1}I^1_{n-2}, I^2_{n-2}). \tag{3.232}$$

Define the subvarieties $Y_{21}, Y_{22} \subset Y_2 \subset \mathbb{P}^{2n-4}(\text{no } DV_5,\ B_0,\ A_0)$:

$$\begin{aligned} Y_{21} &:= \mathcal{V}(I^1_{n-1}, I^1_{n-2}, I^2_{n-2}), \\ Y_{22} &:= \mathcal{V}(I^1_{n-1}, A_{n+1}, I^2_{n-2}) \end{aligned} \tag{3.233}$$

with $Y_3 := Y_{21} \cap Y_{22}$. One has an exact sequence

$$\begin{aligned} \longrightarrow H^{2n-8}_{prim}(Y_{21}) \oplus H^{2n-8}_{prim}(Y_{22}) \longrightarrow H^{2n-8}_{prim}(Y_3) \longrightarrow \\ H^{2n-7}(Y_2) \longrightarrow H^{2n-7}(Y_{21}) \oplus H^{2n-7}(Y_{22}) \longrightarrow . \end{aligned} \tag{3.234}$$

Theorem $A(N = 2n-4,\ k = 3,\ t = 0)$ implies the vanishing of the two leftmost summands. Moreover, by (3.222),

$$Y_{22} := \mathcal{V}(I^1_{n-1}, A_{n+1}, I^2_{n-2}) = \mathcal{V}(A_{n+1}, I^2_{n-2}, A_1 I^3_{n-3}), \tag{3.235}$$

and the defining equations of Y_{22} are independent of B_1. Thus, by *Theorem* $A(N = 2n-4,\ k = 3,\ t = 1)$, we obtain

$$H^{2n-7}(Y_{22}) = 0. \tag{3.236}$$

Now, we change the variables $C_{n-1} := A_{n+1}$ and note that the variety Y'_{21}, the image of Y_{21} under this transformation is exactly the variety appeared in the proof of the WS_n case (was called Z_{n-1}, see Theorem 3.2.3). Thus

$$H^{2n-7}(Y_{21}) \cong \mathbb{Q}(0). \tag{3.237}$$

The sequence (3.234) simplifies to

$$0 \longrightarrow H^{2n-8}_{prim}(Y_3) \longrightarrow H^{2n-7}(Y_2) \longrightarrow \mathbb{Q}(0) \longrightarrow, \tag{3.238}$$

where

$$Y_3 := \mathcal{V}(A_{n+1}, I^1_{n-1}, I^1_{n-2}, I^2_{n-2}). \tag{3.239}$$

We change the notation and consider $Z \subset \mathbb{P}^{2n-5}(\text{no } DV_8)$ defined by

$$Z := \mathcal{V}(I^1_{n-1}, I^1_{n-2}, I^2_{n-2}), \tag{3.240}$$

where $DV_8 := DV_5 \cup \{B_0, A_0, A_{n+1}\}$. To abuse the notation, we write I^i_j for I^i_j after setting $A_{n+1} = 0$ (so, $C_{n-1} = A_{n-2}$). We are interested in

$$H^{2n-8}_{prim}(Z). \tag{3.241}$$

Define $Z_1, Z_2 \subset \mathbb{P}^{2n-5}(\text{no } DV_8)$ by

$$\begin{aligned} Z_1 &:= \mathcal{V}(I^1_{n-1}, I^1_{n-2}), \\ Z_2 &:= \mathcal{V}(I^1_{n-1}, I^2_{n-2}), \end{aligned} \tag{3.242}$$

then $Z = Z_1 \cap Z_2$. We write an exact sequence

$$\begin{aligned} &\longrightarrow H^{2n-8}_{prim}(Z_1) \oplus H^{2n-8}_{prim}(Z_2) \longrightarrow H^{2n-8}_{prim}(Z) \longrightarrow \\ &\qquad H^{2n-7}(\bar{Z}) \longrightarrow H^{2n-7}(Z_1) \oplus H^{2n-7}(Z_2) \longrightarrow, \end{aligned} \tag{3.243}$$

where $\bar{Z} := Z_1 \cup Z_2$. By Lemma 3.2.4, this sequence gives us an isomorphism

$$H^{2n-8}_{prim}(Z) \cong H^{2n-7}(\bar{Z}). \tag{3.244}$$

Using Corollary 1.1.2, we obtain

$$\bar{Z} := \mathcal{V}(I^1_{n-1}, I^1_{n-2} I^2_{n-2}) = \mathcal{V}(I^1_{n-1}, S_{n-2}). \tag{3.245}$$

