Heinz Steinbring

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MATHEMATICS IN TEACHING PROCESSES. THE DISPARITY BETWEEN TEACHER AND STUDENT KNOWLEDGE*

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RÉSUMÉ

L'examen d'épisodes de l'enseignement des mathématiques dans une perspective épistémologique fait voir que la disparité du savoir entre professeurs et élèves ne provient pas tout simplement de ce que les uns savent plus et que les autres savent moins. Les niveaux de compréhension du savoir indépendants et souvent incompatibles, particuliers au professeur et aux élèves, montrent la nécessité de respecter les aspects conceptuels par opposition aux aspects matériels, et la tendance des processus d'enseignement de toujours retomber malgré tout vers une forme du savoir mathématique fortement déterminée par le sujet mathématique et par les méthodes.

RESUMEN

El examen de episodios de la enseñanza de las matemáticas desde una perspectiva epistemológica muestra que la disparidad de saber entre profesores y alumnos no procede solo de lo que los unos saben y los otros conocen menos. Los niveles de comprehensión del saber, independientes y muchas veces incompatibles, particulares de los profesores y los alumnos demuestran que es necesario respetar los aspectos conceptuales frente a los aspectos materiales, y la tendencia de los procesos de ensenanza de reducirse, a pesar de todo, en una forma de saber matematico muy determinada por el tema matemático y los métodos.

ABSTRACT

When analyzing episodes of mathematics instruction from an epistemological perspective, it is seen that the disparity between teacher and student knowledge is not simply due to their knowing more or their knowing less. The independent and frequently incompatible levels of understanding knowledge which are peculiar to teachers and to students show how essential it is to make allowance for conceptual as opposed to material aspects, and how the

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conditions of classroom processes nevertheless always tend to regress to a form of mathematical knowledge strongly determined by subject matter and method.

«What's really angering about instructions of this sort is that teachers imply there's only one way to put mathematics together — their way. And that presumption wipes out all the creativity. Actually there are hundreds of ways to put mathematics together and when they make you follow just one way without showing you the overall problem the instructions become hard to follow in such a way as not to make mistakes. You lose feeling for the work. And not only that, it's very unlikely that they've told you the best way.»
(R.M. Pirsig, Zen and the art of motorcycle maintenance. Slightly modified to mathematics).

1. Introduction: Students yet have to learn what the teacher already knows!

The fact that teacher and student knowledge are different — the topic treated here — touches essential aspects of the basic question: «What is mathematics?» It is well known that this question cannot be answered immediately and in a roundabout way. «The definition of mathematics changes», as Davis and Hersh have remarked (Davis & Hersh 1981, 8). Both historically and epistomologically, and from the perspective of the individual mathematician as well. «Each generation and each thoughtful mathematician within a generation formulates a definition according to his lights.» (Davis & Hersh, 1981, 8). Besides, the answer to that question depends on the perspective and on the interests with which it is approached (cf. Otte 1984, 2).

To begin with, the focus of interest shall be placed on school mathematics which, despite its material relationship with scientific mathematics, is shaped by independent epistemological and social elements because of its institutionalized incorporation in school. Beyond that, the emphasis of analyzing the differences and links between teacher and student knowledge in mathematics instruction will be on epistemological aspects like the means of representation and activity in actual mathematical processes of teaching and learning (cf. Otte, 1984, 2; Steinbring 1985). Perhaps the following thoughts will contribute towards some better understanding of the particularities of school mathematics as compared to scientific mathematics.

I should like to take two somewhat yet unprecise characterizations of mathematical knowledge as a starting point. For one thing, mathematics is a material area of items of information, and for another, there are difficult and complicated conceptualizations in mathematics. It can be stated for the ambient field of mathematics instruction, of teaching and learning mathematics that the material aspects of mathematics are stressed. On many levels, there are such rather more materially oriented characterizations of mathematical knowledge: this begins with the teaching program with its syllabi of subject matter, with the description of various fields of mathematical contents and mathematical routine skills required of the students. In the textbooks, mathematics is ordered in a linear way and presented as a unity in the sense of a ready made product. All the fields, topics, problems, tasks etc. of mathematics belonging to the respective school term must appear here (cf. Davis & Hersh, 1981, 282/3).

It is particularly in the views of those not directly involved in the teaching process which are dominated by a material-quantitative representation of mathematical knowledge. Thus, parents will frequently ask to which page of the textbook instruction has progressed, how far pupils are able to count, or which mathematical procedures and skills their children have acquired up to now.

Students express their own view of mathematical knowledge in a similar way, especially after having acquired a better idea of the mechanisms of the processes of teaching and learning in mathematics instruction after several years of experience in school. We are familiar with student remarks like: «We have not had this yet!», or «Sir, please tell us how this works. You are bound to know it!». Or, even more direct: «The students went further — "Why couldn't the teacher just hand out the solutions, and let the students use them when they wanted them?"» (Davis & Mason, 1986, 1).

For mathematics teachers, too, the mathematical curriculum presents itself predominantly as an extensive stock of subject matter whithin which many topics have to be worked upon and treated; they are confronted with the expectations of parents, students and colleagues who again and again demand that certain elements of knowledge be treated, that the program be realized to a certain degree, for the ultimate problem is that higher grades and other levels of school must be able to found their teaching on prerequisites and stocks of knowledge which must have been exhaustively treated beforehand.

«Present-day mathematics instruction is as a rule dominated by the subject matter, the teacher, in particular, devoting himself entirely to the service of the subject matter and to the duty of conveying it. It is the good teacher who deems it to be his task to inspire life into the dry and abstract things of mathematics, to make them emerge by constructing and developing the edifice of mathematics again and again. This absorbs his whole energy. In doing so, however, he performs like an actor. What happens here is no genuine struggle with the subject matter. His own relationship to mathematics does not find expression. He knows what has to be the result, and the students, too, know that potential results are already predetermined.» (Fischer & Malle, 1985, 330).

Students organize their mathematical knowledge — according to Andelfinger — by means of *material images*. «Each of these material images is characterized, for the student, by a typical smell, a special picture... In the students' minds, these material images are not dissociated, but are fed by a quite palpable repertoire of thought and behaviour in the students who activate, continue or even develop the latter...

Material images are ways and instruments the students use in trying to understand mathematics instruction. Their generation and development serve primarily to communicate in the mathematics classroom, to establish contact with the teacher in his role as a representative of school mathematics.» (Andelfinger, 1985, 58/9).

This dominant conception of school mathematics as of a linear, hierarchical stock of knowledge as found in the environment of school and learning in parents, students, teachers etc., corresponds, in a certain sense, to a widespread social view according to which «knowledge» is primarily conceived of as an encyclopedial stock consisting of multiple facts and details which have to be at one's disposal and which can be evoked and examined.

This «populistic» view of the character of knowledge is reinforced at present by the computer. Computers know more, can store much, have an unlimited potential to keep data, facts, knowledge at disposal and to make them accessible by rapid retrieval. Knowledge — this holds for mathematical knowledge as well — is interpreted, in the first place, as a collection of many individual insights, skills or even solutions. And for mathematics in particular, I think, this conception of ready-made knowledge, of a cumulatively established and accumulated knowledge is especially widespread (cf. Davis & Hersh, 1981, 17ff). Basically, it corresponds to the conception often encountered among non-mathematicians that mathematics is something fully developed and that research will establish nothing new. The exclusive objective of school, from this perspective, is to rediscover and process mathematics and mathematical knowledge already given.

On this basis, the roles of teacher and students with regard to mathematical knowledge are obvious: the teacher knows, and must know, more about mathematics than his students, and he must gradually convey this mathematical knowledge to them. Or, the teacher's status of mathematical knowledge is more advanced than the students', and the teacher will take them «by the hand» and conduct them into the new fields of knowledge.

This extreme version which says that all mathematical knowledge is basically «subject matter» has been repeatedly criticized, be it from an epistemological point of view, be it on the basis of analyses of teaching. «Mathematical knowledge cannot be reduced to a stock of retrievable "facts", but concerns the ability to compute new results. To use Piaget's terms, it is operative rather than figurative... Operative knowledge, therefore, is not associative retrieval of a particular answer but rather knowledge of what to do in order to produce an answer. Operative knowledge is constructive and, consequently, is best demonstrated in situations where something new is generated, something that was not already available to the operator.» (v. Glasersfeld, 1983, 58). This criticism is mainly adressed against the assumption that the mathematics subject matter is simply available or can be «handed over» to the student. Mathematical knowledge must be constructed by the student himself. «...the idea that knowledge can be conveyed — for instance by speech or by other means of communication — collapses in this constructivist perspective. Such a transport cannot be realized, for that which is called knowledge can only exist whithin a mind. At best, a learner can be assisted in constructing something which then proves to be compatible with that which is called knowledge.» (v. Glasersfeld, 1987, 12).

