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**The Theoretical Nature of Probability in the
Classroom**

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Chapter 5

The Theoretical Nature of Probability in the Classroom

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This chapter presents an epistemological analysis of the nature of stochastic knowledge. In particular, the mutual relationship between the elementary concept of probability (in its empirical form of relative frequency and in its theoretical form of Laplace's approach) and the basic idea of chance is demonstrated. An important consequence for teaching elementary probability is that there cannot be a logical and deductive course for introducing the basic concepts and then constructing the theory upon them; developing stochastic knowledge in the classroom has to take into account a holistic and systematic perspective. The concept of *task system* is elaborated as an appropriate curricular means for treating the theoretical nature of stochastic knowledge in the classroom.

1. Approaches towards Teaching

Teaching processes are intended to enable pupils to learn and understand mathematical concepts, methods, calculation techniques, and structural relationships. The first step is to simplify complicated mathematical problems, transforming them into an elementary form to facilitate learning processes. A teaching-learning path through the discipline's structure has to be found which is as easy to understand as possible. This means that teaching mathematics is aligned to carefully selected examples, and that the teacher skilfully attempts to establish new knowledge which has to be learnt on the basis of familiar things. Moreover, there is an effort to avoid ambiguity of mathematical concepts by means of clear characterizations, and to guide and control the process of learning by means of establishing a consistent structure of knowledge.

Teaching mathematical knowledge in school is bound to a context of social communication, within which the pupils are presented with a diversity of requirements relating to subject-specific activities, as well as relating to cooperation with classmates. When the

teacher organizes classroom activities, he/she will directly or indirectly refer to the subject matter of the discipline and to its structure. This helps to determine the mode of work in class: teacher exposition, teacher-pupil interaction, pupils' individual work, group work, project work, and so forth. Practising and repeating stages in the classroom are also closely connected with the demands of the subject. Teacher's activities aim to help pupils assimilate subject knowledge.

Structural approaches

Mathematical knowledge and the teacher's conception of it determines the teaching process to a very high degree. Thus, the educational approach of simplifying the subject matter and explaining it consistently is frequently identified with the logical structure of mathematics. We must, however, remember that a consistent presentation of mathematical knowledge does not necessarily yield the ideal structure for the teaching and learning of mathematics. Teaching and learning certainly cannot be characterized as a logically consistent method. As opposed to a one-sided and structurally oriented method of work, a broad scope of different activities of teaching and learning is required for mathematics education; this is also stated in the Cockcroft report (1982, §243):

"Mathematics teaching at all levels should include opportunities for

- exposition by the teacher;
- discussion between teacher and pupils and between pupils themselves;
- appropriate practical work;
- consolidation and practice of fundamental skills and routines;
- problem solving, including the application of mathematics to everyday situations;
- investigational work.

In setting out this list, we are aware that we are not saying anything which has not already been said many times and over many years. The list which we have given has appeared, by implication if not explicitly, in official reports, ... and the journals and publications of the professional mathematics associations. Yet we are aware that ... there are still many [classes] in which the mathematics teaching does not include even a majority of these elements."

Such exhortations have frequently been made and are basically accepted by mathematics educators. There remains, however, the question why these principles have little effect on real teaching. Good intentions alone do not seem to suffice, so the conditions for realizing different classroom activities must be inquired into.

It may be that classroom activities are closely connected with the conceptions teachers have of the *character of mathematical knowledge*. The epistemological status of mathematical knowledge in the classroom determines to a high degree the methods and activities which are deemed appropriate and suitable for learning mathematics. The unified character of mathematics and its logical structure are in stark contrast to the multitude and heterogeneity of classroom activities. Most teachers consider school mathematics and the curriculum as strictly fixed, firmly predetermined and structured hierarchically. From such a perspective, they organize their classroom activities according to the structure of their own mathematical knowledge. Hence, the teacher hands mathematics down to the pupil like a ready-made object in a simplified if piecemeal way (Steinbring, 1985).

Two teaching strategies fit this line. The teacher may orient himself/herself more towards questions of intuitiveness, of making the subject matter concrete and of illustrating it by means of examples. The intention is to present the subject matter to the pupil in an illustrative way, subdivided into small segments, and in a way which can be concretely represented. This method of teaching is to be found mainly at the primary and at the beginning of the secondary level.

Within the secondary level (Sekundarstufe I, grades 5 to 10), another method of teaching is developed in which the emphasis is on consistent mathematical connections. The mode of justifying and organizing the teaching and learning is aligned to the internal structure of the mathematics curriculum, which must necessarily be organized in a consistent way. This method of teaching does not mean that the logical structure of mathematics is immediately presented to be learnt, but that the focus and orientation of teaching is on bringing it out.

This is borne out by the dominant character of the question-and-answer pattern in direct teaching, the avoidance of experimental and practical work, as well as the almost pervasive three-step teaching pattern of presenting examples, developing mathematical rules or techniques for calculation and then practising these rules by means of further problems or applications (Dörfler and McLone, 1986, p.63).

The two teaching strategies, the intuitive-concrete and the structural method, both start from the assumption that mathematical knowledge and awareness is acquired in a linear, straightforward way. This means that the teacher must initially define basic concepts in order to be able to gradually introduce the further elements of knowledge into this framework. Conceptually clear and fixed foundations are required to ensure that mathe-

mathematical knowledge can be learned according to this view. The basic assumption seems to be that the axiomatic structure already contains, at least in principle, the entire knowledge.

Winter (1983) writes about the consequences of this structural approach in teaching

"The concern for clarity of concepts always runs the risk of degenerating into an obsession with precision. The latter is evident, among other things, 1) in a readiness to provide a lot of definitions (and only few theorems, these being even trivial) which kills motivation; 2) in a tendency to define things which are not systematically used at all; 3) in the obsession of using as many specific terms of the discipline as possible; and 4) in the concern to strive for definite, watertight formulations from the very outset." (p.196/197)

Classroom learning is directed towards the mathematical conclusions drawn from these foundations. This perception of school mathematics leads to the desire to have precise and comprehensive definitions from the beginning for as many concepts as possible - even for the insignificant ones. In due course, the teaching of mathematics runs the risk of aligning classroom activities within the teaching-learning process too directly to the logical mode of mathematics itself. This may quickly transform the process of teaching into a logical, consecutive procedure which excludes the ruptures and contradictions occasioned by the social and communicative context.

Freudenthal (1983, p.41) uses the example of the *reference set* to formulate a similar criticism with regard to the alleged need for a comprehensive set of basic concepts.

"Curriculum developers and textbook authors apparently believe they offer teachers and pupils firm ground under their feet as soon as they restrict mathematical activities to a fixed reference set. This, they think, prevents possible surprises that might be disappointing. Fixed reference sets have been a source of confusion - in particular in probability (which we are obliged to anticipate here) - and have provided, rather than firm ground under their feet, a swamp of contradictions."

