Cyclic coverings, Calabi-Yau manifolds and Complex multiplication

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Contents

Introduction

We search for examples of families of Calabi-Yau manifolds with dense set of complex multiplication fibers and for examples of families of curves with dense set of complex multiplication fibers.

By string theoretical considerations, one is interested in Calabi-Yau manifolds, since Calabi-Yau 3-manifolds provide conformal field theories (CFT). One is especially interested in Calabi-Yau 3-manifolds with complex multiplication, since such a manifold has many symmetries and mirror pairs of Calabi-Yau 3-manifolds with complex multiplication yield rational conformal field theories (RCFT) (see [18]). Moreover S. Gukov and C. Vafa [18] ask for the existence of infinitely many Calabi-Yau manifolds with complex multiplication of fixed dimension n.

For a Calabi-Yau manifold X of dimension n with $n \leq 3$, the condition of complex multiplication is equivalent to the property that for all k the Hodge group of $H^k(X, \mathbb{C})$ is commutative. We will call any family of Calabi-Yau n-manifolds, which has a dense set of fibers satisfying the latter property with respect to the Hodge groups, a $CMCY$ family of n-manifolds. The author uses this condition for technical reasons and hopes that such a $CMCY$ family of n-manifolds in an arbitrary dimension may be interesting for its mathematical beauty, too. Here we will give some examples of $CMCY$ families of 3-manifolds and explain how to construct $CMCY$ families of n-manifolds in an arbitrarily high dimension.

Starting with a family of cyclic covers of $\mathbb{P}^1$ with a dense set of CM fibers, E. Viehweg and K. Zuo [46] have constructed a $CMCY$ family of 3-manifolds. This construction of E. Viehweg and K. Zuo [46] is given by a tower of cyclic coverings, which will be explained in Section 7.3. In Chapter 8 we will give a modified version of a Viehweg-Zuo tower for one of our new examples.

Hence we are interested in the examples of families of curves with a dense set of CM fibers by our search for $CMCY$ families of n-manifolds. But there is an other motivation given by an open question in the theory of curves, too. In [10] R. Coleman formulated the following conjecture:

Conjecture 1. *Fix an integer $g \geq 4$. Then there are only finitely many complex algebraic curves C of genus g such that* $\mathrm{Jac}(C)$ *is of CM type.*

Let $\mathcal{P}_n$ denote the configuration space of $n+3$ points in $\mathbb{P}^1$. One can endow these $n+3$ points in $\mathbb{P}^1$ with local monodromy data and use these data for the construction of a family $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers onto $\mathbb{P}^1$ (see Construction 3.2.1).

The action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1$ yields a quotient $\mathcal{M}_n = \mathcal{P}_n/\mathrm{PGL}_2(\mathbb{C})$. By fixing 3 points on $\mathbb{P}^1$, the quotient $\mathcal{M}_n$ can also be considered as a subspace of $\mathcal{P}_n$.

Remark 2. *In [25] J. de Jong and R. Noot gave counterexamples for $g = 4$ and $g = 6$ to the conjecture above. In [46] E. Viehweg and K. Zuo gave an additional counterexample*

for $g = 6$. The counterexamples are given by families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$ with infinitely many CM fibers. Here we will find additional families $\mathcal{C} \to \mathcal{P}_n$ of cyclic genus 5 and genus 7 covers of $\mathbb{P}^1$ with dense sets of complex multiplication fibers, too.

All new examples $\mathcal{C} \to \mathcal{P}_n$ of the preceding remark have a variation $\mathcal{V}$ of Hodge structures similar to the examples of J. de Jong and R. Noot [25], and of E. Viehweg and K. Zuo [46], which we call pure $(1,n) - VHS$. Let $\mathrm{Hg}(\mathcal{V})$ denote the generic Hodge group of $\mathcal{V}$ and let K denote an arbitrary maximal compact subgroup of $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})$. In Section 4.4 we prove that a pure $(1,n) - VHS$ induces an open (multivalued) period map to the symmetric domain associated with $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})/K$, which yields the dense sets of complex multiplication fibers. We obtain the following result in Chapter 6:

Theorem 3. *There are exactly 19 families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$, which have a pure $(1,n) - VHS$ (including all known and new examples).*

We will use the fact that the monodromy group $\mathrm{Mon}^0(\mathcal{V})$ is a subgroup of the derived group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ and we will study $\mathrm{Mon}^0(\mathcal{V})$. Let ψ be a generator of the Galois group of $\mathcal{C} \to \mathcal{P}_n$ and $C(\psi)$ be the centralizer of ψ in the symplectic group with respect to the intersection pairing on an arbitrary fiber of $\mathcal{C}$. In Chapter 4 we obtain the result, which will be useful for our study of $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ and $\mathrm{Mon}^0(\mathcal{V})$:

Lemma 4. *The monodromy group $\mathrm{Mon}^0(\mathcal{V})$ and the derived Hodge group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ are contained in $C(\psi)$.*

Unfortunely we will not be able to determine $\mathrm{Mon}^0(\mathcal{V})$ for all families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers onto $\mathbb{P}^1$. But we obtain for example the following results in Chapter 5:

Proposition 5. *Let $\mathcal{C} \to \mathcal{P}_n$ be a family of cyclic covers of degree m onto $\mathbb{P}^1$. Then one has:*

1. *If the degree m is a prime number ≥ 3, the algebraic groups $C^{\mathrm{der}}(\psi)$, $\mathrm{Mon}^0(\mathcal{V})$ and $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ coincide.*
2. *If $\mathcal{C} \to \mathcal{P}_{2g+2}$ is a family of hyperelliptic curves, one obtains*
$$\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Hg}(\mathcal{V}) \cong \mathrm{Sp}_{\mathbb{Q}}(2g).$$
3. *In the case of a family of covers onto $\mathbb{P}^1$ with 4 branch points, we need a pure $(1,1) - VHS$ to obtain an open period map to the symmetric domain associated with $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})/K$.*

By our new examples of Viehweg-Zuo towers, we will only obtain $CMCY$ families of 2-manifolds. C. Voisin [48] has described a method to obtain Calabi-Yau 3-manifolds by using involutions on $K3$ surfaces. C. Borcea [8] has independently arrived at a more general version of the latter method, which allows to construct Calabi-Yau manifolds in arbitrary dimension. By using this method, we obtain in Section 7.2:

Proposition 6. *For $i = 1, 2$ assume that $\mathcal{C}^{(i)} \to V_i$ is a $CMCY$ family of n_i-manifolds endowed with the V_i-involution ι_i such that for all $p \in V_i$ the ramification locus $(R_i)_p$ of $\mathcal{C}_p^{(i)} \to \mathcal{C}_p^{(i)}/\iota_i$ consists of smooth disjoint hypersurfaces. In addition assume that V_i has a dense set of points $p \in V_i$ such that for all k the Hodge groups $\mathrm{Hg}(H^k(\mathcal{C}_p^{(i)}, \mathbb{Q}))$ and $\mathrm{Hg}(H^k((R_i)_p, \mathbb{Q}))$ are commutative. By blowing up the singular locus of $\mathcal{C}^{(1)} \times \mathcal{C}^{(2)}/\langle(\iota_1, \iota_2)\rangle$, one obtains a $CMCY$ family of $n_1 + n_2$-manifolds over $V_1 \times V_2$ endowed with an involution satisfying the same assumptions as ι_1 and ι_2.*

Remark 7. *By the preceding proposition, one can apply the construction of C. Borcea and C. Voisin for families to obtain an infinite tower of CMCY families of n-manifolds, which we call a Borcea-Voisin tower.*

Example 8. *The family* $\mathcal{C} \to \mathcal{M}_1$ *given by*

$$\mathbb{P}^2 \ni V(y_1^4 - x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

has a pure $(1,1)-VHS$*. Hence by the construction of Viehweg and Zuo [46], one concludes that the family* $\mathcal{C}_2$ *given by*

$$\mathbb{P}^3 \ni V(y_2^4 + y_1^4 - x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1 \tag{1}$$

is a CMCY family of 2-manifolds.

This family has many $\mathcal{M}_1$*-automorphisms. The quotients by some of these automorphisms yield new examples of CMCY families of 2-manifolds. Moreover there are some involutions on* $\mathcal{C}_2$*, which make this family and its quotient families of* $K3$*-surfaces suitable for the construction of a Borcea-Voisin tower (see Section* 7.4 *for the construction of* $\mathcal{C}_2$*, and for the automorphism group and the quotient families of* $\mathcal{C}_2$ *see Section* 9.3*, Section* 9.4 *and Section* 9.5*).*

Example 9. *The family* $\mathcal{C} \to \mathcal{M}_3$ *given by*

$$\mathbb{P}(2,1,1) \ni V(y_1^3 - x_1(x_1 - x_0)(x_1 - ax_0)(x_1 - bx_0)(x_1 - cx_0)x_0) \to \lambda \in \mathcal{M}_1$$

has a pure $(1,3)-VHS$*. The desingularisation* $\tilde{\mathbb{P}}(2,2,1,1)$ *of the weighted projective space* $\mathbb{P}(2,2,1,1)$ *is given by blowing up the singular locus. By a modification of the construction of Viehweg and Zuo, the family* $\mathcal{W}$ *given by*

$$\tilde{\mathbb{P}}(2,2,1,1) \ni \tilde{V}(y_2^3 + y_1^3 - x_1(x_1 - x_0)(x_1 - ax_0)(x_1 - bx_0)(x_1 - cx_0)x_0) \to \lambda \in \mathcal{M}_3 \tag{2}$$

is a CMCY family of 2-manifolds. The family $\mathcal{W}$ *has a degree 3 quotient, which yields a CMCY family of 2-manifolds. Moreover it has an involution, which makes it and its degree 3 quotient suitable for the construction of a Borcea-Voisin tower (see Chapter 8 for the construction of* $\mathcal{W}$ *and Section* 9.1 *for its degree 3 quotient).*

By using the preceding example, we will obtain (see Section 9.2 for the construction and Section 10.3 for the maximality):

Theorem 10. *Let* $\alpha_{\mathbb{F}_3}$ *denote a generator of the Galois group of a degree 3 cover* $\mathbb{F}_3 \to \mathbb{P}^1$*. The family* $\mathcal{W}$ *has an* $\mathcal{M}_3$*-automorphism* α' *of order 3 such that the quotient* $\mathcal{W} \times \mathbb{F}_3/\langle(\alpha', \alpha_{\mathbb{F}_3})\rangle$ *has a desingularisation, which is a CMCY family* $\mathcal{Q} \to \mathcal{M}_3$ *of 3-manifolds. Moreover the family* $\mathcal{Q}$ *is maximal.*

By using the V. V. Nikulins classification of involutions on $K3$ surfaces [42] and the construction of C. Voisin [48], we obtain in Chapter 11:

Theorem 11. *For each integer* $1 \leq r \leq 11$ *there exists a maximal holomorphic CMCY family of algebraic 3-manifolds with Hodge number* $h^{2,1} = r$*.*

The first three chapters explain well-known facts and yield an introduction of the notations. Chapter 1 is an introduction to Hodge Theory with a special view towards complex multiplication. We consider cyclic covers of $\mathbb{P}^1$ in Chapter 2. Moreover Chapter 3 introduces everything that we need to describe families of cyclic covers of $\mathbb{P}^1$ and their variations of Hodge structures.

In Chapter 4 we consider the Galois group action of a cyclic cover onto $\mathbb{P}^1$ and we state first results for the generic Hodge group of a family $\mathcal{C} \to \mathcal{P}_n$. Moreover we will give a sufficient criterion for the existence of a dense set of CM fibers given by the pure $(1,n) - VHS$. In Chapter 5 we compute $\mathrm{Mon}^0(\mathcal{V})$, which provides many information about $\mathrm{Hg}(\mathcal{V})$. We will see that $\mathrm{Mon}^0(\mathcal{V})$ coincides with $C^{\mathrm{der}}(\psi)$ in infinitely many cases. In Chapter 6 we classify the examples of families of cyclic covers onto $\mathbb{P}^1$ providing a pure $(1,n) - VHS$.

The basic methods of the construction of $CMCY$-families in higher dimension will be explained in Chapter 7. We introduce the Borcea-Voisin tower and the Viehweg-Zuo tower and realize that only a small number of families of cyclic covers of $\mathbb{P}^1$ are suitable to start the construction of a Viehweg-Zuo tower. In Chapter 8 we will give a modified version of the method of E. Viehweg and K. Zuo to construct the $CMCY$ family of 2-manifolds given by (2). We consider the automorphism groups of our examples given by (1) and (2) in Chapter 9. This yields the further quotients of the families given by (1) and (2), which are $CMCY$ families of 2-manifolds. We will see that these quotients are endowed with involutions, which make them suitable for the construction of a Borcea-Voisin tower. Moreover we will construct the family $\mathcal{Q}$ of Theorem 10 in Chapter 9. The next chapter is devoted to the *length* of the Yukawa couplings of our examples families (motivated by the question of rigidity) and the Hodge numbers of their fibers. We finish this chapter with an outlook onto the possibilities to construct $CMCY$ families of 3-manifolds by quotients of higher order. In Chapter 11 we use directly the mirror construction of C. Voisin to obtain maximal holomorphic $CMCY$ families of 2-manifolds, which are suitable for the construction of a holomorphic Borcea-Voisin tower.

Chapter 1

An introduction to Hodge structures and Shimura varieties

In this chapter we recall the general facts about Hodge structures and Shimura varieties, which are needed in the sequel.

1.1 The basic definitions

Definition 1.1.1. Let R be a Ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$. An R-Hodge structure is given by an R-module V and a decomposition

$$V \otimes_R \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

Now let $\mathbb{S} := Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ be the Deligne torus given by the Weil restriction of $\mathbb{G}_{m,\mathbb{C}}$.

Proposition 1.1.2. *Let V be an $\mathbb{R}$-vector space. Each real Hodge structure on V defines by*

$$z \cdot \alpha^{p,q} = z^p \bar{z}^q \alpha^{p,q}$$

for all $\alpha^{p,q} \in V^{p,q}$ an action of $\mathbb{S}$ on $V \otimes \mathbb{C}$ such that one has an $\mathbb{R}$-algebraic homomorphism $h : \mathbb{S} \to \mathrm{GL}(V)$. Moreover by the eigenspace decomposition of $V_{\mathbb{C}}$ with respect to the characters of $\mathbb{S}$, any representation given by an algebraic homomorphism $h : \mathbb{S} \to \mathrm{GL}(V)$ corresponds to a real Hodge structure on V.

Proof. (see [13], 1.1.1[1]) □

From now on let V be a $\mathbb{Q}$-vector space and let

$$h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

[1]Note that P. Deligne writes

$$z \cdot \alpha^{p,q} = z^{-p} \bar{z}^{-q} \alpha^{p,q} \quad \text{instead of} \quad z \cdot \alpha^{p,q} = z^p \bar{z}^q \alpha^{p,q}$$

in [13]. But this is only a matter of the chosen conventions.

be the algebraic homomorphism corresponding to a Hodge structure on V. Note that $\mathbb{S}$ is given by $\mathrm{Spec}(\mathbb{R}[x,y,t]/(t(x^2+y^2)=1))$ and S^1 is the algebraic subgroup given by $\mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2=1))$. This yields

$$S^1(\mathbb{R}) = \{z \in \mathbb{C} : z\bar{z} = 1\} \subset \mathbb{C}^*.$$

We consider the exact sequence

$$0 \to \mathbb{R}^* \overset{\mathrm{id}}{\hookrightarrow} \mathbb{C}^* \overset{z \to z/\bar{z}}{\longrightarrow} S^1(\mathbb{R}) \to 0,$$

which can be obtained by an exact sequence

$$0 \to \mathbb{G}_{m,\mathbb{R}} \overset{w}{\hookrightarrow} \mathbb{S} \to S^1 \to 0 \tag{1.1}$$

of $\mathbb{R}$-algebraic groups.

Remark 1.1.3. The homomorphism given by $h \circ w$ is called weight homomorphism. There exists a $k \in \mathbb{Z}$ such that $V^{p,q} = 0$ for all $p+q \neq k$, if and only if $h \circ w$ is given by $r \to r^k$. A Hodge structure is of weight k, if $h \circ w$ is given by $r \to r^k$.

Remark 1.1.4. By Proposition 1.1.2, any (real) Hodge structure on $V_{\mathbb{R}}$ of weight k determines a unique morphism $h_1 : S^1 \to GL(V_{\mathbb{R}})$ given by

$$S^1 \hookrightarrow \mathbb{S} \overset{h}{\to} \mathrm{GL}(V_{\mathbb{R}}).$$

Since $\mathbb{S} = \mathbb{G}_{m,\mathbb{R}} \cdot S^1$, one can reconstruct h from $h|_{S^1}$ and the weight homomorphism. By using Proposition 1.1.2 again, one can easily see that there is a correspondence between Hodge structures of weight k on $V_{\mathbb{R}}$ and representations $h_1 : S^1 \to \mathrm{GL}(V_{\mathbb{R}})$ given by

$$z \cdot \alpha^{p,q} = z^p \bar{z}^q \alpha^{p,q}$$

for all $\alpha^{p,q} \in V^{p,q}$, which must satisfy $p+q=k$ for all $V^{p,q} \neq 0$.

Example 1.1.5. An integral Hodge structure of weight k is given by

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=r} H^{p,q}(X) \quad \text{with} \quad H^{p,q}(X) = H^q(X, \Omega_X^p)$$

for any compact Kähler manifold X.

Definition 1.1.6. A polarized R-Hodge structure of weight k is given by an R-Hodge structure of weight k on an R-module V and a bilinear form $Q : V \times V \to R$, which is symmetric, if k is even, alternating otherwise, and whose extension on $V \otimes_R \mathbb{C}$ satisfies:

1. The Hodge decomposition is orthogonal for the Hermitian form $i^k Q(\cdot, \bar{\cdot})$.

2. For all $\alpha \in V^{p,q} \setminus \{0\}$ one has

$$i^{p-q}(-1)^{\frac{k(k-1)}{2}} Q(\alpha, \bar{\alpha}) > 0.$$

Example 1.1.7. Let X be a compact Kähler manifold. Recall that for $k \leq dim(X)$ the primitive cohomology $H^k(X,\mathbb{R})_{\text{prim}}$ is the kernel of the Lefschetz operator $H^k(X,\mathbb{R}) \to H^{2n-k+2}(X,\mathbb{R})$ given by

$$\alpha \to \wedge^{n-k+1}(\omega) \wedge \alpha,$$

where $n := dim(X)$, ω denotes the chosen Kähler form and $\alpha \in H^k(X,\mathbb{R})$. By

$$(\alpha,\beta) := \int_X \wedge^{n-k}(\omega) \wedge \alpha \wedge \beta,$$

one obtains a polarization on $H^k(X,\mathbb{Z})_{\text{prim}}$ and hencefore a polarized integral Hodge structure on $H^k(X,\mathbb{Z})_{\text{prim}}$, if $[\omega] \in H^2(X,\mathbb{Z})$ (see [49], 7.1.2)[2].

Definition 1.1.8. Let $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$ be a field and V be a K-vector space. The Hodge group $\text{Hg}_K(V,h)$ of a K Hodge structure (V,h) is the smallest K-algebraic subgroup G of $\text{GL}(V)$ such that

$$h(S^1) \subset G \times_K \mathbb{R}.$$

The Mumford-Tate group $\text{MT}_K(V,h)$ of a K Hodge structure (V,h) is the smallest K-algebraic subgroup G of $\text{GL}(V)$ such that

$$h(\mathbb{S}) \subset G \times_K \mathbb{R}.$$

For simplicity we will write $\text{Hg}(V,h)$ instead of $\text{Hg}_{\mathbb{Q}}(V,h)$ and $\text{MT}(V,h)$ instead of $\text{MT}_{\mathbb{Q}}(V,h)$.

Definition 1.1.9. Let F be a number field. A compact Kähler manifold X of dimension n has complex multiplication (CM) over F, if the Hodge group of the F Hodge structure on $H^n(X,F)$ is a torus. We say that X has complex multiplication, if it has complex multiplication over $\mathbb{Q}$.

There is another concept of complex multiplication: An Abelian variety A is of CM type, if it is isogenous to a fiberproduct of simple Abelian varieties X_i $(i = 1,\ldots,n)$ such that there are fields $K_i \subset \text{End}(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$, which satisfy

$$[K_i : \mathbb{Q}] \geq 2 \cdot dim(X_i).$$

Remark 1.1.10. If the Abelian variety A is of CM type, the fields K_i are CM fields (i.e. a totally imaginary quadratic extension of a totally real number field) and satisfy

$$[K_i : \mathbb{Q}] = 2 \cdot dim(X_i).$$

Proof. (see [29], Theorem 3.1 and Lemma 3.2.) □

Lemma 1.1.11. *An Abelian variety A is of CM type, if and only if $\text{Hg}(H^1(A,\mathbb{Q}))$ is a torus algebraic group.*

Proof. (see [38]) □

Since the Hodge structures on $H^1(C,\mathbb{Q})$ and $H^1(\text{Jac}(C),\mathbb{Q})$ are isomorphic, the relation of our two concepts of complex multiplication is obvious:

Proposition 1.1.12. *A curve C has complex multiplication, if and only if $\text{Jac}(C)$ is of CM type.*

[2]There is a more general definition of a polarized Hodge structure (see [13], 1.1.10). But here we will mainly consider Hodge structures given by the primitive cohomology on a Kähler manifold. Moreover we obtain $H^n(X,\mathbb{R})_{\text{prim}} = H^n(X,\mathbb{R})$, if X is a curve or if X is a Calabi-Yau 3-manifold. Hence in these two cases of our main interest $H^n(X,\mathbb{R}_{\text{prim}})$ is independent by the chosen Kähler form. Moreover by its definition, the corresponding polarization is independent of the Kähler form, if $k = n$. Thus in these two cases the integral polarized Hodge structure depends only on the isomorphism class of X.

1.2 Jacobians, Polarizations and Riemann's Theorem

Let X be a Kähler manifold. Consider the following exact sequence:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

This yields the complex torus

$$\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}),$$

which isomorphic to the Jacobian $\mathrm{Jac}(C)$, if X is a curve C. The theory of Abelian varieties, their Hodge structures and their parameterizing spaces contains several features, which we will need in the sequel.

1.2.1. On the homology $H_1(C, \mathbb{Z})$ of a curve C one can define an intersection pairing. It is compatible with the polarization on$H^1(C, \mathbb{Z})$ by a canonical monomorphism σ, which assigns to each $\gamma \in H_1(C, \mathbb{Z})$ the $\alpha \in H^1(C, \mathbb{C})$, which has the property that

$$\int_C \alpha \wedge \beta = \int_\gamma \beta$$

for all $\beta \in H^1(C, \mathbb{C})$. Thus the homology group $H_1(C, \mathbb{Z})$ is the dual of

$$H^1(C, \mathbb{Z}) = \sigma(H_1(C, \mathbb{Z})).^3$$

By integration over $\mathbb{C}$-valued paths in $H_1(C, \mathbb{C}) := H_1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, the $\mathbb{C}$-valued homology $H_1(C, \mathbb{C})$ is a canonical dual of $H^1(C, \mathbb{C})$. On $H_1(C, \mathbb{C})$ the dual Hodge structure of weight -1 is given by the Hodge filtration

$$0 \subset H^{0,-1}(C) \subset H_1(C, \mathbb{C}) \text{ such that } H^{0,-1}(C) = H^{0,1}(C)^* \text{ and } H^{-1,0}(C) = H^{1,0}(C)^*$$

with $H^{-1,0}(C) = H_1(C, \mathbb{C})/H^{0,-1}(C)$. Moreover one has

$$\sigma \circ h_{-1}(z) = h_1(z) \circ \sigma \text{ for all } z \in S^1(\mathbb{R}),$$

where h_{-1} and h_1 denote the corresponding embeddings

$$h_{-1} : S^1 \to \mathrm{GL}(H_1(C, \mathbb{R})) \text{ and } h_1 : S^1 \to \mathrm{GL}(H^1(C, \mathbb{R})).$$

Thus the Hodge groups of these Hodge structures on $H_1(C, \mathbb{Z})$ and $H^1(C, \mathbb{Z})$ are isomorphic. Hence for a study of the Hodge structure on $H^1(C, \mathbb{Z})$, it is sufficient to consider the corresponding dual Hodge structure on $H_1(C, \mathbb{Z})$.

Next we consider polarizations on Abelian varieties:

Remark 1.2.2. Let $A = W/L$ be a complex g-dimensional torus. There is a canonical isomorphism between $H^2(A, \mathbb{Z})$ and $\mathbb{Z}$-valued alternating forms on $L = H_1(A, \mathbb{Z})$. Moreover for an alternating integral form E on L, there is a line bundle $\mathcal{L}$ on A with $c^1(\mathcal{L}) = E$, if and only if $E(i\cdot, i\cdot) = E(\cdot, \cdot)$. By

$$H(u, v) = E(iu, v) + iE(u, v),$$

[3]Note that $\mathrm{Jac}(C)$ is defined by the quotient of $H^0(\omega_C)^*$ by the period lattice induced by integration over paths in $H_1(C, \mathbb{Z})$. Thus the statement that $H^1(C, \mathbb{Z}) = \sigma(H_1(C, \mathbb{Z}))$ is equivalent to the well-known fact that $\mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$.

we get the corresponding Hermitian form H from E and conversely, given H we obtain E by $E = \Im H$. (See[6], Proposition 2.1.6 and Lemma 2.1.7)

A polarization on an Abelian variety is given by a line bundle $\mathcal{L}$, whose Hermitian form H, which corresponds to its first Chern class E, is positive definite. The alternating form E of the polarization can be given by the matrix

$$\begin{pmatrix} 0 & D_g \\ -D_g & 0 \end{pmatrix}$$

with respect to a symplectic basis of L, where $D_g = \mathrm{diag}(d_1, \ldots, d_g)$ with $d_i | d_{i+1}$ (see [6], 3.§1). The matrix D_g depends on the polarization, and it is called the type of the polarization. The polarization E on A is principal, if $D_g = E_g$.

A positive definite Hermitian form H on W, which has the property that $\Im H$ is an integral alternating form on L, satisfies that $\Im H(i\cdot, i\cdot) = \Im H(\cdot, \cdot)$ resp., is a polarization. Since the Chern class of a line bundle $\mathcal{L}$ is a polarization, if and only if $\mathcal{L}$ is ample (see [6], Proposition 4.5.2.), H yields an ample line bundle. By the Theorem of Chow, A is algebraic in this case. Moreover if A is an Abelian variety, there is a positive definite Hermitian form H on W such that $\Im H$ is integral on L (see [40], §1, too).

Now let V denote a $\mathbb{Q}$-vector space of dimension $2g$, Q be a rational alternating bilinear form on V, and J be a complex structure on $V_{\mathbb{R}}$ (i.e. an automorphism J with $J^2 = -\mathrm{id}$).

Remark 1.2.3. It is a well-known fact that there is a correspondence between Hodge structures h on V of type $(1,0),(0,1)$ and complex structures J on $V_{\mathbb{R}}$ via $h(i) = J$.

Lemma 1.2.4. *The complex structure J on $V_{\mathbb{R}}$ corresponds to a polarized Hodge structure (V, h, Q) of type $(1,0),(0,1)$, if and only if it satisfies*

$$Q(J\cdot, J\cdot) = Q(\cdot, \cdot) \quad \textit{and} \quad Q(J\tilde{v}, \tilde{v}) > 0$$

for all $\tilde{v} \in V_{\mathbb{R}}$.

Proof. Let the complex structure J on $V_{\mathbb{R}}$ be given by a polarized Hodge structure of type $(1,0),(0,1)$ on V. Any $\tilde{v}, \tilde{w} \in V_{\mathbb{R}}$ can be given by

$$\tilde{v} = v + \bar{v} \quad \textit{and} \quad \tilde{w} = w + \bar{w}$$

for some $v, w \in H^{1,0}$, where $H^{1,0}$ and $H^{0,1}$ are totally isotropic with respect to Q. Hence:

$$Q(J\tilde{v}, J\tilde{w}) = Q(iv, -i\bar{w}) + Q(-i\bar{v}, iw) = Q(v, \bar{w}) + Q(\bar{v}, w) = Q(\tilde{v}, \tilde{w})$$

Since the Hermitian form given by $iQ(v, \bar{v})$ is positive definite on $H^{1,0}$, one concludes:

$$Q(J\tilde{v}, \tilde{v}) = Q(iv - i\bar{v}, v + \bar{v}) = Q(iv, \bar{v}) + Q(-i\bar{v}, v) = 2iQ(v, \bar{v}) > 0 \tag{1.2}$$

Conversely assume that $Q(J\cdot, \cdot) > 0$ and $Q(\cdot, \cdot) = Q(J\cdot, J\cdot)$. Thus one has

$$Q(v, v) = Q(Jv, Jv) = Q(iv, iv) = -Q(v, v)$$

$$\text{resp.,} \quad Q(v, v) = Q(Jv, Jv) = Q(-iv, -iv) = -Q(v, v)$$

for all $v \in H^{1,0} := Eig(J, i)$ resp., for all $v \in H^{0,1} := Eig(J, -i)$. Hence $H^{1,0}$ resp., $H^{0,1}$ is isotropic with respect to Q. The same calculation as in (1.2) implies that $iQ(\cdot, \bar{\cdot})$ is positive definite on $H^{1,0}$ and negative definite on $H^{0,1}$. Hence one gets a polarized Hodge structure of type $(1,0),(0,1)$ by Remark 1.2.3. □

By the preceding lemma and an easy calculation using that $z = a + ib \in S^1(\mathbb{R})$ implies $a^2 + b^2 = 1$,[4] we obtain:

Proposition 1.2.5. *A polarized Hodge structure of type* $(1,0),(0,1)$ *on* V *(where* Q *denotes the polarization) induces a faithful symplectic representation*

$$h : S^1 \to \mathrm{Sp}(V_{\mathbb{R}}, Q).$$

Corollary 1.2.6. *Let* (V,h,Q) *be a polarized Hodge structure of type* $(1,0),(0,1)$*. Then*

$$\mathrm{Hg}(V,h) \subset \mathrm{Sp}(V,Q), \quad \textit{and} \quad \mathrm{MT}(V,h) \subset \mathrm{GSp}(V,Q).$$

Theorem 1.2.7 (Riemann). *There is a correspondence between polarized Abelian varieties of dimension* g *and polarized Hodge structures* (L,h,Q) *of type* $(1,0),(0,1)$ *on a torsion-free lattice* L *of rank* $2g$*.*

Proof. Let (L,h,Q) be a polarized Hodge structure on a torsion-free lattice L of rank $2g$. By

$$L \otimes \mathbb{R} \hookrightarrow L \otimes \mathbb{C} \to H^{0,1},$$

one has an isomorphism f of $\mathbb{R}$-vector spaces. The complex structure of the Hodge structure turns $L_{\mathbb{R}}$ into a $\mathbb{C}$-vector space. One has $f(\lambda v) = \bar{\lambda} f(v)$. By f, Q may be considered as (real) alternating form on $H^{0,1}$. But it satisfies $Q(iv,v) < 0$ for all $v \in H^{0,1}$. Hence let $E = -Q$. Lemma 1.2.4 implies that $E(i\cdot, i\cdot) = E(\cdot,\cdot)$ and $E(iv,v) > 0$ for all $v \in H^{0,1}$. Thus the corresponding Hermitian form is positive definite (see Remark 1.2.2) and we have a polarization on the complex torus $H^{0,1}/L$ and hencefore an Abelian variety.

Conversely take a polarized Abelian variety (A,E), where $A = W/L$. Let $Q := -E$. By $J = -i$, one has similar to Lemma 1.2.4 a complex structure corresponding to a polarized Hodge structure of type $(1,0),(0,1)$ on L. Thus we have obviously obtained the desired correspondence. □

Since a polarized rational Hodge structure can be considered as polarized integral Hodge structure with respect to a fixed lattice, if the polarization on this lattice is integral, one concludes by Lemma 1.2.4 and Theorem 1.2.7:

Corollary 1.2.8. *There is a bijection between the sets of polarized Abelian varieties* $A = W/L$ *and complex structures on* $L \otimes \mathbb{R}$ *satisfying*

$$Q(J\cdot, J\cdot) = Q(\cdot,\cdot) \quad \textit{and} \quad Q(Jv,v) > 0$$

for all $v \in L \otimes \mathbb{R}$ *with respect to an integral alternating form* Q *on* L*.*

Remark 1.2.9. The Jacobian $\mathrm{Jac}(C)$ of a curve C is isomorphic to

$$\mathrm{Pic}^0(C) = H^{0,1}(C)/H^1(C,\mathbb{Z}).$$

[4] Let $v, w \in V_{\mathbb{R}}$. The calculation is given by:

$$Q(zv,zw) = a^2Q(v,w) + b^2Q(v,w) + ab(Q(Jv,w) + Q(v,Jw)) =$$

$$= Q(v,w) + ab(Q(Jv,w) + Q(Jv,J(Jw))) = Q(v,w) + ab(Q(Jv,w) + Q(Jv,-w)) = Q(v,w)$$

As in the proof of Riemann's Theorem, the polarization of the integral Hodge structure on $H^1(C,\mathbb{Z})$ can be identified with a polarization on $\mathrm{Jac}(C)$. Since the corresponding intersection form on $H_1(C,\mathbb{Z})$ can be given by the matrix

$$\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$$

with respect to a fixed symplectic basis (follows by [6], Chapter 11, §1 for example), one concludes that this polarization on $\mathrm{Jac}(C)$ is principal.[5]

Remark 1.2.10. Two curves are isomorphic, if their Jacobians are isomorphic as principally polarized Abelian varieties (see [6], Torelli's Theorem 11.1.7).

1.3 Shimura data and Siegel's upper half plane

From now on let (L,h,Q) be a polarized integral Hodge structure of type $(1,0),(0,1)$ on a torsion-free lattice L of rank $2g$ and $V := L\otimes\mathbb{Q}$. For simplicity we assume that Q is given by

$$J_0 = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} \tag{1.3}$$

with respect to a symplectic basis of L.

Now we construct Siegel's upper half plane $\mathfrak{h}_g$ (at present as homogeneous space):

Construction 1.3.1. An embedding $h : S^1 \to \mathrm{Sp}(V,Q)_{\mathbb{R}}$ obtained by a polarized integral Hodge structure (L,h,Q) of type $(1,0),(0,1)$ corresponds via $h(i)$ to a positive complex structure (i.e. a complex structure J such that $Q(Jv,v)>0$) $J\in\mathrm{Sp}(V,Q)_{\mathbb{R}}$ for all $v\in V_{\mathbb{R}}$. By conjugation, $\mathrm{Sp}(V,Q)_{\mathbb{R}}$ acts transitively on the positive complex structures $J\in\mathrm{Sp}(V,Q)_{\mathbb{R}}$ (see [32], page 67[6]) and hencefore it acts transitively on the set of polarized integral Hodge structures (L,h,Q) of type $(1,0),(0,1)$. Let K be the subgroup of $\mathrm{Sp}(V,Q)(\mathbb{R})$, which leaves a fixed $h(S^1)$ stable by conjugation. Then Corollary 1.2.8 allows to identify the set of points of the homogeneous space $\mathfrak{h}_g := \mathrm{Sp}(V,Q)(\mathbb{R})/K$ with the set of principally polarized Abelian varieties of dimension g with symplectic basis.

We want to endow $\mathfrak{h}_g$ with the structure of a Hermitian symmetric domain. But first let us recall some needed facts about groups:

Definition 1.3.2. A Lie algebra $\mathfrak{g}$ is simple, if $dim(\mathfrak{g})>1$ and $\mathfrak{g}$ contains no non-trivial ideals. A connected Lie group G is simple, if its Lie algebra is simple.

Remark 1.3.3. Let G be an algebraic group. The quotient G^{ad} is the image of the adjoint representation of G on its Lie algebra $\mathfrak{g}$. It is a well-known fact that G has the following algebraic subgroups:

[5] Following [6], Chapter 11 the principal polarization is $c_1(\Theta)$, where the Theta divisor Θ is obtained as the image of the Abel-Jacobi map $C^{g-1}\to\mathrm{Jac}(C)$ via $(p_1,\ldots,p_{g-1})\to\mathcal{O}_C(p_1+\ldots+p_{g-1}-(g-1)p_0)$ for an arbitrary $p_0\in C$.

[6] A positive complex structure in the sense of our notation is a negative complex structure in the sense of the notation of [32], and vice versa (Here J is negative, if $Q(Jv,v)>0$ for all $v\in V_{\mathbb{R}}$.). But via $J\leftrightarrow -J$ we have a correspondence between negative and positive complex structures commuting with the actions of $\mathrm{Sp}(V,Q)_{\mathbb{R}}$ on positive and negative complex structures.

The derived group G^{der} of G is the subgroup of G generated by its commutators. By $Z(G)$, we denote the center of G. The Radical $R(G)$ is the maximal connected normal solvable subgroup of G. Its unipotent radical $R_u(G)$ is given by

$$R_u(G) := \{g \in R(G) | g \text{ is unipotent}\}.$$

Definition 1.3.4. Let G be an algebraic group. The group G is a reductive, if $R_u(G) = \{e\}$, and semisimple, if $R(G) = \{e\}$.

Proposition 1.3.5. *Let G be a connected algebraic group. It is reductive, if and only if it is the almost direct product of a torus and a semisimple group. These groups can be given by $Z(G)$ and G^{der}.*

Proof. (see [43], Chapter **I**. §3 for the first statement and [9] **IV**. 14.2 for the second statement) □

Remark 1.3.6. It is a well-known fact that the Lie algebras of an $\mathbb{R}$-algebraic group G and the Lie group $G(\mathbb{R})$ coincide. Moreover G is semisimple, if and only if its Lie algebra $\mathfrak{g}$ is a direct sum of simple Lie algebras.

Remark 1.3.7.

1. Let G be a reductive $\mathbb{Q}$-algebraic group with largest commutative quotient T. One has the obvious exact sequences:

$$1 \to G^{\mathrm{der}} \to G \to T \to 1$$

$$1 \to Z(G) \to G \to G^{\mathrm{ad}} \to 1$$

$$1 \to Z(G^{\mathrm{der}}) \to Z(G) \to T \to 1$$

2. The exact sequences induce a natural isogeny $G^{\mathrm{der}} \to G^{\mathrm{ad}}$ with kernel $Z(G^{\mathrm{der}})$ (see [12], 1.1.)

Lemma 1.3.8. *If G is a semisimple connected Lie group with trivial center, then it is isomorphic to a direct product of simple adjoint groups.*

Proof. By the assumptions and [23], **II**. Corollary 5.2, G coincides with its adjoint group $G^{\mathrm{ad}} \cong G/Z(G)$. Since the Lie algebra $\mathfrak{g}$ of G is the direct sum of simple Lie algebras, $\mathfrak{g}$ is the Lie algebra of a certain direct product of simple groups, too. Without loss of generality one can assume that these simple Lie groups have trivial centers. Recall that the adjoint group depends only on the Lie algebra. Thus this product of simple groups is isomorphic to its adjoint, which is the adjoint of G coinciding with G. □

Let $\mathfrak{g}$ be a complex Lie algebra. By $\mathbb{R} \hookrightarrow \mathbb{C}$, $\mathfrak{g}$ can be considered as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ with complex structure J given by the scalar multiplication with i. A real form of $\mathfrak{g}^{\mathbb{R}}$ is a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}^{\mathbb{R}}$ such that

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus J\mathfrak{g}_0.$$

A real form is called compact, if its (real) adjoint group is a compact Lie group.

Now let $\mathfrak{g}$ be a semisimple real Lie algebra. Any involution ι on $\mathfrak{g}$ endows $\mathfrak{g}$ with a decomposition into two eigenspaces $\mathfrak{t} = \mathrm{Eig}(\iota, 1)$ and $\mathfrak{p} = \mathrm{Eig}(\iota, -1)$. The involution ι is called a Cartan involution, if $\mathfrak{u} := \mathfrak{t} + i\mathfrak{p}$ is a compact real form of the complexified semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. An involutive algebraic automorphism $\tilde{\iota}$ on a connected $\mathbb{R}$-algebraic group G is a Cartan involution on G, if the (real) Lie subgroup of $G(\mathbb{C})$ corresponding to $\mathfrak{u} := \mathfrak{t} + i\mathfrak{p}$ is compact, where again $\mathfrak{t} = \mathrm{Eig}(\tilde{\iota}, 1), \mathfrak{p} = \mathrm{Eig}(\tilde{\iota}, -1) \subset Lie(G)$.

Proposition 1.3.9. *A connected $\mathbb{R}$-algebraic group is reductive, if and only if it has a Cartan involution. Any two Cartan involutions are conjugate by an inner automorphism.*

Proof. (See [43], **I**. 4.3) □

Example 1.3.10. The group $\mathrm{Sp}_{\mathbb{R}}(V,Q) \cong \mathrm{Sp}_{\mathbb{R}}(2g)$ is reductive. The inner automorphism of $\mathrm{Sp}_{\mathbb{R}}(2g)$ given by J_0 (see (1.3)) is a Cartan involution (see [23], **VIII**. Exercise B.2).

Since $\mathrm{Sp}(V,Q)$ is defined for the alternating form Q given by the matrix J_0, one can easily calculate that J_0 satisfies that $Q(J_0v,v) > 0$ for all $v \in V_{\mathbb{R}}$. Hence by Construction 1.3.1, the complex structure J_0 corresponds to a point of $\mathfrak{h}_g$. Moreover one can easily see that the Cartan involution of J_0 fixes exactly its isotropy subgroup K with respect to the action of $\mathrm{Sp}(V,Q)(\mathbb{R})$ on $\mathfrak{h}_g$. Since all points of $\mathfrak{h}_g$ correspond to complex structures conjugate to J_0, the corresponding involutions, which are conjugate by inner automorphisms, are the Cartan involutions fixing the respective isotropy subgroups.

Definition 1.3.11. Let M be a $\mathcal{C}^\infty$ manifold. A Riemannian structure on M is a symmetric tensor field Q of type $(0,2)$, which yields a positive definite non-degenerate bilinear form on $T_p(M)$ for all $p \in M$.

Definition 1.3.12. Let M be a connected $\mathcal{C}^\infty$ manifold with an almost complex structure J. A Riemannian structure g on M is a Hermitian structure, if $g(J\cdot, J\cdot) = g(\cdot,\cdot)$.

Definition 1.3.13. Let D be a connected complex manifold with a Hermitian structure. It is a Hermitian symmetric space, if each point is an isolated fixed point of an involutive holomorphic isometry on D. Let $\mathrm{Hol}(D,g)$ denote the Lie group of holomorphic isometries.

Moreover let $\mathrm{Hol}(D,g)^+$ be a non-compact semisimple Lie group endowed with an involution ι, which induces by its differential a Cartan involution on $Lie(\mathrm{Hol}(D,g))$, and $K_\iota \subset \mathrm{Hol}(D,g)^+$ be the subgroup, on which ι acts as id. The Hermitian symmetric space D is a Hermitian symmetric domain, if the isotropy group K of one point $p \in D$ satisfies $K_\iota^+ \subseteq K \subseteq K_\iota$.

Definition 1.3.14. A bounded symmetric domain D is an open, bounded, connected submanifold D of $\mathbb{C}^N$, which has the property that each $p \in D$ is an isolated fixed point of an involutive holomorphic diffeomorphism onto itself.

Theorem 1.3.15. *Each bounded symmetric domain D can be equipped with a unique Hermitian metric (called Bergman metric), which turns D into a Hermitian symmetric domain. Conversely each Hermitian symmetric domain has a holomorphic diffeomorphism onto a bounded symmetric domain.*

Proof. The correspondence between Hermitian symmetric domains and bounded symmetric domains is given by [23], Theorem **VIII**, 7.1. The uniqueness of the Bergman metric follows from the fact that each holomorphic diffeomorphism between bounded symmetric domains is an isometry with respect to the Bergman metric (see [23], Proposition **VIII**, 3.5.). □

Definition 1.3.16. A Shimura datum (G,h) is given by a reductive $\mathbb{Q}$-algebraic group G and a conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ of algebraic groups satisfying:

1. The inner automorphism of $(\mathrm{ad} \circ h)(i)$ on $G_{\mathbb{R}}^{\mathrm{ad}}$ is a Cartan involution.
2. The adjoint group G^{ad} does not have any direct $\mathbb{Q}$-factor H such the Cartan involution of (1) is trivial on $H_{\mathbb{R}}$.

3. The representation $(\mathrm{ad} \circ h)(\mathbb{S})$ on $Lie(G)_{\mathbb{C}}$ corresponds to a Hodge structure of the type $(1,-1) \oplus (0,0) \oplus (-1,1)$.

Example 1.3.17. The connected algebraic group $\mathrm{GSp}_{\mathbb{Q}}(2g)$ is the almost direct product of its central torus $\mathbb{G}_{m,\mathbb{Q}}$ and its simple derived group $\mathrm{Sp}_{\mathbb{Q}}(2g)$. Hence it is reductive. By Construction 1.3.1 and Example 1.3.10, we have a conjugacy class of complex structures, which corresponds to a conjugacy class of homomorphisms $h : \mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}(2g)$ satisfying the condition (1) of a Shimura datum. The adjoint group $\mathrm{GSp}_{2g}(\mathbb{Q})^{\mathrm{ad}} = \mathrm{Sp}_{2g}(\mathbb{Q})^{\mathrm{ad}}$ has only one direct simple factor on which the Cartan involution above is not trivial. Hence condition (2) of the Shimura datum is satisfied. Since the center of $\mathrm{GSp}_{\mathbb{R}}(2g)$ is given by $\mathbb{G}_{m,\mathbb{R}}$ (see [32], page 66), the kernel of the adjoint representation on $Lie(\mathrm{GSp}_{\mathbb{R}}(2g))$ of any $h(\mathbb{S})$ in the conjugacy class is given by $\mathbb{G}_{m,\mathbb{R}}$. Since $h(a+ib) = aE_{2g} + bJ$, each $g \in \mathrm{GSp}_{2g}(\mathbb{R})$ commutes with J, if and only if it commutes with each element of $\mathbb{S}(\mathbb{R})$. Hence on the complexified eigenspace $(\mathfrak{p}_0)_{\mathbb{C}}$ with eigenvalue -1 with respect to the Cartan involution, $\mathbb{S}$ acts by the characters $z/\bar{z}$ and $\bar{z}/z$. This corresponds to a Hodge structure of the type $(1,-1) \oplus (0,0) \oplus (-1,1)$ on $Lie(\mathrm{GSp}_{\mathbb{R}}(2g))$. Hence condition (3) is satisfied and the conjugacy class of $h_0 : \mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}(2g)$ with $h_0(i) = J_0$ is a Shimura datum.

Remark 1.3.18. Let (G,h) be a Shimura datum. Since $G^{\mathrm{ad}} = G/Z(G)$ and K is the centralizer of $h(\mathbb{S})$, one has $G(\mathbb{R})^+/(K(\mathbb{R}) \cap G(\mathbb{R})^+) = G^{\mathrm{ad}}(\mathbb{R})/\mathrm{ad}_G(K(\mathbb{R}))$.

The Cartan involution $int(\mathrm{ad} \circ h)(i)$ fixes exactly $\mathrm{ad}_G(K)$. By [43], **I**, Corollary 4.5, the subgroups of a connected $\mathbb{R}$-algebraic reductive group on which a Cartan involution acts as id are maximal compact. Hence $\mathrm{ad}_G(K(\mathbb{R}))$ is a maximal compact subgroup.

Remark 1.3.19. Let (L,h,Q) be a polarized integral Hodge structure of type $(1,0),(0,1)$ with corresponding complex structure $J \in Sp_{2g}(\mathbb{R})$, where Q is given by (1.3). The Cartan involution corresponding to J leaves $\mathrm{Hg}(L,h)_{\mathbb{R}} \subset \mathrm{Sp}_{\mathbb{R}}(2g)$ stable. Hence by [43] Theorem **I**. 4.2, the group $\mathrm{Hg}(L,h)_{\mathbb{R}}$ has a Cartan involution and $\mathrm{Hg}(L,h)$ is reductive.

Next we need to recall the definition of a variation of Hodge structures (VHS):

Definition 1.3.20. Let D be a complex manifold and R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$. A variation $\mathcal{V}$ of R-Hodge structures of weight k over D is given by a local system $\mathcal{V}_R$ of R-modules of finite rank and a filtration $\mathcal{F}^\bullet$ of $\mathcal{V}_{\mathcal{O}_D}$ by holomorphic subbundles such that:

1. Griffiths transversality condition holds.

2. $(\mathcal{V}_{R,p}, \mathcal{F}_p^\bullet)$ is an R-Hodge structure of weight k for all $p \in D$.

The variation $\mathcal{V}$ of Hodge structures is polarized, if there is a flat (i.e. locally constant) bilinear form Q on $\mathcal{V}_R$ such that $(\mathcal{V}_{R,p}, \mathcal{F}_p^\bullet, Q_p)$ is a polarized R-Hodge structure of weight k for all $p \in D$.

Theorem 1.3.21. *Let $h : \mathbb{S} \to G$ be a Shimura datum, $t \in \mathbb{N}$ and K denote the centralizer of $h(\mathbb{S})$. Then each connected component D^+ of $D = G(\mathbb{R})/K(\mathbb{R})$ has a unique structure of a Hermitian symmetric domain. These domains are isomorphic, where the connected component of the group of holomorphic isometries is given by the quotient of $G^{\mathrm{ad}}(\mathbb{R})$ by its direct compact factors. Each representation $\rho : G_{\mathbb{R}} \to \mathrm{GL}_{\mathbb{R}}(t)$ yields a holomorphic variation $(\mathbb{R}^t, \rho \circ h)_{h \in D}$ of Hodge structures on D.*

Proof. (See [13], 2.1.1.) □

Remark 1.3.22. The Lie group $\mathrm{GSp}_{2g}(\mathbb{R})$ has two connected components. One component consists of matrices with positive determinant and the other consists of matrices with negative determinant. Hence the corresponding homogeneous space D parametrizing the elements of the conjugacy class has two connected components. Since $\mathrm{GSp}_{2g}(\mathbb{R})^+$ is a product of $\mathrm{Sp}_{2g}(\mathbb{R})$ and $\mathbb{G}_m(\mathbb{R})^+$, where $\mathbb{G}_m(\mathbb{R})^+$ is contained in the stabilizers of all points, the corresponding connected homogeneous space may be identified with $\mathfrak{h}_g$ such that the preceding Theorem endows $\mathfrak{h}_g$ with the structure of a Hermitian symmetric domain. By the natural representation of $\mathrm{GSp}_{\mathbb{R}}(2g)$ on $\mathbb{R}^{2g}$, $\mathfrak{h}_g$ is endowed with the natural holomorphic variation of Hodge structures of type $(1,0),(0,1)$.

1.4 The construction of Shimura varieties

In the preceding section we have seen that a Shimura datum yields a bounded symmetric domain. This is the first step of the construction of a Shimura variety. For completeness we sketch the construction of a Shimura variety in this section. But later we will only need to use the language of Shimura data and their associated bounded symmetric domains.

Definition 1.4.1. Let G be a $\mathbb{Q}$-algebraic group. An arithmetic subgroup Γ of $G(\mathbb{Q})$ is a group, which is commensurable with $G(\mathbb{Z})$.

A subgroup Γ of a connected Lie group H is arithmetic, if there is a $\mathbb{Q}$-algebraic group G, an arithmetic subgroup Γ_0 of $G(\mathbb{Q})$ and a surjective homomorphism $\eta : G(\mathbb{R})^+ \to H$ of Lie groups with compact kernel such that $\eta(\Gamma_0) = \Gamma$.

The second step of the construction of a Shimura variety is given by the following theorem:

Theorem 1.4.2 (of Baily and Borel). *Let D be a bounded symmetric domain, and Γ be an arithmetic subgroup of $\mathrm{Hol}(D)^+$. Then the quotient $\Gamma\backslash D$ can be endowed with a structure of a complex quasi-projective variety. This structure is unique, if Γ is torsion-free.*

Proof. (see [13], 2.1.2. (or [4] for the construction of the structure of a complex variety)) □

Next one needs the ring of finite adèles[7], which is given by

$$\mathbb{A}^f = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \mathbb{Z}_p,$$

where p runs over all prime numbers. Hence $\mathbb{A}^f$ is the subring of $\prod \mathbb{Q}_p$ consisting of the (a_p) such that $a_p \in \mathbb{Z}_p$ for almost all a_p. Now let (G,h) be a Shimura datum, which gives the bounded symmetric domain D^+ by a connected component of the conjugacy class D of h and K be a compact open subgroup of $G(\mathbb{A}^f)$.

Definition 1.4.3. Let G be a $\mathbb{Q}$-algebraic group. A principal congruence subgroup of $G(\mathbb{Q})$ is

$$\Gamma(n) := \{g \in G(\mathbb{Z}) | g \equiv E_g \bmod n\}$$

for some $n \in \mathbb{N}$. A congruence subgroup of $G(\mathbb{Q})$ is a subgroup Γ containing $\Gamma(n)$ such that $[\Gamma : \Gamma(n)] < \infty$ for some $n \in \mathbb{N}$.

[7] One reason for the introduction of adèle rings is given by the fact that one wants to have canonical models of Shimura varieties over number fields in number theory. We do not use this here, but we write it down for completeness

Lemma 1.4.4. *Let K be a compact open subgroup of $G(\mathbb{A}^f)$. Then $\Gamma := K \cap G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{Q})$.*

Proof. (see [32], Proposition 4.1) □

The Shimura variety $\mathrm{Sh}_K(G,h)$ is given by the double quotient

$$\mathrm{Sh}_K(G,h) := G(\mathbb{Q})\backslash D \times G(\mathbb{A}^f)/K := G(\mathbb{Q})\backslash(D \times (G(\mathbb{A}^f)/K)).$$

Proposition 1.4.5. *Let K be a compact open subgroup of $G(\mathbb{A}^f)$, $C := G(\mathbb{Q})\backslash G(\mathbb{A}^f)/K$, and $\Gamma_{[g]} = gKg^{-1} \cap G(\mathbb{Q})^+$ for some $[g] \in C$. Then one has*

$$\mathrm{Sh}_K(G,h) = \bigsqcup_{[g]\in C} \Gamma_{[g]}\backslash D^+.$$

Proof. (see [32], Lemma 5.13) □

Hence the preceding proposition and the Theorem of Baily and Borel endow $\mathrm{Sh}_K(G,h)$ with the structure of an algebraic variety. By [32], Proposition 3.2, the surjection $G \to G^{\mathrm{ad}}$ maps a congruence subgroup of G onto an arithmetic subgroup of G^{ad}. Now we consider compact open subgroups with the property that the resulting arithmetic subgroups on $G^{\mathrm{ad}}(\mathbb{R}) = \mathrm{Hol}(D^+, g) = \mathrm{Hol}(D^+)$ are torsion-free. Recall that the structure of a complex quasi-projective variety on the quotient of a bounded symmetric domain by a torsion-free arithmetic group is unique. If $K' \subset K$, we have a natural morphism

$$\mathrm{Sh}_{K'}(G,h) \to \mathrm{Sh}_K(G,h). \tag{1.4}$$

By the projective limit running over all compact open $K \subset G(\mathbb{A}^f)$ proving a torsion-free arithmetic group on $G^{ad}(\mathbb{R})$, which is given via (1.4), we obtain the Shimura variety[8]

$$\mathrm{Sh}(G,h) = \varprojlim \mathrm{Sh}_K(G,h).$$

1.5 Shimura varieties of Hodge type

Now we know how to construct a Shimura variety. Hence next we construct the Shimura varieties resp., Shimura data, which we will need.

Definition 1.5.1. A Shimura datum (G,h) is of Hodge type, if there is an embedding $\rho : G \hookrightarrow \mathrm{GSp}_{2g,\mathbb{Q}}$ such that one has the Shimura datum of Example 1.3.17 by

$$\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \xrightarrow{\rho_{\mathbb{R}}} \mathrm{GSp}_{2g,\mathbb{R}}.$$

A Shimura variety SH is of Hodge type, if it is obtained by a Shimura datum (G,h) of Hodge type.

From now on we will write K, if mean the algebraic subgroup of $G_{\mathbb{R}}$ given by the centralizer of $h(\mathbb{S})$ resp., $h(S^1)$. Moreover for simplicity we will write K instead of $K(\mathbb{R})$ by an abuse of notation. In the respective situation the respective meaning will be clear.

[8] Some authors denote only $\mathrm{Sh}(G,h)$ as Shimura variety.

Construction 1.5.2. Let (V, h, Q) be a polarized $\mathbb{Q}$-Hodge structure of type $(1,0),(0,1)$. The conjugacy class of the representation $h : \mathbb{S} \to \mathrm{GSp}_{2g,\mathbb{R}}$ is the Shimura datum of Example 1.3.17. Hence the adjoint representation of $\mathbb{S}$ on $Lie(\mathrm{MT}(V,h))_{\mathbb{C}} \subset Lie(\mathrm{GSp}(V,E))_{\mathbb{C}}$ induces a Hodge structure of the same type (or of the type $(0,0)$). Moreover the same arguments imply that the inner automorphism corresponding to $(\mathrm{ad} \circ h)(i)$ is a Cartan involution on $\mathrm{MT}^{\mathrm{ad}}_{\mathbb{R}}(V,h)$. Hence $\mathrm{MT}(V,h)$ is reductive. Thus it remains to show that $\mathrm{MT}(V,h)^{\mathrm{ad}}$ does not have any non-trivial direct $\mathbb{Q}$-factor H on which the Cartan involution is trivial:

Let H be a simple direct $\mathbb{Q}$-factor of $\mathrm{MT}(V,h)^{\mathrm{ad}}$ with trivial Cartan involution. We have a surjection

$$s : \mathrm{MT}(V,h) \xrightarrow{\mathrm{ad}} \mathrm{MT}(V,h)^{\mathrm{ad}} \xrightarrow{pr_H} H,$$

which is obviously a homomorphism of $\mathbb{Q}$-algebraic groups. Hence the kernel $\tilde{K}$ of s is a $\mathbb{Q}$-algebraic group. The complex structure J, which satisfies that $\mathrm{ad}(J)$ is the Cartan involution, satisfies that all elements of the adjoint group $H_{\mathbb{R}}$ commute with $\mathrm{ad}(J)$. Hence J is contained in $\tilde{K}_{\mathbb{R}}$. Thus $Lie(H)_{\mathbb{C}}$ is contained in the Lie sub-algebra of $Lie_{\mathbb{C}}(\mathrm{MT}(V,h))$ on which $\mathbb{S}$ acts by the character 1. Hence one has $h(\mathbb{S}) \subset \tilde{K}_{\mathbb{R}}$, which implies $\tilde{K} = \mathrm{MT}(V,h)$ resp., $H = \{e\}$.

Hence we obtain a Shimura datum $h : \mathbb{S} \to MT(V,h)_{\mathbb{R}}$ of Hodge type.

