Heinz Steinbring

The Relation between Social and Conceptual Conventions in Everyday Mathematics Teaching

In: Luciana Bazzini (ed.): Proceedings of the Second Italian German Bilateral Symposium on Didactics of Mathematics, Haus Ohrbeck (Georgsmarienhütte), 21 - 26 April 1992, Bielefeld 1994
S. 369-383

THE RELATION BETWEEN SOCIAL AND CONCEPTUAL CONVENTIONS IN EVERYDAY MATHEMATICS TEACHING

Heinz Steinbring

ABSTRACT.

The paper analyzes the specific epistemological role of mathematical symbols by using and comparing the Odgen and Richards' triangle of meaning with the epistemological triangle of "Symbol, Concept and Object". The micro-analysis of a teaching episode about the meaning of the decimal point and the role of appended zeros to decimal numbers illustrates the dominating tendency in classroom interaction to conceive of mathematical symbols and signs as pure names for concrete objects. This denotation function of mathematical signs hinders the conceptual development of symbols. The adequate understanding of the symbols role requires to interpret single mathematical symbols as systems of signs with their operations. The construction of such systems depends on the interplay between social and conceptual conventions and conditions.

> "A remarkable feature of all mathematics, making the access to it so difficult for the layman, is its copious use of <u>symbols</u>." (Weyl 1990,88/9).

1. Introduction: Ogden and Richards' triangle of meaning – or what is the nature of the relation between symbol and referent?

Where exactly lies the difficulty for the layman and student in coping with mathematical symbols and signs? Is it sufficient to know the correct translation of the abstract signs and to learn the mathematical symbols like a vocabulary? (cf. Davis & Hersh 1981,122ff). Are symbols used in mathematics like a language, or are they representatives for the objects proper of mathematical analysis? (cf. Jahnke & Otte 1981, 75ff).

The specific use of mathematical signs and symbols in the everyday mathematics class with the interactive patterns and mechanisms leads to an exaggerated confinement to formal signs and the rules of manipulating them may be observed, in which the sign's referent loses its own independent meaning in the process of methodical abstraction. In the classroom, mathematical symbols are treated like mythical signs.

> "The practice of mathematics, particularly in school, is ... induced by automatization, by the algorithm expressed in a formula as a procedure for calculating, to identify sign and signified, or, if the threefold distinction between concept, sign and object is made ..., to identify the sign and the object while neglecting the conceptual aspects which are independent of them." (Otte 1984, 19).

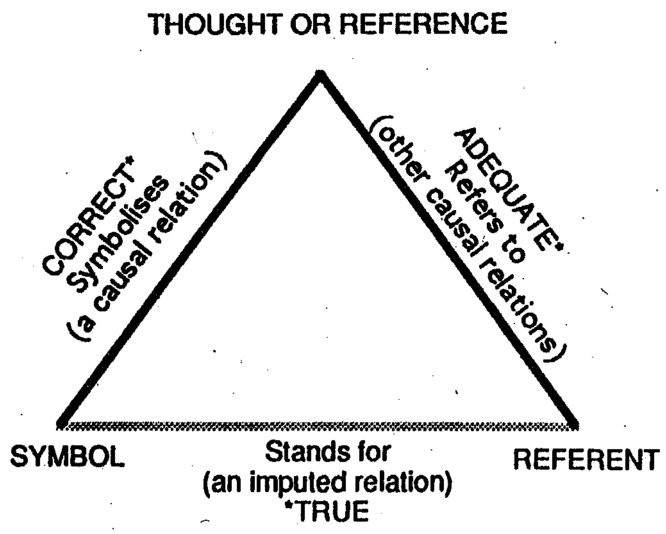
Which alternative descriptions permit a more differentiated consideration of the relationship between symbol and referent? In their book "The Meaning of Meaning" (Odgen & Richards 1923), Odgen and Richards formulate the central proposition that the relationship between symbol and referent is never established in a direct way, but is of *indirect* nature.

> "... for the analysis of the senses of 'meaning' with which we are chiefly concerned, it is desirable to begin with the relations of thoughts, words and things as they are found in cases of reflective speech uncomplicated by emotional, diplomatic, or other disturbances; and with regard to these, the indirectness of the relations between words and things is the feature which first deserves attention." (Odgen & Richards 1923, 10).

