On Shimura Curves in the Schottky locus

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Contents

Introduction

In this thesis we deal with the occurrence of Shimura varieties in the Schottky locus. By Shimura variety we mean a Shimura variety of Hodge type which is an étale covering of a certain moduli space of abelian varieties with prescribed Mumford-Tate group and a suitable level structure as defined in [Mu66].

Let $\mathscr{A}_{g,1}$ be the moduli space over $\mathbb{C}$ of g-dimensional principally polarized abelian varieties and let $\mathscr{M}_g$ be the moduli space of curves of genus g. The Torelli map

$$j : \mathscr{M}_g \longrightarrow \mathscr{A}_{g,1}$$

which assigns to a curve C its principally polarized Jacobian J is an immersion and we consider $\mathscr{M}_g$ as a subspace of $\mathscr{A}_{g,1}$ which we will call the open Schottky locus. Let $\mathscr{M}_g^c$ be the Zariski closure of $\mathscr{M}_g$ in $\mathscr{A}_{g,1}$. $\mathscr{M}_g^c$ is called the Schottky locus. The letter "c" stands for closure as well as for compact because the boundary of $\mathscr{M}_g^c$ consists of the images under the Torelli map of singular stable curves whose Jacobian is still compact, e. g. two smooth curves meeting in exactly one point.

The question is whether there are Shimura varieties U in $\mathscr{A}_{g,1}$ which lie in the Schottky locus $\mathscr{M}_g^c$ or not. Of course, there are, e. g. families of trees of elliptic curves. But these are trivial examples since they lie completely in the boundary of $\mathscr{M}_g^c$. So, the better question is whether there are Shimura varieties U in $\mathscr{A}_{g,1}$ lying in the Schottky locus $\mathscr{M}_g^c$ which intersect the open Schottky locus $\mathscr{M}_g$ non-trivially. We are referring to the second question if we speak about Shimura varieties in the Schottky locus.

A special property about Shimura varieties is that they contain a dense set of CM-points. A CM-point of $\mathscr{A}_{g,1}$ is a point whose corresponding abelian variety admits complex multiplication. The André-Oort conjecture states that the converse should also be true, see [An89] and [Oo94]. More precisely, any subvariety U of $\mathscr{A}_{g,1}$ containing a Zariski-dense set of CM-points is supposed to be a Shimura Variety.

So the occurrence of Shimura varieties in the Schottky locus $\mathscr{M}_g^c$ is linked with the occurrence of CM-points in $\mathscr{M}_g$. In 1987 Coleman made the following conjecture.

Conjecture 1 (Coleman [Co87]) *For $g \gg 0$, the set of CM-points in the moduli space of curves $\mathscr{M}_g$ is finite.*

Coleman actually suggested that this could be true for $g \geq 4$ while it clearly fails for $g \leq 3$ since then $\mathscr{M}_g$ and $\mathscr{A}_{g,1}$ have the same dimension. But de Jong and Noot constructed counter-examples for $g = 4$ and $g = 6$ in [dJN91]. Nevertheless, the Coleman conjecture suggests that for large g there are no Shimura varieties in the Schottky locus.

In [Ha99] Hain studied families of compact Jacobians over locally symmetric domains U satisfying an additional technical condition. Based on his methods, de Jong and Zhang [dJZ06] were able to exclude certain types of Shimura varieties U. However, they did not handle the case $\dim(U) = 1$.

So, we focus our attention to Shimura curves, i. e. one-dimensional Shimura varieties. More precisely, we will look at Shimura curves U' which are étale covers of some curve U in the moduli stack $\mathscr{A}_{g,1}$. Let $A \xrightarrow{f} Y$ be a semistable family of abelian varieties over a projective curve Y, $U = Y - S$ the smooth locus and $V = f^{-1}(U)$ so that $V \xrightarrow{f} U$ is an abelian scheme. Consider the Higgs bundle (E, θ) given by taking the graded sheaf of the Deligne extension of $R^1 f_* \mathbb{C}_V \otimes \mathcal{O}_U$ where $R^1 f_* \mathbb{C}_V$ is the weight 1 variation of Hodge structures. We have a decomposition $E = F \oplus N$ into an ample part F and a flat part N. Following [VZ03] we say that the Higgs field is *maximal* if

$$\theta^{1,0} : F^{1,0} \longrightarrow F^{0,1} \otimes \Omega^1_Y(\log S)$$

is an isomorphism, and that the Higgs field is strictly maximal if additionally $N = 0$. Then Viehweg and Zuo showed the following theorem.

Theorem 2 (Viehweg, Zuo [VZ04]) *Assume that each irreducible and non-unitary sub-variation $\mathbb{V}$ of Hodge structures in $R^1 f_* \mathbb{C}_V$ has a strictly maximal Higgs field. Then there is an étale covering $U' \to U$ such that U' is a Shimura curve and $f' : V' \to U'$ is the corresponding universal family.*

Möller showed in [Mö05] that the converse is also true, namely if $V \to U$ is the universal family of a Shimura curve, then its Higgs field is strictly maximal. So we have a characterization of Shimura curves by the maximality of the Higgs field of the corresponding universal family.

The notion of strict maximality was further extended to higher weight variations of Hodge structures and it turns out that it is of numerical nature [VZ06] since it is equivalent to the case that certain Arakelov type inequalities actually become equalities. We discuss this for families of abelian varieties in section C.5.

Combining the results of Viehweg and Zuo [VZ06] which say that a Shimura curve U in $\mathscr{M}_g$ has to be non-compact with the techniques of Möller [Mö06] shows that U has also to be a Teichmüller curve. Then from [Mö05] it follows that there are no such curves in $\mathscr{M}_g$ unless $g = 3$.

Theorem 3 (Möller, Viehweg, Zuo [MVZ05]) *For $g \geq 2$ the moduli space of curves $\mathscr{M}_g$ does not contain any compact Shimura curves, and it contains a non-compact Shimura curve if and only if $g = 3$.*

Observe that this result deals with the occurrence of Shimura curves in $\mathscr{M}_g$ rather than $\mathscr{M}_g^c$. So it does not answer the question if there are Shimura curves in the Schottky locus.

Returning to a family of abelian varieties $A \to Y$ with strict maximal Higgs field, the analysis of the structure of the weight one variation of Hodge structures $R^1 f_* \mathbb{C}_V$ in [VZ04] yields the following result about the structure of $A \to Y$.

Theorem 4 (Viehweg, Zuo [VZ04]) *If $S \neq \emptyset$ consists of an even number of points, and if $V \to U$ admits a strict maximal Higgs field, then there is an étale covering $Y' \to Y$ such that $A' \to Y'$ is Y'-isogenous to a product*

$$E \times_{Y'} \ldots \times_{Y'} E \times_{\mathbb{C}} B$$

where $B/\mathbb{C}$ is an abelian variety and $E \to Y'$ is a modular family of elliptic curves.

Hence, if there is such a Shimura curve covering a curve in the Schottky locus, there must be a corresponding family of curves $C' \to Y'$ whose family of Jacobians has a decomposition as described in the theorem above. Remember that the Coleman conjecture predicts that such families of curves should not exist. We prove a result in this direction. The test case for this prediction is that Y is rational. Then we do not have to care about étale coverings since there are no étale coverings of $\mathbb{P}^1_{\mathbb{C}}$ except for automorphisms. Further we assume that there is no constant part. So we have to deal with a family of curves $C \to Y$ of genus g whose Jacobian is Y-isogenous to the g-fold product of a modular family of elliptic curves $E \to Y$. We show that the genus g of such a family is bounded. More generally, we show the following theorem for arbitrary base curves Y.

Theorem 5 *Let $C \to Y$ be a family of curves of genus g whose Jacobian $J \to Y$ is Y-isogenous to the g-fold product of a non-isotrivial family of elliptic curves $E \to Y$ which can be defined over a number field. Then the genus g is bounded, i. e. there is a constant $d = d(E \to Y)$ depending only on $E \to Y$ such that g is smaller than d.*

Mind that modular families of elliptic curves can be defined over number fields. The numerical nature of the maximality of the Higgs field shows that for $Y = \mathbb{P}^1_{\mathbb{C}}$ and $J \to \mathbb{P}^1_{\mathbb{C}}$ with exactly 4 singular fibers, we will have an isogeny from $E \times \ldots \times E \times_{\mathbb{C}} B$ to J as in Theorem 4. We discuss this in the last section. Since there are only six semistable families of elliptic curves over $\mathbb{P}^1_{\mathbb{C}}$ having 4 singular fibers, we may immediately conclude the following corollary.

