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Mathematical Understanding in Classroom Interaction – the Interrelation of Social and Epistemological Constraints

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1) Introduction: Understanding as the deciphering of social and epistemological signs

The notion of understanding a mathematical concept or problem plays an important role in any educational consideration of school mathematics, be it in research investigations, be it in practical curricular constructions for the teaching of mathematics. To make possible, to support and to improve mathematical understanding seems to be one of the central objectives of any theoretical or practical endeavor in the didactics of mathematics.

There is a lot of reflection about how to understand understanding, how to classify different types and degrees of understanding (Pirie, 1988; Pirie & Kieren, 1989; Schroeder, 1987; Skemp, 1976), how to conceptually define understanding (Maier, 1988; Sierpinska, 1990a, 1990b; Vollrath, 1993) and many models describing ideal and everyday processes of understanding mathematics have been developed (Herscovics & Bergeron, 1983; Hiebert & Carpenter, 1992). Our first step in approaching the puzzling concept of mathematical understanding is to list a number of seemingly contradictory attributes which are associated with this notion.

* Understanding is conceived of as an expanding process, gradually improving the comprehension of a concept or problem step by step and there is never an absolute understanding; on the other side, when being confronted with a new, unsolved problem that seems to be totally incomprehensible, sometimes there is a sudden and complete understanding without any further doubt.

* Understanding requires to relate the new knowledge to the knowledge already known, to integrate the unknown into the known; but on the other hand, understanding a new mathematical concept often requires to comprehend it in itself, without any formal reduction to concepts already understood.
* Every fruitful process of understanding is based on the active treatment and negotiation of the learning subject; on the other hand, the understanding of fundamental mathematical ideas, concepts and theorems cannot be reinvented completely by every student and has to be passively accepted by him.
* Understanding is an individual and personal quality of the learning subject in every mathematical learning process; on the other hand, the understanding of abstract and general mathematical ideas often needs the social support and stabilization.

How these seemingly contradictory statements about the conception of understanding can be better explained or even solved? The central perspective which will be taken in the following is on problems of understanding school mathematics in the course of everyday teaching and learning processes; this is an important framing, because student's learning in school always is embedded in social and content dependent situations, subjecting the processes of understanding to specific conditions and intentions. In some way one could argue that the school environment is responsible for the described contradictions of understanding, because there is traditionally the combination of individual and common apprehension on the social side, and the construction of all new knowledge on the basis of the old knowledge on the content side.

The kernel of the problem of understanding school mathematics could be specified as follows: The students have to decipher signs and symbols! There are two different types of signs and symbols: mathematical signs and symbols (ciphers, letters, variables, graphics, diagrams, visualizations, etc.), and social signs and symbols (during the classroom interaction: hidden hints, remarks, reinforcements, confirmations, rejections, etc. made by the teacher, and comments and remarks of similar types made by other students). Being able to correctly decipher all signs of these two types would guarantee understanding. But the deciphering of the social and epistemological signs and symbols in classroom discourse is not an easy one, it is not a simple translation according to fixed universal rules and dictionaries. Both types of signs are intentional, they are referring to something other, and what they refer to may even change during the discourse or underlying a decisive shift of meaning.

For example, when introducing negative numbers with the idea of the debt model wherein an amount of debt is represented by a number of red counters (the black counters represent the positive numbers), the mathematical signs (red and black counters) are used with some kind of concrete reference; for more elaborated operations in this model the "rule of compensation" is needed, which represents the zero by a number of pairs of red and black counters. On the observable surface, the same sort of signs is used here, i.e. red and black counters with concrete reference. But in fact, simply the combination of the same number of red and black counters has drastically changed its referential function; even if traditional school mathematics pretends that these are still concrete objects and also are methodically used this way, from an epistemological perspective they have become true mathematical symbols. A concrete zero would have been represented by "nothing", by no black or red counter; this kind of representing the zero by a (unlimited) number of pairs, expresses a relation representing the zero, that is a true symbol. The concretely chosen combination of pairs is not the zero (as 5 red counters are the minus 5), but it represents symbolically the zero. (cf. Steinbring, 1993a). The mathematical sign of counters has totally changed its intentional reference.

And also the social signs given by the teacher during a classroom discourse, have no fixed and direct meaning but have to be deciphered relative to the specific context. The introduction of the "impossible event" during a course in elementary probability, for instance, may produce different referential interpretations for the students than those aimed at by the teacher. Whereas the students always have in mind some kind of concrete impossibility, the teacher tries to develop the idea of the mathematical impossible event, compared with the other elementary events of a sample space. And all his remarks and hints (social signs) he is giving in the interaction have a twofold referential meaning. The teacher, for example, asks: "How would we describe this with an adjective?", provoking the answer of a student: "...the uncertain event." And again the teacher: "The uncertain one? We shall simply say: the impossible event. And now my question: What kind of a subset is this, if I speak of an impossible event?" But for the students, this impossibility remains real and concrete, they answer directly: "That won't work at all!" All the teacher's remarks remain open for different intentional references, they cannot be made strictly unambiguous. (cf. Steinbring, 1991a). This means, the adequate deciphering of social and epistemological signs in a mathematical discourse remains a very difficult task.

The openness of the intentional references of epistemological and social signs and symbols in interactive mathematical processes is at the heart of the problems with the concept of understanding mathematics. This openness is responsible for the contradictory statements made about understanding. Because finding the "correct" intentional references for social and epistemological signs, always depends on individual and common insights, on personal activity and some passivity, on constructing completely new relations and using already known referential interpretations of mathematical concepts, and may provide complete, sudden understanding of a problem and at the same time give an idea of how to come to even more and more deeper understanding.

In the following we will relate our notion of mathematical understanding to conceptions discussed in the educational literature; in some way the following two positions which emphasize the linking of the learning subject with the mathematical content reflect aspects important for our formation of the concept of understanding. The first definition points to the external and corresponding internal representation of mathematical knowledge: "A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections." (Hiebert & Carpenter, 1992, p. 67). The second description tries to relate epistemological constraints of mathematical knowledge to the active role of the learner: "It is proposed to conceive of understanding as an act (of grasping the meaning) and not as a process or way of knowing. ... Relations between the notions of understanding and epistemological obstacle are found; it is argued that understanding as an act and the act of overcoming an obstacle can be regarded as complementary images of the same mental reality." (Sierpinska, 1990a, p. 24).