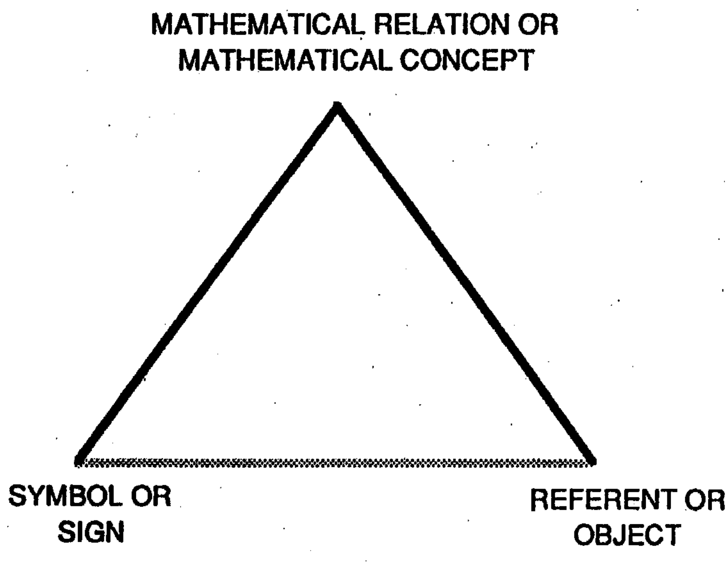
> "Between the symbol and the referent there is no relevant relation other than the indirect one, which consists in its being used by someone to stand for a referent. Symbol and Referent, that is to say, are not connected directly (and when, for grammatical reasons, we imply such a relation it will merely be an imputed, as opposed to a real, relation) but indirectly round the two sides of the triangle." (Odgen & Richards 1923, 11/2).

And below:

> "The root of the Trouble will be traced to the superstition that words are in some way parts of things or always imply things corresponding to them, ... The fundamental and most prolific fallacy is, in other words, that the base of the triangle given above is filled in." (Odgen & Richards 1923, 14/5).



From this perspective the question arises as to the relationship between thought / reference in the triangle of meaning and the mathematical concept as it is situated in the epistemological triangle of "sign, concept and object" (cf. Steinbring 1989, 1991a, 1991b):



In the following we shall use a particular teaching episode to discuss differences and similarities between these two triangles of meaning; in particular there will arise the question for the epistemological peculiarities of mathematical meaning in establishing a relation between sign and referent.

2. Dealing with mathematical meaning in the classroom – a teaching episode

The analyzed teaching episode is about problems of understanding the meaning of decimal numbers. In this case, the number figures already known must be reinterpreted; the figures to the right of the point must be conceived of differently, while the figures to the left of the *point* seem to maintain their old meaning. Besides, a new sign is introduced, the point; and in particular the zero acquires new aspects of meaning, being totally indispensable in some cases and negligible in others, seemingly dependent on which place it occupies in the string of figures.

The basic methodical orientation for introducing decimal numbers (cf. Steinbring 1991c) is the following: decimal numbers are embedded in the context of simple concepts of quantity in order to develop a natural notion avoiding any rupture with old ideas. In this introduction, erroneous ideas arise as well. From the domain of quantities, an interpretation of the decimal point is unconsciously taken over according to which the point first of all serves to separate the quantity and after that also the number into two different quantities or numbers: One quantity stands to the left of the point (for instance D–Mark or kilometer) and another, the "smaller" quantity (Pfennig or meter) to the right of the point. In case of such implicit interpretations of the points role, the students have great difficulties in comprehending the "connection" between the two numerical signs left and right to the point. (cf. Günther 1987, Padberg 1991).

Subsequent to a more or less natural introduction of the "form" of decimal numbers, the technical apparatus for operating correctly with decimal numbers is explained by means of a reductive model. If fractions are already known, it seems possible, as frequently reported in the educational literature, to interpret decimal numbers as mere coding forms or notations of specific fractions. In this case, the students' work consists mainly in translating different coded cipher strings into one another: translating fractions into decimal numbers, and decimal numbers into fractions. In this perspective, decimal numbers are not assigned conceptual meaning of their own, and in consequence some authors do not speak of decimal *numbers*, but of numbers in decimal notation.