Corollary 6 *There is a natural number c such that for any family of curves $C \to \mathbb{P}^1_{\mathbb{C}}$, whose Jacobian $J \to \mathbb{P}^1_{\mathbb{C}}$ has no constant part and 4 singular fibers, the genus of the fibers of $C \to \mathbb{P}^1_{\mathbb{C}}$ is bounded by c.*

We will prove Theorem 5 by reducing the problem to characteristic p. Therefore, we will study in the first chapter families of curves $C \to Y$ defined over a base curve $Y/\mathbb{F}_q$. We will see that the genus g of the fibers is bounded if the Jacobian $J \to Y$ is Y-isogenous to the g-fold product of a non-isotrivial family of elliptic curves $E \to Y$. Moreover, this bound will only depend on $E \to Y$. We do this by counting the singularities δ in the fibers of $C \to Y$. Combining the Weil conjectures for the fibers with the Sato-Tate conjecture about the distribution of Frobenius traces in a family of elliptic curves $E \to Y$ will yield a lower bound for δ. On the other hand, the geometry of the total space C of $C \to Y$ will give an upper bound. We will see that for large genus g the lower bound will exceed the upper bound. Thus, the genus has to be bounded.

In the second chapter, we discuss the above situation when Y is defined over a number field. Here, we will reduce the family of curves $C \to Y$ to characteristic p. Then, the problem is to conclude that the family degenerates in characteristic 0 if it degenerates in characteristic p. We do this by characterizing the reducibility of curves with compact Jacobian in terms of the existence of certain idempotent endomorphisms on the Jacobian. Then we show that these endomorphisms will lift from characteristic p to characteristic 0. Hence, reducibility in characteristic p will carry over to characteristic 0.

In the final chapter, we regard the situation of Theorem 5. We will reduce the situation from $\mathbb{C}$ to a number field by studying Galois-representations on torsion points of families of elliptic curves, and using fine moduli spaces of curves and abelian varieties with level-structures to descend from the complex numbers to a number field. Then we will prove the results announced above.

Chapter A

Bounding the genus in characteristic p

In this chapter we want to prove that the genus g of a curve C defined over the function field K of a curve Y defined over a finite field $\mathbb{F}_q$ is bounded if the Jacobian J of C is isogenous over K to the g-fold product of a single non-isotrivial elliptic curve E over K, and, that this bound depends only on E.

We will proceed as follows. In the first section we will discuss the problem of curves with split Jacobian, i. e. curves whose Jacobian is rationally isogenous to a product of elliptic curves, for curves defined over finite fields $\mathbb{F}_q$. This case was studied by Serre for arbitrary products and we will present his results. Further we will give explicit bounds for products of a single elliptic curve. Except for this last result we will not use anything from this section in what follows. So the content is mostly for the interested reader who wants to know what happens in general in the finite field case.

In the second section we will turn to curves over function fields in its geometric incarnation as semistable families of curves $X \to Y$. There we derive an upper bound for the number of singularities δ in the fibers of $X \to Y$ expressed in terms of the degree of the push-forward of the relative dualizing sheaf $\omega_{X/Y}$.

Then, in the third section, we relate this degree to the height of the Jacobian. From this we will get a more explicit bound in the case of a split Jacobian. In particular, we will be able to estimate the asymptotic growth of the upper bound in terms of the genus g.

The final conclusion comes in the fourth section where we use our results from the first section to give a lower bound in terms of g for the number of singularities in the fibers of $X \to Y$. Since we will see that the lower bound exceeds the upper bound for high genus g we can conclude that the genus of C is bounded.

A.1 Curves over finite fields

Let C be a smooth geometrically connected curve defined over a finite field $\mathbb{F}_q$. We want to show that the genus g of C is bounded if the Jacobian J of C is isogenous over $\mathbb{F}_q$ to a product of elliptic curves.

The idea is to study the Galois representation induced by C and its connection with the Weil conjectures. If $N_{q^n}(C)$ denotes the number of $\mathbb{F}_{q^n}$-rational points of C then we have

$$N_{q^n}(C) = q^n + 1 - \mathrm{Tr}\big(F_C^n \,\big|\, H^1(\bar{C}, \mathbb{Q}_\ell)\big)$$

where F_C denotes the q-th power Frobenius endomorphism of $C/\mathbb{F}_q$. Thus the number of rational points is determined by the Galois action on the ℓ-adic cohomology of C. If J is the Jacobian of C, then we have an isomorphism $H^1(\bar{C}, \mathbb{Q}_\ell) \cong H^1(\bar{J}, \mathbb{Q}_\ell)$ compatible with the Galois action. So we get

$$\mathrm{Tr}\big(F_C^n \,\big|\, H^1(\bar{C}, \mathbb{Q}_\ell)\big) = \mathrm{Tr}\big(F_J^n \,\big|\, H^1(\bar{J}, \mathbb{Q}_\ell)\big)$$

where F_J denotes the Frobenius endomorphism on $J/\mathbb{F}_q$. Let J be isogenous over $\mathbb{F}_q$ to a product $A = E_1 \times \ldots \times E_g$ of elliptic curves $E_i/\mathbb{F}_q$. Since isogenies become isomorphisms on ℓ-adic cohomology, we have isomorphisms

$$H^1(\bar{C}, \mathbb{Q}_\ell) \cong H^1(\bar{J}, \mathbb{Q}_\ell) \cong H^1(\bar{A}, \mathbb{Q}_\ell) = \bigoplus_{i=1}^{g} H^1(\bar{E}_i, \mathbb{Q}_\ell)$$

again compatible with the Galois actions. So we may write

$$\mathrm{Tr}\big(F_C^n \,\big|\, H^1(\bar{C}, \mathbb{Q}_\ell)\big) = \sum_{i=1}^{g} \mathrm{Tr}\big(F_{E_i}^n \,\big|\, H^1(\bar{E}_i, \mathbb{Q}_\ell)\big)$$

and, therefore,

$$N_{q^n}(C) = q^n + 1 - \sum_{i=1}^{g} \mathrm{Tr}\big(F_{E_i}^n \,\big|\, H^1(\bar{E}_i, \mathbb{Q}_\ell)\big).$$

Since the number of rational points is a non-negative integer, this restricts which elliptic curves E_i may occur. We demonstrate this with an easy example.

Example A.1.1 Let $q = 2$ and $E/\mathbb{F}_2$ be the elliptic curve given by

$$E: \; y^2 + xy = x^3 + x.$$

E has 4 $\mathbb{F}_2$-rational points. Thus by the Weil conjectures the trace of the Frobenius is given by

$$4 = N_2(E) = 2 + 1 - \mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big).$$

and, therefore, $\mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) = -1$. It follows that the trace of F_E^3 is

$$\begin{aligned}
\mathrm{Tr}\big(F_E^3 \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) &= \mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big)^3 - 3q \cdot \mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) \\
&= -1 + 6 \;=\; 5.
\end{aligned}$$

Now let $C/\mathbb{F}_2$ be a curve such that its Jacobian J is $\mathbb{F}_2$-isogenous to the g-fold product of E, so that g is the genus of C. Since $H^1(\bar{C}, \mathbb{Q}_\ell) \cong \bigoplus_{i=1}^g H^1(\bar{E}, \mathbb{Q}_\ell)$ as Galois-modules the number of $\mathbb{F}_8$-rational points is

$$N_8(C) = 2^3 + 1 - g \cdot \mathrm{Tr}\big(F_E^3 \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) = 9 - 5g.$$

In particular because of the non-negativity of $N_8(C)$, we get the estimate

$$g \leq \frac{9}{5}.$$

Therefore, the genus is bounded by 1 and thus the only curve of positive genus such that its Jacobian is $\mathbb{F}_2$-isogenous to the g-fold product of E is the curve E itself.

So we see how the Weil conjectures restrict the structure of a Jacobian. Or to say it in another way, how they prevent an arbitrary abelian variety A from being a Jacobian. Namely the quantities

$$N_{q^n} := q^n + 1 - \mathrm{Tr}\big(F_A^n \,\big|\, H^1(\bar{A}, \mathbb{Q}_\ell)\big)$$

have to be non-negative for all n. If we fix g and q, then this would give only a finite number of conditions for A since by the Riemann hypothesis for abelian varieties over finite fields we have

$$\big|\mathrm{Tr}\big(F_A^n \,\big|\, H^1(\bar{A}, \mathbb{Q}_\ell)\big)\big| \leq 2g \cdot (q^n)^{\frac{1}{2}}$$

where g is the dimension of A. But, if we let g grow with q fixed, then we get more and more conditions an abelian variety has to satisfy to be a Jacobian. If g is large enough in comparison to q, then a product of elliptic curves cannot fulfill any longer all these conditions as Serre [Se97] has observed based on the work of Tsfasman [Ts92] and Tsfasman and Vlăduţ [TV97]. We will sketch the proof following Serre's exposition in [Se97].