From this perspective, knowledge is no objective, ready-made concept; knowledge must be constructed. This perspective, however, still does not question the misconception of knowledge as something which is of cumulative character, as a stock composed of many individual facts and elements; it only emphasizes that there is no objective field of subject matter, but that there are different, subjective fields of mathematical knowledge for each individual.

The fact that acquisition of knowledge is not immediately and abruptly realized by the teacher's transmission to his students, but rather is a time-consuming process, is apparent from analyses of real teaching-learning processes in the classroom. There is evidence of a complicated contrast between the characterization of mathematics as a product or structure and the characterization of teaching and learning as a social process. These conflicting perspectives or even *two different cultures* of mathematics instruction are basically made compatible with one another in a relatively brutal way. Under the «pressure» of the linearity and quantitatively sequential form of mathematical subject matter, the social process of teaching-learning is deprived of its essential social character and degenerates to some kind of routinized communicative pattern of events. Bauersfeld has formulated this quite pointedly as follows: «The mathematical logic of an ideal teaching-learning process... becomes replaced by the social logic of this type of instruction.» (Bauersfeld 1988, 38).

This destructive dissolution of the contrast between process and structure naturally has an impact on the different character of knowledge in teachers and students. Not only the fact that the teacher knows more and that the students will always lag behind the teacher with his own knowledge, but rather the social pattern of teaching-learning processes thus reduced confines the students to attain all their knowledge only *via the teacher*. Knowledge is not independent of the teacher: it is filtered by his predeterminations, helps, indications, his choices, etc. Knowledge passes through the teacher to the student, and he does not only determine the scope of the knowledge taught, but also provides decisive orientation for the student with regard to what are legitimate conceptions of mathematical knowledge.

Basically, it is only natural for the teacher to know more than his students. The things demanded for the teacher's professional activity include not only more mathematical knowledge, but also multiple forms of didactical, psychological, pedagogical knowledge, practical classroom experience, etc. It is not only a matter of quantitative extra knowledge, but that there is an intended qualitative difference between teacher knowledge and student knowledge. What should and might this qualitative extra knowledge for teachers comprise? There are few accepted forms and materials for making this extra knowledge accessible to teachers, and in in-serve training, the quantitative aspect of extra knowledge for teachers will in most cases prevail once again. During in-service training seminars, teachers themselves tend to ask for mathematical knowledge immediately relevent for everyday teaching. This pattern is basically congruent with the sardonic exaggeration that the only advantage the teacher's status of knowledge has on that of the students is some pages in the textbook.

In didactical analyses, the problem of qualitative differences between teacher and student knowledge is frequently discussed on the basis of the dialectics between *old and new knowledge* (cf. Chevallard 1985, Jahnke 1978, Otte 1986, Seeger 1988, Steinbring 1988). The teacher already disposes of the new knowledge the students yet have to learn. The epistemological compromises arising from the tension between the fact that new knowledge is reduced to things familiar to the students and the teacher's obligation to teach actually new knowledge again and again will lead to two separate levels of knowledge for teachers and students. The teacher is in possession of mathematical theory, he is aware of conceptual linkages; the student calculates and works out problems (cf. Chevallard 1985, chapters 6 & 7). A typical example of such qualitative differences is *proof* in the mathematics classroom (cf. Jahnke 1978). The teacher has control of proofs he demonstrates in the classroom, and in most cases it is sufficient if his students can follow him.

Dealing with these fundamental differences in the actual classroom situation is rather ambiguous; an «algorithmization of knowledge» (Chevallard) establishes a make-believe identity between teacher and student knowledge to support the conviction that all mathematicsl knowledge is linear, uniform, and situated on a single structural level given at the outset. Teachers will organize the development of mathematics in the classroom along the lines of their own ready-made knowledge, in a way similar to that of textbooks. «The presentation in textbooks is often "backward". The discovery process is eliminated from the description and is not documented. After the theorem and its proof have been worked out, by whatever path and by whatever means, the whole verbal and symbolic presentation is rearranged, polished, and reorganized according to the canons of the logico-deductive method.» (Davis & Hersh, 1981, 282). What is presented to the students as a seemingly original and universal from of description and development of knowledge, however, is itself the result of long and even contradictory efforts. To shape the development of mathematics knowledge in school accordingly proves to be difficult and extremely demanding. In principle, this would require a changed conception of school mathematics, together with its consequences for the actual process of teaching and learning, as Wittmann and Müller (1988) have shown with regard to mathematical proof, for instance.

In the following, I should like to confine myself, in treating the roblem of the different character of teacher knowledge and student knowledge, to the subject matter of school mathematics. This is a deliberate choice not to oppose the broad scope of professional knowledge demanded of the teacher to the «limited» knowledge which is at the students' disposal. Rather, I shall attempt to describe, from an epistemological perspective, where this relationship between material and conceptual aspects of knowledge is found within mathematical teaching-learning processes, where it becomes inevitable and essential, and how classroom mechanisms affect this precarious balance of subject matter and concept.

2. How does the teacher use his «lead in knowledge» for the students' process of learning? — Classroom episodes

An exemplary analysis of two transcribed teaching episodes will be used to elaborate the gaps between teacher knowledge and student knowledge, and the difficulty of «bridging» these gaps. In particular, it will be shown that the teacher's additional knowledge is mostly made accessible to the students only in a reduced form, in a form which appears to promise him immediate help, but which actually prevents him from really developing the new knowledge himself. The maxim frequently voiced: «The student must be fetched where he stands!» has its limitations; if it is interpreted too strictly and too simply, it will in a certain way create obstacles to learning.

2.1. Methodical remedies of the teacher for introducing new knowledge — Description of two episodes

The analysis of the teaching episodes uses a schema of structurization for subdividing the course of teaching into different *phases*. A phase is characterized by the *opening*: the teacher poses a question or a problem to be tackled by the students, and by the *closing*: students seem to be able to answer the question and sometimes the teacher reconfirms this by repeating the answer in his «corrected» manner. Clearly, the course of a whole mathematics lesson normally cannot be structurized as a sequence of phases one following directly the other. The phase-structure is more complex. When treating a problem, a new question may arise causing the opening of a *sub-phase* within the present phase. The closing of the old phase normally has to wait for the closing of the sub-phase or even of other sub-sub-phases (cf. Steinbring 1990). In this way the structurization of transcribed mathematics lessons by means of *phases* leads to a hierarchical and linear pattern: On the micro level, there are phases following in a linear way one another, and on the macro level there are overlapping phases containing a complex of sub-phases. The interaction in the classroom changes from macro to micro levels and back. The chosen episodes to be analysed in the following represent short sections of a lesson whose structure of phases display a linear succession.

Analysis and description of the teaching episode: «The impossible event»

The present teaching episode (cf. annex) can be subdivided into the following four phases:

1. Phase (1-8): Set theoretical language in probability
The teacher's intention is to enable the students to speak of probability in *set theoretical terms*. In order to do so, he starts from a probabilistic game situation and asks the students to name the events occurring upon tossing a die. The set brackets used by the teacher make the students associate the context of «set theory»: the teacher now calls the listed elementary events of the die *fundamental set* in the sense of a simple definition.

2. Phase (9-19): «Definition»: events are subsets
The students exercise their ability of translating probabilistic notions into set theoretical terms. With regard to the probability game treated, the students are to write down all the events occurring in set theoretical language. The «result of work» of this phase is formulated by the teacher as follows: events are «defined» as *subsets* in the context of the set theoretical way of speaking.

3. Phase (20-28): «Introduction» of the certain event
The teacher tries to introduce the «pathological» sets (in particular the *fundamental set* and the *empty set*) for the sake of completeness. In this third phase, the teacher approaches the introduction of the fundamental set directly by means of a task. The gist of his question (N°20) is: «Which subset describes the event: "Toss a number smaller than, or equal to six!"?» The students' answer which refers to their concrete representations, however, is reformulated by the teacher into the desired result by means of a terminological convention he introduces: the total set of the outcomes is called *fundamental set* at this point, and the students are reminded that the fundamental set itself is a *subset*.

