Evaluation of the Approximation Order by Positive Linear Operators

Am Fachbereich Mathematik der

Universität Duisburg-Essen

zur Erlangung des akademischen Grades eines

Doctor în Matematică

angefertigte Dissertation

von

Paula Anamaria Piţul

aus

Sibiu (Hermannstadt), Rumänien

Referenten:

Prof. Dr. Octavian Agratini - Cluj
Prof. Dr. Petru Blaga - Cluj
Prof. Dr. Sorin Gal - Oradea
Prof. Dr. dr.h.c. Heiner Gonska - Duisburg
Prof. Dr. Ioan Raşa - Cluj

Tag der öffentlichen Verteidigung an der Babeş-Bolyai-Universität in Cluj-Napoca:
28. September 2007.

Universitatea Babeş-Bolyai
Facultatea de Matematică şi Informatică

Paula Anamaria PIŢUL

Evaluarea ordinului de aproximare prin operatori liniari şi pozitivi

Teză de doctorat

Conducători ştiinţifici:
Prof. Univ. Dr. P. BLAGA
Prof. Dr. Dr. h. c. H. GONSKA

CLUJ-NAPOCA
2007

To my family

Contents

Introduction

In Approximation Theory positive linear operators play an important role; a fact demonstrated by the vast literature available on this topic.
One simple and old example - its exact year of origin is not recorded - is the *piecewise linear operator*. It has been used in approximately computing the value of a logarithm. The method was to interpolate two neighbouring entries of the logarithmic table. Now in the computer era this approach has become obsolete.
The key moment in the development of Approximation Theory was in 1885 when Karl Weierstrass [157] presented the first proof of his (fundamental) theorem on approximation by algebraic or trigonometric polynomials. This was a long and complicated proof and provoked many famous mathematicians to find simpler and more instructive proofs. We list some of the great mathematicians that relate their names to this most celebrated theorem: Carl Runge (1885), Henri Lebesgue (1908), Edmund Landau (1908), Charles de la Vallée-Poussin (1908), Lipot Fejér (1916) and, of course, Sergej N. Bernstein (1912). On this occasion the (now) very well-known *Bernstein polynomials* were constructed:

$$B_{n,k}(f;x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

for any $f \in C[0,1],\ x \in [0,1]$. Via these polynomials S. N. Bernstein succeeded to give the most elegant and short proof of Weierstrass's theorem. A complete overview on the existing additional proofs can be found in A. Pinkus's article [119] or in the monograph [150].
The importance of these remarkable operators could not have been anticipated in the first half of this century. Their relevance became obvious starting with the contributions of *Paul de Faget Casteljau* at Citroën and *Pierre Bézier* at Renault who had been using Bernstein polynomials as a very useful tool for their industrial design.
We want to emphasize the fact that in the development of the theory of approximation by positive linear operators the Romanian mathematicians brought very important contributions. Tiberiu Popoviciu founded in Cluj-Napoca a remarkable school of thought in Numerical Analysis and Approximation Theory. Some of many other remarkable Romanian mathematicians in this field are: D. D. Stancu, O. Agratini, P. Blaga, Gh. Coman, S. G. Gal, A. Lupaş, R. Păltănea, I. Raşa, R. Trâmbiţaş etc.

Although we do not focus our present work only on the Bernstein operators, we often consider them for comparison reasons or we use them as building blocks in order to obtain new operators. The material in this thesis is divided into five chapters.

In the **first chapter** we concisely present preliminary notions and auxiliary results that will be used throughout this thesis. Our main instruments we use in providing quantitative estimates are: the *modulus of continuity* ω_1 (see (1.5)) , its *least concave majorant* $\tilde{\omega}_1$ (see (1.9)) and the *moduli of smoothness* ω_2 (see (1.6)) or even of higher order $\omega_k,\ k \geq 3$ (see (1.11)). To put everything into a correct historical perspective we mention that ω_1 appeared already in Dunham Jackson's thesis [75] in 1912, a thesis that laid the foundation for the *Quantitative Approximation Theory*, as we know it today. Studies about $\tilde{\omega}_1$ can be found among others in the works of V.K. Dzjadyk [41] and N.P. Korneičuk [89]. Definitions of moduli of smoothness of higher order can be found in the book of L. L. Schumaker [137].
For some estimates in terms of different moduli of smoothness we need a liant, more exactly a special smoothing function that was constructed by V. V. Zhuk in [159]. Therefore we present in this chapter its definition (see (1.12)) and some of its relevant properties, see Section 1.5. Supplementary results on "smoothing of functions by smoother ones" can be found in Lemma 1.28.
Another important instrument in Approximation Theory by positive linear operators is *Peetre's K-functional*, named after its author J. Peetre who introduced it in 1963 in [116]. It represents another means to measure the smoothness of a function in terms of how well it can be approximated by smoother functions. Its definition is given at (1.19) and its most important properties are collected in Lemma 1.30. We mark the fact that there is a close relationship between the K-functional and the moduli of smoothness in Theorem 1.31, a special emphasis is given to *Brudnyĭ's representation theorem*, see Lemma 1.32, which enables us to represent the first order K-functional via $\tilde{\omega}_1$.
In Section 1.7 we present in chronological order some *quantitative Bohman-Korovkin* type theorems, starting with Shisha's & Mond's result 1.35 only in terms of ω_1 and continuing with Gonska's direct estimates via $\tilde{\omega}_1$ in Theorem 1.36.
Many of the known operators (including the Bernstein operator) reproduce also linear functions; it was desirable that this property should also be reflected in a pointwise estimate of the concerned operator. Such a requirement could not meet estimates given in terms of ω_1 or $\tilde{\omega}_1$. Therefore, it was advantageous to measure the degree of approximation by means of ω_2, as it annihilates linear functions. The first estimates involving ω_2 were established by H. Esser [42] in 1976, and later in 1984 improved by H. Gonska in [57]. The latter one was refined by R. Păltănea [111] in 1995 as far as the constants are concerned. In this thesis we shall often refer to the latter result as *Păltănea's theorem*, see Theorem 1.38.

It is also worthwhile to mention that the first uniform estimates in terms of ω_1 for the Bernstein operators were established by T. Popoviciu [123] in 1934 and in 1942 in [124] he gave a second solution. Its result can be summarized in the following:

$$|B_n(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_1 \left(\frac{1}{\sqrt{n}} \right),\ f \in C[0,1],\ x \in [0,1].$$

In 1961 the *exact value of the constant* in front of ω_1 was computed by P. C. Sikkema [141], namely $c = \frac{4306+837\sqrt{6}}{5832} \approx 1,089$. Moreover, T. Popoviciu - see [126] or [127] - observed that the method applied for the Bernstein operators can be easily extended to any positive linear operator L_n reproducing constant functions:

$$|L_n(f;x) - f(x)| \leq 2 \cdot \omega_1(\sqrt{\sup\{L_n((x-t)^2;x), x \in [0,1]\}}),$$

which is a precursor of Shisha's & Mond's result, see [140].
With regard to ω_2 we mention that Y. A. Brudnyĭ [24] showed that there exists a constant $C > 0$, such that

$$\|B_n f - f\|_\infty \leq C \cdot \omega_2 \left(f; \frac{1}{\sqrt{n}} \right),\ f \in C[0,1].$$

The pointwise version was given by Jia-ding Cao [26]:

$$|B_n(f;x) - f(x)| \leq C \cdot \omega_2 \left(f; \sqrt{\frac{x(1-x)}{n}} \right),\ x \in [0,1],\ f \in C[0,1].$$

The first concrete constants which can appear in both estimates were given by H. Gonska in [54] or [58]. These result were optimized and it was proven that the constant in the pointwise estimate can be chosen as $\frac{11}{8} = 1,375$, cf. [113, Corollary 4.1.2], and in the uniform one the constant can be replaced by $\frac{12}{11} \approx 1,0909$, cf. [113, Corollary 4.1.6], or even 1 cf. [113, Theorem 4.2.1].
Besides the degree of approximation we are also interested in investigating some shape-preservation properties of some selected positive operators, for this purpose we present in Section 1.8 some relevant parts (for us) of the *Theory of totally positive kernels* cited from Karlin's exhaustive work [81].

The **second chapter** is dedicated to some rational type discretely defined mappings called NURBS-functions from "non-uniform rational B-splines". They have their roots in CAGD: <u>C</u>omputer <u>A</u>ided <u>G</u>eometric <u>D</u>esign. Farin cites in his book [44] the thesis of Vesprille [154] and articles by Tiller [153] and Piegl & Tiller [117] as early papers on the subject. The standard source on this method is now the book by Piegl & Tiller [118]. Further monographs on the subject are those by Fiorot & Jeannin [50] and by Farin [43]. NURBS are today in use in commercially available software libraries such as SISL from SINTEF in Oslo (see, e.g., [143]). Another noteworthy source that gives a very instructive insight in the history of CAGD is [46].

From our point of view the abbreviation NURBS is an unfortunate acronym. The term is misleading since it suggests that one is exclusively dealing with non-uniform knot spacing, which is not true. We thus prefer the term *rational B-spline function*. They constitute a generalization of Schoenberg's variation-diminishing splines. Adapted to the context of approximation (of functions) theory which we discuss here, the generalization noted by $R_{\Delta_n,k}$ can be given as in (2.2).

In the first section we present all the 5 special cases covered by definition (2.2). We mention that all five methods considered play a fundamental role in CAGD. In order to gain a better overview on all particular cases and their relationship we depict the so-called *NURBS-graph* in Figure 2.1.

In Subsection 2.1.3 we are interested among other things in answering the question if $R_{\Delta_n,k}$ reproduces linear functions. We are able to give only a partial answer, for $k = 3$, based on an identity proven by G. Tachev in [152]. Due to some specific dimension arguments this method works only for $k \leq 3$ and it is largely exposed on p. 24. On the other hand, it is possible to prove a global statement regarding linear preservation for a special case of $R_{\Delta_n,k}$, namely the *rational Bernstein operators* $R_{1,k}$. Regarding this aspect see Proposition 2.12.

The approximation theoretical knowledge about the spline methods mentioned is in contrast to their importance in applications and to the many experimental results available. Therefore, in the following two subsections we start to discuss rational B-spline functions from the viewpoint of quantitative Approximation Theory. The estimates are given in terms of $\tilde{\omega}_1$ (Proposition 2.13 and Theorem 2.15) and ω_1 and ω_2 (Proposition 2.20).

In Section 2.2 we define and study a new family of *(modified) rational Bernstein operators* that, in comparison to the classical one, reproduces also linear functions. Their definition is given at (2.23) and one can observe that it depends on two sets of strictly positive weights $\{\bar{w}_i : 0 \leq i \leq n\}$ and $\{w_j : 0 \leq j \leq n-1\}$ and on the abscissae set $\{\bar{x}_i : 0 \leq i \leq n\}$. However, as will be further seen the three sets are inter-correlated, as we impose the conditions in Theorems 2.28 and 2.30 that $\bar{R}_n$ reproduces constant as well linear functions. More shape-preservation properties, like retaining the positivity, monotonocity and convexity or the variation-diminishing properties are proven in Proposition 2.34 and Corollary 2.35.

In Subsections 2.2.2 and 2.2.3 we state some convergence results for a specific class of denominators (Theorem 2.38) and we give some error estimates in terms of moduli of smoothness for continuous and C^2 functions (Theorem 2.41 and Theorem 2.42). Supplementary results on this topic can be found in [121].

In the last two sections of this chapter we study two types of modified operators. The first one is the so called *BLaC-wavelet operator* as it was introduced by G. P. Bonneau [20]. The abbreviation "BLaC" is derived from "Blending of Linear and Constant", which is a suggestive name as one can see from the definition of its fundamental functions at (2.36–2.37). The reproduction of constant functions is shown among others in Proposition 2.45 and in Subsection 2.3.1 some error bounds are given in terms of ω_1 and ω_2.
Finally, in the last section we study one possible modification of the Bernstein operators given by *King's operators.* J.P. King [86] defined this interesting (and somewhat exotic) sequence of linear and positive operators $V_n : C[0,1] \to C[0,1]$. The definition of this mapping is recalled at (2.42). One main difference between B_n and V_n is that the latter is a non-polynomial operator reproducing constant and quadratic functions, but not linear functions. These facts are being highlighted in Theorem 2.52 and in the subsequent remark. In Subsection 2.4.1 we establish some quantitative estimates via ω_1 and ω_2.

In the **third chapter** we deal with some special positive linear operators. Most of them are defined by means of the *Beta function* $B(p,q)$ with $p,q > 0$. Their general definition cf. (3.3) is

$$\mathbb{B}_n^{(\alpha,\lambda)} := \tilde{\mathbb{B}}_\alpha \circ B_n \circ \tilde{\mathbb{B}}_\lambda,\ \alpha,\ \lambda > 0,$$

where $\tilde{\mathbb{B}}_\alpha$ and $\tilde{\mathbb{B}}_\lambda$ represent a modification of Lupaş's *Beta operators of the second kind*, see (3.2). All the particular cases covered by (3.3) are depicted in Table 3.1. Among these are the *genuine Bernstein-Durrmeyer operators* U_n, that were independently introduced by W. Chen [28]in 1987, and by T. N. T. Goodman & A. Sharma [74] later in 1991. They possess many interesting properties and were therefore investigated by many authors, noteworthy is [115]. A detailed overview and many references can be found in [80]. Another famous operator hidden in the definition (3.3) is $S_n^{<\alpha,0,0>}$. They were introduced by D. D. Stancu in 1968 in [144] and were further investigated in the subsequent papers [145], [146] and [147]. Also many other authors studied them intensively, see e.g., the survey of B. Della Vecchia [35] and the references therein.
In this thesis we shall also focus our attention upon another *Beta-type operator*, which, however, does not fit exactly into the scheme from above, namely a multi-parameter *general Stancu operator* $S_n^{<\alpha,\beta,\gamma>}$. Its compact writing mode, for $\alpha \geq 0$, $0 \leq \beta \leq \gamma$, and any $f \in C[0,1]$ and $x \in [0,1]$, is given in (3.8–3.9):

$$S_n^{<\alpha,\beta,\gamma>}(f;x) = \tilde{\mathbb{B}}_\alpha B_n \left(f \circ \left(\frac{n}{n+\gamma} e_1 + \frac{\beta}{n+\gamma} \right); x \right).$$

In Section 3.2 we show that $\tilde{\mathbb{B}}_\alpha$, their generalizations $\mathbb{B}_n^{(\alpha,\lambda)}$ and $S_n^{<\alpha,\beta,\gamma>}$ preserve convexity - in the spirit of T. Popoviciu [123], [125] - up to any order (see Example 3.8

and Remark 3.9). For this purpose we make use of the powerful tool that represents *total positivity*, a result proven by A. Attalienti & I. Raşa (see here Theorem 3.6) and the fact that a finite product of the same order convex operators is also convex, see Proposition 3.5.
In the following section we compute the rate of convergence of the composite Beta-type operators and of $S_n^{<\alpha,\beta,\gamma>}$. The estimates are given in terms of ω_1 and ω_2 and the technique we employ is a standard one: we use Theorem 1.38 or an appropriate K-functional, see Theorem 3.11, Theorem 3.13. The rates of convergence of the special cases can be taken from Corollary 3.14.
In this chapter we are interested not only in direct estimates but also in *simultaneous approximation*, as one can read in Section 3.4. We mention that for the first time Bl. Sendov & V. Popov formulated in [139] a (non-quantitative) *Korovkin type* theorem for the Banach space $C^r[K],\ K=[a,b]$. Later, G.I. Kudrjavcev [91] (for $r=1$) and H.-B. Knoop & P. Pottinger [87] (for the more general case $r\geq 1$) were the first who proved estimates for simultaneous approximation involving ω_1, in the spirit of Shisha's & Mond's theorem from [140]. In 1984 H. Gonska generalized the result of Knoop & Pottinger by measuring the degree of (simultaneous) approximation in terms of ω_2, the second order modulus of smoothness, see [57]. D. P. Kacsó improved this last assertion by employing Păltănea's Theorem 1.38, see [77] or [79]. We shall slightly generalize her result in Theorem 3.15.
To have a historical background we mention that the first quantitative estimate for simultaneous approximation by Bernstein operators was proved by T. Popoviciu [122] in 1937 and was in terms of the first order modulus of continuity of $D^r f$. A very good historical review on estimates for Bernstein operators can be found in [8]. More applications on simultaneous approximation can be found in the following subsections 3.4.1–3.4.3.
In the context of simultaneous approximation another natural question had risen and has been studied during the recent years: whether simultaneous approximation processes also preserve global smoothness of the derivatives of an r-times differentiable function f. This aspect is studied for $\mathbb{B}_n^{(\alpha,\lambda)}$ and some instances of $S_n^{<\alpha,\beta,\gamma>}$ in Section 3.5 and the subsections therein. The first assertion was obtained by C. Cottin & H. Gonska, see Theorem 2.2 in [33]. More information on this subject can be found in the recent book of G. A. Anastassiou & S. G. Gal [6].

In **Chapter 4** our aim is to study the behavior of the powers of L_n having the following layout: $n\in\mathbb{N}$ is fixed and m goes to infinity. In other words, the operators considered are *over-iterated*. For any positive linear operator $L_n : C[0,1]\to C[0,1],\ n\in\mathbb{N}$, we define inductively the powers of L_n by

$$L_n^0 := Id,\ L_n^1 := L_n \text{ and } L_n^{m+1} := L_n\circ L_n^m,\ m\in\mathbb{N}.$$

In the subsequent three sections of this chapter we describe three methods to investigate the over-iteration of L_n:

1. the contraction principle,
2. a general quantitative method,
3. a method that uses the *spectral properties* of the operator.

The *contraction principle* represents a general method to investigate the behavior of the over-iteration of a fixed operator, see e.g., [11], [12]. The assertions in Section 4.1 were inspired by a recent result of O. Agratini & I. Rus [4] (see also [132]) who proved convergence for over-iteration of certain general discretely defined operators. In the sequel we prove a generalization of the first theorem in [4] also for a whole class of summation-type operators, see their definition at (4.1) and Theorem 4.1.
One advantage of the method is that it can be applied for many known summation type operators, cf. Subsection 4.2.2. On the other hand, the proof is restricted to a *fixed* operator L_n and its iterates L_n^m. Furthermore, the proof is only valid for operators having a contraction constant $c < 1$. In the following section we show that there are cases, where we do have $c = 1$, but still convergence of the iterates takes place.
In Section 4.2 we prove general inequalities for the iterates of positive linear operators preserving linear functions, which are given in the spirit of the paper by S. Karlin & Z. Ziegler [82] and were obtained for classical Bernstein operators in a slightly weaker form first in [54]. The results of this section are gathered in Theorems 4.6, 4.8 and Corollary 4.9.
Due to this general assertion, we are able to prove in Subsection 4.2.1 the convergence of the over-iterates of (4.1) and to provide a full quantitative version of it. Our estimate is given in terms of the second order modulus. However, due to the use of the contraction constant some pointwise information is lost, see Proposition 4.13 and Corollary 4.14.
Both of the two following subsections have an applicative character. In Subsection 4.2.2 we consider a group of operators to which both methods, the *contraction principle* and the *quantitative method* work. The advantage of the latter one is that we immediately obtain the degree of approximation. In Subsection 4.2.3 we consider some classes of operators to which the approach via the contraction principle is not applicable for two reasons: the Beta-type operators (implicitly Lupaş's Beta operators of second kind) are not discretely defined and for the Schoenberg spline operators one cannot find a contraction constant $c < 1$.
In the last section of this chapter we propose a method to study the behavior of the over-iterates of those operators for which both the contraction principle and

the *quantitative method* fail. Our method uses some *spectral properties* of the operators considered (general Stancu operators, Kantorovich operators, (generalized) Durrmeyer operators), such as: the unique representation of a polynomial operator w.r.t. the basis of its eigenfunctions and the fact that the corresponding eigenvalues are strictly less than 1. A similar technique was used in the recent paper of Sh. Cooper & Sh. Waldron [32] for the iterates of Bernstein operators and also in the paper of S. Ostrovska [109] for the iterates of q-Bernstein operators.

Chapter 5 has an eclectic character. In Section 5.1 we estimate the Peano remainder (from the Taylor expansion) by means of the modulus of continuity of the n-th derivative of a function f (Theorem 5.2) and the least concave majorant of the modulus (Theorem 5.3), avoiding in this way the "o" notation.
As an application we prove in Section 5.2 a *quantitative variant of the classical Voronovskaja theorem* for operators reproducing linear functions.
We recall the following very well-known result that describes the asymptotic behavior of Bernstein polynomials:
If f is bounded on $[0,1]$, differentiable in some neighborhood of x and has a second derivative $f''(x)$ for some $x \in [0,1]$, then

$$\lim_{n\to\infty} n \cdot [B_n(f,x) - f(x)] = \frac{x(1-x)}{2} \cdot f''(x).$$

If $f \in C^2[0,1]$, the convergence is uniform. It was first proven in 1932 by
E. V. Voronovskaja [156], but we find it also in the book of DeVore and Lorentz [38, p. 307].
We mention that S. N. Bernstein [14] generalized the uniform version of it in an article that follows directly after that of Voronovskaja, such as:
If $q \in \mathbb{N}$ is even, $f \in C^q[0,1]$, then uniformly in $x \in [0,1]$,

$$n^{q/2}\left\{B_n(f;x) - f(x) - \sum_{r=1}^{q} B_n((e_1-x)^r;x)\cdot\frac{f^{(r)}(x)}{r!}\right\} \to 0,\ n\to\infty.$$

More on this topic one can find in the recent work [62]. We will further deal in this thesis with the simplified version, namely $s = 2$. In Theorem 5.8 we prove a *quantitative description* of Voronovskaja's result in terms of $\tilde{\omega}_1$.
This general result is followed in the subsequent subsection by some applications, among others for Bernstein operators (Proposition 5.10) and Beta operators of the second kind (3.2).
In the first chapters we were mainly interested in determining the rate of convergence of a positive linear operator towards the identity operator, by means of different instruments (K-functionals and/or different moduli of smoothness). In Section 5.3

we widen our research and compare the convergence velocity between two positive linear operators. The means remain the same: K-functionals and different types of moduli of smoothness. In the first four results (Theorem 5.25 and the Corollaries 5.26, 5.27, 5.28) from Subsection 5.3.1 we study the rate of approximation for the difference of two positive operators that agree on the first n moments. These estimates are given in terms of $\tilde{\omega}_1$ and for $f \in C^n[0,1]$. Assertions for every function $f \in C[0,1]$ are obtained by means of moduli of smoothness of higher orders and by employing a result of H. Gonska [59]. For this purpose see Theorem 5.29.
The following two subsections have an applicative character. We give estimates for differences between different positive linear operators, like B_{n+1}, the $(n+1)-th$ Bernstein operator and $\bar{\mathbb{B}}_n$ Lupaş's Beta operator of the second kind, Proposition 5.31. Finally, in Subsection 5.3.3 we try to answer a question formulated by A. Lupaş in the article [97], regarding an estimate for the *commutator*:

$$[B_n, \bar{\mathbb{B}}_n] := B_n \circ \bar{\mathbb{B}}_n - \bar{\mathbb{B}}_n \circ B_n = U_n - S_n^{<1/n,0,0>},$$

where U_n are the genuine Bernstein-Durrmeyer operators and $S_n^{<1/n,0,0>}$ are some special Stancu operators. This is done in Proposition 5.37 after proving that the two operators agree up to the third moments, Lemma 5.36.
Towards the end, we list ten problems to which we have not yet found an appropriate or complete answer during the preparation of this thesis.

Acknowledgement. Finally, I would like to thank to all my colleagues from the Department of Mathematical Computer Science at University Duisburg-Essen who supported me in my work. My special thanks are dedicated to my Ph.D. advisors Prof. Univ. Dr. Petru Blaga and Prof. Dr. Dr. h. c. Heiner Gonska, from whom I have learnt a lot. I also thank for their valuable support: Prof. Univ. Dr. Sorin G. Gal, Dr. Privatdozent Daniela Kacsó, Prof. Univ. Dr. Alexandru Lupaş, Prof. Univ. Dr. Ioan Raşa and Prof. Dr. Paul Sablonnière.

Notations and symbols

In this work we shall often make use of the following symbols:

$:=$	is the sign indicating equal by definition". a:=b" indicates that a is the quantity to be defined or explained, and b provides the definition or explanation. b=:a" has the same meaning.
$\mathbb{N}$	the set of natural numbers,
$\mathbb{N}_0$	the set of natural numbers including zero,
$\mathbb{R}$	the set of real numbers,
$\mathbb{R}_+$	the set of positive real numbers,
$\overset{\circ}{X}$	the interior of the set X,
$[a, b]$	a closed interval,
(a, b)	an open interval.
	Let X be an interval of the real axis.
$\mathcal{F}(X)$	the set of all *real-valued* functions defined on X.
$B(X)$	the set of all real-valued and *bounded* functions defined on X.
$L^p(X)$	the class of the *p-Lebesgue integrable* functions on X, $p \geq 1$.
$\\|f\\|_p$	is the norm on $L^p(X)$ defined by $\\|f\\|_p := \left(\int_X \|f(x)\|dx\right)^{1/p}$, $p \geq 1$.
$C(X)$	the set of all real-valued and *continuous* functions defined on X.
$C[a, b]$	the set of all real-valued and *continuous* functions defined on the compact interval $[a, b]$.
	For $f \in B(X)$ or $f \in C(X)$
$\\|f\\|_\infty$	is the *Chebyshev norm* or *sup-norm*, namely $\\|f\\|_\infty := \sup\{\|f(x)\| : x \in X\}$.
$C^r[a, b]$	the set of all real-valued, *r-times continuously differentiable* function, $(r \in \mathbb{N})$.
$\mathrm{Lip}_\tau M$	the set of all $C[a, b]-$ functions that verify the *Lipschitz condition*: $\|f(x_2) - f(x_1)\| \leq M\|x_2 - x_1\|^\tau$, $\forall x_1, x_2 \in [a, b], 0 < \tau \leq 1$, $M > 0$.
$\prod_n$	$(\prod_n[a, b]$, $n \in \mathbb{N}_0)$ the linear space of all real polynomials with the degree at most n.
e_n	denotes the $n-$th *monomial* with $e_n : [a, b] \ni x \mapsto x^n \in \mathbb{R}$, $n \in \mathbb{N}_0$.
	For a function $f : X \to \mathbb{R}$, X an interval of the real axis we have:
$\Delta_h^k f(x)$	is the finite difference of order $k \in \mathbb{N}$, step $h \in \mathbb{R}\backslash\{0\}$ and starting point $x \in X$. A computing formula: $\Delta_h^k f(x) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x + ih)$, $x + ih \in X$, $i = 0, \ldots, k$, $h \in \mathbb{R}$, $h \neq 0$.
D^r or $f^{(r)}$	$r-$th *derivative* of the function $f \in C^r[a, b]$.
$[x_0, \ldots, x_m; f]$	$m-$th divided difference of $f \in \mathcal{F}(X)$ on the not necessarily distinct

	knots $x_0, \ldots, x_m \in X$.
$(a)_b$	are the *falling factorials* denoted by the *Pochhammer symbol*. $(a)_b := \prod_{i=0}^{b-1}(a-i),\ a \in \mathbb{R},\ b \in \mathbb{N}_0$, where $\prod_{i=0}^{-1} := 1$.
$y^{[m,h]}$	the *factorial power* of step $h \in \mathbb{R}$ defined by: $y^{[m,h]} := \prod_{i=0}^{m-1}(y-ih)$, $m \in \mathbb{N}_0$. As above $\prod_{i=0}^{-1} := 1$.

Chapter 1

Preliminary and auxiliary results

1.1 Positive linear operators

In this section we will give some basic definitions and some elementary properties concerning *positive and linear operators*. For more information on this topic see [30] or [150].

Definition 1.1 Let X, Y be two linear spaces of real functions. The mapping $L : X \to Y$ is called *linear operator* if $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$, $\forall f, g \in X$ and $\forall \alpha, \beta \in \mathbb{R}$.
If $\forall f \geq 0$, $f \in X \Rightarrow Lf \geq 0$, then L is a *positive operator*. X, Y are one of the spaces mentioned before.

Remark 1.2 a) The set $\mathcal{L}(X, Y) := \{L : X \to Y \mid L$ is a linear operator $\}$ is a real vector space.

b) In order to highlight the argument of the function $Lf \in Y$ we use the notation $L(f; x)$ but also in some rare cases $(Lf)(x)$.

Some elementary inequalities are recalled in the following:

Property 1.3 *Let $L : X \to Y$ be a positive and linear operator.*

(i) If f, $g \in X$ with $f \leq g$ then $Lf \leq Lg$. (monotonocity)

(ii) $\forall f \in X$ we have $|Lf| \leq L|f|$.

Definition 1.4 Let $L : X \to Y$, where $X \subseteq Y$ are two linear *normed spaces* of real functions. To each operator L we can assign a non-negative number $\|L\|$ defined by

$$\|L\| := \sup_{\substack{f \in X \\ \|f\|=1}} \|Lf\| = \sup_{\substack{f \in X \\ 0<\|f\|\leq 1}} \|Lf\|.$$

By convention, if X is the zero linear space, any operator L which map X to Y must be the *zero operator* and is assigned the *zero norm.*

It can be easily verified that $\|\cdot\|$ satisfies all the properties of a norm and hence is called *the operator norm.*
Choosing $X = Y = C[a, b]$ the following can be stated regarding the continuity and the operator norm:

Corollary 1.5 *If* $L : C[a, b] \to C[a, b]$ *is linear and positive then* L *is also continuous and* $\|L\| = \|Le_0\|$.

The next result provides a neccessary and sufficient condition for the convergence of a positive linear operator towards the identity operator. It was independently discovered and proved by three mathematicians in three consecutive years: T. Popoviciu [126] in 1951, H. Bohman [19] in 1952 and P. P. Korovkin [90] in 1953.
This classical result of approximation theory is mostly known under the name of *Bohman-Korovkin theorem*, because T. Popoviciu's contribution in [126] remained unknown for a long time.

Theorem 1.6 *Let* $L_n : C[a, b] \to C[a, b]$ *be a sequence of positive linear operators. If* $\lim\limits_{n\to\infty} L_n e_i = e_i$, $i = 0, 1, 2$, *uniformly on* $[a, b]$, *then* $\lim\limits_{n\to\infty} L_n f = f$ *uniformly on* $[a, b]$ *for every* $f \in C[a, b]$.

Remark 1.7 Due to the above result the monomials e_j, $j = 0, 1, 2$, play an important role in the approximation theory of linear and positive operators on spaces of continuous function. They are often called *Korovkin test-functions.*
This elegant and simple result has inspired many mathematicians to extend the last theorem in different directions, generalizing the notion of sequence and considering different spaces. In this way a special branch of approximation theory arose, called Korovkin-type approximation theory. A complete and comprehensive exposure on this topic can be found in [5].
Throughout this paper we will focus on quantitative versions of Theorem 1.6, which will be presented in one of the following subsections.

Example 1.8 Maybe the best-known and celebrated positive operators are the *Bernstein operators*, introduced by S. N. Bernstein [13] in 1912 in order to prove Weierstrass's fundamental theorem, see [157]. For any $f \in C[0, 1]$, $n \in \mathbb{N}$ and $x \in [0, 1]$, they are given by

$$B_n(f; x) := \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where the polynomials

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ 0 \le k \le n, \tag{1.2}$$

form the *Bernstein basis*. To be formally correct we set for $k < 0$ or $k > n$ that $p_{n,k} := 0$. It is not difficult to define the Bernstein operators on an arbitrarily compact interval $[a, b],\ a < b$. Throughout this paper we shall come back many times on the properties of these operators and their generalizations.

1.2 A Hölder-type inequality for positive linear operators

In many estimates the *Cauchy-Schwarz inequality* is employed:

$$(L(fg))^2 \le L\left(f^2\right) L\left(g^2\right),\ f, g \in C[a, b]. \tag{1.3}$$

The disadvantage is that for certain positive operators such estimate creates disastrous upper bounds. For this reason here we prove a Hölder-type inequality for positive linear operators which - at least in principle - provides extra flexibility and reduces to the inequality of Cauchy-Schwarz in case $p = q = 2$. For simplicity we restrict ourselves to the case $[a, b] = [0, 1]$.

Theorem 1.9 *Let $L : C[0, 1] \to C[0, 1]$ be a positive linear operator, $Le_0 = e_0$. For $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, f \in C[0, 1], x \in [0, 1]$ one has*

$$L(|fg|; x) \le L(|f|^p; x)^{\frac{1}{p}} \cdot L(|g|^q; x)^{\frac{1}{q}}.$$

Proof. For x fixed we consider the linear functional

$$A(f) = L(f; x), f \in C[0, 1].$$

(i) Suppose $A(|f|^p) > 0$ and $A(|g|^q) > 0$. Then define $\alpha := \frac{|f|}{A(|f|^p)^{1/p}}$, $\beta := \frac{|g|}{A(|g|^q)^{1/q}}$. By Young's inequality we know that

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q},\ \alpha, \beta \ge 0.$$

Hence

$$\frac{|fg|}{A(|f|^p)^{1/p} \cdot A(|g|^q)^{1/q}} \le \frac{1}{p} \frac{|f|^p}{A(|f|^p)} + \frac{1}{q} \cdot \frac{|g|^q}{A(|g|^q)}.$$

Applying the positive functional A to both sides of this inequality shows that

$$\frac{A(|fg|)}{A(|f|^p)^{1/p} \cdot A(|g|^q)^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

from which the desired inequality follows.
(ii) Suppose that $A(|f|^p) = 0$. As a positive linear functional, with $A(e_0) = 1$, A can be represented as $A(h) = \int_0^1 h d\mu$, where μ is a probability measure on $[0, 1]$. So we have $\int_0^1 |f|^p d\mu = 0$, which entails $|f|^p = 0$ on *supp* μ a.e. Then $|f \cdot g| = 0$ on *supp* μ a.e., so that $A(|fg|) = \int_0^1 |fg| d\mu = 0$. Thus the inequality of Theorem 1.9 is valid also in this case. □

1.3 Moments of higher order for positive linear operators: inequalities and a recurrence formula

For positive linear operators $L : C[a, b] \to C[a, b]$, the following quantities play an important role. The *moments* of order $n, n \geq 0$, namely

$$L((e_1 - x)^n; x) := L((e_1 - x)^n)(x), x \in [a, b],$$

and for $n \geq 1$ also the *absolute moments* of odd order n, that is

$$L(|e_1 - x|^n; x) = L(|e_1 - x|^n)(x), x \in [a, b].$$

As one can see, e.g., in Subsection 1.7 very important are the *first absolute* moments $L(|e_1 - x|; x)$ and the *second order* moments $L((e_1 - x)^2; x)$. In most of the cases it is a difficult task to compute the first absolute moment, therefore the Cauchy-Schwarz inequality is used to estimate as follows:

$$L(|e_1 - x|; x) \leq \sqrt{L(e_0^2; x)} \cdot \sqrt{L((e_1 - x)^2; x)}. \tag{1.4}$$

But sometimes this approximation is to harsh. We mention in the following some alternative ways.

Proposition 1.10 *If L, p, q, f and x are given as in Theorem 1.9, and let $0 \leq n = n_1 + n_2$ be a decomposition of the non-negative number n with $n_1, n_2 \geq 0$. Then*

$$L(|e_1 - x|^n; x) \leq L(|e_1 - x|^{n_1 \cdot p}; x)^{\frac{1}{p}} \cdot L(|e_1 - x|^{n_2 \cdot q}; x)^{\frac{1}{q}}.$$

For the case $n = 1, n = n_1 + n_2 = 0 + 1, p = q = 2$, this reduces to (1.4).

Remark 1.11 Note that in Proposition 1.10 the quantities p, q, n_1 and n_2 may depend on $x \in [0,1]$. That is, under the assumptions of Theorem 1.9 we have for x fixed that

$$L(|e_1 - x|^n; x) \leq \inf_{\substack{p,q>1, \frac{1}{p}+\frac{1}{q}=1 \\ n_1,n_2 \geq 0; n_1+n_2=n}} \{L(|e_1 - x|^{n_1 \cdot p}; x)^{\frac{1}{p}} \cdot L(|e_1 - x|^{n_2 q}; x)^{\frac{1}{q}}\}.$$

Another way to relate moments of different orders to each other is described in:

Proposition 1.12 *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator such that $Le_0 = e_0$ and $1 \leq s < r$. Then*

$$L(|e_1 - x|^s; x)^{\frac{1}{s}} \leq L(|e_1 - x|^r; x)^{\frac{1}{r}}, x \in [0,1].$$

Proof. Let $r > s \geq 1, p := \frac{r}{s} > 1$. If A is given as above, then $A(|f|^s) \leq A(|f|^{ps})^{\frac{1}{p}} = A(|f|^r)^{\frac{s}{r}}$, so that

$$A(|f|^s)^{\frac{1}{s}} \leq A(|f|^r)^{\frac{1}{r}}, f \in C[0,1], 1 \leq s < r.$$

In particular, for $f(t) := |t - x|, t \in [0,1], x$ fixed, this means

$$L(|e_1 - x|^s; x)^{\frac{1}{s}} \leq L(|e_1 - x|^r; x)^{\frac{1}{r}}, 1 \leq s < r.$$

□

Example 1.13 *(i) For a positive linear operator $L : C[0,1] \to C[0,1]$ with $Le_0 = e_0$ one has*

$$L(|e_1 - x|; x) \leq L((e_1 - x)^2; x)^{\frac{1}{2}} \leq L(|e_1 - x|^3; x)^{\frac{1}{3}} \leq L((e_1 - x)^4; x)^{\frac{1}{4}} \leq \ldots.$$

(ii) An alternative way to bound the third term via Cauchy-Schwarz is

$$L(|e_1 - x|^3; x)^{\frac{1}{3}} \leq L((e_1 - x)^2; x)^{\frac{1}{6}} \cdot L((e_1 - x)^4; x)^{\frac{1}{6}}.$$

In [70] it is shown that for some operators the approach from (ii) is the better one. Further we shall prove a recurrence formula for moments of higher order.

Proposition 1.14 *For a linear operator L and $k \in \mathbb{N}_0$ one has*

$$L((e_1 - x)^k; x) = L(e_k; x) - \sum_{l=0}^{k-1} \binom{k}{l} x^{k-l} L((e_1 - x)^l; x).$$

Proof. Write

$$\begin{aligned} L(e_k;x) &= L((e_1-x+x)^k;x) \\ &= L(\sum_{l=0}^{k}\binom{k}{l}x^{k-l}\cdot(e_1-x)^l;x) \\ &= \sum_{l=0}^{k}\binom{k}{l}x^{k-l}\cdot L((e_1-x)^l;x) \\ &= L((e_1-x)^k;x)+\sum_{l=0}^{k-1}\binom{k}{l}x^{k-l}\cdot L((e_1-x)^l;x), \end{aligned}$$

which implies the representation of the k-th moment. □

Remark 1.15 (i) Note that the equality of Proposition 1.14 holds without the assumption $Le_i = e_i, i \in \{0,1\}$.

(ii) The proposition means that $L((e_1-x)^k;x)$ can be computed if we know $L(e_k;x)$ and the lower order moments $L((e_1-x)^l;x), 0 \le l \le k-1$.

Corollary 1.16 *For a linear operator L with $Le_i = e_i, i \in \{0,1\}$, the third and fourth moments can be computed as*

$$\begin{aligned} L((e_1-x)^3;x) &= L(e_3;x)-x^3-3xL((e_1-x)^2;x), \\ L((e_1-x)^4;x) &= L(e_4;x)-x^4-\{4x\cdot L((e_1-x)^3;x)+6x^2\cdot L((e_1-x)^2;x)\}. \end{aligned}$$

Proof. It is a immediate consequence of Proposition 1.14. □

The facts exposed and proved in this section will be used also in Subsections 5.1 and 5.3. For more information on this topic see also [71].

1.4 Different types of moduli of smoothness

The main tools to measure the degree of convergence of positive linear operators towards the identity operator are the *moduli of smoothness* of first and second order. For $f \in C[a,b]$ and $\delta \ge 0$ we have

$$(1.5)\quad \omega_1(f;\delta) := \sup\{|f(x+h)-f(x)| : x, x+h \in [a,b],\ 0 \le h \le \delta\};$$
$$(1.6)\quad \omega_2(f;\delta) := \sup\{|f(x+h)-2f(x)+f(x-h)| : x, x\pm h \in [a,b],\ 0 \le h \le \delta\}.$$

The first modulus of smoothness (continuity) has a long history. It appeared already in 1911 in the Ph. D. thesis of D. Jackson [75], the work that laid the basis for what is known today as *Quantitative Approximation Theory.*

ω_1 inherits its name from the first part of the following property:

Proposition 1.17 *Let $f \in C[a,b]$ and $\delta > 0$.*

a) If $\lim_{\delta \to 0^+} \omega_1(f;\delta) = 0$, then f is continuous on $[a,b]$.

b) The following equivalence holds: $f \in \mathrm{Lip}_\tau M$ iff $\omega_1(f;\delta) \le M \cdot \delta^\tau$, where $0 < \tau \le 1$ and $M > 0$.

A useful modification represents the *least concave majorant* of $\omega_1(f;\cdot)$ given by

$$(1.7) \qquad \tilde{\omega}(f;\varepsilon) = \begin{cases} \sup\limits_{\substack{0 \le x \le \varepsilon \le y \le b-a \\ x \ne y}} \frac{(\varepsilon - x)\omega(f,y) + (y-\varepsilon)\omega(f,x)}{y-x} & \text{for} \quad 0 \le \varepsilon \le b-a, \\ \tilde{\omega}(f, b-a) = \omega(f, b-a) & \text{if} \quad \varepsilon > b-a. \end{cases}$$

The definition of $\tilde{\omega}(f,\cdot)$ shows that

$$(1.8) \qquad \omega(f;\cdot) \le \tilde{\omega}(f;\cdot) \le 2 \cdot \omega_1(f;\cdot).$$

For some further properties of $\tilde{\omega}(f;\cdot)$ see, e.g., V.K. Dzjadyk [41, p. 153ff] or [60]. It was shown by N.P. Korneičuk [89, p. 670] that for any $\varepsilon \ge 0$ and $\xi > 0$ the function $\omega(f;\cdot)$ and its least concave majorant $\tilde{\omega}(f;\cdot)$ are related by the inequality

$$(1.9) \qquad \tilde{\omega}(f;\xi \cdot \varepsilon) \le (1+\xi) \cdot \omega(f;\varepsilon),$$

and that this inequality cannot be improved for each $\varepsilon > 0$ and $\xi = 1, 2 \ldots$.

Remark 1.18 One can construct an (abstract) modulus of continuity by taking into consideration the following known property: Any non-decreasing, subadditive mapping $\Omega : [0,\infty) \to \mathbb{R}$ such that $\Omega(0) = 0$ is the modulus of continuity of its own.

In this spirit and having further applications in mind (see Section 5.1) we present the following example.

Example 1.19 *Let $n \ge 0$ and $0 < \varepsilon \le \frac{1}{2}$, so that also $\frac{\varepsilon}{n+1} \le \frac{1}{2}$. Then let*

$$(1.10) \qquad \Omega(t) = \begin{cases} \frac{n+1}{2\varepsilon} \cdot t, & 0 \le t \le \frac{\varepsilon}{n+1}; \\ \frac{1}{2}, & \frac{\varepsilon}{n+1} \le t \le 1 - \frac{\varepsilon}{n+1}; \\ \frac{n+1}{2\varepsilon} \cdot (t-1) + 1, & 1 - \frac{\varepsilon}{n+1} \le t \le 1. \end{cases}$$

In order to show that Ω is indeed a modulus of continuity, note that the function is continuous, non-decreasing and such that $\Omega(0) = 0$. It can be seen by inspection that Ω is also subadditive, so that Ω is a (non-concave) modulus. As expected Ω is

the modulus of continuity of itself, that is $\omega(\Omega(\cdot);\delta) = \Omega(\delta), 0 \le \delta \le 1$ *(see [93], p. 43).*
Moreover, for $\frac{\varepsilon}{n+1} \le t \le 1$*, the graph of* $\tilde{\Omega}$ *differs from that of* Ω *in the sense that there we have*

$$\tilde{\Omega}(t) = \frac{1}{2(n+1-\varepsilon)}((n+1)(t+1) - 2\varepsilon).$$

Hence $\Omega(\varepsilon) = \frac{1}{2} = \tilde{\Omega}(\frac{\varepsilon}{n+1})$.

Most of the error estimates in this work are given in terms of the two moduli of smoothness or in term of $\tilde{\omega}_1$. However in the last chapter we give estimates, where moduli of higher order are involved. Therefore we give the definition of $\omega_k,\ k \in \mathbb{N}$, as given in 1981 by L. L. Schumaker in his book [137]:

Definition 1.20 For $k \in \mathbb{N},\ \delta \in \mathbb{R}_+$ and $f \in C[a,b]$ the *modulus of smoothness* of order k of is defined by

$$(1.11) \qquad \omega_k(f;\delta) := \sup\{|\Delta_h^k f(x)| \mid 0 \le h \le \delta,\ x, x+kh \in [a,b]\}.$$

Remark 1.21 For clarity sometimes we will write $\omega_k(f;\delta;[a,b])$.
It is obvious for $\delta \ge \frac{b-a}{k}$ one has $\omega_k(f;\delta) = \omega_k(f;\frac{b-a}{k})$.

We collect in the following proposition some useful properties of ω_k:

Property 1.22 (see [150])

1) $\omega_k(f;0) = 0$.

2) $\omega_k(f;\cdot)$ *is a positive, continuous and non-decreasing function on* $\mathbb{R}_+$.

3) $\omega_k(f;\cdot)$ *is sub-addititive, i.e.,* $\omega_1(f;\delta_1+\delta_2) \le \omega_1(f;\delta_1)+\omega_1(f;\delta_2),\ \delta_i \ge 0,\ i = 1,2$.

4) $\forall \delta \ge 0,\ \omega_{k+1}(f;\delta) \le 2\omega_k(f;\delta)$.

5) If $f \in C^1[a,b]$ *then* $\omega_{k+1}(f;\delta) \le \delta \cdot \omega_k(f';\delta),\ \delta \ge 0$.

6) If $f \in C^r[a,b]$ *then* $\omega_r(f;\delta) \le \delta^r \sup_{\delta \in [a,b]} |f^{(r)}(\delta)|$.

7) $\forall \delta > 0$ *and* $n \in \mathbb{N},\ \omega_k(f;n\delta) \le n^k \omega_k(f;\delta)$.

8) $\forall \delta > 0$ *and* $r > 0,\ \omega_k(f;r\delta) \le (1+[r])^k \omega_k(f;\delta)$*, where* $[a]$ *is the integer part of* a.

9) If $\delta \ge 0$ *is fixed, then* $\omega_k(f;\cdot)$ *is a seminorm on* $C[a,b]$.

Corollary 1.23 (see [150])

1) $\forall \delta > 0, \omega_{k+r}(f;\delta) \le 2^r \omega_k(f;\delta),\ k, r \in \mathbb{N}.$

2) $\forall 0 < \delta \le 1,\ \omega_{k+1}(f;\delta^k) \le \omega_k(f;\delta).$

1.5 Zhuk's function and its applications

Some of the estimates in terms of different moduli of smoothness can be elegantly proven by using as an intermediate a special smoothing function that was constructed by V. V. Zhuk in [159]. Therefore we find it instructive to present here its definition and its relevant properties, see also [67].
Zhuk's approach was the following: For $f \in C[a,b]$ he first defined the extension $f_h : [a-h, b+h] \to \mathbb{R}$, with $h > 0$, by

$$f_h(x) := \begin{cases} P_-(x), & a-h \le x \le a, \\ f(x), & a \le x \le b, \\ P_+(x), & b < x \le b+h, \end{cases}$$

where $P_-, P_+ \in \prod_1$ are *the best approximants* to f on the indicated intervals.
Then Zhuk defined its function $Z_h f(\cdot)$ (sometimes also denoted by $f_{2,h}(\cdot)$) by means of the *second order Steklov means*

$$Z_h f(x) := \frac{1}{h} \cdot \int_{-h}^{h} \left(1 - \frac{|t|}{h}\right) f_h(x+t)dt,\ x \in [a,b]. \tag{1.12}$$

It can be shown that $Z_h f \in W_{2,\infty}[a,b]$, where

$$W_{2,\infty}[a,b] := \{f \in C[a,b] : f' \textit{ absolutely continuous},\ \|f''\|_{L_\infty} < \infty\},\ \text{and} \tag{1.13}$$
$$\|f\|_{L_\infty[a,b]} = \|f\|_{L_\infty} = \operatorname{vrai\,sup}\{|f''(x)| : x \in [a,b]\}. \tag{1.14}$$

The following estimates were proven in [159] Lemma 1 (or [67] Lemma 2.1)

Lemma 1.24 *Let* $f \in C[a,b],\ 0 < h \le \frac{1}{2}(b-a)$. *Then*

$$\|f - Z_h f\|_\infty \le \frac{3}{4} \cdot \omega_2(f;h), \tag{1.15}$$
$$\|(Z_h f)''\|_{L_\infty} \le \frac{3}{2} \cdot h^{-2} \cdot \omega_2(f;h). \tag{1.16}$$

Supplementary estimates for lower order derivatives of $Z_h f$ are given in

Lemma 1.25 (see Lemma 2.4 in [67]) *Let f, h and $Z_h f$ be given as in Lemma 1.24. Then*

$$\|(Z_h f)'\|_\infty \leq \frac{1}{h} \cdot \left[2 \cdot \omega_1(f;h) + \frac{3}{2} \cdot \omega_2(f;h)\right], \tag{1.17}$$

$$\|Z_h f\|_\infty \leq \|f\|_\infty + \frac{3}{4} \cdot \omega_2(f;h). \tag{1.18}$$

Corollary 1.26 *As an immediate consequence of the latter lemma, one has the simpler inequalities*

$$\|(Z_h f)'\|_\infty \leq \frac{5}{h} \cdot \omega_1(f;h), \text{ and } \|Z_h f\|_\infty \leq 4 \cdot \|f\|_\infty.$$

As an application of of the upper inequalities the authors proved in [67] the following

Lemma 1.27 (see Lemma 4.1 in [67]) *Let $g \in W_{2,\infty}$ and the polynomial $B_n g$, where B_n is the Bernstein operator defined on $[a,b]$. Then for any $\varepsilon > 0$ and a sufficiently large n the following inequalities hold:*

$$\|g - B_n g\|_\infty < \varepsilon, \; \|B_n g\|_\infty \leq \|g\|_\infty, \; \|(B_n g)'\|_\infty \leq \|g'\|_\infty,$$

and

$$\|(B_n g)''\|_\infty \leq \|g''\|_{L_\infty}.$$

In other words, the latter lemma affirms that functions in $W_{2,\infty}[a,b]$ can be approximated well by functions in $C^2[a,b]$, while "retaining important differential characteristics", see [67].

Supplementary results on "smoothing of functions by smoother ones" can be found in Lemma 3.1 in [59]. Having further applications in mind, we shall present this assertion below:

Lemma 1.28 *Let $I = [0,1]$ and $f \in C^r(I), r \in \mathbb{N}_0$. For any $h \in (0,1]$ and $s \in \mathbb{N}$ there exists a function $f_{h,r+s} \in C^{2r+s}(I)$ with*

(i) $\|f^{(j)} - f^{(j)}_{h,r+s}\|_\infty \leq c \cdot \omega_{r+s}(f^{(j)};h)$ *for* $0 \leq j \leq r$,

(ii) $\|f^{(j)}_{h,r+s}\|_\infty \leq c \cdot h^{-j} \cdot \omega_j(f;h)$, *for* $0 \leq j \leq r+s$,

(iii) $\|f^{(j)}_{h,r+s}\|_\infty \leq c \cdot h^{-(r+s)} \cdot \omega_{r+s}(f^{(j-r-s)};h)$, *for* $r+s \leq j \leq 2r+s$.

Here the constant c depends only on r and s.