This reference set cannot be an invariable foundation for mathematical knowledge. On the contrary, in teaching, the reference set must be constructed and adapted according to the actual state of the knowledge in class.

Teaching within this narrow conception of the nature of school mathematics is not consonant with the genuine process of learning theoretical knowledge. In the structural method, knowledge is unequivocally defined beforehand both by the basic concepts and their representation and by the mathematical methods of inferring new knowledge. In contrast, processes of learning require the organization not only of the development of

knowledge, but simultaneously of its modes of representation and of the opportunities for active engagement.

Processes in the mathematics classroom thus have to face the problem that there are no ready-made basic concepts from which, in principle, we can derive all mathematical knowledge. Teaching has to consider knowledge as dependent on the available means of representation and activity and on contexts. The knowledge to be taught has to be developed at the same time as the means of representing it, and the situations to become active with it. The meaning of mathematical knowledge is not immediately obvious; it is dependent on object-related representations, and on the means of working on these representations. This meaning is developed by actively dealing with objects and their specific epistemological nature; Freudenthal (1983) introduces the notion of *mental objects* to express that.

Beyond that, teaching must also take into consideration how mathematical knowledge can be made effective and meaningful in general education.

"... scientific concepts always include general aspects, the theoretical objects of teaching are neither visible nor palpable. Their transmission therefore cannot be carried out like, say, throwing a ball. The teaching of theoretical concepts must take this into consideration and requires, consequently, a conception of the role of the activity, of the intuition, of the language, and of the logic of concept-development in mathematics instruction. Another aspect is related to the problem of general education and of its social importance, or its function in the development of personality." (Otte and Reiß, 1979, p.124).

2. The Theoretical Nature of Stochastic Knowledge

The tendency of strictly organizing teaching according to ideas about the character of mathematical knowledge also dominates stochastics instruction. The consequences are even worse than in other branches of mathematics education as stochastic knowledge has a specific theoretical character which will be discussed here. The role of this theoretical character of stochastic knowledge is considered for the teaching process and its organization. It implies that the basic concepts cannot completely define probability, but are conversely formed only while developing probability theory so that the teaching process must be organized differently. It is inappropriate to start from ready-made con-

cepts, gradually adding further knowledge. Conversely, it is necessary to begin with meaningful situations which facilitate the development of concepts.

Approaches to teaching probability

The two main methods of teaching are again based on intuitiveness and on consistency. The intuitive approach focuses on experiments, ideal games of chance, and real situations together with their conditions. Frequencies, numerical proportions and symmetries are used to define probability. After this intuitive and experimental phase of teaching, there is a distinct progression to statistical methods and concepts.

Somewhat opposed to this intuitive mode of teaching is a method which places emphasis on the internal consistency of the concepts, and on the inner foundation of the stochastics curriculum. This means that the teacher introduces a number of basic concepts: outcome, result, event, elementary event, certain and impossible event, sample space, etc. These concepts are intended to provide a solid foundation for subsequent development. The specific intention is to give clear definitions of concepts, free of doubts and ambiguities. This method of teaching almost always uses the concept of classical Laplacean probability to underpin the mathematical structure of the conceptual connections.

Frequently, there is a temporal succession between these two basic types of teaching stochastics. The teacher begins, in the initial years, with concrete work. Later, the theory is developed consistently, loosely using the preliminary intuitions of the outset. However, there is no discussion about the conceptual link between the two approaches; the fact that either part is only a different perspective of the same probability concept is not discussed.

As a consequence, probability and statistics are separated in the classroom. From the view of probability theory based on the classical Laplacean approach, statistics runs the risk of becoming a collection of methodological recipes without theoretical connection; moreover, the Laplacean conception lacks reference to real situations which are not equally likely and thus lacks the full range of meaningful applications.

In spite of these different perspectives in the two basic types of teaching, both start from the idea that mathematical knowledge can be gradually enlarged and integrated into the overall structure of the curriculum by adding small elements in a piecemeal way. This is best illustrated by an analysis of the use of the basic ideas of *probability, randomness and chance experiments*.

Usually, one of the two possibilities, relative frequency or classical probability, is used as early as possible. The first approach is through ideas of stabilization based on the empirical law of large numbers, in order to define the concept of probability as the limit of relative frequencies and subsequently reducing the concept of equiprobability to that. Alternatively, the intention is to refer exclusively to the concept of classical probability by restrictions to symmetrical experiments. In both cases the objective is to have a clear and mathematically complete definition of probability as soon as possible to provide a firm grounding.

Before the concept of probability can be explicitly defined, however, the object of probability calculus, in this case the chance experiment and randomness, must be explained. Thus, explaining the concept of the chance experiment is the beginning of any stochastics instruction. Any experiment which cannot be predicted or determined, is dependent on chance. Experiments with several outcomes whose appearances cannot be forecast, are accordingly called chance experiments. Another definition of chance experiment is that it is simply an experiment in which the results are dependent on chance. A chance experiment is thus characterized negatively in comparison to a deterministic experiment or with reference to the probability concept which is still to be explained. Andelfinger (1983) uses the expression "self-referentially" for this latter kind of argument.

In order to illuminate the initially vague ideas of randomness and chance experiments, various concepts, such as result, outcome, event, sample space are introduced. There is no discussion, however, of what is to be understood by chance, how the concept of chance becomes meaningful in real life, and how chance can be specified mathematically (by means of the concept of stochastic independence, for instance). The chance experiment is an introductory idea which is supported mathematically and linguistically in many ways, without developing its precise meaning and an underlying representation in the course of teaching.

These modes of teaching stochastics start from a vague notion of the object, that is of randomness and the chance experiment, whilst using a definite, mathematically precise definition of probability as soon as possible. These two basic concepts - *probability and randomness* - are no longer questioned later, on the contrary, they form the invariable basis from which mathematical conclusions are drawn in the course of teaching. This means that the teacher defines new concepts on this basis, performs certain calculations and attempts to solve problems by means of established methods and procedures.

Theoretical nature of probability

The two epistemological characterizations of probability by means of relative frequencies or by means of the classical definition can be interpreted as different ways of introducing and defining the same concept. The aim of defining basic concepts in a universal and absolute way leads to *vicious circles*. This is because what is to be defined must already be assumed for a universal definition. A well-known example is the concept of equiprobability, which becomes circular in the case of its universal interpretation: the definition of equiprobability is based on equal likelihood of possibilities.

As a basic concept, the concept of relative frequency is not free of such circularities, either. Relative frequencies, or the limit of relative frequencies, can be interpreted as probability only if they refer to a stochastic object such as a stochastic collective. The definition of such a collective, however, assumes that concepts like chance and independence, for instance, are used simultaneously. Here, too, conceptual circularities in the foundation of probability are evident, which means that concepts must be assumed which can only be developed and explained by means of the theory; a detailed analysis is in v. Harten and Steinbring (1984).