The teaching episode* "*What does the zero at the end mean?*" can be structured into the following phases:

1. Phase (1–9): Presentation of the mathematical problem

The teacher reminds the students of a problem from the last lesson and presents the following task of written addition of three decimal numbers:

$$\begin{array}{r} 2.37 \\ 13.731 \\ \underline{0.2} \end{array}$$

The expected calculation is based on the following methodical rule: place point beneath point and then add the decimal numbers like natural numbers; besides, the students need a notion about how to deal with the "empty" places in the algorithmic operation. This is contained in the "rule of appending zeros" (Postel 1991, 8), or an equivalent rule: zeros can be appended to the right of a finite decimal number, or also be left out.

2. Phase (10–36): Attempts at spontaneous foundations offered by the students

For the task of the written calculation, there is a first suggestion of "reformulating" the task. Completing zeros are appended, resulting in the following task:

$$\begin{array}{r} 2.370 \\ 13.731 \\ \underline{0.200} \end{array}$$

Are these zeros permitted or not? Does this change the task? The students spontaneously offer two types of reasoning. The students spontaneously offer two types of reasoning. The <u>first:</u> In the frame of the written calculation, the zeros are of no consequence, for they have no effect on the addition (No. 15, 20, 31, 32, "...for zero

---

* The complete transcript of the teaching episode What does the zero at the end mean? can be obtained from the author.

plus zero will be zero."). The second type of argument: If the zeros have no effect on the operation, they have also no conceptual meaning. And: If zeros are appended to the right side, the decimal number, respectively the number "right" to the decimal point gets bigger (No. 17, 18, 23, 25). The type of argument: Numbers with more figures are bigger than numbers with fewer figures (the point is not taken into account); or, the decimal point separates the decimal number into two numbers, and if zeros are appended to the "right" number, it will become bigger (37 becomes 370, No. 25).

The teacher sums up these two points of view (32–36) and emphasizes the meaning of the point for the correct argument; in this way she opens the next phase.

3. Phase (37–80): Transforming the problem into a new context

The students are to do this "...by not just imagining the numbers, but... please, well, as measures of length... "(37, 38). The teacher would like to have the decimal numbers interpreted as kilometers; she transforms the task once more and obtains:

2.37 Km

13.731 Km

0.2 Km

This now shall serve to examine the relevance of appending zeros to the right of the decimal numbers.

3.1 Phase (42–54): *A "simplified" starting problem: What does 13.731 mean?*

Directly continuing in this new context with the number 2.37 studied before as an example seems to be too difficult (50, 51), and the teacher now intentionally chooses the middle number 13.731 (Km). The desired answer is promptly given, for, this number with its "complete" form, can only be translated into 13 kilometers and 731 meters.

3.2 Phase (55–83): *Three proposals concerning the meaning of 2.37*

The translation of 2.37 into the domain of the quantities "kilometers and meters" yields the three (combinatorically) possible proposals:

2 kilometers and 37 meters (No.. 58, 71)

2 kilometers and 370 meters (No.. 64, 76)

2 kilometers and zero, three, seven meters (No.. 79)

The proposal 2Km 37m is read directly from the sign–structure. That the 37 represents 370 meters is tentatively justified by the necessity of appending a zero to the right (62, 64), and also by comparison with the complete number 13.731 (76, 77). It can not be seen how serious the third proposal is; perhaps it was made only for reasons of completeness.

4. Phase (84–128): *The interactive elaboration of an accepted justification*

In this phase, the teacher tries to elicit arguments why the various proposals might be correct or false. The students at first repeat the two "solutions" 37 and 370 meters, but the teacher wants to know *why* these are correct. When a student says 370 meters, the teacher signals that this is correct by saying: "You are going for the correct thing, the one I am aiming at, too..." (94). The place of the decimal point plays an important part, she says, but she wants the students to present further arguments for the three possibilities.