1.6 K-functionals and their relationship to the moduli

In 1963 J. Peetre introduced in [116] an expression called *Peetre's K-functional*, which represents another important instrument to measure the smoothness of a function in terms of how well it can be approximated by smoother functions. Although it is possible to define the K-functional in a very general context, for the applications we have in mind in the current paper, it is sufficient for us to consider the following definition:

Definition 1.29 For any $f \in C[a,b]$, $\delta \geq 0$ and integer $s \geq 1$ we call

$$
\begin{aligned}
(1.19) \qquad K_s(f;\delta)_{[a,b]} &:= K(f;\delta;C[a,b],C^s[a,b]) \\
&:= \inf\{\|f-g\|_\infty + \delta \cdot \|g^{(s)}\|_\infty : g \in C^s[a,b]\},
\end{aligned}
$$

Peetre's K-functional of order s.

Whenever there is no doubt about the interval of definition of f we shall use for $K_s(f;\delta)_{[a,b]}$ the abbreviation $K_s(f;\delta)$.
It is clear that the quantity in (1.19) reflects some approximation properties of f: the inequality $K_s(f;\delta) < \varepsilon,\ \delta > 0$ implies that f can be approximated with error $\|f-g\|_\infty < \varepsilon$ in $C[a,b]$ by an element $g \in C^s[a,b]$, whose norm is not to large, $\|g^{(s)}\|_\infty < \frac{\varepsilon}{\delta}$.
The following lemma collects some of the properties of $K_s(f;\cdot)$. They were proven by P.L. Butzer & H. Berens [25], but they can also be found in more recent work on approximation theory as in: [137], [38] and [60].

Lemma 1.30 (see Proposition 3.2.3 in [25]) *Let $K_s(f;\cdot)$ be defined as in (1.19).*

1) The mapping $K_s(f;\delta) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous especially at $\delta = 0$, i.e.,

$$\lim_{\delta \to 0^+} K_s(f;\delta) = 0 = K_s(f;0).$$

2) For each fixed $f \in C[a,b]$ the application $K_s(f;\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonically increasing and concave function.

3) For arbitrary $\lambda, \delta \geq 0$, and fixed $f \in C[a,b]$, one has the inequality

$$K_s(f;\lambda\cdot\delta) \leq \max\{1,\lambda\} \cdot K_s(f;\delta).$$

4) For arbitrary $f_1,\ f_2 \in C[a,b]$ we have $K_s(f_1+f_2;\delta) \leq K_s(f_1;\delta)+K_s(f_2;\delta),\ \delta \geq 0$.

5) *For each $\delta \geq 0$ fixed, $K_s(\cdot;\delta)$ is a seminorm on $C[a,b]$, such that*

$$K_s(f;\delta) \leq \|f\|_\infty,$$

for all $f \in C[a,b]$.

6) *For a fixed $f \in C[a,b]$ and $\delta \geq 0$ the identity $K_s(|f|;\delta) = K_s(f;\delta)$ is true.*

The following theorem establishes the close relationship between the K-functional and the moduli of smoothness. K_s and ω_s are related by the following *equivalence relation*, see H. Johnen [76]:

Theorem 1.31 *There exist constants C_1 and C_2, depending only on s and $[a,b]$ such that*

$$C_1 \cdot \omega_s(f;\delta) \leq K_s(f;\delta^s) \leq C_2 \cdot \omega_s(f;\delta), \tag{1.20}$$

for all $f \in C[a,b]$ and $\delta > 0$.

In general there are no sharp constants known in the above (double) inequality. However, there are two exceptional cases for $s = 1,2$. We present them below.
The following lemma known as *Brudnyĭ's representation theorem* establishes the connection between $K_1(f;\delta)_{[a,b]}$ and the least concave majorant defined at (1.7).

Lemma 1.32 *Every function $f \in C[a,b]$ satisfies the equality*

$$K_1\left(f,\delta;C[a,b],C^1[a,b]\right) = \frac{1}{2}\cdot\tilde{\omega}_1(f;2\delta),\ \ \delta \geq 0. \tag{1.21}$$

More details and also proofs of the above lemma can be found in many different sources, as for example: in the article of B. S. Mitjagin & E. M. Semenov [105], or in the book by R. T. Rockafellar [130], or in the monograph of R. A. DeVore & G. G. Lorentz [38, p. 175].
Also for the case $s = 2$ there is something known about the constants in front of the moduli of smoothness. Thus, H. Gonska proved in [54] p. 31 the following

Lemma 1.33 *Let $f \in C[a,b]$ and $0 \leq \delta$. Then we have*

$$\frac{1}{4}\cdot\omega_2(f;\delta) \leq K_2\left(f,\frac{\delta^2}{2};C[a,b],C^2[a,b]\right) \text{ and}$$

$$K_2(f,\delta^2;C[a,b],C^2[a,b]) \leq \left(\frac{3}{2} + 2\cdot\max\left\{1,\frac{\delta^2}{(b-a)^2}\right\}\right)\cdot\omega_2(f;\delta).$$

In another context, but also very useful for our next applications is the following:

Lemma 1.34 *For any $f \in C[a,b]$ and $\delta \geq 0$ the following identity holds,*

$$K(f;\delta;C[a,b],C^2[a,b]) = K(f;\delta;C[a,b],W_{2,\infty}[a,b]), \tag{1.22}$$

where the K-functional on the right hand side can be defined in an analogous way to the other one.

Proof. It is trivial to see that $C^2[a,b] \subset W_{2,\infty}[a,b]$ implies $K(f;\delta;C[a,b],W_{2,\infty}[a,b]) \leq K(f;\delta;C[a,b],C^2[a,b])$. In order to prove the inverse inequality let $\varepsilon > 0$ be fixed and $g \in W_{2,\infty}[a,b]$. Obviously we have $B_n g \in C^2[a,b]$ and furthermore $\|(B_n g)''\|_\infty \leq \|g''\|_{L_\infty}$, see Lemma 1.27. Having this in mind, for a sufficiently large $n \in \mathbb{N}$ and $0 \leq \delta$ the following inequality holds:

$$\begin{aligned} K(f;\delta;C[a,b],C^2[a,b]) &\leq \|f - B_n g\|_\infty + \delta \cdot \|(B_n g)''\|_\infty \\ &\leq \|f - g\|_\infty + \|g - B_n g\|_\infty + \delta \cdot \|(B_n g)''\|_\infty \\ &\leq \|f - g\|_\infty + \varepsilon + \delta \cdot \|g''\|_{L_\infty}. \end{aligned}$$

This implies, by passing on the right hand side to the *infimum* for all functions in $W_{2,\infty}[a,b]$ that

$$K(f,\delta;C[a,b],C^2[a,b]) \leq K(f,\delta;C[a,b],W_{2,\infty}[a,b]) + \varepsilon, \ \varepsilon > 0.$$

But ε was arbitrarily chosen, so letting $\varepsilon \to 0$ we arrive at the desired inequality. □

1.7 General quantitative theorems on $C[a,b]$

In this section we present in chronological order some *quantitative Bohman-Korovkin* type theorems (see 1.6). This direct estimates are given by means of different moduli of smoothness.
One of the first estimates only in terms of ω_1 were given by R. Mamedov [99] for the case $Le_0 = e_0$, and later O. Shisha & B. Mond [140] obtained the following more general result. Let $K = [a,b]$ and $K' \subseteq K$ be also compact and let $L : C(K) \to C(K')$ be a positive linear operator.

Theorem 1.35 *If $f \in C(K)$, then for every $x \in K'$ and for every $h > 0$, the following holds:*

$$\begin{aligned} |L(f;x) - f(x)| &\leq |f(x)| \cdot |L(e_0;x) - 1| \\ &+ \left(L(e_0;x) + \frac{\sqrt{L(e_0;x) \cdot L((e_1 - x)^2;x)}}{h} \right) \cdot \omega_1(f;h). \end{aligned}$$

It is also possible to give direct estimates via $\tilde{\omega}_1$ as in the following result, see H. Gonska [56] or [58]:

Theorem 1.36 *For L defined as above also reproducing constant functions the following inequality holds:*

$$|L(f;x) - f(x)| \leq \max\left\{1, \frac{1}{\delta} \cdot L(|e_1 - x|;x)\right\} \cdot \tilde{\omega}_1(f;h),$$

for all $f \in C(K)$, $x \in C(K')$ and $h > 0$.

Due to (1.8) and with the same assumptions as above we have

Corollary 1.37 *For any $f \in C(K)$, $x \in K'$ and $h > 0$ there holds*

$$|L(f;x) - f(x)| \leq 2 \cdot \max\left\{1, \frac{1}{h} \cdot L(|e_1 - x|;x)\right\} \cdot \omega_1(f;h).$$

Due to the fact that ω_2 annihilates linear functions, it is advantageous to measure the degree of approximation by means of this modulus of smoothness. The first estimates involving ω_2 were established by H. Esser [42] in 1976, and later in 1984 improved by H. Gonska in [57]. The latter one was refined by R. Păltănea [111] in 1995 as far as the constants are concerned. In the sequel we shall often refer to the following result as *Păltănea's theorem*:

Theorem 1.38 *For any $f \in C(K)$, all $x \in K'$ and $0 < h \leq \frac{1}{2}length(K)$ we have*

$$\begin{aligned} (1.23)\quad |L(f;x) - f(x)| &\leq |L(e_0;x) - 1| \cdot |f(x)| + |L(e_1 - x;x)| \cdot \frac{1}{h}\omega_1(f,h) \\ &+ \left(L(e_0;x) + \frac{1}{2} \cdot \frac{1}{h^2} L((e_1 - x)^2;x)\right)\omega_2(f,h). \end{aligned}$$

Remark 1.39 The condition $0 < h \leq \frac{1}{2}length(K)$ can be eliminated for operators which preserve linear functions.

It is possible to improve the latter inequality by substituting the term in front of ω_2 with $\frac{1}{h^s}L(|(e_1 - x)^s|;x)$, with an integer $s \geq 2$, see e.g., in [113, Corollary 2.2.1].

1.8 On totally positive kernels

The theory of totally positive functions plays an fundamental role in many fields of mathematics, among others also in Approximation Theory. By means of totally positive kernels one can easily investigate some shape-preservation properties of positive linear operators, see e.g., Section 3.2. Therefore, in the sequel we present

and select some basic definitions and some properties that are relevant for us. A general survey of the theory of totally positive kernels and its several applications can be found in the book of S. Karlin [81].
According to [81, p. 11] we have

Definition 1.40 A real function $K : X \times Y \to \mathbb{R}$, where X and Y are intervals or sets of positive integers, is called *(strictly) totally positive kernel* if

$$\begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \dots & K(x_1, y_m) \\ K(x_2, y_1) & K(x_2, y_2) & \dots & K(x_2, y_m) \\ \vdots & \vdots & \vdots & \vdots \\ K(x_m, y_1) & K(x_m, y_2) & \dots & K(x_m, y_m) \end{vmatrix} (>) \geq 0,$$

for all $m \geq 1$ and any selections $x_1 < x_2 < \ldots < x_m,\ y_1 < y_2 < \ldots < y_m,\ x_i \in X,\ y_i \in Y$. In particular, we have $K(x, y) \geq 0$ for all $(x, y) \in X \times Y$.

If both X and Y are finite sets, then K can be considered a matrix, in which case it is allowed to speak about *totally positive matrices.*
The following example is noteworthy, see [81, p. 287].

Example 1.41 *For any $n \geq 1$ fixed and all sequences $\mathcal{T}_n = \{0 \leq x_1 < x_2 < \ldots < x_n \leq 1\}$ the matrix $(p_{n,i}(x_j))_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n}}$ is totally positive. In other words, the Bernstein basis forms a totally positive system.*

There are different ways to combine two totally positive kernels in order to produce a new totally positive kernel. Chapter 3 of [81] is dedicated to this topic of constructing a variety of interesting kernels with sign-regularity properties.
In order to use it later, in the sequel we present a simplified version of Theorem 1.1 in [81, p. 99].

Theorem 1.42 *a) If $K(x, y)$ is a totally positive kernel on $X \times Y$, and $\phi(x),\ \psi(y)$ are nonzero positive functions for $x \in X$ and $y \in Y$, respectively, and if $L(x, y) = \phi(x) \cdot \psi(y) \cdot K(x, y)$, then $L(x, y)$ is also totally positive on $X \times Y$.*
b) Let $K(x, y)$ be totally positive ($x \in X,\ y \in Y$), and let $u = \phi^{-1}(x)$ and $v = \psi^{-1}(y)$ each define a strictly increasing function transforming X and Y into U and V, respectively, where ϕ^{-1} and ψ^{-1} are the inverse functions of ϕ respectively ψ. Consider

$$L(u, v) := K[\phi(u), \psi(v)],\ u \in U,\ v \in V.$$

Then $L(u, v)$ is also totally positive.

As a consequence of Theorem 1.42 b) we can state (see (1.5) in [81, p. 100])

Corollary 1.43 *The kernel*

$$K(x,y) = e^{\phi(x)\cdot\psi(y)},\ x \in X,\ y \in Y$$

is totally positive provided the function ϕ and ψ are strictly increasing on X and Y, respectively.

The property of total positivity is strongly related with the *variation-diminishing property* as one can see from the following

Lemma 1.44 (see Theorem 3.1 in [81, Chapter 1]) *Let L be a linear operator, reproducing constant functions, of the form*

$$L(f;x) := \int_Y K(x,y)\cdot f(y)d\sigma(y),\ f \in D(L),\ x \in X, \tag{1.24}$$

where X is a real interval, Y is a real interval or a set of positive integers, depending if L is a continuous or discrete operator. Suppose that K is a totally positive function defined on $X \times Y$, $d\sigma(y)$ is a $\sigma-$finite measure on Y and finally, the domain $D(L)$ of L is a linear space of real functions defined on a real interval $I \supseteq Y$. Supplementary, we suppose that the integral on the right-hand side is absolutely convergent. Under these assumptions the operator L has the variation-diminishing property, i.e.,

$$S^-(Lf) \leq S^-(f) \text{ on } X.$$

More exactly, having a function g defined on I the symbol $S^-(g)$ means

$$S^-(g) = S^-[g(t)] = \sup_{\mathcal{T}}\{g(t_1),\ g(t_2), \dots, g(t_m)\},$$

where $\mathcal{T} := \{t_1 < t_2 < \dots t_m,\ t_i \in I,\ i = 1, \dots, m,\ m \geq 1\}$ and $S^-(x_1, x_2, \dots, x_m)$ is the number of sign changes of the indicated sequence, zero terms being discarded.

Remark 1.45 The latter definition of *variation-diminishing property* was introduced by I. J. Schoenberg [136].

Other important applications of the *total positivity* concern shape-preserving properties, i.e., preservation of *monotonocity and (classical) convexity*, as presented in the following:

Theorem 1.46 (see Theorem 3.4 (a) and Theorem 3.5 (a) in [81, Chapter 6]) *Let L be given as above and K be a totally positive function.*

a) If L reproduces constant functions, then L transforms increasing functions into increasing functions.

b) If L reproduces constant functions and $Le_1(x) = ax + b,\ x \in X,\ a > 0$ and b real, then L maps convex functions into convex functions.

Additional results involving total positivity will be discussed and proved in Section 3.2.

Chapter 2

On rational type operators and some special cases

2.1 Rational B-spline operators

In Computer Aided Geometric Design (CAGD) so-called NURBS ("non-uniform rational B-splines") were introduced. G. Farin cites in his book [44] the thesis of K. Vesprille [154] and articles by W. Tiller [153] and L. Piegl & W. Tiller [117] as early papers on the subject. The standard source on this method is now the book by L. Piegl & W. Tiller [118]. Further monographs on the subject are those by j. Fiorot & P. Jeannin [50] and by G. Farin [43]. NURBS are today in use in commercially available software libraries such as SISL from SINTEF in Oslo (see, e.g., [143]).

2.1.1 Definition and some special cases

The abbreviation NURBS is an unfortunate acronym. The term is misleading since it suggests that one is exclusively dealing with non-uniform knot spacing which is not true. We thus prefer the term rational B-spline function. They constitute a generalization of Schoenberg's variation-diminishing splines. Adapted to the context of approximation (of functions) theory which we discuss here, this generalization is as follows. Many of the results that will be presented in the sequel can be also found in [64].

Definition 2.1 *Let $\Delta_n : \ 0 = x_0 < x_1 < \ldots < x_n = 1$, $n \in \mathbb{N}$, be a finite partition of the interval $I = [0, 1]$, $k \in \mathbb{N}$. We extend this partition by*

$$x_{-k} = \ldots = x_{-1} = x_0 = 0,$$
$$x_n = x_{n+1} = \ldots = x_{n+k} = 1.$$

Define "nodes" (Greville abscissae = evaluation parameters) by

$$\xi_{j,k} := \frac{x_{j+1} + \ldots + x_{j+k}}{k}, \ -k \le j \le n-1.$$

To each Greville abscissa associate a weight $w_{j,k} > 0$. *Putting*

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, x_{j+1}, \ldots, x_{j+k+1}](\cdot - x)_+^k, \tag{2.1}$$

for $f \in \mathbb{R}^{[0,1]}$ *we define*

$$\begin{aligned} R_{\Delta_n,k}(f;x) &:= \frac{\sum\limits_{j=-k}^{n-1} w_{j,k} \cdot f(\xi_{j,k}) \cdot N_{j,k}(x)}{\sum\limits_{j=-k}^{n-1} w_{j,k} \cdot N_{j,k}(x)} \\ &=: \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot R_{j,k}(x), \quad 0 \le x < 1, \ \textit{and} \\ R_{\Delta_n,k}(f;1) &:= \lim_{\substack{x \to 1 \\ x<1}} R_{\Delta_n,k}(f;x). \end{aligned} \tag{2.2}$$

$R_{\Delta_n,k}$ *is the rational B-spline operator and* $R_{\Delta_n,k}(f;\cdot)$ *is a rational B-spline function.*

Remark 2.2 Throughout this thesis we shall use the following convention: in the case of an equidistant knot distribution the symbol $\{\Delta_n, k\}$ is replaced by the simpler one $\{n, k\}$.

For special choices of the weights of k and n we obtain interesting particular cases:

Case 1: Suppose that $w_{j,k} = w > 0$ for $-k \le j \le n-1$. Then

$$\begin{aligned} R_{\Delta_n,k}(f;x) &= \frac{w \cdot \sum\limits_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x)}{w \cdot \sum\limits_{j=-k}^{n-1} N_{j,k}(x)} \\ &= \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x) \\ &= S_{\Delta_n,k}(f;x), \ x \in [0,1]. \end{aligned} \tag{2.3}$$

The latter is the famous (polynomial) variation - diminishing Schoenberg spline. It was introduced by Schoenberg and Greville in 1965 (see [135]).

Case 2: Suppose that $w_{j,k} = w > 0, \ k = 1, \ n \in \mathbb{N}$. Then the "knots" are given as

$$x_{-1} = x_0 < x_1 < \ldots < x_n = x_{n+1},$$

and the "nodes" are

$$\xi_{j,1} = x_{j+1},\ -1 \le j \le n-1.$$

The fundamental functions are now $N_{j,1},\ -1 \le j \le n-1$, and the operator $S_{\Delta_n,1}$ describes piecewise linear interpolation at the points

$$0 = x_0 < x_1 < \ldots < x_n = 1.$$

The following representation is known, due to T. Popoviciu, see [128] on p. 151:

$$\begin{aligned}(2.4)\ S_{\Delta_n,1}(f;x) &= f(x_0) + (x - x_0)[x_0, x_1; f] \\ &+ \sum_{k=2}^{n} \frac{x_k - x_{k-2}}{2}(|x - x_{k-1}| + x - x_{k-1}) \cdot [x_{k-2}, x_{k-1}, x_k; f].\end{aligned}$$

Case 3: Suppose that $w_{j,k} = w > 0,\ n = 1,\ k \in \mathbb{N}$. Then the "knots" are given as

$$\begin{aligned}x_{-k} &= \ldots = x_0 = 0,\\ x_1 &= \ldots = x_{1+k} = 1,\end{aligned}$$

so there are no knots in $(0, 1)$.
For the "nodes" one has

$$\xi_{-k,k} = 0,\ \xi_{-k+1,k} = \frac{1}{k}, \ldots, \xi_{0,k} = 1\ \text{ (equidistant)}.$$

For the fundamental functions one gets from the Mansfield identity:

$$N_{j,k}(x) = \binom{k}{j+k} \cdot x^{j+k}(1-x)^{-j},\ -k \le j \le 0 = n-1.$$

Hence

$$\begin{aligned}\sum_{j=-k}^{0} f(\xi_{j,k}) \cdot N_{j,k}(x) &= \sum_{j=-k}^{0} f\left(\frac{j+k}{j}\right) \cdot \binom{k}{j+k} \cdot x^{j+k}(1-x)^{-j} \\ (2.5) \qquad &= \sum_{j=0}^{k} f\left(\frac{j}{k}\right) \cdot \binom{k}{j} \cdot x^j(1-x)^{k-j} \\ &= B_k(f;x),\ 0 \le x \le 1.\end{aligned}$$

The latter is the Bernstein polynomial of degree k.

Case 4: Suppose that the weights are not identical, but again $n = 1,\ k \in \mathbb{N}$.
Writing $p_{k,j}(x) := \binom{k}{j} \cdot x^j(1-x)^{k-j}$ we arrive at

$$(2.6) \qquad R_{1,k}(f;x) := R_{\Delta_1,k}(f;x) = \frac{\sum_{j=0}^{k} w_{j,k} \cdot f\left(\frac{j}{k}\right) \cdot p_{k,j}(x)}{\sum_{j=0}^{k} w_{j,k} \cdot p_{k,j}(x)}.$$

This is a rational Bernstein function.
All five methods considered play a fundamental role in CAGD.

2.1.2 NURBS-graph

For a better overview on all particular cases and their relationship we shall depict the so-called *NURBS-graph*.

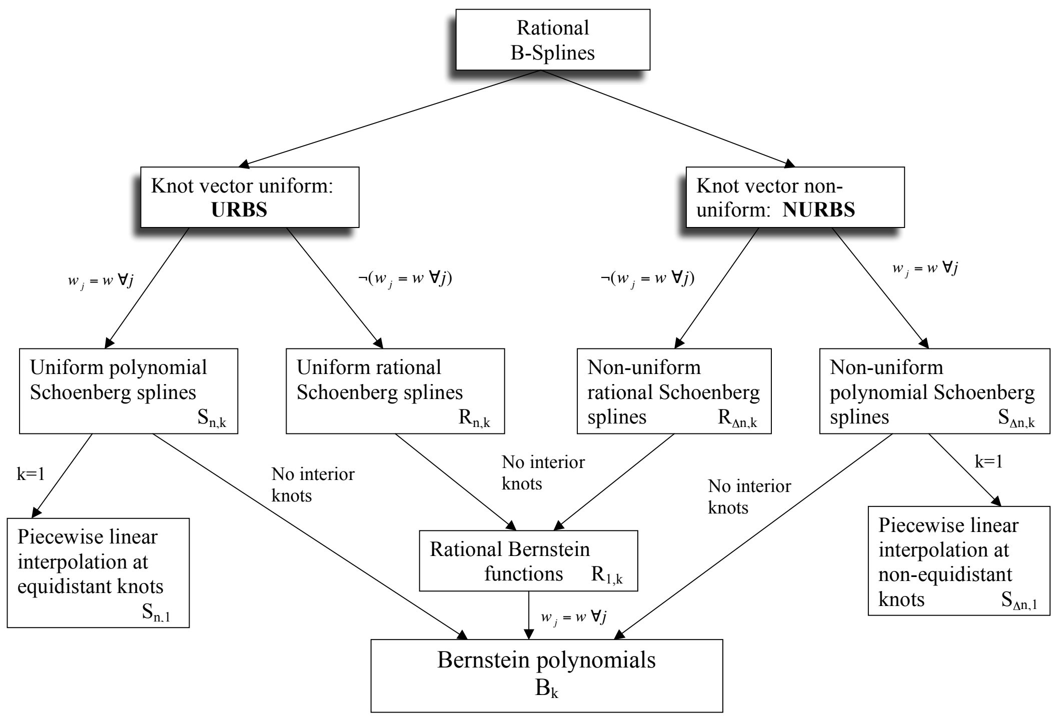

Figure 2.1: NURBS-graph

2.1.3 (Shape-preservation) properties and some negative results about linear precision

We gather in the following some fundamental properties of $R_{\Delta_n,k}$:

Proposition 2.3 *(i)* $R_{\Delta_n,k}$ *is a positive linear operator reproducing constant functions.*

(ii) Both the numerator and the denominator are splines of degree k *and in* $C^{k-1}[0,1]$.

(iii) $R_{\Delta_n,k}$ *interpolates* f *at the endpoints.*

(iv) $R_{\Delta_n,k}$ *is discretely defined, i.e., it depends only on the* $n+k$ *values* $f(\xi_{j,k})$, $-k \leq j \leq n-1$ *(and on the weights* $w_{j,k}$ *associated with* $\xi_{j,k}$*).*

Regarding the fundamental functions $R_{j,k}(x) = \frac{w_{j,k} N_{j,k}(x)}{Q_n(x)}$, $-k \le j \le n-1$, where $Q_n(x) := \sum_{j=-k}^{n-1} w_{j,k} \cdot N_{j,k}(x)$, we can prove the following:

Proposition 2.4 *For any fixed $k \ge 1$ the kernel*

$$[0,1] \times \{-k, \dots, n-1\} \ni (x, j) \mapsto R_{j,k}(x) \in \mathbb{R}$$

is totally positive. Hence, the operator $R_{\Delta_n,k}$ posseses the variation-diminishing property.

Proof. It is well-known that for any fixed $k \ge 1$ the kernel

$$[0,1] \times \{-k, \dots, n-1\} \ni (x, j) \mapsto N_{j,k}(x) \in \mathbb{R}$$

is totally positive, see, e.g., Theorem 4.1 in [81, Chapter 10] or [21]. The weights $w_{j,k}$ and $Q_n(x)$ are both positive, and thus thanks to Theorem 1.42 part a) we arrive at the desired result. The second part follows immediately from Lemma 1.44. □
Hence,

Corollary 2.5 *$R_{\Delta_n,k}$ transforms increasing functions into increasing functions.*

Proof. It is a consequence of the latter proposition and Theorem 1.46 issue a). □

An important issue regarding the operator $R_{\Delta_n,k}$ is, if it does not preserve linear functions. In [152] the following conjecture was formulated:

Conjecture 2.6 *The operator $R_{\Delta_n,k}$ reproduces linear functions, if and only if all weight numbers are equal.*

In the last cited paper was proved that the conjecture is true for $k = 1, 2$ for arbitrary partitions of the interval $[0, 1]$. For both of the cases the following representation was used (see [152, Theorem 2.1]):

Theorem 2.7 *For $n, k \ge 1$ the following equality holds:*

$$R_{\Delta_n,k}(e_1; x) - e_1(x) = \frac{2}{Q_n(x)} \sum_{j=-k}^{n-1} \sum_{i=j+1}^{n-1} (w_{j,k} - w_{i,k})(\xi_{j,k} - \xi_{i,k}) \cdot N_{j,k}(x) \cdot N_{i,k}(x), \tag{2.7}$$

for $x \in [0, 1]$.

By using a different approach that also involves the latter representation we were able to prove the conjecture also for the case $k = 3$. We shall present it in the following:

a) It must be verified, if the $\binom{4}{2} = 6$ products of two different B-splines that live" on $[0, x_1]$, are linearly independent, i.e.,

$$\sum_{j=-3}^{0} \sum_{i=j+1}^{0} a_{i,j} N_{j,3}(x) \cdot N_{i,3}(x) = 0 \Leftrightarrow a_{i,j} = 0,\ -3 \leq j < i \leq 0.$$

b) If a) holds, then if we relate to (2.7) we arrive at

$$R_{\Delta_n,3}(e_1; x) - e_1(x) = 0,\ x \in [0, x_1] \Leftrightarrow (w_j - w_i)(\xi_j - \xi_i) = 0,\ -3 \leq j < i \leq 0,$$

where $w_{l,3} =: w_l$ and $\xi_{l,3} =: \xi_l$. Whence and due to $\xi_j - \xi_i < 0,\ -3 \leq j < i \leq 0$ we arrive at $w_{-3} = w_{-2} = w_{-1} = w_0 =: w$.

c) Further, it will be proved by the *induction principle* that the remaining weight numbers are also equal to w. Thus, we will show the implication

$$w_{-3} = \ldots = w_{l-1} = w \Rightarrow w_l = w,\ 1 \leq l \leq n-1.$$

Suppose that $R_{\Delta_n,3}(e_1; x) - e_1(x) = 0$ on $x \in [x_l, x_{l+1}]$. On this interval relation (2.7) reads:

$$(w - w_l) N_l(x) \left[\sum_{i=l-3}^{l-1} (\xi_i - \xi_l) N_i(x) \right] = 0.$$

It is obvious now that $w_l = w$, because $N_l(x) \neq 0$ and also $[\cdot] \neq 0$ on $[x_l, x_{l+1}]$.

d) By the induction principle it was proven that $R_{\Delta_n,3}(e_1; x) - e_1(x) = 0$ on $[0, 1]$ iff $w_i = w$ for all $i = -3, \ldots, n-1$. □

We have proved issue a) by brute force method and with the help of the computer algebra system *Mathematica 5.0* and we shall present it here for the equidistant knot sequence $x_i = \frac{i}{n},\ i = 0, \ldots, n$, for simplicity sake.
Proof of issue a) for k=3. We assume that

$$aN_{-3,3} \cdot N_{-2,3} + bN_{-3,3} \cdot N_{-1,3} + cN_{-3,3} \cdot N_{0,3} + dN_{-2,3} \cdot N_{-1,3} + eN_{-2,3} \cdot N_{0,3} + fN_{-1,3} \cdot N_{0,3} = 0 \quad (2.8)$$

on the specified interval, the coefficients are real numbers.
Case 1: Here we consider $n \geq 4$. In this context, the *piecewise polynomials* of degree 3 the $N_{i,3}$ are:

$$N_{-3,3}(x) = n^3 \left(\frac{1}{n^3} - \frac{3x}{n^2} + \frac{3x^2}{n} - x^3 \right), \quad (2.9)$$

$$N_{-2,3}(x) = n^3 \left(\frac{3x}{n^2} - \frac{9x^2}{2n} + \frac{7x^3}{4} \right), \quad (2.10)$$

$$N_{-1,3}(x) = n^3\left(\frac{3x^2}{2n} - \frac{11x^3}{12}\right), \tag{2.11}$$

$$N_{0,3}(x) = n^3 \cdot \frac{x^3}{6}. \tag{2.12}$$

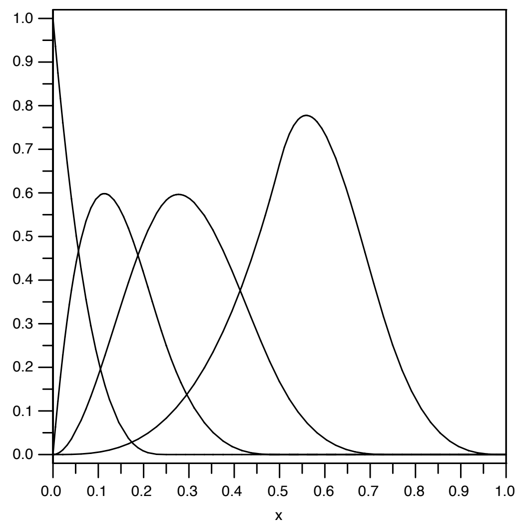

Figure 2.2: Cubic B-splines that "live" on $\left[0, \frac{1}{n}\right]$, $n = 4$

Substituting these identities into (2.8) and reordering them according to powers of x we arrive at:

$$\begin{aligned} & n^6\left(-\frac{7a}{4} + \frac{11b}{12} - \frac{c}{6} - \frac{77d}{48} + \frac{7e}{24} - \frac{11f}{72}\right)x^6 \\ + \; & n^5\left(\frac{39a}{4} - \frac{17b}{4} + \frac{c}{2} + \frac{27d}{4} - \frac{3e}{4} + \frac{f}{4}\right)x^5 \\ + \; & n^4\left(-\frac{87a}{4} + \frac{29b}{4} - \frac{c}{2} - \frac{19d}{2} + \frac{e}{2}\right)x^4 + n^3\left(\frac{97a}{4} - \frac{65b}{12} + \frac{c}{6} + \frac{9d}{2}\right)x^3 \\ + \; & n^2\left(-\frac{27a}{2} + \frac{3b}{2}\right)x^2 + 3anx = 0, \end{aligned}$$

for all $x \in \left[0, \frac{1}{n}\right]$. This is equivalent with: all the coefficients of the 6−th degree polynomial are equal to 0. It can be easily seen that $a = b = 0$. The remaining coefficients are: $c = d = e = f = 0$. This can be justified by

$$\begin{vmatrix} \frac{n^3}{6} & \frac{9n^3}{2} & 0 & 0 \\ -\frac{n^4}{2} & -\frac{19n^4}{2} & \frac{n^4}{2} & 0 \\ \frac{n^5}{2} & \frac{27n^5}{4} & -\frac{3n^5}{4} & \frac{n^5}{4} \\ -\frac{n^6}{6} & -\frac{77n^6}{48} & \frac{7n^6}{24} & -\frac{11n^6}{72} \end{vmatrix} = \frac{5n^{18}}{2304} \neq 0.$$

Case 2: $n = 1$ corresponds to *rational Bernstein functions*, see (2.6). This case will be largely discussed (in a more general context) a little further below (in Proposition 2.12).
Case 3: For $n = 2$, $N_{-1,3}$ and $N_{0,3}$ have a different form as in Case 1 (they have both multiple knots in 1). Simple computation give us $N_{-1,3}(x) = 16\left(\frac{3x^2}{8} - \frac{x^3}{2}\right)$ and $N_{0,-3}(x) = 2x^3$ for $x \in [0, \frac{1}{2}]$. $N_{-3,3}$ and $N_{-2,3}$ can be obviously obtained from (2.9) and (2.10) by substituting $n = 2$. All four B-splines that live on $\left[0, \frac{1}{2}\right]$ are depicted in the following figure, however on $[0, 1]$:

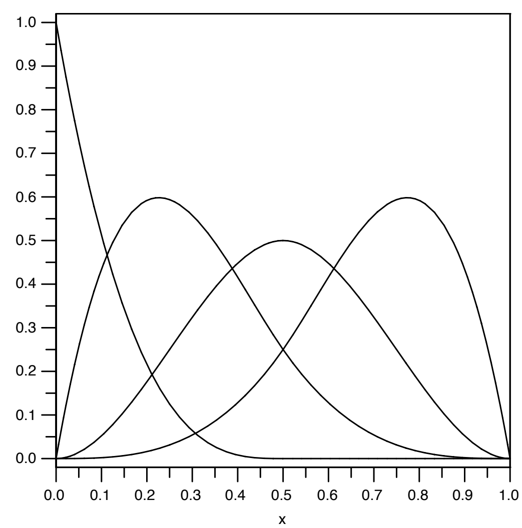

Figure 2.3: Cubic B-splines that "live" on $\left[0, \frac{1}{n}\right]$, $n = 2$

Applying the same strategy as in the first case and we obtain the identity:

$$\begin{aligned} 0 &= 6ax + (-54a + 6b)x^2 + (194a - 44b + 2c + 36d)x^3 \\ &+ (-348a + 120b - 12c - 156d + 12e)x^4 \\ &+ (312a - 144b + 24c + 228d - 36e + 12f)x^5 \\ &+ (-112a + 64b - 16c - 112d + 28e - 16f)x^6, \end{aligned}$$

for all $x \in \left[0, \frac{1}{2}\right]$. Right away one can see that $a = b = 0$. Regarding the rest of the coefficients we see from

$$\begin{vmatrix} 2 & 36 & 0 & 0 \\ -12 & -156 & 12 & 0 \\ 24 & 228 & -36 & 12 \\ -16 & -112 & 28 & -16 \end{vmatrix} = 1152 \neq 0,$$

which means $c = d = e = f = 0$.

Case 4: For $n = 3$ all the B-splines that are not equal with zero on $\left[0, \frac{1}{3}\right]$ can be obtained from (2.9–2.12) by substituting $n = 3$. Therefore, is no need to treat this case separately. Thus, we arrive at the desired assertion. Moreover, we have implicitly proved Conjecture 2.6 for $k = 3$ on an equidistant partition. □

Remark 2.8 An unfortunate disadvantage of this method is that it cannot be extended for $k \geq 4$. For these cases the argument of linear independency fails, i.e., there are too many B-spline products on $[0, x_1]$ in comparison with the dimension of the space $\prod_{2k}$.

We shall focus now our attention, up to the end of this subsection, on the special case of *rational Bernstein* operators. We want to prove that $R_{1,k}$ has the *total variation-diminishing property (TV)*, namely

Proposition 2.9 *For a function f with bounded (total) variation in $I = [0, 1]$ we have*

$$TV(R_{1,k}f) \leq TV(f),$$

where for a function g the symbol $TV(g)$ means:

$$TV(g) =: \sup\{TV_\sigma(g),\ \sigma \text{ a subdivision of } I\}$$

and $TV_\sigma(g)$ represents

$$TV_\sigma(g) := \sum_{i=0}^{k} |g(s_{i+1}) - g(s_i)|,\ for\ \sigma = \{0 = s_0 < s_1 < \ldots < s_k < s_{k+1} = 1\}.$$

As it is well-known that for a function g differentiable having an integrable derivative, its total variation is equal to:

$$TV(g) = \int_0^1 |g'(t)|dt.$$

Therefore we first need the following representation:

Lemma 2.10 *The first derivative of $R_{1,k}$ is given by*

$$(R_{1,k}f)'(x) = \sum_{i=0}^{k-1} \sigma_{i,k}(x) \left(f\left(\frac{i+1}{k}\right) - f\left(\frac{i}{k}\right)\right), \tag{2.13}$$

where the functions $\sigma_{i,k}, 0 \leq i \leq k-1$, are defined by

$$\sigma_{i,k}(x) = \frac{(k!)^2}{(2k-2)!} \frac{1}{N^2} \sum_{p=i+1}^{i+k} \omega_{i,p} p_{2k-2,p-1}(x) \tag{2.14}$$

with the coefficients

$$\omega_{i,p} := \sum_{(j,l)\in K(i,p)} \frac{(l-j)}{p(2k-p)} \binom{p}{j} \binom{2k-p}{k-j} w_{j,k} w_{l,k}, \tag{2.15}$$

for the set of indices

$$K(i,p) := \{(j,l) : j+l=p,\ 0 \le j \le i,\ i+1 \le l \le k\}.$$

We denoted $N := \sum_{j=0}^{k} w_{j,k} \cdot p_{k,j}(x)$.

Proof. According to a result of M. S. Floater [51, Proposition 3], the derivative $(R_{1,k}f)'$ can be written

$$(R_{1,k}f)'(x) = \sum_{i=0}^{k-1} \sigma_{i,k}(x) \left(f\left(\frac{i+1}{k}\right) - f\left(\frac{i}{k}\right) \right),$$

where the functions $\sigma_{i,k}$ are defined by

$$\sigma_{i,k}(x) = \frac{1}{x(1-x)N^2} \sum_{j=0}^{i} \sum_{l=i+1}^{k} (l-j) p_{k,j}(x) p_{k,l}(x) w_{j,k} w_{l,k}.$$

As $1 \le i+1 \le j+l \le i+k \le 2k-1$, there is always a factor $x(1-x)$ in the product $p_{k,j}(x)p_{k,l}(x)$, so the expression can be slightly simplified as follows:

$$\frac{(l-j)}{x(1-x)} p_{k,j}(x) p_{k,l}(x) = (l-j) \binom{k}{j} \binom{k}{l} x^{j+l-1} (1-x)^{2k-j-l-1}$$

$$= (l-j) \frac{\binom{k}{j}\binom{k}{l}}{\binom{2k-2}{j+l-1}} p_{2k-2,j+l-1}(x)$$

$$= \frac{(k!)^2}{(2k-2)!} \frac{(l-j)}{(j+l)(2k-j-l)} \binom{2k-j-l}{k-j} \binom{j+l}{l} p_{2k-2,j+l-1}(x).$$

Now, as $p = j+l$ varies from $i+1$ to $i+k$, we can collect all the terms which are coefficients of $p_{2k-2,p-1}$ and we obtain

$$\sigma_{i,k}(x) = \frac{1}{N^2} \frac{(k!)^2}{(2k-2)!} \sum_{p=i+1}^{i+k} \omega_{i,p} p_{2k-2,p-1}(x),$$

where, using the set of indices $K(i,p) := \{(j,l) : j+l=p,\ 0 \le j \le i,\ i+1 \le l \le k\}$:

$$\omega_{i,p} := \sum_{(i,j)\in K(i,p)} \frac{(l-j)}{p(2k-p)} \binom{p}{j} \binom{2k-p}{k-j} w_{j,k} w_{l,k},$$

which is the desired result. ∎

Now we can proceed in writing down

Proof of Proposition 2.9. Comparing the two expressions for $(R_{1,k})'$:

$$(R_{1,k}f)'(x) = \sum_{j=0}^{k} \mu_j'(x) w_j f\left(\frac{j}{k}\right) = \sum_{i=0}^{k-1} \sigma_i(x) \left(f\left(\frac{i+1}{k}\right) - f\left(\frac{i}{k}\right)\right),$$

where for simplicity, we denoted $\mu_j(x) := \frac{p_{k,j}(x)}{N}$, and we omitted the double index for σ and the weights, we deduce

$$\begin{aligned}
\sigma_0(x) &= -\mu_0'(x)w_0 \\
\sigma_0(x) - \sigma_1(x) &= \mu_1'(x)w_1 \\
&\vdots \\
\sigma_{k-2}(x) - \sigma_{k-1}(x) &= \mu_{k-1}'(x)w_{k-1} \text{ and} \\
\sigma_{k-1}(x) &= \mu_k'(x)w_k.
\end{aligned}$$

By induction we can easily prove that $\sigma_j(x) = -(\mu_0'(x)w_0 + \ldots + \mu_j'(x)w_j)$ with $j = 0, \ldots, k-2$ and $\sigma_{k-1}(x) = \mu_k'(x)w_k$. On the other hand, $\int_0^1 \sigma_j(x)dx = (\mu_0(0)w_0 + \ldots + \mu_j(0)w_j) - (\mu_0(1)w_0 + \ldots + \mu_j(1)w_j) = \frac{w_0}{w_0} - 0 = 1,\ j = 0, \ldots, k-2$ and $\int_0^1 \sigma_{k-1}(x)dx = \mu_k(1)w_k - \mu_k(0)w_0 = \frac{w_k}{w_k} - 0 = 1$. Now, since the functions σ_j are positive (see (2.14)),

$$\begin{aligned}
\int_0^1 |(R_{1,k}f)'(x)|dx &\leq \sum_{j=0}^{k-1} |f\left(\frac{j+1}{k}\right) - f\left(\frac{j}{k}\right)| \int_0^1 \sigma_j(x)dx \\
&= \sum_{j=0}^{k-1} |f\left(\frac{j+1}{k}\right) - f\left(\frac{j}{k}\right)| \leq TV(f).
\end{aligned}$$

and we obtain the desired result. ∎

Corollary 2.11 *The first derivatives at the endpoints are proportional with the slopes of the control polygon at those points. More exactly,*

$$(R_{1,k}f)'(0) = \frac{kw_1}{w_0}\left[f\left(\frac{1}{k}\right) - f(0)\right] \text{ and } (R_{1,k}f)'(1) = \frac{kw_{k-1}}{w_k}\left[f(1) - f\left(\frac{k-1}{k}\right)\right].$$

Proof. The proof is straightforward, if we substitute into (2.13)with $x = 0$ and $x = 1$, respectively. ∎

The situation is similar for the rational Bézier curves, see for instance [43, (7.29) and (7.30)].

In comparison with the general $R_{\Delta_n,k}$, for $R_{1,k}$ it is possible to prove a global statement regarding linear preservation:

Proposition 2.12 *The rational Bernstein operator reproduces linear functions if and only if all weights are equal.*

Proof. Since $R_{1,k}e_0 = e_0$ and $R_{1,k}$ is linear, it suffices to consider the function $e_1(x) = x$.
We have

$$\begin{aligned} R_{1,k}(e_1;x) - x &= \frac{\sum_{j=0}^{k} w_j \frac{j}{k} p_{k,j}(x)}{\sum_{j=0}^{k} w_j p_{k,j}(x)} - x \cdot \frac{\sum_{j=0}^{k} w_j p_{k,j}(x)}{\sum_{j=0}^{k} w_j p_{k,j}(x)} \\ &=: \frac{1}{N} \cdot \left\{ \sum_{j=0}^{k} w_j \frac{j}{k} p_{k,j}(x) - x \cdot \sum_{j=0}^{k} w_j p_{k,j}(x) \right\} \\ &=: \frac{T_1 - T_2}{N}. \end{aligned}$$

We raise the degree of T_1 from k to $k+1$ and get

$$T_1 = \sum_{j=0}^{k+1} \left[\frac{j}{k+1} \cdot w_{j-1} \cdot \frac{j-1}{k} + \left(1 - \frac{j}{k+1}\right) \cdot w_j \cdot \frac{j}{k} \right] \cdot p_{k+1,j}(x).$$

Here we put $w_{-1} := w_0$ and $w_{k+1} := w_k$ to be formally correct.
T_2 can be written as

$$\begin{aligned} T_2 &= \sum_{j=0}^{k} w_j \binom{k}{j} \cdot x^{j+1}(1-x)^{k-j} \\ &= \sum_{j=0}^{k} w_j \frac{j+1}{k+1} \binom{k+1}{j+1} \cdot x^{j+1}(1-x)^{k-j} \\ &= \sum_{j=1}^{k+1} w_{j-1} \frac{j}{k+1} \binom{k+1}{j} \cdot x^{j}(1-x)^{k+1-j} \\ &= \sum_{j=0}^{k+1} w_{j-1} \frac{j}{k+1} \binom{k+1}{j} \cdot x^{j}(1-x)^{k+1-j} \\ &= \sum_{j=0}^{k+1} w_{j-1} \frac{j}{k+1} \cdot p_{k+1,j}(x). \end{aligned}$$

Combining the representations of T_1 and T_2 we obtain

$$R_{1,k}(e_1;x) - x$$

$$= \frac{1}{N} \cdot \sum_{j=0}^{k+1} \left[\left(\frac{j-1}{k} - 1 \right) w_{j-1} \frac{j}{k+1} + \left(1 - \frac{j}{k+1} \right) w_j \frac{j}{k} \right] \cdot p_{k+1,j}(x)$$

$$= \frac{1}{N} \cdot \sum_{j=0}^{k+1} \frac{j}{k+1} \cdot \frac{k+1-j}{k} (w_j - w_{j-1}) \cdot p_{k+1,j}(x)$$

$$= x(1-x) \cdot \frac{1}{N} \cdot \sum_{j=0}^{k-1} (w_{j+1} - w_j) \cdot p_{k-1,j}(x).$$

Hence $R_{1,k}(e_1;x) - x = 0$ for all $x \in [0,1]$ if and only if $w_{j+1} - w_j = 0$ for $0 \leq j \leq k-1$, i.e., $w_0 = w_1 = \ldots = w_k$. □

For the sake of completeness we mention that for rational Bernstein-Bézier <u>curves</u> the situation is somewhat different; see [45].

2.1.4 Degree of approximation by $R_{\Delta_n,k}$ (and some of its particular cases) in terms of $\tilde{\omega}_1$

The approximation theoretical knowledge about the spline methods mentioned is in contrast to their importance in applications and to the many experimental results available. Therefore, in the present section we start to discuss rational B-spline functions from the viewpoint of quantitative approximation theory. To that end we use Theorem 1.36.

Proposition 2.13 *Let $R_{\Delta_n,k}$ be given as above. Define*

$$w_{\Delta_n,k}^{\min} := \min\{w_{j,k} : -k \leq j \leq n-1\} > 0,$$

$$w_{\Delta_n,k}^{\max} := \max\{w_{j,k} : -k \leq j \leq n-1\} > 0,$$

and the "weight ratio" by

$$\rho_{\Delta_n,k} := \frac{w_{\Delta_n,k}^{\max}}{w_{\Delta_n,k}^{\min}} \geq 1.$$

Then for $f \in C[0,1]$ there holds

$$\|R_{\Delta_n,k} f - f\| \leq \rho_{\Delta_n,k} \cdot \tilde{\omega}_1 \left(f; \sqrt{\min\left\{ \frac{1}{2k}, \frac{(k+1) \cdot \|\Delta_n\|^2}{12} \right\}} \right). \quad (2.16)$$

Proof. All that remains is to give an estimate for $R_{\Delta_n,k}(|e_1 - x|;x)$. We have

$$R_{\Delta_n,k}(|e_1 - x|;x) = \frac{\sum_{j=-k}^{n-1} w_{j,k} \cdot |\xi_{j,k} - x| \cdot N_{j,k}(x)}{\sum_{j=-k}^{n-1} w_{j,k} \cdot N_{j,k}(x)}$$

$$
\begin{aligned}
&\le \frac{\sum_{j=-k}^{n-1} w_{\Delta_n,k}^{\max} \cdot |\xi_{j,k}-x| \cdot N_{j,k}(x)}{\sum_{j=-k}^{n-1} w_{\Delta_n,k}^{\min} \cdot N_{j,k}(x)} \\
&= \rho_{\Delta_n,k} \cdot \sum_{j=-k}^{n-1} |\xi_{j,k}-x| \cdot N_{j,k}(x) \\
&= \rho_{\Delta_n,k} \cdot S_{\Delta_n,k}(|e_1-x|;x) \\
&\le \rho_{\Delta_n,k} \cdot \sqrt{S_{\Delta_n,k}((e_1-x)^2;x)}.
\end{aligned}
$$

For the latter quantity Marsden [101] proved the uniform estimate

$$
S_{\Delta_n,k}((e_1-x)^2;x) \le \min\left\{\frac{1}{2k},\ \frac{(k+1)\cdot\|\Delta_n\|^2}{12})\right\},\ 0 \le x \le 1. \tag{2.17}
$$

Hence we conclude from Theorem 1.36 that

$$
\begin{aligned}
&|R_{\Delta_n,k}(f;x)-f(x)| \\
\le\ & \max\left\{1,\ \frac{1}{h}\cdot\rho_{\Delta_n,k}\cdot\sqrt{\min\left\{\frac{1}{2k},\ \frac{(k+1)\cdot\|\Delta_n\|^2}{12}\right\}}\right\}\cdot\tilde{\omega}_1(f;h),\ \forall h>0
\end{aligned}
$$

and putting $h = \sqrt{\min\left\{\frac{1}{2k},\ \frac{(k+1)\cdot\|\Delta_n\|^2}{12}\right\}}$ leads to

$$
|R_{\Delta_n,k}(f;x)-f(x)| \le \rho_{\Delta_n,k}\cdot\tilde{\omega}_1\left(f;\sqrt{\min\left\{\frac{1}{2k},\ \frac{(k+1)\cdot\|\Delta_n\|^2}{12}\right\}}\right).
$$

The right hand side is independent of x, and thus we arrive at our claim. □

Corollary 2.14 *(i) If all weights equal $w > 0$, then for Schoenberg's variation-diminishing spline operator we get*

$$
\|S_{\Delta_n,k}f - f\| \le \tilde{\omega}_1\left(f;\sqrt{\min\left\{\frac{1}{2k},\ \frac{(k+1)\cdot\|\Delta_n\|^2}{12}\right\}}\right).
$$

Similar inequalities were given by Marsden in [101].

(ii) For the Bernstein operators the above reduces to

$$
\|B_k f - f\| \le \tilde{\omega}_1\left(f;\frac{1}{\sqrt{2k}}\right).
$$

This is also a classical inequality similar to the one given by T. Popoviciu [123].

Further, we consider URBS - uniform rational B-splines. In this case much better information is available than in the general case.
Thus we can state

Theorem 2.15

$$|R_{n,k}(f;x) - f(x)| \leq \rho_{\Delta_n,k} \cdot \tilde{\omega}_1 \left(f; \sqrt{\frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}} \right). \tag{2.18}$$

Proof. One has to proceed as in the general case and to take into consideration the available pointwise refinement of (2.17). Thus, according to [16, Theorem 2] in the case of $S_{n,k}$ for $n \geq 1,\ k \geq 1,\ x \in [0,1]$ we have

$$S_{n,k}((e_1 - x)^2; x) \leq 1 \cdot \frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}. \tag{2.19}$$

□

Remark 2.16 It was noted in [17, Remark 3.6] that the upper bound of (2.19) matches Marsden's uniform order in all cases $k, n \geq 1$ and is hence a pointwise refinement.