The first definition claims the integration of the new mathematical knowledge into existing networks of mental representations; the second description interprets understanding as an "act of overcoming an epistemological obstacle", an "act of grasping the new meaning" and in this way points to the limits of integrating the new knowledge into existing networks of representations. From our perspective of understanding as the deciphering of social and epistemological signs referring to a variable intentional context of reference, some modifications have to be considered with regard

to the presented conceptions of understanding. The signs do not only refer to fixed elements in the networks of representation, but also to <u>relations</u> to be constructed in the referential fields (this makes the network of representations and the process of integration more complicated, and even may lead to radical changes and reconstruction of networks of representations which this forces the true symbol function of the signs (cf. Steinbring, 1993b)). The active construction of meaning and of overcoming an epistemological obstacle is necessarily embedded in a social frame of conventions, generalizations, rules, methodological procedures and socially accumulated and accepted knowledge, being at everybody's disposal.

For analyzing mathematical understanding in interactive processes we propose to expand the perspective of investigation with regard to the following two dimensions:

<u>mathematical knowledge:</u>

social aspects <—> epistemological aspects

<u>participating persons:</u>

Students' understanding <—> teacher's understanding
(of mathematics and of teacher's intentions) (of students' understanding of mathematics).

In this way, conceiving of mathematical understanding as the deciphering of social and epistemological signs and symbols necessarily leads to a reciprocal process of understanding between teachers and students, everyone interactively trying to decipher social and epistemological signs communicated by other persons.

Deciphering signs and symbols	social/ conventional ◄►	epistemological
Teacher		
↕ Student		

The next section deals with the relationship between social/conventional and the epistemological constraints of the mathematical knowledge, and the section which follows will analyze this problem of understanding the development of mathematical knowledge embedded in classroom interaction. The investigation is based on specific examples of understanding mathematical knowledge and on exemplary cases of classroom episodes.

2) Mathematical knowledge and understanding

When trying to decipher mathematical signs one has to construct a relationship between the mathematical signs (and their network of representation) already constructed and the new signs still to be understood. What kind of relationship is this? Is it a reduction, an equilibrium, a dependency? And what kind of epistemological constraints and social conventions are involved in this relationship?

In a first example we will analyze the understanding of written symbols with the help of procedures describing the correct mathematical operations. In most cases, the understanding of decimal fractions is explained by describing the correct algorithms for the elementary mathematical operations of addition, subtraction, multiplication and division. For this description one may find the use of the
<u>rule of shifting the point</u>: The multiplication (division) of a decimal fraction by 10, 100, 1000, ... is done by shifting the point one, two, three, ... places to the right (left).
This rule describes the transformation of the decimal fractions into natural numbers. It is used to "define" the mathematical operations, for instance, addition and multiplication of decimal fractions:

"*Rule of addition and subtraction:*
Example:

$$\begin{array}{ccccc} 2{,}743 & + & 3{,}85 & = & 6{,}593 \\ \downarrow \cdot 1000 & & \downarrow \cdot 1000 & & \uparrow :1000 \\ 2743 & + & 3850 & = & 6593 \end{array}$$

Both terms of the sum are multiplied by such a power of 10 (here 1000) which gives natural numbers, which can be added. Afterwards, the enlargement is undone by division by 1000. Also the law of distributivity is used.

Rule of multiplication:
Example:

$$\begin{array}{ccccc} 3{,}45 & \cdot & 2{,}3 & = & 7{,}935 \\ \downarrow \cdot 100 & & \downarrow \cdot 10 & & \uparrow :1000 \\ 345 & \cdot & 23 & = & 7935 \end{array}$$

The theorem is used: If a factor is multiplied, the result also is multiplied accordingly." (Postel, 1991, p. 19).

This way of introducing the arithmetical operations for the decimal fractions clearly shows that, first, the decimal fractions are transformed into natural numbers, in this domain the known arithmetical operations are performed, and afterwards the result is re transformed to a decimal fraction, which is declared as the result of the new operation. Sure, the rule of shifting the point, should contain the new conceptual knowledge of decimal fractions, but in most cases this rule (or a similar version) is made conceptually vain by taking it as a technical device for counting the correct place of the point by reading this off from the surface of the written symbols.

This kind of explaining the decimal fraction by technically explaining the arithmetical rules also is inherent in the usual manner of lining up the two decimal fractions by putting the points in a line and then doing the already known operation. "Suppose the students have represented the procedure in such a way that they have connected the mechanics of aligning digits with the combining of quantities measured with same unit (ones, tens, hundreds, and so on). When these students encounter addition and subtraction with decimal fractions, they are in a good position to connect the frequently taught procedure – line up the decimal points – with combining quantities measured with the same unit. If they build the connection, the addition and subtraction procedure becomes part of the existing network, the network becomes enriched, and adding and subtracting decimals is understood." (Hiebert & Carpenter, 1992, p. 69).

This reductive explanation of the arithmetical operations of decimal fractions with the help of the known algorithms of written procedures for the arithmetical operations displays two important aspects:
<u>First</u>, the new concept of decimal fractions is not used itself, but it is transformed into natural numbers students already are able to operate with.
<u>Second</u>, the rule of transforming decimal fractions into natural numbers, which should conceptually regulate the use of the new sign of point (and in this way clarify an important conceptual aspect of the decimal fraction) is devaluated to a meaningless counting scheme.

When using the technique of lining up the numbers, the counting of the correct position of the point is even superfluous for addition and subtraction, thus really suggesting that decimal fractions are some kind of natural numbers and not something conceptually new. This kind of understanding decimal fractions as natural numbers is known from analyzing students errors. A widely used implicit idea of decimal fractions is the understanding that the point separates the given number with a point into two natural numbers (cf. Wellenreuther & Zech, 1990). For instance, the following types of transformations and arithmetical operations could be observed according to this perspective:

1.25h —> 85 minutes
5.3 + 2.42 —> 7.45
18.27 : 9 —> 2.3
0.2 · 0.4 —> 0.8

This way of understanding and of manipulating decimal fractions is in principle justified on the same basis of conventional rules as described before:

First, the decimal fractions are transformed into natural numbers for performing the required operations.

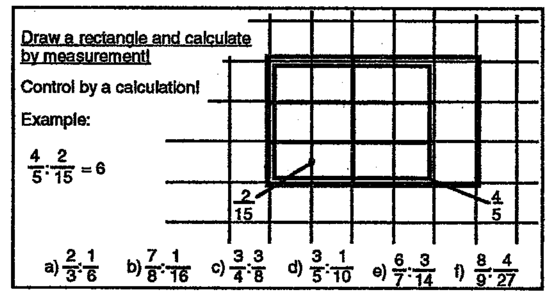
Second, the rule of transforming decimal fractions into natural numbers is a meaningless scheme: there is one natural number left and one right to the point.