Lemma 1.5.3.

$$\mathrm{Hg}(V,h) = (\mathrm{MT}(V,h) \cap \mathrm{SL}(V))^0$$

Proof. By the natural multiplication, we have a morphism

$$m : \mathrm{Hg}(V,h) \times \mathbb{G}_{m,\mathbb{Q}} \to \mathrm{MT}(V,h)$$

with finite kernel. The Zariski closure Z of $m(\mathrm{Hg}(V,h) \times \mathbb{G}_{m,\mathbb{Q}})$ in $\mathrm{MT}(V,h)$ is an algebraic subgroup of $\mathrm{MT}(V,h)$. Moreover one has that $h(\mathbb{S}) \subset Z_{\mathbb{R}} \subset \mathrm{MT}_{\mathbb{R}}(V,h)$. Hence $Z = \mathrm{MT}(V,h)$.

Since all homomorphisms $f : G \to G'$ of algebraic groups over algebraically closed fields satisfy $f(G) = \overline{f(G)}$ (see [1], Satz 2.1.8), we have the equality

$$\mathrm{Hg}_{\bar{\mathbb{Q}}}(V,h) \cdot \mathbb{G}_{m,\bar{\mathbb{Q}}} = Z_{\bar{\mathbb{Q}}} = \mathrm{MT}_{\bar{\mathbb{Q}}}(V,h).$$

Now let $M \in \mathrm{MT}(V,h)(\bar{\mathbb{Q}}) \cap \mathrm{SL}(V)(\bar{\mathbb{Q}})$. It is given by a product $N \cdot M_1$ with $N \in \mathbb{G}_m(\bar{\mathbb{Q}})$ and $M_1 \in \mathrm{Hg}(V,h)(\bar{\mathbb{Q}})$. Since $\mathrm{Hg}(V,h)(\bar{\mathbb{Q}}) \subset \mathrm{SL}(V)(\bar{\mathbb{Q}})$, one concludes $N \in \mathbb{G}_m(\bar{\mathbb{Q}}) \cap \mathrm{SL}(V)(\bar{\mathbb{Q}}) = \mu_n(\bar{\mathbb{Q}})$, where $\dim V = n$. If and only if $N \in Hg(V,h)(\bar{\mathbb{Q}})$, one has $M \in \mathrm{Hg}(V,h)(\bar{\mathbb{Q}})$. Hence by the fact that $\mu_n(\bar{\mathbb{Q}})$ is finite, one obtains the statement. □

Remark 1.5.4. For a polarized Hodge structure of weight 1 of a curve of genus g, we have a natural embedding $\mathrm{Hg}(V,h) \subset \mathrm{Sp}_{\mathbb{Q}}(2g)$. Since $\mu_{2g}(\bar{\mathbb{Q}})$ is not a subgroup of $Sp_{2g}(\bar{\mathbb{Q}})$ for $g > 1$ and for $g = 1$ one has $\mu_2 \subset h(S^1)$, we obtain the equality

$$\mathrm{Hg}(V,h) = \mathrm{MT}(V,h) \cap \mathrm{SL}(V)$$

only in the case of a genus one curve.

Remark 1.5.5. Now assume that (V,h,Q) is a polarized $\mathbb{Q}$-Hodge structure of type $(1,0),(0,1)$. Since $\mathrm{MT}(V,h)$ is reductive and $\mathrm{MT}(V,h)^{\mathrm{der}}$ is semisimple in this case, one concludes by Lemma 1.5.3 that $\mathrm{MT}(V,h)^{\mathrm{der}} = \mathrm{Hg}(V,h)^{\mathrm{der}}$. Thus one has that

$\mathrm{MT}(V,h)^{\mathrm{ad}}(\mathbb{R}) = \mathrm{Hg}(V,h)^{\mathrm{ad}}(\mathbb{R})$. Hence by the preceding construction $\mathrm{Hg}(V,h)^{\mathrm{ad}}(\mathbb{R})$ is the identity component of the holomorphic isometry group of a Hermitian symmetric domain. The isotropy group of a point is given by a maximal compact subgroup of $\mathrm{Hg}(V,h)^{\mathrm{ad}}(\mathbb{R})$ fixed by the Cartan involution on $\mathrm{Hg}(V,h)^{\mathrm{ad}}_{\mathbb{R}}$ of the corresponding complex structure $J \in \mathrm{Hg}(V,h)(\mathbb{R})$. Hence one can consider $h|_{S^1} : S^1 \to \mathrm{Hg}(V,h)_{\mathbb{R}}$ as Shimura datum, too.

Now we construct the holomorphic family of Jacobians over $\mathrm{Hg}(V,h)(\mathbb{R})/K$ corresponding to the VHS induced by the embedding $\mathrm{Hg}(V,h) \hookrightarrow \mathrm{Sp}_{\mathbb{Q}}(2g)$, where (V,h) is of type $(1,0),(0,1)$.[9]

Construction 1.5.6. Let (L,h,Q) be a polarized $\mathbb{Z}$-Hodge structure of type $(1,0),(0,1)$ with $V := L_{\mathbb{Q}}$ as before and $\{v_1,\ldots,v_g,w_1,\ldots,w_g\}$ be a symplectic basis of L with respect to Q. For example it may be given on $L := H^1(C,\mathbb{Z})$, where C is a curve of genus g. One has that $\mathrm{Hg}(V,h) \subset \mathrm{Sp}(V,Q)$. Let $K \subset \mathrm{Hg}(V,h)(\mathbb{R})^+$ be the centralizer of $h(S^1(\mathbb{R}))$. Thus $\mathrm{Hg}(V,h)(\mathbb{R})^+/K$ is a Hermitian symmetric domain as we have seen. Consider the linearly independent set $B = \{[w_1],\ldots,[w_g]\} \subset H^{0,1}$, which generates the real subvector space W. Now iW is obviously generated by $\{[Jw_1],\ldots,[Jw_g]\}$. The principal polarization H of the Abelian variety $A = H^{0,1}/L$ is given by the corresponding alternating form $E = -Q$ as in the proof of Theorem 1.2.7. Since E vanishes on W, the principal polarization H given by $H = E(i.,.) + iE(.,.)$ vanishes on the complex vector space $W \cap iW$, too. Hence $W \cap iW = 0$. Thus the fact that $Span_{\mathbb{R}}(v,Jv)$ is mapped to $Span_{\mathbb{C}}([v])$ implies that B is a $\mathbb{C}$-basis of $H^{0,1}$. Hence the period matrix of the corresponding Abelian variety may be given by (Z,E_g), where the columns of Z are given by the $[v_i]$ in their coordinates with respect to B.

Thus the embedding $H^{1,0} \hookrightarrow V_{\mathbb{C}}$ is given by the matrix $(-E_g,Z^t)^t$. Since we have a holomorphic variation of Hodge structures, this matrix varies holomorphically. Thus the period matrices of the corresponding Abelian varieties vary holomorphically, too. Hence the corresponding action of L on $H^{0,1} \times \mathrm{Hg}(V,h)(\mathbb{R})/K$ is holomorphic and we obtain a holomorphic family of Abelian varieties over $\mathrm{Hg}(V,h)(\mathbb{R})/K$.

Now recall that our main interest is not the theory of Shimura varieties, but families with a dense set of CM points defined below:

Definition 1.5.7. Let D be a complex manifold and $\mathcal{V}$ be a holomorphic variation of rational Hodge structures. A point $p \in D$ is a CM point with respect to $\mathcal{V}$, if $\mathcal{V}_p$ has a commutative Hodge group.

Let $\mathcal{X} \to D$ be a holomorphic family of complex manifolds. A point $p \in D$ is a CM point with respect to $\mathcal{X}$, if $\mathcal{X}_p$ is a CM fiber resp., $\mathcal{X}_p$ has a complex multiplication.

Now we give criteria for dense sets of CM points, which imply that the family of Abelian varieties over $\mathrm{Hg}(V,h)(\mathbb{R})/K$ of Construction 1.5.6 has a dense set of CM fibers:

Lemma 1.5.8. *Let (G,h) denote a Shimura datum. If the connected component $G(\mathbb{R})/K$ contains a CM point with respect to a VHS induced by some closed embedding $G \to \mathrm{GL}(W)$ for some $\mathbb{Q}$-vector space W, then the set of CM points of the same type with respect to the same VHS is dense in D.*

[9] This construction and the rest of this section are similar to [39], §1 with some technical differences.

Proof. We have two cases. Assume that G is a $\mathbb{Q}$-algebraic torus. In this case $G(\mathbb{R})/K$ consists of one point. The fact that we have a closed embedding $G \hookrightarrow \mathrm{GL}(W)$ implies that the Hodge group of the Hodge structure over this point is a subtorus of the torus G.

In the other case G is not a $\mathbb{Q}$-algebraic torus. By the assumptions, we have a CM point in $G(\mathbb{R})/K$ with respect to the VHS induced by some closed embedding $G \to \mathrm{GL}(W)$. This implies that G contains a $\mathbb{Q}$-algebraic torus T such that the conjugacy class of $h : \mathbb{S} \to G_{\mathbb{R}}$ contains an element, which factors through $T_{\mathbb{R}}$. By our preceding construction, the stabilizer of the CM point $[s_0]_K \in G(\mathbb{R})/K$ is given by $s_0 K s_0^{-1}$. Thus one can replace K by $s_0 K s_0^{-1}$. In this case the fact that the VHS is unduced by an embedding $G \hookrightarrow \mathrm{GL}(W)$ implies that the Hodge group of the Hodge structure over $[e]$ is a subtorus of T. Hence $[e]$ is a CM point with respect to this VHS, and any $s \in G(\mathbb{Q}) \subset G(\mathbb{R})$ has the property that it is mapped to a CM point, too. By the Real Approximation Theorem, $G(\mathbb{Q})$ lies dense in the manifold $G(\mathbb{R})$ for all connected affine $\mathbb{Q}$-algebraic groups G. Since the quotient map is continuous, the set of CM points in $G(\mathbb{R})/K$ is dense. □

Theorem 1.5.9. *Let (G, h) denote a Shimura datum. The set of CM points with respect to the VHS induced by some closed embedding $G \to \mathrm{GL}(W)$ for some $\mathbb{Q}$-vector space W is dense in $G(\mathbb{R})/K$.*

Proof. By the preceding lemma, we have only to show that there exists one CM point on $G(\mathbb{R})/K$. By the closed embedding $G \to \mathrm{GL}(W)$, each $\mathbb{Q}$-algebraic torus of G can be identified with a $\mathbb{Q}$-algebraic torus of $\mathrm{GL}(W)$. Thus the existence of a CM point is equivalent to the statement that there is a $h : \mathbb{S} \to G_{\mathbb{R}}$ in this VHS, which factors through a $\mathbb{Q}$-algebraic torus of G.

Now let T be a maximal ($\mathbb{Q}$-algebraic) torus of G. The centralizers of the maximal tori (resp., the Cartan subgroups) of a reductive group are the maximal tori (see [9], IV. 13.17.). The torus $T_{\mathbb{R}}$ is contained in a maximal torus T_M of $G_{\mathbb{R}}$, which has the property that each point of T_M is contained in the centralizer of $T_{\mathbb{R}}$ resp., in the centralizer of T. Thus the torus $T_{\mathbb{R}}$ is in fact maximal in $G_{\mathbb{R}}$.

The Cartan subgroups, i.e. the centralizers of the maximal tori, which are the maximal tori in our case, are conjugate (see [9], IV. 12.1.). The stabilizer of the point given by h in $G_{\mathbb{R}}/K$ is the centralizer K of $h(\mathbb{S})$. The center of K, which is a torus contained in a maximal torus T_1, contains obviously $h(\mathbb{S})$ resp., we have a maximal torus T_1 containing $h(\mathbb{S})$, where $T_1 \subset K$. Thus by the fact that T_1 is conjugate to $T_{\mathbb{R}}$ by some element s_0 and our preceding construction, the Hodge group of $s_0 \circ K \in G(\mathbb{R})/K$ is a subtorus of T. Hence $s_0 \circ K$ is a CM point. □

Chapter 2

Cyclic covers of the projective line

2.1 Description of a cyclic cover of the projective line

Let us first repeat some known facts about Galois covers of $\mathbb{P}^1$.

Definition 2.1.1. Let T_1, T_2, and S be topological spaces resp., complex manifolds resp., algebraic varieties. The coverings $f_1 : T_1 \to S$ and $f_2 : T_2 \to S$, which are morphisms in the respective category, are called equivalent, if there is an isomorphism $g : T_1 \to T_2$ in the respective category such that $f_1 = f_2 \circ g$.

Proposition 2.1.2. *Let G be a finite group, and $S := \{a_1, \ldots, a_n\} \subset \mathbb{A}^1 \subset \mathbb{P}^1$. There is a correspondence between the following objects:*

1. *The isomorphism classes of Galois extensions of $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(x)$ with Galois group G and branch points contained in S.*
2. *The equivalence classes of (non-ramified) Galois coverings $f : R \to \mathbb{P}^1 \setminus S$ of topological spaces with deck transformation group isomorphic to G.*
3. *The normal subgroups in the fundamental group $\pi_1(\mathbb{P}^1 \setminus S)$ with quotient isomorphic to G.*

Proof. (see [50], Theorem 5.14) □

Remark 2.1.3. We will need to understand the correspondence of the preceding Proposition. The correspondence between (1) and (2) is given by the facts that a Galois covering $f : R \to \mathbb{P}^1 \setminus S$ (of topological spaces) yields a covering $f : \bar{R} \to \mathbb{P}^1$ of compact Riemann surfaces, and any morphism of compact Riemann surfaces corresponds to an embedding of their function fields.

The correspondence between (2) and (3) is given by the path lifting properties of coverings of Hausdorff spaces. Take $b \in R$. Let $p = f(b)$, and $\gamma \in \pi_1(\mathbb{P}^1 \setminus S, p)$, and $f^*(\gamma(0)) = b$. Then $f^*(\gamma(1)) = g \cdot b$ for some $g \in G \cong \mathrm{Deck}(R/(\mathbb{P}^1 \setminus P))$. This induces a homomorphism $\Phi_b : \pi_1(\mathbb{P}^1 \setminus S, p) \to G$ and hencefore a kernel of this homomorphism, which is a normal subgroup.

Remark 2.1.4. Let $f : R \to \mathbb{P}^1$ be a Galois covering with branch points $a_1, \ldots, a_n$. One can choose $\gamma_1, \ldots, \gamma_n \in \pi_1(\mathbb{P}^1 \setminus P)$ such that each γ_k is given by a loop running counterclockwise "around" exactly one a_k. Hence one has that

$$\gamma_n = \gamma_1^{-1} \ldots \gamma_{n-1}^{-1}$$

and we conclude that

$$\Phi_b(\gamma_n) = \Phi_b(\gamma_1)^{-1} \dots \Phi_b(\gamma_{n-1})^{-1}.$$

From now on we consider only irreducible cyclic covers of $\mathbb{P}^1$. An irreducible cyclic cover can be given by a prime ideal

$$(y^m - (x-a_1)^{d_1} \cdot \ldots \cdot (x-a_n)^{d_n}) \subset \mathbb{C}[x,y].$$

First this ideal defines only an affine curve in $\mathbb{A}^2$, which has singularities, if there are some $d_k > 1$. But there exists a unique smooth projective curve birationally equivalent to this affine curve. By the natural projection onto the x-axis, one obtains a cyclic cover of the birationally equivalent projective smooth curve onto $\mathbb{P}^1$.

Remark 2.1.5. Let us consider the cover given by

$$y^m = (x-a_1)^{d_1} \cdot \ldots \cdot (x-a_n)^{d_n},$$

and fix a $k_0 \in \{1, \dots n\}$. By an automorphism of $\mathbb{P}^1$, one can put a_{k_0} onto 0. Let $\mu_{k_0} = \frac{d_{k_0}}{m} \in \mathbb{Q}$, and D a small disc centered in 0, which does not contain any other a_k with $k \neq k_0$. Take any point $p \in \partial D$ and remove the line $[0,p]$. The topological space $D \setminus [0,p]$ is simply connected. Hence one can define root functions $z \to z^{\mu_{k_0}}$ on this space.

These functions on $D \setminus [0,p]$ are given by:

$$z^{\mu_{k_0}} = |z|^{\mu_{k_0}} \exp(\frac{2\pi i t d_{k_0}}{m} + 2\pi i \frac{\ell}{m}) \quad (\text{with} \quad \ell = 0,1,\dots,m-1 \quad \text{and} \quad z = |z| \exp(2\pi i t))$$

Since the cover is given by $y^m = x^{d_{k_0}}$ resp., $y = x^{\mu_{k_0}}$ over a small disc around 0, we may lift a closed path around 0 to some path with starting point $(z, z^{\mu_{k_0}})$ and ending point $(z, e^{2\pi i \mu_{k_0}} z^{\mu_{k_0}})$.

Definition 2.1.6. Let $e^{2\pi i \mu_{k_0}}$ and d_{k_0} be given by Remark 2.1.5. Then $e^{2\pi i \mu_{k_0}}$ is the local monodromy datum of d_{k_0}.

Lemma 2.1.7. *Assume that $d_1, \dots, d_n < m$. Let the (non-singular projective) curve C be given by*

$$y^m = (x-a_1)^{d_1} \cdot \ldots \cdot (x-a_n)^{d_n}.$$

Then the Galois group G is $\mathbb{Z}/(m)$, and the covering $C \to \mathbb{P}^1$ is given by the kernel of the homomorphism given by $\gamma_k \to d_k \in \mathbb{Z}/(m)$. If and only if m does not divide $\sum_{k=1}^{n} d_k$, the point ∞ is a branch point and

$$\gamma_\infty \to -\sum_{k=1}^{n} d_k \quad mod \quad m.$$

Proof. The last statement of the lemma follows by the preceding rest of the lemma and the Remark 2.1.4.

The Galois group and $\mathbb{Z}/(m)$ are obviously isomorphic. Let us remove the ramification points of C. Then we obtain a Riemann surface R. Now take a small loop γ_k around p_k, which starts and ends in $p \in \mathbb{P}^1$. Now take a point $b \in R$ with $f(b) = p$. The definition of R and Remark 2.1.5 imply that the lifting $f^*(\gamma_k)$ of the path γ_k starting in b ends in the point $d_k \cdot b$. Hence the statement follows from Proposition 2.1.2 resp., Remark 2.1.3. $\square$

Let $d \in \mathbb{Z}$ and $1 < m \in \mathbb{N}$. The residue class of d in $\mathbb{Z}/(m)$ is denoted by $[d]_m$.

Remark 2.1.8. Let $G = \mathbb{Z}/(m)$, and $[d]_m \in \mathbb{Z}/(m)^*$. We consider the kernels of the monodromy representations of the covers locally given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}$$

and

$$y^m = (x - a_1)^{[dd_1]_m} \cdot \ldots \cdot (x - a_n)^{[dd_n]_m}.$$

By the preceding lemma, these kernels coincide. Hence we conclude that both covers are equivalent.

2.2 The local system corresponding to a cyclic cover

Now let us assume that our cover $\pi : C \to \mathbb{P}^1$ is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n},$$

where m divides $d_1 + \ldots + d_n$ and ∞ is not a branch point. Moreover let

$$S := \{a_1, \ldots, a_n\}.$$

Definition 2.2.1. Let X be a complex algebraic manifold, $\mathcal{L}$ an invertible sheaf on X and

$$D = \sum b_k D_k$$

a normal crossing divisor on X such that $\mathcal{L}^m = \mathcal{O}(D)$ and $b_k < m$ for each k. Then by $\mathcal{L}$ and D, one can construct a cyclic cover of degree m onto X (see [16], §3). The number b_k is called the branch index of D_k with respect to this cyclic cover.

Example 2.2.2. In the case of

$$X = \mathbb{P}^1, \quad D = \sum_{k=1}^{n} d_k a_k, \quad \mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(\frac{1}{m} \sum_{k=1}^{n} d_k),$$

the cyclic cover of the preceding definition is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}.$$

Next we describe the local system $\pi_*(\mathbb{C})|_{\mathbb{P}^1 \setminus S}$ and its monodromy.

Lemma 2.2.3. *Let V be a $\mathbb{C}$-vector space of dimension n, and X be an arcwise connected and locally simply connected topological space with $x \in X$. Then the monodromy representation provides a bijection between the set of isomorphism classes of local systems of stalk V and the set of representations*

$$\pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C}),$$

modulo the action of $\mathrm{Aut}_{\mathbb{C}}(V)$ by conjugation.

Proof. (see [49], Remark 15.12) □

Since $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^*$ is commutative, we can conclude:

Corollary 2.2.4. *The monodromy yields a bijection between the set of isomorphism classes of rank one local systems on $\mathbb{P}^1 \setminus S$ and the set of representations*

$$\pi_1(\mathbb{P}^1 \setminus S) \to \mathrm{GL}_1(\mathbb{C}).$$

The Galois group of our covering curve is isomorphic to $\mathbb{Z}/(m)$ and generated by a map ψ, which is given by $(x, y) \to (x, e^{2\pi i \frac{1}{m}} y)$ with respect to the above affine curve contained in $\mathbb{A}^2$, which is birationally equivalent to the covering curve. Hence a character χ of this group is determined by $\chi(\psi)$ with $\chi(\psi) \in \{e^{2\pi i \frac{j}{m}} | j = 0, 1, \ldots, m-1\}$. Thus the character group is isomorphic $\mathbb{Z}/(m)$ and we identify the character, which maps ψ to $e^{2\pi i \frac{j}{m}}$, with $j \in \mathbb{Z}/(m)$.[1]

Let D be an arbitrary disc contained in $\mathbb{P}^1 \setminus S$. The preimage of D is given by the disjoint union of discs D_r with $r = 0, 1, \ldots, m-1$ such that $\psi(D_r) = D_{[r+1]_m}$. The vector space $\pi_* \mathbb{C}_C|_{\mathbb{P}^1 \setminus S}(D)$ has the basis $\{v_j | j = 0, 1, \ldots, m-1\}$, where

$$v_j := (e^{\frac{2\pi j(m-1)}{m}}, \ldots, e^{\frac{2\pi j}{m}}, 1),$$

and the r-th. coordinate denotes the value of the corresponding section of $\pi^{-1}(D)$ on D_r. By the push-forward action, each v_j is an eigenvector with respect to the character given by j. Since D is arbitrary, one can glue the local eigenspaces, and obtain an eigenspace decomposition

$$\pi_* \mathbb{C}_C|_{\mathbb{P}^1 \setminus S} = \bigoplus_{j=0}^{m-1} \mathbb{L}_j$$

into rank 1 local systems, where $\mathbb{L}_j$ is the eigenspace with respect to the character given by $j \in \mathbb{Z}/(m)$. Hence the monodromy representation $\rho : \pi_1(\mathbb{P}^1 \setminus S) \to GL_m(\mathbb{C})$ has the corresponding decomposition

$$\rho = (\rho_0, \rho_1, \ldots, \rho_{m-1}) : \pi_1(\mathcal{X}) \to \prod_{i=0}^{m-1} \mathrm{GL}_1(\mathbb{C}),$$

where

$$\rho_j : \pi_1(\mathbb{P}^1 \setminus S) \to \mathrm{GL}_1(\mathbb{C})$$

is the monodromy representation of $\mathbb{L}_j$ for all $j = 0, 1, \ldots, m-1$.

Let us recall that our cyclic cover C is given by

$$y^m = (x - a_1)^{d_1} \ldots (x - a_n)^{d_n},$$

where ∞ is not a branch point. Now let $x \in \mathbb{P}^1 \setminus S$, and $x \in D$, where D is a sufficiently small open disc as above. Take a counterclockwise loop γ_k around a_k with $\gamma_k(0) = \gamma_k(1) = x$ and cover the loop with a finite number of (sufficiently) small discs. The continuation of $\tilde{s}$ on the unification of these discs leads to a multi-section. By Remark 2.1.5, the possible liftings $\gamma_k^{(r)}$ of the loop γ_k are paths with starting point $\gamma_k^{(r)}(0) = y_r$, where $y_r \in D_r$ and ending point $\gamma_k^{(r)}(1) = y_{[d_k + r]_m}$. This implies that the monodromy representation of $\mathbb{L}_j$ maps γ_k to $e^{\frac{2\pi j d_k}{m}}$. Hence we conclude:

[1] These two identifications with $\mathbb{Z}/(m)$ are obviously not canonical, but useful for the description of $\pi_* \mathbb{C}_C|_{\mathbb{P}^1 \setminus S}$ by using our explicit equation for $\pi : C \to \mathbb{P}^1$ as we will see a little bit later.

Theorem 2.2.5. *Let the cyclic cover $\pi : C \to \mathbb{P}^1$, which is not branched over ∞, be given by*

$$y^m = (x - a_1)^{d_1} \dots (x - a_n)^{d_n}. \tag{2.1}$$

Then the local system $\pi_*\mathbb{C}|_{\mathbb{P}^1 \setminus S}$ is given by the monodromy representation

$$\gamma_k \to \{(x_j)_{j=0,1\dots,m-1} \to (e^{\frac{2\pi i j d_k}{m}} x_j)_{j=0,1\dots,m-1}\}.$$

Remark 2.2.6. One can consider $\pi_*(\mathbb{Q}(e^{2\pi i \frac{1}{m}}))|_{\mathbb{P}\setminus S}$, too. Since a generator ψ of $\mathrm{Gal}(C;\mathbb{P}^1)$ satisfies $\psi^m = 1$, the minimal polynomial of its action on $\pi_*(\mathbb{Q}(e^{2\pi i \frac{1}{m}}))|_{\mathbb{P}\setminus S}$ decomposes into linear factors contained in $\mathbb{Q}(e^{2\pi i \frac{1}{m}})[x]$. Hence the eigenspace decomposition is defined over $\mathbb{Q}(e^{2\pi i \frac{1}{m}})$.

Each local system L of $\mathbb{C}$-vector spaces on any topological space X has a dual local system $L^\vee$ given by the sheafification of the presheaf

$$U \to \mathrm{Hom}_{\mathbb{C}}(L, \mathbb{C}).$$

Proposition 2.2.7. *One has*

$$\mathbb{L}_j^\vee = \bar{\mathbb{L}}_j.$$

Furthermore the monodromy representation $\mu_{\mathbb{L}_j^\vee}$ of $\mathbb{L}_j^\vee$ is given by $\mu_{\mathbb{L}_j^\vee}(\gamma_s) = \overline{\mu_{\mathbb{L}_j}(\gamma_s)}$ for all $s \in S$.

Proof. (see [15], Proposition 2) □

Hence by the respective monodromy representations, we obtain for all $j = 1, \dots, m-1$:

Corollary 2.2.8.

$$\mathbb{L}_j^\vee = \mathbb{L}_{m-j}$$

Let $r|m$. We consider the $\mathbb{C}$-algebra endomorphism Φ_r of $\mathbb{C}[x, y]$ given by $x \to x$ and $y \to y^r$. The (non-singular) curve C is birationally equivalent to the affine variety given by $\mathrm{Spec}(\mathbb{C}[x, y]/I)$, where

$$I = (y^m - (x - a_1)^{d_1} \dots (x - a_n)^{d_n}).$$

By Φ_r, we obtain the prime ideal

$$\Phi_r^{-1}(I) = (y^{\frac{m}{r}} - (x - a_1)^{d_1} \dots (x - a_n)^{d_n}).$$

Let C_r be the irreducible projective non-singular curve birationally equivalent to the affine variety given by $\mathrm{Spec}(\mathbb{C}[x, y]/\Phi_r^{-1}(I))$.

Remark 2.2.9. By the equation above, we have a cover $\pi_r : C_r \to \mathbb{P}^1$ of degree $\frac{m}{r}$. The homomorphism Φ_r induces a cover $\phi_r : C \to C_r$ of degree r such that

$$\pi = \pi_r \circ \phi_r.$$

Proposition 2.2.10.

$$(\pi_r)_*\mathbb{C}_{C_r}|_{\mathbb{P}^1\setminus S} = \bigoplus_{j=0}^{\frac{m}{r}-1} \mathbb{L}_{r\cdot j} \subset \pi_*\mathbb{C}_C|_{\mathbb{P}^1\setminus S}.$$

Proof. Let $m_0 := \frac{m}{r}$. By Theorem 2.2.5, the monodromy representation of the local system $(\pi_r)_*\mathbb{C}_{C_r}|_{\mathbb{P}^1\setminus\{a_1,\dots,a_n\}}$ is given by

$$\gamma_k \to \{(x_j)_{j=0,1\dots,\frac{m}{r}-1} \to (e^{\frac{2\pi i j d_k}{m_0}} x_j)_{j=0,1\dots,\frac{m}{r}-1} = (e^{\frac{2\pi i j r d_k}{m}} x_j)_{j=0,1\dots,\frac{m}{r}-1}\}.$$

By the respective monodromy representations of the local systems $\mathbb{L}_j$, this yields the statement. □

2.3 The cohomology of a cover

In this section we discuss some known facts about the eigenspace decomposition of the Hodge structure of a curve C with respect to a cyclic cover $\pi : C \to \mathbb{P}^1$. The main reference for this section is given by §3 of the book [16] of H. Esnault and E. Viehweg. Section 2 of the essay [14] of P. Deligne and G. D. Mostow contains additional information about our case.

Let $\pi : C \to \mathbb{P}^1$ be given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}$$

such that ∞ is not a branch point,

$$S = \{a_1, \ldots, a_n\}, \quad D = d_1 a_1 + \ldots + d_n a_n \quad \text{and} \quad \mathcal{L}^{(j)} := \mathcal{O}_{\mathbb{P}^1}(j\frac{d_1 + \ldots + d_n}{m} - \sum_{k=1}^{n+3}[\frac{j}{m} \cdot d_k]).$$

Moreover let the generator ψ of the Galois group of π be given by $(x, y) \to (x, e^{2\pi i \frac{1}{m}} y)$ with respect to the explicit equation above, which yields π.

We fix some new notation: Let $[q]_1 := q - [q]$ for all $q \in \mathbb{Q}$. Moreover we define

$$S_j := \{a \in S | [j\mu_a]_1 \neq 0\}.$$

Proposition 2.3.1. *The sheaves $\pi_*(\mathcal{O})$ and $\pi_*(\omega)$ have a decomposition into eigenspaces with respect to the Galois group representation, which are given by the sheaves $\mathcal{L}^{(j)^{-1}}$ and*

$$\omega_j := \omega_{\mathbb{P}^1}(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}} \quad \textit{with} \quad D^{(j)} := \sum_{a \in S_j} a$$

for $j = 0, 1, \ldots, m-1$ such that ψ acts via pull-back by the character $e^{2\pi i \frac{j}{m}}$ on $\mathcal{L}^{(j)^{-1}}$ resp., ω_j.

Proof. The eigenspace decomposition of $\pi_*(\mathcal{O})$ follows by [16], Corollary 3.11. Moreover [16], Lemma 3.16, d) yields the decomposition of $\pi_*(\omega)$ into the claimed sheaves. Since $\mathcal{L}^{(j)^{-1}}$ is an eigenspace with respect to the Galois group representation, ω_j is an eigenspace of the same eigenvalue. □

Remark 2.3.2. One has obviously $h^0(\omega_0) = 0$. By [16], 2.3, c), one concludes that

$$\omega_{\mathbb{P}^1}(\log D^{(j)}) = \omega_{\mathbb{P}^1}(D^{(j)})$$

for $j = 1, \ldots, m-1$. Hence for $j = 1, \ldots, m-1$ we obtain

$$h^0(\omega_j) = h^0(\mathcal{O}_{\mathbb{P}^1}(-2 + deg(D^{(j)}) - j\frac{d_1 + \ldots + d_{n+3}}{m} + \sum_{k=1}^{n+3}[\frac{j}{m} \cdot d_k]))$$

$$= -1 + |S_j| + \sum_{a \in S_j}(-j\mu_a + [j\mu_a]) = -1 + \sum_{a \in S_j}(1 - [j\mu_a]_1).$$

But here we want to determine our eigenspaces on $\pi_*(\omega_C)$ with respect to the push-forward action. Thus we put $\omega^{(j)} := \omega_{[m-j]_m}$, and we obtain

$$h_j^{1,0}(C) := h^0(\omega^{(j)}) = h^0(\omega_{[m-j]_m}) = -1 + \sum_{a \in S_j}(1 - [(m-j)\mu_a]_1) = -1 + \sum_{a \in S_j}[j\mu_a]_1.$$

Moreover let $H_j^{0,1}(C)$ denote the vector space of antiholomorphic 1-forms on C with respect to the corresponding character of the Galois group action. Since the push-forward action of the Galois group respects the alternating form of the polarization of the Hodge structure on $H^1(C, \mathbb{Z})$, one concludes that $H^{0,1}_{[m-j]_m}(C)$ is the dual of $H_j^{1,0}(C)$. Thus:

Proposition 2.3.3. *We have the eigenspace decomposition*

$$H^1(C, \mathbb{C}) = \bigoplus_{j=1}^{m-1} H_j^1(C, \mathbb{C}) \quad \textit{with} \quad H_j^{1,0}(C) \oplus H_j^{0,1}(C) = H_j^1(C, \mathbb{C}).$$

Moreover by $h_j^{0,1}(C) = h^{1,0}_{[m-j]_m}(C)$ and the preceding calculations, one concludes:

Proposition 2.3.4. *We have*

$$h_j^{1,0}(C) = \sum_{s \in S_j}[j\mu_s]_1 - 1, \text{ and } h_j^{0,1}(C) = \sum_{s \in S_j}(1 - [j\mu_s]_1) - 1.$$

The preceding two propositions imply:

Corollary 2.3.5.

$$h_j^1(C, \mathbb{C}) = |S_j| - 2$$

2.4 Cyclic covers with complex multiplication

Let us now search for examples of covers of $\mathbb{P}^1$ with complex multiplication. The family given by

$$\begin{aligned} \mathbb{P}^2 \ni V(y^m - x_1(x_1 - x_0)(x_1 - a_1x_0)\dots(x_1 - a_{m-3}x_0)) \\ \to (a_1, \dots, a_{m-3}) \in (\mathbb{A}^1 \setminus \{0,1\})^{m-3} \setminus \{a_i = a_j | i \neq j\} \end{aligned}$$

has obviously a fiber isomorphic to the Fermat curve $\mathbb{F}_m$, which is given by $V(y^m + x^m + 1)$ and has complex multiplication (see [19] and [27]). For another family with a fiber with complex multiplication, we must work a little bit.

Lemma 2.4.1. *If (V, h_1) and (W, h_2) are two $\mathbb{Q}$-Hodge structures of weight k, then*

$$\mathrm{Hg}(V \oplus W, h_1 \oplus h_2) \subset \mathrm{Hg}(V, h_1) \times \mathrm{Hg}(W, h_2) \subset \mathrm{GL}(V) \times \mathrm{GL}(W) \subset \mathrm{GL}(V \oplus W),$$

and the projections

$$\mathrm{Hg}(V \oplus W) \to \mathrm{Hg}(V), \quad \textit{and} \quad \mathrm{Hg}(V \oplus W) \to \mathrm{Hg}(W)$$

are surjective.

Proof. (see [46], Lemma 8.1) □

Lemma 2.4.2. *Let $V \subset W$ be a rational sub-Hodge structure of a polarized Hodge structure W. Then we have a direct sum decomposition*

$$W = V \oplus V',$$

where V' is also a rational sub-Hodge structure of W.

Proof. (see [49], Lemma 7.26) □

Lemma 2.4.3. *A curve C, which is covered by the Fermat curve $\mathbb{F}_m$ given by $V(x^m + y^m + z^m) \subset \mathbb{P}^2$ for some $1 \leq m \in \mathbb{N}$, has complex multiplication.*

Proof. A covering $\mathbb{F}_m \to C$ yields an injective vector space homomorphism

$$H^1(C, \mathbb{Q}) \to H^1(\mathbb{F}_m, \mathbb{Q}),$$

which extends to an embedding of Hodge structures (see [49], 7.3.2 for more details). This embedding induces a direct sum decomposition into two rational sub-Hodge structures of $H^1(\mathbb{F}_m, \mathbb{Q})$ (see Lemma 2.4.2). Hence by Lemma 2.4.1 and the fact that $\mathbb{F}_m$ has complex multiplication, one obtains the statement. □

Theorem 2.4.4. *Let $0 < d_1, d < m$, and ξ_k denote a primitive k-th. root of unity for all $k \in \mathbb{N}$. Then the curve C, which is given by*

$$y^m = x^{d_1} \prod_{i=1}^{n-2} (x - \xi_{n-2}^i)^d,$$

is covered by the Fermat curve $\mathbb{F}_{(n-2)m}$ given by $V(y^{(n-2)m} + x^{(n-2)m} + 1)$ and has complex multiplication.

Proof. Let C be the curve, which is given by

$$y^m = x^{d_1} \prod_{i=1}^{n-2} (x - \xi_{n-2}^i)^d,$$

and $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ be the morphism, which is given by $y \to yx^{d_1}$ and $x \to x^m$. By a little abuse of notation, we denote by $C \cap \mathbb{A}^2$ the singular affine curve given by the equation above, which is birationally equivalent to C. The corresponding homomorphism $\phi^* : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ sends the ideal, which defines $C \cap \mathbb{A}^2$, to the ideal generated by

$$y^m x^{m \cdot d_1} - x^{m \cdot d_1} \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d.$$

This is contained in the ideal generated by

$$y^m - \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d. \tag{2.2}$$

Let $m_0 := \frac{m}{gcd(m,d)}$, and $d_0 := \frac{d}{gcd(m,d)}$. It is obvious that

$$y^m - \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d = \prod_{j=0}^{gcd(m,d)-1} (y^{m_0} - \xi_{gcd(m,d)}^j \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^{d_0}).$$

Now we take the curve C_1, which is given by

$$y^{m_0} = \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^{d_0}.$$

By the definitions of m_0 and d_0, and Remark 2.1.8, the curve C_1 is given by

$$y^{m_0} = \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i),$$

too. Hence this curve irreducible, and ϕ induces a cover $C_1 \to C$ resp., ϕ^* induces a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[C \cap \mathbb{A}^2] \to \mathbb{C}[C_1 \cap \mathbb{A}^2]$. By $x \to x$ and $y \to y^{n-2\frac{m}{m_0}}$, we get a cover of the Fermat curve $\mathbb{F}_{(n-2)m}$ given by $V(y^{(n-2)m} + x^{(n-2)m} + 1)$ onto C_1. Now we use the composition of these covers $\mathbb{F}_{(n-2)m} \to C_1$ and $C_1 \to C$, and Lemma 2.4.3. This yields the statement. □

Chapter 3

Some preliminaries for families of cyclic covers

3.1 The theoretical foundations

We want to study the variations of Hodge structures (VHS) of the families of cyclic covers onto $\mathbb{P}^1$, which will be constructed in the next section. Hence let us first make some general observations about the relation between their monodromy groups and Hodge groups resp., Mumford-Tate groups, which will be needed for the calculation (of the derived group) of the generic Hodge group defined below.

Proposition 3.1.1. *Let F be a totally real number field, W be a complex connected algebraic manifold, $\mathcal{A} \to W$ be a family of Abelian varieties and $\mathcal{V}$ be its polarized variation of F-Hodge structures of weight 1 on W. Then there is a countable union $W' \subset W$ of subvarieties such that all $\mathrm{MT}(\mathcal{V}_p)$ coincide (up to conjugation by integral matrices) for all (closed) $p \in W \setminus W'$. Moreover one has $\mathrm{MT}(\mathcal{V}_{p'}) \subset \mathrm{MT}(\mathcal{V}_p)$ for all (closed) $p' \in W'$ and $p \in W \setminus W'$.*

Proof. (see [34], Subsection 1.2) □

The preceding Proposition motivates the definition of the generic Mumford-Tate group of a polarized variation $\mathcal{V}$ of $\mathbb{Z}$-Hodge structures of weight k on a non-singular connected algebraic variety W given by $\mathrm{MT}(\mathcal{V}) = \mathrm{MT}(\mathcal{V}_p)$ for all (closed) $p \in W \setminus W'$.

Since the image of the embedding $\mathrm{SL}(\mathcal{V}_{\mathbb{Q},p}) \hookrightarrow \mathrm{GL}(\mathcal{V}_{\mathbb{Q},p})$ is independent with respect to the chosen coordinates on $\mathcal{V}_{\mathbb{Q},p}$, Lemma 1.5.3 allows us to define the generic Hodge group $Hg(\mathcal{V}) := (\mathrm{MT}(\mathcal{V}) \cap \mathrm{SL}(\mathcal{V}))^0$ such that $\mathrm{Hg}(\mathcal{V}) = \mathrm{Hg}(\mathcal{V}_p)$ for all (closed) $p \in W \setminus W'$.

Definition 3.1.2. Let $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$ be a field and $\mathcal{V} = (\mathcal{V}_K, \mathcal{F}^\bullet, Q)$ be a polarized variation of K Hodge structures on a connected complex manifold D. Then $\mathrm{Mon}^0_K(\mathcal{V})_p$ denotes the connected component of identity of the Zariski closure of the monodromy group in $\mathrm{GL}((\mathcal{V}_K)_p)$ for some $p \in D$. For simplicity we write $\mathrm{Mon}^0(\mathcal{V})_p$ instead of $\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_p$

Theorem 3.1.3. *Keep the assumptions and notations of Proposition* 3.1.1. *One has that $\mathrm{Mon}^0_F(\mathcal{V})_p$ is a subgroup of $\mathrm{MT}^{\mathrm{der}}_F(\mathcal{V}_p)$ for all $p \in W \setminus W'$. Moreover for a variation of $\mathbb{Q}$ Hodge structures one has that $\mathrm{Mon}^0(\mathcal{V})_p$ is a normal subgroup of $\mathrm{MT}^{\mathrm{der}}(\mathcal{V}_p)$ and*

$$\mathrm{Mon}^0(\mathcal{V})_p = \mathrm{MT}^{\mathrm{der}}(\mathcal{V}_p)$$

for all $p \in W \setminus W'$, if $\mathcal{V}_{\mathbb{Q}}$ has a CM point.

Proof. (see [35], Theorem 1.4 for the statement about the variations of $\mathbb{Q}$ Hodge structures and [34], Properties 7.14 for the statement about the variations of F Hodge structures) $\square$

Corollary 3.1.4. *Keep the assumptions of Theorem* 3.1.3. *Then the group* $\mathrm{Mon}^0(\mathcal{V})$ *is semisimple.*

Proof. The Lie subalgebra $Lie(\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ of $Lie(\mathrm{MT}^{der}_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ is an ideal. Hence the algebra $Lie(\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ consists of the direct sum of simple subalgebras of $Lie(\mathrm{MT}^{\mathrm{der}}_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$. Thus $\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}}$ and hence $\mathrm{Mon}^0(\mathcal{V})$ is semisimple. $\square$

3.2 Families of covers of the projective line

Let S be some $\mathbb{C}$-scheme. Recall that the covers $c_1 : V_1 \to \mathbb{P}^1_S$ and $c_2 : V_2 \to \mathbb{P}^1_S$ are equivalent, if there is a S-isomorphism $j : V_1 \to V_2$ such that $c_1 = c_2 \circ j$.

In this section we construct a family of cyclic covers onto $\mathbb{P}^1$ such that all equivalence classes of covers with a fixed number of branch points with fixed branch indeces are represented by some of its fibers. For us it is sufficient to start with a space, which is not a moduli scheme, but whose closed points "hit" all equivalence classes of covers of $\mathbb{P}^1$ with Galois group $G = (\mathbb{Z}/m, +)$ and a fixed number of branch points with fixed branch indeces.

We can start with the space

$$(\mathbb{P}^1)^{n+3} \supset \mathcal{P}_n := (\mathbb{P}^1)^{n+3} \setminus \{z_i = z_j | i \neq j\},$$

which parameterizes the injective maps $\phi : N \to \mathbb{P}^1$, where $N := \{s_1, \ldots, s_{n+3}\}$. Thus a point $q \in \mathcal{P}_n$ corresponds to an injective map $\phi_q : N \to \mathbb{P}^1$.[1] One can consider $\mathcal{P}_n$ as configuration space of $n+3$ ordered points, too.

We endow the points $s_k \in N$ with some local monodromy data $\alpha_k = e^{2\pi i \mu_k}$, where

$$\mu_k \in \mathbb{Q}, \quad 0 < \mu_k < 1 \quad \text{and} \quad \sum_{k=1}^{n+3} \mu_k \in \mathbb{N}.$$

Now we construct a family of covers of $\mathbb{P}^1$ by these local monodromy data:

Construction 3.2.1. Let m be the smallest integer such that $m\mu_k \in \mathbb{N}$ for $k = 1, \ldots, n+3$, and $D_k \subset \mathbb{P}_{\mathcal{P}_n} := \mathbb{P}^1 \times \mathcal{P}_n$ be the prime divisor given by

$$D_k = \{(a_k, a_1, \ldots, a_k, \ldots, a_{n+3})\}.$$

Let D be the divisor

$$D := \sum_{k=1}^{n+3} m\mu_k D_k \sim mD_0 \quad \text{with} \quad D_0 := (\sum_{k=1}^{n+3} \mu_k) \cdot (\{0\} \times \mathcal{P}_n).$$

By the sheaf $\mathcal{L} := \mathcal{O}_{\mathbb{P}_{\mathcal{P}_n}}(D_0)$ and the divisor D, we obtain an irreducible cyclic cover $\mathcal{C}$ of degree m onto $\mathbb{P}_{\mathcal{P}_n}$ as in [16], §3 (where irreducible means that the covering variety is

[1] The set N is some arbitrary finite set, where the set S of the preceding chapter is a concrete set $S \subset \mathbb{P}^1$ given by $S = \phi_q(N)$ for some $q \in \mathcal{P}_n$.

irreducible). By $\pi : \mathcal{C} \to \mathbb{P}^1 \times \mathcal{P}_n \stackrel{pr_2}{\to} \mathcal{P}_n$, this cyclic cover yields a family of irreducible cyclic covers of degree m onto $\mathbb{P}^1$.

Suppose that r divides m. By taking the quotient of the subgroup of order r of the Galois group of the cyclic cover $\mathcal{C} \to \mathbb{P}^1 \times \mathcal{P}_n$, one gets a family $\pi_r : \mathcal{C}_r \to \mathcal{P}_n$ of cyclic covers of degree $\frac{m}{r}$ onto $\mathbb{P}^1$. Let $\phi_r : \mathcal{C} \to \mathcal{C}_r$ denote the quotient map. One has

$$\pi = \pi_r \circ \phi_r.$$

Remark 3.2.2. Without loss of generality one may assume that $q := (a_1, \ldots, a_{n+3}) \in \mathcal{P}_n$ is contained in $\mathbb{A}^{n+3}$, too. Thus the fiber $\mathcal{C}_q$ is given by the equation

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_{n+3})^{d_{n+3}}$$

with $d_k = m\mu_k$. By Remark 2.1.5, the local monodromy datum α_k describes the lifting of a path γ_k around $a_k \in \mathbb{P}^1$.[2] One checks easily that each equivalence class of cyclic covers of degree m with $n+3$ branch points and fixed branch indexes $d_1, \ldots, d_{n+3}$ is represented by some fibers of $\mathcal{C}$. Moreover for $C = \mathcal{C}_q$ the quotient C_r of Remark 2.2.9 is given by the fiber $(\mathcal{C}_r)_q$.

A family of smooth algebraic curves over $\mathbb{C}$ determines a proper submersion $\tau : X \to Y$ in the category of differentiable manifolds (follows by [49], Proposition 9.5). By the Ehresmann theorem, we obtain that over any contractible submanifold W of Y the family is diffeomorphic to $X_0 \times W$, where X_0 is the fiber of some point $0 \in W$. This fact has some consequences for the monodromy representation of its variation of integral Hodge structures.

It is a well-known fact that $R^1\tau_*(\mathbb{Z})$ is the sheaf associated to the presheaf

$$V \to H^1(\tau^{-1}(V), \mathbb{Z}|_{\pi^{-1}(V)}) \quad (\forall \ \ V \in Top(\mathcal{P}_n)).$$

Moreover we have

$$H^1(X_0, \mathbb{Z}) = H^1(X_W, \mathbb{Z}) = (R^1\tau_*(\mathbb{Z}))(W)$$

for some contractible $W \subset \mathcal{P}_n$ with $0 \in W$, which implies that $R^1\tau_*(\mathbb{Z})$ is a local system (see [49], 9.2.1).

By using these facts, one can easily ensure that the monodromy group of the VHS of a family of curves can be calculated over any arbitrary field $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$:

Lemma 3.2.3. *Let K be a field with $char(K) = 0$. Moreover let $\tau : X \to Y$ be a holomorphic family of curves. Then we obtain*

$$R^1\tau_*(K) = R^1\tau_*(\mathbb{Z}) \otimes_{\mathbb{Z}} K.$$

Proof. By [22], **III**, Proposition 8.1, the sheaf $R^1\tau_*(K)$ is given by the sheafification of the presheaf

$$V \to H^1(\tau^{-1}(V), K|_{\tau^{-1}(V)}) \quad (\forall \ \ V \in Top(Y)).$$

Hence by the description of the cohomology of a compact manifold by Čech complexes (see [49], 7.1.1), this presheaf is given by

$$V \to H^1(\tau^{-1}(V), \mathbb{Z}|_{\tau^{-1}(V)}) \otimes_{\mathbb{Z}} K \quad (\forall \ \ V \in Top(Y)).$$

By the fact that a local section of $\mathbb{Z}$ or K on a connected component of V resp., $\tau^{-1}(V)$ is constant, one does not need to differ between the locally constant sheaves given by $\mathbb{Z}$ resp., K on X or Y for the computation of $R^1\tau_*(K)$. Hence by using [22], **III**, Proposition 8.1 for $R^1\tau_*(\mathbb{Z})$, one obtains the desired identification. □

[2] This circumstance explains the term "local monodromy datum".

By the fact that the integral cohomology of a curve does not have torsion, one concludes:

Corollary 3.2.4. *Keep the assumptions of Lemma* 3.2.3. *Then the monodromy representations of* $R^1\tau_*(\mathbb{Z})$ *and* $R^1\tau_*(K)$ *coincide.*

Remark 3.2.5. Recall the we have an eigenspace decomposition of

$$H^1(\mathcal{C}_0, \mathbb{C}) = H^1(\mathcal{C}_0, \mathbb{Z}) \otimes \mathbb{C}$$

with respect to the Galois group action. By $H^1(\mathcal{C}_0, \mathbb{C}) = (R^1\pi_*(\mathbb{C}))(W)$ for some contractible $W \subset \mathcal{P}_n$ with $0 \in W$, we obtain an eigenspace decomposition of $(R^1\pi_*(\mathbb{C}))(W)$. Since we have this decomposition over all contractible $W \subset \mathcal{P}_n$, we can glue these eigenspaces, which yields a decomposition of the whole sheaf $R^1\pi_*(\mathbb{C})$ into eigenspaces with respect to the Galois group action.

Recall that we have an identification between the characters of the Galois group of some fiber and the elements $j \in \mathbb{Z}/(m)$. This identification allows a compatible identification between the characters of the Galois group of the family and the elements $j \in \mathbb{Z}/(m)$. Let $\mathcal{L}_j$ denote the eigenspace of $R^1\pi_*(\mathbb{C})$ with respect to the character j.

Remark 3.2.6. Let $0 \in \mathcal{P}_n$. We have a monodromy action $\rho_{\mathcal{C}}$ by diffeomorphisms on the fiber $\mathcal{C}_0$, which is induced by the glueing diffeomorphisms of the locally constant family of manifolds given by $\mathcal{C}$. Since these glueing differeomorphisms induce the glueing homomorphisms of $R^1\pi_*(\mathbb{Z})$ in the obvious natural way, the monodromy representation ρ of $R^1\pi_*(\mathbb{Z})$ is given by

$$\rho(\gamma)(\eta) = (\rho_{\mathcal{C}}(\gamma))_*(\eta) \quad (\forall \quad \eta \in H^1(\mathcal{C}_0, \mathbb{Z})).$$

Remark 3.2.7. Since each glueing diffeomorphism of the locally constant family of manifolds corresponding to $\mathcal{C}$ respects intersection form, Remark 3.2.6 implies that the monodromy group of $R^1\pi_*(\mathbb{C})$ respects the polarization of the Hodge structures. Assume that $H^1_j(\mathcal{C}_q, \mathbb{C}) = (\mathcal{L}_j)_q$ satisfies that $H^{1,0}_j(\mathcal{C}_q) = n_1$ and $H^{0,1}_j(\mathcal{C}_q)_2$. This means that the variation of integral polarized Hodge structure endows $(\mathcal{L}_j)_q$ with an Hermitian form with signature (n_1, n_2). Hence the monodromy group of this eigenspace is contained in $U(n_1, n_2)$. In this sense we say that $\mathcal{L}_j$ is of type (n_1, n_2).

3.3 The homology and the monodromy representation

In this section we study the monodromy representation of $\pi_1(\mathcal{P}_n)$ on the dual of $R^1\pi_*(\mathbb{C})$ given by the complex homology. This will yield corresponding statements for the monodromy representation of $R^1\pi_*(\mathbb{C})$.

In the case of the configuration space $\mathcal{P}_n$ of $n+3$ points, we make a difference between these different points. One says that the points are "colored" by different "colors". Moreover one can identify its fundamental group with the subgroup of the braid group on $n+3$ strands in $\mathbb{P}^1$, which is given by the braids leaving the strands invariant (see [20], Chapter **I**. 3.). This subgroup of the braid group is called the colored braid group. An element of this group is for example given by the Dehn twist T_{k_1,k_2} with $1 \leq k_1 < k_2 \leq n+3$. The Dehn twist T_{k_1,k_2} is given by leaving a_{k_2} "run" counterclockwise around a_{k_1}.

Now we consider a fiber $C = \mathcal{C}_q$ of $\mathcal{C}$. Recall that C is a cyclic cover of $\mathbb{P}^1$ described in Chapter 2.

Consider the eigenspace $\mathbb{L}_j$, which can be extended from a local system on $\mathbb{P}^1 \setminus S$ to a local system on $\mathbb{P}^1 \setminus S_j$ with $S_j = \{a_1, \ldots, a_{n_j+3}\}$. For simplicity one may without loss of generality assume that $a_{n_j+3} = \infty$ and $a_k \in \mathbb{R}$ such that $a_k < a_{k+1}$ for all $k = 1, \ldots, n_{j+2}$. Here we assume that δ_k is the oriented path from a_k to a_{k+1} given by the straight line.

Construction 3.3.1. Let ζ be a path on $\mathbb{P}^1$. Assume without loss of generality that $\zeta((0,1))$ is contained in a simply connected open subset U of $\mathbb{P}^1 \setminus S$. Otherwise we decompose ζ into such paths. It has m liftings $\zeta^{(0)}, \ldots, \zeta^{(m-1)}$ to C such that $\psi(\zeta^{(\ell)}) = \zeta^{([\ell-1]_m)}$. By the tensorproduct of $\mathbb{C}$ with the free Abelian group generated by the paths on C, one obtains the vector space of $\mathbb{C}$-valued paths on C. Now let $c \in \mathbb{C}$ and take the linear combination of $\mathbb{C}$-valued paths on C given by

$$\hat{\zeta} = c\zeta^{(0)} + \ldots + ce^{2\pi i \frac{jr}{m}}\zeta^{(r)} + \ldots + ce^{2\pi i \frac{j(m-1)}{m}}\zeta^{(m-1)}.$$

By the assumptions, one verifies easily that $\psi(\hat{\zeta}) = e^{2\pi i \frac{j}{m}}\hat{\zeta}$. Moreover by the local sections given by $c, \ldots, ce^{2\pi i \frac{jr}{m}}, \ldots, ce^{2\pi i \frac{j(m-1)}{m}}$ on the corresponding sheets over U containing the different $\zeta^{(\ell)}((0,1))$, one obtains a corresponding section $\tilde{c} \in \mathbb{L}_j(U)$. In this sense we have a $\mathbb{L}_j$-valued path $\tilde{c} \cdot \zeta$ on $\mathbb{P}^1$.

Remark 3.3.2. Consider the (oriented) path δ_k. Let e_k be a non-zero local section of $\mathbb{L}_j$ defined over an open set containing $\delta_k((0,1))$. The $\mathbb{L}_j$-valued path $e_k \cdot \delta_k$ yields an element $[e_k \cdot \delta_k]$ of the homology group of $H_1(C, \mathbb{C})$, which is represented by the corresponding linear combination of paths in C lying over δ_k. It has the character j with respect to the Galois group representation. Let $H_1(C, \mathbb{C})_j$ denote the corresponding eigenspace.

Definition 3.3.3. A partition of S_j into some disjoint sets $S^{(1)} \cup \ldots \cup S^{(\ell)} = S_j$ is stable with respect to the local monodromy data μ_k of $\mathbb{L}_j$, if

$$\sum_{a_k \in S^{(1)}} \mu_k \notin \mathbb{N}, \ldots, \sum_{a_k \in S^{(\ell)}} \mu_k \notin \mathbb{N}.$$

Theorem 3.3.4. *Assume that $S_j = \{a_i : i = 1, \ldots, n_j + 3\}$ has the stable partition $\{a_1, \ldots, a_{\ell+1}\}, \{a_{\ell+2}, \ldots, a_{n_j+3}\}$ for some $1 \leq \ell \leq n_j + 1$. Then the eigenspace $H_1(C, \mathbb{C})_j$ of the complex homology group of C has a basis given by*

$$\mathcal{B} = \{[e_k\delta_k] : k = 1, \ldots, \ell\} \cup \{[e_k\delta_k] : k = \ell + 2, \ldots, n_j + 2\}.$$

Proof. By [30], Lemma 4.5, one has that $\{[e_k\delta_k] : k = 1, \ldots, n_j+1\}$ is a basis of $H_1(C, \mathbb{C})_j$. Hencefore $\{[e_k\delta_k] : k = 1, \ldots, n_j + 2\}$ is linearly dependent.

One can compute a non-trivial linear combination, which yields 0, in the following way: Choose a non-zero section of $\mathbb{L}_j$ over

$$U = \mathbb{P}^1 \setminus (\bigcup_{k=1}^{n_j+2} \delta_k).$$

This yields a linear combination of the sheets over U, on which ψ acts by j. By the boundary operator ∂, one gets the desired non-trivial linear combination of $\mathbb{L}_j$-valued paths, which is equal to 0. Now let α_k denote the local monodromy datum of $\mathbb{L}_j$ around $a_k \in S_j$ for all $k = 1, \ldots, n_j+3$. By the local monodromy data, one can easily compute this linear combination. This computation yields that $\{\delta_1, \ldots, \delta_\ell\} \cup \{\delta_{\ell+2}, \ldots, \delta_{n_j+2}\}$ is linearly independent, if and only if $\{a_1, \ldots, a_{\ell+1}\}, \{a_{\ell+2}, \ldots, a_{n_j+3}\}$ is a stable partition. □

Let α_k denote the local monodromy datum of $\mathbb{L}_j$ around $a_k \in S_j$ for all $k = 1, \ldots, n_j + 3$. One has up to a certain normalization of the basis vectors $[e_1\delta_1], \ldots [e_1\delta_{n_j+1}]$ the following description of the monodromy representation:

The Dehn twist $T_{k,k+1}$ leaves obviously δ_ℓ invariant for all $|k - \ell| > 1$. Moreover by following a path representing $T_{k,k+1}$, one sees that the monodromy action of $T_{k,k+1}$ on $H_1(C, \mathbb{C})_j$ (induced by push-forward) is given by

$$[e_{k-1}\delta_{k-1}] \to [e_{k-1}\delta_{k-1}] + \alpha_k(1 - \alpha_{k+1})[e_k\delta_k],$$

$$[e_k\delta_k] \to \alpha_k\alpha_{k+1}[e_k\delta_k]$$

$$\text{and} \quad [e_{k+1}\delta_{k+1}] \to [e_{k+1}\delta_{k+1}] + (1 - \alpha_k)[e_k\delta_k].$$

Hence the monodromy representation is given by:

Proposition 3.3.5. *The monodromy representation of $T_{\ell,\ell+1}$ on $H_1(C, \mathbb{C})_j$ is given with respect to the basis $\{[e_k\delta_k] | k = 1, \ldots n_j + 1\}$ of $H_1(C, \mathbb{C})_j$ by the matrix with the entries:*

$$M_{\ell,\ell+1}(a, b) = \begin{cases} 1 & : \quad a = b \quad \textit{and} \quad a \neq \ell \\ \alpha_\ell\alpha_{\ell+1} & : \quad a = b = \ell \\ \alpha_\ell(1 - \alpha_{\ell+1}) & : \quad a = \ell \quad \textit{and} \quad b = \ell - 1 \\ 1 - \alpha_\ell & : \quad a = \ell \quad \textit{and} \quad b = \ell + 1 \\ 0 & : \quad \textit{elsewhere} \end{cases}$$

Remark 3.3.6. The monodromy representation of Proposition 3.3.5 corresponds to an eigenspace in the local system given by the direct image of the complex homology. By integration over $\mathbb{C}$-valued paths, this eigenspace is the dual local system of $\mathcal{L}_{m-j}$. By the cup-product, $\mathcal{L}_j$ is the dual of $\mathcal{L}_{m-j}$, too. Hence Proposition 3.3.5 yields the monodromy representation of $\mathcal{L}_j$.