Fix a prime power q and let $(C_\lambda) = (C_\lambda)_{\lambda \in \mathbb{N}}$ be a sequence of curves $C_\lambda/\mathbb{F}_q$ of positive genus g_λ. We always assume that g_λ goes to infinity as λ grows. We are interested in what happens with the number $N_{q^n}(C_\lambda)$ of $\mathbb{F}_{q^n}$-rational points for high genus g_λ. To have some control over these values we will only consider special sequences of curves, so called *asymptotically exact* sequences.

Definition A.1.2 (asymptotically exact sequences of curves)
The sequence of curves (C_λ) is called *asymptotically exact* if the limits

$$\nu_n := \lim_{\lambda \to \infty} N_{q^n}(C_\lambda)/g_\lambda$$

exist for all natural numbers n.

The following proposition is very important since it shows that it is no restriction to consider only asymptotically exact sequences of curves.

Proposition A.1.3 (asymptotically exact sequences exist)
Any sequence of curves (C_λ) contains an asymptotically exact subsequence of curves.

Proof. By the Weil conjectures the values $N_{q^n}(C_\lambda)/g_\lambda$ are bounded by 0 and $q^n + 1 + 2 \cdot q^{n/2}$. So we can find a subsequence $(C^{(1)}_{\lambda_k})$ of (C_λ) such that the limit $\nu_1 := \lim_{k\to\infty} N_{q^1}(C^{(1)}_{\lambda_k})/g_{\lambda_k}$ exists. Now take a subsequence $(C^{(2)}_{\lambda_k})$ of $(C^{(1)}_{\lambda_k})$ such that $\nu_2 := \lim_{k\to\infty} N_{q^2}(C^{(2)}_{\lambda_k})/g_{\lambda_k}$ exists and so on. Finally taking the diagonal $(C^{(n)}_{\lambda_n})$ we obtain an asymptotically exact subsequence of (C_λ). □

In particular, if there would be curves of arbitrary high genus with split Jacobian, then there would be an asymptotically exact sequence of such curves. In fact, we will show that there is no such sequence.

Given a curve $C/\mathbb{F}_q$ let $\{\pi_1, \overline{\pi_1}, \ldots, \pi_g, \overline{\pi_g}\}$ be the set of eigenvalues of the Frobenius of C acting on $H^1(\bar{C}, \mathbb{Q}_\ell)$. Set

$$x_i := (\pi_i + \overline{\pi_i})/q^{1/2}.$$

for $i = 1, \ldots, g$. The x_i are the *normalized Frobenius eigenvalue traces.* If the Jacobian of C is isogenous to a product of elliptic curves E_i, then the x_i are the normalized Frobenius traces of the E_i. Because of the Riemann hypothesis for curves over finite fields the x_i all lie in the interval $\Omega = [-2, 2]$. We want to compute the normalized eigenvalue traces of the higher power Frobenii F_C^n from the x_i by plugging them into suitable polynomials Y_n.

Definition A.1.4 (the polynomials Y_n)
We define polynomials $X_n \in \mathbb{Z}[x]$ recursively by

$$\begin{aligned} X_m &= 0 \quad \text{for } m < 0 \\ X_0 &= 1 \\ X_n &= x \cdot X_{n-1} - X_{n-2} \quad \text{for } n > 0. \end{aligned}$$

and polynomials $Y_n \in \mathbb{Z}[x]$ by

$$Y_n = X_n - X_{n-2}$$

for all integers n.

Up to some change of variables the polynomials X_n are the n-th Chebyshev polynomials. For the Y_n the relations

$$(\pi_i^n + \overline{\pi_i}^n)/q^{n/2} = Y_n(x_i)$$

hold for all natural numbers n. In particular, we can write

$$\begin{aligned} N_{q^n}(C) &= q^n + 1 - \operatorname{Tr}\big(F_C^n \,\big|\, H^1(\bar{C}, \mathbb{Q}_\ell)\big) \\ &= q^n + 1 - \sum_{i=1}^{g} (\pi_i^n + \overline{\pi_i}^n) \\ &= q^n + 1 - q^{n/2} \cdot \sum_{i=1}^{g} Y_n(x_i). \end{aligned}$$

So we know the number of $\mathbb{F}_{q^n}$-rational points of C for all natural numbers n if we know the values x_i. We are interested in the question what values $x_1, \ldots, x_g \in \Omega$ occur for high genus curves C. Or more precisely, what is the limit distribution of the x_i for an asymptotically exact sequence of curves.

Therefore, we will first recall some definitions and facts about measures and distributions following [BoInt1]. Let Ω be a locally compact space, e.g. $\Omega = [-2, 2]$. By $\mathscr{K}(\Omega, \mathbb{R})$ we denote the set of continuous functions $f : \Omega \to \mathbb{R}$ with compact support.

Definition A.1.5 (measure μ, space of measures)
A *measure* μ on Ω is a continuous linear form on $\mathscr{K}(\Omega, \mathbb{R})$. If $f \in \mathscr{K}(\Omega, \mathbb{R})$ is a function, the value of μ at f is denoted by $\langle f, \mu \rangle$. $\mathscr{M}(\Omega, \mathbb{R})$ denotes the space of measures on Ω.

The measures defined above are more precisely called real-valued measures. Since we do not consider any other-valued measures no confusion will arise.

Example A.1.6

1. *The Dirac measure.* Let $x \in \Omega$ be an arbitrary point. Then the map
$$f \ \mapsto \ \langle f, \delta_x \rangle := f(x)$$
on $\mathscr{K}(\Omega, \mathbb{R})$ defines the Dirac measure δ_x at the point x on Ω.

2. *The Lebesgue measure.* The Lebesgue measure on $\Omega = \mathbb{R}$ is given by
$$f \ \mapsto \ \int_{\mathbb{R}} f(x)\ dx, \quad f \in \mathscr{K}(\Omega, \mathbb{R})$$
where $\int_{\mathbb{R}} f(x)dx$ denotes the usual Lebesgue integral. It is well defined since f has compact support.

Let μ be a measure on Ω and $U \subset \Omega$ be an open subset. Then we have an inclusion $\mathscr{K}(U, \mathbb{R}) \hookrightarrow \mathscr{K}(\Omega, \mathbb{R})$ of real linear spaces by extending functions by zero outside U. Restricting the linear form μ on $\mathscr{K}(\Omega, \mathbb{R})$ to the subspace $\mathscr{K}(U, \mathbb{R})$ we obtain a measure $\mu_{|U}$ on U called the *restriction* of μ to U. This enables us to define the support of a measure.

Definition A.1.7 (support of a measure)
Let μ be a measure on a locally compact space Ω. The *support* of μ is the complement of the largest open subset $U \subset \Omega$ such that the restriction of μ to U is zero.

Example A.1.8

1. Let $\Omega \subset \mathbb{R}$ and $x \in \Omega$ be a point. Then the support of the Dirac-measure δ_x is exactly $\{x\}$.

2. The support of the Lebesgue measure on $\mathbb{R}$ is equal to $\mathbb{R}$.

Since we are interested in distributions of points the following measures are particular important for us.

Definition A.1.9 (positive measures of mass 1)
A *positive measure of mass* 1 on a compact space Ω is a measure μ on Ω satisfying the the two properties below.

(1) Positivity: $\langle f, \mu \rangle \geq 0$ for any function $f \in \mathscr{K}(\Omega, \mathbb{R})$ with $f \geq 0$.

(2) Mass 1: $\langle \mathbf{1}, \mu \rangle = 1$ where $\mathbf{1}$ is the constant function on Ω with value 1.

Example A.1.10 1. *The measure associated to a finite set.* Let g be a natural number and $\mathcal{X} = \{x_1, \ldots, x_g\} \subset \Omega$ a finite set with g elements. Define

$$\delta_{\mathcal{X}} := \frac{1}{g} \sum_{i=1}^{g} \delta_{x_i}$$

where δ_{x_i} is the Dirac measure at x_i. $\delta_{\mathcal{X}}$ is a positive measure of mass 1 on Ω. If $\Omega \subset \mathbb{R}$, its support is the finite set $\mathcal{X}$.