4. Phase (29-55): «Introduction» of the impossible event
The teacher pursues the introduction of the *impossible event* as well as the set theoretical characterization as an *empty set* by

developing several questions. He begins with relatively simple enquiries after the set «which is a bit out of the ordinary», which «has been left out», «which actually does not occur», «which we shall practically never write» (as the teacher says in his statements 29-36, 39). This general formulation of the question elicits litte response from the students; in particular, the students introduce other subsets which seem pertinent to them in the present discussion.

Responding to that, the teacher (in statements 45-48) does not directly refute a proposal made by a student. The student has proposed to consider the event «number less than, or equal to one» (N°45). The teacher reinterprets this proposal somewhat by saying: «Toss the die to obtain a number *less than* one?» (N°48). The abstract enquiry after the impossible event is made concrete here by a reformulation provided by the teacher, and reduced to a very special point. To this, the students also respond, according to their representation, with «...won't work», «the uncertain event», etc. The teacher takes up these student responses, but reformulates them in terms of the correct «characterization»: «We shall simply say: the impossible event.» (N°54) The final statement, however, makes clear that this does not work in the student's comprehension (N°55).

This brief teaching episode shows quite clearly how the teacher attempts, on the basis of his own complete knowledge and in a more or less traditional way, to elicit knowledge from the students by questioning, or to develop knowledge in the questioning mode, as the familiar euphemism says. The «application» of set theory to probability is realized in the frame of a general and comprehensive «mathematical language» useful for several fields of mathematics. From this perspective, the classroom dialogue is intended to give a set theoretical description for concepts of probability such as event, outcomes, etc. The concrete things the students are familiar with, namely the game, the events, the die etc. are to be decontextualized in a certain way, and described in the langugage of set theory.

On the basis of his more or less complete «knowledge» about the system of subsets with their extreme cases, the teacher attempts to use the students' knowledge as a starting point. He asks them skilful questions, he focusses his questioning to certain points of emphasis in order to evoke the correct answers. Nevertheless, the convinction at the close of this teaching episode is that teacher and students basically do not talk about the same things, and that the students do accept the things the teacher seemingly has obtained from them by questioning as something predetermined by the teacher. The students have not become aware of this wider aspect of the set system which alone is apt to make them see why «empty set» and «fundamental set» are considered to be subsets of the system. Seen from the concrete situation, the impossible event, which does not happen, as the students say, is superfluous. It does not occur, it is not written down, etc. Thus, there remains in fact a discrepancy between the knowledge of the teacher and the knowledge of the students. The students accept this knowledge which seems to have been obtained by astute questioning as a way of speaking, as a vocabulary for their concrete situation. The teacher conveys the impression of having guided the students from the concrete siutation to the general structure of representing sets. This observable discrepancy does not have any consequences, there seems to be agreement on the surface of the interaction.

Mathematical knowledge constitutes an *interdependent form between material contexts of reference and symbolic representations*. In this episode we are faced with a wide span between a very concrete, locally confined game situation with dice, game board, concrete designations etc. on the one hand, and the rather abstract, generalizing view of set theory on the other (a set theory with its extreme cases which have their rather abstract meaning only within the system of set theoretical calculus and never have a direct, concrete meaning). Here it seems to be obvious that an immediate bridging of this wide distance is impossible.

The following teaching episode shows that this problem of object and symbol, of things given and their general mathematical structure can never be avoided in teaching processes nor be neutralized. Learning mathematics is centered around this relationship between context and structure, if learning is understood to be a social activity which will always require individual, personal aspects of understanding and meanings. The example which follows shows that there can be no dissolution of this relationship despite a seemingly close connection between context and structure, and hence there can be no automatized learning or algorithmic production of knowledge.

Analysis and description of the teaching episode: «Possibilities of the sum of the pips of two dice»

This episode again can be subdivided into four phases (see annex):

1. Phase (1-15): Routine work
In this first phase of the episode, there is a rather rapid stating of the possibilities for the sum of the pips of two dice in a dialogue between the teacher and the students. This is done for the sums 7, 8 and 9. Students name the various possibilities, for instance, as follows: three and three, four and three, etc., or as three plus four, four plus three, etc.

2. Phase (16-29): A conceptual difficulty arises
The routine activity in the classroom is interrupted as one student is in doubt why four plus five and five plus four must be listed twice. The student says (Nos. 16 and 17): «Five plus four is already listed!», and another student responds and says: «No!... Four plus five!» This newly arising difficulty of understanding is then taken up and reinforced by the teacher, he even insists on this question to prevent the students from simply passing it over. Thus, the teacher states at one point that five plus four are nine and that four plus five also yields nine. Besides, he asks: «Why, eh, is two plus two happen not written down twice, as in the case of the other sums?» (N°22). During this phase, the students argue at first within an additive frame (cf. Krummheuer 1988), saying that the «result» in sums is equal, while the order of the summands is different. This different order of summands is no longer discernible in the case of two and two. At the close of the second phase, the teacher is content with the following statement of a student as justification (29): «.... for one plus seven, there is still another possibility. For two, it is always the same.» It seems that this notion of *possibility* with regard to the erroneous justification within the frame of addition is accepted by the teacher.

3. Phase (30-37): Did the students understand the justification?
The students are asked to justify, by repeating the arguments just given, which possibilities are there for the sum of pips nine. One of the students again answers in terms of «sum», «result», etc. In a way, he transforms the justification into a rule of calculating. And he says that three and three need not be written down twice. The teacher is not satisfied with this explanation. Probably as a methodical help, he asks the students to assume two dice of different colour, a red and a black one. The student statements show, however, that they continue arguing in the frame of addition and result with the red and black dice as well. Thus, one of the students says: «If you toss the other way around, the result is the same» (N°35). And another student answers: «If you now take the six and add the two... you have eight. If if you turn this around, I mean the numbers, two plus six, then this yields... that is the other way around, if you have four plus four, and turn this around, this again yields...» (N°37). This demonstrates to the teacher that his offer of making a colour distinction between the two dice does not initiate a better insight and justification in the students.

4. Phase (38-48): The teacher's justification
The teacher now tries to give a justification by directly referring to a red and black die with their respective outcomes. At first, the teacher benevolently accepts a superfical justification given by one of his students: «Exactly the same thing the other way around.» (N°41). And the teacher interprets this student statement in the sense of a justification initiated by the coloured dice. That this justification, however, has not been understood by most of the students is evident in the contribution of one of them who again talks of task, calculation, result, thus expressing the «foolishness» (N°43) of the method presented. For some students, it seems that the difference on the level of symbolic notation is the distinctive characteristic which makes them consider "two plus seven" and "seven plus two" as different and "two plus two" as identical to "two plus two" *turned around.* «What matters is the notation!» (N°47), one student says. At this point, one has the compelling feeling, the teacher «surrenders» for the time being, leaving the gap

between the students' understanding and his own knowledge open. «Well, I think we'll have to do this again in another lesson.» (N°48).

This teaching episode shows, on the one hand, that there is not so large a distance here between the substantial context of two dice, the game etc. on the one hand, and the coding of this situation in signs in the shape of tuples, sums, possibilities, etc. on the other. Besides, it becomes evident how difficult and complex it is to maintain a balance between context and structure within mathematical processes of teaching and learning, and how precariously the epistemological tight-rope walk takes place between the concrete representation on the one hand, and the generalization and mathematical coding on the other. During this teaching episode, the following formulations are used to characterize the dice for distinguishing them: «first die, second die», «the one die, the other die», «red die, black die». It may even be guessed that the way the teacher holds the two dice in his hands leads to representations in the students' mind like «die in left hand, die in right hand». Basically, the students are confronted with a multitude of characteristics which can be used to distinguish between the two dice.

The crucial problem of understanding emerges during this episode at the point where one of the students says that this sum has already been named: «Five plus four is already listed!», and another student responds: «No!... Four plus five!». This problem formulation is emphasized and maintained by the teacher, and he also tries to «offer» a solution, suggesting to use dice of different colour. Why does this methodical «trick» fail to assist the student in accepting the justification? The answer could be that the multitude of possible characteristics of two dice is counterproductive as a red die held in the left hand and a black die held in the right hand designate two distinctive features, which, if «swapped», makes possible a distinction between «red two plus black two» and «black two plus red two». From the viewpoint of the technico-conceptual structure of probability, this is a case of constructing the «suitable» probability space. By methodically introducing more and more distinctions between the two dice, probability spaces of higher dimensions are constructed. Within the context of the game situation, however, only one distinction should be admissible (this is simply a case of two dice!), and this is what should be seen and understood by the students.