Chapter 4

The Galois group decomposition of the Hodge structure

In this chapter we make some general observations of the VHS of $\mathcal{C} \to \mathcal{P}_n$ and its generic Hodge group. Moreover we will give an upper bound for the Hodge group and a sufficient criterion for dense sets of complex multiplication fibers.

4.1 The Galois group representation on the first cohomology

Let $\pi : C \to \mathbb{P}^1$ be a cyclic cover of degree m. The elements of $\mathrm{Gal}(\pi)$ act as $\mathbb{Z}$-module automorphisms on $H^1(C, \mathbb{Z})$. This induces a faithful representation

$$\rho^1 : \mathrm{Gal}(\pi) \to \mathrm{GL}(H^1(C, \mathbb{Q})). \tag{4.1}$$

By the Galois group representation of a cyclic cover of degree m, we have the following eigenspace decomposition:

$$H^1(C, \mathbb{Q}) \otimes \mathbb{Q}(\xi) = H^1(C, \mathbb{Q}(\xi)) = \bigoplus_{i=1}^{m-1} H^1_j(C, \mathbb{Q}(\xi))$$

Recall that $\pi : C \to \mathbb{P}^1$ is given by some fibers of a family $\pi : \mathcal{C} \to \mathcal{P}_n$. The monodromy representation of $R^1\pi_*(\mathbb{C})$ has a decomposition into subrepresentations on the different eigenspaces. In general there is not a $\mathbb{Q}(\xi)$ structure on $H^1(C, \mathbb{Q})$, which turns $H^1(C, \mathbb{Q})$ into a $\mathbb{Q}(\xi)$-vector space. But in this section we will see that $H^1(C, \mathbb{Q})$ has a direct sum decomposition into sub-vector spaces with different $\mathbb{Q}(\xi^r)$ structures, where $r|m$. Moreover we will see that the monodromy representation respects the different $\mathbb{Q}(\xi^r)$ structures, which we will study.

Let ψ denote a generator of $\mathrm{Gal}(\pi)$ as in Chapter 2. The characteristic polynomial of $\rho^1(\psi)$ decomposes into the product of the minimal polynomials of the different ξ^r, where $r|m$ and ξ is a m-th. primitive root of unity. By [28], Satz 12.3.1., we have a decomposition of $H^1(C, \mathbb{Q})$ into subvector spaces $N^1(C_r, \mathbb{Q})$ [1] such that the $\mathbb{Q}$-vector space automorphism

[1] In the next section we will see that there is a correspondence between the covers C_r and the subvector spaces $N^1(C_r, \mathbb{Q})$, which justifies this notation.

$\rho^1(\psi)|_{N^1(C_r,\mathbb{Q})}$ is (up to conjugation) given by a matrix

$$\begin{pmatrix} M & & 0 \\ & \ddots & \\ 0 & & M \end{pmatrix},$$

where M is the $k \times k$ matrix given by

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & -p_0 \\ 1 & 0 & \dots & 0 & -p_1 \\ 0 & 1 & \ddots & 0 & -p_2 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -p_{k-1} \end{pmatrix},$$

where $x^k + p_{k-1}x^{k-1} + \ldots + p_1 x + p_0$ is the minimal polynomial of ξ^r. We call a $\mathbb{Q}$-vector space with such an automorphism of the form $\mathrm{diag}(M, \ldots, M)$ a $\mathbb{Q}(\xi^r)$-structure. By $\xi^r \cdot v := g(v)$, this defines a scalar multiplication of $\mathbb{Q}(\xi^r)$, which turns $N^1(C_r, \mathbb{Q})$ into a $\mathbb{Q}(\xi^r)$-vector space. We obtain:

Proposition 4.1.1. *The direct sum decomposition*

$$H^1(C, \mathbb{Q}) = \bigoplus_{r|m} N^1(C_r, \mathbb{Q})$$

is a direct sum of $\mathbb{Q}(\xi^r)$ structures on $H^1(C, \mathbb{Q})$.

Next we consider the trace map

$$\mathrm{tr} : H^1_j(C, \mathbb{Q}(\xi)) \to H^1(C, \mathbb{Q}) \text{ given by } v \to \sum_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi);\mathbb{Q})} \gamma v,$$

which will be one of our main tools in this chapter. By the Galois group action, the vector space $N^1(C_r, \mathbb{Q}(\xi^r))$ decomposes into eigenspaces $H^1_j(C, \mathbb{Q}(\xi^r)))$ such that

$$H^1_j(C, \mathbb{Q}(\xi)) = H^1_j(C, \mathbb{Q}(\xi^r)) \otimes_{\mathbb{Q}(\xi^r)} \mathbb{Q}(\xi).$$

Lemma 4.1.2. *Let $r|m$ and $r = \gcd(j, m)$. Then $\mathrm{tr}|_{H^1_j(C,\mathbb{Q}(\xi^r))}$ is a monomorphism.*

Proof. Let $f \in H^1_j(C, \mathbb{Q}(\xi^r)) \setminus \{0\}$. We need some Galois theory. By the fact that $\mathbb{Q}(\xi^r)$ is a Galois extension of $\mathbb{Q}$, the group $\Gamma_r := \mathrm{Aut}(\mathbb{Q}(\xi); \mathbb{Q}(\xi^r))$ is a normal subgroup of $(\mathbb{Z}/(m))^* \cong \Gamma := \mathrm{Gal}(\mathbb{Q}(\xi); \mathbb{Q})$, which is the kernel of the epimorphism $\Gamma \to \mathrm{Gal}(\mathbb{Q}(\xi^r); \mathbb{Q})$ given by $\gamma \to \gamma|_{\mathbb{Q}(\xi^r)}$ for all $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi); \mathbb{Q})$. Hence we obtain that

$$\mathrm{tr}(f) = \sum_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi);\mathbb{Q})} \gamma f = \sum_{[\gamma] \in \Gamma/\Gamma_r} [\gamma] \sum_{\gamma \in \Gamma_r} \gamma f = \sum_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})} \gamma |\Gamma_r| f.$$

Since ψ is acts by an integral matrix, one has $\gamma \circ \psi = \psi \circ \gamma$ for all $\gamma \in \Gamma$. This implies that

$$\gamma(\xi^r)\gamma(f) = \gamma(\xi^r f) = (\gamma \circ \psi)(f) = \psi(\gamma f). \tag{4.2}$$

Thus $\gamma(f) \in H^1_{j_0 j}(C, \mathbb{Q}(\xi))$, where $j_0 \in (\mathbb{Z}/(m))^*$ corresponds to γ. By the fact that we have a direct sum of eigenspaces, we conclude that

$$\mathrm{tr}(f) = \sum_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})} \gamma |\Gamma_r| f \neq 0.$$

□

Now we consider the restriction of the trace map to

$$R := \bigoplus_{r|m} H^1_r(C, \mathbb{Q}(\xi^r)).$$

Proposition 4.1.3. *The trace map* $\mathrm{tr}|_R : R \to H^1(C, \mathbb{Q})$ *is an isomorphism of* $\mathbb{Q}$*-vector spaces.*

Proof. Let

$$v := \sum_{r|m} v_r \in R$$

with $v_r \in H^1_r(C, \mathbb{Q}(\xi^r))$. By the proof of the preceding lemma, we know that

$$\mathrm{tr}(v_r) = \sum_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})} \gamma|\Gamma_r|v \in \bigoplus_{j \in (\mathbb{Z}/(\frac{m}{r}))^*} H^1_j(C, \mathbb{Q}(\xi)).$$

These ξ^{jr} with $j \in (\mathbb{Z}/(\frac{m}{r}))^*$ are exactly the $\frac{m}{r}$-th. primitive roots of unity. Thus they are the elements with order $\frac{m}{r}$ in the multiplicative group generated by ξ. Hence by the fact that we have a direct sum of eigenspaces, we conclude that $\mathrm{tr}(v) = 0$ implies that $\mathrm{tr}(v_r) = 0$ for all r with $r|m$. By the preceding lemma, this implies that $v_r = 0$ for all r with $r|m$ and hencefore $v = 0$. Hence the map $\mathrm{tr}|_R$ is injective, and we have only to verify that $\dim_{\mathbb{Q}}(R) = \dim_{\mathbb{Q}}(H^1(C, \mathbb{Q}))$:

$$\dim_{\mathbb{Q}} R = \sum_{r|m} \dim_{\mathbb{Q}(\xi)}(H^1_r(C, \mathbb{Q}(\xi))) \cdot [\mathbb{Q}(\xi^r); \mathbb{Q}]$$

$$= \sum_{r|m} \dim_{\mathbb{Q}(\xi)}(H^1_r(C, \mathbb{Q}(\xi))) \cdot \sharp\{\text{primitive } \frac{m}{r}\text{-th. roots of unity}\}$$

$$= \sum_{j=1}^{m-1} \dim_{\mathbb{Q}(\xi)}(H^1_j(C, \mathbb{Q}(\xi))) = \dim_{\mathbb{Q}(\xi)}(H^1(C, \mathbb{Q}(\xi))) = \dim_{\mathbb{Q}}(H^1(C, \mathbb{Q}))$$

□

Remark 4.1.4. We know that the monodromy representation fixes $H^1(C, \mathbb{Q})$ and each $H^1_j(C, \mathbb{Q}(\xi))$ invariant. By the fact that

$$N^1(C_r, \mathbb{Q}) = N^1(C_r, \mathbb{Q}(\xi)) \cap H^1(C, \mathbb{Q}),$$

we conclude that the monodromy representation fixes $N^1(C_r, \mathbb{Q}))$, too.

Proposition 4.1.5. *The monodromy representation* ρ *on* $N^1(C_r, \mathbb{Q})$ *is given by*

$$\rho(\omega) = \begin{pmatrix} \gamma_1 M_\omega & & \\ & \ddots & \\ & & \gamma_k M_\omega \end{pmatrix},$$

where M_ω *denotes the image of* ω *in the monodromy of* $H^1_r(C, \mathbb{Q}(\xi^r))$*, and* $\{\gamma_1, \ldots, \gamma_k\} = \mathrm{Gal}(\mathbb{Q}(\xi^r); \mathbb{Q})$.

Proof. Since $\rho(\gamma)$ leaves the eigenspaces invariant, it acts by $\mathrm{diag}(M_1, \ldots, M_k)$, where each M_ℓ with $1 \leq \ell \leq k$ describes the action of $\rho(\omega)$ on $\gamma_\ell H^1_r(C, \mathbb{Q}(\xi^r))$. Let $j_\gamma \in (\mathbb{Z}/(\frac{m}{r}))^*$ and γ correspond. The description of the $M_1, \ldots, M_k$ follows from the facts that each $\rho(\omega)$ commutes with each $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r); \mathbb{Q})$, and that $\gamma H^1_r(C, \mathbb{Q}(\xi^r)) = H^1_{rj_\gamma}(C, \mathbb{Q}(\xi^r))$ (see (4.2)) for all $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r); \mathbb{Q})$. □

Now let N_ω denote the restriction of $\rho(\omega)$ on $N^1(C_r, \mathbb{Q})$ and $v \in N^1(C_r, \mathbb{Q})$ given by $v = \mathrm{tr}(w)$ for some $w \in H^1_r(C, \mathbb{Q}(\xi^r))$. By the preceding proposition, we have:

$$N_\omega(v) = N_\omega([\mathbb{Q}(\xi); \mathbb{Q}(\xi^r)] \sum_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r); \mathbb{Q})} \gamma w) = [\mathbb{Q}(\xi); \mathbb{Q}(\xi^r)] \sum_{i=1}^{k} \gamma_i M_\omega(\gamma_i(w))$$

$$= [\mathbb{Q}(\xi); \mathbb{Q}(\xi^r)] \sum_{i=1}^{k} \gamma_i(M_\omega(w)) = tr(M_\omega(w))$$

The trace map $H^1_r(C, \mathbb{Q}(\xi^r)) \to N^1(C_r, \mathbb{Q})$ is an isomorphism of $\mathbb{Q}(\xi^r)$-vector spaces with respect to the $\mathbb{Q}(\xi^r)$ structure on $N^1(C_r, \mathbb{Q})$. Thus one has:

Proposition 4.1.6. *The monodromy representation on $N^1(C_r, \mathbb{Q})$ is a representation on a $\mathbb{Q}(\xi^r)$-vector space given by the $\mathbb{Q}(\xi^r)$ structure, which coincides up to the trace map with the monodromy representation on $H^1_r(C, \mathbb{Q}(\xi^r))$.*

We will need a decomposition of $H^1(C, \mathbb{R})$ into a direct sum of certain sub-vector spaces fixed by the Galois group representation. This decomposition is defined over $\mathbb{Q}(\xi^j)^+ = \mathbb{Q}(\xi^j) \cap \mathbb{R}$ and given by the sub-vector spaces

$$\Re\mathbb{V}(j) := (H^1_j(C, \mathbb{Q}(\xi)) \oplus H^1_{m-j}(C, \mathbb{Q}(\xi))) \cap H^1(C, \mathbb{Q}(\xi^j)^+).$$

Since the monodromy representation fixes

$$H^1_j(C, \mathbb{Q}(\xi)), \quad H^1_{m-j}(C, \mathbb{Q}(\xi)) \quad \text{and} \quad H^1(C, \mathbb{Q}(\xi^j)^+),$$

it fixes $\Re\mathbb{V}(j)$, too.

Remark 4.1.7. One has that $\mathrm{tr} : H^1_j(C, \mathbb{Q}(\xi^j)) \to N^1(C_j, \mathbb{Q})$ coincides with the composition

$$H^1_j(C, \mathbb{Q}(\xi^j)) \xrightarrow{\mathrm{tr}} \Re\mathbb{V}(j) \xrightarrow{\mathrm{tr}} N^1(C_j, \mathbb{Q}).$$

Hence the latter trace map $\Re\mathbb{V}(j) \xrightarrow{\mathrm{tr}} N^1(C_j, \mathbb{Q})$ induces a $\mathbb{Q}(\xi^j)^+$-structure on $N^1(C_j, \mathbb{Q})$, which is compatible with the $\mathbb{Q}(\xi^j)$-structure via $\mathbb{Q}(\xi^j)^+ \hookrightarrow \mathbb{Q}(\xi^j)$. Thus by the preceding results about the monodromy representation on $N^1(C_j, \mathbb{Q})$, the monodromy representation on $N^1(C_j, \mathbb{Q})$ is a $\mathbb{Q}(\xi^j)^+$-vector space representation with respect to the $\mathbb{Q}(\xi^j)^+$-structure.

Remark 4.1.8. In the case of $H^1_{\frac{m}{2}}(C, \mathbb{Q}(\xi^{\frac{m}{2}})$ one gets that $\mathbb{Q}(\xi^{\frac{m}{2}}) = \mathbb{Q}(-1) = \mathbb{Q}$. In other terms: The monodromy group on $H^1_{\frac{m}{2}}(C, \mathbb{Q}(\xi^{\frac{m}{2}})$ is the monodromy group on the rational vector space $N^1(C_{\frac{m}{2}}, \mathbb{Q})$.

4.2 Quotients of covers and Hodge group decomposition

In this section we consider our quotient families $\pi_r : \mathcal{C}_r \to \mathcal{P}_n$ of covers, and their Hodge groups. Moreover we will explain the notation $N^1(C_r, \mathbb{Q})$ and show that the decomposition of $H^1(C, \mathbb{Q})$ into these $\mathbb{Q}(\xi^r)$ structures is a decomposition into rational sub-Hodge structures. Recall that $\mathcal{C}_r$ is given by a quotient of the subgroup of order r of the Galois group of $\mathcal{C}$ (see Construction 3.2.1).

Let C and C_r denote a fiber of $\mathcal{C}$ and the corresponding fiber of $\mathcal{C}_r$ over the same point. The natural cover $\phi_r : C \to C_r$ induces an embedding of Hodge structures, which gives a direct sum decomposition of $H^1(C, \mathbb{Q})$ into two rational sub-Hodge structures (see [49], 7.3.2. and [49], Lemma 7.26).

The Hodge structure on $H^1(C_r, \mathbb{Q})$ is the sub-Hodge structure of $H^1(C, \mathbb{Q})$ fixed by $\mathrm{Gal}(\phi_r)$. Hence the eigenspaces of $H^1(C_r, \mathbb{C})$ with respect to the Galois group π_r can be identified with the eigenspaces of $H^1(C, \mathbb{C})$, on which $\mathrm{Gal}(\phi_r)$ acts trivial. Thus one obtains

$$H^1(C_r, \mathbb{C}) = \bigoplus_{j=1}^{\frac{m}{r}-1} H^1_{jr}(C, \mathbb{C}) \hookrightarrow \bigoplus_{j=1}^{m-1} H^1_j(C, \mathbb{C}) = H^1(C, \mathbb{C}).$$

Recall that every eigenspace $\mathcal{L}_j$ of $R^1\pi_*(\mathbb{C})$ is a local system. We consider the eigenspace $(\mathcal{L}_j)_{\mathcal{C}_r}$ of $R^1(\pi_r)_*(\mathbb{C})$ with the character j and the eigenspace $\mathcal{L}_{jr}$ of $R^1\pi_*(\mathbb{C})$. Proposition 2.2.10 tells us that the local monodromy data of $(\mathbb{L}_j)_{C_r}$ and $\mathbb{L}_{rj}$ coincide. By Proposition 3.3.5, these monodromy data determine the dual monodromy representations of the eigenspaces of the dual VHS given by the homology. Thus we obtain:

Proposition 4.2.1. *The local systems $(\mathcal{L}_j)_{\mathcal{C}_r}$ and $\mathcal{L}_{jr}$ coincide.*

The following statements will explain the notation "$N^1(C_r, \mathbb{Q})$". One has that

$$N^1(C_r, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{j \in (\mathbb{Z}/\frac{m}{r})^*} H^1_{jr}(C, \mathbb{C}).$$

Since each $H^1_{jr}(C, \mathbb{C}) \subset N^1(C_r, \mathbb{C})$ has a decomposition into

$$H^{1,0}_j(C_r) \oplus H^{0,1}_j(C_r), \quad \text{where } \overline{H^1_j(C_r, \mathbb{C})} = H^1_{m-j}(C_r, \mathbb{C}) \subset N^1(C_r, \mathbb{C}),$$

each $N^1(C_r, \mathbb{Q})$ is a rational sub-Hodge structure of $H^1(C, \mathbb{Q})$. Moreover each $N^1(C_r, \mathbb{Q})$ is the maximal sub-Hodge structure of $H^1(C_r, \mathbb{Q})$, which is orthogonal (with respect to the polarization) to each sub-Hodge structure of $H^1(C_r, \mathbb{Q})$ given by a quotient $H^1(C_{r'}, \mathbb{Q})$ with $r < r' < m$, $r|r'$ and $r'|m$. By using Lemma 2.4.1, we have the result:

Proposition 4.2.2. *We have a decomposition*

$$H^1(C, \mathbb{Q}) = \bigoplus_{r|m} N^1(C_r, \mathbb{Q})$$

into rational Hodge structures and a natural embedding

$$\mathrm{Hg}(C) \hookrightarrow \prod_{r|m} \mathrm{Hg}(N^1(C_r, \mathbb{Q}))$$

such that the natural projections

$$\mathrm{Hg}(C) \to \mathrm{Hg}(N^1(C_r, \mathbb{Q}))$$

are surjective for all r.

Remark 4.2.3. Note that the preceding section yields a corresponding statement about the Zariski closures of the monodromy group of $R^1\pi_*(\mathbb{Q})$ and the restricted representations monodromy representations on the different $N^1(C_r, \mathbb{Q})$. These two facts will play a very important role.

4.3 Upper bounds for the Mumford-Tate groups of the direct summands

The different $N^1(C_r, \mathbb{Q})$ on the fibers induce a decomposition of $R^1\pi_*(\mathbb{Q})$ into a direct sum of local systems $\mathcal{N}^1(\mathcal{C}_r, \mathbb{Q})$. Now we consider the induced variations $\mathcal{V}_r$ of rational Hodge structures on the local systems $\mathcal{N}^1(\mathcal{C}_r, \mathbb{Q})$. Let Q_r denote the alternating form on $N^1(C_r, \mathbb{Q})$ obtained by the restriction of the intersection form Q of the curve C. One has that each element of $\rho(\pi_1(\mathcal{P}_n))$ commutes with the Galois group. The same holds true for the image of the homomorphism

$$h : \mathbb{S} \to \mathrm{GSp}(H^1(C, \mathbb{R}), Q)$$

corresponding to the Hodge structure of an arbitrary fiber. Since the Galois group respects the intersection form, its representation on $N^1(C_r, \mathbb{Q})$ is contained in $\mathrm{Sp}(N^1(C_r, \mathbb{Q}), Q_r)$. Let $C_r(\psi)$ denote the centralizer of the Galois group in $\mathrm{Sp}(N^1(C_r, \mathbb{Q}), Q_r)$ and $GC_r(\psi)$ denote the centralizer of the Galois group in $\mathrm{GSp}(N^1(C_r, \mathbb{Q}), Q_r)$. One concludes:

Proposition 4.3.1. *The centralizer $GC_r(\psi)$ contains the generic Mumford-Tate group* $\mathrm{MT}(\mathcal{V}_r)$. *Moreover the centralizer $C_r(\psi)$ contains the generic Hodge group* $\mathrm{Hg}(\mathcal{V}_r)$ *and* $\mathrm{Mon}^0(\mathcal{V}_r)$.

We write

$$C(\psi) := \prod_{r|m} C_r(\psi).$$

Remark 4.3.2. If $r \neq \frac{m}{2}$, the preceding proposition yields some information. In the case $r = \frac{m}{2}$ the elements of the Galois group act as the multiplication with 1 or -1 on $N^1(C_{\frac{m}{2}}, \mathbb{Q})$. Since id resp., $-$id is contained in the center of $\mathrm{Sp}(N^1(C_{\frac{m}{2}}, \mathbb{Q}), Q_{\frac{m}{2}})$, this proposition does not give any new information in this case.

Now let us assume that $r \neq \frac{m}{2}$. We describe $C_r(\psi)$ by its $\mathbb{R}$-valued points. Let ξ^j be a $\frac{m}{r}$-th. primitive root of unity such that $H^1_j(C, \mathbb{C}) \subset N^1(C_r, \mathbb{C})$, $v \in H^1_j(C, \mathbb{C})$ and $M \in C_r(\psi)(\mathbb{R})$. Then one gets

$$\psi M(v) = M(\psi v) = M(\xi^j v) = \xi^j M(v).$$

Thus M leaves each $H^1_j(C, \mathbb{C})$ invariant.

For our description of $C_r(\psi)$ we introduce the trace map

$$\mathrm{tr} : \mathrm{GL}(H^1_j(C,\mathbb{C})) \to \mathrm{GL}(\Re\mathbb{V}(j)_{\mathbb{R}})$$

given by

$$\mathrm{GL}(H^1_j(C,\mathbb{C})) \ni N \to N \times \bar{N} \in \mathrm{GL}(H^1_j(C,\mathbb{C})) \times \mathrm{GL}(H^1_{m-j}(C,\mathbb{C})), \tag{4.3}$$

where $\bar{N}$ denotes the matrix, which satisfies that $\bar{N}\bar{v} = \overline{Nv}$ for all $v \in H^1_j(C,\mathbb{C})$. Recall that we have a fixed complex structure. Thus one checks easily that $N \times \bar{N}$ leaves $\Re\mathbb{V}(j)_{\mathbb{R}}$ invariant. Hence we consider it as a real matrix.

For the Hermitian form $H(\cdot,\cdot) := iE(\cdot,\bar{\cdot})$ and $v, w \in H^1_j(C,\mathbb{C})$ one obtains

$$H(v,w) = iE(v,\bar{w}) = iE(Mv, M\bar{w}) = iE(Mv, \overline{Mw}) = H(Mv, Mw).$$

Thus the matrix $M|_{\Re\mathbb{V}(j)_{\mathbb{R}}}$ is contained in $\mathrm{tr}(\mathrm{U}(H^1_j(C,\mathbb{C}), H|_{H^1_{m-j}(C,\mathbb{C})})$.

Assume conversely that $M \in \mathrm{GL}(N^1(C_r,\mathbb{C}))$ satisfies that

$$M|_{\Re\mathbb{V}(j)_{\mathbb{R}}} \in \mathrm{tr}(\mathrm{U}(H^1_j(C,\mathbb{C}), H|_{H^1_j(C,\mathbb{C})}))$$

for each $\frac{m}{r}$-th. primitive root of unity ξ^j. Since M leaves all eigenspaces $H^1_j(C,\mathbb{C})) \subset N^1(C_r,\mathbb{C})$ invariant, M commutes with the Galois group representation on $N^1(C_r,\mathbb{R})$. Now let $N \in \mathrm{GL}(H^1_j(C,\mathbb{C}))$ be the matrix with $\mathrm{tr}(N) = M|_{\Re\mathbb{V}(j)_{\mathbb{R}}}$. One has that

$$iE(v,\bar{w}) = iE(Nv, \overline{Nw}) \Leftrightarrow E(v,\bar{w}) = E(Nv, \overline{Nw})$$

for all $v, w \in H^1_j(C,\mathbb{C})$. By the fact that E is an alternating form, one gets

$$E(\bar{v}, w) = E(\overline{Nv}, Nw),$$

too. Since each element of $\Re\mathbb{V}(j)_{\mathbb{C}}$ can be given by $v_1 + \bar{v}_2$ and $w_1 + \bar{w}_2$ with $v_1, v_2, w_1, w_2 \in H^1_j(C,\mathbb{C})$, one concludes that

$$E(v_1 + \bar{v}_2, w_1 + \bar{w}_2) = E(v_1, \bar{w}_2) + E(\bar{v}_2, w_1) = E(Nv_1, \overline{Nw_2}) + E(\overline{Nv_2}, Nw_1)$$

$$= E(Mv_1, M\bar{w}_2) + E(M\bar{v}_2, Mw_1) = E(M(v_1 + \bar{v}_2), M(w_1 + \bar{w}_2)).$$

Thus M is contained in the symplectic group. Altogether we conclude:

Theorem 4.3.3. *If $r \neq \frac{m}{2}$, the group $C_r(\psi)(\mathbb{R})$ is isomorphic to the direct product of the Lie groups given by the $\mathbb{R}$-valued points of unitary groups over the spaces $\Re\mathbb{V}(j)_{\mathbb{R}} \subset N^1(C_r,\mathbb{R})$ induced by the trace maps and the unitary groups $\mathrm{U}(H^1_j(C,\mathbb{C}), H|_{H^1_j(C,\mathbb{C})})$.*

Recall the definition of the type (a,b) of an eigenspace $\mathcal{L}_j$ in Remark 3.2.5. If there is an eigenspace of $N^1(C_r,\mathbb{C})$ of type (a,b) with $a > 0$ and $b > 0$, we call $N^1(C_r,\mathbb{Q})$ general. Otherwise we call it special. Now assume that $N^1(C_r,\mathbb{Q})$ is special. In this case $h(\mathbb{S})$ is contained in the center of $GC_r(\psi)_{\mathbb{R}}$, and $h(S^1)$ is contained in the center of $C_r(\psi)_{\mathbb{R}}$. Thus one concludes:

Remark 4.3.4. Assume that $N^1(C_r,\mathbb{Q})$ is special. Then the center $Z(GC_r(\psi))$ of $GC_r(\psi)$ contains $\mathrm{MT}(\mathcal{V}_r)$. Moreover the center $Z(C_r(\psi))$ of $C_r(\psi)$ contains $\mathrm{Hg}(\mathcal{V}_r)$.

Remark 4.3.5. One has that $C_r(\psi)_{\mathbb{R}}$ consists of $\mathrm{U}(s)^t$ for some $s, t \in \mathbb{N}_0$, if $N^1(C_r,\mathbb{Q})$ is special. Thus in this case the monodromy group is a discrete sub-group of the compact group $\mathrm{U}(s)^t$. Hence it is finite and $\mathrm{Mon}^0(\mathcal{V}_r)$ is trivial in this case.

4.4 A criterion for complex multiplication

In this short section we find a sufficient condition for the existence of a dense set of CM fibers of a family of cyclic covers. By technical reasons, we do not consider the family $\mathcal{C} \to \mathcal{P}_n$, but a family over the space $\mathcal{M}_n$, which can be considered as the quotient

$$\mathcal{M}_n = \mathcal{P}_n / \mathrm{PGL}_2(\mathbb{C}).$$

But one has an embedding $\iota_{a,b,c} : \mathcal{M}_n \to \mathcal{P}_n$, too. Its image is the subspace of $\mathcal{P}_n$, which parameterizes the maps $\phi : N \to \mathbb{P}^1$ satisfying $\phi(a) = 0$, $\phi(b) = 1$ and $\phi(c) = \infty$ for some fixed $a, b, c \in N$ (compare to [14], 3.7).

Remark 4.4.1. One can move 3 arbitrary branch points of a fiber of $\mathcal{C} \to \mathcal{P}_n$ to 0, 1 and ∞. Hence one has that all fibers of the geometric points of $\mathcal{P}_n$ occur as fibers of the restricted family $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$, too. Hence the generic Hodge groups and the generic Mumford-Tate groups of the both families coincide.

4.4.2. Each curve C with $g(C) > 1$ has at most $84(g-1)$ automorphisms (see [22], **IV**. Exercise 2.5). Thus C can have only finitely many cyclic covers onto $\mathbb{P}^1$ with different Galois groups. Moreover, there is an automorphism α of $\mathbb{P}^1$, if the Galois groups of the covers of $\mathcal{C}_{p_1}$ and $\mathcal{C}_{p_2}$ can be conjugate by an isomorphism ι such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_{p_1} & \xrightarrow{\iota} & \mathcal{C}_{p_2} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\alpha} & \mathbb{P}^1 \end{array}$$

Thus C occurs only as finitely many fibers of $\mathcal{C}_{\mathcal{M}_n}$, if $g(C) \geq 2$.

Recall that we have defined the type of an eigenspace $\mathcal{L}_j$ in Remark 3.2.5.

Definition 4.4.3. Let $\mathcal{C} \to \mathcal{P}_n$ be a family of cyclic covers onto $\mathbb{P}^1$ and C denote an arbitrary fiber. The family $\mathcal{C}$ has a pure $(1,n) - VHS$, if it has only one eigenspace $\mathcal{L}_j$ of type $(1,n)$ such that $\mathcal{L}_{m-j}$ is of type $(n,1)$ with respect to the Galois group representation, and all other eigenspaces are of type $(a,0)$ or of type $(0,b)$ for some $a, b \in \mathbb{N}_0$.

Theorem 4.4.4. *Let $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$ be a family of cyclic covers onto $\mathbb{P}^1$ and C be a fiber with $g(C) \geq 2$ as before. Assume that $\mathcal{C}$ has a pure $(1,n) - VHS$. Then the family $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$ has a dense set of complex multiplication fibers.*

Proof. We have to show that over an arbitrary open simply connected subset W of $\mathcal{M}_n(\mathbb{C})$ there are infinitely many CM points of the VHS of $\mathcal{C}_{\mathcal{M}_n}$. Let $q_0 \in W$ and $\mathcal{L}_j$ be the eigenspace of type $(1,n)$. We have a trivialization

$$R^1\pi_*(\mathbb{C})|_W = H^1(\mathcal{C}_{q_0}, \mathbb{C}) \times W \text{ such that } \mathcal{L}_j|_W \cong H^1_j(\mathcal{C}_{q_0}, \mathbb{C}) \times W.$$

Let $q \in W$ and $\varpi_q^{(j)} \in H^{1,0}_j(\mathcal{C}_q) \setminus \{0\}$. By the holomorphic VHS of the family, one obtains a holomorphic "fractional period" map

$$p : W \to \mathbb{P}(H^1_j(\mathcal{C}_{q_0}, \mathbb{C})) \text{ via } q \to [\varpi_q^{(j)}].$$

By the assumptions, the integral Hodge structure depends uniquely on the class $[\varpi_q^{(j)}] \in \mathbb{P}(H^1_j(\mathcal{C}_{q_0}, \mathbb{C}))$. Since for each fiber there are only finitely many isomorphic fibers (see

4.4.2) and two curves have isomorphic polarized integral Hodge structures, if and only if they are isomorphic, the fibers of p have the dimension 0. Thus [41], Chapter **VII**, Proposition 4 and the fact that $\dim W = \dim \mathbb{P}(H_j^1(\mathcal{C}_{q_0}, \mathbb{C}))$ tell us that p is open.

The natural embedding $C(\psi) \hookrightarrow \mathrm{GL}(H^1(\mathcal{C}_{q_0}, \mathbb{C}))$ induces a holomorphic variation of Hodge structures over the bounded symmetric domain associated with $C(\psi)(\mathbb{R})/K$. This VHS depends uniquely on the fractional VHS on the eigenspace $H_j^1(\mathcal{C}_{q_0}, \mathbb{C})$ of type $(1, n)$. Hencefore this VHS yields a holomorphic injection $\varphi : C(\psi)(\mathbb{R})/K \to \mathbb{P}(H_j^1(\mathcal{C}_{q_0}, \mathbb{C}))$.

Note that $C(\psi)(\mathbb{R})/K$ parameterizes the integral Hodge structures of type $(1, 0), (0, 1)$ on $H^1(\mathcal{C}_{q_0}, \mathbb{C})$, whose Hodge group is contained in $C(\psi)$. Hence altogether the map $\varphi^{-1} \circ p$, which assigns to each fiber $\mathcal{C}_q$ its integral Hodge structure, is open. Since the set of CM points on $C(\psi)(\mathbb{R})/K$ is dense (see Theorem 1.5.9), this yields the desired statement. $\square$

Chapter 5

The computation of the Hodge group

In this chapter we try to compute the derived group of the generic Hodge group of a family $\mathcal{C} \to \mathcal{P}_n$. For infinitely many examples we will not be able to do this. But we will get many information and in infinitely many examples we will obtain

$$\mathrm{MT}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Mon}^0(\mathcal{V}) = C^{\mathrm{der}}(\psi).$$

Recall that $\mathcal{P}_n$ is the configuration space of $n+3$ points and $\mathcal{M}_n = \mathcal{P}_n/\mathrm{PGL}_2(\mathbb{C})$. Finally we will see that a family $\mathcal{C} \to \mathcal{M}_1$ induces an open period map

$$p : \mathcal{M}_1(\mathbb{C}) \to \mathrm{MT}^{\mathrm{ad}}(\mathcal{V})/K,$$

if and only if it has a pure $(1,1) - VHS$.

5.1 The monodromy group of an eigenspace

Let $j \in \{1, \ldots, m-1\}$. Then we have an eigenspace $\mathcal{L}_j$ in the variation of Hodge structures of a family $\mathcal{C} \to \mathcal{P}_n$ of cyclic degree m covers onto $\mathbb{P}^1$. There are $p, q \in \mathbb{N}_0$ such that the eigenspace $H^1_j(C, \mathbb{C})$ of an arbitrary fiber C is of type (p,q), where (p,q) is the signature of the restricted polarization of the latter eigenspace. The type of $\mathcal{L}_j$ is given by the type of $H^1_j(C, \mathbb{C})$. The embedding $\mathbb{R} \hookrightarrow \mathbb{C}$ allows to consider $H^1_j(C, \mathbb{C})$ as $\mathbb{R}$-vector space. Let $\mathrm{Mon}^0(\mathcal{L}_j)$ denote the identity component of the Zariski closure of the monodromy group of $\mathcal{L}_j$ in $\mathrm{GL}_{\mathbb{R}}(H^1_j(C, \mathbb{C}))$.

We show in this section:

Theorem 5.1.1. *Let $\mathcal{L}_j$ be of type (p,q) with $p, q \geq 1$. Moreover assume that $j \neq \frac{m}{2}$ or $p = q = 1$. Then*

$$\mathrm{Mon}^0(\mathcal{L}_j) = \mathrm{SU}(p,q).$$

If $p = 0$ or $q = 0$, the statement of the preceding theorem does not hold true in general as one can conclude by Remark 4.3.5.

We give a proof of Theorem 5.1.1 by induction over the integer given by $p + q$.

By the following lemma, we start the proof of Theorem 5.1.1:

Lemma 5.1.2. *If $\mathcal{L}_j$ is of type $(1,1)$, its monodromy group contains infinitely many elements.*

Proof. There are two cases: In the first case there are some local monodromy data α_1 and α_2 of the eigenspace $\mathbb{L}_j$ in $(\pi_q)_*(\mathbb{C}_C)|_{\mathbb{P}^1\setminus S_j}$ for the fiber $C := \mathcal{C}_q$ of some arbitrary $q \in \mathcal{P}_n$ such that $\alpha_1\alpha_2 = 1$. In this case the Dehn twist $T_{1,2}$ yields a unipotent triangular matrix (follows by Proposition 3.3.5) and we are done.

Otherwise each Dehn twist $T_{k,\ell}$ provides a semisimple matrix, where its eigenvalues are given by 1 and a m-th. root of unity. Note that the matrices induced by the Dehn twists $T_{1,2}$ and $T_{2,3}$ do not commute. In the considered case $\{a_1, a_2\}, \{a_3, a_4\}$ is a stable partition. Hence one can choose the basis $\mathcal{B} = \{[e_1\gamma_1], [e_3\gamma_3]\}$ of $H_1^j(C, \mathbb{C})$. By the fact that these two cycles do not intersect each other, this basis is orthogonal with respect to the Hermitian form induced by the intersection form. Hence by normalization, this basis is orthonormal with respect to the Hermitian form such that the Hermitian form is without loss of generality given by $\mathrm{diag}(1, -1)$ with respect to $\mathcal{B}$. The matrix induced by $T_{1,2}$ is given by $\mathrm{diag}(\xi, 1)$ with respect to $\mathcal{B}$, where ξ is a m-th. root of unity. Since the matrix A of $T_{2,3}$ with respect to $\mathcal{B}$ does not commute with $\mathrm{diag}(\xi, 1)$, it is not a diagonal matrix. Now we compute the commutator

$$K = A \cdot \mathrm{diag}(\xi, 1) \cdot A^{-1} \cdot \mathrm{diag}(\bar{\xi}, 1).$$

One can replace A by a non-diagonal matrix in $\mathrm{SU}(1,1)$ and the matrix $\mathrm{diag}(\xi, 1)$ by $\mathrm{diag}(e, \bar{e}) \in \mathrm{SU}(1,1)$, where $e^2 = \xi$, for the computation of K. By [43], page 59, one has a description of the matrices in $\mathrm{SU}(1,1)(\mathbb{R})$ such that

$$K = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mathrm{diag}(e, \bar{e}) \begin{pmatrix} \bar{a} & -b \\ -\bar{b} & a \end{pmatrix} \mathrm{diag}(\bar{e}, e) = \begin{pmatrix} a\bar{a} - e^{-2}b\bar{b} & ab - e^2ab \\ \bar{a}\bar{b} - e^{-2}\bar{a}\bar{b} & a\bar{a} - e^2b\bar{b} \end{pmatrix}.$$

Hence

$$tr(K) - 2 = 2a\bar{a} - 2\Re(e^2)b\bar{b} - 2 = 2a\bar{a} - 2\Re(e^2)b\bar{b} - a\bar{a} + b\bar{b} - 1$$
$$\geq (a\bar{a} - |\Re(e^2)|b\bar{b}) + (b\bar{b} - |\Re(e^2)|b\bar{b}) - 1 \geq a\bar{a} - |\Re(e^2)|b\bar{b} - 1 \geq 0.$$

If the eigenvalues of K would be roots of unity (if it is not unipotent), one would have $|tr(K)| < 2$. Hence by the fact that $tr(K) \geq 2$, one concludes that K is unipotent or has eigenvalues v with $|v| \neq 1$. In both cases K has infinite order. □

For the proof of Theorem 5.1.1 we need to recall some facts about complex simple Lie algebras. The complex simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ will be very important:

Remark 5.1.3. The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is given by

$$\mathfrak{sl}_n(\mathbb{C}) = \{M \in M_{n\times n}(\mathbb{C}) : \mathrm{tr}(M) = 0\}.$$

The Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ is given by

$$\mathfrak{h} = \{\mathrm{diag}(a_1, \ldots, a_n) : \sum_{i=1}^{n} a_i = 0\}.$$

Each root space is given by the matrices $(a_{i,j})$, which have exactly one entry $a_{i_0,j_0} \neq 0$ for a fixed pair (i_0, j_0) with $i_0 \neq j_0$.

We want to show a statement about unitary groups, and not about special linear groups. The fact, which makes $\mathfrak{sl}_n(\mathbb{C})$ interesting for us, is given by the following remark:

Remark 5.1.4. We can obviously embed $\mathfrak{su}_{p,q}(\mathbb{R})$ into $\mathfrak{sl}_{p+q}(\mathbb{C})$, since $\mathrm{SU}(p,q)(\mathbb{R})$ is a Lie subgroup of $\mathrm{SL}_{p+q}(\mathbb{C})$. Moreover $i\mathfrak{su}_{p,q}(\mathbb{R})$ is a subvector space of $\mathfrak{sl}_{p+q}(\mathbb{C})$ (considered as real vector space). One has that

$$\mathfrak{su}_{p,q}(\mathbb{C}) = \mathfrak{su}_{p,q}(\mathbb{R}) \oplus i\mathfrak{su}_{p,q}(\mathbb{R}) = \mathfrak{sl}_{p+q}(\mathbb{C}).$$

(see [17], page 433)

Moreover we need to compare the monodromy group of $\mathcal{L}_j$ with the monodromy groups of some of its restrictions over certain subspaces of $\mathcal{P}_n$.

Remark 5.1.5. Consider some embedding $\iota_{a,b,c} : \mathcal{M}_n \hookrightarrow \mathcal{P}_n$. By the holomorphic diffeomorphism

$$\mathrm{PGL}_2(\mathbb{C}) \times \iota_{a,b,c}(\mathcal{M}_n)(\mathbb{C}) \ni M \times q \to M(q) \in \mathcal{P}_n(\mathbb{C}),$$

we have that

$$\mathrm{PGL}_2(\mathbb{C}) \times \mathcal{M}_n \cong \mathcal{P}_n \quad \text{and} \quad \pi_1(\mathrm{PGL}_2(\mathbb{C})) \times \pi_1(\mathcal{M}_n) \cong \pi_1(\mathcal{P}_n),$$

where $\pi_1(\mathrm{PGL}_2(\mathbb{C})) \cong \mathbb{Z}/(2)$ (compare [14], 3.7).

For technical reasons, we need to introduce an additional subspace of $\mathcal{P}_n$:

$$\mathcal{P}_n^{(a_k)} = \{q \in \mathcal{P}_n | \phi_q(a_k) = \infty\}$$

Let G_T denote the group of triangular matrices given by

$$G_T = \{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) | a \neq 0\}.$$

We have obviously an embedding $\iota_{a,b,c} : \mathcal{M}_n \hookrightarrow \mathcal{P}_n^{(a_{n+3})}$ such that we get a holomorphic diffeomorphism

$$G_T \times \iota_{a,b,c}(\mathcal{M}_n)(\mathbb{C}) \ni M \times q \to M(q) \in \mathcal{P}_n^{(a_{n+3})}(\mathbb{C}).$$

Hence we have that

$$G_T \times \mathcal{M}_n \cong \mathcal{P}_n^{(a_{n+3})} \quad \text{and} \quad \pi_1(G_T) \times \pi_1(\mathcal{M}_n) \cong \pi_1(\mathcal{P}_n^{(a_{n+3})}),$$

where $\pi_1(G_T) \cong \mathbb{Z}/(2)$.

The space $\mathcal{P}_n^{(a_{n+3})}$ has a natural interpretation as configuration space of $n+2$ points on $\mathbb{R}^2$. Its fundamental group is the colored braid group on $n+2$ strands in $\mathbb{R}^2$.

Lemma 5.1.6. *The fundamental group of the configuration space of $n+2$ points on $\mathbb{R}^2$ is generated by the Dehn twists T_{k_1,k_2} with $1 \leq k_1 < k_2 \leq n+2$.*

Proof. (see [20], Chapter **I**. 4) □

5.1.7. By the preceding results, the monodromy groups of $\mathcal{L}_j$, $(\mathcal{L}_j)_{\mathcal{M}_n}$ and $(\mathcal{L}_j)_{\mathcal{P}_n^{(a_{n+3})}}$ are commensurable. Hencefore their $\mathbb{R}$-Zariski closures have the same connected component of identity. Thus we do not need to distinguish between them and we will call them simply $\mathrm{Mon}^0(\mathcal{L}_j)$.

Again assume that $\mathcal{L}_j$ is of type $(1,1)$. By Lemma 5.1.6, the monodromy group $\rho_j(\pi_1(\mathcal{P}_1^{(a_4)}))$ of $(\mathcal{L}_j)_{\mathcal{P}_1^{(a_4)}}$ is generated by the matrices $\rho_j(T_{k,\ell})$ for $k,\ell \in \{1,2,3\}$. For each Dehn twist T one can choose a suitable numbering of the branch points such that $T = T_{1,2}$. Hence by Proposition 3.3.5, one concludes that the generators of the monodromy group are contained in the group given by

$$\{M \in GL_2(\mathbb{C}) | \det(M)^m = 1\}.$$

Since $\mathrm{Mon}^0(\mathcal{L}_j)$ is contained in $\mathrm{U}(1,1)$, one concludes that $\mathrm{Mon}^0(\mathcal{L}_j) \subseteq \mathrm{SU}(1,1)$. Thus the complexification of the Lie algebra of $\mathrm{Mon}^0(\mathcal{L}_j)$ is contained in $\mathfrak{sl}_2(\mathbb{C})$. Note that the real Zariski closure $\mathrm{Mon}^0(\Re\mathbb{V}(j)_\mathbb{R})$ is isomorphic to $\mathrm{Mon}^0(\mathcal{L}_j)$ and $\mathrm{Mon}^0(\Re\mathbb{V}(j)_\mathbb{R})$ is a quotient of the semisimple group $\mathrm{Mon}^0_\mathbb{R}(\mathcal{V}_r)$. Thus by the kernel, which is semisimple, we have an exact sequence of algebraic groups. This yields an exact sequence of semisimple Lie algebras such that $\mathrm{Mon}^0(\mathcal{L}_j)$ must be semisimple. One has that $\mathrm{Mon}^0_\mathbb{C}(\mathcal{L}_j) \subseteq \mathrm{SU}_\mathbb{C}(1,1)$. Since $\mathfrak{su}_{1,1}(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$ is the smallest semisimple non-trivial complex Lie algebra (see [17], §14.1, Step 3) and $\mathrm{Mon}^0(\mathcal{L}_j)$ is infinite by Lemma 5.1.2, one concludes:

Proposition 5.1.8. *If $\mathcal{L}_j$ is of type $(1,1)$, then $\mathrm{Mon}^0(\mathcal{L}_j) = \mathrm{SU}(1,1)$.*

Recall that we want to give a proof of Theorem 5.1.1 by induction. The following construction explains our method to compare the monodromy groups of eigenspaces of different type, which we will need for the induction:

Construction 5.1.9 (Collision of points). Let $\mathbb{L}_j$ be an eigenspace in the cohomology of a fiber $C = \mathcal{C}_q$ with the local monodromy data α_k on a_k. Now let

$$b := \{a_{n_j+2}, a_{n_j+3}\} \quad \text{and} \quad P = \{\{a_1\}, \ldots, \{a_{n_j+1}\}, b\}$$

be a stable partition of $N = \{a_1, \ldots, a_{n_j+3}\}$. Let $\phi_P : P \to \mathbb{P}^1$ be some embedding and the local system $\mathbb{L}(P)_j$ on $\mathbb{P}^1 \setminus \phi_P(P)$ have the local monodromy data

$$\alpha_b = \alpha_{a_{n_j+2}}\alpha_{a_{n_j+3}} \quad \text{and otherwise} \quad \alpha_{\{a_k\}} = \alpha_{a_k}.$$

By Construction 3.2.1, these monodromy data allow the construction of a family of cyclic covers

$$\pi(P) : \mathcal{C}(P) \to \mathcal{P}_{n_j-1}.$$

The higher direct image sheaf $R^1\pi(P)_*(\mathbb{C})$ has an eigenspace with respect to the character given by 1, which we denote by $\mathcal{L}(P)_j$.[1] By the description of the respective monodromy representations in Proposition 3.3.5, we can identify the monodromy group of $(\mathcal{L}(P)_j)_{\mathcal{P}^{(b)}_{n_j-1}}$ with the subgroup of the monodromy group of $(\mathcal{L}_j)_{\mathcal{P}_n^{(a_{n_3})}}$ generated by the Dehn twists $T_{a_{k_1},a_{k_2}}$ with $k_1, k_2 \leq n_j + 1$.

Remark 5.1.10. The local system $\mathcal{L}(P)_j$ is in general not the j-th. eigenspace of a family of irreducible covers of degree m obtained by a collision of two branch points of a family of irreducible covers of degree m. The problem is given by the irreducibility of the resulting family obtained by collision. Take for example the family $\mathcal{C} \to \mathcal{P}_2$ with generic fibers given by

$$y^4 = (x-a_1)(x-a_2)(x-a_3)^2 \cdot \ldots \cdot (x-a_5)^2.$$

[1] This definition may seem to be a little bit odd. But it is motivated by some reasons, which should become clearer by Remark 5.1.10.

By the collision of a_1 and a_2, one does not obtain an irreducible family of degree 4 covers. But the resulting local system $\mathcal{L}(P)_1$ is the eigenspace with respect to the character 1 on the higher direct image sheaf of the family $\mathcal{C}(P) \to \mathcal{P}_1$ with generic fibers given by

$$y^2 = (x - a_1) \cdot \ldots \cdot (x - a_4).$$

Now let $\mathcal{L}_j$ be of type (p, q) with $p, q > 0$. By the collision of two points and Proposition 2.3.4, one gets an eigenspace of type $(p, q-1)$ or of type $(p-1, q)$, if there is a suitable corresponding stable partition. A little bit later we will see that this construction yields an induction step such that the statement of Theorem 5.1.1 for local systems of type $(p, q-1)$ (if $p, q-1 \geq 1$) and of type $(p-1, q)$ (if $p-1, q \geq 1$) implies the statement of Theorem 5.1.1 for local systems of type (p, q).

For the application of the step of induction we will need a pair of stable partitions such that the resulting two eigenspaces satisfy the assumptions of Theorem 5.1.1. Moreover one can assume that for each fiber S_j contains at least 5 different points. Otherwise $\mathcal{L}_j$ is of type $(1, 1)$ or unitary. By the following technical lemma, we start to show that there exists a suitable pair of stable partitions, if the assumptions of Theorem 5.1.1 are satisfied and if S_j contains at least 5 points:

Lemma 5.1.11. *Assume that $j \neq \frac{m}{2}$. Then there is an $a_k \in S_j$ with $\mu_k \neq \frac{1}{2}$.*

Proof. Assume that all $a_k \in S_j$ satisfy $\mu_k = \frac{1}{2}$ and $j \neq \frac{m}{2}$. One has that $\mathcal{C}_r$ (with $r = \gcd(m, j)$) is a family of irreducible cyclic covers onto $\mathbb{P}^1$ of degree $\frac{m}{r} > 2$ given by $\mu_1, \ldots, \mu_{n+3}$ in the sense of Construction 3.2.1. By the assumption that all $a_k \in S_j$ satisfy $\mu_k = \frac{1}{2}$, each branch point has the same branch index $\frac{m}{2r}$, which divides the degree $\frac{m}{r}$. Since we assume that $j \neq \frac{m}{2}$, one concludes that the branch indeces given by $\frac{m}{2r}$ are not 1. Thus $\mathcal{C}_r$ is not a family of irreducible cyclic covers. Contradiction! □

Next we show that a $\mu_k \neq \frac{1}{2}$ yields two stable partitions:

Lemma 5.1.12. *Assume that S_j contains at least 5 different points such that there is an $a_k \in S_j$ with $\mu_k \neq \frac{1}{2}$. Then there are some pairwise different $\mu_h, \mu_i, \mu_s, \mu_t \in S_j$ such that*

$$\mu_h + \mu_i \neq 1, \quad \textit{and} \quad \mu_s + \mu_t \neq 1.$$

Proof. Assume that each pair $h, i' \in \{1, \ldots, n+3\}$ with $h \neq i'$ satisfies $\mu_h + \mu_{i'} = 1$. This implies that $\mu_h = \mu_{i'} = \frac{1}{2}$ for each pair h, i'. But this contradicts the assumptions of this lemma. Hence by the assumptions, there must be a pair (h, i') such that $\mu_h + \mu_{i'} \neq 1$.

Now consider $S'_j := S_j \setminus \{a_h, a'_i\}$. Let us assume that each pair $a_{s'}, a_{t'} \in S'_j$ with $s' \neq t'$ satisfies $\mu_{s'} + \mu_{t'} = 1$. Since $|S'_j| \geq 3$, one concludes that $\mu_{s'} = \mu_{t'} = \frac{1}{2}$. Since $\mu_h = \frac{1}{2}$ or $\mu_{i'} = \frac{1}{2}$ would contradict the assumptions in this case, one concludes that $\mu_h, \mu_{i'} \neq \frac{1}{2}$. Hence put $i := s', s := i', t := t'$, and we are done in this case.

If there are $a_{s'}, a_{t'} \in S'_j$ with $s' \neq t'$ and $\mu_{s'} + \mu_{t'} \neq 1$, we put $i := i', s := s', t := t'$, and we are done.

□

By Lemma 5.1.11 and Lemma 5.1.12, one concludes immediately:

Corollary 5.1.13. *Assume that S_j contains at least 5 different points and $j \neq \frac{m}{2}$. Then there are some pairwise different $\mu_h, \mu_i, \mu_s, \mu_t \in S_j$ such that*

$$\mu_h + \mu_i \neq 1, \quad \textit{and} \quad \mu_s + \mu_t \neq 1.$$

Remark 5.1.14. The condition that

$$\mu_h + \mu_i \neq 1, \quad \text{and} \quad \mu_s + \mu_t \neq 1$$

implies that

$$(\mu_h \neq \frac{1}{2} \quad \text{or} \quad \mu_i \neq \frac{1}{2}) \quad \text{and} \quad (\mu_s \neq \frac{1}{2} \quad \text{or} \quad \mu_t \neq \frac{1}{2}).$$

Hencefore the resulting eigenspace obtained by the collision of a_h and a_i resp., a_s and a_t satisfies that there is a local monodromy datum $\mu_k \neq \frac{1}{2}$. Hence the resulting eigenspace is not a middle part $\mathcal{L}_{\frac{m}{2}}$ of the VHS of the family obtained by the respective collision of two points. One must only ensure that the resulting eigenspaces are not of type $(a, 0)$ resp., $(0, b)$ in order to satisfy the assumptions of Theorem 5.1.1 in this case.

5.1.15. Assume that $\mathcal{L}_j$ is of type $(1, n)$ with $n > 1$. By Proposition 2.3.4, one calculates that

$$\sum_{i=1}^{n+3} \mu_i = 2$$

in this case. One can choose the indeces such that

$$\mu_1 \leq \ldots \leq \mu_{n+3}.$$

Hence one has

$$\mu_1 + \mu_3 \leq \mu_2 + \mu_4 \leq \mu_3 + \mu_5.$$

By the fact that

$$(\mu_2 + \mu_4) + (\mu_3 + \mu_5) < 2 \quad \text{and} \quad \mu_2 + \mu_4 \leq \frac{1}{2}((\mu_2 + \mu_4) + (\mu_3 + \mu_5)),$$

one has

$$\mu_1 + \mu_3 \leq \mu_2 + \mu_4 < 1.$$

Since the local systems with respect to the corresponding stable partitions of the collision of a_1 and a_3 resp., the collision of a_2 and a_4 are of type $(1, n-1)$ as one can calculate by Proposition 2.3.4, one can apply the induction hypothesis for these partitions.

Now let $\mathcal{L}_j$ be of type $(n, 1)$. Then the monodromy representation of $\mathcal{L}_j$ is the complex conjugate of the monodromy representation of $\mathcal{L}_{m-j}$, which is of type $(1, n)$ in this case. Hence first the induction step yields the statement for all $\mathcal{L}_j$ of type $(1, n)$. Then we have the statement for all $\mathcal{L}_j$ of type $(n, 1)$, too.

Assume that $\mathcal{L}_j$ is of type (p, q) with $p, q \geq 2$ and satisfies the assumptions of Theorem 5.1.1. By Corollary 5.1.13, one has a pair of stable partitions. Remark 5.1.14 and the fact that $p, q \geq 2$ imply that the corresponding eigenspaces satisfy the assumptions of Theorem 5.1.1, too.

Now we must only prove and explain the step of induction:

One has without loss of generality the stable partitions

$$P_1 = \{\{a_1\}, \ldots, \{a_{n+1}\}, \{a_{n+2}, a_{n+3}\}\}, \quad \text{and} \quad P_2 = \{\{a_1, a_2\}, \{a_3\}, \ldots, \{a_{n+3}\}\}.$$

Here we assume without loss of generality that $a_k \in \mathbb{R}$ and $a_k < a_{k+1}$ such that δ_k is the oriented path from a_k to a_{k+1} given by the straight line.

