Convex–geometric, homological and combinatorial properties of graded ideals

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Pooja Singla

aus
Indien

Referent: Prof. Dr. Herzog
Koreferent: Prof. Dr. Welker

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Dedicated to my parents

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Contents

Introduction

Given a graded ideal I in a polynomial ring, there are several other graded ideals associated to it e.g. graded reduction ideals or generic initial ideals. These ideals play a fundamental role in investigating several homological, algebraic, geometric and combinatorial properties of I. One main aim of this thesis is to understand and explore some of such relations. Another problem that we address in this thesis is the multiplicity conjecture.

We describe the organization of the thesis explaining the background and the motivation for the results. We will also state the main results of each chapter here.

In the first chapter, we start with the consideration of the regularity function of the powers of a monomial ideal in a polynomial ring. Quite generally, Cutkosky–Herzog–Trung [CHT99] and independently Kodiyalam [Ko99] showed that for a graded ideal I in a polynomial ring $A = K[x_1, \ldots, x_n]$, the regularity of I^t is a linear function $pt + c$ for large enough t. Also the coefficient p of the linear function is known and it is given by the $\min\{\theta(J) : J \text{ is a reduction ideal of } I\}$, (see [Ko99]). Here $\theta(J)$ denotes the maximum of the degrees of elements in $G(J)$(the set of minimal monomial generators of J) and a graded ideal $J \subset I$ is said to be a *reduction ideal* of I if there exists some integer m such that $JI^m = I^{m+1}$.

For a monomial ideal $I \subset A$, we give a convex geometric interpretation for the slope p of the above linear function: let S be a set of monomials in A. We denote by $\Gamma(S) \subset \mathbb{N}^n$ the set of exponents of the monomials in S. Let $\text{conv}(I)$ denote the convex hull of the elements of the set $\Gamma(I)$ in $\mathbb{R}^n$ and let $\text{ext}(I)$ denote the extreme points of the convex set $\text{conv}(I)$. Now let J be the monomial ideal which is determined by the property that $\Gamma\big(G(J)\big) = \text{ext}(I)$. We show in Proposition 1.2.1 that the ideal J is the unique minimal monomial reduction ideal of I, that is, J is a reduction ideal of I and there exists no proper monomial ideal $L \subset J$ such that L is again a reduction ideal of I. It turns out that $p = \theta(J)$. In other words, $p = \max\{\deg x^a \colon a \in \text{ext}(I)\}$. Hence we have:

Theorem 1. *Let $I \subset A = K[x_1, \dots, x_n]$ be a monomial ideal. Let J be the unique minimal monomial reduction ideal of I. Then $\operatorname{reg}(I^t) = pt + c$ for $t \gg 0$ where $p = \max\{\deg g : g \in G(J)\}$.*

We call a monomial ideal J to be an extremal ideal if it is its own minimal monomial reduction ideal. The next goal of the first chapter is to determine the structure of the reduced fiber ring $\mathcal{F}(J)_{\mathrm{red}}$ of an extremal ideal J. For a graded ideal L in the polynomial ring $A = K[x_1, \dots, x_n]$, the fiber ring $\mathcal{F}(L)$ is defined to be $\mathcal{R}(L)/\mathfrak{m}\mathcal{R}(L) = \bigoplus_{n \geq 0} L^n/\mathfrak{m}L^n$ where $\mathcal{R}(L)$ is the Rees ring and $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of A. The main motivation to study the structure of the reduced fiber ring of an extremal ideal is to determine the dimension of the fiber ring of an arbitrary monomial ideal. Let $I \subset A$ be a monomial ideal and $J \subset I$ be its minimal monomial reduction. Then J is an extremal ideal, and $\dim \mathcal{F}(I) = \dim \mathcal{F}(J) = \dim \mathcal{F}(J)_{\mathrm{red}}$. So as far as dimension is concerned it is enough to consider the reduced fiber ring $\mathcal{F}(J)_{\mathrm{red}}$ of the extremal ideal J, whose structure is in general much simpler than that of $\mathcal{F}(I)$. We have the following:

Theorem 2. *Let $J \subset A = K[x_1, \dots, x_n]$ be an extremal ideal and let $\mathcal{F}_c$ denote the set of compact faces of $\operatorname{conv}(J)$. For each $F \in \mathcal{F}_c$, we put $K[F] = K[x^{a_j}t\colon a_j \in F]$. Then we have*

$$\mathcal{F}(J)_{\mathrm{red}} \cong \varprojlim_{F \in \mathcal{F}_c} K[F].$$

As an application to Theorem 2, we get in the particular case of monomial ideals a result of Carles Bivia–Ausina [Au03] on the analytic spread of a Newton non–degenerate ideal. Let $\overline{L}$ denote the integral closure of an ideal L. Using convex geometric arguments, we show

Theorem 3. *Let $I \subset A$ be a monomial ideal and $J \subset I$ be its minimal monomial reduction ideal. Then*

$$\overline{I^\ell} = J\overline{I^{\ell-1}}$$

where ℓ is the analytic spread of I.

If in Theorem 3, we assume that I^a is integrally closed for $a \leq \ell - 1$, we obtain that $I^\ell = JI^{\ell-1}$, and that I is a normal ideal.

In the second chapter, we consider the growth of the graded Betti numbers of large enough powers of a graded ideal. Let $I = (f_1, \dots, f_s) \subset A = K[x_1, \dots, x_r]$ be a graded ideal with $\deg f_i = d_i$ and $d_1 \leq \cdots \leq d_s$. Kodiyalam in [Ko93] showed that the total Betti number $\beta_i(I^n)$ is a polynomial function for $n \gg 0$. Let P_i be the polynomials such that $P_i(n) = \beta_i(I^n)$ for $n \gg 0$. We prove in Theorem 2.2.4 that

$$\deg P_{i+1} \leq \deg P_i \quad \text{for } i \geq 0.$$

If $d_1 = \cdots = d_s = d$, then all graded Betti numbers $\beta_{i,i+dn+j}(I^n)$ are polynomial functions and $N(I^n) = N(I^{n+1})$ for $n \gg 0$ where

$$N(I^n) = \{(i,j) \; : \; \beta_{i,i+j+dn}(I^n) \neq 0\}.$$

In general, for given $a, c \in \mathbb{Z}$ the graded Betti number $\beta_{i,i+an+c}(I^n)$ is a quasi polynomial function for $n \gg 0$.

Next, we discuss the regularity function of the power products $I_1^{n_1} I_2^{n_2} \cdots I_m^{n_m}$, $n_i \geq 0$ of the graded ideals $I_1, \ldots, I_m$ in the polynomial ring $K[x_1, \ldots, x_r]$. It is known that for large enough n_i, the regularity function for these power products is a multi–linear function of the form $\sum_{i=1}^m a_i n_i + c$ for some integers a_i and c (see [CHT99, Remark after Corollary 3.5]). We determine the coefficient a_i of this function. In case of monomial ideals, we give a convex geometric interpretation of the a_i. In fact we show that these coefficients are determined by the minimal monomial reduction ideals of the factors I_i.

In the third chapter, we study rigidity properties of graded Betti numbers of a graded ideal when passing to its generic initial ideal. The homological properties of a graded ideal $I \subset A = K[x_1, \ldots, x_n]$ and its generic initial ideal $\operatorname{Gin}(I)$ are closely related. Let $\beta_i^A(M) = \dim_K \operatorname{Tor}_i^A(K, M)$ and $\beta_{i,j}^A(M) = \dim_K \operatorname{Tor}_i^A(K, M)_j$ denote respectively the ith total and (i, j)th graded Betti number of a finitely generated graded A–module M. By definition, the generic initial ideal $\operatorname{Gin}(I)$ is, after performing a generic change of coordinates, the initial ideal of I with respect to the reverse lexicographic order. Here we consider the reverse lexicographic order induced by $x_1 > \cdots > x_n$.

The following inequality of graded Betti numbers is well–known:

$$\beta_{i,j}(S/I) \leq \beta_{i,j}\big(S/\operatorname{Gin}(I)\big),$$

for all i, j (see [Co04, Theorem 1.1]). In his paper [Co04] Conca asked whether the equality $\beta_i(S/I) = \beta_i\big(S/\operatorname{Gin}(I)\big)$ for some $i \geq 1$ of the total Betti numbers implies $\beta_j(S/I) = \beta_j\big(S/\operatorname{Gin}(I)\big)$ for all $j \geq i$. This question of Conca was positively answered in 2004 by Conca, Herzog and Hibi in [CoHHi04].

We extend this result of Conca–Herzog–Hibi to graded Betti numbers. We show the following:

Theorem 4. *Let $\beta_{i,i+k}^A(A/I) = \beta_{i,i+k}^A\big(A/\operatorname{Gin}(I)\big)$ for some $i > 1$ and $k \geq 0$, then*

$$\beta_{q,q+k}^A(A/I) = \beta_{q,q+k}^A\big(A/\operatorname{Gin}(I)\big) \qquad \text{for all} \quad q \geq i.$$

We also prove the similar result for generic initial ideals over an exterior algebra. The following stronger property is true in the exterior algebra: If the graded Betti numbers $\beta_{i,i+k}^E(E/J) = \beta_{i,i+k}^E\big(E/\operatorname{Gin}(J)\big)$ for some $i > 1$ and $k \geq 0$, then one has

$$\beta_{q,q+k}^E(E/J) = \beta_{q,q+k}^E\big(E/\operatorname{Gin}(J)\big) \qquad \text{for all} \quad q \geq 1.$$

Let R be either a polynomial ring over a field K with $\operatorname{char}(K) = 0$ or an exterior algebra over an infinite field and I a graded ideal of R. The above property leads us to ask when a graded ideal $I \subset R$ satisfies $\beta_{i,i+k}^R(R/I) = \beta_{i,i+k}^R\big(R/\operatorname{Gin}(I)\big)$ for all $i \geq 1$, where we fix an integer $k \geq 0$. We will prove the following result answering this question.

Theorem 5. *Let R be either a polynomial ring over a field K with* $\operatorname{char}(K) = 0$ *or an exterior algebra over an infinite field, $I \subset R$ a graded ideal and $k \geq 0$ an integer. The following conditions are equivalent.*

(i) $\beta^R_{i,i+k}(R/I) = \beta^R_{i,i+k}\big(R/\operatorname{Gin}(I)\big)$ *for all $i \geq 1$;*

(ii) $I_{\langle k \rangle}$ *and* $I_{\langle k+1 \rangle}$ *have a linear resolution;*

(iii) $\beta^R_{1,k+1}(R/I) = \beta^R_{1,k+1}\big(R/\operatorname{Gin}(I)\big)$ *and* $\beta^R_{1,k+2}(R/I) = \beta^R_{1,k+2}\big(R/\operatorname{Gin}(I)\big)$,

where $I_{\langle k \rangle}$ denotes the ideal of R generated by all homogeneous elements in I of degree k.

The above result is a generalization of [AHHi00, Theorem 1.1], where it was shown that $\beta^R_{i,j}(R/I) = \beta^R_{i,j}\big(R/\operatorname{Gin}(I)\big)$ for all i, j if and only if I is componentwise linear. At the end, we study the Cancellation Principle for generic initial ideals [G98]. We find a relationship between our results for Betti numbers of a graded ideal in a polynomial ring and the Cancellation Principle for generic initial ideals.

In the fourth chapter, we consider the problem of finding a natural class of spheres whose Stanley–Reisner rings satisfy the multiplicity conjecture.

To state the multiplicity conjecture we need to first introduce some terminology. Let $R = \sum_{i=0}^{\infty} R_i$ be a homogeneous Cohen–Macaulay algebra of dimension d over a field $R_0 = K$ with embedded dimension $n = \dim_K R_1$ and write $R = A/I$, where $A = K[x_1, \ldots, x_n]$ is the polynomial ring in n variables over K and I is a graded ideal of A. Let $H(R, i) = \dim_K R_i$, $i = 0, 1, 2, \ldots$, denote the Hilbert function of R and $F(R, \lambda) = \sum_{i=0}^{\infty} H(R, i)\lambda^i$ the Hilbert series of R. It is known that $F(R, \lambda)$ is a rational function of λ of the form

$$F(R, \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_\ell \lambda^\ell}{(1-\lambda)^d},$$

with each $h_i > 0$. The multiplicity $e(R)$ of R is

$$e(R) = h_0 + h_1 + \cdots + h_\ell.$$

Now, we consider the graded minimal free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow A \longrightarrow R \longrightarrow 0$$

of R over A, where $F_i = \bigoplus A(-j)^{\beta_{i,j}}$ with $\beta_{i,j} \geq 0$. Let

$$m_i = \min\{j : \beta_{i,j} \neq 0\}, \qquad M_i = \max\{j : \beta_{i,j} \neq 0\}.$$

The multiplicity conjecture due to Herzog, Huneke and Srinivasan says that

$$\frac{\prod_{i=1}^{p} m_i}{p!} \leq e(R) \leq \frac{\prod_{i=1}^{p} M_i}{p!}.$$

Let Δ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$ of dimension $d-1$ and $K[\Delta] = A/I_\Delta$, where $A = K[x_1, \ldots, x_n]$, its Stanley–Reisner ring. Suppose that Δ is a ball, i.e., the geometric realization $|\Delta|$ is a ball. Let $\partial\Delta$ denote the boundary complex of Δ and suppose that each vertex of Δ belongs to $\partial\Delta$. Thus $\partial\Delta$ is a sphere, i.e., the geometric realization $|\partial\Delta|$ is a sphere, of dimension $d-2$ on $[n]$. Each face of $\partial\Delta$ is called a boundary face of Δ and each face of $\Delta \setminus \partial\Delta$ is called an inside face of Δ. Let $1 \leq m-1$ be the smallest dimension of a nonface of Δ. We assume the following two assumptions on the simplicial complex Δ:

(A1) Δ has a minimal inside face of dimension $d-m$ and has no minimal inside face of dimension less than $m-1$;

(A2) the h–vector of $\partial\Delta$ is unimodal.

We prove the following:

Theorem 6. *Let Δ be a ball and $\partial\Delta$ be its boundary complex. Suppose that the sphere $\partial\Delta$ satisfies the assumptions* (A1) *and* (A2)*. Then the Stanley–Reisner ring $A/\partial\Delta$ satisfies the multiplicity conjecture.*

A linear ball is a ball whose Stanley–Reisner ring has a linear resolution. Let Δ be a linear ball and $m-1$ be the smallest dimension of a nonface of Δ and suppose that $2 \leq m \leq (d+1)/2$. It is shown that Δ satisfies (A1) and (A2), hence in particular, the Stanley–Reisner ring of the sphere which is the boundary complex of Δ satisfies the multiplicity conjecture (Corollary 4.1.4).

A class of shellable balls satisfying (A1) and (A2) arises from determinantal ideals. And one of the natural classes of shellable linear balls arises from the polarization of a power of the graded maximal ideal.

The results in Chapter 1, Chapter 3 and Chapter 4 are to be published in [Si07], [MSi07] and [HS98] respectively.

CHAPTER 1

Minimal monomial reductions and the reduced fiber ring of an extremal ideal

Let I be a monomial ideal in a polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K. Let $G(I)$ denote the unique minimal set of monomial generators of I.

In [CHT99] Cutkosky–Herzog–Trung and in [Ko99] independently Kodiyalam showed that for a graded ideal I in a polynomial ring $A = K[x_1, \ldots, x_n]$, the regularity of the ideal I^t is a linear function $pt + c$ for large enough t. Also Kodiyalam showed in [Ko99] that the coefficient p is given by:

$$p = \min\{\theta(J) : J \text{ is a reduction ideal of } I\}.$$

Here $\theta(J)$ denotes the maximum of the degrees of elements in $G(J)$. We call a graded ideal $J \subset I$ to be a *reduction ideal* of I if there exists some integer m such that $JI^m = I^{m+1}$.

In Section 1.2 we give a convex geometric interpretation for this coefficient p for any monomial ideal $I \subset A$: let S be any set of monomials in A. We denote by $\Gamma(S) \subset \mathbb{N}^n$ the set of exponents of the monomials in S. Now let J be the monomial ideal which is determined by the property that $\Gamma(G(J)) = \operatorname{ext}(I)$, where $\operatorname{ext}(I)$ denotes the extreme points of the convex set $\operatorname{conv}(I)$. Here $\operatorname{conv}(I)$ denotes the convex hull of the elements of the set $\Gamma(I)$ in $\mathbb{R}^n$. This convex set is commonly called the *Newton polyhedron* of I. We show in Proposition 1.2.1 that the ideal J is the unique minimal monomial reduction ideal of I, that is, J ia a reduction ideal of I and there exists no proper monomial ideal $L \subset J$ such that L is again a reduction ideal of I. It turns out that $p = \theta(J)$. In other words, $p = \max\{\deg x^a \colon a \in \operatorname{ext}(I)\}$.

We call a graded reduction ideal L of I to be a *Kodiyalam reduction* of I if $\theta(L) = p$. Thus the ideal J generated by monomials whose exponents belong to $\mathrm{ext}(I)$ is a Kodiyalam reduction of I.

We call a monomial ideal L to be an *extremal ideal* if $\Gamma(G(L)) = \mathrm{ext}(L)$. In other words, L is an extremal ideal if L is its own minimal monomial reduction. Notice that each squarefree monomial ideal is an extremal ideal. Let $\mu(L)$ denote the number of generators in a minimal generating set of a graded ideal L. It is easy to see that $\mu(\mathrm{Rad}\, I)$ is bounded above by $|\,\mathrm{ext}(I)|$ for any monomial ideal $I \subset A$.

In Section 1.3 we describe the faces of $\mathrm{conv}(I^m)$ for a monomial ideal I, and compare the supporting hyperplanes and the faces of $\mathrm{conv}(I^{n_1})$ and $\mathrm{conv}(I^{n_2})$ for two positive integers n_1, n_2.

In Section 1.4 we determine the structure of the reduced fiber ring $\mathcal{F}(L)_{\mathrm{red}}$ of an extremal ideal L. For any graded ideal $L \subset A = K[x_1, \ldots, x_n]$, the fiber ring $\mathcal{F}(L)$ is defined to be $\mathcal{R}(L)/\mathfrak{m}\mathcal{R}(L) = \bigoplus_{n\geq 0} L^n/mL^n$ where $\mathcal{R}(L)$ is the Rees ring and $\mathfrak{m} = (x_1, \ldots, x_n) \subset A$ is the graded maximal ideal of A. The main motivation to study the structure of the reduced fiber ring of an extremal ideal is to determine the dimension of the fiber ring of an arbitrary monomial ideal. Let $I \subset A$ be a monomial ideal and $J \subset I$ be its minimal monomial reduction. Then J is an extremal ideal, and $\dim \mathcal{F}(I) = \dim \mathcal{F}(J) = \dim \mathcal{F}(J)_{\mathrm{red}}$. So as far as dimension is concerned it is enough to consider the reduced fiber ring $\mathcal{F}(J)_{\mathrm{red}}$ of the extremal ideal J, whose structure is in general much simpler than that of $\mathcal{F}(J)$.

Let $\mathcal{F}_c$ denote the set of all compact faces of the Newton polyhedron $\mathrm{conv}(I)$ of I. It is shown in Lemma 1.3.1 that for each $F \in \mathcal{F}_c$, we have $F = \mathrm{conv}\{a_{j_1}, \ldots, a_{j_t}\}$ where $F \cap \mathrm{ext}(I) = \{a_{j_1}, \ldots, a_{j_t}\}$. For each $F \in \mathcal{F}_c$, we denote by $K[F]$ the semi–group ring $K[x^{a_j}t \colon a_j \in F]$. As the main result of Section 3 we will show in Theorem 1.4.9 that $\mathcal{F}(J)_{\mathrm{red}} \cong \varprojlim_{F\in\mathcal{F}_c} K[F]$. As an application, we get in the particular case of monomial ideals a result of Carles Bivia–Ausina [Au03] on the analytic spread of a Newton non–degenerate ideal.

Let $\overline{L}$ denote the integral closure of an ideal L. In Section 1.5, using convex geometric arguments, we show in Theorem 1.5.1 that $\overline{I^\ell} = J\overline{I^{\ell-1}}$ where ℓ is the analytic spread of I. If we assume that I^a is integrally closed for $a \leq \ell - 1$, then as a corollary of Theorem 1.5.1, we obtain that $I^\ell = JI^{\ell-1}$, and that I is a normal ideal.

1.1 Some preliminaries on the convex geometry of monomial ideals

Let I be a monomial ideal in a polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K. We denote by $G(I)$ the unique minimal set of monomial generators of I.

For a monomial $u = x^a = x_1^{a(1)} \cdots x_n^{a(n)} \in A$, we denote by $\Gamma(u)$ the exponent vector $(a(1), \ldots, a(n))$ of u. Similarly, if S is any set of monomials in A, we set $\Gamma(S) = \{\Gamma(u) \colon u \in S\}$.

We denote the convex hull of $\Gamma(I)$ by $\text{conv}(I)$. Here $\Gamma(I) = \{a : x^a \in I\}$. Recall that $\text{conv}(I)$ is a polyhedron. A polyhedron can be defined as the intersection of finitely many closed half spaces. A polyhedron may also be thought of as the sum of a polytope (which is the convex hull of a finite set of points) and the positive cone generated by a finite set of vectors. Indeed, these two notions are equivalent, (see [Zi95, Theorem 1.2]).

Suppose that $G(I) = \{x^{a_1}, \dots, x^{a_s}\}$, then

$$\text{conv}(I) = \text{conv}\{a_1, a_2, \dots, a_s\} + \mathbb{R}^n_{\geq 0},$$

see [ReV99, Lemma 4.3]. Here the positive cone $\mathbb{R}^n_{\geq 0}$ denotes the set of vectors $u \in \mathbb{R}^n$ such that $u(i) \geq 0$ for all $i = 1, \dots, n$. It follows that $\text{conv}(I)$ is a polyhedron. It is called the *Newton polyhedron* of I.

Let $H_i = \{v \in \mathbb{R}^n \mid \langle v, u_i \rangle = c_i\}$ where $u_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$ for $i = 1, \dots, m$ be the hyperplanes in $\mathbb{R}^n$ such that $\text{conv}(I) = \{v \in \mathbb{R}^n | \ \langle v, u_i \rangle \geq c_i, \ i = 1, \dots, m\}$. We observe

Lemma 1.1.1. *The vectors u_i belong to $\mathbb{R}^n_{\geq 0}$ for $i = 1, \dots, m$.*

Proof. Let e_j denote the canonical unit vector $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 being at jth place. We prove that $\langle e_j, u_i \rangle = u_i(j) \geq 0$ for all i, j. Let $a \in \Gamma(I)$, then $a + te_j \in \text{conv}(I)$ for all j and $t \in \mathbb{R}_{\geq 0}$. Hence $\langle a + te_j, u_i \rangle \geq c_i$ for all i, j. Suppose $\langle e_{j_0}, u_{i_0} \rangle < 0$ for some j_0 and i_0. Then we have $\langle a + te_{j_0}, u_{i_0} \rangle < c_{i_0}$ for $t \gg 0$, which is a contradiction. □

The zero dimensional faces of a convex set $X \in \mathbb{R}^n$ are called exposed points. A point $a \in X$ is said to be an *extreme point*, provided all $b, c \in X$, $0 < \lambda < 1$, and $a = \lambda b + (1 - \lambda)c$ imply $a = b = c$(see [Gr66]).

We denote the extreme points of $\text{conv}(I)$ by $\text{ext}(I)$ and the exposed points of $\text{conv}(I)$ by $\exp(I)$. We have the following

Proposition 1.1.2. *Let I be a monomial ideal in a polynomial ring $A = K[x_1, \dots, x_n]$ over a field K. Then, $a \in \exp(I)$ implies $x^a \in G(I)$.*

Proof. Let $a \in \exp(I)$ and $H = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\} \subset \mathbb{R}^n$ be a supporting hyperplane of $\text{conv}(I)$ such that $H \cap \text{conv}(I) = \{a\}$. Notice that $u \in (\mathbb{R}_{\geq 0}/\{0\})^n$.

Let $G(I) = \{x^{a_1}, \dots, x^{a_s}\}$. Then $\text{conv}(I) = \text{conv}\{a_1, a_2, \dots, a_s\} + \mathbb{R}^n_{\geq 0}$. Therefore $a = \sum_{i=1}^s k_i a_i + v$ where $\sum_{i=1}^s k_i = 1$, $k_i \geq 0$, $v \in \mathbb{R}^n_{\geq 0}$. Now, since $\langle a_i, u \rangle \geq c$ and $\langle w, u \rangle > 0$ for any $0 \neq w \in \mathbb{R}^n_{\geq 0}$, $\langle a, u \rangle = c$ implies $a = a_i$ for some i. □

Remark 1.1.3. For any closed convex set $X \subset \mathbb{R}^n$, one has $\exp(X) \subset \text{ext}(X)$ and $\text{ext}(X) \subset \text{cl}(\exp(X))$ where $\text{cl}(\exp(X))$ denotes the closure of X in $\mathbb{R}^n$ with respect to usual topology (see [Gr66, Statement 3 and 9, Section 2.4]). In case $X = \text{conv}(I)$, one has $\exp(I)$ is a finite set. Therefore $\text{cl}(\exp(I)) = \exp(I)$, and hence $\exp(I) = \text{ext}(I) \subset \Gamma(G(I))$.

1.2 Minimal monomial reduction ideals

In this section we show that for a monomial ideal $I \subset A = K[x_1, \ldots, x_n]$, there exists a unique minimal monomial reduction ideal J of I. We also show that the minimal monomial reduction ideal J of a monomial ideal I is a Kodiyalam reduction of I.

Let $L \subset A = K[x_1, \ldots, x_n]$ be a graded ideal. A graded ideal $N \subset L$ is said to be a *reduction ideal of* L, if there exists a positive integer m such that $NL^{m-1} = L^m$. Let $\bar{I}$ denote the integral closure of an ideal I. It is known that $N \subset L$ is a reduction ideal of L if and only if $\overline{N} = \overline{L}$ (see [BH96, Exercise 10.2.10(c)]).

Now let $I \subset K[x_1, \ldots, x_n]$ be a monomial ideal. We call a monomial ideal $J \subset I$ a *minimal monomial reduction ideal of* I if there exists no proper monomial ideal $J' \subset J$ such that J' is a reduction ideal of I. For a monomial ideal one has

$$\Gamma(\bar{I}) = \operatorname{conv}(I) \cap \mathbb{N}^n$$

(see [Ei95, Exercise 4.22]). Hence a monomial ideal $J \subset I$ is a reduction ideal of I if and only if $\operatorname{conv}(J) = \operatorname{conv}(I)$. We have the following:

Proposition 1.2.1. *Let I be a monomial ideal in a polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K with $\operatorname{ext}(I) = \{a_1, \ldots, a_r\}$. Then the ideal $J = (x^{a_1}, \ldots, x^{a_r})$ is the unique minimal monomial reduction ideal of I.*

Proof. In order to show that J is a reduction ideal of I, we need to prove that $\operatorname{conv}(I) = \operatorname{conv}(J)$. For any monomial ideal $L \subset A$, we know that

$$\operatorname{conv}(L) = \operatorname{conv}(\Gamma(G(L))) + \mathbb{R}^n_{\geq 0},$$

[ReV99, Lemma 4.3]. Also it follows easily from [Sc98, Section 8.9] that

$$\operatorname{conv}(I) = \operatorname{conv}(\operatorname{ext}(I)) + \mathbb{R}^n_{\geq 0}. \tag{1.1}$$

Now since $\Gamma(G(J)) = \operatorname{ext}(I)$, we get that $\operatorname{conv}(I) = \operatorname{conv}(J)$.

Again, it is also easy to see that J is the unique minimal monomial reduction of I. In fact, let L be any other monomial reduction ideal of I. We show that $J \subset L$. We have $\operatorname{conv}(I) = \operatorname{conv}(L)$, and so $\operatorname{ext}(L) = \operatorname{ext}(I)$. By Lemma 1.1.2, we have $\operatorname{ext}(L) = \exp(L) \subset \Gamma(G(L))$. Hence we get $\Gamma(G(J)) \subset \Gamma(G(L))$. □

For the following corollary, we need to define the notion of a supporting hyperplane and a face of the convex set $\operatorname{conv}(I)$.

We say $H = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$ is a *supporting hyperplane* of $\operatorname{conv}(I)$ if $\operatorname{conv}(I) \subset H_+ = \{v \in \mathbb{R}^n \mid \langle v, u \rangle \geq c\}$ and $\operatorname{conv}(I) \cap H \neq \emptyset$.

A set $F \subset \operatorname{conv}(I)$ is called a *face* of $\operatorname{conv}(I)$, if either $F = \emptyset$ or $F = \operatorname{conv}(I)$ or if there exists a supporting hyperplane H of $\operatorname{conv}(I)$ such that $F = \operatorname{conv}(I) \cap H$. We call F to be a *proper face* of $\operatorname{conv}(I)$ if $F \neq \operatorname{conv}(I)$ and $F \neq \emptyset$.

Let F be a proper face of $\operatorname{conv}(I)$. Let $H = \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ be a supporting hyperplane of $\operatorname{conv}(I)$ such that $F = H \cap \operatorname{conv}(I)$. It may be observed that F is a compact face of $\operatorname{conv}(I)$ if and only if the vector $u \in (\mathbb{R}_{\geq 0}\backslash\{0\})^n$ i.e. $u(j) > 0$ for all $j = 1, \ldots, n$.

For a nonnegative integer m, we denote by $J^{[m]}$ the ideal generated by the monomials $x^{ma_1}, \ldots, x^{ma_r}$. We get the following:

Corollary 1.2.2. *The ideal $J^{[m]}$ is the unique minimal monomial reduction ideal of I^m for all m.*

Proof. Let us fix an m, and denote by J_m the unique monomial reduction ideal of I^m. First notice that $J^{[m]}$ is a monomial reduction ideal of I^m. Indeed, as $J^{[m]}$ is a monomial reduction ideal of J^m and J^m is a monomial reduction ideal of I^m, we have $J^{[m]}$ is a reduction ideal of I^m. Therefore $J_m \subset J^{[m]}$, by Theorem 1.2.1.

Next, we claim that $\operatorname{ext}(I^m) \supset \{ma_1, \ldots, ma_r\}$ which in turn will will imply that $J^{[m]} \subset J_m$, by Theorem 1.2.1. Let $H_i = \{v \in \mathbb{R}^n \mid \langle v, u_i\rangle = c_i\}$ be a supporting hyperplane of $\operatorname{conv}(I)$ such that $H_i \cap \operatorname{conv}(I) = \{a_i\}$ for $i = 1, \ldots, r$. Let mH_i denote the hyperplane given by the set $\{v \in \mathbb{R}^n \mid \langle v, u_i\rangle = mc_i\}$, $i = 1, \ldots, r$. We will show that mH_i is a supporting hyperplane of $\operatorname{conv}(I^m)$ with $mH_i \cap \operatorname{conv}(I^m) = \{ma_i\}$. This will imply the above claim.

It is clear that $ma_i \in mH_i \cap \operatorname{conv}(I^m)$. Now, let $a \in \Gamma(I^m)$ be an arbitrary element. Then $a = \sum_{j=1}^m a_{i_j} + v$ where $a_{i_j} \in \{a_1, \ldots, a_s\}$ and $v \in \mathbb{N}^n$. As the vector $u_i \in (\mathbb{R}_{\geq 0}/\{0\})^n$, it follows that $\langle a, u_i\rangle \geq mc_i$, and is equal to mc_i if and only if $a = ma_i$. □

Let I be a graded ideal in a polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K. The *ith regularity* of an ideal I is defined to be $\operatorname{reg}_i(I) = \max\{j : \operatorname{Tor}_i^A(I, K)_{i+j} \neq 0\}$ and the Castelnuovo–Mumford regularity of I is defined to be $\operatorname{reg}(I) = \max_i\{\operatorname{reg}_i(I)\}$.

Cutkosky–Herzog–Trung [CHT99] and independently Kodiyalam [Ko99] have shown that $\operatorname{reg}(I^t) = pt + c$ for $t \gg 0$. Also the coefficient of the linear function is known and it is given by

$$p = \min\{\theta(J) : J \text{ is a reduction ideal of } I\},$$

see [Ko99]. Here $\theta(J)$ denotes the maximum of the degrees of elements in $G(J)$. We define a graded reduction ideal J of I to be a *Kodiyalam reduction* if $\theta(J) = p$.

More generally, it is shown in [CHT99] that $\operatorname{reg}_i(I^t) = p_i t + q_i$ for $t \gg 0$ are linear functions. From the arguments in Kodiyalam's paper [Ko99] it follows immediately that $p_0 = p$.

Corollary 1.2.3. *Let I be a monomial ideal in $K[x_1, \ldots, x_n]$, then the minimal monomial reduction ideal J of I is a Kodiyalam reduction.*

Proof. The proof is inspired by the arguments in ([Ko99, Proposition 4]). By the very definition of p, we have $\theta(J) \geq p$. We now show that $\theta(J) \leq p$. Since for any monomial reduction ideal L, we have $\Gamma(G(J)) = \operatorname{ext}(I) = \operatorname{ext}(L) \subset \Gamma(G(L))$.