A special case of URBS functions is given by rational Bernstein functions $R_{1,k}f$, defined at (2.6). In this case we have

$$|R_{1,k}(f;x) - f(x)| \leq \rho_{\Delta_1,k} \cdot \tilde{\omega}_1 \left(f; \sqrt{\frac{2x(1-x)}{k}} \right). \tag{2.20}$$

The best constant is obtained if the "weight ratio" $\rho_{\Delta_1,k} = 1$, that is, if all weights are equal. This is the case of the polynomial Bernstein operator.

2.1.5 Degree of approximation by some particular cases of $R_{\Delta_n,k}$ in terms of ω_1 and ω_2

For some special cases of $R_{\Delta_n,k}$, e.g., $R_{n,1}$ $(k = 1)$ and the rational Bernstein functions $R_{1,k}(f;\cdot)$ inequalities (2.16), (2.18) or (2.20) from the previous section can be brought into a more adequate form involving first and second order moduli.
In [152] G. Tachev proved by involving relation (2.7) and by applying Theorem 1.38 the following:

Proposition 2.17 *For $f \in C[0,1],\ 0 < h \leq \frac{1}{2}$, there holds*

$$\begin{aligned} |R_{n,1}(f;x) - f(x)| &\leq \frac{1}{N}\left|\sum_{j=-1}^{n-1}(w_j - w_{j+1})N_{j,1}(x)\cdot N_{j+1,1}(x)\right|\cdot\frac{1}{nh}\cdot\omega_1(f;h) \\ &+ \left(1 + \frac{1}{2h^2}\rho_{n,1}\cdot\frac{\min\{2x(1-x),\frac{1}{n}\}}{n}\right)\cdot\omega_2(f;h), \end{aligned} \tag{2.21}$$

with $N := \sum_{j=-1}^{n-1} w_{j,1}\cdot N_{j,1}(x)$.

Example 2.18 *Choosing $w_{j,1} = 1 + \frac{c(j+1)}{n},\ -1 \leq j \leq n,\ c \geq 0,\ \alpha > 0$ and letting $h := \frac{1}{n}$ in (2.21) the author proved in [152] that*

$$|R_{n,1}(f;x) - f(x)| \leq \frac{c(n+1)}{4n^\alpha}\cdot\omega_1\left(f;\frac{1}{n}\right) + \left(1 + \frac{\rho_{n,1}}{2}\right)\cdot\omega_2\left(f;\frac{1}{n}\right).$$

It can be easily seen that for $\alpha > 0$ sufficiently large $\rho_{n,1} \approx 1$.

If all weights equal $w > 0$ we arrive at $S_{n,1}$, the piecewise linear interpolator on equidistant knots. In this case due to A. Lupaş we have a beautiful representation of the second moments, namely

$$S_{n,1}((e_1 - x)^2;x) = \frac{(nx - [nx])(1 + [nx] - nx)}{n^2}, \tag{2.22}$$

where $[a]$ means the integer part of $a \in \mathbb{R}$. The formula can be found in [96, p. 46]. Thus, we can state

Corollary 2.19 *For any $f \in C[0,1],\ x \in [0,1]$ and $n \geq 1$ the estimate*

$$|S_{n,1}(f;x) - f(x)| \leq \frac{3}{2}\cdot\omega_2\left(f, \frac{\sqrt{(nx - [nx])(1 + [nx] - nx)}}{n}\right)$$

holds.

It is also interesting to discuss the rational Bernstein case. The upper bound of the second moments can be computed as follows

$$\begin{aligned} R_{1,k}((e_1 - x)^2;x) &= \frac{1}{N}\cdot\sum_{j=0}^{k} w_j\cdot\left(\frac{j}{k} - x\right)^2\cdot p_{k,j}(x) \\ &\leq \rho_{1,k}\cdot B_k((e_1 - x)^2;x) \\ &= \rho_{1,k}\cdot\frac{x(1-x)}{k}. \end{aligned}$$

Thus we can apply Theorem 1.38 to arrive at

Proposition 2.20 *For $f \in C[0,1],\ x \in [0,1],\ 0 < h \leq \frac{1}{2}$, there holds*

$$\begin{aligned} |R_{1,k}(f;x) - f(x)| &\leq \frac{x(1-x)}{N} \cdot |\sum_{j=0}^{k-1} (w_{j+1} - w_j) \cdot p_{k-1,j}(x)| \cdot h^{-1} \cdot \omega_1(f;h) \\ &\quad + \left(1 + \frac{1}{2} \cdot h^{-2} \cdot \rho_{1,k} \cdot \frac{x(1-x)}{k}\right) \cdot \omega_2(f;h). \end{aligned}$$

In particular, for $h = \sqrt{\frac{x(1-x)}{k}}$, this implies

$$\begin{aligned} |R_{1,k}(f;x) - f(x)| &\leq \frac{\sqrt{k}}{N} \cdot \sqrt{x(1-x)} \cdot |\sum_{j=0}^{k-1} (w_{j+1} - w_j) \cdot p_{k-1,j}(x)| \\ &\quad \cdot \omega_1\left(f; \sqrt{\frac{x(1-x)}{k}}\right) \\ &\quad + \left(1 + \frac{1}{2} \cdot \rho_{1,k}\right) \cdot \omega_2\left(f; \sqrt{\frac{x(1-x)}{k}}\right) \\ &\leq \sqrt{k} \cdot \sqrt{x(1-x)} \cdot \frac{\max\{|w_{j+1} - w_j|\}}{\min\{w_j\}} \cdot \omega_1\left(f; \sqrt{\frac{x(1-x)}{k}}\right) \\ &\quad + \left(1 + \frac{1}{2} \cdot \rho_{1,k}\right) \cdot \omega_2\left(f; \sqrt{\frac{x(1-x)}{k}}\right). \end{aligned}$$

Corollary 2.21 *If*

$$\frac{\max\{|w_{j+1} - w_j|\}}{\min\{w_j\}} \leq c \cdot \frac{1}{k},\ c \geq 0,\ k = 1, 2, \ldots,$$

then

$$\begin{aligned} |R_{1,k}(f;x) - f(x)| &\leq c \cdot \sqrt{\frac{x(1-x)}{k}} \cdot \omega_1\left(f; \sqrt{\frac{x(1-x)}{k}}\right) \\ &\quad + \left(1 + \frac{1}{2} \cdot \rho_{1,k}\right) \cdot \omega_2\left(f; \sqrt{\frac{x(1-x)}{k}}\right). \end{aligned}$$

Example 2.22 *(i) If, with $c \geq 0,\ w_{j,k} := w_j = 1 + \frac{c \cdot j}{k},\ 0 \leq j \leq k$, then – with the same c – the assumptions of the corollary are satisfied. Moreover, $\rho_{1,k} = c + 1$.*

(ii) In the Bernstein polynomial case we have $c = 0$, so $\rho_{1,k} = 1$. Hence here the latter estimate reads

$$|B_k(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2\left(f; \sqrt{\frac{x(1-x)}{k}}\right).$$

Remark 2.23 The constant in front of ω_2 can be replaced by $\frac{11}{8} = 1,375$, see [113, Corollary 4.1.2].

Further in the sequel we give some examples illustrating the impact of the weights on the behavior of rational Bernstein functions.

Example 2.24 *Here we show that with inappropriate choices of the weights not even for the function $e_1(x) = x$ uniform convergence can be expected.*
Indeed, for $0 < w = w_j,\ 0 \le j \le k-1$, and $w_k > w$ to be determined later we have

$$\begin{aligned}
R_{1,k}(e_1;x) - x &= \frac{1}{\sum\limits_{j=0}^{k} w_j \cdot p_{k,j}(x)} \cdot x(1-x) \cdot \sum_{j=0}^{k-1} (w_{j+1} - w_j) p_{k-1,j}(x) \\
&= \frac{1}{w \cdot \sum\limits_{j=0}^{k-1} p_{k,j}(x) + w_k \cdot p_{k,k}(x)} \cdot x(1-x)(w_k - w) p_{k-1,k-1}(x) \\
&= \frac{1}{w(1 - p_{k,k}(x)) + w_k \cdot p_{k,k}(x)} \cdot x(1-x)(w_k - w) p_{k-1,k-1}(x) \\
&= \frac{1}{w + (w_k - w) \cdot p_{k,k}(x)} \cdot x(1-x) \cdot (w_k - w) \cdot p_{k-1,k-1}(x) \\
&= x(1-x) \cdot \frac{(w_k - w) \cdot p_{k-1,k-1}(x)}{w + (w_k - w) \cdot p_{k,k}(x)}.
\end{aligned}$$

Hence

$$\begin{aligned}
R_{1,k}(e_1;\frac{1}{2}) - \frac{1}{2} &= \frac{1}{4} \cdot \frac{(w_k - w) \cdot (\frac{1}{2})^{k-1}}{w + (w_k - w) \cdot (\frac{1}{2})^{k}} \\
&\ge \frac{1}{4} \cdot \frac{(w_k - w) \cdot (\frac{1}{2})^{k-1} + w - w}{w + (w_k - w) \cdot (\frac{1}{2})^{k-1}} \\
&= \frac{1}{4} \cdot \left(1 - \frac{w}{w + (w_k - w) \cdot (\frac{1}{2})^{k-1}}\right).
\end{aligned}$$

Now choose w_k such that $w_k - w = 2^{k-1}$ and arrive at

$$R_{1,k}(e_1;\frac{1}{2}) - \frac{1}{2} \ \ge\ \frac{1}{4} \cdot \left(1 - \frac{w}{w+1}\right) = \frac{1}{4} \cdot \frac{1}{w+1} \neq 0,\ \forall k.$$

Thus

$$R_{1,k}(e_1;\frac{1}{2}) \not\to \frac{1}{2} \ \text{ for } k \to \infty. \qquad \square$$

Example 2.25 *While the last example showed that a "wrong" choice of the weights can lead to divergence, the next illustration indicates that the approximation might be better if the weights are adjusted to the function.*

This can be expected from the trivial relationship

$$\inf\{\|R_{1,k}f - f\|_\infty : (w_0, \dots, w_k) \in \mathbb{R}_+^{k+1}\} \le \|B_k f - f\|_\infty.$$

Here we consider the function

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} \le x \le 1. \end{cases}$$

Hence

$$B_2(f; x) = 2x(1-x),$$

and with $w_0 = w_2 = 1$ *and* $w_1 > 0$ *we have*

$$R_{1,2}(f; x) = \frac{2w_1 \cdot x(1-x)}{(1-x)^2 + 2w_1 \cdot x(1-x) + x^2}.$$

It can be seen by inspection that the approximation of f *is better for* $w_1 = 2$ *or* $w_1 = 3$ *(for example), than it is for* $w_1 = 1$ *(the Bernstein polynomial of* f*).*

Example 2.26 *In the last example we illustrated the fact that choosing non–equal weights can lead to better approximations. Here we show that these can be even best possible.*
We look again at $R_{1,2}$ *associated to the weight sequence* $(w_0, w_1, w_2) = (1, w_1, 1)$ *and consider the function*

$$g_{w_1}(x) = \frac{w_1 \cdot x(1-x) + x^2}{(1-x)^2 + 2w_1 \cdot x(1-x) + x^2}.$$

For $(w_0, w_1, w_2) = (1, 1, 1)$ *we have* $g_1(x) = x$*, so in this case* $R_{1,2}(g_1, x) = g_1(x) = x$*.*
But even for all $w_1 > 0$ *it is true that* $g_{w_1}(0) = 0,\ g_{w_1}(\frac{1}{2}) = \frac{1}{2},\ g_{w_1}(1) = 1,$ *so that also in this case* $R_{1,2}(g_{w_1}, x) = g_{w_1}(x)$*.*
Hence with $(w_0, w_1, w_2) = (1, w_1, 1)$*,* $R_{1,2}$ *has* e_0 *and* g_{w_1} *as eigenfunctions with respect to the eigenvalue* 1*.*

2.2 Some modified rational Bernstein operator

In the latter section we have seen that one main drawback of the *rational Bernstein* is that they do not reproduce linear functions. We shall try in this section to avoid this disadvantage by constructing a specific class of rational Bernstein operators. For more information one can read [121].

2.2.1 Definition and some (shape-preservation) properties

Definition 2.27 *For $f \in C[0,1]$ and $x \in [0,1]$ we define*

$$\bar{R}_n(f;x) := \frac{\sum_{i=0}^{n} \bar{w}_i f(\bar{x}_i) p_{n,i}(x)}{\sum_{j=0}^{n-1} w_j p_{n-1,j}(x)} = \frac{P_n f(x)}{Q_{n-1}(x)}, \tag{2.23}$$

where the weights and the abscissae $\bar{w}_i,\ \bar{x}_i,\ i = 0,\dots,n$ will be determined as functions of $w_j,\ j = 0,\dots,n-1$ that we assume to be strictly positive. $p_{n,i}$ is the Bernstein basis, see (1.1).

Theorem 2.28 *The operator $\bar{R}_n$ reproduces constant functions if and only if the weights $\bar{w}_i,\ 0 \le i \le n$, are defined by*

$$\begin{aligned} \bar{w}_0 &= w_0, \quad \bar{w}_n = w_{n-1} \text{ and} \\ \bar{w}_i &= \frac{i}{n} w_{i-1} + \left(1 - \frac{i}{n}\right) w_i,\ j = 1,\dots,n-1. \end{aligned} \tag{2.24}$$

Proof. In order that $\bar{R}_n e_0 = e_0$, i.e., that $\bar{R}_n$ be exact on constants, we must have:

$$\sum_{i=0}^{n} \bar{w}_i p_{n,i}(x) = \sum_{j=0}^{n-1} w_j p_{n-1,j}(x).$$

As we have, respectively,

$$(1-x)p_{n-1,j}(x) = \left(1 - \frac{j}{n}\right) p_{n,j}(x) \text{ and } x p_{n-1,j}(x) = \frac{j+1}{n} p_{n,j+1}(x),$$

we increase by 1 the degree of the right hand side polynomial by multiplying it by $x + (1-x) = 1$ (degree elevation) to get

$$\sum_{j=0}^{n-1} w_j[(1-x)+x] \cdot p_{n-1,j}(x) = \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) w_j p_{n,j}(x) + \sum_{j=0}^{n-1} \frac{j+1}{n} w_j p_{n,j+1}(x).$$

Now, the denominator can be written as

$$w_0 p_{n,0}(x) + \sum_{j=1}^{n-1} \left[\frac{j}{n} w_{j-1} + \left(1 - \frac{j}{n}\right) w_j\right] p_{n,j}(x) + w_{n-1} p_{n,n}(x)$$

and by equating with the numerator, we obtain the desired result. □

Remark 2.29 We emphasize the fact that since the weights w_j are strictly positive, the weights $\bar{w}_i$ are also strictly positive, whence the *positivity* of the operator $\bar{R}_n$. It has to be noticed that $\bar{R}_n f$ is a particular case of the classical rational Bernstein approximant (introduced in the latter section) since, by degree raising in the denominator, it can be written as

$$\bar{R}_n f = \frac{\sum_{i=0}^{n} \bar{w}_i f(\bar{x}_i) p_{n,i}}{\sum_{i=0}^{n} \bar{w}_i p_{n,i}}, \tag{2.25}$$

with the specific choice of weights given above. However, the choice of abscissae of control points is also fundamental for the linear preservation:

Theorem 2.30 *The operator $\bar{R}_n$ reproduces linear functions if and only if the abscissas $\bar{x}_i$ of the numerator $P_n f$ are defined by $\bar{x}_0 = 0,\ \bar{x}_n = 1$ and*

$$\bar{x}_i = \frac{i}{n} \cdot \frac{w_{i-1}}{\bar{w}_i} = \frac{i w_{i-1}}{i w_{i-1} + (n-i) w_i},\ 1 \le i \le n-1. \tag{2.26}$$

Proof. In order that $\bar{R}_n e_1 = e_1$, i.e., that $\bar{R}_n$ be exact on linear functions, we must have

$$\sum_{i=0}^{n} \bar{w}_i \bar{x}_i p_{n,i}(x) = x \sum_{j=0}^{n-1} w_j p_{n-1,j}(x) = \sum_{j=1}^{n} w_{j-1} \frac{j}{n} p_{n,j}(x),$$

from which we deduce the desired result. □

As we want the sequence $\bar{x}_i$ to be increasing we must have:

Property 2.31

$$\frac{w_{i-1} w_{i+1}}{w_i^2} < \left(1 + \frac{1}{i}\right)\left(\frac{n-i}{n-i-1}\right),\ 1 \le i \le n-2.$$

Proof. From $\frac{i w_{i-1}}{i w_{i-1} + (n-i) w_i} < \frac{(i+1) w_i}{(i+1) w_i + (n-i-1) w_{i+1}}$, $0 \le i \le n-1$, we easily arrive after simplification at

$$i(n-i-1) w_{i-1} w_{i+1} < (i+1)(n-i) w_i^2,\ 1 \le i \le n-2 \qquad .\square$$

Remark 2.32 From now on we assume that the positive weights satisfy the above property.

Given the operator $\bar{R}_n f = \frac{P_n f}{Q_{n-1}}$ as in (2.23), where the weights w_j, $\bar{w}_i$ and the abscissas $\bar{x}_i$ are determined as above, we collect some of its basic properties:

Proposition 2.33 *(i) $\bar{R}_n$ is a positive linear operator reproducing constant and linear functions.*

(ii) Both the numerator and the denominator are polynomials of degree n and $n-1$, respectively.

(iii) $\bar{R}_n$ interpolates f at the endpoints and has the convex hull property.

(iv) $\bar{R}_n$ is discretely defined, i.e., it depends on the n positive values w_j, $j = 0, \ldots, n-1$.

By denoting

$$\rho_{i,n}(x) := \frac{\bar{w}_i \cdot p_{n,i}(x)}{Q_{n-1}}, \; 0 \leq i \leq n, \tag{2.27}$$

one can easily prove in analogy with Proposition 2.4 and with the help of Example 1.41 the following:

Proposition 2.34 *For any fixed $n \geq 1$ the kernel*

$$[0,1] \times \{0, \ldots, n\} \ni (x, i) \mapsto \rho_{i,n}(x) \in \mathbb{R}$$

is totally positive. Hence, the operator $\bar{R}_n$ posseses the variation-diminishing property.

Hence,

Corollary 2.35 *The operator $\bar{R}_n$ retains positivity, monotonicity and convexity of a function $f \in C[0,1]$. Moreover, when f is convex, we have $\bar{R}_n(f;x) \geq f(x)$, $x \in [0,1]$.*

Proof. It was already shown above that $\bar{R}_n$ is a positive operator. The preservation of monotonicity and convexity follows from Theorem 1.46. If f is convex, then by the *inequality of Jensen* and due to the already proven identities

$$\sum_{i=0}^{n} \rho_{i,n}(x) = 1, \quad \sum_{i=0}^{n} \bar{x}_i \rho_{i,n}(x) = x,$$

we have:

$$\bar{R}_n(f;x) = \sum_{i=0}^{n} f(\bar{x}_i) \cdot \rho_{n,i}(x) \geq f\left(\sum_{i=0}^{n} \bar{x}_i \cdot \rho_{n,i}(x)\right) = f(x). \qquad \square$$

Theorem 2.36 *For any function f with bounded total variation on $[0,1]$ the operator $\bar{R}_n$ possesses the total variation diminishing property.*

Proof. In analogy to the proof of Lemma 2.10 one can find the following representation of the derivative $(\bar{R}_n f)'$, namely

$$(\bar{R}_n f)'(x) = \sum_{i=0}^{n-1} \sigma_{i,n}(x)(f(\bar{x}_{i+1}) - f(\bar{x}_i)), \tag{2.28}$$

where the functions $\sigma_{i,n}, 0 \leq i \leq n-1$, are defined by

$$\sigma_{i,n}(x) = \frac{(n!)^2}{(2n-2)!} \frac{1}{Q_{n-1}^2(x)} \sum_{p=i+1}^{i+n} \omega_{i,p} p_{2n-2,p-1}(x) \tag{2.29}$$

with the coefficients

$$\omega_{i,p} := \sum_{(j,l)\in K(i,p)} \frac{(l-j)}{p(2n-p)} \binom{p}{j} \binom{2n-p}{n-j} \bar{w}_{j,n} \bar{w}_{l,n}, \tag{2.30}$$

and the set of indices

$$K(i,p) := \{(j,l) : \, j + l = p, \, 0 \leq j \leq i, \, i+1 \leq l \leq n\}.$$

Comparing the two following expressions of $(\bar{R}_n f)'$ we can write

$$(\bar{R}_n f)'(x) = \sum_{i=0}^{n} \rho_i'(x) f(\bar{x}_i) = \sum_{j=0}^{n-1} \sigma_j(x)(f(\bar{x}_{j+1}) - f(\bar{x}_j)),$$

where we omitted again the double indexing.
Using the fact that $\sum_{i=0}^{n} \rho_i = 1$ and therefore, $\sum_{i=0}^{n} \rho_i' = 0$, we immediately deduce

$$\sigma_j = -\sum_{i=0}^{j} \rho_i', \quad \text{for} \quad 0 \leq j \leq n-1.$$

Moreover, from the expressions of the rational basis functions ρ_i, we know that $\rho_i(0) = \rho_i(1) = 0$ for $1 \leq j \leq n-1$. In addition, $\rho_0(0) = \rho_n(1) = 1$ and $\rho_0(1) = \rho_n(0) = 0$. Hence we obtain

$$\int_0^1 \sigma_j = \sum_{i=0}^{j} (\rho_i(0) - \rho_i(1)) = \rho_0(0) = 1, \quad \text{for} \quad 0 \leq j \leq n-1.$$

Now, since the functions σ_j are positive,

$$\begin{aligned} TV((\bar{R}_n f)') = \int_0^1 |(\bar{R}_n f)'(x)| dx \;&\leq\; \sum_{j=0}^{n-1} |f(\bar{x}_{j+1}) - f(\bar{x}_j)| \int_0^1 \sigma_j(x) dx \\ &=\; \sum_{j=0}^{n-1} |f(\bar{x}_{j+1}) - f(\bar{x}_j)| \leq TV(f). \qquad (2.31) \end{aligned}$$

□

The following is obvious from (2.28).

Property 2.37 *The first derivatives at the endpoints are exactly the slopes of the control polygon at those points. More exactly,*

$$(\bar{R}_n f)'(0) = \frac{f(\bar{x}_1) - f(0)}{\bar{x}_1} \text{ and } (\bar{R}_n f)'(1) = \frac{f(1) - f(\bar{x}_{n-1})}{1 - \bar{x}_{n-1}}.$$

2.2.2 Convergence of $\bar{R}_n$ for a specific class of denominators

Let us assume that there exists a fixed strictly positive continuous function φ defined on $[-2, 2]$ such that:

$$w_i^{(n)} = \varphi\left(\frac{i}{n-1}\right), \quad 0 \le i \le n-1.$$

(we add an upper index n to the weight because it depends on n). The question then arises: for which functions φ does Property 2.31 hold? In that case, the corresponding inequality can be written

$$i(n-i-1)\varphi\left(\frac{i-1}{n-1}\right)\varphi\left(\frac{i+1}{n-1}\right) < (i+1)(n-i)\varphi^2\left(\frac{i}{n-1}\right).$$

Setting $h = \frac{1}{n-1}$ and $x = ih$, we obtain

$$x(1-x)\varphi(x-h)\varphi(x+h) < (x+h)(1-x+h)\varphi(x)^2.$$

Using Taylor's expansions we have

$$\varphi(x+h) = \varphi(x) + h\varphi'(x) + \frac{h^2}{2}\varphi''(x) + O(h^3),$$

$$\varphi(x-h) = \varphi(x) - h\varphi'(x) + \frac{h^2}{2}\varphi''(x) + O(h^3),$$

we obtain

$$x(1-x)\left[\varphi(x)^2 + h^2(\varphi(x)\varphi''(x) - \varphi'(x)^2) + O(h^3)\right] < (x+h)(1-x+h)\varphi(x)^2,$$

or equivalently, with $a_1(x) := \varphi(x)^2 + x(1-x)\left(\varphi(x)\varphi''(x) - \varphi'(x)^2\right)$:

$$0 < \varphi(x)^2 + ha_1(x) + O(h^2).$$

Therefore, Property 2.31 is satisfied for any twice differentiable function φ, provided h is sufficiently small, i.e., n is large enough.
Given $f \in C[0,1]$, we now study the uniform convergence of $\bar{R}_n f$ to f when $n \to \infty$. The denominator of $\bar{R}_n$ can be written as $B_{n-1}\varphi$, so that $\lim_{n\to\infty} B_{n-1}\varphi = \varphi$. In that case we have the following result:

Theorem 2.38 *Let be given a positive continuous function φ defining $Q_{n-1} = B_{n-1}\varphi$. Then, for any $f \in C[0,1]$, the sequence of rational approximants $\bar{R}_n f$ converges uniformly to f when $n \to \infty$.*

Proof. We give a direct proof without using Bohman-Korovkin's Theorem 1.6. Setting $\varphi_n(x) = \varphi\left(\frac{nx}{n-1}\right)$, $x \in [-1,1]$,

$$\begin{aligned} \bar{\varphi}_n(x) &= x\varphi_n(x-\frac{1}{n}) + (1-x)\varphi_n(x) \\ &= x\varphi\left(\frac{nx-1}{n-1}\right) + (1-x)\varphi\left(\frac{nx}{n-1}\right), \end{aligned} \tag{2.32}$$

$$\begin{aligned} \theta_n(x) &= \frac{x\varphi_n(x-\frac{1}{n})}{x\varphi_n(x-\frac{1}{n}) + (1-x)\varphi_n(x)} \\ &= \frac{x\varphi_n(x-\frac{1}{n})}{\bar{\varphi}_n(x)}, \end{aligned} \tag{2.33}$$

with $x \in [0,1]$. Thus, we have respectively

$$\bar{w}_i^{(n)} = \frac{i}{n}w_{i-1}^{(n)} + (1-\frac{i}{n})w_i^{(n)} = \frac{i}{n}\varphi_n\left(\frac{i-1}{n}\right) + (1-\frac{i}{n})\varphi_n\left(\frac{i}{n}\right) = \bar{\varphi}_n\left(\frac{i}{n}\right),$$

and

$$\bar{x}_i = \frac{i}{n}\frac{w_{i-1}^{(n)}}{\bar{w}_i^{(n)}} = \frac{i}{n}\frac{\varphi_n\left(\frac{i-1}{n}\right)}{\frac{i}{n}\varphi_n\left(\frac{i-1}{n}\right) + (1-\frac{i}{n})\varphi_n\left(\frac{i}{n}\right)} = \theta_n\left(\frac{i}{n}\right).$$

Then the numerator of $\bar{R}_n$ can be written as

$$\sum_{i=0}^{n}\left(\frac{i}{n}\varphi_n\left(\frac{i-1}{n}\right) + \left(1-\frac{i}{n}\right)\varphi_n\left(\frac{i}{n}\right)\right) f\left(\frac{i}{n}\cdot\frac{\varphi_n\left(\frac{i-1}{n}\right)}{\frac{i}{n}\varphi_n\left(\frac{i-1}{n}\right) + \left(1-\frac{i}{n}\right)\varphi_n\left(\frac{i}{n}\right)}\right) p_{n,i}(x),$$

which is equal to $B_n(\psi_n; x)$, where

$$\psi_n(x) = \bar{\varphi}_n(x) f\left(\theta_n(x)\right), \text{ with } x \in [0,1].$$

When $n \to +\infty$, φ_n and $\bar{\varphi}_n$ both converge uniformly to φ and θ_n converges uniformly to x, thus $\psi_n(x)$ converges uniformly to $\varphi(x)f(x)$, so $B_n\psi_n(x)$ also converges to $\varphi(x)f(x) =: \psi(x)$. It becomes obvious from

$$\begin{aligned} |B_n(\psi_n; x) - \psi(x)| &\leq |B_n(\psi_n; x) - B_n(\psi; x)| + |B_n(\psi; x) - \psi(x)| \\ &\leq \|\psi_n - \psi\|_\infty + \|B_n\psi - \psi\|_\infty. \end{aligned}$$

As the denominator converges to $\varphi(x)$, we see that $\bar{R}_n f \to f,\ f \in C[0,1]$, as $n \to \infty$. □

2.2.3 Degree of approximation

In this subsection we shall give two error estimates, the first for $f \in C[0,1]$ and the second for $f \in C^2[0,1]$. Let us define the following ratio associated with the continuous function φ:

$$\rho = \rho(\varphi) = \frac{M}{m} = \frac{\max\{\varphi(x),\ x \in [0,1]\}}{\min\{\varphi(x),\ x \in [0,1]\}}.$$

First we shall prove the following lemma:

Lemma 2.39 *The following majoration holds:*

$$\bar{R}_n((e_1 - x)^2; x) \le \rho \left[\frac{1}{16m^2} \omega_1^2 \left(\varphi; \frac{1}{n-1} \right) + \frac{1}{2m} \omega_1 \left(\varphi; \frac{1}{n-1} \right) + \frac{x(1-x)}{n} \right],$$

for any $x \in [0,1]$.

Proof: The function has the following expression:

$$\bar{R}_n((e_1 - x)^2; x) = \frac{\sum_{i=0}^{n} \bar{w}_i (\bar{x}_i - x)^2 p_{n,i}(x)}{\sum_{j=0}^{n-1} w_i p_{n-1,j}(x)}.$$

As $w_j = \varphi\left(\frac{j}{n-1}\right) \ge m$ for all $0 \le j \le n-1$ and $\bar{w}_i = \frac{i}{n} w_{i-1} + (1 - \frac{i}{n}) w_i \le M$ for all $0 \le i \le n$, we deduce

$$\bar{R}_n((e_1 - x)^2; x) \le \rho \sum_{i=0}^{n} (\bar{x}_i - x)^2 p_{n,i}(x) = \rho B_n((\theta_n(t) - x)^2; x), \tag{2.34}$$

where the function θ_n was defined at (2.33).
Therefore we obtain

$$|\theta_n(t) - t| = t(1-t) \frac{|\varphi_n(t - \frac{1}{n}) - \varphi_n(t)|}{t\varphi_n(t - \frac{1}{n}) + (1-t)\varphi_n(t)} \le \frac{1}{m} t(1-t) \left| \varphi\left(\frac{nt-1}{n-1}\right) - \varphi\left(\frac{nt}{n-1}\right) \right|$$

and finally

$$|\theta_n(t) - t| \le \frac{1}{m} t(1-t) \omega_1(\varphi; \frac{1}{n-1}).$$

This implies that

$$B_n((\theta_n(t) - t)^2; x) \le \frac{1}{m^2} \omega_1^2(\varphi; \frac{1}{n-1}) B_n(t^2(1-t)^2; x).$$

As it is known that for any function $g \in C[0,1]$, we have $\|B_n g\|_\infty \leq \|g\|_\infty$, therefore, with $g(t) = t^2(1-t)^2$, we obtain $B_n(t^2(1-t)^2; x) \leq \frac{1}{16}$ and furthermore

$$B_n((\theta_n(t)-t)^2; x) \leq \frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}).$$

With these preparations we are able now to find an upper bound for $\bar{R}_n((e_1 - xe_0)^2; x)$ as follows.

$$\begin{aligned}
& B_n((\theta_n(t)-x)^2; x) = B_n((\theta_n(t)-t+t-x)^2; x) \\
\leq\ & B_n((\theta_n(t)-t)^2; x) + 2|B_n((\theta_n(t)-t)(t-x); x)| + B_n((t-x)^2; x) \\
\leq\ & \frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}) + 2B_n(|\theta_n(t)-t| \cdot |t-x|; x) + \frac{x(1-x)}{n} \\
\leq\ & \frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}) + 2B_n(|\theta_n(t)-t|; x) + \frac{x(1-x)}{n} \\
\leq\ & \frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}) + 2\sqrt{B_n((\theta_n(t)-t)^2; x)} + \frac{x(1-x)}{n} \\
\leq\ & \frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}) + \frac{1}{2m}\omega_1(\varphi; \frac{1}{n-1}) + \frac{x(1-x)}{n}.
\end{aligned}$$

Combining this with (2.34) we arrive at

$$\begin{aligned}
(2.35) \qquad & \bar{R}_n((e_1-x)^2; x) \\
\leq\ & \rho\left[\frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}) + \frac{1}{2m}\omega_1(\varphi; \frac{1}{n-1}) + \frac{x(1-x)}{n}\right].\square
\end{aligned}$$

Remark 2.40 In the result proven in the latter lemma the Bernstein case is also hidden. Indeed by considering the associated weight function φ to be a constant function our rational operators reduces to the classical Bernstein operator, i.e., $\bar{R}_n \equiv B_n$ and obviously $\rho = 1$. In this particular situation inequality (2.35) reads as follows:

$$|B_n((e_1-x)^2; x)| \leq (=)\frac{x(1-x)}{n},$$

a very well known identity.

Taking $h := \sqrt{\frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}) + \frac{1}{2m}\omega_1(\varphi; \frac{1}{n-1}) + \frac{x(1-x)}{n}}$ in the inequalities of Theorems 1.35 and 1.38 and recalling that $\frac{x(1-x)}{n} \leq \frac{1}{4n}$, we obtain the following two error estimates:

Theorem 2.41 *For all $f \in C[0,1]$ and $x \in [0,1]$, there holds*

$$\begin{aligned}
|\bar{R}_n f(x) - f(x)| \ &\leq\ (1+\sqrt{\rho}) \cdot \omega_1\left(f; \sqrt{\frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}) + \frac{1}{2m}\omega_1(\varphi; \frac{1}{n-1}) + \frac{1}{4n}}\right), \\
|\bar{R}_n f(x) - f(x)| \ &\leq\ \left(1+\frac{1}{2}\rho\right) \cdot \omega_2\left(f; \sqrt{\frac{1}{16m^2}\omega_1^2(\varphi; \frac{1}{n-1}) + \frac{1}{2m}\omega_1(\varphi; \frac{1}{n-1}) + \frac{1}{4n}}\right).
\end{aligned}$$

By Peano's kernel theorem (see e.g., [34] or [38]), if $f \in C^2[0,1]$, we have

$$\bar{R}_n f(x) - f(x) = \int_0^1 k_n(x,t) f''(t) dt,$$

where $k_n(x,t) = \bar{R}_n[(\cdot - t)_+](x) - (x-t)_+$ which is positive since the function $(\cdot - t)_+ : x \longrightarrow (x-t)_+$ is convex and $\bar{R}_n$ is shape preserving. Therefore, we obtain

$$\bar{R}_n f(x) - f(x) = f''(\theta) \int_0^1 k_n(x,t) dt = \frac{1}{2} f''(\theta)(\bar{R}_n e_2(x) - e_2(x)).$$

Using the Lemma 2.39, we get the following

Theorem 2.42 *For $f \in C^2[0,1]$ and $\varphi \in C[-1,1]$, there holds:*

$$\|\bar{R}_n f - f\|_\infty \le \frac{\rho}{2} \cdot \|f''\|_\infty \left[\frac{1}{16m^2} \omega_1^2 \left(\varphi; \frac{1}{n-1}\right) + \frac{1}{2m} \omega_1 \left(\varphi; \frac{1}{n-1}\right) + \frac{1}{4n} \right].$$

Moreover, if φ is a C^1 function, we obtain:

$$\|\bar{R}_n f - f\|_\infty \le \frac{\rho}{2} \cdot \|f''\|_\infty \left[\frac{1}{16m^2} \cdot \frac{\|\varphi'\|_\infty^2}{(n-1)^2} + \frac{1}{2m} \cdot \frac{\|\varphi'\|_\infty}{n-1} + \frac{1}{4n} \right].$$

The latter estimate reads as follows: under strong "smoothness" conditions for the function $f \in C^2[0,1]$ and for $\varphi \in C^1[-1,1]$ - the associated weight function - the achieved approximation order is $\mathcal{O}(\frac{1}{n})$.

Remark 2.43 A *qualitative version* of Voronovskaja's theorem for $\bar{R}_n$ ca be found in [121].

2.3 A modification of $S_{\Delta_n,1}$: the BLaC-wavelet operator

G. P. Bonneau introduced (see, e.g., [20]) the so called *BlaC-wavelet operator*, where "BLaC" is derived from "Blending of Linear and Constant", which is a suggestive name as we shall see in the following. First we need some preliminaries.
For the real parameter $0 < \Delta \le 1$ consider the *scaling function* $\varphi_\Delta : \mathbb{R} \to [0,1]$ given by

$$\varphi_\Delta(x) := \begin{cases} \frac{x}{\Delta}, & 0 \le x < \Delta, \\ 1, & \Delta \le x < 1, \\ -\frac{1}{\Delta} \cdot (x - 1 - \Delta), & 1 \le x < 1 + \Delta, \\ 0, & \text{else}. \end{cases} \tag{2.36}$$

Remark 2.44 The two extreme situations are obtained for $\Delta = 1$ and $\Delta \to 0$, when φ_Δ reduces to B-spline functions of first order, the *hat-functions*, and to *piecewise constant* functions, respectively. The gap in between can be smoothly covered by letting Δ be in the interval $(0, 1]$.

Furthermore, for $i = -1, \dots, 2^n - 1,\ n \in \mathbb{N}$, we define by dilation and translation of φ_Δ the following family of (fundamental) functions:

$$\varphi_i^n(x) := \varphi_\Delta(2^n x - i), \quad x \in [0, 1]. \tag{2.37}$$

In Figure 2.4 the functions $\varphi_i^n,\ i = -1, \dots, 2^n - 1$, with a parameter $0 < \Delta < 1$ are depicted. Notice that the support of $\varphi_0^n, \dots, \varphi_{2^n-2}^n$ is fully inside $[0, 1]$, whereas φ_{-1}^n and $\varphi_{2^n-1}^n$ can be viewed as "incomplete".

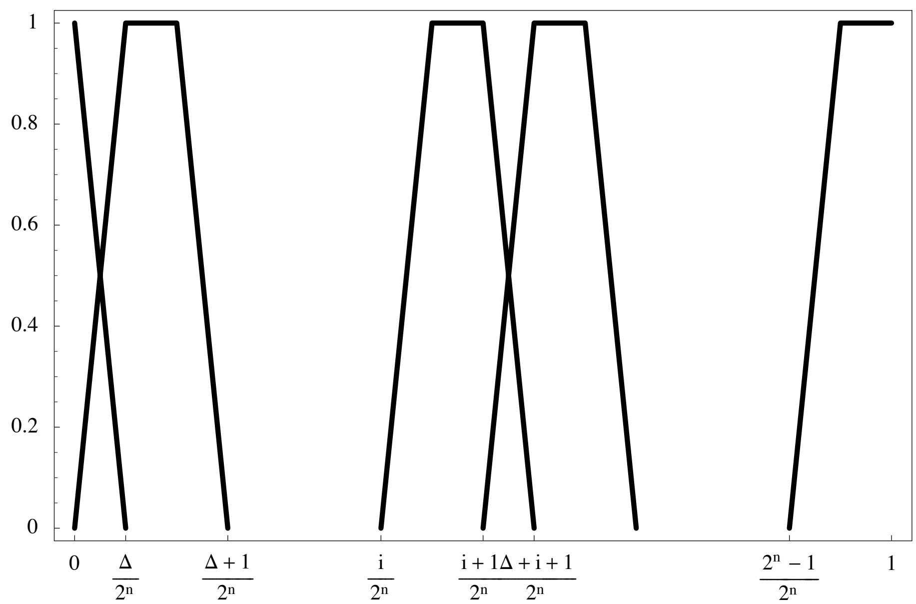

Figure 2.4: Fundamental functions

Also of great relevance are the midpoints η_i^n of the support line of each φ_i^n. Thus, for $i = 0, \dots, 2^n - 2$, we have

$$\eta_i^n := \frac{i}{2^n} + \frac{1}{2} \cdot \frac{1 + \Delta}{2^n},$$

and for $i \in \{-1, 2^n - 1\}$ we set

$$\eta_{-1}^n := 0 \text{ and } \eta_{2^n-1}^n := 1.$$

Equipped with these notations we can introduce the following operator. Concerning this topic the reader is directed to [108] or [63].

For $f \in C[0,1]$ and $x \in [0,1]$ the *BLaC operator* is given by

$$BL_n(f;x) := \sum_{i=-1}^{2^n-1} f(\eta_i^n) \cdot \varphi_i^n(x). \tag{2.38}$$

We first list some elementary facts.

Proposition 2.45

(i) $BL_n : C[0,1] \to C[0,1]$ *is positive and linear;*

(ii) BL_n *interpolates* f *at the points* η_i^n, $i = -1, \dots, 2^n - 1$ *(thus also at the endpoints* 0 *and* 1*);*

(iii) $\sum\limits_{i=-1}^{2^n-1} \varphi_i^n(x) = 1$, *i.e.,* BL_n *reproduces constant functions.*
Hence $\|BL_n\| = 1$.

Proof. (i) This is obvious from the definition and the positivity of φ_i^n.

(ii) One can easily observe that $\varphi_i^n(\eta_j^n) = \delta_{i,j}$ (the Kronecker symbol) for $i, j = -1, \dots, 2^n - 1$. Thus $BL_n(f;\eta_j^n) = f(\eta_j^n) \cdot \varphi_j^n(\eta_j^n) = f(\eta_j^n)$, for $j = -1, \dots, 2^n - 1$.

(iii) For $x = 1$ we have $\sum\limits_{i=-1}^{2^n-1} \varphi_i^n(1) = \varphi_{2^n-1}^n(1) = 1$.
Let $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, $k \in \{0, \dots, 2^n - 1\}$. We discuss separately:
Case 1: For $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$, we have

$$\begin{aligned} \sum_{i=-1}^{2^n-1} \varphi_i^n(x) &= \varphi_{k-1}^n(x) + \varphi_k^n(x) = \varphi_\Delta(2^n x - (k-1)) + \varphi_\Delta(2^n x - k) \\ &= -\frac{1}{\Delta}(2^n x - k - \Delta) + \frac{2^n x - k}{\Delta} = 1. \end{aligned}$$

Case 2: For $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right)$ we get $\sum\limits_{i=-1}^{2^n-1} \varphi_i^n(x) = \varphi_k^n(x) = 1$, due to the definition of φ_Δ.
Hence $\sum\limits_{i=-1}^{2^n-1} \varphi_i^n(x) = 1$ for all $x \in [0,1]$. □

2.3.1 Quantitative estimates

In the present subsection we investigate the degree of approximation by the BLaC-operator BL_n. We establish next two quantities statements, one in terms of ω_1, the second one involving both ω_1 and ω_2.
Thus we have

Proposition 2.46 *For any* $f \in C[0,1] \to C[0,1]$ *and* $x \in [0,1]$ *there holds*

$$|BL_n(f;x) - f(x)| \leq 2 \cdot \omega_1\left(f; \frac{1}{2^n}\right). \tag{2.39}$$

Proof. First we prove that

$$|BL_n(|e_1 - x|;x)| \leq \frac{1}{2^n}, \text{ for all } x \in [0,1].$$

We have $BL_n(|e_1 - x|;x) = \sum_{i=-1}^{2^n-1} |\eta_i^n - x| \cdot \varphi_i^n(x)$. We assume that $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, $k \in \{0,\ldots,2^n-1\}$. This excludes only $x = 1$ in which case we have $BL_n(|e_1 - 1|;1) = 0$.
Case 1: For $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$, we get

$$\begin{aligned} BL_n(|e_1 - x|;x) &= (x - \eta_{k-1}^n) \cdot \varphi_{k-1}^n(x) + (\eta_k^n - x) \cdot \varphi_k^n(x) \\ &= (x - \eta_{k-1}^n) \cdot \varphi_{k-1}^n(x) + (\eta_k^n - x) \cdot (1 - \varphi_{k-1}^n(x)) \\ &\leq \max\{\eta_k^n - x, x - \eta_{k-1}^n\} \leq (\eta_k^n - x + x - \eta_{k-1}^n) = \eta_k^n - \eta_{k-1}^n. \end{aligned}$$

Thus, for $k = 0$ we have $BL_n(|e_1 - x|;x) \leq \eta_0^n - \eta_{-1}^n = \frac{1}{2} \cdot \frac{1+\Delta}{2^n} \leq \frac{1}{2^n}$. For $k > 0$ we get $BL_n(|e_1 - x|;x) \leq \eta_k^n - \eta_{k-1}^n = \frac{k}{2^n} - \frac{k-1}{2^n} = \frac{1}{2^n}$.
Case 2: $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right)$. Then

$$BL_n(|e_1 - x|;x) = |\eta_k^n - x| \cdot \varphi_k^n(x) = |\eta_k^n - x| \leq \frac{1-\Delta}{2^{n+1}} \leq \frac{1}{2^n}.$$

Thus $BL_n(|e_1 - x|;x) \leq \frac{1}{2^n}$, for all $x \in [0,1]$. Applying Corollary 1.37 with $\delta = \frac{1}{2^n}$ yields the estimate (2.39). □

Proposition 2.47 *For any* $f \in C[0,1] \to C[0,1]$, all $x \in [0,1]$ *and* $0 < \delta < \frac{1}{2}$ *the following inequality holds:*

$$|BL_n(f;x) - f(x)| \leq \frac{1-\Delta}{2^{n+1}} \cdot \frac{1}{\delta} \cdot \omega_1(f;\delta) + \left[1 + \frac{1}{2 \cdot \delta^2} \cdot \frac{1}{2^{2n}}\right] \cdot \omega_2(f;\delta). \tag{2.40}$$

Proof. In order to apply Păltănea's Theorem 1.38 we have to find suitable upper bounds for $BL_n(e_1 - x;x)$ and for $BL_n((e_1 - x)^2;x)$. In both cases the approach is the same as for $BL_n(|e_1 - x|;x)$. First note that $BL_n(e_1 - 1;1) = 0$ and $BL_n((e_1 - 1)^2;1) = 0$. We consider again two cases:
Case 1: $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$, $k \in \{0,\ldots,2^n-1\}$.
First we deal with the case $k = 0$. Here we have

$$BL_n(e_1 - x;x) = (\eta_{-1}^n - x) \cdot \varphi_{-1}^n(x) + (\eta_0^n(x) - x) \cdot \varphi_0^n(x)$$

and after some elementary computations we obtain in this case

$$BL_n(e_1 - x; x) = \frac{x(1-\Delta)}{2\Delta} \leq \frac{\Delta}{2^n} \cdot \frac{1-\Delta}{2\Delta} = \frac{1-\Delta}{2^{n+1}}.$$

For $1 \leq k \leq 2^n - 1$ we write successively:

$$\begin{aligned}
BL_n(e_1 - x; x) &= (\eta_{k-1}^n - x) \cdot \varphi_{k-1}^n(x) + (\eta_k^n - x) \cdot \varphi_k^n(x) \\
&= \frac{1}{2^{n+1}} \cdot \frac{1}{\Delta} \big[(2k - 1 + \Delta - 2^{n+1}x)(-2^n x + k + \Delta) \\
&+ (2k + 1 + \Delta - 2^{n+1}x) \cdot (2^n x - k)\big] \\
&= \frac{1}{2^{n+1}} \cdot \frac{1}{\Delta} [(2^n x - k) \cdot (2 - 2\Delta) + \Delta(-1 + \Delta)] \\
&= \frac{1}{2^{n+1}} \cdot \frac{1-\Delta}{\Delta} [2(2^n x - k) - \Delta] \\
&\leq \frac{1}{2^{n+1}} \cdot \frac{1-\Delta}{\Delta} \left[2\left(2^n \cdot \frac{k+\Delta}{2^n} - k\right) - \Delta\right] = \frac{1-\Delta}{2^{n+1}}.
\end{aligned}$$

We proceed in a similar way for the second moments. Hence we get

$$\begin{aligned}
BL_n((e_1 - x)^2; x) &= (x - \eta_{k-1}^n)^2 \cdot \varphi_{k-1}^n(x) + (\eta_k^n - x)^2 \cdot \varphi_k^n(x) \\
&\leq \max\{(x - \eta_{k-1}^n)^2, (\eta_k^n - x)^2\} \leq \big(\max\{(x - \eta_{k-1}^n), (\eta_k^n - x)\}\big)^2 \\
&\leq \left(\frac{1}{2^n}\right)^2 = \frac{1}{2^{2n}}.
\end{aligned}$$

Case 2: $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right)$, $k \in \{0, \ldots, 2^n - 1\}$. For the first moment we arrive at

$$|BL_n(e_1 - x; x)| \leq BL_n(|e_1 - x|; x) \leq \frac{1-\Delta}{2^{n+1}},$$

and for the second moment we have

$$BL_n((e_1 - x)^2; x) = (x - \eta_k^n)^2 \cdot \varphi_k^n(x) = (x - \eta_k^n)^2 \cdot 1 \leq \left(\frac{1-\Delta}{2^{n+1}}\right)^2 \leq \frac{1}{2^{2n}}.$$

Thus, we proved that for all $x \in [0, 1]$

$$|BL_n(e_1 - x; x)| \leq \frac{1-\Delta}{2^{n+1}} \text{ and } BL_n((e_1 - x)^2; x) \leq \frac{1}{2^{2n}}.$$

An application of Theorem 1.38 gives statement (2.40). □

Proposition 2.48 *For the particular choice $\delta = \frac{1}{2^n}$, $n \geq 1$, the estimate (2.40) becomes*

$$|BL_n(f; x) - f(x)| \leq \frac{(1-\Delta)}{2} \cdot \omega_1\left(f; \frac{1}{2^n}\right) + \frac{3}{2} \cdot \omega_2\left(f; \frac{1}{2^n}\right). \tag{2.41}$$

Remark 2.49 BL_n is an approximation operator, i.e., BL_nf converges uniformly towards f, $f \in C[0,1]$ as $n \to \infty$, see (2.41). For $\Delta = 1$, i.e., for *piecewise linear interpolation* at $0, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{2^n-1}{2^n}, 1$ the first term in (2.41) vanishes and we obtain a well-known inequality for polygonal line interpolation at the knots listed above. In fact, it was our aim to obtain for the first moments of the operator an upper bound involving the term $1-\Delta$, in order to have it vanish for the piecewise linear interpolators.

2.4 A modification of B_n: King type operators

In [86] J.P. King defined the following interesting (and somewhat exotic) sequence of linear and positive operators $V_n : C[0,1] \to C[0,1]$ which generalize the classical Bernstein operators B_n.

$$V_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1-r_n(x))^{n-k} f\left(\frac{k}{n}\right) \tag{2.42}$$

for all $f \in C[0,1]$, $0 \le x \le 1$, where $r_n : [0,1] \to [0,1]$ are continuous functions. We list some of their properties.

Proposition 2.50 *If $\{V_n\}_{n\in\mathbb{N}}$ are the operators defined in (2.42) we have*

$$\begin{aligned} V_n(e_0;x) &= e_0(x) \\ V_n(e_1;x) &= r_n(x) \quad \text{and} \\ V_n(e_2;x) &= \frac{r_n(x)}{n} + \frac{n-1}{n}(r_n(x))^2. \end{aligned} \tag{2.43}$$

The equation $V_n(e_1;x) = r_n(x)$ shows that the classical Bernstein operator B_n, which is obtained for $r_n(x) = x$, is the unique mapping of the form (2.42) which reproduces linear functions.

Theorem 2.51 *One has $\lim_{n\to\infty} V_nf(x) = f(x)$ for each $f \in C[0,1]$, $x \in [0,1]$, if and only if $\lim_{n\longrightarrow\infty} r_n(x) = x$.*

Choosing the "right" r_n function, J. P. King proved the following:

Theorem 2.52 *Let $\{V_n^*\}_{n\in\mathbb{N}}$ be the sequence of operators defined in (2.42) with*

$$r_n^*(x) := \begin{cases} r_1^*(x) = x^2, & n = 1, \\ r_n^*(x) = -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & n = 2,3,\ldots \end{cases} \tag{2.44}$$

Then:

(i) $V_n^*(e_2;x) = e_2(x),\ n \in \mathbb{N};\ x \in [0,1]$,

(ii) $V_n^*(e_1;x) \neq e_1(x)$,

(iii) $\lim_{n\to\infty} V_n^*(f;x) = f(x)$ *for each* $f \in C[0,1]$.