The reasoning presented by one girl student: "370 is correct, because otherwise there would be a zero behind the point, or nothing at all." (98, 99) is at once made more precise by the teacher by writing down the code 2.037 which is hidden in the argument given by this girl. This also excludes the possible interpretation zero, three, seven meters. For the three proposals made, the "appropriate" coding forms of the decimal numbers are now obvious, and therefore the 37 in 2.37 only can mean 370 meters. This type of justification is of indirect character; by comparing all possible coding forms on the syntactical level, one should conclude that 2.37 must be translated into 2 kilometers and 370 meters.

In a second attempt using the comparison to 13.731 (13Km 731m), a structural justification based on analogy is given. On the basis of the "complete" number 13.731, all students immediately agree that the seven here must be interpreted as 700 (meters): hence, the three in 2.37 (T: "... we have written all these figures correctly beneath one another..." (118,119)) must also be taken as 300 (meters) for structural reasons. And the corresponding interpretation holds for the seven in 2.37.

The teaching episode closes with the clear affirmation of the teacher for all students what the correct interpretation is.

During this episode, the task of written addition has undergone changes; starting with adding decimal numbers, the second type of task changed to a syntactical form of filling in additional zeros, and then the numbers have been interpreted as concrete quantities, i.e. kilometers and meters. The elaboration of the meaning of these quantities now consists in translating kilometers into kilometers and meters. For the "complete" quantity of 13.731Km, this can be done without any problems. The difficulty in translating 2.37 Km —> 2 Km 370 m shows, that it is *necessary* in this case to append a zero to the right.

The fine distinction between the equivalent quantities 0.37Km and 370m which consists in the fact that the zero appended to the right is necessary in one case, and irrelevant in the other, is not negotiated in the arguments or the justifications. Is the problem in these two cases, 0.37Km and 37..m the same, one of appending zeros to the right? In this way, the original problem of the episode: *Are zeros appended to the right of decimal numbers relevant or not?* is not sufficiently answered, the reference to kilometers and meters perhaps even resulting in the impression that the zeros are always necessary to understand the meaning of decimal numbers. It must be assumed that the "result" of this interaction leaves many students with an unclear and contradictory idea about the original problem.

3. Social conventionalization of mathematical signs and mythical thinking – the relationship between "thought / reference" and the mathematical concept

The immediate goal in the present teaching episode is not only to execute the algorithm of written addition. The tasks' modification aims mainly at the comprehension of decimal numbers. This intention is explicitly stated by the teacher. She says: "... and that is why it is important that we come back again to this meaning of the point and the meaning of the numbers, of the digits before and behind the point, ..." (34-36).

But in spite of the concept meaning aimed at, the justifications given in interaction are more or less technical. Some of these refer to algorithmic aspects, others argue by means of syntactical comparisons. This type of justification is also put forward by the teacher: "So, if we now say we have written all these figures correctly beneath one another, what must the three here mean?" (118,119). This raises the question which type of meaning is constituted in interaction?

Let us again use Ogden and Richards' triangle of meaning to analyze the thoughts / references of the two symbols "point" and "zero" established in the classroom. There are two domains of referents both for "point" and "zero": first the familiar numbers, the arithmetic used till now, and second quantities of kilometers and meters introduced by the teacher.

The thought / reference for the symbol "point" can be described as follows. *Procedural aspect:* point beneath point, *conceptual aspect*: the point separates the decimal number or the quantity into two numbers or quantities. For the symbol "zero", the following thoughts / references are established. *Procedural aspect*: zero plus zero equals zero, *conceptual aspect*: zero means nothing, and its opposite, appended zeros make numbers bigger.

The reference for both symbols "point" and "zero" is mainly defined by a local, structural comparison in interaction; the symbols are representatives for empirical objects, not for theoretical relationships. In the comparison, these symbols are endowed with the same meaning as in the simple, complete exemplary case. No conflict arises between the negotiated empirical meaning for "point" and "zero" and the already existing ideas "point separates", "zero means nothing" and "zeros enlarge numbers"; these are locally integrated.

While the classroom discourse attains a "solution", accepted by students and teacher, there remain doubts, whether an adequate mathematical meaning of "point" and "zero" has been developed. The signs of "point" and "zero" are essentially used as names for objects, they are not extended to mathematical symbols which can serve to facilitate the operative use of mathematical relationships. Use and meaning of the two symbols are implicitly defined by social convention and made unambiguous in social interaction.