Therefore, it is enough to find a monomial reduction ideal L such that $\theta(L) \leq p$. Notice that $\operatorname{ext}(I) = \operatorname{ext}(L)$, as $L \subset I$ being a reduction ideal of I we have $\operatorname{conv}(I) = \operatorname{conv}(L)$.

Consider the minimal monomial generating system of I, given by $f_1, \ldots, f_s$ where $\deg f_i = d_i$ for all i and $d_1 \leq \cdots \leq d_s$. Let j be the largest integer such that $f_j^k \notin \mathfrak{m}I^k$ for any k where $\mathfrak{m}$ is the maximal graded ideal in A. Then $\operatorname{reg}_0(I^t) \geq d_j t$ for all t. Set $L = (f_1, \ldots, f_j)$ and $P = (f_{j+1}, \ldots, f_s)$. Clearly, L is a monomial ideal with $\theta(L) = d_j$. We claim that L is a reduction ideal of I. By the very choice of j, $P^t \subset \mathfrak{m}I^t$ for some t. Then $I^t = (L+P)^t = L(L+P)^{t-1} + P^t \subset LI^{t-1} + \mathfrak{m}I^t$. Hence by Nakayama's lemma, it follows that L is a reduction ideal of I. Now as $\theta(L) = d_j$ and $d_j t \leq pt + q_0$ for $t \gg 0$. We have $d_j \leq p$. Hence $\theta(L) \leq p$. □

We call a monomial ideal L to be an *extremal ideal*, if $G(L) = \operatorname{ext}(L)$. In other words, a monomial ideal L is an extremal ideal if it is the minimal monomial reduction of itself. In particular, the ideal J in Theorem 1.2.1 is an extremal ideal.

Remarks 1.2.4. 1. Every squarefree monomial ideal is an extremal ideal. Let $N \subset A$ be a squarefree monomial ideal and let $x^a \in G(N)$ be a monomial generator. We show that $a \in \operatorname{ext}(N)$. As N is squarefree, for all i, one has $a(i) = 1$ or $a(i) = 0$. Let $r \leq n$ be the cardinality of i's such that $a(i) = 1$. We define a vector $u \in \mathbb{N}^n$ given by $u(i) = 1$ if $a(i) = 1$ and $u(i) = n+1$ if $a(i) = 0$. We claim that the hyperplane $S = \{v \in \mathbb{R}^n : \langle v, u\rangle = r\}$ is a supporting hyperplane of $\operatorname{conv}(N)$ with $S \cap \operatorname{conv}(N) = \{a\}$, which will imply that $a \in \operatorname{ext}(N)$. Clearly, $S \cap \operatorname{conv}(N) \supset \{a\}$. Let $b \in \operatorname{conv}(N) = \operatorname{conv}(\Gamma(G(N)) + \mathbb{R}^n_{\geq 0}$ with $b \neq a$ be an arbitrary element. We claim that $\langle b, u\rangle > r$. Notice that it is enough to consider $b \in \Gamma(G(N))$. Since $x^b \in G(N)$, we notice that there exists an i such that $b(i) = 1$ and $a(i) = 0$. Hence $\langle b, u\rangle \geq n+1$ and therefore $\langle b, u\rangle > r$. Hence the claim.

Let $\mu(L) = |G(L)|$. We have the following:

2. Let $I \subset A$ be a monomial ideal. Then we have $\mu(\operatorname{Rad} I) \leq |\operatorname{ext}(I)|$. Indeed, let $J \subset I$ be the minimal monomial reduction ideal of I. Then one has $\operatorname{Rad} J = \operatorname{Rad} I$. Hence $\mu(\operatorname{Rad} I) = \mu(\operatorname{Rad} J) \leq \mu(J) = |G(J)| = |\operatorname{ext}(I)|$.

1.3 A description of the faces of $\operatorname{conv}(I^m)$

Let $I = (x^{a_1}, x^{a_2}, \ldots, x^{a_s}) \subset A = K[x_1, \ldots, x_n]$ be a monomial ideal. We may assume that $\operatorname{ext}(I) := \{a_1, \ldots, a_r\}$ is the set of extreme points of the convex hull of I after a proper rearrangement of generators. Then $J = (x^{a_1}, x^{a_2}, \ldots, x^{a_r})$ is the minimal monomial reduction ideal of I, see Theorem 1.2.1.

Next we consider the set of faces of $\operatorname{conv}(I)$. Let $\mathcal{F}$ denote the set of proper faces and $\mathcal{F}_c \subset \mathcal{F}$ denote the set of compact faces of $\operatorname{conv}(I)$. Let $F \in \mathcal{F}$ and $S := \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ be a supporting hyperplane of $\operatorname{conv}(I)$ such that $S \cap \operatorname{conv}(I) = F$. It may be observed that $F \in \mathcal{F}_c$ if and only if the vector $u \in (\mathbb{R}_{\geq 0} \backslash \{0\})^n$. For $j = 1, \ldots, n$, we define $e_j = (0, \ldots, 0, 1, \ldots, 0) \in \mathbb{R}^n$ to be the unit vectors, 1 being at jth place. With this notation, we have

Lemma 1.3.1. *Let $F \in \mathcal{F}$ be a face of* conv(I)*, and let $S = \{v \in \mathbb{R}^n \colon \langle v, u\rangle = c\}$ be a supporting hyperplane of* conv(I) *such that $F = S \cap \operatorname{conv}(I)$. Then $F \cap \operatorname{ext}(I) \neq \emptyset$, and*

$$F = \operatorname{conv}\{a_{j_1}, \ldots, a_{j_t}\} + \sum_{\{j \colon u(j)=0\}} \mathbb{R}_{\geq 0} e_j,$$

where $F \cap \operatorname{ext}(I) = \{a_{j_1}, \ldots, a_{j_t}\}$.

Proof. Let $a \in \operatorname{conv}(I)$. Then $a = \sum_i^r k_i a_i + v$ with $\sum k_i = 1$, $k_i \geq 0$, $v \in \mathbb{R}^n_{\geq 0}$ by Equation 1.1. Suppose $F \cap \operatorname{ext}(I) = \emptyset$. Then $\langle a_i, u\rangle > c$ for all $i = 1, \ldots, r$. Therefore, we have $\langle a, u\rangle > c$. Hence $F = S \cap \operatorname{conv}(I) = \emptyset$, a contradiction.

Let $F \cap \operatorname{ext}(I) = \{a_{j_1}, \ldots, a_{j_t}\}$. First we suppose that F is a compact face, then we have $u \in (\mathbb{R}_{\geq 0} \backslash \{0\})^n$. As $\langle a_i, u\rangle > c$ for all $a_i \in \operatorname{ext}(I) \backslash \{a_{j_1}, \ldots, a_{j_t}\}$ and $\langle v, u\rangle > 0$ for all $0 \neq v \in \mathbb{R}^n_{\geq 0}$, we notice that $\langle a, u\rangle = c$ if and only if $a \in \operatorname{conv}\{a_{j_1}, \ldots, a_{j_t}\}$. Hence $F = \operatorname{conv}\{a_{j_1}, \ldots, a_{j_t}\}$.

Now, let F be an noncompact face and let $Z = \{j : u(j) = 0\}$. Notice that the set $Z \neq \emptyset$. As $\langle a_i, u\rangle > c$ for all $a_i \in \operatorname{ext}(I) \backslash \{a_{j_1}, \ldots, a_{j_t}\}$ and $\langle v, u\rangle \geq 0$ for all $v \in \mathbb{R}^n_{\geq 0}$ with $\langle v, u\rangle = 0$ if and only if $v \in \sum_{j \in Z} \mathbb{R}_{\geq 0} e_j$, we see that $\langle a, u\rangle = c$ if and only if $a \in \operatorname{conv}\{a_{j_1}, \ldots, a_{j_t}\} + \sum_{\{j \colon u(j)=0\}} \mathbb{R}_{\geq 0} e_j$. ☐

As an immediate consequence of Lemma 1.3.1 we obtain

Corollary 1.3.2. *Let $S = \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ be a hyperplane. Then S is a supporting hyperplane of* conv(I) *if and only if $\langle a_i, u\rangle \geq c$ for all $a_i \in \operatorname{ext}(I)$ and $\langle a_j, u\rangle = c$ for some $a_j \in \operatorname{ext}(I)$.*

Lemma 1.3.3. *Let $S = \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ where $u \in \mathbb{R}^n$, $c \in \mathbb{R}$, be a hyperplane, and let $n_1, n_2 \geq 1$ two integers and $q = n_2/n_1$. Then S is a supporting hyperplane of* conv(I^{n_1}) *if and only if $qS = \{v \in \mathbb{R}^n \mid \langle v, u\rangle = qc\}$ is a supporting hyperplane of* conv(I^{n_2})*.*

Proof. By Corollary 1.2.2, we have $\operatorname{ext}(I^m) = (ma_1, \ldots, ma_r)$ for all $m \geq 1$. Hence S is a supporting hyperplane of $\operatorname{conv}(I^{n_1})$ if and only if for all $n_1 a_i \in \operatorname{ext}(I^{n_1})$ we have $\langle n_1 a_i, u\rangle \geq c$ and $\langle n_1 a_j, u\rangle = c$ for some $n_1 a_j \in \operatorname{ext}(I^{n_1})$. This is the case if and only if $\langle n_2 a_i, u\rangle = \langle (n_2/n_1) n_1 a_i, u\rangle = q\langle n_1 a_i, u\rangle \geq qc$ and $\langle n_2 a_j, u\rangle = q\langle n_1 a_j, u\rangle = qc$ which is equivalent to say that qS is a supporting hyperplane of $\operatorname{conv}(I^{n_2})$. ☐

Let $\mathcal{F}$ be the set of proper faces of conv(I). For each $F \in \mathcal{F}$, we choose a hyperplane $S = \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ with $F = S \cap \operatorname{conv}(I)$. Then by Lemma 1.3.3, for a nonnegative integer m, the hyperplane mS is a supporting hyperplane of conv(I^m). We set $mF = mS \cap \operatorname{conv}(I^m)$. It is easy to see that this definition does not depend on the choice of S. Indeed,

$$mF = \operatorname{conv}\{ma_{j_1}, \ldots, ma_{j_t}\} + \sum_{\{j \colon u(j)=0\}} \mathbb{R}_{\geq 0} e_j$$

if $F \cap \operatorname{ext}(I) = \{a_{j_1}, \ldots, a_{j_t}\}$. We denote by $m\mathcal{F}$ the set of proper faces of conv(I^m). As an immediate consequence of Lemma 1.3.3 we get

Corollary 1.3.4. *The map from $\mathcal{F}$ to $m\mathcal{F}$ given by $F \mapsto mF$ is bijective.*

1.4 The structure of the reduced fiber ring of an extremal ideal

The main result of this section is Theorem 1.4.9 which gives us the structure of the reduced fiber ring of an extremal ideal. We proceed gradually towards it preparing the ground to prove it. We will use all the notation from previous section.

Recall that a monomial ideal $L \subset A = K[x_1, \dots, x_n]$ is said to be an extremal ideal if $\Gamma(G(L)) = \operatorname{ext}(L)$. In other words an extremal ideal is the minimal monomial reduction of itself, see Proposition 1.2.1.

The main motivation to study the structure of the reduced fiber ring $\mathcal{F}(J)_{\text{red}}$ of an extremal ideal is to determine the dimension of the fiber ring $\mathcal{F}(I)$ for a monomial ideal I. As one notices that $\dim \mathcal{F}(I) = \dim \mathcal{F}(J) = \dim \mathcal{F}(J)_{\text{red}}$, therefore it is enough to consider the reduced fiber ring $\mathcal{F}(J)_{\text{red}}$ as far as the dimension is concerned. We will see that in general the structure of the reduced fiber ring of an extremal ideal is more simple than that of the original fiber ring.

For the proof of Theorem 1.4.3 we shall need the following:

Lemma 1.4.1. *Let $a = \sum_{i=1}^{r} l_i a_i$ where l_i are nonnegative integers, $\sum l_i = m$ and $\operatorname{ext}(I) = \{a_1, \dots, a_r\}$. If $\{a_i : l_i \neq 0\} \not\subset F$ for some $F \in \mathcal{F}$, then $a \notin mF$.*

Proof. Let $S = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$ be a supporting hyperplane of $\operatorname{conv}(I)$ such that $S \cap \operatorname{conv}(I) = F$. Then $mS = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = mc\}$ is a supporting hyperplane of $\operatorname{conv}(I^m)$ such that $mS \cap \operatorname{conv}(I^m) = mF$.

Suppose that $a \in mF$. Then we have $\langle a, u \rangle = mc$. As $\{a_i : l_i \neq 0\} \not\subset F$, there exists at least one j such that $\langle a_j, u \rangle > c$ which implies $\langle a, u \rangle > mc$, a contradiction. □

Remark 1.4.2. From the above lemma, it follows that if the set $\{a_i : l_i \neq 0\} \not\subset F$ for all $F \in \mathcal{F}$, then $a = \sum_{i=1}^{r} l_i a_i \notin G$ for all $G \in m\mathcal{F}$. Indeed, as for every $G \in m\mathcal{F}$ there exists $F \in \mathcal{F}$ such that $G = mF$, by Corollary 1.3.4.

The following theorem is crucial for our study of the structure of the reduced fiber ring of an extremal ideal.

Theorem 1.4.3. *Let J be an extremal ideal with $G(J) = \{f_1, \dots, f_r\}$ and $f_j = x^{a_j}$ for $j = 1, \dots, r$. Let $Z = \{a_{j_1}, \dots, a_{j_t}\}$ be a nonempty subset of $\Gamma(G(J))$. Then the following conditions are equivalent:*

(1) *$Z \subset F$ for some compact face $F \in \mathcal{F}$;*

(2) *For all $l_i \geq 0$ one has $f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \in G(J^m)$ where $0 < m = \sum_{i=1}^{t} l_i$;*

(3) *For all $l_i \gg 0$ one has $f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \in G(J^m)$ where $0 < m = \sum_{i=1}^{t} l_i$.*

Proof. (1) $\Rightarrow$ (2) Suppose there exists integers $l_i \geq 0$ such that the monomial $f' = f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \notin G(J^m)$ where $m = \sum l_i$. Then there exists a monomial $g \in G(J^m)$ such that $f' = hg$ with $\deg h > 0$. Let $S := \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$ be a supporting hyperplane such that $F = S \cap \mathrm{conv}(J)$. Notice that as F is a compact face, the vector u belongs to $(\mathbb{R}_{\geq 0} \backslash \{0\})^n$. Now since the set $Z \subset F$, $\langle a_{j_k}, u \rangle = c$ for all $k = 1, \ldots, t$. Then we have $\langle \Gamma(f'), u \rangle = mc$, but since $\langle \Gamma(h), u \rangle > 0$ and $\langle \Gamma(g), u \rangle \geq mc$, one has $\langle \Gamma(hg), u \rangle > mc$, a contradiction.

(2) $\Rightarrow$ (3) trivial.

(3) $\Rightarrow$ (1) Suppose $Z \not\subset F$ for all compact faces $F \in \mathcal{F}$, then we prove that for all $l_i \gg 0$ we have $f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \notin G(J^m)$ where $m = \sum_{i=1}^t l_i$.

Let $f = f_{j_1} \cdots f_{j_t}$. We will show that $f^{m_0} = f_{j_1}^{m_0} \cdots f_{j_t}^{m_0} \notin G(J^{m_0 t})$ for some positive integer m_0. From which it clearly follows that $f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \notin G(J^m)$ for all $l_i \geq m_0$ where $m = \sum l_i$.

Notice that in order to show that $f^m \notin G(\overline{J^{mt}})$ for some m, it is enough to show that $f^k \notin G(\overline{J^{kt}})$ for some k. Indeed, if $f^k \notin G(\overline{J^{kt}})$ for some k, then $f^k = gh$ where $h \in G(\overline{J^{kt}})$ and $\deg g > 0$. Now as $h \in G(\overline{J^{kt}})$, $h^{k_1} \in J^{ktk_1}$ for some k_1 which implies $f^{kk_1} = g^{k_1} h^{k_1} \notin G(J^{ktk_1})$.

We assumed that $Z \not\subset F$ for all compact faces $F \in \mathcal{F}$, but nevertheless Z may be a subset of a noncompact face in $\mathcal{F}$. We divide the proof in two cases depending on whether Z is a subset of some noncompact face or not.

Case 1: First we assume that $Z \not\subset F$ for all faces (compact or noncompact) $F \in \mathcal{F}$. Suppose $f^m \in G(\overline{J^{mt}})$ for all m. Without loss of generality, let $x_1 | f$. Since $f \in G(\overline{J^t})$, $g = f/x_1 \notin \overline{J^t}$. Hence $\Gamma(f) \in \mathrm{conv}(J^t)$ and $\Gamma(g) \notin \mathrm{conv}(J^t)$. Let l be the line segment joining $\Gamma(f)$ and $\Gamma(g)$. Since $\Gamma(f) \in l \cap \mathrm{conv}(J^t)$, the intersection of l and $\mathrm{conv}(I)$ is a convex set. Let $l_0 = l \cap \mathrm{conv}(J^t)$ be the line segment joining $\Gamma(f)$ and p with $p \in tF$ where F is a face of $\mathrm{conv}(J)$, see Corollary 1.3.4. Notice that $p \neq \Gamma(f)$, see Remark 1.4.2. Hence $\Gamma(f) = p + v$ where $0 < \|v\| < 1$. Now for any m, consider the line segment joining $\Gamma(f^m)$ and $\Gamma(g^m)$, we denote this line segment by ml. We have $\Gamma(f^m) = mp + mv$ where $mp \in mtF$ and mtF is a face of $\mathrm{conv}(J^{mt})$. Again as $f^m \in G(\overline{J^{mt}})$, $f^m/x_1 \notin \overline{J^{mt}}$. Notice that $\Gamma(f^m/x_1)$ and mp lie on ml, and since $\Gamma(f^m/x_1) \notin \mathrm{conv}(J^{mt})$ and $mp \in \mathrm{conv}(J^{mt})$, we have $\|mv\| = \|mp - \Gamma(f^m)\| \leq \|\Gamma(f^m) - \Gamma(f^m/x_1)\| = 1$ for all m, a contradiction.

Case 2: Now assume that $Z \subset G$ for some noncompact face $G \in \mathcal{F}$ and that $\{a_{j_1}, \ldots, a_{j_t}\} \not\subset F$ for all compact faces $F \in \mathcal{F}$. We prove that $f^m \notin G(\overline{J^{mt}})$ for some $m = m_0$ by induction on $\dim G$. If $\dim G = 1$, then $f \notin G(\overline{J^t})$, because it follows from Lemma 1.3.1 that the only point on tG which corresponds to a generator of $\overline{J^t}$, is an extremal point of $\mathrm{conv}(J^t)$ and certainly $a = a_{j_1} + \cdots + a_{j_t}$ is not an extremal point of $\mathrm{conv}(J^t)$, see Corollary 1.2.2. Now let $\dim G = p > 1$. We may assume that $\{a_{j_1}, \ldots, a_{j_t}\} \not\subset G'$ for any proper face G' of G. As if $\{a_{j_1}, \ldots, a_{j_t}\} \subset G'$ for some proper face G' of G, then G' is a noncompact face of G with $\dim G' < \dim G$ and we are through by induction.

Let $S := \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$ be a supporting hyperplane of $\mathrm{conv}(J)$ such that $S \cap \mathrm{conv}(J) = G$. Since G is a noncompact face, there exists j such that $u(j) = 0$.

Consider $a_\lambda := a_{j_1} + \cdots + a_{j_t} - \lambda(0, \ldots, 1, \ldots, 0)$, 1 being at jth place, $\lambda \geq 0$. Notice that there exists $\lambda_0 > 0$ such that $a_{\lambda_0} \notin \operatorname{conv}(J)$. Let l_0 be the line segment joining a and a_{λ_0}. As $a \in l_0 \cap tG$, the intersection of l_0 with tG is a nonempty convex set. Let $l = l_0 \cap tG$ be the line segment joining a and $a_{\lambda'}$ where $a_{\lambda'}$ lies on some proper face tG' of tG and $\lambda' > 0$, as $\dim G' < \dim G$. Also $a_{\lambda'} < a$, so we have $a = a_{\lambda'} + w$, with $\|w\| = \lambda' > 0$. For any positive integer m, $ma_{\lambda'} \in mtG'$ and $\|ma - ma_{\lambda'}\| = m\|a - a_{\lambda'}\| = m\|w\| > 0$. Let for $m = m_0$, $m\|w\| \geq 1$. Then for $m = m_0$, ma and $ma - (0, \ldots, 1, \ldots, 0)$ lies on mtG, 1 being at jth place, so that $\Gamma(f^m/x_j) \in mtG$ which implies $f^m/x_j \in \overline{J^{mt}}$ and hence $f^m \notin G(\overline{J^{mt}})$ for $m = m_0$. □

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_r]$ be a bigraded polynomial ring over a field K with $\deg x_i = (1, 0)$ and $\deg y_j = (d_j, 1)$. Let φ be the surjective homomorphism from S to $\mathcal{R}(J) = K[x_1, \ldots, x_n, f_1t, \ldots, f_rt]$, given by $x_i \mapsto x_i$ and $y_j \mapsto f_jt$ so that $S/L \cong \mathcal{R}(J)$ where $L = \operatorname{Ker}\varphi$. Notice that the ideal L is generated by the binomials of type $g_1h_1 - g_2h_2$ where g_1, g_2 are monomials in x_i and h_1, h_2 are monomials in y_j.

Now, we consider the fiber ring $\mathcal{F}(J) = \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J)$ of the extremal ideal J where $\mathfrak{m} = (x_1, \ldots, x_n) \subset A$. Then we have $\mathcal{F}(J) \cong S/(L, \mathfrak{m}) \cong T/D$ and hence $\mathcal{F}(J)_{\text{red}} \cong T/\operatorname{Rad} D$ where D is the image of the ideal L in $T = S/\mathfrak{m}$, and $T = K[y_1, \ldots, y_r]$. Let $\psi = \varphi \otimes S/\mathfrak{m}\colon T \to \mathcal{F}(J)$ be the induced epimorphism. We have $D = \operatorname{Ker}\psi$. Notice that the ideal D is generated by monomials and homogeneous binomials in the y_j. In fact, if $g_1h_1 - g_2h_2$ is a generator of L, then its image in T is a monomial, if one of the g_i belongs to $\mathfrak{m}$, and otherwise it is a homogeneous binomial. We have the following lemma:

Lemma 1.4.4. *Let $b = b_1 - b_2$ be a homogeneous binomial generator of the ideal D with $b_1 = y_{i_1}^{l_1} \cdots y_{i_u}^{l_u}$, $b_2 = y_{j_1}^{m_1} \cdots y_{j_v}^{m_v}$ and $\sum_{i=1}^u l_i = \sum_{j=1}^v m_j = t$. If the set $\{a_{i_1}, \ldots, a_{i_u}\} \subset G$ for some $G \in \mathcal{F}_c$, then the set $\{a_{j_1}, \ldots, a_{j_v}\} \subset G$.*

Proof. As $b = b_1 - b_2 \in D$, we have $\psi(b) = 0$, i.e. $\psi(b_1) = \psi(b_2)$. Therefore we have $x^{l_1 a_{i_1}} \cdots x^{l_u a_{i_u}} = x^{m_1 a_{j_1}} \cdots x^{m_v a_{j_v}}$ with $\sum_{p=1}^u l_p a_{i_p} = \sum_{k=1}^v m_k a_{j_k}$. Let S be a supporting hyperplane of $\operatorname{conv}(J)$ such that $S \cap \operatorname{conv}(J) = G$ and let S be given by the set $\{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ for some $u \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

We have $\langle \sum_{k=1}^v m_k a_{j_k}, u\rangle = \langle \sum_{p=1}^u l_p a_{i_p}, u\rangle = tc$. Suppose $\{a_{j_1}, \ldots, a_{j_v}\} \not\subset G$, then there exists at least one $k_0 \in \{1, \ldots, v\}$ such that $a_{j_{k_0}} \notin G$. Since $\langle a_{j_k}, u\rangle \geq c$ for all k, it follows that $\langle a_{j_{k_0}}, u\rangle > c$ which in turn implies that $\langle \sum_{k=1}^v l_k a_{j_k}, u\rangle > tc$, a contradiction. □

We denote by $\mathcal{F}_c$ the set of compact faces, and by $\mathcal{F}_{mc}$ the set of maximal compact faces of $\operatorname{conv}(J)$. Let $F \in \mathcal{F}_{mc}$; we set $P_F = (y_j : a_j \notin F)$ and we denote by B_F the kernel of $\theta_F : K[y_j : a_j \in F] \to K[F] := K[f_jt : a_j \in F]$ where $\theta_F(y_j) = f_jt$. In the following proposition we consider the ideals P_F and B_F in the polynomial ring T. With the notation introduced we have

Proposition 1.4.5. $\operatorname{Rad} D = (\bigcap_{F \in \mathcal{F}_{mc}} P_F, \sum_{F \in \mathcal{F}_{mc}} B_F) = \bigcap_{F \in \mathcal{F}_{mc}} (P_F, B_F)$.

Proof. For the proof we proceed in several steps.

1. Step: Let f be a monomial in T. We claim that $f \in \operatorname{Rad} D$ if and only if $f \in \bigcap_{F \in \mathcal{F}_{mc}} P_F$.

We may assume that f is a squarefree monomial. Let $f = y_{j_1} \ldots y_{j_k} \in \operatorname{Rad} D$ with $j_1 < j_2 < \cdots < j_k$. Then $f^{n_0} \in D$ for some integer n_0, and hence $\psi(f^{n_0}) = 0$. This implies that $x^{n_0 a_{j1}} \cdots x^{n_0 a_{jk}} \in \mathfrak{m} J^{n_0 k}$. Hence $x^{n a_{j1}} \cdots x^{n a_{jk}}$ is not a minimal generator of J^{nk} for all $n \geq n_0$. Now Theorem 1.4.3 implies that $\{a_{j1}, \ldots, a_{jk}\} \not\subset F$ for all compact face $F \in \mathcal{F}$. This shows that $f \in \bigcap_{F \in \mathcal{F}_{mc}} P_F$.

Now, let the monomial $f \in \bigcap_{F \in \mathcal{F}_{mc}} P_F$. Then $\{a_{j_1}, \ldots, a_{j_k}\} \not\subset F$ for all $F \in \mathcal{F}_{mc}$. This implies that $\{a_{j_1}, \ldots, a_{j_k}\} \not\subset F$ for all compact faces F of $\operatorname{conv}(J)$. Now by Theorem 1.4.3, we conclude that there exists an integer m such that the monomial $(x^{a_{j_1}} \cdots x^{a_{j_k}})^m \in \mathfrak{m} J^{km}$. Since $\psi(f^m) = (x^{a_{j_1}} \cdots x^{a_{j_k}})^m$ it follows that $f^m \in D$, and hence $f \in \operatorname{Rad} D$.

2. Step: $D \subset (\bigcap_{F \in \mathcal{F}_{mc}} P_F, \sum_{F \in \mathcal{F}_{mc}} B_F)$.

It follows from the first step that all monomial generators in D belong to the ideal $(\bigcap_{F \in \mathcal{F}_{mc}} P_F, \sum_{F \in \mathcal{F}_{mc}} B_F)$. Now let $b = b_1 - b_2$ be a homogeneous binomial generator of D with $b_1 = y_{i_1}^{l_1} \cdots y_{i_u}^{l_u}$, $b_2 = y_{j_1}^{m_1} \cdots y_{j_v}^{m_v}$ and $\sum_{i=1}^{u} l_i = \sum_{j=1}^{v} m_j = t$. Since $b \in D$, we have $\psi(b) = 0$, i.e. $\psi(b_1) = \psi(b_2)$. Therefore we get that the monomial $x^{l_1 a_{i_1}} \cdots x^{l_u a_{i_u}} = x^{m_1 a_{j_1}} \cdots x^{m_v a_{j_v}}$, and so $\sum_{p=1}^{u} l_p a_{i_p} = \sum_{k=1}^{v} m_k a_{j_k}$. We show that $b \in \sum_{F \in \mathcal{F}_{mc}} B_F$, if $b \notin \bigcap_{F \in \mathcal{F}_{mc}} P_F$. In fact, if $b \notin \bigcap_{F \in \mathcal{F}_{mc}} P_F$, then one of the b_i, say $b_1 \notin \bigcap_{F \in \mathcal{F}_{mc}} P_F$. This implies that $\{a_{i_1}, \ldots, a_{i_u}\} \subset G$ for some compact face $G \in \mathcal{F}_{mc}$ and then from Lemma 1.4.4, $\{a_{j_1}, \ldots, a_{j_v}\} \subset G$. Hence $b = b_1 - b_2 \in B_G$.

3. Step: $\sum_{F \in \mathcal{F}_{mc}} B_F \subset D$.

Notice that $B_F = \operatorname{Ker} \theta_F$ and $D = \operatorname{Ker} \psi$. Certainly, $\operatorname{Ker} \theta_F \subset \operatorname{Ker} \psi$ for each $F \in \mathcal{F}_{mc}$. Hence $\sum_{F \in \mathcal{F}_{mc}} B_F \subset D$.

4. Step: $\bigcap_{F \in \mathcal{F}_{mc}} (P_F, B_F) = (\bigcap_{F \in \mathcal{F}_{mc}} P_F, \sum_{F \in \mathcal{F}_{mc}} B_F)$.

For each $F \in \mathcal{F}_{mc}$, let $Q_F = (P_F, B_F)$ and let $M = \bigcap_{F \in \mathcal{F}_{mc}} P_F$, $B = \sum_{F \in \mathcal{F}_{mc}} B_F$. In order to show that $(M, B) = \bigcap_{F \in \mathcal{F}_{mc}} Q_F$, we proceed in the following steps:

(i) First we show $(M, B) \subset \bigcap_{F \in \mathcal{F}_{mc}} Q_F$. Clearly, for each $F \in \mathcal{F}_{mc}$, $M \subset Q_F$. Now we also prove that $B \subset Q_F$ for all $F \in \mathcal{F}_{mc}$. Take $b = b_1 - b_2 \in B$ with $b_1 = y_{i_1}^{l_1} \cdots y_{i_u}^{l_u}$, $b_2 = y_{j_1}^{m_1} \cdots y_{j_v}^{m_v}$ and $\sum_{i=1}^{u} l_i = \sum_{j=1}^{v} m_j = t$. Suppose that $b \notin B_F$, then we prove $b \in P_F$. As $b \notin B_F$, it implies that for one of the b_i, say for b_1, there exists $y_{i_p} | b_1$ such that $a_{i_p} \notin F$. Once we show that there exists also some $k \in \{1, \ldots, v\}$ such that $y_{j_k} | b_2$ and $a_{j_k} \notin F$, then it will imply that $b_1, b_2 \in P_F$ and hence $b \in P_F$. Suppose this is not the case, then $\{a_{j_1}, \ldots, a_{j_v}\} \subset F$. But then by Lemma 1.4.4, we have $\{a_{i_1}, \ldots, a_{i_v}\} \subset F$ which is a contradiction. Hence we have $(M, B) \subset \bigcap_{F \in \mathcal{F}_{mc}} Q_F$.

(ii) Notice that for each $F \in \mathcal{F}_{mc}$, Q_F is a prime ideal. Indeed, Q_F being the kernel of the surjective map $\pi_F : K[y_1, \ldots, y_r] \to K[f_i t : a_i \in F]$ given by $\pi_F(y_j) = f_j t$, if $a_j \in F$ and $\pi_F(y_j) = 0$, if $a_j \notin F$, the assertion follows.

(iii) We claim that $\{Q_F : F \in \mathcal{F}_{mc}\}$ is the set of all minimal prime ideals containing (M, B). Let P be a prime ideal containing (M, B), then it implies that $P \supset M = \bigcap_{\mathcal{F}_{mc}} P_F$ and so $P \supset P_G$ for some $G \in \mathcal{F}_{mc}$. Also, $P \supset B = \sum B_F$. Hence $P \supset Q_G$.

(iv) We claim (M, B) is a radical ideal, that is, $\mathrm{Rad}(M, B) = (M, B)$. This amounts to prove that for all Q_F, $(M, B)T_{Q_F} = Q_F T_{Q_F}$. Fix $G \in \mathcal{F}_{mc}$, the set $\{y_i : a_i \in G\} \subset T \backslash Q_G$ and hence all y_i such that $a_i \in G$ are invertible in T_{Q_G}. For all maximal compact faces $F \neq G$ there exists at least one $y_j \in P_F$ such that $y_j \in G$, as otherwise $P_F \subset P_G$ which implies $F \supset G$, a contradiction. Hence for all $F \neq G$, $P_F T_{Q_G} = T_{Q_G}$. Therefore, we have $(M, B)T_{Q_G} = (\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F)T_{Q_G} = (P_G, \sum_{F\in\mathcal{F}_{mc}} B_F)T_{Q_G} = (P_G, B_G)T_{Q_G} = Q_G T_{Q_G}$.