Let $q \in \mathcal{P}_n$. We consider the monodromy representation with respect to the basis $\mathcal{B}$ of $(\mathcal{L}_j)_q$ given by

$$\mathcal{B} = \{[e_1\delta_1], \ldots, [e_n\delta_n], [e_{n+2}\delta_{n+2}]\}.$$

One has obviously that $\mathrm{Mon}^0(\mathcal{L}_j(P_1))$ leaves $\langle [e_1\delta_1], \ldots, [e_n\delta_n]\rangle$ invariant and fixes all vectors in $\langle [e_{n+2}\delta_{n+2}]\rangle$. Now let U_1 be a small open neighborhood of the identity in $\mathrm{Mon}^0(\mathcal{L}_j(P_1))(\mathbb{R})$ such that the "inverse"

$$\log : U_1 \to Lie(\mathrm{Mon}^0(\mathcal{L}_j(P_1)))$$

of the exponential map is defined on U_1. By Remark 5.1.4 and the induction hypothesis, $\log(U_1)$ generates a Lie algebra, whose complexification L_1 is with respect to $\mathcal{B}$ given by the matrices

$$\begin{pmatrix} a_{1,1} & \ldots & a_{1,n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n,1} & \ldots & a_{n,n} & 0 \\ 0 & \ldots & 0 & 0 \end{pmatrix}, \quad \text{where} \quad N := \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{pmatrix}$$

is an arbitrary $n \times n$ matrix with $\mathrm{tr}(N) = 0$. Note that $\mathrm{Mon}^0(\mathcal{L}_j(P_2))$ fixes all vectors in $\langle [e_1\delta_1]\rangle$ and leaves $\langle [e_3\delta_3], \ldots, [e_{n+2}\delta_{n_2}]\rangle$ invariant. Hence in a similar way $\log(U_2)$ ($e \in U_2 \subset \mathrm{Mon}^0(\mathcal{L}_j(P_2))(\mathbb{R})$) generates a Lie algebra. Its complexification L_2 is given by the matrices

$$\begin{pmatrix} 0 & v \\ 0 & N \end{pmatrix},$$

where N is again an arbitrary $n \times n$ matrix with $\mathrm{tr}(N) = 0$ and

$$v = (v_1, \ldots, v_n)$$

is a vector depending on N. It is easy to see that L_1 and L_2 generate $\mathfrak{sl}_{n+1}(\mathbb{C})$.

Since $\mathrm{Mon}^0(\mathcal{L}_j)$ is contained in $\mathrm{SU}(p,q)$ and $\mathfrak{su}_{p,q} \otimes \mathbb{C} \cong \mathfrak{sl}_{n+1}(\mathbb{C})$, the group $\mathrm{Mon}^0(\mathcal{L}_j)$ is isomorphic to $\mathrm{SU}(p,q)$.

5.2 The Hodge group of a general direct summand

The VHS of a family $\mathcal{C} \to \mathcal{P}_n$ has a decomposition into rational subvariations $\mathcal{V}_r$ of Hodge structures, which where introduced in Section 4.3. Recall that $\mathcal{V}_r$ is general, if its monodromy group is infinite. Otherwise we call it special. Let $r \neq \frac{m}{2}$, $\mathcal{V}_r$ be general and $\mathcal{L}_j \subset \mathcal{V}_r$ in this section. Moreover recall that $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)_q$ denotes the connected component of identity of the Zariski closure of the monodromy group in $\mathrm{GL}(((\mathcal{V}_r)_{\mathbb{R}})_q)$ for some $q \in \mathcal{P}_n$. Since $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)_{q_1}$ and $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)_{q_2}$ are conjugated, we write $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ instead of $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)_q$ for simplicity.

Remark 5.2.1. The group $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ does not need to be equal to $\mathrm{Mon}^0(\mathcal{V}_r) \times_{\mathbb{Q}} \mathbb{R}$. But it satisfies $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r) \subseteq \mathrm{Mon}^0(\mathcal{V}_r) \times_{\mathbb{Q}} \mathbb{R}$. Hence $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ yields a lower bound for $\mathrm{Mon}^0(\mathcal{V}_r)$. Thus one obtains $C_r^{\mathrm{der}}(\psi) = \mathrm{Mon}^0(\mathcal{V}_r)$, if $C_r^{\mathrm{der}}(\psi)_{\mathbb{R}} = \mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$.

By the preceding section, we know that $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{Mon}^0(\mathcal{L}_j)$ can be considered as the projection onto some $\mathrm{SU}(a,b)$, if $\mathcal{L}_j$ is of type (a,b) with $a, b > 0$. Otherwise one can

use induction with the corresponding stable partitions again. We only consider the start of induction:

Assume that $S_j = 4$. hence one has without loss of generality $\mathcal{C}_r \to \mathcal{P}_1$. By our assumptions, there is an eigenspace $\mathcal{L}_{j_2}$ in $\mathcal{N}^1(\mathcal{C}_r, \mathbb{C})$ of type $(1,1)$, whose monodromy group is infinite. Since the monodromy group of $\mathcal{L}_j$ is conjugated to the monodromy of $\mathcal{L}_{j_2}$ by some $\gamma \in Gal(\mathbb{Q}(\xi^r); \mathbb{Q})$, it is infinite, too. One concludes similarly to the preceding section that $\mathrm{Mon}^0(\mathcal{L}_j) = \mathrm{SU}(2)$ (since $\mathfrak{su}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$ by [17], page 433, too). The rest of the proof is an induction analogue to the induction of the preceding section.

By the preceding considerations, one has:

Proposition 5.2.2. *Assume that $\mathcal{V}_r$ is general. Then the image of the natural projection $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{GL}(\Re\mathbb{V}(j)_{\mathbb{R}})$ is given by the special unitary group induced by the trace map and the special unitary group $\mathrm{SU}(H^1_j(C,\mathbb{C}), H|_{H^1_j(C,\mathbb{C})})$ described in Section 4.3.*

Moreover we know that $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ is contained in $C^{\mathrm{der}}_r(\psi)_{\mathbb{R}}$, which is given by a direct product of certain groups $\mathrm{SU}(a,b)$. Either $\mathrm{Mon}^0(\mathcal{V}_r) = C^{\mathrm{der}}_r(\psi)$ or it is given by a proper subgroup. We want to examine the conditions of the case $\mathrm{Mon}^0(\mathcal{V}_r) \neq C^{\mathrm{der}}_r(\psi)$. This will yield information and some criteria for the structure of $\mathrm{Mon}^0(\mathcal{V}_r)$.

First let us make a simple, but very useful observation:

Remark 5.2.3. Let $G_1, \ldots, G_t$ be connected simple Lie groups and $N \subset G_1 \times \ldots \times G_t =: G$ be a normal connected subgroup. One has that $Lie(G)$ is a direct sum of the simple ideals $Lie(G_1), \ldots, Lie(G_t)$, which implies that each ideal is a sum of certain $Lie(G_i)$ (see [23], **II**. Corollary 6.3). Since the normal connected subgroups of G and the ideals of $Lie(G)$ correspond (follows by [17], Proposition 8.41 and [17], Exercise 9.2), one obtains that

$$N = G_1 \times \ldots \times G_{t_0} \times \{e\} \times \ldots \times \{e\}$$

for some $t_0 \leq t$ with respect to a suitable numbering.

The decomposition of the rational Hodge structure $N^1(C_r, \mathbb{Q})$ into the $\mathbb{Q}(\xi^r)^+$-Hodge structures $\Re\mathbb{V}(j)$ yields a decomposition of the variation $\mathcal{V}_r$ of rational Hodge structures into the variations $\Re\mathcal{V}(j)$ of $\mathbb{Q}(\xi^r)^+$-Hodge structures.

By technical reasons, we consider the semisimple adjoint group $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r)$ instead of $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ first. By Remark 5.2.3, one concludes that $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r)$ is isomorphic to the direct product of $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j)_{\mathbb{R}})$ and the kernel K_j of the natural projection $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j)_{\mathbb{R}})$. Moreover one has:

Lemma 5.2.4. *Let $G_1, \ldots, G_t$ be simple adjoint Lie groups and G be a semisimple subgroup of $G_1 \times \ldots \times G_t$ such that each natural projection*

$$G \hookrightarrow G_1 \times \ldots \times G_t \xrightarrow{pr_j} G_j$$

is surjective. One has $G \neq G_1 \times \ldots \times G_t$, if and only if there are some $j_1, j_2 \in \{1, \ldots, t\}$ with $j_1 \neq j_2$ such that G contains a simple subgroup G' isomorphically mapped onto G_{j_1} and G_{j_2} by the natural projections.

Proof. The "if" part is easy to see. The "only if" part follows by induction. □

Note that we have a natural embedding

$$\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r) \hookrightarrow \prod_{j\in\mathbb{Z}/\frac{m}{r}, j\leq\frac{m}{2}} \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j)_{\mathbb{R}}).$$

Thus the preceding lemma and our assumption that $\mathrm{Mon}^0(\mathcal{V}_r) \neq C_r^{\mathrm{der}}(\psi)$ imply that there is a direct simple factor of $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r)$, which isomorphically mapped onto $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1)_{\mathbb{R}})$ and $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_2)_{\mathbb{R}})$ for some j_1 and j_2 with $j_2 \neq j_1$ and $m - j_1$. By Remark 5.2.3, $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r)$ is a direct product of the kernel of the both projections and this direct simple factor.

Thus the natural projections onto $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1)_{\mathbb{R}})$ and $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_2)_{\mathbb{R}})$ yield an isomorphism

$$\alpha^{\mathrm{ad}} : \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \to \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_2)_{\mathbb{R}}).$$

Moreover one concludes that the image $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}})$ of the projection

$$\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \times \mathrm{Mon}^{ad}(\Re\mathcal{V}(j_2)_{\mathbb{R}})$$

is given by the graph of α^{ad}.

5.2.5. For the image $\mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}})$ of the projection

$$\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \times \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}})$$

this implies that the natural projections

$$p_1 : \mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}}) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}})$$

and

$$p_2 : \mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}}) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}})$$

are isogenies. Since

$$\mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}})(\mathbb{C}) = \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}})(\mathbb{C}) = \mathrm{SL}_{a+b}(\mathbb{C}),$$

where (a, b) is the type of $\mathcal{L}_{j_1}$, and the Lie group $\mathrm{SL}_{a+b}(\mathbb{C})$ is simply connected (see [17], Proposition 23.1), the induced isogenies of Lie groups of $\mathbb{C}$-valued points are isomorphisms. Hence the isogenies p_1 and p_2 are isomorphisms.

Hence our assumption implies the existence of an isomorphism

$$\alpha : \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}}),$$

which satisfies that $\mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}})$ is given by $\mathrm{Graph}(\alpha)$.

5.3 A criterion for the reaching of the upper bound

In this section we give a necessary criterion for the existence of an isomorphism α. This yields a sufficient condition that $\mathrm{Mon}^0(\mathcal{V}_r)$ reaches the upper bound $C_r^{\mathrm{der}}(\psi)$. In addition we will see that $\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Mon}^0(\mathcal{V}_1)$ reaches the upper bound, if the degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is a prime number > 2.[2]

[2]For $m = 2$ we will later see that $\mathrm{Mon}^0(\mathcal{V})$ reaches the upper bound as well.

We say that a Dehn twist T is semisimple (with respect to $\mathcal{V}_r$), if the monodromy representation ρ_j of one (and hence of all) $\mathcal{L}_j \subset \mathcal{V}_r$ yields a semisimple matrix $\rho_j(T)$. By the trace map (see (4.3)), we can identify $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j)_{\mathbb{R}})$ and $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_j)$. Thus $\mathrm{Mon}^0(\Re\mathcal{V}(j_1,j_2)_{\mathbb{R}})$ is equal to $\mathrm{Graph}(\alpha)$, if and only if one has a corresponding isomorphism $\alpha^{\mathrm{ad}} : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_1) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_2)$ such that $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1} \oplus \mathcal{L}_{j_2})$ is given by $\mathrm{Graph}(\alpha^{\mathrm{ad}})$. By an abuse of notation, we will write α instead of α^{ad} from now on.

First let us formulate a sufficient criterion for the non-existence of α in the case $\mathcal{C} \to \mathcal{P}_1$:

Proposition 5.3.1. *Let $\mathcal{V}_r$ be general and $\mathcal{L}_{j_1}, \mathcal{L}_{j_2} \subset \mathcal{V}_r$ be of type (a,b), where $a+b=2$. Then there does not exist an isomorphism $\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_1) \to \mathrm{Mon}^{ad}(\mathcal{L}_2)$ such that $P\rho_{j_2} = \alpha \circ P\rho_{j_1}$, if there is a semisimple Dehn twist T such that the non-trivial eigenvalue z_2 of $\rho_{j_2}(T)$ is not contained in $\{z_1, \bar{z}_1\}$, where z_1 denotes the non-trivial eigenvalue of $\rho_{j_1}(T)$.*

Proof. Assume that $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_1)$ and $\mathrm{Mon}^{ad}(\mathcal{L}_2)$ are isomorphic and T satisfies the assumptions of this proposition. Thus $\rho_{j_1}(T)$ generates a finite commutative subgroup FT of $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1})$, which is contained in a maximal torus G of $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1})$. Our assumption that $a+b=2$ implies that $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1}) \cong \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2})$ is isomorphic to $\mathrm{PU}(1,1)$ or $\mathrm{PU}(2)$. Note that the maximal tori of reductive groups are conjugate and in the case of $\mathrm{PU}(1,1)$ resp., $\mathrm{PU}(2)$ the $\mathbb{R}$-valued points of $G(\mathbb{R})$ can (up to conjugation) be given by the diagonal matrices in $\mathrm{PU}(1,1)$ resp., $\mathrm{PU}(2)$. Thus one checks easily that G is unique and that the Lie group $G(\mathbb{R})$ is isomorphic to $S^1(\mathbb{R})$. Hence one can identify $FT(\mathbb{R})$ with some $\langle \xi^s \rangle \subset S^1(\mathbb{R})$. Now let $1 \neq \zeta \in \langle \xi^s \rangle$ satisfy the property that there is a closed interval on $S^1(\mathbb{R})$ with end points 1 and ζ, which does not contain any other element of $\langle \xi^s \rangle$. Hence there is a closed interval I on T with ending points $[\mathrm{diag}(1,1)]$ and $[\mathrm{diag}(\zeta,1)] \in FT$, which does not contain any other element of FT.

Now assume such an isomorphism α exists. Note that we have an identification $\alpha(G)(\mathbb{R}) = S^1(\mathbb{R})$, too. But our assumptions imply that

$$\alpha(\mathrm{diag}(\zeta,1)) \notin \{\mathrm{diag}(\zeta,1), \mathrm{diag}(\bar{\zeta},1)\}.$$

Hence by our identification $\alpha(G)(\mathbb{R}) = S^1(\mathbb{R})$, one obtains that

$$\alpha(\zeta) \notin \{\zeta, \bar{\zeta}\}.$$

Thus $\alpha(I) \subset \alpha(G)(\mathbb{R})$ is not a connected interval, which does not contain any other element of $\langle \xi^s \rangle$ except of 1 and $\alpha(\zeta)$. But α must be a homeomorphism on the $\mathbb{R}$-valued points. Contradiction! □

By the preceding proposition, we can use certain semisimple Dehn twists for the study of the generic Hodge group. Hence we make some observations about the orders and the existence of semisimple Dehn twists:

Lemma 5.3.2. *Let $j \neq \frac{m}{2}$ and $v|\frac{m}{r}$, where*

$$1 \neq v, \quad r := \gcd(m,j) \quad \textit{and} \quad 1,2 \neq \frac{m}{rv}.$$

Then there exists a Dehn twist $T \in \pi_1(\mathcal{P}_n)$ such that $\rho_j(T) \in \mathrm{Mon}(\mathcal{L}_j)$ is semisimple and $|\langle \rho_j(T) \rangle|$ does not divide v.

Proof. One can replace $\mathcal{C}$ by $\mathcal{C}_r$ and choose a suitable collection of local monodromy data for $\mathcal{C}$ such that $j = 1$. By an isomorphism $\langle \xi \rangle \cong \mathbb{Z}/(m)$, the non-trivial eigenvalues of the semisimple Dehn twists T_{k_1,k_2} correspond to some elements $[b_{k_1,k_2}] \in \mathbb{Z}/(m)$, where $b_{k_1,k_2} := d_{k_1} + d_{k_2}$ and d_{k_1} and d_{k_2} denote the branch indeces of a_{k_1} and a_{k_2}.

Assume that each semisimple Dehn twist satisfies that its order divides some v with $v|m$. This implies that $\frac{m}{v}|b_{k_1,k_2}$ for all b_{k_1,k_2}. Hence for all $k = 1, \ldots, n+3$ one has that $\frac{m}{v}$ divides

$$2d_k = (d_k + d_{k_1}) + (d_k + d_{k_2}) - (d_{k_1} + d_{k_2}) = b_{k,k_1} + b_{k,k_2} - b_{k_1,k_2}.$$

Since there does not exist any integer $N \neq 1$, which divides each d_k, one has that $\frac{m}{v}$ divides 2. This implies that $\frac{m}{v} = 1$ or $\frac{m}{v} = 2$. □

For the formulation of our criterion in the higher dimensional case we need the following lemma:

Lemma 5.3.3. *Let $q \in \mathcal{P}_n$. Assume that we have a stable partition*

$$P := \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4, \ldots, a_{n_j+3}\}\}$$

with respect to the local monodromy data of $(\mathcal{L}_j)_q$ such that we can define the eigenspace $\mathcal{L}_j(P)$ over $\mathcal{P}_1$ with $b = \{a_4, \ldots, a_{n_j+3}\}$ as in Construction 5.1.9. *Then the monodromy group $\rho_j(P)(\pi_1(\mathcal{P}_1))$ of $\mathcal{L}_j(P)$ has a subgroup of finite index generated by $\rho_j(T_{1,2})$ and $\rho_j(T_{2,3})$.*

Proof. The stability of the partition ensures that $\alpha_b = \alpha_{a_4} \ldots \alpha_{n_j+3} \neq 1$. It is a well-known fact that $\pi_1(\mathcal{M}_1(\mathbb{C}))$ is generated by the two loops around 0 and 1, where we identify $\mathbb{A}^1 \setminus \{0, 1\} = \mathcal{M}_1$. By the embedding $\mathcal{M}_1 \to \mathcal{P}_1$ given by

$$a_1 = 0, \quad a_3 = 1, \quad a_4 = \infty,$$

we can identify the generators of $\pi_1(\mathcal{M}_1(\mathbb{C}))$ with the Dehn twists $T_{1,2}$ and $T_{2,3}$. The statement follows from the fact that the monodromy group of $\mathcal{L}_j(P)|_{\mathcal{M}_1}$ has finite index in the monodromy group of $\mathcal{L}_j(P)$. □

Proposition 5.3.4. *Let $\mathcal{L}_{j_1}, \mathcal{L}_{j_2} \subset (\mathcal{V}_1)_{\mathbb{C}}$ with $j_1 \neq j_2$ and $j_1 \neq m - j_2$. Assume that we have a stable partition*

$$P := \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4, \ldots, a_{n+3}\}\}$$

such that the monodromy group of $\mathcal{L}_{j_1}(P)$ or $\mathcal{L}_{j_2}(P)$ is infinite. Let $\mathrm{Mon}^0(\mathcal{L}_{j_1}(P))$ and $\mathrm{Mon}^0(\mathcal{L}_{j_2}(P))$ be not isomorphic or $T_{k,\ell}$ be a semisimple Dehn twist with $k, \ell \in \{1, 2, 3\}$ such that the non-trivial eigenvalue z_2 of $\rho_{j_2}(T_{k,\ell})$ is not contained in $\{z_1, \bar{z}_1\}$, where z_1 denotes the non-trivial eigenvalue of $\rho_{j_1}(T_{k,\ell})$. Then

$$\mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}}) = \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \times \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}}).$$

Proof. By Lemma 5.3.3 and the fact that the monodromy group of $\mathcal{L}_j|_{\mathcal{M}_n}$ has finite index in the monodromy group of $\mathcal{L}_j$, one concludes that the group generated by $\rho_{j_1}(T_{1,2})$ and $\rho_{j_1}(T_{2,3})$ resp., $\rho_{j_2}(T_{1,2})$ and $\rho_{j_2}(T_{2,3})$ has finite index in the monodromy representation of $\mathcal{L}_{j_1}(P)$ resp., $\mathcal{L}_{j_2}(P)$. Hencefore an isomorphism

$$\alpha : \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}})$$

yields an isomorphism

$$\alpha(P) : \mathrm{Mon}^0(\mathcal{L}_{j_1}(P)) \to \mathrm{Mon}^0(\mathcal{L}_{j_2}(P)).$$

Thus one only needs to apply Proposition 5.3.1. □

Now let us first define the condition for the reaching of the upper bound and then write down the obvious theorem:

Definition 5.3.5. Assume that one has for each $\mathcal{L}_{j_1}, \mathcal{L}_{j_2} \subset \mathcal{V}_r$ with $j_1 \neq j_2, m - j_2$ and $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{L}_{j_1}) \cong \mathrm{Mon}^0_{\mathbb{R}}(\mathcal{L}_{j_1})$ a stable partition

$$P := \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4, \ldots, a_{n_j+3}\}\}$$

(with respect to a suitable enumeration) such that the monodromy group of $\mathcal{L}_{j_1}(P)$ or $\mathcal{L}_{j_2}(P)$ is infinite and one of the following conditions is satisfied:

1. $\mathrm{Mon}^0(\mathcal{L}_{j_1}(P))$ and $\mathrm{Mon}^0(\mathcal{L}_{j_2}(P))$ are not isomorphic.
2. There is a semisimple Dehn twist $T_{k,\ell}$ with $k, \ell \in \{1, 2, 3\}$ such that the non-trivial eigenvalue z_2 of $\rho_{j_2}(T_{k,\ell})$ is not contained in $\{z_1, \bar{z}_1\}$, where z_1 denotes the non-trivial eigenvalue of $\rho_{j_1}(T_{k,\ell})$.

We call $\mathcal{V}_r$ very general in this case.

A direct summand $\mathcal{V}_r$ is exceptional, if it is general, but not very general.

By Proposition 5.3.4, one concludes:

Theorem 5.3.6. *If $\mathcal{V}_r$ is very general, $\mathrm{Mon}^0(\mathcal{V}_r)$ reaches the upper bound $C^{\mathrm{der}}(\psi)$.*

Theorem 5.3.7. *If the degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is a prime number $m > 2$, $\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Mon}(\mathcal{V}_1)$ reaches the upper bound.*

Proof. By the preceding theorem, we have only to show that $\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Mon}(\mathcal{V}_1)$ is very general. Note that Lemma 5.3.2 implies that there is a semisimple Dehn twist for $m > 2$.

Assume that we are in the case of a family $\mathcal{C} \to \mathcal{P}_1$, and that $j_1 \neq j_2, m - j_2$. Since $\mathbb{Z}/(m)$ is a field in our case, one has that each semisimple Dehn twist satisfies that the non-trivial eigenvalue of $\rho_{j_2}(T)$ is not contained in $\{z_1, \bar{z}_1\}$, where z_1 denotes the non-trivial eigenvalue of $\rho_{j_2}(T)$. Thus in this case the statement follows from Proposition 5.3.1.

Otherwise we have to find a stable partition P as in Proposition 5.3.4. One has without loss of generality the semisimple Dehn twist $T_{1,2}$. Moreover assume without loss of generality that $d_1 + d_2 = m - 1$. One has two cases: Either there is some a_3 such that

$$P = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4, \ldots, a_{n+3}\}\}$$

is the desired stable partition or one has that

$$d_3 = \ldots = d_{n+3} = 1.$$

Since in the case $m = 3$ there is nothing to show, one can otherwise assume that $m > 3$ and take the stable partition

$$P = \{\{a_3\}, \{a_4\}, \{a_5\}, \{a_1, a_2, a_6, \ldots, a_{n+3}\}\}.$$

□

5.4 The exceptional cases

At this time the author does not see a possibility to calculate the monodromy group of the VHS of an arbitrary family $\mathcal{C} \to \mathcal{P}_n$. Hencefore we consider mainly a family $\mathcal{C} \to \mathcal{P}_1$.

5.4.1. Let ρ_{j_1} and ρ_{j_2} denote the monodromy representations of $\mathcal{L}_{j_1}, \mathcal{L}_{j_2} \subset \mathcal{V}_r$. Proposition 3.3.5 yields a description of $\rho_{j_1}(T)$ and $\rho_{j_2}(T)$ for some Dehn twist T. By this description, the entries of the matrices $\rho_{j_1}(T)$ and $\rho_{j_2}(T)$ differ by some $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})$. By its action on $\langle \xi^r \rangle \cong \mathbb{Z}/(\frac{m}{r})$, each γ can be identified with some $[v] \in (\mathbb{Z}/(\frac{m}{r}))^*$ such that $[\frac{j_1}{r}v]_{\frac{m}{r}} = [\frac{j_2}{r}]_{\frac{m}{r}}$. One has a subgroup $H_1(\gamma)$ of $\langle \xi^r \rangle$ consisting of roots of unity fixed by γ and a subgroup $H_2(\gamma)$ of $\langle \xi^r \rangle$ consisting of roots of unity, on which γ acts by complex conjugation. Since $j_1 \neq j_2, m - j_2$, one has that γ is neither given by the complex conjugation nor by the identity. Thus $H_1(\gamma)$ resp., $H_2(\gamma)$ is given by $\{1\}$ or some proper subgroup of $\langle \xi^r \rangle$ generated by $\xi^{rt_1(\gamma)}$ resp., $\xi^{rt_2(\gamma)}$, where $1 \neq t_1(\gamma)$ and $1 \neq t_2(\gamma)$ divide $\frac{m}{r}$.

For the rest of this section we consider only families $\mathcal{C} \to \mathcal{P}_1$ of degree m with an exceptional part $\mathcal{V}_r$. Assume without loss of generality that $\mathcal{V}_1$ is exceptional and $j_1 = 1$. Let γ correspond to v. For simplicity we write t_1 and t_2 instead of $t_1(\gamma)$ and $t_2(\gamma)$, and H_1 and H_2 instead of $H_1(\gamma)$ and $H_2(\gamma)$.

Lemma 5.4.2. *Let $\mathcal{C} \to \mathcal{P}_1$ be a family of degree m covers such that $\mathcal{V}_1$ is exceptional. Then one is without loss of generality in one of the following cases:*

1. *(Complex case) $t_1 | d_1 + d_2$, $t_1 | d_2 + d_3$ and $t_2 | d_1 + d_3$, where t_1 does not divide $d_1 + d_3$.*
2. *(Separated case) $t_1 = 2$ and 2 divides $d_1 + d_2$, $d_2 + d_3$ and $d_1 + d_3$.*

Proof. If $\mathcal{V}_1$ is exceptional, then d_1+d_2, d_2+d_3 and d_1+d_3 are divided by t_1 or t_2. Assume that t_1 (resp., t_2) divides d_1+d_2, d_2+d_3 and d_1+d_3. Hence one has $t_1 = 2$ (resp., $t_2 = 2$) as in the proof of Lemma 5.3.2. Otherwise one has only to choose a suitable enumeration such that one is in the complex case. □

Remark 5.4.3. It can occur that one is in the complex case and the separated case with respect to the same eigenspaces (up to complex conjugation). Consider the family $\mathcal{C} \to \mathcal{P}_1$ of degree 12 covers given by

$$d_1 = 5, \quad d_2 = 1, \quad d_3 = 11, \quad d_4 = 7.$$

Let $v = 5$. Then one has $t_1 = 3$ and $t_2 = 2$ such that $3|d_1 + d_2$, $3|d_2 + d_3$ and $2|d_1 + d_3$. Now let $v = 7$. In this case one has $t_1 = 2$ and 2 divides $d_1 + d_2$, $d_2 + d_3$ and $d_1 + d_3$. By 5.4.10, we will see that there is an isomorphism $\alpha : \mathrm{Mon}^0(\Re\mathcal{V}(1))_{\mathbb{R}} \to \mathrm{Mon}^0(\Re\mathcal{V}(5))_{\mathbb{R}}$.

On the other hand consider the family $\mathcal{C} \to \mathcal{P}_1$ of degree 12 covers given by

$$d_1 = 11, \quad d_2 = 1, \quad d_3 = 11, \quad d_4 = 1.$$

Again by the same arguments, we are in the complex case and the separated case at the same time. But in this case the existence of a suitable isomororphism $\alpha : \mathrm{Mon}^0(\Re\mathcal{V}(1))_{\mathbb{R}} \to \mathrm{Mon}^0(\Re\mathcal{V}(5))_{\mathbb{R}}$ is not known to the author at this time.

5.4.4. Assume that the direct summand $\mathcal{V}_1$ is separated with respect to $[v]_m \in (\mathbb{Z}/(m))^*$ for a family $\mathcal{C} \to \mathcal{P}_1$ of degree m covers. One has $[v2] = [2]$ in each separated case. This implies that $[2][v-1] = [0]$. Hencefore one has $[v] = [\frac{m}{2}+1] \in (\mathbb{Z}/(m))^*$ in each separated case. Hence $v \in (\mathbb{Z}/(m))^*$ is an involution. The fact that $[v] = [\frac{m}{2}+1] \in (\mathbb{Z}/(m))^*$ implies that $\frac{m}{2}+1$ is odd. Hence 4 divides m. In the separated case $r_1 = 2$ divides each $d_k + d_\ell$. Thus $\mathcal{V}_1$ is separated, if and only if $4|m$ and each d_k is odd.

Hencefore there are infinitely many cases of families $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is separated. At this time the author can not give an isomorphism $\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_1) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{\frac{m}{2}+1})$ for each separated example.

By the preceding point we have classified and described all examples $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is separated. Hence we consider only the case of a family $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is complex for the rest of this section.

Lemma 5.4.5. *Assume that $\mathcal{V}_1$ is complex. Then one has:*

$$\ell := \mathrm{lcm}(t_1, t_2) = \begin{cases} m & : \quad m \ \textit{is odd} \\ \frac{m}{2} & : \quad m \ \textit{is even} \end{cases}$$

Proof. If m is odd, $H_1 \cap H_2 = \{1\} = \{\xi^m\}$. If m is even, $H_1 \cap H_2 = \{1, -1\} = \langle \xi^{\frac{m}{2}} \rangle$. □

Lemma 5.4.6. *Assume that $\mathcal{V}_1$ is complex. Then one has that $t_1 t_2 = m$ or $t_1 t_2 = \frac{m}{2}$. Moreover one has that $t_1 t_2 = m$, if m is odd, and $t_1 t_2 = \frac{m}{2}$, if $2|m$, but 4 does not divide m.*

Proof. If m is odd, one has $\ell = \mathrm{lcm}(t_1, t_2) = m$. Hence one has $t_1' t_2' g = m$ for $g := \gcd(t_1, t_2)$ and $t_i = g t_i'$. Hence $|H_1| = t_2'$ and $|H_2| = t_1'$. If $g > 2$, there is a semisimple Dehn twist, whose order does not divide $t_1' t_2'$ (follows from Lemma 5.3.2). But this can not occur by our assumption that $\mathcal{V}_1$ is complex. Hence $g = 1$, since $g = 2$ is not possible for m odd.

If m is even, one has $\ell = \mathrm{lcm}(t_1, t_2) = \frac{m}{2}$. Hence one has $t_1' t_2' g = \frac{m}{2}$ for $g := \gcd(t_1, t_2)$ and $t_i = g t_i'$. If $g > 2$, there is a semisimple Dehn twist, whose order does not divide $t_1' t_2'$. Hence one has $g = 1$ or $g = 2$. Thus $t_1 t_2 = m$ or $t_1 t_2 = \frac{m}{2}$.

Now assume that $2|m$, but 4 does not divide m. Then one has that $\frac{m}{2} = \mathrm{lcm}(t_1, t_2)$ is odd. Hence one can not have that $g = 2$ in this case. Thus $g = 1$ and $t_1 t_2 = \frac{m}{2}$. □

Example 5.4.7. In the case $4|m$ both $t_1 t_2 = m$ and $t_1 t_2 = \frac{m}{2}$ can occur. Let $m = 24$ and take $v = 5$ for the corresponding automorphism of $\mathbb{Q}(\xi)$. In this case one has $t_1 = 6$ and $t_2 = 4$ such that $t_1 t_2 = 24 = m$.

Now let $m = 24$ and take $v = 7$. In this case one has $t_1 = 4$ and $t_2 = 3$ such that $t_1 t_2 = 12 = \frac{m}{2}$.

Proposition 5.4.8. *Assume $\gamma \in Gal(\mathbb{Q}(\xi); \mathbb{Q})$ yields an example of a complex case. Then γ is an involution.*

Proof. Let $[v] \in \mathbb{Z}/(m)^*$ correspond to γ. One has that $t_1 t_2 = m$ or $t_1 t_2 = \frac{m}{2}$. Since one has that $[vt_1]_m = [t_1]_m$ and $[vt_2]_m = -[t_2]_m$, one gets that

$$(v-1)t_1 \in (m) \quad \text{and} \quad (v+1)t_2 \in (m).$$

This implies that $t_2|(v-1)$ and $t_1|(v+1)$ or (if $t_1 t_2 = \frac{m}{2}$) that $2t_2|(v-1)$ and $2t_1|(v+1)$. Hence in each case one obtains that

$$v^2 - 1 = (v-1)(v+1) \in (m).$$

□

Theorem 5.4.9. *Let $\mathcal{C} \to \mathcal{P}_1$ be a family of degree m covers. Then $\mathcal{V}_1$ is complex, if and only if the fibers of $\mathcal{C}$ have the branch indeces $d_1, \ldots, d_4$ with $2m = d_1 + \ldots + d_4$ such that*

$$[vd_2]_m = [d_1 + d_2 + d_3]_m, \quad [vd_1]_m = [-d_3]_m, \quad [vd_3]_m = [-d_1]_m$$

or

$$[vd_2]_m = [d_1 + d_2 + d_3 + \frac{m}{2}]_m, \quad [vd_1]_m = [-d_3 + \frac{m}{2}]_m, \quad [vd_3]_m = [-d_1 + \frac{m}{2}]_m$$

for some v with $[v^2]_m = [1]_m$ and $[v]_m \notin \{[1]_m, [m-1]_m\}$.

Proof. The condition $2m = d_1 + \ldots + d_4$ ensures that $\mathcal{V}_1$ is not special.

By an abuse of notation, each integer z denotes the residue class $[z]_m$ in this proof. Assume that $\mathcal{V}_1$ is complex. Hence by Lemma 5.4.2, one has that

$$2vd_2 = v((d_1+d_2)-(d_1+d_3)+(d_2+d_3)) = (d_1+d_2)+(d_1+d_3)+(d_2+d_3) = 2(d_1+d_2+d_3),$$

$$2vd_1 = v((d_1+d_2)+(d_1+d_3)-(d_2+d_3)) = (d_1+d_2)-(d_1+d_3)-(d_2+d_3)) = -2d_3,$$

$$2vd_3 = v(-(d_1+d_2)+(d_1+d_3)+(d_2+d_3)) = -(d_1+d_2)-(d_1+d_3)+(d_2+d_3)) = -2d_1.$$

Hence one has two cases:

$$vd_2 = d_1 + d_2 + d_3 \quad \text{or} \quad vd_2 = d_1 + d_2 + d_3 + \frac{m}{2}$$

In the first case (resp., the second case) the fact that $v(d_1 + d_2) = d_1 + d_2$ implies that $vd_1 = -d_3$ (resp., $vd_1 = -d_3 + \frac{m}{2}$). Moreover in the first case (resp., the second case) the fact that $v(d_2 + d_3) = d_2 + d_3$ implies that $vd_3 = -d_1$ (resp., $vd_3 = -d_1 + \frac{m}{2}$). Hence we have obtained the claimed equations.

Assume conversely that the family $\mathcal{C} \to \mathcal{P}_1$ satisfies one of the two systems of equations of this theorem. Then one can easily calculate that $\mathcal{V}_1$ is complex. □

5.4.10. Let $\mathcal{C} \to \mathcal{P}_1$ be a family of degree m covers. Assume that d_1, d_2, d_3 satisfy the first system of equations of Theorem 5.4.9 with respect to some v with $[v^2] = [1]_m$, which satisfies that $[v]_m \notin \{[1]_m, [m-1]_m\}$. Moreover let $j \in (\mathbb{Z}/(m))^*$ such that $\mathcal{L}_j \subset \mathcal{V}_1$ with monodromy representation ρ_j. Now we calculate that $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not reach the upper bound $C_1^{\mathrm{der}}(g)_{\mathbb{Q}(\xi)^+}$ in this case.

Let $a_1 = 0$, $a_3 = 1$ and $a_4 = \infty$. The fundamental group of the corresponding copy of $\mathcal{M}_1$ is generated by $T_{1,2}$ and $T_{2,3}$. One obtains that

$$\rho_j(T_{1,2}) = \begin{pmatrix} \xi^{jd_1+jd_2} & 1-\xi^{jd_1} \\ 0 & 1 \end{pmatrix}, \quad \rho_j(T_{2,3}) = \begin{pmatrix} 1 & 0 \\ \xi^{jd_2} - \xi^{jd_2+jd_3} & \xi^{jd_2+jd_3} \end{pmatrix}.$$

Let $\gamma_v \in \mathrm{Gal}(\mathbb{Q}(\xi); \mathbb{Q})$ denote the automorphism corresponding to $[v]$. The monodromy representation of $\mathcal{L}_{jv}$ is given by

$$\rho_{jv}(T_{1,2}) = \begin{pmatrix} \xi^{jd_1+jd_2} & 1-\xi^{-jd_3} \\ 0 & 1 \end{pmatrix}, \quad \rho_{jv}(T_{2,3}) = \begin{pmatrix} 1 & 0 \\ \xi^{jd_1+jd_2+jd_3} - \xi^{jd_2+jd_3} & \xi^{jd_2+jd_3} \end{pmatrix}.$$

One calculates easily that

$$\frac{1-\xi^{jd_1}}{1-\xi^{-jd_3}} \cdot \frac{\xi^{jd_2} - \xi^{jd_2+jd_3}}{\xi^{jd_1+jd_2+jd_3} - \xi^{jd_2+jd_3}} = \frac{\xi^{jd_2} - \xi^{jd_2+jd_3} - \xi^{jd_1+jd_2} + \xi^{jd_1+jd_2+jd_3}}{\xi^{jd_1+jd_2+jd_3} - \xi^{jd_2+jd_3} - \xi^{jd_1+jd_2} + \xi^{jd_2}} = 1.$$

Hence there is a $z \in \mathbb{Q}(\xi)$ such that $\gamma_v|_{<\rho_j(T_{1,2}),\rho_j(T_{2,3})>}$ coincides with $\alpha|_{<\rho_j(T_{1,2}),\rho_j(T_{2,3})>}$, where α is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & zb \\ z^{-1}c & d \end{pmatrix}.$$

Thus by Lemma 5.2.4, the group $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not attain its upper bound in this case. In addition one calculates easily that α is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \sqrt{z^{-1}} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{z^{-1}} & 0 \\ 0 & \sqrt{z} \end{pmatrix}.$$

Thus the monodromy representations of $\mathcal{L}_j$ and $\mathcal{L}_{jv}$ coincide up to conjugation such that $\mathcal{L}_j$ and $\mathcal{L}_{jv}$ are isomorphic for each $j \in (\mathbb{Z}/(m))^*$.

Corollary 5.4.11. *There are infinitely many families $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is complex and $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not reach its upper bound.*

Proof. Let $p, q \in \mathbb{N}$ such that $\gcd(p,q) = 1$ with $p, q \notin \{1,2\}$ and $m := pq$. Hence $\mathbb{Z}/(m) = \mathbb{Z}/(p) \times \mathbb{Z}/(q)$. Let $v < m$ correspond to $(1,-1) \in \mathbb{Z}/(p) \times \mathbb{Z}/(q)$. Thus we get $[v^2] = [1]_m$ and $[v]_m \notin \{[1]_m, [m-1]_m\}$. One has that

$$d_1 = v, \quad d_2 = 1, \quad d_3 = m - 1$$

satisfies the first system of equations of Theorem 5.4.9, which quaranties by 5.4.10 that $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not reach its upper bound. Since there are infinitely many possible choices for $p, q \in \mathbb{N}$ such that $\gcd(p,q) = 1$ with $p, q \notin \{1,2\}$, one obtains infinitely many families $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is exceptional. □

5.5 The Hodge group of a universal family of hyperelliptic curves

If the middle part $\mathcal{V}_{\frac{m}{2}}$ is of type $(1,1)$, one obtains $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}}) = \mathrm{Sp}_{\mathbb{Q}}(2)$, since $\mathrm{Sp}_{\mathbb{R}}(2) \cong \mathrm{SU}(1,1)$, and $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_{\frac{m}{2}}) = \mathrm{SU}(1,1)$ as one has by Theorem 5.1.1.

By Proposition 3.3.5, each Dehn twist $T_{\ell,\ell+1}$ yields a unipotent subgroup of $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}})$ isomorphic to $\mathbb{G}_a$. Its corresponding subvector space of the Lie algebra is generated by

$$A_{\ell,\ell+1}(a,b) = \begin{cases} -1 & : \quad a = \ell \quad \text{and} \quad b = \ell - 1 \\ 1 & : \quad a = \ell \quad \text{and} \quad b = \ell + 1 \\ 0 & : \quad \text{elsewhere} \end{cases}.$$

Now we consider the middle part of type $(2,2)$. Hence we are in the case of the genus 2 curves. For $\ell = 1, \ldots, 4$ the matrices $A_{\ell,\ell+1}$ generate a 4 dimensional vector space. Moreover by $[A_{i,i+1}, A_{i+1,i+2}]$ for $i = 1,2,3$, we get the 3 additional linearly independent matrices

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

By

$$[A_{2,3},[A_{3,4},A_{4,5}]] \text{ resp., } [[A_{1,2},A_{2,3}],A_{3,4}],$$

we obtain the two further linearly independent matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the Lie algebra has at least dimension 9. Moreover one checks easily that

$$[[A_{1,2},A_{2,3}],[A_{3,4},A_{4,5}]] = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

is a tenth linearly independent matrix. Thus the well-known fact that $Sp_{\mathbb{Q}}(4)$ has dimension 10 implies:

Proposition 5.5.1. *If $\mathcal{V}_{\frac{m}{2}}$ is of type $(2,2)$, then $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}}) \cong \mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$.*

Note that the quotient of $\mathrm{Sp}_4(\mathbb{R})$ by its maximal compact subgroup is Siegel's upper half plane $\mathfrak{h}_2$, which has dimension 3. Since $\mathcal{M}_3$ has dimension 3, one concludes for the restricted family $\mathcal{C}_{\mathcal{M}_3} \to \mathcal{M}_3$ of genus 2 curves:

Corollary 5.5.2. *The family $\mathcal{C}_{\mathcal{M}_3} \to \mathcal{M}_3$ of genus 2 curves has a dense set of CM fibers.*

Proof. One has (similarly to the proof of Theorem 4.4.4) that the holomorphic period map $p : \mathcal{M}_3 \to \mathfrak{h}_2$ has fibers of dimension 0. Since $\dim(\mathfrak{h}_2) = \dim(\mathcal{M}_3) = 3$, one concludes that p is open. Hence the statement follows from the fact that $\mathfrak{h}_2$ has a dense set of CM points. □

We will use Proposition 5.5.1 and the calculations, which yield this proposition, to show the following theorem by induction:

Theorem 5.5.3. *If $\mathcal{V}_{\frac{m}{2}}$ is of type (g,g), then $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}}) \cong \mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$.*

Corollary 5.5.4.

$$\mathrm{Hg}(\mathcal{V}_{\frac{m}{2}}) = \mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}}) \quad \textit{and} \quad \mathrm{MT}(\mathcal{V}_{\frac{m}{2}}) = \mathrm{GSp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$$

It is a well-known fact that $\dim(\mathrm{Sp}_{\mathbb{Q}}(2g)) = 2g^2 + g$.[3] Hence one gets

$$\dim(\mathrm{Sp}_{\mathbb{Q}}(2g+1)) = 2(g+1)^2 + g + 1 = (2g^2+g) + (4g+3).$$

We will show by induction that for each $g \in \mathbb{N}$ the matrices $A_{\ell,\ell+1}$ generate a Lie algebra, which has at least the same dimension as $\mathfrak{sp}_{2g}(\mathbb{Q})$. This yields Theorem 5.5.3. Since we have shown the statement for $g = 1, 2$, we will only give the induction step:

Recall that we have defined $\mathbb{L}_j$-valued paths $[e_k\delta_k]$ in Section 3.3. We consider a middle part of type $(g+1, g+1)$ with respect to the basis $\mathcal{B} = \{[e_1\delta_1], \ldots, [e_{2g+2}\delta_{2g+2}]\}$. The Dehn twists $T_{\ell,\ell+1}$ for $\ell = 1, \ldots, 2g$ yield the monodromy group G_1 of a middle part of type (g,g). Hencefore by the induction hypothesis, they yield a group isomorphic to $\mathrm{Sp}_{2g}(\mathbb{Q})$.

[3] Otherwise one has a description of $\mathfrak{sp}_{2g}(\mathbb{C})$ in [17], page 239. By this description, one can easily determine its dimension.

Remark 5.5.5. One has the obvious embedding of $G_1 \hookrightarrow \mathrm{GL}(N^1(C_{\frac{m}{2}}, \mathbb{Q}))$ with respect to the basis $\mathcal{B}_1 := \{[e_1\delta_1], \ldots, [e_{2g}\delta_{2g}], [e_{2g+2}\delta_{2g+2}], [e_{2g+3}\delta_{2g+3}]\}$ such that

$$G_1 \ni A \to \begin{pmatrix} A & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \in \mathrm{GL}(N^1(C_{\frac{m}{2}}, \mathbb{Q})).$$

Moreover this embedding of G_1 into $\mathrm{GL}(N^1(C_{\frac{m}{2}}, \mathbb{Q}))$ is given by

$$G_1 \ni A \to \begin{pmatrix} A & v & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \in \mathrm{GL}(N^1(C_{\frac{m}{2}}, \mathbb{Q})),$$

with respect to the basis $\mathcal{B}$, where $v^t = (v_1, \ldots, v_{2g})$ is a vector depending on A.

Since we consider the embedding with respect to the latter basis, we want to understand v, which is possible, if we understand the base change between the bases of the preceding remark.

Lemma 5.5.6. *Let $C \to \mathbb{P}^1$ be a hyperelliptic curve of genus $g+1$. One has (up to a suitable normalization)*

$$\sum_{i=0}^{g+1} [e_{2i+1}\delta_{2i+1}] = 0.$$

Proof. Let $\zeta \in H_2(C, \mathbb{C})$ be a nontrivial linear combination of the closures of the sheets of $\mathbb{P}^1 \setminus S$, on which ψ acts via push-forward by the character $1 \in \mathbb{Z}/(2)$. One has that $\partial\zeta$ represents a linear combination of $[e_1\delta_1], \ldots, [e_{2g+1}\delta_{2g+3}] \in H_1(C, \mathbb{C})_1$, which is equal to zero. Recall that over $\delta_1 \cup \ldots \cup \delta_{2g+3}$ the glueing of these sheets depends on the local monodromy data determined by the branch indeces of the branch points a_k. Since each a_k has the local monodromy datum -1, this linear combination is (up to a suitable normalization of $[e_1\delta_1], \ldots, [e_{2g+1}\delta_{2g+3}]$) given by

$$\sum_{i=0}^{g+1} [e_{2i+1}\delta_{2i+1}] = 0.$$

□

5.5.7. By the preceding lemma, the matrices of base change between the bases $\mathcal{B}$ and $\mathcal{B}_1$ are given by

$$M_{\mathcal{B}}^{\mathcal{B}_1}(\mathrm{id}) = \begin{pmatrix} 1 & & & & & -1 \\ & \ddots & & & & \vdots \\ & & 1 & & & -1 \\ & & & 1 & & 0 \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix} \quad \text{and} \quad M_{\mathcal{B}_1}^{\mathcal{B}}(\mathrm{id}) = \begin{pmatrix} 1 & & & & -1 & \\ & \ddots & & & \vdots & \\ & & 1 & & -1 & \\ & & & 1 & 0 & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

such that

$$\begin{pmatrix} A & v & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} = M_{\mathcal{B}}^{\mathcal{B}_1}(\mathrm{id}) \cdot \begin{pmatrix} A & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \cdot M_{\mathcal{B}_1}^{\mathcal{B}}(\mathrm{id}).$$

Thus one calculates easily that $v_1 = 0$, if $a_{1,1} = 1$ and $a_{1,j} = 0$ for $2 \leq j \leq 2g$ and $A = (a_{i,j})$. The exponential map exp is a diffeomorphism on a neighborhood of 0. Hence by the definition

$$\exp(m) = 1 + m + \frac{m^2}{2} + \frac{m^3}{6} + \ldots,$$

one concludes that each $(m_{i,j}) \in Lie(G_1)$ satisfies that $m_{1,2g+1} = 0$, if $m_{1,j} = 0$ for all $j = 1, \ldots, 2g$, which will play a very important role later. Otherwise exp would yield a matrix with $a_{1,1} = 1$, $a_{1,j} = 0$ for $2 \leq j \leq 2g$ and $v_1 \neq 0$ as one can calculate by the fact that each $(m_{i,j}) \in Lie(G_1)$ satisfies that $m_{i,j} = 0$ for $i > 2g$.

Lemma 5.5.8. *Let $i_0 \leq 2g$ and $j_0 < 2g$ be integers such that $i_0 - j_0 > 0$. In the Lie algebra $Lie(G_1)$ one finds an element $(x_{i,j}^{(i_0,j_0)})$ with $x_{i_0,j_0}^{(i_0,j_0)} \neq 0$ and $x_{i,j}^{(i_0,j_0)} = 0$, if $i > i_0$ or $j < j_0$ or $i = 1$.*

Proof. Let $k_0 := i_0 - j_0 > 0$. We show the statement by induction over k_0. Each pair (i_0, j_0) with $i_0 - j_0 = k_0 = 1$ is given by $(i_0, i_0 - 1)$. By A_{i_0,i_0+1}, such an element is given for each $(i_0, i_0 - 1)$.

Now let (i_0, j_0) be a pair with $k_0 := i_0 - j_0 > 1$ and assume that the statement is satisfied for $k_0 - 1, \ldots, 1 > 0$. Hence one has $(x_{i,j}^{(i_0,j_0+1)}), A_{j_0+1,j_0+2} \in Lie(G_1)$. By

$$(x_{i,j}^{(i_0,j_0)}) := [(x_{i,j}^{(i_0,j_0+1)}), A_{j_0+1,j_0+2}],$$

one obtains the desired element of $Lie(G_1)$, since one has the entry

$$x_{i_0,j_0}^{(i_0,j_0)} = x_{i_0,j_0+1}^{(i_0,j_0+1)} \cdot (A_{j_0+1,j_0+2})_{j_0+1,j_0} \neq 0.$$

□

Moreover the Dehn twists $T_{2n-1,2n}, \ldots, T_{2g+2,2g+3}$ generate a group G_2 isomorphic to the monodromy group of a middle part of type $(2, 2)$, which has dimension 10. One can easily compare the matrices of $Lie(G_2)$ with the above explicitely given matrices of a middle part of type $(2, 2)$: "The restriction of the matrices of $Lie(G_2)$ to the lower right corner looks like the matrices of the Lie algebra of the monodromy group of a middle part of type $(2, 2)$."

Since the vectors

$$A_{2g-1,2g}, \quad A_{2g,2g+1} \quad \text{and} \quad [A_{2g-1,2g}, A_{2g,2g+1}]$$

are contained in $Lie(G_1) \cap Lie(G_2)$, both Lie algebras yield together a $2g^2 + g + 7$-dimensional vector space of matrices $(x_{i,j})$, whose entries $x_{i,j}$ vanish for $j < 2g - 3$ and $i > 2g$. Hence by using

$$[A_{2g+1,2g+2}, (x_{i,j}^{(2g,j_0)})] \quad \text{and} \quad [[A_{2g+1,2g+2}, A_{2g+2,2g+3}], x_{i,j}^{(2g,j_0)}]$$

for $j_0 < 2g - 3$, one has $4g - 6$ additional linearly independent vectors. Thus we have altogether $(2g^2 + g) + (4g + 1)$ linearly independent vectors. Hence 2 remaining linearly independent vectors are to find. Since $x_{i,j}^{(i_0,j_0)} = 0$ for $i = 1$, in the constructed vector space of matrices $(m_{i,j})$ the coordinate $m_{1,2g+1}$ depends uniquely on the vectors in $Lie(G_1)$ such that $m_{1,2g+1} = 0$, if $m_{1,j} = 0$ for all $j = 1, \ldots, 2g$ as we have seen in 5.5.7. Let

$$Lie(G_1) \ni (y_{i,j}) = [A_{1,2}, [A_{2,3}, [\ldots [A_{2g-1,2g}, A_{2g,2g+1}] \ldots]].$$

One checks easily that

$$y_{1,2g+1} \neq 0.$$

Now the matrix

$$(y'_{i,j}) = [(y_{i,j}), [A_{2g+1,2g+2}, A_{2g+2,2g+3}]]$$

satisfies $y'_{1,2g+1} \neq 0$, $y'_{i,j} = 0$ for $i, j \leq 2g$ and $y'_{1,2g+2} = 0$. Thus we have found a new vector not contained in the vector space, which we have constructed by $Lie(G_1)$, $Lie(G_2)$ and some Lie brackets at the present.

Note that all matrices $(x_{i,j})$, which we have found, satisfy $x_{1,2g+2} = 0$. But

$$(z_{i,j}) := [(y_{i,j}), A_{2g+1,2g+2}]$$

satisfies $z_{1,2g+2} \neq 0$. Hencefore we are done.

5.6 The complete generic Hodge group

By this section, we finish our calculation (of the derived group) of the generic Hodge group and obtain the final result:

Theorem 5.6.1. *One has*

$$\mathrm{Mon}^0(\mathcal{V}) = \prod_{r|m} \mathrm{Mon}^0(\mathcal{V}_r)$$

in the following cases:

1. *The degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is odd.*
2. *$\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.*

Corollary 5.6.2. *Assume that $\mathcal{C} \to \mathcal{P}_n$ satisfies one of the following conditions:*

1. *The degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is odd.*
2. *$\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.*

Then one has

$$\mathrm{MT}^{der}(\mathcal{V}) = \mathrm{Hg}^{der}(\mathcal{V}) \supseteq \prod_{r|m} \mathrm{Mon}^0(\mathcal{V}_r).$$

By Theorem 2.4.4, one has a CM-fiber, if the fibers of $\mathcal{C} \to \mathcal{P}_n$ have $n+1$ branch points with the same branch index d. Thus by the fact that this implies the equality of $\mathrm{Mon}^0(\mathcal{V})$ and $\mathrm{MT}^{\mathrm{der}}(\mathcal{V})$ (see Theorem 3.1.3), one concludes:

Corollary 5.6.3. *Let the fibers of $\mathcal{C} \to \mathcal{P}_n$ have $n+1$ branch points with the same branch index d and $\mathcal{C} \to \mathcal{P}_n$ satisfy one of the following conditions:*

1. *The degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is odd.*
2. *$\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.*

Then

$$\mathrm{MT}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{V}) = \prod_{r|m} \mathrm{Mon}^0(\mathcal{V}_r).$$

Since $C_r^{\mathrm{der}}(g)$ is an upper bound for $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V}_r)$, one concludes finally:

Corollary 5.6.4. *Assume that $\mathcal{C} \to \mathcal{P}_n$ satisfies one of the following conditions:*

1. *The degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is odd.*
2. *$\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.*

If all $\mathcal{V}_r$ except of the middle part are very general or special, one has

$$\mathrm{MT}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Mon}^0(\mathcal{V}) = \prod_{r|m} \mathrm{Mon}^0(\mathcal{V}_r).$$

Recall that we search for families $\mathcal{C} \to \mathcal{P}_n$ with dense set of complex multiplication fibers. One obtains dense set of complex multiplication fibers, if one has an open (multivalued) period map

$$p : \mathcal{M}_n(\mathbb{C}) \to \mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$$

given by the VHS. Hence for our applications we need to know $\mathrm{MT}^{\mathrm{der}}(\mathcal{V})$ and the dimension of $\mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$, but not $\mathrm{MT}(\mathcal{V})$ itself. Let us first prove Theorem 5.6.1. After this proof we will see that the (multivalued) period map of a family $\mathcal{C} \to \mathcal{M}_1$ onto $\mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$ is open, if and only if one has a $(1,1) - VHS$.

For the proof of Theorem 5.6.1 we use the same methods as before. One has that $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V})$ is the direct product of the kernel of the natural projection

$$p_1 : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_{r_1})$$

and an adjoint semisimple group G_{r_1} isomorphic to $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_{r_1})$. Moreover one has that

$$\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}) = \prod_{r|m} \mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_r),$$

if and only if each G_{r_1} is contained in the kernels of the natural projections onto all $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_{r_2})$ with $r_1 \neq r_2$.

We give a proof of Theorem 5.6.1 by contradiction. Thus we assume that

$$\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}) \neq \prod_{r|m} \mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r). \quad \text{This implies} \quad \mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}) \neq \prod_{r|m} \mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r).$$

Hence some G_{r_1} is not contained in the kernel of the projection onto $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_{r_2})$ for some $r_2 \neq r_1$. Since all simple direct factors of G_{r_1} resp., G_{r_2} project isomorphically onto some $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1})$ resp., $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2})$, one gets an isomorphism

$$\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1}) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2}),$$

which respects the respective projective monodromy representations. But by the following proposition, the isomorphism α can not exist, if the assumptions of Theorem 5.6.1 are satisfied. This yields the proof of Theorem 5.6.1.

Proposition 5.6.5. *Assume that $r_1 := \gcd(m, j_1) \neq r_2 := \gcd(m, j_2)$. Moreover assume that one of the following cases holds true:*

1. *m is odd.*

2. $\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.

Then an isomorphism

$$\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1}) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2}),$$

which respects the respective projective monodromy representations, can not exist.

Proof. Assume without loss of generality that $r_1 < r_2$. This implies $\frac{m}{r_1} > \frac{m}{r_2}$. There are two cases: Either $2r_1 \neq r_2$ or $2r_1 = r_2$.

If m is odd, one has $\frac{r_1}{2} \neq g := \gcd(\frac{m}{r_1}, \frac{m}{r_2})$. Hence by Lemma 5.3.2, one finds a Dehn twist T such that $P\rho_{j_1}(T)$ is semisimple and the order of $P\rho_{j_1}(T)$ does not divide g. One has that $P\rho_{j_2}(T)$ is either unipotent or semisimple. If $P\rho_{j_2}(T)$ is semisimple, its order divides $\frac{m}{r_2}$. But the order of $P\rho_{j_1}(T)$ does not divide $\frac{m}{r_2}$. If $P\rho_{j_2}(T)$ is unipotent, its order infinite. But $P\rho_{j_1}(T)$ has finite order. However $P\rho_{j_1}(T)$ and $P\rho_{j_2}(T)$ do not have the same order. Hence such an isomorphism $\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1}) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2})$, which respects the respective projective monodromy representations, can not exist in this case.