The circularity within the definition of the concept of probability can be more generally characterized as the mutual dependence of object (to which the probability refers) and concept (which is introduced on the basis of the object): one is able to speak of probability only if one knows the concept of the chance experiment and randomness; vice versa, chance can only be understood and mathematically specified after the concept of probability has been introduced.

The introduction of probability via relative frequencies while using concrete chance experiments is matched by the teaching method of developing concepts in an operationalized form. This means that concepts are not explicitly defined, but characterized by their use and application. Such an operationalized definition of the concept is expected to create a deeper and closer relationship with the pupils' subjective learning activities and learning processes.

Operationalizing a mathematical concept, however, cannot completely reveal its epistemological meaning. The instructions for use which are intended to define the concept might produce results which differ strongly from each other. For a chance experiment, the relative frequency obtained from a series of experiments, is usually not *identical* with the underlying probability, as different values occur in repetitions of the experiment, even if the conditions remain the same. The ways of operatively using a concept

cannot characterize the concept's meaning; they have to be compared and related to find invariant epistemological aspects of the concept.

If the classical concept of probability is reduced to equiprobability in school, processes of abstraction and idealization play a substantial role in its development. This method abstracts from certain empirical properties; it assumes symmetry of given objects in order to introduce ideas and properties in a mathematically ideal way. This way of idealizing is used to introduce and describe concepts in other fields of mathematics instruction, too.

For stochastics, however, this procedure raises the problem of how the random character of chance situations can be modelled in such an ideal representation. The concept of probability is not simply a fixed quantitative relationship, as it results from ideal symmetry but this relationship is at the same time a 'random quantity' which varies in repeated experiments. Probability refers not only to an ideal structure, but has to reflect the real conditions of the random number generator considered. The concept of probability is thus not just a result of abstraction and idealization of empirical properties, it should also offer possibilities of modelling deviations from this ideal structure.

If taken absolutely, the frequentist and the classical approach to define the concept of probability result in circularities in their epistemological characterization. Their educational derivations in the shape of developing the concept by abstraction or developing the concept operatively are also tainted by one-sidedness and show that the respective opposite aspect is required in order to understand the concept completely. Teaching stochastics cannot simply be based on explicit definitions of probability and chance which are provided so comprehensively that they contain, in principle, all the mathematical consequences. Rather, the concept of probability and the concept of chance must be developed while being mutually related to each other. In either case, one must start with limited and relative definitions and exemplary explanations of randomness and probability, developing them step-by-step by means of applying them to one another and of explaining them.

The central idea of developing concepts in this fashion is now illustrated by the concepts of chance and probability. What might be a first understanding, a simple definition of chance? One could begin with the following idea: a sequence of 0s and 1s is a random sequence if no pattern can be distinguished in the succession of numbers. On the basis of such an implicit idea of the character of chance, the stochastic thinking of children can be developed.

Such a definition, however, is neither explicit nor sufficiently clear-cut. For any sequence, it cannot be stated with certainty whether it is random or not. Are the figures presented only the beginning of the sequence, which will recur later? If two sequences are compared, which sequence is random, or 'more' random? What can this mean mathematically? Just think of pseudo-random numbers as created by computer-algorithms; even the digits of π behave like a random sequence though they are generated by an algorithm.

The consequence for the definition is that randomness and the linked question of whether a sequence is random, cannot be decided a priori once and for all; conversely, chance is defined according to the method available to test the random character. Hence, a sequence is judged to be random if it satisfies all known tests of randomness. This does not define the concept of randomness universally since chance can only be analyzed according to the known statistical methods. The definition of randomness is thus dependent on the developmental level of the stochastics' course (see v.Harten and Steinbring, 1983).

The circularities in defining the concepts of probability and randomness conflict with teachers' intentions to create a solid foundation of knowledge by precise definitions of the basic concepts. The epistemological character of stochastic knowledge and concepts thus inhibits teachers' preferences to start with explicit universal definitions. The basic concepts cannot be provided, once and for all; rather, they must be developed step-by-step within the process of teaching. Jahnke (1978) and Steinbring (1980) use the term *theoretical character of mathematical knowledge* to denote this dependency of the concept and its meaning on the theory's level of development. In this view, the concepts are not simply the foundation for the theory, but conversely, it is only the theory which explains and develops the meaning of fundamental mathematical concepts.

Teaching processes in stochastics thus cannot be aligned only to the mathematical structure of the concepts, they must shape the development of the concepts in accordance with the epistemological status of stochastic knowledge. It is necessary to create an educational framework in which the mutual relationship between probability and randomness, as well as between the various forms of representing probability allow for concepts and theory to unfold in unison. The teaching process must address the paradox that clarification and an understanding of the fundamental concepts can only be achieved by the fully developed theory in the further course of teaching. An important condition necessary to meet this requirement in the classroom is to avoid developing stochastics exclusively by means of relative frequency or via classical symmetry.

Objects, signs and concepts

The common method in the classroom of making calculations and solving problems on the basis of seemingly solid and final definitions of chance and probability is not sufficient even for teaching elementary probability. The connections with combinatorics or Boolean algebra are internal means of justifying stochastic concepts, stressing the aspect of mathematical signs and calculations, but at the same time neglecting stochastic model representations and interpretations of real situations of chance. This exclusive emphasis on mathematical formulae and calculation techniques hinders stochastic interpretation and thinking. Randomness is not automatically contained in the mathematical signs: beyond those, stochastics must be interpreted according to its content; this requires a stochastic situation as object to which these models refer.

The requirement to consider concrete random events and situations is not the same as stating that the empirical situation itself already provides the comprehensive meaning and explanation of stochastics. Randomness is not simply an empirical property of a given real situation, it is simultaneously a theoretical concept which must be developed, refined, and analyzed.

Neither the empirical situation by itself (the object or the problem), nor the mathematical sign by itself (the model, the calculus) can express what randomness and probability mean. Steinbring (1984, p.83) argues that both an autonomous object and a model or calculus, and the relations between them are necessary to understand and develop chance and probability.

The situation is illustrated by a triangle of object, sign and concept. We will show how this diagram of relations in fig.1 describes the particular character of mathematical knowledge and gives orientations for organizing learning processes in the classroom. First, however, we explain how the specific difficulties encountered in stochastic thinking make the theoretical character of knowledge inevitable.

Fig.1: A triangle of relations between object, sign and concept

The previous epistemological analysis of the basic concepts demonstrates the necessity for a *dynamic* perspective on knowledge development. Knowledge cannot be determined completely on the basis of invariable concepts. Knowledge and its meaning has to be developed and this requires a flexible interpretation of these basic concepts. Such a dynamic perspective on knowledge is not only appropriate with respect to the epistemology, but also with respect to the student; the acquisition of mathematical knowledge can only be understood as a developmental process. It is a question of the relation between the student's pre-experiences and subjective notions and the mathematical structures and concepts. Such a broad subjective perspective for teaching probability should not be confused with the conception of *subjectivist* probability; but in a very elementary way important aspects of subjectivist intuitions are contained in this approach.