For the students, however, the die is not simply red or black, but it is also the first and second, the one and the other, the die in the right and the die in the left hand, it is tossed as the first or as the second, etc. For the students, this renders the situation even more multi-faceted and obscure. It could be said that the teacher compresses all his own pertinent knowledge about the total system of the two dice into his methodical reduction to a pair consisting of a red and black die. He is basically aware of all the outcomes of tossing the two dice, and he also knows that two dice must be distinguished (perhaps even that this is a case of distinguishable and not of indistinguishable objects of the kind occurring, for instance, in other statistics). The students, however, can take up and use the methodical assistance he offers only within their own context. They are asked to continue enumerating and justifying for individual outcomes why some can occur only once, while others can occur twice. Apparently, the problems of development and understanding arising within this local mode of work can only be solved by a global view on the entire system which is independent and thus able to contrast the context-bound local methods in order to advance understanding.

Why the choice of a red and a black die? Why the focus on one distinctive criterion despite the fact that there is an infinite number of different characteristics of two concrete dice? Such and similar questions cannot be answered simply from the local, concrete situation; even they already require a — primitive — global systemic point of view, as it is represented in an exemplary fashion in the matrix of all the possibilities. The students, however, proceed locally by coping with the individual cases as required by the curriculum, and in agreement with a linear-quantitative conception of mathematical knowledge. For justifications, students must rely — if at all — on additional indications which can be found, for the content oriented side, on the surface of characteristics of terminology, and for the side of social interaction in teacher responses, reinforcements, or non-acceptances.

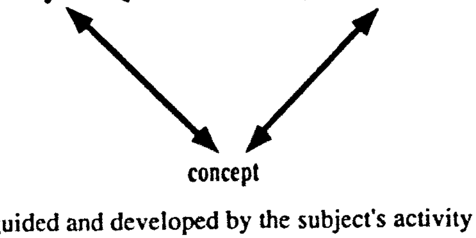
The justification of the present problem which has been methodically «concealed» in the red and black dice by the teacher is intended to start directly from the local, context-bound knowledge of the students. In this reduced form, however, it prevents the students from developing a global, systemic point of view. It is evident that introducing a red and a black die is no help, but rather becomes, in this form, an additional difficulty for the students, besides the various distinctive criteria for two dice which are already subjectively present.

2.2. Epistemological frames for constituting mathematical meaning in classroom interaction

One of the starting points for our considerations was that the difference between teacher and student knowledge is mainly seen *quantitatively*. The teacher has progressed further in the linear-hierarchically structured mathematical knowledge, and the student must follow him. The first episode indeed seems to confirm our initial considerations: the quantitative distance between the teacher's knowledge and the students' knowledge just appears to be too great here. By contrast, the second episode calls attention to the fact that the difference in knowledge must not be sought in a quantitative distance. In this episode, the teacher has approached his knowledge as far as possible to that of the students. Despite this fact, the students' progress in learning or in understanding seems to be almost impossible. What is the cause of the distance between teacher and student knowledge here?

Our comparative analysis of the two episodes has shown: epistemologically, a rather complex systemic relational structure of knowledge emerges in the real process of teaching and learning new knowledge: object-sign-concept. This system, or *epistemological triangle*, is developed, organized, and specified by the activity of the respective learner-subject (cf. Steinbring 1989).

Figure 1.



The distinction between "object" and "sign" is in different forms a well known conceptual means in the philosophy of language. It is used for analyzing the relationships between "words", "objects" and "notions"; the markings of the corners in the triangle have taken various forms, for instance, "sign", "signified" and "significant" (for more details of the different forms this triangle has adopted see Eco, 1977, 30). When using this triangle in mathematics for analysing the structure of mathematical concepts, a mayor difference is that the relation between "sign" and "object" becomes open and variable: there is no pre-defined connection between a certain "sign" and a fixed "object", mathematical "signs" may refer to very different objects.

The differentiation between mathematical symbols and referents has been used in educational studies for analyzing processes of abstraction and the structure of conceptual fields. Kaput in his «Symbol systems theory» (Kaput 1987) focusses on the growing abstraction of concepts: «We asserting that, although symbol systems act as reference fields for one another, ultimately, chains of reference for symbol systems get back to the primary reference fields, such as the natural number system, which are taken to be mathematical structures. These primary referents are not other symbol systems *universally* expressible further in terms of shared symbols relative to a given language community, but rather entities with shared features in a given cultural community. We say "cultural community" rather than "language community" because such a structure seems to be readily shared across language communities — the way "3" is written or spoken seems to be unimportant to its meaning.» (Kaput 1987, 173). The source of the meaning for abstract concepts lies in the primary reference fields.

Vergnaud emphasizes the distinction between "situations" and "symbolic representations" for elaborating an object of research unique for mathematics education. «It is a scientific challenge of our time to promote the study of learning and teaching mathematics as a well-defined and interesting scientific field in its own, not reducable to mathematics, to psychology, to linguistics, to sociology or to any other science.» (Vergnaud 1982, 31). «.... I consider that psychologits and maths educators must not study too small-sized objects, because they would not understand the complex process by which children and adolescents master, or don't master, mathematics. A "conceptual field" is *a set of situations, the mastering of which requires a variety of concepts, procedures and symbolic representations tightly connected with one another.*» (Vergnaud 1982, 36). More precisely, this fundamental, complex object of educational research is defined as follows: «An interactive conception of concept formation considers a concept as a triplet (S, I, ζ).

S: set of situations that make the concept meaningful

I: set of invariants that constitute the concept

ζ: set of symbolic representations used to represent the concept, its properties and the situations it refers to.» (Vergnaud 1982, 36)

«One difficulty for researchers is that a single concept does not refer to only one type of situation, and a single situation cannot be analyzed with only one concept. Therefore, we must study conceptual fields. A conceptual field is defined as a set of situations, the mastering of which requires mastery of several concepts of different natures. For instance, the conceptual field of multiplicative structures consists of all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide.» (Vergnaud 1988, 141). The notion of *"conceptual field"* illustrates the complex and variable relations between "situations" and "concepts", and according to Vergnaud it constitutes a basic unit for educational research into students' learning processes of mathematical concepts.

Our use of the epistemological triangle in the following will take into account a further epistemological feature of processes for constituting mathematical meaning in interactive teaching/learning situations. This triangle does not describe the complex epistemological structure of mathematical concepts *explicitely* from an objective and «external» perspective; this triangle posses a kind of «implicit and axiomatic» structure. Ultimately, elements on the "object"-level as well as elements on the "sign"-level do not exist *per se* or in its on right; they only can live in an interdependent relation between each other controlled by conceptual aspects. This understanding of the epistemological triangle implies the following consequences.

There is not an a-priori fixed regulation, which elements of the complex conceptual structure belong to the "object"-level and which to the "sign"-level, the same element sometimes may belong to the "object"-level and then in other circumstances to the "sign"-level. In this way the die in elementary probability theory sometimes is an "object" of investigation, and then it might be a simple "model" belonging to the "sign"-level for analysing other chance experiences.

Further it is important to notice that elements on the "object"-level as well as elements on the "sign"-level do not express in an immediate manner mathematical meaning. Mathematical signs as well as aspects of the reference situation are specific material «facts» which are only capable to provide *mathematical* meaning *intentionally*. The sign itself has no mathematical meaning, only *in its intention* to some context; and elements of the "object"-level only provide mathematical meaning *in the intention* to display a relational structure hidden in the reference situation. It is this *meaning by intention* by which mathematical signs as well as aspects of the reference situation must be endowed in order to become productive elements in the epistemological triangle.

The creation of *meaning by intention* associated to "signs" and to "objects" requires the involvement of the active knowing subject in the development of mathematical knowledge; in his struggle with the new knowledge this knowing subject has to take decisions about *intentional meaning* and he has to organize the emerging new relations between "object", "sign" and "concept". When using this triangle as a means for analysing everyday teaching episodes, it becomes obvious that only the multi-layered interplay between social conditions, individual aspects and the complex structure of the mathematical knowledge in question together will constitute for one specific situation an explicit application and show how an interpretation of the epistemological triangle will look like. The complex epistemological structure of mathematical concepts is not simply to a certain extend *given*, at the same time it has to be *constituted* in social processes of negotiation and development.

