Local cohomology of bigraded modules

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Dedicated to my parents, my wife and my son Ali

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Contents

Preface

Local cohomology is an extremely useful technique in commutative algebra and algebraic geometry. In this thesis we study local cohomology in various graded situations. Let P_0 be a Noetherian ring, $P = P_0[y_1, \ldots, y_n]$ be the polynomial ring over P_0 with the standard grading and $P_+ = (y_1, \ldots, y_n)$ the irrelevant graded ideal of P. Then for any finitely generated graded P-module M, the local cohomology modules $H^i_{P_+}(M)$ are naturally graded P-modules and each graded component $H^i_{P_+}(M)_j$ is a finitely generated graded P_0-module. The main topic of this thesis is to study the structure of the P_0-modules $H^i_{P_+}(M)_j$ and their asymptotic behavior for $j \ll 0$.

This thesis is organized as follows: We recall in the first chapter some basic definitions and known facts about graded rings and graded modules, local cohomology, Castelnuovo-Mumford regularity and Hilbert function, Stanley-Reisner rings, spectral sequences, tameness local cohomology and, graded and bigraded local cohomology. In Chapter 2 we consider the case that $P_0 = K[x_1, \ldots, x_m]$ is a polynomial ring, so that the K-algebra P is naturally bigraded with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. In this situation, if M is a finitely generated bigraded P-module, then each of the modules $H^i_{P_+}(M)$ is a bigraded P-module, and each

$$H^i_{P_+}(M)_j = \bigoplus_k H^i_{P_+}(M)_{(k,j)}$$

is a finitely generated graded P_0-module whose grading is given by $(H^i_{P_+}(M)_j)_k = H^i_{P_+}(M)_{(k,j)}$. Thus it makes sense to investigate the regularity and Hilbert function of the graded P_0-modules $H^i_{P_+}(M)_j$. We show that if M is a finitely generated bigraded P-module such that the dimension of M/P_+M over P_0 is at most one, then there exists an integer c such that, $-c \leq \operatorname{reg} H^i_{P_+}(M)_j \leq c$ for all i and all j. The rest of this chapter is devoted to study the local cohomology of a hypersurface ring $R = P/fP$ where $f \in P$ is a bihomogeneous polynomial. We show that

$$\dim_K H^n_{P_+}(R)_{(k,j)} \leq \dim_K H^n_{P_+}(R)_{(k,j-1)} \quad \text{for all} \quad k, j.$$

In other words, the Hilbert series of the P_0-module $H^n_{P_+}(R)_j$ is a nonincreasing function of j. In the special case that $H^n_{P_+}(R)_j$ has finite length the regularity of $H^n_{P_+}(R)_j$ is also a nonincreasing functions in j. Next we compute the regularity of $H^i_{P_+}(R)_j$ for a special class of hypersurfaces. For the computation we use in an essential way a result of Stanley and J. Watanabe. They showed that a monomial complete intersection has the strong Lefschetz property. Using these facts the regularity and the Hilbert function of $H^i_{P_+}(P/f^r_\lambda P)_j$ can be computed explicitly. Here $r \in \mathbb{N}$ and $f_\lambda = \sum_{i=1}^n \lambda_i x_i y_i$ with $\lambda_i \in K$. As a consequence we see that $H^{n-1}_{P_+}(P/f^r P)_j$ has a linear resolution and its Betti numbers can be computed. Finally we show that for any bigraded hypersurface ring $R = P/fP$ for which the ideal $I(f)$ generated by all coefficients of f is $\mathfrak{m}$-primary where $\mathfrak{m}$ is the graded maximal ideal of P_0, the regularity of $H^i_{P_+}(R)_j$ is linearly bounded in j.

In Chapter 3 we prove a duality theorem for local cohomology of bigraded modules. We let R be a standard bigraded K-algebra with bigraded irrelevant ideals P generated by all elements of degree $(1,0)$, and Q generated by all elements of degree $(0,1)$ and M be a bigraded finitely generated R-module. We define the bigraded Matlis-dual of M to be $M^\vee$ where the (i,j)th bigraded component of $M^\vee$ is given by $\mathrm{Hom}_K(M_{(-i,-j)},K)$.

We establish the following duality theorem:

Theorem. Let R be a standard bigraded K-algebra with irrelevant bigraded ideals P and Q, and let M be a finitely generated bigraded R-module. Then there exists a convergent spectral sequence

$$E^2_{i,j} = H^{m-j}_P\big(H^i_{R_+}(M)^\vee\big) \underset{j}{\Longrightarrow} H^{i+j-m}_Q(M)^\vee$$

of bigraded R-modules, where m is the minimal number of homogeneous generators of P and R_+ is the unique graded maximal ideal of R.

We show that the above spectral sequence degenerates when M is Cohen-Macaulay and obtain for all k the following isomorphims of bigraded R-modules

$$H^k_P\big(H^s_{R_+}(M)^\vee\big) \cong H^{s-k}_Q(M)^\vee$$

where $s = \dim M$, see Corollary 3.12. We let R_0 be the K-subalgebra of R which is generated by the elements of bidegree $(1,0)$ and set $N = H^s_{R_+}(M)^\vee$. Then N is again an s-dimensional Cohen-Macaulay module and by the above isomorphism we obtain for all j the isomorphisms of graded R_0-modules

$$H^k_{P_0}(N_j) \cong \big(H^{s-k}_Q(M)_{-j}\big)^\vee \tag{1}$$

where P_0 is the graded maximal ideal of R_0. Here we used, that $H^k_P(N)_j \cong H^k_{P_0}(N_j)$ for all k and j.

Brodmann and Hellus [4] raised the question whether the modules $H^k_Q(M)$ are tame if M is a finitely generated graded R-module. In other words, whether for each k there exists an integer j_0 such that either $H^k_Q(M)_j = 0$ for all $j \le j_0$, or else $H^k_Q(M)_j \neq 0$ for all $j \le j_0$. In various cases this problem has been answered in the affirmative, see [3], [4], [24], [18], [20] and [2] for a survey on this problem. In case M is Cohen-Macaulay the tameness problem translates, due to (1), to the following question: Given a finitely generated bigraded R-module N. Does there exist an integer j_0 such that $H^k_{P_0}(N_j) = 0$ for all $j \ge j_0$, or else $H^k_{P_0}(N_j) \neq 0$ for all $j \ge j_0$? More generally, one would expect that for a finitely generated graded R_0-module W and a finitely generated bigraded R-module N there exists for all k an integer j_0 such that $\mathrm{Ext}^k_{R_0}(N_j, W) = 0$ for all $j \ge j_0$, or else $\mathrm{Ext}^k_{R_0}(N_j, W) \neq 0$ for all $j \ge j_0$. However this is not the case as has been recently shown by Cutkosky and Herzog, see [10]. Their example also provides a counterexample to the general tameness problem. To show this, Proposition 3.11 of this chapter is used. On the other hand, in Chapter 5 we are going to show that tameness holds for all local cohomology modules of a ring with monomial relations and with respect to monomial prime ideals.

We use our duality to give a new proofs of known cases of the tameness problem and also to add a few new cases in which tameness holds, see Corollary 3.10, 3.14 and 3.18. The duality is also used in the Corollaries 3.15 and 2.4 to prove some algebraic properties of the modules $H^k_Q(M)_j$ in case M is Cohen-Macaulay.

In Chapter 4 we consider R be a standard graded K-algebra where $(R_0, \mathfrak{m}_0)$ is a local ring with the graded irrelevant ideal $R_+ = \bigoplus_{i>0} R_i$ and M be a graded finitely generated R-module. We define the graded Matlis-dual of M to be $M^\vee$ where the kth graded component of $M^\vee$ is given by $\operatorname{Hom}_{R_0}(M_k, E_{R_0}(R_0/\mathfrak{m}_0))$. We establish the following duality theorem which is inspired by the duality theorem in Chapter 3:

Theorem. Let R be a standard graded K-algebra where $(R_0, \mathfrak{m}_0)$ is a local ring and R_+ the graded irrelevant ideal of R, and let M be a finitely generated graded R-module. Then there exists a convergent spectral sequence

$$E^2_{i,j} = H^{m-j}_{\mathfrak{m}_0}\big(H^i_{\mathfrak{m}}(M)^\vee\big) \underset{j}{\Longrightarrow} H^{i+j-m}_{R_+}(M)^\vee$$

of graded R-modules, where m is the minimal number of homogeneous generators of $\mathfrak{m}_0$ and $\mathfrak{m} = \mathfrak{m}_0 + R_+$ is the unique graded maximal ideal of R.

By using this theorem and a similar proof we obtain all the application results as stated in Chapter 3.

Chapter 5 gives very explicit combinatorial formulas for the Hilbert series of local cohomology modules of the rings defined by monomial relations, with respect to a monomial prime ideal. We first consider the squarefree case, since from a combinatorial point of view this is the more interesting case and also since in this case the formula which we obtain is more simple. As a generalization of Hochster's formula [6], we compute (Theorem 5.3) the Hilbert series of local cohomology of the Stanley-Reisner ring $K[\Delta]$ with respect to a monomial prime ideal. With the choice of the monomial prime ideal, the Stanley-Reisner ring $K[\Delta]$, and hence also all the local cohomology of it, can be given a natural bigraded structure. In Proposition 5.7 we give a explicit formula for the K-dimension of the bigraded components of the local cohomology modules. Using this formula we deduce that the local cohomology of $K[\Delta]$ with respect to a monomial prime ideal is always tame.

In [26] Takayama generalized Hochster's formula to any graded monomial ideal which is not necessarily squarefree. As a generalization of Takayama's result we compute the Hilbert series of local cohomology of monomial ideals with respect to monomial prime ideals and observe that again all these modules are tame. It is known however by the result of Cutkosky and Herzog [10] that in general not all local cohomology modules are tame.

The results in Chapter 2, 3 and 5 have been published in [21], [15] and [22], respectively.

1 Preliminaries

In this chapter we collect some basic facts which will be used throughout of this thesis. In this thesis all rings are assumed to be Noetherian, commutative and with identity unless otherwise stated.

1.1 Graded rings and graded modules

In this section we introduce some basic concepts concerning graded rings and modules.

Definition 1.1. Let $(G, +)$ be an abelian group. A ring R is called G-graded if there exists a family of $\mathbb{Z}$-modules R_g, $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ as a $\mathbb{Z}$-module with $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. Let R be a graded ring. An R-module M is called G-graded if there exists a family of $\mathbb{Z}$-modules M_g, $g \in G$ such that $M = \bigoplus_{g \in G} M_g$ as a $\mathbb{Z}$-module with $R_g M_h \subseteq M_{g+h}$ for all $g, h \in G$.

We call $u \in M$ homogeneous of degree g if $u \in M_g$ for some $g \in G$ and set $\deg(u) = g$. For $g \in G$ we say that M_g is a homogeneous component of M. An ideal $I \subset R$ is G-graded if $I = \bigoplus_{g \in G} I_g$ with $I_g = I \cap R_g$.

Definition 1.2. Let R be a G-graded ring and M, N are G-graded R-modules. An R-linear map $\varphi : M \to N$ is said to be graded (or homogeneous) of degree h for some $h \in G$ if $\varphi(M_g) \subseteq N_{g+h}$ for all $g \in G$. We call φ homogeneous if it is homogeneous of degree 0.

Definition 1.3. Let R be a G-graded ring, M be a G-graded R-module and $g \in G$. We define $M(g)$ to be the G-graded R-module M by shifting its grading g steps. More formally, $M(g)$ is isomorphic to M as a module and has grading defined by

$$M(g)_h = M_{g+h} \quad \text{for all} \quad h \in G.$$

We sometimes call $M(g)$ the gth twist of M. Note that, if $\varphi : M \to N$ is homogeneous of degree h, then the induced map $\widetilde{\varphi} : M(-h) \to N$ is homogeneous.

If G equals $\mathbb{Z}$, $\mathbb{Z}^2$ or $\mathbb{Z}^n$, we say that R is a graded, a bigraded or a $\mathbb{Z}^n$-graded ring and M is a graded, a bigraded or a $\mathbb{Z}^n$-graded R-module. We observe that every bigraded module M has a natural graded structure by setting

$$M_i = \bigoplus_{\substack{(a,b) \in \mathbb{Z}^2 \\ a+b=i}} M_{(a,b)}.$$

Analogously every $\mathbb{Z}^n$-graded module M has a natural graded structure by setting

$$M_i = \bigoplus_{\substack{u \in \mathbb{Z}^n \\ |u|=i}} M_u.$$

In particular the following rings will be considered in this thesis:

(i) Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K. Then S has a graded structure induced by $\deg(x_i) = 1$. We also view S as a $\mathbb{Z}^n$-graded by setting $\deg(x_i) = \varepsilon_i$ where ε_i denotes the ith unit vector of $\mathbb{Z}^n$.

(ii) Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be the polynomial ring in $n+m$-variables. Then S has a bigraded structure induced by $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$. Note that S also has a $\mathbb{Z}^n \times \mathbb{Z}^m$-bigraded structure.

The above polynomial rings are usually called (standard) graded, bigraded and $\mathbb{N}^n$-graded polynomial rings.

From now on by a graded ring (module) we mean a $\mathbb{Z}$-graded ring (module). In the following we recall some known facts:

Lemma 1.4. *Let R be a nonnegatively graded ring and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a graded R-module. For any integer k, we set $M_{\geq k} = \bigoplus_{i \geq k} M_i$. Then $M_{\geq k}$ is a graded submodule of M. In particular, this shows that $R_+ = R_{\geq 1}$ is a graded ideal of R.*

Lemma 1.5. *Let R be a graded ring and M a Noetherian (Artinian) graded R-module. Then M_j is a Noetherian (respectively, Artinian) R_0-module for all j.*

Theorem 1.6. *A graded ring R is Noetherian if and only if R_0 is Noetherian and R is finitely generated as an algebra over R_0.*

If M is an R-module, we denote the length of M as an R-module by $\ell_R(M)$ or simply by $\ell(M)$.

Theorem 1.7. *Let R be a graded ring and M a graded R-module. Then*

$$\ell_R(M) = \ell_{R_0}(M) = \sum_j \ell_{R_0}(M_j).$$

From this theorem we immediately obtain

Corollary 1.8. *Let R be a graded ring and M a graded R-module. Then M has finite length as an R-module if and only if each M_j has finite length as an R_0-module and $M_j = 0$ for all but finitely many j.*

Lemma 1.9. *Let R be a nonnegatively graded ring and M an Artinian graded R-module. Then $M_j = 0$ for $j \gg 0$.*

Lemma 1.10. *Let R be a nonnegatively graded ring which is finitely generated as an R_0-algebra, $R_+ = \bigoplus_{i>0} R_i$ and M a finitely generated graded R-module. Then*

$$M_j \subset (R_+)^k M \quad \textit{for all} \quad k \geq 1 \quad \textit{and} \quad j \gg 0.$$

1.2 Local cohomology

Let R be a Commutative ring, I an ideal of R, and M an R-module. We set

$$\Gamma_I(M) = \{x \in M : I^k x = 0 \quad \text{for some} \quad k \geq 0\}.$$

Then one has that $\Gamma_I(-)$ defines a functor which is covariant, left exact and additive.

Definition 1.11. The ith local cohomology functor, denoted by $H^i_I(-)$, is defined as the ith right derived functor of $\Gamma_I(-)$. In other words, if $\mathbb{F}^\bullet$ is an injective resolution of M, then $H^i_I(M) = H^i(\Gamma_I(\mathbb{F}^\bullet))$ for all $i \geq 0$.

Thus one computes these local cohomology modules by taking an injective resolution of M, then applying $\Gamma_I(-)$, and taking cohomology. Since $\Gamma_I(-)$ is left exact, we have that $H^0_I(M) \cong \Gamma_I(M)$. We also note that $H^i_I(M) = 0$ for $i < 0$.

Observe that if J is another ideal with the same nilradical as I, then $\Gamma_I = \Gamma_J$. Hence $H^i_J(M) = H^i_I(M)$ for all i and for all R-module M.

Another elementary but important property of the local cohomology modules is that every element in $H^i_I(M)$ is killed by a power of I. This follows at once from the definition. There are two very important alternative definitions of local cohomology when the base ring R is Noetherian. For the first alternate definition we rewrite $\Gamma_I(M)$ as the union of the submodules $0 :_M I^n$ where $0 :_M I^n$ denotes the set of elements of M annihilated by I^n, i.e.,

$$\Gamma_I(M) = \bigcup_n (0 :_M I^n).$$

Identify $(0 :_M I^n) = \operatorname{Hom}_R(R/I^n, M)$, so that

$$H^0_I(M) = \Gamma_I(M) = \varinjlim_{n \geq 0} \operatorname{Hom}_R(R/I^n, M),$$

and we define the higher local cohomology groups as the right derived functors of $H^0_I(-)$, i.e., $H^i_I(M)$ is the ith cohomology module of the complex obtained by applying $H^0_I(-)$ to an injective resolution of M. The functor $H^0_I(-)$ is left exact, and so for any short exact sequence of modules we get a long exact sequence in local cohomology. Since local cohomology is the derived functor, it is universal among sequence of functors with this property. On the other hand, the functors $\varinjlim_{n \geq 0} \operatorname{Ext}^i_R(R/I^n, M)$ behave in a similar way, taking short exact sequence to long exact sequence. They have the same universal property as the local cohomology, so they are in fact naturally isomorphic:

$$H^i_I(M) \cong \varinjlim_{n \geq 0} \operatorname{Ext}^i_R(R/I^n, M).$$

The second identification is worth a theorem

Let R be a commutative Noetherian ring, I and ideal of R, and M an R-module. Suppose that $x_1, \dots, x_n \in R$ such that $\sqrt{I} = \sqrt{(x_1, \dots, x_n)}$. We denote by $C(M)^\bullet$ the (extended) Čech complex of M with respect to $x_1, \dots, x_n$ defined as follows:

$$C(M)^\bullet : 0 \to C(M)^0 \to C(M)^1 \to \cdots \to C(M)^n \to 0$$

with

$$C(M)^t = \bigoplus_{1 \leq i_1 < \cdots < i_t \leq n} M_{x_{i_1} \dots x_{i_t}},$$

and where the differentiation $C(M)^t \to C(M)^{t+1}$ is given on the component

$$M_{x_{i_1} \dots x_{i_t}} \to M_{x_{j_1} \dots x_{j_{t+1}}}$$

to be the homomorphism

$$(-1)^{s-1} nat : M_{x_{i_1} \dots x_{i_t}} \to (M_{x_{i_1} \dots x_{i_t}})_{x_{j_s}},$$

if $\{i_1, \dots, i_t\} = \{j_1, \dots, \hat{j}_s, \dots, j_{t+1}\}$ and 0 otherwise. Then by [5, Theorem 5.1.19] we have the following well known functorial isomorphism

$$H_I^i(M) \cong H^i(M \otimes_R C^\bullet) \cong H^i\big(C(M)^\bullet\big) \cong H^i(x^\infty; M) \quad \text{for all} \quad i \geq 0,$$

where $C^\bullet$ denotes the Čech complex of R with respect to $x_1, \dots, x_n$ and $H^i(x^\infty; M)$ denotes the ith cohomology of koszul complex $K^\bullet(x_1, \dots, x_n; M)$.

In the following we recall some well known properties of local cohomology.

Proposition 1.12. (a) *(Independence of Base). Let $\varphi : R \to S$ be a ring homomorphism of Noetherian rings, I an ideal of R, and M an S-module. Then $H_I^i(M) \cong H_{IS}^i(M)$ as S-modules where the first local cohomology is computed over the ring R.*

(b) *Let Λ be a directed set and $\{M_\lambda\}_{\lambda \in \Lambda}$ a direct system of R-modules. Then*

$$\varinjlim_{\lambda} H_I^i(M_\lambda) \cong H_I^i(\varinjlim_{\lambda} M_\lambda).$$

(c) *If φ is a flat homomorphism, then*

$$H_I^i(M) \otimes_R S \cong H_{IS}^i(M \otimes_R S).$$

In particular, local cohomology commutes with localization and completion.

(d) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M an R-module, then*

$$H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}\widehat{R}}^i(\widehat{R} \otimes_R M).$$

Moreover, if M is finitely generated we have

$$H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}\widehat{R}}^i(\widehat{M}).$$

Here $\widehat{M}$ denotes the $\mathfrak{m}$-adic completion of M.

Let $(R, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, residue field K, and $E_R(K)$ be the injective hull of the residue field K. We define the Matlis dual of an R-module M to be the module $M^\vee = \operatorname{Hom}_R(M, E_R(K))$. In the following we state Matlis duality theorem which gives us a one to one arrow reversing correspondence between finitely generated modules over the completion of a Noetherian local ring R and Artinian modules over R. We denote the completion of R by $\widehat{R}$.

Theorem 1.13. *(Matlis Duality). Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field K and $E_R(K)$ the injective hull residue field K.*

(a) *Any Artinian R-module N is isomorphic to a submodule of $E_R(K)^r$ for some integer r.*

(b) *There is a one to one correspondence between finitely generated $\widehat{R}$-modules and Artinian R-modules. This correspondence is given as follows: if M is finitely generated then $M^\vee = \operatorname{Hom}_{\widehat{R}}(M, E_R(K))$ is Artinian. If N is Artinian, then $N^\vee = \operatorname{Hom}_R(N, E_R(K))$ is finitely generated over $\widehat{R}$. Furthermore, $L^{\vee\vee} = L$ if L is either a finitely generated module over $\widehat{R}$ or an Artinian R-module. Moreover, $E_R(K)$ and $\widehat{R}$ are Matlis duals.*

There are three basic vanishing results. The first is an immediate consequence of the idenification of Koszul cohomology with local cohomology. Let I be an ideal in a Noetherian ring. We set $\operatorname{ara}(I)$ equal to the least integer n such that I can be generated by n equations up to radical.

Theorem 1.14. *Let R be a Noetherian ring and let I be an ideal of R. Then $H_I^i(M) = 0$ for all $i > \operatorname{ara}(I)$ and for all R-modules M.*

This theorem gives a lower bound for $\operatorname{ara}(I)$.

Theorem 1.15. *Let R be a Noetherian ring, I be an ideal of R, and M a finitely generated R-module. Then $H_I^i(M) = 0$ for $i < \operatorname{grade}(I, M)$ and for $i > \dim(M)$.*

Theorem 1.16. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, and M be a non-zero, finitely generated R-module of dimension n and depth t. Then*

(a) *$H_{\mathfrak{m}}^i(M)$ is Artinian for all $i \in \mathbb{N}_0$,*

(b) *$H_{\mathfrak{m}}^i(M) \neq 0$ for $i = t$ and $i = n$. Moreover, for $n > 0$, $H_{\mathfrak{m}}^n(M)$ is not finitely generated.*

One can generalize part (a) for $i = n$ to non local situations. In fact, if M be a non zero finitely generated R-module of finite dimension n and I be an ideal of R, then the R-module $H_I^n(M)$ is Artinian.

1.3 Castelnuovo-Mumford regularity and Hilbert function

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and M be a finitely generated graded R-module. By Hilbert's syzygy theorem, M has a graded free resolution over R of the form

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where $F_i = \bigoplus_{j=1}^{t_i} R(-a_{ij})$ for some integers a_{ij} and $k \leq n$. Then the Castelnuovo-Mumford regularity $\operatorname{reg}(M)$ of M is the nonnegative integer

$$\operatorname{reg} M \leq \max_{i,j}\{a_{ij} - i\}$$

with equality holding if the resolution is minimal. Thus one may view $\operatorname{reg}(M)$ as a measure for the complexity of the minimal graded free resolution of M.

Example 1.17. Let $R = K[x_1, \ldots, x_n]$, $\mathfrak{m}$ be the graded maximal ideal of R, and $K = R/\mathfrak{m}$ the residue class field. It is well-known that the Koszul complex gives the minimal graded free R-resolution of K of the form

$$0 \to R(-n)^{\binom{n}{n}} \to R(-n+1)^{\binom{n}{n-1}} \to \cdots \to R(-2)^{\binom{n}{2}} \to R(-1)^{\binom{n}{1}} \to R \to K \to 0.$$

See [6, Section 1.6]. We see that $\operatorname{reg}(K) = 0$.

In the following we give the second important definition of Gastelnuovo-Mamford regularity by using local cohomology. Let $\mathfrak{m} = \bigoplus_{i>0} R_i$ be the graded maximal ideal consisting of all elements of degree greater than 0 which is called the irrelevant ideal. Since $H^i_{\mathfrak{m}}(M)$ is a graded Artinian R-module, then $H^i_{\mathfrak{m}}(M)_j = 0$ for all i and $j \gg 0$. The Gastelnuovo-Mamford regularity is a measure of this vanishing degree. We set

$$a_i(M) = \max\{j : H^i_{\mathfrak{m}}(M)_j \neq 0\},$$

if $H^i_{\mathfrak{m}}(M) \neq 0$ and $a_i(M) = 0$, if $H^i_{\mathfrak{m}}(M) = 0$. Then

$$\operatorname{reg}(M) = \max_i\{a_i(M) + i\} = \max\{i + j : H^i_{\mathfrak{m}}(M)_j \neq 0\}.$$

If M is an Artinian graded R-module, then $H^0_{\mathfrak{m}}(M) = M$, and hence

$$\operatorname{reg}(M) = \max\{j : M_j \neq 0\}.$$

We also use the following characterization of regularity: Let M be a finitely generated R-module. We say M has an q-linear resolution if

$$\operatorname{Tor}_i^R(M, K)_{i+j} = 0 \quad \text{for all} \quad j \neq q.$$

We consider the minimal graded free resolution

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0,$$

of M with

$$F_i = \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{ij}^R(M)} \quad \text{where} \quad \beta_{ij}^R(M) = \dim_K \operatorname{Tor}_i^R(M,K)_j.$$

Here $\beta_{ij}^R(M)$ are the graded Betti numbers of M. Then M has a q-linear resolution if and only if $\beta_{ij}^R(M) = 0$ for $j \neq q$. Therefore M has a free resolution of the form

$$\begin{aligned} 0 \to \bigoplus R(-q-k)^{\beta_{kk+q}^R(M)} \to \cdots \to \bigoplus R(-q-2)^{\beta_{22+q}^R(M)} \to \\ \bigoplus R(-q-1)^{\beta_{11+q}^R(M)} \to \bigoplus R(-q)^{\beta_{0q}^R(M)} \to M \to 0. \end{aligned}$$

For an integer q we define the truncation $M_{\geq q}$ of M as the graded R-module with graded components: $(M_{\geq q})_i = M_i$ if $i \geq q$ and otherwise 0. Thus by [7, Proposition 3.1] $\operatorname{reg}(M)$ is the least integer q such that $M_{\geq q}$ is nonzero and has a q-linear free resolution, i.e.,

$$\operatorname{reg}(M) = \min\{q : M_{\geq q} \quad \text{has a linear resolution}\}.$$

In other words, the module M has an q-linear (or simply linear) resolution if

$$\operatorname{reg}(M) = \operatorname{indeg}(M) = q,$$

where $\operatorname{indeg}(M) = \min\{i : M_i \neq 0\}$ and is called the initial degree of M. Note that the initial degree of M equals to the least degree of a homogeneous generator of M, i.e., $\operatorname{indeg}(M) = \min\{j : \beta_{0j}^R(M) \neq 0\}$. We recall the following well-known properties of regularity.

Lemma 1.18. *Suppose that*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of finitely generated graded R-modules. Thus we have

(a) $\operatorname{reg} M' \leq \max\{\operatorname{reg} M, \operatorname{reg} M'' + 1\}$*;*

(b) $\operatorname{reg} M \leq \max\{\operatorname{reg} M', \operatorname{reg} M''\}$*;*

(c) $\operatorname{reg} M'' \leq \max\{\operatorname{reg} M' - 1, \operatorname{reg} M\}$*;*

(d) *If M' is finite length, then* $\operatorname{reg} M = \max\{\operatorname{reg} M', \operatorname{reg} M''\}$*.*

The following result is due to [13, Lemma 1.6], see also [7].

