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Mathematical Concepts in Didactical Situations as Complex Systems: The Case of Probability

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MATHEMATICAL CONCEPTS IN DIDACTICAL SITUATIONS AS COMPLEX SYSTEMS: THE CASE OF PROBABILITY

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1. Introduction

A systems theoretical characterization of mathematics education as a complex fundamental science is faced by the following problem: one has to study the individual components of this system while at the same time keeping the system as a whole in mind, that is the interrelations between the components of this system. Which are possible components of the system "mathematics education"? If we define mathematics education as an applied basic science, three main components can be identified: research, development, and practice (cf. H.G. Steiner, 1984).

For each of these three subsystems, the interdisciplinary character of mathematics education has to be taken into account in a different way. The diversity of its interdisciplinary references is well known: pedagogy, psychology, the social sciences, philosophy, theory of science, history and, of course, mathematics itself, to mention only the most important ones. No matter whether research, development of materials, or classroom practice are concerned, these interdisciplinary elements of the system and their co-operation are fundamental for any work in mathematical education.

The subject matter of mathematics occupies a special position among all the elements or components of the system "mathematics education" thus characterized - regardless of any individual assessment. If mathematics is neglected, there will just remain the separate scientific disciplines with their specific methodology of research - in the absence of an integrating element. On the other hand, a conception of the role of mathematics which subordinates the subject matter to the social aspects is not appropriate to the special problem of how to learn mathematics.

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We start from the hypothesis that the contents (i.e. school mathematics in this case) constitutes the crucial subsystem within the overall system of mathematics education (cf. AG Mathematiklehrerbildung, p.205f).

In the clarification process of this thesis it is necessary to develop an understanding of what can be structurally meant by the "mathematical subject matter" in the sense of a system's approach, and how this subject matter actually functions as an element of the system. With regard to the subject matter it is necessary to take a novel point of view which permits to express the specific features of mathematics education.

If the subject matter treated in the classroom, is seen more or less as a ready-made product provided by the discipline, it cannot function as a genuine system element in the total system of mathematics education: it will remain an independent and closed component unrelated to other elements of the system; in its resistance to change, mathematics will remain an independent system outside of the total system; in this way, an integrative interrelation of all system components will be impossible.

Under such conditions, it is hence irrelevant whether the subject matter is assigned an inferior role as compared to the other reference disciplines, or whether the subject matter will have a prominent role in mathematics education. The importance of a novel characterization of the subject matter for a system theoretical analysis is evident.

Our hypothesis will indeed only make sense if the subject matter is not simply considered as a structure given from outside. In the frame of a system's approach, the subject matter must become a "variable" element of the system. School mathematics must integrate and reflect the manifold relationships to the other elements of the system. As compared with "pure mathematics", there will be, for school mathematics, changes, deviations, novel meanings, and an emphasis on particular modes of working and interpreting. School mathematics will become, in a certain way, a different type of mathematics.

In an exemplary way the following is to show how this new interpretation of school mathematics is to be understood. The mathematical concept shall serve as an example. More precisely: we shall refer to the concept of probability in particular. The mathematical concept, here, is conceived of as the germ of the mathematical contents, according to our system's approach, the concept is understood to be a subsystem within the total system of mathematics. This interpretation, however, is not simply oriented towards the structural aspects of the discipline, rather, it attempts to include the diversity of interdisciplinary references of mathematics education by always considering the mathematical concept in the context of didactical situations.

2. On the Basic Understanding of School Mathematics

The mathematical concepts of the curriculum (of the lower secondary level: grades 5 to 10) form the technical network of the mathematical domain of knowledge. Two fundamental types of concepts can be distinguished here. On the one hand, there are singular fundamental concepts which run through the entire curriculum; among these, there are the number concept, the geometrical concepts of plane and space, the concept of function, structural concepts of algebra, or the concept of probability. On the other hand, there is a great variety of certain local specifications of these basic concepts which frequently appear only in a few places of the curriculum and are mostly elaborated to a limited extent only. Among these, there are, for example, the following concepts: binomial theorem, center of a triangle, prism, median, irrational number etc.

This characterization expresses the fact that school mathematics does not view its mathematical concepts simply as knots in the network of equal and mutually explanatory relationships, but tends to distinguish between essential concepts and derived ones.

This distinction is also evident in the introduction resp. definition of concepts in mathematics instruction. The special local concepts are frequently explained as special cases of general concepts. Such a reduction cannot be practised in defining the fundamental concepts, sometimes it is even impossible in the school curriculum. The intention to emphasize the structural aspect of school mathematics in the course of the "New Math" was accompanied by the attempt to introduce one single fundamental concept for the mathematics curriculum, namely the set concept. This concept seemed immediately comprehensible and self-explaining, and at the same time suitable to serve as a basis to derive the definitions of all the other mathematical concepts. In principle, this was also an attempt to achieve a unified and strict definition of the conception of school mathematics. The definition of the function concept as a special relation resp. as a special subset of the cross product of two sets is an example for that.

Even if the learning conditions at school and the mathematical abilities of a large majority of pupils prevented this attempt from being realized in pure form, this conception is of central significance as an important idea about the prospective nature of concepts and domains of knowledge in school mathematics.

Changes and modifications are mainly of pragmatic rather than of fundamental character. The ideal of a rigorous explanation of concepts in school mathematics is approximated as far as pupils and teaching permit. In case of those concepts deemed important, the demand for precision will be greater, for less important ones, explanations by means of examples will do. And in cases requiring a precise definition of the concept which cannot be used in school mathematics, or is too difficult for it, something like a didactical substitute for the concept is constructed. Thus, the operator concept, for instance, is a school version of the concept of fraction, for which the concept of fraction field would seem to be technically too difficult for an instruction in general education.

Whether mathematical concepts of the school curriculum can be introduced quite "strictly" in the sense of set theory, or whether an exemplary or in other ways particularly school-proper form of definition must be used - in any case, there is a specific conception of the nature of mathematical concepts in school mathematics.