2. *The Sato-Tate measure.* Let $\Omega = [-2, 2]$ and $\mu_\infty(x)$ be the differential $\mu_\infty(x) = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}\ dx = \frac{2}{\pi}\sin^2\varphi\ d\varphi$ where $x = 2 \cdot \cos\varphi$ for $0 \le \varphi \le \pi$. The Sato-Tate measure μ_∞ on $\Omega = [-2, 2]$ is given by

$$f \ \mapsto\ \langle f, \mu_\infty \rangle := \int_\Omega f(x)\ \mu_\infty(x).$$

It is a positive measure of mass 1 whose support is the whole space Ω.

Now let $(\mathcal{X}_\lambda)$ be a sequence of finite sets $\mathcal{X}_\lambda \subset \Omega$ of cardinality g_λ, e.g. $\mathcal{X}_\lambda = \{x_{\lambda,1}, \ldots, x_{\lambda,g_\lambda}\} \subset [-2, 2]$ the sets of normalized Frobenius eigenvalue traces of curves C_λ of genus g_λ defined over $\mathbb{F}_q$ forming a sequence of curves (C_λ). We want to study the "limit distribution" $\lim_{\lambda\to\infty} \delta_{\mathcal{X}_\lambda}$ (if it exists).

Definition A.1.11 (μ-equidistribution)
We say that the sequence $(\mathcal{X}_\lambda)$ is *μ-equidistributed* with respect to a measure μ on Ω if the measures $\delta_{\mathcal{X}_\lambda}$ converge to μ in $\mathscr{M}(\Omega, \mathbb{R})$ with respect to the topology of point-wise convergence.

In other words, $(\mathcal{X}_\lambda)$ is μ-equidistributed if $\lim_{\lambda\to\infty}\langle f, \delta_{\mathcal{X}_\lambda}\rangle = \langle f, \mu\rangle$ holds for all functions $f \in \mathscr{K}(\Omega, \mathbb{R})$. To justify the term "$\mu$-equidistributed", we remark that if $A \subset \Omega$ is a subset with μ-negligible boundary, then the probability that an element $x_{\lambda,i} \in \mathcal{X}_\lambda$ lies in A is asymptotically $\mu(A) := \langle 1_A, \mu\rangle$ the mass of A with respect to μ. For this use [BoInt1, p.IV.87, prop.22] with $F = \mathbb{R}$ and $f = 1_A$ the characteristic function of A.

Now we can state the main theorem for asymptotically exact sequences of curves which is due to Tsfasman and Vlăduţ and describes the limit distribution of normalized Frobenius eigenvalue traces for such sequences.

Let (C_λ) be a sequence of curves C_λ of positive genus g_λ defined over $\mathbb{F}_q$ with $g_\lambda \to \infty$. Let $\mathcal{X} = \{x_{\lambda,1}, \ldots, x_{\lambda,g_\lambda}\} \subset \Omega = [-2, 2]$ be the set of normalized Frobenius eigenvalue traces of C_λ. So we have a sequence $(\mathcal{X}_\lambda)$ of finite sets and its associated sequence of measures $(\delta_{\mathcal{X}_\lambda})$.

Theorem A.1.12 (the distribution of Frobenius eigenvalues)

(a) *For a sequence of curves (C_λ) defined over $\mathbb{F}_q$ the following statements are equivalent:*

(i) *The sequence (C_λ) is asymptotically exact, i.e. the limits*

$$\nu_n = \lim_{\lambda\to\infty} N_{q^n}(C_\lambda)/g_\lambda$$

exist for all natural numbers n.

(ii) *There is a positive measure μ of mass 1 on Ω such that the sequence $(\mathcal{X}_\lambda)$ is μ-equidistributed.*

(b) *Let (C_λ) be an asymptotically exact sequence of curves and μ its associated measure as in (a). Then the Fourier coefficients $a_n(\mu)$ of μ are given by*

$$\begin{aligned} a_0(\mu) &= 1 \quad \textit{and} \\ a_n(\mu) &= -q^{-n/2} \cdot \nu_n \end{aligned}$$

for $n \geq 1$, i.e. using the parameterization $x = 2\cos\varphi$ of $[-2,2]$, the measure μ is given by the differential

$$\mu(\varphi) = \frac{1}{\pi} F(\varphi)\, d\varphi$$

where $F(\varphi) = 1 - \sum_{n=1}^{\infty} q^{-n/2}\nu_n \cdot \cos n\varphi$, so that $\langle f, \mu\rangle = \frac{1}{\pi}\int_\Omega f(\varphi)F(\varphi)\, d\varphi$ for any function $f \in \mathscr{K}(\Omega, \mathbb{R})$. F is normally convergent and the support of μ equals Ω.

Proof: We follow Serre's proof in [Se97, p.92].

(a) (ii)$\Rightarrow$(i) Let $(\mathcal{X}_\lambda)$ be μ-equidistributed. So by definition of μ-equidistribution we have $\lim_{\lambda\to\infty}\langle Y_n, \delta_{\mathcal{X}_\lambda}\rangle = \langle Y_n, \mu\rangle$ for all n since $Y_n \in \mathscr{K}(\Omega, \mathbb{R})$. It follows that

$$\begin{aligned} \nu_n &= \lim_{\lambda\to\infty} N_{q^n}(C_\lambda)/g_\lambda \\ &= \lim_{\lambda\to\infty} \left(\frac{1+q^n}{g_\lambda} - \frac{q^{\frac{n}{2}}}{g_\lambda} \cdot \sum_{i=1}^{g_\lambda} Y_n(x_{\lambda,i}) \right) \\ &= \lim_{\lambda\to\infty} \left(-\frac{q^{\frac{n}{2}}}{g_\lambda} \cdot \sum_{i=1}^{g_\lambda} \delta_{x_{\lambda,i}}(Y_n) \right) \\ &= -\lim_{\lambda\to\infty} q^{\frac{n}{2}} \langle Y_n, \delta_{\mathcal{X}_\lambda}\rangle \\ &= -q^{\frac{n}{2}} \langle Y_n, \mu\rangle. \end{aligned}$$

In particular, the limits ν_n exist for all n and, therefore, the sequence (C_λ) is asymptotically exact.

(i)$\Rightarrow$(ii) Let (C_λ) be asymptotically exact, i.e. the limits

$$\nu_n = \lim_{\lambda\to\infty} N_{q^n}(C_\lambda)/g_\lambda$$

exist for all n. So by the same computation as above we have

$$\lim_{\lambda\to\infty} \langle Y_n, \delta_{\mathcal{X}_\lambda}\rangle = -q^{-n/2}\nu_n$$

for all natural numbers n. In particular these limits exist. Since the Y_n form a basis of the space of real polynomials, $\lim_{\lambda\to\infty}\langle P, \delta_{\mathcal{X}_\lambda}\rangle$ exists for all polynomials P. By defining $\mu(P) := \lim_{\lambda\to\infty}\langle P, \delta_{\mathcal{X}_\lambda}\rangle$ we get a positive linear functional on the space of polynomials with $\mu(1) = 1$ which extends by continuity to a positive measure of mass 1 on Ω, see [BoInt1, p.III.15, Prop.9]. Therefore, we have $\lim_{\lambda\to\infty}\langle f, \delta_{\mathcal{X}_\lambda}\rangle = \langle f, \mu\rangle$ for all functions $f \in \mathscr{K}(\Omega, \mathbb{R})$, so that $(\mathcal{X}_\lambda)$ is μ-equidistributed.

(b) If μ is any measure on Ω, then its Fourier coefficients are given by $a_n(\mu) = \langle Y_n, \mu\rangle$. For our given measure μ the computation in (a), therefore, shows that

$$a_n(\mu) = \langle Y_n, \mu\rangle = -q^{n/2}\nu_n$$

for $n \geq 1$ and $a_0(\mu) = \langle Y_0, \mu\rangle = \langle 1, \mu\rangle = 1$. In particular, $a_n(\mu) \leq 0$ for all $n \geq 1$. From this it follows, using an approximation argument, that $\sum_{n\geq 1} |a_n(\mu)| \leq 1$. So, F is normally convergent.

Important for us is the computation of the support of μ. Since μ is induced by the differential $\frac{1}{\pi}F\ d\varphi$ the support of μ is the support of the function $F = 1 - \sum_{n\geq 1} |a_n(\mu)| \cos\ n\varphi$. As mentioned above the inequality $\sum_{n\geq 1} |a_n(\mu)| \leq 1$ holds. If the inequality is strict, then F is everywhere non-zero. So assume that $\sum_{n\geq 1} |a_n(\mu)| = 1$. Then there is a natural number m such that $a_m(\mu) \neq 0$. Hence for F to be zero, cos $m\varphi$ must be zero. But this can only happen for finitely many $0 \leq \varphi \leq \pi$. So F has at most finitely many zeroes. It follows that the support of μ is the whole space Ω. □

We deduce several corollaries. In particular we will see that high-dimensional Jacobians can't be highly non-simple. The first corollary from which the other ones will follow says that the Frobenius eigenvalues are densely distributed in Ω.