Since by (iii) we have $\mathrm{Rad}(M, B) = \bigcap_{F\in\mathcal{F}_{mc}} Q_F$ it follows then that the ideal $(M, B) = \bigcap_{F\in\mathcal{F}_{mc}} Q_F$. Now by Step 1, Step 2 and Step 3, one has

$$D \subset \Big(\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F\Big) \subset \mathrm{Rad}\, D.$$

Finally by Step 4, we have $(\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F) = \bigcap_{F\in\mathcal{F}_{mc}}(P_F, B_F)$ which is a radical ideal. Hence we have $\mathrm{Rad}\, D = (\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F) = \bigcap_{F\in\mathcal{F}_{mc}}(P_F, B_F)$. □

We denote by $\mathrm{Min}(R)$ the set of the minimal prime ideals of a ring R.

Corollary 1.4.6. *Let $I \subset A$ be a monomial ideal. Then there is an injective map*

$$\mathcal{F}_{mc} \to \mathrm{Min}(\mathcal{F}(I)).$$

This map is bijective if I is an extremal ideal.

Proof. Let J be the minimal monomial reduction ideal of I. Then J is an extremal ideal. From above proposition we have $\mathcal{F}(J)_{\mathrm{red}} \cong T/\bigcap_{F\in\mathcal{F}_{mc}}(P_F, B_F)$ where (P_F, B_F) is a prime ideal for each $F \in \mathcal{F}_{mc}$. Hence there is a bijective map

$$\rho_1 \colon \mathcal{F}_{mc} \to \mathrm{Min}(\mathcal{F}(J))$$

given by $F \mapsto (P_F, B_F)/D$.

As the fiber ring $\mathcal{F}(I)$ is integral over $\mathcal{F}(J)$, for each prime ideal $P \in \mathrm{Min}(\mathcal{F}(J))$ there exists a minimal prime ideal $Q \in \mathrm{Min}(\mathcal{F}(I))$ such that $P = Q \cap \mathcal{F}(J)$. Therefore there exists an injective map ρ_2 from $\mathrm{Min}(\mathcal{F}(J))$ to $\mathrm{Min}(\mathcal{F}(I))$, and hence $\rho = \rho_2 \circ \rho_1 \colon \mathcal{F}_{mc} \to \mathrm{Min}(\mathcal{F}(I))$ is the desired injective map. Finally, if I is extremal, then $I = J$ and $\rho = \rho_1$ is a bijection. □

Next corollary gives us a combinatorial characterization of the fact when the fiber ring of an extremal ideal J is a domain.

Corollary 1.4.7. *Let $J = (x^{a_1}, \ldots, x^{a_r})$ be an extremal ideal. Then the following conditions are equivalent:*

(1) *The fiber ring* $\mathcal{F}(J)$ *is a domain;*

(2) *The reduced fiber ring* $\mathcal{F}(J)_{\mathrm{red}}$ *is a domain;*

(3) $|\mathcal{F}_{mc}| = 1$.

Proof. (1) $\Rightarrow$ (2) is obvious, and (2) $\iff$ (3) follows from Corollary 1.4.6.
(3) $\Rightarrow$ (1): Let $|\mathcal{F}_{mc}| = 1$. Then it follows by Proposition 1.4.5 that the ideal $\operatorname{Rad} D = (B_F, P_F)$ where $F \in \mathcal{F}_{mc}$. Notice that as there is only one maximal compact face F, the ideal P_F is the zero ideal. Hence $(P_F, B_F) = B_F$. Also by Step 3 in the proof of Proposition 1.4.5 we have $B_F \subset D$. Therefore we have $\operatorname{Rad} D = D = B_F$ which is a prime ideal. Hence $\mathcal{F}(J) \cong T/D$ is a domain. □

By the above corollary the fiber ring of an extremal ideal J is a domain if and only if there is only one maximal compact faces of $\operatorname{conv}(J)$. But in general the property of being reduced cannot be characterized in terms of combinatorial properties of $\operatorname{conv}(J)$, as the the following simple example demonstrates:

Example 1.4.8. Consider the two extremal ideals given by $J_1 = (x^6, x^2y, xy^2, y^6)$ and $J_2 = (x^8, x^6y, x^2y^7, y^{12})$ in the polynomial ring $A = K[x, y]$. It is easy to see that $\operatorname{conv}(J_1)$ and $\operatorname{conv}(J_2)$ have the same face lattices. Nevertheless the fiber ring of the ideal J_1 given by $\mathcal{F}(J_1) \cong K[y_1, y_2, y_3, y_4]/(y_1y_4, y_2y_4, y_1y_3)$ is reduced while the fiber ring of the ideal J_2 given by $\mathcal{F}(J_2) \cong K[y_1, y_2, y_3, y_4]/(y_1y_4, y_2y_4^2, y_2^2y_4 - y_1y_3^2, y_1^2y_3)$ is not reduced.

Next, we define an inverse system of semigroup rings $K[F]$ for $F \in \mathcal{F}_c$ where $K[F] = K[f_it\colon a_i \in F]$ with $f_i = x^{a_i}$. For $G \subset F$, we define the ring homomorphism $\pi_{GF} : K[F] \to K[G]$ such that $\pi_{GF}(f_it) = f_it$, if $a_i \in G$ and $\pi_{GF}(f_it) = 0$, otherwise. Notice that π_{GF} is well defined. To see this, we need to show that if the monomial $f_{i_1}f_{i_2}\cdots f_{i_k}t^k = f_{j_1}f_{j_2}\cdots f_{j_k}t^k$ where $\{a_{i_1}, \ldots, a_{i_k}\}, \{a_{j_1}, \ldots, a_{j_k}\} \subset F$, then $\pi_{GF}(f_{i_1}f_{i_2}\cdots f_{i_k}t^k) = \pi_{GF}(f_{j_1}f_{j_2}\cdots f_{j_k}t^k)$. If $\pi_{GF}(f_{i_1}\cdots f_{i_k}t^k) = 0$, then $\{a_{i_1}, \ldots, a_{i_k}\} \not\subset G$. Since $y_{i_1}\cdots y_{i_k} - y_{j_1}\cdots y_{j_k} \in D$ it follows from Lemma 1.4.4 that $\{a_{j_1}, \ldots, a_{j_k}\} \not\subset G$, too. Hence $\pi_{GF}(f_{j_1}\cdots f_{j_k}t^k) = 0$. On the other hand, if $\pi_{GF}(f_{i_1}\cdots f_{i_k}t^k) \neq 0$, then $\pi_{GF}(f_{j_1}\cdots f_{j_k}t^k) \neq 0$, and so

$$\pi_{GF}(f_{i_1}\cdots f_{i_k}t^k) = f_{i_1}\cdots f_{i_k}t^k = f_{j_1}\cdots f_{i_k}t^k = \pi_{GF}(f_{j_1}\cdots f_{i_k}t^k).$$

Hence $\pi_{GF}(f_{i_1}\cdots f_{i_k}t^k) = \pi_{GF}(f_{j_1}\cdots f_{j_k}t^k)$ in both cases.

Also we may notice that for $H \subset G \subset F$ and $F \in \mathcal{F}_c$, one has $\pi_{HG} \circ \pi_{GF} = \pi_{HF}$. Hence the inverse system is well defined.

Theorem 1.4.9. $F(J)_{\mathrm{red}} \cong \varprojlim_{F \in \mathcal{F}_c} K[F]$.

Proof. For each $F \in \mathcal{F}_c$ consider the ring homomorphism π_F from $K[y_1, \ldots, y_r]$ to $K[F]$ given by $\pi_F(y_j) = f_jt$, if $a_j \in F$ and $\pi_F(y_j) = 0$, if $a_j \notin F$.

Notice that $\operatorname{Ker}\pi_F$ is equal to the ideal $Q_F := (B_F, P_F)$. We define the map

$$\pi : K[y_1, \ldots y_r] \longrightarrow \bigoplus_{F\in\mathcal{F}_c} K[F],$$

given by $\pi = (\pi_F)_{F\in\mathcal{F}_c}$. We have $\operatorname{Ker}\pi = \bigcap_{F\in\mathcal{F}_c} Q_F = \bigcap_{F\in\mathcal{F}_c}(B_F, P_F)$. We claim that for all $G \subset F$ one has $Q_F \subset Q_G$. Indeed, for all $G \subset F$, $P_F \subset P_G$ and by the proof of Proposition 1.4.5, Step 4(i), we have $B_F \subset (B_G, P_G)$. It follows that

$$\operatorname{Ker}\pi = \bigcap_{F\in\mathcal{F}_{mc}} Q_F.$$

Therefore, Proposition 1.4.5 implies that $\operatorname{Ker}\pi = \operatorname{Rad} D$. Thus we have

$$K[y_1, \ldots, y_r]/\operatorname{Ker}\pi \cong F(J)_{\text{red}}.$$

It remains to show that $\operatorname{Im}(\pi) = \varprojlim_{F\in\mathcal{F}_c} K[F]$. Since $\pi_{GF} \circ \pi_F = \pi_G$ for all $G \subset F$, we have $\operatorname{Im}(\pi) \subset \varprojlim_{F\in\mathcal{F}_c} K[F]$.

Now let $v = (m_F)_{F\in\mathcal{F}_c} \in \varprojlim_{F\in\mathcal{F}_c} K[F]$. We may assume that for each $F \in \mathcal{F}_c$, the element m_F is a monomial in $K[F]$ since all homomorphisms in the inverse system are multigraded. For each $F \in \mathcal{F}_c$, we choose $g_F \in K[y_1, \ldots, y_r]$ such that $\pi_F(g_F) = m_F$ and with the property that whenever $m_F = m_G$ in $K[x_1, \ldots, x_n, t]$ then it implies $g_F = g_G$. (Notice that for each $F \in \mathcal{F}$, the K–algebra K[F] can be naturally embedded in the K–algebra $K[x_1, \ldots, x_n, t]$).

Let $Z = \{m_F : m_F \neq 0, F \in \mathcal{F}_c\} = \{m_1, \ldots, m_l\}$. For each $i = 1, \ldots, l$, we define the set $A_i = \{F \in \mathcal{F}_c : m_F = m_i\}$. We claim that for each A_i one has $\bigcap_{F\in A_i} F \in A_i$. Fix an i, and notice that it is enough to show that for any $F, G \in A_i$ we have $F \cap G \in A_i$. Let $m_F = f_{i_1} \cdots f_{i_p} t^p = f_{j_1} \cdots f_{j_p} t^p = m_G$. Then it follows by Lemma 1.4.4 that the sets $\{a_{i_1}, \ldots, a_{i_p}\}, \{a_{j_1}, \ldots, a_{j_p}\} \subset F \cap G = H$. Therefore $\pi_{HF}(m_F) = m_F$ and $\pi_{HG}(m_G) = m_G$. Also as $v = (m_F)_{F\in\mathcal{F}_c} \in \varprojlim_{F\in\mathcal{F}_c} K[F]$ we have $\pi_{HF}(m_F) = m_H = \pi_{HG}(m_G)$. Hence $m_G = m_F = m_H$, so $H \in A_i$. Hence $H_i = \bigcap_{F\in A_i} F \in A_i$, $i = 1, \ldots, l$.

For each i, we choose a monomial $g_{H_i} \in K[y_1, \ldots, y_r]$ such that $\pi_{H_i}(g_{H_i}) = m_{H_i}$. For all $F \in A_i$, we define $g_F = g_{H_i}$, $i = 1, \ldots, l$ and for all $F \in \mathcal{F}_c \setminus \bigcup_{i=1}^{l} A_i$, we define $g_F = 0$. Notice that for all $F \in \mathcal{F}_c$, we have $\pi_F(g_F) = m_F$. Indeed, let $F \in \mathcal{F}_c$. If $F \in \mathcal{F}_c \setminus \bigcup_{i=1}^{l} A_i$, then $g_F = 0 = m_F$ and we have $\pi_F(g_F) = m_F$. If $F \in A_i$ for some i, then as we have $\pi_{H_iF} \circ \pi_F = \pi_{H_i}$ and $\pi_{H_i}(g_F) = m_{H_i} = m_F$, it follows by the very definition of the map π_{H_iF} that $\pi_F(g_F) = m_F$. Moreover, by our choice of the g_F we also have $g_F = g_G$ whenever $m_F = m_G$.

Now let $S = \{g_F : F \in \mathcal{F}_{mc}\}$, and let $g = \sum_{g_F\in S} g_F$. We claim that $\pi(g) = v$, i.e. $\pi_G(g) = m_G$ for all $G \in \mathcal{F}_c$. Notice that it is enough to show that $\pi_G(g) = m_G$ for all $G \in \mathcal{F}_{mc}$. In fact, if $H \in \mathcal{F}_c$ there exists $G \in \mathcal{F}_{mc}$ such that $H \subset G$, and since $\pi_G(g) = m_G$, we have $\pi_H(g) = \pi_{HG}(\pi_G(g)) = \pi_{HG}(m_G) = m_H$.

Now let $G \in \mathcal{F}_{mc}$. We claim that $\pi_G(g_F) = 0$ for all $g_F \neq g_G$, so that we have $\pi_G(g) = m_G$, as asserted.

To prove this claim, let $g_F = y_{i_1} \cdots y_{i_p}$ and suppose that $\pi_G(g_F) \neq 0$. Then we have $\{a_{i_1}, \ldots, a_{i_p}\} \subset G \cap F$. Let $H = G \cap F$, then $H \in \mathcal{F}_c$. Since $v \in \varprojlim_{F \in \mathcal{F}_c} K[F]$ and H is a common face of F and G, we have $\pi_{HF}(m_F) = m_H = \pi_{HG}(m_G)$. As $\{a_{i_1}, \ldots, a_{i_p}\} \subset H$, we have $0 \neq m_F = \pi_{HF}(m_F) = m_H = \pi_{HG}(m_G) = m_G$. Hence $g_F = g_G$, a contradiction. □

The analytic spread ℓ of an ideal I in a Noetherian local ring $(R, \mathfrak{m})$ is given by the Krull dimension of the fiber ring $\mathcal{F}(I)$ of I. It has been shown by Carles Bivia–Ausina [Au03] that the analytic spread of a non–degenerate ideal $I \subset \mathbb{C}[[x_1, \ldots, x_n]]$ is equal to the $c(I) + 1$ where

$$c(I) = \max\{\dim F : F \text{ is a compact face of} \operatorname{conv}(I)\}.$$

Next we show that for monomial ideals this result is an immediate consequence of our structure theorem (Theorem 1.4.9).

Corollary 1.4.10. *Let $I \subset A = K[x_1, \ldots, x_n]$ be a monomial ideal. Let ℓ denote the analytic spread of the ideal I. Then*

$$\ell = c(I) + 1 = \max\{\dim F : F \textit{ is a compact face of } \operatorname{conv}(I)\} + 1.$$

Proof. Let J be the minimal monomial reduction ideal of the monomial ideal I. We have $\ell = \dim \mathcal{F}(I) = \dim \mathcal{F}(J) = \dim \mathcal{F}(J)_{\text{red}}$. By Theorem 1.4.9, we have $\mathcal{F}(J)_{\text{red}} = \varprojlim_{F \in \mathcal{F}_c} K[F] \subset \bigoplus_{F \in \mathcal{F}_c} K[F]$. Hence it follows that

$$\dim(\mathcal{F}(J)) \leq \max\{\dim K[F] : F \in \mathcal{F}_c\}.$$

As $\dim K[F] = \dim F + 1$, we have $\ell \leq c(I) + 1$. To show $\ell \geq c(I) + 1$, we notice that the canonical homomorphisms

$$\bar{\pi}_G \colon \varprojlim_{F \in \mathcal{F}_c} K[F] \to K[G]$$

are surjective for all $G \in \mathcal{F}_c$. Indeed, if m is a monomial in the semigroup ring $K[G]$ and $v = (m_F)_{F \in \mathcal{F}_c} \in \varprojlim_{F \in \mathcal{F}_c} K[F]$ with

$$m_F = \begin{cases} m, & \text{if } \operatorname{supp}(m) \subset F, \\ 0, & \text{if } \operatorname{supp}(m) \not\subset F, \end{cases}$$

then $\bar{\pi}_F(v) = m$. Here $\operatorname{supp}(m)$ of some monomial $m = x_1^{a_1} \cdots x_n^{a_n} \in A$ is defined to be $\operatorname{supp}(m) = \{a_i : a_i \neq 0\}$.
It follows that $\dim F(J) \geq \dim K[F]$ for all $F \in \mathcal{F}_{mc}$. Therefore we have $\ell \geq c(I)+1$, as desired. □

1.5 On the reduction number of a monomial ideal

In this section, we consider the reduction number of a monomial ideal $I \subset A$ with respect to its minimal monomial reduction ideal J. We show in Corollary 1.5.3 that if I^m is integrally closed for $m \leq \ell$ then I is normal and the reduction number of I with respect to J is less than $\ell - 1$. Here ℓ denotes the analytic spread of the monomial ideal I and the reduction number of an ideal I with respect to J is defined to be the minimum of t such that $JI^t = I^{t+1}$.

Theorem 1.5.1. *Let $I \subset A = K[x_1, \ldots, x_n]$ be a monomial ideal and J its minimal monomial reduction ideal. Let ℓ be the analytic spread of I. Then*

$$\overline{I^m} = J\overline{I^{m-1}} \quad \textit{for all} \quad m \geq \ell.$$

Proof. We may assume I is a proper ideal, and let $I = (x^{a_1}, x^{a_2}, \ldots, x^{a_s})$ where $f_i = x^{a_i} = x_1^{a_i(1)} x_2^{a_i(2)} \cdots x_n^{a_i(n)}$ for $i = 1, \ldots, s$. Without loss of generality, let $J = (x^{a_1}, x^{a_2}, \ldots, x^{a_r})$ be the minimal monomial reduction ideal of I so that we have $\text{ext}(I) = \{a_1, \ldots, a_r\}$. Let $m \geq \ell$, we show $\overline{I^m} \subset J\overline{I^{m-1}}$, the other inclusion being trivial. Let $x^b \in \overline{I^m} = \overline{J^m}$ where $x^b = x_1^{b(1)} \cdots x_n^{b(n)}$.

For the proof we consider the following two cases:

Case 1. $b \in F$ where F is a face of $\text{conv}(I^m)$.

First we claim that $b = b_1 + v$ where $b_1 \in G$ for some compact face G of $\text{conv}(I^m)$ and $v \in \mathbb{R}^n_{\geq 0}$. If F is a compact face, then we take $v = 0$ and $b_1 = b$. Now let F be a noncompact face. We prove the claim by induction on $\dim F$. If $\dim F = 1$, then clearly $b = ma_i + v$ where $v \in \mathbb{R}^n_{\geq 0}$ for some $a_i \in \text{ext}(I)$. Now let $\dim F = t > 1$. Let $S = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$ (where $u = (u(1), \ldots, u(n)) \in \mathbb{R}^n$, $c \in \mathbb{R}$) be a supporting hyperplane of $\text{conv}(I^m)$ such that $S \cap \text{conv}(I^m) = F$. Since F is an noncompact face there exists $u(j)$ such that $u(j) = 0$. Consider $b_\lambda := b - \lambda(0, \ldots, 1, \ldots, 0)$, 1 being at jth place, $\lambda \geq 0$. Notice that there exists $\lambda_0 > 0$ such that $b_{\lambda_0} \notin \text{conv}(I^m)$. Let l_0 be the line segment joining b and b_{λ_0}. The intersection of l_0 with F, is nonempty and therefore is a convex set. It follows that $l = l_0 \cap F$ is a line segment joining b and $b_{\lambda'}$ where $b_{\lambda'}$ lies on some proper face F' of F and $\lambda' \geq 0$. Therefore $b = b_{\lambda'} + w$ with $b_{\lambda'} \in F'$ and $w \in \mathbb{R}^n_{\geq 0}$. By induction, $b_{\lambda'} = b_1 + w'$ where $b_1 \in G$ for some compact face G and $w' \in \mathbb{R}^n_{\geq 0}$. Hence $b = b_1 + v$ with $v = w + w' \in \mathbb{R}^n_{\geq 0}$. Hence the claim.

As G is a compact face, we have $\dim G < \ell$ by Corollary 1.4.10. Now since $b_1 \in G$, there exists $p \leq \ell$ affinely independent vectors $\{a_{i_1}, \ldots, a_{i_p}\} \subset \text{ext}(I)$ such that $b_1 = \sum_{j=1}^{p} k_j a_{i_j}$ with $\sum k_i = m$. Since $p \leq \ell \leq m$, there exists $t \in \{1, \ldots, p\}$ such that $k_t \geq 1$. Hence $b_1 - a_{j_t} = \sum_{\substack{j=1 \\ j \neq t}}^{p} k_j a_{j_i} + (k_t - 1)a_{j_t} \in \text{conv}(I^{m-1})$. Now $b - a_{i_t} = b_1 - a_{i_t} + v \in \text{conv}(I^{m-1}) \cap \mathbb{N}^n = \Gamma(\overline{I^{m-1}})$. Hence $b \in \Gamma(J\overline{I^{m-1}})$.

Case 2. $b \notin F$ for any face F of $\text{conv}(I^m)$.

Let $f = x^b$. We may assume that $f \in G(\overline{J^m})$(as in order to show $\overline{J^m} \subset J\overline{I^{m-1}}$, it is enough to show that $G(\overline{J^m}) \subset J\overline{I^{m-1}}$). Without loss of generality, let $x_1 | f$. Since

$f \in G(\overline{J^m})$, $g = f/x_1 \notin \overline{J^m}$. Hence $b \in \text{conv}(I^m)$ and $\Gamma(g) \notin \text{conv}(I^m)$. Let l be the line segment joining b and $\Gamma(g)$. Then l intersects $\text{conv}(I^m)$ at some point $a \in F$ where F is a face of $\text{conv}(I^m)$. Hence, $b = a + v$ where $v \in \mathbb{R}^n_{\geq 0}$. Now by the proof of first case, we may write $a = a_1 + v_1$ such that $a_1 \in G$ for some compact face G of $\text{conv}(I^m)$ and $v_1 \in \mathbb{R}^n_{\geq 0}$. Hence $b = a_1 + w$ where $w = v + v_1 \in \mathbb{R}^n_{\geq 0}$. Now as in the above case, we get that $x^b \in J\overline{I^{m-1}}$. □

Remark 1.5.2. There is a related result by Wiebe. He shows that for the maximal graded ideal $\mathfrak{m}$ in a positive normal affine semigroup ring S of dimension d one has $\overline{\mathfrak{m}^{n+1}} = \mathfrak{m}\overline{\mathfrak{m}^n}$ for all $n \geq d-2$, and that $\overline{\mathfrak{a}^{n+1}} = \mathfrak{a}\overline{\mathfrak{a}^n}$ for all $n \geq d-1$ if $\mathfrak{a} \subset S$ is an integrally closed ideal, see [W06, Theorem 2.1].

Corollary 1.5.3. *Let I^a be integrally closed for all $a \leq \ell - 1$, then $I^\ell = JI^{\ell-1}$ and I is normal, i.e. I^a is integrally closed for all a.*

Proof. By the above theorem, we have $\overline{I^\ell} \subset J\overline{I^{\ell-1}}$. And since $\overline{I^{\ell-1}} = I^{\ell-1}$, we see that $\overline{I^\ell} \subset JI^{\ell-1}$. Hence $I^\ell = JI^{\ell-1}$.

Also, $\overline{I^\ell} = J\overline{I^{\ell-1}} = JI^{\ell-1} \subset I^\ell \subset \overline{I^\ell}$. Hence $\overline{I^\ell} = I^\ell$. By applying induction on a, one has $\overline{I^a} = I^a$ for all a. □

Remarks 1.5.4. (a) Corollary 1.5.3 is a generalization of a result by Reid, Roberts and Vitulli [ReRV03, Proposition 2.3] . They proved that if $I \subset A = K[x_1, \ldots, x_n]$ is a monomial ideal and I^m is integrally closed for $m \leq n-1$, then I is a normal ideal.

(b) In Corollary 1.5.3, once we assume that the monomial ideal I is normal, then the bound on the reduction number with respect to monomial reductions can be obtained as a consequence of a theorem by Valabrega–Valla [VaVa78] and the improved version of the Briancon–Skoda theorem due to Aberbach and Huneke [AbHu93]. In fact, if I is a normal monomial ideal, then $R(I)$ is Cohen–Macaulay and hence the associated graded ring $G(I)$ is Cohen–Macaulay. Thus by Valabrega–Valla [VaVa78] and Aberbach–Huneke [AbHu93], the reduction number of I with respect to monomial reductions is less than the analytic spread ℓ of I. I am thankful to Prof. Verma for this remark.

CHAPTER 2

Graded Betti numbers and the regularity function

In this chapter we study graded Betti numbers of powers of a graded ideal and the regularity function for the powers products of graded ideals.

Let $A = K[x_1, \ldots, x_r]$ be a standard graded polynomial ring and let $I \subset A$ be a graded ideal. Let $\beta_{i,j}(I) = \dim_K \mathrm{Tor}_i(K, I)_j$ and $\beta_i(I) = \dim_K \mathrm{Tor}_i(K, I)$ denote the graded Betti number and the total Betti number of the ideal I, respectively. In [Ko93], Kodiyalam proved that the total Betti number $\beta_i(I^n)$ of the graded ideal I^n is a polynomial function for $n \gg 0$. Let P_i be the polynomial such that $P_i(n) = \beta_i(I^n)$ for $n \gg 0$.

In Section 2.2 we show that $\deg P_{i+1} \leq \deg P_i$. Quite generally, we show that for given $a, c \in \mathbb{Z}$, the graded Betti number $\beta_{i,i+an+c}(I^n)$ of the graded ideal I^n is a quasi polynomial function for $n \gg 0$. For a graded ideal I generated in degree d (by which we mean that all generators of I are of degree d) we show that all graded Betti numbers $\beta_{i,i+dn+j}(I^n)$ are polynomial functions and $N(I^n) = N(I^{n+1})$ for $n \gg 0$ where

$$N(I^n) = \{(i, j) \ : \ \beta_{i,i+j+dn}(I^n) \neq 0\}.$$

In Section 2.3 we consider the regularity function of the power products of graded ideals. Let $I_1, \ldots, I_m$ be graded ideals in the polynomial ring A. Let the regularity function of the power products $I_1^{n_1} I_2^{n_2} \cdots I_m^{n_m}$ be given by the multi–linear function $\sum_{i=1}^{m} a_i n_i + c$ (see [CHT99, Remark after Corollary 3.5]). In Theorem 2.3.4 we determine the coefficient a_i of this function. In case of monomial ideals, we give a convex geometric interpretation of the a_i.

2.1 Diagonal submodules

In this section we recall some facts about diagonal submodules. Let $S = K[x_1, ..., x_n]$ be a bigraded algebra over a field K with $\deg(x_i) = (a_i, b_i)$ for $i = 1, \dots, n$, where a_i, b_i are nonnegative integers. Let M be a finitely generated bigraded S–module.

We define the set

$$\Delta := \{(cs, ds) : \ s \in \mathbb{N}\}$$

which we call the (c, d)–diagonal of $\mathbb{N}^2$. The diagonal subalgebra of S along Δ is defined as the $\mathbb{N}$–graded algebra

$$S_\Delta := \bigoplus_{s \in \mathbb{N}} S_{(cs,ds)}.$$

Similarly, we can define the Δ–submodule of the bigraded submodule M as

$$M_\Delta := \bigoplus_{s \in \mathbb{N}} M_{(cs,ds)}.$$

By construction M_Δ is an $\mathbb{N}$–graded S_Δ–module.

Lemma 2.1.1. *Let Δ be a (c, d)–diagonal of $\mathbb{N}^2$. Then S_Δ is a finitely generated $\mathbb{N}$–graded K–algebra and M_Δ is a finitely generated $\mathbb{N}$–graded S_Δ–module.*

Proof. We may assume that S is a bigraded polynomial ring. Indeed, if S is an arbitrary bigraded K–algebra generated by $y_1, \dots, y_n$ with y_i of bidegree (a_i, b_i), then there is a surjective homomorphism $A \to S$ of bigraded algebras where A is the polynomial ring in the variables $x_1, \dots, x_n$ with bidegree $x_i = (a_i, b_i)$. This induces a surjective K–algebra homomorphism $A_\Delta \to S_\Delta$. Hence if we prove that A_Δ is finitely generated K–algebra, then so is S_Δ.

The algebra S_Δ is the K–vector space spanned by all monomials of S whose exponent vectors $(z_1, \dots, z_n)$ satisfy the following system of linear equations in n indeterminates

$$a_1 z_1 + a_2 z_2 + \cdots + a_n z_n = cs,$$

$$b_1 z_1 + b_2 z_2 + \cdots + b_n z_n = ds$$

with $s \in \mathbb{N}$. We replace s by an indeterminate z_{n+1} to get the following system of homogeneous linear equations in $n + 1$ indeterminates :

$$a_1 z_1 + a_2 z_2 + \cdots + a_n z_n - c z_{n+1} = 0,$$

$$b_1 z_1 + b_2 z_2 + \cdots + b_n z_n - d z_{n+1} = 0.$$

The set of integral solutions of this system of equations is a subgroup G of $\mathbb{Z}^{n+1}$, and $L = G \cap \mathbb{R}_+^{n+1}$ is the set of solutions in $\mathbb{N}^{n+1}$. Here $\mathbb{R}_+^{n+1}$ is the rational cone of vectors of $\mathbb{R}^{n+1}$ with nonnegative entries. Gordon's Lemma implies that L is a finitely generated (normal) semigroup. Thus the corresponding affine semigroup ring $K[L]$ is a finitely generated K–algebra, and so is S_Δ, since $K[L]$ and S_Δ are isomorphic K–algebras. Indeed, the isomorphism is induced by the map which assigns to each element $v = (v_1, \ldots, v_n, v_{n+1}) \in L$ the monomial $x_1^{v_1} \ldots x_n^{v_n} \in S_\Delta$.

We now prove M_Δ is a finitely generated $\mathbb{N}$–graded S_Δ–module. Let $F \to M$ be an epimorphism of bigraded S–modules where F is a free bigraded S–module, that is, $F = \bigoplus_{(l,m)} S(-l, -m)$. Notice that it is enough to show that $S(-l, -m)_\Delta$ is a finitely generated S_Δ–module for any $(l, m) \in \mathbb{N}^2$. We have

$$S(-l, -m)_\Delta = \bigoplus_s S(sc - l, sd - m).$$

We consider the following system of equations in $n + 1$ indeterminates :

$$a_1 z_1 + \ldots + a_n z_n - c z_{n+1} = -l,$$

$$b_1 z_1 + \ldots + b_n z_n - d z_{n+1} = -m.$$

Let T be the set of solutions of the above system in $\mathbb{Z}^{n+1}$. Then for any $a \in T$ one has $T = a + G$.

Let $I \subset K[x_1, \ldots, x_{n+1}]$ be the monomial ideal whose generators are given by monomials $x^a = x_1^{a(1)} \cdots x_{n+1}^{a(n+1)}$ where $a = \big(a(1), \ldots, a(n+1)\big) \in T \cap \mathbb{N}^{n+1}$. As I is finitely generated, there exist $a_1, \ldots, a_m \in T \cap \mathbb{N}^{n+1}$ such that $I = (x^{a_1}, x^{a_2}, \ldots, x^{a_m})$. We claim that

$$T \cap \mathbb{N}^{n+1} = \bigcup_{i=1}^{m} (a_i + G \cap \mathbb{N}^{n+1}).$$

In fact, suppose $a \in T \cap \mathbb{N}^{n+1}$, then $a = a_i + b$, for some i and some $b \in \mathbb{N}^{n+1}$. Since $a, a_i \in T$, it follows that $b = a - a_i \in G$, as desired.

Hence $K[T \cap \mathbb{N}^{n+1}]$ is a finitely generated graded module over $K[L]$ (with $L = G \cap \mathbb{N}^{n+1}$ as before). Here the sth graded component of $K[T \cap \mathbb{N}^{n+1}]$ is the K–vector space spanned by all $a \in T \cap \mathbb{N}^{n+1}$ whose $(n+1)$th component is equal to s.

