Abstract

Let P_0 be a Noetherian ring, $P = P_0[y_1, \ldots, y_n]$ be the polynomial ring over P_0 with the standard grading and $P_+ = (y_1, \ldots, y_n)$ the irrelevant graded ideal of P. Then for any finitely generated graded P-module M, the local cohomology modules $H^i_{P_+}(M)$ are naturally graded P-modules and each graded component $H^i_{P_+}(M)_j$ is a finitely generated P_0-module. Our aim is to study the structure of the P_0-modules $H^i_{P_+}(M)_j$ and their asymptotic behavior for $j \ll 0$.

We consider the case that $P_0 = K[x_1, \ldots, x_m]$ is a polynomial ring, so that the K-algebra P is naturally bigraded with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. In this situation, if M is a finitely generated bigraded P-module, then each $H^i_{P_+}(M)_j$ is a finitely generated graded P_0-module whose grading is given by $(H^i_{P_+}(M)_j)_k = H^i_{P_+}(M)_{(k,j)}$. We investigate the regularity and Hilbert function of the graded P_0-modules $H^i_{P_+}(M)_j$. We show that if M is a finitely generated bigraded P-module such that the dimension of M/P_+M over P_0 is at most one, then there exists an integer c such that, $-c \leq \operatorname{reg} H^i_{P_+}(M)_j \leq c$ for all i and all j. We also show that for any bigraded hypersurface ring $R = P/fP$ for which the ideal $I(f)$ generated by all coefficients of f is $\mathfrak{m}$-primary where $\mathfrak{m}$ is the graded maximal ideal of P_0, the regularity of $H^i_{P_+}(R)_j$ is linearly bounded in j.

Next we prove the following duality theorem for local cohomology of bigraded modules. Let R be a standard bigraded K-algebra with irrelevant bigraded ideals P and Q, and let M be a finitely generated bigraded R-module. We define the bigraded Matlis-dual of M to be $M^\vee$ where the (i, j)th bigraded component of $M^\vee$ is given by $\operatorname{Hom}_K(M_{(-i,-j)}, K)$. Then there exists a convergent spectral sequence

$$E^2_{i,j} = H^{m-j}_P\big(H^i_{R_+}(M)^\vee\big) \underset{j}{\Longrightarrow} H^{i+j-m}_Q(M)^\vee$$

of bigraded R-modules, where m is the minimal number of homogeneous generators of P and R_+ is the unique graded maximal ideal of R.

The above spectral sequence degenerates when M is Cohen-Macaulay and we have for all k the following isomorphims of bigraded R-modules

$$H^k_P\big(H^s_{R_+}(M)^\vee\big) \cong H^{s-k}_Q(M)^\vee$$

where $s = \dim M$.

Brodmann and Hellus raised the question whether the modules $H^k_Q(M)$ are tame if M is a finitely generated graded R-module. In other words, whether for each k there exists an integer j_0 such that either $H^k_Q(M)_j = 0$ for all $j \leq j_0$, or else $H^k_Q(M)_j \neq 0$ for all $j \leq j_0$. Cutkosky and Herzog has recently given a counterexample to this problem. Our duality theorem is used to give this example. We also use the duality theorem to give a new proofs of known cases of the tameness problem and also to add a few new cases in which tameness holds.

We also establish the following duality theorem which is inspired by our duality theorem. Let R be a standard graded K-algebra where $(R_0, \mathfrak{m}_0)$ is a local ring with

the graded irrelevant ideal R_+ and let M be a finitely generated graded R-module. We define the graded Matlis-dual of M to be $M^\vee$ where the kth graded component of $M^\vee$ is given by $\operatorname{Hom}_{R_0}(M_k, E_{R_0}(R_0/\mathfrak{m}_0))$. Then there exists a convergent spectral sequence

$$E^2_{i,j} = H^{m-j}_{\mathfrak{m}_0}\big(H^i_{\mathfrak{m}}(M)^\vee\big) \underset{j}{\Longrightarrow} H^{i+j-m}_{R_+}(M)^\vee$$

of graded R-modules, where m is the minimal number of homogeneous generators of $\mathfrak{m}_0$ and $\mathfrak{m} = \mathfrak{m}_0 + R_+$ is the unique graded maximal ideal of R.

By using this theorem and a similar proof we obtain all the application results as stated in the bigraded case.

Finally, we give a very explicit combinatorial formulas for the Hilbert series of local cohomology modules of the rings defined by monomial relations, with respect to a monomial prime ideal. We first consider the squarefree case. As a generalization of Hochster's formula, we compute the Hilbert series of local cohomology of the Stanley-Reisner ring $K[\Delta]$ with respect to a monomial prime ideal. As a consequence, we obtain a explicit formula for the K-dimension of the bigraded components of the local cohomology modules. Using this formula we deduce that the local cohomology of $K[\Delta]$ with respect to a monomial prime ideal is always tame.

Takayama generalized Hochster's formula to any graded monomial ideal which is not necessarily squarefree. As a generalization of Takayama's result we compute the Hilbert series of local cohomology of monomial ideals with respect to monomial prime ideals and observe that again all these modules are tame. It is however known by the result of Cutkosky and Herzog that in general not all local cohomology modules are tame.