One can easily compute $S_{n-2} = A_1 A_2 \dots A_{n-2}$. Define $Z_3, Z_4 \subset \bar{Z} \subset \mathbb{P}^{2n-5}$ by

$$\begin{aligned} Z_3 &:= \mathcal{V}(I^1_{n-2}, A_{n-2}), \\ Z_4 &:= \mathcal{V}(I^1_{n-2}, S_{n-3}) \end{aligned} \tag{3.246}$$

and $Z_5 := Z_3 \cap Z_4$. One has an exact sequence

$$\begin{aligned} &\longrightarrow H^{2n-8}(Z_5) \longrightarrow H^{2n-7}(\bar{Z}) \longrightarrow \\ &\qquad H^{2n-7}(Z_3) \oplus H^{2n-7}(Z_4) \longrightarrow H^{2n-7}(Z_5) \longrightarrow \end{aligned} \tag{3.247}$$

Since

$$I^1_{n-2} = C_{n-1} I^1_{n-2} - A_{n-2} I^1_{n-3} \tag{3.248}$$

with $C_{n-1} = A_{n-2}$,

$$Z_5 := Z_3 \cap Z_4 = \mathcal{V}(I^1_{n-2}, A_{n-2}, S_{n-3}) = \mathcal{V}(A_{n-2}, S_{n-3}). \tag{3.249}$$

The defining polynomials of Z_5 are independent of B_1 and B_2, *Theorem A*$(N = 2n-5,\ k = 2,\ t = 2)$ implies

$$H^i_{prim}(Z_5) = 0 \quad \text{for } \ i \leq 2n-6. \tag{3.250}$$

Similarly,

$$Z_3 := \mathcal{V}(I^1_{n-1}, A_{n-2}) = \mathcal{V}(A_{n-2}) \quad \text{and} \ \ H^{2n-7}(Z_3) = 0. \tag{3.251}$$

The sequence (3.247) now yields

$$H^{2n-7}(\bar{Z}) \cong H^{2n-7}(Z_4) \cong H^{2n-7}(\mathcal{V}(S_{n-3}, I^1_{n-1})). \tag{3.252}$$

From now the proof is very similar to that of Theorem 11.9 in [BEK]. We will analyse the spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0<\ldots<i_p} H^q\big(\mathcal{V}(A_{i_0}, \ldots, A_{i_p}, I^1_{n-1})\big) \Rightarrow H^{p+q}\big(\mathcal{V}(S_{n-3}, I^1_{n-1})\big). \tag{3.253}$$

First, we have to compute $H^q(\mathcal{V}(A_{i_0}, \ldots, A_{i_p}, I^1_{n-1}))$. For each j, $0 \leq j \leq p+1$, define $n_j := i_j - i_{j-1}$, where $i_{-1} := 0$ and $i_{p+2} : -1$. We have the partition $n-1 = \sum_0^{p+1} n_j$. Computing modulo the ideal $\mathcal{J}$ generated by $A_{i_0}, \ldots, A_{i_p}$, we can factor

$$I^1_{n-1} \equiv I^1_{n_0} I^{i_0+1}_{n_1} \ldots I^{i_{p-1}+1}_{n_p} I^{i_p+1}_{n_{p+1}} \mod \mathcal{J}. \tag{3.254}$$

Each $I^{i_{j-1}+1}_{n_j}$ is a homogeneous function of $\mathbb{P}^{2n_j-2}$ for $j < p+1$ and that of $\mathbb{P}^{2n_j-3}$ for $j = p+1$. If $n_j = 1$, $I^k_1 = B_k$ is a homogeneous function on $\mathbb{P}^0$.