Since $K[L] \cong S_\Delta$, one may consider $K[T \cap \mathbb{N}^{n+1}]$ as an S_Δ–module. Now the map which assigns to each element $a = (a_1, \ldots, a_{n+1}) \in T \cap \mathbb{N}^{n+1}$ the monomial $x_1^{a_1} \cdots x_n^{a_n}$ establishes an isomorphism $K[T \cap \mathbb{N}^{n+1}] \cong S(-l, -m)_\Delta$ of $\mathbb{N}$–graded S_Δ–modules. Hence, $S(-l, -m)_\Delta$ is a finitely generated $\mathbb{N}$–graded module over S_Δ. □

2.2 Graded Betti numbers of powers of ideals

In this section, we study graded Betti numbers of powers a graded ideal $I \subset A = K[x_1, \ldots, x_r]$. We first show the following:

Theorem 2.2.1. *Let $S = K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ be a bigraded polynomial ring over a field K with $\deg x_i = (1, 0), i = 1, \ldots, r$ and $\deg y_j = (0, 1), j = 1, \ldots, s$. Let M be a finitely generated bigraded module over S. Put $M^{(n)} = \bigoplus_i M_{(i,n)}$. Then $\beta_{i,i+j}(M^{(n)})$ is a polynomial function in n for $n \gg 0$.*

Proof. Let $A = K[x_1, \dots, x_r]$ be the polynomial subring of S generated over K by the variables $x_1, \dots, x_n$, and $\mathfrak{m}$ be the graded maximal ideal of A. We have (see [CHT99, Lemma 3.3])

$$\operatorname{Tor}_i^A(K, M^{(n)})_a \cong \operatorname{Tor}_i^S(S/\mathfrak{m}S, M)_{(a,n)}, \text{ for all } a,\ n \text{ and } i \geq 0. \tag{2.1}$$

Then,

$$\beta_{i,i+j}(M^{(n)}) = \dim_K \operatorname{Tor}_i^A(K, M^{(n)})_{i+j} = \dim_K \operatorname{Tor}_i^S(S/\mathfrak{m}S, M)_{(i+j,n)}.$$

Let $T = S/\mathfrak{m}S = K[y_1, \dots, y_s]$ and $\Delta = (0, 1)$. Now as, $\operatorname{Tor}_i^S(T, M)$ is a finitely generated S–module, Lemma 2.1.1 applied to $\operatorname{Tor}_i^S(T, M)(i + j, 0)$ implies that

$$[\operatorname{Tor}_i^S(T, M)(i + j, 0)]_\Delta = \bigoplus_n \operatorname{Tor}_i^S(S/\mathfrak{m}S, M)_{(i+j,n)}$$

is a finitely generated graded $S_\Delta = T$–module. Since the induced grading on T is the standard grading, we conclude $\dim_K \operatorname{Tor}_i^S(S/\mathfrak{m}S, M)_{(i+j,n)}$ is a polynomial function for $n \gg 0$. □

Let $A = K[x_1, \dots, x_r]$ be a standard graded polynomial ring over a field K. Let $I \subset A$ be a graded ideal, minimally generated by the homogeneous elements $f_1, \cdots, f_s$ with $\deg f_i = d$ for $i = 1, \dots, s$. In such a case, we say that the ideal is generated in one degree.

Let $S = K[x_1, \dots, x_r, y_1, \dots, y_s]$ be a bigraded polynomial ring over the field K with $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$, and let $R(I)$ be the Rees ring of I. Notice that $R(I)$ may be viewed as a standard bigraded K–algebra with $\deg x_i = (1, 0)$ and $\deg f_j t = (0, 1)$. Thus $R(I)$ is a finitely generated S–module via the homogeneous surjective map $S \to R(I)$ mapping $x_i \mapsto x_i$ and $y_j \mapsto f_j t$.

As a refinement of a result of Kodiyalam [Ko93], who showed that for a graded ideal I, the function $\beta_i(I^n)$ is a polynomial function for $n \gg 0$, we obtain as an immediate consequence of Theorem 2.2.1 the following

Corollary 2.2.2. *Let I be generated in one degree. Then $\beta_{i,i+dn+j}(I^n)$ is a polynomial function in n for $n \gg 0$ for all i and j.*

Proof. We just need to notice that $R(I)^{(n)} = I^n(-dn)$ for all n. □

In the next theorem we show that the Betti diagram of the graded ideal I^n becomes stable for large enough n.

Theorem 2.2.3. *Suppose that I is generated in degree d. For every $n \in \mathbb{N}$, we define*

$$N(I^n) = \{(i, j) \colon \beta_{i,i+j+dn}(I^n) \neq 0\}.$$

Then, $N(I^n) = N(I^{n+1})$ for $n \gg 0$.

Proof. We have

$$\begin{aligned}\beta_{i,i+j+dn}(I^n) &= \dim_K \operatorname{Tor}_i^S(S/\mathfrak{m}S, R(I))_{(i+j+dn,n)}\\ &= \dim_K \operatorname{Tor}_i^S(S/\mathfrak{m}S, R(I))(i+j,0)_{(dn,n)}\end{aligned}$$

Therefore, $h(n) = \beta_{i,i+j+dn}(I^n)$ is the Hilbert function of the graded T–module $M_{ij} = [\operatorname{Tor}_i^S(S/\mathfrak{m}S, R(I))(i+j,0)]_\Delta$ where Δ is the $(d,1)$ diagonal.

Now, for every i,j the Krull dimension of the module M_{ij} is either equal to zero or not equal to zero. Hence, $\dim_K(M_{ij})_n$ is equal to zero or not equal to zero for $n \gg 0$. Therefore, $N(I^n) = N(I^{n+1})$, for $n \gg 0$. □

Now let $I \subset A$ be an arbitrary graded ideal in A. Here the generators of I need not to have all the same degree. We want to give another addendum to the theorem [Ko93, Theorem 1] of Kodiyalam.

Let $f_1, \ldots, f_s$ be a (minimal) homogeneous system of generators of the ideal I with $\deg f_i = d_i$ for $i = 1, \ldots, s$, and let $S = K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ be the bigraded polynomial ring $\deg x_i = (1,0)$ for $i = 1, \ldots, r$, and $\deg y_j = (d_j, 1)$ for $j = 1, \ldots, s$. Then the K–algebra homomorphism $S \to R(I)$ induced by $x_i \mapsto x_i$ and $y_j \mapsto f_j t$ is a surjective homomorphism of bigraded K–algebras (provided we assign to an element $ft^k \in R(I)$ the natural bidegree $(\deg f, k)$. Thus $R(I)$ may be viewed a bigraded S–module.

Now let M be any finitely generated bigraded S–module. As before we define

$$M^{(n)} = \bigoplus_i M_{(i,n)}.$$

Then $M^{(n)}$ is a graded $A = K[x_1, \ldots, x_r]$–module. In case $M = R(I)$ we have as before $R(I)^{(n)} = I^n$.

Since $S/\mathfrak{m}S$ and M are bigraded S–modules, the $S/\mathfrak{m}S$–modules $\operatorname{Tor}_i^S(S/\mathfrak{m}S, M)$ inherit a natural bigraded structure, and in analogy to formula (2.1) we have

$$\operatorname{Tor}_i^A(K, M^{(n)}) = \operatorname{Tor}_i^S(S/\mathfrak{m}S, M)^{(n)}.$$

The bigraded K–algebra $T = S/\mathfrak{m}S = K[y_1, \ldots, y_r]$ with $\deg y_j = (d_j, 1)$ may as well be viewed as standard graded polynomial ring over K, just by disregarding the first component of the bidegrees. Similarly a bigraded S–module N becomes a graded T–module, with nth graded component $N_n = N^{(n)}$ for all n.

Theorem 2.2.4. *With the assumptions and notation introduced we have*

(a) (Kodiyalam) *There exist polynomials P_i^M such that*

$$P_i^M(n) = \beta_i(M^{(n)}) \quad \textit{for all} \quad n \gg 0.$$

(b) $\deg P_{i+1}^M \leq \deg P_i^M$ *for all $i \geq 0$.*

Proof. The discussions preceding the theorem show that

$$\beta_i(M^{(n)}) = \dim_K \operatorname{Tor}_i^S(T, M)_n,$$

where $T = S/\mathfrak{m}S$. The natural isomorphism $\operatorname{Tor}_i^S(T, M) \cong H_i(x, M)$ is an isomorphism of graded T–modules. Here $H_i(x, M)$ denotes the ith Kosul homology of M with respect to $x = x_1, \ldots, x_r$. In particular, $\beta_i(M^{(n)}) = \dim_K H_i(x, M)_n$. Since M is a finitely generated S–module, it follows that $H_i(x, M)$ is a finitely generated graded T–module. Therefore its Hilbert function $h(n) = \dim_K H_i(x, M)_n$ is a polynomial function for $n \gg 0$, whose degree equals $\operatorname{Krulldim} H_i(x, M) - 1$.

Since $\beta_i(M^{(n)}) = h(n)$ for all n, statement (a) follows. Moreover, (b) will follow once we have shown that $\operatorname{Krulldim}(H_{i+1}(x, M)) \leq \operatorname{Krulldim}(H_i(x, M))$ for all i. Suppose this is not the case, then there exists some prime ideal $P \in \operatorname{Supp}(H_{i+1}(x, M))$ such that $P \notin \operatorname{Supp}(H_i(x, M))$. Therefore, $H_{i+1}(x, M_P) = H_{i+1}(x, M)_P \neq 0$, and $H_i(x, M_P) = H_i(x, M)_P = 0$. This contradicts the rigidity of Koszul homology, see [BH96, 1.6.31]. □

In the following theorem we need the fact, proved by Cutkosky, Herzog and Trung [CHT99] and by Kodiyalam [Ko93], that for any graded ideal $I \subset A = K[x_1, \ldots, x_n]$ the regularity of I^n is a linear function on n for all $n \gg 0$. More precisely,

$$\operatorname{reg}(I^n) = pn + q \quad \text{for all } n \gg 0$$

with $p \geq d$, where d the initial degree of I, that is, the lowest degree of a generator of I.

Theorem 2.2.5. *Let $I \subset A = K[x_1, \ldots, x_r]$ be a graded ideal with initial degree d, and suppose that $\operatorname{reg}(I^n) = pn + q$ for $n \gg 0$. Then*

(a) *for all $a, c \in \mathbb{Z}$, the function $\beta_{i,i+an+c}(I^n)$ is a quasi polynomial in n for all $n \gg 0$;*

(b) *if $a < d$ or $a > p$, then $\beta_{i,i+an+c}(I^n) = 0$ for all c and $n \gg 0$.*

Proof. (a) As above let I be minimally generated by $(f_1, \ldots, f_s)$ with $\deg f_j = d_j$ and $d = d_1 \leq d_2 \leq \cdots \leq d_s$. Let $S = K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ be a bigraded polynomial ring with $\deg x_i = (1, 0)$ and $\deg y_j = (d_j, 1)$. Then the Rees ring $R(I)$ is a bigraded S–module, and we have

$$\beta_{(i,i+an+c)}(I^n) \;=\; \dim_K \operatorname{Tor}_i^A(K, R(I)^{(n)})_{i+an+c} = \dim_K \operatorname{Tor}_i^S(S/\mathfrak{m}S, R(I))_{(i+an+c,n)}$$

Let Δ be the $(a, 1)$–diagonal of $\mathbb{N}^2$. Let $N = \operatorname{Tor}_i^S(S/\mathfrak{m}S, R(I))(i + c, 0)$ and let $T = S/mS$. As N is finitely generated T–module, Lemma 2.1.1 yields that N_Δ is a finitely generated module over the positively graded K–algebra T_Δ. Hence, since $\beta_{(i,i+an+c)}(I^n) = \dim_K(N_\Delta)_n$ is the Hilbert function of N_Δ, assertion (a) follows from [BH96, Theorem 4.4.3].

(b) Let first $a < d$, then $\beta_{i,i+an} = 0$ for all n. Let $c \in \mathbb{Z}$ be any integer, then there exists a natural number n_0 such that $(d-a)n > c$ i.e. $an + c < dn$ for all $n > n_0$. Hence, $\beta_{i,i+an+c}(I^n) = 0$ for all $n > n_0$.

Now let $a > p$, as $\operatorname{reg}(I^n) = pn + q$, we have $\beta_{i,i+j}(I^n) = 0$ for all $j > pn + q$ and and all $n \gg 0$. Let $c \in \mathbb{Z}$ be any integer, then again there exists some integer n_0, such that $(a-p)n + c > q$, that is, $an + c > pn + q$ for all $n > n_0$. Hence again, $\beta_{i,i+an+c}(I^n) = 0$ for $n > n_0$. □

Remarks 2.2.6. (a) The converse of part (*b*) is not true, that is, there exist integres i, a, c such that $\beta_{i,i+an+c}(I^n) = 0$ for $d \leq a \leq p$ and all n. For example, if we take the ideal $I = (x^2, y^4)$ in a polynomial ring $A = k[x,y]$ and $a = 2, c = 1$, then $\beta_{0,0+2n+1}(I^n) = 0$, for all n.

(b) In the above theorem, if we take $a = d_1$, then $\beta_{(i,i+an+c)}$ is in fact a polynomial function for $n \gg 0$, as in this case T_Δ is a standard graded polynomial ring. Hence the Hilbert function of the finitely generated module N_Δ over T_Δ is a polynomial function for $n \gg 0$. But in general, $\beta_{(i,i+an+c)}$ is only a quasi polynomial, as the following example shows: Consider the ideal $I = (x, y^3, z^4)$ in the polynomial ring $A = K[x,y,z]$, then the graded Betti number $\beta_{0,0+3n+1}(I^n)$ is a quasi polynomial function. Indeed, let $P_1(n) = (n+2)/3$, $P_2(n) = (n+1)/3$ and $P_3(n) = n/3$ for $n = 1, 2 \ldots$, then $\beta_{0,0+3n+1}(I^n)$ is equal to $P_1(n)$ if $n = 3k-2$ and is equal to $P_2(n)$ if $n = 3k-1$, and is equal to $P_3(n)$ if $n = 3k$ for $k = 1, 2, \ldots$.

2.3 The regularity of power products of graded ideals

In this section, we discuss the regularity of the power products $I_1^{n_1} I_2^{n_2} \cdots I_m^{n_m}$ of the graded ideals $I_1, \ldots, I_m$ in a polynomial ring $A = K[x_1, \ldots, x_r]$. It is known that for large enough n_i, the regularity function for these power products is a multi–linear function of the form $\sum_{i=1}^m a_i n_i + c$, (see [CHT99, Remark after Corollary 3.5]). We determine the coefficient a_i of this function. In case of monomial ideals, we give a convex geometric interpretation of the a_i. In fact we show that these coefficients are determined by the minimal monomial reduction ideals of the factors I_i.

For each $i = 1, \ldots, m$, we define $p_i = \min\{\theta(J) : JI_i^n = I_i^{n+1} \text{for some } n\}$. Let $J_1, \ldots, J_m$ be Kodiyalam reduction ideals of $I_1, \ldots, I_m$ respectively i.e. $J_1, \ldots, J_m$ are graded reduction ideals of $I_1, \ldots, I_m$ respectively with $\theta(J_i) = p_i, i = 1, \ldots, m$. We first notice that $J_1 \cdots J_m$ is a reduction ideal of $I_1 \cdots I_m$. Therefore the multi–Rees ring of $I_1, \ldots, I_m$ denoted by $R = R(I_1, \ldots, I_m) = \bigoplus_{(n_1,\ldots,n_m)\in\mathbb{N}^m} I_1^{n_1} \cdots I_m^{n_m}$ is a finitely generated module over the multi–Rees ring of $J_1, \ldots, J_m$ denoted by $R' = R'(J_1, \ldots, J_m)$. Let for each $i = 1, \ldots, m$, $J_i = (f_{i1}, \ldots, f_{is_i})$, with $\deg(f_{ij}) = d_{ij}, j = 1, \ldots, s_i$. We consider the $\mathbb{N}^{m+1}$ graded polynomial ring

$$S = K[x_1, \ldots, x_r, y_{11}, \ldots, y_{1s_1}, \ldots, y_{i1}, \ldots, y_{is_i}, \ldots, y_{m1}, \ldots, y_{ms_m}],$$

with $\deg x_i = (1, 0, \ldots, 0) \in \mathbb{N}^{m+1}$ and $\deg y_{ij} = (d_{ij}, 0, \ldots, 1, \ldots, 0) \in \mathbb{N}^{m+1}$ where 1 being at $(i+1)$th place, $j = 1, \ldots, s_i$ and $i = 1, \ldots, m$.

The multi–Rees ring R' can be considered as a quotient module of S, so R' is finitely generated $\mathbb{N}^{m+1}$ graded module over S and as R is finitely generated module over R', therefore R is finitely generated module over S.

Let E be a finitely generated $\mathbb{N}^{m+1}$–graded S module and $(n_1, \ldots, n_m) \in \mathbb{N}^m$, we define

$$E_{n_1 \ldots n_m} = \bigoplus_{a \in \mathbb{N}} E_{(a, n_1, \ldots, n_m)}.$$

Notice that $E_{n_1 \ldots n_m}$ is an $\mathbb{N}$–graded A module. Now let $(n_1, \ldots, n_m) \in \mathbb{N}^m$ and let $(a_0, \ldots, a_m) \in \mathbb{N}^{m+1}$. Then we have

$$\begin{aligned} S(-a_0, -a_1, \ldots, -a_m)_{n_1 \ldots n_m} &= \bigoplus_{a \in \mathbb{N}} S(-a_0, -a_1, \ldots, -a_m)_{(a, n_1, \ldots, n_m)} \\ &= \bigoplus_{a \in \mathbb{N}} S_{(a - a_0, n_1 - a_1, \ldots, n_m - a_m)} \\ &= S_{n_1 - a_1 \ldots n_m - a_m}(-a_0) \\ &\cong \bigoplus_{\sum_{j=1}^{s_i} c_{ij} = n_i - a_i} A(-a_0) y_{11}^{c_{11}} \cdots y_{1s_1}^{c_{s1}} \cdots y_{m1}^{c_{m1}} \cdots y_{ms_m}^{c_{ms_m}} \\ &\cong \bigoplus_{\sum_{j=1}^{s_i} c_{ij} = n_i - a_i} A(-\sum_{i=1}^{m} \sum_{j=1}^{s_i} d_{ij} c_{ij} - a_0). \end{aligned}$$

Theorem 2.3.1. *Let E be a finitely generated $\mathbb{N}^{m+1}$–graded S module. Then there exists a constant $c \in \mathbb{Z}$ such that* $\operatorname{reg}(E_{n_1 \ldots n_m}) \leq p_1 n_1 + p_2 n_2 + \cdots + p_m n_m + c$ *where* $n_j \in \mathbb{N}$, $j = 1, \ldots, m$.

Proof. Consider an $\mathbb{N}^{m+1}$–graded free S–resolution of E given by

$$\mathbf{F} : 0 \to F_t \longrightarrow F_{t-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \to 0,$$

where $F_u = \bigoplus_{v=1}^{k_u} S(-a_{0uv}, -a_{1uv}, \ldots, -a_{muv})$, for $u = 1, \ldots, t$. From the above exact sequence, we get the following exact sequence

$$\mathbf{F}_{n_1 \ldots n_m} : 0 \to (F_t)_{n_1 \ldots n_m} \longrightarrow \cdots \longrightarrow (F_0)_{n_1 \ldots n_m} \longrightarrow E_{n_1 \ldots n_m} \to 0.$$

Since the modules $(F_u)_{n_1 \ldots n_m}$ are free A–modules (as observed in the discussion before the theorem), $\mathbf{F}_{n_1 \ldots n_m}$ is a free A–resolution of the module $E_{n_1 \ldots n_m}$. Again we have

$$(F_u)_{n_1 \ldots n_m} \cong \bigoplus_{v=1}^{k_u} \bigoplus_{\sum_{j=1}^{s_i} c_{ij} = n_i - a_{iuv}} A(-\sum_{i=1}^{m} \sum_{j=1}^{s_i} d_{ij} c_{ij} - a_{0uv}),$$

with maximal shift in $(F_u)_{n_1 \ldots n_m}$ being

$$\max_v \{p_1(n_1 - a_{1uv}) + p_2(n_2 - a_{2uv}) + \cdots + p_m(n_m - a_{muv}) + a_{0uv}\}.$$

Hence $\operatorname{reg}(E_{n_1 \ldots n_m}) \leq p_1 n_1 + p_2 n_2 + \cdots + p_m n_m + c$. □

Remark 2.3.2. The proof of above theorem is based on the arguments of Kodiyalam (see [Ko99, Theorem 1]). Kodiyalm proves the above result for bigraded modules.

Corollary 2.3.3. *Let* $I_1, \ldots, I_m$ *be graded ideals in the polynomial ring* A*. Then for* $(n_1, \ldots, n_m) \in \mathbb{N}^m$, $\operatorname{reg}(I_1^{n_1} \ldots I_m^{n_m}) \leq p_1 n_1 + \cdots + p_m n_m + c$ *for some* $c \in \mathbb{Z}$ *.*

Proof. We replace E in the above theorem by the multi–graded Rees ring $R(I_1, \ldots, I_m)$ of $I_1, \ldots, I_m$. □

Theorem 2.3.4. *Let* $I_1, \ldots, I_m$ *be graded ideals in the polynomial ring* A *and let* $p_i = \min\{\theta(J) : JI_i^{\,n} = I_i^{n+1}$ *for some* $n\}$*. Then* $\operatorname{reg}(I_1^{n_1} \ldots I_m^{n_m}) = p_1 n_1 + \cdots + p_m n_m + c$*, for* $n_j \gg 0$ *and for some* $c \in \mathbb{Z}$*.*

Proof. We follow the line of arguments as in [Ko99, Proposition 4]. We know (see [CHT99, Remark after Corollary 3.5])

$$\operatorname{reg}(I_1^{n_1} \ldots I_m^{n_m}) = a_1 n_1 + \cdots + a_m n_m + c,$$

for $n_j \geq l_j$(say) , $j = 1, \ldots, m$ and for some $c \in \mathbb{Z}$. By the previous corollary we know that $a_j \leq p_j$, hence it remains to show that $a_j \geq p_j$ for all j.

We prove $a_1 \geq p_1$. Let $M = I_2^{l_2} \cdots I_m^{l_m}$, then we have $\operatorname{reg}(I_1^{n_1} M) = a_1 n_1 + c'$, for $n_1 \geq l_1$ where $c' = c + a_2 l_2 + \cdots + a_m l_m$. Let $I_1 = (f_1, \ldots, f_s)$ such that $\deg f_i = d_i$ and $d_1 \leq \cdots \leq d_s$. Let j be the largest integer such that $f_j^n M \notin mI^n M$ for all n, then $\operatorname{reg}(I^n M) \geq d_j n$ for all n. Hence $d_j \leq a_1$. Let $J = (f_1, \ldots, f_j)$ and $K = (f_{j+1}, \ldots, f_m)$. By the very choice of j, there exists an integer n such that $K^n M \subseteq mI^n M$. Consider $I^{n+1} M = (J + K)^n M = J(J + K)^{n-1} M + K^n M \subseteq JI^n M + mI^n M$, by Nakayama lemma we have $I^n M = JI^{n-1} M$. Further it implies that $I^n = JI^{n-1}$. Hence J is a reduction ideal of I, with $\theta(J) = d_j$. By the very definition of p_1, we have $d_j \geq p_1$ and together with $d_j \leq a_1$, we have $a_1 \geq p_1$. Similarly, one can show that $a_i \geq p_i$ for all i. □

Remark 2.3.5. In the case of monomial ideals we have a convex geometric interpretation for the numbers p_i. Let $I_1, \ldots, I_m$ be monomial ideals. Then $\operatorname{reg}(I_1^{n_1} \ldots I_m^{n_m}) = p_1 n_1 + \cdots + p_m n_m + c$ for large enough n_i where $p_i = \max\{\deg x^a : a \in \operatorname{ext}(I_i)\}$, see Proposition 1.2.1 and Corollary 1.2.3.

CHAPTER 3

Rigidity of linear strands and generic initial ideals

In this chapter we study rigidity properties of graded Betti numbers of a graded ideal when passing to its generic initial ideal.

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field K with $\operatorname{char}(K) = 0$ and $I \subset S$ a graded ideal. Let $\beta_i^S(M) = \dim_K \operatorname{Tor}_i^S(K, M)$ and $\beta_{i,j}^S(M) = \dim_K \operatorname{Tor}_i^S(K, M)_j$ denote respectively the ith total and (i, j)th graded Betti number of a finitely generated graded S–module M.

In Section 3.1 we give an upper bound for graded Betti numbers of a finitely generated graded S–module M in terms of its generic graded annihilator numbers. A generic graded annihilator number $\alpha_{p,j}(M)$, $p = 1, \ldots, n$ is the vector space dimension of the jth graded component $(A_p)_j$ of the finite length S module A_p. Here $A_p = (y_1 \ldots, y_{p-1})M :_M y_p/(y_1, \ldots, y_{p-1})$ and $y_1, \ldots, y_n$ is a sequence of generic linear forms for the module M. The results in Section 1 are refinements of the results in [CoHHi04, §1].

The generic initial ideal $\operatorname{Gin}(I)$ plays a fundamental role in investigating various homological, algebraic, combinatorial and geometric properties of I. By definition, the generic initial ideal $\operatorname{Gin}(I)$ is, after performing a generic change of coordinates, the initial ideal of I with respect to the reverse lexicographic order. Here we consider the reverse lexicographic order induced by $x_1 > \cdots > x_n$.

In Section 3.2 we study the graded Betti numbers of a graded ideal in comparison with its generic initial ideal. The following inequality of the graded Betti numbers is well–known:

$$\beta_{i,j}(S/I) \leq \beta_{i,j}\big(S/\operatorname{Gin}(I)\big),$$

for all i, j (see for example [Co04, Theorem 1.1]). Equality holds for all i and j if and only if I is componentwise linear (see [AHHi00, Theorem 1.1]). In his paper [Co04] Conca asked whether the equality $\beta_i(S/I) = \beta_i\big(S/\operatorname{Gin}(I)\big)$ for some $i \geq 1$ of the total Betti numbers implies $\beta_j(S/I) = \beta_j\big(S/\operatorname{Gin}(I)\big)$ for all $j \geq i$. This question of Conca was positively answered in 2004 by Conca, Herzog and Hibi in [CoHHi04].

One of the main results of Section 2 is to extend this result of Conca–Herzog–Hibi to graded Betti numbers. In Corollary 3.2.3 we show the following: If for some $i > 1$ and $k \geq 0$, we have $\beta^S_{i,i+k}(S/I) = \beta^S_{i,i+k}\big(S/\operatorname{Gin}(I)\big)$, then

$$\beta^S_{q,q+k}(S/I) = \beta^S_{q,q+k}\big(S/\operatorname{Gin}(I)\big) \quad \text{for all } q \geq i.$$

In Section 3.3 we consider the Betti numbers and the generic graded annihilator numbers in the case of an exterior algebra. We also recall some basic facts about Cartan complexes.

In Section 3.4 we study the rigidity property of graded Betti numbers of graded ideals over an exterior algebra. Let K be an infinite field and V an n–dimensional K–vector space with basis $e_1, \dots, e_n$ and $E = \bigoplus_{k=0}^n \bigwedge^k V$ the exterior algebra of V. For a graded ideal $J \subset E$, we write $\operatorname{Gin}(J)$ for the generic initial ideal of J with respect to the reverse lexicographic order induced by $e_1 > \cdots > e_n$ and denote by $\beta^E_{i,j}(E/J)$ the (i,j)th graded Betti number of E/J over E. Somewhat surprisingly, the following stronger property is true in the exterior algebra:

If $\beta^E_{i,i+k}(E/J) = \beta^E_{i,i+k}\big(E/\operatorname{Gin}(J)\big)$ for some $i > 1$ and $k \geq 0$, then one has

$$\beta^E_{q,q+k}(E/J) = \beta^E_{q,q+k}\big(E/\operatorname{Gin}(J)\big) \qquad \text{for all} \quad q \geq 1.$$

Let R be either a polynomial ring over a field K with $\operatorname{char}(K) = 0$ or an exterior algebra over an infinite field and I a graded ideal of R. The above property leads us to ask when a graded ideal $I \subset R$ satisfies $\beta^R_{i,i+k}(R/I) = \beta^R_{i,i+k}\big(R/\operatorname{Gin}(I)\big)$ for all $i \geq 1$, where we fix an integer $k \geq 0$.

In Section 3.5 we give an answer to the above question. We prove that the graded Betti number $\beta^R_{i,i+k}(R/I) = \beta^R_{i,i+k}\big(R/\operatorname{Gin}(I)\big)$ for all $i \geq 1$ if and only if the graded Betti numbers $\beta^R_{1,k+1}(R/I) = \beta^R_{1,k+1}\big(R/\operatorname{Gin}(I)\big)$ and $\beta^R_{1,k+2}(R/I) = \beta^R_{1,k+2}\big(R/\operatorname{Gin}(I)\big)$ if and only if the ideals $I_{\langle k\rangle}$ and $I_{\langle k+1\rangle}$ have a linear resolution. Here $I_{\langle k\rangle}$ denotes the ideal of R generated by all homogeneous elements in I of degree k.

This result is a generalization of [AHHi00, Theorem 1.1], where it was shown that $\beta^R_{i,j}(R/I) = \beta^R_{i,j}\big(R/\operatorname{Gin}(I)\big)$ for all i, j if and only if I is componentwise linear.

In the last section we study the Cancellation Principle for generic initial ideals [G98]. We find the relation between our results for Betti numbers of a graded ideal in a polynomial ring and the Cancellation Principle for generic initial ideals. The results in this section are closely related to the results in Section 3.1.

3.1 An upper bound for the graded Betti numbers

In this section, we will give an upper bound for graded Betti numbers in terms of generic graded annihilator numbers, which were introduced in [CoHHi04]. Note that most of the results in this section are refinements of the results in [CoHHi04, §1]. Though these results seem to be somewhat technical, they are of crucial importance for the proof of one of our main theorems in the next section.

Let $S = K[x_1, \ldots, x_n]$ be the standard graded polynomial ring over an arbitrary field K and $\mathfrak{m} = (x_1, \ldots, x_n)$ the graded maximal ideal. Let M be a finitely generated graded S–module. For each nonnegative integer i, the modules $\operatorname{Tor}_i^S(K, M)$ are finitely generated K–vector spaces. The numbers $\beta_i^S(M) = \dim_K \operatorname{Tor}_i^S(K, M)$ and $\beta_{i,j}^S(M) = \dim_K \operatorname{Tor}_i^S(K, M)_j$ are called *Betti numbers* and *graded Betti numbers* of M, respectively. As $\beta_{i,j}^S$ are invariants under base field extensions, from now on we may assume the field K to be infinite.

Let $y_1, \ldots, y_n$ be a sequence of generic linear forms for the module M. For each $p = 1, \ldots, n$, the modules

$$A_p = (y_1, \ldots, y_{p-1})M :_M y_p/(y_1, \ldots, y_{p-1})$$

are $\mathbb{N}$–graded S–modules of finite length. We define $\alpha_p(M) = \dim_K A_p$ which we call the *generic annihilator numbers* of M. We denote by $\alpha_{p,j}(M)$ the vector space dimension of the jth graded component $(A_p)_j$ of A_p which we call the *generic graded annihilator numbers* of M.

Let $H_i(p, M)$ be the Koszul homology $H_i(y_1, \ldots, y_p; M)$ of the partial sequence $y_1, \ldots, y_p$. We set $h_i(p, M) = \dim_K H_i(p, M)$ and $h_{ij}(p, M) = \dim_K H_i(p, M)_j$. We omit M and simply write $\beta_{i,j}^S$, β_i^S, $\alpha_{i,j}$, α_i, $H_i(p)_j$, $H_i(p)$, $h_{ij}(p)$, $h_i(p)$ for the above defined terms, if the module under consideration is fixed. Then we have the following long exact sequence (see [BH96, Corollary 1.6.13]):

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_i(p-1) & \xrightarrow{\varphi_{i,p-1}} & H_i(p-1) & \longrightarrow & H_i(p) & \longrightarrow & H_{i-1}(p-1) \\ \cdots & \longrightarrow & H_0(p-1) & \xrightarrow{\varphi_{0,p-1}} & H_0(p-1) & \longrightarrow & H_0(p) & \longrightarrow & 0. \end{array} \tag{3.1}$$

In the above sequence $\varphi_{i,p-1}$ is the multiplication map on $H_i(p-1)$ with multiplication by $\pm y_p$. One may notice that A_p is given by the kernel of the map $\varphi_{0,p-1}$. Hence we get the following exact sequences with all the maps of degree zero:

$$0 \longrightarrow \operatorname{Im} \varphi_{1,p-1} \longrightarrow H_1(p-1) \longrightarrow H_1(p) \longrightarrow A_p(-1) \longrightarrow 0$$

for all p, and

$$0 \longrightarrow \operatorname{Im} \varphi_{i,p-1} \longrightarrow H_i(p-1) \longrightarrow H_i(p) \longrightarrow H_{i-1}(p-1)(-1) \longrightarrow \operatorname{Im} \varphi_{i-1,p-1} \longrightarrow 0,$$

for all p and $i > 1$.

Let $\delta_{i,j,k} = \dim_K(\operatorname{Im}\varphi_{i,j})_k$. From the above exact sequences, we obtain the following equations for each integer $k \geq 0$:

$$h_{1k}(p) = h_{1k}(p-1) + \alpha_{p,k-1} - \delta_{1,p-1,k}, \tag{3.2}$$

and for all $i > 1$,

$$h_{i,i+k}(p) = h_{i,i+k}(p-1) + h_{i-1,i-1+k}(p-1) - \delta_{i,p-1,i+k} - \delta_{i-1,p-1,i+k}. \tag{3.3}$$

By using (3.2) and (3.3), we obtain

Proposition 3.1.1. *For all nonnegative integers $i \geq 1$ and k, one has*

$$\begin{aligned} h_{i,i+k}(p) \quad &= \quad \sum_{j=1}^{p-i+1} \binom{p-j}{i-1} \alpha_{j,k} \qquad\qquad (3.4) \\ &\quad - \sum_{(a,b)\in A_{i,p}} \left[\binom{p-b-1}{i-a} \delta_{a,b,a+k} + \binom{p-b-1}{i-a-1} \delta_{a,b,a+k+1} \right], \end{aligned}$$

where the set $A_{i,p} = \big\{(a,b) \in \mathbb{N}^2 : 1 \leq b \leq p-1 \text{ and } \max\{i-p+b, 1\} \leq a \leq i\big\}$.