This way of concept definition is exclusively orientated according to the discipline of pure mathematics, resp. to the scientific structure. Even if this mode of representing mathematical concepts cannot be pervasive for schools, the understanding of school mathematics will always be dominated by the image obtained from pure mathematics - despite the various particularities of concept definitions and ways of presenting mathematical knowledge specific for schools.

The formation of school mathematics according to the type of pure mathematics is also evident from other problems of school mathematics. Thus, for instance, it is common to understand the generalization of a mathematical concept, or mathematical abstraction, as the omission of specific features. This view corresponds to the definition of concepts on the basis of set theory: the abstract concept of number, for example, emerges, according to this interpretation, from the fact that the child is presented a variety of different sets (of objects), from which the abstract number concept is gradually derived by neglecting superfluous characteristics. Critics, however, do not see mathematical abstraction as the omission of properties, but, conversely, as an extension of interpretative and operatively application-oriented aspects of the concept (cf. Jahnke, 1984).

A second point at which the specific character of school mathematics becomes obvious is the mathematical proof. Proof is seen as the touchstone which distinguishes mathematics from calculations. Frequently, a proof is understood, according to proving in pure mathematics, as a logically consistent derivation of mathematical facts from known relationships. In school mathematics, there is until now no conception which understands proving as a diverse and explorative activity aiming at finding and developing new knowledge (cf. Jahnke, 1978). Proving, in school mathematics, is establishing a logical connection between known facts - at least, these are familiar to the teacher (and the pupil will believe them without any proof whatever). A codification of mathematics knowledge in school according to structural mathematics corresponds to this conception of proof, too.

Our considerations illustrate the thesis that school mathematics (of the lower secondary level) is a very special type of pure mathematics. The particular conditions of learning in school (social, organizational/ administrative, psychological, and pedagogical) modify this claim in part, but do not question the orientation towards pure mathematics in principle. The emergence of this conception is closely linked to Bourbaki's fundamentalist view of mathematics. Its educational impact was mainly due to Bruner's, in part also to Piaget's work (cf. AG Mathematiklehrerbildung 1981, chapter 4). In a critical analysis of such a fundamentalist interpretation of school mathematics, Otte gives the following rêsumê:"Fundamentalist initiatives stress the importance of the discipline's structure and of the fundamental concepts it requires. They are based on the idea that the activities of the mathematical scientist and the pupil in mathematics instruction are identical ... School mathematics and scientific mathematics are not different from one another. True, the latter is a bit more complicated, but equal to school mathematics in principle. Both are determined by Bourbaki's three mother structures: group, topology, and order, school mathematics establishes the foundations for the elaboration of scientific mathematics, tending to find its only justification in this perspective." (M. Otte, 1982, p.13/14).

The school mathematics version of pure mathematics, no matter whether it is called fundamentalist or elementary, seems to offer advantages not only from a scientific, but also from a practical view. "There is another reason why elementarism has such a strong position in school mathematics. The reason is that it seems to make communication between teachers and pupils simple and unequivocal. In the elementarist view, all concepts have a fixed meaning, there seem to be no different perceptions of meaning in teachers and pupils. Lessons take their orderly and preassigned course." (H.N. Jahnke, 1984, p.34).

In its effort to strive after pure mathematics, school encounters fundamental problems. Particularly the presumed copy of pure mathematics shows essential differences to the discipline. There are great differences in the ways mathematical concepts and concept definitions are handled in school and in science.

For mathematics instruction, the complete and precise definition of a mathematical concept is the final result of the learning process in principle. The concept presented in the definition is a definite and locally confined fact. Pedagogical guidance towards the concept and particular motivations for the learning process may be necessary, or the development of the concept may have several steps. But despite all methodological activities of introducing and testing the concept (e.g. by applications), the completely elaborated mathematical concept will be the ultimate goal of teaching. As opposed to that, the mathematical concept and its definition are understood in the opposite way in science. Here, the definition of the concept is the starting point of the research process. Concept definitions are considered to be implicit resp. axiomatic characterizations which are semantically open, and which will be filled with ever new meanings and interpretations in the course of research activities. This openness of concept development is fundamental in the sense that a concept will never be final and complete; in principle, every concept can be developed further.

This basically different character of science and teaching produces, for school mathematics, the serious reductionism of equating the mathematical concept with its definition. School mathematics, indeed, has no possibility of keeping the relationship between the concept and its respective definition strictly open, as it is practised in science. The scientific procedure of controlling a concept axiomatically, and of developing its contents within the process of research activity, is not applicable in general education. Concepts of school mathematics will require a meaning with regard to contents in a relatively concrete way. As opposed to science, one must frequently be able to say, in the mathematics classroom, what is to be understood by a certain mathematical concept.

The question what a concept really is must in part be answered differently in school and science. The answer of science points to the concept's scientific potential of development, to its manifold relationships with other concepts, and to the new methods this concept introduces into the process of research. Such an answer, however, cannot be provided by mathematics instruction. Its usual answer to the question of the concept's meaning is too overhasty: the mathematical concept acquires its meaning by its definition. This answer seems legitimate in view of the fact that the fundamental concept of school mathematics, i.e. the concept of set, is self-evident, all other concepts being more or less specifications of this concept.

In the classroom, too, it is necessary to distinguish between concept and definition. The meanings of mathematical concepts in the curriculum do not simply result from their definitions, neither they are the result of the individual pupils' research activities. In the classroom, mathematical concepts acquire their meaning from a variety of intersubjective social and object-related activities on the part of the pupils, by means of a diversification of conceptual means of working and of representation. If this distinction between concept and definition is not made, this would lead in the end to an overmethodization of classroom activities. All learning activities are no longer directed towards the proper content of the concept, but are exclusively oriented towards a special form of definition. This results in the bewailed artificiality of mathematics instruction which precludes an access to mathematics for a majority of pupils (cf. Andelfinger, 1984).