The epistemological analysis of knowledge is very helpful for a better understanding of the relation between subjective intuitions and objective random situations and their mathematical models. A dynamic perspective on knowledge helps to soften the strict alternative between subjective intuition and objective structure. If one regards the relational triangle from a developmental perspective, the subjective aspects of the learning process become more apparent. Beginning with very personal judgements about the given random situation, comparing the empirical situation with their intended models will hopefully lead to generalizations, more precise characterizations, and deeper insights.

The many paradoxes of probability theory illustrate the difficult relation between subjective intuitions and objective structures of probability. Paradoxes often show a contradiction between modelling and objects with respect to mathematical calculation and intuitive representation. These contradictions between calculation and representation result in part from a subjectively perceived difference between the mathematical result on the one hand and the intuitive representation of the objective situation on the other.

The difference between object and mathematical model cannot simply be regarded as an error or as a defect to be corrected; the difference between object and model is a matter of principle in stochastics. Model and object can never be made completely congruent; one can look for models which are adapted to the real situation in a better way. The implicit assumptions of the model must be checked as to whether they are justified for the real situation; there is also the question of which of the situation's concrete features should be additionally taken into consideration if the model does not fit satisfactorily.

The discrepancy between a real situation and its model holds for all applied mathematics. For probability, however, this remaining basic difference is a genuine object of theoretical analysis. Stochastics really is a mathematical theory about errors, deviations

and interrelationships between an idealized forecast and real fluctuating behavior. This difference and this interrelationship between object and model together propel the development of stochastics. In this respect, in any stochastic analysis there is always a certain amount of subjective judgment involved, which cannot be completely eliminated though one wants precise information about its extent.

3. Didactic Means to Respect the Theoretical Nature of Probability

The previous considerations have shown that this relational triangle of object, sign and concept is a useful tool to characterize stochastic knowledge avoiding a one-sided emphasis either on mathematical calculations or on empirical-concrete aspects. This section explores how the relational triangle develops educational orientations which enable the teacher to bring the theoretical character of stochastics to the open in the classroom. Hence, it analyzes educational means which permit a description of the actual process of developing mathematical knowledge.

Interrelations between mathematics and exemplary applications

The significance of the relationship between empirical stochastic situations and modeling can also be traced in the historical development of probability theory. Loève (1978) shows that the development of the concept of probability has been advanced in particular by the tension between empirical application and mathematical description. These concepts were thus established as a result, or as descriptions of this relationship, at the same time serving to analyse this relationship and to develop it further. Steinbring (1980) applies the principle of complementarity to describe this dynamic relation between a theory and its applications.

The principle means that stochastic concepts must be simultaneously understood as elements of mathematics and as objects related to a context of application. Neither a mathematical definition nor an interpretation of exemplary relationships and applications alone establish the probability concept; it is the dual interplay between both sides that counts. In the sense of this *complementarity*, the concept of probability should be understood as a relationship between mathematical description and exemplary cases of application. This principle prevents conceptual reduction, be it to mathematical calculations, or to empirical intuitions; it excludes philosophically one-sided conceptions of probabili-

ty like a pure objectivist or a pure subjectivist interpretation. Dinges (1981) also opposes such a conceptual reduction:

"Stochastics in school must organically combine ideas from different philosophical traditions, in particular

- stochastics as the mathematics of mass phenomena
- stochastics as the logic of uncertainty
- statistics as the technique which transforms data into insights
- stochastics as decision theory." (p.51)

Like probability, the concept of randomness bears these complementary elements. Random situations and chance experiments are concrete applications or real situations of chance like random number generators, but theoretical methods and procedures are required to describe and analyze randomness with regard to its mathematical relationships. Even a concept as elementary as equiprobability simultaneously needs the ideal mathematical interpretation of equally distributed in the arithmetical sense, and the empirical representation of statistically equally distributed, as it emerges in concrete experiments.

Equidistribution as a model does not mean that actual results should exactly be equally distributed; if this really happens it is a sufficient reason to mistrust the experiment. The model of exact equidistribution is required as a reference to decide whether observed frequencies can be judged to stem from such a uniform distribution, or if deviations are too large for such a claim. In the case of a loaded die, the usual variability of equidistribution is essential to give an idea about the extent of falsification which is possible on the basis of the observed values. This example shows the complementary nature of stochastic concepts. The concept of equidistribution establishes a relationship between a real stochastic situation and its modelling, a relationship between an empirical, statistical, and an ideal, arithmetical equal distribution. The concept of 'equally distributed' really has this double meaning and neither of these two aspects may be ignored.

In an analogous way, the concept of the expected value mediates between an empirical situation and its mathematical modelling. The notions of mean values and values of dispersion, which characterize a distribution are related to the conceptual neighbourhood of expected values. These latter are ideal values which must always be considered in relation to empirical values and concrete data; they serve as an ideal point of reference for empirical random quantities.

Bernoulli's law of large numbers was the first mathematical theorem on the connection between relative frequency and classical probability. It reveals, in its logical structure,

the complementary relationship between a stochastic situation and its mathematical model; at the same time, it serves to analyze and specify this relationship. It refers to an experiment which has two outcomes 0 and 1. The theorem states:

If p is the probability of a 1, with $0 < p < 1$, and H_n/n is the theoretical relative frequency of 1 in n independent trials, then it holds

$$\forall \varepsilon > 0 \quad \forall \eta > 0 \quad \exists n_0: \quad P(|H_n/n - p| < \varepsilon) \geq 1 - \eta \quad \forall n \geq n_0$$

The reflexive statement in Bernoulli's theorem says that there is a very great probability that relative frequencies in a stochastic experiment will come as close to the probability as desired if the number of trials increases. Altogether we have three variable quantities in the theorem; first, the precision of the statement considered which is measured by ε ; second, the certainty with which the statement holds which is measured by η ; and third, the number of trials necessary which is given by n_0 . These three parameters are mutually dependent on one another, it is possible to fix two and then to calculate the third.

The need to use probability in order to even formulate how relative frequencies 'converge' to the underlying probability is an indicator of the circularity of the concept's definition and of the complementarity between the empirical situation and its mathematical model. A detailed educational analysis of Bernoulli's theorem is given in Biehler and Steinbring (1982) or in v. Harten and Steinbring (1984).

All stochastic concepts examine the mutual relationship between object and mathematical model, they basically emerge in this connection, serving to analyze and explore this difficult linkage further. The concept of probability thus cannot be defined once and for all in a definite form, nor does the concept of chance represent a universal basis which serves to derive all further mathematical statements by logic. The meaning of the concept and its relationship to applications emerge only from the development of the theory and the refinement and elaboration of this relationship between object and model. This dialectics of empirical content-specific and mathematical model-specific aspects establishes what we call the 'theoretical character' of stochastic knowledge.