Lemma 1.19. *Let $\mathbb{F}_\bullet$ be a graded complex of finite free R-modules*

$$\cdots \to \bigoplus_j R(-j)^{\beta_{2j}} \to \bigoplus_j R(-j)^{\beta_{1j}} \to \bigoplus_j R(-j)^{\beta_{0j}} \to 0.$$

We set $b_i(\mathbb{F}_\bullet) = \max\{j : \beta_{ij} \neq 0\}$. If $\dim H_i(\mathbb{F}_\bullet) \leq i$ for $i \geq 1$, then

$$\operatorname{reg}\big(H_0(\mathbb{F}_\bullet)\big) \leq \max_i\{b_i(\mathbb{F}_\bullet) - i\}.$$

Let R be a graded ring and M be a graded R-module. Suppose that $\ell_{R_0}(M_i) < \infty$ for all i. The numerical function

$$H_M(i) = \ell_{R_0}(M_i) \quad \text{for all} \quad i \in \mathbb{Z}$$

is called the Hilbert function of M and

$$\mathrm{Hilb}(M, t) = \sum_{i \in \mathbb{Z}} \ell_{R_0}(M_i) t^i$$

or simply $\mathrm{Hilb}(M)$, is called the Hilbert series of M. The most important class of graded modules which have Hilbert function (Series) are those that are finitely generated over a graded (nonnegatively graded) Noetherian ring in which R_0 is Artinian.

Example 1.20. Let $R = K[x_1, \ldots, x_d]$ be a polynomial ring over a field K and $\deg x_i = 1$ for $i = 1, \ldots, d$. Then one has

$$H_R(i) = \dim_K R_i = \binom{i+d-1}{d-1},$$

and for $d \geq 1$,

$$\mathrm{Hilb}(R, i) = \sum_{i=0}^{\infty} \dim_K R_i t^i = \sum_{i=0}^{\infty} \binom{i+d-1}{d-1} t^i = \frac{1}{(1-t)^d}.$$

Theorem 1.21. *(Hilbert). Let R is generated over R_0 by elements of degree* 1, *i.e., $R = R_0[R_1]$ and M be a finitely generated graded R-module of dimension d. Then $H_M(i)$ agrees for large i with a polynomial of degree $d - 1$.*

This polynomial denoted by $P_M(i)$, is called the Hilbert polynomial of M. It is well known that for any nonzero finitely generated graded module over a standard graded ring there exists a unique $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_M(1) \neq 0$ such that the Hilbert series of M is always a rational function of the form

$$\mathrm{Hilb}(M, t) = \frac{Q_M(t)}{(1-t)^d},$$

where d denotes the Krull-dimension of M. See [6, Corollary 4.1.8].

1.4 Stanley-Reisner rings

A simplicial complex Δ over a set of vertices $V = \{v_1, \ldots, v_n\}$ is a collection of subsets of V, with the property that $\{v_i\} \in \Delta$ for all i, and if $F \in \Delta$ then all subsets of F are also in Δ (including the empty set). An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called vertices and edges, respectively, and $\dim \emptyset = -1$. The maximal faces of Δ under inclusion are called facets of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets.

Definition 1.22. Let Δ be a simplicial complex on the vertex set $V = \{v_1, \ldots, v_n\}$, and K be a field. The Stanley-Reisner ring of the complex Δ is the graded K-algebra

$$K[\Delta] = K[X_1, \ldots, X_n]/I_\Delta,$$

where I_Δ is the ideal generated by all monomials $X_{i_1}X_{i_2}\cdots X_{i_k}$ such that

$$\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \notin \Delta.$$

On the other hand, any quotient of a polynomial ring over an ideal generated by square free monomials of degree greater than 1, is the Stanley-Reisner ring of a simplicial complex.

The following notions will be crucial in Chapter 5 for the analysis of the local cohomology of a Stanley-Reisner ring.

Definition 1.23. Let Δ be a simplicial complex and F a subset of the vertex set of Δ. The star of F is the set $\operatorname{st}_\Delta F = \{G \in \Delta : F \cup G \in \Delta\}$, and the link of F is the set $\operatorname{lk}_\Delta F = \{G : F \cup G \in \Delta, F \cap G = \emptyset\}$.

We write st and lk instead of st_Δ and lk_Δ (for short). We see that $\operatorname{st} F$ is a subcomplex of Δ, $\operatorname{lk} F$ a subcomplex of $\operatorname{st} F$, and that $\operatorname{st} F = \operatorname{lk} F = \emptyset$ if $F \notin \Delta$.

1.5 Spectral sequence

In this section we briefly give some basic facts and results concerning spectral sequence due to [23] and [12].

Let R be a Noetherian ring. A spectral sequence is a collection of R-modules (or more generally, objects of an abelian category) $\{E^r_{p,q}\}$ for all $r \in \mathbb{N}$, $p, q \in \mathbb{Z}$, equipped with maps

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that

$$\cdots \to E^r_{p+r,q-r+1} \xrightarrow{d^r_{p+r,q-r+1}} E^r_{p,q} \xrightarrow{d^r_{p,q}} E^r_{p-r,q+r-1} \to \cdots$$

is a chain complex, and the E^{r+1}'s are its homology, i.e.,

$$E^{r+1}_{p,q} \cong \operatorname{Ker}(d^r_{p,q})/\operatorname{Im}(d^r_{p+r,q-r+1}).$$

Note that what we have defined above is a homology spectral sequence. Cohomology spectral sequence are identical, except that all the arrows go in the other direction. The terms a cohomological spectral sequence are written $E^{p,q}_r$.

A double complex is a collection of objects $C_{i,j}$ for all integers i and j together with two differentials d^{I} and d^{II} with $d^{\mathrm{I}2} = d^{\mathrm{II}2} = 0$. d^{I} is assumed to decrease i, and d^{II} is assumed to decrease j. Furthermore, we assume that the differentials anticommute, so that $d^{\mathrm{I}}d^{\mathrm{II}} + d^{\mathrm{II}}d^{\mathrm{I}} = 0$.

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & C^{i+1,j} \arrow[u] \arrow[r, "d^{\mathrm{I}}"] & C^{i,j} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{i+1,j+1} \arrow[u, "d^{\mathrm{II}}"] \arrow[r, "d^{\mathrm{I}}"] & C^{i,j+1} \arrow[u, "d^{\mathrm{II}}"'] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

We want to compute the iterated homologies $H^{\mathrm{I}}_i\big(H^{\mathrm{II}}_j(C_{\bullet,\bullet})\big)$ and $H^{\mathrm{II}}_j\big(H^{\mathrm{I}}_i(C_{\bullet,\bullet})\big)$. We will do this by filtering our double complex in two different ways. Here are our filtration:

$$(C^{\mathrm{I}}_{i,j})_p = \begin{cases} 0 & \text{if } i < p \\ C_{i,j} & \text{if } i \geq p, \end{cases}$$

and

$$(C^{\mathrm{II}}_{i,j})_p = \begin{cases} 0 & \text{if } j < p \\ C_{i,j} & \text{if } j \geq p. \end{cases}$$

We define the total complex $\mathrm{tot}(C_{\bullet,\bullet})$ to be the complex whose n'th term is $\bigoplus_{i+j=n} C_{i,j}$ and whose differential is $d^{\mathrm{I}} + d^{\mathrm{II}}$. This is a complex because d^{I} and d^{II} are anticommuting differentials. The two filtrations on $C_{i,j}$ give two filtrations on the total complex:

$${}^{\mathrm{I}}\,\mathrm{tot}(C_{\bullet,\bullet})^n_p = \bigoplus_{\substack{i+j=n \\ i<p}} C_{i,j} \quad \text{and} \quad {}^{\mathrm{II}}\,\mathrm{tot}(C_{\bullet,\bullet})^n_p = \bigoplus_{\substack{i+j=n \\ j<p}} C_{i,j}.$$

These spectral sequences give information about the iterated homologies. The E^0, E^1 and E^2 terms of the I filtration are given by ${}^{\mathrm{I}}E^0_{p,q} = C_{p,q}$, ${}^{\mathrm{I}}E^1_{p,q} = H^{\mathrm{II}}_q(C_{p,\bullet})$ and ${}^{\mathrm{I}}E^2_{p,q} = H^{\mathrm{I}}_p\big(H^{\mathrm{II}}_q(C_{\bullet,\bullet})\big)$. Using the other filtration gives us a different spectral sequence with a similar E^2 term: ${}^{\mathrm{II}}E^2_{p,q} = H^{\mathrm{II}}_q\big(H^{\mathrm{I}}_p(C_{\bullet,\bullet})\big)$. What remains is to find a relationship between these two spectral sequences. It will turn out that as r increases, the two sequences will become similar enough to allow useful comparisons.

Most interesting spectral sequence are upper right quadrant, meaning that $E^r_{p,q} = 0$ if p or $q < 0$. If this is the case then for any p, q both $d^r_{p,q}$ and $d^r_{p+r,q-r+1}$ are 0 for sufficiently large r since the target or source is out of the upper right quadrant, so that there exists an integer r_0 such that for all $r > r_0$ we have $E^r_{p,q} = E^{r+1}_{p,q} = \cdots$. This is called $E^\infty_{p,q}$, and in this case we say the spectral sequences collapses at E^r. We write

$$E^r_{p,q} \underset{p}{\Longrightarrow} E^\infty_{p,q}$$

which means $E^r_{p,q}$ is isomorphic to $E^\infty_{p,q}$ for large r.

Definition 1.24. A filtration of a graded module $H = \{H_i : i \in \mathbb{Z}\}$ is a family of graded submodules $\{\mathcal{F}^p H : p \in \mathbb{Z}\}$ with $\mathcal{F}^{p-1}H \subset \mathcal{F}^p H$ for all p.

Definition 1.25. Let H be a graded module. A spectral sequence $\{E^r\}$ converges to H and denoted by $E^2_{p,q} \underset{p}{\Longrightarrow} H_n$ where $n = p+q$, if there is some bounded filtration $\{\mathfrak{N}^p H\}$ of H

$$0 = \mathfrak{N}^s H_n \subset \mathfrak{N}^{s+1} H_n \subset \cdots \subset \mathfrak{N}^t H_n = H_n,$$

such that

$$E^\infty_{p,q} \cong \mathfrak{N}^p H_n / \mathfrak{N}^{p-1} H_n \quad \text{for all} \quad p, q.$$

In most spectral sequence, E^∞ term is not naturally a doubly graded object. Instead, there are usually E^∞_n terms which come with a natural filtration $\mathcal{F}^\bullet E^\infty_n$. In these cases, we set

$$E^\infty_{p,q} = \operatorname{gr}_p E^\infty_{p+q} = \mathcal{F}^p E^\infty_{p+q} / \mathcal{F}^{p-1} E^\infty_{p+q}.$$

We define convergence in the same way as before, but we write

$$E^r_{p,q} \underset{p}{\Longrightarrow} E^\infty_n$$

to mean that whenever $p + q = n$, $E^r_{p,q}$ converges to $E^\infty_{p,q}$. The simplest situation in which we can determine convergence is when the spectral sequence degenerates. We say that the spectral sequence degenerates at E^r if for any $s \geq r$ the differential d_s is zero. The spectral sequence also converges if $E^r_{p,q}$ vanishes for all p less than some p_0 and for all q less than some q_0. If p_0 and q_0 can be chosen to be zero, this is called a first quadrant spectral sequence. Similarly, the spectral sequence also converges if $E^r_{p,q}$ vanishes for all p greater than some p_0 and for all q greater than some q_0.If p_0 and q_0 can be chosen to be zero, this is called a third quadrant spectral sequence.

Theorem 1.26. *Let $C_{\bullet,\bullet}$ be a first or third quadrant double complex, and let $\{{}^{\mathrm{I}}E^r\}$ and $\{{}^{\mathrm{II}}E^r\}$ be the spectral sequence determined by the first and second filtrations of $C_{\bullet,\bullet}$, respectively.*

(a) *For all p, q, we have ${}^{\mathrm{I}}E^\infty_{p,q} = {}^{\mathrm{I}} E^r_{p,q}$ and ${}^{\mathrm{II}}E^\infty_{p,q} = {}^{\mathrm{II}} E^r_{p,q}$ for large r*

(b) ${}^{\mathrm{I}}E^2_{p,q} = H^{\mathrm{I}}_p(H^{\mathrm{II}}_q(C_{\bullet,\bullet})) \underset{p}{\Longrightarrow} H_n(\operatorname{tot}(C_{\bullet,\bullet}))$ *and*

${}^{\mathrm{II}}E^2_{p,q} = H^{\mathrm{II}}_q(H^{\mathrm{I}}_p(C_{\bullet,\bullet})) \underset{p}{\Longrightarrow} H_n(\operatorname{tot}(C_{\bullet,\bullet}))$.

Let F be a double complex, sequence $\{E^r\}$ is called a usual spectral sequence of F if it is the spectral sequence arising from either the first or second filtration of $\operatorname{tot}(F)$.

A usual spectral sequence $\{E^r\}$ of a first or third quadrat double complex collapses if $E^2_{p,q} = 0$ for $q \neq 0$. Thus, collapsing says the bigraded module E^2 can have nonzero terms only on the p-axis.

Lemma 1.27. *Let $\{E^r\}$ be a spectral sequence of a double complex F, where F is first or third quadrant. If $\{E^r\}$ collapses, then*

$$E^\infty_{p,q} = E^2_{p,q} \quad \textit{for all} \quad p, q \quad \textit{and} \quad H_i(\operatorname{tot}(F)) \cong E^2_{i,0}.$$

There is a similar result to Lemma 1.27 if a spectral sequence collapses on the q-axis.

A refinement of the notation for convergence is given: We write

$$E^r_{p,q} \underset{p}{\Longrightarrow} H_{p+q}(\operatorname{tot} F)$$

to mean that the spectral sequence containing the terms $E^{p,q}_r$ converges and that writing $H_{p+q}(\operatorname{tot} F)^p$ for the pth level in the associated filtration of $H_{p+q}(\operatorname{tot} F)$,

$$E^\infty_{p,q} = H_{p+q}(\operatorname{tot} F)^p / H_{p+q}(\operatorname{tot} F)^{p-1}.$$

Theorem 1.28. *Assume that* $E^2_{p,q} \underset{p}{\Longrightarrow} H_{p+q}(G)$, *where* $\{E^r\}$ *is a first quadrant spectral sequence. Then the following hold:*

(a) *for all* i, *there is an epimorphism* $E^2_{0,i} \to E^\infty_{0,i}$, *and a monomorphism* $E^\infty_{i,0} \to E^2_{i,0}$;

(b) *for all* i, *there is a monomorphism* $E^\infty_{0,i} \to H_i(G)$ *and an epimorphism* $H_i(G) \to E^\infty_{i,0}$;

(c) *there is an exact sequence*

$$H_2(G) \to E^2_{2,0} \xrightarrow{d_2} E^2_{0,1} \to H_1(G) \to E^2_{1,0} \to 0.$$

One has the dual version above theorem. Here we only mention the last part:

Theorem 1.29. *Assume that* $E^{p,q}_2 \underset{p}{\Longrightarrow} H^{p+q}(G)$, *where* $\{E_r\}$ *is a third quadrant spectral sequence. Then there is an exact sequence*

$$0 \to E^{1,0}_2 \to H^1(G) \to E^{0,1}_2 \xrightarrow{d_2} E^{2,0}_2 \to H^2(G).$$

Example 1.30. Let $R \to S$ be a ring homomorphism, M and S-module and N and R-module. Then $\operatorname{Ext}^j_R(S,N)$ naturally is an S-module for all j. We have a convergent spectral sequence

$$\operatorname{Ext}^i_S\big(M, \operatorname{Ext}^j_R(S,N)\big) \underset{i}{\Longrightarrow} \operatorname{Ext}^{i+j}_R(M,N).$$

Thus we have the following five term exact sequence

$$\begin{aligned} 0 \to \operatorname{Ext}^1_S\big(M, \operatorname{Hom}_R(S,N)\big) \to \operatorname{Ext}^1_R(M,N) \to \operatorname{Hom}_S\big(M, \operatorname{Ext}^1_R(S,N)\big) \to \\ \operatorname{Ext}^2_S\big(M, \operatorname{Hom}_R(S,N)\big) \to \operatorname{Ext}^2_R(M,N). \end{aligned}$$

Lemma 1.31. *Let* F *be a double complex and* $G = \operatorname{tot}(F)$.

(a) *Suppose that* $H^i(G)^p = H^i(G)^{p+1}$ *for all* $p \neq j$. *Then* $H^i(G) = H^i(G)^j$;

(b) *Suppose that $H^i(G)^p = H^i(G)^{p+1}$ for all $p \neq j, k$ with $j < k$. Then we obtain an exact sequence*

$$0 \to E_\infty^{k,1-k} \to H^i(G) \to E_\infty^{j,1-j} \to 0$$

where

$$E_\infty^{k,1-k} = H^i(G)^k / H^i(G)^{k+1} \quad \text{and} \quad E_\infty^{j,1-j} = H^i(G)^j / H^i(G)^{j+1}$$

Corollary 1.32. (a) *Suppose that $E_2^{i,j} = 0$ for all $j \neq q$. Then $H^i(G) = E_2^{i-q,q}$.*

(b) *Suppose that $E_2^{i,j} = 0$ for all $i \neq p$. Then $H^i(G) = E_2^{p,i-p}$.*

Similar results can be obtained for the homology.

1.6 Tameness and local cohomology

Definition 1.33. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded Noetherian ring, so that R_0 is a Noetherian ring, R is a graded R_0-algebra and $R = R_0[x_1, \ldots, x_n]$ with finitely many elements $x_1, \ldots, x_n \in R_1$. Let N be a graded R-module. The R-module N is called *tame*, if there exists an integer j_0 such that either

$$N_j = 0 \quad \text{for all} \quad j \leq j_0, \quad \text{or} \quad N_j \neq 0 \quad \text{for all} \quad j \leq j_0.$$

Example 1.34. Any finitely generated R-module M is tame. Indeed, we set $M_{\geq -j} = \bigoplus_{i \geq -j} M_i$. Since R is a nonnegatively graded ring and M a graded R-module, then by Lemma 1.4, $M_{\geq -j}$ are the graded submodules of M which increase by inclusion. We set $N_j = M_{\geq -j}$. Since M is finitely generated, there exists an integer j_0 such that

$$N_{j_0} = N_{j_0 + r} \quad \text{for all} \quad r > 0.$$

Hence $M_{-j} = 0$ for all $j > j_0$, i.e., there exists an integer j_0 such that $M_j = 0$ for all $j \leq j_0$.

Example 1.35. By the similar way as Example 1.34 one has that any graded Artinian R-module is also tame.

Now let $R_+ = \bigoplus_{i > 0} R_i$ denotes the irrelevant graded ideal of R and M be a finitely generated graded R-module. Let $H^i_{R_+}(M)$ denotes the ith local cohomology module of M with support in R_+. The R-modules $H^i_{R_+}(M)$ carry a natural grading [5, Chapter 12]. Let $H^i_{R_+}(M)_j$ denotes the jth graded component of $H^i_{R_+}(M)$. According to [5, Proposition 15.1.15] we can say: The R_0-modules $H^i_{R_+}(M)_j$ are finitely generated for all $i \geq 0$ and all $j \in \mathbb{Z}$, and $H^i_{R_+}(M)_j = 0$ for all i and $j \gg 0$. Brodmann and Hellus [4] raised the question whether the local cohomology modules $H^i_{R_+}(M)$ are tame. In other words, whether for each i there exists an integer j_0 such that either $H^i_{R_+}(M)_j = 0$ for all $j \leq j_0$, or else $H^i_{R_+}(M)_j \neq 0$ for all $j \leq j_0$. In various cases this problem has been answered in the affirmative, see [3], [4], [24], [18], [20] and [2] for a survey on this problem. Tameness problem holds for the top local cohomology $H_{R_+}^{\mathrm{cd}(R_+,M)}(M)$ where $\mathrm{cd}(R_+,M)$ denotes the cohomological dimension of M with respect to R_+ and also a few cases in general. For the other cases the base ring R_0 has to have dimension ≤ 2. The argument for case $\dim R_0 = 0$, is obvious. Indeed, in this case the R-modules $H^i_{R_+}(M)$ are Artinian [5, Theorem 17.1.9], and therefore by Example 1.35, $H^i_{R_+}(M)$ is tame. In Section 3 we reprove some known results on this problem when $\dim R_0 = 1$ and in Section 5 we are going to show that the local cohomology modules of a ring with monomial relations with respect to monomial prime ideals are always tame.

However it has been shown by Cutkosky and Herzog [10] that the tameness problem can fail if $\dim R_0 = 3$. Indeed, they proved: There exists a standard graded ring R where R_0 is a graded (or local) normal K-algebra of dimension 3 with isolated singularity and finitely generated R-module M such that for $j \gg 0$ we have

$$H^2_{R_+}(M)_j = \begin{cases} K^2 & j \text{ is even,} \\ 0 & j \text{ is odd.} \end{cases}$$

More recently, various examples of strange behavior of local cohomology have been constructed. See [8].

1.7 Graded and bigraded local cohomology

Let P_0 be a Noetherian ring, and let $P = P_0[y_1, \ldots, y_n]$ be the polynomial ring over P_0 in the variables $y_1, \ldots, y_n$. In this section we introduce the basic facts concerning graded and bigraded local cohomology and give a description of the local cohomology of a graded (bigraded) P-module from its graded (bigraded) P-resolution. We let $P_j = \bigoplus_{|b|=j} P_0 y^b$ where $y^b = y_1^{b_1} \ldots y_n^{b_n}$ for $b = (b_1, \ldots, b_n)$, and where $|b| = \sum_i b_i$. Then P is a standard graded P_0-algebra and P_j is a free P_0-module of rank $\binom{n+j-1}{n-1}$. In most cases we assume that P_0 is either a local ring with residue class field K, or $P_0 = K[x_1, \ldots, x_m]$ is the polynomial ring over the field K in the variables $x_1, \ldots, x_m$. We always assume that all P-modules considered here are finitely generated and graded. In case that P_0 is a polynomial ring, then P itself is bigraded, if we assign to each x_i the bidegree $(1, 0)$ and to each y_j the bidegree $(0, 1)$. In this case we assume that all P-modules are even bigraded. Observe that if M is bigraded, and if we set

$$M_j = \bigoplus_i M_{(i,j)}.$$

Then $M = \bigoplus_j M_j$ is a graded P-module and each graded component M_j is a finitely generated graded P_0-module, with grading $(M_j)_i = M_{(i,j)}$ for all i and j. Now let $S = K[y_1, \ldots, y_n]$. Then $P = P_0 \otimes_K K[y_1, \ldots, y_n] = P_0 \otimes_K S$. Let $P_+ = \bigoplus_{j>0} P_j$ be the irrelevant graded ideal of the P_0-algebra P. Next we want to compute the graded P-modules $H^i_{P_+}(P)$. Observe that there are isomorphisms of graded P-modules

$$\begin{aligned} H^i_{P_+}(P) &\cong \varinjlim_{k\geq 0} \operatorname{Ext}^i_P\left(P/(P_+)^k, P\right) \\ &\cong \varinjlim_{k\geq 0} \operatorname{Ext}^i_{P_0\otimes_K S}\left(P_0\otimes_K S/(y)^k, P_0\otimes_K S\right) \\ &\cong P_0\otimes_K \varinjlim_{k\geq 0} \operatorname{Ext}^i_P\left(S/(y)^k, S\right) \\ &\cong P_0\otimes_K H^i_{(y)}(S). \end{aligned}$$

Since $H^i_{S_+}(S)=0$ for $i\neq n$, we get

$$H^i_{P_+}(P)=\begin{cases} P_0\otimes_k H^n_{(y)}(S) & \text{for } i=n,\\ 0 & \text{for } i\neq n.\end{cases}$$

Let M be a graded S-module. We write $M^\vee=\operatorname{Hom}_K(M,K)$ and consider $M^\vee$ a graded S-module as follows: for $\varphi\in M^\vee$ and $f\in S$ we let $f\varphi$ be the element in $M^\vee$ with

$$f\varphi(m)=\varphi(fm)\quad\text{for all}\quad m\in M,$$

and define the grading by setting $(M^\vee)_j:=\operatorname{Hom}_K(M_{-j},K)$ for all $j\in\mathbb{Z}$. Let ω_S be the canonical module of S. Note that $\omega_S=S(-n)$, since S is a polynomial ring in n indeterminates. By the graded version of the local duality theorem, see [5, Example 13.4.6] we have $H^n_{S_+}(S)^\vee=S(-n)$ and $H^i_{S_+}(S)=0$ for $i\neq n$. Applying again the functor $(\_)^\vee$ we obtain

$$H^n_{S_+}(S)=\operatorname{Hom}_K(S(-n),K)=\operatorname{Hom}_K(S,K)(n).$$

We can thus conclude that

$$H^n_{S_+}(S)_j=\operatorname{Hom}_k(S,K)_{n+j}=\operatorname{Hom}_K(S_{-n-j},K)\quad\text{for all}\quad j\in\mathbb{Z}.$$

Let $S_l=\bigoplus_{|a|=l}Ky^a$. Then

$$\operatorname{Hom}_K(S_{-n-j},K)=\bigoplus_{|a|=-n-j}Kz^a,$$

where $z\in\operatorname{Hom}_K(S_{-n-j},K)$ is the K-linear map with

$$z^a(y^b)=\begin{cases} z^{a-b} & \text{if } b\leq a,\\ 0 & \text{if } b\nleq a.\end{cases}$$

Here we write $b\leq a$ if $b_i\leq a_i$ for $i=1,\dots,n$. Therefore $H^n_{S_+}(S)_j=\bigoplus_{|a|=-n-j}Kz^a$, and this implies that

$$H^n_{P_+}(P)_j=P_0\otimes_K H^n_{(y)}(S)_j=\bigoplus_{|a|=-n-j}P_0z^a. \tag{2}$$

Hence we see that $H^n_{P_+}(P)_j$ is free P_0-module of rank $\binom{-j-1}{n-1}$. Moreover, if P_0 is graded

$$H^n_{P_+}(P)_{(i,j)} = \bigoplus_{|b|=-n-j} (P_0)_i z^b = \bigoplus_{\substack{|a|=i \\ |b|=-n-j}} Kx^a z^b.$$

The next theorem describes how the local cohomology of a graded P-module can be computed from its graded free P-resolution

Theorem 1.36. *Let M be a finitely generated graded P-module. Let $\mathbb{F}$ be a graded free P-resolution of M. Then we have graded isomorphisms*

$$H^{n-i}_{P_+}(M) \cong H_i\big(H^n_{P_+}(\mathbb{F})\big).$$

Proof. Let

$$\mathbb{F} : \cdots \to F_2 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0.$$

Applying the functor $H^n_{P_+}$ to $\mathbb{F}$, we obtain the complex

$$H^n_{P_+}(\mathbb{F}) : \cdots \to H^n_{P_+}(F_2) \to H^n_{P_+}(F_1) \to H^n_{P_+}(F_0) \to 0.$$

We see that

$$H^n_{P_+}(M) = \operatorname{Coker}\big(H^n_{P_+}(F_1) \to H^n_{P_+}(F_0)\big) = H_0\big(H^n_{P_+}(\mathbb{F})\big),$$

since $H^i_{P_+}(N) = 0$ for each $i > n$ and all finitely generated P-modules N. We define the functors:

$$\mathcal{F}(M) := H^n_{P_+}(M) \quad \text{and} \quad \mathcal{F}_i(M) := H^{n-i}_{P_+}(M).$$

The functors $\mathcal{F}_i$ are additive, covariant and strongly connected, i.e., for each short exact sequence $0 \to U \to V \to W \to 0$ one has the long exact sequence

$$0 \to \cdots \to \mathcal{F}_i(U) \to \mathcal{F}_i(V) \to \mathcal{F}_i(W) \to \mathcal{F}_{i-1}(U) \to \cdots \to \mathcal{F}_0(V) \to \mathcal{F}_0(W) \to 0.$$

Moreover, $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_i(F) = H^{n-i}_{P_+}(F) = 0$ for all $i > 0$ and all free P-modules F. Therefore, the theorem follows from the dual version of [5, Theorem 1.3.5]. □

Note that if M is a finitely generated bigraded P-module. Then $H^n_{P_+}(M)$ with natural grading is also a finitely generated bigraded P-module, and hence in Theorem 1.36 we have bigraded isomorphisms

$$H^{n-i}_{P_+}(M) \cong H_i\big(H^n_{P_+}(\mathbb{F})\big).$$

2 Regularity of local cohomology of bigraded algebras

In this chapter we study algebraic properties of the graded components of local cohomology of a bigraded K-algebra. Let P_0 be a Noetherian ring, $P = P_0[y_1, \ldots, y_n]$ be the polynomial ring over P_0 with the standard grading and $P_+ = (y_1, \ldots, y_n)$ the irrelevant graded ideal of P. Then for any finitely generated graded P-module M, the local cohomology modules $H^i_{P_+}(M)$ are naturally graded P-modules and each graded component $H^i_{P_+}(M)_j$ is a finitely generated P_0-module. In case $P_0 = K[x_1, \ldots, x_m]$ is a polynomial ring, the K-algebra P is naturally bigraded with $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$. In this situation, if M is a finitely generated bigraded P-module, then each of the modules $H^i_{P_+}(M)_j$ is a finitely generated graded P_0-module. We are interested in the Hilbert functions and the Castelnuovo-Mumford regularity of these modules. In Section 2.1 we use a result of Gruson, Lazarsfeld and Peskine on the regularity of reduced curves, in order to show that the regularity of $H^i_{P_+}(M)_j$ as a function in j is bounded provided that $\dim_{P_0} M/P_+M \leq 1$. The rest of this chapter is devoted to study the local cohomology of a hypersurface ring $R = P/fP$ where $f \in P$ is a bihomogeneous polynomial. In Section 2.2 we prove that the Hilbert function of the top local cohomology $H^n_{P_+}(R)_j$ is a nonincreasing function in j. If moreover, the ideal $I(f)$ generated by all coefficients of f is $\mathfrak{m}$-primary where $\mathfrak{m}$ is the graded maximal ideal of P_0, then by a result of Katzman and Sharp the P_0- module $H^n_{P_+}(R)_j$ is of finite length. In particular, in this case the regularity of $H^n_{P_+}(R)_j$ is also a nonincreasing function in j. In the following section we compute the regularity of $H^i_{P_+}(R)_j$ for a special class of hypersurfaces. For the computation we use in an essential way a result of Stanley and J. Watanabe. They showed that a monomial complete intersection has the strong Lefschetz property. Stanley used the hard Lefschetz theorem, while Watanabe representation theory of Lie algebras to prove this result. Using these facts the regularity and the Hilbert function of $H^i_{P_+}(P/f^r_\lambda P)_j$ can be computed explicitly. Here $r \in \mathbb{N}$ and $f_\lambda = \sum_{i=1}^n \lambda_i x_i y_i$ with $\lambda_i \in K$. As a consequence we are able to show that $H^{n-1}_{P_+}(P/f^r P)_j$ has a linear resolution and its Betti numbers can be computed. We use these results in the last section to show that for any bigraded hypersurface ring $R = P/fP$ for which $I(f)$ is $\mathfrak{m}$-primary, the regularity of $H^i_{P_+}(R)_j$ is linearly bounded in j.