Now assume that we are in the case of a family $\mathcal{C} \to \mathcal{P}_1$, where 6 does not divide m. There is a Dehn twist T such that $P\rho_{j_1}(T)$ is semisimple. If $P\rho_{j_1}(T)$ and $P\rho_{j_2}(T)$ do not have the same order, one can argue as above. Otherwise all semisimple Dehn twists have the same order. Hence one must have $2r_1 = r_2$. The nontrivial eigenvalue of $\rho_{j_2}(T)$ is given by the square of the nontrivial eigenvalue ξ of $\rho_{j_1}(T)$. Note that the corresponding maximal tori are isomorphic to S^1, where $S^1_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}}$. Thus its character group is isomorphic to $\mathbb{Z}$. Hence the induced map of the corresponding maximal tori can be an isomorphism, only if one has $\xi^2 = \xi^{-1} = \bar{\xi}$. In this case ξ would be a primitive cubic root of unity, which implies that 3 divides m. Since we have that $2r_1 = r_2$, 6 would divide m. But by the assumptions, this is not possible. □

Remark 5.6.6. If $2r_1 = r_2$, there are many additional cases, in which α can not exist. These obvious cases are given, if for a Dehn twist T the order of the semisimple matrix $\rho_{r_1}(T)$ does not divide $\frac{m}{2r_1}$, if $\rho_{r_1}(T)$ is semisimple and $\rho_{j_2}(T)$ is unipotent or if $\mathcal{L}_{j_1}$ and $\mathcal{L}_{j_1}$ are of type (a_1, b_1) and (a_2, b_2) such that

$$(a_1, b_1) \neq (a_2, b_2) \quad \text{and} \quad (a_1, b_1) \neq (b_2, a_2).$$

But in the case of the family $\mathcal{C} \to \mathcal{P}_1$ of degree 6 covers given by the local monodromy data

$$d_1 = d_2 = 1, \quad d_3 = d_4 = 5$$

nothing of them holds true with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$.

Now let us finish this chapter and state the final result about the period map:

Theorem 5.6.7. *In the case of a family $\mathcal{C} \to \mathcal{M}_1$ the period map*

$$p : \mathcal{M}_1(\mathbb{C}) \to \mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$$

is open, if and only if one has a pure $(1,1) - VHS$.

Proof. As we have seen in the proof Theorem 4.4.4, the period map is open, if one has a pure $(1,1) - VHS$.

For the other direction assume that the period map is open and there are up to complex conjugation at least two different eigenspaces, which are not unitary.

Lemma 5.6.8. *Assume that we have a family $\mathcal{C}_{\mathcal{M}_1} \to \mathcal{M}_1$. Only if all $\mathcal{V}_r$ except for exactly one $\mathcal{V}_{r_0}$ are special, the period map*

$$p : \mathcal{M}_1(\mathbb{C}) \to \mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$$

can be open.

Proof. Assume that r_1 and r_2 divide m such that $r_1 \neq r_2$ and $\mathcal{V}_{r_1}$ and $\mathcal{V}_{r_2}$ are not special. If $2r_1 \neq r_2$ or if there is a Dehn twist, whose finite order with respect to $\mathcal{V}_{r_1}$ does not divide $\frac{m}{r_2} = \frac{m}{2r_1}$, the same arguments as in the proof of Proposition 5.6.5 imply that

$$\dim(\mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K) > 1 = \dim(\mathcal{M}_1).$$

Hencefore the period map can not be open.

Otherwise assume without loss of generality that $r_1 = 1$ and all semisimple Dehn twists have an order dividing $\frac{m}{2}$. This implies that all d_k are odd and the degree m is even. Hence $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}})$ is isomorphic to $\mathrm{Sp}_{\mathbb{Q}}(2)$, where its monodromy representation sends all Dehn twists to unipotent matrices. Thus $\dim(\mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K) > 1$. □

By Lemma 5.6.8, these two eigenspaces, which are not unitary, must be contained in the same $\mathcal{V}_{r_0}$, which must be exceptional. Hence assume without loss of generality that $\mathcal{V}_{r_0} = \mathcal{V}_1$.

In the separated case, the fact that all d_k are odd (compare to 5.4.4) implies that $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_{\frac{m}{2}}) = \mathrm{Sp}_{\mathbb{R}}(2)$. Hence by Lemma 5.6.8, we have a contradiction.

In the complex case Lemma 5.4.2 implies without loss of generality that

$$t_1 | d_1 + d_2, \quad t_1 | d_2 + d_3, \quad t_1 | d_1 + d_4, \quad t_1 | d_3 + d_4.$$

This implies that t_1 divides each d_k or that t_1 does not divide any d_k. Thus t_1 does not divide any d_k. Hence $\mathcal{C}_{\frac{m}{t_1}}$ is a family of covers with 4 branch points, where $\rho_{\frac{m}{t_1}}(T_{1,2})$ and $\rho_{\frac{m}{t_1}}(T_{2,3})$ are unitary. Hence $\mathcal{V}_{\frac{m}{t_1}}$ has an infinite monodromy group resp., it is not special. Thus by Lemma 5.6.8, we have a contradiction. □

Chapter 6

Examples of families with dense sets of complex multiplication fibers

6.1 The necessary condition $SINT$

By Theorem 4.4.4, one has a sufficient criterion for a dense set of CM fibers of a family $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$. This criterion is satisfied, if $\mathcal{C}$ has a pure $(1,n) - VHS$ (i.e. its VHS contains one eigenspace of type $(1,n)$, a complex conjugate eigenspace of type $(n,1)$ and otherwise only eigenspaces of the type $(a,0)$ and $(0,b)$ for some $a, b \in \mathbb{N}_0$).

Remark 6.1.1. Assume that the family $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of degree m has a pure $(1,n)$-VHS and that $\mathcal{L}_{j_0}$ is the eigenspace of type $(1,n)$. Let $j_0 \notin (\mathbb{Z}/(m))^*$. Then we have $1 < r_0 := gcd(j_0, m)$. By Section 4.2, the family $\mathcal{C}_{r_0}$ has a pure $(1,n)$-VHS, too.

Definition 6.1.2. A pure $(1,n) - VHS$ is primitive, if $j_0 \in (\mathbb{Z}/(m))^*$. Otherwise it is a derived pure $(1,n) - VHS$ with the associated primitive pure $(1,n) - VHS$ induced by $\mathcal{C}_{r_0}$, where $\mathcal{C}_{r_0}$ is given by the preceding remark.

Hence first we search for families with a primitive pure $(1,n) - VHS$. Later we will look for families with a derived pure $(1,n) - VHS$. It is helpful to have a necessary condition to find the families with a primitive pure $(1,n) - VHS$. In [14] P. Deligne and G. D. Mostow have formulated the following integral condition INT:

Definition 6.1.3. A local system on $\mathbb{P}^1 \setminus S$ of monodromy $(\alpha_s)_{s \in S}$ with $\alpha_s = exp(2\pi i \mu_s)$ and $\mu_s \in \mathbb{Q}$ for all $s \in S$ satisfies the condition INT, if:

1. $0 < \mu_s < 1$ for all $s \in S$.
2. We have for all $s, t \in S$: $(1 - \mu_s - \mu_t)^{-1}$ is an integer, if $s \neq t$ and $\mu_s + \mu_t < 1$.
3. $\sum \mu_s = 2$.

One can identify the local monodromy data, which yield the family $\mathcal{C} \to \mathcal{P}_n$ by Construction 3.2.1, with the local monodromy data of the eigenspace $\mathbb{L}_1$ of some fiber $\mathcal{C}_q$ for an arbitrary $q \in \mathcal{P}_n$. Hence one can formulate the condition INT for the local monodromy data of the family. For the latter data we give a corresponding stronger integral condition $SINT$:

Definition 6.1.4. A family $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers onto $\mathbb{P}^1$ given by the local monodromy data given by $\mu_k \in \mathbb{Q}$ around $s_k \in N$ satisfies $SINT$, if we have:

1. $\mu_{k_1} + \mu_{k_2} = 1$ or $(1 - \mu_{k_1} - \mu_{k_2})^{-1} \in \mathbb{Z}$ for all $s_{k_1}, s_{k_2} \in N$ with $s_{k_1} \neq s_{k_2}$.

2. $\sum \mu_s = 2$.

Remark 6.1.5. The reader checks easily that for a family $\mathcal{C} \to \mathcal{P}_1$ the conditions INT and $SINT$ are equivalent. Moreover by the list on [14], page 86, each family $\mathcal{C} \to \mathcal{P}_n$ with $n \geq 2$, which satisfies INT, satisfies $SINT$, too.

At the present the author can not explain this fact. We use $SINT$ instead of INT, since this yields a stronger and hencefore a more helpful condition.

By the following theorem, we have our helpful necessary condition for families $\mathcal{C}$, which have a primitive pure $(1,n) - VHS$:

Theorem 6.1.6. *If the family $\mathcal{C} \to \mathcal{P}_n$ has a primitive pure $(1,n) - VHS$, its local monodromy data can be given rational numbers satisfying $SINT$.*

For the proof of Theorem 6.1.6 we first reduce the situation to the case of a family $\mathcal{C} \to \mathcal{P}_1$ of covers with only 4 branch points. That means we will consider a pair of branch points of a fiber of $\mathcal{C} \to \mathcal{P}_n$, where $\mathcal{C}$ has a primitive pure $(1,n) - VHS$, as a pair of branch points with the same branch indeces of a fiber of a family $\mathcal{C}(P) \to \mathcal{P}_1$, which has a primitive pure $(1,1) - VHS$. The following lemma will make it possible in almost all cases:

Lemma 6.1.7. *Assume that $\mathcal{C}$ is given by local monodromy data on at least 5 points, where one does not have $\mu_3 = \ldots = \mu_{n+3} = \frac{1}{2}$. Then there exists a stable partition P with $\{a_1\}, \{a_2\} \in P$ such that $|P| = 4$.*[1]

Proof. One can without loss of generality assume that $\mu_1 + \mu_2 \leq 1$. Otherwise we take the local monodromy data of $\mathcal{L}_{m-1}$.

Now assume that such a stable partition P with $\{a_1\}, \{a_2\} \in P$ does not exist. Hence one must have $\mu_1 + \mu_2 + \mu_k = 1$ for all $3 \leq k \leq n+3$. Otherwise one obtains the stable partition

$$P = \{\{a_1\}, \{a_2\}, \{a_k\}, \{a_3, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+3}\}\}$$

Thus one must have

$$\mu := \mu_3 = \ldots = \mu_{n_j+3}.$$

Since

$$P = \{\{a_1\}, \{a_2\}, \{a_3, a_4\}, \{a_5, \ldots, a_{n_j+3}\}\}$$

is not a stable partition by our assumption, too, one has

$$2\mu = \mu_3 + \mu_4 = 1. \text{ Hence } \mu = \frac{1}{2}.$$

□

6.1.8. The family of irreducible cyclic covers of $\mathbb{P}^1$ given by the local monodromy data

$$\mu_1 = \mu_2 = \frac{1}{4}, \quad \mu_3 = \mu_4 = \mu_5 = \frac{1}{2}$$

[1] Since the assumptions of this lemma are sufficient, we do not restrict to the interesting case of a family with a primitive pure $(1,n) - VHS$.

has a primitive pure $(1,2)-VHS$. Moreover it is easy to calculate that this family satisfies $SINT$.

But this is the only example of a family $\mathcal{C} \to \mathcal{P}_n$ with a primitive pure $(1,n)-VHS$ with $n>1$, which does not satisfy the assumptions of Lemma 6.1.7: It is very easy to see that this is the only degree 4 example with a primitive pure $(1,n)-VHS$ for $n>1$, which contradicts the assumptions of Lemma 6.1.7. If $m>4$, $\mathcal{L}_3$ must be unitary. But in this case the condition that

$$n+3>4 \text{ and } [3\mu_3]_1 = \ldots = [3\mu_{n+3}]_1 = \frac{1}{2}$$

and Proposition 2.3.4 imply that

$$h_3^{1,0}(C) \geq (\sum_{k\geq 3}[3\mu_k]_1) - 1 = (\sum_{k\geq 3}\frac{1}{2}) - 1 > 0$$

and

$$h_3^{0,1}(C) \geq \sum_{k\geq 3}(1-[3\mu_k]_1) - 1 = (\sum_{k\geq 3}\frac{1}{2}) - 1 > 0.$$

Thus $\mathcal{L}_3$ is not unitary.

6.1.9. Assume that $\mathcal{C} \to \mathcal{P}_n$ has a primitive pure $(1,n)-VHS$. Hence $\mathcal{L}_1$ is without loss of generality the eigenspace of type $(1,n)$. For our application of Lemma 6.1.7 we must check that the collision of Lemma 6.1.7 resp., its corresponding stable partition yields a family $\mathcal{C}(P) \to \mathcal{P}_1$, which has a primitive pure $(1,1)-VHS$. The family $\mathcal{C}(P)$ is given by $N=P$ with the local monodromy data

$$\alpha_{\{a_k,\ldots,a_\ell\}} = \alpha_k \cdot \ldots \cdot \alpha_\ell \quad (\forall \quad \{a_k,\ldots,a_\ell\} \in P)$$

as in Construction 3.2.1. The fibers of $\mathcal{C}(P)$ have the degree m', where m' divides m. For $j=1,\ldots,m'-1$ and $q \in \mathcal{M}_1$, the eigenspace $\mathbb{L}_j(P)$ in the Hodge structure of $\mathcal{C}(P)_q$ with the character j is given by the local monodromy data

$$[j\mu_1]_1, \quad [j\mu_2]_1, \quad [j\mu_3+\ldots+j\mu_k]_1, \quad [j\mu_{k+1}+\ldots+j\mu_{n+3}]_1.$$

If the eigenspace $\mathcal{L}_j$ in the VHS of $\mathcal{C}$ is of type $(0,a)$, Proposition 2.3.4 implies that its local monodromy data satisfy

$$[j\mu_1]_1 + \ldots + [j\mu_{n+3}]_1 = 1.$$

Hence one has that

$$[j\mu_1]_1 + [j\mu_2]_1 + [j\mu_3+\ldots+j\mu_k]_1 + [j\mu_{k+1}+\ldots+j\mu_{n+3}]_1 = 1,$$

too. Thus by Proposition 2.3.4, $\mathbb{L}_j(P)$ is of type $(0,a')$.

If $\mathcal{L}_j$ is of type $(a,0)$, $\mathcal{L}_{m-j}$ is of type $(0,a)$. The dual eigenspace $\mathbb{L}_j(P)^\vee$ of $\mathbb{L}_j(P)$ is given by

$$[(m-j)\mu_1]_1, \ [(m-j)\mu_2]_1, \ [(m-j)\mu_3+\ldots+(m-j)\mu_k]_1, \ [(m-j)\mu_{k+1}+\ldots+(m-j)\mu_{n+3}]_1.$$

The same arguments as above tell us that $\mathbb{L}_j(P)^\vee$ is of type $(0,a')$. Thus $\mathbb{L}_j(P)$ is of type $(a',0)$.

The restricted family $\mathcal{C}_{\mathcal{M}_1}(P) \to \mathcal{M}_1$ of cyclic covers with 4 different branch points has a non-trivial variation of Hodge structures. This follows from the fact that each fiber of $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$ is isomorphic to only finitely many other fibers (compare to 4.4.2). Hencefore the eigenspaces $\mathbb{L}_1(P)$ and $\mathbb{L}_{m'-1}(P)$ are of type $(1,1)$. In addition one concludes that $m'=2$ or $m'=m$.

Now we are without loss of generality in the case of a family $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$. For the proof of Theorem 6.1.6 we need the following lemma:

Lemma 6.1.10. *Let C and C' be curves and $\gamma : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$ be an isomorphism of principally polarized Abelian varieties. Then there exists a unique isomorphism $f : C \to C'$ such that*

$$\pm\gamma \circ \alpha_p = \alpha_{f(p)} \circ f$$

for each $p \in C$, where α_p and $\alpha_{f(p)}$ denote the respective Abel-Jacobi maps.

Proof. By [31], Theorem 12.1, for each $p \in C$ and $p' \in C'$ there is a unique isomorphism $f : C \to C'$ and a unique $c \in \mathrm{Jac}(C')$ such that

$$\pm\gamma \circ \alpha_p + c = \alpha_{p'} \circ f.$$

Since $(\gamma \circ \alpha_p)(p) = 0 \in \mathrm{Jac}(C')$ and $(\alpha_{p'} \circ f)(p) = [f(p) - p'] \in \mathrm{Jac}(C')$, one has $c = 0$ for $p' = f(p)$. □

By the next proposition, we will apply Lemma 6.1.10 for our proof of Theorem 6.1.6:

Proposition 6.1.11. *Let $q_1, q_2 \in \mathcal{P}_n$ and $\mathcal{C} \to \mathcal{P}_n$ be a family of cyclic covers. Assume there is an isomorphism between the polarized integral Hodge structures of the fibers $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_2}$, which respects the eigenspace decompositions of $H^1(\mathcal{C}_{q_1}, \mathbb{C})$ and $H^1(\mathcal{C}_{q_1}, \mathbb{C})$. Then there is an isomorphism $\iota : \mathcal{C}_{p_1} \to \mathcal{C}_{p_2}$ and an isomorphism $\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}_{p_1} & \xrightarrow{\iota} & \mathcal{C}_{p_2} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\alpha} & \mathbb{P}^1 \end{array}$$

Proof. Let γ be an isomorphism of polarized Hodge structures respecting the eigenspace decompositions of $H^1(\mathcal{C}_{q_1}\mathbb{C})$ and $H^1(\mathcal{C}_{q_1}\mathbb{C})$. Then there exists a suitable pair (ψ_1, ψ_2) of generators of the Galois groups of $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_1}$ such that

$$\gamma \circ (\psi_1)_* = (\psi_2)_* \circ \gamma.$$

For simplicity we write ψ instead of ψ_1 and ψ_2.

By the exponential exact sequence, an isomorphism $\gamma : H^1(\mathcal{C}_{q_1}, \mathbb{Z}) \to H^1(\mathcal{C}_{q_2}, \mathbb{Z})$ of polarized Hodge structures commuting with the action of ψ on these integral Hodge structures induces an isomorphism $\gamma' : \mathrm{Jac}(\mathcal{C}_{q_1}) \to \mathrm{Jac}(\mathcal{C}_{q_2})$ commuting with ψ_*. In other terms one has

$$\gamma' \circ \psi_* = \psi_* \circ \gamma'$$

for the Jacobians.

By Lemma 6.1.10, one obtains a unique isomorphism $\mathcal{C}_{p_1} \xrightarrow{\iota} \mathcal{C}_{p_2}$ such that

$$\iota \circ \psi = \psi \circ \iota.$$

Thus one obtains the desired automorphism α. □

6.1.12. Now assume that $\mathcal{C} \to \mathcal{P}_n$ has a primitive pure $(1,n) - VHS$. Moreover one can without loss of generality assume that $\mathcal{L}_1$ is the eigenspace of type $(1,n)$. Choose $s_1, s_2 \in N$.

If $\mu_1 + \mu_2 = 1$, there remains nothing to prove with respect to these two points for Theorem 6.1.6.

Otherwise we let the branch points collide as in Lemma 6.1.7, if we are not in the only exceptional case, which satisfies $SINT$ as we have seen in 6.1.8. Thus we can restrict to the case $\mathcal{C} \to \mathcal{P}_1$. Assume that all 4 branch points of a fiber of $\mathcal{C} \to \mathcal{P}_1$ have pairwise different branch indeces. Hencefore there will not be an isomorphism α as in Proposition 6.1.11 for different fibers. Hence Proposition 6.1.11 implies that the fractional period map according to $\mathcal{L}_1|_{\mathcal{M}_1}$ is injective. Now choose the embedding $\mathcal{M}_1 \hookrightarrow \mathbb{P}^1$ corresponding to

$$p_1 = 0, \quad p_3 = 1, \quad p_4 = \infty.$$

By [30], Section 4, one can identify the fractional period map concerning $\mathcal{L}_1$ with some multivalued map, which is called Schwarz map. The Schwarz map is the composition of the multivalued map studied by P. Deligne and G. D. Mostow in [14], which is defined by some integrals, with the natural map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}$. By [14], 9.6 and the preceding description of the fractional period map, there exists a sufficiently small neighborhood U of $0 \in \mathbb{P}^1 \setminus \mathcal{M}_1$ such that the fractional period map concerning $\mathcal{L}_1$ is (up to a biholomorphic map) given by $x \to x^{1-\mu_1-\mu_2}$ on $U \setminus \{0\}$. Hence the injectivity of the period map implies that $(1-\mu_1-\mu_2)^{-1} \in \mathbb{Z}$. This yields $SINT$.

6.1.13. Now we have the problem that we can not directly apply Proposition 6.1.11 as before, if we assume that there are 4 branch points, where exactly two of them have the same branch index: Let p_1 and p_4 have the same branch index and p_3 run around p_2, where

$$p_1 = 0, \quad p_2 = 1, \quad p_4 = \infty.$$

The automorphism $x \to x^{-1}$ interchanges 0 and ∞ and leaves a basis of neighborhoods of $1 \in \mathbb{P}^1 \setminus \mathcal{M}_1$ invariant. We have obviously the same problem, if we let p_1 run around p_4. But for all other pairs $k_1, k_2 \in \{1,2,3,4\}$ with $k_1 \neq k_2$ and the coordinates

$$k_1 = 0, \quad k_3 = 1, \quad k_4 = \infty,$$

Proposition 6.1.11 implies that the multivalued period map is injective on $U \setminus \{0\}$, where U is a sufficiently small neighborhood of $0 \in \mathbb{P}^1 \setminus \mathcal{M}_1$. Thus k_1 and k_2 satisfy the integral condition

$$1 - \mu_{k_1} - \mu_{k_2} = 0 \ \text{ or } \ (1 - \mu_{k_1} - \mu_{k_2})^{-1} \in \mathbb{Z}.$$

Hence one must ensure that the remaining pairs satisfy this latter condition, in order to show that $SINT$ is satisfied:

Let us change the enumeration and assume that $\mu_1 = \mu_2$. By Proposition 6.1.11, we can have

$$1 - \mu_1 - \mu_2 = \frac{1}{\ell} \ \text{ or } \ 1 - \mu_1 - \mu_2 = \frac{2}{\ell} \ \text{ or } \ 1 - \mu_1 - \mu_2 = 0$$

for some odd $\ell \in \mathbb{Z}$. Note that $1 - \mu_1 - \mu_2 = -(1 - \mu_3 - \mu_4)$, if $\mu_1 + \ldots + \mu_4 = 2$. Hence we only have to exclude the second case $(1 - \mu_1 - \mu_2) = \frac{2}{\ell}$. First assume that m is odd. In this case $m - 2d_1$ is odd and the second case can not occur. Hence assume that m is even and let $m = 2^s r$, where $r = k \cdot \ell$ is odd. If the second case holds true, one has

$$\frac{2^s k\ell - 2d_1}{2^s k\ell} = \frac{2}{\ell} \Leftrightarrow 2^{s-1}k\ell - d_1 = 2^s k \Leftrightarrow d_1 = 2^{s-1}k(\ell - 2).$$

If $s \geq 2$, one has that $d_1 = d_2$ is even. Since $\mathcal{C}_2$ must have a trivial VHS, one has without loss of generality that $d_3 = 2^{s-1}k\ell$. Since we have

$$2m = d_1 + \ldots + d_4,$$

which is even, where d_1, d_2, d_3 are even, too, d_4 must be even. But in this case the cover is not irreducible. Hence we must have $s = 1$. Since $\mathcal{C}_2$ must have a trivial VHS, one has without loss of generality $d_3 = k\ell$. Since we have

$$2m = d_1 + \ldots + d_4,$$

which is even, where d_1, d_2, d_3 are odd, one must have that d_4 is odd, too. But in this case $\mathcal{C}_{k\ell}$ is the family of elliptic curves and we do not have a primitive pure $(1,1) - VHS$. Hencefore the second case is excluded.

Remark 6.1.14. If we have 4 branch points and more than exactly two of them have the same branch index, one can have the additional simple cases

$$\mu_1 = \mu_2 = \mu_3 \quad \text{or} \quad \mu_1 = \mu_2, \quad \mu_3 = \mu_4.$$

For these very simple cases one can directly calculate all occurring examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$. Then one can verify by their local monodromy data that Theorem 6.1.6 holds true in these cases as we will do now.

Remark 6.1.15. One must without loss of generality have

$$d_1 < d_4 \quad \text{resp.}, \quad d_1 = d_2 = d_3 < d_4 \quad \text{or} \quad d_1 = d_2 < d_3 = d_4$$

in the simple cases, if $m > 2$. Otherwise we would obtain

$$d_1 = d_2 = d_3 = d_4 = \frac{m}{2},$$

which implies that $\mathcal{C}$ is not irreducible, if $m > 2$.

Lemma 6.1.16. *Assume that the family $\mathcal{C} \to \mathcal{P}_1$ with the branch indeces $d_1 = d_2 = d_3 \neq d_4$ has a primitive pure $(1,1) - VHS$. Then the degree m is odd and satisfies $m \leq 9$. Moreover one has without loss of generality that $d_1 = d_2 + d_3 = 1$.*

Proof. By the assumptions we have that $2m = 3d_1 + d_4$. Hence $g = \gcd(m, d_1) = 1$ divides d_4, too, which implies by the irreducibility of the fibers of $\mathcal{C}$ that $g = 1$. Thus if m is even, we have that $d_1 = d_2 = d_3$ and d_4 are odd. But then $\mathcal{C}_{\frac{m}{2}}$ would be the family of elliptic curves such that $\mathcal{L}_{\frac{m}{2}}$ is of type $(1,1)$. Contradiction! Hence m must be odd.

It remains to show that $m \leq 9$. Since $\gcd(m, d_1) = 1$, the fibers are without loss of generality given by

$$y^m = x(x-1)(x-\lambda)$$

such that $\mathcal{L}_{[\frac{m}{2}]}$ is of type $(1,1)$ as one can calculate by Proposition 2.3.4. By Proposition 2.3.4, one can calculate the type of $\mathcal{L}_{\frac{m-3}{2}}$ by its local monodromy data, too. For this local system one gets that

$$3[\frac{m-3}{2m}]_1 + [\frac{(m-3)(m-3)}{2m}]_1$$

$$= 3\frac{m-3}{2m} + (m-3)\frac{m-3}{2m} - [\frac{(m-3)(m-3)}{2m}] = \frac{m-3}{2} - [\frac{(m-3)(m-3)}{2m}].$$

Now let us assume that $9 < m$. Since m must be odd, we obtain

$$3[\frac{m-3}{2m}]_1 + [\frac{(m-3)(m-3)}{2m}]_1 = \frac{m-3}{2} - [\frac{m-6}{2} + \frac{9}{2m}] = \frac{m-3}{2} - \frac{m-7}{2} = 2.$$

This result and Proposition 2.3.4 imply that $\mathcal{L}_{\frac{m-3}{2}}$ is of type $(1,1)$ in this case, too. Hence we do not have a pure $(1,1) - VHS$, if $9 < m$. □

Remark 6.1.17. In the case of the preceding lemma one obtains all examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$ by $m = 5, 7, 9$, which satisfy $SINT$ as one can calculate easily, too.

Remark 6.1.18. If we are in the second simple case $d_1 = d_2 \neq d_3 = d_4$, one obtains

$$d_1 + d_3 = d_2 + d_4 = m.$$

By the fact that $d_1 \neq d_3$, one concludes that $\mu_1, \mu_3 \neq \frac{1}{2}$. Hence the local monodromy data of $\mathcal{L}_2$ satisfy $[2\mu_i]_1 \neq 0$ for all $i = 1, \ldots, 4$. Moreover one has

$$[2\mu_1]_1 + [2\mu_3]_1 = [2\mu_2]_1 + [2\mu_4]_1 = 1.$$

Hence $\mathcal{L}_2$ is of type $(1,1)$ and $\mathcal{C}$ can have a primitive $(1,1) - VHS$, only if $m = 3$. Thus the only possible case is given by

$$\mu_1 = \mu_2 = \frac{1}{3} \text{ and } \mu_3 = \mu_4 = \frac{2}{3},$$

which satisfies $SINT$ as one can easily verify.

6.2 The application of $SINT$ for the more complicated cases

In the preceding section we have seen that $SINT$ is a necessary condition for families $\mathcal{C} \to \mathcal{P}_n$ with a primitive pure $(1,n) - VHS$. In addition we have given all examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$, which do not satisfy that at most two branch points have the same branch index. Here we calculate all examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$, which satisfy that at most two branch points have the same branch index.

By technical reasons, we will sometimes assume $m \geq 4$. Note that the only possible case of a family $\mathcal{C} \to \mathcal{P}_1$ of degree 3 covers with a pure $(1,1) - VHS$ is given by Remark 6.1.18, where the only possible case of degree 2 covers is given by the elliptic curves. Thus this assumption does not provide any restriction for the more complicated cases.

Note that in the case of a family $\mathcal{C} \to \mathcal{P}_1$ the condition $SINT$ is equivalent to INT.

Remark 6.2.1. By [14], 14.3, one can describe all families of covers $\mathcal{C} \to \mathcal{P}_1$, whose local monodromy data satisfy INT, such that there is not any pair $k_1, k_2 \in \{1,2,3,4\}$ with $k_1 \neq k_2$ satisfying $\mu_{k_1} + \mu_{k_2} = 1$, in the following way: Let $(p,q,r) \in \mathbb{N}^3$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and $1 < p \leq q \leq r < \infty$. Then in the case of 4 branch points these solutions of INT for covers can be given by:

$$\mu_1 = \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{r}), \quad \mu_2 = \frac{1}{2}(1 - \frac{1}{p} + \frac{1}{q} - \frac{1}{r}),$$
$$\mu_3 = \frac{1}{2}(1 + \frac{1}{p} - \frac{1}{q} - \frac{1}{r}), \quad \mu_4 = \frac{1}{2}(1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{r})$$

We have that

$$\mu_1+\mu_2=1-\frac{1}{p},\quad \mu_1+\mu_3=1-\frac{1}{q},\quad \mu_2+\mu_3=1-\frac{1}{r}.$$

Thus p ,q, and r divide the degree m of the cover. This fact and the equations, which use p, q, and r for the definition of the different μ_i, imply that we have

$$m=\mathrm{lcm}(p,q,r)\ \text{ or }\ m=2\cdot\mathrm{lcm}(p,q,r).$$

If we are in the case of a family with a primitive pure $(1,1)-VHS$ such that all local monodromy data satisfy $\mu_{k_1}+\mu_{k_2}\neq 1$ and at most two branch points have the same index, we are in the case of Remark 6.2.1 with the additional condition $p<r$. Hence let us first consider this case. Later we will consider families with at most two branch points with the same branch index and some $\mu_{k_1}+\mu_{k_2}=1$, which is the last remaining subcase.

Now let $\ell:=\mathrm{lcm}(p,q,r)$.

Lemma 6.2.2. *Let $\mathcal{C}\to\mathcal{P}_1$ be given by p,q,r as in Remark* 6.2.1*, where $p<r$, and have a primitive pure $(1,1)-VHS$. Then one has*

$$\frac{1}{p}=\frac{1}{q}+\frac{1}{r}.$$

Proof. Since $p|\ell$ resp., $p|m$, we have the family $\mathcal{C}_p$, which must have a trivial VHS. This implies that there is a d_{i_0} with $\frac{m}{p}|d_{i_0}$, which implies that $\frac{\ell}{p}|d_{i_0}$. Since

$$d_{i_0}=\ell\pm\frac{\ell}{p}\pm\frac{\ell}{q}\pm\frac{\ell}{r}\ \text{ or }\ 2d_{i_0}=\ell\pm\frac{\ell}{p}\pm\frac{\ell}{q}\pm\frac{\ell}{r},$$

one concludes that $\frac{\ell}{p}|(\frac{\ell}{q}\pm\frac{\ell}{r})$. From the fact that $\frac{\ell}{p}\geq\frac{\ell}{q}$ and $\frac{\ell}{p}>\frac{\ell}{r}$, one obtains

$$\frac{\ell}{p}=\frac{\ell}{q}+\frac{\ell}{r}.\ \text{ Hence }\ \frac{1}{p}=\frac{1}{q}+\frac{1}{r}.$$

□

Lemma 6.2.3. *Let $\mathcal{C}\to\mathcal{P}_1$ be a family with a primitive pure $(1,1)-VHS$, which is given by p,q,r as in Remark* 6.2.1*, where $p<r$. Then the family $\mathcal{C}$ and the eigenspace $\mathbb{L}_1$ are given by the local monodromy data*

$$\mu_1=\frac{1}{2}-\frac{1}{q},\quad \mu_2=\frac{1}{2}-\frac{1}{r},\quad \mu_3=\frac{1}{2},\quad \mu_4=\frac{1}{2}+\frac{1}{p}.$$

Proof. By Lemma 6.2.2, we have

$$\frac{1}{p}=\frac{1}{q}+\frac{1}{r}.$$

This equation and Remark 6.2.1, this imply that $\mathcal{C}$ and $\mathbb{L}_1$ have the local monodromy data

$$\mu_1=\frac{1}{2}-\frac{1}{q},\quad \mu_2=\frac{1}{2}-\frac{1}{r},\quad \mu_3=\frac{1}{2},\quad \mu_4=\frac{1}{2}+\frac{1}{p}.$$

□

Remark 6.2.4. Let $\mathcal{C} \to \mathcal{P}_1$ is a family of covers of degree $m \geq 4$ with a primitive pure $(1,1) - VHS$ satisfying the assumptions of Lemma 6.2.3. Moreover assume that $3 \notin (\mathbb{Z}/(m))^*$. Hence the assumption that $\mathcal{C} \to \mathcal{P}_1$ has primitive pure $(1,1) - VHS$ implies that the family $\mathcal{C}_3$ must have a trivial VHS. Thus all fibers of $\mathcal{C}_3$ must be isomorphic. Hence they are ramified over at most 3 points. By Lemma 6.2.3, one concludes that

$$0 = [\frac{1}{2} - \frac{3}{q}]_1, \quad 0 = [\frac{1}{2} - \frac{3}{r}]_1 \text{ or } 0 = [\frac{1}{2} + \frac{3}{p}]_1.$$

Since $\mu_4 = \frac{1}{2} + \frac{1}{p} < 1$, one concludes that $2 < p \leq q \leq r$. Thus one has

$$p = 6, \quad q = 6 \text{ or } r = 6.$$

Hence one can determine all examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$ in this case as we will do now:

6.2.5. Keep the assumptions of Remark 6.2.4. In the case $p = 6$ one has that $[3\mu_4]_1 = 0$. One can have $q = 7, 8, 9, 10, 11, 12$, where $q = 12$ implies that

$$\frac{1}{6} = \frac{1}{p}, \quad \frac{1}{q} = \frac{1}{r} = \frac{1}{12},$$

which leads to a family with a primitive pure $(1,1) - VHS$. Now we verify that $q = 7, 8, 9, 10, 11$ do not lead to a family with a primitive pure $(1,1) - VHS$: One must have that $\mathbb{L}_5$ is unitary. It has the local monodromy data

$$\mu_3 = \frac{1}{2} \text{ and } \mu_4 = [\frac{1}{2} + \frac{5}{6}]_1 = \frac{1}{3}.$$

Hence one must have that

$$\frac{1}{6} \geq \mu_1 = [\frac{1}{2} - \frac{5}{q}]_1,$$

which is satisfied for $q = 10, 11$, but not for $q = 7, 8, 9$. For $q = 10$ we have that

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q} = \frac{1}{15}.$$

This leads to a family given by the local monodromy data

$$\mu_1 = \frac{4}{10}, \quad \mu_2 = \frac{13}{30}, \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{2}{3}.$$

One calculates easily that the eigenspace $\mathcal{L}_7$ in the VHS of this family is given by

$$\mu_1 = \frac{4}{5}, \quad \mu_2 = \frac{1}{30}, \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{2}{3}.$$

Hence this family has not a pure $(1,1) - VHS$.

For $q = 11$ we have that

$$\frac{1}{p} - \frac{1}{q} = \frac{5}{66}.$$

Hence the equation of Lemma 6.2.2 can not be satisfied in this case.

6.2.6. Keep the assumptions of Remark 6.2.4. Moreover assume that $q = 6$. In this case we can have $p = 3, 4, 5$, where $p = 3$ implies

$$\frac{1}{p} = \frac{1}{3}, \quad \frac{1}{q} = \frac{1}{r} = \frac{1}{6},$$

which yields an example of a family with a primitive $(1,1) - VHS$. For $p = 4$ resp., $p = 5$ Lemma 6.2.2 and Lemma 6.2.3 yield a family of covers given by the local monodromy data

$$\mu_1 = \frac{1}{3}, \quad \mu_2 = \frac{5}{12} \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{3}{4}$$

resp.,

$$\mu_1 = \frac{1}{3}, \quad \mu_2 = \frac{14}{30} \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{7}{10}.$$

Hence one can easily verify that $\mathcal{L}_5$ is an eigenspace of type $(1,1)$ in both cases. Thus $p = 4, 5$ do not lead to a primitive pure $(1,1) - VHS$.

6.2.7. Keep the assumptions of Remark 6.2.4. Moreover assume that $r = 6$. In this case Lemma 6.2.2 implies that

$$\frac{1}{p} \geq \frac{2}{r} = \frac{1}{3}.$$

Hence one has $p = 2$ or $p = 3$, where $p = 2$ would imply that $\mu_4 = 1$, and $p = 3$ yields the same example of a family with a primitive pure $(1,1) - VHS$ as in 6.2.6.

Now we have considered the subcase given by $3 \notin (\mathbb{Z}/(m))^*$. We start the consideration of the subcase given by $3 \in (\mathbb{Z}/(m))^*$ by the following lemma:

Lemma 6.2.8. *Let $\mathcal{C} \to \mathcal{P}_1$ be a family with a primitive pure $(1,1) - VHS$, which satisfies that each $\mu_{k_1} + \mu_{k_2} \neq 1$. Then one has $m > 4$.*

Proof. We know that one must have $m \geq 4$ in the considered case. Thus we must only exclude $m = 4$. Since for a family $\mathcal{C}$ of degree 4 covers with a primitive pure $(1,1) - VHS$ the family $\mathcal{C}_2$ must have a trivial VHS, one has without loss of generally $d_1 = 2$. By the assumption that each $\mu_{k_1} + \mu_{k_2} \neq 1$, one concludes that d_1, d_2, d_3 are not equal to 2. Hence d_1, d_2, d_3 are odd. But this contradicts our assumptions, which imply that we have the even sum

$$2m = d_1 + \ldots + d_4.$$

□

Remark 6.2.9. Keep the assumption of Lemma 6.2.3. If $m > 4$, the eigenspace $\mathbb{L}_3$ is not of type $(1,1)$. Assume that 3 is a unit in $\mathbb{Z}/(m)$. Thus Lemma 6.2.8 implies for the local monodromy data of $\mathbb{L}_3$ that $\sum \mu_i = 3$ or $\sum \mu_i = 1$. Recall that $\mu_3 = \frac{1}{2}$. Hence $\sum \mu_i = 3$ implies that

$$\mu_1, \mu_2, \mu_4 > \frac{1}{2}.$$

By Lemma 6.2.3, one concludes that

$$\mu_1 = \frac{3}{2} - \frac{3}{q} \quad \text{and} \quad \mu_2 = \frac{3}{2} - \frac{3}{r}.$$

By Lemma 6.2.2, this implies that

$$\mu_4 = 3 - \mu_1 - \mu_2 - \mu_3 = 3 - \frac{3}{2} + \frac{3}{q} - \frac{3}{2} + \frac{3}{r} - \frac{1}{2} = -\frac{1}{2} + \frac{3}{p} + \frac{3}{q} = -\frac{1}{2} + \frac{3}{p}$$

in this case. This implies that $p, q, r < 6$.

In the case $\sum \mu_i = 1$ one gets $\mu_1, \mu_2, \mu_4 < \frac{1}{2}$. By Lemma 6.2.2 and Lemma 6.2.3, this implies that

$$\mu_1 = \frac{1}{2} - \frac{3}{q}, \quad \mu_2 = \frac{1}{2} - \frac{3}{r} \text{ and } \mu_4 = -\frac{1}{2} + \frac{3}{p}$$

such that $p < 6$ and $q, r > 6$.

Remark 6.2.10. The case $p, q, r < 6$ does not yield any example of a family with a primitive pure $(1,1) - VHS$, since no triple $(p, q, r) \in \mathbb{N}^3$ with $2 \leq p \leq q \leq r < 6$ satisfies both

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{r} \text{ and } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

as one can check by calculation for each example.

6.2.11. Assume that we are in the case $p < 6$ and $q, r > 6$. Since $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, one has $\frac{1}{p} \leq 2\frac{1}{q}$ such that $6 < q \leq 2p$ and $3 < p < 6$. Hence one has two cases: $p = 4$ or $p = 5$. Thus by using that $\frac{1}{p} - \frac{1}{q} = \frac{1}{r}$, one calculates that only the examples given by

$$p = 4, \quad q = r = 8 \text{ and } p = 5, \quad q = r = 10$$

have a primitive pure $(1,1) - VHS$ in this case.

Now we consider the last remaining case of a family $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$. In this case there are at most 2 branch indeces equal and one has some $\mu_{k_1} + \mu_{k_2} = 1$.

Lemma 6.2.12. *Let $\mathcal{C} \to \mathcal{P}_1$ a family of cyclic covers. If there are $k_1, k_2 \in \{1,2,3,4\}$ such that $d_{k_1} + d_{k_2} = m$ with $d_1 \leq d_2 \leq d_3 \leq d_4$, then one has*

$$d_1 + d_4 = m \text{ and } d_2 + d_3 = m.$$

Proof. (quite easy to see) □

Remark 6.2.13. By the preceding lemma, we have that $d_1 + d_4 = d_2 + d_3 = m$, if there are $k_1, k_2 \in \{1,2,3,4\}$ such that $d_{k_1} + d_{k_2} = m$ with $d_1 \leq d_2 \leq d_3 \leq d_4$. Hence if $d_1 + d_3 = m$ resp., $d_3 = d_4$, one gets $d_1 = d_2$, too. But this contradicts the assumption that at most 2 branch indexes are equal. Hence by $SINT$, one gets

$$\mu_1 + \mu_2 = 1 - \frac{1}{p} < 1, \quad \mu_1 + \mu_3 = 1 - \frac{1}{q} < 1, \quad \mu_2 + \mu_3 = 1$$

with $p, q \in \mathbb{N}$ and $p \leq q$. Hence one obtains similarly to Remark 6.2.1 with $\frac{1}{p} + \frac{1}{q} < 1$:

$$\mu_1 = \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{q}), \quad \mu_2 = \frac{1}{2}(1 - \frac{1}{p} + \frac{1}{q}),$$
$$\mu_3 = \frac{1}{2}(1 + \frac{1}{p} - \frac{1}{q}), \quad \mu_4 = \frac{1}{2}(1 + \frac{1}{p} + \frac{1}{q})$$

Lemma 6.2.14. *Assume that the local monodromy data of Remark* 6.2.13 *yield a family of degree $m \geq 4$ with a primitive pure $(1,1) - VHS$. Then one has $p = q$ and m is even.*

Proof. In the case of Remark 6.2.13 the eigenspace $\mathbb{L}_2$ is given by the local monodromy data

$$\mu_1 = 1 - \frac{1}{p} - \frac{1}{q}, \quad \mu_2 = [1 - \frac{1}{p} + \frac{1}{q}]_1, \quad \mu_3 = [\frac{1}{p} - \frac{1}{q}]_1, \quad \mu_4 = \frac{1}{p} + \frac{1}{q}.$$

Thus in this case $\mathcal{L}_2$ is of type $(1,1)$, if and only if $p < q$. Hence one can obtain a primitive pure $(1,1) - VHS$, only if $p = q$. Now $p = q$ implies that $\mu_2 = \mu_3 = 0$ for the local monodromy data of $\mathbb{L}_2$. Hence the family of covers has an even degree. □

Proposition 6.2.15. *Assume that the local monodromy data of Remark* 6.2.13 *yield a family of degree $m \geq 4$ with a primitive pure $(1,1) - VHS$. Then $p = q \leq 6$.*

Proof. By the preceding lemma, the assumptions imply that $p = q$. Hence by Remark 6.2.13, we have:

$$\mu_1 = \frac{p-2}{2p}, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{p+2}{2p} \tag{6.1}$$

If $p > 6$, then $\mathbb{L}_3$ has the local monodromy given by

$$\mu_1 = \frac{p-6}{2p}, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{p+6}{2p}.$$

Hence Proposition 2.3.4 implies that $\mathcal{L}_3$ is of type $(1,1)$ in this case. □

Lemma 6.2.16. *Assume that the local monodromy data of Remark* 6.2.13 *yield a family of degree $m \geq 4$ with a primitive pure $(1,1) - VHS$. Then p must be even.*

Proof. Assume that p is odd. Since $gcd(p-2, 2p) = 1$ in this case, one gets a family of degree $2p$ with branch indeces

$$d_1 = p - 2, \quad d_2 = d_3 = p, \quad d_4 = p + 2.$$

Thus all branch indeces are odd, and $\mathcal{C}_p$ is a family of elliptic curves such that $\mathcal{L}_p$ is of type $(1,1)$. Contradiction! □

Remark 6.2.17. Keep the assumptions of the preceding lemma. Since one must have $\mu_1 > 0$, the preceding proposition and (6.1) imply that

$$3 \leq p \leq 6.$$

Since $p = q$ must be even, one can only have $p = 4$ and $p = 6$.

1. For $p = 4$ one obtains the example of a family with a primitive pure $(1,1) - VHS$ given by
$$\mu_1 = \frac{1}{4}, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{3}{4}.$$
2. If $p = 6$ one has the example of a family with a primitive pure $(1,1) - VHS$ given by
$$\mu_1 = \frac{1}{3}, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{2}{3}.$$

6.3 The complete lists of examples

In this section we give the complete lists of examples of families $\mathcal{C} \to \mathcal{P}_n$ with primitive $(1,n)$-variations or derived pure $(1,n)$-variations of Hodge structures.

By our preceding calculations, we get the following complete list of families of covers $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$, where "ref" denotes the number of the preceding remark, lemma, proposition or point yielding the respective example:

number	degree	branch points with branch index	genus	ref
1	2	1 1 1 1	1	(known)
2	3	1 2 2 1	2	6.1.18
3	4	1 2 2 3	2	6.2.17, (1)
4	5	1 3 3 3	4	6.1.17
5	6	1 4 4 3	3	6.2.6, 6.2.7
6	6	2 3 3 4	2	6.2.17, (2)
7	7	2 4 4 4	6	6.1.17
8	8	2 5 5 4	5	6.2.11
9	9	3 5 5 5	7	6.1.17
10	10	3 6 6 5	6	6.2.11
11	12	4 7 7 6	7	6.2.5

We will later see that each derived pure $(1,n)-VHS$ is in fact a derived pure $(1,1)-VHS$. In the next section we will verify that we get the following complete list of families of covers $\mathcal{C} \to \mathcal{P}_1$ with a derived pure $(1,1) - VHS$, where N_{r_0} means the number of $\mathcal{C}_{r_0}$ in the preceding list, which has the corresponding primitive pure $(1,1) - VHS$:

degree	branch points with branch index	genus	r_0	N_{r_0}
4	1 1 1 1	3	2	1
6	1 1 1 3	4	3	1
6	1 2 2 1	4	2	2

Note that any family $\mathcal{C} \to \mathcal{P}_n$ with a primitive pure $(1,n) - VHS$ satisfies $SINT$, which implies INT. Hence by consulting the list of [14] on page 86, which contains all examples satisfying INT for $n \geq 2$, (and the calculation of the types of the eigenspaces of the corresponding covers), we have the following complete list of families of covers with a primitive pure $(1,n) - VHS$ for $n > 1$:

degree	branch points with branch index	genus
3	2 1 1 1 1	3
4	2 2 2 1 1	3
5	2 2 2 2 2	6
6	3 3 3 2 2	4
3	1 1 1 1 1 1	4

In [10] R. Coleman formulated the following conjecture:

Conjecture 6.3.1. *Fix an integer $g \geq 4$. Then there are only finitely many complex algebraic curves C of genus g such that* $\mathrm{Jac}(C)$ *is of CM type.*

Remark 6.3.2. J. de Jong and R. Noot [25] resp., E. Viehweg and K. Zuo [46] have given counterexamples of families with infinitely many CM fibers for $g = 4, 6$. In our lists here we have counterexamples for $g = 5, 7$.

J. de Jong and R. Noot resp., E. Viehweg and K. Zuo needed to show first the existence of one fiber with CM for the proofs that their examples of families have infinitely many CM fibers. In the proof of Theorem 4.4.4, which implies that the examples of this section have dense set of complex multiplication fibers, we did not need to show the existence of one CM fiber first.

But by the fact that our examples $\mathcal{C} \to \mathcal{M}_n$ with a dense set of CM fibers satisfy that $n+1$ branch points have the same branch index, Theorem 2.4.4 yields the CM-type of one CM fiber and hencefore by Lemma 1.5.8, the CM-type of a dense set of CM fibers.

6.4 The derived variations of Hodge structures

In this section we determine the families of cyclic covers with a derived pure $(1,n)-VHS$ and verify that the list of examples in the preceding section is complete.

Remark 6.4.1. Assume that the family $\mathcal{C}$ of degree dm covers has a derived pure $(1,n)-VHS$ induced by $\mathcal{C}_d$. Let

$$d = p_1^{n_1} \cdot \ldots \cdot p_t^{n_t}$$

be the decomposition of d into its prime factors. Then there exists a family of covers of degree $p_1 m$ with a derived pure $(1,n)-VHS$. Hence there are two cases to consider first: d is a prime number and divides m, or d is a prime number and does not divide m.

Lemma 6.4.2. *Let p be a prime number. Assume that d is a prime number such that $\gcd(d,p) = 1$. Then a family $\mathcal{C}$ of covers of degree $p \cdot d$ with a derived pure $(1,n) - VHS$ induced by $\mathcal{C}_d$ can not exist, if all Dehn twists yield semisimple matrices with respect to the monodromy representation of $\mathcal{L}_d$.*

Proof. Since $\mathcal{C}_p$ must have a trivial VHS, there exists a d_2 such that d divides d_2. Moreover there is a d_1 such that d does not divide d_1. Hence $\gcd(d, d_1 + d_2) = 1$. By the fact that $\mathcal{C}_d$ has the property that its local monodromy data satisfy $\mu_1 + \mu_2 \neq 1$, one concludes that $\gcd(p, d_1 + d_2) = 1$, too. Hence $[d_1 + d_2]_{dp}$ is a unit in $\mathbb{Z}/(dp)$. Thus there exists a $d_0 \in (\mathbb{Z}/(dp))^*$ such that $d_0[d_1 + d_2]_{dp} = 1$. One obtains that the sum of the integers of $\{1, \ldots, p-1\}$ representing $[d_0 d_1]_{dp}$ and $[d_0 d_2]_{dp}$ is given by $dm + 1$. By Proposition 2.3.4, one concludes that $\mathcal{L}_{d_0}$ is not of type $(0, n+1)$. Moreover the fact that the local monodromy data of $\mathcal{L}_{d_0}$ satisfy

$$\mu_1 + \mu_2 = \frac{dp+1}{dp}, \quad \mu_3 \leq \frac{dp-1}{dp}, \quad \mu_4 + \ldots + \mu_{n+3} < n$$

tells us that

$$\mu_1 + \ldots + \mu_{n+3} < n + 2.$$

Hence one concludes by Proposition 2.3.4 that $\mathcal{L}_{d_0}$ is not of type $(n+1, 0)$, too. □

Lemma 6.4.3. *Let $m = 2^t p$, where $p \neq 2$ is a prime number and $t \geq 1$. Assume that d is a prime number such that $\gcd(d,m) = 1$. Then a family $\mathcal{C}$ of degree $m \cdot d$ covers with a derived pure $(1,n) - VHS$, which is induced by $\mathcal{C}_d$, can not exist.*

Proof. Since $\mathcal{C}_2$ must have a trivial VHS, one has $d_1 = \ldots = d_n = 2^{t-1}dp$. By the fact that $\mathcal{C}_p$ must have a trivial VHS, we obtain that $2^t d$ divides n different branch indeces. Since there must be at least two different branch indeces, which are not divided by d, d_{n+2} and d_{n+3} are not divided by d. By the fact that $d_1 = \ldots = d_n = 2^{t-1}dp$ is not divided by $2^t d$, one must have $n = 1$ and that $2^t d$ divides d_2. Moreover the facts that

$$d_1 + \ldots + d_4 \in (m) = (2^t pd) \text{ and } 2|d_2$$

imply without loss of generality that 2 does not divide d_3. We have two cases: Either $p|d_3$ or this does not hold true. In the first case one has that 2, p and d do not divide $d_2 + d_3$. Hence $d_2 + d_3$ is a unit, and again we use the argument that there is a $d_0 \in (\mathbb{Z}/(dm))^*$ such that $[d_0 d_2 + d_0 d_3] = 1$.

In the other case d_3 yields a unit of $\mathbb{Z}/(dm)$. Hence we have without loss of generality $d_3 = 1$. Thus $g := \gcd(dm, d_1 + d_3) \in \{1, 2\}$. If $g = 1$, we are done again. Otherwise we must have $t = 1$, if $g = 2$. Hence

$$[(pd - 2)(d_1 + d_3)]_{dm} = pd + pd - 2 = dm - 2$$

such that $\mathcal{L}_{pd-2}$ is neither of type $(0, n + 1)$ nor of type $(n + 1, 0)$, since the fact that $2^t d$ divides d_2 implies that $[(pd - 2)d_2]_{dm} \neq [1]_{dm}$. □

Lemma 6.4.4. *Let p be a prime number and $m = p^t$ with $t \geq 2$. Assume that d is a prime number such that $\gcd(d, p) = 1$. Then there can not be a family $\mathcal{C}$ of degree $m \cdot d$ covers with a derived pure $(1, n) - VHS$, which is induced by $\mathcal{C}_d$.*

Proof. Since $\mathcal{C}_p$ must have a trivial VHS, one concludes without loss of generality that dp^{t-1} divides $d_1, \ldots, d_n$. Since d and p divide

$$dp^t = d_1 + \ldots + d_{n+3},$$

too, p resp., d does not divide at least two different elements of $\{d_{n+1}, d_{n+2}, d_{n+3}\}$. Hence there is an element of $\{d_{n+1}, d_{n+2}, d_{n+3}\}$, which is not divided by both d and p. Without loss of generality d_{n+1} is a unit in $\mathbb{Z}/(2^t d)$. Hence one has without loss of generality $[d_1 + d_{n+1}]_{dm} = [1]_{dm}$. □

There are only few remaining examples, which do not satisfy the assumptions of the preceding lemmas. One of these examples is considered in the following lemma:

Lemma 6.4.5. *Let $d \neq 3$ be a prime number. There can not be a family of covers of degree $3d$ with a derived pure $(1, 2) - VHS$ induced by $\mathcal{C}_d$ given by the local monodromy data*

$$\mu_1 = \ldots = \mu_4 = \frac{1}{3}, \quad \mu_5 = \frac{2}{3}.$$

Proof. Let $\gcd(d, 3) = 1$ and $\mathcal{C}$ be a family of degree $3d$ with a derived pure $(1, 2) - VHS$. Since $\mathcal{C}_3$ should have a trivial VHS, one has with a new enumeration $d|d_1$ and $d|d_2$. Moreover one has without loss of generality that d_3 and d_4 are not divided by d. Hence d divides neither $d_1 + d_3$ nor $d_2 + d_4$. Moreover the local monodromy data of $\mathcal{C}_d$ tell us that 3 does not divide $d_1 + d_3$ or $d_2 + d_4$. Hence without loss of generality $d_1 + d_3$ is a unit in $\mathbb{Z}/(3d)$ such there is a $d_0 \in (\mathbb{Z}/(3d))^*$ with the property that $[d_0 d_1 + d_0 d_3]_{3d} = 1$, which implies that $\mathbb{L}_{d_0}$ is of type $(1, 2)$ or of type $(2, 1)$. □

The reader checks easily that all examples of families with a primitive pure $(1,n)-VHS$ satisfy with two exceptions the assumptions of one of the preceding lemmas. These two exceptions yield examples of families with a derived pure $(1,n)-VHS$ as we will see now.

6.4.6. Now we consider the case of the elliptic curves. Let d be a prime number with $\gcd(d,2)=1$ and $\mathcal{C}$ be a family of degree $d\cdot 2$ covers with a derived pure $(1,1)-VHS$ induced by $\mathcal{C}_d$. Thus $d_1,\ldots,d_4$ must be odd. Without loss of generality we have $d_4=d$, since $\mathcal{C}_2$ must have a trivial VHS. Since $d_3=d$ would imply that $\mathbb{L}_1$ is of type $(1,1)$, one has that $d_1,d_2,d_3\in(\mathbb{Z}/(2d))^*$. We have two cases. Either $d_1=d_2$ or this does not hold true. In the first case we put $d_1=d_2=d-2$. One has

$$2d<d_1+d_2+d_4<2\cdot 2d$$

such that $\mathbb{L}_1$ is of type $(1,1)$, if $4<d$. Thus one can have $d=3$. In this case one has a family of degree 6 covers, where $d_4=3$. Hence one must have

$$\mu_1=\mu_2=\mu_3=\frac{1}{6},\quad \mu_4=\frac{1}{2}.$$

In the second case, one puts $d_3=d-2$. This implies that $d_3+d_4=2d-2$. Since $d_1\neq d_2$, one can not have $d_1=d_2=1$ such that $\mathbb{L}_1$ is of type $(1,1)$ in this case.

6.4.7. Now we consider the case number 2 in the list of examples with a primitive pure $(1,1)-VHS$. Let d be a prime number with $\gcd(d,3)=1$ and $\mathcal{C}$ be a family of degree $d\cdot 3$ covers with a derived pure $(1,1)-VHS$ induced by $\mathcal{C}_d$. Assume without loss of generality that d divides d_1 and $d_1+\ldots+d_4=3d$. We have 2 cases: Either d divides d_2, d_3 or d_4, or d does not divide d_2, d_3 and d_4. In the first case one has without loss of generality that d divides d_2. Since d divides d_1 and $d_1+\ldots+d_4=3d$, one concludes that $d_1=d_2=d$. This implies that $\mathcal{L}_2$ is of type $(1,1)$ such that $d=2$. In addition one concludes that

$$d_1=d_2=2,\quad d_3=d_4=1.$$

In the second case one has that 3 does not divide $d(d_1+d_i)$ for exactly one $k\in\{2,3,4\}$, which follows by the branch indeces in the case number 2. Hence 3 does not divide d_1+d_k. Moreover d does not divide d_1+d_k, too. Hence $d_1+d_k\in(\mathbb{Z}/(3d))^*$.

Proposition 6.4.8. *Let d be a prime number, which divides m and $\mathcal{C}$ be a family of covers of degree md. Assume a Dehn twist yields a semisimple matrix of maximal order m with respect to the monodromy representation of $\mathcal{L}_d$. Then $\mathcal{C}$ can not have a derived pure $(1,n)-VHS$ induced by $\mathcal{C}_d$.*

Proof. Assume without loss of generality that $\rho_d(T_{1,2})$ yields a matrix of order m. In this case $[d(d_1+d_2)]\in\mathbb{Z}/(dm)$ has the order m. Hence the fact that d divides m implies that $d_1+d_2\in(\mathbb{Z}/(dm))^*$. □

Remark 6.4.9. One can easily check that the assumptions of the preceding proposition are satisfied for all examples of families with a primitive pure $(1,n)-VHS$ except of the case of elliptic curves. In this case we have in fact an example of a family of degree 4 covers with a derived pure $(1,1)-VHS$. Without loss of generality we have

$$d_1+\ldots+d_4=4.$$

Hence the only possibility is given by

$$d_1=\ldots=d_4=1.$$

6.4.10. In the case of the elliptic curves we have families of degree 6 and degree 4 covers with a derived pure $(1,1) - VHS$. Hence one must check that there is not a family of degree 8, 12 or 18 covers with derived pure $(1,1) - VHS$ in this case.