A strict orientation of school mathematics according to the image of pure mathematics thus entails problematical shifts in the understanding of mathematical concepts, because of special conditions in school. The transposition of other characteristics of pure mathematics to school mathematics, however, is not without problems, either. The idea of a unified object of pure mathematics, e.g. in the frame of an axiomatically founded structure, is expressed, in school mathematics, in the set concept. The above discussion has shown how questionable this correspondence is. But even the assumption that the subject of research can be defined by itself and without limitations in science is contested. "The question what is the subject shows, in the case of pure mathematics, the special character of this science insofar as the subject of mathematics, the contents of the mathematical activity, may in no way be defined absolutely and independent of the means of mathematical activity." (M. Otte, 1984, p.I)

Moreover, pure mathematics is characterized by manifold effective and universal methods of proving, of treating and solving problems as well as of deriving new mathematical relationships. In the course of the historical development, these methods were standardized and classified; they have more and more become the specific internal tool of the mathematician. School mathematics, too, stresses the use of intramathematical methods of proving and problem-solving. If at all, the extramathematical context of intramathematical work, for instance its reference to physics, experimental sciences and measuring practice, in short, its quasi-empirical character, is understood as mere propaedeutics. In the effort to be on firm ground, mathematics instruction has increasingly thematized the logical and propositional foundations of mathematics in order to make methods of deriving, proving and solving waterproof. Accordingly, there was a tendency to de-emphasize extramathematical ways of working and arguing.

Such a characterization of school mathematics as a special type of pure mathematics raises the question whether the negative effects were unknown, or whether they provoked counter-measures. Mathematics education has indeed used many approaches to counteract a too strict structural orientation in school mathematics, and to integrate the mathematical subject matter into the social context of the learning situation. There were many reform proposals to change mathematics instruction. Among these were:

- application-oriented mathematics instruction
- problem-oriented mathematics instruction
- project-oriented mathematics instruction
- mathematization and modelling in the classroom
- problem-solving skills in mathematics instruction
- pupil-oriented mathematics instruction
- mathematics instruction integrating other subjects

and many more. All these proposals have in common that they intend to emphasize the particularities of learning situation and pupil activities in the classroom. By and large, however, it can be noted that almost all these proposals first require additional activities and procedures in mathematics instruction. The mathematical subject matter is not questioned as to its basic structure, rather, the intention is to wrap it up better for the pupil. There is no fundamental criticism of the goal of mathematics instruction, but only proclamation of new paths to attain it - paths which are believed to be better adapted to the pupil. Thus, even application- or problem-oriented approaches frequently stress that mathematical knowledge must first be developed in order to be able to carry out applications or solve problems afterwards. Besides, there is, for many teachers, a contradiction between pedagogical intentions of pupil-oriented teaching and the mathematical subject matter, which, in their opinion, cannot be easily overcome. In their view, systematically structured school mathematics and social learning are incompatible.

The majority of reform proposals for mathematics instruction underestimates the fact that an effective interaction between the mathematical subject matter and the special conditions of the learning situations can only be established if there is an intentional generalization and expansion of working methods and means of representation, as well as of meanings of knowledge and interpretations of concepts in school mathematics. The codification of school mathematical knowledge, and the selection of admissible mathematical working methods and means of representation must not be aligned to the present established image of pure mathematics only. Mathematical concepts and working methods in general education must make allowance for the broad scope of historical developments, of considerations of philosophy and theory of science, and for the tension between pure and applied mathematics as well as for new mathematical developments (such as EDA). Not the confinement to special techniques of pure mathematics, but only the development of the mathematical activity of knowing as a general social activity on the basis of the diversity of these aspects will permit to establish an effective interaction between the pupils' activity in the process of learning and the activity of knowing in mathematics.

Only in this way school mathematics will become an actual and effective subsystem of mathematics education; only in this way a genuine interaction with other subsystems will be established.

In the following section, the concept of probability shall serve as an example to discuss how this concept receives its specific meaning only from a diversity of working methods and means of representation which enables the teacher to relate them to the pupils' learning activities.

3. Probability - A New Paradigm for Mathematical Concepts in the School Curriculum?

Comparative analysis will focus on examples which yield an insightful contrast to familiar facts. Probability theory is such an example as compared to the traditional school curriculum.

The probability concept has a curious property: as soon as one tries to characterize it unambiguously by means of a definite mathematical definition, nothing stochastical will be left. It is no longer a matter of chance. There remain certain mathematical facts and well-determined computations, namely set theory, fractional calculus resp. percentages, combinatory rules and counting methods, aspects of elementary geometry, simple algorithms, etc.

There is a certain contradiction between the rigour required of a mathematical definition, and the random character of probability. This is the point which expresses the fact that probability cannot be reduced to its mathematical definition. An understanding of the concept cannot be based simply on the definition, as school mathematics does with its mathematical concepts. This is the first particularity of stochastics which creates problems for the teachers. They feel that an understanding of the concept will not be possible for their pupils simply by the definition, no matter how careful the subject is introduced and mathematically elaborated. The difficulty is not how to recapituale a definition, but how to interpret it appropriately, and to apply it to specific situations. If this is attempted, new and different interpretations of probability will result. An exhaustive and definite clarification of the concept, the usual aim of teaching, is fundamentally impossible in the case of the probability concept. The question "What is probability?" cannot be answered along the same pattern used for the other concepts of school mathematics.

This difference between the mathematical characterization of probability and its particular stochastical meaning is the reason for many didactical difficulties raised by this unusual subject matter in school. We shall take this as a starting point to find a new perspective with regard to mathematical concepts in the school curriculum. Our objective, however, is not simply to plead for stochastics education. Rather, we intend to analyze the diversity of the mathematical activity of knowing, and to model the mathematical concept as a relational system by contrasting traditional mathematical concepts to probability, with respect to considerations of the discipline, of history, of theory of science, and of didactics and teaching practice.