2.1 Regularity of the graded components of local cohomology for modules of small dimension

Let M be a finitely generated bigraded P-module, thus $H^i_{P_+}(M)_j$ is a finitely generated graded P_0-module. Let $f_{i,M}$ be the numerical function given by

$$f_{i,M}(j) = \operatorname{reg} H^i_{P_+}(M)_j$$

for all j. In this section we show that $f_{i,M}$ is bounded provided that M/P_+M has Krull dimension ≤ 1. There are some explicit examples which show that the condition $\dim_{P_0} M/P_+M \leq 1$ is indispensable. We postpone the example to Section 2.4. First one has the following

Lemma 2.1. *Let M be a finitely generated graded P-module. Then*

$$\dim_{P_0} M_i \leq \dim_{P_0} M/P_+M \quad \textit{for all} \quad i.$$

Proof. Let $r = \min\{j : M_j \neq 0\}$. We prove the lemma by induction on $i \geq r$. Let $i = r$. Note that

$$M/P_+M = M_r \oplus M_{r+1}/P_1M_r \oplus \cdots .$$

It follows that M_r is a direct summand of the P_0-module M/P_+M, so that $\dim_{P_0} M_r \leq \dim_{P_0} M/P_+M$. We now assume that $i > r$ and $\dim_{P_0} M_j \leq \dim_{P_0} M/P_+M$, for $j = r, \ldots, i-1$. We will show that $\dim_{P_0} M_i \leq \dim_{P_0} M/P_+M$. We consider the exact sequence of P_0-modules

$$0 \to P_1M_{i-1} + \cdots + P_{i-r}M_r \to M_i \xrightarrow{\varphi} (M/P_+M)_i \to 0.$$

By the induction hypothesis, one easily deduces that

$$\dim_{P_0} \sum_{j=1}^{i-r} P_jM_{i-j} \leq \dim_{P_0} M/P_+M,$$

and since $(M/P_+M)_i$ is a direct summand of M/P_+M, it also has dimension $\leq \dim_{P_0} M/P_+M$. Therefore, by the above exact sequence, $\dim M_i \leq \dim_{P_0} M/P_+M$, too. □

The following lemma is needed for the proof of the next proposition

Lemma 2.2. *Let M be a finitely generated graded P-module. Then there exists an integer i_0 such that*

$$\operatorname{Ann}_{P_0} M_i = \operatorname{Ann}_{P_0} M_{i+1} \quad \textit{for all} \quad i \geq i_0.$$

Proof. Since $P_1M_i \subseteq M_{i+1}$ for all i and M is a finitely generated P-module, there exists an integer t such that $P_1M_i = M_{i+1}$ for all $i \geq t$. This implies that $\operatorname{Ann}_{P_0} M_t \subseteq \operatorname{Ann}_{P_0} M_{t+1} \subseteq \cdots$. Since P_0 is Noetherian, there exists an integer k such that $\operatorname{Ann}_{P_0} M_{t+k} = \operatorname{Ann}_{P_0} M_i$ for all $i \geq t+k = i_0$. □

Proposition 2.3. *Let M be a finitely generated graded P-module. Then*

$$\dim_{P_0} H^i_{P_+}(M)_j \leq \dim_{P_0} M_j \quad \textit{for all} \quad i \quad \textit{and} \quad j \gg 0.$$

Proof. Let $P_+ = (y_1, \dots, y_n)$. Then by [5, Theorem 5.1.19] we have

$$H^i_{P_+}(M) \cong H^i\big(C(M)^\bullet\big) \quad \text{for all} \quad i \geq 0$$

where $C(M)^\bullet$ denote the (extended) Čech complex of M with respect to $y_1, \dots, y_n$ defined as follows:

$$C(M)^\bullet : 0 \to C(M)^0 \to C(M)^1 \to \cdots \to C(M)^n \to 0$$

with

$$C(M)^t = \bigoplus_{1 \leq i_1 < \cdots < i_t \leq n} M_{y_{i_1} \cdots y_{i_t}},$$

and where the differentiation $C(M)^t \to C(M)^{t+1}$ is given on the component

$$M_{y_{i_1} \cdots y_{i_t}} \to M_{y_{j_1} \cdots y_{j_{t+1}}}$$

to be the homomorphism

$$(-1)^{s-1} nat : M_{y_{i_1} \cdots y_{i_t}} \to (M_{y_{i_1} \cdots y_{i_t}})_{y_{j_s}},$$

if $\{i_1, \dots, i_t\} = \{j_1, \dots, \hat{j}_s, \dots, j_{t+1}\}$ and 0 otherwise. We set $\mathcal{I} = \{i_1, \dots, i_t\}$ and $y_{\mathcal{I}} = y_{i_1} \dots y_{i_t}$. For $m/y^k_{\mathcal{I}} \in M_{y_{\mathcal{I}}}$, m homogeneous, we set $\deg m/y^k_{\mathcal{I}} = \deg m - \deg y^k_{\mathcal{I}}$. Then we can define a grading on $M_{y_{\mathcal{I}}}$ by setting

$$(M_{y_{\mathcal{I}}})_j = \{m/y^k_{\mathcal{I}} \in M_{y_{\mathcal{I}}} : \deg m/y^k_{\mathcal{I}} = j\} \quad \text{for all} \quad j.$$

In view of Lemma 2.2 there exists an ideal $I \subseteq P_0$ and an integer j_0 such that $\operatorname{Ann}_{P_0} M_j = I$ for all $j \geq j_0$. We now claim that $I \subseteq \operatorname{Ann}_{P_0}(M_{y_{\mathcal{I}}})_j$ for all $j \geq j_0$. Let $a \in I$ and $m/y^k_{\mathcal{I}} \in (M_{y_{\mathcal{I}}})_j$ for some integer k. We may choose an integer l such that

$$\deg m + \deg y^l_{\mathcal{I}} = \deg m y^l_{\mathcal{I}} = t \geq j_0.$$

Thus $am/y^k_{\mathcal{I}} = amy^l_{\mathcal{I}}/y^{k+l}_{\mathcal{I}} = 0$, because $my^l_{\mathcal{I}} \in M_t$. Thus we have

$$\dim_{P_0}(M_{y_{\mathcal{I}}})_j = \dim_{P_0} P_0/\operatorname{Ann}(M_{y_{\mathcal{I}}})_j \leq \dim_{P_0} P_0/I = \dim_{P_0} M_j.$$

Since $H^i_{P_+}(M)_j$ is a subquotient of the jth graded component of $C(M)^i$, the desired result follows. □

Now we can state the main result of this section as follows

Theorem 2.4. *Let M be a finitely generated bigraded P-module such that*

$$\dim_{P_0} M/P_+M \leq 1.$$

Then for all i the functions $f_{i,M}(j) = \operatorname{reg} H^i_{P_+}(M)_j$ are bounded.

In a first step we prove the following

Proposition 2.5. *Let M be a finitely generated bigraded P-module with*

$$\dim_{P_0} M/P_+M \leq 1.$$

Then the function $f_{n,M}(j) = \operatorname{reg} H^n_{P_+}(M)_j$ is bounded above.

Proof. By the bigraded version of Hilbert's syzygy theorem, M has a bigraded free resolution of the form

$$\mathbb{F} : 0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0$$

where $F_i = \bigoplus_{k=1}^{t_i} P(-a_{ik}, -b_{ik})$. Applying the functor $H^n_{P_+}(-)_j$ to this resolution yields a graded complex of free P_0- modules

$$H^n_{P_+}(\mathbb{F})_j : 0 \to H^n_{P_+}(F_k)_j \to \cdots \to H^n_{P_+}(F_1)_j \to H^n_{P_+}(F_0)_j \to H^n_{P_+}(M)_j \to 0.$$

Theorem 1.36, together with Proposition 2.3, Lemma 2.1 and our assumption imply that for $j \gg 0$ we have

$$\dim_{P_0} H_i\big(H^n_{P_+}(\mathbb{F})_j\big) = \dim_{P_0} H^{n-i}_{P_+}(M)_j \leq \dim_{P_0} M/P_+M \leq 1 \leq i \quad \text{for all} \quad i \geq 1.$$

Moreover we know that

$$H^n_{P_+}(M) = H_0\big(H^n_{P_+}(\mathbb{F})\big).$$

Then by a theorem of Lazardsfeld [13, Lemma 1.6], see also [7, Theorem 12.1], one has

$$\operatorname{reg} H^n_{P_+}(M)_j = \operatorname{reg} H_0\big(H^n_{P_+}(\mathbb{F})\big)_j \leq \max\{b_i\big(H^n_{P_+}(\mathbb{F})_j\big) - i \quad \text{for all} \quad i \geq 0\}$$

where $b_i\big(H^n_{P_+}(\mathbb{F})_j\big)$ is the maximal degree of the generators of $H^n_{P_+}(F_i)_j$. Note that

$$H^n_{P_+}(F_i)_j = \bigoplus_{k=1}^{t_i} \bigoplus_{|a|=-n-j+b_{ik}} P_0(-a_{ik})z^a.$$

Thus we conclude that

$$\operatorname{reg} H^n_{P_+}(M)_j \leq \max_{i,k}\{a_{ik} - i\} = c \text{ for } j \gg 0,$$

as desired. □

Next we want to give a lower bound for the functions $f_{i,M}$. We first prove

Proposition 2.6. *Let*

$$\mathbb{G} : 0 \to G_p \xrightarrow{d_p} G_{p-1} \to \cdots \to G_1 \xrightarrow{d_1} G_0 \to 0,$$

be a complex of free P_0-modules, where $G_i = \bigoplus_j P_0(-a_{ij})$ for all $i \geq 0$. Let $m_i = \min_j\{a_{ij}\}$. Then

$$\operatorname{reg} H_i(\mathbb{G}) \geq m_i.$$

Proof. Since $H_i(\mathbb{G}) = \operatorname{Ker} d_i / \operatorname{Im} d_{i+1}$ and $\operatorname{Ker} d_i \subseteq G_i$ for all $i \geq 0$, it follows that

$$\begin{aligned} \operatorname{reg} H_i(\mathbb{G}) &\geq \text{ largest degree of generators of } \mathrm{H}_i(\mathbb{G}) \\ &\geq \text{ lowest degree of generators of } \mathrm{H}_i(\mathbb{G}) \\ &\geq \text{ lowest degree of generators of } \operatorname{Ker} d_i \\ &\geq \text{ lowest degree of generators of } \mathrm{G}_i \\ &= m_i, \end{aligned}$$

as desired. □

Corollary 2.7. *Let M be a finitely generated bigraded P-module. Then for each i, the function $f_{i,M}$ is bounded below.*

Proof. Let $\mathbb{G}$ be the complex $H^n_{P_+}(\mathbb{F})_j$ in the proof of Proposition 2.5, then the assertion follows from Proposition 2.6. □

Proof of Theorem 2.4. Because of Corollary 2.7 it suffices to show that for each i, $f_{i,M}$ is bounded above.

There exists an exact sequence $0 \to U \to F \xrightarrow{\varphi} M \to 0$ of finitely generated bigraded P-modules where F is free. This exact sequence yields the exact sequence of P_0-modules

$$0 \to H^{n-1}_{P_+}(M)_j \to H^n_{P_+}(U)_j \to H^n_{P_+}(F)_j \xrightarrow{\varphi} H^n_{P_+}(M)_j \to 0.$$

Let $K_j := \operatorname{Ker} \varphi$. We consider the exact sequences

$$0 \to K_j \to H^n_{P_+}(F)_j \to H^n_{P_+}(M)_j \to 0$$

$$0 \to H^{n-1}_{P_+}(M)_j \to H^n_{P_+}(U)_j \to K_j \to 0.$$

Thus we have

$$\operatorname{reg} K_j \leq \max\{\operatorname{reg} H^n_{P_+}(F)_j, \operatorname{reg} H^n_{P_+}(M)_j + 1\} \tag{3}$$

$$\operatorname{reg} H^{n-1}_{P_+}(M)_j \leq \max\{\operatorname{reg} H^n_{P_+}(U)_j, \operatorname{reg} K_j + 1\}. \tag{4}$$

Let $F = \bigoplus_{i=1}^k P(-a_i, -b_i)$, then

$$H^n_{P_+}(F)_j = \bigoplus_{i=1}^{k} \bigoplus_{|a|=-n-j+b_i} P_0(-a_i) z^a.$$

Therefore, $\operatorname{reg} H^n_{P_+}(F)_j = \max_i\{a_i\}$. By Proposition 2.5, the functions $f_{n,M}$ and $f_{n,U}$ are bounded above, so that, by the inequalities (3) and (4), $f_{n-1,M}$ is bounded above. To complete our proof, for $i > 1$ we see that

$$H^{n-i}_{P_+}(M)_j \cong H^{n-i+1}_{P_+}(U)_j.$$

Thus $f_{n-i,M} = f_{n-i+1,U}$ for $i > 1$. By induction on $i > 1$ all $f_{i,M}$ are bounded above, as required. □

2.2 Hilbert function of the components of the top local cohomology of a hypersurface ring

Let R be a hypersurface ring. In this section we want to show that the Hilbert function of the P_0-module $H^n_{P_+}(R)_j$ is a nonincreasing function in j. Let $f \in P$ be a bihomogeneous form of degree (a, b). Write

$$f = \sum_{\substack{|\alpha|=a \\ |\beta|=b}} c_{\alpha\beta} x^\alpha y^\beta \quad \text{where} \quad c_{\alpha\beta} \in K.$$

We may also write $f = \sum_{|\beta|=b} f_\beta y^\beta$ where $f_\beta \in P_0$ with $\deg f_\beta = a$. The monomials y^β for which $|\beta| = b$ are ordered lexicographically induced by $y_1 > y_2 > \cdots > y_n$. We consider the hypersurface ring $R = P/fP$. From the exact sequence

$$0 \to P(-a, -b) \xrightarrow{f} P \to P/fP \to 0,$$

we get an exact sequence of P_0-modules

$$\bigoplus_{|c|=-n-j+b} P_0(-a)z^c \xrightarrow{f} \bigoplus_{|c|=-n-j} P_0 z^c \to H^n_{P_+}(R)_j \to 0.$$

We also order the bases elements z^c lexicographically induced by $z_1 > z_2 > \cdots > z_n$. Applying f to the bases elements we obtain $fz^c = \sum_{|\beta|=b} f_\beta z^{\beta - c}$, where $z^{\beta-c} = 0$ if $c \not\leq \beta$ componentwise. With respect to these bases the map of free P_0-modules is given by a $\binom{-j-1}{n-1} \times \binom{-j+b-1}{n-1}$ matrix which we denote by U_j. This matrix also describes the image of this map as submodule of the free module F_j where $F_j = \bigoplus_{|c|=-n-j} P_0 z^c$, so that $H^n_{P_+}(R)_j$ is just $\operatorname{Coker} f = F_j/U_j$. Note that $H^n_{P_+}(R)_j = 0$ for all $j > -n$.

Let B_d denote the set of all monomials of degree d in the indeterminates $z_1, \ldots, z_n$. Let $h = \sum_{v \in B_{-n-j}} h_v v \in U_j$ where $h_v \in P_0$ for all v. Then $h_u u$ is called the *initial term* of h if $h_u \neq 0$ and $h_v = 0$ for all $v > u$, and we set $\operatorname{in}(h) = h_u u$. The polynomial $h_u \in P_0$ is called the *initial coefficient* and the monomial u is called the *initial monomial* of h.

Now for a monomial $u \in B_{-n-j}$ we denote $U_{j,u}$ the set of elements in U_j whose initial monomial is u, and we denote by $I_{j,u}$ the ideal generated by the initial coefficients of the elements in $U_{j,u}$.

Note that

$$U_j \setminus \{0\} = \bigcup_{u \in B_{-n-j}} U_{j,u}.$$

We fix the lexicographical order introduced above, and let $\operatorname{in}(U_j)$ be the submodule generated by $\{\operatorname{in}(h) : h \in U_j\}$. Then

$$\operatorname{in}(U_j) = \bigoplus_{u \in B_{-n-j}} I_{j,u} u. \tag{5}$$

Proposition 2.8. *With the above notation we have*

$$I_{j,u} = I_{j-1,z_1u} \quad \text{for all} \quad j \leq -n \quad \text{and} \quad u \in B_{-n-j}.$$

Proof. Let $h_0 \in I_{j,u}$. Then there exists $h \in U_j$ such that $h = h_0u+$lower terms. We set $k = -n-j+b$, for short. Since h is in the image of f, we may also write $h = \sum_{|c|=k} f_c f z^c$ where $f_c \in P_0$ and $fz^c = \sum_{\beta\leq c} f_\beta z^{c-\beta}$. We define $g = \sum_{|c|=k} f_c f z^{c+e_1}$ where $fz^{c+e_1} = \sum_{\beta \leq c+e_1} f_\beta z^{c+e_1-\beta}$ and $e_1 = (1, 0, \dots, 0)$. We see that $g \in U_{j-1}$. We may write

$$g = \sum_{|c|=k} f_c \sum_{\beta\leq c} f_\beta z^{c+e_1-\beta} + \sum_{|c|=k} f_c \sum_{\substack{\beta \nleq c \\ \beta \leq c+e_1}} f_\beta z^{c+e_1-\beta}.$$

Thus we conclude that $g = z_1h + h_1$ where

$$h_1 = \sum_{|c|=k} f_c \sum_{\substack{\beta \nleq c \\ \beta \leq c+e_1}} f_\beta z^{c+e_1-\beta}.$$

We now claim that h_1 does not contain z_1 as a factor. For each $\alpha \in \mathbb{N}^n$ we denote by $\alpha(i)$ the ith component of α. Assume that $(c+e_1-\beta)(1) > 0$ for some β appearing in the sum of h_1. Then $c(1) \geq \beta(1)$. Moreover, if $i > 1$, then $(c+e_1-\beta)(i) \geq 0$ implies that $c(i) \geq \beta(i)$. Hence $c(i) \geq \beta(i)$ for all i, a contradiction. It follows that $\text{in}(g) = \text{in}(h)z_1$. Therefore $h_u \in I_{j-1,z_1u}$.

Conversely, suppose $h_0 \in I_{j-1,z_1u}$. Then there exists $g \in U_{j-1}$ such that $g = h_0z_1u+$ lower terms. We may write $g = \sum_{|c|=k} f'_c f z^{c+e_1}$ where $f'_c \in P_0$ and $fz^{c+e_1} = \sum_{\beta\leq c+e_1} f_\beta z^{c+e_1-\beta}$. Thus

$$g = \sum_{|c|=k} f'_c \sum_{\beta\leq c} f_\beta z^{c+e_1-\beta} + \sum_{|c|=k} f'_c \sum_{\substack{\beta \nleq c \\ \beta \leq c+e_1}} f_\beta z^{c+e_1-\beta}.$$

As above we see that $g = z_1f'+$ lower terms, where $f' = \sum_{|c|=k} f'_c f z^c$. We see that $f' \in U_j$, and that $\text{in}(f')z_1 = \text{in}(g) = h_0z_1u$. Therefore, $\text{in}(f') = h_0u$, and hence $h_0 \in I_{j,u}$. □

Let M and N be graded P_0-modules. We denote by $\text{Hilb}(M) = \sum_{i\in\mathbb{Z}} \dim_K M_i t^i$ the Hilbert series of M, as defined before. We write $\text{Hilb}(M) \leq \text{Hilb}(N)$ when $\dim_K M_i \leq \dim_K N_i$ for all i.

Let F be a free P_0-module with basis $\beta = \{u_1, \dots, u_r\}$. Let U be a graded submodule of F. For $f \in U$, we write $f = \sum_{i=1}^r f_iu_i$ where $f_i \in P_0$. We set $\text{in}(f) = f_ju_j$ where $f_j \neq 0$ and $f_i = 0$ for all $i < j$. We also set $\text{in}(U)$ be the submodule of F generated by all $\text{in}(f)$ such that $f \in U$. Let I be a homogeneous ideal of P_0. We say that set of homogeneous elements of P_0 forms a K-basis for P_0/I if its image forms a K-basis for P_0/I. Now we can state the following result which is related to a theorem of Macaulay [6, Theorem 4.2.3], see also [6, Corollary 4.2.4]. For the convenience of the reader we include its proof.

Lemma 2.9. *With notation as above we have*

$$\operatorname{Hilb}(F/U) = \operatorname{Hilb}\big(F/\operatorname{in}(U)\big).$$

Proof. As in (5) we have $\operatorname{in}(U) = \bigoplus_{i=1}^{r} I_{u_i}u_i$ where I_{u_i} is the ideal generated by all $f_i \in P_0$ such that there exists $f \in F$ with $\operatorname{in}(f) = f_iu_i$. Thus we have $F/\operatorname{in}(U) = \bigoplus_{i=1}^{r} P_0/I_{u_i}$. For each j let β_j be a set of homogeneous elements $h_{ij} \in P_0$ which forms a K-basis of P_0/I_{u_j}. Then $\beta = \{\beta_1u_1, \ldots, \beta_ru_r\}$ is a homogeneous K-basis of $F/\operatorname{in}(U)$. To complete our proof we will show that β is also a K-basis of F/U. We first show that the elements of β in F/U are linearly independent. Suppose that in F/U, we have $\sum_{i,j} a_{ij}h_{ij}u_j = 0$ with $a_{ij} \in K$. Thus $\sum_{j=1}^{r}(\sum_i a_{ij}h_{ij})u_j \in U$. We set $h_j = \sum_i a_{ij}h_{ij}$, so that $h_1u_1 + \cdots + h_ru_r \in U$. If all $h_j = 0$, then $a_{ij} = 0$ for all i and j, as required. Assume that $h_j \neq 0$ for some j, and let k be the smallest integer such that $h_k \neq 0$. It follows that $h_ku_k + h_{k+1}u_{k+1} + \cdots \in U$, so that $h_k \in I_k$, and hence $\sum_i a_{ik}h_{ik} = 0$ modulo I_k. Since h_{ik} are part of a K-basis of P_0/I_k, it follows that $a_{ik} = 0$ for all i, and hence $h_k = 0$, a contradiction.

Now we want to show that each element in F/U can be written as a K-linear combination of elements of β. Let $f + U \in F/U$ where $f \in F$. Thus there exists $f_i \in P_0$ such that $f = \sum_{i=1}^{r} f_iu_i$. Since $f_1 + I_{u_1} \in P_0/I_{u_1}$, there exists $\lambda_{i1} \in K$ such that $f_1 + I_{u_1} = \sum_i \lambda_{i1}(h_{i1} + I_{u_1})$, so that $f_1 = \sum_i \lambda_{i1}h_{i1} + h_{u_1}$ for some $h_{u_1} \in I_{u_1}$. Hence

$$f = \sum_i \lambda_{i1}h_{i1}u_1 + h_{u_1}u_1 + \sum_{i=2}^{r} f_iu_i.$$

We set

$$f' = f - \sum_i \lambda_{i1}h_{i1}u_1 = h_{u_1}u_1 + \sum_{i=2}^{r} f_iu_i.$$

Since $h_{u_1} \in I_{u_1}$, there exist $g_2, \ldots, g_r \in P_0$ such that $h_{u_1}u_1 + \sum_{i=2}^{r} g_iu_i \in U$. Therefore, $h_{u_1}u_1 = -\sum_{i=2}^{r} g_iu_i$ modulo U. Hence it follow that

$$f' = -\sum_{i=2}^{r} g_iu_i + \sum_{i=2}^{r} f_iu_i = \sum_{i=2}^{r} f_i'u_i \quad \text{modulo} \quad U.$$

Here $f_i' = -g_i + f_i$ for $i = 2, \ldots, r$. By induction on the number of summands, we may assume that $\sum_{i=2}^{r} f_i'u_i$ is a linear combination of elements of β modulo U. Since f differs from f' only by a linear combination of elements of β, the assertion follows. $\square$

Now we are able to prove that the Hilbert series of the P_0-module $H^n_{P_+}(R)_j$ is a nonincreasing function in j.