Means of representation and activities

The idea of complementarity prohibits the hierarchical method of teaching by beginning with content-related representations of applications, and then passing on to the symbols, formulae and mathematical modelling which represent an abstraction of the real situation. Usually, the empirical situation is intended to facilitate modelling, which is then

followed by generalization. In principle, the interrelationship between object and model must always be kept in mind throughout the teaching process. The dialectics of object and model requires a tempered approach.

At the outset there should be a situation in which the relationship between intuitive representations and mathematical modelling statements is a crucial element which promotes development. Only those stochastic situations showing a multitude of different aspects are appropriate to stimulate pupils to investigate ideas. The need for subjective experiences and intuitions can be met by such problem situations which can then be examined and, if necessary, modified by comparison with a stochastic model, developed for the particular random situation.

As the process of learning begins with relatively complex situations, certain assumptions are made implicitly at first. These are assumptions about the stochastic character of the situation such as equiprobability, or the independence of experiments. In the course of teaching, these assumptions have to be dealt with, questioning the appropriateness of the modelling implicitly assumed at first, and improving and changing it if necessary. Analyzing and developing the implicit assumptions are vital elements of concept development which is governed by the tension between concrete stochastic events and their mathematical modelling.

If we agree that the mutual relationship between object and model in stochastics cannot be ignored, the question remains as to how to develop this relationship mathematically. Historically, a multitude of conceptual means of description and presentation have been developed to characterize the relationship between concrete individual situations and their stochastic modelling. Closer analysis reveals two categories of representation to exemplify the tension between a situation and its stochastic model.

The distribution category contains all the diagrams relating to the processing of frequencies. Relevant for the classroom are tables, tally sheets, pie charts, stem-and-leaf displays, column graphs, histograms, polygons, etc. The other category includes different types of tree diagrams which mainly deal with probability calculus. For school, the following forms are relevant: tables, complete and reduced tree diagrams, 'infinite' tree diagrams, Pascal's triangle and the probability abacus.

These types of representation are used as an auxiliary means to process data, or as a method leading to the mathematical formula for calculation. With regard to the relationship between a situation and its model, however, it is appropriate to consider distributions and tree diagrams as being complementary to each other. This means, for instance,

that the concept of distribution can also be used to calculate and represent probabilities; conversely, Pascal's triangle from the tree diagram category can be used to explain the binomial distributions.

Comparing different representations leads to new insights into the relations between a concrete situation and its stochastic modelling. Each representation treats this relationship in a specific way; each means of representation reflects the complementarity of stochastic concepts. The different modes of application which these representations permit, are an expression of these complementary aspects. They enable us to build a broad spectrum of relations between a concrete situation and its mathematical model.

Thus, the tree diagram may be interpreted both as a flow chart for carrying out a concrete experiment, and as a characterization of all the possibilities occurring in the shape of a model. Hence, the tree diagram can be used to describe specific experiments; it can also be used as a scenario for a large number of idealized experiments; it may also serve to derive the path rules (addition and multiplication rules) and to establish stochastic models like the binomial distribution.

The different variants of the concept of distribution can be used to represent, order, and relate data. It is possible to make absolute or relative comparisons, either by areas or lines. Different scales or certain transformations such as square or logarithmic functions may reveal structures in the data. Distributions characterize simultaneously the overall system of the observed quantities and show specific salient features like symmetry, means, variances, etc. The educational role of graphical representations is dealt with in Biehler (1985). These specific characteristics of distributions and the analysis of random situations enable us to compare empirically observed distributions and their possible mathematical modelling.

For mathematics instruction, the means of representation and activity are of fundamental importance as they allow the development of concepts and the formation of new knowledge. The different modes permit the first generalization beyond the present level of knowledge. At the same time, for their concrete shape, they are an expression of a limited point of view, i.e. they indicate a potential framework of the development and of possible steps. The different modes of representation are an expression of the fact that the acquisition of knowledge, on the one hand, proceeds in stages and leaps and bounds, while the interrelationship between these stages is at the same time potentially and tentatively established. This conflicts with a conception of the development of science in which *addition* of new elements of knowledge is possible on the basis of a given structure.

Thus, the means of representation and activity is a 'mediator' between the subjective individual growth of knowledge and the curricular, epistemological development of knowledge. They do not only describe a potential curricular process of development but they also offer opportunities of connecting pupils' individual learning to this curricular process.

The relational triangle of object, sign and concept, the idea of complementarity, and the discussion of representation and activity prove the following distinctions useful: knowledge has to be distinguished from the forms to represent it; knowledge has to be distinguished from applications or interpretations of it. Hence, the teaching process requires more than simply organizing the development of knowledge. At the same time teaching has to organize the forms of representing this knowledge and the opportunities to become active with it.

Forms of activity in the classroom might not directly refer to stochastic knowledge but they must respect the complex structure of object, sign and concept. Therefore, in preparing and conducting their teaching, teachers have to search for suitable means of activity and representation. For theoretical knowledge is not directly accessible; to acquire it, one has to use mediating concepts. The frequent use of means of activity and representation in teaching is an indicator of the theoretical character of knowledge. The diagram in fig.2 illustrates the underlying relationships in an effort to find a balance between the knowledge to be learnt and the activities in the classroom which are to serve that purpose.

Our analysis of stochastic knowledge reveals that this knowledge is theoretical and complex; its logically structured representation cannot claim to cover the whole meaning and explanatory power of stochastics. This conception of the character of stochastics knowledge and the development of means of representation and different modes of use and activities form the basis required to achieve successful learning processes. Logically structured and hierarchical exposition of knowledge is not sufficient.

Fig.2: Balance between activities and representations

4. On the Didactic Organization of Teaching Processes

Practical teaching requires a variety of concrete and effective aids and suggestions for *possible* approaches in the classroom. As has been discussed, the theoretical character of probability should bear a strong influence on educational approaches in the classroom. On the one hand, the demands of the subject matter require the teacher to develop special ways of teaching and planning; on the other hand, the concrete conditions in classroom and the crucial role of the teacher within the learning process have to be respected.

The role of teachers

Stochastics is a topic which is new for most teachers. Attempts to introduce new subject matter in mathematics instruction and to make concrete suggestions for implementation in the classroom are often ambiguous about the teacher's role. Above all, the teacher is considered to be a deliverer of the mathematical knowledge. This leads to an effort to provide the new materials in a detailed and ready-made way to ensure that it can be immediately integrated into teaching. Too often, educators and researchers as well as

teachers work on the basis of such a limited understanding. Teachers implicitly expect in-service training to provide immediately effective material, while this attitude induces educators and researchers to develop suggestions which can be directly applied in the classroom.

There are various suggestions which stress the internal mathematical consistency and logical hierarchy of knowledge which belong in this realm of so called 'teacher proof' curricula for stochastics instruction. One approach is mathematically based on Boolean algebra and makes an exaggerated use of set theory. The other, more frequent approach stresses Laplacean probability and combinatorial aspects, as these facilitate a mathematically tidy presentation of the subject matter. The elegance and structure of these proposals suggest to the teacher that implementation of the curriculum will create no problems in the classroom.