3.1. Concept and Meaning

The question of the probability concepts' meaning has set off violent controversies in the course of its historical development, led to fundamental philosophical disputes, influenced the discussion about foundations in mathematics at the turn of the century, initiated considerations in the philosophy of science about the status of scientific concepts, and has produced controversal points of view in the educational debate. The answers to these questions are ambiguous, contradictory, and of totally different status. Among the various controversial statements about probability, the following are well known:

- Probability is a characteristic of the knowing subject who constructs this concept while coping with the environment. As opposed to that: Probability is a property of certain objective mechanisms functioning like random generators.
- Probability is a concept of observable reality; it is interpreted

as a limit of relative frequencies. As opposed to that: Probability is an ideal concept; it is constituted by certain ideal symmetries.

- Probability is an auxiliary concept which the knowing subject consults when his knowledge is incomplete. If there is sufficient knowledge about the situation concerned, probability is superfluous. As apposed to that: All knowledge about reality is, in the last instance, subject to uncertainties: hence, the probability concept is the most fundamental concept of scientific insight.
- The probability concept makes sense only in special mass experiments; it cannot be applied to the individual case. As opposed to that: Probability is a concept of everyday knowledge, which is of immediate use and can be applied to all kinds of situations.
- Probability is a theoretical concept, its meaning cannot be directly inferred, it will result from the construction of stochastical models and their application. As opposed to that: Probability is an empirical concept: its meaning results from carrying out and evaluating a statistical experiment.

Controversial views of this kind have caused probability to be split up into a colourful diversity of independent conceptions. There are types of axiomatic probability, of comparative probability, frequency interpretations of probability, classical (Laplacean) probabilities, logical probability, information theory approaches to probability, foundations of probability on theories of complexity, probability as subjective evidencies, etc. (cf. T.L. Fine, 1973).

Each of these variants confers a specific meaning upon probability, interpreting it according to its own requirements. In view of this variety, a homogeneous interpretation of the concept is impossible.

If we consider on the other hand, the respective structures of the mathematical calculus we can observe similarities and coincidences. The mathematical devices of all probability concepts are comparable and can be related to one another on the background of a common structure - e.g. the axioms of Kolmogorov.

Some mathematicians and educators may have been therefore motivated to ask no longer for extramathematical aspects of probability, but simply to emphazise the axiomatic system or to take the basic rules of the probability calculus. This is an attempt to solve the problem of meaning by denying its existence.

Which are the effects of the diversity of meanings of probability on the mathematical curriculum? General mathematics education must not be based on just one single interpretation of the probability concept. "Stochastics, in school, must organically combine ideas from different philosophical traditions, in particular

- stochastics as the mathematics of mass phenomena
- stochastics as the logic of uncertainty
- statistics as technique which transforms data into insights
- stochastics as decision theory." (Dinges, 1981, p.51)

For mathematics education as well, the conclusion has to be that there is no universal and unique interpretation of probability. This concept consists, on the one hand, of a mathematical structure (e.g. an axiomatic system, or implicitly defining equations, or rules of calculation), and refers, on the other hand, to most diverse and specific interpretation with regard to its contents (cf. v.Harten/Steinbring, 1984, p.64/65). As the mathematical structure of the concept permits different interpretations, the meaning of the probability concept cannot be the exclusive result of intramathematical considerations. The interpretation of the probability concept becomes an independent problem which is additional to mathematics.

School mathematics tends to clarify its other basic concepts exclusively by mathematical definitions. This clarification mostly consists of conferring a direct and empirical meaning on concepts. Thus, for instance, the number concept acquires at first empirical meaning by the activity of counting real objects, and is then obtained by generalizing from the enumeration of equipotent sets. In a similar way, there is correspondence between empirical meaning and concept structure of the discipline in the case of the function concept, or of the basic concepts of geometry.

The probability concept cannot be subjected to this simple schema. Its meaning is neither exhausted by the mathematical structure alone - which would reduce it, in the school curriculum, to Boolean algebra or combinatorics -, nor does this concept acquire its meaning by immediate empirical experience or experiment - which would quickly turn stochastics into a collection of recipes of simple calculating techniques. In contrast to the traditional concepts of the school curriculum, the probability concepts explicitly require, besides their mathematical definition, a frame of reference for interpretations as to their contents and applications. The meaning of probability is neither automatically supplied by the logical structure of the subject matter, nor by the application.

This particularity of stochastics has been pointed out by well-known mathematicians. Thus Kolmogorov, in his foundation of probability, puts his considerations on the topic of "The Relationship to the World of Experience" beside his formulation of the axioms of probability. He discusses a specifically application-oriented interpretation of probability which is aligned to the interpretation of frequency (Kolmogorov, 1933, p.3f.). In a similar vein, Feller stresses in the introduction to his book the fact, that instructions for a meaningful understanding of this concept and for its use are necessary besides the formal mathematical definition of probability. Feller says:"In a rough way we may characterize this concept by saying that our probabilities do ... refer ... to possible outcomes of a conceptual experiment. Before we speak of probabilities, we must agree on an idealized model of a particular conceptual experiment such as tossing a coin, sampling kangaroos on the moon, observing a particle under diffusion, counting the number of telephone calls" (Feller, 1968, p.4).

The diversity of the probability concept in scientific discussions can also be found in the classroom. Thus, for instance, younger pupils who have not yet become accustomed to the standards of academic arguments, will draw on multiple explanations of contents when asked for the reasons of probabilities in concrete experiments of chance. To the question:"What is the more probable result of tossing an (ideal) die, a 4 or 6?" sixth formers, for instance, gave answers of a kind that, also appear in scientific debates. Some pupils supposed this could not be said because it depended on chance, other pupils saw no differences between 4 and 6 because of the die's assumed conditions of symmetry, while one pupil suggested to make one thousand tosses in order to see whether 4 and 6 appear with equal frequency. One could say that the pupils chose individually certain perspectives of interpreting probability: they used the physical conditions of the chance device as their criterion, or they interpreted probability in the sense of a repeatable mass experiment, or they considered probability to be a purely subjective quantity depending on chance.