Theorem 2.10. *Let $R = P/fP$ be a hypersurface ring. Then*

$$\operatorname{Hilb}\big(H^n_{P_+}(R)_{j-1}\big) \geq \operatorname{Hilb}\big(H^n_{P_+}(R)_j\big) \quad \textit{for all} \quad j \leq -n.$$

Proof. Let $F_j = \bigoplus_{u \in B_{-n-j}} P_0 u$ where $u = z_1^{a_1} \dots z_n^{a_n}$ with $\sum_{i=1}^n a_i = -n-j$. In view of (5) we have $F_j / \operatorname{in}(U_j) = \bigoplus_{u \in B_{-n-j}} P_0 / I_{j,u}$. By Lemma 2.9 we know that F_j/U_j and $F_j/\operatorname{in}(U_j)$ have the same Hilbert function. Thus Proposition 2.8 implies that for all $j \leq -n$ we have

$$\begin{aligned}
\operatorname{Hilb}(H^n_{P_+}(R)_j) &= \operatorname{Hilb}(F_j/U_j) = \sum_i \dim_K \Big(\bigoplus_{u \in B_{-n-j}} P_0/I_{j,u} \Big)_i t^i \\
&= \sum_i \sum_{u \in B_{-n-j}} \dim_K \big(P_0/I_{j,u}\big)_i t^i \\
&= \sum_i \sum_{u \in B_{-n-j}} \dim_K \big(P_0/I_{j-1,z_1 u}\big)_i t^i \\
&= \sum_i \sum_{\substack{v \in B_{-n-j+1} \\ a_1 > 0}} \dim_K \big(P_0/I_{j-1,v}\big)_i t^i \\
&\leq \sum_i \sum_{v \in B_{-n-j+1}} \dim_K \big(P_0/I_{j-1,v}\big)_i t^i \\
&= \sum_i \dim_K \Big(\bigoplus_{v \in B_{-n-j+1}} P_0/I_{j-1,v} \Big)_i t^i = \operatorname{Hilb}\big(H^n_{P_+}(R)_{j-1}\big),
\end{aligned}$$

as desired. □

Corollary 2.11. *Let R be the hypersurface ring P/fP such that the P_0-module $H^n_{P_+}(R)_j$ has finite length for all j. Then*

$$\operatorname{reg} H^n_{P_+}(R)_{j-1} \geq \operatorname{reg} H^n_{P_+}(R)_j \quad \textit{for all} \quad j \leq -n.$$

Proof. The assertion follows from the fact that

$$\operatorname{reg} H^n_{P_+}(R)_j = \deg \operatorname{Hilb}(H^n_{P_+}(R)_j).$$

□

Now one could ask when P_0-module $H^n_{P_+}(R)_j$ is of finite length. To answer this question we need some preparation. Let A be a Noetherian ring and M be a finitely generated A-module with presentation

$$A^m \xrightarrow{\varphi} A^n \to M \to 0.$$

Let U be the corresponding matrix of the map φ and $I_{n-i}(U)$ for $i = 0, \dots, n-1$ be the ideal generated by the $(n-i)$-minors of matrix U. Then $\operatorname{Fitt}_i(M) = I_{n-i}(U)$ is called the ith Fitting ideal of M. We use the convention that $\operatorname{Fitt}_i(M) = 0$ if $n-i > \min\{n,m\}$, and $\operatorname{Fitt}_i(M) = A$ if $i \geq n$. In particular, we obtain $\operatorname{Fitt}_r(M) = 0$ if $r < 0$, $\operatorname{Fitt}_0(M)$ is generated by the n-minors of U, and $\operatorname{Fitt}_{n-1}(M)$ is generated by all entries of U. Note that $\operatorname{Fitt}_i(M)$ is an invariant on M, i.e., independent of the presentation. By [12, Proposition 20.7] we have $\mathrm{Fitt}_0(M) \subseteq \operatorname{Ann} M$ and if M can be generated by r elements, then $(\operatorname{Ann} M)^r \subseteq \mathrm{Fitt}_0(M)$. Thus we can conclude that $\sqrt{\mathrm{Fitt}_0(M)} = \sqrt{\operatorname{Ann} M}$. Therefore

$$\dim M = \dim A/\operatorname{Ann} M = \dim A/I_n(U). \tag{6}$$

Now we can state the following

Proposition 2.12. *Let R be the hypersurface ring P/fP, and $I(f)$ the ideal generated by all the coefficients of f. Then $\dim_{P_0} H^n_{P_+}(R)_j \leq \dim P_0/I(f)$. In particular, if $I(f)$ is $\mathfrak{m}$-primary where $\mathfrak{m} = (x_1, \ldots, x_m)$, then P_0-module $H^n_{P_+}(R)_j$ is of finite length for all j.*

Proof. Note that $H^n_{P_+}(R)_j = 0$ for $j > -n$. Therefore we may suppose that $j \leq -n$. As we have already seen, $H^n_{P_+}(R)_j$ has P_0-presentation

$$P_0^{n_1}(-a) \xrightarrow{\varphi} P_0^{n_0} \to H^n_{P_+}(R)_j \to 0,$$

where $n_0 = \binom{-j-1}{n-1}$ and $n_1 = \binom{-j+b-1}{n-1}$. In view of (6) we have $\dim_{P_0} H^n_{P_+}(R)_j = \dim P_0/I_{n_0}(U_j)$ where U_j is the corresponding matrix of the map φ. By [18, Lemma 1.4] we have $\sqrt{I(f)} \subseteq \sqrt{I_{n_0}(U_j)}$. It follows that $\dim_{P_0} H^n_{P_+}(R)_j \leq \dim P_0/I(f)$. Since $I(f)$ is $\mathfrak{m}$-primary, it follows that $\dim P_0/I(f) = 0$. Therefore $\dim_{P_0} H^n_{P_+}(R)_j = 0$, and hence $H^n_{P_+}(R)_j$ has finite length, as required. ☐

2.3 Regularity of the graded components of local cohomology for a special class of hypersurfaces

Let $A = \bigoplus_{i=0}^n A_i$ be a standard graded Artinian K-algebra where K is a field of characteristic 0. We say that A has the weak Lefschetz property (WLP) if there is a linear form l of degree 1 such that the multiplication map $A_i \xrightarrow{l} A_{i+1}$ has maximal rank for all i. This means the corresponding matrix has maximal rank, i.e., l is either injective *or* surjective. Such an element l is called a weak Lefschetz element on A. We also say that A has the strong Lefschetz property if there is a linear form l of degree 1 such that the multiplication map $A_i \xrightarrow{l^k} A_{i+k}$ has maximal rank for all i and k. Such an element l is called a strong Lefschetz element on A. For an algebra A as above, we say that A has the strong Stanley property (SSP) if there exists $l \in A_1$ such that $l^{n-2i} : A_i \to A_{n-i}$ is bijective for $i = 0, 1, \ldots, [n/2]$.

Remark 2.13. (a) It is easy to see that if A has the WLP and if A has the standard grading, then the Hilbert function of A is unimodal, i.e., there exists an integer m $(0 \leq m \leq n)$ such that

$$\dim A_0 \leq \dim A_1 \leq \cdots \leq \dim A_m \geq \dim A_{m+1} \geq \cdots \geq \dim A_n.$$

(b) The set of all weak Lefschetz elements on A is a Zariski-open subset of the affine space A_1, and the same holds for the set of all strong Lefschetz elements on A.

(c) Let $a_1, \dots, a_n$ be the integers such that $a_i \geq 1$ and assume that $\operatorname{char} K = 0$. Then $A = K[x_1, \dots, x_n]/(x_1^{a_1}, \dots, x_n^{a_n})$ has the strong Lefschetz property. See [25] and [27].

(d) If A has the (SSP), then the Hilbert function of A is unimodal and symmetric, i.e., $\dim A_i = \dim A_{n-i}$ for all $i = 0, 1, \dots, [n/2]$.

Theorem 2.14. *Let $r \in \mathbb{N}$, $f_\lambda = \sum_{i=1}^{n} \lambda_i x_i y_i$ with $\lambda_i \in K$ and $n \geq 2$, and assume that* $\operatorname{char} K = 0$. *Then there exists a Zariski open subset $V \subset K^n$ such that for all $\lambda = (\lambda_1, \cdots, \lambda_n) \in V$ one has*

$$\operatorname{reg} H^n_{P_+}\left(P/f^r_\lambda P\right)_j = -n - j + r - 1.$$

Proof. We first prove the theorem in the case that $f = f_{(1,\dots,1)} = \sum_{i=1}^{n} x_i y_i$, and set $R = P/f^r P$. From the exact sequence

$$0 \to P(-r,-r) \xrightarrow{f^r} P \to R \to 0,$$

we get an exact sequence of P_0-modules,

$$\bigoplus_{|b|=-n-j+r} P_0(-r)z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P_+}(R)_j \to 0. \tag{7}$$

Note that $H^n_{P_+}(R)_j$ is generated by elements of degree 0 and the ideal generated by the coefficients of f is $\mathfrak{m}$-primary. By Proposition 2.12, we need only to show that

(a) $\left[H^n_{P_+}(R)_j\right]_{-n-j+r-1} \neq 0$, and

(b) $\left[H^n_{P_+}(R)_j\right]_{-n-j+r} = 0.$

Let $k = -n - j$ for short. For the proof of (a), we take the $(k+r-1)$th component of the exact sequence (7), and obtain the exact sequence of K-vector spaces

$$\bigoplus_{\substack{|a|=k-1\\|b|=k+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k+r-1\\|b|=k}} Kx^a z^b \to \left[H^n_{P_+}(R)_j\right]_{k+r-1} \to 0.$$

We set

$$V_{\alpha,\beta} := \bigoplus_{\substack{|a|=\alpha\\|b|=\beta}} Kx^a z^b.$$

Hence one has $\dim_K V_{k-1,k+r} = \binom{n+k-2}{k-1}\binom{n+k+r-1}{k+r}$ which is less than $\dim_K V_{k+r-1,k} = \binom{n+k+r-2}{k+r-1}\binom{n+k-1}{k}$ for $n \geq 2$. Thus f^r is not surjective, so (a) follows. For the proof of (b), we take the $(k+r)$th component of the exact sequence (7), and obtain the exact sequence of K-vector spaces

$$\bigoplus_{\substack{|a|=k\\|b|=k+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k+r\\|b|=k}} Kx^a z^b \to \left[H^n_{P_+}(R)_j\right]_{k+r} \to 0.$$

Note that $\dim_K V_{k,k+r} = \dim_K V_{k+r,k}$. We are going to show that f^r is an isomorphism, then we are done. We fix $c \in \mathbb{N}_0^n$ such that $c = (c_1, \ldots, c_n)$ where $c_i \geq 0$. We set

$$V^c_{\alpha,\beta} := \bigoplus_{\substack{|a|=\alpha\\|b|=\beta\\a+b=c}} Kx^a z^b \quad \text{and} \quad A^c_i := \bigoplus_{\substack{|a|=i\\a\leq c}} Kx^a.$$

We define $\varphi : V^c_{k,k+r} \longrightarrow A^c_k$ by setting $\varphi(x^a z^b) = x^a$. Note that φ is an isomorphism of K-vector spaces. Let $A^c = \bigoplus_{i=0}^{|c|} A^c_i$. We can define an algebra structure on A^c. For $x^s, x^t \in A^c$ we define

$$x^s x^t = \begin{cases} x^{s+t} & \text{if } s+t \leq c, \\ 0 & \text{if } s+t \not\leq c. \end{cases}$$

A K-basis of A^c is given by all monomials x^a with $a \leq c$. It follows that

$$A^c = K[x_1, \ldots, x_n]/(x_1^{c_1+1}, \ldots, x_n^{c_n+1}).$$

Now we see that the map

$$V_{k,k+r} = \bigoplus_{|c|=2k+r} V^c_{k,k+r} \xrightarrow{f^r} \bigoplus_{|c|=2k+r} V^c_{k+r,k} = V_{k+r,k}$$

is an isomorphism if and only if restriction map $f' := f^r|_{V^c_{k,k+r}} : V^c_{k,k+r} \longrightarrow V^c_{k+r,k}$ is an isomorphism for all c with $|c| = 2k+r$.

For each such c we have a commutative diagram

$$\begin{tikzcd} V^c_{k,k+r} \arrow[r, "f'"] \arrow[d] & V^c_{k+r,k} \arrow[d] \\ A^c_k \arrow[r, "l^r"] & A^c_{k+r}, \end{tikzcd}$$

with $l = x_1 + x_2 + \ldots + x_n \in A^c_1$ and where $A^c_k \xrightarrow{l^r} A^c_{k+r}$ is multiplication by l^r in the K-algebra A^c. Since the socle degree of A^c equals $s = 2k+r$, we have $k+r = s-k$. Therefore the multiplication map $l^r : A_k \to A_{s-k}$ with $r = s-2k$ is an isomorphism by the strong Stanley property of the algebra A^c, see [27, Corollary 3.5]

Now if we replace f by f_λ, then the corresponding linear form in the above commutative diagram is the form $l_\lambda = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$. It is known that the property of l_λ to be a weak Lefschetz element is an open condition, that is, there exists a Zariski open set $V \subset K^n$ such that l_λ is a weak Lefschetz element. This open set is not empty since $\lambda = (1, \ldots, 1) \in V$. Since any weak Lefschetz element satisfies (SSP), we can replace in the above proof f by f_λ for each $\lambda \in V$, and obtain the same conclusion. □

Remark 2.15. It is now the time that to show Theorem 2.4 may fail without the assumption that $\dim_{P_0} M/P_+M \leq 1$. In case of Theorem 2.14 we have $M = R = P/f_\lambda^r P$, and so $M/P_+M = P_0$. Therefore in that case $\dim_{P_0} M/P_+M = \dim_{P_0} P_0 = n \geq 2$, and in fact $f_{n,R}$ is not bounded.

Now in the Theorem 2.14, we want to compute the Hilbert function of the P_0-module $H^n_{P_+}(R)_j$.

Corollary 2.16. *With the assumption of Theorem* 2.14*, we have*

$$\dim_K \big(H^n_{P_+}(R)_j\big)_i = \begin{cases} \binom{n+i-1}{i}\binom{-j-1}{-n-j} & \text{if } i \leq r, \\ \binom{n+i-1}{i}\binom{-j-1}{-n-j} - \binom{n+i-r-1}{i-r}\binom{-j+r-1}{-n-j+r} & \text{if } r \leq i \leq -n-j+r-1. \end{cases}$$

Proof. We set $-n-j = k$, for short. We take ith component of exact sequence (7), and obtain the exact sequence of K-vector space

$$\bigoplus_{\substack{|a|=i-r\\|b|=k+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=i\\|b|=k}} Kx^a z^b \to \big[H^n_{P_+}(R)_j\big]_i \to 0.$$

If $i \leq r$, from the above exact sequence we see that

$$\dim_K \big(H^n_{P_+}(R)_j\big)_i = \dim_K V_{i,k} = \binom{n+i-1}{i}\binom{-j-1}{-n-j}.$$

Now let $r \leq i \leq -n-j+r-1$. First one has $\dim_K V_{i-r,k+r} < \dim_K V_{i,k}$. We claim that f^r is injective, then we are done. We see that the map

$$V_{i-r,k+r} = \bigoplus_{|c|=i+k} V^c_{i-r,k+r} \xrightarrow{f^r} \bigoplus_{|c|=i+k} V^c_{i,k} = V_{i,k}$$

where $f^r(V^c_{i-r,k+r}) \subset V^c_{i,k}$ is injective if and only if restriction map $f' := f^r|_{V^c_{i-r,k+r}} : V^c_{i-r,k+r} \longrightarrow V^c_{i,k}$ is injective for all c with $|c| = i+k$.

For each such c we have a commutative diagram

$$\begin{tikzcd} V^c_{i-r,k+r} \arrow[r, "f'"] \arrow[d] & V^c_{i,k} \arrow[d] \\ A^c_{i-r} \arrow[r, "l^r"] & A^c_i, \end{tikzcd}$$

with $l = x_1 + x_2 + \ldots + x_n \in A^c_1$. Since $i < -n-j+r$, then $i < |c| - (i-r)$ and by the weak Lefschetz property the algebra A^c is unimodal. Therefore $\dim_K A^c_{i-r} \leq \dim_K A^c_i$. The strong Lefschetz property implies that the map l^r is injective, and hence f' is injective, as required. ☐

Corollary 2.17. *Assume that* $\operatorname{char} K = 0$*. Then with the notation of Theorem* 2.14*, we have*

$$\operatorname{reg} H^{n-1}_{P_+}\big(P/f_\lambda^r P\big)_j = -n-j+r+1.$$

Proof. We consider the exact sequence of P_0-modules

$$0 \to H^{n-1}_{P_+}(R)_j \to \bigoplus_{|b|=-n-j+r} P_0(-r)z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P_+}(R)_j \to 0. \qquad (8)$$

where $R = P/f^r_\lambda P$. It follows that $H^{n-1}_{P_+}(R)_j$ is the second syzygy module of $H^n_{P_+}(R)_j$. Let

$$\cdots \to \bigoplus_{j=1}^{t_2} P_0(-a_{1j}) \to \bigoplus_{j=1}^{t_1} P_0(-a_{0j}) \to H^{n-1}_{P_+}(R)_j \to 0$$

be the minimal graded free resolution of $H^{n-1}_{P_+}(R)_j$. We combine two above resolutions, and obtain a graded free resolution for $H^n_{P_+}(R)_j$ of the form

$$\cdots \to \bigoplus_{j=1}^{t_1} P_0(-a_{0j}) \xrightarrow{d_0} \bigoplus_{|b|=-n-j+r} P_0(-r)z^b \xrightarrow{f^r} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P_+}(R)_j \to 0.$$

We choose a basis element $h \in \bigoplus_{j=1}^{t_1} P_0(-a_{0j})$ of degree a_{0j}. Thus

$$d_0(h) = \sum_{|b|=-n-j+r} h_b z^b,$$

where $h_b \in P_0$ with $\deg h_b = a_{0j} - r$. Because the free resolution is minimal, at least one $h_b \neq 0$, so that $r < a_{0j}$ and hence $r - 1 \leq a_{0j} - 2$. Thus we have

$$\operatorname{reg} H^n_{P_+}(R)_j = \max_{i,j}\{0, r-1, a_{ij} - i - 2\} = \max_{i,j}\{a_{ij} - i - 2\}.$$

Theorem 2.14 implies that

$$\operatorname{reg} H^{n-1}_{P_+}(R)_j = \max_{i,j}\{a_{ij} - i\} = -n - j + r + 1.$$

□

Corollary 2.18. *Assume that* $\operatorname{char} K = 0$. *Then with the notation of the Theorem 2.14 the P_0-module $H^{n-1}_{P_+}\big(P/f^r_\lambda P\big)_j$ has a linear resolution.*

Proof. Taking the kth component of the exact sequence (8), we obtain the exact sequence of K-vector spaces

$$0 \to \big[H^{n-1}_{P_+}(R)_j\big]_k \to \bigoplus_{\substack{|a|=k-r \\ |b|=-n-j+r}} Kx^a z^b \xrightarrow{f^r} \bigoplus_{\substack{|a|=k \\ |b|=-n-j}} Kx^a z^b \to \big[H^n_{P_+}(R)_j\big]_k \to 0.$$

For k we distinguish several cases. Let $k = -n - j + r + 1$. One has

$$\dim_K V_{k-r,-n-j+r} > \dim_K V_{k,-n-j}.$$

This implies that

$$\left[H^{n-1}_{P_+}(R)_j\right]_k \neq 0 \quad \text{for all} \quad k \geq -n-j+r+1,$$

since $H^{n-1}_{P_+}(R)_j$ is torsion-free.

Let $k=-n-j+r$. Then $\dim_K V_{k-r,-n-j+r} = \dim_K V_{k,-n-j}$, so that $\left[H^{n-1}_{P_+}(R)_j\right]_k = 0$. Finally let $k < -n-j+r$. We claim that

$$\dim_K V_{k-r,-n-j+r} = \binom{n+k-r-1}{k-r}\binom{-j+r-1}{-n-j+r}$$

is less than

$$\dim_K V_{k,-n-j} = \binom{n+k-1}{k}\binom{-j-1}{-n-j}.$$

Indeed,

$$\binom{n+k-r-1}{k-r}\binom{-j+r-1}{-n-j+r} < \binom{n+k-1}{k}\binom{-j-1}{-n-j} \quad \text{if and only if}$$

$$\prod_{i=1}^{r} \frac{-j+r-i}{-n-j+r-i+1} < \prod_{i=1}^{r} \frac{n+k-i}{k-i+1}.$$

Since

$$\frac{-j+r-i}{-n-j+r-i+1} < \frac{n+k-i}{k-i+1} \quad \text{for all} \quad i=1,\dots,r \quad \text{if and only if}$$

$$k(n-1) < (-n-j+r)(n-1),$$

the claim is clear. Thus by Corollary 2.17 the regularity of $H^{n-1}_{P_+}(R)_j$ is equal to the least integer k such that $\left[H^{n-1}_{P_+}(R)_j\right]_k \neq 0$. This means that P_0-module

$$H^{n-1}_{P_+}(R)_j = \left[H^{n-1}_{P_+}(R)_j\right]_{\geq -n-j+r+1}$$

has a $(-n-j+r+1)$-linear resolution, and its resolution is the form

$$\cdots \to P_0^{\beta_3}(n+j-r-2) \to P_0^{\beta_2}(n+j-r-1) \to H^{n-1}_{P_+}(R)_j \to 0.$$

□

Combining the above resolution with the exact sequence

$$0 \to H^{n-1}_{P_+}(R)_j \to P_0^{\beta_1}(-r) \to P_0^{\beta_0} \to H^{n}_{P_+}(R)_j \to 0,$$

we obtain a graded free resolution for $H^{n}_{P_+}(R)_j$ of the form

$$\cdots \to P_0^{\beta_3}(n+j-r-2) \to P_0^{\beta_2}(n+j-r-1) \to P_0^{\beta_1}(-r) \to P_0^{\beta_0} \to H^{n}_{P_+}(R)_j \to 0.$$

In this resolution we know already the Betti numbers

$$\beta_0 = \binom{-j-1}{-n-j} \quad \text{and} \quad \beta_1 = \binom{-j+r-1}{-n-j+r}.$$

Next we are going to compute the remaining Betti numbers and also the multiplicity of $H^n_{P_+}(R)_j$. For this we need to prove the following extension of the formula of Herzog and Kühl [6].

Proposition 2.19. *Let M be a finitely generated graded Cohen-Macaulay P_0-module of codimension s with minimal graded free resolution*

$$0 \to P_0^{\beta_s}(-d_s) \to \cdots \to P_0^{\beta_1}(-d_1) \to P_0^{\beta_0} \to M \to 0.$$

Then

$$\beta_i = (-1)^{i+1}\beta_0 \prod_{j \neq i} \frac{d_j}{(d_j - d_i)}.$$

Proof. We consider the square matrix A of size s and the following $s \times 1$ matrices of X and Y:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_s \\ \vdots & \vdots & \vdots & \vdots \\ d_1^{s-1} & d_2^{s-1} & \cdots & d_s^{s-1} \end{pmatrix}, \quad X = \begin{pmatrix} -\beta_1 \\ \beta_2 \\ \vdots \\ (-1)^s \beta_s \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -\beta_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

With similar arguments as in the proof of Lemma 1.1 in [16] one has

$$\sum_{i=1}^{s} (-1)^i \beta_i d_i^k = \begin{cases} 0 & \text{for } 1 \leq k < s, \\ (-1)^s s! e(M) & \text{for } k = s, \end{cases}$$

Note that $\sum_{i=1}^{s}(-1)^i\beta_i = \beta_0$. Thus we can conclude that $AX = Y$. Now we can apply Cramer's rule for the computation of β_i. We replace the ith column of A by Y, then we expand the determinant $|A|$ of A along to the Y, we get $\beta_i = -\beta_0\,|A'|\,/\,|A|$ where A' is the matrix

$$\begin{pmatrix} d_1 & \cdots & d_{i-1} & d_{i+1} & \cdots & d_s \\ d_1^2 & \cdots & d_{i-1}^2 & d_{i+1}^2 & \cdots & d_s^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_1^{s-1} & \cdots & d_{i-1}^{s-1} & d_{i+1}^{s-1} & \cdots & d_s^{s-1} \end{pmatrix},$$

of size $s-1$. A is a Vandermonde matrix whose determinant is $\prod_{1 \leq j < i \leq s}(d_i - d_j)$. We also note that

$$|A'| = \prod_{j \neq i} d_j \prod_{\substack{1 \leq t < k \leq s \\ t \neq i}} (d_k - d_t),$$

so the desired formula follows. □

We also have the following generalization of a formula of Huneke-Miller [17].

Proposition 2.20. *With the assumption of Proposition* 2.19, *we have*

$$e(M) = \frac{\beta_0}{s!}\prod_{i=1}^{s} d_i.$$

Proof. We consider the square matrix

$$M = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_{s-1} d_{s-1} & \beta_s d_s \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_{s-1} d_{s-1}^2 & \beta_s d_s^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_1 d_1^s & \beta_2 d_2^s & \cdots & \beta_{s-1} d_{s-1}^s & \beta_s d_s^s \end{pmatrix} \tag{9}$$

of size s.

We will compute the determinant $|M|$ of M in two different ways. First we replace the last column of M by the alternating sum of all columns of M. The resulting matrix will be denoted by M'. It is clear that $|M| = (-1)^s|M'|$. Moreover, due to [16, Lemma 1.1], the last column of M' is the transpose of the vector $(0, \dots, 0, (-1)^s se(M))$. Thus if we expand M' with respect to the last column we get

$$|M| = (-1)^s|M'| = s!e(M)|N|$$

where N is the matrix

$$N = \begin{pmatrix} \beta_1 d_1 & \beta_2 d_2 & \cdots & \beta_{s-1} d_{s-1} \\ \beta_1 d_1^2 & \beta_2 d_2^2 & \cdots & \beta_{s-1} d_{s-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_1 d_1^{s-1} & \beta_2 d_2^{s-1} & \cdots & \beta_{s-1} d_{s-1}^{s-1} \end{pmatrix}$$

of size $s-1$. Thus

$$|M| = s!e(M)\prod_{i=1}^{s-1}\beta_i\prod_{i=1}^{s-1} d_i|V(d_1, \dots, d_{s-1})| \tag{10}$$

where $V(d_1, \dots, d_{s-1})$ is the Vandermonde matrix of size $s-1$ whose determinant is $\prod_{1\le j<i\le s-1}(d_i - d_j)$. On the other hand, directly from (9) we get

$$|M| = \prod_{i=1}^{s}\beta_i\prod_{i=1}^{s} d_i|V(d_1, \dots, d_s)| \tag{11}$$

where $V(d_1, \dots, d_s)$ is the Vandermonde matrix of size s whose determinant is $\prod_{1\le j<i\le s}(d_i - d_j)$. In view of (10) and (11) we get the desired formula. $\square$

Now we are able to compute all Betti numbers and the multiplicity of $H^n_{P_+}(R)_j$. We recall that its resolution is the form

$$\begin{aligned} 0 \to P_0^{\beta_n}(j-r+1) \to P_0^{\beta_{n-1}}(j-r+2) \to \cdots \to P_0^{\beta_3}(n+j-r-2) \to \\ P_0^{\beta_2}(n+j-r-1) \to P_0^{\beta_1}(-r) \to P_0^{\beta_0} \to H^n_{P_+}(R)_j \to 0, \end{aligned}$$

$$\text{where} \quad \beta_0 = \binom{-j-1}{-n-j} \quad \text{and} \quad \beta_1 = \binom{-j+r-1}{-n-j+r}.$$

Corollary 2.21. *With the above notation we have*

$$\beta_i = \frac{(-1)^i r(n-1)!\beta_0\beta_1}{(i-2)!(n-i)!(-n-j+r+i-1)(n+j-i+1)} \quad \textit{for all} \quad i \geq 2,$$

and

$$e\big(H^n_{P_+}(R)_j\big) = \frac{r(-j+r-1)!\beta_0}{n!(-n-j+r)!}.$$

Proof. The assertion follows from Proposition 2.20 and Proposition 2.19. □

2.4 Linear bounds for the regularity of the graded components of local cohomology for hypersurface rings

In this section for a bihomogenous polynomial $f \in P$ we want to give a linear bound for the function $f_{i,R}(j) = \operatorname{reg} H^i_{P_+}(R)_j$ where $R = P/fP$. First we prove the following

Proposition 2.22. *Let R be the hypersurface ring P/fP where $f = \sum_{i=1}^n f_i y_i$ with $f_i \in P_0$. Suppose that $\deg f_i = d$ and that $I(f)$ is $\mathfrak{m}$-primary. Then there exists an integer q such that for $j \ll 0$ we have*

(a) $\operatorname{reg} H^n_{P_+}(R)_j \leq (-n-j+1)d + q,$ *and*

(b) $\operatorname{reg} H^{n-1}_{P_+}(R)_j \leq (-n-j+1)d + q + 2.$

Proof. (a) From the exact sequence $0 \to P(-d,-1) \xrightarrow{f} P \to R \to 0$, we get exact sequence P_0-modules

$$\bigoplus_{|b|=-n-j+1} P_0(-d)z^b \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P_+}(R)_j \to 0. \tag{12}$$

We first assume that $f_i = x_i$. Theorem 2.14 implies that $\operatorname{reg} H^n_{P_+}(R)_j = -n-j$. We set $k = -n-j$. Thus we can get the surjective map of K-vector spaces

$$\bigoplus_{\substack{|a|=k \\ |b|=k+1}} Kx^a z^b \longrightarrow \bigoplus_{\substack{|a|=k+1 \\ |b|=k}} Kx^a z^b.$$

Replacing x_i by f_i, we therefore get a surjective map

$$\begin{aligned}\bigoplus_{|b|=k+1}\left(I(f)^k\right)_{dk}z^b = \quad & \bigoplus_{\substack{|a|=k\\|b|=k+1}} K f_1^{a_1}\dots f_n^{a_n}z^b \xrightarrow{f} \\ & \bigoplus_{\substack{|a|=k+1\\|b|=k}} K f_1^{a_1}\dots f_n^{a_n}z^b = \bigoplus_{|b|=k}\left(I(f)^{k+1}\right)_{d(k+1)}z^b.\end{aligned}$$

Since $I(f)$ is $\mathfrak{m}$-primary by [11, Theorem 2.4] there exists an integer q such that

$$\operatorname{reg}\left(P_0/I(f)^{k+1}\right) = (k+1)d+q \quad \text{for} \quad k \gg 0.$$

We set $l=(k+1)d+q$. Then for $l \gg 0$ we have

$$(P_0)_{l+1} = \left(I(f)^{k+1}\right)_{l+1}.$$

We take the $(l+1)$th component of the exact sequence (12) and consider the following diagram

$$\begin{tikzcd}
\bigoplus_{|b|=k+1}(P_0)_{l-d+1}z^b \arrow[r] & \bigoplus_{|b|=k}(P_0)_{l+1}z^b \arrow[r] & \left[H^n_{P_+}(R)_j\right]_{l+1} \arrow[r] & 0 \\
\bigoplus_{|b|=k+1}\left(I(f)^k\right)_{l-d+1}z^b \arrow[r] \arrow[u] & \bigoplus_{|b|=k}\left(I(f)^{k+1}\right)_{l+1}z^b \arrow[r] \arrow[u, equal] & 0, &
\end{tikzcd}$$

in which left-hand vertical homomorphism is inclusion. Thus we conclude that

$$\left[H^n_{P_+}(R)_j\right]_{l+1} = 0,$$

so that $\operatorname{reg} H^n_{P_+}(R)_j \le l = (k+1)d+q$, as required.

