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A METHOD OF GENERATING INTEGRAL RELATIONS BY THE SIMULTANEOUS SEPARABILITY OF GENERALIZED SCHRÖDINGER EQUATIONS*

DIETER SCHMIDT† AND GERHARD WOLF†

Abstract. One of the most important methods in the theory of special functions of mathematical physics is that of generating integral relations for these functions by the simultaneous separability of the 3-dimensional wave equation in different orthogonal coordinate systems. In the present paper it will be shown that a consequent application of this principle of simultaneous separability to more general partial differential equations and higher dimensions yields various types of integral relations for the solutions of a wide class of ordinary differential equations which especially contains all second-order equations of Fuchsian type.

Introduction. Let D be a domain (nonvoid open connected set) in the k-dimensional complex vector space $\mathbb{C}^k$ with $\mathbb{N} \ni k \geqq 2$ and let $p = (p_\kappa)\colon D \to \mathbb{C}^k$ and $q\colon D \to \mathbb{C}$ be analytic functions. In this paper we consider the generalized Schrödinger equation

$$Aw := \Delta w + p(x)^t \cdot \operatorname{grad} w + q(x)w = 0, \tag{1}$$

where Δ denotes the Laplace operator; grad, the gradient and $p(x)^t$ the transpose of $p(x)$.

In § 1 we introduce several orthogonal curvilinear coordinate systems, namely ellipsoidal, sphero-conal and special forms of spherical and rectangular coordinates. We give the representations of the operator A in terms of these coordinates, which directly imply sufficient conditions for separability. The most interesting result of § 1 is that the "special" Schrödinger operator A with coefficients

$$\begin{aligned} p_\kappa(x) &= \alpha x_\kappa + \beta_\kappa / x_\kappa, \qquad \kappa = 1, \cdots, k, \\ q(x) &= \gamma \cdot \sum_{\kappa=1}^{k} x_\kappa^2 + \delta + \sum_{\kappa=1}^{k} (\varepsilon_\kappa / x_\kappa^2), \end{aligned} \tag{2}$$

where α, β_κ, γ, δ, ε_κ are complex parameters, separates simultaneously in all four coordinate systems specified above and that its separation yields a wide class of ordinary differential equations especially containing all second-order equations of Fuchsian type and some of their confluent forms.

Since the results of § 1 can be readily verified, we have merely stated the facts and omitted all the proofs. The proofs are essentially the same as in the 3-dimensional case and can be carried out by direct computation or, more elegantly, by the use of Lie theory ([12], [18], [5], [10]). It also can be shown that the sufficient conditions for separability and simultaneous separability stated in § 1 are necessary, too.

In the first part of § 2 we establish a general principle to obtain $(k-1)$-linear integral relations for the solutions of k ordinary linear differential equations occurring with the separation of a k-dimensional partial differential operator and which are thus linked by $k-1$ separation parameters. Such theorems, in a more or less abstract formulation, are well known in multiparameter eigenvalue theory ([1], [2], [16]). We have restricted ourselves to a special formulation which enables us to meet the various situations of § 1. Furthermore, we have restricted our formulation of the integral relations only to proper integrals, since, in this paper, we particularly want to point out the more formal aspects of the method. The corresponding relations with improper integrals can be obtained in the same way.

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† Fachbereich Mathematik, Universität Essen–Gesamthochschule, 4300 Essen 1, West Germany.

In the second part of § 2 the most important applications of the above stated principle to different situations of § 1 are discussed. Of course, one can always apply the above stated principle whenever our general Schrödinger operator separates in one coordinate system; however, then there is the problem of finding suitable solutions of the partial differential equation which can serve as kernels. In the case of our special Schrödinger operator this problem can be solved due to its simultaneous separability. Separation in one coordinate system yields product solutions in terms of these variables, which then may serve as nontrivial kernels for integral relations obtained by separation in another coordinate system. This method yields various types of integral relations for the solutions of the special ordinary differential equations (14), (22.1), (22.2), (29.1), (29.2), and (34). Only the most interesting cases, especially those which lead to new types of integral relations, are discussed here.

Explicit examples and applications of our integral relations, especially with regard to special functions of mathematical physics, will be treated in a later paper.

The present paper was stimulated by a series of papers of Leitner and Meixner [7], [8], [9] as well as by the papers of Erdélyi [4] and Sleeman [15].

In [7], [8], [9] Leitner and Meixner made an approach to a unifying concept for generating integral relations for the special functions of mathematical physics by studying the simultaneous separability of the 3-dimensional Schrödinger equation (1) with $p=0$. This concept was carried on in the thesis of Turner [17], which was initiated by Leitner. Their investigations were restricted to those pairs of coordinate systems which share a common coordinate to be separated out. Hence, their integral relations were linear.

In earlier papers Lambe and Ward [6] and Erdélyi [4] obtained linear integral relations and equations for Heun polynomials and Heun functions by the simultaneous separability of the 3-dimensional special Schrödinger equation (1) where $q=0$ and p is given in (2) with $\alpha=0$ in terms of sphero-conal and spherical coordinates, which share the common coordinate r. Later on, Sleeman [15] obtained quadratic integral relations and equations for the solutions of the Heun equation by the simultaneous separability of the same Schrödinger equation in terms of ellipsoidal and spherical coordinates.

1. Separability of Schrödinger equations in k-dimensional orthogonal coordinate systems.

1.1. General orthogonal coordinates. Let G be a domain in the k-dimensional complex vector space $\mathbb{C}^k$ with $\mathbb{N}\ni k\geqq 2$ and

$$\mathbb{C}^k\supset G\ni z=(z_\kappa)\mapsto x=\phi(z)=(\phi_\kappa(z))\in\mathbb{C}^k$$

be an analytic transformation. We call ϕ "orthogonal" if

$$\sum_{\kappa=1}^{k}\frac{\partial\phi_\kappa}{\partial z_\rho}\cdot\frac{\partial\phi_\kappa}{\partial z_\sigma}=\delta_{\rho\sigma}\cdot g_\rho,\qquad \rho,\sigma\in\{1,\cdots,k\},\tag{3}$$

where $\delta_{\rho\sigma}$ denotes the Kronecker symbol and the g_ρ are analytic functions satisfying

$$g_\rho(z)\neq 0,\qquad z\in G;\quad \rho=1,\cdots,k.\tag{4}$$

If $w:D\to\mathbb{C}$ is an analytic function with domain $D\subset\mathbb{C}^k$ and $\tilde{w}:=w\circ\phi$, our Schrödinger operator A in terms of the new variable $z=(z_1,\cdots,z_k)$ becomes

$$\begin{aligned}Aw=\tilde{A}\tilde{w}&:=\sum_{\kappa=1}^{k}\frac{1}{g_\kappa}[\cdots]+(q\circ\phi)\cdot\tilde{w},\\ [\cdots]&=\frac{\partial^2\tilde{w}}{\partial z_\kappa^2}+\frac{1}{2}\left(\sum_{\substack{\rho=1\\ \rho\neq\kappa}}^{k}\frac{1}{g_\rho}\frac{\partial g_\rho}{\partial z_\kappa}-\frac{1}{g_\kappa}\frac{\partial g_\kappa}{\partial z_\kappa}+\varphi_\kappa\right)\frac{\partial\tilde{w}}{\partial z_\kappa},\end{aligned}\tag{5}$$

where

$$\varphi_\kappa = 2 \cdot \sum_{\rho=1}^{k} \frac{\partial \phi_\rho}{\partial z_\kappa} \cdot (p_\rho \circ \phi), \qquad \kappa = 1, \cdots, k. \tag{5.1}$$