Remark 2.53 Since $V_n^* e_1 = r_n^*$, it is clear that V_n^* is not a polynomial operator.

J. P. King also gave quantitative estimates for V_n^* in terms of the classical first order modulus $\omega_1(f;\cdot)$ using a result of O. Shisha & B. Mond [140] in the current paper it is given as Theorem 1.35.

Theorem 2.54 *For* $\{V_n^*\}_{n\in\mathbb{N}}$ *defined in (2.42) we have*

$$|V_n^*(f;x) - f(x)| \leq 2\omega_1\left(f; \sqrt{2x(x - V_n^*(e_1;x))}\right), \quad f \in C[0,1],\ x \in [0,1]. \tag{2.45}$$

Remark 2.55 From the fact that $V_n^*(e_1;x) = r_n^*(x)$ and $x \geq r_n^*(x)$ the square root in (2.45) indeed represents a real number.

From Theorem 2.54 one can easily obtain that V_n^* interpolates f at the endpoints:

Proposition 2.56 *With* $\{V_n^*\}_{n\in\mathbb{N}}$ *from (2.42) we have* $V_n^*(f;0) = f(0)$ *and* $V_n^*(f;1) = f(1)$, *i.e.,* V_n^* *interpolates at the endpoints 0 and 1.*

Proof. We put $\alpha_n(x) := \sqrt{2x(x - V_n^*(e_1;x))}$. For $x = 0$ we have $\alpha_n(0) = 0$, so $\omega_1(f;\alpha_n(0)) = 0$. That means $V_n^*(f;0) = f(0)$. For $x = 1$ we have $V_n^*(e_1;1) = r_n^*(1)$, and if we insert in (2.44) the value 1, we obtain $r_n^*(1) = 1$. That leads us again to $\omega_1(f;\alpha_n(1)) = 0$ and $V_n^*(f;1) = f(1)$. □

Remark 2.57 For a linear and positive operator $L : C[0,1] \to C[0,1]$ with $Le_i = e_i,\ i = 0,1$, it is known that L interpolates f in 0 and 1. This follows easily, if we insert $x = 0$ and $x = 1$ in

$$|L(f;x) - f(x)| \leq 2 \cdot \omega_1(f; L(|e_1 - x|;x)),$$

see e.g., [99] or here Corollary 1.37. We observe now, with the help of the operators introduced by J. P. King, that the above property is only necessary and not sufficient. Indeed, the V_n^*, $n \in \mathbb{N}$, interpolate f in 0 and 1, they are linear and positive, but $V_n^* e_1 \neq e_1$.

2.4.1 Quantitative estimates

Păltănea's Theorem 1.38 reads as follow for V_n^*:

Proposition 2.58 *Let V_n^* be the operators defined as above. Then for any $f \in C[0,1]$ the following estimate holds*

$$\begin{aligned} |V_n^*(f;x) - f(x)| &\leq (x - r_n^*(x)) \cdot \frac{1}{h}\omega_1(f;h) + \\ &+ \left(1 + \frac{1}{h^2}x(x - r_n^*(x))\right)\omega_2(f;h). \end{aligned}$$

and for $h := \sqrt{x - r_n^*(x)}$ *we arrive at*

$$|V_n^*(f;x) - f(x)| \leq \sqrt{x - r_n^*(x)} \cdot \omega_1(f; \sqrt{x - r_n^*(x)}) + (1+x)\omega_2(f; \sqrt{x - r_n^*(x)}).$$

Remark 2.59 If $f \in C^1[0,1]$ then due to the fact that $\omega_1(f;h) = O(h)$ and also $\omega_2(f;h) = O(h)$ we have the approximation order $O(\sqrt{x - r_n^*(x)})$, when $n \to \infty$. For $f \in C^2[0,1]$ having similar properties for the moduli $\omega_1(f;h) = O(h)$ and $\omega_2(f;h) = O(h^2)$ we obtain $O(x - r_n^*(x))$, $n \to \infty$.

2.4.2 Polynomial operators of King's type

In the following we shall concentrate on the question: Can we find *polynomial* operators of the form (2.42) that reproduce e_2? The answer is negative.
Indeed, by the last two equations of (2.43) and the condition $V_n(e_2;x) = x^2$, r_n must be a polynomial of first degree. We put $r_n(x) = ax + b$ and we get:

$$x^2 = \frac{n-1}{n}a^2x^2 + \left(\frac{a}{n} + \frac{2(n-1)ab}{n}\right)x + \left(\frac{b}{n} + \frac{n-1}{n}b^2\right).$$

This leads to the equations:

$$\begin{cases} 1 = & \frac{n-1}{n}a^2, \\ 0 = & \frac{a}{n} + \frac{2(n-1)ab}{n}, \\ 0 = & \frac{b}{n} + \frac{n-1}{n}b^2. \end{cases}$$

So $a = \pm\sqrt{\frac{n}{n-1}}$ and $b = 0$ or $b = \frac{1}{1-n}$. But for these values the second equation is not satisfied. One open question remains: Can we find another type of linear and positive polynomial operators L for which $Le_2 = e_2$?

2.4.3 General case

In the sequel we want to "optimize" the second moments $V_n((e_1-x)^2;x),\ x\in[0,1]$, of the general V_n and study in this case which properties remain.
The second moments are in the general case

$$\begin{aligned}\alpha_n^2(x)=V_n((e_1-x)^2;x) &= \frac{r_n(x)}{n}+\frac{n-1}{n}(r_n(x))^2-2xr_n(x)+x^2= \\ &= \frac{1}{n}r_n(x)(1-r_n(x))+(r_n(x)-x)^2, \end{aligned} \tag{2.46}$$

where $0\le r_n(x)\le 1$ are continuous functions. We want to find r_n so that α_n^2 is minimal.
We define $g_x:[0,1]\to[0,1],\ x\in[0,1]$ a fixed parameter, by $g_x(y):=\frac{1}{n}y(1-y)+(y-x)^2$. We can write $g_x(y)=\left(1-\frac{1}{n}\right)y^2+\left(\frac{1}{n}-2x\right)y+x^2$. Because $1-\frac{1}{n}>0,\ n=2,3,\ldots,$ the function g_x admits a minimum point:

$$y_{min}=-\frac{\frac{1}{n}-2x}{2-\frac{2}{n}}=\frac{2nx-1}{2n-2}.$$

We need $0\le y_{min}\le 1$, which means $\frac{1}{2n}\le x\le 1-\frac{1}{2n},\ n=2,3,\ldots$
We define $r_n^{min}:[0,1]\to[0,1]$ by

$$r_n^{min}(x):=\begin{cases}0, & x\in\left[0,\frac{1}{2n}\right),\\ \frac{2nx-1}{2n-2}, & x\in\left[\frac{1}{2n},1-\frac{1}{2n}\right],\\ 1, & x\in\left(1-\frac{1}{2n},1\right].\end{cases} \tag{2.47}$$

Theorem 2.60 *The function r_n^{min} defined in (2.47) yields the minimum value for α_n^2.*

Proof. For $x\in\left[\frac{1}{2n},1-\frac{1}{2n}\right]$ this was proven before. It remains to show the above affirmation for $x\in\left[0,\frac{1}{2n}\right)$ and $x\in\left(1-\frac{1}{2n},1\right]$.
First case: $x\in\left[0,\frac{1}{2n}\right)\Rightarrow r_n^{min}(x)=0$ and we have to prove that $g_x(y)\ge g_x(0)$ for each $y\in[0,1]$ or $\frac{1}{n}y(1-y)+(y-x)^2\ge x^2$ for each $x\in[0,1]$. But the latter is equivalent to $\frac{1}{2n}+y\left(\frac{1}{2}-\frac{1}{2n}\right)\ge x$, which is true due to our choice of x.
Second case: $x\in\left(1-\frac{1}{2n},1\right]\Rightarrow r_n^{min}(x)=1$ and we have to prove that $g_x(y)\ge g_x(1)$ for each $y\in[0,1]$ or $\frac{1}{n}y(1-y)+(y-x)^2\ge(1-x)^2$. This means $\left(1-\frac{1}{2n}\right)-(1-y)\left(\frac{1}{2}-\frac{1}{2n}\right)\le x$, which is again true due to our choice of x. □
The operators V_n defined via r_n^{min} we denote by V_n^{min}.

Proposition 2.61 *For the (minimal) second moments α_n^2 of V_n^{min} we have the representation*

$$\alpha_n^2(x) = \begin{cases} x^2, & x \in \left[0, \frac{1}{2n}\right), \\ \frac{1}{n-1}\left(x(1-x) - \frac{1}{4n}\right), & x \in \left[\frac{1}{2n}, 1-\frac{1}{2n}\right], \\ (1-x)^2, & x \in \left(1-\frac{1}{2n}, 1\right]. \end{cases}$$

Proof. This follows immediately from the general form

$$\frac{1}{n} r_n(x)(1-r_n(x)) + (r_n(x)-x)^2$$

and the above representation of $r_n^{min}(x)$. □

Using Păltănea‘s theorem 1.38 again we arrive at

$$\begin{aligned} |V_n^{min}(f;x) - f(x)| &\leq |x - r_n^{min}(x)| \cdot \frac{1}{h} \cdot \omega_1(f;h) + \\ &+ \left(1 + \frac{1}{2} \cdot \frac{1}{h^2} \cdot \alpha_n^2(x)\right) \cdot \omega_2(f;h), \quad h > 0. \end{aligned}$$

For $h = |\alpha_n(x)|$ we obtain

$$|V_n^{min}(f;x) - f(x)| \leq \frac{|x - r_n^{min}(x)|}{|\alpha_n(x)|} \cdot \omega_1(f; |\alpha_n(x)|) + \frac{3}{2} \cdot \omega_2(f; |\alpha_n(x)|).$$

Note that $|x - r_n^{min}(x)| = |V_n^{min}(e_1 - x; x)| \leq V_n^{min}(|e_1 - x|; x) \leq \sqrt{V_n^{min}((e_1-x)^2; x)} = |\alpha_n(x)|$, and thus $\frac{|x - r_n^{min}(x)|}{|\alpha_n(x)|} \leq 1, \quad x \in [0,1]$. □

Remark 2.62 (i) From the definition of r_n^{min} we have $\lim_{n\to\infty} r_n^{min}(x) = x$ and from Theorem 2.51 $\lim_{n\to\infty} V_n(f;x) = f(x)$.

The latter fact is also a consequence of our second application of Theorem 1.38 for V_n^{min}.

(ii) V_n^{min} does not reproduce e_2. Starting from (2.43) we see that $V_n^{min}(e_2;x) = 0 \neq x^2, \quad x \in \left(0, \frac{1}{2n}\right)$.

(iii) The interpolation properties at the endpoints remain. Indeed, $V_n^{min}(f;0) = \binom{n}{0} \times (1 - r_n(0))^n f(0) = f(0)$, and $V_n^{min}(f;1) = \binom{n}{n} f(\frac{n}{n}) = f(1)$.

(iv) For $f \in C^1[0,1]$ we have, with a constant c independent of x,

$$|V_n^{min}(f;x) - f(x)| \leq c \cdot (|x - r_n^{min}(x)| + |\alpha_n(x)|) =$$

$$= c \cdot \begin{cases} 2x, & x \in \left[0, \frac{1}{2n}\right), \text{ hence } O\left(\frac{1}{n}\right), \\ \frac{|\frac{1}{2}-x|}{n-1} + \sqrt{\frac{1}{n-1}\left(x(1-x) - \frac{1}{4n}\right)}, & x \in \left[\frac{1}{2n}, 1 - \frac{1}{2n}\right], \text{ hence } O\left(\frac{1}{\sqrt{n}}\right), \\ 2(1-x), & x \in \left(1 - \frac{1}{2n}, 1\right], \text{ hence } O\left(\frac{1}{n}\right). \end{cases}$$

So the degree of approximation is better close to the endpoints, a fact shared by the Bernstein operators where $r_n(x) = x$.

(v) If $f \in C^2[0,1]$, then

$$|V_n^{min}(f;x) - f(x)| \leq c \cdot (|x - r_n^{min}(x)| + |\alpha_n^2(x)|) =$$

$$= c \cdot \begin{cases} x + x^2, & x \in \left[0, \frac{1}{2n}\right), \\ \frac{|\frac{1}{2}-x|}{n-1} + \frac{1}{n-1}\left(x(1-x) - \frac{1}{4n}\right), & x \in \left[\frac{1}{2n}, 1 - \frac{1}{2n}\right], \\ (1-x) + (1-x)^2, & x \in \left(1 - \frac{1}{2n}, 1\right]. \end{cases}$$

So for C^2-functions we get a global degree of approximation of order $O\left(\frac{1}{n}\right)$ which is also the case for the classical Bernstein operators.

Chapter 3

Selected results for some general Beta-type operators

The aim of this chapter is to establish some quantitative estimates for some special linear positive operators. Most of them are defined by means of special functions, namely by the *Beta function* $B(p, q)$ with $p, q > 0$. The subject of this sequel is not only the classical quantitative estimates, but also simultaneous approximation, and also we try to answer the question about the *global smoothness preservation* of the operators considered.

3.1 Definitions and some relevant particular cases

A. Lupaş introduced in his German Ph. D. thesis two types of *Beta operators*, both with remarkable properties:

1) the *Beta operator of the first kind* [95, p. 37] defined by

$$\mathbb{B}_n(f;x) = \frac{1}{B(nx+1, n+1-nx)} \cdot \int_0^1 t^{nx}(1-t)^{n(1-x)} f(t)dt, \quad f \in C[0,1], \text{ and} \tag{3.1}$$

2) the *Beta operator of the second kind* [95, p. 63] given by

$$\bar{\mathbb{B}}_n(f;x) := \begin{cases} f(0), & x = 0, \\ \frac{1}{B(nx, n-nx)} \cdot \int\limits_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t)dt, & 0 < x < 1, \\ f(1), & x = 1, \end{cases} \tag{3.2}$$

for any $f \in C[0,1]$, $n \in \mathbb{N}$.

We recall the definition of the *Beta function*:

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt,\ p,q>0.$$

One important advantage of the last one is that it reproduces linear functions. In this work we are mostly interested in the second type of Beta operator and its generalizations. In this context, we introduce the following composite Beta-type operator. If $f \in C[0,1]$, $x \in [0,1]$, and α and λ are strictly positive real numbers, then we denote by

$$\mathbb{B}_n^{(\alpha,\lambda)}(f;x) := (\tilde{\mathbb{B}}_\alpha \circ B_n \circ \tilde{\mathbb{B}}_\lambda)(f;x), \tag{3.3}$$

where B_n is the $n-$th Bernstein operator and $\tilde{\mathbb{B}}_\alpha$ respectively $\tilde{\mathbb{B}}_\lambda$ are instances of the same modification of (3.2):

$$\tilde{\mathbb{B}}_\tau(f;x) := \begin{cases} f(0), & x=0, \\ \frac{1}{B(\frac{x}{\tau},\frac{1-x}{\tau})} \cdot \int\limits_0^1 t^{\frac{x}{\tau}-1}(1-t)^{\frac{1-x}{\tau}-1} \cdot f(t)dt, & 0<x<1, \\ f(1), & x=1, \end{cases} \tag{3.4}$$

for any $\tau > 0$.

In the following we shall adopt the following conventions: $\tilde{\mathbb{B}}_0 := Id$ and $B_\infty := Id$, where Id is the identity operator.

It is important to observe that

Proposition 3.1 *If $f \in C[0,1]$ and $\alpha > 0$, then $\tilde{\mathbb{B}}_\alpha f \in C[0,1]$.*

Proof. Let $\alpha > 0$ be fixed, $n \geq 1$ natural and $f \in C[0,1]$ then $\tilde{\mathbb{B}}_\alpha f \in B([0,1])$. We have

$$\|\tilde{\mathbb{B}}_\alpha(B_n f) - \tilde{\mathbb{B}}_\alpha f\|_\infty \leq \|\tilde{\mathbb{B}}_\alpha\|_\infty \cdot \|B_n f - f\|_\infty = \|B_n f - f\|_\infty.$$

Thus it follows that

$$\lim_{n\to\infty} \tilde{\mathbb{B}}_\alpha(B_n f) = \tilde{\mathbb{B}}_\alpha f \text{ uniformly.}$$

As $\tilde{\mathbb{B}}_\alpha B_n f \in C[0,1]$ it follows that $\tilde{\mathbb{B}}_\alpha f \in C[0,1]$. □

We could not find a satisfactory answer regarding the *derivability*. Hence,

Conjecture 3.2 *If $f \in C^1[0,1]$, then $\tilde{\mathbb{B}}_\alpha f \in C^1[0,1]$? Or more generally, if $f \in C^r[0,1]$, then $\tilde{\mathbb{B}}_\alpha f \in C^r[0,1]$, $r \geq 1$ a natural number?*

In order to obtain an overview of all particular cases covered by Definition (3.3) we depict them in the following table:

	α	n	λ	Notations/Observations
1	$\neq 0$	$\neq \infty$	$\neq 0$	$\mathbb{B}_n^{(\alpha,\lambda)}$ see its definition above.
2	$\neq 0$	$\neq \infty$	$\frac{1}{n}$	They were studied by Z. Finta in [48], [49]. For these operators we use the notation $\mathbb{B}_n^{(\alpha,1/n)} =: F_n^\alpha$ and call them *Finta's operators*. Explicit representation: $F_n^\alpha(f;x) = f(0)w_{n,0}^{(\alpha)}(x) + f(1)w_{n,n}^{(\alpha)}(x) + \sum_{k=1}^{n-1} w_{n,k}^{(\alpha)}(x) \cdot \int_0^1 (n-1)p_{n-2,k-1}(t)f(t)dt,$ where (3.5) $w_{n,k}^{(\alpha)}(x) := \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}.$
3	$\neq 0$	$\neq \infty$	0	They were introduced by D. D. Stancu in 1968 in [144]. They were further investigated in the subsequent papers [145], [146] and [147]. They were studied by a long line of authors, see e.g., the survey of B. Della Vecchia [35] and the references therein. An alternative notation used in this work is $\mathbb{B}_n^{(\alpha,0)} =: S_n^{<\alpha,0,0>}$. Discret representation is: (3.6) $S_n^{<\alpha,0,0>}(f;x) = \sum_{k=0}^{n} w_{n,k}^{(\alpha)}(x) \cdot f\left(\frac{k}{n}\right), \quad x \in [0,1],$ where the polynomials $w_{n,k}^{(\alpha)}$ are the same as in (3.5).
4	$\frac{1}{n}$	$\neq \infty$	0	It is obviously a subcase of the previous one, namely $\mathbb{B}_n^{(1/n,0)} =: S_n^{<1/n,0,0>}$. Appears also in [144]. Admits the following compact representation: $S_n^{<1/n,0,0>}(f;x) = \frac{2n!}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} (nx)_k (n-nx)_{n-k} \cdot f\left(\frac{k}{n}\right),$ where $x \in [0,1]$ and $(a)_b$ is the *Pochhammer symbol*, see the table of notations.
5	0	$\neq \infty$	$\neq 0$	$\mathbb{B}_n^{(0,\lambda)} =: U_n^\lambda$ a possible generalization of row 6.
6	0	$\neq \infty$	$\frac{1}{n}$	For this choice of the parameters we arrive at the *genuine Bernstein-Durrmeyer operators* denoted by U_n, these were independently introduced by W. Chen [28] in 1987,

	α	n	λ	Notations/Observations
				and by T. N. T. Goodman & A. Sharma [74] later in 1991. They possess many interesting properties and were therefore investigated by many authors, noteworthy is [115]. For a detailed overview and many references the reader is guided to the recent work [80]. Can be explicitly written: $U_n(f;x) := f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt$ (3.7) with $x \in [0,1]$. $p_{n,k}$ is the Bernstein basis, see (1.2).
7	$\neq 0$	∞	$\neq 0$	No special notation needed $\mathbb{B}_\infty^{(\alpha,\beta)} =: \tilde{\mathbb{B}}_\alpha \circ \tilde{\mathbb{B}}_\lambda$.
8	0	∞	$\neq 0$	Reduces to $\mathbb{B}_\infty^{(0,\lambda)} =: \tilde{\mathbb{B}}_\lambda$, see (3.4).
9	0	∞	$\frac{1}{n}$	Are Lupaş's well-known $\bar{\mathbb{B}}_n$, see (3.2).
10	0	$\neq \infty$	0	It is obvious that $\mathbb{B}_n^{(0,0)} =: B_n$ are the Bernstein operators, i.e., $B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot f\left(\frac{k}{n}\right)$.
11	0	∞	0	This is the identity operator.

Table 3.1: An overview

Remark 3.3 In this thesis we shall also focus our attention upon another *Beta-type operator*, which, however, does not fit exactly into the scheme from above, namely a multi-parameter *general Stancu operator* given, for $\alpha \geq 0$ and $0 \leq \beta \leq \gamma$, by:

$$S_n^{<\alpha,\beta,\gamma>}(f;x) := \sum_{k=0}^{n} w_{n,k}^{(\alpha)}(x) f\left(\frac{i+\beta}{n+\gamma}\right), \tag{3.8}$$

for any $f \in C[0,1]$ and $x \in [0,1]$. The $w_{n,k}^{(\alpha)}$ are defined at (3.5).

We find them defined in [148] by D. D. Stancu. However, they appear there in a little more general form: they depend upon a fourth parameter $p \in \mathbb{N}$. The disadvantage of choosing $p > 0$ is that the domain of definition depends on n, namely $f \in C\left[0, \frac{p}{n}\right]$. Therefore we will restrict ourselves to (3.8). The interested reader can find a well structured overview on all known Stancu operators in [104, Table 4.3, p. 111]. See also [68].
The compact variant of (3.8) is:

$$S_n^{<\alpha,\beta,\gamma>}(f;x) = \tilde{\mathbb{B}}_\alpha B_n\left(f \circ \left(\frac{n}{n+\gamma}e_1 + \frac{\beta}{n+\gamma}\right); x\right) \tag{3.9}$$

$$= \begin{cases} f\left(\frac{\beta}{n+\gamma}\right), & x = 0 \\ \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot B_n\left(f \circ \left(\frac{n}{n+\gamma}e_1 + \frac{\beta}{n+\gamma}\right); t\right) dt, & x \in (0,1) \\ f\left(\frac{n+\beta}{n+\gamma}\right), & x = 1, \end{cases} ,$$

which is very similar to the formula in [144, p. 1182].
Evidently, due to the above convention $\tilde{\mathbb{B}}_0 := Id$, for $\alpha = 0$ we arrive at

$$S_n^{<0,\beta,\gamma>}(f;x) = B_n\left(f \circ \left(\frac{n}{n+\gamma}e_1 + \frac{\beta}{n+\gamma}e_0\right); x\right). \tag{3.10}$$

This decomposition can also be found in [129].
An implication deriving from the representations from above is that neither $S_n^{<\alpha,\beta,\gamma>}$ nor $S_n^{<0,\beta,\gamma>}$ interpolate the function f at both of the endpoints. On the other hand, $S_n^{<\alpha,0,0>}$ does interpolate f in $\{0,1\}$.

Remark 3.4 Also nowadays new modifications of Stancu operators are invented and investigated. For example, in the recent paper [106] the authors define a more "flexible" extension of $S_n^{<0,\beta,\gamma>}$. Flexible, in the sense that $0 \leq \beta \leq \gamma$ are not fixed numbers, but infinite sequences, which depend on n, the degree and k, the summation index.

3.2 Preservation of higher order convexity by $\tilde{\mathbb{B}}_\alpha$ and $\mathbb{B}_n^{(\alpha,\lambda)}$

In the present section we shall prove that $\tilde{\mathbb{B}}_\alpha$ and their generalizations preserve convexity up to any order. First we need some basic definitions and some preliminary results.
Let $K = [a,b]$ be a compact interval of the real axis and $K' \subset K$ also compact. We consider the Banach space $X = C^r(K)$ with the norm $\|g\|_X := \max\limits_{0\leq j\leq r}(\|D^j g\|_K)$. Here $\|\cdot\|_K$ is the Chebyshev norm in $C(K) := C^0(K)$.
Let $\mathcal{K}_K^i := \{f \in C(K) : [x_0,\dots,x_i;f] \geq 0 \text{ for any } x_0 < \dots < x_i \in \mathcal{K}\}$, where $[x_0,\dots,x_i;f]$ is an $i-$th order *divided difference* of f. In other words, the class $\mathcal{K}_K^i$ represents the set of all $i-$convex functions on K, a definition that was also given by T. Popoviciu (see [123], [125]). Note that $\mathcal{K}_K^0$ is the set of all positive functions, $\mathcal{K}_K^1$ is the set of non-decreasing functions, and $\mathcal{K}_K^2$ are the usual convex functions. Very often instead of $\mathcal{K}_K^i$ the notation $\mathcal{C}(e_0,e_1,\dots,e_{i-1})$ is used.
An operator $L : V \to C(K')$, $V \subset C(K)$ can be verified to be *r-convexity preserving* or *convex of order* $r-1$, $r \in \mathbb{N}\cup\{0\}$, if the following holds

$$f \in \mathcal{K}_K^r \cap V \text{ implies } Lf \in K_{K'}^r,$$

compare also with A. Lupaş [94].
H.-B. Knoop & P. Pottinger [87] have slightly weakened the notion of convex operators by *almost convex* operators: an operator $L : V \to C(K')$ is called *almost convex* of order $r-1,\ r \geq 1$ if there exist $p \geq 0$ integers $i_j,\ 1 \leq j \leq p$, satisfying $0 \leq i_1 < \ldots < i_p < r$ such that

$$f \in \left(\bigcap_{j=1}^{p} \mathcal{K}_K^{i_j}\right) \cap \mathcal{K}_K^r \cap V \text{ implies } Lf \in \mathcal{K}_{K'}^r.$$

For $p = 0$ we put $\bigcap\limits_{j=1}^{p} \mathcal{K}_K^{i_j} := V$ and in this case L is convex of order $r-1$.
Relatively to the composition of two (almost) convex operators the following can be said

Proposition 3.5 *If $A, B : C(K) \to C(K)$ are both (almost) convex of order $r-1$, then $A \circ B$ is also (almost) convex of order $r-1$.*

Proof. Let f be a function in $\mathcal{K}_K^r$. It means that $Bf \in \mathcal{K}_K^r$ and moreover, $A(Bf) \in \mathcal{K}_K^r$. □

It is obvious that the above assertion remains true for a finite product of (almost) convex operators.

A common way to verify if an operator is (almost) convex of order $r-1$, employs the *differential operator*. More exactly: $Lf \in C^r(K')$ is an element of $\mathcal{K}_{K'}^r$ iff $D^r Lf \geq 0$. Of course, this approach is only possible when $Lf \in C^r[a,b]$. Therefore, another possible and useful alternative to study the convexity of a certain order is via *total positivity*. Having future applications in mind we focus more on this aspect.
A. Attalienti & I. Raşa proved the following very instructive theorem, see [7, Theorem 2.3]:

Theorem 3.6 *If L is a positive linear operator of the form (1.24), which additionally to the assumptions in Lemma 1.44 satisfies also the following:*

(i) $L(D(L) \cap C(I)) \subset C(X)$.

(ii) There exists an integer $r \geq 2$ such that for each $k = 0, 1, \ldots, r$ the power function e_k belongs to $D(L)$ and Le_k is a polynomial of degree k with leading coefficient $a_k > 0$.

Then we have:

a) The operator L is $r-$convexity preserving on $D(L) \cap C(I)$.

b) $L(D(L)\cap C(I)\cap \mathrm{Lip_r M})\subset \mathrm{Lip}_r(a_rM)$ *for any* $M\geq 0$.

c) If $f\in D(L)\cap C^r(I)$ *has a bounded derivative of order* r*, i.e.,* $\|f^{(r)}\|:=\sup_{x\in I}|f^{(r)}(x)|<\infty$*, then* $Lf\in C^{r-2}(\overset{\circ}{X})$ *and* $(Lf)^{r-2}$ *has a right derivative which is right-continuous on* $\overset{\circ}{X}$ *and a left derivative which is left-continuous on* $\overset{\circ}{X}$*. Finally, if* $Lf\in C^r(X)$ *too, then* $\|(Lf)^r\|\leq a_r\cdot\|f^{(r)}\|$.

Remark 3.7 Due to [7, Remark 2.4], a) and b) still hold if $r\geq 1$ in (ii). Moreover, an inspection of the proof shows, in order to preserve $r-$convexity one needs to require the preservation of the polynomials in (ii) up to the degree $r-1$, which in this case is cf. with Theorem 1.46.

The previous theorem generalizes Theorem 3.3 from Chapter 6 Section 3 in [81], where the interested reader can find an exhaustive theory on *totally positive kernels* and its several applications.

Example 3.8 The upper theorem provides us an accessible mean to prove that $\tilde{\mathbb{B}}_\alpha$, and implicitly also Lupaş's Beta operators of the second kind (3.2) preserve the convexity up to any order, without using the differential operator, which in the light of Conjecture 3.2 would be somewhat hard. Earlier in 1992 J. A. Adell, F. G. Badía & J. de la Cal have already shown in their joint work [1] (see also [2]) by means of probabilistical methods that these two type of Beta operators preserve the monotonicity and the classical convexity. For the Beta operators of the first kind defined at (3.1) the same property holds, see [7].

Totally positive kernel: According to (1.24), for any fixed $\alpha>0$ the corresponding kernel is given by

$$(0,1)\times(0,1)\ni(x,t)\mapsto\frac{t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}}{B(\frac{x}{\alpha},\frac{1-x}{\alpha})}\in\mathbb{R},$$

or, equivalently, by

$$(0,1)\times(0,1)\ni(x,t)\mapsto e^{\frac{x}{\alpha}\cdot(\ln t-\ln(1-t))}e^{\frac{1}{\alpha}\cdot\ln(1-t)}\frac{1}{t(1-t)\cdot B(\frac{x}{\alpha},\frac{1-x}{\alpha})}\in\mathbb{R}.$$

The functions $x\mapsto\frac{x}{\alpha},\ x\in(0,1),\ \alpha>0$ and $t\mapsto\ln t-\ln(1-t),\ t\in(0,1)$ are strictly increasing on $(0,1)$. Thus, due to Corollary 1.43 the kernel $(0,1)\times(0,1)\ni(x,t)\mapsto e^{\frac{x}{\alpha}\cdot(\ln t-\ln(1-t))}$ is totally positive. Furthermore, the functions $t\mapsto\frac{e^{\frac{\ln(1-t)}{\alpha}}}{t(1-t)},\ t\in(0,1)$ and $x\mapsto\frac{1}{B(\frac{x}{\alpha},\frac{1-x}{\alpha})},\ x\in(0,1)$ are strictly positive functions on the indicated domains. Thus, on behalf of Theorem 1.42 part a) the kernel

$$(0,1)\times(0,1)\ni(x,t)\mapsto\frac{t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}}{B(\frac{x}{\alpha},\frac{1-x}{\alpha})}$$

is totally positive on $(0,1) \times (0,1)$.

The leading coefficient a_k: Direct computations, using the properties of the *Beta function*, yields for any $\alpha > 0$ and $k \geq 1$

$$\tilde{\mathbb{B}}_\alpha e_k = \prod_{i=0}^{k-1} \left(\frac{1}{\alpha i + 1} e_1 + \frac{\alpha i}{\alpha i + 1} e_0 \right) \in \prod_k . \tag{3.11}$$

Obviously we also have $\tilde{\mathbb{B}}_\alpha e_0 = e_0$. The leading coefficient of these polynomials are $a_k := \prod\limits_{i=0}^{k-1} \frac{1}{\alpha i+1} > 0$, for $k \geq 1$ and $a_0 = 1$, for $k = 0$.

The conclusion: The modified Beta operators preserve convexity of any order and map Lipschitz classes into Lipschitz classes. This affirmation is also true for Lupaş's Beta operators of the second kind as $\tilde{\mathbb{B}}_{\frac{1}{n}} =: \bar{\mathbb{B}}_n$. □

Remark 3.9 It is well-known that the Bernstein operators B_n retain convexity up to any order, an aspect to which we shall come back in Subsection 3.4.3. However, due to Proposition 3.5 and the accumulated knowledge about the Beta-type operators $\tilde{\mathbb{B}}_\alpha$ we can now affirm that $\mathbb{B}_n^{(\alpha,\lambda)}$, and subsequently all the operators listed in Table 3.1, including the general $S_n^{<\alpha,\beta,\gamma>}$, $\alpha \geq 0$, $0 \leq \beta \leq \gamma$, have high shape-preserving properties: they retain the convexity up to any order.

3.3 Degree of approximation via moduli of smoothness and via K-functionals

In this section we shall compute the rate of convergence of the composite Beta-type operators $\mathbb{B}_n^{(\alpha,\lambda)}$ and of $S_n^{<\alpha,\beta,\gamma>}$, see (3.3) and (3.8). The estimates are given in terms of ω_1 and ω_2 and the technique we employ is a standard one: we use Theorem 1.38 or an appropriate K-functional.
Therefore, we need the following results:

Lemma 3.10 *For $n \geq 1$ natural the operators $\mathbb{B}_n^{(\alpha,\lambda)}$ are positive linear and polynomial type operators reproducing linear functions, i.e., $\mathbb{B}_n^{(\alpha,\lambda)}(e_0;x) = 1$ and $\mathbb{B}_n^{(\alpha,\lambda)}(e_1 - x;x) = 0$. Their second moments can be computed by*

$$\mathbb{B}_n^{(\alpha,\lambda)}((e_1 - x)^2;x) = x(1-x) \cdot \left(1 - \frac{1}{(1+\alpha)(1+\lambda)} \cdot \frac{n-1}{n}\right), \tag{3.12}$$

where $x \in [0,1]$.

Proof. Positivity and linearity are easy to verify. As $\mathbb{B}_n f \in \prod_n$ for any $f \in C[0,1]$, then in combination with (3.11) we have $\mathbb{B}_n^{(\alpha,\lambda)} f \in \prod_n$. $\tilde{\mathbb{B}}_\alpha e_1 = e_1$, is a fact that follows from (3.11). The same property obviously holds for B_n, whence we obtain $\mathbb{B}_n^{(\alpha,\lambda)} e_1 = e_1$.
Before we compute the second moments of $\mathbb{B}_n^{(\alpha,\lambda)}$ we recall the following basic results: $B_n((e_1-x)^2;x) = \frac{x(1-x)}{n}$ and $\mathbb{B}_\tau((e_1-x)^2;x) = \frac{\tau x(1-x)}{1+\tau}$, $\tau > 0$. The first relation was already used by T. Popviciu as early as 1942 (see [124], cf. also [127]), and the second one can be easily obtained by elementary computations (for $\tau = \frac{1}{n}$ see [95]). Further, we apply the recurrence formula for second moments proved in [61, Theorem 3.3], a generalization of the identity shown by D. Kacsó in [78]. Namely, for three linear operators $P_i,\ i = 1,2,3$, with $P_i e_j = e_j,\ i = 1,2,3$, and $j = 0,1$, one has

$$\left(\prod_{i=1}^{3} P_i\right)((e_1-x)^2;x) = P_1(P_2(P_3((e_1-u)^2;u);v);x) + P_1(P_2((e_1-v)^2;v);x) + P_1((e_1-x)^2;x). \tag{3.13}$$

For our operators the above relation reads

$$\begin{aligned}
\mathbb{B}_n^{(\alpha,\lambda)}((e_1-x)^2;x) &= \tilde{\mathbb{B}}_\alpha\left(B_n\left(\frac{\lambda u(1-u)}{1+\lambda};v\right);x\right) + \tilde{\mathbb{B}}_\alpha\left(\frac{v(1-v)}{n};x\right) \\
&+ \frac{\alpha x(1-x)}{1+\alpha} \\
&= \tilde{\mathbb{B}}_\alpha\left(\frac{\lambda}{1+\lambda}B_n(u-u^2;v);x\right) + \frac{1}{n}\tilde{\mathbb{B}}_\alpha(v-v^2;x) + \frac{\alpha x(1-x)}{1+\alpha} \\
&= \frac{\lambda}{(1+\lambda)(1+\alpha)}x(1-x)\left(1-\frac{1}{n}\right) + \frac{1}{n(1+\alpha)}x(1-x) \\
&+ \frac{\alpha x(1-x)}{1+\alpha} \\
&\vdots \\
&= x(1-x)\frac{n(\alpha+\lambda+\alpha\cdot\lambda)+1}{n(1+\alpha)(1+\lambda)}.
\end{aligned}$$

□

For the particular cases of $\mathbb{B}_n^{(\alpha,\lambda)}$ we deduce:

L	$L((e_1-x)^2;x)$
F_n^α	$\frac{x(1-x)}{1+\alpha}\cdot\left(\alpha+\frac{2}{n+1}\right)$
$S_n^{<\alpha,0,0>}$	$x(1-x)\frac{n\alpha+1}{n(1+\alpha)}$
$S_n^{<1/n,0,0>}$	$x(1-x)\frac{2}{n+1}$
U_n^λ	$x(1-x)\frac{n\lambda+1}{n(1+\lambda)}$
$U_n =: U_n^{1/n}$	$x(1-x)\frac{2}{n+1}$
$\tilde{\mathbb{B}}_\alpha \circ \tilde{\mathbb{B}}_\lambda$	$x(1-x)\frac{\alpha+\lambda+\alpha\cdot\lambda}{(1+\alpha)(1+\lambda)}$
$\tilde{\bar{\mathbb{B}}}_\alpha$	$x(1-x)\frac{\alpha}{1+\alpha}$
$\bar{\mathbb{B}}_n$	$\frac{x(1-x)}{n+1}$
B_n	$\frac{x(1-x)}{n}$

Table 3.2: Second moments

The first and second moments for $S_n^{<\alpha,\beta,\gamma>}$ were already computed in [104, p.122/126] or [68]. The situation is a little bit different as above, as they do not reproduce linear functions. This can be read from:

$S_n^{<\alpha,\beta,\gamma>}((e_1-x);x)$	$\frac{\beta}{n+\gamma}-\frac{\gamma}{n+\beta}x$
$S_n^{<\alpha,\beta,\gamma>}((e_1-x)^2;x)$	$\left[\frac{n(n-1)}{(n+\gamma)^2(1+\alpha)}-\frac{2n}{n+\gamma}+1\right]\cdot x^2 + \left[\frac{(1+2\beta)n}{(n+\gamma)^2}+\frac{(n-1)n\alpha}{(n+\gamma)^2(1+\alpha)}-\frac{2\beta}{n+\gamma}\right]\cdot x+\frac{\beta^2}{(n+\gamma)^2}$

Table 3.3: First and second moments of $S_n^{<\alpha,\beta,\gamma>}$

In terms of a K-functional we arrive at the following pointwise inequality:

Theorem 3.11 *For all $f \in C[0,1]$ we have*

$$|\mathbb{B}_n^{(\alpha,\lambda)}(f;x)-f(x)| \leq 2\cdot K_2\left(f; x(1-x)\cdot\left(1-\frac{1}{(1+\alpha)(1+\lambda)}\cdot\frac{n-1}{n}\right)\right),$$

where K_2 is Peetre's second order K-functional, see (1.19) and $x \in [0,1]$.

Proof. The (standard) method is to consider the Taylor expansion with integral remainder. Let $x \in [0,1]$, $n \in \mathbb{N}$ be fixed and $g \in C^2[0,1]$. Thus we have,

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-s)g''(s)ds.$$

Hence, by $\mathbb{B}_n^{(\alpha,\lambda)} e_i = e_i$, $i = 0,1$, we obtain

$$\mathbb{B}_n^{(\alpha,\lambda)}(g;x) - g(x) = \mathbb{B}_n^{(\alpha,\lambda)}\left(\int_x^t (t-s)g''(s)ds; x\right).$$

In view of Lemma 3.10

$$\begin{aligned}
|\mathbb{B}_n^{(\alpha,\lambda)}(g;x)-g(x)| &\le \mathbb{B}_n^{(\alpha,\lambda)}\left(\left|\int_x^t |t-s|\cdot|g''(s)|ds\right|;x\right)\\
&\le \mathbb{B}_n^{(\alpha,\lambda)}((t-x)^2;x)\cdot\|g''\|_\infty\\
&= x(1-x)\cdot\left(1-\frac{1}{(1+\alpha)(1+\lambda)}\cdot\frac{n-1}{n}\right)\cdot\|g''\|_\infty.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|\mathbb{B}_n^{(\alpha,\lambda)}(g;x)-g(x)| &\le |\mathbb{B}_n^{(\alpha,\lambda)}(f-g;x)-(f-g)(x)|+|\mathbb{B}_n^{(\alpha,\lambda)}(g;x)-g(x)|\\
&\le 2\|f-g\|_\infty + x(1-x)\left(1-\frac{1}{(1+\alpha)(1+\lambda)}\frac{n-1}{n}\right)\|g''\|_\infty.
\end{aligned}$$

Taking in the above inequality the infimum over all $g\in C^2[0,1]$ we arrive at the desired inequality. □

Remark 3.12 If we use the equivalence relation between K_2 and ω_2, see relation (1.20), the latter inequality can be continued:

$$|\mathbb{B}_n^{(\alpha,\lambda)}(f;x)-f(x)|\le C\omega_2\left(f;\sqrt{x(1-x)\cdot\left(1-\frac{1}{(1+\alpha)(1+\lambda)}\cdot\frac{n-1}{n}\right)}\right),$$

where C is an absolute constant. However the disadvantages of this approach are obviously: we do not obtain sharp constant.

Therefore, for the next pointwise estimate we apply Theorem 1.38 both for $S_n^{<\alpha,\beta,\gamma>}$ and $\mathbb{B}_n^{(\alpha,\lambda)}$:

Theorem 3.13 *For $f\in C[0,1]$, and $x\in[0,1]$, there holds*

$$\begin{aligned}
a)\qquad & |S_n^{<\alpha,\beta,\gamma>}(f;x)-f(x)|\le\frac{1}{h}\cdot\left|\frac{\beta}{n+\gamma}-\frac{\gamma}{n+\beta}\cdot x\right|\cdot\omega_1(f,h)\\
+\ & \left\{1+\frac{1}{2h^2}\left\{\left[\frac{n(n-1)}{(n+\gamma)^2(1+\alpha)}-\frac{2n}{n+\gamma}+1\right]\cdot x^2\right.\right.\\
+\ & \left.\left.\left[\frac{(1+2\beta)n}{(n+\gamma)^2}+\frac{n\alpha(n-1)}{(n+\gamma)^2(1+\alpha)}-\frac{2\beta}{n+\gamma}\right]\cdot x+\frac{\beta^2}{(n+\gamma)^2}\right\}\right\}\cdot\omega_2(f,h),
\end{aligned}$$

with $0<h\le\frac{1}{2}$.

$$b)|\mathbb{B}_n^{(\alpha,\lambda)}(f;x)-f(x)|\le\left[1+\frac{1}{2h^2}x(1-x)\cdot\left(1-\frac{1}{(1+\alpha)(1+\lambda)}\cdot\frac{n-1}{n}\right)\right]\omega_2(f;h),$$

$h>0$.

For an adequate choice of $h > 0$ the following pointwise error estimate are successively obtained, with no claim that the constants in front of ω_2 are optimal:

Corollary 3.14 *For $f \in C[0,1]$ and $x \in [0,1]$, we have*

a) $|\mathbb{B}_n^{(\alpha,\lambda)}(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{x(1-x)\left(1 - \frac{1}{(1+\alpha)(1+\lambda)} \cdot \frac{n-1}{n}\right)}\right)$,

b) $|F_n^{\alpha}(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{\frac{x(1-x)}{1+\alpha}\left(\alpha + \frac{2}{n+1}\right)}\right)$. *Compare with [48, Theorem 1].*

c) $|S_n^{<\alpha,0,0>}(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{x(1-x)\frac{n\alpha+1}{n(1+\alpha)}}\right)$,

d) $|S_n^{<1/n,0,0>}(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{x(1-x)\frac{2}{n+1}}\right)$,

e) $|U_n^{\lambda}(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{x(1-x)\frac{n\lambda+1}{n(1+\lambda)}}\right)$,

f) $|U_n(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{x(1-x)\frac{2}{n+1}}\right)$. *Studies were made in [80, Theorem 37, p. 51] regarding the lower bound of the constant in front of ω_2 in uniform estimates.*

g) $|(\tilde{\mathbb{B}}_\alpha \circ \tilde{\mathbb{B}}_\lambda)(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{x(1-x)\frac{\alpha+\beta+\alpha\cdot\lambda}{(1+\alpha)(1+\lambda)}}\right)$,

h) $|\tilde{\mathbb{B}}_\alpha(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{x(1-x)\frac{\alpha}{1+\alpha}}\right)$,

i) $|\bar{\mathbb{B}}(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{\frac{x(1-x)}{1+n}}\right)$,

j) $|B_n(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{\frac{x(1-x)}{n}}\right)$. *See also Remark 2.23.*

3.4 Degree of simultaneous approximation

Bl. Sendov & V. Popov formulated for the first time in [139] a (non-quantitative) *Korovkin type* theorem for the Banach space $C^r[K]$, $K = [a,b]$. Later, G. I. Kudrjavcev [91] (for $r = 1$) and H.-B. Knoop & P. Pottinger [87] (for the more general case $r \geq 1$) were the first who proved estimates for simultaneous approximation involving ω_1, in the spirit of Shisha's and Mond's theorem from [140]. In 1984 H. Gonska generalized the result of Knoop & Pottinger by measuring the degree of (simultaneous) approximation in terms of ω_2, the second order modulus of smoothness, see [57]. D. P. Kacsó improved this last assertion by employing Păltănea's Theorem 1.38, see [77] or [79]. We shall recall her result, but first let us prove the following

Theorem 3.15 *Let $r \in \mathbb{N}$ and the operator $L : C^r(K) \to C^r(K')$ be almost convex of order $r-1$. If L is degree reducing, i.e., $L(\prod_{r-1}) \subseteq \prod_{r-1}$, then for all $f \in C^r(K)$, $x \in K'$, $0 < h \leq \frac{1}{2}$length(K) and $s \geq 2$ even the following holds:*

$$\begin{aligned} |D^r L(f;x) - D^r f(x)| &\leq \left|\frac{1}{r!} D^r L(e_r;x) - 1\right| \cdot |D^r f(x)| + \frac{1}{h} \cdot \gamma_L(x) \cdot \omega_1(D^r f; h) \\ &+ \left[D^r L\left(\frac{1}{r!}e_r; x\right) + \frac{1}{2h^s} \cdot \beta_L(x)\right] \cdot \omega_2(D^r f; h), \end{aligned} \tag{3.14}$$

where

$$\gamma_L(x) := \left|D^r L\left(\frac{1}{(r+1)!}e_{r+1} - \frac{1}{r!}x \cdot e_r; x\right)\right| \text{ and} \tag{3.15}$$

$$\beta_L(x) := D^r L\left(\sum_{i=0}^{s} (-1)^{s-i} \frac{s!}{(s-i)!(i+r)!} x^{s-i} e_{r+i}; x\right). \tag{3.16}$$

Proof. In the following we shall adapt the proof from Theorem 2.1 in [87]. Consider $I_r : C(K) \to C^r(K)$ defined by $(I_r f)(x) = \int\limits_a^x \frac{(x-t)^{r-1}}{(r-1)!} \cdot f(t)dt$. Let $Q : C(K) \to C(K')$ be $Q := D^r \circ L \circ I_r$. Since L is almost convex of order $r-1$, it follows that Q is a linear and positive (convex of order -1) operator. Since $L(I_r D^r f - f) \in \prod_{r-1}$ and $L(\prod_{r-1}) \subseteq \prod_{r-1}$, we have $L(I_r D^r f - f) \in \prod_{r-1}$. It follows $D^r L I_r D^r f = D^r L f$, hence $Q D^r f = D^r L f$, for all $f \in C^r(K)$.
We apply now Theorem 1.38 for an arbitrary function $g \in C(K)$, $s \geq 2$ even and for any $0 < h \leq \frac{1}{2}$length(K):

$$\begin{aligned} |Q(g;x) - g(x)| &\leq |Q(e_0;x) - 1| \cdot |g(x)| + \frac{1}{h} \cdot |Q(e_1 - x; x)| \cdot \omega_1(g;h) \\ &+ \left[Q(e_0;x) + \frac{1}{2h^s} Q\left((e_1 - x)^s; x\right)\right] \cdot \omega_2(g;h). \end{aligned} \tag{3.17}$$

Putting $g = D^r f$ for $f \in C^r(K)$ and taking into account that $QD^r f = D^r L f$, the left hand side in (3.17) is equal to $|D^r L(f;x) - D^r f(x)|$. Furthermore, from $L(\prod_{r-1}) \subseteq \prod_{r-1}$ we also conclude that

$$Q((e_1 - x)^s; x) = D^r L\left(\sum_{i=0}^{s} (-1)^{s-i} \frac{s!}{(s-i)!(i+r)!} x^{s-i} e_{r+i}; x\right) = \beta_L(x),$$

$$
\begin{aligned}
Q((e_1-x);x) &= D^r L\left(\frac{1}{(r+1)!}e_{r+1}-\frac{1}{r!}x\cdot e_r;x\right)\\
&= \pm\gamma_L(x).
\end{aligned}
$$

Note that $Q(e_0;x) = D^r L\left(\frac{1}{r!}e_r;x\right)$. Substituting these quantities into (3.17) we arrive at the desired estimate. □

It is obviously that by choosing $s=2$ in (3.14) we obtain D. Kacsó's result; see with Theorem 3 in [77]. In the sequel we shall only consider $s=2$.

In the following subsections we shall compute/recall the degree of approximation for some selected Beta-type operators, namely for U_n^α and U_n, for some instances of $S_n^{<\alpha,\beta,\gamma>}$ and also for B_n.

3.4.1 Estimates for general Stancu operators

The first results in simultaneous approximation by Stancu operators (more exactly for $S_n^{<\alpha,0,0>},\ \alpha=\alpha(n)=o(1/n)$) were provided by G. Mastroianni & M. R. Occorsio in 1978 in their joint work [102]. Their estimates were given in terms of ω_1. Later, in 1996 O. Agratini proved in [3] that under appropriate assumptions on the three involved parameters, $D^r S_n^{<\alpha,\beta,\gamma>}f,\ f\in C^r[0,1],\ 0\le r\le n$ converges uniformly toward $D^r f$. The degree of approximation was computed in terms of ω_1. In this sequel we shall often relate to some results obtained in [102] and integrate some of their notations in the sequel.

In the following we will refine the known results, namely we will compute the degree of simultaneous approximation by $S_n^{<\alpha,0,0>}$ and $S_n^{<0,\beta,\gamma>}$ via ω_1 and ω_2. In order to avoid long computations we make use of Zhuk's function $Z_h f$, see its definition in Section 1.5, and we ignore for the moment Theorem 3.15.

Corollary 3.16 *Let $r\in\mathbb{N}\cup\{0\},\ n\ge r+2,\ f\in C^r[0,1],\ x\in[0,1],\ 0<h\le\frac{1}{2}$ and the positive parameter α. Then*

$$
\begin{aligned}
|D^r S_n^{<\alpha,0,0>}(f;x) &- D^r f(x)| \le \left(\frac{1}{(1-\alpha n)^r}-\beta_{n,r}^\alpha\right)\cdot\|D^r f\|_\infty+\frac{1}{h}\cdot\frac{2r}{n}\cdot\omega_1(D^r f;h)\\
&+ 3\left[1+\frac{1}{2}\left(\frac{1}{(1-\alpha n)^r}-\beta_{n,r}^\alpha\right)+\frac{r}{2n}\cdot\frac{1}{h}+\frac{\delta_{n,r}(x)}{4}\cdot\frac{1}{h^2}\right]\cdot\omega_2(D^r f;h),
\end{aligned}
$$

where $\beta_{n,r}^\alpha := \frac{(n)_r}{n^r}\cdot\frac{1}{(\alpha(n-1)+1)\ldots(\alpha(n-r)+1)}$ and $\delta_{n,r}(x) := \frac{(1+\alpha(n-r))x(1-x)}{(1+\alpha)(n-r)}+\frac{3r}{n}$.