First we check that there is not a family $\mathcal{C}$ of degree 8 covers with a derived pure $(1,1) - VHS$. Otherwise one has such a family $\mathcal{C}$ of degree 8 covers such that $\mathcal{C}_2$ is the family of degree 4 covers with a derived pure $(1,1) - VHS$. This implies that each d_k satisfies $[d_k]_4 = [1]_4$ or each d_k satisfies $[d_k]_4 = [3]_4$. Moreover one has without loss of generality that $d_1 + \ldots + d_4 = 8$. Hence it is not possible that each d_k satisfies $[d_k]_4 = [3]_4$. Thus the only possibility is (up to the numbering) given by

$$d_1 = d_2 = d_3 = 1, \quad d_4 = 5.$$

But in this case $\mathcal{L}_3$ is of type $(1,1)$. Thus there can not exist a family of degree 8 covers with a derived pure $(1,1) - VHS$.

There can not be a family of degree 12 covers with a derived pure $(1,1) - VHS$ induced by $\mathcal{C}_6$. Otherwise one has that $\mathcal{C}_3$ the example of degree 4 covers with a derived pure $(1,1) - VHS$. Thus one concludes that

$$[d_1]_4 = \ldots = [d_4]_4 = [1]_4 \text{ or } [d_1]_4 = \ldots = [d_4]_4 = [3]_4.$$

Since one has without loss of generality that $d_1 + \ldots + d_4 = 12$, the only possibilities are given by

$$d_1 = d_2 = 5, \quad d_3 = d_4 = 1 \text{ and } d_1 = 9, \quad d_2 = d_3 = d_4 = 1.$$

In the first case $\mathcal{L}_2$ is of type $(1,1)$ and in the second case $\mathcal{L}_5$ is of type $(1,1)$.

There can not be a family of degree 18 covers with a derived $(1,1) - VHS$ induced by $\mathcal{C}_9$. Otherwise one has that $\mathcal{C}_3$ is the example of degree 6 covers with a derived pure $(1,1) - VHS$ induced by the elliptic curves. Thus one concludes that

$$[d_1]_6 = \ldots = [d_3]_6 = [1]_6 \text{ and } [d_4]_6 = [3]_6$$

or

$$[d_1]_6 = \ldots = [d_3]_6 = [5]_6 \text{ and } [d_4]_6 = [3]_6.$$

Since one has without loss of generality that $d_1 + \ldots + d_4 = 18$, the only possibilities are given by:

$$d_1 = 13, \quad d_2 = d_3 = 1, \quad d_4 = 3$$

$$d_1 = d_2 = 7, \quad d_3 = 1, \quad d_4 = 3$$

$$d_1 = 7, \quad d_2 = d_3 = 1, \quad d_4 = 9$$

$$d_1 = d_2 = d_3 = 1, \quad d_4 = 15$$

$$d_1 = d_2 = d_3 = 5, \quad d_4 = 3.$$

One has that $\mathcal{L}_5$ is of type $(1,1)$ in case 1, $\mathcal{L}_2$ is of type $(1,1)$ in case 2, $\mathcal{L}_5$ is of type $(1,1)$ in case 3, $\mathcal{L}_7$ is of type $(1,1)$ in case 4 and $\mathcal{L}_2$ is of type $(1,1)$ in case 5.

6.4.11. It remains to show that there can not exist a degree 12 cover with a derived $(1,1) - VHS$ induced by the degree 3 cover given by

$$d_1 = d_2 = 1, \quad d_3 = d_4 = 2.$$

Otherwise one has such a family $\mathcal{C}$ of degree 12 covers such that $\mathcal{C}_2$ is the family of degree 6 covers with a derived pure $(1,1) - VHS$ by the degree 3 example above. Thus one concludes that

$$[d_1]_6 = [d_2]_6 = [2]_6 \text{ and } [d_3]_6 = [d_4]_6 = [1]_6$$

or

$$[d_1]_6 = [d_2]_6 = [4]_6 \text{ and } [d_3]_6 = [d_4]_6 = [5]_6$$

Since one has without loss of generality that $d_1 + \ldots + d_4 = 12$, the only possibilities are given by

$$d_1 = 8, \quad d_2 = 2, \quad d_3 = d_4 = 1 \text{ and } d_1 = d_2 = 2, \quad d_3 = 7, \quad d_4 = 1.$$

One has that $\mathcal{L}_5$ is of type $(1,1)$ in the first case and one has that $\mathcal{L}_3$ is of type $(1,1)$ in the second case.

Chapter 7

The construction of Calabi-Yau manifolds with complex multiplication

7.1 The basic construction and complex multiplication

Now we have finished our considerations on Hodge structures of cyclic covers of $\mathbb{P}^1$. We start with the second part, which is devoted to the construction of families of Calabi-Yau manifolds with dense set of complex multiplication fibers.

In the works of C. Borcea [7], [8], of E. Viehweg and K. Zuo [46] and of C. Voisin [48] the methods to obtain higher dimensional Calabi-Yau manifolds contain one common basic construction. In this section we describe this construction and explain how it yields complex multiplication. For this construction we need Kummer coverings. Let $A - B$ be a principal divisor with $(f) = A - B$ for some $f \in \mathbb{C}(X)$. The Kummer covering given by $\mathbb{C}(X)(\sqrt[m]{\frac{A}{B}})$ is nothing but the normalization of X in $\mathbb{C}(X)(\sqrt[m]{f})$.

Let V_1 and V_2 be irreducible complex algebraic manifolds and $\mathcal{A}$ resp., $\mathcal{B}$ be a bundle of irreducible algebraic manifolds with universal fiber A resp., B over V_1 resp., V_2. Moreover let $\mathcal{Z}$ resp., Σ be a cyclic Galois cover of $\mathcal{A}$ resp., a cyclic Galois cover of $\mathcal{B}$ of degree m over V_1 resp., V_2 ramified over a smooth divisor. We assume that the irreducible components of these ramification divisors intersect each fiber of $\mathcal{Z}$ resp., Σ transversally in smooth subvarieties of codimension 1. Thus we assume that $\mathcal{Z}$ and Σ are given by Kummer coverings of the kind

$$\mathbb{C}(W) = \mathbb{C}(X)(\sqrt[m]{\frac{D_1 + \ldots + D_k}{D_0^m}}),$$

where $D_1, \ldots, D_k$ are (reduced) smooth prime divisors, which do not intersect each other.

Example 7.1.1. By a cyclic degree 2 cover $S \to R$ of surfaces (or in general algebraic varieties), one has an involution on S. Let us assume that the surface S is a smooth $K3$ surface. Moreover assume that there exists an involution ι on S, which acts via pull-back by -1 on $\Gamma(\omega_S)$. It has the property that it fixes at most a divisor D, whose support consists of smooth curves, which do not intersect each other (see [48], 1.1.). Moreover by [48], 1.1., to give an involution ι on S, which acts by -1 on $\Gamma(\omega_S)$, is the same as to give

a cyclic degree 2 cover $S \to R$ of smooth surfaces. In this case R is rational, if and only if $D \neq 0$.

We consider the following commutative diagram, which yields the basic construction:

$$\begin{array}{ccccccc} \mathcal{Z} \times \Sigma & \xrightarrow{\gamma} & \mathcal{Y}' & \xrightarrow{\alpha} & \mathcal{A} \times \mathcal{B} & \longrightarrow & V_1 \times V_2 \\ \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \zeta} & & \\ \widetilde{\mathcal{Z} \times \Sigma} & \xrightarrow{\tilde{\gamma}} & \tilde{\mathcal{Y}} & \xrightarrow{\tilde{\alpha}} & \hat{\Pi} & & \end{array} \tag{7.1}$$

First we explain the upper line of this diagram: The cyclic covers $\mathcal{Z}$ and Σ can locally be described by equations of the type

$$y^m = \prod_{i=1,\dots,k} f_i(x_1, \dots, x_j)$$

over any open affine set $\mathbb{A}$ of $\mathcal{A}$ resp., $\mathcal{B}$, where f_i is the (reduced) equation of D_i in $\mathbb{A}$. The Galois transformations are given by

$$(y, x_1, \dots, x_j) \xrightarrow{g_k} (e^{2\pi\sqrt{-1}\frac{k}{m}} y, x_1, \dots, x_j)$$

for some $k \in \mathbb{Z}/(m)$. Hence we have a natural identification between $\mathbb{Z}/(m)$ and the Galois groups given by $[k]_m \to g_k$. By the describtion of the covers above in terms of Kummer coverings, this identification is independent of the chosen open affine subset. Now γ is the quotient by

$$G := \langle (1,1) \rangle \quad \subset \quad G' := \mathrm{Gal}(\mathcal{Z}; \mathcal{A}) \times \mathrm{Gal}(\Sigma; \mathcal{B}),$$

and α is the quotient by G'/G. The morphism ζ is given by the blowing up of the fiber product of the supports of the branch divisors of $\mathcal{Z}$ and Σ. Moreover δ is the blowing up along the singular points of $\mathcal{Y}'$, which is given by the intersection locus of the ramification divisors, and β is the blowing up with respect to the corresponding inverse image ideal sheaf. Hence $\tilde{\alpha}$ and $\tilde{\gamma}$ are the unique cyclic covers obtained by the universal property of the blowing up (compare to [22], **II**. Corollary 7.15). By the construction of α, one can easily check that $\tilde{\alpha}$ is not ramified over the exceptional divisor. Hence the branch locus of $\tilde{\alpha}$ is smooth. This implies that $\tilde{\mathcal{Y}}$ is smooth, too. The ramification locus of $\tilde{\gamma}$ is given by the smooth exceptional divisor of β, since G leaves the generators of the inverse image ideal sheaf invariant as one can see by the following remark:

Remark 7.1.2. Now we describe $\widetilde{\mathcal{Z} \times \Sigma}$. A neighborhood of the preimage point $p \in \mathcal{Z} \times \Sigma$ of a singular point can be identified with an open neighborhood of $0 \in \mathbb{C}^2 \times \mathbb{B}$, where $\mathbb{B}$ is a ball of suitable dimension and the Galois group acts via $(x_1, x_2) \to (e^{\frac{2\pi i}{m}} x_1, e^{\frac{2\pi i}{m}} x_2)$ with respect to the coordinates on $\mathbb{C}^2$. Due to [5], **III**. Proposition 5.3, each singular point of $\mathcal{Y}'$ has an analytic neighborhood isomorphic to $V(x^m = y^{m-1}z) \times \mathbb{B}$. Hence locally we have the product of a cover of surfaces with $\mathbb{B}$. One should have $\mathbb{B}$ in mind. But for the description of $\widetilde{\mathcal{Z} \times \Sigma}$, it is sufficient to consider only covers of surfaces. The inverse image ideal sheaf with respect to this cover is generated by $\{x_1^{m-i} x_2^i : i = 0, 1, \dots, m\}$. By the Veronese embedding for relative projective manifolds, one can easily identify the blowing up with respect to this ideal with the blowing up with respect to the ideal generated by $\{x_1, x_2\}$. But this is the blowing up of the origin resp., the preimage point of the singular point. Hence in the general situation $\widetilde{\mathcal{Z} \times \Sigma}$ is given by the blowing up of the reduced preimage $\gamma^{-1}(S)$, where S is the singular locus of $\mathcal{Y}'$.

Now we have described the basic construction. Next we see that this construction yields complex multiplication. We use following fact:

Proposition 7.1.3. *For all $\tilde{a} \in \mathcal{A}$, and $\tilde{b} \in \mathcal{B}$, we have the following tensor product of rational Hodge structures on the fibers:*

$$H^n(\mathcal{Z}_{\tilde{a}} \times \Sigma_{\tilde{b}}, \mathbb{Q}) = \bigoplus_{a+b=n} H^a(\mathcal{Z}_{\tilde{a}}, \mathbb{Q}) \otimes H^b(\Sigma_{\tilde{b}}, \mathbb{Q})$$

such that

$$H^{r,s}(\mathcal{Z}_{\tilde{a}} \times \Sigma_{\tilde{b}}) = \bigoplus_{p+p'=r, q+q'=s} H^{p,q}(\mathcal{Z}_{\tilde{a}}) \otimes H^{p',q'}(\Sigma_{\tilde{b}})$$

Proof. (follows from [49], Theorem 11.38) □

We want to construct higher dimensional varieties with complex multiplication. The first main tool is:

Proposition 7.1.4. *Let h_1 and h_2 be rational polarized Hodge structures. Then $h_3 = h_1 \otimes h_2$ is of CM type, if and only if h_1 and h_2 are of CM type.*

Proof. (see [7], Proposition 1.2) □

By the fact that $\mathcal{Y}'$ is not smooth, but the blowing up $\tilde{\mathcal{Y}}$ is smooth, $\tilde{\mathcal{Y}}$ will be our candidate for a family of Calabi-Yau manifolds with dense set of complex multiplication fibers. Hence we must consider the behavior of the Hodge structures under blowing up:

Lemma 7.1.5. *Let X be an algebraic manifold of dimension n and $\tilde{X}$ be the blowing up X with respect to some submanifold $Z \in X$ of codimension 2. Then $\mathrm{Hg}(H^k(\tilde{X}, \mathbb{Z}))$ is commutative, if and only if $\mathrm{Hg}(H^k(X, \mathbb{Z}))$ and $\mathrm{Hg}(H^{k-2}(Z, \mathbb{Z}))$ are commutative, too.*

Proof. By [49], Theorem 7.31, we have an isomorphism

$$H^k(X, \mathbb{Z}) \oplus H^{k-2}(Z, \mathbb{Z}) \cong H^k(\tilde{X}, \mathbb{Z})$$

of Hodge structures, where $H^{k-2}(Z, \mathbb{Z})$ is shifted by $(1, 1)$ in bi-degree. Since

$$\mathrm{Hg}(H^k(\tilde{X}, \mathbb{Z})) = \mathrm{Hg}(H^k(X, \mathbb{Z}) \oplus H^{k-2}(Z, \mathbb{Z})) \subset \mathrm{Hg}(H^k(X, \mathbb{Z})) \times \mathrm{Hg}(H^{k-2}(Z, \mathbb{Z}))$$

such that the natural projections

$$\mathrm{Hg}(H^k(\tilde{X}, \mathbb{Z})) \to \mathrm{Hg}(H^k(X, \mathbb{Z})) \quad \text{and} \quad \mathrm{Hg}(H^k(\tilde{X}, \mathbb{Z})) \to \mathrm{Hg}(H^{k-2}(Z, \mathbb{Z}))$$

are surjective (compare to [46], Lemma 8.1), we obtain the result. □

Corollary 7.1.6. *Let X be a smooth surface and $\tilde{X}$ be the blowing up of some point $p \in X$. Then X has complex multiplication, if and only if $\tilde{X}$ has complex multiplication, too. Moreover we obtain that*

$$\mathrm{Hg}(H^2(\tilde{X}, \mathbb{Z})) \cong \mathrm{Hg}(H^2(X, \mathbb{Z})).$$

Now we want to consider the behavior of the fibers. Hence for simplicity we assume now that $V_1 = V_2 = \mathrm{Spec}(\mathbb{C})$ in diagram (7.1). By the fact that $\tilde{\mathcal{Y}}$ has the Hodge structure given by the Hodge sub-structure of $\widetilde{\mathcal{Z} \times \Sigma}$ invariant under the Galois group, one concludes:

Theorem 7.1.7. *If for all k the groups $\mathrm{Hg}(H^k(\mathcal{Z},\mathbb{Q}))$, $\mathrm{Hg}(H^k(\Sigma,\mathbb{Q}))$ and $\mathrm{Hg}(H^k(Z_i,\mathbb{Q}))$ are commutative,[1] then $\mathrm{Hg}(H^k(\tilde{\mathcal{Y}},\mathbb{Q}))$ is commutative for all k, too.*

Remark 7.1.8. At first sight the condition that for all k the groups $\mathrm{Hg}(H^k(\mathcal{Z},\mathbb{Q}))$, $\mathrm{Hg}(H^k(\Sigma,\mathbb{Q}))$ and $\mathrm{Hg}(H^k(Z_i,\mathbb{Q}))$ have to be commutative may seem to be a little bit restrictive. But by the Hodge diamond of a Calabi-Yau n-manifold with $n \leq 3$ or the Hodge diamond of a Calabi-Yau n-manifold given by a projective hypersurface, one sees that the condition that all its Hodge groups are commutative is equivalent to the condition that it has complex multiplication. Moreover we will need this condition for an inductive construction of families of Calabi-Yau manifolds with dense set of complex multiplication fibers in arbitrary high dimension in the next section.

7.2 The Borcea-Voisin tower

Recall that we want to construct families of Calabi-Yau manifolds with a dense set of CM fibers. Hence let us now define Calabi-Yau manifolds:

Definition 7.2.1. A Calabi-Yau manifold X of dimension n is a compact Kähler manifold of dimension n such that $\Gamma(\Omega_X^i) = 0$ for all $i = 1,\ldots,n-1$ and $\omega_X \cong \mathcal{O}_X$.

By the construction of the preceding section, which we will use, we need more and we get more than only complex multiplication. Hence let us define, which we will get:

Definition 7.2.2. A $CMCY$ family $\mathcal{X} \to \mathcal{B}$ of n-manifolds is a (smooth) family of Calabi-Yau manifolds of dimension n, which has a dense set of fibers $\mathcal{X}_b$ satisfying the property that $\mathrm{Hg}(H^k(\mathcal{X}_b,\mathbb{Q}))$ is commutative for all k.

In this section the degree m of all cyclic covers, which will occur, is equal to 2. We apply the construction of a Calabi-Yau manifold with an involution by two Calabi-Yau manifolds with involutions by C. Borcea [8]. This yields an iterative construction of $CMCY$ families with involutions in arbitrary high dimension by $CMCY$ families in lower dimension.[2]

Construction 7.2.3. Let $\mathcal{Z}_1 \to \mathcal{M}$ be a $CMCY$ family of n-manifolds covering the A bundle $\mathcal{A}$ with ramification locus R_1, which satisfies the assumptions for $\mathcal{Z}$ in diagram (7.1). Moreover let Σ_i be a $CMCY$ family $\Sigma_i \to \mathcal{M}^{(i)}$ of n_i-manifolds covering the B_i bundle $\mathcal{B}_i$ over $\mathcal{M}^{(i)}$ with ramification locus $R^{(i)}$, which satisfies the assumptions for Σ in diagram (7.1), for all $1 < i \in \mathbb{N}$.

Let us assume that there is a dense subset of points $m^{(i)} \in \mathcal{M}^{(i)}$ resp., $p \in \mathcal{M}$, which have the property that each $\mathrm{Hg}(H^k((\Sigma_i)_{m^{(i)}},\mathbb{Q}))$ and each $\mathrm{Hg}(H^k(R^{(i)}_{m^{(i)}},\mathbb{Q}))$ resp., each $\mathrm{Hg}(H^k((\mathcal{Z}_1)_p,\mathbb{Q}))$ and each $\mathrm{Hg}(H^k((R_1)_p,\mathbb{Q}))$ is commutative.

We define an iterative tower of covers

$$\mathcal{Z}_i \to V^{(i)} := \mathcal{M} \times \mathcal{M}^{(2)} \times \ldots \times \mathcal{M}^{(i)}$$

given by

$$\mathcal{Z}_i = \tilde{\mathcal{Y}}_i,$$

[1] One needs in fact the condition that each $\mathrm{Hg}(H^k(Z_i,\mathbb{Q}))$ is commutative. The argument is similar to the argument in the proof of Proposition 10.3.2.

[2] The construction of C. Borcea is repeated in Proposition 7.2.5. By C. Voisin [48], the same construction was used to construct Calabi-Yau 3-manifolds by $K3$-surfaces with involutions and elliptic curves. This is the reason that our construction here is called "Borcea-Voisin tower".

where $\tilde{\mathcal{Y}}_i$ is obtained from $\tilde{\mathcal{Y}}$ in the diagram (7.1) with $V_1 = V^{(i-1)}$, $V_2 = \mathcal{M}^{(i)}$, $\Sigma = \Sigma_i$ and $\mathcal{Z} = \mathcal{Z}_{i-1}$ for all $i \in \mathbb{N}$. Let us call such a construction Borcea-Voisin tower.

The assumption that we have ramification in codimension 1 on the fibers of a family of Calabi-Yau manifolds leads to the important property that the corresponding involutions act by -1 on the global sections of their canonical sheaves, as we see by the following Lemma:

Lemma 7.2.4. *Let C be a Calabi-Yau manifold and ι be an involution on it. Assume that the points fixed by ι are given by a non-trivial (reduced) effective smooth divisor D. Then ι acts by -1 on $H^0(C, \omega_C)$.*

Proof. By our assumptions, the induced natural cyclic cover $\gamma : C \to C/\iota$ is ramified over a smooth non-trivial divisor D such that C/ι is smooth. Hence one has a cyclic cover of manifolds and one can apply the Hurwitz formula (compare [5], **I**. 16). Since C has a trivial canonical divisor, the Hurwitz formula implies that $\mathcal{O}_C(-D) \cong \gamma^*(\omega_{C/\iota})$. This implies that $\omega_{C/\iota}$ does not contain any global section. Since $\omega_{C/\iota}$ yields the eigenspace for the character 1 of $\gamma_*(\omega_C)$ (see [16], §3), the character of the action of ι on $H^0(C, \omega_C)$ is not given by 1. Thus it is given by -1. □

Proposition 7.2.5. *Assume that $\gamma_1 : C_1 \to M_1$ and $\gamma_2 : C_2 \to M_2$ are cyclic covers of degree 2 with the involutions ι_1 and ι_2 and ramification divisors $D_1 \subset C_1$ and $D_2 \subset C_2$, which consist of disjoint smooth hypersurfaces. Moreover assume that C_1 and C_2 are Calabi-Yau manifolds of dimension n_1 and n_2. Let $\widetilde{C_1 \times C_2}$ denote the blowing up of $C_1 \times C_2$ with respect to $D_1 \times D_2$. Then by the involution on $\widetilde{C_1 \times C_2}$ given by (ι_1, ι_2), one obtains a cyclic cover $\gamma : \widetilde{C_1 \times C_2} \to C$ such that C is a Calabi-Yau manifold.*

Proof. We assume that each C_i is a Calabi-Yau manifold such that $h^{t,0}(C_i) = 0$ for all $t = 1, \ldots, n_i - 1$. By the assumption that one has the ramification divisors D_1 and D_2 and Lemma 7.2.4, the corresponding involution of each γ_i acts by -1 on each ω_{C_i}. Thus one concludes that $h^{j,0}(C) = 0$ for all $j = 1, \ldots, (n_1 + n_2) - 1$.

The canonical divisor $K_{\widetilde{C_1 \times C_2}}$ of $\widetilde{C_1 \times C_2}$ is given by the exceptional divisor E of the blowing up $\widetilde{C_1 \times C_2} \to C_1 \times C_2$. Moreover the ramification divisor R of γ coincides with E. Hence by the Hurwitz formula ([5], **I**.16), we have

$$\mathcal{O}_{\widetilde{C_1 \times C_2}}(R) \cong \mathcal{O}_{\widetilde{C_1 \times C_2}}(K_{\widetilde{C_1 \times C_2}}) = \omega_{\widetilde{C_1 \times C_2}} \cong \gamma^*(\omega_C) \otimes \mathcal{O}_{\widetilde{C_1 \times C_2}}(R).$$

Thus one concludes that $\gamma^*(\omega_C) \cong \mathcal{O}$.

Since ι_1 and ι_2 act by the character -1, the involution (ι_1, ι_2) on $\widetilde{C_1 \times C_2}$ leaves the global sections of $\omega_{\widetilde{C_1 \times C_2}}$ invariant. Now recall that $\gamma_*(\omega_{\widetilde{C_1 \times C_2}})$ consists of a direct sum of invertible sheaves, which are the eigenspaces with respect to the characters of the Galois group action. By [16], §3, the eigenspace for the character 1 is given by ω_C. Thus ω_C has a non-trivial global section. Hence the canonical divisor of C satisfies (up to linear equivalence) $K_C \geq 0$. Thus by the fact that $\gamma^*(\omega_C) \cong \mathcal{O}$, we have the desired result $K_C \sim 0$. □

Altogether one has the following result:

Theorem 7.2.6. *Each family $\mathcal{Z}_i \to \mathcal{M} \times \mathcal{M}^{(2)} \times \ldots \times \mathcal{M}^{(i)}$ obtained by the Borcea-Voisin tower is a CMCY family of $n + n_2 + \ldots + n_i$-manifolds.*

Proof. The statement that each $(\mathcal{Z}_i)_p$ is a Calabi-Yau manifold follows fiberwise by induction. By the assumptions, we have the result for $n = 1$. First by induction, one can show that the ramification loci are given by smooth divisors. By using this fact and the induction hypothesis, one can apply Lemma 7.2.4 such that each involution acts by the character -1 on each $\Gamma(\omega)$. Hence the assumptions of Proposition 7.2.5 are satisfied, which provides the induction step.

Next we want to show the statement about the commutativity of all Hodge groups over a dense subset of the basis. Due to the situation described in diagram (7.1) the connected components of the ramification locus $(R_{i+1})_{p\times m^{(i+1)}}$ of $(\mathcal{Z}_{i+1})_{p\times m^{(i+1)}}$ over $p \times m^{(i+1)} \in V^{(i)} \times \mathcal{M}^{(i+1)}$ are given by the connected components of $(\mathcal{Z}_i)_p \times R^{(i+1)}_{m^{(i+1)}}$ and by the connected components of $(R_i)_p \times (\Sigma_{i+1})_j$, where $(R_i)_p$ is the ramification locus of $(\mathcal{Z}_i)_p$.

Hence it is sufficient to use an inductive argument and to show the following Claim: □

Claim 7.2.7. *Assume that for all k the Hodge group $\mathrm{Hg}(H^k((\mathcal{Z}_i)_p, \mathbb{Z}))$ is commutative and each connected component Z of the ramification locus $(R_i)_p$ satisfies that each $\mathrm{Hg}(H^k(Z, \mathbb{Z}))$ is commutative. In addition we assume that for all k the Hodge group $\mathrm{Hg}(H^k(\Sigma_{i+1})_{m^{(i+1)}}, \mathbb{Z}))$ is commutative and each connected component Z_{i+1} of $R^{(i+1)}_{m^{(i+1)}}$ satisfies that each $\mathrm{Hg}(H^k(Z_{i+1}, \mathbb{Z}))$ is commutative. Then for all k each connected component $\tilde{Z}$ of $(R_{i+1})_{p\times m^{(i+1)}}$ satisfies that each $\mathrm{Hg}(H^k(\tilde{Z}, \mathbb{Z}))$ is commutative and for all k $\mathrm{Hg}(H^k((\mathcal{Z}_{i+1})_{p\times m^{(i+1)}}, \mathbb{Z}))$ is commutative.*

Proof. By the assumptions of this claim and the description of R_{i+1} above, one obtains obviously that the connected components $\tilde{Z}$ of $(R_{i+1})_{p\times m^{(i+1)}}$ satisfy that each $\mathrm{Hg}(H^k(\tilde{Z}, \mathbb{Z}))$ is commutative. Then one must simply use Theorem 7.1.7 and one obtains that each $\mathrm{Hg}(H^k(\mathcal{Z}_{i+1})_{p\times m^{(i+1)}}, \mathbb{Z}))$ is commutative, too. □

7.3 The Viehweg-Zuo tower

By the Borcea-Voisin tower, one can construct $CMCY$ families of manifolds in arbitrary high dimension. But one needs $CMCY$ families of manifolds (in low dimension) with a suitable involution, which can be used to be $\mathcal{Z}_1$ or some Σ_i. One way to obtain some suitable $CMCY$ families of n-manifolds (in low dimension) is given by the Viehweg-Zuo tower, which we introduce now.

E. Viehweg and K. Zuo [46] have constructed a tower of projective algebraic manifolds starting with a family $\mathcal{F}_1$ of cyclic covers of $\mathbb{P}^1$ given by

$$\mathbb{P}^2 \ni V(y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2,$$

which has a dense set of CM fibers. This is one example of a family of cyclic covers, which has a primitive pure $(1, 2) - VHS$ as one can easily verify by using Proposition 2.3.4. Since each of these covers given by the fibers of the family can be embedded into $\mathbb{P}^2$, the fibers of $\mathcal{F}_1$ are the branch loci of the fibers of a family $\mathcal{F}_2$ of cyclic covers onto $\mathbb{P}^2$ of degree 5. Moreover the fibers of $\mathcal{F}_2$, which can be embedded into $\mathbb{P}^3$, are the branch loci of the fibers of a family $\mathcal{F}_3$ of cyclic covers onto $\mathbb{P}^3$, which can be embedded into $\mathbb{P}^4$. The family $\mathcal{F}_3$ is given by

$$\mathbb{P}^4 \ni V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2.$$

Thus the fibers of $\mathcal{F}_3$ are Calabi-Yau 3-manifolds. By an inductive argument, this latter family has a dense set of CM points on the basis given by the dense set of the CM points

of the family of curves we have started with (see [46]). Since only the Hodge group of the Hodge structure on $H^3(X,\mathbb{Q})$ of a projective hypersurface $X \subset \mathbb{P}^4$ can be non-trivial, the family $\mathcal{F}_3$ is a $CMCY$ family of 3-manifolds.

Example 7.3.1. We consider the $CMCY$ family $\mathcal{F}_3$

$$\mathbb{P}^4 \ni V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2$$

constructed by E. Viehweg and K. Zuo. On each fiber $(\mathcal{F}_3)_p$ the involution ι given by

$$\iota(y_3 : y_2 : y_1 : x_1 : x_0) = (y_2 : y_3 : y_1 : x_1 : x_0)$$

leaves the smooth divisor $\mathcal{D}_p$ given by the equation $y_3 = y_2$ invariant. Moreover one has that $\mathcal{D}_p \cong (\mathcal{F}_2)_p$. Therefore there is a dense set of points $p \in \mathcal{M}_2$, which have the property that for all k the Hodge groups of $H^k(\mathcal{D}_p, \mathbb{Q})$ and $H^k((\mathcal{F}_3)_p, \mathbb{Q})$ are commutative. Hence one can use $\mathcal{F}_3$ to be $\mathcal{Z}_1$ or some Σ_i for the construction of a Borcea-Voisin tower of $CMCY$ families of n-manifolds.

Example 7.3.2. Let $\mathbb{F}_d$ denote the Fermat curve of degree $d > 2$. The curve $\mathbb{F}_d$ has complex multiplication (see [27] and [19]). By the construction of E. Viehweg and K. Zuo in [46], one concludes that the Calabi-Yau manifold H_d given by

$$V(\sum_{i=0}^{d-1} x_i^d) \in \mathbb{P}^{d-1}$$

has complex multiplication. Since H_d is a projective hypersurface, this implies that H_d has only commutative Hodge groups. We have the involution ι_a given by

$$(x_{d-1} : \ldots : x_2 : x_1 : x_0) \to (x_{d-1} : \ldots ; x_2 : x_0 : x_1)$$

on H_d. If d is even, one has the additional involution ι_b given by

$$(x_{d-1} : \ldots : x_1 : x_0) \to (x_{d-1} : \ldots : x_1 : -x_0).$$

The involution ι_a resp., ι_b (if it is given on H_d) fixes the points of a smooth divisor on H_d, which is isomorphic to

$$V(\sum_{i=0}^{d-2} x_i^d) \in \mathbb{P}^{d-2}.$$

Therefore by the same arguments as in Example 7.3.1, one can use H_d to be $\mathcal{Z}_1$ or some Σ_i with $\mathcal{M} = \mathrm{Spec}(\mathbb{C})$, resp., $\mathcal{M}^{(i)} = \mathrm{Spec}(\mathbb{C})$ for the construction of a Borcea-Voisin tower of $CMCY$ families of n-manifolds.

We want to start the construction of a Viehweg-Zuo tower (of projective hypersurfaces as in [46] or the construction of a modified version) with a family of cyclic covers $\mathcal{C} \to \mathcal{M}_n$ of $\mathbb{P}^1$ with a dense set of CM fibers. For the smoothness of the higher dimensional fibers of the resulting families, we will need the assumption that the fibers of $\mathcal{C}$ are given by

$$V(y^m + x(x-1)(x-a_1)\ldots(x-a_n)) \in \mathbb{A}^2, \tag{7.2}$$

where m divides $n+3$ such that all branch indeces coincide.

By our preceding results, we have only the following examples of families of cyclic covers onto $\mathbb{P}^1$ with a dense set of CM fibers, which satisfy this assumption:

degree m	number of ramification points of the fibers
2	4
2	6
3	6
4	4
5	5

Remark 7.3.3. The case with $m = 2$ and 4 ramification points is the case of elliptic curves, which has been considered by C. Borcea in [7]. The case with $m = 5$ yields the example by E. Viehweg and K. Zuo in [46].

The case with $m = 3$ is one of the examples of a family of covers onto $\mathbb{P}^1$ with a dense set of CM fibers by J. de Jong and R. Noot [25]. We must a bit work to give a suitable modified construction of a Viehweg-Zuo tower for this example. The next chapter is devoted to this modified construction of a Viehweg-Zuo tower.

In the case of the family $\mathcal{C} \to \mathcal{M}_3$ of genus 2 curves the author does not see a possibility for the construction of a Viehweg-Zuo tower.[3]

The case with $m = 4$ yields the Shimura- and Teichmüller curve of M. Möller [33], which provides the example of the next section.

7.4 A new example

Here we see that the Shimura- and Teichmüller curve of M. Möller yields an example of a Viehweg-Zuo tower. Moreover we will see that the resulting $CMCY$ family of 2-manifolds is endowed with some involutions, which make it suitable for the construction of a Borcea-Voisin tower. In addition we try to decide, which involutions provide isomorphic quotients resp., isomorphic $CMCY$ families by the construction of a Borcea-Voisin tower.

Proposition 7.4.1. *The family $\mathcal{C}_2 \to \mathcal{M}_1$ given by*

$$\mathbb{P}^3 \ni V(y_2^4 + y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

is a $CMCY$ family of 2-manifolds.

Proof. It is well-known that a hypersurface of $\mathbb{P}^3$ of degree 4 is a $K3$-surface.

By [46], Notation 2.2, and Corollary 8.5, we have that λ_0 is a CM-point of $\mathcal{C}_2$, if λ_0 is a CM-point of the family $\mathcal{C}_1 \to \mathcal{M}_1$ given by

$$\mathbb{P}^2 \ni V(y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1.$$

Note that $\mathcal{C}_1$ has in fact a dense set of CM fibers, since it has a derived pure $(1,1) - VHS$ as we have seen. Since only the Hodge group of the Hodge structure on $H^2(X, \mathbb{Q})$ can be non-trivial for a $K3$-surface X (follows by definition resp., by the Hodge diamond of a $K3$-surface), the family $\mathcal{C}_2$ is a $CMCY$ family of 2-manifolds. □

As we will see, this family has some involutions, which make it suitable for the construction of a Borcea-Voisin tower. The following lemma is obvious:

[3] One natural choice for an embedding of the fibers of the family of genus 2 curves is given by the weighted projective space $\mathbb{P}(3,1,1)$. But the canonical divisor of the desingularization of $\mathbb{P}(3,1,1)$ does not allow a natural construction of a Viehweg-Zuo tower as in the case of $\mathbb{P}(2,1,1)$, which we will see in the next chapter for the degree 3 case.

Lemma 7.4.2. *Over the basis $\mathcal{M}_1$ the family $\mathcal{C}_2$ has three involutions given by*

$$\iota_1(y_2 : y_1 : x_1 : x_0) = (-y_2 : y_1 : x_1 : x_0), \quad \iota_2(y_2 : y_1 : x_1 : x_0) = (y_2 : -y_1 : x_1 : x_0),$$

$$\iota_3(y_2 : y_1 : x_1 : x_0) = (-y_2 : -y_1 : x_1 : x_0),$$

which constitute with the identity map a subgroup of the $\mathcal{M}_1$-automorphism group of $\mathcal{C}_2$ isomorphic to the Kleinsche Vierergruppe.

Remark 7.4.3. Over $\mathcal{M}_1$ there are at least the 4 following additional involutions on $\mathcal{C}_2$:

$$\iota_4(y_2 : y_1 : x_1 : x_0) = (y_1 : y_2 : x_1 : x_0), \quad \iota_5(y_2 : y_1 : x_1 : x_0) = (iy_1 : -iy_2 : x_1 : x_0),$$

$$\iota_6(y_2 : y_1 : x_1 : x_0) = (-y_1 : -y_2 : x_1 : x_0), \quad \iota_7(y_2 : y_1 : x_1 : x_0) = (-iy_1 : iy_2 : x_1 : x_0)$$

Theorem 7.4.4. *By the involutions ι_1 and ι_4, the family $\mathcal{C}_2$ can be used to be $\mathcal{Z}_1$ or some Σ_i for the construction of a Borcea-Voisin tower of $CMCY$ families of n-manifolds.*

Proof. The divisor of the fiber $(\mathcal{C}_2)_\lambda$, which is fixed by ι_1 resp., ι_4 is given by $y_2 = 0$ resp., $y_2 = y_1$. Hence both divisors smooth and isomorphic to the fiber $(\mathcal{C}_1)_\lambda$ given by

$$\mathbb{P}^2 \ni V(y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1.$$

We use the same arguments as in the proof of Proposition 7.4.1: If $(\mathcal{C}_1)_\lambda$ has complex multiplication, then $(\mathcal{C}_2)_\lambda$ and the divisor fixed by ι_1 resp., ι_4 have complex multiplication, too. Hence by the fact that $\mathcal{C}_1$ has a dense set of complex multiplication fibers, $\mathcal{C}_2$ and ι_1 resp., $\mathcal{C}_2$ and ι_4 satisfy the assumptions of Construction 7.2.3. □

Remark 7.4.5. By the fact that

$$\iota_2 = \iota_4 \circ \iota_1 \circ \iota_4,$$

the involution ι_2 is suitable for the construction of a Borcea-Voisin tower, too. But according to the construction of C. Voisin [48], this implies that ι_2 yields a $CMCY$ family of 3-manifolds over $\mathcal{M}_1 \times \mathcal{M}_1$, which is isomorphic to the corresponding family obtained by ι_1.

Let α denote the $\mathcal{M}_1$-automorphism of $\mathcal{C}_2$ given by

$$(y_2 : y_1 : x_1 : x_0) \to (iy_2 : y_1 : x_1 : x_0).$$

One calculates easily that

$$\iota_5 = \alpha \circ \iota_4 \circ \alpha^{-1}, \quad \iota_6 = \alpha^2 \circ \iota_4 \circ \alpha^{-2}, \quad \iota_7 = \alpha^{-1} \circ \iota_4 \circ \alpha.$$

Hence one has that $\mathcal{C}_2/\iota_4, \ldots, \mathcal{C}_2/\iota_7$ resp., the resulting $CMCY$ families of 3-manifolds obtained by the method of C. Voisin [48] are isomorphic as $\mathcal{M}_1$-schemes resp., as $\mathcal{M}_1 \times \mathcal{M}_1$-schemes.

Since

$$\iota_3 = \iota_1 \iota_2,$$

the involution ι_3 acts by id on each $\Gamma(\omega_{(\mathcal{C}_2)_\lambda})$ such that it can not be used for the construction of a Borcea-Voisin tower.

Remark 7.4.6. The author does not see a way to conjugate ι_1 into ι_4. Moreover we will see that the fibers of the resulting $CMCY$ families of 3-manifolds constructed with ι_1 and ι_4 according to C. Voisin [48] have the same Hodge numbers. This means that the question for isomorphisms between these two families remains open.

Chapter 8

The degree 3 case

8.1 Prelude

We construct a surface R^1 by a desingularization of the weighted projective space $\mathbb{P}(2,1,1)$ during this section. Our modified construction of a Viehweg-Zuo tower starts with the family $\mathcal{C}$ of curves with a dense set of CM fibers given by

$$R^1 \ni V(y^3 = x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)(x_1 - \gamma x_0)x_0) \to (\alpha, \beta, \gamma) \in \mathcal{M}_3. \quad (8.1)$$

We use such a weighted projective space, since the degree of these covers onto $\mathbb{P}^1$ does not coincide with the sum of branch indeces. In this section we construct a rational 3-manifold R^2 with a natural projection onto R^1. For each fiber $\mathcal{C}_q$ this projection induces a cyclic degree 3 cover of a Calabi-Yau hypersurface of R^2 onto R^1 ramified over $\mathcal{C}_q$. We will later see that these Calabi-Yau hypersurfaces of R^2 yield a $CMCY$ family of 2-manifolds suitable for the construction of a Borcea-Voisin tower.

Recall that the usual projective space $\mathbb{P}^n$ is given by $\mathrm{Proj}(\mathbb{C}[z_n, \ldots, z_1, z_0])$, where each z_j (with $j = 0, \ldots, n$) has the weight 1. Our weighted projective space Q^n is given by $\mathrm{Proj}(\mathbb{C}[y_n, \ldots, y_1, x_1, x_0])$, where each y_j (with $j = 1, \ldots, n$) has the weight 2, and x_0 and x_1 have the weight 1.

First we investigate and describe the projective space Q^n. The following well-known Lemma will be very useful here:

Lemma 8.1.1. *(Veronese embedding) Let R be a graded ring. Then we have*

$$\mathrm{Proj}(R) \cong \mathrm{Proj}(R^{[d]}).$$

Proposition 8.1.2. *The weighted projective space Q^n is isomorphic to the irreducible singular hypersurface in $\mathbb{P}^{n+2}$ given by the equation $z_1 z_3 = z_2^2$. The singular locus of Q^n is given by $V(z_1, z_2, z_3)$.*

Proof. By the Veronese embedding, we have

$$Q^n \cong \mathrm{Proj}(k[x_0^2, x_0x_1, x_1^2, y_1, \ldots, y_n]).$$

Hencefore we obtain a closed embedding of Q^n into $\mathbb{P}^{n+2}$ given by

$$x_0^2 \to z_1, \quad x_0x_1 \to z_2, \quad x_1^2 \to z_3, \quad y_1 \to z_4, \quad \ldots, \quad y_n \to z_{n+3}.$$

We have that $Q^n \setminus V(x_0^2)$ is isomorphic to $\mathbb{A}^{n+1}$. Hence $\dim(Q^n) = n+1$, which implies that its projective cone, which is contained in $\mathbb{A}^{n+3}$, has the dimension $n+2$. By [22], **I**. Proposition 1.13, each irreducible component of dimension $n+2$ of this cone is given by an ideal generated by one irreducible polynomial. The corresponding polynomial of the unique irreducible component of Q^n is

$$f(z_1, z_2, z_3) = z_1 z_3 - z_2^2,$$

since each point $p \in Q^n \subset \mathbb{P}^{n+2}$ satisfies $f(p) = 0$ and f is irreducible. The last statement about the singular locus follows from calculating the partial derivatives of f. □

Let $a_1, \ldots, a_{2m} \in \mathbb{C}$, and $m \in \mathbb{N} \setminus \{1\}$. Then $C_{(n)} \subset Q^n$ is the subvariety, which is given by the homogeneous polynomial

$$y_n^m + \ldots + y_1^m + (x_1 - a_1 x_0) \ldots (x_1 - a_{2m} x_0).$$

It is a very easy exercise to check that this polynomial is irreducible.

Proposition 8.1.3. *There exists a homogenous polynomial $G \in \mathbb{C}[z_1, z_2, z_3]$ of degree m such that $C_{(n)} \subset \mathbb{P}^{n+2}$ is given by the ideal generated by h and f, where*

$$h = z_{n+3}^m + \ldots z_4^m + G.$$

Proof. We can obviously choose a polynomial G such that

$$G(x_0^2, x_0 x_1, x_0^2) = (x_1 - a_1 x_0) \ldots (x_1 - a_{2m} x_0).$$

Now let $h = z_{n+3}^m + \ldots z_4^m + G$, and

$$\phi : \mathbb{C}[z_1, \ldots, z_{n+3}] \to \mathbb{C}[x_0^2, x_0 x_1, x_1^2, y_1, \ldots, y_n]$$

be the homomorphism associated to the closed embedding $Q^n \hookrightarrow \mathbb{P}^{n+2}$, which has the kernel (f). We obtain

$$\phi(h) = y_n^m + \ldots + y_1^m + (x_1 - a_1 x_0) \ldots (x_1 - a_{2m} x_0).$$

Hence $C_{(n)} \subset \mathbb{P}^{n+2}$ is given by the prime ideal

$$\phi^{-1}(\mathcal{I}(C_{(n)})) = (h, f).$$

□

Proposition 8.1.4. *The singular locus of $C_{(n)}$ is given by $C_{(n)} \cap V(z_1, z_2, z_3)$.*

Proof. On $Q^n \setminus V(x_0) \cong \operatorname{Spec}(\mathbb{C}[x_1, y_1, \ldots, y_n])$ the hypersurface $C_{(n)}$ is given by the equation

$$0 = y_n^m + \ldots + y_1^m + (x_1 - a_1) \ldots (x_1 - a_{2m}).$$

By the partial derivatives of the polynomial on the right hand, one can easily check that there are no singularities of $C_{(n)}$ in this affine subset. The same arguments give the same statement for $Q^n \setminus V(x_1)$. Hence all singularities of $C_{(n)}$ are contained in $V = V(z_1, z_2, z_3)$. For all $P \in C_{(n)} \cap V$, the Jacobian matrix of $C_{(n)}$ at P does not have the maximal rank 2, where this is obtained by explicit calculation of the partial derivatives of f and h. □

8.1.5. The variety Q^n has a natural interpretation as degree 2 cover onto the variety given by $\{z_2 = 0\}$ ramified over $\{z_1 = z_2 = 0\}$ and $\{z_2 = z_3 = 0\}$. Hence by blowing up $V = V(z_1, z_2, z_3)$, the proper transform $R^n := \tilde{Q}^n_V$ is the natural degree 2 cover onto the proper transform of $\{z_2 = 0\}$ ramified over the disjoint proper transforms of $\{z_1 = z_2 = 0\}$ and $\{z_2 = z_3 = 0\}$. Thus R^n is non-singular.

Note that the general construction of the blowing up yields a natural embedding of an open subset of R^n into $\mathbb{A}^{n+2} \times \mathbb{P}^2$. Hence the Jacobian matrix at each point of R^n has the maximal rank 3 with respect to this local embedding. The Jacobian matrix of the proper transform $\tilde{C}_n$ of $C_{(n)}$ is given by adding the line of the partial derivatives of h to the Jacobian matrix of R^n. Without loss of generality we are on the open subset $\{y_1 = 1\}$. On the exceptional divisor E the polynomial G vanishes. Thus all points of $\tilde{C}_n \cap E$ satisfy

$$y_n^m + \ldots + y_2^n + 1 = 0.$$

Hence for each $p \in \tilde{C}_n \cap E$ there is a partial derivative $\partial h/\partial y_i(P) \neq 0$. Since all partial derivatives of the equations defining R^n with respect to y_i vanish, the Jacobian matrix of $\tilde{C}_n$ has the maximal rank 4 at each point on the exceptional divisor. Thus $\tilde{C}_n$ is smooth.

Remark 8.1.6. Note that Q^1 has a natural interpretation as projective closure of the affine cone of a rational curve of degree 2 in $\mathbb{P}^2$. By [22], **V**. Example 2.11.4, one has that R^1, which is the blowing up of the unique singular point given by the vertex of the cone, is a rational ruled surface isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(2))$, where the exceptional divisor has the self-intersection number -2.

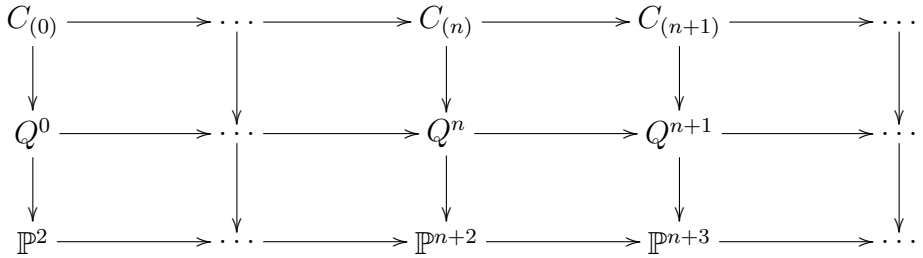
By [22], **II**. Proposition 8.20, one has for $n \geq 1$:

$$\begin{aligned}\omega_{Q^n \setminus V(z_1,z_2,z_3)} &= \omega_{\mathbb{P}^{n+2} \setminus V(z_1,z_2,z_3)} \otimes \mathcal{I}(Q^n \setminus V(z_1, z_2, z_3)) \otimes \mathcal{O}_{Q^n \setminus V(z_1,z_2,z_3)} \\ &= \mathcal{O}_{Q^n \setminus V(z_1,z_2,z_3)}(-(n+1)V(z_4))\end{aligned}$$

By [3], Theorem 2.7 and the fact that the self-intersection number of the exceptional divisor is -2, the pull-back of the canonical divisor of Q^1 with respect to the blowing up morphism is the canonical divisor of R^1. Note that the canonical divisor of Q^1 yields the canonical divisor of $Q^1 \setminus \{s\}$, where s denotes the singular point. Thus:

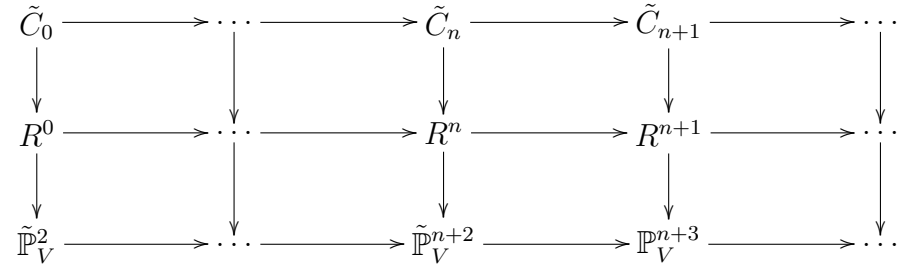
Corollary 8.1.7. *The canonical divisor of R^1 is given by $-2V(z_4)$.*

The following lemma describes the construction of this section. One has the following commutative diagram of closed embeddings:



The ideal sheaf of each blowing up $\tilde{C}_n \to C_{(n)}$ and $R^n \to Q^n$ is generated by z_1, z_2, z_3. Moreover this ideal sheaf is obviously the inverse image ideal sheaf of the ideal sheaf generated by z_1, z_2, z_3 with respect to all embeddings. Hence we obtain by [22], II. Corollary 7.15 for $V := V(z_1, z_2, z_3)$:

Lemma 8.1.8. *We have the commutative diagram*



of closed embeddings.

Remark 8.1.9. Note that $C_{(0)} = \tilde{C}_0$, $C_{(1)} = \tilde{C}_1$ and $Q^0 = R^0$.

Theorem 8.1.10. *The canonical divisor of R^n is given by $-(n+1)\tilde{V}(z_4)$ for $n \geq 1$.*

Proof. By Corollary 8.1.7, we have the statement for $n = 1$.

We use induction for higher n. Let E_n denote exceptional divisor of the blowing up $R^n \to Q^n$. The open subset $R^n \setminus E_n$ is isomorphic to $Q^n \setminus V(z_1, z_2, z_3)$. We know that $-(n+1)\tilde{V}(z_4)$ is the canonical divisor of $Q^n \setminus V(z_1, z_2, z_3)$. Hence we conclude that

$$K_{R^{n+1}} = -(n+2)\tilde{V}(z_4) + zE_{n+1}$$

for some $z \in \mathbb{Z}$. We have that $R^n \sim \tilde{V}(z_4)$ in $Cl(R^{n+1})$. By the induction hypothesis, we have

$$\mathcal{O}_{R^n}(-(n+1)\tilde{V}(z_4)) \cong \omega_{R^n} \cong \mathcal{O}_{R^{n+1}}(\tilde{V}(z_4)) \otimes \omega_{R^{n+1}} \otimes \mathcal{O}_{R^n}$$

such that $z = 0$ and $-(n+2)\tilde{V}(z_4)$ is the canonical divisor of R^{n+1}. □

Since we want to construct a family of Calabi-Yau manifolds, we note:

Theorem 8.1.11. *The hypersurface $\tilde{C}_{m-1} \subset R^{m-1}$ is a Calabi-Yau manifold.*

Proof. By Theorem 8.1.10, $-m\tilde{V}(z_4)$ is the canonical divisor of R^{m-1}. Hence [22], **II**. Proposition 8.20 and $\tilde{C}_{m-1} \sim m\tilde{V}(z_4)$ imply that

$$\omega_{\tilde{C}_{m-1}} = \mathcal{O}_{\tilde{C}_{m-1}}.$$

By the fact that $h^{q,0}$ is a birational invariant of non-singular projective varieties (see [22], page 190), and R^{m-1} is birationally equivalent to $\mathbb{P}^m$, we obtain that $h^{q,0}(R^{m-1}) = 0$ for all $1 \leq q \leq m$. By Hodge symmetry and Serre duality, we obtain that $h^q(R^{m-1}, \mathcal{O}) = 0$ for all $1 \leq q \leq m$ and $h^q(R^{m-1}, \omega) = 0$ for all $0 \leq q \leq m-1$. Since the canonical divisor of R^{m-1} is linearly equivalent to $-\tilde{C}_{m-1}$, we obtain the exact sequence

$$0 \to \omega_{R^{m-1}} \to \mathcal{O}_{R^{m-1}} \to \mathcal{O}_{\tilde{C}_{m-1}} \to 0.$$

This implies that $h^i(\tilde{C}_{m-1}, \mathcal{O}) = 0$ for $1 \leq i < m-1 = \dim(\tilde{C}_{m-1})$. Hence $\tilde{C}_{m-1}$ is a Calabi-Yau manifold. □

8.1.12. The projection $\mathbb{P}^{n+2} \setminus \{(1:0:\ldots:0)\} \to \mathbb{P}^{n+1}$ given by

$$(z_{n+3} : \ldots : z_1) \to (z_{n+2} : \ldots : z_1)$$

induces a cyclic cover $C_{(n+1)} \to Q^n$ of degree m ramified over $C_{(n)}$. The Galois group is generated by

$$(z_{n+3} : z_{n+2} : \ldots : z_1) \to (\xi z_{n+3} : z_{n+2} : \ldots : z_1),$$

where ξ is a primitive m-th. root of unity.

Recall the commutative diagram of Lemma 8.1.8. Let $\mathbb{A}^4$ be given by $\{z_4 = 1\} \subset \mathbb{P}^4$ and $\mathbb{A}^3$ be given by $\{z_4 = 1\} \subset \mathbb{P}^3$. Then the projection above yields a morphism

$$f : \mathbb{A}^4 \times \mathbb{P}^2 \to \mathbb{A}^3 \times \mathbb{P}^2. \tag{8.2}$$

Since the blowing up yields natural embeddings of open subsets of $\tilde{C}_2$ and R^1 into the varieties of (8.2), f induces a rational map $\tilde{C}_2 \to R^1$. Now this rational map $\tilde{C}_2 \to R^1$ is again a cyclic cover of degree m with the Galois group as above (on the open locus of definition). On the complements of the exceptional divisors it coincides with the cyclic cover $C_{(2)} \to Q^1$ above. Hence by glueing, one has a cyclic cover $\tilde{C}_2 \to R^1$ ramified over $C_{(1)}$.

8.2 A modified version of the method of Viehweg and Zuo

The following construction is a modified version of the construction in [46], Section 5. That means here we show that $\tilde{C}_2$ has CM, if $C_{(1)}$ has CM. In the next section we will use the construction of the preceding section to define a family of $K3$-surfaces. In this section we give the argument that this family of $K3$-surfaces will be a $CMCY$ family of 2-manifolds.

For our application, it is sufficient to consider the situation fiberwise and to work with $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ resp., with rational ruled surfaces. Let $\pi_n : \mathbb{P}_n \to \mathbb{P}^1$ denote the rational ruled surface given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ and σ denote a non-trivial global section of $\mathcal{O}_{\mathbb{P}^1}(6)$, which has the six different zero points represented by a point $q \in \mathcal{M}_3$. The sections E_σ, E_0 and E_∞ of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(6))$ are induced by

$$\mathrm{id} \oplus \sigma : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(6), \quad \mathrm{id} \oplus 0 : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(6)$$

$$\text{and} \quad 0 \oplus \mathrm{id} : \mathcal{O}(6) \to \mathcal{O} \oplus \mathcal{O}(6)$$

resp., by the corresponding surjections onto the cokernels of these embeddings as described in [22], **II**. Proposition 7.12.

Remark 8.2.1. The divisors E_σ and E_0 intersect each other transversally over the 6 zero points of σ. Recall that $\mathrm{Pic}(\mathbb{P}_6)$ has a basis given by a fiber and an arbitrary section. Hence by the fact that E_σ and E_0 do not intersect E_∞, one concludes that they are linearly equivalent with self-intersection number 6. Since E_∞ is a section, it intersects each fiber transversally. Thus one has that $E_\infty \sim E_0 - (E_0.E_0)F$, where F denotes a fiber. Hencefore one concludes

$$E_\infty.E_\infty = E_\infty.(E_0 - (E_0.E_0)F) = -(E_0.E_0) = -6.$$

Next we establish a morphism $\mu : \mathbb{P}_2 \to \mathbb{P}_6$ over $\mathbb{P}^1$. By [22], **II**. Proposition 7.12., this is the same as to give a surjection $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(6)) \to \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf

on $\mathbb{P}_2$. By the composition

$$\pi_2^*(\mathcal{O}\oplus\mathcal{O}(6)) = \pi_2^*(\mathcal{O})\oplus\pi_2^*\mathcal{O}(6) \hookrightarrow \bigoplus_{i=0}^{3}\pi_2^*\mathcal{O}(2i) = Sym^3(\pi_2^*(\mathcal{O}\oplus\mathcal{O}(6))) \to \mathcal{O}_{\mathbb{P}_2}(3),$$

where the last morphism is induced by the natural surjection $\pi_2^*(\mathcal{O}\oplus\mathcal{O}(2)) \to \mathcal{O}_{\mathbb{P}_2}(1)$ (see [22], **II**. Proposition 7.11), we obtain a morphism μ^* of sheaves. This morphism μ^* is not a surjection onto $\mathcal{O}_{\mathbb{P}_2}(3)$, but onto its image $\mathcal{L}\subset\mathcal{O}_{\mathbb{P}_2}(3)$. Locally over $\mathbb{A}^1\subset\mathbb{P}^1$ all rational ruled surfaces are given by $\mathrm{Proj}(\mathbb{C}[x])[y_1,y_2]$, where x has the weight 0. Hence we have locally that $\pi_2^*(\mathcal{O}\oplus\mathcal{O}(6)) = \mathcal{O}e_1\oplus\mathcal{O}e_2$. Over $\mathbb{A}^1$ the morphism μ^* is given by

$$e_1 \to y_1^3, e_2 \to y_2^3$$

such that the sheaf $\mathcal{L} = im(\mu^*)\subset\mathcal{O}_{\mathbb{P}_2}(3)$ is invertible. Thus the morphism $\mu:\mathbb{P}_2\to\mathbb{P}_6$ corresponding to μ^* is locally given by the ring homomorphism

$$(\mathbb{C}[x])[y_1,y_2] \to (\mathbb{C}[x])[y_1,y_2] \ \text{ via } \ y_1\to y_1^3 \ \text{ and } \ y_2\to y_2^3.$$

Construction 8.2.2. One has a commutative diagram

$$\begin{array}{ccccc}
\mathcal{Y}' & \xrightarrow{\tau'} & \mathbb{P}_2' & \xrightarrow{\mu'} & \mathbb{P}^1\times\mathbb{P}^1 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta_2} & & \uparrow{\scriptstyle\delta_6} \\
\hat{\mathcal{Y}} & \xrightarrow{\hat\tau} & \hat{\mathbb{P}}_2 & \xrightarrow{\hat\mu} & \hat{\mathbb{P}}_6 \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho_2} & & \downarrow{\scriptstyle\rho_6} \\
\mathcal{Y} & \xrightarrow[\sqrt[3]{\frac{\mu^*E_\sigma}{3\cdot(\mu^*E_0)_{red}}}]{\tau} & \mathbb{P}_2 & \xrightarrow[\sqrt[3]{\frac{E_\infty+6\cdot F}{E_0}}]{\mu} & \mathbb{P}_6 \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi_2} & & \downarrow{\scriptstyle\pi_6} \\
\mathbb{P}^1 & \xrightarrow{\mathrm{id}} & \mathbb{P}^1 & \xrightarrow{\mathrm{id}} & \mathbb{P}^1
\end{array}$$

of morphisms between normal varieties with:

(a) δ, δ_2, δ_6, ρ, ρ_2 and ρ_6 are birational.