In contrast to the widespread misunderstanding that the point separates a quantity, a conceptual notion would be imaginable that the point may establish as a relation between two quantities or numbers. The quantity 13.731Km is not divided into 13Km and 731m; conversely, the decimal numbers raise a conceptual problem of how to make 13Km and 207cm a unique quantity? There is quite a number of possibilities: 13.00207Km, or 1300.207Dm (Deka–meters), or ... . The paradoxical new problem of the point notation is that the figures to the right of the point in the case of 13.731Km must also be interpreted as kilometers and not as meters, despite the fact that there are no "whole" kilometers available. This is a novel and hitherto unknown conceptual aspect of decimal quantities and numbers.

In a similar way, the zero sign must not be conceived of as a mere name; this sign must always be seen together with its position with regard to the other numerical figures. The figures express *number and position* at once, and it is significant for the zero in which relationship it stands with the other figures and the point in the string of ciphers.

The *relational connection* between the figures, the zero and the point is shown in an exemplary manner in the following relational scheme for conceptual aspects of decimal numbers.

T	O	t	h	th	tth	ht h	m
10 Km	K m	H m	D m	m	d m	c m	m m
	13			73 1			
1	3	7	3	1			
	13		73 .1		0	0	
0		13 0		73 0	10		
1. 3						73 100	
	2	3	7				
	2		37				
	2				37 00		
		23 .7					

In this scheme, mathematical relations between positions of the figures, the zero and the point etc. can be explored by variable uses; the decimal number is no mere unambiguous notation, it contains a rich relational concept–structure. The system of underlying relationships with their mutual conditions can be unfolded from this scheme, at the same time, the point itself can be used in this scheme in the sense of a theoretical *self–application*. Decimal numbers now are no longer notations or forms of coding fractions or natural numbers with a point. The structure of decimal numbers conceals a conceptual relational structure which must be unfolded and taken into consideration when using and applying these numbers

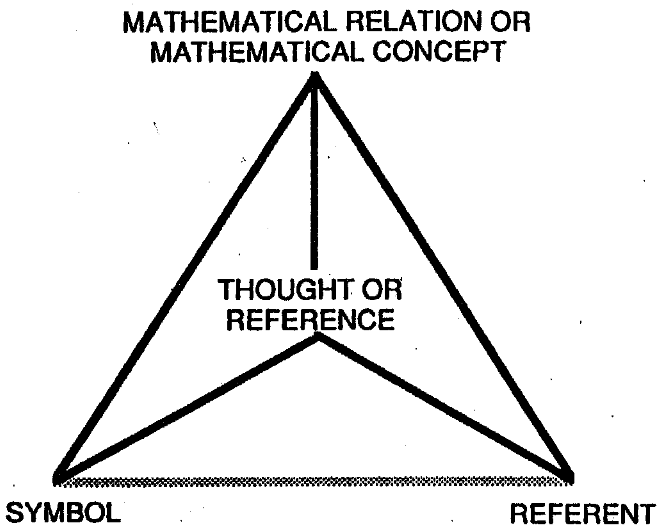
From this scheme we can learn that mathematical signs or symbols, even such simple ones as 17.312, are not exact names for one well defined object, but they represent in principle a whole system of symbols with their operations, and possessing in principle an indefinite number of ways of combinations. For reasons to make the understanding as easy as possible, signs are used in everyday teaching as isolated names for objects; and this is in direct contrast to the systems character of mathematical symbols. The good teaching intentions destroy the epistemological and relational structure of mathematical symbols.

Within this multitude of how to construct a mathematical symbol from the elements of the symbol–system there are some adequate and other inadequate notations, there are some canonical, and some which are socially agreed upon. The particular symbol notation and use in this way depends for instance on

- simplicity
- additional conditions for construction
- social, historical, cultural, personal etc. conditions
- it depends on the means of representation

In this frame there is no reason to say this symbol is mathematically correct or incorrect, they are adequate or inadequate *relative* to mathematical and communicative conditions. For being able to understand this perspective, one has to unfold the underlying structural system of the mathematical symbol and not simply treat it as an unambiguous name. This marks one big difference between the two triangles of meaning.