For the proof (b), we notice that the exact sequence of P_0-modules of (12) breaks into two short exact sequence of P_0-modules

$$0 \to K_j \to \bigoplus_{|b|=k} P_0 z^b \to H^n_{P_+}(R)_j \to 0,$$

$$0 \to H^{n-1}_{P_+}(R)_j \to \bigoplus_{|b|=k+1} P_0(-d)z^b \to K_j \to 0,$$

where $K_j = \operatorname{Im} f$. We see from the first of these sequences that $\operatorname{reg} K_j \le \operatorname{reg} H^n_{P_+}(R)_j + 1$. The second short exact sequence, together with part (a) of this theorem and the fact that $d \le \operatorname{reg} K_j$ implies that

$$\operatorname{reg} H^{n-1}_{P_+}(R)_j \le \max\{d, \operatorname{reg} K_j + 1\} = \operatorname{reg} K_j + 1 \le (-n-j+1)d+q+2,$$

as desired. $\square$

Proposition 2.23. *Let* $\mathbb{N}^n_d = \{\beta \in \mathbb{N}^n : |\beta| = d\}$, $P_0 = K[\{x_\beta\}_{\beta\in\mathbb{N}^n_d}]$ *and* $P = P_0[y_1, \dots, y_n]$. *Let* $R = P/fP$ *where* $f = \sum_{|\beta|=d} x_\beta y^\beta$. *Then*

$$\operatorname{reg} H^n_{P_+}(R)_j \le (-n - j + 1)d - 1.$$

Proof. We set $P_+ = (y_1, \dots, y_n)$ and $P_0 = K[x_1, \dots, x_m]$ where $m = \binom{n+d-1}{d}$, as usual. From the exact sequence

$$0 \to P(-1, -d) \xrightarrow{f} P \to R \to 0,$$

we get the exact sequence of P_0-modules

$$\bigoplus_{|b|=-n-j+d} P_0(-1)(y^b)^* \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0(y^b)^* \to H^n_{P_+}(R)_j \to 0,$$

whose ith graded component is

$$\bigoplus_{\substack{|a|=i-1\\ |b|=-n-j+d}} Kx^a(y^b)^* \xrightarrow{f} \bigoplus_{\substack{|a|=i\\ |b|=-n-j}} Kx^a(y^b)^* \to H^n_{P_+}(R)_{(i,j)} \to 0. \tag{13}$$

Here $(y^b)^* = z^b$ in the notation of Section 1.7. Now we exchange the role of x and y: We may write $f = \sum_{|\beta|=d} y^\beta x_\beta$ and set $Q_+ = (x_1, \dots, x_m)$ and $Q_0 = K[y_1, \dots, y_n]$. From the exact sequence

$$0 \to P(-d, -1) \xrightarrow{f} P \to R \to 0,$$

we get the exact sequence of P_0-modules

$$\bigoplus_{|b|=-m-t+1} Q_0(-d)(x^b)^* \xrightarrow{f} \bigoplus_{|b|=-m-t} Q_0(x^b)^* \to H^m_{Q_+}(R)_t \to 0,$$

whose sth graded component is

$$\bigoplus_{\substack{|a|=s-d\\ |b|=-m-t+1}} Ky^a(x^b)^* \xrightarrow{f} \bigoplus_{\substack{|a|=s\\ |b|=-m-t}} Ky^a(x^b)^* \to H^m_{Q_+}(R)_{(s,t)} \to 0.$$

Applying the functor $\operatorname{Hom}_K(-, K)$ to the above exact sequence and due to the exact sequence (13) we have

$$\begin{aligned} 0 \to H^m_{Q_+}(R)^*_{(s,t)} \to \bigoplus_{\substack{|a|=s\\ |b|=-m-t}} K(y^a)^*x^b \xrightarrow{f} &\bigoplus_{\substack{|a|=s-d\\ |b|=-m-t+1}} K(y^a)^*x^b \\ &\to H^n_{P_+}(R)_{(-m-t+1,-n-s+d)} \to 0. \end{aligned}$$

Therefore

$$H^m_{Q_+}(R)^*_{(s,t)} \cong H^{n-1}_{P_+}(R)_{(-m-t+1,-n-s+d)}.$$

Thus we have

$$0 \to \left[H^{n-1}_{P_+}(R)_{-n-s+d}\right]_{-m-t+1} \to \bigoplus_{\substack{|a|=s \\ |b|=-m-t}} K(y^a)^* x^b$$
$$\xrightarrow{f} \bigoplus_{\substack{|a|=s-d \\ |b|=-m-t+1}} K(y^a)^* x^b \quad \to \left[H^{n}_{P_+}(R)_{-n-s+d}\right]_{-m-t+1} \to 0.$$

We set $j = -n - s + d$. Proposition 2.22 implies that

$$\operatorname{reg} H^n_{P_+}(R)_j \leq (-n-j+1)d + q \quad \text{for some} \quad q.$$

Since $I(f) = (y_1, \ldots, y_n)^d$, thus $\operatorname{reg}\left(P_0/I(f)^{k+1}\right) = (k+1)d - 1$. Hence in Proposition 2.22 we have $q = -1$. □

Now the main result of this section is the following

Theorem 2.24. *Let $P = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$, and $f \in P$ be a bihomogeneous polynomial such that $I(f)$ is $\mathfrak{m}$-primary. Let $R = P/fP$. Then the regularity of $H^n_{P_+}(R)_j$ is linearly bounded.*

Proof. We may write $f = \sum_{|\beta|=d} f_\beta y^\beta$ and let $\deg f_\beta = c$. From the exact sequence

$$0 \to P(-c, -d) \xrightarrow{f} P \to R \to 0,$$

we get the exact sequence of P_0-modules

$$\bigoplus_{|b|=-n-j+d} P_0(-c) z^b \xrightarrow{f} \bigoplus_{|b|=-n-j} P_0 z^b \to H^n_{P_+}(R)_j \to 0.$$

We first assume that $f_\beta = x_\beta$. Proposition 2.23 implies that $\operatorname{reg} H^n_{P_+}(R)_j \leq (-n-j+1)d - 1$. We set $k = (-n-j+1)d$. Thus we get the surjective map of K-vector spaces

$$\bigoplus_{\substack{|a|=k-1 \\ |b|=-n-j+d}} K x^a z^b \longrightarrow \bigoplus_{\substack{|a|=k \\ |b|=-n-j}} K x^a z^b.$$

We proceed as in the proof of Proposition 2.22, and we get $\left[H^n_{P_+}(R)_j\right]_{kd+q'+1} = 0$ for some integer q'. Therefore

$$\operatorname{reg} H^n_{P_+}(R)_j \leq (-n-j+1)d^2 + q'.$$

□

Corollary 2.25. *With the assumption of Theorem* 2.24*, we have*

$$\operatorname{reg} H^{n-1}_{P_+}(R)_j \leq (-n-j+1)d^2 + q' + 2.$$

Proof. For the proof one uses the same argument as in the proof of Proposition 2.22(b). □

3 Local duality for bigraded modules

Let R be a standard bigraded K-algebra with bigraded irrelevant ideals P generated by all elements of degree $(1,0)$, and Q generated by all elements of degree $(0,1)$. We want to relate the local cohomology functors $H^i_P(-)$ and $H^j_Q(-)$ via duality in the category of bigraded modules. In the ordinary local duality theorem Matlis duality establishes isomorphisms between the local cohomology modules of a module and its Ext-groups.

In our situation we have to consider Matlis duality for bigraded modules. Given a bigraded R-module M we define the bigraded Matlis-dual of M to be $M^\vee$ where the (i,j)th bigraded component of $M^\vee$ is given by $\operatorname{Hom}_K(M_{(-i,-j)}, K)$.

In this chapter we establish the following duality theorem:

Theorem. Let R be a standard bigraded K-algebra with irrelevant bigraded ideals P and Q, and let M be a finitely generated bigraded R-module. Then there exists a convergent spectral sequence

$$E^2_{i,j} = H^{m-j}_P\big(H^i_{R_+}(M)^\vee\big) \underset{j}{\Longrightarrow} H^{i+j-m}_Q(M)^\vee$$

of bigraded R-modules, where m is the minimal number of homogeneous generators of P and R_+ is the unique graded maximal ideal of R.

Note that the above spectral sequence degenerates when M is Cohen-Macaulay and one obtains for all k the following isomorphims of bigraded R-modules

$$H^k_P\big(H^s_{R_+}(M)^\vee\big) \cong H^{s-k}_Q(M)^\vee$$

where $s = \dim M$, see Corollary 3.12. We let R_0 be the K-subalgebra of R which is generated by the elements of bidegree $(1,0)$ and set $N = H^s_{R_+}(M)^\vee$. Then N is again an s-dimensional Cohen-Macaulay module and by the above isomorphism we obtain for all j the isomorphisms of graded R_0-modules

$$H^k_{P_0}(N_j) \cong \big(H^{s-k}_Q(M)_{-j}\big)^\vee \tag{14}$$

where P_0 is the graded maximal ideal of R_0. Here we used, that $H^k_P(N)_j \cong H^k_{P_0}(N_j)$ for all k and j.

In case M is Cohen-Macaulay the tameness problem translates, due to (14), to the following question: Given a finitely generated bigraded R-module N. Does there exist an integer j_0 such that $H^k_{P_0}(N_j) = 0$ for all $j \geq j_0$, or else $H^k_{P_0}(N_j) \neq 0$ for all $j \geq j_0$? However this is not the case as has been recently shown by Cutkosky and Herzog, see [10]. Their example also provides a counterexample to the general tameness problem. To show this, Proposition 3.11 of this chapter is used. On the other hand, in Chapter 5 we are going to show that tameness holds for all local cohomology modules of a ring with monomial relations and with respect to monomial prime ideals.

In Section 3.2 we use our duality to give a new proofs of known cases of the tameness problem and also to add a few new cases in which tameness holds, see Corollary 3.10, 3.14 and 3.18. The duality is also used in the Corollaries 3.15 and 2.4 to prove some algebraic properties of the modules $H^k_Q(M)_j$ in case M is Cohen-Macaulay.

3.1 Proof of the duality theorem

Let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over the field K. We set $K[x] = K[x_1, \dots, x_m]$ and $K[y] = K[y_1, \dots, y_n]$ and consider both as a standard graded polynomial rings.

If A is a standard (bi)graded K-algebra, and M a (bi)graded A-module. Then we set $M^\vee = \operatorname{Hom}_K(M, K)$, and view $M^\vee$ as (bi)graded A-module with the (bi)grading

$$(M^\vee)_a = \operatorname{Hom}_K(M_{-a}, K)$$

for $a \in \mathbb{Z}$ (respectively $a \in \mathbb{Z}^2$ in the bigraded case).

The following simple fact is needed for the proof next lemma.

Lemma 3.1. *Let M be a graded $K[x]$-module and N be a graded $K[y]$-module. Then there exists a natural bigraded isomorphism of bigraded S-modules*

$$(M \otimes_K N)^\vee \cong M^\vee \otimes_K N^\vee.$$

Proof. Let $S = K[x] \otimes_K K[y] = K[x, y]$. Note that $M \otimes_K N$ is a bigraded free S-module with the natural bigrading

$$(M \otimes_K N)_{(i,j)} = M_i \otimes_K N_j.$$

Thus we see that

$$\big((M \otimes_K N)^\vee\big)_{(i,j)} = \operatorname{Hom}_K\big((M \otimes_K N)_{(-i,-j)}, K\big) = \operatorname{Hom}_K(M_{-i} \otimes_K N_{-j}, K).$$

By using the universal property of tensor product one has the following natural isomorphism of K-vector spaces

$$\operatorname{Hom}_K(M_{-i} \otimes_K N_{-j}, K) \cong \operatorname{Hom}_K(M_{-i}, K) \otimes_K \operatorname{Hom}_K(N_{-j}, K).$$

Thus we have

$$\begin{aligned}\big((M \otimes_K N)^\vee\big)_{(i,j)} &\cong \operatorname{Hom}_K(M_{-i}, K) \otimes_K \operatorname{Hom}_K(N_{-j}, K)\\ &= (M^\vee)_i \otimes_K (N^\vee)_j\\ &= (M^\vee \otimes_K N^\vee)_{(i,j)}.\end{aligned}$$

So the desired isomorphism follows. □

Lemma 3.2. *Let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over the field K with the irrelevant bigraded ideals $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. Then we have the following isomorphism of bigraded S-modules*

$$H^m_P(\omega_S) \cong H^n_Q(S)^\vee,$$

where ω_S is the bigraded canonical module of S.

Proof. First we notice that there is a natural isomorphism of bigraded S-modules

$$H^m_P(S) \cong H^m_{P_0}(K[x]) \otimes_K K[y].$$

By the graded version of the local duality theorem (see [5, Example 13.4.6]) we have

$$H^m_{P_0}(K[x])^\vee \cong K[x](-m).$$

Thus we see that

$$\begin{aligned} H^m_P(\omega_S) = H^m_P(S(-m,-n)) &= H^m_P(S)(-m,-n) \\ &\cong \big(K[x](-m)^\vee \otimes_K K[y]\big)(-m,-n) \\ &= K[x]^\vee \otimes_K K[y](-n). \end{aligned}$$

On the other hand, using again the local duality theorem, Lemma 3.1 yields

$$\begin{aligned} H^n_Q(S)^\vee \cong \big(K[x] \otimes_K H^n_{Q_0}(K[y])\big)^\vee &\cong \big(K[x] \otimes_K K[y](-n)^\vee\big)^\vee \\ &\cong K[x]^\vee \otimes_K K[y](-n)^{\vee\vee} \\ &\cong K[x]^\vee \otimes_K K[y](-n), \end{aligned}$$

as desired. □

Corollary 3.3. *Let F be a finitely generated bigraded free S-module, and set $F^* = \operatorname{Hom}_S(F, \omega_S)$. Then there exists a natural isomorphism of bigraded S-modules,*

$$H^m_P(F^*) \cong H^n_Q(F)^\vee.$$

Proof. Let $F = \bigoplus_{k=1}^t S(-a_k, -b_k)$. Thus $F^* = \bigoplus_{k=1}^t (\omega_S)(a_k, b_k)$ and hence by Lemma 3.2 we have

$$\begin{aligned} H^m_P(F^*) \cong \bigoplus_{k=1}^t H^m_P(\omega_S)(a_k, b_k) &\cong \bigoplus_{k=1}^t H^n_Q(S)^\vee(a_k, b_k) \\ &\cong H^n_Q\Big(\bigoplus_{k=1}^t S(-a_k, -b_k)\Big)^\vee \\ &\cong H^n_Q(F)^\vee. \end{aligned}$$

□

The previous result can easily be extended as follows

Lemma 3.4. *Let $\mathbb{F}$ be a bounded complex of bigraded free S-modules. We set $\mathbb{F}^* = \operatorname{Hom}_S(\mathbb{F}, \omega_S)$. Then we have a functorial isomorphism*

$$H_P^m(\mathbb{F}^*) \cong H_Q^n(\mathbb{F})^\vee$$

of complexes of bigraded modules.

Proof. In order to prove that the complexes of $H_P^m(\mathbb{F}^*)$ and $H_Q^n(\mathbb{F})^\vee$ are isomorphic, we observe that for any bihomogeneous linear map $\varphi : G \to F$ between finitely generated free bigraded S-modules we obtain the following commutative diagram

$$\begin{tikzcd} H_Q^n(F)^\vee \arrow[r, "\psi_1^\vee"] \arrow[d] & H_Q^n(G)^\vee \arrow[d] \\ H_P^m(F^*) \arrow[r, "\psi_2"] & H_P^m(G^*), \end{tikzcd}$$

where $\psi_1 = H_P^n(\varphi)$ and $\psi_2 = H_Q^m(\varphi^*)$ and where the vertical maps are the isomorphisms given in Corollary 3.3. The commutativity of the diagram results from the fact that all maps in the diagram are functorial. ☐

Proposition 3.5. *Let M be a finitely generated bigraded S-module, P and Q be the irrelevant bigraded ideals of S. Then we have the following convergent spectral sequence*

$$E_{i,j}^2 = H_P^{m-j}\big(\operatorname{Ext}_S^{n+m-i}(M, \omega_S)\big) \underset{j}{\Longrightarrow} H_Q^{i+j-m}(M)^\vee.$$

Proof. Let $(\mathbb{F}, d)$ be a bigraded free resolution of M of length $n+m$, and let $\mathbb{G}$ be the complex of bigraded S-modules with $G_i = \operatorname{Hom}_S(F_{m+n-i}, \omega_S)$ and differential $\partial_i = \operatorname{Hom}_S(d_{m+n-i}, \omega_S)$. Next we choose a bigraded free resolution $\mathbb{C}$ of the complex $\mathbb{G}$. In other words, $\mathbb{C}$ is a double complex C_{ij} of finitely generated bigraded free S-modules with $i, j \geq 0$ such that:

(i) the ith column of $\mathbb{C}$ is a free resolution of G_i for all i, i.e.

$$H_j(C_{i\bullet}) = \begin{cases} G_i & \text{for } j = 0, \\ 0 & \text{for } j > 0. \end{cases}$$

(ii) for each row the image of $C_{i-1,j} \leftarrow C_{i,j}$ is a bigraded free direct summand of the kernel of $C_{i-2,j} \leftarrow C_{i-1,j}$. In particular, the homology of

$$C_{i-2,j} \leftarrow C_{i-1,j} \leftarrow C_{i,j}$$

is a bigraded free S-module for all i and j.

(iii) for each i the complex

$$0 \leftarrow H_i(C_{\bullet,0}) \leftarrow H_i(C_{\bullet,1}) \leftarrow H_i(C_{\bullet,2}) \leftarrow \cdots$$

is a bigraded free resolution of $H_i(\mathbb{G})$.

Now we compute the total homology of the double complex $H^m_P(\mathbb{C})$: Since all G_i are free S-modules, it follows that the complexes

$$0 \leftarrow G_i \leftarrow C_{i,0} \leftarrow C_{i,1} \leftarrow \cdots$$

are all split exact. Hence the complexes

$$0 \leftarrow H^m_P(G_i) \leftarrow H^m_P(C_{i,0}) \leftarrow H^m_P(C_{i,1}) \leftarrow \cdots$$

are again exact.

This implies that the E^1-terms of the double complex $H^m_P(\mathbb{C})$ with respect to the column filtration are

$$E^1_{i,j} = \begin{cases} H^m_P(G_i) & \text{for } j = 0, \\ 0 & \text{for } j > 0. \end{cases}$$

As a consequence, for the E^2-terms of $H^m_P(\mathbb{C})$ we have that $E^2_{i,j} = 0$ for $j > 0$, and that $E^2_{i,0}$ is the ith homology of the complex $H^m_P(\mathbb{G})$. Now we use Lemma 3.4 as well as [21, Theorem 1.1] and obtain

$$E^2_{i,j} = \begin{cases} H^{i-m}_Q(M)^\vee & \text{for } j = 0, \\ 0 & \text{for } j > 0, \end{cases}$$

since $H_i\big(H^m_P(\mathbb{G})\big) = \big(H_{n+m-i}\big(H^n_Q(\mathbb{F})\big)\big)^\vee$. From this it follows that the $(i+j)$th total homology of $H^m_P(\mathbb{C})$ is equal to $H^{i+j-m}_Q(M)^\vee$.

Now we compute the homology of $H^m_P(\mathbb{C})$ using the row filtration. Each row $H^m_P(C_{\bullet j})$ of $H^m_P(\mathbb{C})$ is split exact with homology $H_i\big(H^m_P(C_{\bullet j})\big) = H^m_P\big(H_i(C_{\bullet j})\big)$. In other words, $E^1_{i,j} = H^m_P\big(H_i(C_{\bullet j})\big)$. Hence by property (iii) of the complex $\mathbb{C}$ and by [21, Theorem 1.1] it follows that $E^2_{i,j} = H^{m-j}_P\big(\operatorname{Ext}^{m+n-i}_S(M, \omega_S)\big)$. This yields the desired conclusion. □

Now our main theorem is an easy consequence of Proposition 3.5:

Proof. As R is a standard bigraded K-algebra, it is the homomorphic image of a standard bigraded polynomial ring $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. We may consider R and S as well as a standard graded K-algebras with the unique graded maximal ideal R_+ (resp. S_+), and M as a graded R-module (resp. S-module). Then by the graded local duality theorem we have

$$\operatorname{Ext}^{m+n-i}_S(M, \omega_S) \cong H^i_{S_+}(M)^\vee.$$

Since $H^i_{S_+}(M) \cong H^i_{R_+}(M)$, it follows that

$$H^{m-j}_P\big(H^i_{R_+}(M)^\vee\big) = H^{m-j}_P\big(\operatorname{Ext}^{m+n-i}_S(M, \omega_S)\big).$$

Let $(x) = (x_1, \ldots, x_m)$ and $(y) = (y_1, \ldots, y_n)$ be the irrelevant ideals of S. We note that $H^{m-j}_P\big(\operatorname{Ext}^{m+n-i}_S(M, \omega_S)\big) = H^{m-j}_{(x)}\big(\operatorname{Ext}^{m+n-i}_S(M, \omega_S)\big)$ and that $H^{i+j-m}_Q(M)^\vee = H^{i+j-m}_{(y)}(M)^\vee$. Therefore, Proposition 3.5 yields the desired convergent spectral sequence. □

Corollary 3.6. *Let R be a standard bigraded d-dimensional Cohen-Macaulay K-algebra with irrelevant bigraded ideals P and Q, and let M be a finitely generated bigraded R-module. Then there exists a convergent spectral sequence*

$$E^2_{i,j} = H^{m-j}_P\big(\operatorname{Ext}^{d-i}_R(M,\omega_R)\big) \underset{j}{\Longrightarrow} H^{i+j-m}_Q(M)^\vee$$

of bigraded R-modules, where m is the minimal number of homogeneous generators of P.

Proof. The assertion follows from our main theorem by using the fact that $H^i_{R_+}(M)^\vee = \operatorname{Ext}^{d-i}_R(M,\omega_R)$. ☐

A general version of our duality theorem have recently been constructed. See [9].

3.2 Some applications

In this section, unless otherwise stated, R denotes a standard bigraded K-algebra of dimension d, and M a finitely generated and bigraded R-module.

We note that for the E^2-terms in the spectral sequence of our main theorem we have $E^2_{i,j} = H^{m-j}_P(H^i_{R_+}(M)^\vee) = 0$ if $i < \operatorname{depth} M$ or $i > \dim M$ or $j < 0$ or $j > m$. Thus the possible non-zero E^2-terms are in the shadowed region of the following picture.

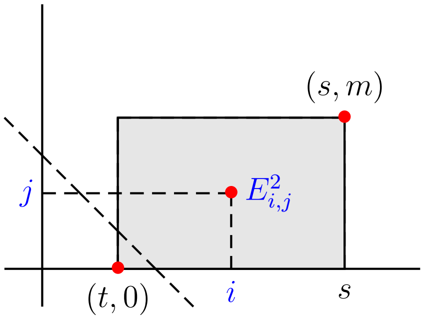


Figure 1

Here $t = \operatorname{depth} M$, $s = \dim M$ and $E^2_{i,j} = H^{m-j}_P\big(H^i_{R_+}(M)^\vee\big)$.

We first observe that the graded local duality theorem is a special case of our main theorem. In fact, if we assume that $P = (0)$, then $m = 0$, and $\mathfrak{m} = Q$ is the unique graded maximal ideal of R. Moreover, $E^2_{i,j} = E^\infty_{i,j} = 0$ for $j \neq 0$ and all i, since $H^k_{(0)}(-) = 0$ if $k \neq 0$. Therefore we have

$$\operatorname{Ext}^{d-i}_R(M,\omega_R) = H^0_{(0)}\big(\operatorname{Ext}^{d-i}_R(M,\omega_R)\big) \cong H^i_{\mathfrak{m}}(M)^\vee.$$

Considering Figure 1 we immediately obtain the following corner isomorphisms

Proposition 3.7. *Let* $\dim M = s$ *and* $\operatorname{depth} M = t$. *Then there are natural isomorphisms*

$$H^m_P\bigl(H^t_{R_+}(M)^\vee\bigr) \cong H^{t-m}_Q(M)^\vee \quad \textit{and} \quad H^0_P\bigl(H^s_{R_+}(M)^\vee\bigr) \cong H^s_Q(M)^\vee.$$

Moreover for $i < t-m$ *we have* $H^i_Q(M) = 0$.

Definition 3.8. Let R_0 be a commutative Noetherian ring, R a graded R_0-algebra and N a graded R-module. The R-module N is called *tame*, if there exists an integer j_0 such that either

$$N_j = 0 \quad \text{for all} \quad j \le j_0, \quad \text{or} \quad N_j \ne 0 \quad \text{for all} \quad j \le j_0.$$

In case of a standard bigraded K-algebra R we let R_0 be the K-subalgebra of R generated by all elements of degree $(1,0)$. Then R is a graded R_0-algebra with components $R_j = R_{(*,j)} = \bigoplus_i R_{(i,j)}$. Let N be a bigraded R-module. We may view N as a graded R-module with graded components $N_j = N_{(*,j)} = \bigoplus_i N_{(i,j)}$. Each of the modules N_j is a graded R_0-module, and if N is a finitely generated R-module then each N_j is a finitely generated R_0-module.

Now let M be a finitely generated bigraded R-module. By the definition local cohomology using the Čech complex one has that $H^i_P(M)_j = H^i_{P_0}(M_j)$ where P_0 is the graded maximal ideal of R_0. Since M_j is a finitely generated R_0-module it follows that $H^i_{P_0}(M_j)$ is a graded Artinian R_0-module. Hence we see that $\bigl(H^i_P(M)^\vee\bigr)_j = \bigl(H^i_P(M)_{-j}\bigr)^\vee = H^i_{P_0}(M_{-j})^\vee$ is a finitely generated graded R_0-module for all j. Of course this does not imply that $H^i_P(M)^\vee$ is a finitely generated R-module.