Define linear spaces

$$L_j \subset \mathbb{P}^{2n-p-6}(A_1, \ldots, \widehat{A}_{i_0}, \ldots \widehat{A}_{i_p}, \ldots, A_{n-2}, B_1, \ldots, B_{n-2}) \tag{3.255}$$

and cone maps $\pi : \mathbb{P}^{2n-p-6} \backslash L_j \longrightarrow \mathbb{P}^{2n_j-2}$ for $0 \leq j \leq p$ and $\pi : \mathbb{P}^{2n-p-6} \backslash L_{p+1} \longrightarrow \mathbb{P}^{2n_j-3}$. Then

$$\mathcal{V}(A_{i_0}, \ldots, A_{t_{p+1}}, I^1_{n-1}) = \bigcup \pi_j^{-1}(\mathcal{V}(I^{i_{j-1}+1}_{n_j})) \tag{3.256}$$

In the case $n_j = 1$, $j \leq p$, we have simply $\mathcal{V}(I^{i_{j-1}+1}_{n_j}) \cong L_j$. Set

$$U_j := \mathbb{P}^{2n_j-2} \backslash \mathcal{V}(I^{i_{j-1}+1}_{n_j}) \tag{3.257}$$

for $0 \leq j \leq p$ and $U_{p+1} : \mathbb{P}^{2n_j-3} \backslash \mathcal{V}(I^{i_p+1}_{n_{p+1}})$. Define

$$U := \mathbb{P}^{2n-p-6} \backslash \bigcup_{j=0}^{p+1} \pi^{-1}(\mathcal{V}(I^{i_{j-1}+1}_{n_j})). \tag{3.258}$$

It is easy to see that the map $\prod \pi_j : U \to \prod U_j$ is a $\mathbb{G}_m^{p+1}$-bundle. Using Künneth formula, we get

$$H_c^*\Big(\mathbb{P}^{2n-p-6}\backslash\mathcal{V}(A_{i_0},\ldots,A_{i_p},I_{n-1}^1)\Big) = H_c^*(U) \cong H_c^*(\mathbb{G}_m^{p+1}) \otimes \bigotimes_{j=0}^{p+1} H_c^*(U_j). \tag{3.259}$$

Consider the case when $n_j \geq 1$ for some j, $1 \leq j \leq p$, or $n_{p+1} > 2$. Then, the cohomology groups $H_c^*(U)$ vanish in degrees less than or equal to

$$p+1+\sum_{j=0}^{p}(2n_j-2)+2n_{p+1}-3 = 2n-p-6. \tag{3.260}$$

This implies

$$H^i_{prim}(\mathcal{V}(A_{i_0},\ldots,A_{i_p},I_{n-1}^1)) = 0 \quad \text{for } i \leq 2n-p-7. \tag{3.261}$$

The exceptional case is when $n_j = 1$, $j \leq p$ and $n_{p+1} = 2$. In this case $p-4$, $U \cong \mathbb{G}_m^{p+1}$ and $H^{n-2}(U) \neq 0$ contrary to (3.260). We get

$$E_1^{n-4,q} = H^q(\mathcal{V}(A_1,\ldots,A_{n-3},I_{n-1}^1)) = H^q(\mathcal{V}(\prod_{j=1}^{n-3} B_jB_{n-2}A_{n-2})) \tag{3.262}$$

(here, for I_2^{n-3}, we used the change of variables $B_{n-2} := B_{n-2}-A_{n-2}$). Stratifying $\mathcal{V}(B_1\ldots B_{n-2}A_{n-2})$, using Mayer-Vietoris's sequence and induction, it is easy to compute $(E_1^{n-4,n-3})_{prim} = \mathbb{Q}(0)$, and $E_2^{p,q} = 0$ for $p+q = 2n-7$ and $1 \leq p \leq n-5$. Moreover,

$$E_2^{0,2n-7} = \ker\Big(\bigoplus_{i=1}^{n-3} H^{2n-7}(\mathcal{V}(A_i,I_{n-1}^1)) \longrightarrow \bigoplus_{i_1,i_2} H^{2n-7}(\mathcal{V}(A_{i_1},A_{i_2},I_{n-1}^1))\Big) \tag{3.263}$$