On the other hand, the p_ρ are expressible in terms of the φ_κ. The orthogonality relation (3) directly yields

$$p_\kappa \circ \phi = \frac{1}{2} \sum_{\rho=1}^{k} \frac{1}{g_\rho} \cdot \frac{\partial \phi_\kappa}{\partial z_\rho} \cdot \varphi_\rho, \qquad \kappa = 1, \cdots, k. \tag{5.2}$$

1.2. Ellipsoidal coordinates. Let $a = (a_1, \cdots, a_k) \in \mathbb{C}^k$ be a fixed vector with $a_\kappa \neq a_\rho$ $(\kappa \neq \rho)$. Ellipsoidal coordinates $\xi = (\xi_1, \cdots, \xi_k)$, which are related to rectangular coordinates $x = (x_1, \cdots, x_k)$ by

$$\sum_{\kappa=1}^{k} \frac{x_\kappa^2}{\xi_\rho - a_\kappa} = 1, \qquad \rho = 1, \cdots, k, \tag{6}$$

can be introduced by

$$\mathbb{C}^k \supset G \ni \xi \mapsto x = \phi(\xi) = (\phi_\kappa(\xi)) \in \mathbb{C}^k$$

where $G \subset (\mathbb{C} \backslash \{a_1, \cdots, a_k\})^k$ is a domain and the ϕ_κ are analytic functions with

$$\phi_\kappa(\xi)^2 = (\xi_\kappa - a_\kappa) \cdot \prod_{\substack{\rho=1 \\ \rho \neq \kappa}}^{k} \left(\frac{\xi_\rho - a_\kappa}{a_\rho - a_\kappa} \right), \qquad \kappa = 1, \cdots, k. \tag{7}$$

At each point $\xi \in G$ with $\xi_\rho \neq \xi_\sigma$ $(\rho \neq \sigma)$, ϕ satisfies the orthogonality relations (3) with

$$g_\kappa(\xi) = \tfrac{1}{4} f(\xi_\kappa ; a)^{-1} \cdot \prod_{\substack{\rho=1 \\ \rho \neq \kappa}}^{k} (\xi_\kappa - \xi_\rho), \tag{8}$$

where

$$f(t; \xi) := \prod_{\rho=1}^{k} (t - \xi_\rho). \tag{9}$$

We now introduce the determinant

$$P_k(\xi) := \det \begin{bmatrix} \xi_1^{k-1} & \cdots & \xi_k^{k-1} \\ \vdots & & \vdots \\ \xi_1 & \cdots & \xi_k \\ 1 & \cdots & 1 \end{bmatrix} = \prod_{1 \leqq \sigma < \rho \leqq k} (\xi_\sigma - \xi_\rho) \tag{10}$$

and assume for the following

$$G \subset \{\xi : P_k(\xi) \neq 0\}.$$

Furthermore, let $\overset{\kappa}{\xi}$ denote the $(k-1)$-dimensional vector $(\xi_1, \cdots, \xi_{\kappa-1}, \xi_{\kappa+1}, \cdots, \xi_k)$. Then by (5) the representation of our Schrödinger operator A in ellipsoidal coordinates becomes

$$\tag{11} \begin{aligned} \tilde{A} &= \frac{1}{P_k(\xi)} \sum_{\kappa=1}^{k} (-1)^{\kappa-1} P_{k-1}(\overset{\kappa}{\xi}) \cdot \tilde{A}_\kappa, \\ \tilde{A}_\kappa &= 4 f(\xi_\kappa ; a) \cdot \left[\frac{\partial^2}{\partial \xi_\kappa^2} + \frac{1}{2} \left(\sum_{\rho=1}^{k} \frac{1}{\xi_\kappa - a_\rho} + \varphi_\kappa \right) \frac{\partial}{\partial \xi_\kappa} \right] + \psi_\kappa, \end{aligned}$$

where

$$\varphi_\kappa = \sum_{\rho=1}^{k} \frac{\phi_\rho \cdot (p_\rho \circ \phi)}{\xi_\kappa - a_\rho} \tag{11.1}$$

and ψ_κ are any functions with

$$q \circ \phi = \frac{1}{P_k(\xi)} \cdot \sum_{\kappa=1}^{k} (-1)^{\kappa-1} P_{k-1}(\overset{\kappa}{\xi}) \cdot \psi_\kappa. \tag{11.2}$$

If φ_κ and ψ_κ depend only on the variable ξ_κ, then $\tilde{A}$ may be written in the form

$$\tilde{A} = \frac{1}{P_k(\xi)} \cdot \det \begin{bmatrix} \tilde{A}_1 & \cdots & \tilde{A}_k \\ \xi_1^{k-2} & & \xi_k^{k-2} \\ \vdots & & \vdots \\ \xi_1 & \cdots & \xi_k \\ 1 & \cdots & 1 \end{bmatrix} \tag{12}$$

where $\tilde{A}_\kappa$ is an ordinary differential operator with respect to ξ_κ, which just means that $P_k(\xi) \cdot \tilde{A}$ is separable ([13], [12]). We say: "A separates in ellipsoidal coordinates".

Especially, if p and q have the form (2), we obtain from (11.1)

$$\varphi_\kappa = \alpha + \sum_{\rho=1}^{k} \frac{\beta_\rho}{\xi_\kappa - a_\rho}. \tag{13.1}$$

If we choose

$$\psi_\kappa = \gamma \xi_\kappa^k + \left(\delta - \gamma \cdot \sum_{\rho=1}^{k} a_\rho\right) \cdot \xi_\kappa^{k-1} + \sum_{\rho=1}^{k} \frac{\varepsilon_\rho}{\xi_\kappa - a_\rho} \cdot \prod_{\substack{\sigma=1 \\ \sigma \neq \rho}}^{k} (a_\rho - a_\sigma), \tag{13.2}$$

also (11.2) is satisfied. Therefore, our special Schrödinger operator separates in ellipsoidal coordinates. Now, using well-known facts on separated solutions of separable operators, we can establish the following

PROPOSITION 1. *For $\kappa = 1, \cdots, k$ let $v_\kappa: G_\kappa \to \mathbb{C}$ be analytic with domain $G_\kappa \subset \mathbb{C}\backslash\{a_1, \cdots, a_k\}$, such that $G \subset \times_{\kappa=1}^{k} G_\kappa$. Furthermore, let $w: D \to \mathbb{C}$ be analytic with domain $D \subset \mathbb{C}^k$, such that $\phi(G) \subset D$. Finally, let $w \neq 0$ and*

$$(w \circ \phi)(\xi) = \prod_{\kappa=1}^{k} v_\kappa(\xi_\kappa).$$

Then w is a solution of our special Schrödinger equation

$$Aw = 0,$$

iff there exist separation constants $(\lambda_0, \cdots, \lambda_{k-2}) \in \mathbb{C}^{k-1}$, such that the v_κ, $(\kappa = 1, \cdots, k)$, are solutions of the ordinary differential equation

$$\begin{aligned} &\prod_{\rho=1}^{k} (z - a_\rho) \cdot \left[v'' + \frac{1}{2}\left(\alpha + \sum_{\rho=1}^{k} \frac{1+\beta_\rho}{z - a_\rho}\right) v' \right] \\ &\quad + \frac{1}{4}\left(\sum_{\rho=1}^{k} \frac{\varepsilon_\rho}{z - a_\rho} \prod_{\substack{\sigma=1 \\ \sigma \neq \rho}}^{k} (a_\rho - a_\sigma) + \gamma z^k + \delta' z^{k-1} + \sum_{\rho=0}^{k-2} \lambda_\rho z^\rho \right) v = 0 \end{aligned} \tag{14}$$

where $\delta' = \delta - \gamma \cdot \sum_{\rho=1}^{k} a_\rho$.