The analysis shows that both ways of operating with decimal fractions, i.e. the students' procedure and the "official" one, are justified on the same grounds; when reducing the concept of decimal fractions to natural numbers and their operations and at the same time intending to make the rule of transformation as simple as possible (an intention which necessarily leads to an evacuation of conceptual relations which at the beginning have been contained in the meaning of the rule) then there is no possibility to explain why the students' rule is incorrect. The correctness of rules to be used and applied reduces to external social conventions only, it is the teacher who decides which rule is correct.

This first example of introducing decimal numbers makes clear, that understanding a new mathematical concept cannot be done by reducing it completely to concepts already known. When trying to enhance understanding by describing the operational rules in such a reductive process, then there is the great danger of changing totally the status of the rules from mathematical operations referring to conceptual aspects into formalized conventional recipes. Understanding is not the total integration into the existing network of representations, but for the concept of decimal fractions it is the other way around: Decimal fraction symbols have not to be deciphered as natural number symbols, but they the natural number symbols have to be re-considered as a kind of decimal fraction symbols (cf. Steinbring, 1989, 1991b, 1992).

In a second example we will analyze the understanding of written symbols and mathematical concepts with the help of introducing referential objects for providing mathematical meaning for the new symbols. Similar to the understanding of decimals, the introduction of fractions often is done by transforming the new symbols into natural numbers and performing the elementary arithmetical operations in this number domain. For example, the following operation with fractions $^4/_5 : {}^2/_{15}$ is explained according to the recipe: "Fractions are divided by multiplying with the reciprocal value of the second fraction. Fractions are multiplied by multiplying the denominators and multiplying the numerators." In this way one simply gets arithmetical operations already known: 4·15 and 5·2 leading to the result: $^{60}/_{10}$ or 6. This kind of procedural reduction of the new symbols to known symbols has been explained in the example before. Now the focus is on the construction of referential meaning for the new symbols.

Consider the following problem from a textbook for 6th-grade students:



This problem deals with the division of fractions and tries to use a graphic diagram to mediate in a direct way the meaning of fraction division. This contrast between formula and graphic diagram is suitable to clarify some epistemological aspects between sign and object (or referent) in school mathematics. On the one side, there are mathematical signs connected by some operational symbols, functioning as a little system: On the other side, there is a geometrical reference context, intended to furnish the understanding of the signs and operations. The diagram should support the process of constructing a meaning for the formula. The *relational structures* in the geometrical diagram and the formula are the important aspects and not the signs itself.

In which way can this diagram give meaning to the formula? Is it possible to deduce the idea of the division of fractions from it? Is it adequate to conceive of the elements in this diagram as concrete objects for directly showing the meaning of division?

First of all, one observes that all problems to be tackled have denominators that are a multiple of the denominator of the other fraction. Consequently, the intended explanation with the help of the diagram cannot be universal. A certain type of fractions seems to be presupposed, indicating a first reciprocal interplay between diagram and formula. There are more indications for this interplay: In this representation, a variable comprehension of 1 or the unit is necessary. The big rectangle with the 15 squares once is the unit, used to visualise the proportions of $^{4}/_{5}$ and $^{2}/_{15}$ as four rectangles (with 3 squares each) and as a rectangle of 2 squares respectively. The composition of three squares to a rectangle represents a new unit or 1. When interpreting the operation $^{4}/_{5} : ^{2}/_{15} = 6$, the epistemological meaning of the result "6" changes according to the changes of the unit. How is the 6 represented in the diagram? It cannot be the sextuple of the original rectangle, hence no pure empirical element.

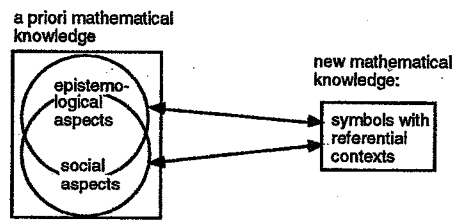
The 6 could mean: In $^{4}/_{5}$ there are 6 times $^{2}/_{15}$ or there are 6 pairs of two squares in $^{4}/_{5}$. Or, interpreting $^{4}/_{5}$ as $^{12}/_{15}$ as implicitly suggested in the diagram itself, the operation modifies to: $^{12}/_{15} : ^{2}/_{15} = 6$. But this is nothing else than the operation: $12:2 = 6$, because the denominator can be taken as a kind of "variable," that is, the 15 could also be 20, or 27, and so forth. In this division, in principle, the half is calculated, a division by 2 is made.

The analysis shows changing interpretations of the unit: First, the unit is represented by the big rectangle of 15 squares, then one single square also represents the unit. The epistemological reason is that a fraction like $^{12}/_{15}$ is not simply and exclusively the relation of the two concrete numbers 12 and 15, but a single representative of a lot of such relations: $^{4}/_{5}$, $^{8}/_{10}$, $^{16}/_{20}$, $^{20}/_{25}$, . . . What is defined as the unit in the diagram is partly arbitrary and made by some convention, and, furthermore, the constraints of the geometrical diagram and of the given numerical sign structure determine partly the choice of the unit. For instance, for this arithmetical problem, it would not be an adequate choice to take the rectangle of 5 x 7 squares as the unit; wheras a rectangle of 6 x 10 squares, or subdivision of the squares into quarters, would be valid.

The intentional variability implicit in the numerical structure of a fraction is partly destroyed in the geometrical diagram used to represent the fraction; this variability has to be restored in the diagram by means of flexibly changing the unit. The concrete single diagram, with its parameters once chosen, has to be conceived of as a "general" diagram.

This shows that the meaning of the new symbols cannot be directly provided by the given referential context in its customary perception. When starting with visual representations of units and parts for displaying the idea of fractions as a means to distribute something, there is then a sudden shift in this relation between the symbols and their referential context when dealing with the division of fractions which cannot be explained within the pre-given frame of empirical things and their concrete distribution. The symbol system: $^{4}/_{5} : ^{2}/_{15}$ does not simply show a greater complexity, but also changes drastically its referential relationship: the old network of representations has changed in a way that now new relations in this network have to be constructed. The diagram no longer displays objects (rectangles of different shapes) but relations between objects (the relation $^{4}/_{5}$ or the "unit" - relation in all possible combinations of geometrical objects).

Understanding new mathematical symbols, being able to decipher epistemological and social signs requires the construction of a relation between the knowledge already known and the new concept (the symbol with its new meaning).