This approach also includes courses in which concepts are developed by means of a universal concrete model. For instance, Freudenthal (1975) fosters his course entirely around the urn model; Bentz and Palm (1980) use the wheel of fortune instead. A concrete stochastic model permits special points of reference, explanations, and modes of work in the classroom. This model may serve to disclose the mathematical structure and logical order and, moreover, as a medium to model real applications. However, if used exclusively, this approach bears the risk that the chosen model itself becomes the sole object of teaching for the conceptual basis of stochastics. This universal model would then be overlaid with its different roles, namely standing for the concept, being a concrete model for a situation, and being a prototype for the modelling process as a whole. Furthermore, this concrete model might shift attention to calculations so that questions of stochastic interpretation, application, and evaluation are in the background.

The intention of preparing teaching material in a perfectly structured and ready made way to by-pass teachers, has failed. The emerging problems of comprehension of students require an active intervention by the teacher and corresponding adjustments of the teaching material. The teacher must actively interfere with this process; he/she must shape, modify, organize and evaluate it. Classroom problems with 'teacher-proof' curricula have led to encourage teachers to develop their own material. Such a view of the teacher's role was also, involuntarily fostered from the perspective of genetically-developmental teaching. This conception of teaching sees the teacher as an active decision-maker and organizer of classroom activities. Only the teacher is familiar with the prior knowledge and abilities of his/her pupils and the conditions in the classroom. However, it is unrealistic to expect teachers to develop materials independently from one another.

Both extreme views on the teacher's role do not improve actual teaching. Neither is the teacher irrelevant with regard to the subject matter taught, nor is it sensible to view the teacher as a curriculum developer. The demands and necessities of practical teaching must be taken into consideration when developing materials and teaching suggestions; it is evident, however, that this consideration for conditions relevant in the classroom means that it is not possible to make suggestions for teaching which by-pass the teacher. Teachers have to acquire special knowledge to professionalize their daily practice. This professional knowledge of teachers must always reflect the mutual relationship between classroom requirements on the one hand, and discipline-specific and educational connections on the other.

For the teacher it is neither sufficient to know the systematic structure of stochastics nor is it sufficient to understand stochastics merely as subject matter to be taught to the pupils. The latter includes knowing the pupil's learning situation and knowing how to facilitate the presentation of the subject matter. Rather, the teacher must develop an idea of the relationship between the structure of the discipline and the pupils' acquisition of knowledge. In a pragmatic sense, this idea encompasses elements which are theoretical and systematic as well as including examples and experiences which are of practical effect and importance in the classroom. The special character of professional teachers' knowledge is formed by this 'mixture' of theoretical and practical elements, which means that this knowledge is neither final nor only theoretically based.

At the same time, it means that this knowledge is neither exclusively pragmatic nor immediately effective. The teacher has to relate the different social, pedagogical and psychological elements, and elements of the subject matter to one another within the teaching/learning process. In doing so, however, he/she has to keep in mind that the classroom process itself is subject to pragmatic decisions with practical effects which cannot be organized according to a systematic modelling of the discipline in the sense of a scientific justification. Classroom approaches will be effective only if they give due weight to the essential role of teachers and their special professional knowledge.

The role of task systems

In previous sections, we have shown that the character of stochastics is necessarily theoretical. Hence, suggestions for teaching must be developed within the context of the teacher's professional activity and teaching practice. These suggestions, however, cannot anticipate the practice of teaching in all its details, thus making the teacher's role super-

fluous. What kind of proposals meet this difficult requirement of containing simultaneously theoretical and systematic elements, and orientations effective for practice? What sort of teaching materials support the teachers' active role, whilst permitting alternatives and deviations without limiting these materials to a one-sided viewpoint?

The *mathematical task* plays an important part both in the classroom and in the teacher's understanding of mathematics instruction. Teachers plan and organize their teaching in line with mathematical tasks. Problems constitute a link between ideas which are effective in the classroom and the practical requirements; the idea and conception of a mathematical task serves to connect theoretical orientations and activities with practical effectiveness (Bromme and Juhl, 1985).

Tasks have a multitude of functions in teaching. They are the smallest units of classroom activity, which enable the teacher to relate the social-communicative and the subject-specific content of mathematics instruction to each other. Mathematics teachers often think and talk about teaching in the form of tasks; tasks are in some way their 'concepts'. However, they use these concepts pragmatically as practitioners; these concepts are highly interwoven with the local requirements, and lose their meaning without this context. Theoretical ideas of these concepts are relatively underdeveloped; the appropriate use of tasks requires at first many practical experiences and this can be observed in what is called a 'good teacher'.

Mathematical tasks which make sense never occur separately. They are always connected to a larger problem context which conveys meaning and coherence to a certain degree. It is in this way that serious mathematical tasks are distinct from puzzles and braintwisters. The context of tasks in the classroom consists mainly of other tasks. An optimal form of presenting several tasks together in the classroom is neither a number of routine exercises nor a haphazard mixture which are vaguely linked to each other. Hence it is important to develop a concept of a system of tasks which is between these extremes.

The most important aspect is to find a *structure of relationships* for tasks which belong together, in short, to interpret tasks as variables. This heuristic image of 'tasks as variables' is a good first approach to systems of tasks. Besides the mathematical context, there are variations of pedagogy, teaching method, means of representation and activity; possibilities of variations of the most diverse kind which may even contradict each other, thus leading to compromises in practical teaching.

What are the features which characterize task systems? What links individual tasks together to form a *system*? On the one hand, these tasks should relate to one *common* object - even if they do so from different perspectives. On the other hand, the tasks should be *analogous* to one another. It is evident that this is not a precise description of the concept of task system, for only working with tasks and solving tasks will show that they refer to the same object and are analogous to each other.

Task systems thus are 'open' to a certain degree; their system character is not simply given a priori, but constituted by the application, that is by using them and working with them. The concept of analogy is of special importance for constructing and describing task systems. Pólya (1967) has advocated the search for analogous problems as a strategy to solve problems. He explains that the crucial element of analogy lies in the similarity of certain types of relationships.

"Analogy has something to do with 'similarity'. The concept of similarity is vague in the sense of everyday language, and in geometry it is more restricted than the concept of analogy. Similar things agree in some aspect, analogous things agree in certain relationships between their respective parts." (p.52)

In order to construct a task analogous to one which has been presented, it is necessary to develop an overview from which this similarity of the type of relationships becomes visible. It is necessary to reach a point of vantage from which it is evident how different types of relations between tasks can be transformed into one another. On the one hand, analogous tasks thus agree in a certain relationship between their respective parts, on the other hand, it is clear that there will always remain a difference between them as well. Both are important characteristics of the systems concept: agreement of the individual parts under a global aspect and distinction between these parts in their local aspects.