Proof. For $f\in C^r[0,1]$ due to relation (2.13) in [102, p. 277] we have the following upper bound for $|D^r S_n^{<\alpha,0,0>}(f;x)-D^r f(x)|$:

$$
(3.18)\qquad \left|D^r S_n^{<\alpha,0,0>}(f;x)-D^r f(x)\right|\le\left(4+\frac{1}{(1-\alpha n)^r}-\beta_{n,r}^\alpha\right)\|D^r f\|_\infty.
$$

Taking into account (2.17) on p. 280 in [102] for a function $g \in C^{r+2}[0,1]$ we arrive at:

$$
\text{(3.19)}\qquad \left|D^r S_n^{<\alpha,0,0>}(g;x) - D^r g(x)\right| \leq \delta_{n,r}(x)\frac{\|D^{r+2}g\|_\infty}{2} + \frac{r}{n}\|D^{r+1}g\|_\infty + \left(\frac{1}{(1-\alpha n)^r} - \beta_{n,r}^\alpha\right)\|D^r g\|_\infty .
$$

Now for any $f \in C^r[0,1]$ and $g \in C^{r+2}[0,1]$ we can write

$$
\begin{aligned}
& \left|D^r S_n^{<\alpha,0,0>}(f;x) - D^r f(x)\right| \\
\leq\ & \left|D^r S_n^{<\alpha,0,0>}((f-g);x) - D^r (f-g)(x)\right| + \left|D^r S_n^{<\alpha,0,0>}(g;x) - D^r g(x)\right| \\
\leq\ & \left(4 + \frac{1}{(1-\alpha n)^r} - \beta_{n,r}^\alpha\right)\|D^r(f-g)\|_\infty + \delta_{n,r}(x)\frac{\|D^{r+2}g\|_\infty}{2} + \frac{r}{n}\|D^{r+1}g\|_\infty \\
+\ & \left(\frac{1}{(1-\alpha n)^r} - \beta_{n,r}^\alpha\right)\|D^r g\|_\infty .
\end{aligned}
$$

We substitute now $g^{(r)} \in C^2[0,1]$ by $B_n(Z_h(f^{(r)})) \in C^2[0,1]$. Due to the inequalities in Lemmas 1.24, 1.25 and 1.27 we arrive for a sufficiently large n, a fixed $\varepsilon > 0$ and $0 < h \leq \frac{1}{2}$ at:

$$
\begin{aligned}
\|(f-g)^{(r)}\|_\infty &\leq \|f^{(r)} - Z_h(f^{(r)})\|_\infty + \|Z_h((f^{(r)})) - B_n(Z_h(f^{(r)}))\|_\infty \\
&\leq \frac{3}{4}\cdot\omega_2(D^r f;h) + \varepsilon, \\
\|(g^{(r)})''\|_\infty &\leq \|(Z_h(f^{(r)}))''\|_{L_\infty} \leq \frac{3}{2}\cdot\frac{1}{h^2}\cdot\omega_2(D^r f;h), \\
\|(g^{(r)})'\|_\infty &\leq \|(Z_h(f^{(r)}))'\|_\infty \leq \frac{1}{h}\left[2\cdot\omega_1(D^r f;h) + \frac{3}{2}\cdot\omega_2(D^r f;h)\right] \text{ and,} \\
\|g^{(r)}\|_\infty &\leq \|Z_h(f^{(r)})\|_\infty \leq \|D^r f\|_\infty + \frac{3}{4}\cdot\omega_2(D^r f;h).
\end{aligned}
$$

We let $\varepsilon \to 0$ and afterwards we substitute the obtained inequalities into (3.20). By regrouping the terms we arrive at the desired estimate. □

We can easily see that we obtain similar assertions as in [3], regarding the uniform convergences: choose $h := \frac{1}{\sqrt{n}}$ and consider $\alpha := \alpha(n) = o(1/n)$. Under these conditions the following inequality holds:

$$
\delta_{n,r}(x) \leq 2x(1-x)\frac{1}{n-r} + \frac{3r}{n} \leq \frac{1}{2(n-r)} + \frac{3r}{n} = \mathcal{O}\left(\frac{1}{n}\right).
$$

With the same assumptions as in Corollary 3.16 we arrive at

Corollary 3.17

$$\|D^r S_n^{<\alpha,0,0>} f - D^r f\|_\infty \le \left(\frac{1}{(1-\alpha n)^r} - \beta_{n,r}^\alpha\right) \cdot \|D^r f\|_\infty + \frac{2r}{\sqrt{n}} \omega_1\left(D^r f; \frac{1}{\sqrt{n}}\right)$$
$$+ \quad 3\left[1 + \frac{n}{2(n-r)} + 3r + \frac{r}{2\sqrt{n}} + \frac{1}{2}\left(\frac{1}{(1-\alpha n)^r} - \beta_{n,r}^\alpha\right)\right] \omega_2\left(D^r f; \frac{1}{\sqrt{n}}\right).$$

Remark 3.18 For $\alpha := \alpha(n) = o(1/n)$ we have also $\lim_{n\to\infty} \frac{1}{(1-\alpha n)^r} - \beta_{n,r}^\alpha = 0$.

We consider in the sequel the special case $S_n^{<0,\beta,\gamma>}$ with $0 \le \beta \le \gamma$ real numbers. For these operators and their derivatives we shall give error estimates using directly Theorem 3.15 and not by making a detour via the Zhuk function as above.
For this simpler situation we are able to give an explicit representation of the images of the monomials $e_k,\ 0 \le k \le n$ under $S_n^{<0,\beta,\gamma>}$, which is similar to the one for the Bernstein operators.

Lemma 3.19 *For any $f \in C[0,1],\ x \in [0,1]$ and $e_k(x) := x^k,\ 0 \le k \le n$ we have*

$$S_n^{<0,\beta,\gamma>}(e_k; x) = \sum_{j=0}^{k} (n)_j \left[\sum_{i=j}^{k} \binom{k}{i} u^i v^{k-i} \frac{S(i,j)}{n^i}\right] x^j, \tag{3.20}$$

where $u := \frac{n}{n+\gamma},\ v := \frac{\beta}{n+\gamma}$ and $S(i,j)$ are the Stirling numbers of the second kind.

Proof. Due to the close relation between $S_n^{<0,\beta,\gamma>}$ and Bernstein operators ($S_n^{<0,\beta,\gamma>}(f;x) = B_n(f \circ (ue_1 + ve_0); x)$) we can write

$$\begin{aligned}
S_n^{<0,\beta,\gamma>}(e_k; x) &= B_n((ue_1 + ve_0)^k; x) = B_n\left(\sum_{i=0}^{k} \binom{k}{i} u^i v^{k-i} e_i; x\right) \\
&= \sum_{i=0}^{k} \binom{k}{i} u^i v^{k-i} B_n(e_i; x) = \sum_{i=0}^{k} \binom{k}{i} u^i v^{k-i} \frac{1}{n^i} \sum_{j=0}^{i} (n)_j S(i,j) x^j \\
&= \sum_{j=0}^{k} (n)_j \left[\sum_{i=j}^{k} \binom{k}{i} u^i v^{k-i} \frac{S(i,j)}{n^i}\right] x^j.
\end{aligned}$$

We used above the expansion in terms of the *Stirling numbers of the second kind* for $B_n e_i$ (see [82]):

$$B_n(e_i; x) = \frac{1}{n^i} \sum_{j=0}^{i} (n)_j \cdot S(i,j) x^j,\ i = 0, \ldots, n \tag{3.21}$$

Thus we have proved (3.20). □

Now we can make statements for quantitative simultaneous approximation for $S_n^{<0,\beta,\gamma>}$:

Corollary 3.20 *Let* $r \in \mathbb{N} \cup \{0\}$, $f \in C^r[0,1]$, $x \in [0,1]$, $0 < h \le \frac{1}{2}$ *and* $n \ge \max\{r+2, r(r+1)\}$. *Then*

$$|D^r S_n^{<0,\beta,\gamma>}(f;x) - D^r f(x)| \le \frac{r(r-1+\gamma)}{n+\gamma} \cdot |D^r f(x)| + \frac{1}{h} \cdot \frac{|\beta + \frac{r}{2}|}{n+\gamma} \cdot \omega_1(D^r f; h)$$
$$+ \quad \left[1 + \frac{1}{2h^2} \cdot \left(\frac{n - r(r+1)}{4(n+\gamma)^2} + 2\frac{(r+\gamma)(\gamma-\beta)}{(n+\gamma)^2} + \frac{12\beta(\beta+1) + r(3r+1)}{12(n+\gamma)^2}\right)\right]$$
$$\cdot \quad \omega_2(D^r f; h).$$

Proof. For the quantities appearing in Theorem 3.15, one has

$$\begin{aligned}
\left|\frac{1}{r!} D^r S_n^{<0,\beta,\gamma>}(e_r;x) - 1\right| &= 1 - \frac{(n)_r}{(n+\gamma)^r} = 1 - \frac{1}{\prod_{i=0}^{r-1} \frac{n+\gamma}{n-i}} = 1 - \frac{1}{\prod_{i=0}^{r-1}\left(1 + \frac{\gamma+i}{n-i}\right)} \\
&\le 1 - \frac{1}{\left(1 + \frac{\gamma+r-1}{n}\right)^r} = 1 - \left(1 - \frac{\gamma+r-1}{n+r-1+\gamma}\right)^r \\
&\le 1 - \left(1 - \frac{r(\gamma+r-1)}{n+r-1+\gamma}\right) \le \frac{r(\gamma+r-1)}{n+\gamma}.
\end{aligned}$$

Also due to relation (3.20) we have

$$\begin{aligned}
\gamma_{S_n^{<0,\beta,\gamma>}}(x) &:= \left|D^r S_n^{<0,\beta,\gamma>}\left(\frac{1}{(r+1)!} e_{r+1} - \frac{1}{r!} x \cdot e_r\right)(x)\right| \\
&= \frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{n+\gamma} | - x(r+\gamma) + \beta + \frac{r}{2}| \le \frac{\beta + \frac{r}{2}}{n+\gamma}.
\end{aligned}$$

Finally, for $\beta_{S_n^{<0,\beta,\gamma>}}(x)$ we have

$$\beta_{S_n^{<0,\beta,\gamma>}}(x) := D^r \left(\frac{2}{(r+2)!} e_{r+2} - \frac{2}{(r+1)!} x \cdot e_{r+1} + \frac{1}{r!} x^2 \cdot e_r\right)(x)$$
$$\begin{aligned}
&= \frac{(n)_r}{(n+\gamma)^r}\left[x^2\left(-\frac{n-r^2-r}{(n+\gamma)^2} + \frac{2\gamma(r+\gamma)}{(n+\gamma)^2}\right) + x\left(\frac{n-r^2-r}{(n+\gamma)^2} - \frac{2\beta(r+\gamma)}{(n+\gamma)^2}\right)\right. \\
&+ \left.\frac{12\beta(\beta+1) + r(3r+1)}{12(n+\gamma)^2}\right] \\
&= \frac{(n)_r}{(n+\gamma)^r}\left[\frac{n-r^2-r}{(n+\gamma)^2} x(1-x) + 2x \cdot \frac{r+\gamma}{(n+\gamma)^2} \cdot (\gamma x - \beta)\right. \\
&+ \left.\frac{12\beta(\beta+1) + r(3r+1)}{12(n+\gamma)^2}\right] \\
&\le \frac{1}{4} \cdot \frac{n-r^2-r}{(n+\gamma)^2} + 2\frac{(r+\gamma)(\gamma-\beta)}{(n+\gamma)^2} + \frac{12\beta(\beta+1) + r(3r+1)}{12(n+\gamma)^2}.
\end{aligned}$$

Hence, we are lead to the desired inequality. $\square$

It is obvious that in the above estimate the "Bernstein case" is hidden. If we substitute $\beta = \gamma = 0$ we obtain a similar estimate to [79]. However, the degree of simultaneous approximation by Bernstein operators will be shortly described in one of the following subsections.

3.4.2 Estimates for $U_n^\alpha = B_n \circ \tilde{\mathbb{B}}_\alpha$

From the representation of U_n^α and due to Remark 3.9 it is obvious that the operators verify the requirements of Theorem 3.15 (with $s = 2$), i.e., they are (almost) convex of any order and are degree reducing. Thus we can prove the following:

Corollary 3.21 *Let $r \in \mathbb{N} \cup \{0\}$, $f \in C^r[0,1]$, $x \in [0,1]$, $0 < h \leq \frac{1}{2}$ and n sufficiently large, e.g., $n \geq r+2$. Then*

$$
\begin{aligned}
&|D^r U_n^\alpha(f;x) - D^r f(x)| \leq \left(1 - \frac{(n)_r}{n^r} \cdot \frac{1}{1^{[r,-\alpha]}}\right) \cdot |D^r f(x)| \\
+ \quad & \frac{1}{h} \cdot \frac{(n)_r}{n^r} \cdot \frac{r(n\alpha+1)}{2n(\alpha r+1)} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot |1-2x| \cdot \omega_1(D^r f; h) \\
+ \quad & \frac{(n)_r}{n^r} \cdot \frac{1}{1^{[r,-\alpha]}} \left\{1 + \frac{1}{2h^2}\left[\left(\frac{(n-r)(n-r-1)}{n^2(\alpha r+1)(\alpha(r+1)+1)} - \frac{2(n-r)}{n(\alpha r+1)} + 1\right) x^2\right.\right. \\
& \frac{1}{\alpha r+1}\left(\frac{(n\alpha+1)(n-r)(r+1)}{n^2(\alpha(r+1)+1)} - \frac{r}{n} - \alpha r\right) x \\
& \left.\left.\frac{r}{(\alpha r+1)(\alpha(r+1)+1)}\left(\frac{3r+1}{12n^2} + \frac{(r+1)\alpha}{2n} + \frac{(3r+5)\alpha^2}{12}\right)\right]\right\} \omega_2(D^r f; h),
\end{aligned}
$$

where $y^{[m,-\alpha]}$ are the factorial power.

Proof. We recall relation (3.11):

$$
\tilde{\mathbb{B}}_\alpha(e_r;x) = \frac{x \ldots (x + \alpha(r-1))}{(\alpha+1)\ldots(\alpha(r-1)+1)} \in \prod_r .
$$

For our further computations we need at least the first three coefficients (in descending order) of the polynomial $\tilde{\mathbb{B}}_\alpha e_r$. By employing the relation of Viéte we arrive at

$$
\begin{aligned}
(3.22) \qquad & \tilde{\mathbb{B}}_\alpha(e_r;x) \\
= \quad & \frac{1}{1^{[r,-\alpha]}} \cdot \left[x^r + \alpha\frac{r(r-1)}{2}x^{r-1} + \alpha^2\frac{(r-2)(r-1)r(3r-1)}{24}x^{r-2} + \ldots\right].
\end{aligned}
$$

But U_n^α means $B_n \circ \tilde{\mathbb{B}}_\alpha$, so after applying the Bernstein operator on (3.22) we arrive at

$$\begin{aligned} U_n^\alpha(e_r;x) &= \frac{1}{1^{[r,-\alpha]}} \cdot \left\{ \frac{(n)_r}{n^r} x^r + \binom{r}{2} \frac{(n)_{r-1}}{n^r} \cdot (\alpha n+1) x^{r-1} \right. \\ &+ \left. \frac{(n)_{r-2}}{n^{r-2}} \cdot \frac{(r-2)(r-1)r}{4} \left[\frac{3r-5}{6n^2} + \frac{\alpha(r-1)}{n} + \frac{\alpha^2(3r-1)}{6} \right] x^{r-2} + \ldots \right\}. \end{aligned}$$

Above we reused relation (3.21) and also the well-known identity

$$S(k,j) = \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} i^k, \ 0 \leq j \leq k,$$

compare with L. Comtet [31].

These preparations were necessary for computing the quantities that appear in Theorem 3.15. Thus one has:

$$\begin{aligned} \left| \frac{1}{r!} D^r U_n^\alpha(e_r;x) - 1 \right| &= 1 - \frac{1}{1^{[r,-\alpha]}} \cdot \frac{(n)_r}{n^r}, \\ \gamma_{U_n^\alpha}(x) &= \frac{(n)_r}{n^r} \cdot \frac{r(n\alpha+1)}{2n(\alpha r+1)} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot |1-2x| \ \textit{and} \\ \beta_{U_n^\alpha}(x) &= \frac{(n)_r}{n^r} \cdot \frac{1}{1^{[r,-\alpha]}} \left\{ \left[\left(\frac{(n-r)(n-r-1)}{n^2(\alpha r+1)(\alpha(r+1)+1)} - \frac{2(n-r)}{n(\alpha r+1)} + 1 \right) x^2 \right.\right. \\ & \quad \frac{1}{\alpha r+1} \left(\frac{(n\alpha+1)(n-r)(r+1)}{n^2(\alpha(r+1)+1)} - \frac{r}{n} - \alpha r \right) x \\ & \quad \left.\left. \frac{r}{(\alpha r+1)(\alpha(r+1)+1)} \left(\frac{3r+1}{12n^2} + \frac{(r+1)\alpha}{2n} + \frac{(3r+5)\alpha^2}{12} \right) \right] \right\}. \end{aligned}$$

Substituting them into (3.14) we arrive at the desired estimate. □

In order to have a more instructive insight into the above estimate we set $\alpha = \alpha(n) := \frac{1}{n}$. Thus we arrive at the *genuine Bernstein-Durrmeyer* operators and the inequality in Corollary 3.21 reads as in the following

Corollary 3.22 *Let $r \in \mathbb{N} \cup \{0\}$, $f \in C^r[0,1]$, $x \in [0,1]$ and $0 < h \leq \frac{1}{2}$. Then*

$$\begin{aligned} & |D^r U_n(f;x) - D^r f(x)| \leq \left(1 - \frac{(n)_r}{(n+r-1)_r} \right) \cdot |D^r f(x)| \\ + & \frac{1}{h} \cdot \frac{(n)_r}{(n+r-1)_r} \cdot \frac{r}{n+r} \cdot \omega_1(D^r f;h) + \frac{(n)_r}{(n+r-1)_r} \\ \cdot & \left[1 + \frac{1}{2h^2} \cdot \frac{2(2r^2+2r-n)x^2 - 2(2r^2+2r-n)x + r(r+1)}{(n+r)(n+r+1)} \right] \cdot \omega_2(D^r f;h) \\ \leq & \frac{r(r-1)}{n} \cdot |D^r f(x)| + \frac{1}{h} \cdot \frac{r}{n+r} \cdot \omega_1(D^r f;h) \\ + & \left[1 + \frac{1}{2h^2} \cdot \frac{\max\left\{\frac{n}{2}, r(r+1)\right\}}{(n+r)(n+r+1)} \right] \cdot \omega_2(D^r f;h). \end{aligned}$$

Choosing in the upper inequality $h := \frac{1}{\sqrt{n+1}}$ we obtain the uniform estimate of

Corollary 3.23

$$\begin{aligned}\|D^r U_n f - D^r f\|_\infty &\le \frac{r(r-1)}{n}\|D^r f\|_\infty + \frac{r}{n+r}\cdot\sqrt{n+1}\omega_1\left(D^r f; \frac{1}{\sqrt{n+1}}\right)\\ &+ \frac{5}{4}\cdot\omega_2\left(D^r f; \frac{1}{\sqrt{n+1}}\right).\end{aligned}$$

These estimates for the U_n's (both the pointwise and the uniform) were computed by D. Kacsó in her recent work [80]. For this purpose she used the explicit representation of the $r-$th derivatives of U_n (see [80, p.60]), namely

$$D^r U_n(f;x) = \frac{(n)_r}{(n+r-2)_r}\sum_{k=0}^{n-r} p_{n-r,k}(x)\cdot\int_0^1 p_{n+r-2,k+r-1}(t)\cdot f^{(r)}(t)dt.$$

3.4.3 Estimates for the Bernstein operators

G. G. Lorentz gave in [92] (p. 12) the following beautiful representation for the derivatives of the *Bernstein polynomials*:

$$D^r B_n(f;x) = \frac{(n)_r}{n^r}\cdot r!\sum_{k=0}^{n-r}\left[\frac{k}{n},\ldots,\frac{k+r}{n};f\right]\binom{n-r}{k}x^k(1-x)^{n-r-k},$$

It immediately implies that B_n is (almost) convex of order $r-1$. D. Kacsó computed the following estimates in [79]:

Corollary 3.24 *Let $r \in \mathbb{N}\cup 0$, $n \ge \max\{r+2, r(r+1)\}$. Then for all $f \in C^r[0,1]$ and $x \in [0,1]$, one has*

$$\begin{aligned}|D^r B_n(f;x) - D^r f(x)| &\le \frac{r(r-1)}{2n}\cdot|D^r f(x)| + \frac{r}{2\sqrt{n}}\cdot\omega_1\left(D^r f; \frac{1}{\sqrt{n}}\right)\\ &+ \frac{9}{8}\cdot\omega_2\left(D^r f; \frac{1}{\sqrt{n}}\right),\end{aligned}$$

and

$$\begin{aligned}|D^r B_n(f;x) - D^r f(x)| &\le \frac{r(r-1)}{2n}\cdot|D^r f(x)| + \omega_1\left(D^r f; \frac{r}{n}\right)\\ &+ \frac{9}{8}\cdot\omega_2\left(D^r f; \frac{1}{\sqrt{n-r}}\right).\end{aligned}$$

3.5 Global smoothness preservation

In the context of simultaneous approximation another natural question had risen and has been studied during the recent years: whether simultaneous approximation processes also preserve global smoothness of the derivatives of an r-times differentiable function f. The first assertion was obtained by C. Cottin & H. Gonska, see [33, Theorem 2.2]. More information on this subject can be found in the recent book of G. A. Anastassiou & S. G. Gal [6].

Proposition 3.25 *Let $r \geq 0$ and $s \geq 1$ be integers, and let K and K' be given as above. Furthermore, let $L : C^r(K) \to C^r(K')$ be a linear operator having the following properties:*

(i) L is almost convex of orders $r-1$ and $r+s-1$,

(ii) L maps $C^{(r+s)}(K)$ into $C^{(r+s)}(K')$,

(iii) $L(\prod_{r-1}) \subseteq \prod_{r-1}$ and $L(\prod_{r+s-1}) \subseteq \prod_{r+s-1}$

(iv) $L(C^r(K)) \nsubseteq \prod_{r-1}$.

Then for all $f \in C^r(K)$ and all $\delta \geq 0$ we have

$$K_s(D^r Lf;\delta)_{K'} \leq \frac{1}{r!} \cdot \|D^r Le_r\| \cdot K_s\left(f^{(r)}; \frac{1}{(r+s)_s} \cdot \frac{\|D^{r+s}Le_{r+s}\|}{\|D^r Le_r\|} \cdot \delta\right)_K .$$

In the above, K_s is the Peetre K-functional of order s, $s \geq 1$, defined by

$$K_s(f;\delta) := K(f;\delta;C[0,1],C^s[0,1]) := \inf\{\|f-g\|_\infty + \delta \cdot \|g^{(s)}\|_\infty : g \in C^s[0,1]\},$$

and $\prod_{-1} := 0$.

In the following subsections we can prove that $S_n^{<\alpha,\beta,\gamma>}$, with $\alpha \geq 0,\ 0 \leq \beta \leq \gamma$, and $\mathbb{B}_n^{(\alpha,\lambda)} = \tilde{\mathbb{B}}_\alpha \circ B_n \circ \tilde{\mathbb{B}}_\lambda$, with $\alpha,\ \lambda \geq 0$ and $n \in \mathbb{N}$, meet the requirements of Proposition 3.25.

3.5.1 Application to general Stancu operators

Theorem 3.26 *Let $r \geq 0,\ s \geq 1$ be fixed integers. Then for all $n \geq r+s,\ f \in C^r[0,1]$ and $\delta \geq 0$, the following estimates hold:*

$$\begin{aligned}
& K_s(D^r S_n^{<\alpha,\beta,\gamma>} f;\delta)_{[0,1]} \\
\leq\ & \frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot K_s\left(D^r f; \frac{(n-r)_s}{(n+\gamma)^s} \cdot \frac{1^{[r,-\alpha]}}{1^{[r+s,-\alpha]}} \cdot \delta\right)_{[0,1]} \\
\leq\ & K_s(D^r f;\delta)_{[0,1]}.
\end{aligned}$$

Proof. In [102] was proved that $S_n^{<\alpha,0,0>}$ are (almost) convex of all orders. The same is true for the more general $S_n^{<\alpha,\beta,\gamma>}$, see e.g., Remark 3.9. The rest of the conditions (ii)–(iv) are easily verified: Since $n \geq r+s$ and both operators map a polynomial of degree i, $0 \leq i \leq n$, into a polynomial of degree i, both conditions (ii)–(iii) are satisfied. Regarding (iv) it is clear that $B_n e_r \in \prod_r \setminus \prod_{r-1}$ and further it follows that $\tilde{\mathbb{B}}_\alpha(B_n e_r) \in \prod_r \setminus \prod_{r-1}$.
In proving inequality (3.26) we use the representation:

$$\begin{aligned} D^r S_n^{<\alpha,\beta,\gamma>}(e_r;x) &= D^r \tilde{\mathbb{B}}_\alpha B_n \left[\left(\frac{n}{n+\gamma}e_1 + \frac{\beta}{n+\gamma}e_0\right)^r ; x\right] \\ &= D^r \tilde{\mathbb{B}}_\alpha \left(\frac{n^r}{(n+\gamma)^r} \cdot \frac{(n)_r}{n^r} \cdot e_r + \text{terms of lower degree}; x\right) \\ &= D^r \left(\frac{n^r}{(n+\gamma)^r} \cdot \frac{(n)_r}{n^r} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot e_r(x) + \text{terms of lower degree}\right) \\ &= r! \cdot \frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{1^{[r,-\alpha]}}. \end{aligned} \tag{3.23}$$

Substituting these expressions into the inequality of Proposition 3.25 yields our estimates. □

Putting in the above $s = 1$ we are lead to the following estimates in terms of ω_1 respectively $\tilde{\omega}_1$, the *least concave majorant* of $\omega_1(f;\cdot)$.

Proposition 3.27 *Let $r \geq 0$ be a fixed integer. Then for all $n \geq r+1$, $f \in C^r[0,1]$ and $\delta \geq 0$ we have*

$$\begin{aligned} \omega_1(D^r S_n^{<\alpha,\beta,\gamma>} f;\delta) &\leq \frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot \tilde{\omega}_1\left(D^r f; \frac{n-r}{(n+\gamma)(\alpha r+1)} \cdot \delta\right) \\ &\leq 1 \cdot \tilde{\omega}_1(D^r f;\delta) \leq 2 \cdot \omega_1(D^r f;\delta). \end{aligned}$$

Moreover, the leftmost inequality is best possible in the sense that for $f = e_{r+1}$ both sides are equal and do not vanish.

Proof. Obviously for $s = 1$ relation (3.23) become

$$K_1(D^r S_n^{<\alpha,\beta,\gamma>} f;\delta) \leq \frac{(n_r)}{(n+\gamma)_r} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot K_1\left(D^r f; \frac{n-r}{n+\gamma} \cdot \frac{1}{\alpha r+1} \cdot \delta\right).$$

If we take into account the known relation between K_1 and $\tilde{\omega}_1$, see (1.21), and the (double) inequality from (1.8), then we arrive to the desired estimate.
The strong statement for $f = e_{r+1}$ can be easily proved by using the property $\omega(c \cdot e_1 + d \cdot e_0;\delta) = |c| \cdot \delta$, $c, d \in \mathbb{R}$ (the same for $\tilde{\omega}_1$). Thus, for $n \geq r+1$, both sides in the leftmost inequality from above are equal to

$$(r+1)! \cdot \frac{(n)_{r+1}}{(n+\gamma)^{r+1}} \cdot \frac{1}{1^{[r+1,-\alpha]}} \cdot \delta > 0$$

for $\delta > 0$. □
Hence it follows

Corollary 3.28 *For a fixed integer $r \geq 0$ the following affirmations are true for all $n \in \mathbb{N}$. If $f^{(r)} \in Lip_\tau M$ for some $M \geq 0$ and some $0 < \tau \leq 1$, then $D^r S_n^{<\alpha,\beta,\gamma>} f$ is in the same Lipschitz class.*

Proposition 3.29 *Let $r \geq 0$ be a fixed integer. Then for all $n \geq r+2$, $f \in C^r[0,1]$ and $\delta \geq 0$ the following estimates in terms of ω_2 hold:*

$$\begin{aligned} \omega_2(D^r S_n^{<\alpha,\beta,\gamma>} f;\delta) &\leq 3 \cdot \frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{1^{[r,-\alpha]}} \\ &\cdot \left[1 + \frac{(n-r)(n-r-1)}{2(n+\gamma)^2(\alpha r+1)(\alpha(r+1)+1)}\right] \omega_2(D^r f;\delta) \\ &\leq \frac{9}{2} \cdot \omega_2(D^r f;\delta) \end{aligned}$$

Proof. From Theorem 3.26 with $s = 2$ we arrive at

$$\begin{aligned} K_2(D^r S_n^{<\alpha,\beta,\gamma>} f;\delta)_{[0,1]} &\leq \frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{1^{[r,-\alpha]}} \\ &\cdot K_2\left(f^{(r)}; \frac{(n-r)(n-r-1)}{(n+\gamma)^2} \frac{1}{(\alpha r+1)(\alpha(r+1)+1)} \cdot \delta\right)_{[0,1]} \\ &\leq K_2(f^{(r)};\delta)_{[0,1]}. \end{aligned}$$

In our further argumentation we shall employ *Zhuk's* function $Z_h f$ (see Section 1.5) and avoid to use the equivalence relations between K_2 and ω_2. This technique provides (generally) better constants. First recall the identity

$$K(f;\delta;C[0,1],C^2[0,1]) = K(f;\delta;C[0,1],W_{2,\infty}[0,1]),$$

proven in Lemma 1.34.
Let now $f \in C^r[0,1]$, $0 < \delta \leq \frac{1}{2}$ be arbitrarily given, and let $|h| \leq \delta$. Further, we write the expression that appears in the definition of $\omega_2(D^r S_n^{<\alpha,\beta,\gamma>} f;\delta)$:

$$\begin{aligned} & |D^r S_n^{<\alpha,\beta,\gamma>}(f;x-h) - 2D^r S_n^{<\alpha,\beta,\gamma>}(f;x) + D^r S_n^{<\alpha,\beta,\gamma>}(f;x+h)| \\ = \; & |\{D^r S_n^{<\alpha,\beta,\gamma>}(f-g;x-h) - 2D^r S_n^{<\alpha,\beta,\gamma>}(f-g;x) + D^r S_n^{<\alpha,\beta,\gamma>}(f-g;x+h)\} \\ & \quad + \{D^r S_n^{<\alpha,\beta,\gamma>}(g;x-h) - 2D^r S_n^{<\alpha,\beta,\gamma>}(g;x) + D^r S_n^{<\alpha,\beta,\gamma>}(g;x+h)\}| \\ =: \; & |\{A\} + \{B\}|, \end{aligned}$$

where $g \in C^r[0,1]$ with $g^{(r)} \in W_{2,\infty}[0,1]$ arbitrarily chosen.

$|A|$ can be estimated from above as follows:

$$|A| \leq 4 \cdot \|D^r S_n^{<\alpha,\beta,\gamma>}(f-g)\|_\infty \leq 4\frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot \|(f-g)^{(r)}\|_\infty.$$

For the absolute value of B we have

$$\begin{aligned} |B| &= |D^r S_n^{<\alpha,\beta,\gamma>}(g;x-h) - 2D^r S_n^{<\alpha,\beta,\gamma>}(g;x) + D^r S_n^{<\alpha,\beta,\gamma>}(g;x+h)| \\ &= 2! \cdot h^2 \cdot \frac{1}{2!}|D^{r+2} S_n^{<\alpha,\beta,\gamma>}(g;\xi)| \ (for\ some\ \xi\ between\ x-h\ and\ x+h) \\ &\leq \|D^{r+2} S_n^{<\alpha,\beta,\gamma>} g\|_\infty h^2 \leq \frac{(n)_{r+2}}{(n+\gamma)^{r+2}} \cdot \frac{1}{1^{[r+2,-\alpha]}} \cdot h^2 \cdot \|g^{(r+2)}\|_{L_\infty}. \end{aligned}$$

We substitute now the function $g^{(r)} \in W_{2,\infty}[0,1]$ by Zhuk's function $Z_h(f^{(r)})$, hence

$$\begin{aligned} \|(f-g)^{(r)}\|_\infty &= \|f^{(r)} - Z_h(f^{(r)})\| \leq \frac{3}{4} \cdot \omega_2(f^{(r)};h), \text{ and} \\ \|g^{(r+2)}\|_{L\infty} &= \|(Z_h(f^{(r)}))''\|_{L_\infty} \leq \frac{3}{2} \cdot \frac{1}{h^2} \cdot \omega_2(f^{(r)};h), \end{aligned}$$

cf. the inequalities within Lemma 1.24. Combining these estimates and taking into account the preceding steps we obtain

$$\begin{aligned} \omega_2(D^r S_n^{<\alpha,\beta,\gamma>} f;\delta) &\leq 4 \cdot \frac{3}{4} \cdot \frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot \omega_2(D^r f;\delta) \\ &+ \frac{3}{2} \cdot \frac{(n)_{r+2}}{(n+\gamma)^{r+2}} \cdot \frac{1}{1^{[r+2,-\alpha]}} \cdot \omega_2(D^r f;\delta) \\ &= 3 \cdot \frac{(n)_r}{(n+\gamma)^r} \cdot \frac{1}{1^{[r,-\alpha]}} \\ &\cdot \left[1 + \frac{(n-r)(n-r-1)}{2(n+\gamma)^2(\alpha r+1)(\alpha(r+1)+1)}\right] \cdot \omega_2(D^r f;\delta) \\ &\leq \frac{9}{2} \cdot \omega_2(D^r f;\delta). \Box \end{aligned}$$

We recall the definition of the *Lipschitz classes* w.r.t. the *second order modulus*:

$$\mathrm{Lip}_\tau^* M := \left\{ f \in C[0,1] : \omega_2(f;\delta) \leq M \cdot \delta^\tau,\ 0 \leq \delta \leq \frac{1}{2} \right\},\ 0 < \tau \leq 2,$$

Proposition 3.29 can be rephrased as follows:

Corollary 3.30 *For a fixed integer $r \geq 0$ the following assertion holds for all $n \in \mathbb{N}$. If $f^{(r)} \in \mathrm{Lip}_\tau^* M$ for some $M \geq 0$ and some $0 < \tau \leq 2$, then*

$$D^r S_n^{<\alpha,\beta,\gamma>} f \in Lip_\tau^*(4.5M).$$

In the recent work [29] one can find similar results for the particular case $S_n^{<\alpha,0,0>}$.

3.5.2 Application to $\mathbb{B}_n^{(\alpha,\lambda)} = \tilde{\mathbb{B}}_\alpha \circ B_n \circ \tilde{\mathbb{B}}_\lambda$

Theorem 3.31 *Let $r \geq 0$, $s \geq 1$ be fixed integers. Then for all $n \geq r+s$, $f \in C^r[0,1]$ and $\delta \geq 0$, the following estimates hold:*

$$\begin{aligned} K_s(D^r\mathbb{B}_n^{(\alpha,\lambda)}f;\delta)_{[0,1]} &\leq \frac{(n)_r}{n^r}\cdot\frac{1}{1^{[r,-\alpha]}\cdot 1^{[r,-\lambda]}}K_s\left(D^rf;\frac{(n-r)_s}{n^s}\frac{1^{[r,-\alpha]}}{1^{[r+s,-\alpha]}}\cdot\frac{1^{[r,-\lambda]}}{1^{[r+s,-\lambda]}}\delta\right)_{[0,1]} \\ &\leq K_s(D^rf;\delta)_{[0,1]}. \end{aligned}$$

Proof. The requirements (i)–(iv) of Proposition 3.25 can be easly verified, by means of Lemma 3.10 and Remark 3.9. The desired estimate is obtained in combination with the inequality in Proposition 3.25 and with:

$$\begin{aligned} D^r\mathbb{B}_n^{(\alpha,\lambda)}(e_r;x) &= D^r\tilde{\mathbb{B}}_\alpha B_n\left[\frac{1}{1^{[r,-\lambda]}}e_r+\ldots;x\right] = D^r\tilde{\mathbb{B}}_\alpha\left[\frac{1}{1^{[r,-\lambda]}}\cdot\frac{(n)_r}{n^r}e_r+\ldots;x\right] \\ &= r!\cdot\frac{1}{1^{[r,-\lambda]}}\cdot\frac{1}{1^{[r,-\alpha]}}\cdot\frac{(n)_r}{n^r}. \end{aligned}$$

□

Using the same technique as for the general Stancu operators we arrive at the following estimates in terms of moduli of continuity.

Proposition 3.32 *Let $r \geq 0$ be a fixed integer, $f \in C^r[0,1]$ and $\delta \geq 0$.*

a) For all $n \geq r+1$ we have

$$\begin{aligned} \omega_1(D^r\mathbb{B}_n^{(\alpha,\lambda)}f;\delta) &\leq \frac{(n)_r}{n^r}\cdot\frac{1}{1^{[r,-\alpha]}\cdot 1^{[r,-\lambda]}}\cdot\tilde{\omega}_1\left(D^rf;\frac{n-r}{n}\cdot\frac{1}{(\alpha r+1)(\lambda r+1)}\delta\right) \\ &\leq 1\cdot\tilde{\omega}_1(D^rf;\delta) \leq 2\cdot\omega_1(D^rf;\delta). \end{aligned}$$

The leftmost inequality is best possible in the sense that for e_{r+1} both sides are equal and do not vanish. More exactly, both sides are equal to

$$(r+1)!\cdot\frac{(n)_{r+1}}{(n)_{r+1}}\cdot\frac{1}{1^{[r+1,-\alpha]}}\cdot\frac{1}{1^{[r+1,-\lambda]}}\cdot\delta > 0$$

for $\delta > 0$.

b) For all $n \geq r+2$ the following estimates in terms of ω_2 hold

$$\begin{aligned} &\omega_2(D^r\mathbb{B}_n^{(\alpha,\lambda)}f;\delta) \leq 3\cdot\frac{(n)_r}{n^r}\cdot\frac{1}{1^{[r,-\alpha]}}\cdot\frac{1}{1^{[r,-\lambda]}} \\ &\cdot\left[1+\frac{(n-r)(n-r-1)}{2n^2(\alpha r+1)(\alpha(r+1)+1)(\lambda r+1)(\lambda(r+1)+1)}\right]\omega_2(D^rf;\delta) \\ &\leq \frac{9}{2}\cdot\omega_2(D^rf;\delta) \end{aligned}$$

In terms of *Lipschitz classes* w.r.t. the first and second order modulus, respectively, the latter proposition can be rephrased

Corollary 3.33 *a) For a fixed integer $r \geq 0$ the following affirmations are true for all $n \in \mathbb{N}$. If $f^{(r)} \in Lip_\tau M$ for some $M \geq 0$ and some $0 < \tau \leq 1$, then $D^r \mathbb{B}_n^{(\alpha,\lambda)} f$ is in the same Lipschitz class.*

b) For a fixed integer $r \geq 0$ the following assertion holds for all $n \in \mathbb{N}$. If $f^{(r)} \in \mathrm{Lip}_\tau^* M$ for some $M \geq 0$ and some $0 < \tau \leq 2$, then

$$D^r \mathbb{B}_n^{(\alpha,\lambda)} f \in \mathrm{Lip}_\tau^*(4.5M).$$

For the sake of completeness, we briefly present the results regarding the degree of smoothness preservation for 3 particular cases of $\mathbb{B}_n^{(\alpha,\lambda)}$.

Application to Finta's operators $F_n^\alpha = \tilde{\mathbb{B}}_\alpha \circ B_n \circ \tilde{\mathbb{B}}_{1/n}$

Theorem 3.34 *Let $r \geq 0$, $s \geq 1$ be fixed integers. Then for all $n \geq r+s$, $f \in C^r[0,1]$ and $\delta \geq 0$, the following estimates hold:*

$$\begin{aligned} & K_s(D^r F_n^\alpha; \delta)_{[0,1]} \\ \leq \; & \frac{(n)_r}{(n+r-1)_r} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot K_s\left(D^r f; \frac{(n-r)_s}{(n+r+s-1)_s} \cdot \frac{1^{[r,-\alpha]}}{1^{[r+s,-\alpha]}} \cdot \delta\right)_{[0,1]} \\ \leq \; & K_s(D^r f; \delta)_{[0,1]}. \end{aligned}$$

In terms of moduli of continuity the above theorem reads as given in

Proposition 3.35 *Let $r \geq 0$ be a fixed integer, $f \in C^r[0,1]$ and $\delta \geq 0$.*

a) For all $n \geq r+1$ we have

$$\begin{aligned} \omega_1(D^r F_n^\alpha f) \; & \leq \; \frac{(n)_r}{(n+r-1)_r} \cdot \frac{1}{1^{[r,-\alpha]}} \cdot \tilde{\omega}_1\left(D^r f; \frac{n-r}{(n+r)(\alpha r+1)}\delta\right) \\ & \leq \; 1 \cdot \tilde{\omega}_1(D^r f; \delta) \leq 2 \cdot \omega_1(D^r f; \delta). \end{aligned}$$

The leftmost inequality is best possible in the sense that for e_{r+1} both sides are equal and do not vanish. More exactly, both sides are equal to

$$(r+1)! \cdot \frac{(n)_{r+1}}{(n+r)_{r+1}} \cdot \frac{1}{1^{[r+1,-\alpha]}} \cdot \delta > 0$$

for $\delta > 0$.

b) For all $n \geq r+2$ the following estimates in terms of ω_2 hold

$$\begin{aligned}\omega_2(D^r F_n^\alpha f;\delta) &\leq 3\cdot \frac{(n)_r}{(n+r-1)_r}\cdot\frac{1}{1^{[r,-\alpha]}}\\ &\cdot \left[1+\frac{(n-r)(n-r-1)}{2(n+r)(n+r+1)(\alpha r+1)(\alpha(r+1)+1)}\right]\omega_2(D^r f;\delta)\\ &\leq \frac{9}{2}\cdot\omega_2(D^r f;\delta)\end{aligned}$$

Application to $U_n^\alpha = B_n \circ \tilde{\mathbb{B}}_\alpha$

Theorem 3.36 *Let $r \geq 0$, $s \geq 1$ be fixed integers. Then for all $n \geq r+s$, $f \in C^r[0,1]$ and $\delta \geq 0$, the following estimates hold:*

$$K_s(D^r U_n^\alpha f;\delta)_{[0,1]} \leq \frac{(n)_r}{n^r}\cdot\frac{1}{1^{[r,-\alpha]}}K_s\left(D^r f;\frac{(n-r)_s}{n^s}\frac{1^{[r,-\alpha]}}{1^{[r+s,-\alpha]}}\delta\right)_{[0,1]} \leq K_s(D^r f;\delta)_{[0,1]}.$$

In terms of moduli of continuity the above theorem reads

Proposition 3.37 *Let $r \geq 0$ be a fixed integer, $f \in C^r[0,1]$ and $\delta \geq 0$.*

a) For all $n \geq r+1$ we have

$$\begin{aligned}\omega_1(D^r U_n^\alpha f;\delta) &\leq \frac{(n)_r}{n^r}\cdot\frac{1}{1^{[r,-\alpha]}}\cdot\tilde{\omega}_1\left(D^r f;\frac{n-r}{n(\alpha r+1)}\cdot\delta\right)\\ &\leq 1\cdot\tilde{\omega}_1(D^r f;\delta) \leq 2\cdot\omega_1(D^r f;\delta).\end{aligned}$$

Moreover, the leftmost inequality is best possible in the sense that for $f = e_{r+1}$ both sides are equal and do not vanish. More exactly, both sides are equal to

$$(r+1)!\cdot\frac{(n)_{r+1}}{n^{r+1}}\cdot\frac{1}{1^{[r+1,-\alpha]}}\cdot\delta > 0$$

for $\delta > 0$.

b) For all $n \geq r+2$ the following estimates in terms of ω_2 hold

$$\begin{aligned}\omega_2(D^r U_n^\alpha f;\delta) &\leq 3\cdot\frac{(n)_r}{n^r}\cdot\frac{1}{1^{[r,-\alpha]}}\cdot\left[1+\frac{(n-r)(n-r-1)}{2n^2(\alpha r+1)(\alpha(r+1)+1)}\right]\omega_2(D^r f;\delta)\\ &\leq \frac{9}{2}\cdot\omega_2(D^r f;\delta)\end{aligned}$$

Remark 3.38 In the above two subsections we implicitly presented the properties of smoothness preservation for the genuine Bernstein-Durrmeyer operator, e.g., consider $U_n^{1/n}$ or F_n^0 and get the corresponding estimates. Compare also with the results of D. Kacsó [80]

Application to Bernstein operators B_n

Regarding the classical Bernstein operators one can find in [33] the following results.

Proposition 3.39 *Let $r \geq 0$ and $s \geq 1$ be fixed integers. Then for all $n \geq r+s$, all $f \in C^r[0,1]$ and all $\delta \geq 0$ the following inequality holds:*

$$K_s(D^r B_n f;\delta)_{[0,1]} \leq \frac{(n)_r}{n^r} \cdot K_s\left(D^r f; \frac{(n-r)_s}{n^s} \cdot \delta\right)_{[0,1]} \leq K_s(D^r f;\delta)_{[0,1]}.$$

Further the authors took into consideration in [33] the two special cases $s = 1, 2$ which will be presented compactly in the following proposition:

Proposition 3.40 *Let $r \geq 0$ be a fixed integer, $f \in C^r[0,1]$ and $\delta \geq 0$.*

a) For all $n \geq r+1$ we have

$$\omega_1(D^r B_n f) \leq \frac{(n)_r}{n^r} \cdot \tilde{\omega}_1\left(D^r f; \frac{n-r}{n} \cdot \delta\right) \leq 1 \cdot \tilde{\omega}_1(D^r f;\delta) \leq 2 \cdot \omega_1(D^r f;\delta).$$

The leftmost inequality is best possible in the sense that for e_{r+1} both sides are equal and do not vanish.

b) For all $n \geq r+2$ we have

$$\omega_2(D^r B_n f;\delta) \leq 3 \cdot \frac{(n)_r}{n^r} \cdot \left[1 + \frac{(n-r)(n-r-1)}{2n^2}\right] \cdot \omega_2(D^r f;\delta) \leq \frac{9}{2} \cdot \omega_2(D^r f;\delta).$$

In particular, for $r = 0$ we have

$$\omega_2(B_n f;\delta) \leq 4\left[1 + \frac{n-1}{2n}\right] \cdot \omega_2(f;\delta) \leq 4.5 \cdot \omega_2(f;\delta).$$

Remark 3.41 The constant 4.5 in front of the last ω_2 can be replaced by 3, according to [112]. Thus, for all $f \in C[0,1]$ and $\delta \in [0,1]$ we have

$$\omega_2(B_n f;\delta) \leq 3 \cdot \omega_2(f;\delta).$$

Chapter 4

Over-iteration for some positive linear operators

For any positive linear operator $L_n : C[0,1] \to C[0,1],\ n \in \mathbb{N}$, we define inductively the powers of L_n by

$$L_n^0 := Id,\ L_n^1 := L_n \text{ and } L_n^{m+1} := L_n \circ L_n^m,\ m \in \mathbb{N}.$$

Our aim is to study the behavior of the powers of L_n having the following layout: $n \in \mathbb{N}$ is fixed and m goes to infinity. In other words, the operators considered are *over-iterated*.
In the subsequent three sections we describe three methods to investigate the over-iteration of L_n:

- the contraction principle,
- a general quantitative method,
- a method that uses the *spectral properties* of the operator.

4.1 The contraction principle

A general method to investigate the behavior of the over-iteration of a fixed operator is via the *contraction principle* (see, e.g., [11], [12]). The following assertions were inspired by a recent result of O. Agratini & I. Rus [4] (see also [132]) who proved convergence for over-iteration of certain general discretely defined operators. In the sequel we prove a generalization of the first theorem in [4] also for a whole class of summation-type operators. They are defined by $L_n : C[0,1] \to C[0,1]$ with

$$L_n(f;x) := \sum_{k=0}^{n} \psi_{n,k}(x) \cdot a_{n,k}(f), \tag{4.1}$$

where $\psi_{n,k}(x) \geq 0$, $a_{n,k}$ are linear positive functionals with $a_{n,k}e_0 = 1$, $k = 0, \ldots, n$, and $a_{n,0}(f) = f(0)$, $a_{n,n}(f) = f(1)$, $f \in C[0,1]$. With the supplementary condition that these operators reproduce linear functions we have the following relations:

$$\sum_{k=0}^{n} \psi_{n,k}(x) = 1, \text{ and } \sum_{i=0}^{n} \psi_{n,k}(x) \cdot a_{n,k}(e_1) = x, \ x \in [0,1].$$

For these we can state the following

Theorem 4.1 *Let L_n, $n \in \mathbb{N}$ fixed, be the operators given above. Define $u_n := \min_{x\in[0,1]} (\psi_{n,0}(x) + \psi_{n,n}(x))$. If $u_n > 0$, then the iterates $(L_n^m f)_{m\geq 1}$ with $f \in C[0,1]$ converge uniformly toward the linear function that interpolates f at the endpoints* 0 *and* 1*, i.e.,*

$$\lim_{m\to\infty} L_n^m(f;x) = f(0) + (f(1) - f(0))x, \ f \in C[0,1].$$

Proof. Consider the Banach space $(C[0,1], \|\cdot\|_\infty)$ where $\|\cdot\|_\infty$ is the Chebyshev norm. Let

$$X_{\alpha,\beta} := \{f \in C[0,1] : f(0) = \alpha, \ f(1) = \beta\}, \ \ \alpha, \beta \in \mathbb{R}.$$

We note that

a) $X_{\alpha,\beta}$ is a closed subset of $C[0,1]$;

b) $C[0,1] = \bigcup_{\alpha,\beta\in\mathbb{R}} X_{\alpha,\beta}$ is a partition of $C[0,1]$;

c) $X_{\alpha,\beta}$ is an invariant subset of L_n for all $\alpha, \beta \in \mathbb{R}$, $n \in \mathbb{N}$, since the reproduction of linear functions implies interpolation of the function at the endpoints, i.e., $L_n(f;0) = f(0)$ and $L_n(f;1) = f(1)$.

Now we show that

$$L_n |_{X_{\alpha,\beta}} : X_{\alpha,\beta} \to X_{\alpha,\beta}$$

is a contraction for all $\alpha, \beta \in \mathbb{R}$.
Let $f, g \in X_{\alpha,\beta}$. We can write

$$\begin{aligned} |L_n(f;x) - L_n(g;x)| &= \left| \sum_{i=1}^{n-1} \psi_{n,k}(x) \cdot a_{n,k}(f-g) \right| \\ &\leq \sum_{i=1}^{n-1} \psi_{n,k}(x) \cdot \|a_{n,k}\| \cdot \|f-g\|_\infty \\ &= (1 - \psi_{n,0}(x) - \psi_{n,n}(x)) \cdot \|f-g\|_\infty \\ &\leq (1-u_n) \cdot \|f-g\|_\infty. \end{aligned} \tag{4.2}$$

Hence $\|L_n f - L_n g\|_\infty \leq (1 - u_n) \cdot \|f - g\|_\infty$ with $u_n > 0$ and thus $L_n \mid_{X_{\alpha,\beta}}$ is contractive.
On the other hand $\alpha + (\beta - \alpha) \cdot e_1 \in X_{\alpha,\beta}$ is a fixed point for L_n. If $f \in C[0,1]$ is arbitrarily given, then $f \in X_{f(0),f(1)}$ and from the contraction principle we have

$$\lim_{m\to\infty} L_n^m f = f(0) + (f(1) - f(0))e_1. \qquad \square$$

Remark 4.2 One advantage of the method is: what we have proven in the latter theorem is true for many known summation type operators, see for example Subsection 4.2.2. On the other hand one can note that the above proof is restricted to a *fixed* operator L_n and its iterates L_n^m. Furthermore, the proof is only valid for operators having a contraction constant $(1 - u_n) < 1$. However, there are cases in which we do not have $u_n > 0$, but still convergence of the iterates takes place, as one can see in the following section.