This relation between the new and the old knowledge cannot be a complete reduction of the new symbols and their operations to the old symbols and their operations. With the intention to base the understanding exclusively on the procedural links between the new and old symbols, there is the danger of converting mathematical rules to meaningless conventional recipes. And when relating the new concepts to the referential objects already used for the known concepts, there also is the danger of simplifying the meaning of the new concepts by making the symbols names for objects (cf. Steinbring, 1988). The new mathematical concept has to be conceived of as a new, proper symbol with a changed referential context: The new intentional references of the symbol which still have to be deciphered forces the transformation and change of the old referential context. This perspective to look from the new concepts towards the old concepts is only possible if there is an interplay between the epistemological and social / conventional aspects of mathematical knowledge.

3) Classroom interaction and understanding

When trying to investigate processes of understanding mathematics in everyday classroom settings at once it becomes obvious that every person's inclination to understand always means to discover the intentional meaning of the epistemological and of the social, conventionalized signs and that he/she has to relate them in some way. Thus, for students it is important to figure out the teacher's intentions with regard to the school mathematical knowledge, and also the teacher has to become clear about the intentions students follow in processes of understanding mathematics. And this reciprocal way of understanding the intentions of the partners for being able to understand the new mathematical knowledge relies on the intricate relation between the "a priori mathematical knowledge" and the "new mathematical knowledge" (in its twofold dimensions), as well as for students as for the teacher.

The following analysis of two short classroom episodes will demonstrate in more detail the dependencies between epistemological and social/conventional aspects of the knowledge in question and how the students' intention to understand relates to the understanding the mathematics teacher has of the students' understanding. The first example concentrates on how the students are pushed to follow the a priori understanding of the teacher which leads to an acceptance of the conventionalized aspects of the knowledge without really understanding the epistemological point of the concept dealt with. The focus of the second example is on how a student understands an epistemological relation of the new mathematical knowledge which is not accepted by the mathematics teacher because she strictly adheres to her fixed a priori mathematical knowledge.

During the first episode dealing with the topic "What is relative frequency?" (see appendix 1), the teacher tries to recall the concept of relative frequency. She starts with the question: "What do we understand by relative frequency, Markus?" (1). Already

the way of formulating "what do we understand ..." indicates that there was some accepted, conventionalized form of noting the concept of relative frequency. The first student, Markus, collects nearly all the necessary elements, i.e. "the number of cases observed", "the number of trials", and he points out in a way of joking that these elements have to be combined.

Then the student Klaus proceeds in another direction, saying "Relative frequency means for example often, it is, hem, a medium value." (6). The teacher rejects his proposal, and the student Frank goes on with the old idea, trying some possibilities for combining the elements already identified: " ... the trials are divided with the cases observed, I think, or multiplied." (9,10). This is strongly approved by the teacher with the statement: "Markus did already say it quite correct, just the decisive word did miss..." (11). This now clarifies the accepted frame for the students to handle the question: The way of searching the needed pieces of a puzzle is expected, the two pieces "the number of cases observed" and "the number of trials" are already found, another decisive piece (the mathematical combination of the two other) is still missing. The very quick proposals "Subtract" and "Take it minus" provoke a severe refutation by the teacher: "That's incredible!" (15), leading to an alternative approach by asking for the way of how this is written down.

The phase (17 – 42) opened by the question: "... you should ask how you will write it down?" (17) introduces the context of fraction and fractional calculus into the discussion. Unlike she expected, the key words "fraction" and "fractional calculus" (24,26) evoke students' contributions as "...subtracted in the fractional calculus." (31), "...one as denominator and the other as numerator, ..."(35,36), "To calculate a fraction!" (37), and "To reduce to the common denominator." (42). The teacher simply intended to point to the link between fraction and division, a seemingly simple transfer which only could be made toward the end of this phase in a funnel like pattern (Bauersfeld, 1978).

The teacher's question: "... what kind of calculation is it then, if you write a fraction?" (38,39) forces the "correct" answer: "Dividing". The last piece of puzzle is found, now it remains to combine all three in the right way (phase 43 – 48). The first offer made by the teacher: "... relative frequency is ... the number of cases divided by ...? (43,44) does not use the detailed vocabulary and consequently the answer of a student does not fit: " ... the number of cases observed." (45). At last the correct formulation is initiated by the repeated complete teacher question: "No, the cases observed divided by ...? Aha!" (46). Finally, relative frequency is also formulated with means of fractional calculus.

The earlier codified official understanding of the concept of relative frequency is represented in the statement: "Relative frequency is the number of cases observed divided by the number of trials." This verbal description often is visualized as a fraction in the following manner:

$$\text{relative frequency} = \frac{\text{number of cases observed}}{\text{number of trials}}.$$

This representation in form of a fraction simply should express the operation of division, no further relations to fractional calculus are implied. The teacher starts the repetition of the concept of relative frequency with this accepted "definition" of the concept; this seems to represent her a priori mathematical knowledge what has to be reconstructed.

Her formulation, saying "What do we understand by relative frequency ...?" (1) is a signal pointing to the fact, that there is nothing completely new to be discovered, but something already introduced and codified. Nevertheless, the student Klaus presents something very open, which could lead to an epistemological and conceptual consideration of this concept. But the teacher strictly refutes this orientation, and she adheres to the course already taken, i.e. searching for the matching pieces of the conventionalized definition. The two main ingredients are already detected, and the teacher signals that only "the decisive word" still misses. This sign can only be partly interpreted by the students: they know to be on the right track, and they have to look for a mathematical operation, but which one?

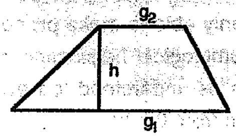
This becomes a meaningless guessing game, and the teacher gives another signal demanding the students to remember how the relative frequency is written down. This produces the framing of a fraction, evoking all sorts of descriptions connected with fractions and fractional calculus. The signal "fraction" and "fractional calculus" the students misleads in some way, and they are not able to simply decipher the analogy of the fraction bar with the operation of division, the teacher is aiming at. This is only possible after some discussion and with a further direct signal: "... what kind of calculation is it then, if you write a fraction?" (38, 39). And also in the end of this episode, the teacher feels forced to make her signal very explicit for getting the correct, expected answers.

The signals given by the teacher in the course of this episode, intend to orientate the students towards the conventional agreed "definition" of the concept of relative frequency. This intention is understood very quickly by most students, and the rejection of Klaus' proposal perhaps leading away from this orientation reinforces the understanding of this intention. The more the accepted answers are restricted to the social/conventional side, the more the students are deprived of some epistemological means of justification. They are only able to make proposals to be approved or rejected by the teacher. The epistemological helplessness of the students becomes obvious when the signal "fraction" is understood too extensively instead of simply reading off the operation of division from the fraction bar.