In a first approximation the following assignment of elements occurring in the classroom interaction during the episodes belonging to the "object"-level or to the "sign"-level can be performed. Within the first teaching episode «The impossible event», for instance, the following elements are on the "object"-level: *game, dice, outcomes, sum of pips*. On the "sign"-level, there are: *set, fundamental set/certain event, subset, empty set/impossible event*. Similarly, this is true for the second episode «Possibilities of the sum of the pips of two dice». The "object"-level here contains: *game, dice, 1st die/2nd die, red die/black die*. And the "sign"-level here contains *«tuples»* in the shape of *«and», plus (3+4), possibility, sum, result, order*. The epistemological prerequisites of concept formation (in both episodes) are to develop and maintain a tension between elements of the "object"-level and the "sign"-level.

A closer look at the phenomena occurring in the real classroom shows that the crucial assumption here is a chronologically organized structure of knowledge. The teacher basically assumes a form of mathematical knowledge he has organized for himself in retrospect and attempts to develop the students' status of knowledge from this perspective. If the process of learning works as organized by the teacher, the students actually seem to follow the path of knowledge step by step just as the teacher has sketched it from his own ordered perspective of knowledge. (cf. figure 2)

Figure 2.

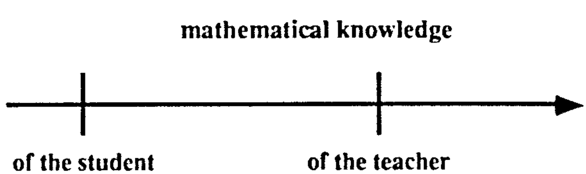
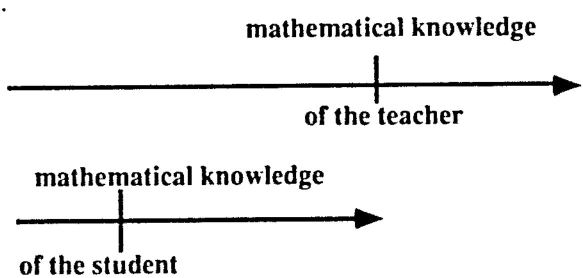


Figure 3.

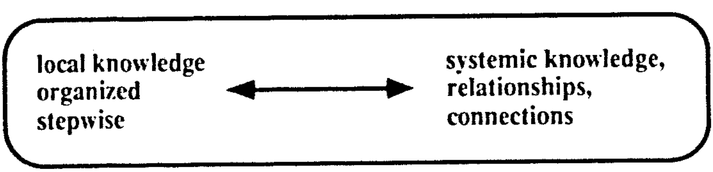


If, however, difficulties of understanding emerge for the students during the process of learning, it is at least evident that student knowledge and teacher knowledge — even despite its respective linear order (which is often only fractional and incomplete in the student) — are situated on totally different levels. The teacher must raise the student's understanding to his own level of knowledge. (cf. figure 3)

The teacher attempts this with teaching methods, by means of a skillful game of questions and answers, or by refusing and reinforcing student suggestions. In our examples, for instance, with the question concerning the event: «Toss a number *smaller than* one!», or by means of the methodical trick of the «red and black dice». These methodical helps are mostly provided with the intention of guiding the students back to the correct path of knowledge, to allow them to surmount too large gaps in knowledge by intermediate steps, to spare them dead ends and detours on the path to mathematical knowledge so consistently prepared. It is often difficult for the participants involved in the learning process to realize that the breakdowns of understanding are not simply caused by a deficit, a deviation, or a mistake with regard to the correct path of learning, but that such breakdowns are an expression of a fundamental problem of developing and understanding theoretical knowledge in its conceptual-systemic structure.

The equilibrium which must be achieved in the learning process between material and symbolic aspects of the concept is subject to a complex interrelationship between local and systemic aspects of knowledge:

Figure 4.

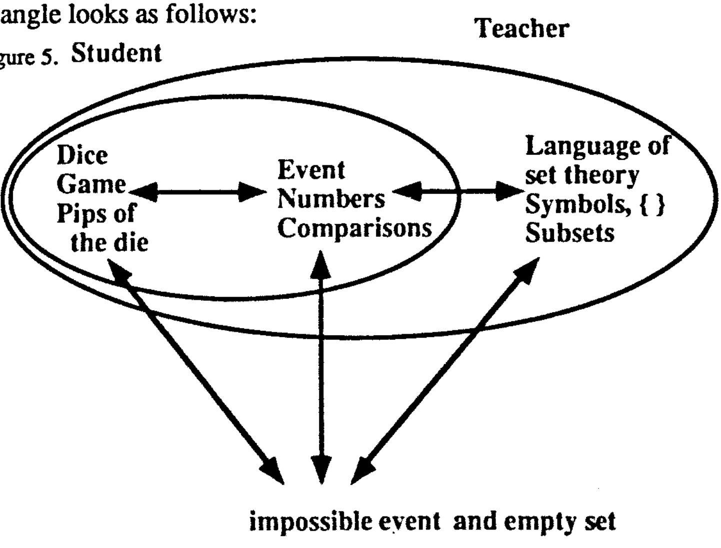


The difficulties and obstacles of comprehension encountered by the students within the process of learning in both teaching episodes are to a large part due to the contradiction resulting from the fact that the process of learning is organized step by step and deductively according to the hierarchy of knowledge with its linear order, whereas new conceptual relationships of knowledge require a systemic-holistic connection, as the new concept is constituted only by its position within the system and is not the result of methodical or logical deductions. The methodical help offered to the students in both episodes by the teacher consisted basically of giving direct hints on the level of local and stepwise arranged knowledge.

Contrary to a superficially observable linear course of knowledge development our analysis shows that teacher and students are coping in *essentially different* ways with their mathematical knowledge. Teacher and students are working in *different epistemological triangles*. Whereas the students locate the relation between "sign" and "object" according to their local perspective closely between elements of the probability game and numbers, operations with numbers and elementary probability notions, the teacher implicitely uses an epistemological triangle expressing his more global perspective and embracing the students' "object-sign" relation as an new "object" of higher order in his triangle.

For the first episode the structure of students' and teacher's triangle looks as follows:

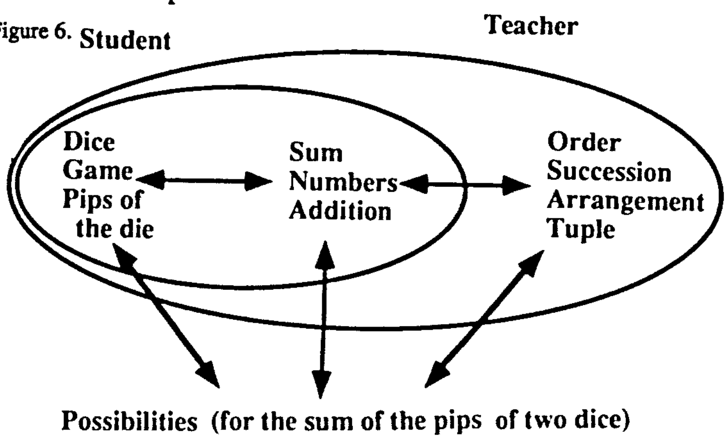
Figure 5.



This diagram explains why the students interpreted the «impossible event» as a concrete empirical impossibility. Within their frame of justification there cannot be a *concept* of "impossible event" or of "empty set". Nevertheless, teacher and students can use the same terms within their frames of argumentation, but with totally different *intentional meaning.*

The structure of both superimposed triangles looks similar for the second episode:

Figure 6.



Within their triangle, the reference to a concrete «black» and «red» die represents an additional difficulty and no help at all on the level of local and stepwise ordered knowledge. Here too, students and teacher using seemingly the same notions, but according to their different frames of argumentation with totally different *intentional meaning.*

Hence, the status of the methodical hints given with reference to the «impossible event» and to «red and black die» is totally different for teacher or students: the teacher assesses it from a systemic point of view, and the students are under the impression of having to integrate these hints locally. And despite the fact that teacher and students act on different levels and argue with different *intentional meanings in their epistemological triangle,* even «talk over each other's heads» for an outside observer, something like formal, external agreement seems to emerge or at least to be forthcoming towards the end. The difference of the *intended meanings* of the concepts aimed at during the episodes remains concealed; to cope constructively with this difference would require taking the fundamental interrelationship between local (material) elements and systemic (conceptual) relationships of knowledge seriously, and not to dissolve it by methodical helps and tricks in favour of the material aspect.