Corollary A.1.13 (Frobenius eigenvalues are dense)
Let (C_λ) be a sequence of curves. Then set $\bigcup_\lambda \mathcal{X}_\lambda$ of all normalized Frobenius eigenvalue traces is dense Ω.

Proof. We may assume that (C_λ) is asymptotically exact. But then by (A.1.12) the sequence $(\mathcal{X}_\lambda)$ is μ-equidistributed where μ is a positive measure with support equal to Ω.

Let $U \subset \Omega$ be a non-empty open subset. We have to show that there is an element $x_{\lambda,i}$ lying in U. Let $f \geq 0$ be a continuous function on Ω with non-empty compact support in U. Such functions always exist. In particular $\langle f, \mu\rangle > 0$ because f and μ are non-zero and non-negative [BoInt1, p.III.28, Prop.9]. Thus

$$\lim_{\lambda\to\infty} \frac{1}{g_\lambda} \sum_i f(x_{\lambda,i}) = \lim_{\lambda\to\infty} \langle f, \delta_{\mathcal{X}_\lambda}\rangle \overset{!}{=} \langle f, \mu\rangle > 0$$

since $(\mathcal{X}_\lambda)$ is μ-equidistributed. So there is an $x_{\lambda,i}$ with $f(x_{\lambda,i}) > 0$ for $\lambda \gg 0$. But then $x_{\lambda,i}$ is contained in the support of f which lies in U. So $\bigcup_\lambda \mathcal{X}_\lambda \cap U \neq \varnothing$ and, therefore, $\bigcup_\lambda \mathcal{X}_\lambda$ is dense in Ω. □

From this we can deduce some structural information about Jacobians $J(C)$ of high genus curves C.

Corollary A.1.14 (Structure of high-dimensional Jacobians)
Let (C_λ) be a sequence of curves defined over $\mathbb{F}_q$ and $d(C_\lambda)$ the dimension of the largest simple abelian $\mathbb{F}_q$-subvariety of $J(C_\lambda)$. Then the values $d(C_\lambda)$ are unbounded, i.e. $d(C_\lambda) \to \infty$ as $\lambda \to \infty$.

Proof. Assume that there is a constant D such that $d(C_\lambda) \leq D$ for all λ, so that $d(C_\lambda)$ is bounded. Then $H^1(\bar{C}_\lambda, \mathbb{Q}_\ell)$ is a direct sum of $Gal\big(\overline{\mathbb{F}_q}/\mathbb{F}_q\big)$-$\mathbb{Q}_\ell$-modules of dimension at most $2D$. So by the Weil conjectures the normalized Frobenius eigenvalue traces $x_{\lambda,i}$ are algebraic integers of degree at most $2D$ and all Galois-conjugates $x_{\lambda,i}^\sigma$ of $x_{\lambda,i}$ satisfy

$$|x_{\lambda,i}^\sigma| \ \leq \ 2.$$

Therefore, the integral equations for all the $x_{\lambda,i}$ have uniformly bounded degrees and coefficients. But there are only finitely many polynomials with integer coefficients such that the degree and the coefficients are bounded. Hence there can occur only finitely many different $x_{\lambda,i}$, contradicting (A.1.13) saying that the set of all $x_{\lambda,i}$ is dense in Ω for sequences of curves. □

In particular there are no curves of arbitrary high genus with split Jacobian over a fixed finite field $\mathbb{F}_q$. Moreover we have the following.

Corollary A.1.15 (Curves with split Jacobian)
Up to isomorphism there are only finitely many curves defined over $\mathbb{F}_q$ such that their Jacobians are $\mathbb{F}_q$-isogenous to a product of elliptic curves.

Proof. By (A.1.14) the genus of such curves is bounded. Since the moduli spaces $\mathscr{M}_g/\mathbb{Z}$ of curves of genus g are of finite type, the corollary follows. □

Let C over $\mathbb{F}_q$ be a curve of genus g with split Jacobian. In the case $q = 2$ we have the bound $g \leq 26$ by [DE02]. The bound is sharp since the modular curve $X(11)/\mathbb{F}_2$ has genus equal to 26 and its Jacobian splits [ES93, p.511].

If the Jacobian $J/\mathbb{F}_q$ of $C/\mathbb{F}_q$ is $\mathbb{F}_q$-isogenous to the g-fold product of a single elliptic curve $E/\mathbb{F}_q$, then we can give the following explicit bounds for the genus g of C. In fact, this is the only part we will use in the later sections.

Proposition A.1.16 (explicit bounds)
Let C be a curve defined over $\mathbb{F}_q$ such that its Jacobian is $\mathbb{F}_q$-isogenous to the g-fold product of an elliptic curve $E/\mathbb{F}_q$. Then the genus g of C is bounded and we have the following explicit bounds:

(a) *If* $\mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) > 0$, *then* $g \leq q + 1$.

(b) *If* $\mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) = 0$, *then* $g \leq \frac{1}{2}q^2$.

(c) *If* $\mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) < -\frac{\pi}{2}q^{1/2}$, *then* $g \leq q^2 + 1$.

(d) *If* $-\frac{\pi}{2}q^{1/2} \leq \mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) < 0$, *then* $g \leq q^3 + 1$.

Proof: As explained in the beginning of the section, we have

$$N_{q^n}(C) = q^n + 1 - g \cdot \mathrm{Tr}\big(F_E^n \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big)$$

since the Jacobian of C is $\mathbb{F}_q$-isogenous to the g-fold product of E over $\mathbb{F}_q$.

(a) Since $\mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big)$ is positive, we get

$$g \leq \frac{q+1}{\mathrm{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big)}.$$

For the other cases, write $\operatorname{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) = z + \bar{z}$ where z is an eigenvalue of F_E acting on $H^1(\bar{E}, \mathbb{Q}_\ell)$. Without loss of generality we may assume that $z = q^{1/2} \cdot e^{i\theta}$ and $0 \leq \theta \leq \pi$. We want to find a natural number n such that $\operatorname{Tr}\big(F_E^n \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) = z^n + \bar{z}^n$ is positive.

(b) If $\operatorname{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) = 0$, then $\theta = \frac{\pi}{2}$. So $\operatorname{Tr}\big(F_E^4 \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) = 2q^2 > 0$. It follows that

$$g \leq \frac{q^4 + 1}{2q^2} = \frac{1}{2}q^2 + \frac{1}{2q^2}.$$

Since g is an integer, we get $g \leq \frac{1}{2}q^2$.

(c) If $\operatorname{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) < -\frac{\pi}{2}q^{1/2}$, then $\frac{3}{4}\pi < \theta < \pi$. So $\operatorname{Tr}\big(F_E^2 \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big)$ is positive and we conclude

$$g \leq \frac{q^2 + 1}{\operatorname{Tr}\big(F_E^2 \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big)} \leq q^2 + 1.$$

(d) In the remaining case the restriction on $\operatorname{Tr}\big(F_E \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big)$ forces $\frac{\pi}{2} < \theta \leq \frac{3}{4}\pi$. So $\operatorname{Tr}\big(F_E^3 \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big) > 0$ and

$$g \leq \frac{q^3 + 1}{\operatorname{Tr}\big(F_E^3 \,\big|\, H^1(\bar{E}, \mathbb{Q}_\ell)\big)} \leq q^3 + 1. \quad \square$$

In the case $q = 2$ we see that the genus g of a curve whose Jacobian is rationally isogenous to the product of a single elliptic curve has to be smaller than 9. This is smaller than the general bound $g \leq 26$.

Also bear in mind that in general over suitable fields in characteristic p there are examples of curves of arbitrary high genus whose Jacobians are rationally isogenous to products of some elliptic curves. See for example [ST67].

A.2 Families of curves

Now let C/K be a smooth, projective, geometrically connected curve defined over the function field K of a smooth, projective, geometrically connected curve $Y/\mathbb{F}_q$. Assume that the Jacobian of C is K-isogenous to a product of elliptic curves. Since C is projective, we may extend C/K to a model $X \xrightarrow{f} Y$ with generic fiber C/K. The fibers of $X \xrightarrow{f} Y$ are curves with split Jacobian defined over finite fields $\mathbb{F}_{q^n}$. Hence, if the genus g of C is large, by the preceding section, fibers over small fields have to be singular. Of course, families of curves $X \xrightarrow{f} Y$ are allowed to have singular fibers, but, we are interested in how many singularities a fibration $X \xrightarrow{f} Y$ may have?