The a priori knowledge of the teacher dominates the course of this episode: first she starts with her fixed understanding of the concept of relative frequency as it already has been "defined" in the class as a special fraction; second she presupposes that her students already had arrived at a similar "definition" as she herself. In this way, the process of understanding simply reduced to something of re-finding a conventionalized description for a concept already given. And all the signals the students are giving during this discussion are simply judged by the teacher according to how they ensured to reach as directly as possible the goal aimed at. The restriction of the interaction on the social/conventional level makes the understanding of a piece of mathematical knowledge which is already judged as definitely given and fixed a priori a process of negotiating the right words, names and rules, and its validity in most cases only can be delivered by the teacher's authority (cf. Steinbring, 1991a).

During the second episode coping with "The area formula for the trapezoid" (see appendix 2), the students are asked to explore in two different ways a new area formula. They already have some experiences, because they have in a similar way discussed the formula for the parallelogram. With her first long statement (1-8) the teacher gives some introducing hints and she signals positively and negatively how the two expected ways of geometrical construction and argumentation might function. For the first solution, she indicates to enlarge the trapezoid and to construct a parallelogram. With reference to the fabrication of the area formula for the parallelogram done the day before, she explicitly points to the way of how to cut off angles and adding them on other sides for bringing about the second solution; here, she emphasizes, it is not possible to cut off a whole angle, as in the case of the parallelogram, but one has still to try to construct a rectangle.

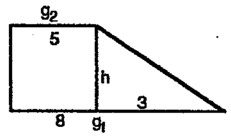
The students perform the expected first solution; now the discussion of a possible second solution starts. A student makes a proposal not yet to be understood in its details: " ... could one not, this line here, the height ... ... make parallel to, to this other line, and then one would have such a quadrangle and on the other side a triangle ..." (11,13,14). The student points to a drawing:



According to her hints given at the beginning, the teacher refuses this proposal, because she understands, a whole angle, i.e. the big triangle should be cut off, which is not possible as she believes: "... the problem is, that one only knows the whole basis line g_1 and not only this little piece there. That's totally unknown. Well, when you really have concrete lengths, you could draw and measure it. But in general I can't do it. How should I then say, g_1 minus some small piece, but how long is this little piece at all? Hence, that does not work!" (17 - 21). By looking at only one triangle to be cut off from the trapezoid, the first impression is that one cannot arrive at a formula, because one cannot determine explicitly the basis line of this triangle. (This approach could lead to a solution, if also the other triangle on the left side is cut off, then the basis lines of both triangles together would be g_1 - g_2 which would be sufficient to know for deducing a formula for the trapezoid.)

The student is not impressed by the teacher's argument, he takes up again his idea, which seems to be different from the idea turned down by the teacher. "Yes, if we now, ehm the longer side .... well, take g_1 minus g_2 ... .. and then, ehm, then one has, there would be a remainder; let's say g_1 would be 5, hence 8 minus 5, giving 3 and then, if one would push now g_2 backwards, in a way giving a right angle, then something would remain there on the right side ... " (23, 26-29). The teacher wants not to follow his idea, but the student continues: "... I do mean something else, there you take 5 times h, then, what is left here, there remains 3 centimeters, that is a slope, yes, then, ehm, you also take the slope ... ... the remainder is 3 times the height then, divided by two ..." (32-34, 35). The teacher interjects: "... I really have understood this, ..." (35), when the student develops his line of argumentation, but she did not understand the student's idea, but intends to explain, that she has understood really the apparent incorrectness of his proposal; with her last contribution the teacher begins to present what she thinks is a correct second solution (37-40).

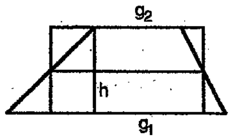
Probably, the student's proposal could be explained by the following diagram; this was not shown in this way by the student, he simply expressed his idea verbally. One change the student makes is to substitute the variables g_1 and g_2 by concrete numbers, but still he is able to use these numbers for developing in principle a correct universal formula. The student has transformed the trapezoid to a rectangle together with a triangle in the following way:



By what he has called " ... if one would push now g_2 backwards ..." the student constructs a rectangle and a rectangular triangle allowing for the calculation of the whole area. And then he determines the areas of both surfaces: 5·h for the rectangle and $^{3 \cdot h}/_2$ as the area for the triangle. If the teacher could accept this, a retranslation of the concrete values 5, 3, and 8 into the corresponding variables could be made giving a general formula for the area of the trapezoid:

$$A = g_2 \cdot h + \frac{(g_1 - g_2) \cdot h}{2} = \frac{2 \cdot g_2 \cdot h + (g_1 - g_2) \cdot h}{2} = \frac{(g_1 + g_2) \cdot h}{2}$$

This is a perfect solution, but unfortunately not in the scope of the teacher, and she is not able or not willing to really uncover the student's argumentation. She has a fixed a priori geometrical construction in mind for the second solution which prevents her understanding of the student's solution. In her last contribution the teacher explains her expectation. The students should have drawn the medium line, just in agreement with the earlier given hint, not to cut off a whole angle but only parts for constructing in this way a rectangle:



When developing this construction of the medium line, same problems might arise as classified generally unsolvable by the teacher, i.e. to determine the length of the basis line of the small triangles to be cut off. From the final complete drawing with the medium line, the length of the medium line can be generally determined as

$$m = \frac{(g_1 + g_2)}{2}.$$

What sorts of understanding can be observed in this short episode, both on the side of the teacher and of the side of this student? Similar to the episode discussed before, also the teacher here starts with her definite a priori knowledge about the admitted geometrical constructions and linked deductions of the area formula for the two solutions sought for. At the beginning, the teacher informs in some way the students about her intentions and her a priori ideas of the mathematical problem. And the first solution is presented just as the teacher expected: the students constructed the right parallelogram with area twice as big as that of the trapezoid, thus developing the correct formula. All the signals the teacher has given before, could be deciphered by the students in the expected way.

When discussing the second solution, this frame of trying to correctly decipher the signals given before and keeping to the further hints, rejections and reinforcements of the teacher changes. One student seems to be very sure about his solution, and he stays presenting it despite serious counter remarks made by the teacher, even criticizing the mathematical correctness and the impossibility of the proposed solution. First, the teacher's reactions intend to signal that the student, as others too, is pursuing a false direction: "... yes, ehm, some others also made this proposal, ..." (15), and: "Hence, that does not work!" (21), or later: "... pay attention, ..." (30). But because the student is not sensible to the teacher's hints, the teacher now directly rejects the student's proposal: "... I really have understood this, ..." (35) expressing in this way the student should definitely stop now presenting his terrible argumentation. At last, the teacher introduces her correct mathematical deduction of the second solution with a loud and distinct "... no!!".