Another interesting issue to take into consideration is that the limiting operator does not necessary need to be equal to B_1 (the linear interpolator at the points 0 and 1), like the following two examples illustrate.

King's operators
The first example represents the *King's operators* [86], see Section 2.4 of this thesis. In [69] it was proved that its over-iterates converge to a parabola:

Theorem 4.3 *If $n \in \mathbb{N}$ is fixed, then for all $f \in C[0,1],\ x \in [0,1]$*

$$\lim_{m\to\infty} (V_n^*)^m(f;x) = f(0) + [f(1) - f(0)] \cdot x^2 = V_1^*(f;x),$$

where $V_1^* f = f(0) + (f(1) - f(0))e_2$.

Rational Bernstein operators
For more complex operators like the *rational Bernstein operator* (see their definition at (2.6)) we can only prove via the contraction principle the existence of the limiting operator as one can see in

Theorem 4.4 *If $k \in \mathbb{N}$ is fixed, then for all $f \in C[0,1],\ x \in [0,1]$,*

$$\lim_{m\to\infty} R_{1,k}^m(f;x) = f^*(x),$$

where $f^* \in C[0,1]$ and $R_{1,k}(f^*;x) = f^*(x)$.

In this case the *contraction constant* is equal to $c := 1 - \frac{w^{min}}{w^{max}} \cdot \frac{1}{2^{k-1}}$, where the positive quantities w^{min} and w^{max} represent the *minimum* respectively the *maximum* of all weights.

Remark 4.5 We have managed to find the *fixed point* for these rational operators only for some particular cases, for some special choices of the associated weight sequence:

- We look now at $R_{1,2}$ $(k=2)$ and at the sequence $(w_{0,2}, w_{1,2}, w_{2,2}) = (1, w_1, 1)$. For
$$g_{w_1}(x) = \frac{w_1 \cdot x(1-x) + x^2}{(1-x)^2 + 2w_1 \cdot x(1-x) + x^2}$$
one can easily prove by direct computations that $R_{1,2} g_{w_1} = g_{w_1}$.

- We have also discovered an interesting link between the rational Bernstein operators and some instances of *Stancu* operators. So, if we take in (2.6) $w_{j,k} := \frac{j}{k}$ or $w_{j,k} := 1 - \frac{j}{k}$ or finally $w_{j,k} := \frac{j}{k}\left(1 - \frac{j}{k}\right)$ then the corresponding rational Bernstein functions are reduced (in this order) to
$$\frac{B_{k+1}(e_1 \cdot f)}{B_{k+1} e_1} = S_k^{<0,1,1>} f; \ \frac{B_{k+1}((e_0 - e_1) \cdot f)}{B_{k+1}(e_0 - e_1)} = S_k^{<0,0,1>} f; \text{ and}$$
$$\frac{B_{k+1}((e_1 - e_2) \cdot f)}{B_{k+1}(e_1 - e_2)} = S_k^{<0,1,2>} f.$$
The over-iterates of these operators are discussed in a larger context in Subsections 4.3.1 and 4.3.2.

4.2 A general quantitative method

In this section we prove general inequalities for the iterates of positive linear operators which are given in the spirit of the paper by S. Karlin & Z. Ziegler [82] and were obtained for classical Bernstein operators in a slightly weaker form first in [54]. In the sequel we will consider again $L_n : C[0,1] \to C[0,1]$. However, relaxing the assumption of the previous section we will consider general positive linear operators which reproduce linear functions. Note that in this section there will be no contraction argument. Supplementary details can also be viewed in [65].
The following estimate holds.

Theorem 4.6 *If L_n is given as above, for $m, n \in \mathbb{N}$ we have*

$$(4.3) \qquad |L_n^m(f;x) - B_1(f;x)| \le \frac{9}{4} \cdot \omega_2\left(f; \sqrt{L_n^m(e_1 \cdot (e_0 - e_1); x)}\right),$$

where $f \in C[0,1]$, $x \in [0,1]$, B_1 is the first Bernstein operator, and $e_i(t) = t^i$, $i \ge 0$.

Proof. For $g \in C^2[0,1]$ arbitrarily chosen we have the following estimate

$$\begin{aligned} |L_n^m(f;x) - B_1(f;x)| &\leq |(L_n^m - B_1)(f-g;x)| + |(L_n^m - B_1)(g;x)| \\ &\leq (\|L_n^m\|_\infty + \|B_1\|_\infty) \cdot \|f-g\|_\infty + |(L_n^m - B_1)(g;x)| \\ &\leq 2 \cdot \|f-g\|_\infty + |(L_n^m - B_1)(g;x)|. \end{aligned}$$

Since both of operators L_n^m and B_1 reproduce linear functions, we have

$$L_n^m(B_1 g) = B_1(B_1 g) \in \prod\nolimits_1,$$

the polynomials of degree ≤ 1. Now we can evaluate

$$\begin{aligned} |(L_n^m - B_1)(g;x)| &= |L_n^m(g;x) - B_1(g;x) - L_n^m(B_1 g;x) + B_1(B_1 g;x)| \\ &= |L_n^m(g - B_1 g;x)| \leq L_n^m(|g - B_1 g|;x) \\ &\leq L_n^m\left(\frac{1}{2} \cdot \|g''\|_\infty \cdot e_1(e_0 - e_1);x\right) \\ &= \frac{1}{2} \cdot \|g''\|_\infty \cdot L_n^m(e_1(e_0 - e_1);x). \end{aligned}$$

Thus

$$|L_n^m(f;x) - B_1(f;x)| \leq 2 \cdot \|f-g\|_\infty + \frac{1}{2}\|g''\|_\infty \cdot L_n^m(e_1(e_0 - e_1);x).$$

We substitute now $g := B_n(Z_h f) \in C^2[0,1]$, where $Z_h f$ is Zhuk's function, see its definition at 1.12. According to the Lemmas 1.24 and 1.27 for a sufficiently large n and a fixed $\varepsilon > 0$ we have

$$\begin{aligned} \|f-g\|_\infty &\leq \|f - Z_h f\|_\infty + \|Z_h f - B_n(Z_h f)\|_\infty \\ &\leq \frac{3}{4} \cdot \omega_2(f;h) + \varepsilon \\ \|g''\|_\infty &\leq \|(Z_h f)''\|_{L_\infty} \leq \frac{3}{2} \cdot h^{-2} \cdot \omega_2(f;h). \end{aligned}$$

Letting $\varepsilon \to 0$ we arrive at

$$|L_n^m(f;x) - B_1(f;x)| \leq 2 \cdot \frac{3}{4} \cdot \omega_2(f;h) + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{h^2} \cdot L_n^m(e_1(e_0 - e_1);x) \cdot \omega_2(f;h),$$

with $h > 0$. If $L_n^m(e_1(e_0 - e_1);x) > 0$ taking $h := \sqrt{L_n^m(e_1(e_0 - e_1);x)}$ yields the desired result. If $L_n^m(e_1(e_0 - e_1);x) = 0$, then $|L_n^m(f;x) - B_1(f;x)| \leq \frac{3}{2} \cdot \omega_2(f;h)$ for all $h \geq 0$. For $h \to 0$ we obtain $L_n^m(f;x) = B_1(f;x)$ for all $x \in [0,1]$. □

Lemma 4.7 *Under the same assumptions on the operator L_n as above, we have*

$$(4.4) \qquad 0 \leq L_n(e_1(e_0 - e_1);x) \leq x(1-x)\left[1 - \min_{x\in(0,1)} \frac{L_n((e_1 - x)^2;x)}{x(1-x)}\right].$$

Proof. Due to the linearity of the operator L_n and the fact that it preserves linear functions, one can easily observe that $L_n(e_1(e_0-e_1);x) = x(1-x) - L_n((e_1-x)^2;x)$. Thus,

$$\begin{aligned} 0 \le L_n(e_1(e_0-e_1);x) &= x(1-x)\left[1 - \frac{L_n((e_1-x)^2;x)}{x(1-x)}\right], \ x \in (0,1), \\ &\le x(1-x)\left[1 - \min_{x\in(0,1)} \frac{L_n((e_1-x)^2;x)}{x(1-x)}\right]. \qquad \square \end{aligned}$$

For our further discussion we will exclude those operators whose second moments have zeros in the interior of the interval, $[0,1]$ in our case.

Theorem 4.8 *Let $L_n : C[0,1] \to C[0,1]$ be positive linear operators which preserve linear functions. We also suppose that there exists $\varepsilon_n > 0$ such that*

$$\varepsilon_n \cdot x(1-x) \le L_n((e_1-x)^2;x), \ x \in [0,1]. \tag{4.5}$$

Then we have

$$0 \le L_n^m(e_1(e_0-e_1);x) \le x(1-x)\cdot(1-\varepsilon_n)^m, \ m \in \mathbb{N}. \tag{4.6}$$

Proof. We will prove the above statement by induction. First we take $m = 1$. Condition (4.5) can be rewritten as $\varepsilon_n \le \frac{L_n((e_1-x)^2;x)}{x(1-x)}$ for $x \in (0,1)$ implying

$$\varepsilon_n \le \min_{x\in(0,1)} \frac{L_n((e_1-x)^2;x)}{x(1-x)}.$$

Thus inequality (4.4) yields

$$L_n(e_1(e_0-e_1);x) \le x(1-x)(1-\varepsilon_n).$$

We assume the relation

$$L_n^m(e_1(e_0-e_1);x) \le x(1-x)(1-\varepsilon_n)^m$$

to be true for a fixed $m \in \mathbb{N}$ and shall prove it for $m+1$. Indeed, we have

$$L_n^{m+1}(e_1(e_0-e_1);x) \le (1-\varepsilon_n)^m \cdot L_n(e_1(e_0-e_1);x) \le x(1-x)\cdot(1-\varepsilon_n)^{m+1}.$$

Hence it follows that the estimate (4.6) is true for all $m \in \mathbb{N}$. $\square$

In case that $\varepsilon_n < 1$ (which occurs often), by combining the above theorem and Theorem 4.6 we get the following

Corollary 4.9 *With the same assumptions on the operator L_n as above and (4.5) we get*

$$|L_n^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2 \left(f; \sqrt{x(1-x)(1-\varepsilon_n)^m}\right), \tag{4.7}$$

$f \in C[0,1],\ x \in [0,1]$.

Note that the operator L_n now is *not* necessarily fixed. We can thus - as was done by Karlin and Ziegler - consider $\lim_{n\to\infty} L_n^{m_n}$ where m_n depends on n such that $\lim_{n\to\infty} (1-\varepsilon_n)^{m_n} \to 0$ and still get uniform convergence towards $B_1 f$.

4.2.1 Discretely defined operators

In the sequel we show that the previous general result implies the convergence assertion of Agratini and Rus, also providing a full quantitative version of it. Our assertion is given in terms of the second order modulus, the best to be expected under the present conditions. However, due to the use of the contraction constant $(1-u_n)$ some pointwise information is lost.

We return to the operators considered in the previous section. For a given partition on $[0,1]$ such that $0 = x_{n,0} < x_{n,1} < \ldots < x_{n,n} = 1$ we specialize the functionals $a_{n,k}$ by assuming

$$a_{n,k}(f) = f(x_{n,k}),\ k = 0, \ldots, n.$$

We obtain

$$L_n(f;x) = \sum_{i=0}^{n} \psi_{n,k}(x) \cdot f(x_{n,k}),\ f \in C[0,1],\ x \in [0,1]. \tag{4.8}$$

Guided by a result of R.P. Kelisky & T.J. Rivlin [83], O. Agratini and I. Rus studied these operators L_n in [4]. It is known that operators L_n of this type have attracted attention for at least 100 years now. We mention here the interesting note of T. Popoviciu [126] who in turn refers to the classical book of É. Borel [22] , see also [23]. (Polynomial) operators of the given type also appear in H. Bohman's now classical paper [19] and in Butzer's problem (see, e.g., [53] and the references cited there for details). Further historical information can be found in A. Pinkus' most interesting work [119]; see also the recent paper of J. Szabados [151].

Lemma 4.10 *As in the first section we assume that the operators (4.8) reproduce linear functions. This implies that $\psi_{n,0}(0) = \psi_{n,n}(1) = 1$.*

Proof. It is known that $L_n e_i = e_i,\ i = 0,1$, implies interpolation at the endpoints of the function, i.e., $L_n(f;0) = f(0)$ and $L_n(f;1) = f(1)$. This means that

$$f(0) = L_n(f;0) = \psi_{n,0} \cdot f(0) + \sum_{i=1}^{n} \psi_{n,k}(0) \cdot f(x_{n,k}) \text{ or}$$

$$(4.9) \quad (1-\psi_{n,0}(0))\cdot f(0) \;=\; \sum_{i=1}^{n}\psi_{n,k}(0)\cdot f(x_{n,k}),\ \text{for all } f\in C[0,1].$$

We define $f \in C[0,1]$ by

$$f(x) := \begin{cases} -\frac{1}{x_{1,n}}\cdot x+1, & x\in[0,x_{1,n}] \\ 0, & x\in(x_{1,n},1]. \end{cases}$$

and substitute it into (4.9). Thus we easily arrive at $\psi_{n,0}(0)=1$. In a similar way we can prove that $\psi_{n,n}(1)=1$. □

Thus the conditions $\psi_{n,0}(0)=\psi_{n,n}(1)=1$ are automatically satisfied. Furthermore, we will give pointwise and uniform estimates for these operators L_n which imply the result of O. Agratini and I. Rus.

First we have

Proposition 4.11 *For $L_n : C[0,1]\to C[0,1]$ defined as in (4.8) one has*

$$(4.10) \qquad L_n^m(e_1(e_0-e_1);x)\le \frac{1}{4}\cdot(1-u_n)^{m-1}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x)),$$

with $m\in\mathbb{N}$, $f\in C[0,1]$ and $x\in[0,1]$. Like in Theorem 4.1 the inequality $u_n = \min\limits_{x\in[0,1]}(\psi_{n,0}(x)+\psi_{n,n}(x))>0$ is assumed.

Proof. We will prove this statement by induction. For $m=1$ we have

$$\begin{aligned} L_n(e_1(e_0-e_1);x) &= L_n(e_1-e_2;x)=\sum_{i=0}^{n}(x_{n,k}-x_{n,k}^2)\cdot\psi_{n,k}(x) \\ &= \sum_{i=1}^{n-1}(x_{n,k}-x_{n,k}^2)\cdot\psi_{n,k}(x)\le\frac{1}{4}\cdot\sum_{i=1}^{n-1}\psi_{n,k}(x) \\ &= \frac{1}{4}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x)). \end{aligned}$$

We suppose now that the relation

$$L_n^m(e_1(e_0-e_1);x)\le\frac{1}{4}\cdot(1-u_n)^{m-1}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x))$$

is true for a fixed $m\in\mathbb{N}$. We show it for $m+1$. We apply on this relation the operator L_n, obtaining

$$\begin{aligned} L_n^{m+1}(e_1(e_0-e_1);x) &\le \frac{1}{4}(1-u_n)^{m-1}L_n(1-\psi_{n,0}-\psi_{n,n};x) \\ &= \frac{1}{4}(1-u_n)^{m-1}L_n\left(\sum_{i=1}^{n-1}\psi_{n,k};x\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}(1-u_n)^{m-1}\sum_{l=0}^{n}\sum_{k=1}^{n-1}\psi_{n,k}(x_{n,l})\psi_{n,l}(x) \\
&= \frac{1}{4}(1-u_n)^{m-1}\sum_{l=1}^{n-1}\sum_{k=1}^{n-1}\psi_{n,k}(x_{n,l})\psi_{n,l}(x) \\
&= \frac{1}{4}(1-u_n)^{m-1}\sum_{l=1}^{n-1}\psi_{n,l}(x)\cdot\sum_{i=1}^{n-1}\psi_{n,k}(x_{n,l}) \\
&\le \frac{1}{4}(1-u_n)^{m}\sum_{l=1}^{n-1}\psi_{n,l}(x) = \frac{1}{4}(1-u_n)^{m}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x)).
\end{aligned}$$

We have thus proved that relation (4.10) is true for any $m \in \mathbb{N}$. ☐

Remark 4.12 Uniformly one has

$$L_n^m(e_1(e_0-e_1)) \le \frac{1}{4}\cdot(1-u_n)^m. \tag{4.11}$$

The following pointwise estimate is a consequence of Theorem 4.6.

Proposition 4.13 *For the operators L_n in (4.1) we have*

$$|L_n^m(f;x)-B_1(f;x)| \le \frac{9}{4}\cdot\omega_2\left(f;\frac{1}{2}\cdot\sqrt{(1-u_n)^{m-1}\cdot(1-\psi_{n,0}(x)-\psi_{n,n}(x))}\right).$$

This inequality reflects the fact that the iterates interpolate B_1f (and f) at $x=0$ and $x=1$.

Corollary 4.14 *The uniform estimate is also easily obtained from Theorem 4.6 and (4.11) as*

$$\|L_n^m f - B_1 f\|_\infty \le \frac{9}{4}\cdot\omega_2\left(f;\frac{1}{2}\cdot\sqrt{(1-u_n)^m}\right).$$

Note that the contraction constant $1-u_n<1$ figures repeatedly in the above inequalities.

4.2.2 Applications I

In this section we consider a group of operators to which both methods, the *contraction principle* and the *quantitative method* are applicable. The advantage of the latter one (which we shall use in the sequel) is that we immediately obtain the degree of approximation.

An instance of general Stancu operators

We start with the operators $S_n^{<\alpha,0,0>} : C[0,1] \to \prod_n$ considered by D. D. Stancu in [144], see here row no. 3 in Table 3.1.
It is obvious that the $S_n^{<\alpha,0,0>}$ satisfy the requirements of Theorem 4.8: they are positive and linear, preserve linear functions and by selecting $\varepsilon_n := \frac{1}{n} \cdot \frac{1+n\alpha}{1+\alpha} < 1$ condition (4.5) is also verified.
Taking into account Corollary 4.9 we arrive at

Proposition 4.15 *Let $S_n^{<\alpha,0,0>}$, $n \in \mathbb{N}$, $\alpha \geq 0$ be a sequence of Stancu operators. For $m \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1]$ we have*

$$|\left[S_n^{<\alpha,0,0>}\right]^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2\left(f; \sqrt{x(1-x)\left(1 - \frac{1}{n} \cdot \frac{1+n\alpha}{1+\alpha}\right)^m}\right).$$

Remark 4.16 It is worthwile to mention that already in 1978 G. Mastroianni & M. R. Occorsio [103] have introduced and investigated the iterates of $S_n^{<\alpha,0,0>}$ by extending a procedure used by R. P. Kelisky & T. J. Rivlin [83] for the Bernstein operators. In the next section we shall focus our attention on the multi-parameter variant of this operator, that do not reproduce linear functions.

The classical Bernstein operators

For $\alpha = 0$ we arrive at the classical Bernstein operators. An early paper on over-iterated Bernstein operators is - besides the one by R.P. Kelisky & T.J. Rivlin - an article of P.C. Sikkema [142]. Using Proposition 4.15 immediately yields

Proposition 4.17 *Let B_n, $n \in \mathbb{N}$, be the sequence of Bernstein operators. For $m \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1]$ we obtain*

$$|B_n^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2\left(f, \sqrt{x(1-x)\left(1 - \frac{1}{n}\right)^m}\right).$$

A similar result was first obtained by H. Gonska in [54] with a constant 4 instead of $\frac{9}{4}$, and as a special consequence of a more general quantitative result for the approximation of finitely defined operators (see [107] and [55] for further details). More information on iterated Bernstein operators can be found in the recent note [73].

The generalized genuine Bernstein-Durrmeyer operators

In the same category fit U_n^λ defined in Table 3.1 row 5, and implicitly its famous particular case U_n. In Table 3.2 we computed its second moments, thus by choosing $\varepsilon_n := \frac{\lambda n+1}{n(1+\lambda)} < 1,\ n > 1$ we get the following error estimation:

Proposition 4.18 *Let U_n^λ be the sequence defined as above. Let $m \in \mathbb{N},\ f \in C[0,1]$ and $x \in [0,1]$. Then we have*

$$|[U_n^\lambda]^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2\left(f; \sqrt{x(1-x)\left(\frac{n-1}{n(\lambda+1)}\right)^m}\right).$$

The genuine Bernstein-Durrmeyer operators

Substituting in the latter proposition $\lambda := \frac{1}{n}$, we arrive at

Proposition 4.19 *Let $n, m \in \mathbb{N},\ f \in C[0,1]$ and $x \in [0,1]$. The following inequality holds*

$$|U_n^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2\left(f; \sqrt{\left(1 - \frac{2}{n+1}\right)^m \cdot x(1-x)}\right).$$

The latter pointwise estimate was earlier established by D. Kacsó in her recent work [80].

Remark 4.20

(i) I. Gavrea and D. H. Mache [52] discussed a certain special case of the general operators (4.1). Restricting ourselves to a special situation, their operators were defined by

$$A_n(f;x) := \sum_{i=0}^{n} \binom{n}{k} x^i (1-x)^{n-i} \cdot a_{n,i}(f). \tag{4.12}$$

Here $a_{n,i} : C[0,1] \to \mathbb{R}$ are positive linear functionals verifying $a_{n,i}e_0 = 1$ and $a_{n,i}e_1 = \frac{i}{n},\ i = 0, \ldots, n$ (the latter condition being our special situation). Hence linear functions are reproduced so that Theorem 4.6 is applicable. We also note that $A_n(f;0) = f(0)$ and $A_n(f;1) = f(1)$, which is true for every positive linear operator reproducing linear functions. This implies that $a_{n,0}(f) = f(0)$ and $a_{n,n}(f) = f(1)$. The special form of the fundamental functions implies that we can take $u_n = \frac{1}{2^{n-1}}$ to arrive - in a way analogous to Proposition 4.13 - at

$$|A_n^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2\left(f; \frac{1}{2} \cdot \sqrt{\left(1 - \frac{1}{2^{n-1}}\right)^{m-1} (1 - (1-x)^n - x^n)}\right).$$

Note that both U_n^λ and U_n have this particular form.

(ii) A further class of positive linear operators which generalize the Bernstein operators was recently introduced by N. Vornicescu [155]. His operators use general knots $0 = x_0 < x_1 < \ldots < x_n = 1$. More specific, his operator is defined by $T : C[0,1] \to \prod$ with

$$T(f;x) = \sum_{i=0}^{n} f(x_i)u_i(x),$$

where $x \in [0,1]$ and u_i are the set of polynomials described by

$$\begin{aligned} u_0(x) &= (1-x)\left[1-(1-\alpha)x\sum_{i=1}^{n-1}\frac{1}{x_i}q_i(x)\right],\ \alpha \in \mathbb{R} \\ u_i(x) &= \frac{1-\alpha}{x_i(1-x_i)}x(1-x)q_i(x),\ \ 1 \le i \le n-1, \text{ and} \\ u_n(x) &= x\left[1-(1-\alpha)(1-x)\sum_{i=1}^{n-1}\frac{1}{1-x_i}q_i(x)\right]. \end{aligned}$$

$\{q_1(x), \ldots, q_{n-1}(x)\}$ is a set of polynomials that must verify $\sum_{i=1}^{n-1} q_i(x) = 1$, for all $x \in [0,1]$. In Lemma 2.1 from [155] it was proved among other that T reproduces linear functions. Hence, the general results from Theorem 4.6 and Proposition 4.13 are in this case also applicable.

4.2.3 Applications II

Here we consider two types of operators to which the *contraction principle* is not applicable. The Beta-type operators in the next subsection are not discretely defined and the Schoenberg spline operators are such that $u_n = 0$, so that the contraction argument fails in this case.

Beta operators of the second kind and there modifications

Here we discuss an example which is not covered by the ansatz of Section 4.1, namely for the Beta operators of the second kind, see (3.2). It is easy to check that all the conditions of Theorem 4.8 are verified. Thus we can set in this case $\varepsilon_n := \frac{1}{n+1} < 1,\ n \ge 1$. Due to Corollary 4.9 we arrive at:

Proposition 4.21 *Let $\bar{\mathbb{B}}_n$ be a sequence of Beta operators of the second kind. Let $m \in \mathbb{N},\ f \in C[0,1]$ and $x \in [0,1]$. Then we have*

$$|\bar{\mathbb{B}}_n^m(f;x) - B_1(f;x)| \le \frac{9}{4}\cdot\omega_2\left(f;\sqrt{x(1-x)\left(1-\frac{1}{n+1}\right)^m}\right).$$

Analogous assertions are obtained for $\tilde{\mathbb{B}}_\alpha$, as they have similar properties with $\bar{\mathbb{B}}_n$:

Proposition 4.22 *For the iterates of $\tilde{\mathbb{B}}_\alpha$ the following inequality holds:*

$$|\tilde{\mathbb{B}}_\alpha^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2 \left(f; \sqrt{x(1-x)\frac{1}{(1+\alpha)^m}} \right),$$

where $m \in \mathbb{N},\ f \in C[0,1]$ and $x \in [0,1]$.

It is obvious that for $\alpha > 0$ the convergence of the process is assured, when $m \to \infty$.

Composite Beta-type operator

In analogy with $\tilde{\mathbb{B}}_\alpha$ we obtain the following, by substituting $\varepsilon_n := 1 - \frac{1}{(1+\alpha)(1+\lambda)} \cdot \frac{n-1}{n} < 1, n \geq 2$ - see relation (3.12) - in Corollary 4.9:

Proposition 4.23 *For $m \in \mathbb{N},\ f \in C[0,1]$ and $x \in [0,1]$ the following estimate holds:*

$$|[\mathbb{B}_n^{(\alpha,\lambda)}]^m(f;x) - B_1(f;x)| \leq \frac{9}{4} \cdot \omega_2 \left(f; \sqrt{x(1-x)\left(\frac{1}{(1+\alpha)(1+\lambda)} \cdot \frac{n-1}{n}\right)^m} \right).$$

Remark 4.24 We shall not discuss further the rest of the particular cases of $\mathbb{B}_n^{(\alpha,\lambda)}$. Information for the behavior of their over-iterates can be obtained from the above inequality, by making the "right" substitution for the parameters $\alpha,\ \lambda$ and n, see Table 3.1.

Schoenberg spline operators on equidistant knots

The contraction principle, very efficient in many cases, is not applicable in the case of Schoenberg splines, since one cannot find a contraction constant strictly less than 1. One motivation for this section is to propose a method that yields relevant results also for the iterates of Schoenberg splines. So far, we succeeded for certain cases with equidistant knots.
Consider in (2.3) the equidistance knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k},\ 2 \leq k \leq n-1$ with

$$\Delta_n : x_{-k} = \ldots = x_0 = 0 < x_1 < x_2 < \ldots < x_n = \ldots x_{n+k} = 1,$$

and $x_i = \frac{i}{n}$ for $0 \leq i \leq n$.
The following proposition provides a possible choice for ε_n.

Proposition 4.25 *For the second moments of the latter operators we have the lower estimate*

$$\min\left\{\frac{2}{21n^2}, \frac{1}{21n(k-1)}\right\} \cdot x(1-x) \le S_{n,k}((e_1-x)^2;x),\ 2 \le k \le n-1.$$

Proof. The following lower bound of the second moments was given in [18] (see also [15]). For $2 \le k \le n-1$ one has

$$\frac{S_{n,k}((e_1-x)^2;x)}{x(1-x)} \ge c_k \cdot \frac{\min\left\{2x(1-x), \frac{k}{n}\right\}}{n(k-1)x(1-x)} \ge c_k \cdot \frac{\min\left\{2, \frac{k}{n} \cdot \frac{1}{x(1-x)}\right\}}{n(k-1)},$$

where $c_k = \frac{9}{88} \ge \frac{1}{10}$ for $k \ge 3$ and $c_2 = \frac{3}{124} \ge \frac{1}{42}$.

We consider now two cases:
First case. For $2k > n$ and $2 \le k \le n-1$ we have $\min\left\{2, \frac{k}{n} \cdot \frac{1}{x(1-x)}\right\} = 2$. Thus,

$$\frac{S_{n,k}((e_1-x)^2;x)}{x(1-x)} \ge \frac{1}{21n(k-1)} \text{ for } n < 2k.$$

Second case. If $n \ge 2k$, then $\min\left\{2, \frac{k}{n} \cdot \frac{1}{x(1-x)}\right\} \ge \frac{4k}{n}$. We have $\frac{S_{n,k}((e_1-x)^2;x)}{x(1-x)} \ge c_k \cdot \frac{4}{n^2} \cdot \frac{k}{k-1}$. This estimate can be carried out further, since $\frac{k}{k-1} \ge 1$ and $c_k \ge \frac{1}{42}$. We arrive at

$$\frac{S_{n,k}((e_1-x)^2;x)}{x(1-x)} \ge \frac{2}{21n^2} \text{ for } n \ge 2k. \ \Box$$

Remark 4.26 The above proposition implies one possible value of

$$\varepsilon_{n,k} = \min\left\{\frac{2}{21n^2}, \frac{1}{21n(k-1)}\right\} < 1,$$

with $2 \le k \le n-1$. One can observe that for $k = 1$ condition (4.5) is not verified, because the second moment of the piecewise linear operator has zeros in the interior of the interval (e.g., see A. Lupaş [96]). It is also clear that $S_{n,1}^m f = S_{n,1} f,\ m \ge 1$.

Now we can easily derive a convergence result for the iterates of the Schoenberg spline operator.

Proposition 4.27 *For $S_{n,k},\ 2 \le k \le n-1$, defined as above we have*

$$|S_{n,k}^m(f;x) - B_1(f;x)| \le \frac{9}{4}\omega_2\left(f; \sqrt{x(1-x)\left(1 - \min\left\{\frac{2}{21n^2}, \frac{1}{21n(k-1)}\right\}\right)^m}\right).$$

Remark 4.28 For $2 \le k \le n-1$ fixed we have $\lim_{m\to\infty}(1-\varepsilon_{n,k})^m = 0$. Thus $\lim_{m\to\infty} S_{n,k}^m f = B_1 f$. An analogous convergence result also holds for more general knot sequences, as shown by H. J. Wenz in [158]. Due to the lack of a suitable lower bound for more general second moments we have not yet been able to give quantitative results in this general case.

4.3 Via the eigenstructure

We propose now a method to study the behavior of the over-iterates of those operators for which neither the *contraction principle*, nor the *quantitative method* is applicable. More exactly this means the operators in question do not have at least one of the following two properties:

A) reproducing linear functions,

B) interpolating the function in 0 and 1,

but they have the property of reproducing constant functions. However, no A) means that the quantitative method fails and lack of B) makes it hard to achieve global results, i.e., for every $x \in [0,1]$, via the contraction principle, see for example [131].

The method uses the unique representation of a polynomial operator w.r.t. the basis of its eigenpolynomials, and in our case, the fact that the corresponding eigenvalues are strictly less than 1. In the frame briefly described the following operators fit: some general Stancu operators, the classical and the generalized Durrmeyer operators and also the Kantorovich operators. The same approach one can find in the recent paper of Sh. Cooper & Sh. Waldron [32] for the iterates of Bernstein operators and also in the paper of S. Ostrovska [109] where, among others, the iterates of q-Bernstein operators are investigated. The reader is also directed to [72].

4.3.1 Bernstein-Stancu operators

First we consider the "less general" $S_n^{<0,\beta,\gamma>}$ with $0 \le \beta \le \gamma$ defined at (3.10). Most of the results presented here will be be reused for the more general case $\alpha > 0$ in the following subsection.

First we recall that G. Călugăreanu determined in [27] the eigenvalues of the Bernstein operator as follows:

Proposition 4.29 *The Bernstein operator B_n has $n+1$ eigenvalues, all of them lie in the interval $(0,1]$ and have the following form*

$$\nu_{n,j} = \frac{(n)_j}{n^j},\ j = 1,\ldots,n, \tag{4.13}$$

and $\nu_{n,0} = 1$. Equivalently it means that the leading coefficient of the n-th degree polynomial $B_n e_j$ is equal to $\nu_{n,j}$, i.e., $B_n e_j = \nu_{n,j} e_j + P_{n,j-1}$ where $P_{n,j-1} \in \prod_{j-1}$, $j = 1,\ldots,n$.

Remark 4.30 An exhaustive research on the *eigenstructure* of the Bernstein operator was done in [32]. There, for example, we find that the *(monic)* eigenpolynomial for $\nu_{n,j}$ is a polynomial $b_{n,j}$ of degree j given by

$$b_{n,j}(x) = x^j - \frac{j}{2}x^{j-1} + \text{lower order terms.}$$

The rest of the coefficients are described in [32] via a recurrence relation that involves the Stirling numbers of the second kind.

Now we can prove the following

Proposition 4.31 *The eigenvalues of* $S_n^{<0,\beta,\gamma>}$ *are* $\lambda_{n,0} = 1$ *and*

$$\lambda_{n,j} = \frac{(n)_j}{(n+\gamma)^j},\ j = 1, \ldots, n.$$

The corresponding (normalized) eigenpolynomials are $q_{n,0} = e_0$ *and* $q_{n,j} = e_j + a_{n,j-1}^{(j)}e_{j-1} + \ldots + a_{n,0}^{(j)}e_0,\ j = 1, \ldots, n,$ *with uniquely determined coefficients.*

Proof. Obviously $S_n^{<0,\beta,\gamma>}e_0 = e_0$, i.e. $\lambda_{n,0} = 1$. Furthermore, we want to prove that there exists $q_{n,j} \in \prod_j$ such that

$$S_n^{<0,\beta,\gamma>}q_{n,j} = \lambda_{n,j} \cdot q_{n,j},\ j = 1, \ldots, n. \tag{4.14}$$

Denoting $u := \frac{n}{n+\gamma}$ and $v := \frac{\beta}{n+\gamma}$ and using the identity (3.10) we arrive at $B_n(q_{n,j} \circ (ue_1 + ve_0)) = \lambda_{n,j} \cdot q_{n,j}$. This can be extended as follows:

$$\begin{aligned} B_n[(ue_1+ve_0)^j &+ a_{n,j-1}^{(j)}(ue_1+ve_0)^{j-1} + \ldots + a_{n,1}^{(j)}(ue_1+ve_0) + a_{n,0}^{(j)}e_0] \\ &= \lambda_{n,j} \cdot (e_j + a_{n,j-1}^{(j)}e_{j-1} + \ldots + a_{n,1}^{(j)}e_1 + a_{n,0}^{(j)}e_0). \end{aligned}$$

As a consequence of Proposition 4.29 and the fact that $B_n(\prod_i) \subseteq \prod_i,\ i = 0, \ldots, n,$ (is *degree reducing*) we get the equation

$$\begin{aligned} u^j\nu_{n,j}e_j &+ c_{n,j-1}^{(j)}e_{j-1} + \ldots + c_{n,1}^{(j)}e_1 + c_{n,0}^{(j)}e_0 \\ &+ a_{n,j-1}^{(j)}[u^{j-1}\nu_{n,j-1}e_{j-1} + c_{n,j-2}^{(j-1)}e_{j-2} + \ldots + c_{n,0}^{(j-1)}e_0] \\ &+ \ldots + a_{n,1}^{(j)}[ue_1 + ve_0] + a_{n,0}^{(j)}e_0 \\ &= \lambda_{n,j}e_j + a_{n,j-1}^{(j)}\lambda_{n,j}e_{j-1} + \ldots + a_{n,1}^{(j)}\lambda_{n,j}e_1 + a_{n,0}^{(j)}\lambda_{n,j}e_0 \end{aligned}$$

Now we have to identify the coefficients in front of the monomials $e_i,\ i = 0, \ldots, j$. First of all $u^j\nu_{n,j} = \lambda_{n,j}$ must be satisfied and thus we arrive at:

$$\lambda_{n,j} = \frac{n^j}{(n+\gamma)^j} \cdot \frac{(n)_j}{n^j} = \frac{(n)_j}{(n+\gamma)^j},\ j = 1, \ldots, n,$$

which are the eigenvalues of the operator $S_n^{<0,\beta,\gamma>}$. In analogy to the Bernstein operator we observe that each two of them are distinct and that all are (strictly) less than 1 except $\lambda_{n,0}$.
Equating now the coefficients in front of the lower degree monomials we obtain the following linear (triangular) system (with n equations and n unknowns $a_{n,i}^{(j)},\ i = 1,\ldots,j-1$):

$$\begin{aligned} a_{n,j-1}^{(j)}(\lambda_{n,j}-\lambda_{n,j-1}) &= c_{n,j-1}^{(j)} \\ a_{n,j-2}^{(j)}(\lambda_{n,j}-\lambda_{n,j-2}) &= c_{n,j-2}^{(j)} + a_{n,j-1}^{(j)} \cdot c_{n,j-2}^{(j-1)} \\ &\vdots \\ a_{n,0}^{(j)}(\lambda_{n,j}-\lambda_{n,0}) &= c_{n,0}^{(j)} + a_{n,j-1}^{(j)} \cdot c_{n,0}^{(j-1)} + \ldots + a_{n,1}^{(j)} \cdot v. \end{aligned}$$

Its determinant is $(\lambda_{n,j}-\lambda_{n,j-1})(\lambda_{n,j}-\lambda_{n,j-2})\ldots(\lambda_{n,j}-\lambda_{n,0}) \neq 0$, and hence there exists a unique (monic) eigenpolynomial $q_{n,j},\ 1 \leq j \leq n$ corresponding to the eigenvalue $\lambda_{n,j}$. □
Now we can state the following result regarding the powers of the operator $S_n^{<0,\beta,\gamma>}$.

Theorem 4.32 *If $n \in \mathbb{N}$ is fixed, then for all $f \in C[0,1],\ x \in [0,1]$*

$$\lim_{m\to\infty} [S_n^{<0,\beta,\gamma>}]^m (f;x) = b_0 e_0(x), \tag{4.15}$$

where $b_0 = b_0(f)$ is a convex combination of the values of the function f that appear in the operator's definition, namely

$$b_0 = \sum_{j=0}^{n} d_j f\left(\frac{j+\beta}{n+\gamma}\right). \tag{4.16}$$

Proof. If $f \in C[0,1]$, then $S_n^{<0,\beta,\gamma>} f \in \prod_n$. Moreover, due to the fact that the eigenpolynomials $\{q_{n,0}, q_{n,1}, \ldots, q_{n,n}\}$ form a basis in $\prod_n$ we can write $S_n^{<0,\beta,\gamma>} f = b_0 q_{n,0} + b_1 q_{n,1} + \ldots + b_n q_{n,n}$. It follows that

$$\begin{aligned} [S_n^{<0,\beta,\gamma>}]^m f &= [S_n^{<0,\beta,\gamma>}]^{m-1}(S_n^{<0,\beta,\gamma>} f) = [S_n^{<0,\beta,\gamma>}]^{m-1}(b_0 q_{n,0} + b_1 q_{n,1} + \ldots + b_n q_{n,n}) \\ &= b_0 \lambda_{n,0}^{m-1} q_{n,0} + b_1 \lambda_{n,1}^{m-1} q_{n,1} + \ldots + b_n \lambda_{n,n}^{m-1} q_{n,n}. \end{aligned}$$

Passing to the limit we get $\lim_{m\to\infty} [S_n^{<0,\beta,\gamma>}]^m f = b_0 e_0$, because $\lambda_{n,j} \in (0,1)$ for $j = 1,\ldots,n$.
Since $S_n^{<0,\beta,\gamma>} f = \sum_{j=0}^{n} p_{n,j} f\left(\frac{j+\beta}{n+\gamma}\right)$ we assume and we will prove that b_0 has the form $b_0 = \sum_{j=0}^{n} d_j f\left(\frac{j+\beta}{n+\gamma}\right)$ with suitable $d_j \in \mathbb{R}$, the same for all f.

In order to simplify the notation we put $a_j := \frac{j+\beta}{n+\gamma}$, $0 \le j \le n$, and we write $S^{(\beta,\gamma)} f := \sum\limits_{j=0}^{n} d_j f(a_j) e_0$. Under these assumptions, taking $f := e_0$ and recalling that $S_n^{<0,\beta,\gamma>} e_0 = e_0$ we also get $S^{(\beta,\gamma)} e_0 = e_0$ which, in combination with the positivity of $S^{(\beta,\gamma)}$, implies

$$d_j \ge 0 \text{ and } d_0 + \ldots + d_n = 1.$$

Further we shall prove the existence of such coefficients that satisfy these conditions. Since $\lim\limits_{m\to\infty} [S_n^{<0,\beta,\gamma>}]^m f = S^{(\beta,\gamma)} f$ and $\lim\limits_{m\to\infty} [S_n^{<0,\beta,\gamma>}]^m f = \lim\limits_{m\to\infty} [S_n^{<0,\beta,\gamma>}]^{m-1} (S_n^{<0,\beta,\gamma>} f) = S^{(\beta,\gamma)}(S_n^{<0,\beta,\gamma>} f)$ we have $S^{(\beta,\gamma)}(S_n^{<0,\beta,\gamma>} f) = S^{(\beta,\gamma)} f$, $f \in C[0,1]$. Carrying out the computation we arrive at

$$\begin{aligned} S^{(\beta,\gamma)}(S_n^{<0,\beta,\gamma>} f) &= \left(\sum_{j=0}^{n} d_j (S_n^{<0,\beta,\gamma>} f)(a_j) \right) e_0 = \left(\sum_{j=0}^{n} d_j \sum_{i=0}^{n} p_{n,i}(a_j) f(a_i) \right) e_0 \\ &= \left(\sum_{i=0}^{n} \left(\sum_{j=0}^{n} d_j p_{n,i}(a_j) \right) f(a_i) \right) e_0, \quad \text{and} \\ S^{(\beta,\gamma)} f &= \left(\sum_{i=0}^{n} d_i f(a_i) \right) e_0. \end{aligned}$$

As a result we obtain the linear system

$$\sum_{j=0}^{n} p_{n,i}(a_j) d_j = d_i, \; i = 0, 1, \ldots, n. \tag{4.17}$$

Consider the matrix

$$T := \begin{pmatrix} p_{n,0}(a_0) & p_{n,1}(a_0) & \ldots & p_{n,n}(a_0) \\ p_{n,0}(a_1) & p_{n,1}(a_1) & \ldots & p_{n,n}(a_1) \\ \ldots & & & \\ p_{n,0}(a_n) & p_{n,1}(a_n) & \ldots & p_{n,n}(a_n) \end{pmatrix}$$

The system of equations (4.17) can be rewritten as

$$T^t \cdot \begin{pmatrix} d_0 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} d_0 \\ \vdots \\ d_n \end{pmatrix} \tag{4.18}$$

The matrix T is *stochastic*, i.e., has non-negative elements and the sum on each row is 1. Consider now the following three cases:
(i) If $0 < \beta < \gamma$, then all the elements of T are strictly positive and the system (4.18) has exactly one positive solution which also satisfies $d_0 + \ldots + d_n = 1$. This

is a fact known from the *Theory of Markov Chains*, for more more information on this issue see [85], Theorem 4.1.6.
(ii) If $\beta = 0$ (4.18) becomes

$$(4.19)\qquad \begin{pmatrix} 1 & p_{n,0}(a_1) & \dots & p_{n,0}(a_n) \\ 0 & p_{n,1}(a_1) & \dots & p_{n,1}(a_n) \\ \dots & & & \\ 0 & p_{n,n}(a_1) & \dots & p_{n,n}(a_n) \end{pmatrix} \cdot \begin{pmatrix} d_0 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} d_0 \\ \vdots \\ d_n \end{pmatrix},$$

where all the elements on the columns $1, 2, \dots, n$ are strictly positive. It is easy to see that this system has exactly one solution which fulfills $d_j \geq 0$ and $d_0 + \dots + d_n = 1$, namely $d_0 = 1$ and $d_1 = \dots = d_n = 0$. In this case $S^{(0,\gamma)} f = f(0)e_0,\ f \in C[0,1]$.
(iii) If $\beta = \gamma$ we find in a similar manner that $S^{(\beta,\beta)} f = f(1)e_0,\ f \in C[0,1]$. □

Example 4.33 *If $n = 1$ (4.18) can be written as*

$$\begin{pmatrix} \frac{1+\gamma-\beta}{1+\gamma} & \frac{\gamma-\beta}{1+\gamma} \\ \frac{\beta}{1+\gamma} & \frac{\beta+1}{1+\gamma} \end{pmatrix} \cdot \begin{pmatrix} d_0 \\ d_1 \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \end{pmatrix}.$$

This leads to $d_0 = 1 - \frac{\beta}{\gamma}$ and $d_1 = \frac{\beta}{\gamma}$. Thus

$$\lim_{m\to\infty} [S_1^{<0,\beta,\gamma>}]^m f = \left[\left(1 - \frac{\beta}{\gamma}\right) f\left(\frac{\beta}{1+\gamma}\right) + \frac{\beta}{\gamma} f\left(\frac{1+\beta}{1+\gamma}\right)\right] e_0.$$

4.3.2 General Stancu operators

In analogy to Proposition 4.31 we can formulate the following for the general $S_n^{<\alpha,\beta,\gamma>}$:

Proposition 4.34 *The eigenvalues of $S_n^{<\alpha,\beta,\gamma>}$ are $\lambda_{n,0}^{\alpha} = 1$ and*

$$\lambda_{n,j}^{\alpha} = \frac{(n)_j}{(n+\gamma)^j} \cdot \frac{1}{1^{[j,-\alpha]}},\ j = 1, \dots, n.$$

The corresponding (normalized) eigenpolynomials are $q_{n,j}(x) = e_j(x) + A_{n,j-1}^{(j)} e_{j-1}(x) + \dots + A_{n,0}^{(j)} e_0(x),\ j = 0, \dots, n$, with uniquely determined coefficients.

Proof. Due to $S_n^{<\alpha,\beta,\gamma>} e_0 = 1 \cdot e_0$ we have $\lambda_{n,0}^{\alpha} = 1$ and $q_{n,0} = e_0$. Like in the previous case we want to prove that there exist $q_{n,j} \in \prod_j$ such that

$$S_n^{<\alpha,\beta,\gamma>}(q_{n,j}; x) = \lambda_{n,j}^{\alpha} \cdot q_{n,j}(x),\ j = 1, \dots, n \text{ and } x \in [0,1].$$

Using the integral representation (3.9) we can write

$$\frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1} \cdot (1-t)^{\frac{1-x}{\alpha}-1} B_n[q_{n,j} \circ (ue_1 + ve_0); t] dt = \lambda_{n,j}^{\alpha} q_{n,j}(x),$$

where $u := \frac{n}{n+\gamma}$ and $v := \frac{\beta}{n+\gamma}$.

This can be expanded into

$$(4.20) \quad \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}$$
$$\cdot \quad B_n[(ue_1+ve_0)^j + A_{n,j-1}^{(j)}(ue_1+ve_0)^{j-1} + \ldots + A_{n,1}^{(j)}(ue_1+ve_0) + A_{n,0}^{(j)}e_0; t]dt$$
$$= \quad \lambda_{n,j}^{\alpha} \cdot (e_j(x) + A_{n,j-1}^{(j)}e_{j-1}(x) + \ldots + A_{n,1}^{(j)}e_1(x) + A_{n,0}^{(j)}e_0(x)),$$

which is equivalent to

$$(4.21) \quad \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}$$
$$\cdot \quad [u^j \nu_{n,j} e_j(t) + B_{n,j-1}^{(j)}e_{j-1}(t) + \ldots + B_{n,1}^{(j)}e_1(t) + B_{n,0}^{(j)}e_0(t)$$
$$+ \quad A_{n,j-1}^{(j)}(u^{j-1}\nu_{n,j-1}e_{j-1}(t) + B_{n,j-2}^{(j-1)}e_{j-2}(t) + \ldots + B_{n,0}^{(j-1)}e_0(t))$$
$$+ \quad \ldots + A_{n,1}^{(j)}(ue_1(t) + ve_0(t)) + A_{n,0}^{(j)}e_0(t)]dt$$
$$= \quad \lambda_{n,j}^{\alpha} \cdot (e_j(x) + A_{n,j-1}^{(j)}e_{j-1}(x) + \ldots + A_{n,1}^{(j)}e_1(x) + A_{n,0}^{(j)}e_0(x)).$$

In order to determine the eigenvalues $\lambda_{n,j}^{\alpha}$ we equate in the above equation the coefficients in front of e_j and get the values

$$\begin{aligned} \lambda_{n,j}^{\alpha} &= u^j \nu_{n,j} \cdot \frac{1}{(\alpha \cdot 0 + 1)(1 \cdot \alpha + 1)(2 \cdot \alpha + 1)\ldots(\alpha \cdot (j-1) + 1)} \\ &= \frac{n(n-1)\ldots(n-j+1)}{(n+\gamma)^j} \cdot \frac{1}{(\alpha \cdot 0 + 1)(1 \cdot \alpha + 1)(2 \cdot \alpha + 1)\ldots(\alpha \cdot (j-1) + 1)}; \end{aligned}$$

All of them are distinct and strictly less than 1 (except $\lambda_{n,0}^{\alpha}$). In computing $\lambda_{n,j}^{\alpha}$ we employed the recurrence formula

$$B\left(\frac{x}{\alpha}+j, \frac{1-x}{\alpha}\right) = \frac{\frac{x}{\alpha}+j-1}{\frac{1}{\alpha}+j-1} \cdot \frac{\frac{x}{\alpha}+j-2}{\frac{1}{\alpha}+j-2} \cdots \frac{\frac{x}{\alpha}}{\frac{1}{\alpha}} B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)$$

and the fact that $S_n^{<\alpha,\beta,\gamma>}$ maps polynomials of degree $i = 0, 1, \ldots, n$ into polynomials of degree i (is *degree reducing*).