(b) π is a family of curves, π_2 and π_d are $\mathbb{P}^1$-bundles.

(c) All the horizontal arrows (except for the ones in the bottom line) are Kummer coverings of degree 3.

Proof. One must only explain δ_6 and ρ_6. Recall that E_σ is a section of $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(6))$, which intersects E_0 transversally in exactly 6 points. The morphism ρ_6 is the blowing up of the six intersection points of $E_0\cap E_\sigma$. The preimage of the six points given by $q\in\mathcal{M}_3$ with respect to $\pi_6\circ\rho_6$ consists of the exceptional divisor $\hat D_1$ and the proper transform $\tilde D_2$ of the preimage of these six points with respect to ρ_6 given by 6 rational curves with self-intersection number -1. The morphism δ_6 is obtained by blowing down $\tilde D_2$. □

Remark 8.2.3. The section σ has the zero divisor given by some $q\in\mathcal{P}_3$. Hence one obtains $\mu^*(E_\sigma)\cong\mathcal{C}_q$, where $\mathcal{C}\to\mathcal{P}_3$ denotes the family of cyclic covers onto $\mathbb{P}^1$ with a pure $(1,3)-VHS$ of degree 3. Since τ is the unique cyclic degree 3 covering of $\mathbb{P}_2\cong R^1$ ramified over $\mu^*(E_\sigma)\cong\mathcal{C}_q$, the surface $\mathcal{Y}$ is isomorphic to some $K3$-surface $\tilde C_2$ of the preceding section.

Recall that $\mathbb{F}_n$ denotes the Fermat curve of degree n.

Proposition 8.2.4. *The surface $\mathcal{Y}$ is birationally equivalent to $\mathcal{C}_q \times \mathbb{F}_3/\langle(1,1)\rangle$.*[1]

Proof. Let $\tilde{E}_\bullet$ denote the proper transform of the section $E_\bullet$ with respect to ρ_6. Then $\hat{\mu}$ is the Kummer covering given by

$$\sqrt[3]{\frac{\tilde{E}_\infty + 6 \cdot F}{\tilde{E}_0 + \hat{D}_1}},$$

where $\hat{D}_1$ denotes the exceptional divisor of ρ_6. Thus the morphism μ' is the Kummer covering

$$\sqrt[3]{\frac{(\delta_6)_*\tilde{E}_\infty + 6 \cdot (\delta_6)_* F}{(\delta_6)_*\tilde{E}_0 + (\delta_6)_*\hat{D}_1}} = \sqrt[3]{\frac{\mathbb{P}^1 \times \{\infty\} + 6 \cdot (P \times \mathbb{P}^1)}{\mathbb{P}^1 \times \{0\} + \Delta \times \mathbb{P}^1}},$$

where Δ is the divisor of the 6 different points in $\mathbb{P}^1$ given by $q \in \mathcal{M}_3$ and $P \in \mathbb{P}^1$ is the point with the fiber F. Since $E_0 + E_\sigma$ is a normal crossing divisor, $\tilde{E}_\sigma$ neither meets $\tilde{E}_0$ nor $\tilde{D}_2$, where $\tilde{D}_2$ is the proper transform of $\pi_6^*(\Delta)$. Therefore $(\delta_6)_*\tilde{E}_\sigma$ neither meets

$$(\delta_6)_*\tilde{E}_0 = \mathbb{P}^1 \times \{0\} \quad \text{nor} \quad (\delta_6)_*\tilde{E}_\infty = \mathbb{P}^1 \times \{\infty\}.$$

Hence one can choose coordinates in $\mathbb{P}^1$ such that $(\delta_6)_*\tilde{E}_\sigma = \mathbb{P}^1 \times \{1\}$.

By the definition of τ, we obtain that $\hat{\tau}$ is given by

$$\sqrt[3]{\frac{\rho_2^*\mu^*(E_\sigma)}{\rho_2^*\mu^*(E_0)}} = \sqrt[3]{\frac{\hat{\mu}^*(\tilde{E}_\sigma)}{\hat{\mu}^*(\tilde{E}_0)}},$$

and τ' is given by

$$\sqrt[3]{\frac{\mu'^*(\mathbb{P}^1 \times \{1\})}{\mu'^*(\mathbb{P}^1 \times \{0\})}}.$$

By the fact that the last function is the third root of the pullback of a function on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to μ', it is possible to reverse the order of the field extensions corresponding to τ' and μ' such that the resulting varieties obtained by Kummer coverings are birationally equivalent. Hence we have the composition of $\beta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\sqrt[3]{\frac{\mathbb{P}^1 \times \{1\}}{\mathbb{P}^1 \times \{0\}}}$$

with

$$\sqrt[3]{\frac{\beta^*(\mathbb{P}^1 \times \{\infty\}) + 6 \cdot (P \times \mathbb{P}^1)}{\beta^*(\mathbb{P}^1 \times \{0\}) + (\Delta \times \mathbb{P}^1)}},$$

which yields the covering variety isomorphic to $\mathbb{F}_3 \times \mathcal{C}_q/\langle(1,1)\rangle$. □

Hence $\tilde{C}_2 \cong \mathcal{Y}$ is birationally equivalent to the algebraic manifold $\hat{\mathcal{Y}}$ in the diagram (7.1) with $\mathcal{Z} = C_{(1)}$ and $\Sigma = \mathbb{F}_3$. Therefore by Corollary 7.1.6, we obtain:

Corollary 8.2.5. *If the curve $\mu^*(E_\sigma)$ has complex multiplication, then the $K3$-surface $\mathcal{Y}$ has only commutative Hodge groups.*

[1] Similarly to [46], Construction 5.2, we show that $\mathcal{Y}'$ is birationally equivalent to $\mathcal{C}_q \times \mathbb{F}_3/\langle(1,1)\rangle$.

8.3 The resulting family and its involutions

8.3.1. Let us summarize the things we have done. By using the Veronese embedding, the weighted projective space $Q^2 = \mathbb{P}_{\mathbb{C}}(2,2,1,1)$ is given by $V(z_1z_3 = z_2^2) \subset \mathbb{P}^4$. Moreover there exists a homogeneous polynomial $G_{(a_1,a_2,a_3)} \in \mathbb{C}[z_1,z_2,z_3]$ of degree 3 such that

$$G_{(a_1,a_2,a_3)}(x^2,x,1) = x(x-1)(x-a_1)(x-a_2)(x-a_3)$$

for each $(a_1,a_2,a_3) \in \mathcal{M}_3$. Let $W \hookrightarrow Q^2 \times \mathcal{M}_3 \overset{pr_2}{\to} \mathcal{M}_3$ be the family with the fibers given by $W_q = V(z_1z_3 - z_2^2,\ \ z_5^3 + z_4^3 + G_q)$ for all $q \in \mathcal{M}_3$. Moreover let $\mathcal{W} \hookrightarrow R^2 \times \mathcal{M}_3 \to \mathcal{M}_3$ be the smooth family obtained by the proper transform of W with respect to the blowing up of $V(z_1,z_2,z_3) \times \mathcal{M}_3$. Since the family $\mathcal{C} \to \mathcal{M}_3$ given by

$$R^1 \ni V(y^3 - x_1(x_1-x_0)(x_1-a_1x_0)(x_1-a_2x_0)(x_1-a_3x_0)x_0) \to (a_1,a_2,a_3) \in \mathcal{M}_3$$

has dense set of complex multiplication fibers, Corollary 8.2.5 implies that $\mathcal{W}$ is a $CMCY$ family of 2-manifolds.

Next we will find and study involutions on $\mathcal{W}$ over $\mathcal{M}_3$ satisfying the assumptions for the construction of a Borcea-Voisin tower.

Remark 8.3.2. We have the involutions on W over $\mathcal{M}_3$ given by

$$\begin{aligned}\gamma^{(1)}(z_5:z_4:z_3:z_2:z_1) &= (z_4:z_5:z_3:z_2:z_1),\\ \gamma^{(2)}(z_5:z_4:z_3:z_2:z_1) &= (\xi z_4:\xi^2 z_5:z_3:z_2:z_1),\\ \gamma^{(3)}(z_5:z_4:z_3:z_2:z_1) &= (\xi^2 z_4:\xi z_5:z_3:z_2:z_1),\end{aligned}$$

where ξ is a fixed primitive cubic root of unity. For simplicity we write γ instead of $\gamma^{(1)}$, too. Since the ideal sheaf of $V(z_1,z_2,z_3) \cap W$ coincides with its inverse image ideal sheaf with respect to $\gamma^{(i)}$ (for all $i = 1,2,3$), each $\gamma^{(i)}$ induces an involution on $\mathcal{W}$ over the basis $\mathcal{M}_3$ denoted by $\gamma^{(i)}$, too.

Remark 8.3.3. We have the $\mathcal{M}_3$-automorphism κ of W given by

$$\kappa(z_5:z_4:z_3:z_2:z_1) = (\xi z_5:z_4:z_3:z_2:z_1) \text{ with}$$

$$\kappa^{-1}(z_5:z_4:z_3:z_2:z_1) = (\xi^2 z_5:z_4:z_3:z_2:z_1)$$

such that by the same argument as in Remark 8.3.2, we obtain an automorphism of $\mathcal{W}$ over $\mathcal{M}_3$ denoted by κ, too. On W and hencefore on $\mathcal{W}$ one has

$$\gamma^{(2)} = \kappa \circ \gamma \circ \kappa^{-1} \ \text{ and } \ \gamma^{(3)} = \kappa^{-1} \circ \gamma \circ \kappa.$$

Hence these involutions act by the same character on the global differential forms of the fibers of $\mathcal{W}$, and all quotients $\mathcal{W}/\gamma^{(i)}$ are isomorphic. Therefore it is sufficient to consider the quotient by γ.

Proposition 8.3.4. *On each fiber of $\mathcal{W}$ the involution γ fixes exactly the points on the divisor given by $V(z_4 = z_5)$ and one exceptional line over one singular point of the corresponding fiber of W.*

Proof. Let $q \in \mathcal{M}_3$ and let S denote the singular locus of W_q. On $W_q \setminus S$ the points fixed by γ are given by the divisor $V(z_4 = z_5)$. Now let us consider the exceptional divisors of the blowing up, which turns W into the family $\mathcal{W}$ of smooth $K3$-surfaces. There are exactly 3 points of S given by $z_1 = z_2 = z_3 = 0$ and $z_4^3 + z_5^3 = 0$. The involution γ fixes $(1 : -1 : 0 : 0 : 0)$ and interchanges the other two singular points. Since the generators of the ideal of the blowing up are invariant under γ, one concludes that each point on the exceptional line over $(1 : -1 : 0 : 0 : 0)$ is fixed by γ. □

Since the divisor on W_q given by $V(z_4 = z_5)$ is isomorphic to $\mathcal{C}_q$ and the projective line providing the fixed exceptional divisor has CM, one has by Corollary 8.2.5:

Theorem 8.3.5. *By the involution γ, the family $\mathcal{W}$ can be used to be some $\mathcal{Z}_1$ or Σ_i in the construction of a Borcea-Voisin tower.*

Chapter 9

Other examples and variations

In this chapter we consider the automorphism groups of our examples of $CMCY$ families. We want to find some new examples of $CMCY$ families of n-manifolds by quotients by cyclic subgroups of these automorphism groups. By using [16], Lemma 3.16, d), one can easily determine the character of the action of these cyclic groups on the global sections of the canonical sheaves of the fibers. In this chapter we state this character with respect to the pull-back action.

9.1 The degree 3 case

Let ξ denote a fixed primitive cubic root of unity. In 8.3.1 we have constructed the $CMCY$ family $\mathcal{W} \to \mathcal{M}_3$ given by

$$R^2 := \tilde{\mathbb{P}}_{\mathbb{C}}(2,2,1,1) \ni \tilde{V}(y_2^3 + y_1^3 + x_1(x_1-1)(x_1-a_1x_0)(x_1-a_2x_0)(x_1-a_3x_0)x_0)$$
$$\to (a_1, a_2, a_3) \in \mathcal{M}_3.$$

First we introduce an $\mathcal{M}_3$-automorphism group $\mathbb{G}_3$ of the family $\mathcal{W}$. The elements $g \in \mathbb{G}_3$ can be uniquely written as a product $g = abc$ with $a \in \langle\alpha\rangle$, $b \in \langle\beta\rangle$ and $c \in \langle\gamma\rangle$, where:

$$\alpha(z_5 : z_4 : z_3 : z_2 : z_1) = (\xi z_5 : z_4 : z_3 : z_2 : z_1),$$
$$\beta(z_5 : z_4 : z_3 : z_2 : z_1) = (z_5 : \xi z_4 : z_3 : z_2 : z_1),$$
$$\gamma(z_5 : z_4 : z_3 : z_2 : z_1) = (z_4 : z_5 : z_3 : z_2 : z_1)$$

The group $\mathbb{G}_3$ contains exactly 18 elements. The action of $\mathbb{G}_3$ on the global sections of the canonical sheaves of the fibers induces a surjection of $\mathbb{G}_3$ onto the multiplicative group of the 6-th. roots of unity. Its kernel is the cyclic group of order 3 generated by $\alpha\beta^{-1}$.

Remark 9.1.1. Since $\alpha\beta^{-1}$ is an $\mathcal{M}_3$-automorphism, one obtains the quotient family $\mathcal{W}/\langle\alpha\beta^{-1}\rangle \to \mathcal{M}_3$. One checks easily that $\alpha\beta^{-1}$ leaves exactly the sections given by $z_5 = z_4 = 0$ invariant. Let $q \in \mathcal{M}_3$. The fiber $(\mathcal{W}/\langle\alpha\beta^{-1}\rangle)_q$ of $\mathcal{W}/\langle\alpha\beta^{-1}\rangle$ has quotient singularities of the type $A_{3,2}$ (see [5], **III.** Proposition 5.3). We blow up the sections of fixed points on $\mathcal{W}$ and call the resulting exceptional divisor E_1. On each connected component of E_1 one has two disjoint sections of fixed points again. But on a fiber the quotient map sends any fixed point onto a singularity of the type $A_{3,1}$.[1] Hence let us blow

[1]For this description consider the corresponding action of the cyclic group on an analytic open neighborhood of a fixed point.

up these latter sections of fixed points with exceptional divisor E_2. The canonical divisor of the resulting fibers $\tilde{\mathcal{W}}_q$ is given by

$$K_{\tilde{\mathcal{W}}_q} = (\tilde{E}_1)_q + 2(E_2)_q,$$

where quotient map φ induced by $\alpha\beta^{-1}$ has ramification on E_2. Thus by the Hurwitz formula, one calculates that $\varphi^*(\omega_q) = \mathcal{O}((\tilde{E}_1)_q)$. Note that the irreducible components of the exceptional curve $(E_1)_q$ have selfintersection-number -1. Since $(E_2)_q$ is the exceptional divisor of the blowing up of two points of each irreducible component of $(E_1)_q$, each irreducible component of $(\tilde{E}_1)_q$ has selfintersection-number -3. By the fact that the quotient map $\varphi : \tilde{\mathcal{W}}_q \to (\tilde{\mathcal{W}}/\langle\alpha\beta^{-1}\rangle)_q$ is not ramified over $\varphi((\tilde{E}_1)_q)$, the irreducible components of $\varphi((\tilde{E}_1)_q)$ have selfintersection-number -1.

From now on let $\mathcal{X} := \tilde{\mathcal{W}}/\langle\alpha\beta^{-1}\rangle$.

Proposition 9.1.2. *One can blow down $\varphi(\tilde{E}_1)$ such that the blowing down morphism $\phi : \mathcal{X} \to \mathcal{Y}$ yields a CMCY family $\mathcal{Y} \to \mathcal{M}_3$ of 2-manifolds.*

Proof. By the construction of the projective family, one has an invertible relatively very ample sheaf $\mathcal{A} := \mathcal{O}_{\mathcal{X}}(D)$ on $\mathcal{X}$. Let P denote some connected component of $\varphi(\tilde{E}_1)$. Note that $\varphi(\tilde{E}_1)$ consists of different copies of $\mathbb{P}^1_{\mathrm{Spec}(R)}$ with $\mathrm{Spec}(R) = \mathcal{P}_n$ such that each invertible sheaf on P is uniquely determined by its degree. Thus the intersection number $\mu_P := D_q.P_q$ is independent of $q \in \mathcal{P}_n$. As in the proof of the Castelnuovo Theorem in [22], **V**. Theorem 5.7 the invertible sheaf

$$\mathcal{L} := \mathcal{A}(\sum_{P \subset \varphi(\tilde{E}_1)} \mu_P P)$$

yields the blowing down morphism on the fibers. Since this $\mathcal{P}_n$-morphism is globally defined, one obtains a global blowing down morphism f such that the resulting family $\mathcal{Y} = f(\mathcal{X})$ is smooth.

By the fact that $\alpha\beta$ acts by the character 1 on $\Gamma(\omega_{\mathcal{W}_q})$, one concludes easily that $\mathcal{Y} \to \mathcal{M}_3$ is a family of $K3$ surfaces. Since $\mathcal{W}$ has a dense set of CM fibers, one concludes that $\mathcal{X} = \tilde{\mathcal{W}}/\langle\alpha\beta\rangle$ and $\mathcal{Y}$ have dense sets of CM fibers, too. □

By the blowing down of $\varphi(\tilde{E}_1)$, we get the following situation:

$$\begin{array}{ccccc}
\tilde{E}_1 \cup E_2 & \xrightarrow{\varphi} & \varphi(\tilde{E}_1 \cup E_2) & \xrightarrow{\phi} & \phi \circ \varphi(E_2) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{\mathcal{W}} & \xrightarrow[\mathrm{mod}\langle\alpha\beta^2\rangle]{\varphi} & \mathcal{X} & \xrightarrow[\mathrm{Bl}(\varphi(\tilde{E}_1))]{\phi} & \mathcal{Y}
\end{array}$$

Proposition 9.1.3. *The $\mathcal{M}_3$-automorphism γ of $\mathcal{W}$ yields an involution on $\mathcal{Y}$, which makes it suitable for the construction of a Borcea-Voisin tower.*

Proof. One has the following commutative diagram:

$$\begin{array}{ccc}
\tilde{\mathcal{W}} & \xrightarrow{\alpha\beta^{-1}} & \tilde{\mathcal{W}} \\
{\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\
\tilde{\mathcal{W}} & \xrightarrow{\alpha^{-1}\beta} & \tilde{\mathcal{W}}
\end{array}$$

Thus γ yields an involution on $\mathcal{X} = \tilde{\mathcal{W}}/\langle\alpha\beta^{-1}\rangle$. By the fact that $\gamma(E_1) = E_1$, it induces an involution on the complement of the sections of $\mathcal{Y}$ obtained by blowing down $\varphi(\tilde{E}_1)$. Since these sections have codimension 2, the involution extends to a holomorphic involution on $\mathcal{Y}$ (by Hartogs' Extension Theorem [49], Theorem 1.25). By the fact that γ acts by -1 on $\Gamma(\omega_{\mathcal{W}_q})$, the same holds true for $\mathcal{X}_q$ and $\mathcal{Y}_q$.

Let $\mathcal{C} \to \mathcal{M}_3$ denote the family of degree 3 covers with a pure $(1,3) - VHS$. We have seen that $\mathcal{W}_q$ has CM, if $\mathcal{C}_q$ has CM. Hencefore $H^k(\mathcal{Y}_q, \mathbb{Q})$ has a commutative Hodge group for all k, if $\mathcal{C}_q$ has CM. Thus the following point describes the ramification divisor of γ_q on $\mathcal{Y}_q$ and ensures that there is a dense set of CM fibers $\mathcal{Y}_q$ such that the ramification divisor of γ_q has CM, too. □

9.1.4. Now we describe the divisor of points of $\mathcal{Y}_q$ fixed by γ for some $q \in \mathcal{M}_3$. Each point of $\mathcal{Y}_q \setminus (\phi \circ \varphi(E_2))$ can be given by the image $[p]$ of a point $p \in \mathcal{W}_q$ with respect to the quotient map according to $\langle\alpha\beta^{-1}\rangle$. One has that a point $[p] \in \mathcal{Y}_q \setminus (\phi \circ \varphi(E_2))$ is fixed by γ, if and only if $\gamma(p) \in \langle\alpha\beta^2\rangle \cdot p$. These points $p \in \mathcal{W}_q$ are exactly given by $\langle\alpha\beta^2\rangle \cdot V(y_2 = y_1)$ and the exceptional divisor of $\mathcal{W}_q \to W_q$.

By the fact that $\langle\alpha\beta^2\rangle \cdot V(y_2 = y_1)$ interchanges all 3 irreducible components of $\langle\alpha\beta^2\rangle \cdot V(y_2 = y_1)$ and all 3 irreducible components of the exceptional divisor of $\mathcal{W} \to W$, one obtains a divisor of fixed points on $\mathcal{Y}_q$ given by $\mathcal{C}_q$ and one copy of $\mathbb{P}^1$. Since γ is given by $(y_2 : y_1) \to (y_1 : y_2)$ on E_1 and $\alpha\beta^2$ is given by $(y_2 : y_1) \to (y_2 : \xi y_1)$ on E_1, γ interchanges each two irreducible components of E_2, which intersect the same irreducible component of $\tilde{E}_1$. Thus the ramification divisor of $\mathcal{Y} \to \mathcal{Y}/\gamma$ given by a family of rational curves and $\mathcal{C}$, where $\mathcal{C}$ denotes the example of a family of degree 3 covers with a pure $(1,3) - VHS$.

9.2 Calabi-Yau 3-manifolds obtained by quotients of degree 3

We have seen that the family $\mathcal{W}$ of $K3$-surfaces given by

$$R^2 := \tilde{\mathbb{P}}_{\mathbb{C}}(2,2,1,1) \ni \tilde{V}(y_2^3 + y_1^3 + x_1(x_1-1)(x_1 - a_1x_0)(x_1 - a_2x_0)(x_1 - a_3x_0)x_0)$$

$$\to (a_1, a_2, a_3) \in \mathcal{M}_3$$

has a dense set of fibers $\mathcal{W}_q$ such that $H^k(\mathcal{W}_q, \mathbb{Q})$ has a commutative Hodge group for all k.

Recall that the canonical divisor of $R^1 \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ is given by $-2\tilde{V}(z_4)$. Now we consider the up to isomorphisms unique cyclic cover of degree 3 given by $\mathcal{W}_q \to R^1$ ramified over $\mathcal{C}_q$, whose Galois group is generated by α. Moreover consider the cyclic degree 3 cover $\mathbb{F}_3 \to \mathbb{P}^1$, where $\mathbb{F}_3 = V(x^3 + y^3 + z^3) \subset \mathbb{P}^2$ denotes the Fermat curve of degree 3 and $\alpha_{\mathbb{F}_3}$ given by

$$(x : y : z) = (x : y : \xi z),$$

is a generator of the Galois group, which acts by the character ξ on $\Gamma(\omega_{\mathbb{F}_3})$.

Let X be a singular variety of dimension n such that each irreducible component of its singular locus S has at least the codimension 2. Then we call X a singular Calabi-Yau n-manifold, if $h^0(X \setminus S, \Omega^k_{X\setminus S}) = 0$ for all $k = 1, \ldots, n-1$ and $\omega_{X\setminus S} \cong \mathcal{O}_{X\setminus S}$. With the notation of diagram (7.1) one gets:

Proposition 9.2.1. *The quotient of $\mathcal{W}\times\mathbb{F}_3$ by $\langle(1,2)\rangle$ yields a family of singular Calabi-Yau 3-manifolds with a dense set of CM fibers.*

Proof. Note that the VHS of the family $\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle$ is the sub-VHS fixed by $\langle(1,2)\rangle$.[2] Since $\mathbb{F}_3$ has complex multiplication, a CM fiber of $\mathcal{W}$ yields a corresponding CM fiber of $\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle$.

Let φ denote the quotient map

$$\varphi : \mathcal{W}\times\mathbb{F}_3 \to \mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle$$

and S denote the singular locus of $\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle$. Over each point, which lies not in the singular locus given by 3 copies of $\mathcal{C}$, one does not have ramification. Hence by the Hurwitz formula, $\varphi^*(\omega_{(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)\setminus S})$ is given by the structure sheaf. Since $\langle(1,2)\rangle$ acts on $\Gamma(\omega_{\mathcal{W}\times\mathbb{F}_3})$ by the character 1, the sheaf $\omega_{(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)\setminus S}$ has global sections. Hence

$$\omega_{(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)\setminus S} = \mathcal{O}_{(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)\setminus S}.$$

In addition the reader checks easily that $\langle(1,2)\rangle$ does not act by the character 1 on a non-trivial sub-vector space of $H^{1,0}(\mathcal{W}\times\mathbb{F}_3)$ or $H^{2,0}(\mathcal{W}\times\mathbb{F}_3)$. Thus $\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle$ is a family of singular Calabi-Yau 3-manifolds. □

9.2.2. Now consider a fiber $(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)_q$ of $\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle$ and its singularities in the complex analytic setting. For the construction of the blowing up of a complex submanifold we refer to [49], 3.3.3. As in [49], 3.3.3 described, one constructs first the blowing up over open sets. The global blowing up is given by glueing the local blowing ups. Here were consider the situation on sufficiently small complex open submanifolds.

The $\mathcal{M}_3$-automorphism α acts on y_2 by ξ. On each fiber $\mathcal{W}_q$ the curve $\mathcal{C}_q$ defines the ramification locus of $\mathcal{W}_q \to R^2$, which is fixed by α. A local parameter $p_{\mathcal{C}_q}$ on $\mathcal{C}_q$ yields a local parameter on $\mathcal{W}_q$ fixed by α. By z, one has a local parameter for the neighborhoods of the ramification points of $\mathbb{F}_3$. On a small open subset, which intersects the ramification locus of

$$\varphi_q : (\mathcal{W}\times\mathbb{F}_3)_q \to (\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)_q,$$

one has the three local parameters given by y_2, $p_{\mathcal{C}_q}$ and z. By the action of $\langle(1,2)\rangle$ on these three local parameters, the singularities $\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle$ are locally given by the product of the 1-ball $\mathbb{B}_1$ with a surface, which has a singularity of the type $A_{3,2}$ (with the notation in [5], **III**. Subsection 5). Let us blow up the family of fixed curves on $\mathcal{W}\times\mathbb{F}_3$ with respect to $\langle(1,2)\rangle$ and let E_1 denote the exceptional divisor. On each connected component of E_1 one has two disjoint families of fixed curves with respect to the action of $\langle(1,2)\rangle$ again. Again this follows from the consideration of the action of $\langle(1,2)\rangle$ on local parameters of a small open subset. On a fiber the quotient map sends any neighborhood of a point on these latter curves onto the product of the 1-ball $\mathbb{B}_1$ with a surface with a singularity of the type $A_{3,1}$. Hence let us blow up these latter two families of curves with exceptional divisor E_2. The canonical divisor of the resulting fibers $(\widetilde{\mathcal{W}\times\mathbb{F}_3})_q$ is given by

$$K_{(\widetilde{\mathcal{W}\times\mathbb{F}_3})_q} = (\tilde{E}_1)_q + 2(E_2)_q,$$

[2]For a short introduction to such orbifolds and their Hodge theory see [11], Appendix A.3.

where quotient map φ by $\langle(1,2)\rangle$ is ramified over E_2. Thus by the Hurwitz formula, one calculates that $\varphi^*(\omega_q) = \mathcal{O}((\tilde{E}_1)_q)$.[3]

On the other hand $\alpha\beta$ acts by the character ξ^2 on $\Gamma(\omega_{\mathcal{W}_q})$ for all $q \in \mathcal{M}_3$. Moreover we have a Galois cover $\mathbb{F}_3 \to \mathbb{P}^1$ of degree 3 with a generator $\alpha_{\mathbb{F}_3}$ given by

$$(x : y : z) = (x : y : \xi z),$$

which acts by the character ξ on $\gamma(\omega_{\mathbb{F}_3})$. Hence $\alpha_2 := (\alpha\beta, \alpha_{\mathbb{F}_3})$ leaves $\Gamma(\omega_{\mathcal{W}_q \times \mathbb{F}_3})$ invariant.

The automorphism α_2 fixes a finite number of points on $\mathcal{W}_q \times \mathbb{F}_3$ given by

$$\{z_5 = z_4 = 0\} \times \{z = 0\},$$

and α_2 fixes in addition the points on the curves given by the fiber product of $\{z = 0\}$ with the exceptional divisor of the blowing up $\mathcal{W}_q \to W_q$. The latter statement about the exceptional divisor of $\mathcal{W}_q \to W_q$ follows from the fact that $\alpha\beta$ fixes the generators of the corresponding ideal sheaf of the blowing up and the singular points of W_q given by

$$(1 : -1 : 0 : 0 : 0), \quad (1 : -\xi : 0 : 0 : 0) \text{ and } (1 : -\xi^2 : 0 : 0 : 0).$$

9.2.3. Now we determine the action of $\alpha\beta$ on the local parameters, whose zero-loci are given by the exceptional divisor $E_{\mathcal{W}_q}$ of $\mathcal{W}_q \to W_q$. The action of $\alpha\beta$ on $W_q \subset \mathbb{P}^4$ is given by

$$(z_5 : z_4 : z_3 : z_2 : z_1) \to (\xi z_5 : \xi z_4 : z_3 : z_2 : z_1) \text{ resp.,}$$
$$(z_5 : z_4 : z_3 : z_2 : z_1) \to (z_5 : z_4 : \xi^{-1} z_3 : \xi^{-1} z_2 : \xi^{-1} z_1).$$

By using the explicit equations for W_q in 8.3.1, one can very easily calculate that $\alpha\beta$ acts by ξ^{-1} on these local parameters.[4]

Hence the singularities of $\mathcal{W}_q \times \mathbb{F}_3/\langle\alpha_2\rangle$, which result by the exceptional divisor of $\mathcal{W}_q \to W_q$, are locally given by the product of $\mathbb{B}_1$ with a singularity of the type $A_{3,2}$.

Now we construct a desingularisation of $\mathcal{W} \times \mathbb{F}_3/\langle\alpha_2\rangle$, which is a $CMCY$ family of 3-manifolds. Let $E_{\mathcal{W}}$ denote the exceptional divisor of $\mathcal{W} \to W$. We start with the blowing up of the family of rational curves given by the fiberproduct of $E_{\mathcal{W}}$ with the points on $\mathbb{F}_3$ fixed by $\alpha_{\mathbb{F}_3}$. This yields the exceptional divisor E_C consisting of 9 rational ruled surfaces. By the same arguments as in 9.2.2, each connected component of E_C contains two families of rational curves of fixed points. The blowing up $\widetilde{\mathcal{W} \times \mathbb{F}_3}$ of these latter families has a quotient

$$\mathcal{R} := \widetilde{\mathcal{W} \times \mathbb{F}_3}/\alpha_2$$

with quotient map given by φ such that on the complement of the isolated sections fixed by φ

$$\varphi^* \omega_q = \mathcal{O}((\tilde{E}_C)_q).$$

[3]The author has searched for an opportunity of a smooth blowing down similar to Proposition 9.1.2. He considered a fiber $\mathcal{W}_q$, which is a family of curves given by

$$\mathcal{W}_q \to R^2 \to \mathbb{P}^1.$$

But here we do not blow up sections of $\mathcal{W}_q \times \mathbb{F}_3 \to \mathbb{P}^1$. Hence here one can not formulate a relative version of the Castelnuovo Theorem as in Proposition 9.1.2.

[4]The singular locus of W_q is contained in $W_q \cap \{z_5 = 1\}$. Thus one can calculate the desingularization with the usual equations $z_i t_j = z_j t_i$ for $i, j = 1, 2, 3$. On $\{t_i = 1\}$ the zero locus of the local parameter z_i yields the exceptional divisor. The local parameter fixed by $\alpha\beta$ can be given by t_1/t_i or t_3/t_i.

9.2.4. Recall that R^1 is a rational ruled surface, where the exceptional divisor E_{R^1} of the blowing up $R^1 \to Q^1$ is a section of $R^1 \to \mathbb{P}^1$ (see Remark 8.1.6). A fiber $\mathcal{W}_q$ can be considered as a family

$$\mathcal{W}_q \xrightarrow{f} R^1 \to \mathbb{P}^1$$

of curves, where f is constructed in 8.1.12. By 8.3.1 and the projection $R^2 \to R^1$, the morphism f extends to a morphism $f : \mathcal{W} \to R^1 \times \mathcal{M}_3$ such that the exceptional divisor $E_{\mathcal{W}}$ of the blowing up $\mathcal{W} \to W$ is send to the exceptional divisor $E_{R^1 \times \mathcal{M}_3} = E_{R^1} \times \mathcal{M}_3$ of the blowing up $R^1 \times \mathcal{M}_3 \to Q^1 \times \mathcal{M}_3$. The following commutative diagram describes the situation:

$$\begin{array}{ccccc}
E_{\mathcal{W}} & \longrightarrow & \mathcal{W} & \longrightarrow & W \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
E_{R^1} \times \mathcal{M}_3 & \longrightarrow & R^1 \times \mathcal{M}_3 & \longrightarrow & Q^1 \times \mathcal{M}_3 \\
\downarrow & & \downarrow & & \\
\mathbb{P}^1 \times \mathcal{M}_3 & \xrightarrow{\mathrm{id}} & \mathbb{P}^1 \times \mathcal{M}_3 & &
\end{array}$$

9.2.5. Thus

$$g : \mathcal{W} \xrightarrow{f} R^1 \times \mathcal{M}_3 \to \mathbb{P}^1 \times \mathcal{M}_3$$

is a family of curves, which has 3 distinguished sections given by the exceptional divisor $E_{\mathcal{W}}$ of $\mathcal{W} \to W$. Moreover by the description of $f : \mathcal{W}_q \to R^1$ as degree 3 cover, one can easily see that the fibers of g are given by the Fermat curve of degree 3 or consist of 3 smooth rational curves intersecting each other in exactly one point, which does not lie on $(E_{\mathcal{W}})_q$. Over $\mathbb{P}^1 \setminus \{\infty\} \times \mathcal{M}_3$ and $\mathbb{P}^1 \setminus \{0\} \times \mathcal{M}_3$ one can embed the restricted family into some copy of $\mathbb{P}^2_{\mathbb{A}^1 \times \mathcal{M}_3}$.

Hencefore we obtain the family

$$\mathcal{W} \times \mathbb{F}_3 \to \mathbb{P}^1 \times \mathcal{M}_3$$

of surfaces, which has sections given by the fiberproduct of the exceptional divisors of $\mathcal{W} \to W$ with the points fixed by $\alpha_{\mathbb{F}_3}$, which do not meet any singular point of a fiber. In addition α_2 is a $\mathbb{P}^1 \times \mathcal{M}_3$-automorphism of this family. Hence by the same arguments as in the proof of Proposition 9.1.2, we can blow down $\varphi(\tilde{E}_C)$ over $(\mathbb{P}^1 \setminus \{\infty\}) \times \mathcal{M}_3$ and $(\mathbb{P}^1 \setminus \{0\}) \times \mathcal{M}_3$. By glueing, we obtain the family $\hat{\mathcal{Q}}$. Note that the singular fibers of $\mathcal{W} \times \mathbb{F}_3 \to \mathbb{P}^1 \times \mathcal{M}_3$ are given by 3 copies of $\mathbb{P}^1 \times \mathbb{F}_3$. Hence by the restriction of the sheaf, which yields the blowing down morphism, to the corresponding copies of $\widetilde{\mathbb{P}^1 \times \mathbb{F}_3}/\langle \alpha_2 \rangle$, one obtains smooth blowing down morphisms on these copies.

Construction 9.2.6. But $\hat{\mathcal{Q}}$ has 18 sections of singular points given by the 18 isolated sections fixed by α_2 on $\widetilde{\mathcal{W} \times \mathbb{F}_3}$. Recall that these sections are given by

$$\{z_5 = z_4 = 0\} \times \{z = 0\}.$$

Let $\mathcal{Q} \to \hat{\mathcal{Q}}$ denote the blowing up of the singular sections of $\hat{\mathcal{Q}}$ and

$$\widetilde{\widetilde{\mathcal{W} \times \mathbb{F}_3}} \to \widetilde{\mathcal{W} \times \mathbb{F}_3}$$

denote the blowing up of these 18 sections. By the same arguments as in Remark 7.1.2, we obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\widetilde{\mathcal{W}}} & \xrightarrow{\tilde{\varphi}} & \mathcal{Q} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{W}} & \xrightarrow{\varphi} & \hat{\mathcal{Q}} \end{array}$$

Note that $\tilde{\varphi}$ is a cyclic cover on the complement of $\tilde{E}_C$. Thus by the Hurwitz formula and the fact that α_2 acts by the character 1 on $\Gamma(\omega_{\mathcal{W}_q \times \mathbb{F}_3})$ for each $q \in \mathcal{M}_3$, one concludes that $\mathcal{Q}$ is a family of Calabi-Yau 3-manifolds.

Proposition 9.2.7. *The family $\mathcal{Q} \to \mathcal{M}_3$ is a CMCY family of 3-manifolds.*

Proof. Note that on each fiber we blow up some points and several copies of $\mathbb{P}^1$, which have CM. Hence by Theorem 7.1.7, we must only apply the facts that $\mathbb{F}_3$ has CM and $\mathcal{W}$ has a dense set of fibers $\mathcal{W}_q$ such that $\mathrm{Hg}(H^k(\mathcal{W}_q, \mathbb{Q}))$ is commutative for all k. □

9.3 The degree 4 case

Consider the $CMCY$ family $\mathcal{C}_2 \to \mathcal{M}_1$ of 2-manifolds given by

$$\mathbb{P}^3 \ni V(y_2^4 + y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

of Section 7.4. In this section we construct quotients of $\mathcal{C}_2$ by cyclic subgroups of its group of $\mathcal{M}_1$-automorphisms, which will be suitable to obtain new $CMCY$ families of 2-manifolds. In the next section we will see that these new examples are endowed with involutions, which make them suitable for the construction of the Borcea-Voisin tower. Hence by the Hurwitz formula and some other obvious reasons, one has:

Claim 9.3.1. *Let C be a $K3$ surface and α be an involution on C, which admits a finite set S of fixed points on C. Then the quotient $\tilde{C}/\alpha$, where $\tilde{C}$ denotes the blowing up of C with respect to the subvariety given by S, is a $K3$ surface, too. Moreover $\tilde{C}/\alpha$ has complex multiplication resp., only commutative Hodge groups, if C has complex multiplication resp., only commutative Hodge groups.*

Now we introduce a group $\mathbb{G}_4$ of $\mathcal{M}_1$-automorphisms of the $CMCY$ family $\mathcal{C}_2 \to \mathcal{M}_1$. The elements $g \in \mathbb{G}_4$ can be uniquely written as a product $g = abc$ with $a \in \langle\alpha\rangle$, $b \in \langle\beta\rangle$, and $c \in \langle\iota_4\rangle$, where:

$$\alpha(y_2 : y_1 : x_1 : x_0) = (iy_2 : y_1 : x_1 : x_0), \quad \beta(y_2 : y_1 : x_1 : x_0) = (y_2 : iy_1 : x_1 : x_0),$$

$$\iota_4(y_2 : y_1 : x_1 : x_0) = (y_1 : y_2 : x_1 : x_0)$$

Therefore the group $\mathbb{G}_4$ contains exactly 32 elements. The action of $\mathbb{G}_4$ on the global sections of the canonical sheaves of the fibers induces a surjection of $\mathbb{G}_4$ onto the multiplicative group of the 4-th. roots of unity.

Its kernel $\mathbb{K}_4$ is a normal subgroup of order 8. It contains the following automorphisms of order 4:

$$\delta(y_2 : y_1 : x_1 : x_0) = (-y_1 : y_2 : x_1 : x_0), \quad \epsilon(y_2 : y_1 : x_1 : x_0) = (iy_2 : -iy_1 : x_1 : x_0),$$

$$\eta(y_2 : y_1 : x_1 : x_0) = (iy_1 : iy_2 : x_1 : x_0)$$

One has that

$$\iota_3 = \delta^2 = \epsilon^2 = \eta^2 = (\alpha\beta)^2.$$

Moreover one checks easily that $\mathbb{K}_4$ is isomorphic to the quarternion group and has the generators δ, ϵ and η. Thus one has

$$\mathbb{K}_4/\langle\iota_3\rangle = (\mathbb{Z}/2)^2. \tag{9.1}$$

One can easily calculate that

$$\alpha\langle\delta\rangle\alpha^{-1} = \langle\eta\rangle.$$

By the fact that $\mathbb{K}_4$ has 2 residue classes with respect to $\langle\delta\rangle$ resp., $\langle\epsilon\rangle$ resp., $\langle\eta\rangle$, one concludes that $\langle\delta\rangle$ resp., $\langle\epsilon\rangle$ resp., $\langle\eta\rangle$ is a normal subgroup of $\mathbb{K}_4$. Since $[\alpha]_{\mathbb{K}_4}$ generates $\mathbb{G}_4/\mathbb{K}_4$ and

$$\alpha\langle\epsilon\rangle\alpha^{-1} = \langle\epsilon\rangle,$$

$\langle\epsilon\rangle$ is a normal subgroup of $\mathbb{G}_4$.

9.3.2. Recall that ι_3 denotes the involution given by

$$\iota_3(y_2 : y_1 : x_1 : x_0) = (-y_2 : -y_1 : x_1 : x_0).$$

Let $\mathcal{C}_{\langle\iota_3\rangle}$ be the $CMCY$ family of 2-manifolds given by the quotient $\tilde{\mathcal{C}}_2/\langle\iota_3\rangle$, where $\tilde{\mathcal{C}}_2$ denotes the blowing up of $\mathcal{C}_2$ with respect to the 8 sections fixed by ι_3. Four sections fixed by ι_3 are given by $(1 : \zeta : 0 : 0)$, where ζ runs through the primitive 8-th. roots of unity. The other 4 sections are given by

$$(0 : 0 : 0 : 1),\ \ (0 : 0 : 1 : 1),\ \ (0 : 0 : \lambda : 1)\ \text{ and }\ (0 : 0 : 1 : 0).$$

Since the generators α, β and ι_4 of $\mathbb{G}_4$ leave the ideal sheaf corresponding to these 8 sections invariant, all automorphisms of $\mathbb{G}_4$ induce automorphisms on $\tilde{\mathcal{C}}_2$. Note that ι_3 commutes with each $\tau \in \mathbb{G}_4$. For each $\tau \in \mathbb{G}_4$ one finds open affine subsets invariant under $\langle\tau, \iota_3\rangle$. On these affine sets the global sections of the structure sheaf invariant under $\langle\tau, \iota_3\rangle$ are contained in $\mathcal{O}^{\langle\iota_3\rangle}$, where τ leaves $\mathcal{O}^{\langle\iota_3\rangle}$ invariant. Hencefore τ induces an automorphism on $\mathcal{C}_{\langle\iota_3\rangle}$. One checks easily that δ, η and ϵ yield involutions on $\mathcal{C}_{\langle\iota_3\rangle}$ leaving only finitely many sections fixed. Thus by using Claim 9.3.1, these involutions yield the $CMCY$ families of 2-manifolds

$$\mathcal{C}_{\langle\delta\rangle} \cong \mathcal{C}_{\alpha\langle\delta\rangle\alpha^{-1}} = \mathcal{C}_{\langle\eta\rangle}\ \text{ and }\ \mathcal{C}_{\langle\epsilon\rangle}.$$

9.4 Involutions on the quotients of the degree 4 example

In Section 7.4 we introduced several $\mathcal{M}_1$-involutions $\iota_1, \ldots, \iota_7$ of $\mathcal{C}_2$. We have seen that ι_3 acts by the character 1 on the global sections of the canonical sheaves of the fibers. Moreover $\iota_1, \iota_2, \iota_4, \ldots, \iota_7$ act by the character -1 on the global sections of the canonical sheaves of the fibers. Here we show that each for each $i = 1, 2, 4, \ldots, 7$ the involution ι_i induces $\mathcal{M}_1$-involutions on the quotient families of 9.3.2, which make them suitable for the construction of a Borcea-Voisin tower,

We fix some new notation: Let C_2 be an arbitrary fiber of $\mathcal{C}_2$, $p \in C_2$, where p is not fixed by ι_3, and F_i denote the curve of fixed points on C_2 with respect to ι_i for all $i = 1, 2, 4, \ldots, 7$.

9.4.1. The involutions ι_1 and ι_2 induce the same involution on $\mathcal{C}_{\langle\iota_3\rangle}$. One has that $\iota_1([p]_{\langle\iota_3\rangle}) = [p]_{\langle\iota_3\rangle}$, if and only if $p \in F_1 \cup F_2$. The involution ι_3 induces an involution on the curve F_1 and on the curve F_2. Each of the covers induced by these involutions has 4 ramification points. Hence by the Hurwitz formula, ι_1 induces an involution on $\mathcal{C}_{\langle\iota_3\rangle}$, which has a divisor of fixed points containing two families of elliptic curves. By [48], 1.1, the ramification divisor of our involution on a fiber of $\mathcal{C}_{\langle\iota_3\rangle}$ has at most one irreducible component of genus $g > 0$ or consists of two elliptic curves. Thus it consists of two elliptic curves. It is quite easy to check that by this involution ι_1, the family $\mathcal{C}_{\langle\iota_3\rangle}$ is suitable for the construction of a Borcea-Voisin tower.

9.4.2. The involutions ι_4 and ι_6 induce the same involution on $\mathcal{C}_{\langle\iota_3\rangle}$. One has that $\iota_4([p]_{\langle\iota_3\rangle}) = [p]_{\langle\iota_3\rangle}$, if and only if $p \in F_4 \cup F_6$. The involution ι_3 induces an involution on the curve F_4 and on the curve F_6. Each of the covers induced by these involutions have 4 ramification points. Hence by the same arguments as in 9.4.1, the involution ι_4 induces an involution on $\mathcal{C}_{\langle\iota_3\rangle}$, which has a divisor of fixed points consisting of two families of elliptic curves. It is quite easy to check that by this involution ι_1, the family $\mathcal{C}_{\langle\iota_3\rangle}$ is suitable for the construction of a Borcea-Voisin tower.

Since $\alpha\iota_4\alpha^{-1} = \iota_5$ and $\alpha\iota_3\alpha^{-1} = \iota_3$, the involutions ι_5 and ι_7 induce up isomorphisms the same involution as ι_4 and ι_6 on $\mathcal{C}_{\langle\iota_3\rangle}$.

Recall the $\mathcal{M}_1$-automorphisms

$$\delta(y_2 : y_1 : x_1 : x_0) = (-y_1 : y_2 : x_1 : x_0), \quad \epsilon(y_2 : y_1 : x_1 : x_0) = (iy_2 : -iy_1 : x_1 : x_0)$$

of $\mathcal{C}_2$ of order 4.

Remark 9.4.3. Now we consider the quotient families $\mathcal{C}_{\langle\delta\rangle}$ and $\mathcal{C}_{\langle\epsilon\rangle}$ in 9.3.2. One checks easily that δ and ϵ act as involutions on the 4 sections given by $(1 : \zeta : 0 : 0)$, where ζ runs through the primitive 8-th. roots of unity, and leave the sections given by

$$(0 : 0 : 0 : 1), \quad (0 : 0 : 1 : 1), \quad (0 : 0 : \lambda : 1), \quad (0 : 0 : 1 : 0)$$

invariant.

Moreover there does not exist a point $p \in C_2$ such that $\delta(p) = \iota_3(p)$ or $\epsilon(p) = \iota_3(p)$. This follows from the facts that $\iota_3 = \delta^2 = \epsilon^2$ and $\delta^2(p) = \delta(p)$ resp., $\epsilon^2(p) = \epsilon(p)$ would imply that δ resp., ϵ is not bijective.

Hencefore either p is contained in one of the 8 sections fixed by ι_3 or $\langle\delta\rangle \cdot p$ and $\langle\epsilon\rangle \cdot p$ contain 4 different elements. For our notation we will assume that p is not fixed by ι_3 as above.

9.4.4. The involutions ι_1 and ι_2 commute with ϵ. Thus the same holds true with respect to the involutions on $\mathcal{C}_{\langle\iota_3\rangle}$ induced by ι_1, ι_2 and ϵ. Hence one concludes that ι_1 and ι_2 induce an involution an $\mathcal{C}_{\langle\epsilon\rangle}$. Since ι_1 and ι_2 induce the same involution on $\mathcal{C}_{\langle\iota_3\rangle}$, the involutions ι_1 and ι_2 induce the same involution on $\mathcal{C}_{\langle\epsilon\rangle}$.

A point $[p]$ on the fiber $C_{\langle\epsilon\rangle}$ of $\mathcal{C}_{\langle\epsilon\rangle}$ is fixed by ι_1, if $\iota_1(p) = \epsilon^i(p)$ for $i = 0, \ldots, 3$. This is exactly satisfied on F_1 and F_2 for $i = 0$ or $i = 2$. The automorphism ϵ yields a quotient of F_1 resp., F_2 of degree 4 fully ramified over 4 points. Hence by the Hurwitz formula, $F_1/\langle\epsilon\rangle$ and $F_2/\langle\epsilon\rangle$ are rational curves.

By the definitions of ι_1 and ϵ, one checks easily that their actions coincide on the exceptional divisor on $\tilde{\mathcal{C}}_2$ over the four sections given by $V(y_2, y_1)$. Moreover by the definitions of ι_1 and ϵ, one checks easily that for each primitive 8-th. root ζ of unity

$$\iota_1(1 : \zeta : 0 : 0) = \epsilon(1 : \zeta : 0 : 0) = (1 : -\zeta : 0 : 0).$$

Both $\mathcal{M}_1$-automorphisms fix the local parameters x_1 and x_2.

Thus altogether the involution ι_1 induces an involution on $\mathcal{C}_{\langle\epsilon\rangle}$, which has a divisor of fixed points consisting of 8 disjoint families of rational curves. It is quite easy to check that $\mathcal{C}_{\langle\epsilon\rangle}$ is suitable for the construction of a Borcea-Voisin tower by this involution.

9.4.5. The involutions $\iota_4, \ldots, \iota_7$ do not commute with ϵ. But one has $\epsilon\iota_i = \iota_i\epsilon^3$ for all $i = 4, \ldots, 7$. Hence ι_i $(i = 4, \ldots, 7)$ induces an involution on $\mathcal{C}_{\langle\iota_3\rangle}$. Since $\iota_5 = \epsilon\iota_4$, $\iota_6 = \epsilon^2\iota_4$ and $\iota_7 = \epsilon^3\iota_4$, these involutions induce the same involution on $\mathcal{C}_{\langle\epsilon\rangle}$.

A point $[p] \in C_{\langle\epsilon\rangle}$ is invariant under ι_4, if $\iota_4(p) = \epsilon^i(p)$ for $i = 0, \ldots, 3$. One has that $\iota_4(p) = (p)$ on F_4, $\iota_4(p) = \epsilon^1(p)$ on F_7, $\iota_4(p) = \epsilon^2(p)$ on F_6 and $\iota_4(p) = \epsilon^3(p)$ on F_5. Note that $\epsilon(F_4) = F_6$, $\epsilon(F_6) = F_4$, $\epsilon^2(F_4) = F_4$ and $\epsilon^2(F_6) = F_6$. Moreover one has $\epsilon(F_5) = F_7$, $\epsilon(F_7) = F_7$, $\epsilon^2(F_5) = F_5$ and $\epsilon^2(F_5) = F_5$. The automorphism $\epsilon^2 = \iota_3$ yields a quotient of F_4, F_5, F_6 resp., F_7 of degree 2 ramified over 4 points, where F_4 and F_6 resp., F_5 and F_7 are mapped onto the same quotient by ϵ. Hence by the Hurwitz formula, the quotient consists of two families of elliptic curves.

By [48], 1.1, the ramification divisor of our involution on $C_{\langle\epsilon\rangle}$ has at most one irreducible component of genus $g > 0$ or consists of two elliptic curves. Thus ι_4 induces an involution on $\mathcal{C}_{\langle\epsilon\rangle}$, which has a divisor of fixed points consisting of 2 families of elliptic curves. It is quite easy to check that this involution makes $\mathcal{C}_{\langle\epsilon\rangle}$ suitable for the construction of a Borcea-Voisin tower.

9.4.6. The involutions ι_4 and ι_6 do not commute with δ. But one has $\delta\iota_4 = \iota_4\delta^3$ and $\delta\iota_6 = \iota_6\delta^3$. Moreover one has

$$\iota_1 = \delta \circ \iota_4, \quad \iota_6 = \delta^2 \circ \iota_4, \quad \text{and} \quad \iota_2 = \delta^3 \circ \iota_4.$$

Hence ι_1, ι_2, ι_4 and ι_6 induce the same involution on $\mathcal{C}_{<\delta>}$.

A point $[p] \in C_{\langle\delta\rangle}$ is invariant under ι_4, if $\iota_4(p) = \delta^i(p)$. This occurs, if and only if

$$p \in F_1 \cup F_2 \cup F_4 \cup F_6.$$

Note that $\delta(F_4) = F_6$ and $\delta(F_1) = F_2$. Moreover δ yields a degree 4 quotient of $F_4 \cup F_6$, and a degree 4 quotient of $F_1 \cup F_2$. Thus the divisor of fixed points contains two families of elliptic curves.

By the same arguments as in 9.4.5, the involution ι_4 induces an involution on $\mathcal{C}_{\langle\delta\rangle}$, which has a divisor of fixed points consisting of 2 families of elliptic curves and makes $\mathcal{C}_{\langle\delta\rangle}$ suitable for the construction of a Borcea-Voisin tower.

9.4.7. The involution ι_5 commutes with δ. One has that $p = \iota_5(p)$, if $p \in F_5$ and $\delta^2(p) = \iota_5(p)$, if $p \in F_7$. Note that δ acts as degree 4 automorphism on F_5 resp., F_7. Each of the corresponding quotient maps is fully ramified over 4 points. By the same arguments as in 9.4.4, the $\mathcal{M}_1$-automorphisms ι_5 and δ act in the same way on the exceptional divisor of $\tilde{\mathcal{C}}_2$. Thus ι_5 induces an involution on $\mathcal{C}_{\langle\delta\rangle}$, which fixes a divisor consisting of 8 families of rational curves. Moreover it is quite easy to check that this involution makes $\mathcal{C}_{\langle\delta\rangle}$ suitable for the construction of a Borcea-Voisin tower.

9.4.8. Since $\alpha\iota_1\alpha^{-1} = \iota_1$ and $\alpha\delta\alpha^{-1} = \eta$, one concludes that the involution induced by ι_1 on $\mathcal{C}_{\langle\eta\rangle}$ coincides up to an isomorphism with the involution induced by ι_1 on $\mathcal{C}_{\langle\delta\rangle}$.

Since $\alpha\iota_5\alpha^{-1} = \iota_6$ and $\alpha\delta\alpha^{-1} = \eta$, one concludes that the involution induced by ι_6 on $\mathcal{C}_{\langle\eta\rangle}$ coincides up to an isomorphism with the involution induced by ι_5 on $\mathcal{C}_{\langle\delta\rangle}$.

9.5 The extended automorphism group of the degree 4 example

The group $\mathbb{G}_4$ of $\mathcal{M}_1$-automorphisms of $\mathcal{C}_2$ does not contain all $\mathcal{M}_1$-automorphisms of $\mathcal{C}_2$. In this section we give an additional group $\mathbb{E}_4$ of $\mathcal{M}_1$-automorphisms such that $\mathbb{G}_4$ and $\mathbb{E}_4$ generate an extended $\mathcal{M}_1$-automorphism group $\bar{\mathbb{G}}_4$. Moreover we will make some remarks about $\bar{\mathbb{G}}_4$ and $\mathbb{E}_4$.

We obtain due to [24], Proposition 9 and the notations of [24], Section 2:

Proposition 9.5.1. *The family $\mathcal{C}_2$ has a group $\mathbb{E}_4$ of $\mathcal{M}_1$-automorphisms consisting of 16 different automorphisms given by $(\alpha\beta)^\nu$ with $\nu = 0, \ldots, 3$ and:*

$$\alpha_\zeta(y_2 : y_1 : x_1 : x_0) = (\zeta y_2 : \zeta y_1 : x_1 - \lambda x_0 : x_1 - x_0), \quad \zeta^4 = (1-\lambda)^2$$

$$\beta_\varsigma(y_2 : y_1 : x_1 : x_0) = (\varsigma y_2 : \varsigma y_1 : x_1 - x_0 : \frac{1}{\lambda}x_1 - x_0), \quad \varsigma^4 = (1-\frac{1}{\lambda})^2$$

$$\gamma_\kappa(y_2 : y_1 : x_1 : x_0) = (\kappa y_2 : \kappa y_1 : \lambda x_0 : x_1), \quad \kappa^4 = \lambda^2$$

The involutions of $\mathbb{E}_4$ are given by $(\alpha\beta)^\nu$, α_ζ, β_ς and γ_κ for $\nu = 2$, $\zeta^2 = 1-\lambda$, $\varsigma^2 = 1-\frac{1}{\lambda}$ and $\kappa^2 = \lambda$. The group $\mathbb{E}_4$ has a subgroup isomorphic to the quarternion group given by $(\alpha\beta)^\nu$, α_ζ, β_ς and γ_κ for $\nu = 0, 2$, $\zeta^2 = -1+\lambda$, $\varsigma^2 = -1+\frac{1}{\lambda}$ and $\kappa^2 = -\lambda$.