The differential equation (14) has the only singular points $a_1, \cdots, a_k$ and ∞. The finite points a_κ are regular singular points with characteristic exponents ν_κ^1, ν_κ^2 determined by

$$\nu_\kappa^1+\nu_\kappa^2=\tfrac{1}{2}(1-\beta_\kappa), \qquad \nu_\kappa^1\cdot\nu_\kappa^2=\tfrac{1}{4}\varepsilon_\kappa, \qquad \kappa=1,\cdots,k. \tag{15.1}$$

If $\alpha=\gamma=\delta=0$, the point ∞ is also a regular singular point and the differential equation is the most general second-order equation of Fuchsian type with $k+1$ singularities [14, p. 136]. The characteristic exponents ν_∞^1, ν_∞^2 of the point ∞ are then determined by

$$\nu_\infty^1+\nu_\infty^2=\frac{1}{2}\cdot\sum_{\kappa=1}^{k}(1+\beta_\kappa)-1, \qquad \nu_\infty^1\nu_\infty^2=\frac{1}{4}\lambda_{k-2}. \tag{15.2}$$

The remaining $k-2$ separation constants $\lambda_0,\cdots,\lambda_{k-3}$ are the so-called "accessory parameters".

We would mention that in the case $k=2$ and $\alpha=\varepsilon_\kappa=0$ equation (14) is a confluent form of the Heun equation [6]. Thus, special cases are the Mathieu equation as well as the spheroidal wave equation. In the case $k=3$ and $\alpha=\gamma=\delta=\varepsilon_\kappa=0$, (14) is the Heun equation ([4], [15]).

1.3. Sphero-conal coordinates. As in the case of ellipsoidal coordinates let $a=(a_1,\cdots,a_k)\in\mathbb{C}^k$ be a vector with $a_\kappa\neq a_\rho$ $(\kappa\neq\rho)$. Sphero-conal coordinates $\zeta=(\zeta_1,\cdots,\zeta_k)$, which are related to rectangular coordinates $x=(x_1,\cdots,x_k)$ by

$$\zeta_1=\sum_{\kappa=1}^{k}x_\kappa^2, \qquad \sum_{\kappa=1}^{k}\frac{x_\kappa^2}{\zeta_\rho-a_\kappa}=0, \qquad \rho=2,\cdots,k, \tag{16}$$

can be introduced by

$$\mathbb{C}^k\supset G\ni\zeta\mapsto x=\phi(\zeta)=(\phi_\kappa(\zeta))\in\mathbb{C}^k,$$

where $G\subset(\mathbb{C}\backslash\{0\})\times(\mathbb{C}\backslash\{a_1,\cdots,a_k\})^{k-1}$ is a domain and the ϕ_κ are analytic functions with

$$\phi_\kappa(\zeta)^2=\zeta_1\cdot\prod_{\rho=1}^{\kappa-1}\left(\frac{\zeta_{\rho+1}-a_\kappa}{a_\rho-a_\kappa}\right)\cdot\prod_{\rho=\kappa+1}^{k}\left(\frac{\zeta_\rho-a_\kappa}{a_\rho-a_\kappa}\right), \qquad \kappa=1,\cdots,k. \tag{17}$$

Using the notations of (9) we find that ϕ satisfies the orthogonality relations (3) with

$$\begin{aligned} g_1(\zeta)&=\frac{1}{4}\frac{1}{\zeta_1},\\ g_\kappa(\zeta)&=-\frac{1}{4}f(\zeta_\kappa;a)^{-1}\cdot\zeta_1\cdot\prod_{\substack{\rho=2\\ \rho\neq\kappa}}^{k}(\zeta_\kappa-\zeta_\rho), \qquad \kappa=2,\cdots,k, \end{aligned} \tag{18}$$

at each point ζ of a region $G\subset\{\zeta:P_{k-1}(\overset{1}{\check{\zeta}})\neq0\}$.

Now by (5) the representation of our Schrödinger operator A in terms of sphero-conal coordinates becomes

$$\begin{aligned} \tilde{A}&=\tilde{A}_1+(\zeta_1\cdot P_{k-1}(\overset{1}{\check{\zeta}}))^{-1}\cdot\sum_{\kappa=2}^{k}(-1)^{\kappa-1}P_{k-2}(\overset{1,\kappa}{\check{\zeta}})\tilde{A}_\kappa,\\ \tilde{A}_1&=4\zeta_1\frac{\partial^2}{\partial\zeta_1^2}+2(k+\varphi_1)\frac{\partial}{\partial\zeta_1}+\psi_1,\\ \tilde{A}_\kappa&=4f(\zeta_\kappa;a)\left[\frac{\partial^2}{\partial\zeta_\kappa^2}+\frac{1}{2}\left(\sum_{\rho=1}^{k}\frac{1}{\zeta_\kappa-a_\rho}+\varphi_\kappa\right)\frac{\partial}{\partial\zeta_\kappa}\right]+\psi_\kappa \qquad \kappa=2,\cdots,k, \end{aligned} \tag{19}$$

where

$$(19.1)\qquad \varphi_1=\sum_{\rho=1}^{k}\phi_\rho\cdot(p_\rho\circ\phi),\qquad \varphi_\kappa=\sum_{\rho=1}^{k}\frac{\phi_\rho\cdot(p_\rho\circ\phi)}{\zeta_\kappa-a_\rho},\qquad \kappa=2,\cdots,k,$$

and ψ_κ are any functions with

$$(19.2)\qquad q\circ\phi=\psi_1+(\zeta_1\cdot P_{k-1}(\overset{1}{\check{\zeta}}))^{-1}\sum_{\kappa=2}^{k}(-1)^{\kappa-1}P_{k-2}(\overset{1,\kappa}{\check{\zeta}})\cdot\psi_\kappa.$$

If φ_κ and ψ_κ depend only on the variable ζ_κ, then $\tilde{A}$ may be written in the form

$$(20)\qquad \tilde{A}=\frac{1}{P_{k-1}(\overset{1}{\check{\zeta}})}\det\begin{bmatrix}\tilde{A}_1 & \tilde{A}_2 & \cdots & \tilde{A}_k\\ 1/\zeta_1 & \zeta_2^{k-2} & & \zeta_k^{k-2}\\ 0 & \vdots & & \vdots\\ \vdots & \zeta_2 & & \zeta_k\\ 0 & 1 & & 1\end{bmatrix}$$

where $\tilde{A}_\kappa$ is an ordinary differential operator with respect to ζ_κ. Therefore, $P_{k-1}(\overset{1}{\zeta})\cdot\tilde{A}$ is separable and we say: "A separates in sphero-conal coordinates".