Proof. We will prove the above formula by induction on p. For $p = 1$, we have from Equation (3.2) and Equation (3.3):

$$h_{i,i+k}(1) = \begin{cases} \alpha_{1,k} & \text{if } i = 1, \\ 0 & i \geq 2. \end{cases}$$

which is what the formula given in the statement of the proposition suggests. Now we assume $p > 1$ and we assume the result to be true for $p - 1$.

Let first $i = 1$. By induction hypothesis and from Equation (3.2), we get :

$$\begin{aligned} h_{1,1+k}(p) &= h_{1,1+k}(p-1) + \alpha_{p,k} - \delta_{1,p-1,1+k} \\ &= \sum_{j=1}^{p-1} \binom{p-1-j}{0} \alpha_{j,k} - \sum_{(a,b)\in A_{1,p-1}} \binom{p-b-2}{1-a} \delta_{a,b,a+k} + \alpha_{p,k} - \delta_{1,p-1,1+k} \\ &= \sum_{j=1}^{p} \alpha_{j,k} - \sum_{(a,b)\in A_{1,p}} \left[\binom{p-b-1}{1-a} \delta_{a,b,a+k} \right] \end{aligned}$$

which is what the formula suggests.

Now let $i > 1$. From Equation (3.3), we have:

$$h_{i,i+k}(p) \quad = \quad h_{i,i+k}(p-1) + h_{i-1,i-1+k}(p-1) - \delta_{i,p-1,i+k} - \delta_{i-1,p-1,i+k}.$$

Note that one has $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$ for all integers $a \geq b \geq 0$. Then, using induction hypothesis, the right hand side of the above equation is a sum of the following three terms:

$$\textstyle\sum_{j=1}^{p-i+1} \left\{ \binom{p-j-1}{i-1} + \binom{p-j-1}{i-2} \right\} \alpha_{j,k} = \sum_{j=1}^{p-i+1} \binom{p-j}{i-1} \alpha_{j,k}, \tag{3.5}$$

$$-\{ \sum_{(a,b)\in A_{i,p-1}} \binom{p-b-2}{i-a}\delta_{a,b,a+k} + \delta_{i,p-1,i+k} + \sum_{(a,b)\in A_{i-1,p-1}} \binom{p-b-2}{i-a-1}\delta_{a,b,a+k}\}, \quad (3.6)$$

and

$$-\{ \sum_{(a,b)\in A_{i,p-1}} \binom{p-b-2}{i-a-1}\delta_{a,b,a+k+1} + \delta_{i-1,p-1,i+k} + \sum_{(a,b)\in A_{i-1,p-1}} \binom{p-b-2}{i-a-2}\delta_{a,b,a+k+1}\}. \quad (3.7)$$

The term (3.6) can be written as:

$$-\{ \sum_{(a,b)\in A_{i,p-1}} \binom{p-b-2}{i-a}\delta_{a,b,a+k} + \sum_{(a,b)\in A_{i,p-1}} \binom{p-b-2}{i-a-1}\delta_{a,b,a+k} + \delta_{i,p-1,i+k}\},$$

which is further equal to

$$-\{ \sum_{(a,b)\in A_{i,p-1}} \binom{p-b-1}{i-a}\delta_{a,b,a+k} + \delta_{i,p-1,i+k}\},$$

which in the end equals

$$-\sum_{(a,b)\in A_{i,p}} \binom{p-b-1}{i-a}\delta_{a,b,a+k}. \quad (3.8)$$

Now we notice that the term (3.7) can be written as:

$$-\{ \sum_{(a,b)\in A_{i,p-1}} \binom{p-b-2}{i-a-1}\delta_{a,b,a+k+1} + \sum_{(a,b)\in A_{i,p-1}} \binom{p-b-2}{i-a-2}\delta_{a,b,a+k+1} + \sum_{b=p-i+1}^{p-2} \delta_{i-p+b,b,i-p+b+k+1} + \delta_{i-1,p-1,i+k}\}.$$

This can be rewritten as:

$$-\{ \sum_{(a,b)\in A_{i,p-1}} \binom{p-b-1}{i-a-1}\delta_{a,b,a+k+1} + \sum_{b=p-i+1}^{p-2} \delta_{i-p+b,b,i-p+b+k+1} + \delta_{i-1,p-1,i+k}\},$$

which then is equal to

$$-\sum_{(a,b)\in A_{i,p}} \binom{p-b-1}{i-a-1}\delta_{a,b,a+k+1}. \quad (3.9)$$

Hence $h_{i,i+k}(p)$ is the sum of (3.5),(3.8) and (3.9), as required. □

Remark 3.1.2. Notice that summing the formula stated in Proposition 3.1.1 over k, gives us back the formula given in the proof of [CoHHi04, Proposition 1.1].

Proposition 3.1.1 implies the following fact.

Corollary 3.1.3. *We have*

(a) $h_{i,i+k}(p) \leq \sum_{j=1}^{p-i+1} \binom{p-j}{i-1}\alpha_{j,k}$.

(b) *For given integers $i \geq 1$ and $p \geq 1$, the following conditions are equivalent:*

(i) $h_{i,i+k}(p) = \sum_{j=1}^{p-i+1} \binom{p-j}{i-1}\alpha_{j,k}$

(ii) $(\operatorname{Im}\varphi_{a,b})_{(a+k)} = 0$ *for all* $(a,b) \in A_{i,p} \setminus \{(i-p+b,b) : b \leq p-1\}$ *and* $(\operatorname{Im}\varphi_{a,b})_{(a+k+1)} = 0$ *for all* $(a,b) \in A_{i,p} \setminus \{(i,b) : b \leq p-1\}$.

(iii) $\big(\mathfrak{m}H_a(b)\big)_{(a+k)} = 0$ *for all* $(a,b) \in A_{i,p} \setminus \big\{(i-p+b,b) : b \le p-1\big\}$ *and* $\big(\mathfrak{m}H_a(b)\big)_{(a+k+1)} = 0$ *for all* $(a,b) \in A_{i,p} \setminus \big\{(i,b) : b \le p-1\big\}$.

Proof. Statement (a) is clear from Proposition 3.1.1. The equivalence of (i) and (ii) follows immediately from Proposition 3.1.1. Indeed, $h_{i,i+k}(p) = \sum_{j=1}^{p-i+1}\binom{p-j}{i-1}\alpha_{j,k}$ if and only if all graded maps appearing in the formula in Proposition 3.1.1 vanish whenever their binomial coefficients are nonzero. And for the equivalence of (ii) and (iii), we may notice that a generic linear form annihilates $\big(H_a(b)\big)_k$ if and only if $\mathfrak{m}$ annihilates $\big(H_a(b)\big)_k$. □

The next corollary is a special case ($p = n$) of the above corollary.

Corollary 3.1.4. (a) $\beta^S_{i,i+k} \le \sum_{j=1}^{n-i+1}\binom{n-j}{i-1}\alpha_{j,k}$ *for all* $i \ge 1$.

(b) *For given* $i \ge 1$ *the following are equivalent:*

(i) $\beta^S_{i,i+k} = \sum_{j=1}^{n-i+1}\binom{n-j}{i-1}\alpha_{j,k}$.

(ii) $(\operatorname{Im}\varphi_{a,b})_{(a+k)} = 0$ *for all* $(a,b) \in A_{i,n} \setminus \big\{(i-n+b,b), b \le n-1\big\}$ *and* $(\operatorname{Im}\varphi_{a,b})_{(a+k+1)} = 0$ *for all* $(a,b) \in A_{i,n} \setminus \big\{(i,b), b \le n-1\big\}$.

(iii) $\big(\mathfrak{m}H_a(b)\big)_{(a+k)} = 0$ *for all* $(a,b) \in A_{i,n} \setminus \big\{(i-n+b,b), b \le n-1\big\}$ *and* $\big(\mathfrak{m}H_a(b)\big)_{(a+k+1)} = 0$ *for all* $(a,b) \in A_{i,n} \setminus \big\{(i,b), b \le n-1\big\}$.

3.2 Graded rigidity of resolutions and linear components

In this section we generalize [CoHHi04, Theorem 2.3] of Conca–Herzog–Hibi. They gave an upper bound of total Betti numbers in terms of generic annihilator numbers, and proved that if the Betti number $\beta^S_i(M)$ for some $i \ge 1$ reaches its upper bound, then the Betti numbers $\beta^S_q(M)$ also reach their upper bounds for all $q \ge i$. We show that if a graded Betti number $\beta^S_{i,i+k}(M)$ for some $i > 1$ reaches its upper bound given in Corollary 3.1.4, then so do all the graded Betti numbers $\beta^S_{q,q+k}(M)$ for $q \ge i$. Here we need the assumption $i > 1$ as we will see later in Remark 3.2.4.

We state the main theorem of this section:

Theorem 3.2.1. *Let* M *be a finitely generated graded* S*–module. Suppose for some* $i > 1$*, we have* $\beta^S_{i,i+k}(M) = \sum_{j=1}^{n-i+1}\binom{n-j}{i-1}\alpha_{j,k}(M)$*. Then*

$$\beta^S_{q,q+k}(M) = \sum_{j=1}^{n-q+1}\binom{n-j}{q-1}\alpha_{j,k}(M) \text{ for all } q \ge i.$$

Before proving the theorem, we recall the following vanishing property of Koszul homology. For a sequence of elements $y_1, \dots, y_r \in S$ and a set $A \subseteq \{1, \dots, r\}$, we set $y_A = \{y_j : j \in A\}$.

Lemma 3.2.2. *Let $I \supseteq (y_1, \dots, y_r)$ and assume that $\big(IH_i(y_A; M)\big)_{i+k} = 0$ for all $A \subseteq \{1, \dots, r\}$ for some i, k. Then $\big(IH_{i+1}(y_A; M)\big)_{i+k+1} = 0$ for all $A \subseteq \{1, \dots, r\}$.*

The proof of Lemma 3.2.2 is the same as [CoHHi04, Corollary 2.2]. Hence we skip the proof.

Proof of Theorem 3.2.1. First we notice that it is enough to prove the claim in the case when $q = i+1$. Therefore we only need to show that $\big(\mathfrak{m}H_a(b)\big)_{a+k} = 0$ for all $(a, b) \in A_{i+1,n} \setminus \big\{(i+1-n+b, b) : b \leq n-1\big\}$ and $\big(\mathfrak{m}H_a(b)\big)_{a+k+1} = 0$ for all $(a, b) \in A_{i+1,n} \setminus \big\{(i+1, b) : b \leq n-1\big\}$, as is clear from Corollary 3.1.3.

By assumption, $\big(\mathfrak{m}H_a(b)\big)_{a+k} = 0$ for all $(a, b) \in A_{i,n} \setminus \big\{(i-n+b, b) : b \leq n-1\big\}$ and $\big(\mathfrak{m}H_a(b)\big)_{a+k+1} = 0$ for all $(a, b) \in A_{i,n} \setminus \big\{(i, b) : b \leq n-1\big\}$. Also a routine computation implies

$$A_{i+1,n} \setminus \big(A_{i,n} \setminus \big\{(i-n+b, b) : b \leq n-1\big\}\big) = \big\{(i+1, b) : b \leq n-1\big\}$$

and

$$\big(A_{i+1,n} \setminus \{(i+1, b) : b \leq n-1\}\big) \setminus \big(A_{i,n} \setminus \{(i, b) : b \leq n-1\}\big) = \big\{(i, b) : b \leq n-1\big\}.$$

Therefore, we need to show that $\big(\mathfrak{m}H_{i+1}(b)\big)_{i+1+k} = 0$ and $\big(\mathfrak{m}H_i(b)\big)_{i+k+1} = 0$ for all $b \leq n-1$. However, from assumption $\big(\mathfrak{m}H_i(b)\big)_{i+k} = 0$ and $\big(\mathfrak{m}H_{i-1}(b)\big)_{i+k} = 0$ for all $b \leq n-1$, now it follows from Lemma 3.2.2 that for all $b \leq n-1$, we have $\big(\mathfrak{m}H_{i+1}(b)\big)_{i+1+k} = 0$ and $\big(\mathfrak{m}H_i(b)\big)_{i+k+1} = 0$. Hence we are done. □

A graded ideal $I \subset S$ generated in degree d is said to have a *linear resolution* if the regularity $\operatorname{reg}(I) = \max\big\{k : \beta^S_{i,i+k}(I) \neq 0\big\}$ of I is equal to d. Also, a graded ideal I is said to be *componentwise linear* if the ideal $I_{\langle k\rangle}$ has linear resolution for each k. A monomial ideal $I \subset S$ is said to be *strongly stable* if $ux_q \in I$ implies $ux_p \in I$ for any $1 \leq p < q \leq n$. Note that generic initial ideals are strongly stable if $\operatorname{char}(K) = 0$, and strongly stable ideals are componentwise linear.

Theorem 3.2.1 has a nice meaning in the special case $M = S/I$ where I is a graded ideal in S. Let $I \subset S$ be a graded ideal and $\operatorname{Gin}(I)$ its generic initial ideal with respect to the reverse lexicographic order. From [CoHHi04, Theorem 1.5], it follows that a graded ideal $I \subset S$ is componentwise linear if and only if the Betti numbers of S/I reaches the upper bound given in Corollary 3.1.4. Also, it is not hard to show that the generic graded annihilator numbers $\alpha_{i,j}(S/I) = \alpha_{i,j}(S/\operatorname{Gin}(I))$ for all i and j (see [CoHHi04, Lemma 2.5]). Then, since $\operatorname{Gin}(I)$ is componentwise linear, we have

$$\beta^S_{i,i+k}\big(S/\operatorname{Gin}(I)\big) = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_{j,k}(S/I) \quad \text{for all } i \text{ and } k. \tag{3.10}$$

This fact and Theorem 3.2.1 immediately imply

Corollary 3.2.3. *Suppose* $\operatorname{char} K = 0$. *Let* $I \subset S$ *be a graded ideal. If for some* $i > 1$ *and* $k \geq 0$, $\beta^S_{i,i+k}(S/I) = \beta^S_{i,i+k}\big(S/\operatorname{Gin}(I)\big)$, *then*

$$\beta^S_{q,q+k}(S/I) = \beta^S_{q,q+k}\big(S/\operatorname{Gin}(I)\big) \text{ for all } q \geq i.$$

Remark 3.2.4. The assumption $i > 1$ in Theorem 3.2.1 (and Corollary 3.2.3) is necessary. In the case when $i = 1$, we notice from the proof that we need to show that $\big(\mathfrak{m}H_2(b)\big)_{2+k} = 0$ and $\big(\mathfrak{m}H_1(b)\big)_{2+k} = 0$ for all $b \leq n-1$. As the set $A_{1,n} \setminus \big\{(1,b),\ b \leq n-1\big\} = \emptyset$, the second equality does not follow. Moreover in the case when $M = R/I$ where $I \subset S$ is a graded ideal, we always have that the graded Betti number $\beta^S_{1,d_0}(R/I) = \beta^S_{1,d_0}\big(R/\operatorname{Gin}(I)\big) = \sum_{j=1}^{n+1} \alpha_{j,d_0-1}(R/I)$ where d_0 is the minimum of the degrees of generators of I. So if Theorem 3.2.1 would have been true for $i = 1$, then it would follow that $\beta^S_{i,i+d_0-1}(R/I) = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_{j,d_0-1}(R/I)$ for all i, which is false in general.

As we see in Remark 3.2.4, Corollary 3.2.3 is false for $i = 1$. However, the following property is true for the first graded Betti numbers.

Corollary 3.2.5. *Suppose* $\operatorname{char} K = 0$. *Let* $I \subset S$ *be a graded ideal. Then, for a given integer* k, *the graded Betti numbers* $\beta^S_{i,i+k}(S/I) = \beta^S_{i,i+k}\big(S/\operatorname{Gin}(I)\big)$ *for all* $i \geq 1$ *if and only if* $\beta^S_{1,k+1}(S/I) = \beta^S_{1,k+1}\big(S/\operatorname{Gin}(I)\big)$ *and* $\beta^S_{1,k+2}(S/I) = \beta^S_{1,k+2}\big(S/\operatorname{Gin}(I)\big)$.

Proof. First, we will show the "if" part. Since $\beta^S_{1,k+1}(S/I) = \beta^S_{1,k+1}\big(S/\operatorname{Gin}(I)\big)$ and $\beta^S_{1,k+2}(S/I) = \beta^S_{1,k+2}\big(S/\operatorname{Gin}(I)\big)$, Corollary 3.1.4 says that $\mathfrak{m}H_1(b)_{1+k} = 0$ and $\mathfrak{m}H_1(b)_{2+k} = 0$ for all $b \leq n-1$. Thus Lemma 3.2.2 says that $\mathfrak{m}H_a(b)_{a+k} = 0$ and $\mathfrak{m}H_a(b)_{a+k+1} = 0$ for all (a,b) with $a \in \mathbb{Z}$ and $b \leq n-1$. Then, by Corollary 3.1.4, we have $\beta^S_{i,i+k}(S/I) = \beta^S_{i,i+k}\big(S/\operatorname{Gin}(I)\big)$ for all $i \geq 1$.

Next, we will show the "only if" part. Since $\beta^S_{1,k+1}(S/I) = \beta^S_{1,k+1}\big(S/\operatorname{Gin}(I)\big)$ follows from the assumption, what we must prove is $\beta^S_{1,k+2}(S/I) = \beta^S_{1,k+2}\big(S/\operatorname{Gin}(I)\big)$. Since $\beta^S_{2,k+2}(S/I) = \beta^S_{2,k+2}\big(S/\operatorname{Gin}(I)\big)$, Corollary 3.1.4 says that $\mathfrak{m}H_a(b)_{(a+k+1)} = 0$ for all $(a,b) \in A_{2,n} \setminus \big\{(2,b) : b \leq n-1\big\} = A_{1,n}$. This fact and Corollary 3.1.4 imply $\beta^S_{1,k+2}(S/I) = \beta^S_{1,k+2}\big(S/\operatorname{Gin}(I)\big)$. ☐

For any monomial $u \in S$, write $m(u)$ for the maximal integer i such that x_i divides u. We recall a result of Eliahou–Kervaire [EK90] which we need in the proof of our next proposition. They proved that if $I \subset S$ is a strongly stable ideal then

$$\beta_{i,i+j}(I) = \sum_{u \in G(I),\ \deg(u)=j} \binom{m(u)-1}{i} \quad \text{for all } i \text{ and } j \tag{3.11}$$

where $G(I)$ is the minimal set of monomial generators of I. Aramova–Herzog–Hibi [AHHi00, Theorem 1.1] proved that a graded ideal I in S with $\operatorname{char}(K) = 0$ is componentwise linear if and only if $\beta^S_{i,j}(I) = \beta^S_{i,j}(\operatorname{Gin}(I))$ for all i, j. We will refine this result in terms of the maximal degree of minimal generators.

Proposition 3.2.6. *Suppose* $\operatorname{char} K = 0$. *Let* $I \subset S$ *be a graded ideal, and let* d *be the maximum of the degrees of the generators of* I. *Then the following conditions are equivalent.*

(i) I *is componentwise linear;*

(ii) $\beta^S_{i,i+k}(I) = \beta^S_{i,i+k}\big(\operatorname{Gin}(I)\big)$ *for all* $i \geq 0$ *and all* $k \leq d$*;*

(iii) $\beta^S_{1,1+k}(I) = \beta^S_{1,1+k}\big(\operatorname{Gin}(I)\big)$ *for all* $k \leq d$*;*

(iv) $\beta^S_{0,k}(I) = \beta^S_{0,k}\big(\operatorname{Gin}(I)\big)$ *for all* $k \leq d+1$.

Proof. (i) $\Rightarrow$ (ii) follows from [AHHi00, Theorem 1.1] and (ii) $\Rightarrow$ (iii) is obvious. On the other hand, we already proved that if $\beta^S_{1,k}(I) = \beta^S_{1,k}\big(\operatorname{Gin}(I)\big)$, then we have $\beta^S_{0,k}(I) = \beta^S_{0,k}\big(\operatorname{Gin}(I)\big)$ in the proof of Corollary 3.2.5. This fact implies (iii)$\Rightarrow$ (iv).

Now we show (iv) $\Rightarrow$ (i). We have $\beta^S_{0,d+1}(I) = \beta^S_{0,d+1}\big(\operatorname{Gin}(I)\big) = 0$, by assumption. Now, since $\operatorname{Gin}(I)$ is strongly stable, by Eliahou–Kervaire formula (3.11) we have $\beta^S_{i,i+d+1}(I) = \beta^S_{i,i+d+1}\big(\operatorname{Gin}(I)\big) = 0$ for all $i \geq 0$. However, the equality of graded Betti numbers $\beta^S_{1,d+2}(I) = \beta^S_{1,d+2}\big(\operatorname{Gin}(I)\big) = 0$ implies the equality $\beta^S_{0,d+2}(I) = \beta^S_{0,d+2}\big(\operatorname{Gin}(I)\big) = 0$ as we see in the proof of Corollary 3.2.5. Then again we have $\beta^S_{i,i+d+2}(I) = \beta^S_{i,i+d+2}\big(\operatorname{Gin}(I)\big) = 0$ for all $i \geq 0$. Arguing inductively, we have $\beta^S_{0,j}(I) = \beta^S_{0,j}\big(\operatorname{Gin}(I)\big)$ for all $j \geq 0$. Then Corollary 3.2.5 implies that $\beta_{i,j}(I) = \beta_{i,j}\big(\operatorname{Gin}(I)\big)$ for all i, j. Hence I is componentwise linear. □

3.3 The Cartan–complex and generic annihilator numbers

In this section, we recall some basic facts about Cartan complex introduced by Cartan and consider generic annihilator numbers in an exterior algebra.

Let K be an infinite field, V an n–dimensional K–vector space with basis $e_1, \dots, e_n$ and $E = \bigoplus_{k=0}^n \bigwedge^k V$ the exterior algebra of V. For any subset $S = \{i_1, \dots, i_d\}$ with $1 \leq i_1 < \cdots < i_d \leq n$, the element $e_S = e_{i_1} \wedge \cdots \wedge e_{i_d} \in E$ is called a *monomial* of E of degree d. Let $v_1, \dots, v_m \in E_1$ be linear forms. The *Cartan complex* $C_\bullet(v_1, \dots, v_m; E)$ of the sequence $v_1, \dots, v_m$ is defined as the complex whose i–chains $C_i(v_1, \dots, v_m; E)$ are the elements of degree i of the free divided power algebra $E\langle x_1, \dots, x_m\rangle$. Thus $C_\bullet(v_1, \dots, v_m; E)$ is the polynomial ring over E in the set of variables

$$x_i^{(j)}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots,$$

modulo the relations

$$x_i^{(j)} x_i^{(k)} = \frac{(j+k)!}{j!k!} x_i^{(k+j)},$$

where we set $x_i^{(0)} = 1$ and $x_i^{(1)} = x_i$ for $i = 1, \dots, m$. The algebra $C_\bullet(v_1, \dots, v_m; E)$ is a free E–module with basis $x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)}$ with $a \in \mathbb{Z}^m$.

The E–linear differential ∂ on $C_\bullet(v_1,\dots,v_m;E)$ is defined by

$$\partial\big(x^{(a)}\big)=\sum_{a_i>0} v_i\cdot x_1^{(a_1)}\cdots x_i^{(a_i-1)}\cdots x_m^{(a_m)}.$$

It is easily verified that $\partial\circ\partial=0$, so that $C_\bullet(v_1,\dots,v_m;E)$ is indeed a complex.

Let $\mathcal{M}$ be the category of finitely generated graded left and right E–module satisfying $ax=(-1)^{\deg(a)+\deg(x)}xa$ for all homogeneous elements $a\in E$ and $x\in M$, where $M\in\mathcal{M}$. The complex $C_\bullet(v_1,\dots,v_m;M)=C_\bullet(v_1,\dots,v_m;E)\otimes_E M$ is called the *Cartan complex of M with respect to* $v_1,\dots,v_m\in E_1$, and its homology $H_\bullet(v_1,\dots,v_m;M)$ is called the *Cartan homology.* We recall two basic properties of the Cartan homology. (See [AHHi97] or [H02] for the detail.)

Lemma 3.3.1. [AHHi97, Theorem 2.2] *Let $v_1,\dots,v_n\in E$ be linearly independent linear forms and $M\in\mathcal{M}$. One has*

$$H_i(v_1,\dots,v_n;M)_j\cong \operatorname{Tor}_i^E(K,M)_j.$$

Lemma 3.3.2. [AHHi97, Corollary 2.4] *Let $v_1,\dots,v_n\in E$ be linear forms and $M\in\mathcal{M}$. For $p=1,2,\dots,n-1$, there exists a long exact sequence*

$$\cdots\xrightarrow{\gamma_{i,p}} H_i(v_1,\dots,v_p;M)\xrightarrow{\eta_{i,p}} H_i(v_1,\dots,v_{p+1};M)\xrightarrow{\psi_{i,p}} H_{i-1}(v_1,\dots,v_{p+1};M)(-1)$$
$$\xrightarrow{\gamma_{i-1,p}} H_{i-1}(v_1,\dots,v_p;M)\xrightarrow{\eta_{i-1,p}} H_{i-1}(v_1,\dots,v_{p+1};M)\longrightarrow\cdots$$

where $\eta_{i,p}$ is the map induced by the inclusion map and the maps $\psi_{i,p}$ and $\gamma_{i,p}$ are defined as follows: If $z=g_0+g_1x_{p+1}+\cdots+g_ix_{p+1}^{(i)}$ is a cycle in $C_i(v_1,\dots,v_{p+1};M)$ with each $g_k\in C_i(v_1,\dots,v_p;M)$, then $\psi_{i,p}\big([z]\big)=[g_1+g_2x_{p+1}+\cdots+g_ix_{p+1}^{(i-1)}]$ and $\gamma_{i,p}\big([z]\big)=[g_0v_{p+1}]$.

Next, we will introduce generic annihilator numbers in the exterior algebra. Let $M\in\mathcal{M}$ and let $v_1,\dots,v_n\in E$ be generic linear forms of M. For $p=1,2,\dots,n$, set

$$A^{(p)}(M)=\big((v_1,\dots,v_{p-1})M:_M v_p\big)/(v_1,\dots,v_p)M \tag{3.12}$$

and

$$\alpha_{p,k}(M)=\dim_K\big(A^{(p)}(M)_k\big).$$

Note that $A^{(p)}(M)=\operatorname{Ker}(\gamma_{0,p-1})$ for $p=2,3,\dots,n$. These numbers $\alpha_{p,k}(M)$ are constant for a generic choice of linear forms $v_1,\dots,v_n\in E_1$, and will be called *exterior generic annihilator numbers of M*. In the rest of this section, we will give the formula to compute the graded Betti numbers of generic initial ideals in the exterior algebra from exterior generic annihilator numbers.

A monomial ideal $J\subset E$ is said to be *strongly stable* if $e_S\in J$ and $j\in S$ implies that $e_{(S\setminus\{j\})\cup\{i\}}\in J$ for all $i<j$ with $i\notin S$. It is known that generic initial ideals are strongly stable ([AHHi97, Proposition 1.7]).

Lemma 3.3.3. *Let $J \subset E$ be a graded ideal. Then one has*

$$\alpha_{p,k}(E/J) = \left|\left\{e_S \in G\big(\operatorname{Gin}(J)\big)_{k+1} : \max(S) = n-p+1\right\}\right| \quad \textit{for } p = 1,2,\ldots,n,$$

where $|A|$ denotes the cardinality of a finite set A and $G\big(\operatorname{Gin}(J)\big)_{k+1}$ is the set of minimal monomial generators of $\operatorname{Gin}(J)$ of degree $k+1$.

Proof. By a generic change of coordinates, we may assume that $\operatorname{in}(J) = \operatorname{Gin}(J)$ and $v_1, v_2, \ldots, v_{p+1} = e_n, e_{n-1}, \ldots, e_{n-p}$. Then, by (3.12), we have

$$A^{(p+1)}(E/J) = \big((e_n, \ldots, e_{n-p+1}) + J :_E e_{n-p}\big)/\big((e_n, \ldots, e_{n-p}) + J\big),$$

where $p = 0, 1, \ldots, n-1$. Set

$$B^{(p+1)}(E/J) = \big((e_n, \ldots, e_{n-p+1}) + \operatorname{in}(J) :_E e_{n-p}\big)/\big((e_n, \ldots, e_{n-p}) + \operatorname{in}(J)\big).$$

Since we consider the reverse lexicographic order induced by $e_1 > \cdots > e_n$, it follows from [AH00, Proposition 5.1] that

$$\operatorname{in}\big((e_n, \ldots, e_{n-p+1}) + J :_E e_{n-p}\big) = \big((e_n, \ldots, e_{n-p+1}) + \operatorname{in}(J) :_E e_{n-p}\big)$$

and

$$\operatorname{in}\big((e_n, \ldots, e_{n-p}) + J\big) = (e_n, \ldots, e_{n-p}) + \operatorname{in}(J).$$

Since $\big((e_n, \ldots, e_{n-p+1}) + J :_E e_{n-p}\big) \supset (e_n, \ldots, e_{n-p}) + J$ and taking initial ideals does not change Hilbert functions, it follows that $B^{(p+1)}(E/J)$ and $A^{(p+1)}(E/J)$ have the same Hilbert function. Thus we have $\alpha_{p,k}(E/J) = \dim_K B^{(p)}(E/J)_k$ for all $k \geq 0$.

Then, to prove the claim, it is enough to show that the set of monomials

$$\{[e_S] \in E/((e_n, \ldots, e_{n-p}) + \operatorname{in}(J)) : \max(S) < n-p,\ e_S \wedge e_{n-p} \in G(\operatorname{in}(J))_{k+1}\} \tag{3.13}$$

forms a K–basis of $B^{(p+1)}(E/J)_k$.

If e_S satisfies the condition of (3.13), then we have $e_S \notin (e_n, \ldots, e_{n-p}) + \operatorname{in}(J)$. Thus the set (3.13) is indeed the set of K–linearly independent monomials belonging to $B^{(p+1)}(E/J)$. Hence we need to prove that any nonzero monomial $e_S \in B^{(p+1)}(E/J)$ of degree k is contained in the set (3.13).

Let $[e_S] \in B^{(p+1)}(E/J)\backslash\{0\}$ be a monomial of degree k. Then we have $e_S \wedge e_{n-p} \in (e_n, \ldots, e_{n-p+1}) + \operatorname{in}(J)$. Also, since $[e_S]$ is not zero, we have $e_S \notin (e_n, \ldots, e_{n-p})$. Thus we have $\max(S) < n-p$ and $e_S \wedge e_{n-p} \in \operatorname{in}(J)$. Since $\operatorname{in}(J) = \operatorname{Gin}(J)$ is strongly stable and $e_S \notin \operatorname{in}(J)$, any monomial $e_T \in E$ of degree k which divides $e_S \wedge e_{n-p}$ does not belongs to $\operatorname{in}(J)$. Thus we have $e_S \wedge e_{n-p} \in G(\operatorname{in}(J))$, and $[e_S]$ is contained in the set (3.13). □

For a monomial $e_S \in E$, let $m(e_S) = \max(S)$. If $J \subset E$ is a strongly stable ideal, then it follows from [AHHi97, Corollary 3.3] that

$$\beta^E_{i,i+k}(E/J) = \sum_{p=k+1}^{n} \sum_{\substack{e_S \in G(J)_{k+1} \\ m(e_S)=p}} \binom{p-1+i-1}{i-1} \quad \text{for all } i \geq 1 \text{ and all } k \geq 0.$$

Since every generic initial ideal is strongly stable, the above equality together with Lemma 3.3.3 imply the next lemma.

Lemma 3.3.4. *Let J be a graded ideal in E. Then one has*

$$\beta^E_{i,i+k}\big(E/\operatorname{Gin}(J)\big) = \sum_{p=1}^{n-k} \binom{n-p+i-1}{i-1} \alpha_{p,k}(E/J)$$

for all $i \geq 1$ and all $k \geq 0$.

3.4 Rigidity of resolutions over the exterior algebra

In this section, we will prove similar results studied in Section 2 for generic initial ideals in the exterior algebra.

Let $M \in \mathcal{M}$. Throughout this section, let $v_1, \ldots, v_n \in E_1$ be generic liner forms and write $H_i(p)_k$, $h_{i,k}(p)$ and $\alpha_{p,k}$ for $H_i(v_1, \ldots, v_p; M)_k$, $\dim_K\big(H_i(v_1, \ldots, v_p; M)_k\big)$ and $\alpha_{p,k}(M)$ respectively. Set $\delta_{i,p,k} = \dim_K\big(\operatorname{Im}(\gamma_{i,p})_k\big)$ for $i > 0$ and $\delta_{0,p,k} = 0$ for all p, k.