It is necessary to develop the *conceptual relationship* of the symbol for establishing an indirect reference between symbol and referent. The thought / reference in Ogden and Richards' triangle of meaning must be differentiated beyond the definition of a local, empirical notion to the level of conceptual, relational abstraction.



Three types of establishing references between symbol and referent are obtained:

- empirical reference with the tendency of identifying symbol and referent; the symbol becomes a socially conventionalized and unambiguous name for the referent (7 = 700, 3 = 300, 7 =70); subjectively and interactively constituted images of representation.
- structural reference in which the connection between symbol and referent is indirectly mediated by syntactical and logical structures on the symbol level and on the referent level; images of representation generated by structural analogies.
- conceptual–systemic reference in which the new conceptual relation which is to be developed generates and organizes the system of theoretical relationships; the mediating conceptual image of representation incorporates the new conceptual relation.

The mathematical concept contains a novel relation which is not exhausted by logical definitions or subjective images of representations. This new relationship permits to overcome the socially conventionalized strictness of sign attribution and to develop a conceptual flexibility for constructing references between symbol and referent.

Problems of understanding, ruptures, mistakes, misunderstandings etc. which appear in everyday mathematics teaching could be analyzed from different perspectives:

- with regard to the "correctness" of denotations, definitions, conventions, logical connections, which are not faultlessly mastered in the frame of verbal communication, and
- from an epistemological point of view according to which every mathematical interaction has always to cope with a specific problem of knowledge justification.

The observable problems of everyday mathematics teaching are not restricted to the level of verbal communication, at the same time they refer to epistemological constraints caused by the theoretical nature of mathematical knowledge. Both perspectives are necessary for analyzing mathematical interactions. Learning requires a social practice of schoolmathematical knowledge (a "social wholeness") (cf. Solomon 1989) for enabling the students to construct their own meanings of mathematical knowledge (cf. Cobb, Yackel & Wood 1992). The teaching of mathematics requires that teachers take consciously the particular epistemology of mathematical knowledge (a "wholeness of mathematical knowledge") for extending the denotation function of signs to mathematical symbols which represent *relations*.

References

Cobb, P., Yackel, E. & Wood, T.: 1992, A constructivist Alternative to the Representational View of Mind in Mathematics Education, in: *JRME,* 23(1, 2-33.

Davis, P.J. & Hersh, R.: 1981, *The Mathematical Experience*, Birkhäuser, Boston.

Günther, K.: 1987, Über das Verständnis der Schüler von Dezimalzahlen und auftretende Schülerfehler, in: *MUP*, 25-40.

Jahnke, H. N. & Otte M.: 1981, On 'Science as a Language', in: Jahnke H. N. & Otte M.: eds., *Epistemological and Social Problems of the Sciences in the Early Nineteenth Century*, Reidel, Dordrecht.

Odgen, C. K. & Richards, F. A.: 1923, *The Meaning of Meaning*, Routledge and Kegan, London.

Otte, M.: 1984, Was ist Mathematik?, *Occasional Paper No. 43*, IDM, Bielefeld.

Padberg, F.: 1991, Problembereiche bei der Behandlung von Dezimalbrüchen – eine empirische Untersuchung an Gymnasialschulen, in: *MU* 2, 39-69.

Solomon, Y.: 1989, *The Practice of Mathematics*, Routledge, London.

Steinbring, H.: 1989, Routine and Meaning in the Mathematics Classroom, in: *For the Learning of Mathematics*, 9, 1, 24-33.

Steinbring, H.: 1991a, The concept of chance in everyday teaching: Aspects of a social epistemology of mathematical knowledge, in: *Educational Studies in Mathematics*, 22, 503–522.

Steinbring, H.: 1991b, Mathematics in teaching processes - The disparity between teacher and student knowledge, in: *Recherches en Didactique des Mathématiques*, vol. 11 no. 1, 1991, 65 – 107.

Steinbring, H.: 1991c, Eine andere Epistemologie der Schulmathematik - Kann der Lehrer von seinen Schülern lernen?, erscheint in: *mathematica didactica*.

Weyl, H.: 1990, *Philosophie der Mathematik und Naturwissenschaft*, Oldenbourg, München.

Heinz Steinbring
IDM, Universität
Postfach 10 01 31
D - 4800 Bielefeld. Germany