In the case of our special Schrödinger operator with coefficients p and q of the form (2) we obtain from (19.1)

$$(21.1)\qquad \varphi_1=\alpha\zeta_1+\beta,\qquad \varphi_\kappa=\sum_{\rho=1}^{k}\frac{\beta_\rho}{\zeta_k-a_\rho},\qquad \kappa=2,\cdots,k,$$

where $\beta=\sum_{\rho=1}^{k}\beta_\rho$. If we choose

$$(21.2)\qquad \psi_1=\gamma\zeta_1+\delta,\qquad \psi_\kappa=\sum_{\rho=1}^{k}\frac{\varepsilon_\rho}{\zeta_\kappa-a_\rho}\prod_{\substack{\sigma=1\\ \sigma\neq\rho}}^{k}(a_\rho-a_\sigma),\qquad \kappa=2,\cdots,k,$$

(19.2) also is satisfied. Thus, our special Schrödinger operator also separates in sphero-conal coordinates. Hence, we can establish the following

PROPOSITION 2. *Let $v_\kappa: G_\kappa\to\mathbb{C}$ $(\kappa=1,\cdots,k)$ be analytic with domains $G_1\subset\mathbb{C}\setminus\{0\}$ and $G_\kappa\subset\mathbb{C}\setminus\{a_1,\cdots,a_k\}$ $(\kappa=2,\cdots,k)$, such that $G\subset\times_{\kappa=1}^{k}G_\kappa$. Further, let $w: D\to\mathbb{C}$ be analytic with domain $D\subset\mathbb{C}^k$, such that $\phi(G)\subset D$. Finally, let $w\neq 0$ and*

$$(w\circ\phi)(\zeta)=\prod_{\kappa=1}^{k}v_\kappa(\zeta_\kappa).$$

Then w is a solution of our special Schrödinger equation

$$Aw=0,$$

iff there exist separation constants $(\lambda_0,\cdots,\lambda_{\kappa-2})\in\mathbb{C}^{k-1}$, such that v_1 is a solution of

$$(22.1)\qquad zv''+\frac{1}{2}(\alpha z+(k+\beta))v'+\frac{1}{4}\left(\gamma z+\delta+\frac{\lambda_{k-2}}{z}\right)v=0$$

where $\beta=\sum_{\kappa=1}^{k}\beta_\kappa$, and the v_κ $(\kappa=2,\cdots,k)$ are solutions of

$$(22.2)\qquad \prod_{\rho=1}^{k}(z-a_\rho)\left[v''+\frac{1}{2}\left(\sum_{\rho=1}^{k}\frac{1+\beta_\rho}{z-a_\rho}\right)v'\right]+\frac{1}{4}\left(\sum_{\rho=1}^{k}\frac{\varepsilon_\rho}{z-a_\rho}\prod_{\substack{\sigma=1\\ \sigma\neq\rho}}^{k}(a_\rho-a_\sigma)+\sum_{\rho=0}^{k-2}\lambda_\rho z^\rho\right)v=0.$$

The differential equation (22.1) has the only singularities 0 and ∞. The point 0 is a regular singular point with characteristic exponents μ_1^1, μ_1^2 determined by

$$\mu_1^1+\mu_1^2=1-\frac{1}{2}\sum_{\kappa=1}^{k}(1+\beta_\kappa),\qquad \mu_1^1\mu_1^2=\frac{1}{4}\lambda_{k-2}. \tag{23}$$

If $\alpha=\gamma=\delta=0$, the point ∞ is also a regular singular point and (22.1) is an Euler equation. Equation (22.1) can always be integrated by confluent hypergeometric functions.

The differential equation (22.2) is the general second-order equation of Fuchsian type already obtained in (14) in the case $\alpha=\gamma=\delta=0$.

1.4. Spherical and rectangular coordinates. Since we want the ordinary differential equations obtained by separation to be in an appropriate "normal form", we use in this paper an algebraic form of spherical coordinates $\eta=(\eta_1,\cdots,\eta_k)$, which are related to rectangular coordinates $x=(x_1,\cdots,x_k)$ by

$$\eta_1=\sum_{\rho=1}^{k}x_\rho^2,\qquad \frac{x_{\kappa-1}^2}{\eta_\kappa}+\frac{\sum_{\rho=\kappa}^{k}x_\rho^2}{\eta_\kappa-1}=0,\qquad \kappa=2,\cdots,k. \tag{24}$$

These can be introduced by

$$\mathbb{C}^k\supset G=\mathop{\times}_{\kappa=1}^{k}G_\kappa\ni\eta\mapsto x=\phi(\eta)=(\phi_\kappa(\eta))\in\mathbb{C}^k,$$

where $G_1\subset\mathbb{C}\backslash\{0\}$, $G_\kappa\subset\mathbb{C}\backslash\{0,1\}$ $(\kappa=2,\cdots,k)$ are domains and the ϕ_κ are analytic functions with

$$\begin{aligned}&\phi_1(\eta)^2=\eta_1\cdot\eta_2,\\ &\phi_\kappa(\eta)^2=\eta_1\cdot\eta_{\kappa+1}\cdot\prod_{\rho=2}^{\kappa}(1-\eta_\rho),\qquad 2\leqq\kappa\leqq k-1,\\ &\phi_k(\eta)^2=\eta_1\cdot\prod_{\rho=2}^{k}(1-\eta_\rho).\end{aligned} \tag{25}$$

From (5) we find, that our general Schrödinger operator A in terms of spherical coordinates has the form

$$\begin{aligned}&\tilde A=\tilde A_1+\frac{1}{\eta_1}\sum_{\kappa=2}^{k}(-1)^{\kappa-1}\cdot\Bigl(\prod_{\rho=2}^{\kappa-1}(\eta_\rho-1)\Bigr)^{-1}\cdot\tilde A_\kappa,\\ &\tilde A_1=4\eta_1\frac{\partial^2}{\partial\eta_1^2}+2(k+\varphi_1)\frac{\partial}{\partial\eta_1}+\psi_1,\\ &\tilde A_\kappa=4\eta_\kappa(\eta_\kappa-1)\Bigl[\frac{\partial^2}{\partial\eta_\kappa^2}+\frac{1}{2}\Bigl(\frac{1}{\eta_\kappa}+\frac{k+1-\kappa}{\eta_\kappa-1}+\varphi_\kappa\Bigr)\frac{\partial}{\partial\eta_\kappa}\Bigr]+\psi_\kappa,\qquad \kappa=2,\cdots,k,\end{aligned} \tag{26}$$

where

$$\begin{aligned}&\varphi_1=\sum_{\rho=1}^{k}\phi_\rho\cdot(p_\rho\circ\phi),\\ &\varphi_\kappa=\frac{1}{\eta_\kappa}\cdot\phi_{\kappa-1}\cdot(p_{\kappa-1}\circ\phi)+\frac{1}{\eta_\kappa-1}\cdot\sum_{\rho=\kappa}^{k}\phi_\rho\cdot(p_\rho\circ\phi),\qquad \kappa=2,\cdots,k,\end{aligned} \tag{26.1}$$

and ψ_κ are any functions with

$$q\circ\phi=\psi_1+\frac{1}{\eta_1}\cdot\sum_{\kappa=2}^{k}(-1)^{\kappa-1}\Bigl(\prod_{\rho=2}^{\kappa-1}(\eta_\rho-1)\Bigr)^{-1}\cdot\psi_\kappa. \tag{26.2}$$

If φ_κ and ψ_κ depend only on the variable η_κ, then $\tilde{A}$ is separable and may be written in the form

$$\tilde{A}=\det\begin{bmatrix}\tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \cdots & \tilde{A}_{k-1} & \tilde{A}_k\\ \dfrac{1}{\eta_1} & 1 & 0 & \cdots & 0 & 0\\ 0 & \dfrac{1}{\eta_2-1} & 1 & & & \\ \vdots & \vdots & & & 1 & 0\\ 0 & 0 & & \cdots & \dfrac{1}{\eta_{k-1}-1} & 1\end{bmatrix} \tag{27}$$

where $\tilde{A}_k$ is an ordinary differential operator with respect to η_κ. We say: "A separates in spherical coordinates".