We denote by $\operatorname{cd}(Q,M)$ the cohomological dimension of M with respect to Q, i.e., the number

$$\operatorname{cd}(Q,M) = \sup\{i \in \mathbb{N}_0 : H^i_Q(M) \ne 0\}.$$

Corollary 3.9. *Let* $N = H^s_{R_+}(M)^\vee$. *Then the following statements hold:*

(a) $\operatorname{cd}(Q,M) < \dim(M)$ *if and only if* $\operatorname{depth}_{R_0} N_j > 0$ *for all* j*;*

(b) *if* $\operatorname{cd}(Q,M) < \dim(M) - 1$, *then* $\operatorname{depth}_{R_0} N_j > 1$ *for all* j.

Proof. We note that $\operatorname{cd}(Q,M) < \dim(M)$, if and only if $H^s_Q(M) = 0$. Hence Proposition 3.7 yields part (a) of the corollary.

For the proof of (b) we notice that $H^i_P(N) = E^\infty_{s,m-i}$ for $i = 0,1$, and that $E^\infty_{s,m-i}$ is a submodule of $H^{s-i}_Q(M)^\vee$ for all i. Thus our assumption implies that $H^i_{P_0}(N_j) = H^i_P(N)_j = 0$ for $i = 0,1$ and all j. This yields the desired conclusion. $\square$

The second statement of the next corollary of is well-known (see [2, Theorem 4.8 (e)]).

Corollary 3.10. *Let* M *be a finitely generated bigraded* R*-module of dimension* s *and depth* t. *Then* $H^{t-m}_Q(M)$ *and* $H^s_Q(M)$ *are tame.*

Proof. We first prove $H_Q^{t-m}(M)$ is tame. We set $N = H_{R_+}^t(M)^\vee$ and $s_0 = \dim_{R_0} N_j$ for $j \gg 0$. Such s_0 exists, see [1, Proposition 2.5]. Note that $s_0 \leq \dim R_0 \leq m$. Thus we have $H_P^m(N)_j = H_{P_0}^m(N_j) = 0$ for $j \gg 0$ if $s_0 < m$ and $H_P^m(N)_j = H_{P_0}^m(N_j) \neq 0$ for $j \gg 0$ if $s_0 = m$. Therefore by Proposition 3.7 there exists an integer j_0 such that either

$$H_Q^{t-m}(M)_j = 0 \quad \text{for all} \quad j \leq j_0, \quad \text{or} \quad H_Q^{t-m}(M)_j \neq 0 \quad \text{for all} \quad j \leq j_0,$$

as desired. In order to prove that $H_Q^s(M)$ is tame, we set $N = H_{R_+}^s(M)^\vee$. Since $H_{R_+}^s(M)$ is a graded Artinian R-module, N is a finitely generated graded R-module. Thus N_j is a finitely generated R_0-module. By [1, Proposition 2.5] the set of associated prime ideals of $\operatorname{Ass}_{R_0}(N_j)$ is constant for large j. If $P_0 \in \operatorname{Ass}_{R_0}(N_j)$ it follows that $H_P^0(N)_j = H_{P_0}^0(N_j) \neq 0$ for large j, and if $P_0 \notin \operatorname{Ass}_{R_0}(N_j)$ then $H_P^0(N)_j = H_{P_0}^0(N_j) = 0$ for large j. Thus in view of Proposition 3.7, $H_Q^s(M)$ is also tame. □

We say that M is a generalized Cohen-Macaulay R-module if $H_{R_+}^i(M)$ has finite length for all $i \neq \dim M$.

Proposition 3.11. *Let M be a generalized Cohen-Macaulay R-module of dimension s. Then we have the following long exact sequence of bigraded R-modules*

$$\begin{aligned} 0 \to H_P^1\big(H_{R_+}^s(M)^\vee\big) &\to H_Q^{s-1}(M)^\vee \to H_{R_+}^{s-1}(M)^\vee \to \\ H_P^2\big(H_{R_+}^s(M)^\vee\big) &\to H_Q^{s-2}(M)^\vee \to H_{R_+}^{s-2}(M)^\vee \to \\ &\cdots \to H_Q^{s-m}(M)^\vee \to H_{R_+}^{s-m}(M)^\vee \to 0. \end{aligned}$$

Moreover, we have the following isomorphisms

$$H_{R_+}^i(M) \cong H_Q^i(M) \quad \textit{for all } i < s-m.$$

Proof. Since M is a generalized Cohen-Macaulay module, we have that $H_{R_+}^i(M)^\vee$ is of finite length for $i \neq s$. Thus by Grothendieck's vanishing theorem [5, Theorem 6.1.2] we see that $E_{i,j}^2 = E_{i,j}^\infty = 0$ for $j = 0, \ldots, m-1$ and $i \neq s$. The following picture will make this clear.

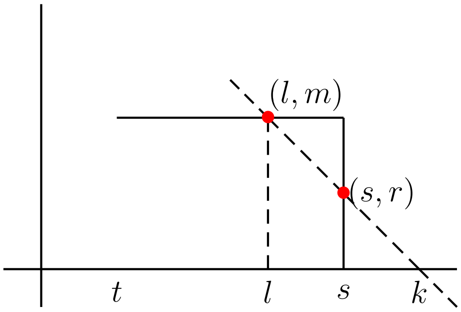


Figure 2

Therefore for all k with $s \leq k < s+m$, we get the following exact sequences

$$0 \to E^\infty_{s,r} \to H^l_Q(M)^\vee \to E^\infty_{l,m} \to 0,$$

$$0 \to E^\infty_{l,m} \to E^2_{l,m} \to E^2_{s,r-1} \to E^\infty_{s,r-1} \to 0,$$

where l and r are defined by the equations $s+r = l+m = k$.

Composing these two exact sequences we get the long exact sequence

$$\begin{gathered} \cdots \to E^2_{s,r} \to H^l_Q(M)^\vee \to E^2_{l,m} \to E^2_{s,r-1} \to \\ H^{l-1}_Q(M)^\vee \to E^2_{l-1,m} \to E^2_{s,r-2} \to \cdots, \end{gathered}$$

which yields the desired exact sequence. Observing that

$$H^0_P\big(H^i_{R_+}(M)^\vee\big) = H^i_{R_+}(M)^\vee$$

for $i \neq s$, since for such i the modules $H^i_{R_+}(M)^\vee$ have finite length. The last statement of the proposition follows similarly. □

Corollary 3.12. *Suppose M is a generalized Cohen-Macaulay module of dimension s. Then the following conditions are equivalent:*

(a) *M is Cohen-Macaulay;*

(b) *$H^k_P\big(H^s_{R_+}(M)^\vee\big) \cong H^{s-k}_Q(M)^\vee$ for all k.*

Proof. (a) ⇒ (b): Since M is Cohen-Macaulay we have $H^i_{R_+}(M)^\vee = 0$ for all $i \neq s$. Therefore it follows from the long exact sequence in Proposition 3.11 that $H^k_P\big(H^s_{R_+}(M)^\vee\big) \cong H^{s-k}_Q(M)^\vee$ for $k = 1, \ldots, m$. The assertion for $k=0$ follows from Proposition 3.7. The assertion is also clear when $k < 0$. Now assume that $k > m$. Then $s-k < s-m$, and hence by Proposition 3.11 it follows that $H^{s-k}_Q(M)^\vee = H^{s-k}_{R_+}(M)^\vee = 0$. On the other hand, we also have $H^k_P\big(H^s_{R_+}(M)^\vee\big) = 0$ because $k > m$.

(b) ⇒ (a): is proved the same way. □

As a generalization of Lemma 3.2 we obtain as an immediate consequence of Corollary 3.12 the following

Remark 3.13. Let R be a bigraded Cohen-Macaulay K-algebra of dimension d. Then

$$H^k_P(\omega_R) \cong H^{d-k}_Q(R)^\vee \quad \text{for all } k.$$

Recall that for a finitely generated graded R-module N one has that $\dim_{R_0} N_j$ as well as $\operatorname{depth}_{R_0} N_j$ is constant for large j, see [1, Proposition 2.5]. In fact, if N is Cohen-Macaulay, then $\lim_{j\to\infty} \operatorname{depth}_{R_0} N_j = \dim N - \dim N/P_0N$ as shown in [14]. We call these constants the limit depth and limit dimension, respectively. Using this fact we have

Corollary 3.14. *Let M be a bigraded Cohen-Macaulay R-module of dimension s. We set $N = H^s_{R_+}(M)^\vee$, and put $t_0 = \lim_{j\to\infty} \operatorname{depth}_{R_0} N_j$ and $s_0 = \lim_{j\to\infty} \dim_{R_0} N_j$. Then the R-modules $H^j_Q(M)$ are tame for all $j \leq s - s_0$ and $j \geq s - t_0$.*

Proof. We see that $H^{s-i}_P(N)_j = H^{s-i}_{P_0}(N_j) \neq 0$ for $j \gg 0$ if $i = s - s_0$ and $i = s - t_0$, and also $H^{s-i}_P(N)_j = H^{s-i}_{P_0}(N_j) = 0$ for $j \gg 0$ if $i < s - s_0$ and $i > s - t_0$. Therefore by Corollary 3.12 we have the desired conclusion. ☐

Corollary 3.15. *Assume R_0 is Cohen-Macaulay and M is a bigraded Cohen-Macaulay R-module of dimension s. We set $N = H^s_{R_+}(M)^\vee$. Then*

(a) *for all k and j we have the following isomorphism of graded R_0-modules*

$$\operatorname{Ext}^{d-k}_{R_0}(N_j, \omega_{R_0}) \cong H^{s-k}_Q(M)_{-j},$$

where $d = \dim R_0$.

(b) $\dim H^{s-k}_Q(M)_{-j} \leq k$ *for all k and j.*

Proof. Corollary 3.12 implies that

$$\big(H^{s-k}_Q(M)_{-j}\big)^\vee \cong \big(H^{s-k}_Q(M)^\vee\big)_j = H^k_{P_0}(N_j).$$

Thus the local duality theorem yields

$$H^{s-k}_Q(M)_{-j} \cong H^k_{P_0}(N_j)^\vee \cong \operatorname{Ext}^{d-k}_{R_0}(N_j, \omega_{R_0}),$$

as desired.

Finally by [6, Corollary 3.5.11(c)] one has $\dim_{R_0} \operatorname{Ext}^{d-k}_{R_0}(N_j, \omega_{R_0}) \leq k$. This proves statement (b). ☐

Let $N \neq 0$ be a graded R_0-module. We set $a(N) = \inf\{i\colon N_i \neq 0\}$ and $b(N) = \sup\{i\colon N_i \neq 0\}$. If $N = 0$ we set $a(N) = \infty$ and $b(N) = -\infty$.

Recall that the *regularity* of N is defined to be

$$\operatorname{reg} N = \max\{b(H^k_{P_0}(N)) + k\colon k = 0, 1, \ldots\}.$$

With the assumptions and notation introduced in Corollary 3.12 we therefore have

$$\operatorname{reg}(N_j) = -\min\{a(H^{s-k}_Q(M)_{-j}) - k\colon k = 0, 1, \ldots\}.$$

In [11] and [19] it is shown that $\operatorname{reg}(N_j)$ is bounded above by a linear function of j. Thus in view of the preceding formula we get

Corollary 3.16. *Let M be a Cohen-Macaulay R-module. Then there exist integers c and d such that $a\big(H^k_Q(M)_j\big) \geq cj + d$ for all k and all j.*

If the dimension and the depth of M differ at most by 1 or $\dim R_0 \leq 1$ one obtains

Proposition 3.17. *The following statements hold:*

(a) *if* $\dim M = s$ *and* $\operatorname{depth} M = s-1$*, then we obtain the long exact sequence*

$$\begin{aligned}\cdots \to H_P^{m-j-2}\bigl(H_{R_+}^{s-1}(M)^\vee\bigr) \to H_P^{m-j}\bigl(H_{R_+}^{s}(M)^\vee\bigr) \to H_Q^{s-m+j}(M)^\vee \to \\ H_P^{m-j-1}\bigl(H_{R_+}^{s-1}(M)^\vee\bigr) \to H_P^{m-j+1}\bigl(H_{R_+}^{s}(M)^\vee\bigr) \to \cdots ;\end{aligned}$$

(b) *if* $\dim R_0 = 0$*, then* $H_{R_+}^i(M) \cong H_Q^i(M)$ *for all* i*;*

(c) *if* $\dim R_0 = 1$*, then for all* i *we have the short exact sequence*

$$0 \to H_P^1\bigl(H_{R_+}^{i+1}(M)^\vee\bigr) \to H_Q^i(M)^\vee \to H_P^0\bigl(H_{R_+}^i(M)^\vee\bigr) \to 0.$$

Proof. We first prove (a). Our hypotheses imply the following exact sequences

$$0 \to E_{s,j}^\infty \to H_Q^{s-m+j}(M)^\vee \to E_{s-1,j+1}^\infty \to 0,$$

$$0 \to E_{s-1,j+1}^\infty \to E_{s-1,j+1}^2 \to E_{s,j-1}^2 \to E_{s,j-1}^\infty \to 0.$$

Putting these two exact sequences together we get the long exact sequence

$$\begin{aligned}\cdots \to E_{s-1,j+2}^2 \to E_{s,j}^2 \to H_Q^{s-m+j}(M)^\vee \to E_{s-1,j+1}^2 \to \\ E_{s,j-1}^2 \to H_Q^{s-m-1+j}(M)^\vee \to E_{s-1,j}^2 \to E_{s,j-2}^2 \to \ldots,\end{aligned}$$

which yields the desired exact sequence.

For the proof (b) we set $N = H_{R_+}^i(M)^\vee$. Since $\dim R_0 = 0$, it follows that N_k is a finitely generated R_0-module of finite length. Thus $H_P^{m-j}(N)_k = H_{P_0}^{m-j}(N_k) = 0$ for all k and $j < m$ and hence $E_{i,j}^2 = 0$ for all i and $j \neq m$. Therefore we have $N = H_P^0(N) \cong H_Q^i(M)^\vee$ for all i, and so $H_{R_+}^i(M) \cong H_Q^i(M)$ for all i. In order to prove (c) we again set $N = H_{R_+}^i(M)^\vee$. Since $\dim R_0 = 1$ it follows that $H_P^{m-j}(N)_k = H_{P_0}^{m-j}(N_k) = 0$ for all k and $j < m-1$ and hence $E_{i,j}^2 = 0$ for all i and $j \neq m, m-1$. Thus for all i we get the exact sequence

$$0 \to E_{i+1,m-1}^\infty \to H_Q^i(M)^\vee \to E_{i,m}^\infty \to 0.$$

Since $E_{i+1,m-1}^\infty = E_{i+1,m-1}^2$ and $E_{i,m}^\infty = E_{i,m}^2$ for all i, the result follows. ☐

As a simple consequence of Proposition 3.17 (b), (c) we obtain the following tameness result due to [2, Theorem 4.5].

Corollary 3.18. *Let* $\dim R_0 \leq 1$*. Then* $H_Q^i(M)$ *is tame for all* i*.*

Proof. First we assume that $\dim R_0 = 0$. Since any Artinian graded R-module is tame, the result follows from Proposition 3.17 (b).

Now we assume that $\dim R_0 = 1$. Let N be a finitely generated bigraded R-module. By Proposition 3.17 (c) it is enough to prove that there exists an integer j_0 such that for $i = 0, 1$ one has:

$$H^i_{P_0}(N_j) = 0 \quad \text{for all} \quad j \geq j_0, \quad \text{or} \quad H^i_{P_0}(N_j) \neq 0 \quad \text{for all} \quad j \geq j_0.$$

We set $t_0 = \lim_{j\to\infty} \operatorname{depth}_{R_0} N_j$ and $s_0 = \lim_{j\to\infty} \dim_{R_0} N_j$. Then $H^0_{P_0}(N_j) \neq 0$ for $j \gg 0$ if $t_0 = 0$, and $H^0_{P_0}(N_j) = 0$ for $j \gg 0$ if $t_0 \neq 0$. Similarly, $H^1_{P_0}(N_j) \neq 0$ for $j \gg 0$ if $s_0 = 1$, and $H^1_{P_o}(N_j)$ for $j \gg 0$ if $s_0 = 0$. ☐

Finally we want to mention two standard 5-term exact sequences arising from our spectral sequence.

Proposition 3.19. *There is a 5-term exact sequence for the corner* $(t, 0)$

$$\begin{aligned} H^{t+2-m}_Q(M)^\vee \to H^{m-2}_P\big(H^t_{R_+}(M)^\vee\big) \to H^m_P\big(H^{d+1}_{R_+}(M)^\vee\big) \to \\ H^{t+1-m}_Q(M)^\vee \to H^{m-1}_P\big(H^t_{R_+}(M)^\vee\big) \to 0, \end{aligned}$$

and a 5-term exact sequence for the corner (s, m)

$$\begin{aligned} H^s_Q(M)^\vee \to H^0_P\big(H^{s-1}_{R_+}(M)^\vee\big) \to H^2_P\big(H^s_{R_+}(M)^\vee\big) \to \\ H^{s-1}_Q(M)^\vee \to H^0_P\big(H^{s-1}_{R_+}(M)^\vee\big) \to 0. \end{aligned}$$

Proof. It can be seen form Figure 1 that one obtains the following exact sequences

$$0 \to E^\infty_{t+1,0} \to H^{t+1-m}_Q(M)^\vee \to E^\infty_{t,1} \to 0,$$

$$0 \to E^\infty_{t,2} \to E^2_{t,2} \xrightarrow{d^2_{t,2}} E^2_{t+1,0} \to E^\infty_{t+1,0} \to 0.$$

Putting these two exact sequences together we get the exact

$$H^{t+2-m}_Q(M)^\vee \to E^2_{t,2} \to E^2_{t+1,0} \to H^{t+1-m}_Q(M)^\vee \to E^2_{t,1} \to 0,$$

which yields the first 5-term sequence.

In the corner (s, m) we have the following exact sequences

$$0 \to E^\infty_{s,m-1} \to H^{s-1}_Q(M)^\vee \to E^\infty_{s-1,m} \to 0,$$

$$0 \to E^\infty_{s-1,m} \to E^2_{s-1,m} \xrightarrow{d^2_{s-1,m}} E^2_{s,m-2} \to E^\infty_{s,m-2} \to 0.$$

Putting these two exact sequences together we get the five term exact sequence

$$H^s_Q(M)^\vee \to E^2_{s-1,m} \to E^2_{s,m-2} \to H^{s-1}_Q(M)^\vee \to E^2_{s-1,m} \to 0,$$

which yields the second 5-term exact sequence. ☐

4 Local duality for graded modules

In this chapter we are going to establish a duality theorem for graded modules where the base ring is local. Let R be a standard graded K-algebra where $(R_0, \mathfrak{m}_0)$ is a local ring and M be a finitely generated graded R-module. We define the graded Matlis-dual of M to be $M^\vee$ where the kth graded component of $M^\vee$ is given by $\operatorname{Hom}_{R_0}(M_k, E_{R_0}(R_0/\mathfrak{m}_0))$.

We have the following duality theorem which is inspired by the local duality theorem in Chapter 3:

Theorem. Let R be a standard graded K-algebra where $(R_0, \mathfrak{m}_0)$ is a local ring. Let $R_+ = \bigoplus_{i>0} R_i$ be the graded irrelevant ideal of R, and M be a finitely generated graded R-module. Then there exists a convergent spectral sequence

$$E^2_{i,j} = H^{m-j}_{\mathfrak{m}_0}\big(H^i_{\mathfrak{m}}(M)^\vee\big) \underset{j}{\Longrightarrow} H^{i+j-m}_{R_+}(M)^\vee$$

of graded R-modules, where m is the minimal number of homogeneous generators of $\mathfrak{m}_0$ and $\mathfrak{m} = \mathfrak{m}_0 + R_+$ is the unique graded maximal ideal of R.

By using this theorem we obtain similar results as stated in Chapter 3.

4.1 Proof of the duality theorem

In a first step we prove the following

Lemma 4.1. *Let $R = R_0[x_1, \ldots, x_n]$ be a standard graded polynomial ring over R_0 where $(R_0, \mathfrak{m}_0)$ is a regular local ring and $R_+ = (x_1, \ldots, x_n)$ be the irrelevant graded ideal of R. Then we have the following isomorphism of graded R-modules*

$$H^m_{\mathfrak{m}_0}(\omega_R) \cong H^n_{R_+}(R)^\vee,$$

where ω_R is the graded canonical module of R and m is the minimal number of homogeneous generators of R_0.

Proof. First by graded version of [6, Corollary 3.3.21] we see that $\omega_R = (\omega_{R_0} \otimes_{R_0} R)(-n)$. Since R_0 is a regular ring it is a Gorenstein ring, and so $\omega_{R_0} = R_0$. Thus $\omega_R = R(-n)$ and hence $H^m_{\mathfrak{m}_0}(\omega_R) = H^m_{\mathfrak{m}_0}(R)(-n)$. Since the inclusion map $f : R_0 \to R$ is a flat homomorphism of Noetherian rings, the graded flat base change property of local cohomology yields the natural isomorphisms of R-modules

$$H^m_{\mathfrak{m}_0}(R)(-n) \cong \big(H^m_{\mathfrak{m}_0}(R_0) \otimes_{R_0} R\big)(-n) = H^m_{\mathfrak{m}_0}(R_0) \otimes_{R_0} R(-n).$$

(See [5, Theorem 4.3.2]). Applying the graded version local duality theorem we have

$$H^m_{\mathfrak{m}_0}(R_0) \cong \operatorname{Hom}_{R_0}\big(\operatorname{Ext}^{\dim R_0 - m}_{R_0}(R_0, R_0), E_{R_0}(R_0/\mathfrak{m}_0)\big).$$

Since R_0 is a regular ring, then $\operatorname{emb}\dim R_0 = \dim R_0 = m$. Thus $H^m_{\mathfrak{m}_0}(R_0) = R_0^\vee$. Therefore we conclude that

$$H^m_{\mathfrak{m}_0}(\omega_R) = R_0^\vee[x_1, \ldots, x_n](-n).$$

On the other hand, by [5, Example 12.4.1] we have the following graded R-isomorphism

$$H^n_{R_+}(R_0[x_1, \ldots, x_n]) \cong R_0[x_1^{-1}, \ldots, x_n^{-1}](n).$$

Therefore we have

$$\begin{aligned} H^n_{R_+}(R)^\vee &\cong \big(R_0[x_1^{-1}, \ldots, x_n^{-1}](n)\big)^\vee \\ &\cong R_0^\vee[x_1, \ldots, x_n](-n). \end{aligned}$$

The last isomorphism follows from the fact that for all j and $a > 0$ we have

$$\begin{aligned} \big[\big(R_0[x_1^{-1}, \ldots, x_n^{-1}](n)\big)^\vee\big]_j &\cong \big(R_0[x_1^{-1}, \ldots, x_n^{-1}]_{n-j}\big)^\vee \cong \big(\bigoplus_{|a|=n-j} R_0 x^{-a}\big)^\vee \\ &\cong \bigoplus_{|a|=-n+j} R_0^\vee x^a \cong R_0^\vee[x_1, \ldots, x_n]_{-n+j} \\ &\cong \big[R_0^\vee[x_1, \ldots, x_n](-n)\big]_j. \end{aligned}$$

□

Corollary 4.2. *Let F by a finitely generated graded free R-module, and set $F^* = \operatorname{Hom}_R(F, \omega_R)$. Then there exists a natural isomorphism of graded R-modules,*

$$H^m_{\mathfrak{m}_0}(F^*) \cong H^n_{R_+}(F)^\vee.$$

Proof. Let $F = \bigoplus_{k=1}^t R(-a_k)$. Thus $F^* = \bigoplus_{k=1}^t (\omega_R)(a_k)$ and hence by Lemma 4.1 we have

$$\begin{aligned} H^m_{\mathfrak{m}_0}(F^*) \cong \bigoplus_{k=1}^t H^m_{\mathfrak{m}_0}(\omega_R)(a_k) &\cong \bigoplus_{k=1}^t H^n_{R_+}(R)^\vee(a_k) \\ &\cong H^n_{R_+}\big(\bigoplus_{k=1}^t R(-a_k)\big)^\vee \\ &\cong H^n_{R_+}(F)^\vee. \end{aligned}$$

□

Corollary 4.2 can easily be extended as follows

Lemma 4.3. *Let $\mathbb{F}$ be a bounded complex of graded free R-modules. We set $\mathbb{F}^* = \operatorname{Hom}_R(\mathbb{F}, \omega_R)$. Then we have a functorial isomorphism*

$$H^m_{\mathfrak{m}_0}(\mathbb{F}^*) \cong H^n_{R_+}(\mathbb{F})^\vee$$

of complexes of graded modules.

Proof. This is proved the same way as Lemma 3.4. $\square$

Proposition 4.4. *Let $R = R_0[x_1, \dots, x_n]$ be a standard graded polynomial ring over R_0 where $(R_0, \mathfrak{m}_0)$ is a regular local ring, M be a finitely generated graded R-module and R_+ be the irrelevant graded ideal of R. Then we have the following convergent spectral sequence*

$$E^2_{i,j} = H^{m-j}_{\mathfrak{m}_0}\big(\operatorname{Ext}^{n+m-i}_R(M, \omega_R)\big) \underset{j}{\Longrightarrow} H^{i+j-m}_{R_+}(M)^\vee,$$

where m is the minimal number of homogeneous generators of $\mathfrak{m}_0$.

Proof. This is proved the same way as Proposition 3.5. $\square$

The following lemma is needed for the proof main theorem.

Lemma 4.5. *Let R be a standard graded K-algebra where $(R_0, \mathfrak{m}_0)$ is a local ring, M be a finitely generated R-module and I an ideal of R. Then for all i*

$$H^i_I(M)^\vee \cong H^i_{\widehat{I}}(\widehat{M})^\vee,$$

where $\widehat{M}$ denotes the $\mathfrak{m}_0$-adic completion of M and $\widehat{I} = I\widehat{R}$.