The fact that the mathematical definition of probability will always include a heuristic conception of interpreting its contents turns stochastics into a special type of application-oriented mathematics. It is too simple to regard the stochastical applications like the traditional ones, which normally emerge only after the theory has been completely developed. Conversely in this case the reference to applications is a prerequisite for developing the mathematical concept. Such a type of pragmatic and immediate application of mathematical concepts (cf. R. Hermann, 1978/79) entails a shift of the problem of meaning for mathematics education. Meanings of probability do not result from elaborated theories; neither can they be the result of a unification of the manifold interpretations of the concepts of probability. The specific meaning of stochastics emerges from the system of activities and means of representation which permits to work stochastically in this experimental and application-oriented sense. For the pupil, meanings of concepts are not the result of a unification of the concept's definition, but become possible by extending the means of representation and activity belonging to the concept.

3.2. Concept and Means

Stochastical thinking can indeed be conceived of as the example of theoretical thinking. How is this to be understood? As compared to other modes of thinking, e.g. the geometrical, algebraical, or mechanical thinking one is forced, in stochastics, to be aware of the difference between mathematical model and real situation from the very beginning. This "distant rationality" (cf. Dinges, 1982) between the mathematical structure and reality emerges, so to say, by itself in stochastical insight. Stochastical statements about real phenomena are no images of empirical truths, but mainly reflect possible theoretical connections in the present situation. This is expressed by the fact that stochastical statements about reality are always afflicted with a so-called probability of error (cf. Dinges, 1977). No matter how this is interpreted with respect to the epistemological status of probability - whether this is considered to be the auxiliary character or the conceptional progress of stochastics - in any case this will prevent the direct identification of the mathematical statement with the real facts.

This particularity is not only a problem in philosophical debates on the relationship between practical statistics and probability theory, or in case of comprehensive applications of stochastics, but emerges also in mathematics education in the form of relationships between the concrete individual case and the stochastical model.

A simple, but illustrative example in mathematics education is provided by constructing an ideal spinner. First, model ideas about mechanical and symmetrical conditions will enter into the construction: subdivision of the circular disk (e.g. beer mat) into sectors of equal size, inserting the axis in the center of the disc, etc. After the spinner has been completed, the pupils, in a second step, have to investigate this device as a concrete individual case. In the course of experimental tests, deviations from the so-called ideal symmetry can be noted, and conclusions as to errors of construction may be drawn. Thus, pupils in the classroom recognized that the experimentally observed differences in the relative frequencies of their devices were due to imprecise fixation of the center point. Correction led to better results. Besides, the pupils also changed the "instructions" for the experiment: the rotating disc was no longer to be held vertically, but horizontally. Experiments in the concrete individual case, and general ideal conceptions about the model case thus led mutually to an improvement of the device, and to first elementary insights into the relationship between mechanical factors and random parameters.

In this example, as in many others, however, this mutual process of knowing cannot be automatized. In the last instance, it will always remain a conscious decision of the subject whether he tends to interpret deviations observed as relevant or as accidental and hence negligible - whether this interpretation occurs on the basis of extensive statistical techniques, or whether it is a case of elementary stochastical relationships in mathematics education. A further example for that is the question whether a die is to be considered loaded or regular, resp. which probability distribution would be assigned to a loaded die. This question does not simply refer to statistical methods; beyond that, mechanical aspects such as the die's center of gravity will play a part. Thus, pupils express, for instance, rather appropriate assumptions about the expected falsifications caused by mechanical defects. It is also interesting to ask pupils to construct themselves a die loaded to a certain extent.

Mainly the younger pupils in most cases will take up the relationship between model and individual case in stochastical situations in an exemplary way for their own reasoning. In doing so, they do not simply depend on general mathematical justifications, but also consider specific aspects of the concrete situations. Simultaneously, it is sometimes very surprising at which point pupils attempt to create a "theory" for themselves about concrete individual cases. It is well known that the tack experiment is considered t h e statistical experiment which does not permit a theoretical model. A pupil however made a plausible model assumption which did permit stochastical predictions. He suggested to investigate tacks with long pins, somewhat shorter ones, normal ones, very short pins, and without pins at all. This model-type variation of the parameter of "length of the tack pin" permitted comparative stochastical statements which were not exclusively based on experimental results.

These examples demonstrate the fundamental relationship in the analysis of stochastical situations: the interaction between the stochastical model and the concrete individual case. Neither does the model result automatically from the experiment of the individual case, nor does it provide a complete explanation for all the concrete phenomena of the individual case. For mathematics education, it is an entirely unfamiliar fact that the empirical details cannot be reduced to a general mathematical structure. This is how stochastics introduces into mathematics education a strict separation of signs and the things signified, i.e. of mathematical modelling description and object. Whereas in the traditional subject matter fields of the school curriculum the mathematical sign and the object are frequently identified.

If we start from the assumption that the relationship between the individual case and the model cannot be dissolved in a reductionist way, this will raise the question how to treat this stochastical relationship in the mathematics teaching. In the history of probability theory, a lot of conceptional means of description and representation have been developed which characterize, in particular, the special type between system and system element. The first concept to be named here is that of distribution (cf. Sačkov, 1978).

A closer look helps to discern two forms of representational means to analyze the stochastical relationship between individual case and model. The category of distribution includes all those diagrams which primarily permit to process the results of frequency experiments. In the classroom, these are:
tables, tally sheets, pie charts, stem-and-leaf displays, column graphs, histograms, polygons, probability distributions, etc.

On the other hand, there are diagrams which refer mainly to the probability calculus; typical for these is the so-called tree diagram. In the classroom, the follwoing forms belong among them:
tables (matrices), complete tree, reduced tree, "infinite" tree, "recursive" tree, Pascal's triangle, probability abacus, etc.

In the classroom, these two types of stochastical means of representation are mostly used strictly according to their intentions either as a help to processing data, or as a step towards the mathematical formula. With regard to the relationship between individual case and model, however, it is appropriate to conceive of distribution and tree diagram as of complementary means of representation. This is to say, for example, that the concept of distribution should also be used for calculating and representing probabilities, and, conversely that means of representation belonging to the tree category such as Pascal's triangle should be used in order to explain certain distributions.