For an integer $j \geq 0$, Lemma 3.3.2 yields the following exact sequence

$$\cdots \xrightarrow{\gamma_{i,p}} H_i(p)_j \xrightarrow{\eta_{i,p}} H_i(p+1)_j \xrightarrow{\psi_{i,p}} H_{i-1}(p+1)_{j-1} \xrightarrow{\gamma_{i-1,p}} H_{i-1}(p)_j \longrightarrow \cdots$$

where $p = 1, 2, \ldots, n-1$. Then, in the same way as Section 1, we have

$$h_{1,k}(p+1) \quad = \quad h_{1,k}(p) + \alpha_{p+1,k-1} - \delta_{1,p,k} \tag{3.14}$$

and, for $i > 1$, we have

$$h_{i,i+k}(p+1) \quad = \quad h_{i,i+k}(p) + h_{i-1,i+k-1}(p+1) - \{\delta_{i,p,i+k} + \delta_{i-1,p,i+k}\}. \tag{3.15}$$

Proposition 3.4.1. *With the same notation as above, one has*

$$\begin{aligned} h_{i,i+k}(p) \quad = \quad & \sum_{j=1}^{p} \binom{p-j+i-1}{i-1} \alpha_{j,k} \\ & - \sum_{s=1}^{i} \sum_{j=1}^{p-1} \binom{p-1-j+i-1-(s-1)}{i-1-(s-1)} \Big\{ \delta_{s,j,s+k} + \delta_{s-1,j,s+k} \Big\}. \end{aligned} \tag{3.16}$$

Proof. The proof is quite similar to the proof of Proposition 3.1.1. So we will skip some detail calculations.

We use induction on p and i. First, we will show the case $p = 1$. Recall that $C_\bullet(v_1; M)$ is the complex

$$\cdots \longrightarrow C_{i+1}(v_1; M) \xrightarrow{\partial} C_i(v_1; M) \xrightarrow{\partial} C_{i-1}(v_1; M) \longrightarrow \cdots$$

with the differential $\partial(x_1^{(i)}) = v_1 x_1^{(i-1)}$. Thus we have

$$H_i(1)_{i+k} \cong \big((M :_M v_1)/v_1 M\big)_k = A^{(1)}(M)_k,$$

and therefore we have $h_{i,i+k}(1) = \alpha_{1,k}$ for all $i \geq 1$ and all $k \geq 0$. This is equal to the formula (3.16).

Second, we will consider the case $i = 1$. Since we already proved $h_{1,1+k}(1) = \alpha_{1,k}$, the equation (3.14) says that

$$h_{1,1+k}(p) \;=\; \{\alpha_{1,k} + \cdots + \alpha_{p,k}\} - \{\delta_{1,1,1+k} + \cdots + \delta_{1,p-1,1+k}\}$$

which is equal to the formula (3.16).

Finally, the formula (3.16) for $i > 1$ and $p > 1$ follows from the equation (3.15) together with the induction hypothesis in the same way as Proposition 3.1.1. □

Next, we will show the following vanishing property of $\mathrm{Im}(\gamma_{i,p})$, which is an analogue of Lemma 3.2.2.

Lemma 3.4.2. *Let $i \geq 1$ be a positive integer. If $\delta_{i,p,k} = 0$ for all $1 \leq p \leq n-1$, then one has $\delta_{i+t,p,k+t} = 0$ for all $1 \leq p \leq n-1$ and all $t \geq 0$.*

Proof. It is enough to prove the claim for $t = 1$. Remark that $\delta_{i,p,k} = 0$ if and only if the map $\eta_{i,p} : H_i(p)_k \to H_i(p+1)_k$ is injective. Let $\partial_\ell^{(p)} : H_{i+1}(p)_{k+1} \to H_i(p)_k$ be the map defined by

$$\partial_\ell^{(p)}\big([g_0 + g_1 x_\ell + g_2 x_\ell^{(2)} + \cdots + g_{i+1} x_\ell^{(i+1)}]\big) = [g_1 + g_2 x_\ell + \cdots + g_{i+1} x_\ell^{(i)}],$$

where $1 \leq \ell \leq p$ and each g_t does not contain the variable $x_\ell^{(s)}$ for all $s \geq 1$. Thus $\partial_p^{(p)}$ is equal to the map $\psi_{i+1,p-1}$ which appears in Lemma 3.3.2. Set $\partial^{(p)} = \bigoplus_{\ell=1}^p \partial_\ell^{(p)}$. Then we have the following commutative diagram.

$$\begin{array}{ccc}
H_{i+1}(p)_{k+1} & \xrightarrow{\partial^{(p)}} & \bigoplus_{k=1}^{p} H_i(p)_k \\
\downarrow f_p & & \downarrow h_p \\
H_{i+1}(p+1)_{k+1} & \xrightarrow{\partial^{(p+1)}} & \bigoplus_{k=1}^{p+1} H_i(p+1)_k
\end{array}$$

where h_p is the map defined by $h_p(z_1, \ldots, z_p) = \big(\eta_{i,p}(z_1), \ldots, \eta_{i,p}(z_p), 0\big)$ and f_p is the map defined by $f_p(z) = \eta_{i+1,p}(z)$.

Then $\partial^{(1)}$ is injective since $\partial^{(1)}\big([g_{i+1}x_1^{(i+1)}]\big) = [g_{i+1}x_1^{(i)}]$. Also, by the assumption, the map $\eta_{i,p} : H_i(p)_k \to H_{i+1}(p+1)_k$ is injective for all $1 \le p \le n-1$. Thus h_p is injective for all $1 \le p \le n-1$. We will show that if $\partial^{(p)}$ is injective then $\partial^{(p+1)}$ is also injective.

Set $u \in \operatorname{Ker}(\partial^{(p+1)})$. Then we have $\partial^{(p+1)}_{p+1}(u) = \psi_{i+1,p}(u) = 0$. Thus, by the long exact sequence in Lemma 3.3.2, there exists $w \in H_{i+1}(p)$ such that we have $\eta_{i+1,p}(w) = f_p(w) = u$. Since $h_p \circ \partial^{(p)}(w) = \partial^{(p+1)} \circ f_p(w) = 0$ and $h_p \circ \partial^{(p)}$ is injective by the induction hypothesis, it follows that $w = 0$ and $\partial^{(p+1)}$ is injective.

Now, we proved that $\partial^{(p)}$ is injective for all $1 \le p \le n-1$. Thus $h_p \circ \partial^{(p)}$ is injective for all $1 \le p \le n-1$. This fact together with the commutative diagram imply that the map $\eta_{i+1,p} : H_{i+1}(p)_{k+1} \to H_{i+1}(p+1)_{k+1}$ is injective for all $1 \le p \le n-1$. Hence we have $\delta_{i+1,p,k+1} = \dim_K\big(\operatorname{Im}(\gamma_{i+1,p})_{k+1}\big) = 0$ for all $1 \le p \le n-1$. □

Proposition 3.4.1 and Lemma 3.4.2 imply the next theorem.

Theorem 3.4.3. *Let $M \in \mathcal{M}$. Suppose that for some $i > 1$ and $k \ge 0$, we have $\beta^E_{i,i+k}(M) = \sum_{j=1}^n \binom{n-j+i-1}{i-1}\alpha_{j,k}(M)$. Then*

$$\beta^E_{q,q+k}(M) = \sum_{j=1}^{n} \binom{n-j+i-1}{i-1}\alpha_{j,k}(M) \quad \text{for all } q \ge 1.$$

Proof. Since all binomial coefficients in the formula (3.16) are nonzero, the assumption says that $\delta_{s,p,s+k} = 0$ and $\delta_{s-1,p,s+k} = 0$ for all $1 \le s \le i$ and all $1 \le p \le n-1$. Then Lemma 3.4.2 says that $\delta_{s,p,s+k} = 0$ and $\delta_{s-1,p,s+k} = 0$ for all $s \ge 1$ and all $1 \le p \le n-1$. Thus, the statement follow from the formula (3.16). □

Next we consider the case $M = E/J$. Lemma 3.3.3 says that, for any graded ideal J of E, one has $\alpha_{j,k}(E/J) = 0$ for $j > n-k$. Thus for any $i \ge 1$ and $k \ge 0$ we have $\sum_{j=1}^n \binom{n-j+i-1}{i-1}\alpha_{j,k}(E/J) = \sum_{j=1}^{n-k} \binom{n-j+i-1}{i-1}\alpha_{j,k}(E/J)$. Then the following corollaries follows from Lemma 3.3.4 and Theorem 3.4.3 in the same way as in Section 2.

Corollary 3.4.4. *Let $J \subset E$ be a graded ideal. If $\beta^E_{i,i+k}(E/J) = \beta^E_{i,i+k}\big(E/\operatorname{Gin}(J)\big)$ for some $i > 1$ and $k \ge 0$, then*

$$\beta^E_{q,q+k}(E/J) = \beta^E_{q,q+k}\big(E/\operatorname{Gin}(J)\big) \text{ for all } q \ge 1.$$

Corollary 3.4.5. *Let $J \subset E$ be a graded ideal. Then, for a given integer k, the graded Betti numbers $\beta^E_{i,i+k}(E/J) = \beta^E_{i,i+k}\big(E/\operatorname{Gin}(J)\big)$ for all $i \ge 1$ if and only if $\beta^E_{1,k+1}(E/J) = \beta^E_{1,k+1}\big(E/\operatorname{Gin}(J)\big)$ and $\beta^E_{1,k+2}(E/J) = \beta^E_{1,k+2}\big(E/\operatorname{Gin}(J)\big)$.*

Remark 3.4.6. Notice that Corollary 3.4.4 above and Corollary 3.2.3 in Section 2 are similar. But as we see Corollary 3.4.4 is relatively more stronger. We give here an example to show that in the case of a polynomial ring one cannot have the stronger result as in Corollary 3.4.4. Consider the monomial ideal given by $I = (x_1x_4^2, x_2^3, x_2^2x_3) \subset S = \mathbb{C}[x_1, x_2, x_3, x_4]$. The minimal graded free resolution of S/I and $S/\operatorname{Gin}(I)$ are given by :

$$0 \longrightarrow S(-7) \longrightarrow S(-4) \oplus S^2(-6) \longrightarrow S^3(-3) \longrightarrow S \longrightarrow S/I \longrightarrow 0,$$

and

$$0 \longrightarrow S(-7) \longrightarrow S^2(-4) \oplus S(-5) \oplus S^2(-6) \longrightarrow$$
$$S^3(-3) \oplus S(-4) \oplus S\ (-5) \longrightarrow S \longrightarrow S/\operatorname{Gin}(I) \longrightarrow 0.$$

From above resolutions, we see that $\beta^S_{2,2+4}(S/I) = \beta^S_{2,2+4}\big(S/\operatorname{Gin}(I)\big) = 2$ and ofcourse then $\beta^S_{3,3+4}(S/I) = \beta^S_{3,3+4}\big(S/\operatorname{Gin}(I)\big) = 1$. But the graded Betti number $\beta^S_{1,1+4}(S/I) = 0 \neq 1 = \beta^S_{1,1+4}\big(S/\operatorname{Gin}(I)\big)$.

In the case of exterior algebra, the notions of regularity, linear resolutions and componentwise linear ideals are defined in the same way as in the case of polynomial ring. In [AHHi00, Theorem 2.1] it was proved that a graded ideal J in E is componentwise linear if and only if J and $\operatorname{Gin}(J)$ have the same graded Betti numbers. Theorem 3.4.4 and Corollary 3.4.5 provide the following new characterization of componentwise linear ideals in the exterior algebra. (See also [NRV07] for other characterizations of componentwise linear ideals.)

Theorem 3.4.7. *A graded ideal J in the exterior algebra E is componentwise linear if and only if $\beta^E_i(E/J) = \beta^E_i\big(E/\operatorname{Gin}(J)\big)$ for some $i \geq 1$.*

Proof. Since $\beta^E_{i,i+k}(E/J) \leq \beta^E_{i,i+k}\big(E/\operatorname{Gin}(J)\big)$ for all $i \geq 1$ and $k \geq 0$, the equality $\beta^E_i(E/J) = \beta^E_i\big(E/\operatorname{Gin}(J)\big)$ implies $\beta^E_{ii+k}(E/J) = \beta^E_{i,i+k}\big(E/\operatorname{Gin}(J)\big)$ for all $k \geq 0$. Then Theorem 3.4.4 and Corollary 3.4.5 say that $\beta^E_i(E/J) = \beta^E_i\big(E/\operatorname{Gin}(J)\big)$ for some $i \geq 1$ if and only if J and $\operatorname{Gin}(J)$ have the same graded Betti numbers. Hence the claim follows. ☐

3.5 Linear components and graded Betti numbers

Throughout this section, we assume that R is either the polynomial ring S over the field K with $\operatorname{char}K = 0$ or the exterior algebra E over an infinite field.

First, we will extend Corollaries 3.2.3 and 3.4.4 to lexsegment ideals and generic initial ideals with respect to any term order. For a strongly stable ideal I in R and for integers $q = 1, \ldots, n$ and $k \geq 0$, let

$$m_{\leq q}(I,k) = \big|\{u \in I : u \text{ is a monomial with } m(u) \leq q \text{ and } \deg(u) = k\}\big|.$$

Lemma 3.5.1. *Let $I \subset R$ be a graded ideal and $I' \subset R$ a strongly stable ideal with the same Hilbert function as I. Assume that I' satisfies $m_{\leq q}(I',d) \leq m_{\leq q}\big(\operatorname{Gin}(I),d\big)$ for all q,d and $\beta^R_{i,i+k}(R/I) = \beta^R_{i,i+k}(R/I')$ for some $i > 1$ and $k \geq 0$.*

(i) *If $R = S$, then one has $\beta^S_{q,q+k}(S/I) = \beta^S_{q,q+k}(S/I')$ for all $q \geq i$.*

(ii) *If $R = E$, then one has $\beta^E_{q,q+k}(E/I) = \beta^E_{q,q+k}(E/I')$ for all $q \geq 1$.*

Proof. We will show the case $R = S$. (The proof for the case $R = E$ is same.) It follows from [Bi93, Proposition 2.3] that, for any strongly stable ideal $J \subset S$, we have

$$\begin{aligned}\beta^S_{i,i+j}(S/J) &= \dim_K J_{j+1}\binom{n-1}{i} \\ &\quad -\sum_{q=i}^{n-1} m_{\le q}(J,j+1)\binom{k-1}{i-1} - \sum_{q=i+1}^{n} m_{\le q}(J,j)\binom{k-1}{i}\end{aligned} \tag{3.17}$$

for all i and j. (A similar formula for graded Betti numbers over the exterior algebra appears in [AHHi97, Theorem 4.4].) Then by (3.17) and the assumption, we have $\beta^S_{i,j}(S/I) \le \beta^S_{i,j}\big(S/\operatorname{Gin}(I)\big) \le \beta^S_{i,j}(S/I')$ for all i, j. Thus, by Corollary 3.2.3, what we must prove is $\beta^S_{q,q+k}\big(S/\operatorname{Gin}(I)\big) = \beta^S_{q,q+k}(S/I')$ for all $q \ge i$. However (3.17) and the assumption imply that $m_{\le q}\big(\operatorname{Gin}(I), k+1\big) = m_{\le q}(I', k+1)$ for all $q \ge i$ and $m_{\le q}\big(\operatorname{Gin}(I), k\big) = m_{\le q}(I', k)$ for all $q \ge i+1$. Hence for all $q \ge i$, we have $\beta^S_{q,q+k}\big(S/\operatorname{Gin}(I)\big) = \beta^S_{q,q+k}(S/I')$ as desired. □

Let $I \subset R$ be a graded ideal. We write $\operatorname{Lex}(I) \subset R$ for the unique lexsegment ideal of R with the same Hilbert function as I defined in [Bi93] (or [AHHi98] for the exterior case) and $\operatorname{Gin}_\sigma(I)$ for the generic initial ideal of I with respect to a term order σ. It is known that $\operatorname{Lex}(I)$ and $\operatorname{Gin}_\sigma(I)$ satisfy the assumption of Lemma 3.5.1 (see [Co04, §5] and [NRV07, §5]). Thus we have

Theorem 3.5.2. *Let $I \subset R$ be a graded ideal, σ a term order and let J be either $\operatorname{Gin}_\sigma(I)$ or $\operatorname{Lex}(I)$. Suppose that $\beta^R_{i,i+k}(R/I) = \beta^R_{i,i+k}(R/J)$ for some $i > 1$.*

(i) *If $R = S$, then one has $\beta^S_{q,q+k}(S/I) = \beta^S_{q,q+k}(S/J)$ for all $q \ge i$.*

(ii) *If $R = E$, then one has $\beta^E_{q,q+k}(E/I) = \beta^E_{q,q+k}(E/J)$ for all $q \ge 1$.*

Next, we consider when a graded ideal J satisfies $\beta^E_{i,i+d}(E/J) = \beta^E_{i,i+d}\big(E/\operatorname{Gin}(J)\big)$ for all $i \ge 1$, where we fix an integer $d \ge 0$. The next lemma follows from [BaS87] and [AH00, Theorem 5.3].

Lemma 3.5.3. *Let $I \subset R$ be a graded ideal. Then, I has a linear resolution if and only if $\operatorname{Gin}(I)$ has a linear resolution.*

We also require the following.

Lemma 3.5.4. [Crystallization Principle] *Let $I \subset R$ be a graded ideal. If I is generated by elements of degree $\le d$ and $\beta^R_{1,d+1}\big(R/\operatorname{Gin}(I)\big) = 0$, then $\operatorname{reg}(I) \le d$.*

The Crystallization Principle was proved by Green [G98, Corollary 2.28] for generic initial ideals over a polynomial ring, however, this fact can also be proved for generic initial ideals over an exterior algebra in the same way.

Proposition 3.5.5. *Let $I \subset R$ be a graded ideal. The following conditions are equivalent.*

(i) $I_{\langle k\rangle}$ *has a linear resolution;*

(ii) $\beta^R_{1,k+1}(R/I) = \beta^R_{1,k+1}\big(R/\operatorname{Gin}(I)\big)$*, that is, the number of elements of degree* $k+1$ *belonging to the set of minimal generators of* I *is equal to that of* $\operatorname{Gin}(I)$*.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Since $\beta^R_{1,k+1}(R/I)$ is the numbers of generators in $G(I)$ of degree $k+1$, we have

$$\begin{aligned}\beta^R_{1,k+1}(R/I) &= \dim_K I_{k+1} - \dim_K(\mathfrak{m}I_{\langle k\rangle})_{k+1}\\ &= \dim_K I_{k+1} - \dim_K(I_{\langle k\rangle})_{k+1}\end{aligned}$$

and

$$\beta^R_{1,k+1}(R/\operatorname{Gin}(I)) = \dim_K(\operatorname{Gin}(I)_{k+1}) - \dim_K\big(\mathfrak{m}\operatorname{Gin}(I_{\langle k\rangle})\big)_{k+1}.$$

Then, from above equations we have $\beta^R_{1,k+1}(R/I) = \beta^R_{1,k+1}\big(R/\operatorname{Gin}(I)\big)$ if and only if $\dim_K(I_{\langle k\rangle})_{k+1} = \dim_K\big(\mathfrak{m}\operatorname{Gin}(I_{\langle k\rangle})\big)_{k+1}$.

Suppose $I_{\langle k\rangle}$ has a linear resolution. Then, by Lemma 3.5.3 $\operatorname{Gin}(I_{\langle k\rangle})$ has a linear resolution. Hence $\dim_K\big(\mathfrak{m}\operatorname{Gin}(I_{\langle k\rangle})\big)_{k+1} = \dim_K\big(\operatorname{Gin}(I_{\langle k\rangle})\big)_{k+1} = \dim_K(I_{\langle k\rangle})_{k+1}$. Hence we have $\beta^R_{1,k+1}(R/I) = \beta^R_{1,k+1}\big(R/\operatorname{Gin}(I)\big)$ as required. On the other hand, if $\beta^R_{1,k+1}(R/I) = \beta^R_{1,k+1}\big(R/\operatorname{Gin}(I)\big)$, then $\dim_K\big(\mathfrak{m}\operatorname{Gin}(I_{\langle k\rangle})\big)_{k+1} = \dim_K\big(\operatorname{Gin}(I_{\langle k\rangle})\big)_{k+1}$. This implies $\beta^R_{1,k+1}\big(R/\operatorname{Gin}(I_{\langle k\rangle})\big) = 0$. Then the Crystallization Principle says that $I_{\langle k\rangle}$ has a linear resolution. □

Now, the following theorem immediately follows from the above proposition together with Corollaries 3.2.5 and 3.4.5.

Theorem 3.5.6. *Let $I \subset R$ a graded ideal and $k \geq 0$ an integer. The following conditions are equivalent.*

(i) $\beta^R_{i,i+k}(R/I) = \beta^R_{i,i+k}\big(R/\operatorname{Gin}(I)\big)$ *for all* $i \geq 1$*;*

(ii) $I_{\langle k\rangle}$ *and* $I_{\langle k+1\rangle}$ *have a linear resolution;*

(iii) $\beta^R_{1,k+1}(R/I) = \beta^R_{1,k+1}\big(R/\operatorname{Gin}(I)\big)$ *and* $\beta^R_{1,k+2}(R/I) = \beta^R_{1,k+2}\big(R/\operatorname{Gin}(I)\big)$*,*

where $I_{\langle k\rangle}$ denotes the ideal of R generated by all homogeneous elements in I of degree k.

Example 3.5.7. Let $I = (x_1^2, x_2^2, x_1x_2x_3^2, x_3^5) \subset S = \mathbb{C}[x_1, x_2, x_3]$. Then we have

$$\operatorname{Gin}(I) = (x_1^2, x_1x_2, x_2^3, x_2^2x_3^2, x_1x_3^4, x_2x_3^5, x_3^6).$$

Then Proposition 3.5.5 says that $I_{\langle k\rangle}$ has a linear resolution for $k = 3, 4, 7, 8, 9, \ldots$. In particular, for $k = 4, 8, 9, 10, \ldots$, we have $\beta^S_{i,i+k}(I) = \beta^S_{i,i+k}\big(\operatorname{Gin}(I)\big)$ for all $i \geq 0$.

3.6 The cancellation principle

Let K be a field of characteristic 0. In this section, we will study the relation between our results in Section 1 and the Cancellation Principle for generic initial ideals, which was considered in [G98]. This observation would help us to understand why we require the assumption $i > 1$ in Corollary 3.2.3 and why we need to consider $I_{\langle k\rangle}$ and $I_{\langle k+1\rangle}$ in Theorem 3.5.6.

First, we recall what is the Cancellation Principle.

Lemma 3.6.1. [G98, Corollary 1.21] *Let I be a graded ideal in S and σ a term order. The minimal free resolution of I is obtained from that of $\mathrm{in}_\sigma(I)$ by cancelling adjacent terms, in other words, there exists integers $\tau_{i,i+k}$ with $1 \leq i \leq n-1$ and $k \geq 0$ such that*

$$\beta^S_{i,i+k}\big(\mathrm{in}_\sigma(I)\big) = \beta^S_{i,i+k}(I) + \tau_{i,i+k} + \tau_{i+1,i+k} \quad \textit{for all } i \geq 0 \textit{ and all } k \geq 0,$$

where we let $\tau_{0,k} = 0$ for all $k \geq 0$.

We refer the reader to [G98, Example 1.35] for further information about the Cancellation Principle.

Let I be a graded ideal in S. Then Lemma 3.6.1 says that there exists integers $c_{i,i+k}(I)$ with $1 \leq i \leq n-1$ and with $k \geq 0$ such that

$$\beta^S_{i,i+k}\big(\mathrm{Gin}(I)\big) = \beta^S_{i,i+k}(I) + c_{i,i+k}(I) + c_{i+1,i+k}(I) \quad \text{for all } i \geq 0 \text{ and all } k \geq 0,$$

where we let $c_{0,k}(I) = 0$ for all $k \geq 0$. It can be easily verified that the integers $c_{i,i+k}(I)$ are uniquely determined for a given ideal I. We will call the integer $c_{i,i+k}(I)$ the $(i, i+k)$*th cancellation number of I*.

Example 3.6.2. Let $I = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_1x_3x_4) \subset S = \mathbb{C}[x_1, x_2, x_3, x_4]$. Then we have $\mathrm{Gin}(I) = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_1x_2x_3, x_1x_3^3)$. The minimal free resolution of I is

$$0 \longrightarrow S(-5) \oplus S(-6) \longrightarrow S^6(-4) \oplus S(-5) \longrightarrow S^6(-3) \longrightarrow I \longrightarrow 0,$$

and that of $\mathrm{Gin}(I)$ is

$$0 \to S^2(-5) \oplus S(-6) \to S^7(-4) \oplus S^2(-5) \to S^6(-3) \oplus S(-4) \to \mathrm{Gin}(I) \to 0.$$

Hence we have $c_{1,4}(I) = 1$, $c_{2,5}(I) = 1$ and all other cancellation numbers of I are 0.

In Section 2, we already proved that (see Proposition 3.1.1 and (3.10))

$$\beta^S_{i,i+k}(I) = \beta^S_{i,i+k}\big(\mathrm{Gin}(I)\big) - \sum_{(a,b)\in A_{i+1,n}} \left[\binom{n-b-1}{i-a+1}\delta_{a,b,a+k-1} + \binom{n-b-1}{i-a}\delta_{a,b,a+k}\right],$$

where $\delta_{a,b,a+k} = \dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k}$ and where $\varphi_{a,b}$ is the map which appears in the long exact sequence (3.1). This formula enables us to write the cancellation numbers in terms of the Koszul homology of generic linear forms.

Lemma 3.6.3. *With the same notation as above, one has*

$$c_{i,i+k}(I) = \sum_{(a,b)\in A_{i+1,n}} \binom{n-b-1}{i-a} \dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k} \quad \textit{for all } i \geq 0 \textit{ and all } k \geq 0.$$

Proof. For all $i \geq 0$ and all $k \geq 0$, we set $C_{i,i+k} = \sum_{(a,b)\in A_{i+1,n}} \binom{n-b-1}{i-a} \dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k}$ and $C'_{i,i+k} = \sum_{(a,b)\in A_{i+1,n}} \binom{n-b-1}{i-a+1} \dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k-1}$. Then we have

$$\beta^S_{i,i+k}(I) = \beta^S_{i,i+k}\big(\mathrm{Gin}(I)\big) - C_{i,i+k} - C'_{i,i+k}.$$

Notice that we only need to show that $C'_{i,i+k} = C_{i+1,i+k}$. Recall that, in the proof of Theorem 3.2.1, we already proved that

$$A_{i+2,n} \setminus \big\{(i+2,b) : b \leq n-1\big\} = A_{i+1,n} \setminus \big\{(i-n+b+1,b) : b \leq n-1\big\}.$$

Now, since the binomial $\binom{n-b-1}{i-a+1} = 0$ for all $(a,b) \in \big\{(i+2,b) : b \leq n-1\big\}$ and for all $(a,b) \in \big\{(i-n+b+1,b) : b \leq n-1\big\}$, we have

$$\begin{aligned}
C_{i+1,i+k} &= \sum_{(a,b)\in A_{i+2,n}} \binom{n-b-1}{i-a+1} \dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k-1} \\
&= \sum_{(a,b)\in A_{i+2,n}\setminus\{(i+2,b):b\leq n-1\}} \binom{n-b-1}{i-a+1} \dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k-1} \\
&= \sum_{(a,b)\in A_{i+1,n}\setminus\{(i-n+b+1,b):b\leq n-1\}} \binom{n-b-1}{i-a+1} \dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k-1} \\
&= \sum_{(a,b)\in A_{i+1,n}} \binom{n-b-1}{i-a+1} \dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k-1} \\
&= C'_{i,i+k}.
\end{aligned}$$

This concludes the proof. □

By using Lemma 3.6.3, we can prove an analogue of Corollaries 3.2.3 and 3.2.5.

Theorem 3.6.4. *Let I be a graded ideal in S. If $c_{i,i+k}(I) = 0$ for some $i \geq 1$ and $k \geq 0$, then one has $c_{q,q+k}(I) = 0$ for all $q \geq i$.*

Proof. It suffices to show the case $q = i+1$. Remark that $\dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k} = 0$ if and only if $\big(\mathfrak{m}H_a(b)\big)_{a+k} = 0$. In the proof of Theorem 3.2.1, we proved that if $\dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k} = 0$ for all $(a,b) \in A_{i+1,n} \setminus \big\{(i+1,b) : b \leq n-1\big\}$, then $\dim_K \big(\mathrm{Im}\varphi_{a,b}\big)_{a+k} = 0$ for all $(a,b) \in A_{i+2,n} \setminus \big\{(i+2,b) : b \leq n-1\big\}$. Then, since $\binom{n-b-1}{i-a+1} = 0$ for any $(a,b) \in \big\{(i+2,b) : b \leq n-1\big\}$, Lemma 3.6.3 says that $c_{i+1,i+1+k}(I) = 0$. □

Corollary 3.6.5. *Let I be a graded ideal in S. Then $c_{i,i+k}(I) = 0$ for all $i \geq 1$ if and only if $I_{\langle k \rangle}$ has a linear resolution.*

Proof. Since the graded Betti number $\beta^S_{0,k+1}(\mathrm{Gin}(I)) = \beta^S_{0,k+1}(I) + c_{1,1+k}(I)$, we have $\beta^S_{0,k+1}(\mathrm{Gin}(I)) = \beta^S_{0,k+1}(I)$ if and only if $c_{1,1+k}(I) = 0$. However, by Theorem 3.6.4, we have $c_{1,1+k}(I) = 0$ if and only if $c_{i,i+k}(I) = 0$ for all $i \geq 1$. Also, by Proposition 3.5.5, we have $\beta^S_{0,k+1}(\mathrm{Gin}(I)) = \beta^S_{0,k+1}(I)$ if and only if $I_{\langle k \rangle}$ has a linear resolution. Thus the assertion follows. □

Observe that Theorems 3.6.4 and Corollary 3.6.5 are stronger than Corollaries 3.2.3 and 3.2.5. Indeed, Corollary 3.2.3 immediately follows from Theorem 3.6.4, since the graded Betti numbers $\beta^S_{i,i+k}(I) = \beta^S_{i,i+k}(\mathrm{Gin}(I))$ if and only if $c_{i,i+k}(I) = 0$ and $c_{i+1,i+k}(I)$=0.

We also remark the next fact which follows from Lemma 3.6.3.

Corollary 3.6.6. *Let I be a graded ideal in S. Assume that $I_{\langle k \rangle}$ has a linear resolution.*

(i) *If $\beta^S_{q,q+k+2}(I) = \beta^S_{q,q+k+2}(\mathrm{Gin}(I))$, then $\beta^S_{q+1,q+k+2}(I) = \beta^S_{q+1,q+k+2}(\mathrm{Gin}(I))$;*

(ii) *If $\beta^S_{q,q+k-1}(I) = \beta^S_{q,q+k-1}(\mathrm{Gin}(I))$, then $\beta^S_{q-1,q+k-1}(I) = \beta^S_{q-1,q+k-1}(\mathrm{Gin}(I))$.*

Proof. By Corollary 3.6.5, we have $c_{\ell,\ell+k}(I) = 0$ for all integers $\ell \geq 1$. Then, we have the graded Betti numbers $\beta^S_{q+1,q+k+2}(\mathrm{Gin}(I)) = \beta^S_{q+1,q+k+2}(I) + c_{q+1,q+k+2}(I)$ and $\beta^S_{q-1,q+k-1}(\mathrm{Gin}(I)) = \beta^S_{q-1,q+k-1}(I) + c_{q,q+k-1}(I)$. On the other hand, if the graded Betti number $\beta^S_{q,q+k+2}(I) = \beta^S_{q,q+k+2}(\mathrm{Gin}(I))$ then we have $c_{q+1,q+k+2}(I) = 0$. Also, if $\beta^S_{q,q+k-1}(I) = \beta^S_{q,q+k-1}(\mathrm{Gin}(I))$ then we have $c_{q,q+k-1}(I) = 0$. Thus the assertion follows. □

As for any graded ideal I, $I_{\langle 1 \rangle}$ always has a linear resolution, it follows that if $\beta^S_{q,q+3}(\mathrm{Gin}(I)) = \beta^S_{q,q+3}(I)$ then we have $\beta^S_{q+1,q+3}(\mathrm{Gin}(I)) = \beta^S_{q+1,q+3}(I)$.

Since it is not difficult to find the Betti numbers of a strongly stable ideal J, one may expect to find all possible Betti numbers of graded ideals I such that $\mathrm{Gin}(I) = J$ by using Betti numbers of J and by considering all possible cancellations. However, this problem is far reaching as pointed out in [G98, Example 1.35].

Thanks: All of the examples that we have presented in this chapter are computed by the computer algebra system CoCoA [CO]. We also mention that computations of generic initial ideals are done by a random choice of matrices.

CHAPTER 4

Linear balls and the multiplicity conjecture

The multiplicity conjecture due to Herzog, Huneke and Srinivasan is one of the most attractive conjectures lying between combinatorics and commutative algebra. In this chapter we consider the problem of finding a natural class of spheres whose Stanley–Reisner rings satisfy the multiplicity conjecture.