So, let k be an algebraically closed field, not necessarily of characteristic p, Y/k a smooth, projective, geometrically connected curve of genus q and $X \to Y$ a family of curves of genus g, i.e. a proper, flat morphism with reduced, connected fibers of dimension 1. We will only look at semistable families of curves.

Definition A.2.1 (semistable family of curves)
A family of curves $X \to Y$ is called *semistable* if

(1) X is relatively minimal and nonsingular.

(2) The fibers X_y have only ordinary double points as singularities.

(3) Any rational nonsingular component E of a fiber X_y meets all other components of X_y in at least two points.

It is not a restriction to consider only semistable families because of the semistable reduction theorem [AW71], saying that any family of curves becomes semistable after a suitable base change.

Therefore, assume that $X \xrightarrow{f} Y$ is semistable. Then $X \xrightarrow{f} Y$ owns a relative dualizing sheaf $\omega_{X/Y}$ which can be described by

$$\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$$

and which is compatible with base changes [Vi95, p.14]. Let be

$$d = \deg\ f_*\omega_{X/Y}$$

and let δ denote the number of ordinary double points in the fibers of $X \xrightarrow{f} Y$.

We will bound the number δ with the help of the value d. First, we express the invariant c_2 of X by invariants of the fibration $X \xrightarrow{f} Y$.

Proposition A.2.2 (computing $c_2(X)$)
Let $X \xrightarrow{f} Y$ be a semistable family of curves of genus g. Then

$$c_2(X) = 4(g-1)(q-1) + \delta$$

where q is the genus of Y and δ the number of ordinary double points in the fibers of f.

Proof. Since $f^*\Omega_Y^1$ is invertible, the sequence

$$0 \longrightarrow f^*\Omega_Y^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/Y}^1 \longrightarrow 0$$

is exact. Therefore, it follows that

$$c_2(X) = c_2(\Omega_X^1) = c_2(\Omega_{X/Y}^1) + c_1(f^*\Omega_Y^1).c_1(\Omega_{X/Y}^1).$$

To compute c_1 and c_2 of $\Omega_{X/Y}^1$, look at the exact sequence

$$0 \longrightarrow \Omega_{X/Y}^1 \longrightarrow \omega_{X/Y} \longrightarrow N \longrightarrow 0$$

where N denotes the cokernel of $\Omega_{X/Y}^1 \to \omega_{X/Y}$. Both sheaves coincide outside the singularities of the fibers of $X \to Y$. Since, by assumption, all singularities are ordinary double points, N is a direct sum of δ local rings of length 1. Hence, $c_1(N) = 0$ and $c_2(N) = -\delta$. It follows that

$$c_1(\omega_{X/Y}) = c_1(\Omega_{X/Y}^1) + c_1(N) = c_1(\Omega_{X/Y}^1)$$

and, $\omega_{X/Y}$ is invertible,

$$c_2(\Omega^1_{X/Y}) = c_2(\omega_{X/Y}) - c_2(N) - c_1(\omega_{X/Y}).c_1(N) = \delta.$$

Therefore,

$$\begin{aligned} c_2(X) &= c_2(\Omega^1_{X/Y}) + c_1(f^*\Omega^1_Y).c_1(\Omega^1_{X/Y}) \\ &= \delta + (2q-2)\cdot(2g-2), \end{aligned}$$

proving the proposition. □

Lemma A.2.3 (computing $(\omega_{X/Y}.\omega_{X/Y})$)
Let $X \xrightarrow{f} Y$ be a semistable family of curves of genus g. Then

$$(\omega_{X/Y}.\omega_{X/Y}) = c_1(X)^2 - 8(g-1)(q-1)$$

where q is the genus of Y.

Proof. As mentioned above, we know that $\omega_{X/Y} = \omega_X \otimes (f^*\Omega^1_Y)^{-1}$. Hence,

$$(\omega_{X/Y}.\omega_{X/Y}) = c_1(\omega_X)^2 - 2\cdot\big(c_1(\omega_X).c_1(f^*\Omega^1_Y)\big) + c_1(f^*\Omega^1_Y)^2.$$

Since the self-intersections of fibers are zero, we have $c_1(f^*\Omega^1_Y)^2 = 0$ and the adjunction formula implies $c_1(\omega_X).c_1(f^*\Omega^1_Y) = (2g-2)\cdot(2q-2)$. So we get $(\omega_{X/Y}.\omega_{X/Y}) = c_1(X)^2 - 2\cdot(2g-2)(2q-2)$. □

Now we can relate the values δ and d via the self-intersection of $\omega_{X/Y}$.

Proposition A.2.4 (relating δ and d)
Let $X \xrightarrow{f} Y$ be a semistable family of curves of genus g. Then

$$(\omega_{X/Y}.\omega_{X/Y}) = 12d - \delta$$

where $d = \deg\ f_*\omega_{X/Y}$ and δ is the number of singularities in the fibers of f.

Proof. By lemma (A.2.3) we have $(\omega_{X/Y}.\omega_{X/Y}) = c_1(X)^2 - 8(g-1)(q-1)$ and proposition (A.2.2) says that $c_2(X) = 4(g-1)(q-1) + \delta$. Combining these two equations delivers

$$(\omega_{X/Y}.\omega_{X/Y}) = c_1(X)^2 + c_2(X) - 4(g-1)(q-1) - \delta - 8(g-1)(q-1).$$

The Noether formula tells us that $c_1(X)^2 + c_2(X) = 12\cdot\chi(\mathcal{O}_X)$, so that

$$(\omega_{X/Y}.\omega_{X/Y}) = 12\cdot\chi(\mathcal{O}_X) - 12(g-1)(q-1) - \delta.$$

We have to compute $\chi(\mathcal{O}_X)$. An application of the Leray spectral sequence shows $\chi(\mathcal{O}_X) = \chi(f_*\mathcal{O}_X) - \chi(R^1f_*\mathcal{O}_X)$. Since by Grothendieck-duality the identity $R^1f_*\mathcal{O}_X \cong (f^*\omega_{X/Y})\check{}$ holds, the Riemann-Roch-theorem on Y enables us to compute $\chi(f_*\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 1-q$ and $\chi(R^1f_*\mathcal{O}_X) = \chi\big((f_*\omega_{X/Y})\check{}\big) = -d + g\cdot(1-q)$. Putting all this together yields

$$\begin{aligned} (\omega_{X/Y}.\omega_{X/Y}) &= 12\big(1-q+d+g\cdot(q-1)\big) - 12(g-1)(q-1) - \delta \\ &= 12d - \delta \end{aligned}$$

finishing the proof. □

To get an upper bound for the number δ of singularities of $X \xrightarrow{f} Y$, we finally have to show that $12d - \delta \geq 0$.

Proposition A.2.5 (upper bound for δ)
Let $X \xrightarrow{f} Y$ be a semistable family of curves of genus $g \geq 2$. Then

$$\delta \leq 12d = 12 \cdot \deg f_*\omega_{X/Y}$$

where δ is the number of singularities in the fibers of f.

Proof. This was proven by Arakelov [Ar71] in characteristic 0 and by Szpiro [Sz78] in characteristic p. Since we are interested in the characteristic p case, we will give Szpiro's proof here. Thus let Y be a curve defined over a finite field.

Combining lemma (A.2.3) and proposition (A.2.4) we see that

$$12d - \delta = (\omega_{X/Y}.\omega_{X/Y}) = c_1(X)^2 - 8(g-1)(q-1),$$

and, therefore,

$$c_1(X)^2 = 12d - \delta + 8(g-1)(q-1).$$

We need to know how $c_1(X)^2$ changes under base extensions. So let $Y' \to Y$ be a finite surjective morphism between nonsingular curves and let X' be the desingularization of $X \times_Y Y'$. Then the analogues values d' and δ' of the fibration $X' \to Y'$ differ from d and δ by multiplication by $\alpha := \deg(Y' \to Y)$, so that

$$c_1(X')^2 = \alpha \cdot (12d - \delta) + 8(g-1)(q'-1)$$

where q' is the genus of Y'.

In particular, if $Y \xrightarrow{F^n} Y$ is the n-th power Frobenius endomorphism, and $X^{(p^n)}$ the desingularization of $X \times_{F^n} Y$, then

$$c_1\big(X^{(p^n)}\big)^2 = p^n(12d - \delta) + 8(g-1)(q-1).$$

Assume that $12d - \delta < 0$. Then for $n \gg 0$, we have $c_1(X^{(p^n)})^2 < 0$. It follows from the classification of surfaces [Mu69, p.329] that $X^{(p^n)}$ contains infinitely many rational curves.