We equate the coefficients in front of $e_i,\ i = 0, \ldots, j-1$ in (4.20) and we obtain a triangular system with the unknowns $A_{n,i}^{(j)},\ i = 1, \ldots, j-1$.

$$\begin{aligned} A_{n,j-1}^{(j)}(\lambda_{n,j}^{\alpha} - \lambda_{n,j-1}^{\alpha}) &= \tilde{B}_{n,j-1}^{(j)} \\ A_{n,j-2}^{(j)}(\lambda_{n,j}^{\alpha} - \lambda_{n,j-2}^{\alpha}) &= \tilde{B}_{n,j-2}^{(j)} + A_{n,j-1}^{(j)} \cdot \tilde{B}_{n,j-2}^{(j-1)} \\ &\vdots \\ A_{n,0}^{(j)}(\lambda_{n,j}^{\alpha} - \lambda_{n,0}^{\alpha}) &= \tilde{B}_{n,0}^{(j)} + A_{n,j-1}^{(j)} \cdot \tilde{B}_{n,0}^{(j-1)} + \ldots + A_{n,1}^{(j)} \cdot v. \end{aligned}$$

Its determinant is $(\lambda_{n,j}^{\alpha}-\lambda_{n,j-1}^{\alpha})(\lambda_{n,j}^{\alpha}-\lambda_{n,j-2}^{\alpha})\dots(\lambda_{n,j}^{\alpha}-\lambda_{n,0}^{\alpha})\neq 0$. The implication is similar to the one in the previous subsection: there exists a unique (monic) eigenpolynomial $q_{n,j}$ of degree j, $1\leq j\leq n$ with the eigenvalue $\lambda_{n,j}^{\alpha}$. Thus we have proved that $S_n^{<\alpha,\beta,\gamma>}(q_{n,j};x)=\lambda_{n,j}^{\alpha}\cdot q_{n,j}(x)$. And this is valid on the whole compact interval $[0,1]$, due to the continuity of $S_n^{<\alpha,\beta,\gamma>}q_{n,j}$. □

About the over-iterates of $S_n^{<\alpha,\beta,\gamma>}$ we can assert the following generalization of Theorem 4.32

Theorem 4.35 *If $n\in\mathbb{N}$ is fixed, then for all $f\in C[0,1]$, $x\in[0,1]$*

$$\lim_{m\to\infty}[S_n^{<\alpha,\beta,\gamma>}]^m(f;x)=b_0^{\alpha}e_0(x), \tag{4.22}$$

where $b_0^{\alpha}=b_0^{\alpha}(f)$ is a convex combination of the function f values that appears in the operator's definition,

$$b_0^{\alpha}=\sum_{j=0}^{n}d_j^{\alpha}f\left(\frac{j+\beta}{n+\gamma}\right). \tag{4.23}$$

Proof. In proving this statement we will use the same "trick" as in the previous subsection. We can write

$$\begin{aligned}
[S_n^{<\alpha,\beta,\gamma>}]^m(f;x) &= [S_n^{<\alpha,\beta,\gamma>}]^{m-1}(S_n^{<\alpha,\beta,\gamma>}(f;x))\\
&= [S_n^{<\alpha,\beta,\gamma>}]^{m-1}(b_0^{\alpha}q_{n,0}+b_1^{\alpha}q_{n,1}+\dots+b_n^{\alpha}q_{n,n};x)\\
&= b_0^{\alpha}(\lambda_{n,0}^{\alpha})^{m-1}q_{n,0}(x)+b_1^{\alpha}(\lambda_{n,1}^{\alpha})^{m-1}q_{n,1}(x)+\dots+b_n^{\alpha}(\lambda_{n,n}^{\alpha})^{m-1}q_{n,n}(x),
\end{aligned}$$

for any $f\in C[0,1]$ and $x\in[0,1]$. Letting $m\to\infty$ we obtain $\lim\limits_{m\to\infty}[S_n^{<\alpha,\beta,\gamma>}]^m(f;x)=b_0^{\alpha}\cdot e_0(x)$. Here we used (again) the representation of the polynomial $S_n^{<\alpha,\beta,\gamma>}\in\prod_n$ with respect to the basis of the eigenpolynomials $\{q_{n,0},q_{n,1},\dots,q_{n,n}\}$. Further we put $S^{(\alpha,\beta,\gamma)}f:=b_0^{\alpha}e_0$ and we will assume and prove that

$$b_0^{\alpha}=d_0^{\alpha}f(a_0)+d_1^{\alpha}f(a_1)+\dots+d_n^{\alpha}f(a_n),$$

where $a_j:=\frac{j+\beta}{n+\gamma}$, $j=0,\dots n$. To justify this we can rely on many arguments presented in the proof of Theorem 4.32. In order to reduce redundancy we shall point out only the important steps.
$S_n^{<\alpha,\beta,\gamma>}e_0=e_0$ leads to $S^{(\alpha,\beta,\gamma)}e_0=e_0$, i.e., $d_j^{\alpha}\geq 0$ and $d_0^{\alpha}+\dots+d_n^{\alpha}=1$. Using the fact that $S^{(\alpha,\beta,\gamma)}(S_n^{<\alpha,\beta,\gamma>}f)=S^{(\alpha,\beta,\gamma)}f$ holds, we get a system of equations similar to (4.17), namely

$$\sum_{j=0}^{n}w_{n,i}^{(\alpha)}(a_j)d_j^{\alpha}=d_i^{\alpha},\ i=0,1,\dots,n. \tag{4.24}$$

The equivalent matrix form is

$$T^t \cdot \begin{pmatrix} d_0^\alpha \\ \vdots \\ d_n^\alpha \end{pmatrix} = \begin{pmatrix} d_0^\alpha \\ \vdots \\ d_n^\alpha \end{pmatrix}, \tag{4.25}$$

where T is stochastic and has the form $T = (w_{n,i}^{(\alpha)}(a_j))_{i,j=0,\ldots,n}$.
(i) If $0 < \beta < \gamma$, then all the elements of T are strictly positive and the system has exactly one positive solution which also satisfies $d_0^\alpha + \ldots + d_n^\alpha = 1$. For more information about stochastic matrix see [85].
(ii)–(iii) The cases $\beta = 0$ or $\beta = \gamma$ can be approached in a similar way as in the preceding proof. For $\beta = 0$ we get $S^{(\alpha,0,\gamma)}(f;x) = f(0)e_0(x),\ x \in [0,1]$ and for the last one $S^{(\alpha,\beta,\beta)}(f;x) = f(1)e_0(x),\ x \in [0,1]$. □

Remark 4.36 For the sake of completeness let us give a brief explanation, why all coefficients d_j^α in (4.23) (respectively the ones in (4.16) for $\alpha = 0$) are the same for all functions f. Consider the two bases, that of *eigenpolynomials* $\{q_{n,0}, \ldots, q_{n,n}\}$, and that of *fundamental Stancu polynomials* $\{w_{n,0}^{(\alpha)}, \ldots, w_{n,n}^{(\alpha)}\}$ of $\prod_n$, see its definition at (3.5). The matrix $\Theta = (\theta_{i,j})_{i,j=0,\ldots,n}$ allows us to pass from one basis to the other and is defined by

$$\begin{aligned} w_{n,0}^{(\alpha)} &= \theta_{0,0} \cdot q_{n,0} + \ldots + \theta_{n,0} \cdot q_{n,n} \\ &\ldots \\ w_{n,n}^{(\alpha)} &= \theta_{0,n} \cdot q_{n,0} + \ldots + \theta_{n,n} \cdot q_{n,n}. \end{aligned}$$

Then the coordinates of $S_n^{<\alpha,\beta,\gamma>} f$ with respect to the two bases are related by

$$\begin{pmatrix} b_0^\alpha \\ \vdots \\ b_n^\alpha \end{pmatrix} = \begin{pmatrix} \theta_{0,0} & \ldots & \theta_{0,n} \\ \ldots & & \\ \theta_{n,0} & \ldots & \theta_{n,n} \end{pmatrix} \cdot \begin{pmatrix} f(a_0) \\ \vdots \\ f(a_n) \end{pmatrix}$$

Thus $d_0^\alpha = \theta_{0,0}, \ldots, d_n^\alpha = \theta_{0,n}$ and they are independent of f. Thus we have discovered this second possibility to determine (algebraically) $d_0^\alpha, \ldots, d_n^\alpha$ by decomposing $w_{n,j}^\alpha$ with respect to the basis $\{q_{n,0}, \ldots, q_{n,n}\}$.

4.3.3 Kantorovich operators

In 1930 L. V. Kantorovich introduced in [84] an operator closely related to the Bernstein operator. It satisfies $K_n(Df) = D(B_{n+1}f)$ for any $f \in C^1[0,1]$. We recall their explicit representation. The *Kantorovich operators* K_n, with $n \in \mathbb{N}$, map any function $f \in L^1[0,1]$ into $C[0,1]$ and they are defined by

$$K_n(f;x) := (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,$$

where $p_{n,k}$ are the *fundamental Bernstein* polynomials.
From the *spectral properties* of the Bernstein operator one can easily derive the spectral properties of the Kantorovich operator.

Corollary 4.37 (see [32, p. 158]) *The eigenvalues of* K_n *are*

$$\lambda_{n,j} = \frac{n!}{(n-j)!} \cdot \frac{1}{(n+1)^j}, \; j = 0, 1 \ldots, n,$$

and the corresponding eigenpolynomials $q_{n,j}$ *of degree* j *are described by*

$$q_{n,j} = Db_{n+1,j+1},$$

where $b_{n+1,j+1}$ *are the eigenpolynomials (of degree* $j+1$*) of the Bernstein operator* B_{n+1}*, see Remark 4.30.*
Moreover, it can be shown that for every f *an integrable function on* $[0,1]$ *the approximant* $K_n f$ *can be decomposed as follows:*

$$K_n f = \sum_{j=0}^{n} \lambda_{n,j} \cdot v_{n,j} \cdot q_{n,j},$$

where $v_{n,0} = \int_0^1 f(t)dt$.

Now we can investigate the behavior of the over-iterates of the Kantorovich operator:

Theorem 4.38 *If* $n \in \mathbb{N}$ *is fixed, then for all* f *integrable on* $[0,1]$ *and* $x \in [0,1]$ *we have*

$$\lim_{m\to\infty} K_n^m(f;x) = \left(\int_0^1 f(t)dt\right) e_0(x). \tag{4.26}$$

Proof. From the last Corollary we deduce that

$$K_n^m f = \sum_{j=0}^{n} (\lambda_{n,j})^m \cdot v_{n,j} \cdot q_{n,j}.$$

The eigenvalues are distinct and $0 < \lambda_{n,j} < 1$, $j = 1, \ldots n$, only $\lambda_{n,0} = 1$. Thus letting $m \to \infty$ it is implied that $\lim_{m\to\infty} K_n^m(f;x) = \left(\int_0^1 f(t)dt\right) e_0(x)$. □

4.3.4 Durrmeyer operators

In his thesis in 1967 J. L. Durrmeyer [40] introduced on $L^2[0,1]$ a modification of the Bernstein operator which has some remarkable properties (e.g., is self-adjoint and commutative).
For f an integrable function on $[0,1]$ the *Durrmeyer operators* are defined by

$$D_n(f;x) := (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,$$

where $p_{n,k}$ are the fundamental Bernstein polynomials.
Due to their interesting properties they were intensively studied by many authors. Therefore we shall cite here only a small subset of all the mathematicians and their works: A. Lupaş [95], M. M. Derrienic ([36], [37]), Z. Ditzian & K. Ivanov [39].
Compare the following result with Theorem 4.38.

Theorem 4.39 *If $n \in \mathbb{N}$ is fixed, then for all f integrable on $[0,1]$ and $x \in [0,1]$ we have*

$$\lim_{m\to\infty} D_n^m(f;x) = \left(\int_0^1 f(t)dt\right) e_0(x). \tag{4.27}$$

Proof. The answer comes easily because the eigenvalues and the eigenpolynomials are well known for these operators (see M.M. Derrienic [37]). Due to the fact that constant functions are reproduced we have $\lambda_{n,0} = 1$ and the other values are $\lambda_{n,j} = \frac{(n+1)!n!}{(n-j)!(n+j+1)!}$, $j = 1,\ldots n$. It is obvious that $\lambda_{n,j} \in (0,1)$ for $j \geq 1$. Their eigenpolynomials are exactly the *Legendre polynomials* normalized in $L^2[0,1]$:

$$P_0^{(0,0)}(x) = e_0(x) \text{ and } P_j^{(0,0)}(x) = \frac{\sqrt{2j+1}}{j!}\cdot\frac{\partial^j}{\partial x^j}(x^j(1-x)^j),\ j = 1,\ldots,n,\ x \in [0,1].$$

Furthermore, they admit the following representation (diagonal form) for any f integrable on $[0,1]$:

$$D_n f = \sum_{j=0}^{n} \lambda_{n,j}\cdot\left(\int_0^1 f(t)P_j^{(0,0)}(t)dt\right) P_j^{(0,0)}. \tag{4.28}$$

Thus we can write

$$\begin{aligned} D_n^m f &= D_n^{m-1}(D_n f) = D_n^{m-1}\Big(\sum_{j=0}^{n} \lambda_{n,j}\cdot\left(\int_0^1 f(t)P_j^{(0,0)}(t)dt\right) P_j^{(0,0)}\Big) \\ &= \sum_{j=0}^{n} (\lambda_{n,j})^m\cdot\left(\int_0^1 f(t)P_j^{(0,0)}(t)dt\right) P_j^{(0,0)}. \end{aligned}$$

Letting $m \to \infty$ and recalling that $0 < \lambda_{n,j} < 1$, $j \geq 1$, we get $\lim\limits_{m\to\infty} D_n^m(f;x) = \left(\int_0^1 f(t)P_0^{(0,0)}(t)dt\right) P_0^{(0,0)}(x)$. But $P_0^{(0,0)} = e_0$, and so we get the desired result. □

4.3.5 Durrmeyer operators with Jacobi weights

The Durrmeyer operators were generalized in the following way:
Let $\omega^{(\alpha,\beta)}(x) = x^\alpha(1-x)^\beta,\ \alpha,\beta > -1$, be the Jacobi weight on the interval $(0,1)$ and let $L^1_{\omega^{(\alpha,\beta)}}(0,1)$ be the *space of Lebesgue-measurable* functions f on $(0,1)$, such that the norm $\|f\|_{\omega^{(\alpha,\beta)}} := \sqrt{\int\limits_0^1 f^2(x)\omega^{(\alpha,\beta)}(x)dx}$ is finite.
The operators $D_n^{(\alpha,\beta)} : L^1_{\omega^{(\alpha,\beta)}}(0,1) \to C[0,1]$ defined by

$$D_n^{(\alpha,\beta)}(f;x) := \sum_{k=0}^{n} p_{k,n}(x) \frac{\int\limits_0^1 p_{k,n}(t)f(t)\omega^{(\alpha,\beta)}(t)dt}{\int\limits_0^1 p_{k,n}(t)\omega^{(\alpha,\beta)}(t)dt}, \tag{4.29}$$

where $p_{n,k}$ is the Bernstein basis, are the *generalized Durrmeyer operators* w.r.t. the *Jacobi weight* $\omega^{(\alpha,\beta)}$.
Due to the fact that for any $f \in C[0,1], D_n^{(\alpha,\beta)} f$ can be represented as a linear combination of *Jacobi polynomials* they are also called *Bernstein-Jacobi* operators. If we take $\alpha = \beta = 0$ we obtain the "classical" Durrmeyer operators, from whom these generalized operators inherit many of their properties, e.g., are self-adjoint and commute with each other. But more interesting results can be found in the literature, here we mention some authors: P. Sablonniére [133] (an unpublished report), R. Păltănea ([110], [113]) and H. Berens & Xu ([9], [10]). The limit case $\alpha,\beta \to -1^+$ provides $\lim\limits_{\alpha,\beta\to-1^+} D_n^{(\alpha,\beta)}(f;x) = U_n(f;x)$, for any fixed $f \in C[0,1],\ x \in [0,1]$, see [113]. The over-iterates of U_n's were already studied in Subsection 4.2.2.
In this case Theorem 4.39 can be reformulated as follows

Theorem 4.40 *If $n \in \mathbb{N}$ is fixed, then for all f integrable on $[0,1]$ and $x \in [0,1]$ we have*

$$\lim_{m\to\infty} [D_n^{(\alpha,\beta)}]^m(f;x) = \left(\int_0^1 f(t) \cdot t^\alpha(1-t)^\beta dt\right) e_0(x). \tag{4.30}$$

Proof. In [110] and also [133] it was proved that for any $f \in C[0,1],\ x \in [0,1]$ we have

$$D_n^{(\alpha,\beta)} f = \sum_{j=0}^{n} \lambda_{n,j}^{(\alpha,\beta)} \cdot \left(\int_0^1 f(t)P_j^{(\alpha,\beta)} t^\alpha(1-t)^\beta dt\right) \cdot P_j^{(\alpha,\beta)},$$

where

$$\lambda_{n,j}^{(\alpha,\beta)} = \frac{\Gamma(n+1)\Gamma(\alpha+\beta+n+2)}{\Gamma(n-j+1)\Gamma(\alpha+\beta+n+j+2)},\ 0 \le j \le n,$$

and $P_j^{(\alpha,\beta)}$ are the *Jacobi polynomials* of degree j with respect to the weight function $t^\alpha(1-t)^\beta,\ t \in [0,1]$.

Using this representation we arrive at

$$
\begin{aligned}
& [D_n^{(\alpha,\beta)}]^m f = [D_n^{(\alpha,\beta)}]^{m-1}(D_n^{(\alpha,\beta)} f) \\
= \; & [D_n^{(\alpha,\beta)}]^{m-1} \left(\sum_{j=0}^{n} \lambda_{n,j}^{(\alpha,\beta)} \cdot \left(\int_0^1 f(t) P_j^{(\alpha,\beta)} t^\alpha (1-t)^\beta dt \right) P_j^{(\alpha,\beta)} \right) \\
= \; & \sum_{j=0}^{n} [\lambda_{n,j}^{(\alpha,\beta)}]^m \cdot \left(\int_0^1 f(t) P_j^{(\alpha,\beta)} t^\alpha (1-t)^\beta dt \right) P_j^{(\alpha,\beta)}.
\end{aligned}
$$

It is easy to check that $\lambda_{n,0}^{(\alpha,\beta)} = 1$ (whence $P_0^{(\alpha,\beta)} = e_0$) and $\lambda_{n,j}^{(\alpha,\beta)} < 1,\ j = 1, \dots n$, for $\alpha, \beta > -1$. Using these facts and letting $m \to \infty$ we get the desired result: $\lim\limits_{m\to\infty} [D_n^{(\alpha,\beta)}]^m f = \left(\int_0^1 f(t) \cdot t^\alpha (1-t)^\beta dt \right) e_0$. □

Remark 4.41 In conclusion, the over-iterates of the operators taken into consideration in this section tend toward a constant function.
This method involving the eigenstructure of the considered operator, can be successfully applied also to other classes of operators, and - what is also important- not only for over-iteration, but also for iteration in general. For more information concerning this the reader is directed to [32] and [109].

Chapter 5

A new form of Peano's theorem and applications to positive linear operators

5.1 About the Peano form

In Marsden's article [100] a certain function s is introduced which arises from Peano's form of the Taylor remainder for univariate functions which are n-times continuously differentiable. In both old books (see, e.g., [47, p. 230] or [88, p. 489]) and new books (cf. [114, p. 84]) this remainder is given using "little o" Landau notation. This unfortunate abbreviation always appears at the end of the story, since hardly any further serious considerations can be based on a little-o-statement unless further information is given concerning "o".
Further we recall Theorem 1.6.6 from Davis' book [34] where the remainder term is attributed to Young.

Theorem 5.1 *Let $f(x)$ be n times differentiable at $x = x_0$. Then*

$$\begin{aligned} f(x) = f(x_0) &+ f'(x_0)(x - x_0) + \ldots + \frac{1}{(n-1)!} f^{(n-1)}(x_0)(x - x_0)^{n-1} \\ &+ \frac{(x - x_0)^n}{n!} \cdot [f^{(n)}(x_0) + \varepsilon(x)] \end{aligned} \tag{5.1}$$

where $\lim_{x \to x_0} \varepsilon(x) = 0$.

From the latter relation the "o" notation is derived, but also Young's form does not exhibit the relation between ε and f.

It is thus the aim of this section to estimate the Peano remainder in a different form by relating it appropriately to the expanded function. This will be done using the

modulus of continuity of the n-th derivative $f^{(n)}$ of the function f and the least concave majorant of the modulus. Details will be given below, but the interested reader is also directed to [70] or to [71].
By using the direct approach we can prove

Theorem 5.2 *For $n \in \mathbb{N}_0$ let $f \in C^n[a,b]$ and $x, x_0 \in [a,b]$. Then for the remainder in Taylor's formula we have*

$$|R_n(f;x_0,x)| \le \frac{|x-x_0|^n}{n!} \cdot \omega_1(f^{(n)}; |x-x_0|; [a,b]).$$

Proof. For a function $f \in C^n[a,b]$, the space of n-times continuously differentiable functions, the remainder in Taylor's formula is given by $(x_0, x \in [a,b], n \in \mathbb{N})$

$$R_n(f;x_0,x) := f(x) - \sum_{k=0}^{n} \frac{1}{k!} \cdot f^{(k)}(x_0) \cdot (x-x_0)^k.$$

Hence, for $n \ge 1$,

$$R_n(f;x_0,x) = R_{n-1}(f;x_0,x) - \frac{1}{n!} f^{(n)}(x_0) \cdot (x-x_0)^n.$$

The remainder $R_{n-1}(f;x_0,x)$ figuring here can be represented in its Lagrange form as

$$f^{(n)}(\xi_x) \cdot \frac{(x-x_0)^n}{n!} \text{ with } \xi_x \text{ between } x \text{ and } x_0.$$

We will denote the closed interval with endpoints x and x_0 by $< x, x_0 >$. So

$$\xi_x \in < x, x_0 > = \begin{cases} [x, x_0], & \text{if} \quad x \le x_0; \\ [x_0, x], & \text{if} \quad x_0 < x. \end{cases}$$

We can thus write

$$R_n(f;x_0,x) = \frac{(x-x_0)^n}{n!} (f^{(n)}(\xi_x) - f^{(n)}(x_0)),$$

or

$$\begin{aligned} |R_n(f;x_0,x)| &\le \tfrac{|x-x_0|^n}{n!} |f^{(n)}(\xi_x) - f^{(n)}(x_0)| \\ &\le \tfrac{|x-x_0|^n}{n!} \cdot \omega(f^{(n)}; |\xi_x - x_0|; < x, x_0 >) \\ &\le \tfrac{|x-x_0|^n}{n!} \cdot \omega(f^{(n)}; |x - x_0|; < x, x_0 >) \\ &\le \tfrac{|x-x_0|^n}{n!} \cdot \omega(f^{(n)}; |x - x_0|; [a,b]). \end{aligned}$$

Since $f^{(n)}$ is continuous on $[a,b]$ we have $\omega(f^{(n)}; |x-x_0|; [a,b]) = o(1)$ for $x \to x_0$. So the above are our first more precise versions of the Taylor remainder in Peano's form. □

An even more precise form will be given in the next theorem via a K-functional.

Theorem 5.3 *For $n \in \mathbb{N}_0$ let $f \in C^n[a,b]$ and $x, x_0 \in [a,b]$. Then for the remainder in Taylor's formula we have*

$$|R_n(f;x_0,x)| \le \frac{|x-x_0|^n}{n!} \cdot \tilde{\omega}(f^{(n)}; \frac{|x-x_0|}{n+1}),$$

where $\tilde{\omega}(f^{(n)};\cdot)$ is the least concave majorant of the modulus of $\omega(f^{(n)};\cdot)$, see (1.7).

Proof. Consider $f \in C^n[a,b]$ first. First (see the latter theorem) we have

$$\begin{aligned} |R_n(f;x_0,x)| &\le \tfrac{|x-x_0|^n}{n!} \cdot \omega(f^{(n)}; |x-x_0|; [a,b]) \\ &\le 2 \cdot \tfrac{|x-x_0|^n}{n!} \|f^{(n)}\|_\infty. \end{aligned}$$

Moreover, for $g \in C^{n+1}[a,b]$ we get - using the Lagrange form of the remainder again - that

$$\begin{aligned} |R_n(g;x_0,x)| &= \tfrac{|x-x_0|^{n+1}}{(n+1)!} \cdot |g^{(n+1)}(\theta_x)|, \ \theta_x \in < x, x_0 >, \\ &\le \tfrac{|x-x_0|^{n+1}}{(n+1)!} \cdot \|g^{(n+1)}\|_\infty. \end{aligned}$$

Keeping f fixed and letting g be arbitrary in $C^{n+1}[a,b]$ we have

$$\begin{aligned} |R_n(f;x_0,x)| &= |R_n(f-g+g;x_0,x)| \\ &\le |R_n(f-g;x_0,x)| + |R_n(g;x_0,x)| \\ &\le \tfrac{2\cdot|x-x_0|^n}{n!} \{\|(f-g)^{(n)}\|_\infty + \tfrac{|x-x_0|}{2(n+1)} \cdot \|g^{(n+1)}\|_\infty\}. \end{aligned}$$

Passing to the infimum over $g \in C^{n+1}[a,b]$ gives

$$\begin{aligned} |R_n(f;x_0,x)| &\le \tfrac{2\cdot|x-x_0|^n}{n!} \cdot K\left(f^{(n)}; \tfrac{|x-x_0|}{2(n+1)}; C[a,b], C^1[a,b]\right) \\ &= \tfrac{|x-x_0|^n}{n!} \cdot \tilde{\omega}\left(f^{(n)}; \tfrac{|x-x_0|}{n+1}\right). \qquad \Box \end{aligned}$$

Example 5.4 *The latter estimate is best possible in the sense that, e.g., for the function $e_{n+1} : [-1,1] \ni x \mapsto x^{n+1}$ equality occurs for $x_0 = 0$. Indeed, for e_{n+1} we have $R_n(e_{n+1};0,x) = x^{n+1}$, and*

$$\begin{aligned} \tfrac{|x-0|^n}{n!} \cdot \tilde{\omega}(e_{n+1}^{(n)}; \tfrac{|x-0|}{n+1}) &= \tfrac{|x|^n}{n!} \cdot \tilde{\omega}((n+1)!e_1; \tfrac{|x|}{n+1}) \\ &= \tfrac{|x|^n}{n!} \cdot (n+1)! \cdot \tfrac{|x|}{n+1} \\ &= |x|^{n+1}, \ \textit{and thus} \\ |R_n(e_{n+1};0,x)| &= \tfrac{|x-0|^n}{n!} \cdot \tilde{\omega}_1(e_{n+1}^{(n)}; \tfrac{|x-0|}{n+1}). \qquad \Box \end{aligned}$$

We use Korneichuk's observation [89] to relate the inequality of the latter theorem to that included in Theorem 5.2. We have

$$\tilde{\omega}(f^{(n)};\frac{|x-x_0|}{n+1}) \le (1+\frac{1}{n+1})\cdot\omega(f^{(n)};|x-x_0|),$$

so the inequality in terms of $\omega(f^{(n)};\cdot)$ which we derive via $\tilde{\omega}(f^{(n)};\cdot)$ is slightly worse than what is obtained using the "direct approach".
We shall further compare the two approaches by means of two well chosen examples:

Example 5.5 *The example $f = e_{n+1}$ from above also shows that the K-functional approach can be better than the direct one. Indeed,*

$$\omega(e_{n+1}^{(n)};|x-0|;[-1,1]) = \omega((n+1)!e_1;|x|;[-1,1]) = (n+1)!|x|,$$

leading to the upper bound

$$|R_n(e_{n+1};0,x)| \le (n+1)\cdot|x|^{n+1},$$

which for $x \neq 0$ is larger than $|x|^{n+1} = \frac{|x|^n}{n!}\cdot\tilde{\omega}(e_{n+1}^{(n)};\frac{|x|}{n+1})$. □

Example 5.6 *Here we give an example of a function f for which*

$$\omega(f^{(n)};|x-x_0|) \le \tilde{\omega}(f^{(n)};\frac{|x-x_0|}{n+1}).$$

This will show that in certain cases the direct approach can lead to a result at least as good as the second one via the K-functional. For this purpose consider the (abstract) moduli of continuity Ω and $\tilde{\Omega}$ constructed in Example 1.19. Suppose further that for $x, x_0 \in [0,1]$ we have $|x-x_0| = \varepsilon$. Furthermore, let $f \in C^n[0,1]$ be such that $f^{(n)}(t) = \Omega(t)$. Then

$$\omega(f^{(n)};|x-x_0|) = \omega(\Omega(\cdot);\varepsilon) = \Omega(\varepsilon) = \tilde{\Omega}(\frac{\varepsilon}{n+1}) = \tilde{\omega}(f^{(n)};\frac{|x-x_0|}{n+1}),$$

which confirms our claim. □

5.2 Voronovskaja's theorem revisited

The result of Voronovskaja for the Bernstein operators is well-known, was first proved in [156] and is given in the book of DeVore and Lorentz [38, p. 307] as follows.

Theorem 5.7 *If f is bounded on $[0,1]$, differentiable in some neighborhood of x and has a second derivative $f''(x)$ for some $x \in [0,1]$, then*

$$\lim_{n\to\infty} n \cdot [B_n(f,x) - f(x)] = \frac{x(1-x)}{2} \cdot f''(x).$$

If $f \in C^2[0,1]$, the convergence is uniform.

In the following we will describe the degree of this uniform convergence:

Theorem 5.8 *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator such that $Le_i = e_i$ for $i = 0,1$. If $f \in C^2[0,1]$ and $x \in [0,1]$, then*

$$\begin{aligned}&|L(f;x) - f(x) - \tfrac{1}{2} \cdot f''(x) \cdot L((e_1 - x)^2;x)| \\ &\leq \tfrac{1}{2} \cdot L((e_1 - x)^2;x) \cdot \tilde{\omega}(f'', \tfrac{1}{3} \cdot \sqrt{\tfrac{L((e_1-x)^4;x)}{L((e_1-x)^2;x)}}).\end{aligned}$$

Proof. For a linear operator $L : C[0,1] \to C[0,1], f \in C^n[0,1]$ and $x \in [0,1]$ we write

$$\begin{aligned} & L(f;x) - f(x) = L(f(t);x) - f(x) \\ = \; & L(\sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(x) \cdot (t-x)^k; x) + \\ & L(f - \sum_{k=0}^{n} \tfrac{1}{k!} \cdot f^{(k)}(x) \cdot (t-x)^k; x) - f(x) \\ = \; & f(x)[L(e_0;x) - 1] + \sum_{k=1}^{n} \tfrac{1}{k!} f^{(k)}(x) \cdot L((e_1 - x)^k; x) + \\ & L(f - \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(x)(e_1 - x)^k; x). \end{aligned}$$

Rearranging terms gives

$$\begin{aligned} & L(f;x) - f(x) - f(x)[L(e_0;x) - 1] - \sum_{k=1}^{n} \tfrac{1}{k!} f^{(k)}(x) \cdot L((e_1 - x)^k; x) \\ & = L(f - \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(x)(e_1 - x)^k; x) \\ & = L(\tfrac{(e_1-x)^n}{n!} \cdot \mu_x(\cdot); x), \end{aligned}$$

where

$$\frac{(t-x)^n}{n!} \cdot \mu_x(t) := f(t) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) \cdot (t-x)^k.$$

As we know from Theorem 5.3,

$$|\frac{(t-x)^n}{n!} \cdot \mu_x(t)| \leq \frac{|t-x|^n}{n!} \tilde{\omega}(f^{(n)}; \frac{|t-x|}{n+1}),$$

where $\tilde{\omega}(f^{(n)}; \frac{|t-x|}{n+1}) = o(1)$ if $t \to x$.
If L reproduces polynomials up to degree $n-1$ the above equality leads to

$$\begin{aligned}&|L(f;x) - f(x) - \tfrac{1}{n!} \cdot f^{(n)}(x) \cdot L((e_1-x)^n;x)|\\ &= |L(\tfrac{(e_1-x)^n}{n!} \cdot \mu_x(\cdot);x)|\end{aligned}$$

Moreover, if L is a *positive* operator and $n = 2$ we are led to the inequality

$$\begin{aligned}&|L(f;x) - f(x) - \tfrac{1}{2} \cdot f''(x) \cdot L((e_1-x)^2;x)|\\ &\leq L(\tfrac{(e_1-x)^2}{2} \cdot |\mu_x(\cdot)|;x)\\ &\leq L(\tfrac{(e_1-x)^2}{2} \cdot \tilde{\omega}(f''; \tfrac{|e_1-x|}{3});x).\end{aligned}$$

For the last expression we will now derive a more convenient upper bound.
For $g \in C^3[0,1]$ arbitrary we write

$$\begin{aligned}&L(\tfrac{(e_1-x)^2}{2} \cdot \tilde{\omega}(f''; \tfrac{|e_1-x|}{3});x)\\ &= L((e_1-x)^2 \cdot K(f''; \tfrac{|e_1-x|}{6}; C^0[0,1], C^1[0,1]);x)\\ &\leq L\left((e_1-x)^2 \cdot \{\|(f-g)''\|_\infty + \tfrac{|e_1-x|}{6} \cdot \|g'''\|_\infty\};x\right)\\ &= L((e_1-x)^2;x) \cdot \|(f-g)''\|_\infty + \tfrac{1}{6} \cdot L(|e_1-x|^3;x) \cdot \|g'''\|_\infty\\ &= L((e_1-x)^2;x) \cdot \left\{\|(f-g)''\|_\infty + \tfrac{1}{6} \cdot \tfrac{L(|e_1-x|^3;x)}{L((e_1-x)^2;x)} \cdot \|g'''\|_\infty\right\}.\end{aligned}$$

Passing back to the inf over $g \in C^3[0,1]$ yields

$$\begin{aligned}&L(\tfrac{(e_1-x)^2}{2} \cdot \tilde{\omega}(f''; \tfrac{|e_1-x|}{3});x)\\ &\leq L((e_1-x)^2;x) \cdot K(\tfrac{1}{6} \cdot \tfrac{f'';L(|e_1-x|^3;x)}{L((e_1-x)^2;x)}; C^0, C^1)\\ &= \tfrac{1}{2} L((e_1-x)^2;x) \cdot \tilde{\omega}(f'', \tfrac{1}{3} \cdot \tfrac{L(|e_1-x|^3;x)}{L((e_1-x)^2;x)}).\end{aligned}$$

Writing $L(|e_1-x|^3;x) = L((e_1-x)^2 \cdot |e_1-x|;x)$ and using the Cauchy-Schwarz inequality for positive linear functionals shows that

$$L(|e_1-x|^3;x) \leq \sqrt{L((e_1-x)^4;x)} \cdot \sqrt{L((e_1-x)^2;x)}.$$

Hence - due to the monotonicity of $\tilde{\omega}(f'', \cdot)$ - we arrive at the desired statement. □

Remark 5.9 In the recent paper [62, Theorem 3.2] we find an estimate that generalizes this inequality, namely for $q \in \mathbb{N}_0$ and $f \in C^q[0,1]$ the following inequality holds:

$$\left|L(f;x) - \sum_{r=0}^{q} L((e_1-x)^r;x) \cdot \frac{f^{(r)}(x)}{r!}\right| \leq \frac{L(|e_1-x|^q;x)}{q!} \cdot \tilde{\omega}_1\left(f^{(q)}; \frac{1}{q+1} \cdot \frac{L(|e_1-x|^{q+1};x)}{L(|e_1-x|^q;x)}\right).$$

$q = 2$ produces the inequality in the proof of Theorem 5.8. Other interesting cases are discussed in [62].

5.2.1 Application to some positive linear operators

Applications of our refined Voronovskaja-type theorem are given here for the classical Bernstein operators, for some selected special cases of the composite Beta-type operator $\mathbb{B}_n^{(\alpha,\lambda)}$ and also for the piecewise linear operator $S_{n,1}$.

Proposition 5.10 *For the Bernstein operators $B_n, n \geq 1$, we have*

$$|n \cdot [B_n(f;x) - f(x)] - \frac{1}{2} \cdot f''(x) \cdot x(1-x)| \leq \frac{x(1-x)}{2} \cdot \tilde{\omega}(f'', \frac{1}{3 \cdot \sqrt{n}}).$$

Proof. For the 4th moments one has the representation (see [150, Lemma 6.24])

$$(5.2) \qquad B_n((e_1 - x)^4; x) = \frac{1}{n^4}[3n^2x^2(1-x)^2 + n\{x(1-x) - 6x^2(1-x)^2\}];$$

for the second ones there holds

$$(5.3) \qquad B_n((e_1 - x)^2; x) = \frac{x(1-x)}{n}.$$

And so

$$\frac{B_n((e_1 - x)^4; x)}{B_n((e_1 - x)^2; x)} = \frac{3}{n}x(1-x) + \frac{1}{n^2}(1 - 6x(1-x)) \leq \frac{1}{n} \text{ for } n \geq 1.$$

This shows that

$$|B_n(f;x) - f(x) - \frac{1}{2} \cdot f''(x) \cdot \frac{x(1-x)}{n}| \leq \frac{x(1-x)}{2n} \cdot \tilde{\omega}(f'', \frac{1}{3 \cdot \sqrt{n}}),$$

and multiplying both sides by n gives the claimed inequality. □

Remark 5.11 We recall that the inequality of Proposition 5.10 was achieved by considering the term $\sqrt{\frac{B_n((e_1-x)^4;x)}{B_n((e_1-x)^2;x)}}$ which replaced the smaller expression $\frac{B_n(|e_1-x|^3;x)}{B_n((e_1-x)^2;x)}$ (see the proof preceeding Theorem 5.8). The numerator of the latter ratio can be estimated as follows close to the endpoints 0 and 1: Let $0 \leq x \leq \frac{1}{n}$. Then

$$\begin{aligned}
B_n(|e_1 - x|^3; x) &= \sum_{j=0}^{n} |\tfrac{j}{n} - x|^3 \cdot p_{nj}(x) \\
&= x^3 \cdot p_{n,0}(x) + \sum_{j=1}^{n} (\tfrac{j}{n} - x)^3 \cdot p_{nj}(x) \\
&= 2x^3 \cdot p_{n,0}(x) + \sum_{j=0}^{n} (\tfrac{j}{n} - x)^3 \cdot p_{nj}(x) \\
&= 2x^3 \cdot (1-x)^n + B_n((e_1 - x)^3; x) \\
&= 2x^3 \cdot (1-x)^n + \tfrac{x(1-x)(1-2x)}{n^2} \\
&= \tfrac{x(1-x)}{n^2}[2n^2x^2(1-x)^{n-1} + 1 - 2x] \\
&\leq \tfrac{3x(1-x)}{n^2} \text{ for } n \geq 1.
\end{aligned}$$

The same inequality is true for $x \in [1-\frac{1}{n},1]$. For $x \in [0,\frac{1}{n}] \cup [1-\frac{1}{n},1]$ this yields

$$\frac{B_n(|e_1-x|^3;x)}{B_n((e_1-x)^2;x)} \le \frac{3x(1-x)}{n^2} \cdot \frac{n}{x(1-x)} = \frac{3}{n},$$

and hence we arrive at

$$\begin{aligned}
&|n \cdot [B_n(f;x)-f(x)] - \tfrac{x(1-x)}{2} \cdot f''(x)| \\
&\le \tfrac{x(1-x)}{2} \cdot \tilde{\omega}(f''; \tfrac{1}{3} \cdot \tfrac{B_n(|e_1-x|^3;x)}{B_n((e_1-x)^2;x)}) \\
&\le \tfrac{x(1-x)}{2} \cdot \tilde{\omega}(f''; \tfrac{1}{n}).
\end{aligned}$$

So close to 0 and 1 an estimate better than the global one in Proposition 5.10 is available. □

There is room for an even better global estimate as it was shown in [62, Theorem 5.1]:

Proposition 5.12 *For* $f \in C[0,1],\ x \in [0,1]$ *and* $n \in \mathbb{N}$ *one has*

$$\left| n[B_n(f;x)-f(x)] - \frac{x(1-x)}{2} \cdot f'' \right| \le \frac{x(1-x)}{2} \cdot \tilde{\omega}_1 \left(f''; \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right).$$

In the context of this subsection it is maybe interesting to collect some information regarding *absolute (odd) moments* of Bernstein operators, especially for the *third absolute moments*.

Remark 5.13 (see [38, p. 304]) For $r = 0,1,\ldots$ one has, uniformly in x,

$$B_n(|e_1-x|^{2r+1};x) = o\left(\frac{1}{n^r}\right),\ n \to \infty.$$

Remark 5.14 For the Bernstein operators the first absolute moments can be written in the form

$$B_n(|e_1-x|;x) = \frac{2}{n}(n-r)\binom{n}{r} x^{r+1}(1-x)^{n-r},$$

where $r := [nx]$ denotes the largest integer not exceeding nx. This was proved by Schurer & Steutel in [138]; for details of the computations see [104, p. 12–20].

Remark 5.15 For the third absolute moments $B_n(|e_1-x|^3;x)$ no explicit representation analogous to the one for $B_n(|e_1-x|;x)$ is known to us. All we know from Remark 5.13 is that there is a null sequence (ε_n) such that

$$\sup_{x\in[0,1]} B_n(|e_1-x|^3;x) \le \varepsilon_n \cdot \frac{1}{n},\ n \in \mathbb{N}.$$

It is thus desirable to have a pointwise inequality of the form

$$B_n(|e_1 - x|^3; x) \le \varepsilon_n(x) \cdot \frac{1}{n},\ n \in \mathbb{N},$$

in which $\varepsilon_n(x) \le \varepsilon_n$ for $x \in [0,1]$.

A first step into this direction can be obtained via the Cauchy-Schwarz inequality:

Proposition 5.16 *For the third absolute moments of the Bernstein operators the following pointwise estimate holds:*

$$B_n(|e_1 - x|^3; x) \le \frac{x(1-x)}{n^{\frac{3}{2}}} \cdot \left(3x(1-x) + \frac{1-6x(1-x)}{n}\right)^{\frac{1}{2}},$$

$x \in [0,1]$.

Proof. Using the right hand side of Example 1.13 (ii) we arrive at:

$$\begin{aligned} B_n(|e_1 - x|^3; x) &\le B_n((e_1-x)^2; x)^{\frac{1}{2}} \cdot B_n((e_1-x)^4; x)^{\frac{1}{2}} \\ &= \tfrac{x(1-x)}{n^{\frac{3}{2}}} \cdot \left(3x(1-x) + \tfrac{1-6x(1-x)}{n}\right)^{\frac{1}{2}} =: A, \end{aligned}$$

which is a better approach as using (i) from Example 1.13. Indeed, we have

$$\begin{aligned} B_n(|e_1 - x|^3; x) &\le B_n((e_1-x)^4; x)^{\frac{3}{4}} \\ &= \tfrac{(x(1-x))^{\frac{3}{4}}}{n^{\frac{3}{2}}} \cdot (3x(1-x) + \tfrac{1-6x(1-x)}{n})^{\frac{3}{4}} =: B. \end{aligned}$$

Dividing A by B gives

$$\begin{aligned} \tfrac{A}{B} &= (x(1-x))^{\frac{1}{4}} \cdot (3x(1-x) + \tfrac{1-6x(1-x)}{n})^{-\frac{1}{4}} \\ &= \left(\tfrac{x(1-x)}{3x(1-x)+\frac{1-6x(1-x)}{n}}\right)^{\frac{1}{4}} \le 1 \text{ for all } x \in [0,1]. \quad \square \end{aligned}$$

Remark 5.17 For $\frac{1}{n} \le x \le 1 - \frac{1}{n},\ n \ge 2$ we obtain $3x(1-x) + \frac{1-6x(1-x)}{n} \le 4x(1-x)$.
Clearly, this inequality is not true if $x \in \left[0, \frac{1}{n}\right) \cup \left(1 - \frac{1}{n}, 1\right]$.
At least for $x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ we have

$$B_n(|e_1 - x|^3; x) \le 2 \cdot \left[\frac{x(1-x)}{n}\right]^{3/2} = 2 \cdot B_n((e_1-x)^2; x)^{3/2}. \qquad \square$$

It is thus legitimate to conjecture that there is an absolute constant c such that for <u>all</u> $x \in [0,1]$ one has

$$B_n(|e_1 - x|^3; x) \le c \cdot B_n((e_1-x)^2; x)^{3/2}.$$

However, this conjecture is wrong. Even more will be shown in the following.

Example 5.18 *Let $n \geq 1$ be fixed. For any $\alpha > 2$ there is no absolute constant c such that*

$$B_n(|e_1 - x|^3; x) \leq c \cdot B_n((e_1 - x)^2; x)^{\frac{\alpha}{2}} \text{ for all } x \in [0,1].$$

Observe that for $\alpha = 2$ the inequality with $c = 1$ is obvious. So in that sense $\alpha = 2$ is a sharp bound. W.l.o.g., for n fixed we consider $x \in \left[0, \frac{1}{n}\right]$ only and write

$$\begin{aligned} B_n(|e_1 - x|^3; x) &= \sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^3 \cdot p_{n,k}(x) \\ &= x^3 \cdot (1-x)^n + \left(\frac{1}{n} - x\right)^3 \cdot n \cdot x(1-x)^{n-1} \\ &+ \sum_{k=2}^{n} \left(\frac{k}{n} - x\right)^3 \binom{n}{k} x^k (1-x)^{n-k}. \end{aligned}$$

Dividing this for $\alpha > 2$ by $B_n((e_1 - x)^2; x)^{\frac{\alpha}{2}} = \left[\frac{x(1-x)}{n}\right]^{\frac{\alpha}{2}}$ for $0 < x \leq \frac{1}{n}$ shows that

$$\begin{aligned} &\lim_{x \to 0+} \frac{B_n(|e_1 - x|^3; x)}{B_n((e_1 - x)^2; x)^{\frac{\alpha}{2}}} \\ = &\lim_{x \to 0+} n^{\alpha/2} \left\{ x^{3-\frac{\alpha}{2}} (1-x)^{n-\frac{\alpha}{2}} + \left(\frac{1}{n} - x\right)^3 n x^{1-\frac{\alpha}{2}} (1-x)^{n-1-\frac{\alpha}{2}} + \ldots \right\}. \end{aligned}$$

The second term tends to infinity if $x \to 0+$ for all $\alpha > 2$, and this confirms our claim. At 1 the situation is analogous. □

Now we return to Theorem 5.8 and we will present *quantitative Voronovskaja theorems* for further classes of positive linear operators. We start with $\bar{\mathbb{B}}_n$, see its definition at (3.2):

Proposition 5.19 *For $\bar{\mathbb{B}}_n$, $n \geq 1$ and $x \in [0,1]$ there holds*

$$|(n+1) \cdot [\bar{\mathbb{B}}_n(f; x) - f(x)] - \tfrac{1}{2} \cdot f''(x) \cdot x(1-x)| \leq \tfrac{1}{2} x(1-x) \cdot \tilde{\omega}(f'', \tfrac{1}{3} \cdot \sqrt{\tfrac{2}{n+3}}).$$

Proof. In [95] it was shown that

$$\overline{\mathbb{B}}_n((e_1 - x)^2; x) = \frac{x(1-x)}{n+1}. \tag{5.4}$$

$$\overline{\mathbb{B}}_n((e_1 - x)^4; x) = \frac{3nx^2(1-x)^2 + 6x(1-x)(3x^2 - 3x + 1)}{(n+1)(n+2)(n+3)}, \tag{5.5}$$

Hence

$$\begin{aligned} \frac{\overline{\mathbb{B}}_n((e_1-x)^4;x)}{\overline{\mathbb{B}}_n((e_1-x)^2;x)} &= \frac{3nx(1-x)+6(3x^2-3x+1)}{(n+2)(n+3)} \\ &\leq \frac{2}{n+3}, \end{aligned}$$

which together with Theorem 5.8 leads us to the desired inequality. □

In the sequel we want to achieve similar results for other two very well-known Beta-type operators. Hence, for the *genuine Bernstein-Durrmeyer operators* -see row 6 in Table 3.1- we can state

Proposition 5.20 *For the U_n the following version of Voronovskaja's formula holds for $f \in C^2[0,1], x \in [0,1], n \geq 1$:*

$$|(n+1)\cdot[U_n(f;x) - f(x)] - f''(x)\cdot x(1-x)| \leq x(1-x)\cdot\tilde{\omega}(f''; \frac{2}{3}\cdot\frac{1}{\sqrt{(n+3)}}).$$

Proof. For the moments in question, due to Theorem 5.8, we have in this case:

$$(5.6)\quad U_n((e_1-x)^2;x) = \frac{2x(1-x)}{n+1}, \text{ see Table 3.2}$$

$$(5.7)\quad U_n((e_1-x)^4;x) = \frac{12x^2(1-x)^2\cdot(n-7)}{(n+1)(n+2)(n+3)} + \frac{24x(1-x)}{(n+1)(n+2)(n+3)}.$$

For the last identity consult [80, Proposition 3.5]. So now

$$\begin{aligned}
&|U_n(f;x) - f(x) - \tfrac{1}{2}\cdot f''(x)\cdot U_n((e_1-x)^2;x)| \\
&= |U_n(f;x) - f(x) - f''(x)\cdot \tfrac{x(1-x)}{n+1}| \\
&\leq \tfrac{1}{2}U_n((e_1-x)^2;x)\cdot\tilde{\omega}(f''; \tfrac{1}{3}\sqrt{\tfrac{U_n((e_1-x)^4;x)}{U_n((e_1-x)^2;x)}}) \\
&= \tfrac{x(1-x)}{n+1}\cdot\tilde{\omega}(f''; \tfrac{1}{3}\sqrt{\tfrac{[12x^2(1-x)^2\cdot(n-7)+24x(1-x)]\cdot(n+1)}{(n+1)(n+2)(n+3)\cdot 2x(1-x)}}) \\
&= \tfrac{x(1-x)}{n+1}\cdot\tilde{\omega}(f''; \tfrac{1}{3}\sqrt{\tfrac{6x(1-x)\cdot(n-7)+12}{(n+2)(n+3)}}), \\
&= \tfrac{x(1-x)}{n+1}\cdot\tilde{\omega}(f''; \tfrac{1}{3}\cdot\sqrt{\tfrac{6}{n+3}}\cdot\sqrt{\tfrac{x(1-x)(n-7)+2}{n+2}}) \\
&\leq \tfrac{x(1-x)}{n+1}\cdot\tilde{\omega}(f''; \tfrac{2}{3}\cdot\tfrac{1}{\sqrt{n+3}}),
\end{aligned}$$

assuring the desired result. □

In [62, Theorem 5.3] we find an improvement of this inequality, namely

Proposition 5.21 *For $f \in C^2[0,1],\ x \in [0,1]$ and $n \in \mathbb{N},\ n \geq 2$, the following*

$$|(n+1)[U_n(f;x) - f(x)] - x(1-x)f''(x)| \leq \tfrac{x(1-x)}{n+1}\cdot\tilde{\omega}_1\left(f''; 4\sqrt{\tfrac{1}{(n+1)^2} + \tfrac{x(1-x)}{n+1}}\right)$$

holds.

Further, we consider one special case of the Stancu operator, namely $S_n^{<1/n,0,0>}$, introduced at row 4 in Table 3.1.

Proposition 5.22 *For Stancu's operators* $S_n^{<1/n,0,0>}, n \geq 1,$ *we have*

$$|(n+1)\cdot[S_n^{<1/n,0,0>}(f;x)-f(x)]-f''(x)\cdot x(1-x)| \leq x(1-x)\cdot\tilde{\omega}(f'';\frac{2}{3}\cdot\frac{1}{\sqrt{n+3}}).$$

Proof. In [98, p. 68] the following representations can be found:

$$(5.8)\quad S_n^{<1/n,0,0>}((e_1-x)^2;x) = \frac{2x(1-x)}{n+1},$$

$$(5.9)\quad S_n^{<1/n,0,0>}((e_1-x)^4;x) = \frac{2x(1-x)[6n(n-7)x(1-x)+13n-1]}{n(n+1)(n+2)(n+3)}.$$

Hence

$$\frac{S_n^{<1/n,0,0>}((e_1-x)^4;x)}{S_n((e_1-x)^2;x)} = \frac{6n(n-7)x(1-x)+13n-1}{n(n+2)(n+3)} \leq \frac{4}{n+3}. \qquad \square$$

Our last application of Theorem 5.8 is for the *piecewise linear interpolant* on equidistant knots, $S_{n,1}$, see (2.4). In this case we have

Proposition 5.23 *Let* $S_{n,1}$ *be given as in Table 3.1 and* $f \in C^2[0,1],\ x \in [0,1]$. *Then*

$$\begin{aligned}|n^2[S_{n,1}(f;x)-f(x)] &- \frac{1}{2}\cdot f''(x)\cdot z_n(x)(1-z_n(x))| \\ (5.10)\qquad &\leq \frac{1}{2}z_n(x)(1-z_n(x))\cdot\tilde{\omega}\left(f'';\frac{1}{3n}\right).\end{aligned}$$

Here $z_n(x) = nx - [nx]$, *where* $[nx]$ *denotes the integer part of* nx.

Proof. Write $z_n(x) := nx - [nx]$. The following representations of the second and the fourth moments of $S_{n,1}$ can be found in [95, p. 46]:

$$(5.11)\quad S_{n,1}((e_1-x)^2;x) = \frac{1}{n^2}z_n(x)(1-z_n(x)), \text{ and}$$

$$(5.12)\quad S_{n,1}((e_1-x)^4;x) = \frac{1}{n^2}z_n(x)(1-z_n(x))[1-3z_n(x)(1-z_n(x))].$$

Substituting these into the inequality of Theorem 5.8 yields the result once we take into account that

$$\frac{S_{\Delta_n}((e_1-x)^4;x)}{S_{\Delta_n}((e_1-x)^2;x)} = \frac{1}{n^2}[1-3z_n(x)(1-z_n(x))] \leq \frac{1}{n^2} \text{ for } x \in [0,1]. \qquad \square$$

Remark 5.24 Non-quantitative versions of Voronovskaja-type results are also known for other cases of Schoenberg's variation diminishing spline operators. It would be of interest to find quantitative statements also for other cases than $S_{n,1}$.

5.3 On differences of positive linear operators

One of the purposes of the previous sections was to compute the rate of convergence of a positive linear operator towards the identity operator, by means of different instruments (K-functionals and/or different moduli of smoothness). In the present section we wish to widen our research and to compare the convergence velocity between two positive linear operators. The means remain the same: K-functionals and different types of moduli of smoothness. The interested reader is guided to [71].