Proof. We first observe that $\widehat{M} = \bigoplus_j \widehat{M_j}$ where $\widehat{M_j}$ is the $\mathfrak{m}_0$-adic completion of the finitely generated R_0-module M_i. In particular, we have that $\widehat{M_j} = \widehat{R}_0 \otimes_{R_0} M_j$ for all j. Therefore $\widehat{M} = \bigoplus_j \widehat{R}_0 \otimes_{R_0} M_j = \widehat{R}_0 \otimes_{R_0} M$. By the graded flat base change property of local cohomology we have

$$H^i_I(M) \otimes_R \widehat{R} \cong H^i_{\widehat{I}}(M \otimes_R \widehat{R}).$$

Since $\widehat{R} \cong \widehat{R}_0 \otimes_{R_0} R$, then we have

$$H^i_I(M) \otimes_R \widehat{R} \cong H^i_I(M) \otimes_{R_0} \widehat{R}_0 \quad \text{and} \quad M \otimes_R \widehat{R} \cong M \otimes_{R_0} \widehat{R}_0 \cong \widehat{M}.$$

Therefore

$$H^i_I(M)_j \otimes_{R_0} \widehat{R}_0 \cong H^i_{\widehat{I}}(\widehat{M})_j \quad \text{for all} \quad j.$$

We set $K = R_0/\mathfrak{m}_0$. For all $j \in \mathbb{Z}$ we have

$$\big[H^k_{\widehat{I}}(\widehat{M})^\vee\big]_j \cong \operatorname{Hom}_{\widehat{R}_0}\big(H^k_{\widehat{I}}(\widehat{M})_{-j}, E_{\widehat{R}_0}(K)\big) \cong \operatorname{Hom}_{\widehat{R}_0}\big(H^k_I(M)_{-j} \otimes_{R_0} \widehat{R}_0, E_{\widehat{R}_0}(K)\big).$$

By [6, Exercise 3.2.14] as an $\widehat{R}_0$-module we have $E_{R_0}(K) \cong E_{\widehat{R}_0}(K)$ and for any finitely generated R_0-module N there exists a natural isomorphism

$$\operatorname{Hom}_{R_0}(N, E_{R_0}(K)) \cong \operatorname{Hom}_{\widehat{R}_0}(\widehat{N}, E_{R_0}(K)).$$

Since $H^k_I(M)_j$ is a finitely generated R_0-module, therefore we have

$$\big[H^k_{\widehat{I}}(\widehat{M})^\vee\big]_j \cong \operatorname{Hom}_{R_0}\big(H^k_I(M)_{-j}, E_{R_0}(K)\big) \cong \big[H^k_I(M)^\vee\big]_j \quad \text{for all} \quad j \in \mathbb{Z},$$

as required. $\square$

Now our main theorem is an easy consequence of Proposition 4.4:

Proof of Theorem. We denote by $\widehat{\mathfrak{m}}$ the graded maximal ideal of $\widehat{R} = \bigoplus_{i\geq 0} \widehat{R}_i$. In other words, $\widehat{\mathfrak{m}} = \widehat{\mathfrak{m}}_0 \oplus \bigoplus_{i>0} \widehat{R}_i$. By Lemma 4.5 we have $H^i_{\mathfrak{m}}(M)^\vee \cong H^i_{\widehat{\mathfrak{m}}}(\widehat{M})^\vee$. We set $N = H^i_{\mathfrak{m}}(M)^\vee$. Since N is a graded $\widehat{R}$-module, the natural homomorphism $R_0 \to \widehat{R_0}$ implies that $H^i_{\mathfrak{m}_0}(N) \cong H^i_{\widehat{\mathfrak{m}}_0}(N)$. Thus

$$H^i_{\mathfrak{m}_0}\big(H^i_{\mathfrak{m}}(M)^\vee\big) \cong H^i_{\widehat{\mathfrak{m}}_0}\big(H^i_{\widehat{\mathfrak{m}}}(\widehat{M})^\vee\big).$$

By using again Lemma 4.5 we have $H^{i+j-m}_{R_+}(M)^\vee = H^{i+j-m}_{\widehat{R}_+}(\widehat{M})^\vee$. Thus we may assume that R_0 is complete. By Cohen's theorem we may choose a regular local ring $(S_0, \mathfrak{n}_0)$ where R_0 is a homomorphic image of S_0 and $\operatorname{emb\,dim} S_0 = \operatorname{emb\,dim} R_0 = m$. It follows that R is the homomorphic image of a graded polynomial ring $S = S_0[x_1, \ldots, x_n]$ where $(S_0, \mathfrak{n}_0)$ is a regular local ring. We set $\mathfrak{n} = \mathfrak{n}_0 + S_+$, and consider M as a finitely generated graded S-module. Then by the graded local duality theorem we have

$$\operatorname{Ext}^{m+n-i}_S(M, \omega_S) \cong H^i_{\mathfrak{n}}(M)^\vee.$$

Since $H^i_{\mathfrak{n}}(M) \cong H^i_{\mathfrak{m}}(M)$ it follows that

$$H^{m-j}_{\mathfrak{m}_0}\big(H^i_{\mathfrak{m}}(M)^\vee\big) \cong H^{m-j}_{\mathfrak{m}_0}\big(\operatorname{Ext}^{m+n-i}_S(M, \omega_S)\big).$$

Therefore, Proposition 4.4 yields the desired convergent spectral sequence. □

Corollary 4.6. *Let R be a standard graded d-dimensional Cohen-Macaulay K-algebra where $(R_0, \mathfrak{m}_0)$ is a local ring and R_+ be the irrelevant graded ideal of R, and let M be a finitely generated graded R-module. Then there exists a convergent spectral sequence*

$$E^2_{i,j} = H^{m-j}_{\mathfrak{m}_0}\big(\operatorname{Ext}^{d-i}_R(M, \omega_R)\big) \underset{j}{\Longrightarrow} H^{i+j-m}_{R_+}(M)^\vee$$

of graded R-modules where m is the minimal number of homogeneous generators of $\mathfrak{m}_0$.

Proof. The assertion follows from our main theorem by using the fact that $H^i_{\mathfrak{m}}(M)^\vee = \operatorname{Ext}^{d-i}_R(M, \omega_R)$. □

Remark 4.7. By arguments which are analogous to the bigraded case one obtains similar applications as in Chapter 3.

5 Tameness of local cohomology of monomial ideals with respect to monomial prime ideals

In this chapter we give an explicit combinatorial formulas for the Hilbert series of local cohomology modules of the rings defined by monomial relations, with respect to a monomial prime ideal. In Section 5.1, as a generalization of Hochster's formula [6], we compute the Hilbert series of local cohomology of the Stanley-Reisner ring $K[\Delta]$ with respect to a monomial prime ideal and then we give a explicit formula for the K-dimension of the bigraded components of the local cohomology modules. Using this formula we deduce that the local cohomology of $K[\Delta]$ with respect to a monomial prime ideal is always tame.

In [26] Takayama generalized Hochster's formula to any graded monomial ideal which is not necessarily squarefree. In Section 5.2, as a generalization of Takayama's result we compute the Hilbert series of local cohomology of monomial ideals with respect to monomial prime ideals and observe that again all these modules are tame. The result proved in this chapter is surprising because recently, Cutkosky and Herzog [10] gave an example which shows that in general not all local cohomology modules are tame.

5.1 Local cohomology of Stanley-Reisner rings with respect to monomial prime ideals

Let K be a field and let $S = K[Y_1, \dots, Y_r]$ be a polynomial ring with the standard grading. For a squarefree monomial ideal $I \subset S$ we set $R = S/I$. We denote by y_i the residue classes of indeterminates Y_i in R for $i = 1, \dots, r$. Thus we have $R = K[y_1, \dots, y_r]$. We may view R as the Stanley-Reisner ring of some simplicial complex Δ with vertices $\{w_1, \dots, w_r\}$.

Let P be any monomial prime ideal of R. We may assume that $P = (y_1, \dots, y_n)$ for some integer $n \leq r$. After this choice of P we view R as a bigraded K-algebra. We rename some of the variables, and set $x_i = y_{n+i}$ for $i = 1, \dots, m$ where $m = r-n$, and assign the following bidegrees: $\deg x_i = (1,0)$ for $i = 1, \dots, m$ and $\deg y_j = (0,1)$ for $j = 1, \dots, n$. We decompose the vertex set of the corresponding simplicial complex Δ accordingly, so that Δ has vertices $\{v_1, \dots, v_m, w_1, \dots, w_n\}$ where vertices $V = \{v_1, \dots, v_m\}$ and $W = \{w_1, \dots, w_n\}$ correspond to the variables of $x_1, \dots, x_m$ and $y_1, \dots, y_n$, respectively. By [5, Theorem 5.1.19] we have

$$H^i_P(R) \cong H^i(C^\bullet) \quad \text{for all} \quad i \geq 0,$$

where $C^\bullet$ is the Čech complex

$$C^\bullet : 0 \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

with

$$C^t = \bigoplus_{1 \leq i_1 < \cdots < i_t \leq n} R_{y_{i_1} \cdots y_{i_t}},$$

and whose differential is composed of the maps

$$(-1)^{s-1} nat : R_{y_{i_1} \cdots y_{i_t}} \longrightarrow (R_{y_{i_1} \cdots y_{i_t}})_{y_{j_s}},$$

if $\{i_1, \ldots, i_t\} = \{j_1, \ldots, \hat{j}_s, \ldots, j_{t+1}\}$ and 0 otherwise. Note that $C^\bullet$ is a $\mathbb{Z}^m \times \mathbb{Z}^n$-bigraded complex. For $(a,b) \in \mathbb{Z}^m \times \mathbb{Z}^n$ and $y = y_{j_1} \ldots y_{j_s}$ with $1 \leq j_1 < \cdots < j_s \leq n$ one defines a $\mathbb{Z}^m \times \mathbb{Z}^n$-bigrading on R_y by setting

$$(R_y)_{(a,b)} = \{r/y^l : \deg r - l \deg y = (a,b)\}. \tag{15}$$

Here r is a bihomogeneous element in R, $l \in \mathbb{Z}$ and deg denotes the multi-bidegree. Given $F = \{w_{j_1}, \ldots, w_{j_s}\} \subseteq W$ and $b \in \mathbb{Z}^n$. We set

$$G_b = \{w_j : w_1 \leq w_j \leq w_n, b_j < 0\} \quad \text{and} \quad H_b = \{w_j : w_1 \leq w_j \leq w_n, b_j > 0\}.$$

The support of b is the set $\operatorname{supp} b = \{w_j : w_1 \leq w_j \leq w_n, b_j \neq 0\}$. Note that $\operatorname{supp} b = G_b \cup H_b$.

We also set $N_a = \{v_i : v_1 \leq v_i \leq v_m, a_i \neq 0\} = \operatorname{supp} a$ for $a \in \mathbb{Z}^m$ and denote by $\mathbb{Z}^m_+$ and $\mathbb{Z}^n_-$ the sets of $\{a \in \mathbb{Z}^m : a_i \geq 0 \quad \text{for} \quad i = 1, \ldots, m\}$ and $\{b \in \mathbb{Z}^n : b_i \leq 0 \quad \text{for} \quad i = 1, \ldots, n\}$, respectively. With the notation introduced one has

Lemma 5.1. *The following statements hold:*

(a) $\dim_K (R_y)_{(a,b)} \leq 1$, *for all* $a \in \mathbb{Z}^m$ *and* $b \in \mathbb{Z}^n$.

(b) $(R_y)_{(a,b)} \cong K$, *if and only if* $F \supset G_b$, $F \cup H_b \cup N_a \in \Delta$ *and* $a \in \mathbb{Z}^m_+$.

Proof. As explained before, we may view the standard graded polynomial ring S as a standard bigraded polynomial ring and then $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ with $m + n = r$ is also naturally bigraded. Thus part (a) follows from [6, Lemma 5.3.6 (a)]. For the proof (b) we set $c = (a,b)$. By [6, Lemma 5.3.6 (b)] we have $F \supset G_c$ and $F \cup H_c \in \Delta$. Thus (15) implies that $a \in \mathbb{Z}^m_+$ and hence $G_c = G_b$. We also note that $H_c = H_b \cup N_a$. □

As a consequence of Lemma 5.1 for $a \in \mathbb{Z}^m_+$, $b \in \mathbb{Z}^n$ and $i \in \mathbb{Z}$ we observe that $(C^i)_{(a,b)}$ has the following K- basis:

$$\{b_F : F \supset G_b, F \cup H_b \cup N_a \in \Delta, |F| = i\}.$$

Therefore since $C^\bullet$ is $\mathbb{Z}^m \times \mathbb{Z}^n$-bigraded complex one obtains for each $(a,b) \in \mathbb{Z}^m \times \mathbb{Z}^n$ a complex

$$(C^\bullet)_{(a,b)} : 0 \to (C^0)_{(a,b)} \to (C^1)_{(a,b)} \to \cdots \to (C^n)_{(a,b)} \to 0,$$

of finite dimensional K-vector spaces

$$(C^i)_{(a,b)} = \bigoplus_{\substack{F \supset G_b \\ F \cup H_b \cup N_a \in \Delta \\ |F| = i}} K b_F.$$

The differential $\partial : (C^i)_{(a,b)} \longrightarrow (C^{i+1})_{(a,b)}$ is given by

$$\partial(b_F) = \sum(-1)^{\partial(F,F')} b_{F'}$$

where the sum is taken over all F' such that $F' \supset F$, $F' \cup H_b \cup N_a \in \Delta$ and $|F'| = i+1$, and where $\partial(F, F') = s$ for $F' = [w_0, \ldots, w_i]$ and $F = [w_0, \ldots, \hat{w}_s, \ldots, w_i]$. Then we describe the (a, b)th component of the local cohomology in terms of this subcomplex:

$$H^i_P(K[\Delta])_{(a,b)} = H^i(C^\bullet)_{(a,b)} = H^i(C^\bullet_{(a,b)}). \tag{16}$$

Let Δ be simplicial complex with vertex set V. An orientation on Δ is a linear order on V. A simplicial complex together with an orientation is an oriented simplicial complex. Let Δ be a oriented simplicial complex of dimension $d-1$, and $F \in \Delta$ an i-face. We write $F = [v_0, \ldots, v_i]$ if $F = \{v_0, \ldots, v_i\}$ with $v_0 < v_1 < \cdots < v_i$, and $F = [\quad]$ if $F = \emptyset$. We define the augumented oriented chain complex of Δ:

$$\widetilde{\mathcal{C}}(\Delta) : 0 \to \mathcal{C}_{d-1} \xrightarrow{\partial} \mathcal{C}_{d-2} \to \cdots \to \mathcal{C}_0 \xrightarrow{\partial} \mathcal{C}_{-1} \to 0$$

where

$$\mathcal{C}_i = \bigoplus_{F \in \Delta, \dim F = i} \mathbb{Z}F \quad \textit{and} \quad \partial F = \sum_{j=0}^{i} (-1)^j F_j$$

for all $F \in \Delta$. Here we set $F_j = [v_0, \ldots, \widehat{v}_j, \ldots, v_i]$ for $F = [v_0, \ldots, v_i]$. Now for an abelian group G, we define the ith reduced simplicial homology $\widetilde{H}_i(\Delta; G)$ of Δ to be the ith homology of the complex $\widetilde{\mathcal{C}}(\Delta) \otimes G$ for all i. See [6, Section 5.3] for details. For an abelian group G, the ith reduced simplicial cohomology of Δ with values in G is defined to be

$$\widetilde{H}^i(\Delta; G) = H^i\big(\operatorname{Hom}_{\mathbb{Z}}(\widetilde{\mathcal{C}}(\Delta), G)\big) \quad \text{for all} \quad i. \tag{17}$$

For $W \subseteq V$ we denote by Δ_W the simplicial complex restricted to W, i.e., the simplicial complex consisting of all faces $F \in \Delta$ whose vertices belong to W.

Now in order to compute $H^i(C^\bullet_{(a,b)})$, we prove the following

Lemma 5.2. *For all $a \in \mathbb{Z}^m_+$ and $b \in \mathbb{Z}^n$ there exists an isomorphism of complexes*

$$(C^\bullet)_{(a,b)} \longrightarrow \operatorname{Hom}_{\mathbb{Z}}\big(\widetilde{\mathcal{C}}(\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a)_W[-j-1]; K\big), \quad j = |G_b|$$

Proof. The assignment $F \mapsto F' = F - G_b$ establishes a bijection between the set

$$\beta = \{F \in \Delta_W : F \supset G_b, F \cup H_b \cup N_a \in \Delta, |F| = i\}$$

and the set

$$\beta' = \{F' \in \Delta_W : F' \in (\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a)_W, |F'| = i - j\}.$$

Here $F' \in (\mathrm{lk}_{\mathrm{st}\, H_b} G_b \cup N_a)_W$, since $F' \cap (G_b \cup N_a) = \emptyset$ and $F' \cup (G_b \cup N_a) \in \mathrm{st}\, H_b$. Therefore we see that

$$\alpha^i : (C^i)_{(a,b)} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(\widetilde{\mathcal{C}}(\mathrm{lk}_{\mathrm{st}\, H_b} G_b \cup N_a)_{i-j-1}; K\right), \quad b_F \mapsto \varphi_{F-G_b}$$

is an isomorphism of vector spaces. Here $\varphi_{F'}$ is defined by

$$\varphi_{F'}(F'') = \begin{cases} 1 & \text{if } F = F'', \\ 0 & \text{otherwise.} \end{cases}$$

□

As a generalization of Hochster's formula [6, Theorem 5.3.8] we prove the following

Theorem 5.3. *Let $I \subset S = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be a squarefree monomial ideal with the natural $\mathbb{Z}^m \times \mathbb{Z}^n$-bigrading. Then the bigraded Hilbert series of the local cohomology modules of $R = S/I = K[\Delta]$ with respect to the $\mathbb{Z}^m \times \mathbb{Z}^n$-bigrading is given by*

$$H_{H^i_P(K[\Delta])}(\mathbf{s}, \mathbf{t}) = \sum_{F \in \Delta_W} \sum_{G \subset V} \dim_K \widetilde{H}_{i-|F|-1}\big((\mathrm{lk}\, F \cup G)_W; K\big) \prod_{v_i \in G} \frac{s_i}{1-s_i} \prod_{w_j \in F} \frac{t_j^{-1}}{1-t_j^{-1}}$$

where $\mathbf{s} = (s_1, \ldots, s_m)$, $\mathbf{t} = (t_1, \ldots, t_n)$, $P = (y_1, \ldots, y_n)$ and Δ is the simplicial complex corresponding to the Stanley-Reisner ring $K[\Delta]$.

Proof. By (16), Lemma 5.2 and (17) we observe that there are isomorphisms of bigraded K-vector spaces

$$\begin{aligned} H^i_P(K[\Delta])_{(a,b)} &\cong H^i\big(\mathrm{Hom}_{\mathbb{Z}}\big(\widetilde{\mathcal{C}}((\mathrm{lk}_{\mathrm{st}\, H_b} G_b \cup N_a)_W)[-j-1]; K\big)\big), \quad j = |G_b| \\ &= H^{i-|G_b|-1}\big(\mathrm{Hom}_{\mathbb{Z}}\big(\widetilde{\mathcal{C}}((\mathrm{lk}_{\mathrm{st}\, H_b} G_b \cup N_a)_W); K\big)\big) \\ &= \widetilde{H}^{i-|G_b|-1}\big((\mathrm{lk}_{stH_b} G_b \cup N_a)_W; K\big), \end{aligned}$$

and therefore by [6, Exercise 5.3.11] we have

$$\dim_K H^i_P(K[\Delta])_{(a,b)} = \dim_K \widetilde{H}_{i-|G_b|-1}\big((\mathrm{lk}_{stH_b} G_b \cup N_a)_W; K\big). \tag{18}$$

If $H_b \neq \emptyset$ by [6, Lemma 5.3.5], $\mathrm{lk}_{stH_b} G_b \cup N_a$ is acyclic, and so $\widetilde{H}_{i-|G_b|-1}\big((\mathrm{lk}_{stH_b} G_b \cup N_a)_W; K\big) = 0$ for all i. If $H_b = \emptyset$, then $stH_b = \Delta$, and so $\mathrm{lk}_{stH_b} G_b \cup N_a = \mathrm{lk}\, G_b \cup N_a$. Thus in this case $\mathrm{supp}(b) = G_b$. We also note that $H_b = \emptyset$ if and only if $b \in \mathbb{Z}^n_-$. In order to simplify notation we will write $s(a)$ and $s(b)$ for the support of $a \in \mathbb{Z}^m_+$ and $b \in \mathbb{Z}^n_-$, respectively and set

$$d(i, s(b), s(a)) = \dim_K \widetilde{H}_{i-|s(b)|-1}\big(\big(\mathrm{lk}_{stH_b} s(b) \cup s(a)\big)_W; K\big).$$

Using these facts and (18) we have

$$\begin{aligned}
H_{H^i_P(K[\Delta])}(\mathbf{s},\mathbf{t}) &= \sum_{a\in\mathbb{Z}^m_+, b\in\mathbb{Z}^n_-} \dim_K H^i_P(K[\Delta])_{(a,b)}\mathbf{s}^a\mathbf{t}^b \\
&= \sum_{b\in\mathbb{Z}^n_-}\left(\sum_{a\in\mathbb{Z}^m_+} d(i,s(b),s(a))\mathbf{s}^a\right)\mathbf{t}^b \\
&= \sum_{b\in\mathbb{Z}^n_-}\left(\sum_{G\subset V}\sum_{\substack{s(a)=G\\ a\in\mathbb{Z}^m_+}} d(i,s(b),s(a))\mathbf{s}^a\right)\mathbf{t}^b \\
&= \sum_{b\in\mathbb{Z}^n_-}\left(\sum_{G\subset V} d(i,s(b),s(a))\sum_{s(a)=G}\mathbf{s}^a\right)\mathbf{t}^b \\
&= \sum_{b\in\mathbb{Z}^n_-}\left(\sum_{G\subset V} d(i,s(b),G)\prod_{v_i\in G}\frac{s_i}{1-s_i}\right)\mathbf{t}^b \\
&= \sum_{F\in\Delta_W}\sum_{\substack{s(b)=F\\ b\in\mathbb{Z}^n_-}}\left(\sum_{G\subset V} d(i,s(b),G)\prod_{v_i\in G}\frac{s_i}{1-s_i}\right)\mathbf{t}^b \\
&= \sum_{F\in\Delta_W}\sum_{G\subset V} d(i,F,G)\prod_{v_i\in G}\frac{s_i}{1-s_i}\prod_{w_j\in F}\frac{t_j^{-1}}{1-t_j^{-1}},
\end{aligned}$$

as desired. Here $\mathbf{s}^a = s_1^{a_1}\dots s_m^{a_m}$ for $a=(a_1,\dots,a_m)$ and $\mathbf{t}^b = t_1^{b_1}\dots t_n^{b_n}$ for $b=(b_1,\dots,b_n)$. We also used the fact that $\sum_{s(a)=G}\mathbf{s}^a = 1$ for $G=\emptyset$ and $\sum_{s(a)=G}\mathbf{s}^a = \prod_{v_i\in G}\frac{s_i}{1-s_i}$ for $G\neq\emptyset$. □

We observe that Hochster's formula [6, Theorem 5.3.8] easily follows from Theorem 5.3. In fact, if we assume that $m=0$, then $G=\emptyset$, $(\operatorname{lk} F\cup G)_W = \operatorname{lk} F$ and $\prod_{v_i\in G} s_i/(1-s_i) = 1$. Moreover, we may consider $\deg Y_j = 1$ for all j. Therefore we get the Hochster's formula.

In view of Theorem 5.3 and (18) we get the following isomorphism of K-vector spaces

Corollary 5.4. *For all $a\in\mathbb{Z}^m_+$ and $b\in\mathbb{Z}^n_-$ we have*

$$H^i_P(K[\Delta])_{(a,b)} \cong \widetilde{H}^{i-|F|-1}\big((\operatorname{lk} F\cup G)_W;K\big),$$

where $F=\operatorname{supp} b$ and $G=\operatorname{supp} a$.

Corollary 5.5. *With the notation of Theorem* 5.3 *one has*

$$H_{H^i_P(K[\Delta])}(\mathbf{s},\mathbf{t}) = H_{H^i_{\mathfrak{m}}(K[\Delta_W])}(\mathbf{t}) + \sum_{F\in\Delta_W}\sum_{\substack{G\subset V\\ G\neq\emptyset}} d(i,F,G)\prod_{v_i\in G}\frac{s_i}{1-s_i}\prod_{w_j\in F}\frac{t_j^{-1}}{1-t_j^{-1}},$$

where $H_{H^i_{\mathfrak{m}}(K[\Delta_W])}(\mathbf{t})$ *is the Hilbert series of ith ordinary local cohomology of* $K[\Delta_W]$ *with respect to the maximal ideal* $\mathfrak{m} = (y_1, \dots, y_n)$ *and where*

$$d(i, F, G) = \dim_K \widetilde{H}_{i-|F|-1}\big((\operatorname{lk} F \cup G)_W; K\big).$$

Proof. By Theorem 5.3 we may write

$$H_{H^i_P(K[\Delta])}(\mathbf{s}, \mathbf{t}) = \sum_{F \in \Delta_W} d(i, F, \emptyset) \prod_{w_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}} \quad +$$
$$\sum_{F \in \Delta_W} \sum_{\substack{G \subset V \\ G \neq \emptyset}} d(i, F, G) \prod_{v_i \in G} \frac{s_i}{1 - s_i} \prod_{w_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}}.$$

Since

$$d(i, F, \emptyset) = \dim_K \widetilde{H}_{i-|F|-1}\big((\operatorname{lk} F \cup \emptyset)_W; K\big) = \dim_K \widetilde{H}_{i-|F|-1}(\operatorname{lk}_{\Delta_W} F; K),$$

Hochster's formula [6, Theorem 5.3.8] implies the desired equality. ☐

In view of Corollary 5.5 we immediately obtain

Corollary 5.6. $H^i_P(K[\Delta]) \neq 0$ *for* $i = \operatorname{depth} K[\Delta_W]$ *and* $i = \dim K[\Delta_W]$.

We are interested in the Hilbert series of $H^i_P(K[\Delta])$ as a $\mathbb{Z} \times \mathbb{Z}$-bigraded algebra. Note that for all $k, j \in \mathbb{Z}$ we have

$$H^i_P(K[\Delta])_{(k,j)} = \bigoplus_{\substack{a \in \mathbb{Z}^m, |a| = k \\ b \in \mathbb{Z}^n, |b| = j}} H^i_P(K[\Delta])_{(a,b)}, \tag{19}$$

where $|a| = \sum_{i=1}^m a_i$ for $a = (a_1, \dots, a_m)$ and $|b| = \sum_{i=1}^n b_i$ for $b = (b_1, \dots, b_n)$.

Using this observation we obtain

Proposition 5.7. *For all i and* $k, j \in \mathbb{Z}$ *one has*

$$\dim_K H^i_P(K[\Delta])_{(k,j)} = \sum_{\substack{F \in \Delta_W \\ G \subset V}} d(i, F, G) \binom{k-1}{|G|-1} \binom{-j-1}{|F|-1},$$

where

$$d(i, F, G) = \dim_K \widetilde{H}_{i-|F|-1}\big((\operatorname{lk} F \cup G)_W; K\big).$$

Proof. We set $|G| = g$ and $|F| = f$. In view of (19) it follows that the Hilbert series of $H^i_P(K[\Delta])$ with respect to the $\mathbb{Z} \times \mathbb{Z}$-bigrading is obtained from Theorem 5.3 by replacing all s_i and t_j by s and t, respectively. Thus we have

$$H_{H^i_P(K[\Delta])}(s, t) = \sum_{F \in \Delta_W} \sum_{G \subset V} d(i, F, G) (\frac{s}{1-s})^g (\frac{t^{-1}}{1-t^{-1}})^f.$$

We note that

$$\frac{1}{(1-s)^i} = \sum_{r=0}^{\infty} \binom{i+r-1}{i-1} s^r \quad \text{for all} \quad i > 0. \tag{20}$$

Expanding $(\frac{s}{1-s})^g$ for $g = 0$ and $g = 1$ and comparing coefficients with (20) we are forced to make the following convention: $\binom{-1}{-1} = 1$, $\binom{i}{-1} = 0$ for all $i \geq 0$ and $\binom{i}{0} = 1$ for all $i \geq 0$. Thus we have

$$H_{H^i_P(K[\Delta])}(s,t) = \sum_{F\in\Delta_W} \sum_{G\subset V} d(i,F,G) \sum_{r=0}^{\infty} \binom{g+r-1}{g-1} s^{r+g} \sum_{h=0}^{\infty} \binom{f+h-1}{f-1} t^{-f-h}.$$

We set $k = r + g$ and $j = -f - h$. Thus $r = k - g$ and $h = -j - f$. Therefore for all k and j with $g \leq k$ and $0 \leq f \leq -j$ the desired formula follows. ☐

Corollary 5.8. *For all i and $j \in \mathbb{Z}$ one has*

$$\dim_K H^i_P(K[\Delta])_{(0,j)} = \dim_K H^i_{\mathfrak{m}}(K[\Delta_W])_j,$$

where $H^i_{\mathfrak{m}}(K[\Delta_W])$ is the ith ordinary local cohomology of $K[\Delta_W]$ with respect to $\mathfrak{m} = (y_1, \ldots, y_n)$.

Proof. By Proposition 5.7 and the fact that $(\operatorname{lk} F)_W = \operatorname{lk}_{\Delta_W} F$ we have

$$\begin{aligned}
\dim_K H^i_P(K[\Delta])_{(0,j)} &= \sum_{F\in\Delta_W} \dim_K \widetilde{H}_{i-|F|-1}\big((\operatorname{lk} F)_W; K\big) \binom{-j-1}{|F|-1} \\
&= \sum_{F\in\Delta_W} \dim_K \widetilde{H}_{i-|F|-1}(\operatorname{lk}_{\Delta_W} F; K) \binom{-j-1}{|F|-1} \\
&= \dim_K H^i_{\mathfrak{m}}(K[\Delta_W])_j.
\end{aligned}$$

The last equality follows from Hochster's formula. ☐

For all $j \in \mathbb{Z}$, we set

$$H^i_P(K[\Delta])_j = \bigoplus_k H^i_P(K[\Delta])_{(k,j)},$$

and consider $H^i_P(K[\Delta])_j$ as a finitely generated graded R_0-module. In the following we show that the Krull-dimension of $H^i_P(K[\Delta])_j$ is constant for $j \ll 0$.