In short, the task is to develop systematical connections between all the stochastical means of representation which permit to carry out the transition from the individual case to the model, and vice versa. Each of the above means of representation thematizes, in a specific way, the relationship between individual case and model. Thus, the tree diagram can be simultaneously interpreted as a plan for carrying out a concrete experiment, and as a model-type characterization of all possibilities occurring. The concept of distribution, for instance, is a form of representation which characterizes the total system of all random variables, and which at the same time provides general theoretical statements about an individual case on the basis of individual specified values of the distribution (e.g. of symmetry, of mean values, or of measures of spread (cf. Steinbring, 1980) etc.). This double role confers a common point of reference to all stochastical means of representation. In the classroom, they must be harmonized and pedagogically organized. It is only the entire system of stochastical means of representation which permits an appropriate stochastical interpretation of the relationship between model and individual case.

In the frame of a didactical organization of the stochastical means of representation, it is advisable to use different forms on different levels: using the diagrams to describe, carry out, and evaluate real random experiments, to simulate experiments, to carry out and evaluate ideal mass experiments, to represent recursive schemes (e.g. binomial coefficients), to derive simple formulae of stochastics (e.g. path rules), to establish a stochastical model (e.g. binomial distribution). The systematical variation of the stochastical means of representation assures that the productive tension between concrete random situation and its stochastical model is maintained. The system of the means of representation thus becomes the bearer of the specific stochastical meaning, which plays a part in the respective application, and it includes the possibility to simultaneously interpret this situation as a model situation and a concrete stochastical case.

The relationship between the individual case and the stochastical model which is fundamental for stochastical thinking introduces the separation between the mathematical sign and the object signified into mathematics education. An appropriate handling of the relationship between individual case and model requires the system of stochastical means of representation (tree diagrams and distributions). The system of these means corresponds to a lot of uses by the knowing subject. In making use of the stochastical means the pupil acquires specific stochastical meaning; the variation of these means admits a meaningful connection between the concrete random situation and its stochastical model. By distinguishing between object and sign, the mathematical concept is attained. This threefold distinction between sign, object and concept will be analyzed in the following section with respect to the process of learning.

3.3. Concept and Learning

The mathematical concept is not identical with its definition; in addition to that, it is distinct from the object of learning. How then is the concept constituted within the process of learning, if it is neither equal to its definition (the mathematical sign), nor to the object?

The concept ermerges from the subject's active grappling with the relationship between the object and the mathematical sign, in mutual application of the respective mathematical definition and the concrete case of application. For the concept of probability, the stochastical means of representation mediate between the concrete random situation and the mathematical model.

Within this context, the dual characterization of the probability concept becomes useful for the acquisition of this concept. On the one hand, there is an experimental characterization of probability as a value estimated on the basis of relative frequencies observed. On the other hand, there is a theoretical interpretation of probability; on the basis of hypotheses (about symmetries, area portions etc.), probability is determined by a relative portion. The probability concept thus contains, at its core, the dual structure of mathematical-modelling and of concrete-experimental aspects. This structure of the concept "corresponds" to the relationship between individual case and model.

This shows how detrimental it would be to attempt to fix the probability concept in school once and for all by a single definition. Rather, stochastics education should be aligned to the following didactical principle:
For a meaningful treatment of stochastics in mathematics education, it is necessary to develop the probability concept simultaneously on the level of experiment and application, and on the level of the mathematical model. Any hasty decision in favor of one of these two characterizations of probability, and against the other one - whether as relative frequency, or as relative portion - entails the danger of reducing probability theory to formal mathematical techniques.

There is a certain contradiction between this principle and the common understanding of school mathematics, according to which one proceeds step-by-step towards the proper mathematical facts, that is to formulae and concepts, by neglecting properties conditioned by the situation while afterwards applying the learned concepts to examples in the exercises. Stochastics in school (in the lower secondary level) is not based on a universal dogma; rather, it requires manifold approaches, and different modes of working.

In correspondence to the system of stochastic means of representation, the pupils' system of activities is created within the process of learning. When coping with the means of representation, and by using them specifically, the pupils carry out activities on various levels: they experiment, collect data, compare, evaluate, assess, measure, decide, make assumptions and construct models, calculate with formulae, give mathematical arguments, etc.

These activities require different means, general means of knowing, and stochastical means of representation. They must not be structured hierarchically in the sense of being subdivided into tentative or marginal and into actual mathematical activities. It is the system of activities as a whole which enables the pupil to grasp the stochastical meaning. The broad scope realized within the system establishes a connection between the pupils' everyday knowledge and the mathematical knowledge.

The example of the Galton quincunx or probability board shall serve to exemplify these considerations. The double role of being at the same time stochastical model and special individual case is particularly evident in Galton's apparatus. Thus, several experiments with this random generator will yield ever different distribution curves which, however, simultaneously show certain equalities, general similarities resp. symmetries. In the special cases, a model type is recognized.

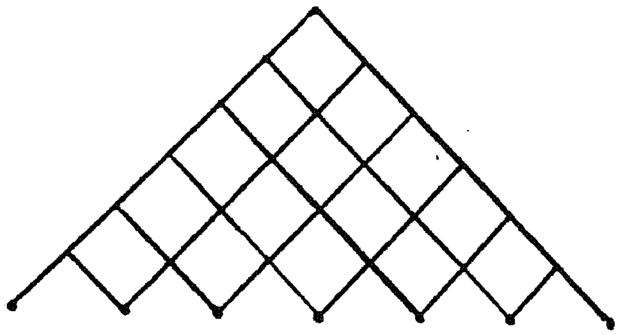
For pupils, this situation offers many opportunities to integrate their everyday knowledge, to develop simple physical arguments, and to think about assumptions concerning the object. Thus, for instance, pupils often assume the significant peak of the balls in the middle to be caused by gravitation. This argument is not easily refuted, if reasons are not exclusively sought on the level of the stochastic model, but the real object is taken serious as well. When the board is tilted backwards, the rolling friction of the balls produces a stronger effect of gravitation: the peak in the middle will be more pronounced than in the normal case of the vertical board in which gravitation can be considered to be statistically negligible.