Two important strategies to construct analogous tasks are varying the problem's constants and change of the model. We will discuss them with the help of the 'birthday problem': 'In a room, a party of n persons has gathered. What is the probability of at least two persons having their birthday on the same day of the year?' Instead of asking for the birthday, one could ask for the birth-month, or the birth-week. Such variations are simplifications of the given problem. The birthweek problem, for instance, can be simulated much faster and more clearly than the original problem. However, part of the surprise at the high probability of repeated birthdays is lost because the coincidence probability is neither related linearly to the number of birth categories nor to the number of persons.

Varying the problem's constants may lead to an easy simulation of it. Simulation is an essential method in stochastics as it not only provides a method of solution, but its actual performance already requires a first model. Concrete drawings from an urn, throwing one or several dice, or coding and noting suitable random numbers from a table of random numbers are experiments which may lead to tree diagrams and a general stochastic model. Therefore, easy access to a feasible simulation of a problem does not only give a numerical result but also a conceptual understanding of the situation.

In the birthday problem, the path does not lead immediately from the simulation to the solution. This is due to the condition of 'at least two persons', a logical peculiarity which is frequently best treated by means of negation. In this case, however, the simulation may also serve to develop a model for the task and the problem posed.

Varying the constants may permit the construction of analogous tasks which open new paths to solution. Is this always the case with this form? Even in the case of the birthday problem, the seemingly harmless variations sketched may rapidly lead to important differences. For instance, the fact that months do not have an equal number of days, makes the assumption of equal probability dubious; it is less probable for a person to have her birthday in February than in August. Changing the wording from 'at least 2 persons' to 'at least 3 persons' leads to new difficulties. This variation goes beyond the initial model, it implies a *change of model*. A further complication is that the calculations frequently become much more difficult.

Such a divergence in the model is mostly an expression for an explanation of certain aspects of content, e.g. stochastic assumptions which suddenly become relevant for the model: these are mainly assumptions with regard to the independence of events like drawing with replacement, or assumptions such as equal probability.

For simple variations of the constants, the original model remains unchanged. Changes of the model in tasks can perhaps be integrated in a hierarchically superior model. At first, however, the tasks are mostly only analogous by way of a common problem object. Changes of the model, however, do not only occur in the case of extreme modifications. For instance, the following phrasing of the birthday problem, which might lead to a false intuitive perception of the situation, brings about an interesting change of the model: 'In a room, n persons are present besides me. What is the probability of at least one person having my birthday?'

These two forms of analogy, i.e. variation and change of the model, also represent a certain type of requirement asked of the learner: tasks within a system which are analo-

gous because of a simple variation of constants, primarily serve purposes of developing and consolidating skills. In the case of a change of the model between analogous tasks, however, the emphasis is on aspects of the mathematical concept and of its development.

Task systems give an opportunity to organize forms of representing knowledge and activities of learning in a way which relates them better to one another. This gives due consideration to the theoretical character of knowledge. Besides these epistemological features of task systems, the relation to practice is important; educational differentiation is a case in point. A system should also contain tasks which provide access to stochastics for pupils with varying aptitudes and inclinations. Tasks within a system may be easy or difficult not only according to the subject matter, but also with respect to teaching. If we consider an entire system of tasks rather than individual tasks, the question whether a task is easy or difficult must be replaced by systematic questions like:

- What is the task's function within the overall system?
- How does its status change if it is removed to another position?
- Can the task be possibly replaced by another one?
- How does a task relate to certain other tasks of a system?
- Which are the different paths through a task system for pupils?
- To what extent are these paths equivalent?

There is also a large repertoire of different types of activities in stochastics: organizing experiments and simulations; obtaining and processing data; measuring, evaluating, and deciding; modelling using formulae and diagrams; these activities offer extra scope for variation of task systems. For a well organized system of tasks to be useful in the classroom, the variations relating to the subject matter and to the activities of learning must supplement and support each other. Such a task system may be used in a flexible way so that the combination of subject matter aspects and appropriate activities of learning offers opportunities for differentiation in the intended learning process.

5. Discussion of an Exemplary Task

As an illustration, parts of a stochastic task system for the initial teaching of probability are presented. The whole system together with a detailed analysis is contained in material for teachers to support their preparation for teaching (Brönstrup et al., 1986). Apart from the tasks and their solution, the material includes hints for the teacher con-

cerning the didactic use of the tasks and their possible variations which are given parallel to the tasks. The rationale behind the development of these materials is discussed in Steinbring (1989). The following task is from v. Harten (1985, pp.61-63).

Didactic framework of the task

Fig.3 shows a board for a game. To play the game you need two dice. Choose a number from 1 to 12 and put your playing piece on the starting field with this number. Now two dice are thrown and the pips are added. If this sum equals your number, then your playing piece moves up one position. Whoever reaches the finish first, has won.

F i n i s h											
1	2	3	4	5	6	7	8	9	10	11	12
S t a r t											

Fig.3: Board for the 'two dice game'

- (a) For each throw, denote the sum of pips in the following table:

Sum of pips	Number of throws with this sum
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
more than 12	

Total number of throws: _____

Fig.4: Tally sheet for the frequency of the various sums

- (b) Would you choose the 1 in this game? - Explain your answer!
- (c) Which number or which numbers are good choices in your opinion? - Explain your answer!

At the beginning, the class might speculate and discuss experiences of winning strategies in chance games. Having played the game several times, these ideas can be checked. The material offers not only opportunities for statistical experiments and the calculation of relative frequencies as estimates for the probabilities, it also offers activities and tasks to explore the possible combinations of two dice in order to calculate the Laplacean probability of the various sums of pips. In an elementary way, the two conceptual aspects of probability are thus related to one another within this task.

There are hints for the teacher to use, vary and develop such tasks in the framework of a didactic task system. These hints link the different levels of professional knowledge. They have been revised after several uses in class.

Pupils' previous knowledge. Specific previous knowledge is not necessary but prior experience of graphical representations is useful.

Concepts involved. This task serves as an introduction to randomness and strategy. It introduces a more difficult random mechanism, the throwing of two dice. A more detailed analysis of this mechanism could be done with a subsequent task from the original system.

Methodical hints. This game should be played in small groups or by two pupils. Problems in understanding the rules of the game are rare; most pupils are familiar with this kind of game. Pupils in the groups may be given different duties: to throw the dice, to note the results, etc.

Pedagogical and technical notes. The tally sheet of this task should be kept by the pupils. It may be the first example of a distribution of results which are not equally probable. Other forms of representation may be introduced such as bar charts or histograms. A comparison with the table of a previous, similar task of the original system (if this task has been dealt with in the class) could be useful. One might use differently coloured dice.

Duration. This task can be carried out in one lesson. Experiences with work in various classes are: 5-10 minutes for explaining the game and the working sheets, 20 minutes for working in groups, and 10 minutes for the discussion of results with the whole class. How long pupils actually work in groups depends on how often they need

to play the game. If a pupil chooses the favourable number 7, he/she needs an average of about 60 throws to reach the goal, deviations of more than 20 throws being rather unlikely. Numbers between 5 and 9 require not more than 90 throws on average.