One can ask for the character of the action of the involutions of $\mathbb{E}_4$ on $\Gamma(\omega_{(\mathcal{C}_2)_q})$ for each $q \in \mathcal{M}_1$ and the possibilities to use these involutions for the construction of Borcea-Voisin towers. For example one has:

Example 9.5.2. One checks easily that $\gamma_{\sqrt{\lambda}}$ resp., $\gamma_{-\sqrt{\lambda}}$ fixes the family curves on $\mathcal{C}_2$ given by

$$x_1 = \sqrt{\lambda}x_0 \ \text{ resp., } \ x_1 = -\sqrt{\lambda}x_0.$$

This family of curves is isomorphic to the constant family with universal fiber given by the Fermat curve $\mathbb{F}_4$ of degree 4, which has the genus 3. Thus it acts by the character -1 on $\Gamma(\omega_{(\mathcal{C}_2)_q})$ for each $q \in \mathcal{M}_1$. Since $\mathbb{F}_4$ has complex multiplication, $\gamma_{\sqrt{\lambda}}$ and $\gamma_{-\sqrt{\lambda}}$ make $\mathcal{C}_2$ suitable for the construction of a Borcea-Voisin tower.

The following claim implies that $\gamma_{\sqrt{\lambda}}$ and $\gamma_{-\sqrt{\lambda}}$ yield isomorphic families by the Borcea-Voisin tower:

Claim 9.5.3. *One can conjugate $\gamma_{\sqrt{\lambda}}$ and $\gamma_{-\sqrt{\lambda}}$ in $\mathbb{E}_4$.*

Proof. There exists some g of order 4 contained in the quarternion subgroup of $\mathbb{E}_4$ such that

$$\gamma_{\sqrt{\lambda}} = (\alpha\beta)g \ \text{ and } \ \gamma_{-\sqrt{\lambda}} = (\alpha\beta)^3 g = (\alpha\beta)(\alpha\beta)^2 g = (\alpha\beta)g^{-1}.$$

It is a well-known fact that there is a g_2 contained in the quarternion group such that

$$g^{-1} = g_2 \circ g \circ g_2^{-1}.$$

Since $(\alpha\beta)$ is contained in the center of $\mathbb{E}_4$, one obtains the result. □

Finally the question for isomorphy between $\mathcal{C}_2/\iota_1$ and $\mathcal{C}_2/\iota_4$ resp., the corresponding $CMCY$ families of 3-manifolds constructed by the method of C. Voisin [48] remains open, since we have:

Remark 9.5.4. By the description of $\mathbb{E}_4$ in Proposition 9.5.1, one checks easily that the generators α, β, ι_4 of $\mathbb{G}_4$ commute with each element of $\mathbb{E}_4$. Hence each element of $\bar{\mathbb{G}}_4$, which is the group generated by $\mathbb{G}_4$ and $\mathbb{E}_4$, can be written as $\kappa\tau$ with $\kappa \in \mathbb{E}_4$ and $\tau \in \mathbb{G}_4$. Thus for each $\sigma \in \mathbb{G}_4$ one obtains

$$(\kappa\tau)^{-1}\sigma(\kappa\tau) = \tau^{-1}\sigma\tau. \tag{9.2}$$

Hence the fact that ι_1 and ι_4 are not conjugate in $\mathbb{G}_4$ implies that ι_1 and ι_4 are not conjugate in $\bar{\mathbb{G}}_4$.

Moreover (9.2) implies that $\gamma_{\sqrt{\lambda}}$ is not conjugate to ι_1 or ι_4 in $\bar{\mathbb{G}}_4$.

Remark 9.5.5. One may search for additional involutions in $\bar{\mathbb{G}}_4$ and try to determine the character of the actions of all involutions on $\Gamma(\omega_{(\mathcal{C}_2)_q})$ for each $q \in \mathcal{M}_1$. In addition one can try to determine the involutions, which are suitable for the construction of a Borcea-Voisin tower and try to repeat the construction of the preceding section for arbitrary induced involutions on suitable quotients by cyclic subgroups of $\bar{\mathbb{G}}_4$.

9.6 The automorphism group of the degree 5 example by Viehweg and Zuo

We consider the $CMCY$ family $\mathcal{F}_3$

$$\mathbb{P}^4 \ni V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2$$

of 3-manifolds constructed by E. Viehweg and K. Zuo. Let ξ denote a fixed primitive 5-th. root of unity. We introduce an $\mathcal{M}_2$-automorphism group $\mathbb{G}_5$ of the family $\mathcal{F}_3 \to \mathcal{M}_2$. The elements $g \in \mathbb{G}_5$ can be uniquely written as a product $g = abcd$ with $a \in \langle\alpha\rangle$, $b \in \langle\beta\rangle$, $c \in \langle\gamma\rangle$ and $d \in S_3$, where:

$$\alpha(y_3 : y_2 : y_1 : x_1 : x_0) = (\xi y_3 : y_2 : y_1 : x_1 : x_0),$$

$$\beta(y_3 : y_2 : y_1 : x_1 : x_0) = (y_3 : \xi y_2 : y_1 : x_1 : x_0),$$

$$\gamma(y_3 : y_2 : y_1 : x_1 : x_0) = (y_3 : y_2 : \xi y_1 : x_1 : x_0),$$

$$d(y_3 : y_2 : y_1 : x_1 : x_0) = (y_{d(3)} : y_{d(2)} : y_{d(1)} : x_1 : x_0)$$

Therefore the group $\mathbb{G}_5$ contains exactly $5 \cdot 5 \cdot 5 \cdot 6 = 750$ elements. The action of $\mathbb{G}_5$ on the global sections of the canonical sheaves of the fibers induces a surjection of $\mathbb{G}_5$ onto the multiplicative group of the 10-th. roots of unity.[5]

Its kernel $\mathbb{K}_5$ is a normal subgroup of order 75. It contains the subgroup $\langle\alpha\beta^{-1}, \beta\gamma^{-1}\rangle$ of automorphisms of order 5. Moreover it contains the cyclic group of order 3 given by the permutations of A_3. Therefore all elements of $\mathbb{K}_5$ are determined.

9.6.1. Let us consider all cyclic groups $\langle g\rangle \subset \mathbb{K}_5$ with $g = abc \neq e$ as above. If $a = e$ or $b = e$ or $c = e$, $\langle g\rangle$ is given by $\langle\alpha\beta^{-1}\rangle$, $\langle\beta\gamma^{-1}\rangle$ or $\langle\alpha\gamma^{-1}\rangle$. These groups are conjugate by $(1,2), (1,3), (2,3) \in S_3$.

Now consider the cyclic group $\langle g\rangle \subset \mathbb{K}_5$ with $g = abc$ and $a, b, c \neq e$. One has that $\langle g\rangle$ contains an element $\alpha\beta^b\gamma^{4-b}$ with $b = 1, 2, 3$. Hence by $e \in S_3$ or $(2,3) \in S_3$, it is conjugate

[5]Note that S_3 is generated by the involutions given by the cycles $(1,2)$ and $(2,3)$, which act by the character -1 on the global sections of the canonical sheaves of the fibers.

to $\langle\alpha\beta\gamma^3\rangle$ or $\langle\alpha\beta^2\gamma^2\rangle$. By the cycle $(1,3) \in S_3$, these both groups are conjugate. By the fact that $\langle\alpha\beta\gamma^3\rangle$ leaves only finitely many points invariant on each fiber, but $\langle\alpha\beta^{-1}\rangle$ leaves a curve invariant on each fiber, both groups can not be conjugate.

Hencefore we have two conjugacy classes of cyclic subgroups $\langle g\rangle \subset \mathbb{K}_5$ with $g = abc \neq e$ represented by $\langle\alpha\beta^{-1}\rangle$ and $\langle\alpha\beta\gamma^3\rangle$.

Claim 9.6.2. *Any automorphism $\tau \in \mathbb{K}_5$, which is not given by*

$$\tau(y_3 : y_2 : y_1 : x_1 : x_0) = (\xi^s y_3 : \xi^t y_2 : \xi^{5-s-t} y_1 : x_1 : x_0)$$

for some $s, t \in \mathbb{Z}$, satisfies $\tau^3 = \mathrm{id}$.

Proof. If τ satisfies the assumptions of the Claim, then τ or τ^{-1} is given by

$$(y_3 : y_2 : y_1 : x_1 : x_0) \to (\xi^s y_1 : \xi^t y_3 : \xi^{5-s-t} y_2 : x_1 : x_0) \tag{9.3}$$

for some $s, t \in \mathbb{Z}$. Hence assume without loss of generality that τ is given by (9.3) and verify the statement by calculation:

$$\tau^3(y_3 : y_2 : y_1 : x_1 : x_0) = \tau^2(\xi^s y_1 : \xi^t y_3 : \xi^{-s-t} y_2 : x_1 : x_0)$$

$$= \tau(\xi^{-t} y_2 : \xi^{s+t} y_1 : \xi^{-s} y_3 : x_1 : x_0) = (y_3 : y_2 : y_1 : x_1 : x_0)$$

□

For each τ as in (9.3) one can easily calculate that $\alpha^{-s}\beta^{-s-t} \circ \tau \circ \alpha^s\beta^{s+t}$ is given by

$$(y_3 : y_2 : y_1 : x_1 : x_0) \to (y_1 : y_3 : y_2 : x_1 : x_0).$$

Therefore all cyclic subgroups of $\mathbb{K}_5$ are up to conjugation determined. Hence:

Proposition 9.6.3. *The family $\mathcal{F}_3$ has up to isomorphisms the following quotient families of Calabi-Yau orbifolds with dense sets of CM fibers:*

$$\mathcal{F}_3/\langle\alpha\beta^4\rangle, \quad \mathcal{F}_3/\langle\alpha\beta\gamma^3\rangle, \quad \mathcal{F}_3/\langle(1,2,3)\rangle$$

Proof. The existence of dense sets of CM fibers follows, since the VHS of a quotient family of $\mathcal{F}_3$ is a sub-VHS of $\mathcal{F}_3$. □

Chapter 10

Examples of $CMCY$ families of 3-manifolds and their invariants

10.1 The *length* of the Yukawa coupling

First let us construct the Yukawa coupling. A little bit later in this short section we will give a motivation to consider it and describe how to calculate its *length* for our examples of $CMCY$ families of 3-manifolds.

Construction 10.1.1. Assume that U is a quasi projective variety and $\mathcal{V}$ is a complex polarized variation of Hodge structures of weight n on U. It is a well-known fact that there exists a suitable finite cover of U such that the pullback of $\mathcal{V}$ has local unipotent monodromy. We replace U by this finite cover. There exists a smooth projective compactification Y of U such that $S := Y \setminus U$ is a normal crossing divisor. Then one can construct the Deligne extension $\mathcal{H}$ of $\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ (i. e., the unique extension such that the Gauß-Manin connection yields the structure of a logarithmic Higgs bundle (F, θ) on the associated graded bundle and the real components of eigenvalues of the residues are contained in $[0, 1)$). The graduation gives a decomposition of F into locally free sheaves $E^{p,n-p}$ and the Gauß-Manin connection induces an $\mathcal{O}_Y$-linear morphism

$$E^{p,n-p} \to E^{n-1,n-p+1} \otimes \Omega^1_Y(\log S),$$

called Higgs field. The Yukawa coupling θ_i (for $i \leq n$) is defined by the composition

$$\theta_i : E^{n,0} \xrightarrow{\theta_{n,0}} E^{n-1,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\theta_{n-1,1}} E^{n-2,2} \otimes \mathrm{Sym}^2\Omega^1_Y(\log S) \xrightarrow{\theta_{n-2,2}} \dots$$

$$\xrightarrow{\theta_{n-i+1,i-1}} E^{n-i,i} \otimes \mathrm{Sym}^i\Omega^1_Y(\log S).$$

Definition 10.1.2. Let $f : V \to U$ be a family with fibers of dimension n as in Construction 10.1.1. The *length* $\zeta(f)$ of the Yukawa coupling is given by

$$\zeta(f) := \min\{i \geq 1; \theta_i = 0\} - 1.$$

We say that the Yukawa coupling has *maximal length*, if $\zeta(f) = n$.

The family $f : V \to U$ is rigid, if there does not exist a non-trivial deformation of f over a nonsingular quasi-projective curve T.

The following proposition yields our motivation to consider the *length* of the Yukawa coupling:

Proposition 10.1.3. *If the Yukawa coupling has* maximal length*, the family is rigid.*

Proof. (see [44], Section 8) □

The statements of the following lemma, which allow the computation of *length* of the Yukawa couplings of our examples of $CMCY$ families of 3-manifolds by their construction, are well-known:

Lemma 10.1.4. *For two variations of Hodge structures* $\mathbb{V}$ *and* $\mathbb{W}$ *on a holomorphic manifold one has*

$$\zeta(\mathbb{V}\otimes\mathbb{W}) = \zeta(\mathbb{V}) + \zeta(\mathbb{W}) \quad \textit{and} \quad \zeta(\mathbb{V}\oplus\mathbb{W}) = \max\{\zeta(\mathbb{V}), \zeta(\mathbb{W})\}.$$

10.2 Examples obtained by degree 2 quotients

Let $\mathcal{Z}_1 \to \mathcal{M}$ be one of the examples of a $CMCY$ family of 2-manifolds, which we have constructed in the preceding chapters, with a suitable involution ι such that it satisfies the assumptions for $\mathcal{Z}_1$ in the construction of a Borcea-Voisin tower. Here we list all examples of $CMCY$ families $\mathcal{Z}_2$ of 3-manifolds obtained by the Borcea-Voisin tower starting with such a family $\mathcal{Z}_1$ and Σ_2 given by the family $\mathcal{E} \to \mathcal{M}_1$ of elliptic curves endowed with its natural involution. By the definition of Calabi-Yau manifolds, Serre duality and Hodge symmetry, all Hodge numbers of the fibers of the resulting $CMCY$ family $\mathcal{Z}_2$ of 3-manifolds are determined by $h^{1,1}$ and $h^{2,1}$.

Claim 10.2.1. *Keep the assumptions above. Let* $(\mathcal{Z}_1)_p \to (\mathcal{Z}_1)_p/\iota$ *be ramified over* N *curves with genus* $g_1, \ldots, g_N$ *for all* $p \in \mathcal{M}$*. Then the fibers of* $\mathcal{Z}_2$ *have the Hodge numbers*

$$h^{1,1} = 11 + 5N - N' \textit{ and } h^{2,1} = 11 + 5N' - N, \quad \textit{where} \quad N' = \sum g_i.$$

Proof. (see [48], Corollaire 1.8) □

Hence for our examples of $CMCY$ families of 3-manifolds obtained by using the Borcea-Voisin tower and $CMCY$ families of 2-manifolds with suitable involutions, we have the following table:

family $\mathcal{Z}_1$	basis $\mathcal{M}$	involution ι	N	N'	$h^{1,1}$	$h^{2,1}$	ζ	reference
$\mathcal{C}_2$	$\mathcal{M}_1$	ι_1	1	3	13	25	2	7.4.4
$\mathcal{C}_2$	$\mathcal{M}_1$	ι_4	1	3	13	25	2	7.4.4
$\mathcal{C}_2$	$\mathcal{M}_1$	$\gamma_{\sqrt{\lambda}}$, $\gamma_{\sqrt{-\lambda}}$	1	3	13	25	2	9.5.2
$\mathcal{C}_{\langle\iota_3\rangle}$	$\mathcal{M}_1$	ι_1	2	2	19	19	2	9.4.1
$\mathcal{C}_{\langle\iota_3\rangle}$	$\mathcal{M}_1$	ι_4	2	2	19	19	2	9.4.2
$\mathcal{C}_{\langle\epsilon\rangle}$	$\mathcal{M}_1$	ι_1	8	0	51	3	2	9.4.4
$\mathcal{C}_{\langle\epsilon\rangle}$	$\mathcal{M}_1$	ι_4	2	2	19	19	2	9.4.5
$\mathcal{C}_{\langle\delta\rangle}$	$\mathcal{M}_1$	$\iota_1 = \iota_4$	2	2	19	19	2	9.4.6
$\mathcal{C}_{\langle\delta\rangle}$	$\mathcal{M}_1$	ι_5	8	0	51	3	2	9.4.7
$\mathcal{W}$	$\mathcal{M}_3$	γ	2	4	17	29	2	8.3.4, 8.3.5
$\mathcal{Y}$	$\mathcal{M}_3$	γ	2	4	17	29	2	9.1.3, 9.1.4

10.3 The Example obtained by a degree 3 quotient and its maximality

In this section we determine the Hodge numbers of the $CMCY$ family $\mathcal{Q}$ of 3-manifolds obtained by Proposition 9.2.7.

Remark 10.3.1. In the case of the $CMCY$ family of Proposition 9.2.7 one has $\zeta = 1$ for the *length* of the Yukawa coupling as one concludes by its construction and using Lemma 10.1.4.

Let X be a complex manifold and γ an automorphism of X of order m. Then $H^k(X,\mathbb{C})_\ell$ denotes the eigenspace of $H^k(X,\mathbb{C})$, on which γ acts via pullback by the character $e^{2\pi i\frac{\ell}{m}}$. For the calculation of the Hodge numbers of this family we will need the following proposition:

Proposition 10.3.2. *Let X be a Kähler manifold of dimension 3. Moreover let φ be an automorphism of X fixing a finite set of some isolated points Z_0 and a finite set Z_1 of disjoint curves such that $\varphi^m = \mathrm{id}$ for some $m \in \mathbb{N}$. Then one has the following eigenspaces:*

$$H^2(\tilde{X}_{Z_1\cup Z_0},\mathbb{Z})_0 \cong H^2(X,\mathbb{Z})_0 \oplus H^0(Z_1,\mathbb{Z}) \oplus H^0(Z_0,\mathbb{Z}),$$
$$H^3(\tilde{X}_{Z_1\cup Z_0},\mathbb{Z})_0 \cong H^3(X,\mathbb{Z})_0 \oplus H^1(Z_1,\mathbb{Z})$$

Proof. Let Y be a Kähler manifold and Z be a submanifold of codimension r. Then the Hodge structure of the blowing up $\tilde{Y}_Z$ along Z is given by

$$H^k(Y,Z) \oplus \bigoplus_{i=0}^{r-2} H^{k-2i-2}(Z,\mathbb{Z}) \cong H^k(\tilde{Y}_Z,\mathbb{Z}),$$

where $H^{k-2i-2}(Z,\mathbb{Z})$ shifted by $(i+1,i+1)$ in bi-degree (see [49], Théorème 7.31).

Thus one has:

$$H^2(\tilde{X}_{Z_1\cup Z_0},\mathbb{Z}) \cong H^2(X,\mathbb{Z}) \oplus H^0(Z_1,\mathbb{Z}) \oplus H^0(Z_0,\mathbb{Z}),$$
$$H^3(\tilde{X}_{Z_1\cup Z_0},\mathbb{Z}) \cong H^3(X,\mathbb{Z}) \oplus H^1(Z_1,\mathbb{Z})$$

Hence it remains to show that $H^0(Z_1,\mathbb{Z})$, $H^0(Z_0,\mathbb{Z})$ and $H^1(Z_1,\mathbb{Z})$ are invariant as sub-Hodge structures by φ. Hencefore one considers the proof of [49], Théorème 7.31. These sub-Hodge structures are given by the image of $j_* \circ (\pi|_{Z_1\cup Z_0})^*(H^0(Z_1\cup Z_0,\mathbb{Z}))$ and $j_* \circ (\pi|_{Z_1\cup Z_0})^*(H^1(Z_1\cup Z_0,\mathbb{Z}))$, where j denotes the embedding of the exceptional divisor E of the blowing up morphism $\pi : \tilde{X}_{Z_1\cup Z_0} \to X$.[1] One has the following commutative diagram:

$$\begin{array}{ccc}
\tilde{X}_{Z_1\cup Z_0} & \xrightarrow{\varphi} & \tilde{X}_{Z_1\cup Z_0} \\
\uparrow{\scriptstyle j} & & \uparrow{\scriptstyle j} \\
E & \xrightarrow{\varphi} & E \\
\downarrow{\scriptstyle \pi|_E} & & \downarrow{\scriptstyle \pi|_E} \\
Z_1\cup Z_0 & \xrightarrow{\varphi} & Z_1\cup Z_0
\end{array}$$

[1] In general one has $\bigoplus_{i=0}^{r-2} j_* \circ h^i \circ (\pi|_{Z_1\cup Z_0})^*$ instead of $j_* \circ (\pi|_{Z_1\cup Z_0})^*$ for $i = 0,\dots,r-2$ in [49], Théorème 7.31, where h denotes the cup-product with $c_1(\mathcal{O}_E(1))$ and the sheaf $\mathcal{O}_E(1)$ of the projective bundle E is described in [49], Subsection 3.3.2. But here the weight of the Hodge structures is to small for $i > 0$.

Since φ acts as the identity on $Z_1 \cup Z_0$, the same holds true for the Hodge structures on $Z_1 \cup Z_0$. Hence by the commutative diagram, the same holds true for the sub-Hodge structures on $\tilde{X}$ given by $j_* \circ (\pi|_{Z_1 \cup Z_0})^*$. □

Proposition 10.3.3. *For all $q \in \mathcal{M}_3$ the action of the cyclic group $\langle \alpha\beta \rangle$ on $\mathcal{W}$ yields an eigenspace decomposition of $H^{1,1}(\mathcal{W}_q)$ of the dimensions*

$$h^{1,1}(\mathcal{W}_q)_0 = 14, \quad h^{1,1}(\mathcal{W}_q)_1 = 3, \quad h^{1,1}(\mathcal{W}_q)_2 = 3.$$

Proof. Let $\tilde{\mathcal{W}} \to \mathcal{W}$ be the blowing up of the six sections fixed by $\alpha\beta$. By the same arguments as in the proof of the preceding proposition, each fiber $\tilde{\mathcal{W}}_q$ has the Hodge numbers

$$h^{2,0} = 1, \quad h^{1,1} = 26, \quad h^{0,2} = 1.$$

Let $M := \tilde{\mathcal{W}}_q / \langle \alpha\beta \rangle$. Now we consider the quotient morphism $\varphi : \tilde{\mathcal{W}}_q \to M$. By the Hurwitz formula, one concludes that

$$\varphi^*(K_M) = -2E - E^{(2)},$$

where E is the exceptional divisor of $\mathcal{W}_q \to W_q$ given by three -2 curves and $E^{(2)}$ is the exceptional divisor of $\tilde{\mathcal{W}}_q \to \mathcal{W}_q$. From [49], Proposition 21.14, we have that $3 \cdot K_M^2 = (\varphi^*(K_M))^2$. Since

$$(\varphi^*(K_M))^2 = (-2E - E^{(2)})^2 = 4 \cdot (-6) - 6 = -30$$

and $c_1(M)^2 = K_M^2$ (see [22], Appendix A, Example 4.1.2), one obtains

$$c_1(M)^2 = K_M^2 = -10.$$

By the Noether formula (compare to [22], Appendix A, Example 4.1.2 and [49], Remarque 23.6), one has

$$\chi(\mathcal{O}_M) = \frac{1}{12}(c_1(M)^2 + c_2(M)) \quad \text{with} \quad c_2(M) - 2 = b_2(M)$$

in our case. From the fact that $\chi(\mathcal{O}_M) = 1$, one calculates that

$$h^{1,1}(\tilde{\mathcal{W}}_q)_0 = b_2(M) = 20.$$

By the fact that the blowing up morphism $\tilde{\mathcal{W}}_q \to \mathcal{W}_q$ has an exceptional divisor consisting of 6 rational curves, we conclude similar to Proposition 10.3.2 that

$$h^{1,1}(\mathcal{W}_q)_0 = h^{1,1}(\tilde{\mathcal{W}}_q)_0 - 6 = 20 - 6 = 14.$$

Since the $K3$ surface $\mathcal{W}_q$ has the Hodge number

$$h^{1,1}(\mathcal{W}_q) = 20 \quad \text{and} \quad h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2,$$

one concludes that

$$h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2 = 3.$$

□

Proposition 10.3.4. *For all $q \in \mathcal{M}_3$ one has*

$$h^{1,1}(\mathcal{Q}_q) = 51.$$

Proof. Since

$$h^{0,0}(\mathcal{W}_q)_0 = h^{0,0}(\mathbb{F}_3)_0 = h^{1,1}(\mathbb{F}_3)_0 = 1, \quad b_1(\mathcal{W}_q) = 0$$

and Proposition 10.3.3 tells us that

$$h^{1,1}(\mathcal{W}_q)_0 = 14,$$

one concludes that $h^{1,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = 15$. Note that α_2 fixes $6 \cdot 3 = 18$ points. Moreover we have an additional exceptional divisor consisting of $3 \cdot 3 \cdot 3 = 27$ rational ruled surfaces. In the construction of $\mathcal{Q}$ we blow down 9 of these families of ruled surfaces. Hence by Proposition 10.3.2,

$$h^{1,1}(\mathcal{Q}_q) = 15 + 18 + 27 - 9 = 51.$$

□

Recall that $\alpha\beta$ acts by the character $e^{2\pi i \frac{2}{3}}$ on the global sections of $\omega_{\mathcal{W}_q}$ for all $q \in \mathcal{P}_n$ and $\alpha_{\mathbb{F}_3}$ acts by the character $e^{2\pi i \frac{1}{3}}$ on the global sections of $\omega_{\mathbb{F}_3}$. Hence one obtains

$$h^{1,0}(\mathbb{F}_3)_1 = h^{0,1}(\mathbb{F}_3)_2 = h^{2,0}(\mathcal{W}_q)_2 = h^{0,2}(\mathcal{W}_q)_1 = 1$$

and

$$h^{1,0}(\mathbb{F}_3)_2 = h^{0,1}(\mathbb{F}_3)_1 = h^{2,0}(\mathcal{W}_q)_1 = h^{0,2}(\mathcal{W}_q)_2 = 0.$$

Note that $b_1(\mathcal{W}_q) = b_3(\mathcal{W}_q) = 0$, $h^{1,1}(\mathcal{W}_q)_0 = 14$ and $h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2 = 3$. Thus

$$H^3(\mathcal{W}_q \times \mathbb{F}_3, \mathbb{C})_0 = \bigoplus_{t=0}^{2} H^2(\mathcal{W}_q, \mathbb{C})_t \otimes H^1(\mathbb{F}_3, \mathbb{C})_{[3-t]_3}.$$

Hence one concludes that

$$H^3(\mathcal{W}_q \times \mathbb{F}_3, \mathbb{C})_0 = (H^{2,0}(\mathcal{W}_q)_2 \oplus H^{1,1}(\mathcal{W}_q)_2) \otimes H^{1,0}(\mathbb{F}_3)_1$$

$$\oplus (H^{1,1}(\mathcal{W}_q)_1 \oplus H^{0,2}(\mathcal{W}_q)_1) \otimes H^{0,1}(\mathbb{F}_3)_2.$$

This implies that

$$H^{2,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = H^{1,1}(\mathcal{W}_q)_2 \otimes H^{1,0}(\mathbb{F}_3)_1 \text{ such that } h^{2,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = 3.$$

Hence by Proposition 10.3.2 and the fact that $b_1(\mathbb{P}^1) = 0$, one concludes easily:

Proposition 10.3.5. *For all $q \in \mathcal{M}_3$ one has*

$$h^{1,2}(\mathcal{Q}_q) = h^{2,1}(\mathcal{Q}_q) = 3.$$

Next we show that $\mathcal{Q}$ is a maximal family of Calabi-Yau manifolds. First let us define maximality. For this definition recall:

Proposition 10.3.6. *Each Calabi-Yau manifold X has a local universal deformation $\mathcal{X} \to B$, where*

$$\dim(B) = h^{2,1}(X).$$

Proof. (see [49], 10.3.2) □

Definition 10.3.7. A family $\mathcal{F} \to Y$ of Calabi-Yau manifolds is maximal in $0 \in Y$, if the universal property of the local universal deformation $\mathcal{X} \to B$ of $\mathcal{F}_0$ yields a surjection of a neighborhood of 0 onto B. The family $\mathcal{F} \to Y$ is maximal, if it is maximal in all $0 \in Y$.

Remark 10.3.8. If the family $\mathcal{F} \to Y$ of Calabi-Yau manifolds is maximal in some $0 \in Y$, its restriction to the complement of a closed analytic subvariety of Y is maximal.

Remark 10.3.9. Since $\mathcal{W}_q$ is birationally equivalent to $\mathbb{F}_3 \times \mathcal{C}_q/\langle(1,1)\rangle$ (see Proposition 8.2.4), one has

$$H^{2,0}(\mathcal{W}_q) = H^{1,0}(\mathbb{F}_3)_1 \otimes H^{1,0}(\mathcal{C}_q)_2,$$

where $\mathcal{C}$ denotes the family of degree 3 covers with a pure $(1,3) - VHS$. Thus by our former notation with respect to the push forward action, the VHS of $\mathcal{W}$ depends uniquely on the fractional VHS of the eigenspace $\mathcal{L}_1$ of the VHS of $\mathcal{C}$.

In Section 9.2 we have seen that $\mathcal{Q}$ is birationally equivalent to a quotient of $\mathcal{W} \times \mathbb{F}_3$. It differs by some blowing up morphism with respect to some families of rational curves and some isolated sections. Thus by similar arguments, the VHS of $\mathcal{Q}$ depends on the VHS of $\mathcal{W}$. Hence the VHS of $\mathcal{Q}$ depends uniquely on the fractional VHS of $\mathcal{L}_1$. Thus the period map of $\mathcal{Q}$ can be considered as a multivalued map to the ball $\mathbb{B}_3$.

The preceding remark tells us the period map of the family $\mathcal{Q} \to \mathcal{M}_3$ is locally injective. Hence by the Torelli theorem for Calabi-Yau manifolds, one concludes:

Theorem 10.3.10. *The family $\mathcal{Q} \to \mathcal{M}_3$ is maximal.*

10.4 Outlook onto quotients by cyclic groups of high order

Recall that we used $K3$ surfaces S and elliptic curves E with cyclic degree m covers $S \to R$ and $E \to \mathbb{P}^1$ to construct Calabi-Yau 3-manifolds by a quotient, where $m = 2, 3$. In this chapter we give an outlook on the possibilities to use of cyclic groups of higher order for the construction of Calabi-Yau 3-manifolds by an elliptic curve and a $K3$-surface.

First the following Lemma shows that there are only finitely many elliptic curves with an action of a cyclic group with order $m > 2$, which could be suitable:

Lemma 10.4.1. *Let E be an elliptic curve, and $f : E \to \mathbb{P}^1$ be a cyclic cover. Then one obtains*

$$m := deg(f) = 2, \quad 3, \quad 4 \quad or \quad 6.$$

For each $m > 2$ there is at most only one elliptic curve having a cyclic cover $f : E \to \mathbb{P}^1$ of degree m.[2]

[2]The well-educated reader knows the automorphism group of the abelian variety given by one elliptic curve. But the quotient map by a cyclic subgroup of this automorphism group is fully ramified at the zero-point. There may be cyclic covers, which are not fully ramified over all branch points. Hence for the proof of this lemma, it is not sufficient to know the automorphism group of this Abelian variety.

Proof. We use Proposition 2.3.4 and Corollary 2.3.5. Let $f : E \to \mathbb{P}^1$ be be a cyclic cover of degree $m > 2$. Moreover if f has n branch points, then $\mathbb{L}_1$ is of type (p, q) with $p + q = n - 2$. Thus there must be at least 2 branch points. If there are 2 branch points, we are in the case of the cover $\mathbb{P}^1 \to \mathbb{P}^1$ given by $x \to x^m$. Since $\mathbb{L}_1$ is of type (p, q) with $p + q = n - 2$, C can be an elliptic curve for $m > 2$, only if $n = 3$.

For $n = 3$ and $m > 2$ we have that $\mathbb{L}_1$ is of type (p, q) with $p + q = 1$. Without loss of generality we assume that $p = 0$ and $q = 1$. Hence by Proposition 2.3.4, one concludes that

$$\mu_1 + \mu_2 + \mu_3 = 1.$$

If $m = 3$, one has only the case of the Fermat curve of degree 3 given by

$$\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}.$$

If $m > 3$, $\mathbb{L}_2$ must be of type $(0, 0)$, which implies without loss of generality that $\mu_1 = \frac{1}{2}$. Hence for $m = 4$ we have only the case of the cover given by

$$\mu_1 = \frac{1}{2}, \quad \mu_2 = \mu_3 = \frac{1}{4}.$$

If $m > 4$, $\mathbb{L}_2$ and $\mathbb{L}_3$ must be of type $(0, 0)$, which implies without loss of generality that $\mu_1 = \frac{1}{2}$ and $\mu_2 = \frac{1}{3}$. Hence we obtain the only additional case given by the degree 6 cover with the local monodromy data

$$\mu_1 = \frac{1}{2}, \quad \mu_2 = \frac{1}{3}, \quad \mu_3 = \frac{1}{6}.$$

□

Let S be a $K3$-surface, E be an elliptic curve and the cyclic groups $\langle \gamma_S \rangle$ and $\langle \gamma_E \rangle$ of order $m > 1$ acting on S and E with the loci F_S and F_E of fixed points such that γ_S and γ_E act by -1 on the global sections of the respective canonical sheaves. The aim is the construction of a Calabi-Yau 3-manifold by a desingularisation of $S \times E/\langle(\gamma_S, \gamma_E)\rangle$. The following proposition tells us that there are singularities on $S \times E/\langle(\gamma_S, \gamma_E)\rangle$, if $m > 2$:

Proposition 10.4.2. *Let $m > 2$. Then γ_S must fix some points.*

Proof. If γ_S does not fix any point, one concludes by the Hurwitz formula that $\varphi_S^*\omega = \mathcal{O}$. Thus the quotient has a canonical sheaf ω with $\omega^{\otimes m} = \mathcal{O}$ for $m > 2$. Moreover it has the Betti number $b_1 = 0$. In addition it must be a minimal model, since a rational -1 curve would (up to linear equivalence) be in the support of the canonical divisor K and forbid any torsion of K. But by the Enriques-Kodaira classification (compare to [5], **VI**), such a minimal model does not exist. □

Remark 10.4.3. The branch points of the degree 4 resp., the degree 6 cover $E \to \mathbb{P}^1$ have different branch indeces. Hence for the degree 4 and degree 6 case this yields some problems to find a desingularisation of

$$S \times E/\langle(\gamma_S, \gamma_E)\rangle,$$

which is a Calabi-Yau manifold.

Chapter 11

Maximal families of $CMCY$ type

In this chapter we use the classification of involutions on $K3$ surfaces by V. V. Nikulin [42]. We will see that certain involutions on the integral cohomology of $K3$ surfaces yield a possibility to construct $CMCY$ families of 3-manifolds with maximal variations of Hodge structures. For each $n \in \mathbb{N}$ with $n \leq 11$ we will obtain a holomorphic maximal $CMCY$ family over a basis of dimension n.

11.1 Facts about involutions and quotients of $K3$-surfaces

In this section we collect some known facts about $K3$ surfaces and their involutions, which we will need in the sequel.

11.1.1. The integral cohomology $H^2(S, \mathbb{Z})$ is a lattice of rank 22. We have the cup-product $(\cdot, \cdot)$ on $H^2(S, \mathbb{Z})$. Let $L := (H^2(S, \mathbb{Z}), (\cdot, \cdot))$. It is a well-known fact that one has the orthogonal direct sum decomposition

$$L \cong (-E_8) \oplus (-E_8) \oplus H \oplus H \oplus H,$$

where $-E_8$ consists of $\mathbb{Z}^8$ endowed with a certain negative definite integral bilinear form and H denotes the hyperbolic plane, i. e. $H = (\mathbb{Z}^2, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see [5], **VIII**. 1 and also [5], **I**. Examples 2.7 for details).

Remark 11.1.2. Let S be a $K3$-surface and $L = H^2(S, \mathbb{Z})$, where L is endowed with an involution ι. Assume that ι corresponds to an involution on S, which acts by the character -1 on $\Gamma(\omega_S)$. Then the involution induces a degree 2 cover $\gamma : S \to R$ onto a smooth surface R. Moreover the divisor of fixed points, which yields the ramification divisor of γ, consists of a disjoint union of smooth curves or it is the zero-divisor. Moreover ι yields integral sub-Hodge structures $H^2(S, \mathbb{Z})_0$ and $H^2(S, \mathbb{Z})_1$ of $H^2(S, \mathbb{Z})$ such that ι acts by $(-1)^i$ on $H^2(S, \mathbb{Z})_i$. Since ι acts by -1 on $\Gamma(\omega_S)$ and

$$H^2(R, \mathbb{Q}) = H^2(S, \mathbb{Q})_0,$$

one has that

$$H^{2,0}(S), H^{0,2}(S) \subset H^2(S, \mathbb{C})_1.$$

Moreover the intersection form has the signature $(2, r)$ on $H^2(S, \mathbb{Z})_1$ (compare to [48], §1 and [48], 2.1).

Remark 11.1.3. Let

$$D = \{[\omega] \in \mathbb{P}(H^2(S, \mathbb{C})_1) | (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}.$$

By the Torelli theorem, each marked $K3$ surface $(S', \phi_{S'})$ endowed with an involution, which yields the the same involution ι on his cohomology lattice, yields a unique one dimensional vector space $H^{2,0}(S') \subset H^2(S, \mathbb{C})_1$ corresponding to some $p \in D$.

11.2 The associated Shimura datum of D

The Hodge structure of a $K3$ surface S with a cyclic degree 2 cover onto a rational surface resp., Enriques surface R has a decomposition into two rational Hodge structures $H^2(S, \mathbb{Q})_1$ and $H^2(S, \mathbb{Q})_0$. We consider $H^2(S, \mathbb{Q})_1$, since the variation of Hodge structures given by $H^2(S, \mathbb{Q})_0$ is trivial.

The Hodge decomposition of $H^2(S, \mathbb{C})$ is orthogonal with respect to the Hermitian form $(\cdot, \bar{\cdot})$. Hencefore the corresponding embedding

$$h : S^1 \to \mathrm{SL}(H^2(S, \mathbb{R})_1)$$

factors trough the special orthogonal group $\mathrm{SO}(H^2(S, \mathbb{R})_1)$ with respect to the symmetric form given by the cup product pairing, where $\mathrm{SO}(H^2(S, \mathbb{R})_1)$ is isomorphic to $\mathrm{SO}(2, r)_{\mathbb{R}}$. Let $\omega \in \omega_S \setminus \{0\}$,

$$\Re\omega := \frac{1}{2}(\omega + \bar{\omega}), \quad \Im\omega := \frac{i}{2}(\omega - \bar{\omega})$$

and $\{v_1, \dots v_r\}$ be a basis of $H^{1,1}(X, \mathbb{R})_1$. One has the basis

$$\{\Re\omega, \Im\omega, v_1, \dots, v_r\}$$

of $H^1(X, \mathbb{R})_1$ such that the intersection form is without loss of generality given by the matrix $\mathrm{diag}(1, 1, -1, \dots, -1)$ with respect to this basis. The subgroup, whose elements are invariant under

$$g \to h(i)gh(i^{-1}),$$

is given by $\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(r))$, where

$$h(i) = h(i^{-1}) = \mathrm{diag}(-1, -1, 1, \dots, 1).$$

Since $h^2(i) = h(-1) = \mathrm{diag}(1, \dots, 1)$, the action of i is an involution. This implies that one has a decomposition of $\mathfrak{so}_{2,r}(\mathbb{R})$ into 2 eigenspaces with respect to the eigenvalues 1 and -1. Hence $h(\sqrt{i})$ yields a complex structure on the eigenspace with eigenvalue -1. The eigenspace for the eigenvalue 1 is given by the Lie algebra of $\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(r))$. Thus we have a decomposition

$$\mathfrak{so}_{2,r}(\mathbb{C}) = \mathfrak{h}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_-$$

such that S^1 acts by the characters $z/\bar{z}$, 1 and $\bar{z}/z$ on the respective complex sub-vector spaces.

We continue our consideration of the involution ι given by

$$\iota(g) = h(i)gh^{-1}(i).$$

The matrices $M_1 \in \mathrm{SO}(2,r)(\mathbb{C})$ with $\bar{M}_1 = \iota(M_1)$ satisfy that

$$\bar{M}_1 = \mathrm{diag}(-1,-1,1,\ldots,1) \cdot M_1 \cdot \mathrm{diag}(-1,-1,1,\ldots,1)$$

$$= \mathrm{diag}(1,1,-1,\ldots,-1) \cdot M_1 \cdot \mathrm{diag}(1,1,-1,\ldots,-1).$$

Since $\mathrm{SO}(2,r)(\mathbb{C})$ is given by the matrices M satisfying

$$M^t \cdot \mathrm{diag}(1,1,-1,\ldots,-1) \cdot M = \mathrm{diag}(1,1,-1,\ldots,-1)$$

$$\Leftrightarrow M^{-1} = \mathrm{diag}(1,1,-1,\ldots,-1) \cdot M^t \cdot \mathrm{diag}(1,1,-1,\ldots,-1),$$

each matrix M_1 satisfies

$$M_1^{-1} = \bar{M}_1^t.$$

Thus M_1 is contained in the compact group $\mathrm{SU}(2+r)$, and one concludes:

Proposition 11.2.1. *Our morphism*

$$h : S^1 \to \mathrm{SO}(H^2(S,\mathbb{R})_1)_{\mathbb{R}}$$

yields a Shimura datum.

Remark 11.2.2. Note that the simple Lie group $\mathrm{SO}(2,r)(\mathbb{R})$ consists of two connected components (see [17], Exercise 7.2). Since the Lie group $\mathrm{SO}(2+r)(\mathbb{C}) \cong \mathrm{SO}(H^2(S,\mathbb{R})_1)(\mathbb{C})$ is connected (see [23], **IX**. Lemma 4.2), the algebraic group $\mathrm{SO}(H^2(S,\mathbb{R})_1)$ is connected, too. Recall that all Cartan involutions of the simple algebraic group $\mathrm{SO}(H^2(S,\mathbb{R})_1)$ are conjugate. The action of S^1 on $H^2(S,\mathbb{R})_1$ is given by its action on $\langle \Re\omega, \Im\omega \rangle$ and S^1 fixes all vectors of $H^{1,1}(S,\mathbb{R})_1$. This implies that all morphisms

$$h : S^1 \to \mathrm{SO}(H^2(S,\mathbb{R})_1),$$

which yield the Hodge structure of a $K3$ surface, satisfy that their images $h(S^1)$ are conjugate. The definition of the Hodge structure on $H^2(S,\mathbb{R})_1$ implies that the $\mathbb{R}$-valued points of the kernel of h are given by $\{1,-1\} \in S^1(\mathbb{R})$. Let $\iota_{S^1} : S^1 \to S^1$ be the involution given by $x \to x^{-1}$. For each morphism h_1 in the conjugacy class of h, there exists exactly one other morphism h_2 with $h_1(S^1) = h_2(S^1)$ and kernel given by $\{1,-1\} \in S^1(\mathbb{R})$, which is given by $h_2 = h_1 \circ \iota_{S^1}$. The conjugation by $\mathrm{diag}(-1,1,-1,1,\ldots,1)$ yields an inner automorphism φ of $\mathrm{SO}(H^2(S,\mathbb{R})_1)$ such that $h_2 = \varphi \circ h_1$. Thus each Hodge structure of a $K3$ surface obtained by some $p \in D$ is obtained by some element of the conjugacy class of our morphism $h : S^1 \to \mathrm{SO}(H^2(S,\mathbb{R})_1)$. Moreover note that the holomorphic VHS over the bounded symmetric domain associated with $\mathrm{SO}(H^2(S,\mathbb{R})_1)(\mathbb{R})^+/K$, which is induced by the natural embedding $\mathrm{SO}(H^2(S,\mathbb{Q})_1) \to \mathrm{GL}(H^2(S,\mathbb{Q})_1)$, is uniquely determined by the variation of the subbundle of rank 1 given by $H^{2,0}$. Since

$$r = \dim(D) = \dim(\mathrm{SO}(H^2(S,\mathbb{R})_1)(\mathbb{R})/K),$$

this VHS yields a biholomorphic map from the bounded symmetric domain associated with $\mathrm{SO}(H^2(S,\mathbb{R})_1)(\mathbb{R})^+/K)$ onto D^+.

The preceding remark and Theorem 1.5.9 imply:

Theorem 11.2.3. *There is a dense set of CM points on D with respect to the VHS on D obtained by Remark* 11.2.2.

11.3 The examples

First we construct a holomorphic family of marked $K3$-surfaces with a global involution over its basis:

Construction 11.3.1. There exists a universal family $u : \mathcal{X} \to B$ of marked analytic $K3$-surfaces, whose basis is not Hausdorff (see [5], **VIII**. 12). Let ϕ denote the global marking of the family $\mathcal{X} \to B$. We consider an involution ι on a marked $K3$ surface (S, ϕ), which acts by -1 on $H^{2,0}(S)$. This involution yields an involutive isometry ι on the lattice L. Thus the involution ι endows $\mathcal{X} \to B$ with a new marking $\iota \circ \phi$. By the universal property of the universal family, this new marking yields an involution of the family:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\iota_{\mathcal{X}}} & \mathcal{X} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ B & \xrightarrow{\iota_B} & B \end{array}$$

Let $\Delta : B \to B \times B$ denote the diagonal embedding. We define

$$B_\iota = \mathrm{Graph}(\iota_B) \cap \Delta(B) \subset B \times B.$$

Note that each point $b \in B_\iota$ has an analytic neighborhood $U \subset B$ such that $\mathcal{X}_U \to U$ is given by the Kuranishi family and yields an injective period map for U. Thus on $U \times U$ the diagonal $\Delta(U)$ and $\mathrm{Graph}(\iota_B|_U)$ are closed analytic submanifolds. Hence B_ι has the structure of an analytic variety, which is not necessarily Hausdorff, and can have singularities. The composition $\Delta \circ u$ allows to consider $\Delta(B)$ as basis of the universal family of the marked $K3$ surfaces. By the restricted family $\mathcal{X}_{B_\iota} \to B_\iota$, we obtain a holomorpic family with a global involution over the basis B_ι. For simplicity we write $\mathcal{X}_\iota \to B_\iota$ instead of $\mathcal{X}_{B_\iota} \to B_\iota$.

Remark 11.3.2. The fibers of $\mathcal{X}_\iota \to B_\iota$ have by the involution ι a cyclic covering onto a projective surface (compare to [48], 2.1). Thus the fibers of $\mathcal{X}_\iota \to B_\iota$ are algebraic.

Proposition 11.3.3. *Assume that for all $b \in B_\iota$ the involution $\iota_{\mathcal{X}_b}$ on $\mathcal{X}_b$ has a locus of fixed points consisting of rational curves. Then the holomorphic family $\mathcal{X}_\iota \to B_\iota$ is due to its global involution suitable for the construction of a holomorphic Borcea-Voisin tower.*

Proof. Let $b_0 \in B_\iota$ and $U \subset B_\iota$ be a small open neighborhood of b_0. The eigenspace decomposition with respect to ι yields a variation of Hodge structures on the eigenspace with respect to -1. The corresponding period map yields an open injection of U into D. By the fact that D has a dense set of CM points, the family $\mathcal{X}_\iota \to B_\iota$ has a dense set of CM fibers. Since the locus of fixed points with respect to $\iota_{\mathcal{X}_b}$ consists of rational curves, this locus of fixed points has complex multiplication, too. Hence $\mathcal{X}_\iota \to B_\iota$ can be used for the construction of a holomorphic Borcea-Voisin tower. □

Assume that $\mathcal{X}_\iota \to B_\iota$ satisfies the assumptions of Proposition 11.3.3. Then let $\mathfrak{X}_\iota \to B_\iota \times \mathcal{M}_1$ denote the family obtained by the holomorphic Borcea-Voisin tower from $\mathcal{X}_\iota \to B_\iota$ and $\mathcal{E} \to \mathcal{M}_1$ denote the family of elliptic curves.

Definition 11.3.4. A family $\mathcal{F} \to Y$ of Calabi-Yau manifolds is maximal in $0 \in Y$, if the universal property of the local universal deformation $\mathcal{X} \to B$ of $\mathcal{F}_0$ yields a surjection of a neighborhood of 0 onto B. The family $\mathcal{F} \to Y$ is maximal, if it is maximal in all $0 \in Y$.

Theorem 11.3.5. *The family $\mathfrak{X}_\iota$ is maximal.*

Proof. By the following lemma, we start to prove Theorem 11.3.5:

Lemma 11.3.6.
$$H^3((\mathfrak{X}_\iota)_{p\times q}) = H^2((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \otimes H^1(\mathcal{E}_q, \mathbb{Q})$$

Proof. Due to Proposition 10.3.2 and the fact that the exceptional divisors consist of some rational curves, one only needs to determine $H^3((\mathcal{X}_\iota)_p \times \mathcal{E}_q, \mathbb{Q})_0$. Since $b_1((\mathcal{X}_\iota)_p) = b_3((\mathcal{X}_\iota)_p) = 0$ and $H^1(\mathcal{E}_q, \mathbb{Q}) = H^1(\mathcal{E}_q, \mathbb{Q})_1$, we are done. □

By using the preceding lemma, we prove the following proposition.

Proposition 11.3.7. *One has that* $\dim(B_\iota \times \mathbb{B}_1)$ *and* $h^{2,1}((\mathfrak{X}_\iota)_{p\times q})$ *coincide.*

Proof. By Proposition 11.3.6,

$$H^3((\mathfrak{X}_\iota)_{p\times q}) = H^2((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \otimes H^{1,0}(\mathcal{E}_q, \mathbb{Q}) \oplus H^2((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \otimes H^{0,1}(\mathcal{E}_q, \mathbb{Q}).$$

Hencefore

$$h^{2,1}((\mathfrak{X}_\iota)_{p\times q}) = h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \cdot h^{1,0}(\mathcal{E}_q, \mathbb{Q}) + h^{2,0}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \cdot h^{0,1}(\mathcal{E}_q, \mathbb{Q})$$

$$= h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 + h^{2,0}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 = h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 + 1.$$

Recall that D^+ is the bounded symmetric domain obtained by $\mathrm{SO}(2,r)^+(\mathbb{R})$, where $r = h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{R})_1$. By [23], **IX**. Table **II**, D has the complex dimension r.[1] Since the period map $p : B_\iota \to D$ of $\mathcal{X}_\iota \to B_\iota$ is locally bijective, one concludes

$$h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 = r = \dim(D) = \dim(B_\iota),$$

which yields the result. □

By the following proposition, we finish the proof of Theorem 11.3.5: □

Proposition 11.3.8. *The period map yields a multivalued map from $\mathcal{M}_1 \times B_\iota$ to the period domain, which is locally injective.*

Proof. Let $\mathbb{B}$ be a small open subset of $\mathcal{M}_1 \times B_\iota$ and let $x_1, x_2 \in \mathbb{B}$. Note tha the period map p on $\mathcal{M}_1 \times B_\iota$ yields different image points $p(x_1)$ and $p(x_2)$, if the classes of $H^{3,0}((\mathfrak{X}_\iota)_{x_1})$ and $H^{3,0}((\mathfrak{X}_\iota)_{x_2})$ in $\mathbb{P}(H^3((\mathfrak{X}_\iota)_{x_1}, \mathbb{C}))$ do not coincide. The respective period maps on B_ι and $\mathcal{M}_1$ are locally injective and depend only on $\omega_{\mathcal{E}_q}$ and $\omega_{(\mathcal{X}_\iota)_p}$. Since

$$H^{3,0}((\mathfrak{X}_\iota)_{p\times q}) \subset H^3((\mathfrak{X}_\iota)_{p\times q}) = H^2((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \otimes H^1(\mathcal{E}_q, \mathbb{Q})$$

is given by $H^{2,0}((\mathcal{X}_\iota)_p) \otimes H^{1,0}(\mathcal{E}_q)$, the period map concerning $\mathfrak{X}_\iota$ is locally injective, too. □

It remains to classify the possible involutions ι on L, which provide our families $\mathcal{X}_\iota \to B_\iota$ with a global involution.

Remark 11.3.9. The involutions on L, which yield involutions on certain $K3$ surfaces, are characterized by the triples of the following integers (compare to [42]):

[1] By [23], **IX**. Table **II**, D has the dimension $2r$ as real manifold.

- The integer t is the rank of the sublattice $\mathrm{Pic}(S)_0$ of the Picard lattice of an arbitrary fiber S of $\mathcal{X}_\iota$, which is invariant under the global involution.
- By the intersection pairing, one obtains a homomorphism $\mathrm{Pic}(S)_0 \to \mathrm{Pic}(S)_0^\vee$. The integer a is given by $(\mathbb{Z}/(2))^a \cong \mathrm{Pic}(S)_0^\vee/\mathrm{Pic}(S)_0$.
- By the morphism $\mathrm{Pic}(S)_0 \to \mathrm{Pic}(S)_0^\vee$, the intersection form on $\mathrm{Pic}(S)_0$ yields a quardratic form q on $\mathrm{Pic}(S)_0^\vee$ with values in $\mathbb{Q}$. The integer δ is 0, if q has only values in $\mathbb{Z}$ and 1 otherwise.

For a fixed triple (t, a, δ) we write $\mathcal{X}_{(t,a,\delta)} \to B_{(t,a,\delta)}$ instead of $\mathcal{X}_\iota \to B_\iota$ and $\mathfrak{X}_{(t,a,\delta)}$ instead of $\mathfrak{X}_\iota$.

Remark 11.3.10. The ramification locus of the fibers with respect to the involution on $\mathcal{X} \to \mathcal{B}_{t,a,\delta}$ is given by two elliptic curves, if $(t, a, \delta) = (10, 8, 0)$, is empty, if $(t, a, \delta) = (10, 10, 0)$, and otherwise given by $C_{N'} + E_1 + \ldots + E_{N-1}$, where $C_{N'}$ is a curve of genus

$$N' = \frac{1}{2}(22 - t - a), \quad \text{and} \quad N = \frac{1}{2}(t - a) + 1.$$

(compare to [42])

Hencefore the triples

$$(t, a, \delta) = (10, 10, 0) \quad \text{and} \quad (t, a, \delta) \quad \text{with} \quad t + a = 22$$

yield the examples of families $\mathcal{X}_{(t,a,\delta)} \to B_{(t,a,\delta)}$ with global involutions over the basis, whose locus of fixed points consists at most of families of rational curves. Hence by Proposition 11.3.3, these triples yield maximal holomorphic $CMCY$ families of 3-manifolds.

11.3.11. By [42], Figure 2, one gets the following complete list of holomorphic maximal $CMCY$ families $\mathfrak{X}_{(t,a,\delta)} \to B_{(t,a,\delta)} \times \mathcal{M}_1$ of 3-manifolds obtained by this method. By Claim 10.2.1, we obtain the Hodge numbers $h^{1,1}$ and $h^{2,1}$ of the fibers of $\mathfrak{X}_{(t,a,\delta)}$.

t	a	δ	N	$h^{1,1}$	$h^{2,1}$
10	10	0	0	11	11
11	11	1	1	16	10
12	10	1	2	21	9
13	9	1	3	26	8
14	8	1	4	31	7
15	7	1	5	36	6
16	6	1	6	41	5
17	5	1	7	46	4
18	4	1	8	51	3
18	4	0	8	51	3
19	3	1	9	56	2
20	2	1	10	61	1

Remark 11.3.12. In [7] there is a construction of Calabi-Yau manifolds of dimension 3 with CM by using 3 elliptic curves with involutions. This construction yields a $CMCY$ family of 3-manifolds over $\mathcal{M}_1 \times \mathcal{M}_1 \times \mathcal{M}_1$. The fibers have the Hodge numbers $h^{1,1} = 51$

and $h^{2,1} = 3$. By similar arguments as in Theorem 11.3.5, this family is maximal. The associated period domain is given by $\mathbb{B}_1 \times \mathbb{B}_1 \times \mathbb{B}_1$.

As we have seen in Section 10.3, the family $\mathcal{Q} \to \mathcal{M}_3$ is a maximal $CMCY$ family of 3-manifolds, whose fibers have the same Hodge numbers $h^{1,1} = 51$ and $h^{2,1} = 3$. The associated period domain is given by $\mathbb{B}_3$

Moreover by Theorem 11.3.5 and the preceding point, we have two additional holomorphic maximal $CMCY$ families of 3-manifolds, whose fibers have the same Hodge numbers $h^{1,1} = 51$ and $h^{2,1} = 3$. The associated period domain is given by $\mathbb{B}_1 \times D$, where D denotes the bounded domain given by $\mathrm{SO}(2,2)(\mathbb{R})/K$

Hence there exist 4 maximal $CMCY$ families of 3-manifolds, whose fibers have the Hodge numbers $h^{1,1} = 51$ and $h^{2,1} = 3$. One can easily check that the example of [7] has a Yukawa coupling of *length* 3, where the Yukawa coupling of the family $\mathcal{Q} \to \mathcal{M}_3$ constructed in Section 9.2 has the *length* 1. Hence there are not any open sets of the respective bases, which allow a local identification of these two families.

By using the involutions on elliptic curves, one gets a local identification between $\mathcal{E} \times \mathcal{E}/\langle(\iota_{\mathcal{E}}, \iota_{\mathcal{E}})\rangle \to \mathcal{M}_1 \times \mathcal{M}_1$, which yields the example of [7], with one of our examples $\mathcal{X}_{(t,a,\delta)} \to B_{(t,a,\delta)}$ with $t = 18$ and $a = 4$. This implies a local identification between the resulting $CMCY$ families of 3-manifolds obtained by the Borcea-Voisin tower.

Remark 11.3.13. It would be interesting to consider the following question: Is the maximal $CMCY$ family $\mathfrak{X}_{(10,10,0)}$ its own mirror family?

Let S denote a $K3$ surface with an involution, which acts by -1 on $\Gamma(\omega_S)$. In [48] the triples (t, a, δ), which yield our families $\mathcal{X}_{(t,a,\delta)} \to B_{(t,a,\delta)}$ satisfying the assumptions of Proposition 11.3.3, do not satisfy the assumptions of the technical Lemma [48], Lemme 2.5. This Lemma guarantees the existence of a hyperbolic plane $H \subset H^2(S, \mathbb{Z})_1$, which is needed for the mirror construction in [48]. Hence these triples (t, a, δ) do not satisfy the assumptions of the Mirror Theorem [48], Théorème 2.17. But by [11], Lemma 4.4.4, there is a hyperbolic plane $H \subset H^2(S, \mathbb{Z})_1$ for these triples, too.

In her construction of a Calabi-Yau 3-manifold ([48], Lemme 1.3) C. Voisin assumes that the involution on the $K3$ surface is not given by the triple $(10, 10, 0)$, since it is easy to see that the resulting 3-manifold is not simply connected in this case. But by Proposition 7.2.5 the resulting 3-manifold satisfies our definition of a Calabi-Yau manifold (Definition 7.2.1) in this case, too.

The mirror of a fiber of $\mathfrak{X}_{(10,10,0)}$ must have the same Hodge numbers $h^{1,1} = h^{2,1} = 11$. By Claim 10.2.1, this implies for an involution on a $K3$ surface:

$$5N - N' = 5N' - N = 0$$

Hence one calculates easily that $N = N' = 0$. Thus by V. V. Nikulins [42] classification of involutions on $K3$ surfaces, the Voisin-Borcea Mirror (in the notation of [11]) of a fiber of $\mathfrak{X}_{(10,10,0)}$ should be obtained by the triple $(10, 10, 0)$, too. Hence the author has the impression that one can consider the maximal $CMCY$ family $\mathfrak{X}_{(10,10,0)}$ of 3-manifolds as its own mirror family, but one must check the details.

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