In a first stage, the pupils carry out experiments with the Galton board, collecting and organizing the data obtained in order to get comparative evaluations and first assumptions for a mathematical model. The answers to questions as to the causes for certain visible phenomena (e.g. symmetry, peak in the middle, etc.) are facilitated by skilful variations of the experiment.

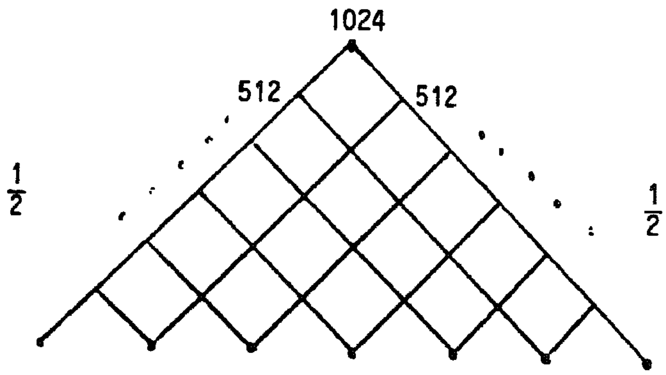
One of the variations of the experiment with the Galton board is tilting the board backwards; in another variation, different experiments can be carried out by tilting the board sideways with various angles (between the horizontal and the board's basis). Evaluative comparison of these experiments requires activities of collecting, organizing, and representing by means of distributions. Subsequently, hypotheses about certain experiments can be formulated, for instance:

- there will be a symmetrical distribution if the board is in a normal position,
- the balls, in this case, will fall with equal probability to the right or the left of the pegs,
- there will be a skewed distribution shifted to one side in case of a Galton board tilted sideways (dependent on the size of the angle),
- the balls, in the latter case, will not fall with equal probability to the right and to the left (dependent on the angle of tilt and on the edge of tilt, the balls will fall with greater probability on one side).

What is illustrated experimentally by the Galton board can be described, in a second stage, by means of an ideal simulation. The most important device for such an activity of simulation is Pascal's triangle.

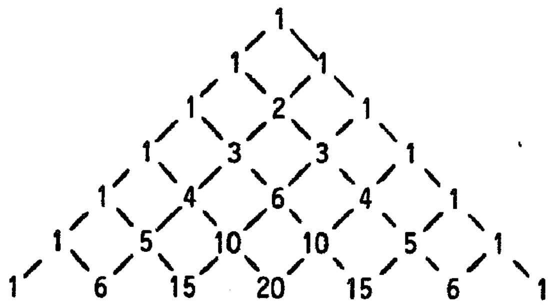


According to the experimentally assumed probabilities, a large number (of experiments) is "pushed" through the diagram. For the normal situation, for instance, half of the numbers is guided left, and the other half right.



The ideal simulation is a link between the real experiment, for which Pascal's triangle constitutes a description here, and a mathematical model of calculation.

In the third stage, simple mathematical activities and routines referring to a variation of Pascal's triangle can be introduced into the process of learning. The first important concept in this connection is the "Number of paths to a final point" (the binomial coefficient). By means of Pascal's triangle, this concept can be founded and calculated in a recursive-experimental way.



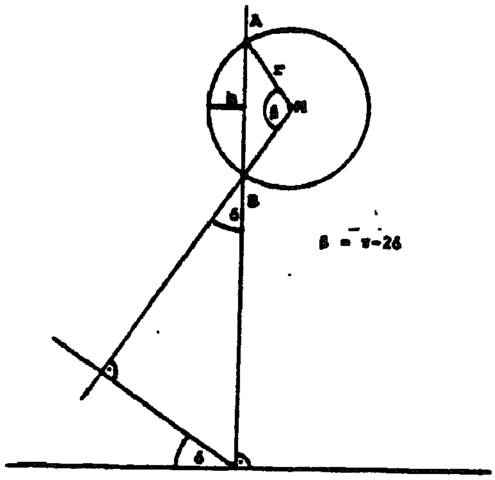
The second important concept refers to the path rules (product and sum rule). These rules are situated between the experimental activities and the mathematical formulae. By means of Pascal's triangle (resp. of corresponding tree diagrams), these rules can be founded by experiment and ideal simulation, and then derived in a formula.

In the fourth stage, the binomial formula

$$B_n(k,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

is discussed as a stochastical model for the Galton board. Among interesting questions are the following:
Is this formula an appropriate model for the normal situation of the Galton board?
What does this formula state about the tilted board?

The Galton board itself prevents attaining an intended final point in the development of the concept, which seems to lead from the example to the abstraction. Thus, for instance, the binomial distribution to $p \neq 1/2$ is not simply as a mathematical formula a sufficient explanation for the tilted board. What is lacking are specific justifications leading to a mathematical model. Such assumptions may be derived from geometrical aspects of this situation.



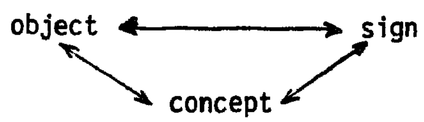
Thus, probability assumptions can be formulated according to the proportions of the balls falling right and left of the peg (cf. the drawing, e.g. distances, disk segments, area proportions, volume proportions). These assumptions then lead, across the binomial distribution, to an ideal description of the model which, however, can be proved to be correct only by comparison with experimental results.

The mathematical formula does not contain in itself the specific character of stochastics. Only if it is conceived of as an element of the overall system of stochastic activities and means of representation which establishes the connection between the concrete random phenomenon and the stochastical model it will become possible to make the stochastic concepts valid. "The game of fortune, or the die, or the generator of chance, or the problem of exactness of measurement determine the concept of 'probability' just as rigorously and as strongly as a corresponding definition" (Otte, 1978, p.20).