Without loss of generality we may assume that $q \geq 2$, because this can be achieved by a base change $Y' \to Y$ and the discussion above showed that base changes don't change the sign of $12d - \delta$. Hence, the infinitely many rational curves have to lie in the finitely many singular fibers of $X^{(p^n)} \to Y$, because the genus of the fibers is at least 2. But this is impossible, contradicting the assumption that $12d - \delta < 0$. So it follows that $\delta \leq 12d$. □

Thus we found an upper bound for the number of singularities of a semistable family of curves. We will compute an example in the next section.

A.3 Heights and abelian varieties

In the previous section we have seen that the number of singularities of a semistable family of curves $X \xrightarrow{f} Y$ is bounded by $12 \deg f_*\omega_{X/Y}$. Now we want to relate this value to the structure of the corresponding family of Jacobians $J \to Y$. In particular, we are interested in the case that J is isogenous to a product of elliptic curves.

We will see that $\deg f_*\omega_{X/Y}$ is intimately connected with the height of the Jacobian $J \to Y$. As always, let Y be a smooth, projective, geometrically connected curve defined over some field k.

Definition A.3.1 (height of a group scheme)
Let $G \to Y$ be a group scheme with zero section $Y \xrightarrow{s} G$. Then

$$h(G) := \deg(s^*\Omega^1_{G/Y})$$

is called the *height* of G over Y.

Remark A.3.2 (heights, products and Cartier duals)
If G_1/Y and G_2/Y are two group schemes, then $h(G_1 \times_Y G_2) = h(G_1) + h(G_2)$. This follows from the fact that $\Omega^1_{G_1 \times G_2/Y}$ is isomorphic to $p_1^*\Omega^1_{G_1/Y} \oplus p_2^*\Omega^1_{G_2/Y}$ where p_i denotes the projection $G_1 \times_Y G_2 \to G_i$.

If G/Y is a finite group scheme with Cartier dual $\widehat{G}/Y$, then the height of G equals the height of $\widehat{G}$ up to sign, namely $h(G) = -h(\widehat{G})$.

Example A.3.3

1. Let $G \to Y$ be an étale group scheme. Then $\Omega^1_{G/Y}$ is zero, so that $h(G) = 0$ follows.
2. Assume that we are working in characteristic p and let $\mu_p \to Y$ be the p-th roots of unity. Then μ_p/Y is no longer étale, but its Cartier dual $\widehat{\mu}_p$ is. Thus we see that $h(\mu_p) = -h(\widehat{\mu}_p) = 0$.

The next proposition is very important for our purposes since it compares the degree of the push-forward of the relative dualizing sheaf of a semistable family of curves $X \to Y$ with the height of its Jacobian $J = \mathrm{Pic}^0(X/Y) \to Y$.

Proposition A.3.4 (height of Jacobians)
Let $X \xrightarrow{f} Y$ be a semistable family of curves and $J \to Y$ its corresponding family of Jacobians. Then

$$\det s^*\Omega^1_{J/Y} \cong \det f_*\omega_{X/Y}.$$

In particular, $h(J) = \deg f_*\omega_{X/Y}$.

Proof. See [Fa83, p.351]. □

Example A.3.5 Let $E(3) \to X(3)$ be the universal family of elliptic curves over the modular curve $X(3)$ defined over $\mathbb{C}$. We want to compute the height of its Jacobian which we will also denote by $E(3) \to X(3)$. For the properties of this family see [Be82].

The total space of $E(3) \to X(3)$ is a rational surface, so $c_1^2 = 0$. We know further that $E(3) \to X(3)$ is semistable and has 4 singular fibers with 3 ordinary double points each. So from the previous section we get

$$12d = c_1^2 - 8(g-1)(q-1) + \delta = 12$$

where d is the degree of the push-forward of the relative dualizing sheaf, $g = 1$ the genus of the fibers, $q = 0$ the genus of $X(3)$ and $\delta = 12$ the number of singularities in the fibers. Therefore, we conclude that $d = 1$. It follows that $h\big(E(3)\big) = 1$.

So we see that the number of singularities δ of $X \xrightarrow{f} Y$ is bounded by $12 \cdot h(J)$. We want to compute the height of $J \to Y$ in the case that J is Y-isogenous to a product of elliptic curves. It is enough to concentrate on semiabelian schemes.

Definition A.3.6 (semiabelian schemes)
A *semiabelian scheme* $A \to Y$ of relative dimension g is a smooth group scheme whose fibers are g-dimensional, connected extensions of abelian varieties by a torus.

Example A.3.7 If $X \to Y$ is a semistable family of curves, then its corresponding family of Jacobians $J \to Y$ is a semiabelian scheme.

The converse is also true.

Theorem A.3.8 (semistable reduction of curves and Jacobians)
A family of curves $X \to Y$ is semistable if and only if its family of Jacobians $J \to Y$ is a semiabelian scheme.

Proof. This can be found in [DM69, p.89]. □

Now we come to the computation of the height of a semiabelian scheme which is isogenous to the product of non-isotrivial elliptic curves. Remember that a family of curves $X \to Y$ is called *isotrivial*, if it becomes constant after a base extension, i.e. there is an extension $Y' \to Y$ such that $X' = X \times_Y Y'$ is isomorphic to $F \times_k Y'$, where F denotes a curve over the base field k.

Proposition A.3.9 (heights and products of elliptic curves)
Let $A \to Y$ be a semiabelian scheme which is isogenous over Y to a product $E_1 \times_Y \cdots \times_Y E_g$ of non-isotrivial families of elliptic curves $E_i \to Y$. Then the identity $h(A) = h(E_1) + \cdots + h(E_g)$ holds.

Proof. Let $E_1 \times_Y \cdots \times_Y E_g \to A$ be an isogeny and N its kernel. Then we have $h(A) = h(E_1 \times \cdots \times E_g) - h(N)$. Since the E_i are non-isotrivial, they are in particular not supersingular. So N is an extension of an étale group scheme by some factors of the form μ_{p^n}.

As discussed in example (A.3.3) both group schemes have height zero. Hence, N has height zero, and, therefore, $h(A) = h(E_1 \times \cdots \times E_g)$. And since the height is compatible with products, we conclude that $h(A) = h(E_1) + \cdots + h(E_g)$. □

Corollary A.3.10 (bounding δ for split Jacobians)
Let $X \to Y$ be a semistable family of curves whose Jacobian $J \to Y$ is Y-isogenous to the g-fold product of a non-isotrivial family of elliptic curves $E \to Y$. Then

$$\delta \leq 12h(E) \cdot g$$

where δ is the number of singularities of $X \to Y$. In particular, the constant $h(E)$ depends only on $E \to Y$, so that δ is linearly bounded by g.

Proof. Using proposition (A.3.4) and (A.3.9) it follows that the number of singularities fulfill $\delta \leq 12d = 12h(J) = 12h(E) \cdot g$. □

This gives us a somehow more explicit bound in the split case since we are able to estimate the the maximal growth of δ if g increases.

Example A.3.11 Let $E(3) \to X(3)$ be the universal family of elliptic curves over the modular curve $X(3)$, and let $C \to X(3)$ be a family of curves such that its Jacobian $J \to X(3)$ is isogenous over $X(3)$ to the g-fold product of $E(3)$ over $X(3)$. Then by example (A.3.5) and corollary (A.3.10)

$$\delta \leq 12 \cdot g$$

where δ is the number of singularities in the fibers of $C \to X(3)$.

A.4 Curves over function fields

Let Y be a smooth, projective, geometrically connected curve defined over some finite field $\mathbb{F}_q$ and $X \to Y$ a semistable family of curves of genus g such that its Jacobian $J \to Y$ is isogenous to the g-fold product of a non-isotrivial family of elliptic curves $E \to Y$.

We learned that the number δ of geometric singularities in the fibration $X \to Y$ is bounded above by $12h(E) \cdot g$. Now we want to derive a lower bound for δ. Proposition (A.1.16) gives us explicit bounds for the genus of a curve with split Jacobian defined over a finite field. Using this we see that there are fibers which are singular, and, that there are more and more of them as g tends to infinity.

Since the bounds in (A.1.16) depend on the traces of the fibers E_y of $E \to Y$, we need to know how the traces are distributed in a family of elliptic curves. The answer is given by the Sato-Tate-conjecture which says that they are asymptotically distributed according to the Sato-Tate-measure (see also (A.1.10)).