Within the *usual* «casting of the parts» between teacher and students in everyday mathematics instruction, the teacher treats the mathematical knowledge «with hindsight», assumes a «final status» which always conveys some form of order and uniformity to knowledge; as opposed to that, the student as a learner at first always is confronted with an ex ante knowledge which presents itself to him as complex and obscure within his process of understanding, not as ready-made knowledge, but as evolving knowledge.

What is the reason for the fact that this interrelationship between the linearity of processes of teaching and learning and the systemic character of mathematical knowledge is so frequently distorted in favour of a linear order of knowledge and instruction in the actual classroom situation? It is important to note that the more the «linearity» of knowledge and instruction in the classroom get the upper hand, the more reduced and limited forms of mathematical knowledge and insight will result.

3. Learning as centering on the teacher — teaching as an expectation adressed to the students

There is a fundamental and difficult interrelationship between the forms of interaction between the teacher and the students in the mathematics classroom on the one hand, and their implicit conceptions and attitudes about the character of the mathematical subject matter on the other. This is a view quite different from that formulated by radical constructivism according to which all knowledge and all meanings are subjective constructs of the participants of the educational process, and there is no knowledge as such, but that knowledge and meanings are always negotiated in interactive performance. A closer inspection of the present teaching episodes (or at other examples of interaction in mathematics instruction) — with regard to these forms of interaction which are reduced in my opinion — may indeed give the impression that it is unnecessary to assume an «objective» mathematics existing outside of these interactional relationships. A mathematics existing externally does not seem to be able to contribute something essential to an understanding of these internal mechanism and forms of communication and of establishing knowledge and meaning.

In the frame of reduced forms of classroom interaction and reduced conceptions about the mathematical subject matter, however, neither of these assumptions can be proved: it is neither absolutely necessary to assume an objective existence of mathematical facts nor to negate the existence of such facts. This contradiction results, among other things, from an extreme position towards a too *absolute* objectivization of knowledge on the one hand, and with regard to its presumed *exclusively* subjective-social constitution on the other. The total externalization and objectivization of knowledge can only be «moderated» by independent subjective reconstructions and by social interaction.

In his article «Multiple perspectives» (Cobb 1990) Cobb points to the paradox of mathematical meaning: «Mathematical meaning can be in the world (mathematico-experiential), in the individual's head (mathematico-cognitive), and in social interaction (mathematico-anthropological). (....) This complementarity that seems endemic to mathematics education theorizing expresses the apparent paradox between mathematics as a personal, subjective construction and as a mind-independent, objective truth. Accounts for students' mathematical learning typically emphasize one extreme or the other. We seem to have a choice between individual students each constructing their lonely, isolated mathematical realities or students mysteriously apprehending preconstructed mathematical knowledge in the world. As with the complementarity implicit in teaching, we cannot resolve the problem once and for all. Rather, we have to learn to cope with it in local situations by reflecting on "the underlying antagonistic relationships and mutual interactions of the two positions" (Steiner 1987, 48).» (Cobb 1990, 214/5).

In more detail, Cobb suggests the following way out of this dilemma: «The most inviting way out that I see is to complement cognitive constructivism with an anthropological perspective that considers that cultural knowledge (including language and mathematics) is continually regenerated and modified by the coordinated actions of members of communities. This characterization of mathematical knowledge is, of course, compatible with findings that indicate that self-evident mathematical practices differ from one community to another..... Furthermore, it captures the evolving nature of mathematical knowledge revealed by historical analysis...» (Cobb 1990, 214). In this respect our analysis partly agrees with the «solution» of the dilemma between the subjective or objective character of mathematical knowledge offered by Cobb.

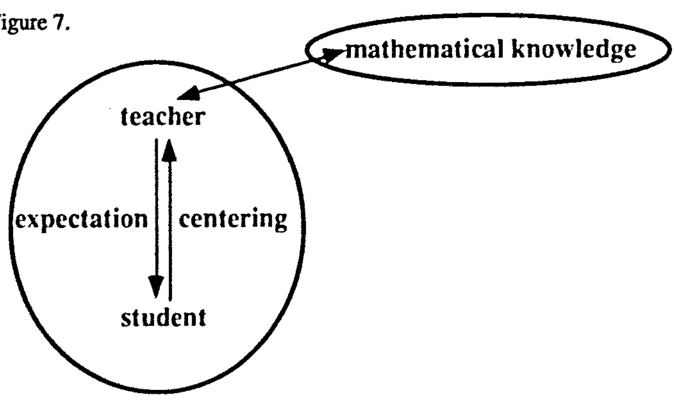
Teaching/learning processes of mathematical knowledge are interactive as well as constructive activities. But according to our conceptual frame, one crucial fact is ignored: When constructing interactively in social processes mathematical knowledge and mathematical meaning, there is not only a development of comprehension, understanding, agreement and including of knowledge into «the physical and intellectual practices taken for granted by specific communities of knowers.» (Cobb 1990, 211). Simultaneously, in social interactions of knowledge construction mathematical concepts are provided with a growing *theoretical* nature: Mathematical knowledge displays a *self-referent* structure, an essential feature of mathematical concepts which calls them into being. This *theoretical* or *self-referent* nature of mathematical knowledge is quite obvious in the social, historical development of many mathematical concepts; a paradigm example in this respect is the history of the probability concept (cf. Steinbring 1980).

The social interactive construction of mathematics in the classroom also provides mathematical concepts with a *self-referent* structure, relative to the social community of teacher and students. The interactive construction produces simultaneously new *epistemological* constraints of knowledge which make knowledge alive and at the same time erects new obstacles for understanding. In our teaching episodes we have observed these *theoretical* and *self-referent* aspects of mathematical knowledge in the tension between *local, context-bound* and *global, systemic* aspects of knowledge. The "impossible event" or the "empty set" is a concept depending in principle on a *self-referent* foundation within a system of higher order. And an adequate justification of "the possibilities of the sum of the pips of two dice" requires an exchange of perspective from a local to a global structure, which cannot be deduced, but which depends too on the *self-referent* nature of mathematical knowledge and meaning.

In interactive processes within the classroom, the emerging theoretical nature of mathematical knowledge is suppressed. On the one hand, the observable disparity between teacher's and students' knowledge is a strong indicator for the theoretical nature of the knowledge in construction; on the other hand, real teaching processes tend to distroy the theoretical nature of mathematical knowledge by declaring this conceptual disparity simply as a quantitative difference in knowledge. This causes the dominant role of the teacher for the mediation of mathematical knowledge in classroom interaction.

A sensitive social interrelationship is observable in real teaching processes which is being reinforced during school years: the students' activities of learning and their opportunities of learning are very strongly determined by the teacher, are *centered* on his person by his preorientation, requirements, and with regard to the test questions he formulates. Vice versa, this is accompanied by the teacher's *expectations* and demands adressed to the students. This mutual social process of «centering on the teacher» and «expectation towards the students» leads to a neglect of mathematical knowledge as a relatively independent quantity. Knowledge is increasingly introduced only by the teacher, is dependent on him and can take on no autonomous corrective or modifying function within the process of teaching and learning.

Figure 7.



The teacher defines the frame of knowledge (set theoretical language or possibilities of the sum of the pips of two dice), he varies questions, introduces methodical helps, formulates new requirements, refuses students' answers and propositions, reinforces possible solutions by transforming and emphasizing them. This interaction between «centering and expectation», as it is evident in our episodes in an exemplary way, is based on the assumption that the mathematical principle is clearly given and ready-made, and at the same time reinforces this assumption again and again.

Here we are faced with a contradiction which places the old controversy about whether mathematical knowledge is objectively given or can only be subjectively constructed into a new frame for teaching: in the mathematics classroom with its mechanisms of teaching and learning and its social environment, the objectivization of knowledge as a prefabricated and given product ordered in a linear way causes that this seemingly totally objective knowledge becomes totally dependent on the teacher as a subject. It is seen that the difference between a strict objectivity of knowledge and its complete subjective dependency is not so large at all.

