Abstract

Given a graded ideal $I$ in a polynomial ring, there are several other graded ideals associated to it e.g. graded reduction ideals or generic initial ideals. These ideals play a fundamental role in investigating several homological, algebraic, geometric and combinatorial properties of $I$. One main aim of this thesis is to understand and explore some of such relations. Another problem that we address in this thesis is the multiplicity conjecture.

Let $I$ be a monomial ideal in a standard graded polynomial ring $A$. Using the convex-geometric properties, we prove that there exists a unique minimal monomial reduction ideal $J$ of $I$ and we show that the maximum degree of a monomial generator of $J$ determines the slope $p$ of the linear function $\text{reg}(I^t) = pt + c$ for $t \gg 0$. We determine the structure of the reduced fiber ring $F(J)_{\text{red}}$ of $J$ and show that $F(J)_{\text{red}}$ is isomorphic to the inverse limit of an inverse system of semigroup rings determined by convex geometric properties of $J$.

Another aim is to consider the homological properties of graded ideals. Let $K$ be a field, $S$ a polynomial ring and $E$ an exterior algebra over $K$, both in a finite set of variables. We study rigidity properties of the graded Betti numbers of graded ideals in $S$ and $E$ when passing to their generic initial ideals.

Through combinatorial properties of squarefree monomial ideals, we study and prove the multiplicity conjecture for a class of spheres. A linear ball is a simplicial complex whose geometric realization is homeomorphic to a ball and whose Stanley–Reisner ring has a linear resolution. It turns out that the Stanley–Reisner ring of the sphere which is the boundary complex of a linear ball satisfies the multiplicity conjecture. A class of shellable spheres arising naturally from commutative algebra whose Stanley–Reisner rings satisfy the multiplicity conjecture will be presented.