Never, the student is willing to quit his argumentation and to enter the intended frame of discussion trying to uncover the a priori fixed geometrical construction the teacher has in mind. No remark of the teacher hinders the student to finish his argumentation, which always is in conflict with the solution the teacher is aiming at. The discussion of the first solution develops according to the expected context, the students refer to the teacher's ideas and try to match their own contributions to the social conventions and expectations. Now there is a rupture of this former implicit agreement, one student presents a complete new and unforeseen solution; the

teacher is unable to decipher really the students proposal and at the same time this is prevented by the fact that she always has in mind her own a priori idea, she wants to push forward and she uses to reject the student's proposal.

During this short episode analyzed, the teacher is not able to move the interaction into the planned frame, nor to reduce the mathematical discussion solely on the social/conventional level; the student violates in some sense the conventions of how to proceed in classroom discussion when developing new knowledge in agreement with the teacher's expectations, requirements and her definition of the accepted context and manners of justifying new knowledge (mostly in a way of reducing it to the teacher's implicit or explicit ideas and demands).

The persistent attitude of the student, based on his conviction to have a mathematical solution which is always put into conflict with the teacher's expectation, is a main reason that here the process of understanding focuses on epistemological aspects of the new knowledge. A really changed perspective from the new knowledge (i.e. the area formula developed in the student's argumentation) is possible toward the a priori knowledge giving new insights for the old knowledge. But the teacher remains unaware of this, she is not able to really understand that this student has made an advanced mathematical understanding of the area formula for the trapezoid and has not simply discovered the already existing solution of the teacher by reducing all his ideas and proposals to this ready made old knowledge, by entering the question-answer game between students and teacher on the social/ conventional level.

4) The necessary equilibrium between social and epistemological aspects in interactive processes of understanding – Consequences

We have started our reflection about the problem of understanding mathematics in interactive processes with the description that understanding is the deciphering of social and epistemological signs and symbols. An important aspect inherent in this idea is the fact, that signs and symbols posses an intentionality, i.e. they are referring to something else which has to be detected and to be constructed, in epistemological as well as in social/ communicative regard. The intentionality of signs and symbols displays an openness which at the same time cannot be arbitrary but has to be framed by epistemological constraints and by social conventions.

The framing of signs and symbols in an interactive mathematical discourse is manifold. From elementary algebra it is known that, for instance, the variables x, y and a, b etc. evoke specific interpretations (often also methodically desired by the teacher); and students also have learned to read correctly the implicit communicative signals the teacher is sending (examples can be found in our two short episode). In this way the negotiation of meaning in classroom interaction at the same time is a process of mutually framing the intentionality of the signs and symbols which are communicated (cf. Bauersfeld, 1983, 1988; Voigt, 1984a, 1985). The feedback one gets on the remarks made, configurates in some instances the intentionality of signs and symbols under discussion.

During classroom interaction when trying to organize the understanding of new mathematical knowledge, not only the reciprocal framing and interpreting of signs and symbols can be observed, this interactive negotiation also is based on a discursive frame structuring and legitimizing the process of understanding and what are accepted forms of understanding. Thus, understanding a mathematical concept or problem in classroom teaching, is not simply a direct, right determination of the intentionality of the mathematical symbols in question, but what could be the right and adequate interpretation also depends on the social and conventional rules and patterns of what is an accepted understanding, which in this way can be reproduced and communicated, that is an understanding which could also be understood by the other partners of the social environment (cf. Maier & Voigt, 1989; Voigt, 1984b).

Accordingly, the understanding of mathematical knowledge is a reciprocal relationship of understanding the new mathematical knowledge in its own right (i.e. integrating the old knowledge structure into the future, new knowledge structure (as in the example of the negative numbers)), and at the same time of organizing and formulating this understanding of new knowledge in the frame of the conventional, legitimized and accepted patterns of social knowledge justification (or of demonstrating the necessity of modifying the social, conventional framework). The student has to understand the new mathematics and at the same time he has to be able to express his mathematical understanding in the setting of the conventional social patterns, descriptions and metaphors. And both requirements have to be fulfilled until it can be "verified" that the student really has understood.

This socially conventionalized discursive frame of understanding is indispensable; but it cannot be conceived of as a fixed organizational scheme according to which

understanding mathematics could be correctly defined or even be deduced. It describes, for instance, what are legitimate patterns of mathematical argumentation, what are accepted forms of proof, what are accepted analogies, what degree of rigour is needed, whether an example is sufficient as a general argument, etc. In this way the conventionalized discursive frame of understanding is open to change and modifications occurring in the course of further mathematical development.

The reciprocal relationship between directly understanding the new mathematical knowledge and organizing it in a socially conventionalized discursive frame of understanding is based on an equilibrium between social and epistemological aspects of mathematical knowledge. Certain intentions of mathematical signs and symbols have to be agreed upon socially, generalizations of mathematical concepts growing in the course of knowledge development have to be socially sanctioned, and the relational structure of this enlarging new knowledge has to be controlled epistemologically. Epistemological changes and reorganizations of mathematical knowledge can only be performed if they are in agreement with the socially conventionalized patterns of argumentation and understanding, which themselves depend on the epistemological character of mathematics in a way that their modifications and changes are subject to evolving new epistemological constraints.

The equilibrium between understanding the new mathematical knowledge and organizing it in a socially conventionalized discursive frame of understanding, to some extend, is destroyed in everyday mathematics teaching. The discursive pattern becomes an interactively constituted methodical ceremonial for generating common understanding of schoolmathematical knowledge. The essence of this methodical ceremonial is to collect the necessary elements (i.e. descriptions and operational signs) for the knowledge under discussion and to combine them for getting the expected methodical metaphor for the mathematical knowledge (cf. Steinbring, 1991c). In our first episode, this was the concept of relative frequency, described by the statement "Relative frequency is the number of cases observed divided by the number of trials.", written down in the shape of a fraction:

$$\text{relative frequency} = \frac{\text{number of cases observed}}{\text{number of trials}}.$$

For the second episode, the teacher expected the construction of the geometrical methodical metaphors of a parallelogram (made of two trapezoids) and a rectangle for developing the area formula; the students are expected to collect all necessary elements, to combine them and to reconstruct what the teacher already knows.

The main reason for the dominance of this methodical ceremonial is the central objective of mathematics teaching to make mathematical understanding (directly) possible, and this simultaneously requires that the teacher already has understood. On the basis of his understanding, i.e. his fixed a priori knowledge, the teacher now organizes the methodical course of his teaching, which, in many cases during everyday teacher-student-interaction, causes the total replacement of epistemological aspects of knowledge by conventionalized communicative strategies. Because the teacher already understood the problem and integrated all knowledge elements into a network of representation, for the students only remain the correct discovering of these elements, a search which is guided and made easy by the teacher's social signs and hints indicating for the students whether they are on the right track and how far they are still away from the goal.