Variations of the task. There are dice in the shape of a dodecahedron, with numbers from 1 to 12; to use such a die could give rise to interesting comparisons to the previous game. Three dice may be thrown and a board may be used with starting numbers from 1 to 18; this game is much more complex and time-consuming.

Classroom observations

This exemplary presentation of a problem belonging to the system is now supplemented and illustrated by observations from a classroom. In the course of their lessons on elementary probability, 6th graders repeatedly played a game of two dice in pairs during the previous lesson, noting their results on the tally sheet provided. In the lesson presented here which yielded the observation results, analysis of the game begins, and a simple theoretical explanation is sought for the unequal probability of winning this game with various starting numbers.

Most of the students recognized that the '1' never occurred as the sum of two dice; it really cannot occur, as they were quick to explain. They also noticed that numbers bigger than 12 are impossible. The sums 6 to 8 were obtained most frequently as they could see from their tally sheet. What is the reason for this? How can this be theoretically explained? This is the subject of the present lesson. Subsequent to a brief recall and a reminder of the experiment from the previous lesson, the teacher hands out a new worksheet to the students. They are to work on the following problem:

Problem: The lower table contains the various possibilities from which the sums of pips 2, 3 and 4 are composed. Please complete the table!

2	1+1		
3	1+2	2+1	
4	2+2	3+1	1+3
5			
6			
7			
8			
9			
10			
11			
12			

Fig.5: Tally sheet for the possibilities of the various sums

During the minutes which follow, the students work individually, or together with their neighbour. One of the students who has solved the problem quickly, is allowed to write his results onto a prepared overhead transparency. Gradually, almost all the students finish their work. Now the discussion of the results begins as a student-teacher dialogue. The possibilities are read aloud by students for each sum of pips and compared to the results on the transparency.

T.: What about 5? Isn't $2+3$ the same as $3+2$?... Frank?

S.: No, 'cause I can turn this around, and then it is something different!

T.: Let's pass to the 6. What possibilities do we have here? Jasmin?

S.: $1+5$, $5+1$, $2+4$, $4+2$, and $3+3$.

S.: I've got something different!

T.: Yes, Nicole, what have you got?

S.: $3+3$, $2+4$, $4+2$, $1+5$, and $5+1$.

T.: Only the order is different here!... But, what about $3+3$? Don't we have to count 2 possibilities for this, too, just as for $1+5$ and $5+1$?

S.: If I turn the $3+3$ around, I'll get the same anyway. This makes no sense, does it?

S.: For $2+4$, you take again the opposite, that is $4+2$, while you won't get anything new for $3+3$. Or you write it down in mirror script, but that makes no sense.

T.: Just remember your game, the two dice! Yes, Christian?

S.: I'll mark the two dice. For instance, I'll mark one with A and the other with B, and you can have a 1 on one die, and a 5 on the other. Or the other way around, the die with B has a 1, and the die with A has a 5.

T.: Very good. And what about $3+3$, Peter?

S.: This occurs only once. For $2+4$ and $4+2$ are two possibilities but with $3+3$ there is only one.

These episodes give rise to many considerations and analyses. This brief section is merely intended to illustrate in an exemplary way the special treatment of the present task in the classroom. The students are very well able to establish all the possible combinations of two numbers for the various sums. The teacher's question why the sum $3+3$ cannot be counted twice like all the other combinations is met with an explanation by reference to the concrete dice which is convincing within the context of the given task. One of the students imagines marked (i.e. distinguishable) dice and is thus able to give a pertinent justification for two distinctive combinations and for a single combination.

This brief episode covered only the beginnings of the classical concept of probability in the shape of 'all possible combinations of the sum of pips of two dice' which is later

formalized by the rule 'number of all favourable events'. Nevertheless, it reveals the fundamental didactic principle: processes of understanding and developing concepts in the framework of a system of problems occur in interrelated work between simple models (symbols, operations, etc.) and contexts (applications, experiments). The students seem to deal with numbers, signs, sums etc.; their work, however, is organized with the previous game in mind, thus being enriched with possibilities of concrete justifications and meanings, as is shown by one of the students referring to two marked dice.

Besides this, the episode illustrates the context-bound character of conceptual learning processes. The concepts involved, like 'combinations', 'number of all possible cases', etc., are not detached from the application context by means of formal designations; they remain within the framework of the conditions determined by the context without losing their conceptual essence. The fact that 1 is not possible as the sum of pips need not be formalized; however, this could also be called an 'impossible event'. Nevertheless, it would be erroneous to expect the students to derive the conceptually abstract idea of the 'impossible event' as an ideal element within the 'system of all the events' from this directly.

Implications for task systems

The teaching experience with the exemplary task presented here is a typical case in point for how processes of understanding concepts can develop according to the principle of interrelated work between simple models and interpretative contexts in stochastics instruction. The conceptual considerations with regard to task systems for stochastics are summarized in the shape of practical rules:

- 1) Task systems in stochastics should contain concrete experiments and simulations. The activities may be addressed to individual pupils or to groups of pupils. The task system is connected together by a common formulation of a problem; the individual tasks may require different capabilities from the pupils. A careful organization and coordination of experiments and of simulations or polls helps to orientate these individual tasks according to common aims of teaching.
- 2) Task systems in stochastics should contain, where possible, a situation requiring a decision. The compulsion to decide, to be subjectively involved in a situation, is an important motivator for pupils.
- 3) In stochastic task systems, diverse random number generators should be used and compared as objects: from dice to the Galton board and random number table; from loaded dice to real stochastic situations. The requirement to compare such

random number generators compels the study of common properties in the shape of variations, or the analysis of differences in the shape of fundamental changes of the model.

- 4) An important educational principle of stochastic task systems is the parallel work both on an empirical/experimental and on a theoretical modelling level. Both levels should be related to each other; undue reliance on one of these two modes of work should be avoided. In this sense, the two sides of probability, that is Laplace's probability and probability as relative frequency, should be related to each other as analogous forms of the same concept without, however, being identified.

The analysis of the concept of task system as a potentially important component of teachers' professional knowledge is an example for those approaches in class which, in our opinion, are effective and useful for teaching stochastics. For mathematical knowledge and its acquisition in school a certain complexity of tasks is necessary, and it is only this feature which offers opportunities of gaining access to theoretical knowledge and of shaping the development of theoretical concepts didactically. Other forms which could organize the necessary learning process in class enhancing the theoretical character of probability are small projects, teaching stories, or even historical texts (Jahnke and Seeger, 1986, p.82). In an exemplary way, this is meant to show that curricular materials for stochastics instruction should not and cannot provide 'ready-made' suggestions for the daily practice of teaching. Rather, teaching approaches have to provide orientations and principles and materials for school stochastics which are consistent with the background of the epistemological difficulties of stochastic knowledge. In this sense, task systems are exemplary approaches to stochastics; their principles of construction reflect the relationships between professional teacher knowledge and the curricular elements of stochastics, between classroom activities and the problems of mathematical knowledge.

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