Theorem 5.9. *For all i there exists an integer j_0 such that for $j \leq j_0$, the Krull-dimension $\dim H^i_P(K[\Delta])_j$ is constant.*

Proof. By Proposition 5.7 and using (20) the $\mathbb{Z}$-graded Hilbert series of $H^i_P(K[\Delta])_j$ is given by

$$\begin{aligned}
H_{H^i_P(K[\Delta])_j}(s) &= \sum_{k=0}^{\infty} \dim_K H^i_P(K[\Delta])_{(k,j)} s^k \\
&= \sum_{k=0}^{\infty} \sum_{\substack{F\in\Delta_W \\ G\subset V}} d(i,F,G) \binom{k-1}{|G|-1} \binom{-j-1}{|F|-1} s^k \\
&= \sum_{\substack{F\in\Delta_W \\ G\subset V, |G|=0}} d(i,F,G) \binom{-j-1}{|F|-1} \sum_{r=-1}^{\infty} \binom{r}{-1} s^{r+1} \\
&+ \sum_{\substack{F\in\Delta_W \\ G\subset V, |G|=1}} d(i,F,G) \binom{-j-1}{|F|-1} \sum_{r=0}^{\infty} \binom{r}{0} s^{r+1} \\
+\cdots&+ \sum_{\substack{F\in\Delta_W \\ G\subset V, |G|=m}} d(i,F,G) \binom{-j-1}{|F|-1} \sum_{i=m-1}^{\infty} \binom{r}{m-1} s^{r+1} \\
&= A_0(j) + \frac{A_1(j)s}{1-s} + \frac{A_2(j)s^2}{(1-s)^2} + \cdots + \frac{A_m(j)s^m}{(1-s)^m} \\
&= \frac{\sum_{r=0}^{m} A_r(j)(1-s)^{m-r}s^r}{(1-s)^m},
\end{aligned}$$

where

$$A_r(j) = \sum_{\substack{F\in\Delta_W \\ G\subset V, |G|=r}} d(i,F,G) \binom{-j-1}{|F|-1}.$$

We may write

$$\sum_{r=0}^{m} A_r(j)(1-s)^{m-r}s^r = \sum_{r=0}^{m} B_r(j)s^r,$$

where $B_r(j)$ is a polynomial with coefficients in $\mathbb{Q}$ of degree at most $n-1$. Therefore we have

$$H_{H^i_P(K[\Delta])_j}(s) = \frac{Q_j(s)}{(1-s)^m}, \tag{21}$$

where $Q_j(s) = \sum_{r=0}^{m} B_r(j)s^r$. We denote by $Q_j(s)^{(k)}$ the kth derivative of $Q_j(s)$ as a function in s and set $R_k(j) = [Q_j(s)^{(k)}](1)$ which is of course a polynomial in j. Here we distinguish two cases: First suppose that $R_k = 0$ for all $k \geq 0$. Then the Taylor expansion of $Q_j(s)$

$$Q_j(s) = \frac{R_0(j)}{0!} + \frac{R_1(j)}{1!}(1-s) + \frac{R_2(j)}{2!}(1-s)^2 + \ldots$$

implies that $Q_j(s) = 0$ for all j. Thus we see that $R_k = 0$ for all $k \geq 0$, which is equivalent to say that $Q_j(s) = 0$ for all j, and which in turn implies that $H^i_P(K[\Delta])_j = 0$ for all j, and we set $\dim H^i_P(K[\Delta])_j = -\infty$. Now we assume that not all $R_k = 0$, and define

$$c = \min\{i : R_i \neq 0\}.$$

Thus $R_k(j) = 0$ for all j and all $k < c$. Since R_c has only finitely many zeroes, it follows that $R_c(j) \neq 0$ for $j \ll 0$, i.e., there exists an integer j_0 such that $R_c(j) \neq 0$ for $j \leq j_0$. Therefore

$$R_k(j) = 0 \quad \text{for} \quad j \leq j_0, \quad \text{if} \quad k < c,$$

and

$$R_k(j) \neq 0 \quad \text{for} \quad j \leq j_0, \quad \text{if} \quad k = c.$$

Thus for $j \leq j_0$ we may write $Q_j(s) = (1-s)^c \widetilde{Q}_j(s)$ where $\widetilde{Q}_j(s)$ is a polynomial in s with $\widetilde{Q}_j(1) \neq 0$. Therefore by (21) and [6, Corollary 4.1.8] we have

$$\dim H^i_P(K[\Delta])_j = m - c \quad \text{for all} \quad j \leq j_0,$$

as desired. □

We recall the following definition as stated in Chapter 1.

Definition 5.10. Let R be a positively graded Noetherian ring. A graded R-module N is called *tame*, if there exists an integer j_0 such that either

$$N_j = 0 \quad \text{for all} \quad j \leq j_0, \quad \text{or} \quad N_j \neq 0 \quad \text{for all} \quad j \leq j_0.$$

Corollary 5.11. *Let $I \subset S = K[Y_1, \ldots, Y_r]$ be a graded squarefree monomial ideal and let Δ be the simplicial complex such that $K[\Delta] = S/I$. Let P be a monomial prime ideal of $K[\Delta]$. Then for all i, the local cohomology modules of $H^i_P(K[\Delta])$ are tame.*

Proof. The assertion follows from Theorem 5.9. □

We end this section with the following example:

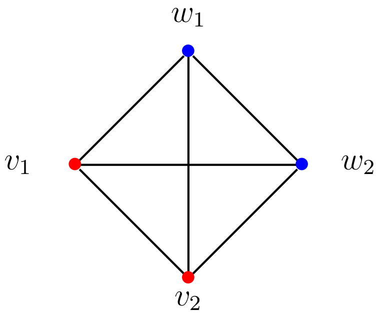
Example 5.12. Let $S = K[X_1, X_2, Y_1, Y_2]$ be a standard bigraded K-algebra and we consider

$$R = \frac{K[X_1, X_2, Y_1, Y_2]}{(X_1X_2Y_1Y_2)}.$$

We denote by x_i and y_j the residue classes of X_i and Y_j for $i, j = 1, 2$. We may view $R = K[x_1, x_2, y_1, y_2]$ as the Stanley-Reisner ring simplicial complex Δ whose facets are given by the sets

$$\{v_1, v_2, w_1\}, \{v_1, w_1, w_2\}, \{w_1, w_2, v_2\}, \quad \text{and} \quad \{w_2, v_2, v_1\}.$$

Here we denote the vertices corresponding to the X_i by v_i and those corresponding to the Y_j by w_j.



We set $f = X_1X_2Y_1Y_2$, $S_0 = K[X_1, X_2]$ and $S_+ = (Y_1, Y_2)$. From the exact sequence

$$0 \to S(-2,-2) \xrightarrow{f} S \to R \to 0,$$

we get the exact sequence of graded S_0-modules

$$0 \to H^1_{S_+}(R)_j \to \bigoplus_{|b|=-j} S_0(-2)z^b \xrightarrow{f} \bigoplus_{|b|=-2-j} S_0 z^b \to H^2_{S_+}(R)_j \to 0.$$

The corresponding matrix of the map f is a $(-j+1) \times (-j-1)$ matrix given by

$$\mathbf{U_j} = \begin{pmatrix} 0 & x_1x_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_1x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & x_1x_2 & 0 \end{pmatrix}.$$

We see that $\dim_{S_0} H^2_{S_+}(R)_j = \dim_{S_0} S_0^{-j-1}/\operatorname{Im} f$, where the $\operatorname{Im} f$ can be described as matrix $\mathbf{U_j}$. By using CoCoA one can check that

$$\dim_{S_0} H^2_{S_+}(R)_j = 1 \quad \text{for all} \quad j \leq -2.$$

We set $R_0 = K[x_1, x_2]$ and $P = (y_1, y_2)$. Since $H^2_{S_+}(R) = H^2_P(R)$ it follows that $\dim_{R_0} H^2_P(R)_j = 1$ for all $j \leq -2$. Next we are going to show that above calculation is true by using our theorem. By Theorem 5.9

$$H_{H^2_P(K[\Delta])_j}(s) = \frac{\sum_{r=0}^{2} A_r(j)(1-s)^{2-r}s^r}{(1-s)^2},$$

where

$$A_r(j) = \sum_{\substack{F \in \Delta_W \\ G \subset V, |G|=r}} d(2, F, G)\binom{-j-1}{|F|-1},$$

and where

$$d(2,F,G)=\dim_K \widetilde{H}_{1-|F|}\big((\operatorname{lk} F\cup G)_W;K\big).$$

Now we compute $A_r(j)$ for $r=0,1,2$. We first compute $A_0(j)$ and for computing $A_r(j)$ for $j=1,2$ we proceed in the same way as the computing $A_0(j)$.

$$\begin{aligned} A_0(j) &= \sum_{F\in\Delta_W} d(2,F,\emptyset)\binom{-j-1}{|F|-1}\\ &= d(2,\emptyset,\emptyset)\binom{-j-1}{-1}+d(2,\{w_1\},\emptyset)\binom{-j-1}{0}\\ &+\ d(2,\{w_2\},\emptyset)\binom{-j-1}{0}+d(2,\{w_1,w_2\},\emptyset)\binom{-j-1}{1}. \end{aligned}$$

Now we compute $d(2,F,\emptyset)$ for $F=\emptyset,\{w_1\},\{w_2\},\{w_1,w_2\}$. We see that

$$d(2,\emptyset,\emptyset)=\dim_K \widetilde{H}_1\big((\operatorname{lk}\emptyset)_W;K\big)=\dim_K \widetilde{H}_1(\Delta_1;K)=0,$$

where $\Delta_1=\{\emptyset,\{w_1\},\{w_2\},\{w_1,w_2\}\}$;

$$d(2,\{w_1\},\emptyset)=\dim_K \widetilde{H}_0\big((\operatorname{lk}\{w_1\})_W;K\big)=\dim_K \widetilde{H}_0(\Delta_2;K)=0,$$

where $\Delta_2=\{\emptyset,\{w_2\}\}$;

$$d(2,\{w_2\},\emptyset)=\dim_K \widetilde{H}_0\big((\operatorname{lk}\{w_2\})_W;K\big)=\dim_K \widetilde{H}_0\big(\Delta_3;K\big)=0,$$

where $\Delta_3=\{\emptyset,\{w_1\}\}$; and

$$d(2,\{w_1,w_2\},\emptyset)=\dim_K \widetilde{H}_{-1}\big((\operatorname{lk}\{w_1,w_2\})_W;K\big)=\dim_K \widetilde{H}_{-1}(\Delta_4;K)=1,$$

where $\Delta_4=\{\emptyset\}$. Thus we conclude that $A_0(j)=\binom{-j-1}{1}$. Similarly one obtains $A_1(j)=2\binom{-j-1}{1}$ and $A_2(j)=0$. Therefore

$$H_{H_P^2(K[\Delta])_j}(s)=\frac{(-j-1)(1+s)}{1-s},$$

and by [6, Corollary 4.1.8] it follows that $\dim_{R_0} H_P^2(K[\Delta])_j=1$ for all $j\le -2$. We can also compute the Hilbert series of $H_P^i(K[\Delta])_j$ for $i=0,1$. We proceed in the same way as above and obtain

$$H_{H_P^1(K[\Delta])_j}(s)=\frac{2s^2}{(1-s)^2}.$$

This implies that $\dim_{R_0} H_P^1(R)_j=2$ for all $j\le -1$. We also see

$$H_{H_P^0(K[\Delta])_j}(s)=0.$$

5.2 Local cohomology of monomial ideals with respect to monomial prime ideals

We recall two results due to Takayama [26]. Let $S = K[Y_1, \dots, Y_n]$ be a polynomial ring with the standard grading. For a monomial ideal $I \subset S$ we set $R = S/I$. We denote by y_i the image of Y_i in R for $i = 1, \dots, n$ and set $\mathfrak{m} = (y_1, \dots, y_n)$, the unique maximal ideal. For a monomial ideal $I \subset S$, we denote by $G(I)$ the minimal set of monomial generators. Let $u = Y_1^{c_1} \cdots Y_n^{c_n}$ be a monomial with $c_j \geq 0$ for all j, then we define $\nu_j(u) = c_j$ for $j = 1, \dots, n$, and $\operatorname{supp}(u) = \{j : c_j \neq 0\}$.

We set $G_b = \{j : b_j < 0\}$ for $b \in \mathbb{Z}^n$. By Takayama we have

Lemma 5.13. *Let $y = y_{i_1} \cdots y_{i_r}$ with $i_1 < \cdots < i_r$ and set $F = \{i_1, \dots, i_r\}$. For all $b \in \mathbb{Z}^n$ we have $\dim_K (R_y)_b \leq 1$ and the following are equivalent*

(a) $(R_y)_b \cong K$,

(b) *$F \supset G_b$ and for all $u \in G(I)$ there exists $j \notin F$ such that $\nu_j(u) > b_j \geq 0$.*

For any $b \in \mathbb{Z}^n$, we define a simplicial complex

$$\Delta_b = \left\{ F - G_b \mid \begin{array}{l} F \supset G_b, \text{ and for all } u \in G(I) \text{ there exists } j \notin F \\ \text{such that } \nu_j(u) > b_j \geq 0 \end{array} \right\}.$$

Theorem 5.14. *Let $I \subset S = K[Y_1, \dots, Y_n]$ be a monomial ideal. Then the multigraded Hilbert series of the local cohomology modules of $R = S/I$ with respect to the $\mathbb{Z}^n$-grading is given by*

$$\operatorname{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t}) = \sum_{F \in \Delta} \sum \dim_K \widetilde{H}_{i-|F|-1}(\Delta_b; K) \mathbf{t}^b$$

where $\mathbf{t} = (t_1, \cdots, t_n)$. The second sum runs over $b \in \mathbb{Z}^n$ such that $G_b = F$ and $b_j \leq \rho_j - 1$ for $j = 1, \dots, n$, with $\rho_j = \max\{\nu_j(u) : u \in G(I)\}$ for $j = 1, \dots, n$, and Δ is the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$.

Note that Takayama's formula can be rewritten as following

$$\begin{aligned} \operatorname{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t}) &= \sum_{F \in \Delta} \dim_K \widetilde{H}_{i-|F|-1}(\Delta_b; K) \sum_{\substack{b \in \mathbb{Z}^n, G_b = F \\ b_j \leq \rho_j - 1 \\ j = 1, \dots, n}} \mathbf{t}^b \\ &= \sum_{F \in \Delta} \dim_K \widetilde{H}_{i-|F|-1}(\Delta_b; K) \prod_{w_j \notin F} \frac{1 - {t_j}^{\rho_j}}{1 - t_j} \prod_{w_j \in F} \frac{{t_j}^{-1}}{1 - {t_j}^{-1}}. \end{aligned}$$

We see that in the squarefree case, this formula together with [26, Corollary 1] implies again Hochster's formula.

Now let $S = K[X_1, \dots, X_m, Y_1, \dots, Y_n]$ be a standard bigraded polynomial ring over K. For a monomial ideal $I \subset S$ we set $R = S/I$. The residue classes of the variables will be denoted by x_i and y_j and set $P = (y_1, \ldots, y_n)$. For monomial $u \in S$ we may write $u = u_1u_2$ where u_1 and u_2 are monomials in X and Y. Let Δ be the simplicial complex corresponding to $\sqrt{I}$. As before we denote the vertices corresponding to the X_i by v_i and those corresponding to the Y_j by w_j. We set $G_b = \{w_j : b_j < 0\}$ for $b \in \mathbb{Z}^n$. With the same arguments as Lemma 5.13 we have

Lemma 5.15. *Let $y = y_{i_1} \cdots y_{i_r}$ with $i_1 < \cdots < i_r$ and set $F = \{i_1, \ldots, i_r\}$. For all $a \in \mathbb{Z}^m$ and $b \in \mathbb{Z}^n$ we have $\dim_K(R_y)_{(a,b)} \leq 1$ and the following are equivalent*

(a) *$(R_y)_{(a,b)} \cong K$;*

(b) *$F \supset G_b$, $a \in \mathbb{Z}^m_+$, and for all $u \in G(I)$ there exists $j \notin F$ such that $\nu_j(u_2) > b_j \geq 0$ or for at least one i, $\nu_i(u_1) > a_i \geq 0$.*

For any $a \in \mathbb{Z}^m_+$ and $b \in \mathbb{Z}^n$, we define a simplicial complex

$$\Delta_{(a,b)} = \left\{ F - G_b \mid \begin{array}{c} F \supset G_b, a \in \mathbb{Z}^m_+ \text{ and for all } u \in G(I) \text{ there exists } j \notin F \\ \text{such that } \nu_j(u_2) > b_j \geq 0, \text{or for at least one i}, \nu_i(u_1) > a_i \geq 0 \end{array} \right\}.$$

As a generalization of Takayama's result we have

Theorem 5.16. *Let $I \subset S = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be a monomial ideal with the natural $\mathbb{Z}^m \times \mathbb{Z}^n$-bigrading. Then the bigraded Hilbert series of the local cohomology modules of $R = S/I$ with respect to the $\mathbb{Z}^m \times \mathbb{Z}^n$-bigrading is given by*

$$\begin{aligned} &\operatorname{Hilb}(H^i_P(R), \mathbf{s}, \mathbf{t}) = \\ &\sum_{F \in \Delta_W} \sum_{G \subset V} D(i, F, G) \prod_{v_i \notin G} \frac{1 - {s_i}^{-\sigma_i}}{1 - (s_i)^{-1}} \prod_{v_i \in G} \frac{s_i}{1 - s_i} \prod_{w_j \notin F} \frac{1 - {t_j}^{\rho_j}}{1 - t_j} \prod_{w_j \in F} \frac{{t_j}^{-1}}{1 - {t_j}^{-1}}, \end{aligned}$$

where $P = (y_1, \ldots, y_n)$, $D(i, F, G) = \dim_K \widetilde{H}_{i-|F|-1}(\Delta_{(a,b)}; K)$, $\mathbf{s} = (s_1, \cdots, s_m)$, $\mathbf{t} = (t_1, \cdots, t_n)$, $\rho_j = \max\{\nu_j(u_2) \colon u \in G(I)\}$ for $j = 1, \ldots, n$, $\sigma_i = \max\{\nu_i(u_1) : u \in G(I)\}$ for $i = 1, \ldots, m$, and Δ is the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$.

Proof. With the same arguments as in the proof theorem 1 in [26] we can show that

$$\operatorname{Hilb}(H^i_P(R), \mathbf{s}, \mathbf{t}) = \sum \sum \dim_K \widetilde{H}_{i-|F|-1}(\Delta_{(a,b)}; K) \mathbf{s}^a \mathbf{t}^b$$

where the first sum runs over all $F \in \Delta_W$, $b \in \mathbb{Z}^n$ such that $G_b = F$ and $b_j \leq \rho_j - 1$ for $j = 1, \ldots, n$, and the second sum runs over $a \in \mathbb{Z}^m$ such that $N_a = G$ and $a_i \geq \sigma_i - 1$ for $i = 1, \ldots, m$. Indeed, if we assume that $b_j > \rho_j - 1$ and $a_i < \sigma_i - 1$, the proof of [26, Theorem 1] shows that $\dim_K \widetilde{H}_{i-|F|-1}(\Delta_{(a,b)}; K) = 0$ for all i. Therefore we may write

$$\operatorname{Hilb}(H^i_P(R), \mathbf{s}, \mathbf{t}) = \sum_{F \in \Delta_W} \sum_{G \subset V} D(i, F, G) \sum_{\substack{a \in \mathbb{Z}^m, N_a = G \\ a_i \geq \sigma_i - 1 \\ i = 1, \ldots, m}} \mathbf{s}^a \sum_{\substack{b \in \mathbb{Z}^n, G_b = F \\ b_j \leq \rho_j - 1 \\ j = 1, \ldots, n}} \mathbf{t}^b,$$

where

$$D(i,F,G)=\dim_K \widetilde{H}_{i-|F|-1}(\Delta_{(a,b)};K).$$

Since

$$\sum_{\substack{a\in\mathbb{Z}^m, N_a=G\\ a_i\geq\sigma_i-1\\ i=1,\dots,m}} \mathbf{s}^a=\prod_{v_i\notin G}\frac{1-s_i{}^{-\sigma_i}}{1-(s_i)^{-1}}\prod_{v_i\in G}\frac{s_i}{1-s_i}$$

and

$$\sum_{\substack{b\in\mathbb{Z}^n, G_b=F\\ b_j\leq\rho_j-1\\ j=1,\dots,n}} \mathbf{t}^b=\prod_{w_j\notin F}\frac{1-t_j{}^{\rho_j}}{1-t_j}\prod_{w_j\in F}\frac{t_j{}^{-1}}{1-t_j{}^{-1}},$$

the desired formula follows. □

We observe that Theorem 5.3 is a special case of Theorem 5.16. In fact, if we assume that $\sigma_i=1$ for $i=1,\dots,m$ and $\rho_j=1$ for $j=1,\dots,n$, then $a\in\mathbb{Z}^m_+$, $b\in\mathbb{Z}^n_-$, $\prod_{v_i\notin G}\frac{1-s_i{}^{-\sigma_i}}{1-(s_i)^{-1}}=1$ and $\prod_{w_j\notin F}\frac{1-t_j{}^{\rho_j}}{1-t_j}=1$. Moreover by the proof of [26, Corollary 1] we have

$$\Delta_{(a,b)}=\mathrm{lk}_{\mathrm{st}\,H_{(a,b)}}G_{(a,b)}=\mathrm{lk}_{\mathrm{st}\,N_a\cup H_b}G_b=\mathrm{lk}_{\mathrm{st}\,N_a}G_b=(\mathrm{lk}\,F\cup G)_W.$$

As a generalization of Proposition 5.7 we prove the following

Proposition 5.17. *For all i and $k,j\in\mathbb{Z}$ one has*

$$\dim_K H^i_P(R)_{(k,j)}=\sum_{\substack{F\in\Delta_W\\ G\subset V}} D(i,F,G)\sum_{r=0}^{\sigma_G}a_G(r)\binom{k+r-1}{|G|-1}\sum_{h=0}^{\rho_F}b_F(h)\binom{h-j-1}{|F|-1},$$

where $\sigma_G=\sum_{v_i\notin G}(\sigma_i-1)$, $\rho_F=\sum_{w_j\notin F}(\rho_j-1)$ and $a_G(r),b_F(h)\in\mathbb{Z}$ for $r=0,\dots,\sigma_G$ and $h=0,\dots,\rho_F$.

Proof. In Theorem 5.16 we replace all s_i by s and all t_j by t, and obtain

$$H_{H^i_P(R)}(s,t)=\sum_{F\in\Delta_W}\sum_{G\subset V}D(i,F,G)P_G(s^{-1})(\frac{s}{1-s})^{|G|}Q_F(t)(\frac{t^{-1}}{1-t^{-1}})^{|F|},$$

where

$$P_G(s^{-1})=\prod_{v_i\notin G}(1+s^{-1}+\cdots+s^{-\sigma_i+1})\quad\textit{and}\quad Q_F(t)=\prod_{w_j\notin F}(1+t^1+\cdots+t^{\rho_j-1})$$

with $\deg P_G(s^{-1})=\sigma_G$ and $\deg Q_F(t)=\rho_F$. We may write

$$P_G(s^{-1})=\sum_{r=0}^{\sigma_G}a_G(r)s^{-r}\quad\text{where}\quad a_G(r)\in\mathbb{Z}\quad\text{for}\quad r=0,\dots,\sigma_G$$

and

$$Q_F(t)=\sum_{h=0}^{\rho_F} b_F(h)t^h \quad \text{where} \quad b_F(h)\in\mathbb{Z} \quad \text{for} \quad h=0,\dots,\rho_F.$$

By setting $|G|=g$ and $|F|=f$, we have

$$H_{H^i_P(R)}(s,t)=\sum_{F\in\Delta_W}\sum_{G\subset V} D(i,F,G)A_G(s)B_F(t),$$

where

$$A_G(s)=\sum_{r=0}^{\sigma_G} a_G(r)\sum_{l=0}^{\infty}\binom{g+l-1}{g-1}s^{l+g-r}$$

and

$$B_F(t)=\sum_{h=0}^{\rho_F} b_F(h)\sum_{\rho=0}^{\infty}\binom{f+\rho-1}{f-1}t^{-f-\rho+h}.$$

We set $k=l+g-r$ and $j=-f-\rho+h$. Then $r=l+g-k$ and $h=j+f+\rho$, and the desired formula follows. □

As generalization of Theorem 5.9 we prove the following

Theorem 5.18. *For all i there exists an integer j_0 such that for $j\le j_0$, the Krull-dimension* $\dim H^i_P(R)_j$ *is constant.*

Proof. By Proposition 5.17 the $\mathbb{Z}$-graded Hilbert series of $H^i_P(R)_j$ is given by

$$\begin{aligned}
H_{H^i_P(R)_j}(s) &= \sum_{k=0}^{\infty}\dim_K H^i_P(R)_{(k,j)}s^k\\
&= \sum_{k=0}^{\infty}\sum_{\substack{F\in\Delta_W\\ G\subset V}} D(i,F,G)\sum_{t=0}^{\sigma_G}a_G(t)\binom{k+t-1}{|G|-1}\sum_{h=0}^{\rho_F}b_F(h)\binom{h-j-1}{|F|-1}s^k\\
&= A_0(j)\sum_{r=-1}^{\infty}\sum_{t=0}^{\sigma_G}a_G(t)\binom{t+r}{-1}s^{r+1}\\
&+ A_1(j)\sum_{r=-1}^{\infty}\sum_{t=0}^{\sigma_G}a_G(t)\binom{t+r}{0}s^{r+1}\\
&+\cdots+ A_m(j)\sum_{r=-1}^{\infty}\sum_{t=0}^{\sigma_G}a_G(t)\binom{t+r}{m-1}s^{r+1}\\
&= A_0(j)+A_1(j)\frac{P_1(s)}{1-s}+A_2(j)\frac{P_2(s)}{(1-s)^2}+\cdots+A_m(j)\frac{P_m(s)}{(1-s)^m}\\
&= \frac{\sum_{r=0}^{m}A_r(j)(1-s)^{m-r}P_r(s)}{(1-s)^m},
\end{aligned}$$

where

$$A_r(j) = \sum_{\substack{F \in \Delta_W \\ G \subset V, |G|=r}} D(i,F,G) \sum_{h=0}^{\rho_F} b_F(h) \binom{h-j-1}{|F|-1},$$

and $P_r(s) \in \mathbb{Z}[s]$ with $\deg P_r(s) = r$. Here we used (20) and that $a_G(0) = 1$ for $G = \emptyset$, $\binom{t+r}{-1} = 1$ for $t+r = -1$ and 0 otherwise, $\binom{t+r}{0} = 1$ for $t + r \geq 0$ and $\binom{t+r}{n} = \binom{t}{n} + r\binom{t}{n-1} + \frac{r(r-1)}{2!}\binom{t}{n-2} + \frac{r(r-1)(r-2)}{3!}\binom{t}{n-3} + \cdots + \frac{r(r-1)(r-2)\ldots(r-n+1)}{n!}\binom{t}{0}$ for $t + r \geq n \geq 1$. We may write

$$\sum_{r=0}^{m} A_r(j)(1-s)^{m-r} P_r(s) = \sum_{r=0}^{m} B_r(j) s^r,$$

where $B_r(j)$ is a polynomial with coefficients in $\mathbb{Q}$ of degree at most $n-1$. Therefore we have

$$H_{H^i_P(R)_j}(s) = \frac{Q_j(s)}{(1-s)^m},$$

where $Q_j(s) = \sum_{r=0}^{m} B_r(j) s^r$. We proceed in the same way as in the proof of Theorem 5.9 and get the desired result. □

Corollary 5.19. *Let $I \subset S = K[Y_1, \ldots, Y_r]$ be any monomial ideal and set $R = S/I$. Let P be a monomial prime ideal in R. Then for all i, the local cohomology modules of $H^i_P(R)$ are tame.*

Proof. The assertion follows from Theorem 5.18. □

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