4. The Mathematical Concept as an Interplay of Experimental and Theoretical Elements

"The practice of mathematics, particularly in school, is ... induced by automatization, by the algorithm expressed in the formula as a procedure of calculating, to identify sign and signified, or, if the threefold distinction between concept, sign and object is made which is in principle necessary, to identify sign and object while neglecting the conceptual aspects which are independent of it" (Otte, 1984, p.19).

Probability in the classroom has proved to be rather resistant to this problem of hasty identification of object, sign, and concept. An identification of the object with the sign is prevented by the particular relationship between the concrete individual case and the stochastical model. That probability plays an independent part is assured by the great diversity of meanings.



There is, however, an important prerequisite for this sophisticated procedure upon learning concepts: the dual structure of the probability concept - its status which is both theoretical and experimental - must not be destroyed by definition, i.e. fixed in favor of one of the two sides. Otherwise, there is the danger here as well that the relational triangle of "object, sign, concept" collapses. In view of the fact that the formal apparatus of mathematics cannot simply express what is specific for stochastics, this danger is perhaps not as great as in other fields of school mathematics.

The fundamental prerequisite for an intended open process of learning, for a real development of concepts in the classroom is the actual consideration for the complementarity of the theoretical and the experimental aspects of the mathematical concept (cf. AG Mathematiklehrerbildung 1981, p. 205ff.). For probability, this means that this concept has to be developed by contrasting these two characterization. This educational activity makes the pupils, too, aware of the fact that the concept is not identical to its definition.

If the mathematical structure does not serve as the crucial orientation for the process of learning, what else guides the acquisition of the mathematical concept in the classroom? Acquisition and understanding of the concept is done by continuous development, and by progressive use of manifold means of representation and activity. The totality of the means of activity and representation permits to encompass the broad scope of concrete/experimental and theoretical/model-type aspects as they are present both in the learning situation and in the mathematical concept. For the process of learning in school, it is necessary to consider the specific means and conditions of the "didactical situation" essential for the acquisition of the concept, and not as irrelevant marginal conditions. "Instead of viewing didactical situations merely as isolated resources to produce the desired behaviour one must consider them ... as link of a chain or as part of a process which constitutes a small artificial genesis of a concept, these processes being both goal and means of teaching" (Brousseau, 1982, p.8).

The means of activity and representation in such a didactical situation permit connections to the communicative and social aspects contained in the general education of mathematics. They assist the pupil in finding an effective personal relationship to mathematical knowledge. Decisive for this is that allowance is made for concrete experimental activity. Contrasting experiment and model, comparing empirical results to theoretical assumptions and predictions starts the development of the concept. This development is no specification of the concept in the frame of set theory, it is a real development which helps the pupil to attain new and hitherto unknown insights, which generates the concept from the contradictions between the individual experimental case and the mathematical model.

The insight that the system of the appropriate means of activity and representation is indispensible for the respective development of the mathematical concept, permits to compare different forms of concept development without identifying them with one another. Concept development in didactical situations can learn much, for instance, from the concept's historical development, but must not try to copy the latter. In the same vein, scientific or psychological aspects of the concept's development are of great interest, but learning concepts cannot be reduced to them. Learning in didactical situations means that the mathematical concept must be created anew by teacher and pupils, not as a copy of an independent scientific specimen of concept, but as a specific concept of school mathematics. This is made possible, and assured, by the specific means of the didactical situation.

Traditional school mathematics considers itself to be a certain type of pure mathematics. This orientation is one of the causes for many reductionist shortcomings in educational epistemology which does not permit an appropriate representation of concept learning as a specific mathematical activity of knowing in mathematics instruction. On the contrary, the system of the means of activity and representation in a didactical situation permits to arrange the concept development in the classroom as an "open" process. Thus, the learning of concepts in the classroom becomes comparable, to a certain extent, to the open, creative process of research in science. Both, school mathematics and scientific mathematics are not based on mere formal structures; both require appropriate means and representations. "... the objectivation of mathematical knowledge, its status as true and objective knowledge, can only be secured by reference to the universe of 'all' conceivable means, and all possible representations. The question as to the objectivity of mathematics can only be formulated on the basis of 'all' possible modes of access to mathematical knowledge" (Otte, 1984, p.II/III).

On this basis, school mathematics and scientific mathematics can be distinguished and separated from each other. The mathematical means of knowing and the activities for pupils in general education are more general, more fundamental, more diverse and manifold than the special and universal techniques as developed in pure mathematics. While a purity of methods and means has prevailed in the discipline due to the division of labor in science which has led, in mathematics, to a complicated diversity of specializations, mathematics instruction, by its system of means, must bridge the distance between everyday knowledge and mathematical knowledge. School mathematics must not be classified and studied according to the model of scientific specialization into subdisciplines.

Hence, there is a decisive change in the epistemological status of school mathematics: mathematical knowledge in the classroom is no professional and disciplinary knowledge. School mathematics is a knowledge comprising general and extramathematical justifications. School mathematics is not abstract and universal, but local and exemplary. For school mathematics, the didactical situation, together with its system of means, is a frame of reference which expresses its specific meaning in a way similar to what the various specific frames of reference do for the concept of probability.

The question concerning the character of school mathematics hence does not raise the (scientific) alternative between pure and applied mathematics. More promising for teaching is the relationship between experimental and theoretical mathematics which, by interaction of these two poles, makes school mathematics a special type of mathematics oriented towards applications.

According to this educational epistemology, frames of development should be elaborated for the concepts of school mathematics, i.e. appropriate systems of means of activity and representation should be constructed which, in principle, require an interaction of experimental and theoretical elements. Testing such systems of concepts in didactical situations is the starting point for establishing educational text materials (for teachers and pupils), for a cooperation with teachers in school, for psychological, social and educational studies, and for many other things. A prerequisite for that is the characterization of mathematical knowledge as a component of the general human activity of knowing. In this way, mathematical knowledge becomes a real and important element in the total system of mathematics education.

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