In the case of our special Schrödinger operator with coefficients p and q of the form (2) we obtain from (26.1)

$$\varphi_1=\alpha\eta_1+\beta,\qquad \varphi_\kappa=\frac{\beta_{\kappa-1}}{\eta_\kappa}+\frac{\sum_{\rho=\kappa}^{k}\beta_\rho}{\eta_\kappa-1},\qquad \kappa=2,\cdots,k, \tag{28.1}$$

where $\beta=\sum_{\rho=1}^{k}\beta_\rho$. If we choose

$$\psi_1=\gamma\eta_1+\delta,\qquad \psi_\kappa=-\frac{\varepsilon_{\kappa-1}}{\eta_\kappa}\quad(2\leqq\kappa\leqq k-2),\quad \psi_k=-\frac{\varepsilon_{k-1}}{\eta_k}+\frac{\varepsilon_k}{\eta_k-1} \tag{28.2}$$

(26.2) also is satisfied. Thus, our special Schrödinger operator separates in spherical coordinates and we can establish

PROPOSITION 3. *Let $v_\kappa: G_\kappa\to\mathbb{C}$ $(\kappa=1,\cdots,k)$ and $w: D\to\mathbb{C}$ be analytic with domain $D\subset\mathbb{C}^k$ such that $\phi(G)\subset D$. Further, let $w\neq 0$ and*

$$(w\circ\phi)(\eta)=\prod_{\kappa=1}^{k}v_\kappa(\eta_\kappa).$$

Then w is a solution of our special Schrödinger equation

$$Aw=0,$$

iff there exist separation constants $(\lambda_0,\cdots,\lambda_{k-2})\in\mathbb{C}^{k-1}$ such that v_1 is a solutions of

$$zv''+\frac{1}{2}(\alpha z+(k+\beta))v'+\frac{1}{4}\Bigl(\gamma z+\delta+\frac{\lambda_{k-2}}{z}\Bigr)v=0 \tag{29.1}$$

with $\beta=\sum_{\rho=1}^{k}\beta_\rho$ and the v_κ $(\kappa=2,\cdots,k)$ are solutions of

$$\begin{aligned} z(z-1)\Bigl[v''+\frac{1}{2}\Bigl(\frac{1+\beta_{\kappa-1}}{z}+\frac{\sum_{\rho=\kappa}^{k}(1+\beta_\rho)}{z-1}\Bigr)v'\Bigr]&\\ +\frac{1}{4}\Bigl(\frac{-\varepsilon_{\kappa-1}}{z}+\frac{\lambda_{k-1-\kappa}}{z-1}+\lambda_{k-\kappa}\Bigr)v&=0\end{aligned} \tag{29.2}$$

where $\lambda_{-1}=\varepsilon_k$.

The differential equation (29.1) is identical with equation (22.1) and thus can always be integrated by confluent hypergeometric functions.

The differential equations (29.2) are of Fuchsian type with the 3 singular points 0, 1, and ∞ and thus can always be integrated by hypergeometric functions. By use of the Riemann P-notation, equations (29.2) may be symbolized by

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ \nu^1_{\kappa-1} & \mu^1_\kappa & -\mu^1_{\kappa-1} & z \\ \nu^2_{\kappa-1} & \mu^2_\kappa & -\mu^2_{\kappa-1} & \end{Bmatrix}, \qquad \kappa = 2, \cdots, k, \tag{30}$$

where the ν^1_κ, ν^2_κ are determined by (15.1) and the μ^1_κ, μ^2_κ by

$$\mu^1_\kappa + \mu^2_\kappa = 1 - \frac{1}{2}\sum_{\rho=\kappa}^{k}(1+\beta_\rho), \qquad \mu^1_\kappa \cdot \mu^2_\kappa = \tfrac{1}{4}\lambda_{k-1-\kappa}, \qquad \kappa = 1, \cdots, k-1, \tag{30.1}$$

and

$$\mu^\rho_k = \nu^\rho_k, \qquad \rho = 1, 2. \tag{30.2}$$

Finally, we give a simple transformation of rectangular coordinates, such that the corresponding ordinary differential equations obtained by separation are also in an appropriate form.

Let $\theta = (\theta_1, \cdots, \theta_k) \in \times_{\kappa=1}^{k} G_\kappa$ with domains $G_\kappa \subset \mathbb{C}\backslash\{0\}$ be related to rectangular coordinates $x = (x_1, \cdots, x_k) = \phi(\theta)$ by

$$x^2_\kappa = \theta_\kappa, \qquad \kappa = 1, \cdots, k. \tag{31}$$

Our general Schrödinger operator in terms of the variable θ then becomes

$$\begin{aligned} \tilde{A} &= \sum_{\kappa=1}^{k} \tilde{A}_\kappa, \\ \tilde{A}_\kappa &= 4\theta_\kappa \frac{\partial^2}{\partial\theta^2_\kappa} + 2(1 + \phi_\kappa \cdot (p_\kappa \circ \phi))\frac{\partial}{\partial\theta_\kappa} + \psi_\kappa, \end{aligned} \tag{32}$$

where the ψ_κ are any functions with

$$q \circ \phi = \sum_{\kappa=1}^{k} \psi_\kappa. \tag{32.1}$$

If φ_κ and ψ_κ depend only on θ_κ, then $\tilde{A}$ is separable and may be written in the form

$$\tilde{A} = \det \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \cdots & \tilde{A}_{k-1} & \tilde{A}_k \\ -1 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & & 1 & 0 \\ 0 & 0 & & \cdots & -1 & 1 \end{bmatrix} \tag{33}$$

where $\tilde{A}_\kappa$ is an ordinary differential operator with respect to θ_κ.

In the case of our special Schrödinger operator we have

$$\phi_\kappa \cdot (p_\kappa \circ \phi) = \alpha\theta_\kappa + \beta_\kappa, \qquad \kappa = 1, \cdots, k, \tag{33.1}$$

and with

$$\psi_\kappa := \gamma\theta_\kappa + \delta_\kappa + \varepsilon_\kappa/\theta_\kappa, \qquad \kappa = 1, \cdots, k, \tag{33.2}$$

where $\sum_{\kappa=1}^{k} \delta_\kappa = \delta$, condition (32.1) is satisfied. Thus, our special Schrödinger operator separates in the coordinates $\theta = (\theta_1, \cdots, \theta_k)$ and we can establish

PROPOSITION 4. *Let $v_\kappa: G_\kappa \to \mathbb{C}$ $(\kappa = 1, \cdots, k)$ and $w: D \to \mathbb{C}$ be analytic with domain $D \subset \mathbb{C}^k$ such that $\phi(G) \subset D$. Further, let $w \neq 0$ and*

$$(w \circ \phi)(\theta) = \prod_{\kappa=1}^{k} v_\kappa(\theta_\kappa).$$

Then w is a solution of our special Schrödinger equation

$$Aw = 0,$$

iff there exist separation constants $(\lambda_1, \cdots, \lambda_k) \in \mathbb{C}^k$ with $\sum_{\kappa=1}^{k} \lambda_\kappa = \delta$ such that for $\kappa = 1, \cdots, k$ v_κ is a solution of

$$zv'' + \frac{1}{2}(\alpha z + (1+\beta_\kappa))v' + \frac{1}{4}\left(\gamma z + \lambda_\kappa + \frac{\varepsilon_\kappa}{z}\right)v = 0. \tag{34}$$

The differential equations (34) are of the same type as equation (22.1) and thus can always be integrated by confluent hypergeometric functions. Obviously, the indices ν_κ^1, ν_κ^2 of the regular singular point 0 of the κth equation (34) are determined by (15.1).