A dilemma arises for real interactive processes of mathematical understanding: the more and better the teacher already has understood the new mathematical knowledge in question (and, of course, the teacher has the strong obligation to understand all the mathematics before), the greater the danger, that the organizing of processes of understanding in mathematics teaching degenerates to the described methodical ceremonial for generating common understanding of schoolmathematical knowledge, which in its essence means a prevention of true mathematical understanding. The first episode shows the degeneration of the concept of relative frequency (indeed some kind of medium value) to a written (and spoken) form of fraction (numerator and denominator): The epistemological relation of this mathematical concept is in no way operationally modeled. And in the second episode, the teacher cannot understand (i.e. integrate into her a priori knowledge) the epistemological reflections the student is developing for finding the area formula.

A re-establishment of the equilibrium between understanding the new mathematical knowledge and organizing it in a socially conventionalized discursive frame of understanding is only possible in everyday mathematics interaction, when the teacher becomes aware that he/she himself/herself has to really understand something in this process, and that not everything is already understood before. In most cases, this understanding the teacher has to construct concerns the problems students have when actually going through an interactive process of understanding, and even the understanding of new <u>mathematical</u> knowledge might be required by the teacher, too. In the first episode, the consideration of the student's remark: "Relative frequency means for example often, it is, hem, a medium value." (6) could have led

to a better mathematical understanding of the concept than the formal translation into the form of a fraction, but this was impossible because of the definite a priori knowledge, the teacher strongly relied on.

The second episode clearly shows, that there really is something - even mathematically - the teacher has to understand, what she did not know before, but this is an understanding she refused because of the obvious differences to her a priori knowledge on which she wanted to base her teaching process methodically. Indeed, the argumentation of the student is not easy to understand and it is not only the collection of already existing elements and its operational combination. It is the actual, developmental construction of mathematical relations using simultaneously some social conventions. The central idea of understanding for the student is the variable changement of the trapezoid to a geometrical surface fulfilling different requirements: Keeping the area unchanged and producing geometrical figures of which the area formulae are already known. At the same time the student uses a convention, widely accepted by students, namely taking concrete numbers instead of variables, but using them without any further restriction. A real mathematical convention, whose acceptability might be negotiated.

The analysis of the student's process of understanding shows the interrelation of social, conventional and epistemological constraints to be developed and related for really producing understanding, i.e. for grasping actively the new meaning. Here, the teacher herself really could get much new understanding, instead of trying to force the student to enter her fixed methodical frame of understanding. In this way, the teacher trying to understand the student's process of understanding could then support the integration of this kind of understanding into the conventional, legitimized and accepted patterns of social knowledge justification for the other students. This would allow to question and to modify the social conventions of codifying the understanding of schoolmathematical knowledge according to new, unforeseen epistemological insights.

References

Bauersfeld, H. (1978). Kommunikationsmuster im Mathematikunterricht - Eine Analyse am Beispiel der Handlungsverengung durch Antworterwartung. In H. Bauersfeld (Eds.), Fallstudien und Analysen zum Mathematikunterricht (pp. 158-170). Hannover: Schroedel.

Bauersfeld, H. (1983). Subjektive Erfahrungsbereiche als Grundlage einer Interaktionstheorie des Mathematiklernens und -lehrens. In H. Bauersfeld et al. (Eds.), Lernen und Lehren von Mathematik (pp. 1 - 56). Köln: Aulis.

Bauersfeld, H. (1988). Interaction, construction and knowledge: alternative perspectives for mathematics education. In D. A. Grouws, T. J. Cooney, & D. Jones (Eds.), Effective mathematics teaching (pp. 27-46). Reston, Virginia: NCTM & Lawrence Erlbaum.

Herscovics, N., & Bergeron, J. C. (1983). Models of understanding. Zentralblatt für Didaktik der Mathematik 15(2), 75 - 83.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. Grouws (Eds.), Handbook of Research in Teaching and Learning Mathematics (pp. 65-97). New York: Macmillan.

Maier, H. (1988). "Verstehen" im Mathematikunterricht - Explikationsversuch zu einem vielverwendeten Begriff. In P. Bender (Eds.), Mathematikdidaktik - Theorie und Praxis. Festschrift für Heinrich Winter (pp. 131-142). Berlin: Cornelsen.

Maier, H., & Voigt, J. (1989). Die entwickelnde Lehrerfrage im Mathematikunterricht (Teil 1 u. 2). mathematica didactica 12(1u. 2/3), 23-55 u. 87-94.

Pirie, S. E. B. (1988). Understanding: Instrumental, relational,intuitive, constructed, formalised ...? How can we know? For the Learning of Mathematics 8(3), 2 - 6.

Pirie, S. E. B., & Kieren, T. (1989). A recursive theory of mathematical understanding. For the Learning of Mathematics 9(3), 7 - 11.

Postel, H. (1991). Konzeptionen zur Behandlung der Dezimalbruchrechnung in der Bundesrepublik Deutschland. Der Mathematikunterricht 37(2), 5-21.

Schroeder, T. L. (1987). Students' understanding of mathematics: A review and synthesis of recent research. In Proceedings of Psychology of Mathematics Education, XI (pp. 332 - 337).

Sierpinska, A. (1990a). Some remarks on understanding in mathematics. For the Learning of Mathematics 10(3), 24-36.

Sierpinska, A. (1990b). Epistemological obstacles and understanding: two useful categories of thought for research into the learning and teaching of mathematics. In Proceedings of the 2nd Bratislava International Symposium on Mathematics Education, August 1990. Bratislava: University of Bratislava.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. Mathematics Teaching (77), 20-26.

Steinbring, H. (1988). Nature du Savoir Mathématique dans la Pratique de l'Enseignant. In C. Laborde (Eds.), Actes du Premier Colloque Franco-Allemand, de Didactique des Mathématiques et de l'Informatique (pp. 307-316). Grenoble: La Pensée Sauvage.

Steinbring, H. (1989). Routine and Meaning in the Mathematics Classroom. For the Learning of Mathematics 9(1), 24-33.

Steinbring, H. (1991a). Mathematics in teaching processes - The disparity between teacher and student knowledge. Recherches en Didactique des Mathématiques 11(1), 65 – 107.

Steinbring, H. (1991b). Eine andere Epistemologie der Schulmathematik - Kann der Lehrer von seinen Schülern lernen? mathematica didactica 14(2/3), 69-99.

Steinbring, H. (1991a). The concept of chance in everyday teaching: Aspects of a social epistemology of mathematical knowledge. Educational Studies in Mathematics 22, 503–522.