Let Δ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$ of dimension $d-1$ and $K[\Delta] = S/I_\Delta$, where $S = K[x_1, \ldots, x_n]$, its Stanley–Reisner ring. Suppose that Δ is a ball, i.e., the geometric realization $|\Delta|$ is a ball. Let $\partial\Delta$ denote the boundary complex of Δ and suppose that each vertex of Δ belongs to $\partial\Delta$. Thus $\partial\Delta$ is a sphere, i.e., the geometric realization $|\partial\Delta|$ is a sphere, of dimension $d-2$ on $[n]$. Each face of $\partial\Delta$ is called a boundary face of Δ and each face of $\Delta \setminus \partial\Delta$ is called an inside face of Δ. Let $1 \leq m-1$ be the smallest dimension of a nonface of Δ. In Section 4.1 in Theorem 4.1.2 we show that the sphere $\partial\Delta$ with the following assumptions satisfies the multiplicity conjecture:

(A1) Δ has a minimal inside face of dimension $d-m$ and has no minimal inside face of dimension less than $m-1$;

(A2) the h–vector of $\partial\Delta$ is unimodal.

A linear ball is a ball whose Stanley–Reisner ring has a linear resolution. Let Δ be a linear ball. Let $m-1$ be the smallest dimension of a nonface of Δ and suppose that $2 \leq m \leq (d+1)/2$. Then the sphere which is the boundary complex of Δ satisfies (A1) and (A2). In particular the Stanley–Reisner ring of the sphere satisfies the multiplicity conjecture (Corollary 4.1.4).

In Section 4.2 we discuss a class of shellable spheres arising from determinantal ideals. These shellable spheres satisfy assumptions (A1),(A2) and hence the multiplicity conjecture is also satisfied. Let $X = (X_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ be an $m \times n$ matrix of indeterminates, where $2 \leq m \leq n$. Write τ for the lexicographic order of the polynomial ring $K[X] = K[\{X_{ij}\}_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}]$ induced by the ordering of the variables

$$X_{11} \geq X_{12} \geq \cdots \geq X_{1n} \geq X_{21} \geq \cdots \geq X_{2n} \geq \cdots \geq X_{m1} \geq \cdots \geq X_{mn}.$$

Let I_r denote the ideal of $K[X]$ generated by all $(r+1)\times(r+1)$ minors of X, where $1 \leq r \leq m-1$. In particular I_{m-1} is the ideal of $K[X]$ generated by all maximal minors of X. It is known that the initial ideal I_r^* of I_r with respect to τ is generated by squarefree monomials. Let Δ_r denote the simplicial complex whose Stanley–Reisner ideal coincides with I_r^*. Theorem 4.2.4 says that, for each $1 \leq r \leq m-1$, the simplicial complex Δ_r is a shellable ball satisfying (A1) and (A2). Moreover Δ_r is a linear ball if and only if $r = m-1$ (Corollary 4.2.6).

In Section 4.3 we produce a natural class of shellable linear balls arising from the polarization of a power of the graded maximal ideal. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal of $S = K[x_1, \ldots, x_n]$, $n \geq 3$. Clearly, each power $\mathfrak{m}^t$ of $\mathfrak{m}$ has a linear resolution. For a given $t \geq 1$, let Δ be the simplicial complex whose Stanley–Reisner ideal coincides with the polarization of $\mathfrak{m}^t$. It is shown (Theorem 4.3.1) that Δ is a shellable linear ball and hence it satisfies the multiplicity conjecture.

4.1 The multiplicity conjecture

The main goal of this section is to prove Theorem 4.1.2 in which we show that a certain class of spheres satisfy the multiplicity conjecture.

First, we recall what the multiplicity conjecture says. Let $R = \sum_{i=0}^{\infty} R_i$ be a homogeneous Cohen–Macaulay algebra over a field $R_0 = K$ of dimension d with embedded dimension $n = \dim_K R_1$. We write $R = S/I$, where $S = K[x_1, \ldots, x_n]$ is the polynomial ring in n variables over K and I is a graded ideal of S. Let $H(R,i) = \dim_K R_i$, $i = 0, 1, 2, \ldots$, denote the Hilbert function of the algebra R and $F(R,\lambda) = \sum_{i=0}^{\infty} H(R,i)\lambda^i$ the Hilbert series of R. It is known that $F(R,\lambda)$ is a rational function of λ of the form

$$F(R,\lambda) = \frac{h_0 + h_1\lambda + \cdots + h_\ell\lambda^\ell}{(1-\lambda)^d},$$

with each $h_i > 0$. The multiplicity $e(R)$ of R is

$$e(R) = h_0 + h_1 + \cdots + h_\ell.$$

Now, we consider the graded minimal free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow R \longrightarrow 0$$

of R over S, where $F_i = \bigoplus S(-j)^{\beta_{i,j}}$ with $\beta_{i,j} \geq 0$. Let

$$m_i = \min\{j : \beta_{i,j} \neq 0\}, \qquad M_i = \max\{j : \beta_{i,j} \neq 0\}.$$

The multiplicity conjecture due to Herzog, Huneke and Srinivasan says that

$$\frac{\prod_{i=1}^{p} m_i}{p!} \leq e(R) \leq \frac{\prod_{i=1}^{p} M_i}{p!}.$$

A nice survey of the multiplicity conjecture and the record of past results in different cases of the conjecture can be found in [HZ06]. For more recent results one may look into [KW07], [MNR07], [NoS07].

Next, we recall the fundamental material on Stanley–Reisner ideals and rings of simplicial complexes. We refer the reader to [BH96], [Hi92], [St95] for further information. Let $[n] = \{1, \ldots, n\}$ be the vertex set and Δ a simplicial complex on $[n]$. Thus Δ is a collection of subsets of $[n]$ such that

(i) $\{i\} \in \Delta$ for all $i \in [n]$, and

(ii) if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$.

Each element $F \in \Delta$ is called a *face* of Δ. The dimension of a face F is $|F| - 1$. Let $d = \max\{|F| : F \in \Delta\}$ and define the dimension of Δ to be $\dim \Delta = d - 1$. A *nonface* of Δ is a subset F of $[n]$ with $F \notin \Delta$.

Let $f_i = f_i(\Delta)$ denote the number of faces of Δ of dimension i. Thus in particular $f_0 = n$. The sequence $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f–vector* of Δ. Letting $f_{-1} = 1$, we define the *h–vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by the formula

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. For each subset $F \subset [n]$, we set

$$x_F = \prod_{i \in F} x_i.$$

The *Stanley–Reisner ideal* of Δ is the ideal I_Δ of S which is generated by those squarefree monomials x_F with $F \notin \Delta$. In other words,

$$I_\Delta = (x_F : F \notin \Delta).$$

The quotient ring $K[\Delta] = S/I_\Delta$ is called the *Stanley–Reisner ring* of Δ. It follows that the Hilbert series of $K[\Delta]$ is

$$F(K[\Delta], \lambda) = (h_0 + h_1\lambda + \cdots + h_d\lambda^d)/(1-\lambda)^d,$$

where $(h_0, h_1, \ldots, h_d)$ is the h–vector of Δ. Thus in particular the multiplicity of $K[\Delta]$ is $\sum_{i=0}^{d} h_i$ $(= f_{d-1})$.

We say that Δ is Cohen–Macaulay (resp. Gorenstein) over K if $K[\Delta]$ is Cohen–Macaulay (resp. Gorenstein). Note that if the geometric realization $|\Delta|$ of Δ is homeomorphic to a ball, then Δ is Cohen–Macaulay over an arbitrary field and if the geometric realization $|\Delta|$ of Δ is homeomorphic to a sphere, then Δ is Gorenstein over an arbitrary field.

Now, let Δ be a simplicial complex on $[n]$ of dimension $d-1$ whose geometric realization $|\Delta|$ is homeomorphic to a manifold. The *boundary complex* $\partial\Delta$ of Δ consists of those faces F of Δ with the property that there is a $(d-2)$–dimensional face F' of Δ with $F \subset F'$ such that F' is contained in exactly one $(d-1)$–dimensional face of Δ. Each face of $\partial\Delta$ is called a *boundary face* and each face of $\Delta \setminus \partial\Delta$ is called an *inside face* of Δ. In particular if Δ is a ball, i.e., $|\Delta|$ is homeomorphic to a ball, of dimension $d-1$, then $\partial\Delta$ is a sphere, i.e., $|\partial\Delta|$ is homeomorphic to a sphere, of dimension $d-2$.

Theorem 4.1.1 (Hochster)**.** *Let Δ be a Cohen–Macaulay complex over a field K of dimension $d-1$ whose geometric realization $|\Delta|$ is a manifold with a nonempty boundary complex $\partial\Delta$, and let ω_Δ be the canonical ideal of $K[\Delta]$. Write J for the ideal of $K[\Delta]$ generated by those monomials $\overline{x_F}$ with $F \in \Delta \setminus \partial\Delta$. Then the following conditions are equivalent:*

(a) *$\omega_\Delta \cong J$ as a $\mathbb{Z}^n$–graded $K[\Delta]$–module;*

(b) *$\partial\Delta$ is a Gorenstein complex over K.*

If the equivalent conditions hold, then $K[\partial\Delta] \cong K[\Delta]/\omega_\Delta$.

Let Δ be a simplicial complex on $[n]$ of dimension $d-1$ whose geometric realization $|\Delta|$ is a ball and $\partial\Delta$ its boundary complex. Assume that every vertex of Δ belongs to $\partial\Delta$. Thus $\partial\Delta$ is a simplicial complex on $[n]$ of dimension $d-2$ whose geometric realization $|\partial\Delta|$ is a sphere. Since $\partial\Delta$ is Gorenstein, it follows that

(P1) The h–vector $h(\partial\Delta) = (h'_0, h'_1, \ldots, h'_{d-1})$ of $\partial\Delta$ is symmetric i.e. $h'_i = h'_{d-1-i}$ for all $i = 0, \ldots, d-1$; see [BH96, Theorem 5.4.2, Theorem 5.6.2].

(P2) The minimal free resolution of the Stanley–Reisner ring of $\partial\Delta$ is symmetric ([Ei95, Corollary 21.16]), i.e. if

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I_{\partial\Delta} \longrightarrow 0$$

is the minimal free resolution of the ring $S/I_{\partial\Delta}$, where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$, $i = 0, \ldots, p$, $p = n - (d-1)$ and $F_0 = S$, then we have $\beta_{i,j} = \beta_{p-i,n-j}$ for all $i = 0, \ldots, p$. In particular, $M_i = n - m_{p-i}$ where $M_i = \max\{j : \ \beta_{i,j} \neq 0\}$ and $m_i = \min\{j : \ \beta_{i,j} \neq 0\}$.

Since $\partial\Delta$ is a simplicial complex on $[n]$ and $\operatorname{core}\partial\Delta = \partial\Delta$ (see the proof of [BH96, Lemma 5.6.4]), we notice that above $F_p = S(-n)^{\beta_{p,n}}$.

(P3) The canonical ideal ω_Δ of the Stanley–Reisner ring $K[\Delta] = S/I_\Delta$ is generated by the monomials $\overline{x_F}$, $F \in \Delta \setminus \partial\Delta$ (see Theorem 4.1.1).

In addition,

(F1) Let
$$0 \longrightarrow F'_{n-d} \longrightarrow \cdots \longrightarrow F'_1 \longrightarrow F'_0 \longrightarrow S/I_\Delta \longrightarrow 0$$
be the minimal free resolution of S/I_Δ with $F'_i = \bigoplus_j S(-j)^{\beta'_{i,j}}$. Then the generators of the canonical module ω_Δ of $K[\Delta]$ are of degrees $n-j$ with $\beta'_{n-d,j} \neq 0$ (see [BH96, Corollary 3.3.9]).

(F2) One has $m_1 < m_2 < \cdots < m_{n-d+1}$.

Now, let $1 \leq m-1$ be the smallest dimension of the nonfaces of Δ. In other words, m is the smallest degree of monomials belonging to $G(I_\Delta)$, the minimal system of monomial generators of I_Δ. Our goal is to show that the Stanley–Reisner ring $K[\partial\Delta] = S/I_{\partial\Delta}$ satisfies the multiplicity conjecture under the following hypothesis (Theorem 4.1.2):

(A1) Δ has a minimal (under inclusion) inside face of dimension $d-m$ and has no minimal inside face of dimension less than $m-1$;

(A2) The h–vector of the boundary complex $\partial\Delta$ is unimodal.

(In general, we say that a finite sequence of real numbers $a_0, \ldots, a_t$ is *unimodal* if
$$a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_t$$
for some $0 \leq j \leq t$.)

Now, we wish to understand the minimal and maximal shifts given by m_i and M_i respectively of the minimal free resolution
$$\mathcal{F}_{\partial\Delta} : 0 \longrightarrow F_{n-d+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow S/I_{\partial\Delta} \longrightarrow 0$$
of $S/I_{\partial\Delta}$ where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$, to calculate the lower and upper bounds of the multiplicity of $S/I_{\partial\Delta}$. First, we consider the minimal free resolution
$$\mathcal{F}_\Delta : 0 \longrightarrow F'_{n-d} \longrightarrow \cdots \longrightarrow F'_1 \longrightarrow S \longrightarrow S/I_\Delta \longrightarrow 0$$
of S/I_Δ where $F'_i = \bigoplus_j S(-j)^{\beta'_{i,j}}$. Let m'_i and M'_i denote the minimal and maximal shifts of the minimal free resolution $\mathcal{F}_\Delta$. Since m is the minimum of the degree of generators of I_Δ, one has $m'_1 = m$. By the assumption (A1) on Δ, there exists a minimal inside face of Δ of dimension $d-m$, hence by Theorem 4.1.1, it follows that the canonical ideal ω_Δ of Δ has a generator of degree $d-m+1$. Therefore $\beta'_{n-d,n-(d-m+1)} \neq 0$, by (F1). As we have $m'_1 = m$ and $m'_{n-d} \leq m+n-d-1$, we get $m'_i = m+i-1$ for $i = 1, \ldots, n-d$, by (F2).

We claim that the minimal shifts in the minimal free resolution $\mathcal{F}_{\partial\Delta}$ of $S/(I_{\partial\Delta})$ are given by $m_i = m + i - 1$ for $i = 1, \ldots, n-d$ and $m_{n-d+1} = n$. Indeed, by assumption (A1), we have that the canonical ideal ω_Δ has no generator of degree less than m. Hence the S–module $I_{\partial\Delta}/I_\Delta$ has no generator of degree less than m (Theorem 4.1.1). From the following short exact sequence

$$0 \longrightarrow I_\Delta \longrightarrow I_{\partial\Delta} \longrightarrow I_{\partial\Delta}/I_\Delta \longrightarrow 0,$$

we get the following long exact sequence

$$\begin{aligned} &\cdots \longrightarrow \mathrm{Tor}_{i+1}(I_{\partial\Delta}/I_\Delta, K) \\ \longrightarrow \mathrm{Tor}_i(I_\Delta, K) \longrightarrow \mathrm{Tor}_i(I_{\partial\Delta}, K) &\longrightarrow \mathrm{Tor}_i(I_{\partial\Delta}/I_\Delta, K) \longrightarrow \cdots \end{aligned}$$

Now, as $\mathrm{Tor}_i(I_\Delta, K)_{i+t} = 0$ and $\mathrm{Tor}_i(I_{\partial\Delta}/I_\Delta, K)_{i+t} = 0$ for $t \leq m-1$ and $i = 1, \ldots, n-d$, from the above long exact sequence we get $\mathrm{Tor}_i(I_{\partial\Delta}, K)_{i+t} = 0$ for $t \leq m-1$ and $i = 1, \ldots, n-d$. Also as $\mathrm{Tor}_{i+1}(I_{\partial\Delta}/I_\Delta, K)_{i+1+m-1} = 0$ and $\mathrm{Tor}_i(I_\Delta, K)_{i+m} \neq 0$, we get $\mathrm{Tor}_i(I_{\partial\Delta}, K)_{i+m} \neq 0$, $i = 1, \ldots n-d$. From here it follows that $m_i = m + i - 1$ for $i = 1, \ldots, n-d$. Since $S/I_{\partial\Delta}$ is Gorenstein and $m_0 = M_0 = 0$, we have $m_{n-d+1} = M_{n-d+1} = n$ by Property (P2).

Now, we need to determine the maximal shifts M_i for $i = 1, \ldots, n-d$ in the minimal free resolution $\mathcal{F}_{\partial\Delta}$ of $S/I_{\partial\Delta}$. Again, as $S/I_{\partial\Delta}$ is Gorenstein, by Property (P2) we have $M_i = n - m_{n-d+1-i} = n - (m + n - d + 1 - i - 1) = d - m + i$ for $i = 1, \ldots, n-d$.

Since $m_1 \leq M_1$, we may observe that $m \leq (d+1)/2$. We would need this inequality later.

Hence, we have now

$$\begin{aligned} L &= \prod_{i=1}^{n-d+1} \frac{m_i}{(n-d+1)!} = \frac{n\prod_{i=1}^{n-d}(m+i-1)}{(n-d+1)!} \quad \text{and} \\ U &= \prod_{i=1}^{n-d+1} \frac{M_i}{(n-d+1)!} = \frac{n\prod_{i=1}^{n-d}(d-m+i)}{(n-d+1)!}. \end{aligned}$$

Next, our goal is to estimate the multiplicity $e(S/I_{\partial\Delta})$ of the ring $S/I_{\partial\Delta}$. Let $h'_0, \ldots, h'_{d-1}$ denote the h–vector of the ring $S/I_{\partial\Delta}$. As the ring $S/I_{\partial\Delta}$ is Cohen–Macaulay, and m is the minimum of the degree of the generators of $I_{\partial\Delta}$, we have $h'_i = h'_{d-1-i} = \binom{n-d+1+i-1}{i} = \binom{n-d+i}{i}$ for $i = 0, \ldots, m-1$. From assumption (A2) and property (P1) we have that the h–vector is symmetric and unimodal, therefore we conclude that $h'_i \geq \binom{n-d+m-1}{m-1}$ for $i = m, \ldots, d-(m+1)$.

Hence

$$e(S/I_{\partial\Delta}) = \sum_{i=1}^{d-1} h_i \geq 2\sum_{i=0}^{m-1}\binom{n-d+i}{i} + (d-2m)\binom{n-d+m-1}{m-1}.$$

Theorem 4.1.2. *Let Δ be a ball and $\partial\Delta$ be its boundary complex. Suppose that the sphere $\partial\Delta$ satisfies the assumptions (A1) and (A2). Then the Stanley-Reisner ring $S/\partial\Delta$ satisfies the multiplicity conjecture i.e.*

$$L \leq e(S/I_{\partial\Delta}) \leq U.$$

For the proof of the theorem, we need to first define cyclic polytopes. Let $C(n, d-1)$ denote the convex hull of any n distinct points in $\mathbb{R}^{d-1}$ on the curve $\{(t, t^2, \ldots, t^{d-1}) \in \mathbb{R}^{d-1}, t \in \mathbb{R}\}$. The polytope $C(n, d-1)$ is called the cyclic polytope of dimension $d-1$. It is known that $C(n, d-1)$ is simplicial (i.e., every proper face is a simplex), and so the boundary of $C(n, d-1)$ defines a simplicial complex which we denote by $\partial C(n, d-1)$ such that $|\partial C(n, d-1)|$ is a sphere of dimension $d-2$. Let $(h_0^*, h_1^*, \ldots, h_{d-1}^*)$ denote the h–vector of $\partial C(n, d-1)$. Then

$$h_i^* = h_{d-1-i}^* = \binom{n-d+i}{i} \text{ for } i = 1, \ldots, \lfloor \frac{d-1}{2} \rfloor,$$

(see [St95, Section 3]). Let $e(\partial C(n, d-1)) = \sum h_i^*$ denote the multiplicity of the Stanley–Reisner ring of the boundary complex $\partial C(n, d-1)$. Notice that we have $h_i' \leq h_i^*$, hence

$$e(S/I_{\partial\Delta}) \leq e\big(\partial C(n, d-1)\big). \tag{4.1}$$

In [THi96], the minimal free resolution of the $\partial C(n, d-1)$ is computed. We have the following [THi96, Theorem 3.2]: If $d-1 \geq 2$ is even, then the maximal shifts M_i^* in the minimal free resolution of $\partial C(n, d-1)$ are given by

$$M_i^* = \frac{d-1}{2} + i \text{ for } i = 1, \ldots, n-d \text{ and } M_{n-d+1}^* = n \tag{4.2}$$

and if $d-1 \geq 3$ is odd, then the maximal shifts M_i^* are as follows:

$$M_i^* = \lfloor \frac{d-1}{2} \rfloor + i + 1 \text{ for } i = 1, \ldots, n-d \text{ and } M_{n-d+1}^* = n. \tag{4.3}$$

Even though the following Lemma 4.1.3 follows from [HS98, Theorem 1.2], we want to give a direct computational proof.

Lemma 4.1.3. *We have*

$$e\big(\partial C(n, d-1)\big) \leq \frac{\prod_{i=1}^{n-d+1} M_i^*}{(n-d+1)!}. \tag{4.4}$$

Proof. Let $U = \frac{\prod_{i=1}^{n-d+1} M_i^*}{(n-d+1)!}$. Let first $d-1 \geq 2$ is even. Then

$$U = \frac{n(\frac{d}{2} + \frac{1}{2})(\frac{d}{2} + \frac{3}{2}) \cdots (n - \frac{d}{2} - \frac{1}{2})}{(n-d+1)!}.$$

We have the multiplicity

$$\begin{aligned}
e\big(\partial C(n,d-1)\big) &= \sum_{i=0}^{d-1} h^* \\
&= 2\Big[\binom{n-d+0}{0}+\cdots+\binom{(n-d)+d/2-3/2}{d/2-3/2}\Big]+\binom{(n-d)+d/2-1/2}{d/2-1/2} \\
&= 2\binom{n-d/2-1/2}{d/2-3/2}+\binom{n-d/2-1/2}{d/2-3/2} \\
&= \frac{2(n-d/2-1/2)\cdots(d/2-1/2)}{(n-d+1)!}+\frac{(n-d/2-1/2)\cdots(d/2+1/2)}{(n-d)!} \\
&= \frac{(n-d/2-1/2)\cdots(d/2+1/2)}{(n-d+1)!}(d-1+n-d+1) \\
&= U.
\end{aligned}$$

Now let $d-1\geq 3$ be odd. Then

$$U=\frac{n\big(\frac{d}{2}+1\big)\cdots\big(\frac{d}{2}+(n-d)\big)}{(n-d+1)!}.$$

And the multiplicity is given by

$$\begin{aligned}
e\big(\partial C(n,d-1)\big) &= \sum_{i=0}^{d-1} h^* \\
&= 2\Big[\binom{n-d+0}{0}+\binom{n-d+1}{1}+\cdots+\binom{n-d+d/2-1}{d/2-1}\Big] \\
&= 2\binom{n-d/2}{d/2-1} \\
&= 2\frac{(n-d/2)\cdots(d/2+1)(d/2)}{(n-d+1)!}.
\end{aligned}$$

We see that $e\big(\partial C(n,d-1)\big)\leq U$ if and only if $d\leq n$ which is true. □

Proof of Theorem 4.1.2. Since $m\leq(d+1)/2$, we have $M_i^*\leq M_i$ both when d is odd and even. Hence, by Equation (4.1) and Equation (4.4), we get

$$e(S/I_{\partial\Delta})\leq\frac{\prod_{i=1}^{n-d+1}M_i}{(n-d+1)!}. \tag{4.5}$$

It remains to show that $e(S/I_{\partial\Delta})\geq L$. Since

$$e(S/I_{\partial\Delta})\geq 2\sum_{i=0}^{m-1}\binom{n-d+i}{i}+(d-2m)\binom{n-d+m-1}{m-1},$$

it is enough to show that

$$2\sum_{i=0}^{m-1}\binom{n-d+i}{i}+(d-2m)\binom{n-d+m-1}{m-1}\geq\frac{n\prod_{i=1}^{n-d}(m+i-1)}{(n-d+1)!}$$

which is to prove

$$2\binom{n-d+m}{m-1}+(d-2m)\binom{n-d+m-1}{m-1}\geq\frac{n\prod_{i=1}^{n-d}(m+i-1)}{(n-d+1)!}.$$

We need to show

$$2(n-d+m)\cdots(m+1)(m)+(d-2m)(n-d+m-1)\cdots(m+1)(m)(n-d+1)$$
$$\geq n(m)(m+1)\cdots(m+n-d-1)$$

which further amounts to prove that $2(n-d+m)+(d-2m)(n-d+1)\geq n$. Notice that it is enough to show that $2(n-d+m)+(d-2m)\geq n$ which is true as $n>d$. □

Corollary 4.1.4. *Let Δ be a linear ball. Let $m-1$ be the smallest dimension of a nonface of Δ with $2\leq m\leq(d+1)/2$. Then the simplicial sphere $\partial\Delta$ satisfies the multiplicity conjecture.*

Proof. We only need to show that the assumptions (A1) and (A2) are satisfied in this case. Since S/I_Δ has a linear resolution, the minimal and maximal shifts in the minimal free resolution of S/I_Δ are given by $m'_i=M'_i=m+i-1$ for $i=1,\ldots,n-d$. Hence Δ has minimal inside faces only of dimension $n-(m+n-d-1)-1=d-m$, by fact (F1) and Theorem 4.1.1. Also, there is no minimal inside face of dimension less than $m-1$ since $d-m\geq m-1$. Hence the assumption (A1) is satisfied. We now show that the h–vector $(h'_0,\ldots,h'_{d-1})$ of $S/I_{\partial\Delta}$ is unimodal. As the Stanley–Reisner ideal I_Δ has linear resolution and $S=K[\Delta]=S/I_\Delta$ is Cohen–Macaulay, we get that the h–vector $(h_0,\ldots,h_d)$ of S/I_Δ is given by $h_i=\binom{n-d+(i-1)}{i}$ for $i=0,\ldots,m-1$ and $h_i=0$ for $i\geq m$.

Now the h–vector of $S/I_{\partial\Delta}$ is equal to (see [St95, p. 137]) :

$$(h_0-h_d,h_0+h_1-h_d-h_{d-1},\ldots,h_0+\cdots+h_{d-1}-h_d-\cdots-h_1).$$

Hence the h–vector of $S/I_{\partial\Delta}$ is given by

$$h'_i=\begin{cases}\binom{n-d+i}{i} & \text{for } i=0,\ldots,m-2;\\[2ex] \binom{n-d+m-1}{m-1} & \text{for } i=m-1,\ldots,d-m;\\[2ex] \binom{n-d+(d-1-i)}{d-1-i} & \text{for } i=d-m+1,\ldots,d-1.\end{cases}$$

Hence the assumption (A2) also holds. □

4.2 Determinantal ideals

In this section, we study simplicial complexes arising from determinantal ideals. It is known that these simplicial complexes are shellable. We prove that the geometric realization of these simplicial complexes are balls and these balls are linear only in the case of the ideal of maximal minors. We show that the boundary complexes of the simplicial complexes arising from certain determinantal ideals satisfy the multiplicity conjecture.

Let $X = (X_{ij})$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $2 \leq m \leq n$ be an $m \times n$ matrix of indeterminates. We denote by $[a_1, \ldots, a_r \,|\, b_1, \ldots, b_r]$, the minor $\det(X_{a_i b_j})$ of X where $i, j = 1, \ldots, r$. Further we define

$$[a_1, \ldots, a_r \,|\, b_1, \ldots, b_r] \leq [a_1^{'}, \ldots, a_s^{'} \,|\, b_1{'}, \ldots, b_s^{'}],$$

if $r \geq s$ and $a_i \leq a_i^{'}$, $b_i \leq b_i^{'}$ for $i = 1, \ldots, s$. Let $\Delta(X)$ denote the poset of minors of X. For $\sigma = [a_1, \ldots, a_r \,|\, b_1, \ldots, b_r] \in \Delta(X)$, we denote by I_σ the ideal generated by all minors $\gamma \not\geq \sigma$. We call such ideals determinantal ideals. Notice that for $\sigma = [1, \ldots, r \,|\, 1, \ldots, r]$, $r \leq m-1$, the ideal I_σ is the ideal generated by all $(r+1) \times (r+1)$ minors of X. For $\sigma = [1, \ldots, r \,|\, 1, \ldots, r]$, $r \leq m-1$, we denote the ideal I_σ by I_r. Note that the ideal I_{m-1} is generated by all maximal minors of X.

Let the symbol τ denote the lexicographic term order on the polynomial ring $S = K[X] = K[X_{ij},\ i = 1, \ldots, m,\ j = 1, \ldots, n]$ induced by the variable order

$$X_{11} \geq X_{12} \geq \cdots \geq X_{1m} \geq X_{21} \geq X_{22} \cdots \geq X_{2m} \geq X_{n1} \geq X_{n2} \geq \cdots \geq X_{mn}.$$

Notice that under the monomial order τ, the initial monomial of any minor of X is the product of the elements of its main diagonal. Such a monomial order is called diagonal order. In [HT92], it is shown that the generators of I_σ form a Gröbner basis and hence I_σ^* of I_σ with respect to the monomial order τ, is generated by squarefree monomials. In other words, $K[X]/I_\sigma^*$ may be viewed as a Stanley–Reisner ring of a certain simplicial complex Δ_σ. For $\sigma = [1, \ldots, r \,|\, 1, \ldots, r]$, $r \leq m-1$, we denote the simplicial complex Δ_σ by Δ_r.

We show in Theorem 4.2.4 that for any $\sigma = [a_1, \ldots, a_r \,|\, b_1, \ldots, b_r] \in \Delta(X)$, the geometric realization $|\Delta_\sigma|$ of the simplicial complex Δ_σ is a shellable ball. By Theorem 4.2.4 and Corollary 4.2.6 together, it follows that the geometric realization $|\Delta_{m-1}|$ of Δ_{m-1} is in fact a shellable linear ball.

According to [HT92], the facets of simplicial complex Δ_σ can be described as follows: its vertex set is the set of coordinate points $V = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. We define a partial order on V by setting $(i,j) \leq (i^{'}, j^{'})$ if $i \geq i^{'}$ and $j \leq j^{'}$. A maximal chain in V will be called a *path*.

Theorem 4.2.1. [HT92, Theorem 3.3] *Let $\sigma = [a_1, \ldots, a_r \,|\, b_1, \ldots, b_r]$, and let $P_i = (a_i, n)$ and $Q_i = (m, b_i)$ for $i = 1, \cdots, r$. Then the facets of Δ_σ are the non–intersecting paths from P_i to Q_i, that is, subsets $C_1 \cup C_2 \cup \cdots \cup C_r$ of V where each C_i is a path with end points P_i and Q_i and where $C_i \cap C_j = \emptyset$ for all $i \neq j$.*

We denote the set of facets of Δ_σ by $\mathcal{F}(\Delta_\sigma)$. The complex Δ_σ has a natural partial order on the set of facets which we recall from [HT92, Theorem 4.9]: Let F_1 and F_2 be two facets of Δ_σ. We write $F_1 = \bigcup_{i=1}^r C_i$ and $F_2 = \bigcup_{i=1}^r D_i$ as unions of non–intersecting paths with end points P_i and Q_i. We say that $F_2 \geq F_1$, if D_i is contained in the upper right side of C_i for all $i = 1, \ldots, r$, that is, if for each $(x, y) \in D_i$ there is some $(u, v) \in C_i$ such that $u \leq x$ and $v \leq y$, where $i = 1, \ldots, r$. This is a partial order on the facets of Δ_σ, and this partial order extended to any linear order gives us a shelling. We fix a linear order and let Σ denote the corresponding shelling. From [BV88, Corollary 5.18], we have $\dim(S/I_\sigma^*) = r(m + n + 1) - \sum_{i=1}^r (a_i + b_i)$.

Before stating the next theorem, we define the notion of a *corner* of a path. Let C be a path in V. A point $(i, j) \in C$ will be called a *corner* of C, if $(i - 1, j)$ and $(i, j - 1)$ belong to C. Let F be a facet of Δ_σ, then we denote by $\mathcal{C}(F)$, the set of corners of the paths in F, and we define $c(F) = |\mathcal{C}(F)|$.

For the proof of Theorem 4.2.4, we need the following lemma from algebraic topology:

Lemma 4.2.2. *Let E_1 be a simplicial complex whose geometric realization $|E_1|$ is a ball of dimension d, and let E_2 be a simplex of dimension d. Let the intersection $E_1 \cap E_2 = \langle G_1, \ldots, G_r \rangle \neq \emptyset$, where $G_1, \ldots, G_r$ are facets of the boundary complexes ∂E_i of E_i, $i = 1, 2$ and $\langle G_1, \ldots, G_r \rangle$ is a proper subset of ∂E_2. Then the geometric realization $|E_1 \cup E_2|$ of $E_1 \cup E_2$ is again a ball.*

The following lemma follows from the proof of [BH92, Theorem 2.4].