2. Integral relations.

2.1. A general principle of generating integral relations. Let $G_\kappa \subset \mathbb{C}$ $(\kappa = 1, \cdots, k)$ be domains and

$$r_\kappa, p_\kappa, q_\kappa, c_\kappa^\rho: G_\kappa \to \mathbb{C}, \qquad \kappa = 1, \cdots, k; \quad \rho = 0, \cdots, k-2, \tag{35}$$

be analytic functions. We then define second-order ordinary differential operators A_κ (with respect to $z_\kappa \in G_\kappa$) by

$$A_\kappa v_\kappa := r_\kappa v_\kappa'' + p_\kappa v_\kappa' + q_\kappa v_\kappa, \qquad \kappa = 1, \cdots, k, \tag{36}$$

and with these the second-order partial differential operator A (with respect to $z = (z_1, \cdots, z_k) \in G := \times_{\kappa=1}^{k} G_\kappa$) by

$$A := \det \begin{bmatrix} A_1 & \cdots & A_k \\ c_1^{k-2} & & c_k^{k-2} \\ \vdots & & \vdots \\ c_1^0 & \cdots & c_k^0 \end{bmatrix} =: \sum_{\kappa=1}^{k} (-1)^{\kappa-1} d_\kappa(\overset{\kappa}{z}) A_\kappa. \tag{37}$$

Since A is separable, we can establish the following theorem.

THEOREM 2.1. *Let*

(i) *$w: G = \times_{\kappa=1}^{k} G_\kappa \to \mathbb{C}$ be an analytic solution of*

$$Aw = 0;$$

(ii) *for $\kappa = 1, \cdots, k-1$, $v_\kappa: G_\kappa \to \mathbb{C}$ be an analytic solution of*

$$A_\kappa v_\kappa + \left(\sum_{\rho=0}^{k-2} \lambda_\rho c_\kappa^\rho\right) v_\kappa = 0,$$

where $(\lambda_0, \cdots, \lambda_{k-2}) \in \mathbb{C}^{k-1}$;

(iii) *for $\kappa = 1, \cdots, k-1$, $\omega_\kappa: G_\kappa \to \mathbb{C}$ be an analytic function with*

$$(\omega_\kappa r_\kappa)' = p_\kappa \cdot \omega_\kappa;$$

(iv) *for* $\kappa=1,\cdots,k-1$, $\mathfrak{C}_\kappa$ *be a path in* G_κ, *such that*

$$\left[\omega_\kappa\cdot r_\kappa\cdot\left(\frac{\partial w}{\partial z_\kappa}\cdot v_\kappa-w\cdot v'_\kappa\right)\right]_{\dot{\mathfrak{C}}_\kappa}=0.$$

Then $v_k\colon G_k\to\mathbb{C}$, *defined by*

$$v_k(z_k):=\int_{\mathfrak{C}_1}\cdots\int_{\mathfrak{C}_{k-1}}d_k(\overset{k}{\check{z}})w(z)\cdot\prod_{\kappa=1}^{k-1}(\omega_\kappa(z_\kappa)v_\kappa(z_\kappa))\,dz_1\cdots dz_{k-1},$$

is an analytic solution of

$$A_kv_k+\left(\sum_{\rho=0}^{k-2}\lambda_\rho c_k^\rho\right)v_k=0.$$

Proof. We consider

$$u\mapsto L_\kappa u:=A_\kappa u+\left(\sum_{\sigma=0}^{k-2}\lambda_\sigma c_\kappa^\sigma\right)u.$$

Condition (iii) implies that $\omega_\kappa\cdot L_\kappa$ is formally self-adjoint

$$w\cdot\omega_\kappa\cdot L_\kappa u-u\cdot\omega_\kappa\cdot L_\kappa w=\frac{\partial}{\partial z_\kappa}\left[\omega_\kappa\cdot r_\kappa\cdot\left(w\cdot\frac{\partial u}{\partial z_\kappa}-u\cdot\frac{\partial w}{\partial z_\kappa}\right)\right].$$

Since $L_\kappa v_\kappa=0$ by (ii), we get

$$\omega_\kappa\cdot v_\kappa\cdot L_\kappa w=\frac{\partial}{\partial z_\kappa}\left[\omega_\kappa\cdot r_\kappa\cdot\left(v_\kappa\cdot\frac{\partial w}{\partial z_\kappa}-v'_\kappa\cdot w\right)\right]$$

and therefore by (iv)

$$\int_{\mathfrak{C}_\kappa}\omega_\kappa(z_\kappa)\cdot v_\kappa(z_\kappa)\cdot(L_\kappa w)(z)\,dz_\kappa=0$$

identically with respect to $\overset{\kappa}{\check{z}}$. On multiplying this by

$$d_\kappa\cdot\prod_{\substack{\rho=1\\ \rho\neq\kappa}}^{k-1}(\omega_\rho\cdot v_\rho),$$

integrating $(k-2)$-times and changing the order of integration, we find

$$(*)\qquad \int_{\mathfrak{C}_1}\cdots\int_{\mathfrak{C}_{k-1}}d_\kappa(\overset{\kappa}{\check{z}})\cdot\prod_{\rho=1}^{k-1}(\omega_\rho(z_\rho)v_\rho(z_\rho))\cdot(L_\kappa w)(z)\,dz_1\cdots dz_{k-1}=0$$

for $\kappa=1,\cdots,k-1$. On the other hand, (37) and (i) imply

$$\sum_{\kappa=1}^{k}(-1)^{\kappa-1}d_\kappa(\overset{\kappa}{\check{z}})(L_\kappa w)(z)=(Aw)(z)=0.$$

On multiplying this by $\prod_{\rho=1}^{k-1}(\omega_\rho v_\rho)$, integrating $(k-1)$ times and using $(*)$ for $\kappa=1,\cdots,k-1$, we see that $(*)$ also holds for $\kappa=k$. Hence, by definition of v_k and by use of the fact that differentiation may be carried out under the integral sign we finally obtain $L_kv_k=0$.

We have formulated Theorem 2.1 only for proper integrals. If we deal with improper integrals—one knows that these play the more important role in applications—we have to take care that all repeated improper integrals, which occur in the definition of v_k and also in (*), converge locally uniformly with respect to the remaining variables and are independent of the order of integration.

A special case of Theorem 2.1 should be pointed out, since it yields a reduction principle.

THEOREM 2.2. *Let $k \geqq 3$ and let the assumptions* (ii) *and* (iii) *of Theorem* 2.1 *be given. Further, let*

$$c_1^\rho = 0, \qquad \rho = 0, \cdots, k-3.$$

Then we can choose

$$d_\kappa(\overset{\kappa}{\check{z}}) =: c_1^{k-2}(z_1) \cdot \tilde{d}_\kappa(\overset{1,\kappa}{\check{z}}), \qquad \kappa = 2, \cdots, k,$$
$$\tilde{A}_\kappa := A_\kappa + \lambda_{k-2} c_\kappa^{k-2},$$

and

$$\tilde{A} := \det \begin{bmatrix} \tilde{A}_2 & \cdots & \tilde{A}_k \\ c_2^{k-3} & & c_k^{k-3} \\ \vdots & & \vdots \\ c_2^0 & \cdots & c_k^0 \end{bmatrix} = \sum_{\kappa=2}^{k} (-1)^{\kappa-2} \tilde{d}_\kappa(\overset{1,\kappa}{\check{z}}) \cdot \tilde{A}_\kappa.$$

Now, let

(i′) $\tilde{w}: \tilde{G} := \times_{\kappa=2}^{k} G_\kappa \to \mathbb{C}$ *be an analytic solution of*

$$\tilde{A}\tilde{w} = 0;$$

(iv′) *for $\kappa = 2, \cdots, k-1$, $\mathfrak{C}_\kappa$ be a path in G_κ, such that*

$$\left[\omega_\kappa \cdot r_\kappa \cdot \left(\frac{\partial \tilde{w}}{\partial z_\kappa} \cdot v_\kappa - \tilde{w} \cdot v_\kappa'\right)\right]_{\mathfrak{C}_\kappa} = 0$$

and $\mathfrak{C}_1$ be any path in G_1.