Mechanisms in the communication and interaction within the mathematics classroom have been analyzed in various ways in the educational literature. Among the mechanisms described are the «funnel pattern» («Trichtermuster», Bauersfeld 1978), the «didactical contract» («contract didactique», Brousseau 1986), and the «circularity of teaching practice» (Steinbring 1988). The familiar metaphor that «teaching is conveying knowledge» (cf. Cooney 1988) sheds light on central aspects of these classroom mechanisms, too.

These explanatory models of mechanisms of interaction in mathematics education may at first be considered as being relatively independent of the question concerning the existence and nature of mathematical knowledge. It is our conviction, however, that these modellings of circular mechanisms in communication always get into conflict with concealed and implicit conceptions of mathematics. To a certain degree these mechanisms like the funnel, the didactical contract, the idea of the conveyance of knowledge are inevitable. Their negative essence will only result from a reduced conception of mathematical knowledge. Or, in other words: these inevitable phenomena in classroom interaction cannot be changed, varied, let alone be improved on the basis of socio-communicative parameters. Every purely interaction-oriented optimization would again — if based on a linear understanding of mathematics — introduce even more methodical tricks and helps into mathematics instruction, thus producing even more mechanisms and circularities of communicative interaction.

For what is required in the shape of representations about the character of school mathematical knowledge in order to be able to overcome the reduced forms of social routines and interaction patterns observable in the classroom? The knowledge which is to be learned cannot be handed over by the teacher, it must be constructed by the student. Hence, the teacher must provide opportunities of learning for the student which promote such an active construction of knowledge. And the form of such opportunities of learning cannot be chosen at will, as every student has his own, very particular representation of knowledge. Developing and using opportunities of learning in the classroom process requires that epistemological conditions of school mathematics be analyzed and taken into account in a way even more differentiated as that required for teaching according to the «handing down» metaphor.

The analysis of the two teaching episodes has shown that the difficulties of teaching and learning are caused here by particular epistemological constraints as well. A dynamical conception of knowledge — as it is appropriate for processes of teaching and learning — is an expression of the difficult relationship between systemic-holistic structure and stepwise hierarchical elements of knowledge. This epistemological condition affects the teaching process directly, be it even in the reduced form observed that students remain on the level of local knowledge ordered in a linear way, while the teacher attempts to organize learning in a methodical-deductive way starting from the respective interrelationship, be it that the intention is to modify such reduced interactions.

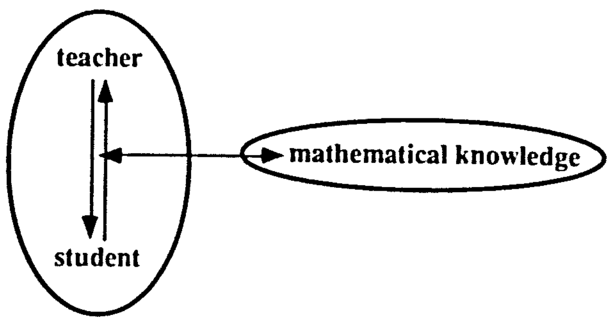
To vary and change these socio-communicative mechanisms within the process of learning requires varying and changing the conception of mathematical knowledge. In view of the prevailing «linearity» of teaching which corresponds to a «linearity» of mathematical knowledge, only the additional characterization of knowledge as a relational system instead of a linear order of learning would seem to be of help in case of emerging difficulties of understanding. This duality of linearity of teaching and systemic character of knowledge is not only evident in the first of the episodes with its global relationships, but emerges — in in the second episode — even at entirely local points of the process of learning.

A conception of the character of school mathematical knowledge thus modified would contribute towards establishing a new balance of subjective and objective elements of mathematical knowledge in processes of learning. From this persepctive, mathematical knowledge is no longer an objective product somehow made absolute, but rather contains, as knowledge, the two aspects, subjective and objective elements. And only this persepctive can enable us to shake off the domination of routine communication in classroom interaction or at least to limit it.

The reestablished balance between the subjective and the objective elements of mathematical knowledge is required if the intention is to take away the teacher's exclusive function as the only one who posses knowledge, decides about it, evaluates it, etc. This is the only way to ensure that knowledge is put to discussion, interpretation, evaluation, to avoid that it is predetermined, produced or ready-made to be skilfully taught by means of social mechanisms in the classroom. The special epistemological prerequesites of mathematical knowledge which may have an impeding effect on understanding and which cannot be communicated, conveyed, or methodically evaded by the teacher, require that students be not strictly dependent on the teacher within the social process of teaching and learning. Students must be enabled to cope with the knowledge «on their own»; the knowledge must be put at their direct disposal, it does not «belong» exclusively to the teacher. This requirement should not be mistunderstood: processes of teaching and learning are of social and interactive character, and there is no «pure» knowledge in them, but there are always social attitudes and conceptions concerning this knowledge as well. The difficult epistemological conditions in mathematical knowledge, however, show that this knowledge acquires autonomous characteristics during its socio-historical constitution which cannot remain entirely arbitrary individually and subjectively. The relationship between socio-subjective and epistemological aspects becomes more complicated in processes of teaching and learning if the intention is to modify reduced forms of teaching; such processes cannot be reduced neither to objective, hierarchically ordered knowledge nor to subjective constructions.

Any intention to change the mode of developing knowledge in processes of teaching and learning thus should not be shaped alone by existing examples in the classroom, replacing the concrete teacher by a generalized one, as Voigt has suggested: «By negotiation of meaning the students can learn to argue with themselves, i.e. to argue with a virtual "generalized" teacher in one's mind — gaining autonomy.» (Voigt 1989, 13). Within a socially organized process of teaching and learning, mathematical knowledge itself should become the object for negotiation of meanings. The many layers of mathematical knowledge between conventional definitions, subject matter, conceptual connections and epistemological conditions cannot be maintained by social and subjective forms alone.

Figure 8.



In order to ensure that learning can occur, it is necessary to make the conceptual difficulties which are manifest in the system of "object", "sign", and "concept" a topic instead of smoothing them out or avoiding them, with the best of intentions, by methodical simplifications. A characterization of mathematical knowledge thus modified could contribute to understanding communicative principles like the funnel pattern, the didactical contract and the conveyance metaphor for mathematical knowledge in a form not reduced and negative, but productive for teaching.

With regard to the mathematical subject matter and its meaning, the analysis of the different character of teacher and student knowledge has revealed the following: a quantitative difference between the things teacher and students know exists of course. Together with the idea, however, that mathematics is in principle some quantitative, material stock of knowledge, this perspective often has a negative effect on actual teaching. It prevents to become aware of the fact that there is an additional difference between teacher knowledge and student knowledge as well, and that this difference is not only due to the fact that the teacher is in control of theory, theoretical connections and proof while the the students merely master routine skills and methods of calculation.

The «chronological» position of teachers and students with regard to knowledge is not the same: the student is often in a situation «before» knowledge, and the teacher mostly feels his position towards knowledge to be one of hindsight, something ready-made. Or, to express it with a (not too exact) comparison to Kuhn's conception: the student always seems to be in a state of revolution, while the teacher is more a normal scientist of school mathematics. Because of their respective position within the didactical system of education, the epistemological view students and teachers take on the same mathematical knowledge is basically different. An independent epistemological view of the student on knowledge is frequently not perceived or accepted as necessary for learning; at the latest, it fades away at the end of the lesson after consensus about the facts of knowledge treated has been achieved somehow. The consequence of this would basically be to assume that teachers and students are first of all different individuals having their own contexts, ideas of meaning, and activities.

If teaching is understood as a process in which students acquire more mathematical knowledge with the teacher's help, a process intended to reduce the difference between teacher knowledge and student knowledge, the paradoxical conclusion is that the *differences* between the mathematical knowledge of teachers and students can only be reduced if the fundamental *disparity* between student knowledge and teacher knowledge gains more importance.