Steinbring, H. (1992). The relation between social and conceptual conventions in everyday mathematics teaching (Manuscript). Bielefeld: IDM.

Steinbring, H. (1993a). Symbole, Referenzkontexte und die Konstruktion mathematischer Bedeutung - am Beispiel der negativen Zahlen im Unterricht (Manuskript). Bielefeld: IDM.

Steinbring, H. (1993b). Epistemology of Mathematical Knowledge and Teacher–Learner Interaction. The Journal of Mathematical Behavior.

Voigt, J. (1984a). Interaktionsmuster und Routinen im Mathematikunterricht - Theoretische Grundlagen und mikroethnographische Falluntersuchungen. . Weinheim : Beltz .

Voigt, J. (1984b). Der kurztaktige, fragend-entwickelnde Mathematikunterricht. mathematica didactica 7, 161 - 186.

Voigt, J. (1985). Patterns and Routines in Classroom Interaction. Recherches en Didactique des Mathématiques 6(1), 69-118.

Vollrath, H.-J. (1993). Paradoxien des Verstehens von Mathematik. Journal für Mathematik-Didaktik 14(1), 35 - 58.

Wellenreuther, M., & Zech, F. (1990). Kenntnisstand und Verständnis in der Dezimalbruchrechnung am Ende des 6. Schuljahres. mathematica didactica 13(3/4).

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Appendix 1: Transcript 1, grade 7: "What is relative frequency?"

T.: What do we understand by relative frequency, Markus?
S.: Relative frequency is, if you take the number of the cases observed together with the number of trials, well you throw all into one pot, and stir it up.
S.: Hahaha, good appetite!
T.: Klaus?
S.: Relative frequency means for example often, it is, hem, a medium value.
T.: Yes, what....? Then it's better to say nothing!
T.: Relative frequency, Frank?
S.: Hem, relative frequency does mean, when the trials are divided with the cases observed, I think, or multiplied.
T.: Markus did already say it quite correct, just the decisive word did miss,
T.: ... but now you do know it, Markus?
S.: Subtract.
S.: Take it minus.
T.: That's incredible! Silvia?
S.: In a chance experiment, the number of cases observed, when you ....
T.: Hem, Markus, you should ask how you will write it down?
S.: Oh yes, OK., well, relative frequency is, when you, the number of trials with the number of cases observed, well, if you then ...
T.: But how you write it down? How do you write it down?
T.: Come on, write it on the blackboard!
S.: Shortly writing down.
T.: Thank you, Ulli.
S.: Writing as a fraction.
T.: Aha, as a fraction, so, what is it then, what kind of calculus?
S.: Fractional calculus.
T.: Well, a complete sentence, Markus!
S.: Hem, well relative frequency is ...
T.: Let it aside.
S.: ... the number of, hem, the number of cases observed with the number of trials, yes, subtracted in the fractional calculus.
T.: Subtracted in the fractional calculus!!! Markus, that's incredible!
T.: Ulli, formulate it reasonably!

S.: If you take that, the number of the, the cases observed, yes, and the number of trials, hem, well, if you then, one as denominator and the other as numerator, hem, I don't know what you mean.
S.: To calculate a fraction!
T.: Either to write as a fraction, or what kind of calculation is it then, if you write a fraction?
S.: Dividing.
T.: Yes, that's what I think too. Ok., yes. Nobody doesn't know it any more?!
S.: To reduce to the common denominator.
T.: Well, relative frequency is, and Markus has said it correctly, the number of cases divided by ...?
S.: ... the number of cases observed.
T.: No, the cases observed divided by ...? Aha! Or, if you formulate it as Ulli said, the cases observed as numerator and the number of trials as ...?
S.: Denominator.

Appendix 2: Transcript 2, grade 8, "The area formula for the trapezoid"

Some properties of the trapezoid are discussed in the class and compared with the parallelogram and the rectangle. The students have been asked by the teacher to explore the area formula of the trapezoid in partner work. She gives the following hints.:

T.: The problem is to find a formula for the area ... there are two possibilities to come to a formula for the area ... you can, eh, enlarge this trapezoid ... eh .. but one can also draw it in some other way, or extend it, that one gets a parallelogram afterwards ... the other possibility is, that you indeed try, as we have done it yesterday with the parallelogram, that somehow one divides it, cutting off angles and adding them on the other side, so that one gets a rectangle at the end ... No, not a whole angle, that does not work ... only parts! You should try it in your groups!

Subsequent to the group work of the students, the proposals for the solutions are discussed. The first proposal totally is in accordance with the expectations of the teacher. Some students have turned up side down a copy of the trapezoid and have put it along side to the old trapezoid, having now a parallelogram from what they know the area formula; they use it correctly for developing the area formula of the trapezoid getting in this way: $A = \frac{(g_1 + g_2) \cdot h}{2}$. In the following the second proposal for getting the formula is discussed.

T.: ... OK., the second possibility, even if I would like to stop teaching here .. Ehm, the second possibility ... .
S.: .. could one not, this line here, the height ...
S.: ... could you please speak a bit louder ...
S.: ... make parallel to, to this other line, and then one would have such a quadrangle and on the other side a triangle ...
T.: ... yes, ehm, some others also made this proposal, ... only the problem is if I would draw a line of height here, yes, and this triangle is cut off there, with all these proposals and possibilities the problem is, that one only knows the whole basis line g_1 and not only this little piece there. That's totally unknown. Well, when you really have concrete lengths, you could draw and measure it. But in general I can't do it. Hoe should I then say, g_1 minus some small piece, but how long is this little piece at all? Hence, that does not work! .... Hem, I would like to have another possibility ... Jochen!
S.: Yes, if we now, ehm the longer side .... well, take g_1 minus g_2 ...
T.: ... aha ...
S.: .. and then, ehm, then one has, there would be a remainder; let's say g_1 would be 5, hence 8 minus 5, giving 3 and then, if one would push now g_2 backwards, in a way giving a right angle, then something would remain there on the right side ...
T.: ... pay attention, now you are again coming up with pushing backwards g_2, later we will deal with it ...
S.: ... I do mean something else, there you take 5 times h, then, what is left here, there remains 3 centimeters, that is a slope, yes, then, ehm, you also take the slope ...
T.: ... I really have understood this, ...
S.: ... the remainder is 3 times the height then, divided by two ...
T.: ... no!! A second possibility, ehm, one tries to construct a rectangle, and I have seen how some of you have used the drawing triangle and they have moved it in this way ... and at this one point ..... I will draw here, in this trapezoid something like a medium line .... but first of all one has to get this idea, OK!