Then $w: G \to \mathbb{C}$, defined by

$$w(z) := v_1(z_1) \cdot \tilde{w}(\overset{1}{\check{z}}),$$

satisfies (i) *and* (iv) *of Theorem* 2.1 *and $v_k: G_k \to \mathbb{C}$, defined in Theorem* 2.1, *becomes*

$$v_k(z_k) = \gamma \cdot \int_{\mathfrak{C}_2} \cdots \int_{\mathfrak{C}_{k-1}} \tilde{d}_k(\overset{1,\kappa}{\check{z}}) \cdot \tilde{w}(\overset{1}{\check{z}}) \cdot \prod_{\kappa=2}^{k-1} (\omega_\kappa(z_\kappa) v_\kappa(z_\kappa))\, dz_2 \cdots dz_{k-1},$$

where

$$\gamma = \int_{\mathfrak{C}_1} c_1^{k-2}(z_1)\omega_1(z_1)v_1(z_1)^2\, dz_1.$$

We notice that in the case of $c_1^{k-2} \cdot \omega_1 \cdot v_1 \neq 0$ one can always find a path $\mathfrak{C}_1$ in G_1 such that $\gamma \neq 0$.

2.2. Integral relations for special functions. Let $a_1, \cdots, a_k$ be different points in $\mathbb{C}$ and $\nu_\kappa^1, \nu_\kappa^2$ ($\kappa = 1, \cdots, k$), α, γ, δ and λ_ρ ($\rho = 0, \cdots, k-2$) be complex parameters. Let $\delta' = \delta - \gamma \cdot \sum_{\rho=1}^{k} a_\rho$.

Our aim is to get integral relations for the solutions of the differential equation

$$\prod_{\rho=1}^{k} (z - a_\rho) \cdot \left[v'' + \left(\frac{\alpha}{2} + \sum_{\rho=1}^{k} \frac{1 - \nu_\rho^1 - \nu_\rho^2}{z - a_\rho}\right) v'\right]$$
$$(38) \qquad + \left(\sum_{\rho=1}^{k} \frac{\nu_\rho^1 \nu_\rho^2}{z - a_\rho} \cdot \prod_{\substack{\sigma=1 \\ \sigma \neq \rho}}^{k} (a_\rho - a_\sigma) + \frac{\gamma}{4} z^k + \frac{\delta'}{4} z^{k-1} + \sum_{\rho=0}^{k-2} \lambda_\rho z^\rho\right) v = 0,$$

by applying the methods of § 2.1 to the situations of §§ 1.2 and 1.3 where (38) occurs in connection with the separation of the special Schrödinger operator in ellipsoidal and sphero-conal coordinates.

As kernels for the integral relations we shall use the product solutions of the special Schrödinger equation in terms of spherical and rectangular coordinates to be found by Propositions 3 and 4.

2.2.1. Kernels in terms of spherical coordinates. Let $\mu_\kappa^1, \mu_\kappa^2$ ($\kappa = 1, \cdots, k-1$) be complex parameters with

$$(39) \qquad 1-\mu_\kappa^1-\mu_\kappa^2 = \sum_{\rho=\kappa}^{k} (1-\nu_\rho^1-\nu_\rho^2), \qquad \kappa = 1, \cdots, k-1.$$

The kernels in terms of spherical coordinates to be found by Proposition 3 are then of the form

$$(40) \qquad V(\eta) = R(\eta_1)\cdot K(\overset{1}{\eta}), \qquad K(\overset{1}{\eta}) = \prod_{\kappa=2}^{k} K_\kappa(\eta_\kappa),$$

where R is a solution of

$$(40.1) \qquad zv'' + \left(\frac{\alpha}{2}z + (1-\mu_1^1-\mu_1^2)\right)v' + \left(\frac{\gamma}{4}z + \frac{\delta}{4} + \frac{\mu_1^1\mu_1^2}{z}\right)v = 0,$$

and

$$(40.2) \qquad K_\kappa \in P\begin{Bmatrix} o & 1 & \infty & \\ \nu_{\kappa-1}^1 & \mu_\kappa^1 & -\mu_{\kappa-1}^1 & z \\ \nu_{\kappa-1}^2 & \mu_\kappa^2 & -\mu_{\kappa-1}^2 & \end{Bmatrix}, \qquad \kappa = 2, \cdots, k,$$

with $\mu_k^\rho = \nu_k^\rho$ ($\rho = 1, 2$).

The solutions of (40.1) may be written in terms of confluent hypergeometric functions. We have to distinguish 3 cases. Let

$$(41) \qquad \tilde{\alpha} := (\alpha^2 - 4\gamma)^{1/2}, \qquad \tilde{\delta} := \delta - \alpha\cdot\sum_{\rho=1}^{k}(1-\nu_\rho^1-\nu_\rho^2).$$

Then one easily finds [3, vol. II], that the solutions of (40.1) are given by the following.

Case 1. $\tilde{\alpha} \neq 0$.

$$(40.1.1) \quad R(z) = \exp\left(\frac{1}{4}(\tilde{\alpha}-\alpha)z\right)\cdot z^{\mu_1^1}\cdot {}_1\mathfrak{F}_1\left(\frac{\tilde{\delta}}{2\tilde{\alpha}} + \frac{1}{2}(1+\mu_1^1-\mu_1^2);\, 1+\mu_1^1-\mu_1^2;\, -\frac{\tilde{\alpha}}{2}z\right),$$

Case 2. $\tilde{\alpha} = 0$, $\tilde{\delta} =: -\tau^2 \neq 0$.

$$(40.1.2) \quad \begin{aligned} R(z) &= \exp(-\tfrac{1}{4}\alpha z)\cdot z^{\mu_1^1}\cdot \exp(\tau z^{1/2}) \\ &\quad \cdot {}_1\mathfrak{F}_1(\tfrac{1}{2}+\mu_1^1-\mu_1^2;\, 1+2\mu_1^1-2\mu_1^2;\, -2\tau z^{1/2}), \end{aligned}$$

Case 3. $\tilde{\alpha} = \tilde{\delta} = 0$.