5.3.1 General inequalities

In the sequel we give some more general results concerning the issue in question. We start with:

Theorem 5.25 *Let $A, B : C[0,1] \to C[0,1]$ be positive linear operators such that*

$$(A-B)((e_1-x)^i;x) = 0 \text{ for } i = 0,1,\dots,n \text{ and } x \in [0,1].$$

Then for $f \in C^n[0,1]$ there holds

$$|(A-B)(f;x)| \le \frac{1}{n!}(A+B)(|e_1-x|^n;x)\cdot\tilde{\omega}(f^{(n)};\frac{1}{n+1}\frac{(A+B)(|e_1-x|^{n+1};x)}{(A+B)(|e_1-x|^n;x)}).$$

Proof. Using the Taylor expansion with quantitative Peano remainder, proven in the first section of this chapter, see Theorem 5.3, we first have

$$\begin{aligned}&|(A-B)(f;x)| = |(A-B)(f(t);x)|\\&= |(A-B)(\tfrac{(t-x)^n}{n!}\cdot\mu_x(t);x)|.\end{aligned}$$

Here we defined

$$\frac{(t-x)^n}{n!}\mu_x(t) := f(t) - \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x)\cdot(t-x)^k.$$

Hence

$$\begin{aligned}&|(A-B)(f;x)|\\&\le (A+B)(\tfrac{|t-x|^n}{n!}\cdot\tilde{\omega}(f^{(n)};\tfrac{|t-x|}{n+1});x)\\&= (A+B)(2\cdot\tfrac{|t-x|^n}{n!}\cdot K(f^{(n)};\tfrac{|t-x|}{2(n+1)});x)\\&\le (A+B)(\tfrac{2\cdot|t-x|^n}{n!}\cdot\{\|(f-g)^{(n)}\|_\infty + \tfrac{|t-x|}{2(n+1)}\cdot\|g^{(n+1)}\|_\infty\};x),\, g\in C^{n+1}[0,1] \text{ arbitrary},\\&= (A+B)(\tfrac{2\cdot|t-x|^n}{n!};x)\cdot\|(f-g)^{(n)}\|_\infty + (A+B)(\tfrac{|t-x|^{n+1}}{(n+1)!};x)\cdot\|g^{(n+1)}\|_\infty\\&= (A+B)(\tfrac{2\cdot|t-x|^n}{n!};x)\cdot\{\|(f-g)^{(n)}\|_\infty + \tfrac{1}{2(n+1)}\tfrac{(A+B)(|t-x|^{n+1};x)}{(A+B)(|t-x|^n;x)}\cdot\|g^{(n+1)}\|_\infty\}.\end{aligned}$$

Passing back to infimum over $g \in C^{n+1}[0,1]$, and using Brudnyĭ's representation theorem, Lemma 1.32, again shows that

$$\begin{aligned} |(A-B)(f;x)| &\le (A+B)(\tfrac{2\cdot|t-x|^n}{n!};x)\cdot\tfrac{1}{2}\cdot\tilde{\omega}(f^{(n)};\tfrac{1}{n+1}\cdot\tfrac{(A+B)(|t-x|^{n+1};x)}{(A+B)(|t-x|^n;x)}) \\ &= \tfrac{1}{n!}(A+B)(|t-x|^n;x)\cdot\tilde{\omega}(f^{(n)};\tfrac{1}{n+1}\cdot\tfrac{(A+B)(|t-x|^{n+1};x)}{(A+B)(|t-x|^n;x)}). \end{aligned}$$

□

Corollary 5.26 *With $L := A+B$ we have for $\underline{n+1 \text{ odd}}$*

$$\frac{L(|t-x|^{n+1};x)}{L(|t-x|^n;x)} \le \frac{\sqrt{L((t-x)^{2n};x)}\cdot\sqrt{L((t-x)^2;x)}}{L((t-x)^n;x)};$$

so that the bound in Theorem 5.25 can be modified accordingly.

Proof. Write

$$\begin{aligned} &L(|t-x|^{n+1};x) = L(|t-x|^n\cdot|t-x|;x) \\ &\le \sqrt{L(|t-x|^{2n};x)}\cdot\sqrt{L(|t-x|^2;x)} = \sqrt{L((t-x)^{2n};x)}\cdot\sqrt{L((t-x)^2;x)} \end{aligned}$$

which arises from the Cauchy-Schwarz inequality. □

If n is odd the absolute moment $L(|t-x|^n;x)$ appears in the denominator. The operators A and B are such that $A(e_0,x) = B(e_0,x), x \in [0,1]$. We assume now that $A(e_0,x) = B(e_0,x) = 1, x \in [0,1]$.
So $L := \frac{1}{2}(A+B)$ reproduces constant functions. Hence by Hölder's inequality for positive linear operators we have proven for $1 \le s < r$ that

$$L(|e_1-x|^s;x)^{\frac{1}{s}} \le L(|e_1-x|^r;x)^{\frac{1}{r}},$$

see Proposition 1.12. Thus

$$(A+B)(|e_1-x|^n;x) = 2\cdot L(|e_1-x|^n;x) \ge 2\cdot\{L((e_1-x)^{n-1};x)^{\frac{n}{n-1}}\}.$$

Under these conditions we have

Corollary 5.27 *If under the assumptions of Theorem 5.25 n is odd, we also get*

$$\begin{aligned} &|(A-B)(f;x)| \\ &\le \tfrac{1}{n!}(A+B)(|e_1-x|^n;x)\cdot\tilde{\omega}(f^{(n)};\tfrac{1}{2(n+1)}\cdot\tfrac{(A+B)((e_1-x)^{n+1};x)}{\{\frac{1}{2}(A+B)((e_1-x)^{n-1};x)\}^{\frac{n}{n-1}}} \\ &= \tfrac{1}{n!}\cdot(A+B)(|e_1-x|^n;x)\cdot\tilde{\omega}(f^{(n)};\tfrac{2^{\frac{1}{n-1}}}{n+1}\cdot\tfrac{(A+B)((e_1-x)^{n+1};x)}{(A+B)((e_1-x)^{n-1};x)^{\frac{n}{n-1}}}). \end{aligned}$$

Note that the moments inside $\tilde{\omega}(f^{(n)};\cdot)$ are now both of even order and can thus be evaluated conveniently. The absolute moment in front of $\tilde{\omega}(f^{(n)};\cdot)$ can also be estimated using Hölder's inequality.

Corollary 5.28 *If A and B are given as in Theorem 5.25, then for $g \in C^{n+1}[0,1], x \in [0,1]$, there holds*

$$|(A-B)(g;x)| \leq \frac{1}{(n+1)!}(A+B)(|t-x|^{n+1};x) \cdot \|g^{(n+1)}\|_\infty.$$

The question remains how to estimate the difference for all functions in $C[0,1]$. So we will carry the result over from $C^{n+1}[0,1]$ to $C[0,1]$. In order to do so we use moduli of smoothness of higher order, see its definition at (1.11), and employ Lemma 1.28 for $r=0$ and $s=n+1$. We obtain thus for $h \in (0,1]$ and $f \in C[0,1]$ functions $f_{h,n+1}$ with

$$\|f-f_{h,n+1}\|_\infty \leq c \cdot \omega_{n+1}(f;h), \quad \|f_{h,n+1}^{(n+1)}\|_\infty \leq c \cdot h^{-(n+1)} \cdot \omega_{n+1}(f;h).$$

In this context we can state

Theorem 5.29 *If A and B are given as in Theorem 5.25, also satisfying $Ae_0 = Be_0 = e_0$, then for all $f \in C[0,1], x \in [0,1]$ we have*

$$|(A-B)(f;x)| \leq c_1 \cdot \omega_{n+1}(f; \sqrt[n+1]{\frac{1}{2}(A+B)(|e_1-x|^{n+1};x)}).$$

Here c_1 is an absolute constant independent of f, x and A and B.

Proof. Let $f \in C[0,1]$ be fixed and $g = f_{h,n+1}, 0 < h \leq 1$, be given as above. Then, with the constant c from Lemma 1.28,

$$\begin{aligned}
&|(A-B)(f;x)| \leq |(A-B)(f-g;x)| + |(A-B)(g;x)| \\
&\leq (\|A\|+\|B\|) \cdot \|f-g\|_\infty + \tfrac{1}{(n+1)!} \cdot (A+B)(|e_1-x|^{n+1};x) \cdot \|g^{(n+1)}\|_\infty \\
&\leq 2 \cdot c \cdot \omega_{n+1}(f;h) + c \cdot \tfrac{1}{(n+1)!} \cdot (A+B)(|e_1-x|^{n+1};x) \cdot \tfrac{1}{h^{n+1}} \cdot \omega_{n+1}(f;h).
\end{aligned}$$

If $(A+B)(|e_1-x|^{n+1};x) = 0$, then – $h > 0$ being arbitrary – we also have $|(A-B)(f;x)| = 0$.
Otherwise we put $h = \sqrt[n+1]{\frac{1}{2} \cdot (A+B)(|e_1-x|^{n+1};x)} \leq 1$ to arrive at

$$|(A-B)(f;x)| \leq c_1 \cdot \omega_{n+1}(f; \sqrt[n+1]{\tfrac{1}{2}(A+B)(|e_1-x|^{n+1};x)},$$

where $c_1 = 2 \cdot c + c \cdot \frac{2}{(n+1)!} c \cdot (2 + \frac{2}{(n+1)!})$. □

5.3.2 Estimates for the differences of some positive operators

This subsection is dedicated to some concrete applications of the theoretical results presented above. We start by estimating the difference between B_{n+1}, the $(n+1)$-th Bernstein operator and Lupaş operator $\bar{\mathbb{B}}_n$:

Proposition 5.30

$$\begin{aligned} |(B_{n+1}-\bar{\mathbb{B}}_n)(f;x)| &\leq \frac{x(1-x)}{n+1}\cdot\tilde{\omega}\left(f'';\sqrt{\frac{(n+1)(6nx(1-x)+7)}{18n^2}}\right),\ f\in C^2[0,1] \\ &\leq \frac{x(1-x)}{3n\sqrt{n+1}}\sqrt{\frac{6nx(1-x)+7}{2n}}\cdot\|f'''\|_\infty,\ f\in C^3[0,1]. \end{aligned}$$

Proof. Corollary 5.26 can be applied for the two operators (with $n=2$), as its second moments agree, see e.g., (5.3) and (5.4). Consequently we obtain

$$(B_{n+1}+\bar{\mathbb{B}}_n)((e_1-x)^2;x)=\frac{2x(1-x)}{n+1},$$

and from (5.2) and (5.5) we arrive at

$$\begin{aligned} (B_{n+1}+\bar{\mathbb{B}}_n)((e_1-x)^4;x) &= \left(\frac{3(n-1)}{(n+1)^3}+\frac{3}{(n+2)(n+3)}\right)x^2(1-x)^2 \\ &\quad+\left(\frac{1}{(n+1)^3}+\frac{6}{(n+1)(n+2)(n+3)}\right)x(1-x) \\ &\leq \left(\frac{3}{n^2}+\frac{3}{n^2}\right)x^2(1-x)^2+\left(\frac{1}{n^3}+\frac{6}{n^3}\right)x(1-x) \\ &= \frac{x(1-x)}{n^2}\cdot\frac{6nx(1-x)+7}{n}. \end{aligned}$$

Using the above mentioned corollary and properties of $\tilde{\omega}$, see Section 1.4, we obtain the desired inequalities. □

For all $f\in C[0,1]$ Theorem 5.29 implies the following

Proposition 5.31

$$\begin{aligned} |(B_{n+1}-\bar{\mathbb{B}}_n(f;x))| &\leq c\cdot\omega_3\left(f;\sqrt[3]{\frac{1}{2}(B_{n+1}+\bar{\mathbb{B}}_n)(|e_1-x|^3;x)}\right) \\ &\leq c\cdot\omega_3\left(f;\sqrt[6]{\frac{x^2(1-x)^2}{n^3}\cdot\frac{6nx(1-x)+7}{n}}\right). \end{aligned}$$

Proof. The first inequality is a direct consequence of Theorem 5.29. The second one can be obtained via Cauchy-Schwarz, see item (ii) in Example 1.13, where L is replaced in this case by $L := \frac{(B_{n+1}+\bar{\mathbb{B}}_n)}{2}$. Involving parts of the proof of the previous proposition we get to the desired result. $\square$

One further application of Theorem 5.29 for the case $n = 1$ is the following:

Proposition 5.32

$$|(B_n - U_n)(f;x)| \leq c \cdot \omega_2(f; \sqrt[2]{\frac{3x(1-x)}{2n}}).$$

Proof. Taking the corresponding second moments from Table 3.2 for the operators involved we see that $\frac{1}{2} \cdot (B_n + U_n)((e_1 - x)^2; x) \leq \frac{3}{2n} x(1-x)$, which implies the claim. $\square$

Another interesting operator which has certain similarity with U_n is

$$D_n := B_n \circ B_{n+1}.$$

Therefore we shall investigate in the following the difference

$$D_n - U_n := B_n \circ B_{n+1} - U_n = B_n \circ B_{n+1} - B_n \circ \bar{\mathbb{B}}_n = B_n \circ (B_{n+1} - \bar{\mathbb{B}}_n),$$

hence providing further applications of Theorems 5.25 and 5.29 for $n = 2$.

Proposition 5.33

$$\begin{aligned} |(D_n - U_n)(f;x)| &\leq \tfrac{2x(1-x)}{n+1} \tilde{\omega}(f'', \sqrt{\tfrac{(n+1)(8nx(1-x)+13)}{12n^3}}), f \in C^2[0,1], \\ &\leq \tfrac{x(1-x)}{n\sqrt{n+1}} \sqrt{\tfrac{8nx(1-x)+13}{3n}} \|f'''\|, f \in C^3[0,1]. \end{aligned}$$

Proof. All operators involved reproduce linear functions, so

$$(D_n - U_n)((e_1 - x)^i; x) = 0 \text{ for } i = 0, 1.$$

By rewriting formula (3.13) for the composition of two operators we can easily derive that

$$B_{n+1}((e_1 - x)^2; x) = \bar{\mathbb{B}}_n((e_1 - x)^2; x) = \frac{x(1-x)}{n+1}, \tag{5.13}$$

and hence also $(D_n - U_n)((e_1 - x)^2; x) = 0$. Thus Theorem 5.25 is applicable with $n = 2$, once we have estimated

$$(D_n + U_n)(|e_1 - x|^3; x) \leq \sqrt{(D_n + U_n)((e_1 - x)^2; x)} \cdot \sqrt{(D_n + U_n)((e_1 - x)^4; x)},$$

which obviously follows from the Cauchy-Schwarz inequality. The fourth moments of D_n were computed in [66] and are equal to:

$$D_n((e_1-x)^4;x) = \frac{1}{n^2(n+1)^3}\cdot\{12(n^3-6n^2+4n-1)x^2(1-x)^2+(15n^2-9n+2)x(11-x)\}.$$

Consequently we obtain

$$\begin{aligned}(D_n+U_n)((e_1-x)^2;x) &= \tfrac{4x(1-x)}{n+1},\\ (D_n+U_n)((e_1-x)^4;x) &= (\tfrac{12(n^3-6n^2+4n-1)}{n^2(n+1)^3}+\tfrac{12(n-7)}{(n+1)(n+2)(n+3)})x^2(1-x)^2\\ &\quad +(\tfrac{15n^2-9n+2}{n^2(n+1)^3}+\tfrac{24}{(n+1)(n+2)(n+3)})x(1-x)\\ &\le (\tfrac{12}{n^2}+\tfrac{12}{n^2})x^2(1-x)^2+(\tfrac{15}{n^3}+\tfrac{24}{n^3})x(1-x)\\ &= \tfrac{x(1-x)}{n^2}\tfrac{24nx(1-x)+39}{n}.\end{aligned}$$

This leads us to the desired inequalities. □

An application of Theorem 5.29 yields

Proposition 5.34

$$\begin{aligned}&|(D_n-U_n)(f;x)| \le c\cdot\omega_3(f;\sqrt[3]{\frac{1}{2}(D_n+U_n)(|e_1-x|^3;x)})\\ \le\ & c\cdot\omega_3\left(f;\sqrt[6]{\frac{x^2(1-x)^2}{(n+1)n^3}\cdot(24nx(1-x)+39)}\right).\end{aligned}$$

Remark 5.35 For the difference $D_n - S_n^{<1/n,0,0>}$ similar estimates can be given, since the second moments of both operators are the same (see Row 4 in Table 3.2 and formula (5.13)) and the structures of the second and fourth moments are analogous to the cases considered before.

5.3.3 Estimates for the commutator of positive linear operators: Lupaş's problem

The last application of this section is motivated by Problem 3 in A. Lupaş's article [97]. One of the questions raised by him was to give an estimate for the *commutator*

$$[B_n,\bar{\mathbb{B}}_n] := B_n\circ\bar{\mathbb{B}}_n-\bar{\mathbb{B}}_n\circ B_n = U_n - S_n^{<1/n,0,0>}. \tag{5.14}$$

First we prove the following lemma:

Lemma 5.36 *For U_n and $S_n^{<1/n,0,0>}$ with $x\in[0,1]$ we have*

$$(U_n-S_n^{<1/n,0,0>})((e_1-x)^i;x) = 0 \ for\ i=0,1,2,3.$$

Proof. The affirmation for $i = 0, 1$, is trivial, as both operators reproduce linear functions. The second moments of the two operators are equal as one can extract from Table 3.2.
The third moments of U_n are computed in [80, Proposition 3.5] and are equal to

$$U_n((e_1 - x)^3; x) = \frac{6x(1-x)(1-2x)}{(n+1)(n+2)}.$$

A possible way to compute the third moments for $S_n^{<1/n,0,0>}$ is via Corollary 1.16. In [98, p. 68] it is shown that

$$S_n^{<1/n,0,0>}(e_3; x) = x^3 + \frac{6x(1-x)}{(n+1)(n+2)} + \frac{6nx^2(1-x)}{(n+1)(n+2)}$$

Hence, by Corollary 1.16 we find that

$$\begin{aligned} S_n^{<1/n,0,0>}((e_1 - x)^3; x) &= \tfrac{6x(1-x)}{(n+1)(n+2)} + \tfrac{6nx^2(1-x)}{(n+1)(n+2)} - 3x \cdot \tfrac{2x(1-x)}{n+1} \\ &= \tfrac{6x(1-x)(1-2x)}{(n+1)(n+2)}, \end{aligned}$$

whence obtaining the desired identities. □

We are now in the state to estimate the commutator of B_n and $\bar{\mathbb{B}}_n$ and thus give a solution to Lupaş' problem.

Proposition 5.37 *For any $f \in C[0,1]$ and $x \in [0,1]$ we have*

$$|[B_n, \bar{\mathbb{B}}_n](f; x)| = |(S_n^{<1/n,0,0>} - U_n)(f; x)| \le c_1 \cdot \omega_4(f; \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}).$$

Here c_1 is an absolute constant independent of n, f and x.

Proof. All that remains to be done is to add the fourth moments of $S_n^{<1/n,0,0>}$ and U_n, see for this purpose the relations (5.9) and (5.7). We thus arrive at

$$(S_n^{<1/n,0,0>} + U_n)((e_1 - x)^4; x) = \tfrac{2x(1-x)\cdot[12n(n-7)x(1-x)+25n-1]}{n(n+1)(n+2)(n+3)} \le \tfrac{6x(1-x)}{n(n+1)}.$$

Substituting this into Theorem 5.29 with $n = 3$ gives the desired inequality. □

Remark 5.38 We mention that by a similar approach the *commutator*

$$[B_n, \tilde{\mathbb{B}}_\alpha] := B_n \circ \tilde{\mathbb{B}}_\alpha - \tilde{\mathbb{B}}_\alpha \circ B_n = U_n^\alpha - S_n^{<\alpha,0,0>} \tag{5.15}$$

can be estimated, see the definitions in Table 3.1.

We shall not carry out here all the computations. We will only prove that

$$U_n^{\alpha}((e_1 - x)^i; x) = S_n^{<\alpha,0,0>}((e_1 - x)^i; x),\ i = 0, 1, 2, 3.$$

The identities up to the (inclusively) second moments are valid, as it is visible from Lemma 3.10 and Table 3.2. In order to evaluate the *third moments* of the two operators we need the following ingredients:

a) the decomposition formula into *simpler* operators, see (5.15);

b) the image of e_3 by B_n is equal to

$$B_n e_3(x) = \frac{(n-1)(n-2)}{n^2} x^3 + \frac{3(n-1)}{n^2} x^2 + \frac{x}{n^2}, \tag{5.16}$$

see (3.21), and finally,

c) the recurrence formula for the third moments proven at Corollary 1.16.

Thus, after carrying out some elementary computations we arrive at:

$$\begin{aligned} U_n^{\alpha}((e_1 - x)^3; x) &= S_n^{<\alpha,0,0>}((e_1 - x)^3; x) \\ &= \frac{(\alpha n + 1)(2\alpha n + 1)(1 - 2x)}{(1+\alpha)(1+2\alpha)n^2} x(1-x). \end{aligned}$$

10 open problems

In this final section we propose a list of problems which has risen during the preparation of the present work and to which we have not yet found an appropriate or complete answer.

Problem 1: We start with Tachev's Conjecture 2.6 which states:
The operator $R_{\Delta_n,k}$ see (2.2) reproduces linear functions if and only if all weights number are equal.
We recall that the cases $k = 1, 2, 3$ were already verified in [152] respectively here in Subsection 2.1.3. My colleague M. Wozniczka has almost solved this problem for any $k \geq 1$ natural.

Problem 2: *To develop a suitable approximation theory for Schoenberg spline defined over an partition that also accepts interior knots of higher multiplicity.* One attempt in this direction was made in [120].

Problem 3: We have seen in Subsection 2.2 that the modified rational Bernstein operators $\bar{R}_n$ shares many beautiful (shape-preserving) properties with the classical *Bernstein operators*: reproduction of linear functions; preservation of the positivity, monotonocity and convexity; has the variation-diminishing property etc. In this context, the following problem has risen:
If f is a convex function, then the sequence $(\bar{R}_n f)_{n\geq 1}$ is decreasing.
We have performed some experiments in *Mathematica 5.0* and for the chosen examples the problem has a positive answer.

Problem 4: We recall Conjecture 3.2:
If $f \in C^1[0,1]$, then $\tilde{\mathbb{B}}_\alpha f \in C^1[0,1]$? Or more generally, if $f \in C^r[0,1]$, then $\tilde{\mathbb{B}}_\alpha f \in C^r[0,1]$, $r \geq 1$ a natural number.

Problem 5: In Table 3.1 Row 7 appears the composite operator $\tilde{\mathbb{B}}_\alpha \circ \tilde{\mathbb{B}}_\lambda$ with α and λ positive. In this context we wondered, *if the operator can be written under the following form:*

$$\tilde{\mathbb{B}}_\alpha \circ \tilde{\mathbb{B}}_\lambda = \tilde{\mathbb{B}}_{f(\alpha,\lambda)},$$

where $f(\alpha, \lambda)$ represents an expression depending on the two constants.

Problem 6: Motivated by the previous work of A. Lupaş, we have stepped on the following interesting mapping, which employs both the Beta operator of the second kind and the piecewise linear interpolant at equidistant knots, i.e.,

$$\mathbb{L}_n := \bar{\mathbb{B}}_n \circ S_{n,1}. \tag{5.17}$$

We observed that this operator seems to be a very good approximate and (maybe) a non-trivial decomposition of the Bernstein operator B_n. Therefore, we can think that it is interesting to make further research on this matter.

Problem 7: *Is it possible to improve for example for the Bernstein operators the known simultaneous estimates by choosing in Theorem 3.15 instead of $s = 2$ a higher value, maybe $s = 4$?*

Problem 8: In Subsection 3.4.1 we provided simultaneous estimates in terms of ω_1 and ω_2 for the instances $S_n^{<\alpha,0,0>}$ and $S_n^{<0,\beta,\gamma>}$. *Maybe it is useful to find analogous results for the more general $S_n^{<\alpha,\beta,\gamma>}$, see its definition at (3.8).*

Problem 9: In Theorem 4.35 we have studied the behavior of the over-iterates of $S_n^{<\alpha,\beta,\gamma>}$ and the following question has naturally appeared:
How can we determine the degree of approximation for $\left|[S_n^{<\alpha,\beta,\gamma>}]^m - S^{(\alpha,\beta,\gamma)}\right|$?

Problem 10: This last proposed problem refers to the possibility to give *quantitative Voronovskaja theorems* for a larger class of operators, e.g., for $S_{\Delta_n,k}$ with $k \geq 1$ and Δ_n an arbitrary partition of $[0, 1]$.

Bibliography

[1] J. Adell, F. G. Badía & J. de la Cal, Beta-type operators preserve shape properties, Stochastic Process. Appl. **48** (1993), 1–8.

[2] J.A. Adell, F. G. Badía, J. de la Cal & F. Plo, On the property of monotonic convergence for Beta operators, J. Approx. Theory **84** (1996), 61–73.

[3] O. Agratini, On simultaneous approximation by Stancu-Bernstein operators. In: "Approximation and optimization" (Proc. of Int. Conf. on Approximation and Optimisation), Cluj-Napoca: Transilvania 1997, 157–162.

[4] O. Agratini & I. A. Rus, Iterates of a class of discrete linear operators, Comment. Math. Univ. Carolinae **44** (2003), 555–563.

[5] F. Altomare & M. Campiti, "Korovkin-type Approximation Theory and its Applications". Berlin - New-York: de Gruyter Studies in Mathematics 1994.

[6] G. A. Anastassiou & S. G. Gal, "Approximation Theory, Moduli of Continuity and Global Smoothness Preservation". Boston: Birkhäuser 2000.

[7] A. Attalienti & I. Raşa, Total positivity: an application to positive linear operators and to their limiting semigroups, Preprint 2006.

[8] C. Badea, I. Badea & H. Gonska, Improved estimates on simultaneous approximation by Bernstein operators, Anal. Numér. Théor. Approx. **22** (1993), 1-22.

[9] H. Berens & Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights. In: Approximation Theory and Functional Analysis (ed. by C.K. Chui). Boston: Acad. Press 1991, 25–46.

[10] H. Berens & Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights: The cases $p = 1$ and $p = \infty$. In: Approximation, Interpolation and Summation (Israel Math. Conf. Proc., ed. by S. Baron and D. Leviatan), 1991, 51–62.

[11] I. S. Beresin & P. Zhidkov, Numerische Methoden II, Berlin: VEB Deutscher Verlag der Wissenschaften 1971.

[12] V. Berinde, ”Iterative Approximation of Fixed Points”, Baia Mare: Editura Efemeride 2002.

[13] S. N. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités, Comm. Soc. Math. Kharkow **13** (1912/13), 1–2. [Also appears in Russian translation in Bernstein's Collected Works.]

[14] S. N. Bernstein, Complément à l'article de E. Voronovskaja ”Détermination de la forme asymptotique de l'approximation des fonctions par les polyômes de M. Bernstein”. C. R. (Dokl.) Acad. Sci. URSS A (1932), No.4, 89–92.

[15] L. Beutel, On a lower bound of the second moment of the quadratic Schoenberg operator. In:”Mathematical Analysis and Approximation Theory” (Proc. of the 5th Romanian-German Seminar on Approx. Theory and its Applications), Sibiu: Burg Verlag 2002, 37–44.

[16] L. Beutel, H. Gonska, D. Kacsó & G. Tachev, On variation-diminishing Schoenberg operators: new quantitative statements. In: ”Multivariate Approximation and Interpolation with Applications (ed. by M. Gasca), Monogr. Academia Ciencias de Zaragoza **20** (2002), 9–58.

[17] L. Beutel, H. Gonska, D. Kacsó & G. Tachev, Variation-diminishing splines revisited. In: ”Proc. Int. Sympos. on Numerical Analysis and Approximation Theory” (Cluj–Napoca, May 9–11, 2002, ed. by Radu T. Trâmbiţaş), 54–75.) Cluj–Napoca: Presa Universitară Clujeană 2002.

[18] L. Beutel, H. Gonska, D. Kacsó & G. Tachev, On the second moments of variation-diminishing splines, J. Concr. Appl. Math. **2** (2004), 91–117.

[19] H. Bohman, On approximation of continuous and of analytic functions, Ark. Mat. **2** (1952), 43–56.

[20] G. P. Bonneau, Multiresolution analysis with non-nested spaces. In: ”Tutorials on Multiresolution in Geometric Modelling” (Lectures European School on Principles of Multiresolution in Geometric Modelling, Munich University of Technology, August 2001; ed. by A. Iske, E. Quak and M. S. Floater), 147–163. Berlin: Springer 2002.

[21] C. de Boor, Total positivity of the spline collocation matrix, Ind. Univ. J. Math. **25** (1976), 541–551.

[22] É. Borel, Leçons sur les fonctions de variables réelles et les développements en séries de polynômes. Paris: Gauthier–Villars 1905 [2nd edition (1928)].

[23] É. Borel, Sur l'interpolation des fonctions continues par des polynômes, Verh. d. 3. intern. Math. Kongr. Heidelberg (1905), 229–232.

[24] Yu. A. Brudnyĭ, On a certain method for approximation of bounded functions, given on a segment (Russian). In: "Studies in Contemporary Problems in Constructive Theory of Functions" (Proc. Second All-Union Conf. Baku 1962; ed. by I.I. Ibragimov), Baku: Izdat. Akad. Nauk Azerbaidz̧an (1965), 40–45.

[25] P. L. Butzer & H. Berens, "Semi-groups of Operators and Approximation". Berlin-Heidelberg-New-York: Springer-Verlag 1967.

[26] Jia-ding Cao, On linear approximation methods (Chinese), Acta Sci. Natur. Univ. Fudan **9** (1964), 43–52.

[27] G. Călugăreanu, On operators of S. N. Bernstein. Spectra of operators (Romanian), Gaz. Mat. Ser. A **71** (1966), 448–451.

[28] W. Chen, On the modified Bernstein-Durrmeyer operator. In: Report of the Fifth Chinese Conference on Approximation Theory, Zhen Zhou, China, 1987.

[29] V. A. Cleciu, Aproximare simultană cu ajutorul operatorilor uni şi multidimensionali de tip discret, Ph. D. Thesis, Cluj-Napoca: Babeş Bolyai University 2006.

[30] Gh. Coman, "Analiză Numerică". Cluj-Napoca: Ed. Libris 1995.

[31] L. Comtet, "Analyse combinatoire II". Paris: Presses Universitaires de France 1970.

[32] Sh. Cooper & Sh. Waldron, The eigenstructure of the Bernstein operator, J. Approx. Theory **105** (2000), 133–165.

[33] C. Cottin & H. Gonska, Simultaneous approximation and global smoothness preservation, Rend. Circ. Mat. Palermo (2) Suppl. **33** (1993), 259–279.

[34] P. J. Davis, "Interpolation and Approximation". New York: Dover 1975.

[35] B. Della Vecchia, On the approximation of functions by means of the operators of D. D. Stancu, Studia Univ. Babeş-Bolyai, Mathematica **37** (1992), 3–36.

[36] M.M. Derrienic, Sur l'approximation des fonctions d'une ou plusieurs variables par de polynômes de Bernstein modifiés et application au problème des moments. Thèse de 3e cycle, Université de Rennes, 1978.

[37] M. M. Derriennic, Sur l'approximation des fonctions intégrables sur $[0,1]$ par des polynômes de Bernstein modifiés, J. Approx. Theory **31** (1981), 325–343.

[38] R. A. DeVore & G. G. Lorentz, "Constructive Approximation". Berlin-Heidelberg-New-York: Springer-Verlag 1993.

[39] Z. Ditzian & K. Ivanov, Bernstein-type operators and their derivatives, J. Approx. Theory **56** (1989), 72–90.

[40] J. L. Durrmeyer, Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments. Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.

[41] V.K. Dzjadyk, "Introduction to the Theory of Uniform Approximation of Functions by Polynomials (Russian)", Moscow: Izdat. Nauka 1977.

[42] H. Esser, On pointwise convergence estimates for positive linear operators on $C[a,b]$, Indag. Math. **38** (1976), 189–194.

[43] G. Farin, "NURBS Curves and Surfaces". Boston: AK Peters 1995.

[44] G. Farin, "Curves and Surfaces for Computer Aided Geometric Design – A Practical Guide". New York: Academic Press 1996 (4th ed.).

[45] G. Farin, D. Jung, Linear precision of rational Bézier curves, Computer Aided Geometric Design **12** (1995), 431–433.

[46] G. Farin, "A history of curves and surfaces in CAGD. Handbook of computer aided geometric design" Amsterdam: North-Holland 2002.

[47] G. M. Fichtenholz, "Differential- und Integralrechnung I". Berlin: VEB Deutscher Verlag der Wissenschaften 1964.

[48] Z. Finta, Quantitative estimates for some linear and positive operators, Studia Univ. Babeş-Bolyai, Mathematica **47** (2002), 71–84.

[49] Z. Finta, Direct and converse theorems for generalized Bernstein-type operators, Serdica Math. J. **30** (2004) 33–42.

[50] J. Fiorot, P. Jeannin, "Rational Curves and Surfaces". Chichester: Wiley 1992.

[51] M.S. Floater, Derivatives of rational Bézier curves, Comput. Aided Geom. Design **9** (1992), 161–174.

[52] I. Gavrea & D. H. Mache, Generalization of Bernstein-type approximation methods. In: Approximation Theory (Proc. IDoMAT 95; ed. by M. W. Müller et al.). Berlin: Akademie Verlag 1995, 115–126.

[53] I. Gavrea, H. Gonska & D. Kacsó, Variation on Butzer's problem: characterization of the solutions, Comput. Math. Appl. **34** (1997), 51–64.

[54] H. Gonska, Quantitative Aussagen zur Approximation durch positive lineare Operatoren, Ph. D. Thesis, Duisburg: Universität Duisburg 1979.

[55] H. Gonska, On Mamedov estimates for the approximation of finitely defined operators. In: "Approximation Theory III" (Proc. Int. Sympos. Austin 1980; ed. by E. W. Cheney). New York - San Francisco - London: Acad. Press 1980, 443–448.

[56] H. Gonska, On approximation in spaces of continuous functions, Bull. Austral. Math. Soc. **28** (1983), 411–432.

[57] H. Gonska, Quantitative Korovkin-type theorems on simultaneous approximation, Math. Z. **186** (1984), 419–433.

[58] H. Gonska, On approximation by linear operators: improved estimates, Anal. Numér. Théor. Approx. **14** (1985), 7–32.

[59] H. Gonska, Degree of approximation by lacunary interpolators: (0, ..., R-2, R) interpolation, Rocky Mountain J. Math., **19** (1989), 157–171.

[60] H. Gonska, The rate of convergence of bounded linear processes on spaces of continuous functions, Automat. Comput. Appl. Math. **7** (1998), 38-97.

[61] H. Gonska, On the composition and decomposition of positive linear operators. In: "Approximation theory and its applications" (Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 31, Kiev, 1999). Kiev: Natsīonal. Akad. Nauk Ukraïni, Īnst. Mat. 2000, 161–180.

[62] H. Gonska, On the degree of approximation in Voronovskaja's theorem. To appear in: Studia Univ. "Babeş-Bolyai", Mathematica (2007).

[63] H. Gonska, D. Kacso, O. Nemitz & **P. Piţul**, Piecewise linear interpolation revisited: BLaC-wavelets, Studia Univ. "Babeş-Bolyai", Mathematica **51** (2006), 105–115.

[64] H. Gonska, D. Kacsó & **P. Piţul**, On rational B-spline functions. In: "Constructive Theory of Functions - Varna 2005" (ed. by B. Bojanov). Sofia: Marin Drinov Academic Publ. House 2006, 145–157.

[65] H. Gonska, D. Kacsó & **P. Piţul**, The degree of convergence of over-iterated positive linear operators, J. Appl. Funct. Anal. **1** (2006), 403–423.

[66] H. Gonska, D. Kacsó & I. Raşa, On genuine Bernstein-Durrmeyer operators. To appear in: Results in Mathematics **50** (2007), xx–xx.

[67] H. Gonska & R. K. Kovacheva, The second order modulus revisited: remarks, applications, problems, Conf. Sem. Mat. Univ. Bari **257** (1994), 1–32.

[68] H. Gonska & J. Meier, Quantitative theorems on approximation by Bernstein-Stancu operators, Calcolo **21** (1984), 317–335.

[69] H. Gonska & **P. Piţul**, Remarks on an article of J. P. King, Comment. Math. Univ. Carolinae **46** (2005), 645–652.

[70] H. Gonska, **P. Piţul** & I. Raşa, On Peano's form of the Taylor remainder, Voronovkaja's theorem and the commutator of positive linear operators. In: " Numerical Analysis and Approximation Theory" (Proc. of the Int. Conf. on Numerical Analysis and Approximation Theory July 4-8, 2006, Cluj-Napoca, Romania). Cluj-Napoca: Casa Cărţii de Ştiinţă 2006, 55–80.

[71] H. Gonska, **P. Piţul** & I. Raşa, On differences of positive linear operators, Carpathian J. Math. **22** (2006), 65–78.

[72] H. Gonska, **P. Piţul** & I. Raşa, Over-iterates of Bernstein-Stancu operators, Calcolo **44** (2007), 117–125.

[73] H. Gonska & I. Raşa, The limiting semigroup of the Bernstein iterates: Degree of convergence, Acta Math. Hung., **111 (1–2)** (2006), 119–130.

[74] T. N. T. Goodman & A. Sharma, A Bernstein type operator on the simplex, Math. Balkanica **5** (1991), 129–145.

[75] D. Jackson, Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung, Ph.D. Thesis, Göttingen: Georg-August Univ. of Göttingen 1911.

[76] H. Johnen, Inequalities connected with moduli of smoothness, Mat. Vesnik **9 (24)** (1972), 289–303.

[77] D. P. Kacsó, On the degree of simultaneous approximation. In: "RoGer 2000–Braşov" (Proc. 4th Romanian-German Seminar on Approximation Theory and its Applications, Braşov 2000; ed. by H. H. Gonska, D. Kacsó and L. Beutel), 62–69.

[78] D. P. Kacsó, Discrete Jackson-type operators via a Boolean sum approach, J. Comput. Anal. Appl. **3** (2001), 399–413.

[79] D. P. Kacsó, Simultaneous approximation by almost convex operators. In: Rend. Circ. Mat. Palermo (2) Suppl. **68**, part II (2002), 523–538.

[80] D. P. Kacsó, Certain Bernstein-Durrmeyer type operators preserving linear functions, "Habilitation" thesis, Duisburg: Universität Duisburg-Essen 2006.

[81] S. Karlin, "Total Positivity". Stanford, California: Stanford University Press 1968.

[82] S. Karlin & Z. Ziegler, Iteration of positive approximation operators, J. Approx. Theory **3** (1970), 310–339.

[83] R. P. Kelisky & T. J. Rivlin, Iterates of Bernstein polynomials, Pacific J. Math. **21** (1967), 511–520.

[84] L. V. Kantorovich, Sur certains developpements suivant les polynômes de la forme de S. Bernstein I, II, Dokl. Akad. Nauk. SSSR (1930), 563–568, 595–600.

[85] J. G. Kemeny & J. L. Snell, "Finite Markov Chains". New-York: Springer Verlag 1976.

[86] P. J. King, Positive linear operators which preserve x^2, Acta Math. Hungar. **99** (2003), 203–208.

[87] H.-B. Knoop & P. Pottinger, Ein Satz vom Korovkin-Typ für C^k-Räume, Math. Z. **148** (1976), 23–32.

[88] A.N. Kolmogorov & S.V. Fomin, "Reelle Funktionen und Funktionalanalysis". Berlin: VEB Deutscher Verlag der Wissenschaften 1975.

[89] N.P. Korneičuk, On best constants in Jackson's inequality for continuous periodic functions (Russian), Mat. Zametki, **32** (5) (1982), 669–674.

[90] P. P. Korovkin, On convergence of linear positive operators in the space of continuous functions, Dokl. Akad. Nauk SSSR **90** (1953), 961–964.

[91] G.I. Kudrjavcev, On the order of approximation by some sequences of operators of class M (Russian), Naučn. Tr. Novosib. Gos. Ped. Inst. **94** (1974), 20–23.

[92] G. G. Lorentz, "Bernstein Polynomials". Toronto: Univ. Press, 1953.

[93] G. G. Lorentz, "Approximation of Functions". New-York: Chelsea, 1986.

[94] A. Lupaş, Some properties of the linear positive operators (I), Mathematica (Cluj) **9**, No. 32 (1967), 77–83.

[95] A. Lupaş, Die Folge der Betaoperatoren, Ph.D. Thesis, Stuttgart: Universität Stuttgart 1972.

[96] A. Lupaş, Contribuţii la teoria aproximării prin operatori liniari, Ph.D. Thesis, Cluj-Napoca: Babeş-Bolyai University 1975.

[97] A. Lupaş, The approximation by means some positive linear operators. In: "Approximation Theory" (Proc. Int. Dortmund Meeting on Approximation Theory 1995 ed. by M. W. Müller et al.). Berlin: Akademie Verlag 1995, 201–229.

[98] L. Lupaş and A. Lupaş, Polynomials of binomial type and approximation operators, Studia Univ. Babes-Bolyai Math. **32** (4) (1987), 61–70.

[99] R. G. Mamedov, On the order of approximation of functions by sequences of linear positive operators (Russian), Dokl. Akad. Nauk SSSR **128** (1959), 674–676.

[100] M. J. Marsden, An identity for spline functions with applications to variation-diminishing spline approximation, J. Approx. Theory **3** (1970), 7–49.

[101] M.J. Marsden, On uniform spline approximation, J. Approx. Theory **6** (1972), 249–253.

[102] G. Mastroianni & M. R. Occorsio, Sulle derivate dei polinomi di Stancu, Rend. Accad. Sci. Fis. Mat. Napoli **45**, IV Ser. (1978), 273–281.

[103] G. Mastroianni & M. R. Occorsio, Una generalizatione dell'operatore di Stancu, Rend. Accad. Sci. Fis. Mat. Napoli **45**, IV Ser. (1978), 495–511.

[104] J. Meier, Zur Approximation durch gewöhnliche und modifizierte Bernstein-Operatoren unter besonderer Berücksichtigung quantitativer Aussagen und asymptotischer Formeln, Diploma thesis, Duisburg: Universität Duisburg 1982.

[105] B. S. Mitjagin & E. M. Semenov, Lack of interpolation of linear operators in spaces of smooth functions, Math. USSR-Izv. **11** (1977), 1229-1266.

[106] C. Mortici & I. Oancea, A non-smooth extension for the Bernstein-Stancu operators and an application, Studia Univ. Babeş-Bolyai, Mathematica **51** (2006), 69–81.

[107] J. Nagel, Sätze Korovkinschen Typs für die Approximation linearer positiver Operatoren, Ph. D. Thesis, Essen: Universität Essen 1978.

[108] O. Nemitz, BLaC-Wavelets und nicht ineinander geschachtelte Wavelets. Diploma thesis, University of Duisburg-Essen 2003.

[109] S. Ostrovska, q-Bernstein polynomials and their iterates, J. Approx. Theory **123** (2003), 232–255.

[110] R. Păltănea, Sur un opérateur polynomial defini sur l'ensemble des fonctions intégrables. In: "Itinerant Seminar on Functional Equations, Approximation and Convexity" (Cluj-Napoca, 1983), 101–106. Cluj-Napoca: Babeş-Bolyai University 1983.

[111] R. Păltănea, Best constants in estimates with second order moduli of continuity. In: Approx. Theory, (Proc. Int. Dortmund meeting on Approx. Theory 1995, ed. by M. W. Müller et al.). Berlin: Akad. Verlag 1995, 251–275.

[112] R. Păltănea, On the transformation of the second order modulus by Bernstein operators, L'Analyse Numér. et la Th. de l'Approx. **27** (1998), 309–313.

[113] R. Păltănea, "Approximation Theory using Positive Linear Operators". Boston: Birkhäuser 2004.

[114] R. Păltănea & E. Păltănea, "Analiză matematică". Braşov: Editura Universităţii Transilvania 2003.

[115] P. E. Parvanov & B. D. Popov, The limit case of Bernstein's operators with Jacobi - weights, Math. Balkanica **8** (1994), no. 2–3, 165–177.

[116] J. Peetre, A theory of interpolation of normed spaces". In: "Notas de Matemática" **39**, Rio de Janeiro: Instituto de Matemática Pura e Aplicada, Conselho Nacional de Pesquisas, 1968.

[117] L. Piegl & W. Tiller, Curve and surface constructions using rational B–splines, Computer Aided Design **19** (9) (1987), 485–498.

[118] L. Piegl & W. Tiller, "The NURBS Book". Berlin: Springer 1995.

[119] A. Pinkus, Weierstrass and approximation theory, J. Approx. Theory **107** (2000), 1–66.

[120] **P. Piţul**, Second moments of quadratic Schoenberg-Splines with knots of multiplicity two in the interior of the definition interval. In: Mathematical Analysis and Approximation Theory (Proc. of the 6th Romanian-German Seminar on Approximation Theory and its Applications June 3–6 2004, Băişoara, Romania). Cluj-Napoca: Mediamira Science Publisher 2005, 173–180.

[121] **P. Piţul** & P. Sablonnière, A family of univariate rational Bernstein operators, submitted for publication at J. Approx. Theory.

[122] T. Popoviciu, "Despre cea mai bună aproximaţie a funcţiilor continue prin polinoame". Cluj-Napoca: Inst. Arte Grafice Ardealul 1937.

[123] T. Popoviciu, Sur l'approximation des fonctions convexes d'ordre supérieur, Mathematica (Cluj) **10** (1934), 49–54.

[124] T. Popoviciu, Sur l'approximation des fonctions continues d'une variable réelle par des polynomes, Ann. Sci. Univ. Iaşi, Sect. I, Math **28** (1942), 208.

[125] T. Popoviciu, Notes sur les fonctions convexes d'ordre supérieur (IX), Bull. Math. de la Soc. Roum. des Sci. **43** (1942), 85–141.

[126] T. Popoviciu, Asupra demonstraţiei teoremei lui Weierstrass cu ajutorul polinoamelor de interpolare, Lucrările Ses. Gen. Şt. Acad. Române din 1950 **1–4** (1950), translated into English by D. Kacsó, On the proof of Weierstrass' theorem using interpolation polynomials, East J. Approx. **4** (1998), 107–110.

[127] T. Popoviciu, Über die Konvergenz von Folgen positiver Operatoren, An. Şt. Univ. Al.I. Cuza Iaşi (N.S.) **17** (1971), 123–132.

[128] T. Popoviciu, "Curs de Analiză Matematică", Partea III-a Continuitate. Cluj-Napoca: Babeş-Bolyai University 1974.

[129] I. Raşa & T. Vladislav, Some properties of Bernstein and Stancu operators, In: Approximation and Optimization Vol. 1 (Proc. Int. Conf. on Approximation and Optimization ICAOR; ed. by D. D. Stancu et al.). Transilvania Press, Cluj-Napoca 1997, 345–350.

[130] R. T. Rockafellar, "Convex Analysis". Princeton: Princeton University Press, 1970.

[131] I. A. Rus, Iterates of Stancu operators, via contraction principle, Studia Univ. Babeş-Bolyai, Mathematica **47** (2002), 101–104.

[132] I. A. Rus, Iterates of Bernstein operators, via contraction principle, J. Math. Anal. Appl. **292** (2004), 259–261.

[133] P. Sablonnière, Opérateurs de Bernstein-Jacobi et polynômes orthogonaux, Publication A.N.O. 37, Université des Sciences et Techniques de Lille 1981.

[134] P. Sablonnière, Positive Bernstein-Sheffer operators, J. Approx. Theory **83** (1995), 330–341.

[135] I.J. Schoenberg, On spline functions, with a supplement by T.N.E. Greville. In: "Inequalities" (Proc. Symposium Wright–Patterson Air Force Base, August 1965, ed. by O. Shisha). New York: Acad. Press 1967, 255–291.

[136] I. J. Schoenberg, On variation-diminishing approximation methods. In: "On Numerical Approximation" (Proc. Sympos. conducted by the MRC; ed. by R. E. Langer). Madison, Wisconsin: Univ. of Wisconsin Press 1959 249–274.

[137] L. L. Schumaker, "Spline Functions: Basic Theory". New-York: John Wiley & Sons, 1981.

[138] F. Schurer & F. W. Steutel, On the degree of approximation of functions in $C^1[0,1]$ by Bernstein polynomials, T.H. Report 75-WSK-07, Eindhoven University of Technology 1975.

[139] Bl. Sendov & V. Popov, The convergence of the derivatives of positive linear operators (Russian), C.R. Acad. Bulgare Sci. **22** (1969), 507–509.

[140] O. Shisha & B. Mond, The degree of convergence of linear positive operators, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 1196-1200.

[141] P. C. Sikkema, Der Wert einiger Konstanten in der Theorie der Approximation mit Bernstein–Polynomen, Num. Math. **3** (1961), 107–116.

[142] P. C. Sikkema, Über Potenzen von verallgemeinerten Bernstein-Operatoren, Mathematica Cluj **8 (31)** (1966), 173–180.

[143] SISL – The SINTEF Spline Library Reference Manual (version 4.4). http://www.sintef.no/upload/IKT/9011/geometri/sisl/manual.pdf

[144] D. D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rev. Roum. Math. Pures et Appl. **8** (1968), 1173–1194.

[145] D. D. Stancu, Use of probabilistic methods in the theory of uniform approximation of continuous functions, Rev. Roum. Math. Pures et Appl. **14** (1969), 673–691.

[146] D. D. Stancu, Approximation properties of a class of linear positive operators, Studia Univ. Babeş-Bolyai, Math.-Mech. **15** (1992), 33–38.

[147] D. D. Stancu, On the remainder of approximation of functions by means of a parameter dependent linear polynomial operator, Studia Univ. Babeş-Bolyai, Math.-Mech. **16** (1992), 59–66.

[148] D. D. Stancu, Approximation of functions by means of some new classes of positive linear operators. In: "Numerische Methoden der Approximationstheorie" (Proc. Conf. Math. Res. Inst. Oberwolfach 1971; ed. by L. Collatz et al.). Basel: Birkhäuser Verlag 1972, 187–203.

[149] D. D. Stancu, Approximation properties of a class of multiparameter positive linear operators. In: Approximation and Optimization Vol. 1 (Proc. Int. Conf. on Approximation and Optimization ICAOR; ed. by D. D. Stancu et al.). Cluj-Napoca: Transilvania Press 1997, 203–218.

[150] D. D. Stancu, O. Agratini, Gh. Coman & R. Trâmbiţaş, "Analiză Numerică şi Teoria Aproximării" vol. I, Cluj-Napoca: Presa Universitară Clujeană 2001.

[151] J. Szabados, Discrete linear interpolatory operators, Surv. Approx. Theory **2** (2006), 53–60.

[152] G. T. Tachev, Approximation by rational spline functions, Calcolo **43** (2006), 279–286.

[153] W. Tiller, Rational B-splines for curve and surface representation, IEEE Comput. Graphics and Appl. **3** (6) (1983), 61–69.

[154] K. Vesprille, "Computer aided design application of the rational B-spline approximation form". Ph.D. Thesis, Syracuse University, 1975.

[155] N. Vornicescu, A class of linear positive operators. In: Mathematical Analysis and Approximation Theory (Proc. of the 5th Romanian–German Seminar on Approx. Theory and its Applications), Sibiu: Burg Verlag 2002, 309–315.

[156] E. V. Voronovskaja, Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de M. Bernstein, Dokl. Akad. Nauk SSSR, **4** (1932), 86–92.

[157] K. G. Weierstrass, Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen, Sitzungsber. Akad. Berlin (1885), 633–639, 789–805. [Also in: "Mathematische Werke", Vol. 3, 1–37, Berlin: Mayer & Müller 1903.]

[158] H. J. Wenz, On the limits of (linear combinations of) iterates of linear operators, J. Approx. Theory **89** (1997), 219–237.

[159] V. V. Zhuk, Functions of the Lip 1 class and S. N. Bernstein's polynomials (Russian), Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1** (1989), 25–30.