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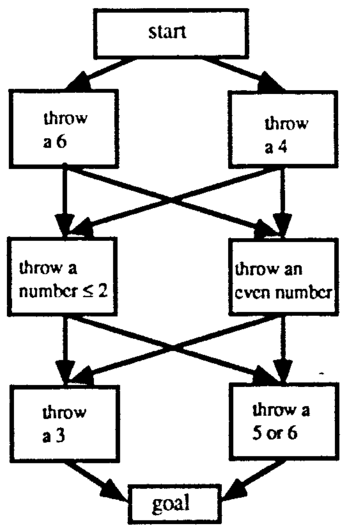
ANNEX

Teaching episode from a 6th form lesson on probability. Topic: *«The impossible event»*

In the lesson before, pupils have played the following game, discussing it with the teacher afterwards:

1. Here, you see a game board. For the game, you will need a die.



At the beginning of each game, a token (e.g. a colored hat) is placed on one of the above fields «throw a 6» or «throw a 4». You may choose one of these two fields. After you have completed the task inscribed in the field, you may advance to the next field in the direction of the arrows (again, you have a choice between two fields). The first player to reach the goal wins.

1 T.: Which are the possibilities of tossing the die? Just name them!
Teacher hands a die to one of the students.
2 T.: Just toss the die!
Student tosses the die.
3 T.: A one! We can begin by writing this down. What are the other possible events?
Teacher notes outcome on the blackboard.
4 S.: Two.
5 T.: Yes.
Students shout: three, four, five,...
6 T.: ...and six. Now we have written down all the possibilites, and we can go and just place these brackets here...
Teacher notes set brackets on the blackboard.
...what do we have before us?
7 S.: A set.
8 T.: Right! And this set, we are going to call this the fundamental set.
Teacher writes this down on the blackboard. Students write it into their workbooks.
9 T.: Now let us look at this sheet *(worksheet with exercise 1)* again, taking out one of the squares. Read it aloud, Monika!
10 S.: Toss a number smaller than, or equal to two.
11 T.: My question to you now is: we have written down here the possible outcomes of a game of dice. Can anybody tell me how I can formulate what is written in the square on the worksheet by means of the set theoretical way of writing?
12 S.: Write down the one and the two.
13 T.: Right! In this case, we may go and write this as follows:
Teacher writes on the blackboard {1,2}.
14 T.: Now take a look at the other squares and tell me how they can be described as subsets. Anke, just take any of the squares!
15 S.: Toss a three.
16 T.: Toss a three. How can you write this? This is now written in German language, how could you write it in a mathematical language?
17 S.: In this case, we must write the brackets and insert the three into it.

18 T.: Yes, this would be one of the possibilities we have.
The students are now asked to write all the squares as subsets in silent work. After that, they are read aloud by one of the students.
19 T.: We shall name such subsets events.
Teacher writes corresponding sentence on the blackboard.
20 T.: Now you should write down that subset which is described by the following task: toss a number smaller than, or equal to six.
21 S.: Here!!
22 T.: Toss a number smaller than, or equal to six! You are asked now to describe this as a subset!
23 S.: Then it is all the numbers.
24 T.: Good! This is also an event then. Question to you now: How may we call this event?
25 S.: All the numbers.
26 T.: You are right! This are practically all the numbers.
27 S.: Fundamental set.
28 T.: This is the fundamental set. This fundamental set, which is also a subset... if you remember set theory, we have said that the fundamental set is a subset, too. We shall now designate it as the certain event.
Teacher writes this down on the blackboard.
29 T.: There also was a subset which created some difficulties for us. Who remembers this subset, which also was a bit out of the ordinary?
30 T.: Have you written down all the subsets which we can form now?
31 S.: Where only one number is included.
32 T.: Which one have you written down? Read it aloud!
33 S.: Open bracket four close bracket, open bracket six close bracket.
34 T.: Good! But have you written down all the subsets?
35 S.: There is quite a row of others we can form.
36 T.: Yes, we can form quite a row of others. But there is a subset which we shall practically never write, but which we want nevertheless at least to mention.
37 S.: Open bracket two comma four close bracket.
38 T.: This we have already.
Students become restless and raise objections.
39 T.: Well, if we haven't written it down. But I mean one

which is written on the margin just like this certain event, that is one, two, three, four, five, and six in brackets.
40 S.: Open bracket one, three, five close bracket.
41 T.: Now these would be...
42 S.: ...the odd numbers.
43 T.: Yes, I think we've had this.
44 S.: No! We've had the even ones!
45 S.: Number less than, or equal to one.
46 T.: This could be done. But how would you write this as a set?
47 S.: One in brackets.
48 T.: Right, yes! If I now say: toss a die to obtain a number *less than* one?
49 S.: ...won't work!
50 T.: But this is an event, too! However, as you've already correctly said, this event...
51 S.: ...won't work!... Won't work!
52 T.: Yes. How would we describe this with an adjective?
52 S.: ...certain,...
53 S.: ...the uncertain event.
54 T.: The uncertain one? We shall simply say: the impossible event. And now my question: What kind of a subset is this, if I speak of an impossible event?
55 S.: That won't work at all!

Teaching episode from a 6th form lesson on probability. Topic: *«Possibilities of the sum of the pips of two dice.»*

In the classroom dialogue, the number of possibilities for the sums of pips of two dice is discussed. The possibilities of the sums 2, 3, 4, 5 and 6 have been enumerated by the students.
1 S.: For seven?
2 T.: Jasmine?
3 S.: One and six, two and five, three and four, four and three, five and two, six and one.
4 S.: I have something different!
5 T.: How do you have it?
6 S.: Three plus four, four plus three, five plus two, two plus five, six plus one, one plus six.
7 T.: Yes, you have another order, hm?

8 S.: Ah, ...yes.
9 T.: Frank!
10 S.: Four plus four, six plus two, two plus six, three plus five, five plus three.
11 T.: What? three plus five... and? Holger!... Frank!... Well no, I thought something had come for the nine as well: Take a another look at this!
12 S.: Six plus three, four plus five...
13 T.: Maren!
14 S.: Four plus five and five plus four.
15 S.: That's not right at all!
16 T.: Five plus four is already listed!
17 S.: No!... Four plus five!
18 T.: Holger!
19 S.: Four plus five, three plus six.
20 T.: But this will yield the same. Four plus five are nine and five plus four are also nine... Olaf!
21 S.: ...then, ehm, tosses, eh, the one die tosses a four, and the other tosses a five. And then you have five on the first and four on die two. Then you toss again, and there is again four and five, and the five on the two and the four on the one.
22 T.: Yes. Or we could imagine this with a red or a black die. But then I do not understand this. Why, eh, is two plus two happen not written down twice, as in the case of the other sums?
23 S.: No!... No!!!,... ridiculous.
24 T.: Andreas?
25 S.: This makes no sense, to write two plus two down again.
26 T.: Well, this makes as little sense as the other thing... Heiko?
27 S.: You can write... If you write the two plus two the other way around, you will again have two plus two.
28 T.: Yes, Dirk?
29 S.: Or, for instance... for one plus seven, there is still another possibility. For two, it is always the same.
30 T.: I think it is clear. Volker, will you please correct this for the nine?
31 S.: As nine is five and four, then you do the opposite number, four and five. Only for numbers like in case of the six, the three and three, yo need not write down twice.
32 T.: Yes, but why? This has already been explained just now. Christoph!

33 S.: Three plus three...
34 T.: No, just now this was a mathematical justification... Don't shout in class! Stefanie, explain this to us! As a hint, imagine a red and a black die.
35 S.: ...a two and a two yields four. If you toss the other way around, the result is the same.
36 T.: Stefanie, again with two different numbers! Take four plus one, or something like it!
37 S.: Well. If you now take the six and add the two... you have eight. If if you turn this around, I mean the numbers, two plus six, then this yields... that is the other way around, if you have four plus four, and turn this around, this again yields...
38 T.: Now watch it! I'll explain again.
39 S.: It remains the same task nevertheless!
40 T.: Yes, let me... Holger, I'll explain it to you again... Now, again a red die and a black die... Now, on the red die there is the one, and on the black die there is the six... One possibility... Now show me another possibility, Holger!
41 S.: Exactly the same thing the other way around!
42 T.: Yes, isn't it?... One on the black and six on the red... Hold it, don't shout in class!... Heiko?
43 S.: I know that this is two different exercises, but this is foolish to do them, as this is the same result!... This is the very same result, but with two different calculations, this makes no sense!
44 S.: ...this is also the same, only that it is written the other way around.
45 S.: That's what is the important thing!
46 T.: Andreas, what is the important thing?
47 S.: What matters is the notation!
48 T.: Well, I think we'll have to do this again in another lesson. I'll bring along other things then. We shall be able to try this again. It hasn't become so clear today with this one example... Yes, come on, let us continue!