$$(40.1.3) \qquad R(z) = \exp(-\tfrac{1}{4}\alpha z)\cdot(c_1 z^{\mu_1^1} + c_2 z^{\mu_1^2}), \qquad c_1, c_2 \in \mathbb{C},$$

where ${}_1\mathfrak{F}_1(a;c;z)$ denotes any solution of the confluent hypergeometric equation

$$zu''+(c-z)u'-au=0.$$

Obviously, the K_κ ($\kappa=2,\cdots,k$) are given by

$$\begin{aligned}K_\kappa(z)&=z^{\nu^1_{\kappa-1}}\cdot(1-z)^{\mu^1_\kappa}\\ &\cdot{}_2\mathfrak{F}_1(\nu^1_{\kappa-1}+\mu^1_\kappa-\mu^1_{\kappa-1},\nu^1_{\kappa-1}+\mu^1_\kappa-\mu^2_{\kappa-1};1+\nu^1_{\kappa-1}-\nu^2_{\kappa-1};z)\end{aligned}\tag{40.2.1}$$

with $\mu^\rho_k=\nu^\rho_k$ ($\rho=1,2$), where ${}_2\mathfrak{F}_1(a,b;c;z)$ denotes any solution of the hypergeometric differential equation

$$z(1-z)u''+(c-(a+b+1)z)u'-abu=0.$$

2.2.2. Kernels in terms of rectangular coordinates. Let τ_κ ($\kappa=1,\cdots,k$) be complex parameters with

$$\sum_{\kappa=1}^k\tau^2_\kappa+\tilde{\delta}=0\tag{42}$$

where $\tilde{\delta}$ is given by (41). The kernels in terms of rectangular coordinates to be found by Proposition 4 are then of the form

$$W(\theta)=\prod_{\kappa=1}^k W_\kappa(\theta_\kappa),\tag{43}$$

where W_κ is a solution of

$$zv''+\left(\frac{\alpha}{2}z+(1-\nu^1_\kappa-\nu^2_\kappa)\right)v'+\left(\frac{\gamma}{4}z+\frac{\alpha(1-\nu^1_\kappa-\nu^2_\kappa)-\tau^2_\kappa}{4}+\frac{\nu^1_\kappa\nu^2_\kappa}{z}\right)v=0.\tag{43.1}$$

In the same way as in § 2.2.1 for (40.1) one finds that the W_κ are given by the following.

Case 1. $\tilde{\alpha}\neq0$.

$$\begin{aligned}W_\kappa(z)&=\exp\left(\frac{1}{4}(\tilde{\alpha}-\alpha)z\right)\cdot z^{\nu^1_\kappa}\\ &\cdot{}_1\mathfrak{F}_1\left(-\frac{\tau^2_\kappa}{\tilde{\alpha}}+\frac{1}{2}(1+\nu^1_\kappa-\nu^2_\kappa);1+\nu^1_\kappa-\nu^2_\kappa;-\frac{\tilde{\alpha}}{2}z\right).\end{aligned}\tag{43.1.1}$$

Case 2. $\tilde{\alpha}=0$.

$$\begin{aligned}W_\kappa(z)&=\exp\left(-\frac{\alpha}{4}z\right)\cdot z^{\nu^1_\kappa}\\ &\cdot\exp(\tau_\kappa z^{1/2})\cdot{}_1\mathfrak{F}_1(\tfrac{1}{2}+\nu^1_\kappa-\nu^2_\kappa;1+2\nu^1_\kappa-2\nu^2_\kappa;-2\tau_\kappa z^{1/2}),\qquad \tau_\kappa\neq0,\\ W_\kappa(z)&=\exp\left(-\frac{\alpha}{4}z\right)\cdot(c_1z^{\nu^1_\kappa}+c_2z^{\nu^2_\kappa}),\qquad c_1,c_2\in\mathbb{C},\qquad \tau_\kappa=0,\end{aligned}\tag{43.1.2}$$

where $\tilde{\alpha}$ is given by (41) and ${}_1\mathfrak{F}_1(a;c;z)$ denotes any solution of the confluent hypergeometric equation.

2.2.3. Two types of integral relations. Let $G_\kappa\subset\mathbb{C}\backslash\{a_1,\cdots,a_k\}$ ($\kappa=1,\cdots,k$) be domains and

$$\omega(z)=\exp\left(\frac{\alpha}{2}z\right)\prod_{\rho=1}^k(z-a_\rho)^{-\nu^1_\rho-\nu^2_\rho}.\tag{44}$$

Application of Theorem 2.1 in connection with § 1.2 then yields the next theorem.

THEOREM 2.3. *Let* $k \geqq 2$.

(i) *Let* $w : \times_{\kappa=1}^{k} G_\kappa \to \mathbb{C}$ *denote either the function* V *in* (40) *or the function* W *in* (43) *in terms of ellipsoidal coordinates.*

(ii) *Let* $v_\kappa : G_\kappa \to \mathbb{C}$ $(\kappa = 1, \cdots, k-1)$ *be solutions of* (38).

(iii) *Let for* $\kappa = 1, \cdots, k-1$, $\mathfrak{C}_\kappa$ *be a path in* G_κ *such that*

$$\left[\omega(z_\kappa) \cdot \prod_{\rho=1}^{k} (z_\kappa - a_\rho) \cdot \left(\frac{\partial w}{\partial z_\kappa}(z) v_\kappa(z_\kappa) - v_\kappa'(z_\kappa) w(z)\right)\right]_{\mathfrak{C}_\kappa} = 0$$

identically with respect to $\overset{\kappa}{\check{z}}$.

Then $v_k : G_k \to \mathbb{C}$ *defined by*

$$v_k(z_k) = \int_{\mathfrak{C}_1} \cdots \int_{\mathfrak{C}_{k-1}} P_{k-1}(\overset{k}{\check{z}}) \cdot w(z) \cdot \prod_{\kappa=1}^{k-1} (\omega(z_\kappa) v_\kappa(z_\kappa))\, dz_1 \cdots dz_{k-1}$$

is an analytic solution of (38).

Special cases of integral relations of this type are for $k = 2$ the well-known (linear) integral relations for Mathieu and spheroidal wave functions ([11], [3, vol. III]), and for $k = 3$ the (quadratic) integral relations for Heun functions found by Sleeman [15]. It should be noted that there is a mistake in [15]: the operator in (4.2) of [15] and therefore the following kernels have to depend also on δ and ε.

Application of Theorems 2.1 and 2.2 in connection with § 1.3 yields the following theorem.

THEOREM 2.4. *Let* $k \geqq 3$ *and* $\alpha = \gamma = \delta = 0$.

(i) *Let* $\tilde{w} : \times_{\kappa=2}^{k} G_\kappa \to \mathbb{C}$ *denote the function* K *in* (40) *with* $\mu_1^1 \cdot \mu_1^2 = \lambda_{k-2}$ *in terms of sphero-conal coordinates.*

(ii) *Let* $v_\kappa : G_\kappa \to \mathbb{C}$ $(\kappa = 2, \cdots, k-1)$ *be solutions of* (38).

(iii) *Let for* $\kappa = 2, \cdots, k-1$, $\mathfrak{C}_\kappa$ *be a path in* G_κ *such that*

$$\left[\omega(z_\kappa) \cdot \prod_{\rho=1}^{k} (z_\kappa - a_\rho) \cdot \left(\frac{\partial \tilde{w}}{\partial z_\kappa}(\overset{1}{\check{z}}) v_\kappa(z_\kappa) - v_\kappa'(z_\kappa) \tilde{w}(\overset{1}{\check{z}})\right)\right]_{\mathfrak{C}_\kappa} = 0$$

identically with respect to $\overset{1,\kappa}{\check{z}}$.

Then $v_k : G_k \leftarrow \mathbb{C}$ *defined by*

$$v_k(z_k) = \int_{\mathfrak{C}_2} \cdots \int_{\mathfrak{C}_{k-1}} P_{k-2}(\overset{1,k}{\check{z}}) \cdot \tilde{w}(\overset{1}{\check{z}}) \cdot \prod_{\kappa=2}^{k-2} (\omega(z_\kappa) v_\kappa(z_\kappa))\, dz_2 \cdots dz_{k-1}$$

is an analytic solution of (38).

Special cases of integral relations of this type for $k = 3$ are the (linear) integral relations for Heun functions found by Erdélyi [4] and Lambe and Ward [6].

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