Lemma 4.2.3. *Let $\Delta_\sigma = \langle F_1, \ldots, F_t \rangle$ be the simplicial complex with Stanley–Reisner ideal I_σ where $F_1, \ldots, F_t$ is the shelling order Σ. Let $\Delta_i = \langle F_1, \ldots, F_i \rangle$ and let $G = F_k \setminus \{v\}$ for some $v \in F_k$, $k \leq i$. Then $G \subset F_\ell$ for some $\ell < k$ if and only if $v \in \mathcal{C}(F_k)$. If the equivalent conditions hold then F_ℓ is uniquely determined.*

Theorem 4.2.4. *For any $\sigma = [a_1, \ldots, a_r \,|\, b_1, \ldots, b_r] \in \Delta(X)$, the geometric realization $|\Delta_\sigma|$ of the simplicial complex Δ_σ is a shellable ball of dimension $r(m + n + 1) - \sum_{i=1}^r (a_i + b_i) - 1$.*

Proof. The fact that the dimension of the simplicial complex Δ_σ is $r(m + n + 1) - \sum_{i=1}^r (a_i + b_i) - 1$ follows from [BV88, Corollary 5.18]. Let $\Delta_\sigma = \langle F_1, \ldots, F_t \rangle$ where $F_1, \ldots, F_t$ is the shelling order Σ. Let $\Delta_i = \langle F_1, \ldots, F_i \rangle$. We prove that $|\Delta_i|$ is a ball by induction on i. Assume that $|\Delta_{i-1}|$ is a ball, we will show that $|\Delta_i|$ is a ball. We have $\Delta_i = \Delta_{i-1} \cup \langle F_i \rangle$, let $\Delta_{i-1} \cap \langle F_i \rangle = \langle G_1, \ldots, G_r \rangle$. Notice that G_j are codimension one faces of Δ_{i-1} as Δ_σ is shellable. By Lemma 4.2.2, we notice that $|\Delta_i|$ is a ball (assuming that $|\Delta_{i-1}|$ is a ball), if the following two conditions are satisfied:

1. Each G_j is a subset of exactly one F_k for $k \leq i - 1$, which in turn implies that $G_j \in \partial \Delta_{i-1}$,

2. $G_1, \ldots, G_r$ is a proper subset of the boundary complex $\partial\langle F_i\rangle$ of $\langle F_i\rangle$.

The first condition follows from Lemma 4.2.3. For the second condition, we define $G_v = F_i \setminus \{v\}$ where $v \notin \mathcal{C}(F_i)$ (Notice that such a v exists as not all points in F_i are corner points of F_i). Then again from Lemma 4.2.3, there exists no F_j, $j \leq i-1$ such that $G_v = F_j \cap F_i$. Hence $G_v \subset \partial\langle F_i\rangle$ and $G_v \neq G_j$ for $j = 1, \ldots, r$. □

Remark 4.2.5. We learned from Ezra Miller that Theorem 2.4 can also be deduced as a special case of [KM04, Theorem 3.7] and [KMY05, Theorem 4.4].

An ideal $I \subset S$ generated in degree d is said to have a linear resolution if in the minimal free resolution of I, one has the maximal shifts $M_i = d + i$ for all i. It is known that the ideal I_{m-1} generated by the maximal minors of matrix X has a linear resolution. In fact, the Eagon–Northcott complex gives a minimal free resolution for I_{m-1}, see [BV88, Theorem 2.16]. We have the following :

Corollary 4.2.6. *Let Δ_r be the simplicial complex with the Stanley–Reisner Ideal I_r^*. Then $|\Delta_r|$ is a linear ball if and only if $r = m-1$.*

Proof. First we show that $|\Delta_{m-1}|$ is a linear ball i.e. we show that the Stanley Reisner ideal I_{m-1}^* has a linear resolution. As stated before, we know that the ideal I_{m-1} has a linear resolution. Moreover, the ring S/I_{m-1} is Cohen–Macaulay, see [BV88, Theorem 2.8]. Now as Δ_{m-1} is shellable, the ring S/I_{m-1}^* is also Cohen–Macaulay. From here it follows, that the Stanley–Reisner ideal I_{m-1}^* also has a linear resolution. Indeed, note that S/I_{m-1} and S/I_{m-1}^* have the same Hilbert function. Let $\dim S/I_{m-1} = \dim S/I_{m-1}^* = d$. Let $y_1, \ldots, y_d$ and $y_1', \ldots, y_d'$ be the maximal regular sequences of linear forms in S/I_{m-1} and in S/I_{m-1}^*, respectively. Then $\overline{S}/\overline{I_{m-1}}$ is zero dimensional (here $\bar{\ }$ denotes modulo the sequence $(y_1, \ldots, y_d)$) and has a linear resolution. This is only possible if $\overline{I_{m-1}}$ is a power of the maximal ideal of $\overline{S}$. Now the zero dimensional ring $\overline{S}/\overline{I_{m-1}^*}$ (here $\bar{\ }$ denotes modulo the sequence $(y_1', \ldots, y_d')$) has the same Hilbert function as $\overline{S}/\overline{I_{m-1}}$. This is only possible if $\overline{I_{m-1}^*}$ is the same power of the maximal ideal as $\overline{I_{m-1}}$. In particular, $\overline{I_{m-1}^*}$ has linear resolution, and therefore I_{m-1}^* has a linear resolution.

Now we show that I_r^* does not have a linear resolution for $r \neq m-1$. Notice that it is enough to show that I_r does not have linear resolution for $r \neq m-1$, since $\beta_{i,j}(I_r^*) \geq \beta_{i,j}(I_r)$. The a–invariant of the ring S/I_r is equal to $-nr$ i.e. the minimum of the degree of generators of the canonical module of S/I_r is given by nr, see [BH92, Corollary 1.5]. As the projective dimension of S/I_r is given by $(m-r)(n-r)$[BV88, Corollary 5.18], we have $M_{(m-r)(n-r)}(S/I_r) = nm - rn$ by (F1) in the first section. Hence $M_{(m-r)(n-r)-1}(I_r) - (m-r)(n-r) + 1 = nm - rn - (m-r)(n-r) + 1 = r(m-r) + 1$ and $M_0(I_r) = r+1$. Hence for $r \neq m-1$, the ideal I_r does not have a linear resolution. □

The Stanley–Reisner ring $S_\sigma = K[\Delta_\sigma]$ being Cohen–Macaulay, admits a graded canonical module ω_σ. In [BH92], the a–invariant of S_σ which is the negative of the least degree of canonical module ω_σ is computed. Next, we want to determine the degree of all the generators of ω_σ for $\sigma = [1, \dots, r \,|\, 1, \dots, r]$, $r \leq m-1$. First we need the following lemma:

Lemma 4.2.7. *Let $\Delta_\sigma = \langle F_1, \dots, F_t \rangle$ be the simplicial complex with Stanley–Reisner ideal I_σ and $F_1, \dots, F_t$ be the shelling order Σ. Let $\Delta_i = \langle F_1, \dots, F_i \rangle$. Then the boundary complex of Δ_i is given by*

$$\partial(\Delta_i) = \big\{ G \in \Delta_i : \; F_k \setminus G \not\subset \mathcal{C}(F_k) \textit{ for all } k \leq i \textit{ with } G \subset F_k \big\}.$$

Proof. It is enough to show that the set of facets of $\partial(\Delta_i)$ is given by

$$\mathcal{F}(\partial(\Delta_i)) = \big\{ G \in \Delta_i : F_k \setminus G = \{v\}, v \notin \mathcal{C}(F_k) \text{ for all } k \leq i \text{ with } G \subset F_k \big\}.$$

Indeed, if we assume the above statement to be true, then the boundary complex is the set:

$$\big\{ H \in \Delta_i : \; H \subset G \text{ for some } G \in \mathcal{F}\big(\partial(\Delta_i)\big) \big\},$$

which is further equal to the set

$$\big\{ H \in \Delta_i : \; H \subset G, \; F_k \setminus G = \{v\}, \; v \notin \mathcal{C}(F_k) \text{ for all } k \leq i \text{ with } G \subset F_k \big\}.$$

The above set is equal to

$$\big\{ H \in \Delta_i : \; F_k \setminus H \not\subset \mathcal{C}(F_k) \text{ for all } k \leq i \text{ with } H \subset F_k \big\},$$

as in the statement of the lemma.

Let $\mathcal{S} = \big\{ G \in \Delta_i : F_k \setminus G = \{v\}, v \notin \mathcal{C}(F_k) \text{ for all } k \leq i \text{ with } G \subset F_k \big\}$. By Lemma 4.2.3, we have $\mathcal{S} \subset \mathcal{F}\big(\partial(\Delta_i)\big)$. Now let $G \notin \mathcal{S}$ be of codimension one. It follows that G is of the form $F_k \setminus \{v\}$ where $v \in \mathcal{C}(F_k)$ for some $k \leq i$. Again by Lemma 4.2.3, there exists $\ell < k$ such that $G \subset F_\ell$. Hence $G = F_\ell \cap F_k$, which implies $G \notin \partial(\Delta_i)$. □

In Theorem 4.2.4, we have shown that the geometric realization $|\Delta_\sigma|$ of Δ_σ is a ball and therefore the geometric realization $|\partial_\sigma|$ of ∂_σ is a sphere. It is known that simplicial spheres are Gorenstein over any field, see [BH96, Corollary 5.6.5]. Hence we may apply Theorem 4.1.1 to compute ω_σ. Before stating the next corollary, we define the notion of a *non–flippable* path. Let D be a path from a to b. Let $v \in D$ such that $\{v + (1,0), v + (0,1)\} \in D$ and neither $v + (1,0)$ nor $v + (0,1)$ is a corner point of D. Then v can be flipped to get a path $D' = (D \setminus \{v\}) \cup \{v + (1,1)\}$. We call such an interchange of the point v to $v + (1,1)$ *a flip*. Notice that the new path D' obtained after a flip from D has the following property: $\mathcal{C}(D) \subset \mathcal{C}(D')$. We call a path D to be a *flippable* path if D could be flipped to get a new path D', otherwise we call D to be a *non–flippable* path. Hence, a non–flippable path D from a to b is a path which has the following property: for all $v \in D$ such that $\{v + (0,1), v + (1,0)\} \subset D$, one has either $v + (0,1)$ or $v + (1,0)$ is a corner point of

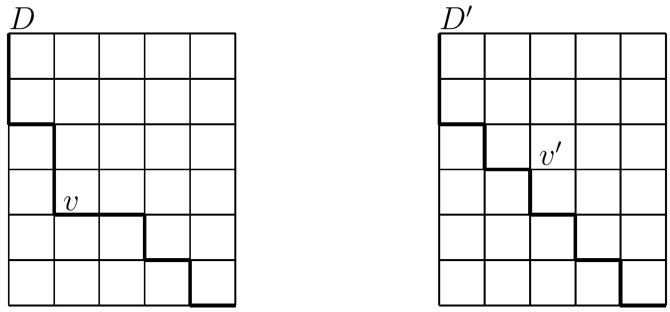

Figure 4.1: A flippable path D and a non–flippable path D' where $D' = (D \setminus \{v\}) \cup \{v'\}$.

D. Equivalently, one may notice that a path D from a to b is a non–flippable path if for a path D' from a to b with $\mathcal{C}(D') \supset \mathcal{C}(D)$, one has $D' = D$.

We call a facet $F = \bigcup_i C_i$ of the simplicial complex Δ_σ a *non–flippable facet*, if each C_i is a non–flippable path, otherwise we call F a *flippable facet*. Notice that a facet F of Δ_σ is non-flippable if for each facet F' of Δ_σ with $\mathcal{C}(F') \supset \mathcal{C}(F)$, one has $F' = F$. We denote the set of non-flippable facets of Δ_σ by $\mathcal{NF}(\Delta_\sigma)$. Let F, F' be two facets of Δ_σ with $\mathcal{C}(F) \subset \mathcal{C}(F')$. Then F' is obtained from F by finite number of flips. One has:

Lemma 4.2.8. *Let F, F' be two facets of Δ_σ, then the following two conditions are equivalent:*

(a) $\mathcal{C}(F) \subset \mathcal{C}(F')$,

(b) $F' \setminus \mathcal{C}(F') \subset F \setminus \mathcal{C}(F)$.

For a given subset Z of $[m] \times [n]$ we denote by X_Z, the monomial $\prod_{(i,j)\in Z} X_{ij}$. We have :

Corollary 4.2.9. *Let ω_σ be the canonical ideal of $K[\Delta_\sigma]$ and $\mathcal{M}$ denote the set $\{F \setminus \mathcal{C}(F) : F \in \mathcal{NF}(\Delta_\sigma)\}$. Then the minimal set of generators of ω_σ is given by $G(\omega_\sigma) = \{X_G : G \in \mathcal{M}\}$.*

Proof. By Theorem 4.2.4 and Theorem 4.1.1, it is enough to show that $\mathcal{M}$ is the set of the minimal inside faces (under inclusion) of Δ_σ.

By Lemma 4.2.7, we know that the set of inside faces of the simplicial complex Δ_σ is given by $\mathcal{S} = \{F \setminus Z : F \in \mathcal{F}(\Delta_\sigma), Z \subset \mathcal{C}(F)\}$. Therefore each minimal inside face G is of the form $F \setminus \mathcal{C}(F)$, $F \in \mathcal{F}(\Delta_\sigma)$.

Let $F \in \mathcal{NF}(\Delta_\sigma)$. Suppose $G = F \setminus \mathcal{C}(F)$ is a not a minimal inside face. Then there exists $G' \subset G$ such that $G' = F' \setminus \mathcal{C}(F')$ is a minimal inside face. By Lemma 4.2.8, it follows $\mathcal{C}(F') \supset \mathcal{C}(F)$, a contradiction.

Now, let $G = F \setminus \mathcal{C}(F)$ be a minimal inside face. Suppose $F \notin \mathcal{NF}(\Delta_\sigma)$, then there exists a facet F' such that $\mathcal{C}(F') \supset \mathcal{C}(F)$. Again, by Lemma 4.2.8, it follows then $F' \setminus \mathcal{C}(F') \subset F \setminus \mathcal{C}(F)$, a contradiction. $\square$

In general, to give the explicit expressions of multi–degrees of the generators of canonical ideal ω_σ may not be possible. But we would like to give all possible total degrees of the generators of the canonical ideal ω_σ for $\sigma = [1, \dots, r \,|\, 1, \dots, r]$, $r \leq m-1$. In this case, I_σ is the ideal generated by all $r+1 \times r+1$ minors of X. For $\sigma = [1, \dots, r \,|\, 1, \dots, r]$, we denote I_σ by I_r, ω_σ by ω_r and Δ_σ be Δ_r.

From Corollary 4.2.9, it follows that $|F| - c(F)$, $F \in \mathcal{NF}(\Delta_\sigma)$ are the total degrees of the generators of the canonical ideal ω_σ. We call the corners of the a non–flippable facet $F \in \mathcal{NF}(\Delta_\sigma)$ the *non–flippable corners*. In the case of the simplicial complex Δ_r, we will show that the number t of the non–flippable corners could be any integer between r and $r(m-r)$.

Proposition 4.2.10. *Let Δ_r be the simplicial complex with the Stanley–Reisner ideal I_r^*. Then there exists a non–flippable facet F of the simplicial complex Δ_r with t corners if and only if $r \leq t \leq r(m-r)$.*

Proof. We will construct a non–flippable facet for any given number of corners between r and $r(m-r)$. As any facet F of Δ_σ is a disjoint union of r paths C_i from (i,n) to (m,i), we notice that the minimum number of non–flippable corner for any path C_i is one and the maximum is $(m-r)$. Hence minimum and maximum number of possible total non–flippable corners are r and $r(m-r)$ respectively. As a path C_i is determined by its corners, we define the non–flippable corners for each path. For r corners, we define C_i such that $\mathcal{C}(C_i) = (i+1, i+1)$ such that $F = C_1 \cup \cdots \cup C_r$ is a non–flippable facet with r corners; see Figure 4.2.

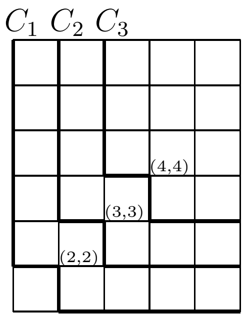

Figure 4.2: A non–flippable facet with $r = 3$ corners.

One can write any $r \leq t \leq r(m-r)$ as $t = r + p(m-r-1) + q$ for $0 \leq p \leq r$ and $0 \leq q < (m-r-1)$. For any such t, we define the corners of the path C_i as follows: For $0 \leq k \leq p-1$, the path C_{r-k} has corners at

$$\big(r-(k-1), n-(k+1)\big), \big(r-(k-2), n-(k+2)\big), \dots, \big(r-(k-m+r), n-(k+m-r)\big).$$

The path C_{r-p} has corners at

$$(r-p, r-p+q), (r-p+1, r-p+q-1), \dots, (r-p+q, r-p),$$

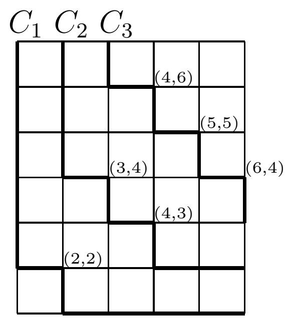

Figure 4.3: A non–flippable facet with $t = r + p(m - r - 1) + q$ corners with $m = 6, n = 7, r = 3$ and $p = 1, q = 1$.

and for $1 \leq i \leq r - p - 1$, the path C_i has corner at $(i+1, i+1)$. Now $F = \bigcup_{i=1}^{r} C_i$ is a non–flippable facet with exactly $t = r + p(m - r - 1) + q$ corners; see Figure 4.3.

□

Corollary 4.2.11. *The canonical ideal ω_r has a minimal generator of degree t if and only if $rn \leq t \leq r(n + m - r - 1)$.*

Proof. We have $\dim R/I_r = |F| = r(m + n) - r^2$, [BV88, Corollary 5.18]. Now by Corollary 4.2.9 and from Proposition 4.2.10, follows the result. □

Next, we want to consider the boundary complex ∂_r of the simplicial complex Δ_r. We want to show that the Stanley–Reisner ring S/I_{∂_r} satisfies the multiplicity conjecture. The geometric realization $|\partial_r|$ of the boundary complex ∂_r is a sphere of dimension $r(m + n) - r^2 - 1$. Therefore the Stanley–Reisner ring S/I_{∂_r} is a Gorenstein ring, see [BH96, Corollary 5.6.5]. Hence, the boundary complex ∂_r satisfies properties (P1), (P2), (P3) of Section 1 and by Theorem 4.1.1, we have $S/I_{\partial_r} = K[\Delta_r]/(\omega_r)$.

Theorem 4.2.12. *The Stanley–Reisner ring S/I_{∂_r} satisfies the multiplicity conjecture.*

Proof. We need to show that assumptions (A1) and (A2) are satisfied, see Theorem 4.1.2. As the generators of the canonical ideal ω_r of Δ_r has degrees t where $rn \leq t \leq r(m + n - r - 1)$, there exists a minimal inside face of dimension $r(m + n - r - 1) - 1 = \dim R/I_{\partial_r} - (r + 1)$ and there is no inside face of dimension less than $r + 1$, see Theorem 4.1.1. Hence assumption (A1) is satisfied.

For Assumption (A2), we need to show that h–vector of S/I_{∂_r} is unimodal. Let the h–vector of the simplicial complex Δ_r be given by $\left(h_0, \dots, h_{r(m+n)-r^2}\right)$, then the h–vector $\left(h'_0, \dots, h'_{r(m+n)-r^2-1}\right)$ of the boundary complex ∂_r is given by (see [St95, Page 137]):

$$h_0 - h_{r(m+n)-r^2}, \dots, h_0 + \cdots + h_{r(m+n)-r^2-1} - h_{r(m+n)-r^2} - \cdots - h_1.$$

By [BH92, Theorem 2.4] we have that h_i calculates the number of facets F of Δ_r with number of corners $c(F) = i$ and from Corollary 4.2.10, we get that the maximal number of corners possible are $r(m-r)$, hence $h_t = 0$ for all $r(m-r)+1 \le t \le r(m+n) - r^2$. Then it follows that the h–vector of S/I_{∂_r} is given by

$$h'_i = \begin{cases} h'_{r(m+n)-r^2-1-i} = \sum_{j=0}^{i} h_j & \text{for } i = 0, \dots, r(m-r); \\ \sum_{j=0}^{r(m-r)} h_j & \text{for } j = r(m-r)+1, \dots, nr-2; \end{cases}$$

Hence h–vector of S/I_{∂_r} is unimodal. □

In the remaining part of this section, we compare the Stanley–Reisner ideal I^*_{m-1} of Δ_{m-1} with its $(I^*_{m-1})^\vee$. We will see in Theorem 4.2.13 that the dual ideal $(I^*_{m-1})^\vee$ is again the initial ideal of the ideal of the maximal minors of a certain matrix.

Let Δ be a simplicial complex on the vertex set $[n]$ and $I_\Delta \subset K[X_1, \dots, X_n]$ be the corresponding Stanley–Reisner ideal. There is another simplicial complex $\Delta^\vee$ associated to Δ which is called the *Alexander dual* of Δ. The Alexander dual is defined by the simplicial complex $\Delta^\vee = \{[n] \setminus F : F \notin \Delta\}$. It is easy to see that the complement of the minimal non–faces of the simplicial complex Δ define the facets of the dual complex $\Delta^\vee$ and vise–versa. Hence, the Stanley Reisner ideal $I_{\Delta^\vee}$ is equal to the ideal $\big(X_{i_1} \cdots X_{i_k} : [n] \setminus \{i_1, \dots, i_k\} \in \mathcal{F}(\Delta)\big)$. One may write $I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$ where $P_F = (X_i : i \notin F)$. Therefore the monomials $X_{P_F} = \prod_{X_i \in P_F} X_i$, $F \in \mathcal{F}(\Delta)$ form a set of minimal generators of $I_{\Delta^\vee}$. From here it follows that a monomial g is a minimal generator of $I_{\Delta^\vee}$ if and only if $\mathcal{S} = \{X_i : \ X_i | \ g\}$ is a vertex cover of the set of minimal generators $G(I_\Delta)$ of I_Δ (We call a set of indeterminates $\mathcal{S} \subset \{X_1, \dots, X_n\}$ to be vertex cover of a set of monomials $\{m_1, \dots, m_k\}$ if for all m_i there exists some $X_j \in S$ such that $X_j | \ m_i$).

Let $X = (X_{ij})$ be a matrix of indeterminates of order $m \times n$. We call a matrix $Y = (Y_{ij})$ of indeterminates of order $(n-m+1) \times n$ a dual of the matrix X if $Y_{i,j+i-1} = X_{j,j+i-1}$ for $i = 1, \dots, n-m+1$ and $j = 1, \dots, m$. Notice that if Y is a dual of X, then X is a dual of Y. For example, if

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \end{pmatrix}$$

is a matrix of order 3×4 then a dual matrix Y of order 2×4 can be defined as follows:

$$Y = \begin{pmatrix} X_{11} & X_{22} & X_{33} & Y_{14} \\ Y_{21} & X_{12} & X_{23} & X_{34} \end{pmatrix}.$$

Let again I^*_{m-1} denote the initial ideal of the ideal of maximal minors of an $m \times n$ matrix $X = (X_{ij})$ of indetermiantes and Δ_{m-1} be the simplicial complex with Stanley–Reisner ideal I^*_{m-1}. We denote the Alexander dual of the simplicial complex Δ_{m-1} by $\Delta_{m-1}^{\vee}$ and the corresponding Stanley–Reisner ideal by $(I_{m-1}^*)^{\vee}$. Let $Y = (Y_{ij})$ be a dual matrix of X. Let J_{n-m} denote the ideal of the maximal minors of the matrix Y and the initial ideal of J_{n-m} be denoted by J_{n-m}^* (notice J_{n-m}^* does not depend upon the choice of the dual matrix Y). We define a polynomial ring $T = K[X_{ij}, Y_{kj} : \ 1 \le i \le m, 1 \le k \le n-m+1, 1 \le j \le n]$. Then we have:

Theorem 4.2.13.

$$(I_{m-1}^*)^{\vee}T = J_{n-m}^*T.$$

Proof. First we show that the ideal J_{n-m}^*T is contained in the ideal $(I_{m-1}^*)^{\vee}T$. Let $g = Y_{1j_1}Y_{2j_2}\cdots Y_{n-m+1,j_{n-m+1}}, \ j_1 < j_2 < \cdots < j_{n-m+1}$ be a minimal generator of the ideal J_{n-m}^*. As $Y_{1j} = X_{jj}, Y_{2j+1} = X_{jj+1}, \ldots, Y_{n-m+1,j+n-m} = X_{jj+n-m}$ for $j = 1, \ldots, m$, the monomial g is of the form $X_{i_1,i_1}X_{i_2,i_2+1}\cdots X_{i_{n-m+1},i_{n-m+1}+n-m}$ for some $1 \le i_1 \le i_2 \le \cdots \le i_{n-m+1} \le m$. We need to show that the set S given by $\{X_{i_1,i_1}, X_{i_2,i_2+1}, \ldots, X_{i_{n-m+1},i_{n-m+1}+n-m}\}$ is a vertex cover for $G(I_{m-1}^*)$. Let

$$h = X_{1,1+t_1}X_{2,2+t_2}\cdots X_{m,m+t_m}, \ 0 \le t_1 \le t_2 \le \cdots \le t_m \le n-m$$

be a minimal generator of I_{m-1}^*. We show that there exists $X_{i,j} \in S$ such that $X_{i,j} | \ h$. Suppose the contrary, then $X_{i_k,i_k+(k-1)}$ does not divide h for any $k = 1, \ldots, n-m+1$ which implies $t_{i_k} > k-1$ for $k = 1, \ldots, n-m+1$, in particular $t_{i_{n-m+1}} > n-m$ which is a contradiction.

To show that $(I_{m-1}^*)^{\vee}T \subset J_{n-m}^*T$, we need to show that if S is a minimal vertex cover of $G(I_{m-1}^*)$, then $\prod_{X_{ij}\in S} X_{ij}$ is a generator of J_{n-m}^*. Since, the monomials $\prod_{i=1}^{m} X_{i,i+k}, \ k = 0, \ldots, n-m$ are minimal generators of $G(I_{m-1}^*)$, we get that the subset of the form $S' = \{X_{i_1,i_1}, X_{i_2,i_2+1}, \ldots, X_{i_{n-m+1},i_{n-m+1}+n-m}\}$ is contained in any minimal vertex cover S of $G(I_{m-1}^*)$. Also one may notice that, we must have $1 \le i_1 \le i_2 \le \cdots \le i_{n-m+1} \le m$. Now, the generators of J_{n-m}^* are exactly of the form $\prod_{X_{ij}\in S'} X_{ij}$, hence $(I_{m-1}^*)^{\vee}T \subset J_{n-m}^*T$. □

Corollary 4.2.14. *The Stanley Reisner Ideal I_{m-1}^* has linear quotients.*

Proof. By above theorem and Theorem 4.2.4 we get that the simplicial complex $\Delta_{m-1}^{\vee}$ gives the triangulation of a shellable linear ball. Now it follows from Theorem 1.4 [HHiZ04] that I_{m-1}^* has linear quotients. □

4.3 Polarization of the powers of a maximal ideal

Let $S = K[x_1, \ldots, x_n], n \ge 3$ be a standard graded polynomial ring over the field K and let $\mathfrak{m} = (x_1, \ldots, x_n) \subset S$ denote the maximal graded ideal.

Let $u = \prod_{i=1}^{n} x_i^{a_i}$ be a monomial in S. Then the squarefree monomial given by

$$u^P = \prod_{i=1}^{n}\prod_{j=1}^{a_i} x_{ij} \in K[x_{11}, \ldots, x_{1a_1}, \ldots, x_{n1}, \ldots, x_{na_n}]$$

is called the *polarization* of u. Let $I = \mathfrak{m}^t$ be the tth power of the maximal ideal. Let $G(I) = \{u_1, \ldots, u_m\}$, then the squarefree monomial ideal $I^P = (u_1^P, \ldots, u_m^P) \subset K[x_{11}, \ldots, x_{1t}, \ldots, x_{n1}, \ldots, x_{nt}]$ is called the *polarization* of I.

Let $\Gamma = \{a \in \mathbb{N}^n \ : \ x^a \notin I\}$ be the multicomplex associated to the ideal I. The detailed information about multicomplexes can be found in [HP06]. In our case, Γ is a shellable multicomplex, see [HP06, Theorem 10.5] and all the elements of Γ are its facets. Clearly, Γ consists of those $a \in \mathbb{N}^n$ such that $\sum a(k) \leq t-1$. We define a partial order on the facets of Γ as follows: Let a, b be any two facets of Γ, we say $a < b$ if $\sum_{k=1}^n a(k) \leq \sum_{k=1}^n b(k)$. This partial order extended to any total order gives us a shelling. We fix a total order and we call the respective shelling Σ. Let $\mathcal{F}(\Gamma) = \{a_1, \ldots, a_m\}$ be the set of the facets of Γ in the shelling order Σ. Let Δ be the simplicial complex with the Stanley–Reisner ideal I^P and let $\mathcal{F}(\Delta)$ be the set of facets of Δ. By [Dr93], it follows that Δ is shellable. Furthermore by [Ja07, Lemma 3.7] and [HP06, Proposition 10.3] together, it follows that there is a bijection between $\mathcal{F}(\Gamma)$ and $\mathcal{F}(\Delta)$ given by

$$\theta : \mathcal{F}(\Gamma) \to \mathcal{F}(\Delta), \ a_k \mapsto F_{a_k}.$$

Here given the facet $a_k = \big(a_k(1), \ldots, a_k(n)\big)$ of Γ, the facet F_{a_k} of Δ is defined to be $\{x_{ij}, \ i = 1, \ldots, n, j = 1, \ldots, t, j \neq a_k(i) + 1\}$. Also, $F_{a_1}, \ldots, F_{a_m}$ is a shelling order of the facets of the simplicial complex Δ.

We have the following:

Theorem 4.3.1. *The geometric realization $|\Delta|$ of the simplicial complex Δ is a shellable linear ball.*

Proof. We already know that $\Delta = \langle F_{a_1}, \ldots, F_{a_m} \rangle$ is a shellable simplicial complex. Note that the Stanley–Reisner ideal $I_\Delta = I^P$ has a linear resolution because the graded Betti numbers of a monomial ideal and its polarization are the same, and $I = \mathfrak{m}^t$ obviously has a linear resolution. Let $\Delta_k = \langle F_{a_1}, \ldots, F_{a_k} \rangle$. We will prove $|\Delta_k|$ is a ball by induction on k as in Theorem4.2.4. The assertion is obvious for $k = 1$. Assume that $|\Delta_{k-1}|$ is a ball, we will show that $|\Delta_k|$ is a ball where the simplicial complex $\Delta_k = \Delta_{k-1} \cup \langle F_{a_k} \rangle$. Let $\Delta_{k-1} \cap \langle F_{a_k} \rangle = \{G_1, \ldots, G_r\}$ where $G_1, \ldots, G_r$ are codimension one faces of F_{a_k}. By Lemma 4.2.2, we notice that $|\Delta_k|$ is a ball (assuming that $|\Delta_{k-1}|$ is a ball) if the following two conditions are satisfied:

1. Each G_ℓ is a subset of exactly one F_{a_i} for $i \leq k-1$, which in turn implies that $G_\ell \in \partial\Delta_{k-1}$,

2. $G_1, \ldots, G_r$ is a proper subset of the boundary complex ∂F_{a_k} of F_{a_k}.

Let $a_k = (s_1, \ldots, s_n)$ where $\sum s_i \leq t-1$. Then

$$F_{a_k} = \{x_{ij}, \ i = 1, \ldots, n, j = 1, \ldots, t, j \neq s_i + 1\}.$$

Suppose $G_\ell = F_{a_k} \setminus \{x_{i_\ell j_\ell}\}$ where $1 \leq i_\ell \leq n$ and $1 \leq j_\ell \leq t$. Then clearly, $G_\ell = F_{a_k} \cap F_{a_{p_\ell}}$ where $a_{p_\ell} = (s_1, \ldots, s_{i_\ell - 1}, j_\ell - 1, s_{i_\ell + 1}, \ldots, s_n)$ and also $G_\ell \not\subset F_{a_q}$ for any $q \leq k-1$, $q \neq p_\ell$.

For the second condition, let $1 \leq q \leq n$ be the minimum integer such that $s_q < t-1$. Let $G = F_{a_k} \setminus \{x_{qt}\}$. Suppose $G \subset F_{a_j}$ for some $j \leq k-1$, then it would imply that $a_j = (s_1, \ldots, s_{q-1}, t-1, s_{q+1}, \ldots, s_n)$. Since $\sum a_j(i) \geq t$, we have $a_j \notin \Gamma$, a contradiction. Hence $G \notin \{G_1, \ldots, G_r\}$ and G is a facet of the boundary complex ∂F_{a_k}.

□

Now by the above theorem and Corollary 4.1.4, we have the following:

Corollary 4.3.2. *The simplicial sphere $\partial\Delta$ satisfies the multiplicity conjecture.*

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