By (3.260), $E_2^{0,2n-7} = 0$ as well. Now, consider the sequences

$$E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \tag{3.264}$$

for $r \geq 2$ and $p+q = 2n-7$. The group to the left vanishes by (3.260), the group in the middle vanishes for $p \neq n-4$. For $p-4$ the group to the right vanishes because we have only $n-3$ components. Thus, $E_{r+1}^{p,q} = E_r^{p,q} = E_\infty^{p,q}$. Finally, we get

$$H^{2n-7}(Z_4) = \mathbb{Q}(0). \tag{3.265}$$

By (3.239), (3.240), (3.244) and (3.247), the sequence (3.238) takes the form

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow H^{2n-7}(Y_2) \longrightarrow \mathbb{Q}(0) \longrightarrow, \tag{3.266}$$

Together with (3.217) and (3.231), this gives us the exact sequence

$$0 \longrightarrow \mathbb{Q}(-2) \longrightarrow H^{2n-3}(Y) \longrightarrow \mathbb{Q}(-2) \longrightarrow . \tag{3.267}$$

Consequently,

$$H^{2n-3}(Y) \cong \mathbb{Q}(-2)^{\oplus i} \tag{3.268}$$

for $i = 1$ or $i = 2$. We return to the sequence (3.202), and using (3.204), we get

$$0 \longrightarrow \mathbb{Q}(-1) \longrightarrow H^{2n-3}(V_3 \backslash V_{31}) \longrightarrow H^{2n-3}(Y) \longrightarrow . \tag{3.269}$$

By (3.194) and (3.199), we can rewrite this sequence

$$0 \longrightarrow \mathbb{Q}(-1) \longrightarrow H^{2n-2}_{prim}(V) \longrightarrow H^{2n-3}(Y) \longrightarrow . \tag{3.270}$$

From this, one can describe $H^{2n-2}_{prim}(V)$:

$$\mathrm{gr}\,{}_2^W(H^{2n-2}_{prim}(V)) = \mathbb{Q}(-1), \;\; \mathrm{gr}\,{}_4^W(H^{2n-2}_{prim}(V)) = \mathbb{Q}(-2)^{\oplus j} \tag{3.271}$$

and all other gr^i_W are zero. Here $0 \leq j \leq i$, thus j equals 0, 1 or 2. Now, using the exact sequence (3.179) we get finally

$$\mathrm{gr}\,{}_6^W(H^{2n-2}_{prim}(X)) = \mathbb{Q}(-3) \quad \text{and} \;\; \mathrm{gr}\,{}_8^W(H^{2n-2}_{prim}(X)) = \mathbb{Q}(-4)^{\oplus d}, \tag{3.272}$$

where $d = 0$, 1 or 2, and all other $gr_i^W = 0$.

□

We are almost sure that we always have $\mathrm{gr}\,{}^W_{min}(X) = \mathbb{Q}(-3)$ for X being a graph hypersurface of $WS_n \times WS_m$, $n, m \geq 4$, but we are not sure that this can be done with our technique even in the case of $WS_4 \times WS_4$.

It is very interesting to understand, what can the (first nontrivial weight piece of) middle dimensional cohomology for gluings of primitively log divergent graphs be in general. Consider the very simple series of examples of such gluings, namely the WS's glued "in a strip". Fix some $m > 2$ and choose m graphs $\Delta_1, \dots, \Delta_m$, where $\Delta_i \cong WS_{n_i}$ for $n_i \geq 3$, $1 \leq i \leq m$. As in Theorem 2.2.5 , a-edges are defined to be the "spokes" while the b-edges are the other (boundary) edges. Define $\Gamma_1 := \Delta_1$ and $\Gamma_{k+1} := \Gamma_k \times \Delta_{k+1}$, the gluing of the graphs Γ_k and Δ_{k+1} along a b-edge of Γ_k that belongs to Δ_k and some b-edge of Δ_{k+1}, $k \leq m$. For any graph Γ from this series we hope that the minimal weight piece is still of Tate type

$$\mathrm{gr}^W_{min}(X_\Gamma) \cong \mathbb{Q}(-m-1)^{\oplus d} \tag{3.273}$$

for some $d = d(\Gamma) \geq 1$.

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