If y is an $\mathbb{F}_{q^n}$-rational point of Y, we denote by $\Theta(y)$ the angle of a Frobenius eigenvalue of the fiber E_y. I.e. the eigenvalues of the Frobenius acting on $H^1(\bar{E}_y, \mathbb{Q}_\ell)$ are given by $q^{n/2} \cdot e^{\pm\Theta(y)\cdot i}$.

Theorem A.4.1 (Sato-Tate-conjecture)
Let $E \to Y$ be a non-isotrivial family of elliptic curves and a and b two numbers between 0 and π. Then

$$\lim_{n\to\infty} \frac{\#\{y \in Y(\mathbb{F}_{q^n}) \bigm| a \leq \Theta(y) \leq b\}}{q^n} = \frac{2}{\pi} \int_a^b \sin^2 \varphi \, d\varphi.$$

Proof. This was proven by Deligne in [De80, p.212, (3.5.7)]. □

If $\Theta(y) < \frac{\pi}{2}$, then the trace $\operatorname{Tr}\big(F_E \bigm| H^1(\bar{E}, \mathbb{Q}_\ell)\big)$ is positive. So by the Sato-Tate-conjecture asymptotically half of the fibers over $\mathbb{F}_{q^n}$-rational points have positive trace. These fibers have to be singular if $g > q^n + 1$. In this case, a fiber has at least $g/(q^n + 1)$ singularities. Since Y has approximately q^n $\mathbb{F}_{q^n}$-rational points for large n, half of them with positive trace, we get $\frac{q^n}{2} \cdot \frac{g}{q^n+1} \approx \frac{1}{2}g$ singularities. If $g > q^{n+1}$, we will get additional $\frac{1}{2}g$ singularities and so on. Hence, for increasing g we can collect as often as we want $\frac{1}{2}g$ singularities. But this means that the total number δ of singularities is not linearly bounded by g. We will give a more precise proof of this fact.

Proposition A.4.2 (lower bound for δ in the split case)
Let $X \to Y$ be a semistable family of curves of genus g whose Jacobian $J \to Y$ is isogenous over Y to the g-fold product of a non-isotrivial family of elliptic curves $E \to Y$. Then there is a constant $\mathcal{C} > 0$ depending only on $E \to Y$ such that

$$\mathcal{C} \cdot \frac{\log g}{\log \log g} \cdot g \leq \delta$$

where δ is the number of singularities in the geometric fibers of $X \to Y$. In particular, the number δ is not linearly bounded above by g.

Proof. Our intention is not to give the best possible lower bound, but a lower bound which is not linearly bounded by g, and which is easy to compute.

After enlarging q if necessary, using the Sato-Tate-conjecture, we may assume that $\#\{y \in Y(\mathbb{F}_{q^n}) \,|\, 0 \leq \Theta(y) < \frac{\pi}{2}\} > \frac{1}{4}q^n$. If $g > q^n + 1$, then a fiber over an $\mathbb{F}_{q^n}$-rational point y of Y with $\Theta(y) < \frac{\pi}{2}$ has to be singular by (A.1.16). Its Jacobian is either isogenous to the g-fold product of a single elliptic curve or a torus. In the toric case the curve has at least g singularities. In the compact case the curve is a chain of smooth curves each of genus less or equal to $q^n + 1$. Such a curve will have at least $\left\lfloor \frac{g}{q^n+2} \right\rfloor$ singularities. Underestimating the number of singularities we can say that in any case we have at least $\frac{g}{2q^n}$ singularities. So the total number of singularities we get from these fibers is at least

$$\frac{1}{4}q^n \cdot \frac{g}{2q^n} = \frac{1}{8}g$$

singularities.

There is one point we have to take care of. If m is a natural number dividing n, then $Y(\mathbb{F}_{q^m}) \subset Y(\mathbb{F}_{q^n})$. So saying that we get $\frac{1}{8}g$ singularities from the $\mathbb{F}_{q^m}$-rational fibers and additional $\frac{1}{8}g$ singularities from the $\mathbb{F}_{q^n}$-rational fibers is not fully correct because we possibly count some singularities more than once. To deal with this problem we will only consider extensions $\mathbb{F}_{q^e}$ of prime degree e.

Hence assume that $g-1 > q^2, q^3, q^5, q^7, q^{11}, \ldots, q^e, \ldots$ where the exponents e are prime numbers. How many $q^e < g-1$ with e prime are there? It is the number of primes e with $e < \log_q(g-1)$. So by the prime number theorem there is a constant $\mathcal{C}_1 > 0$ such that there are at least $\mathcal{C}_1 \frac{\log_q(g-1)}{\log \log_q(g-1)}$ such primes e ($\mathcal{C}_1$ is a little bit less than 1 if g is large). So we get not less than

$$\mathcal{C}_1 \frac{\log_q(g-1)}{\log \log_q(g-1)} \cdot \frac{1}{8}g$$

singularities up to multiply counted ones.

Thus, we have to deal with the singularities we counted more than once, namely the ones coming from fibers defined over $\mathbb{F}_q$-rational points because $Y(\mathbb{F}_q) \subset Y(\mathbb{F}_{q^n})$ for all n. Using a bad estimate for $\#Y(\mathbb{F}_q)$ we assume that there are at most $2q$ $\mathbb{F}_q$-rational points (q not too small, enlarge if necessary). Then we counted at most $2q \cdot \frac{g}{2q^e} = \frac{g}{q^{e-1}}$ points too often for each prime e. So an upper bound for the total error is

$$\sum_{e \text{ prime}} \frac{g}{q^{e-1}} \leq 2g.$$

Therefore, the corrected total number of singularities we counted is

$$\left(\frac{\mathcal{C}_1}{8}\frac{\log_q(g-1)}{\log\log_q(g-1)} - 2\right) g.$$

Remember that we enlarged q to apply the Sato-Tate-conjecture for $E \to Y$. So for our original q, we can say that there is a constant $\mathcal{C} > 0$ depending only on $E \to Y$ such that there are at least $\mathcal{C}\frac{\log g}{\log\log g} g$ singularities. □

Now we have an upper bound and a lower bound for the number δ of singularities of a semistable family of curves $X \to Y$ whose Jacobian is isogenous to the g-fold product of a non-isotrivial family of elliptic curves $E \to Y$. From these two bounds follows the main theorem of this chapter.

Let $K = k(Y)$ be the function field of the curve Y defined over $k = \mathbb{F}_q$.

Theorem A.4.3 (the genus of a curve with split Jacobian is bounded) *Let C/K be a smooth, projective, geometrically connected curve of genus g whose Jacobian is K-isogenous to the g-fold product of a single elliptic curve E/K. Then the genus of C is bounded, i.e. there is a constant $\mathcal{C} > 0$ depending only on E/K such that g is smaller than $\mathcal{C}$.*

Proof. Without loss of generality we may assume that E/K has semistable reduction everywhere. If not, we can achieve this after a base extension $\operatorname{Spec} K' \to \operatorname{Spec} K$ using the semistable reduction theorem [SGA7I, p.351, (3.6)].

Let $X \to Y$ be a minimal projective model of C/K and $J \to Y$ its family of Jacobians. By assumption the general fiber of $J \to Y$ is isogenous over K to the g-fold product of E/K and E/K has semistable reduction. Hence, $J \to Y$ is a semiabelian scheme [SGA7I, p.333, (2.2.6)] and, therefore, by (A.3.8) the family of curves $X \to Y$ is semistable. So by (A.4.2) and (A.3.10) the number δ of singularities in the geometric fibers of $X \to Y$ satisfies

$$\mathcal{C}_0 \frac{\log g}{\log\log g} \cdot g \leq \delta \leq 12h(E) \cdot g$$

where the constant $\mathcal{C}_0 > 0$ depends only on E/K. But then g cannot be arbitrarily large, since the left hand side is not linearly bounded by g. So there is a constant $\mathcal{C} > 0$ depending only on E such that g is smaller than $\mathcal{C}$. □

Chapter B

Bounding the genus in characteristic 0

In this chapter we want to derive results in characteristic 0 analogously to the first chapter's situation in characteristic p. Namely, we want to show that if Y is a curve, defined over a number field F, with function field K, and C is a curve over K whose Jacobian J is K-isogenous to the g-fold product of a non-isotrivial elliptic curve E over K, then the genus g of C is bounded where the bound depends only on E. We will achieve this by reducing our situation from characteristic 0 to characteristic p.

For this we start in the first section with the characterization of reducible curves in terms of its Jacobians. More precisely, we show that a curve is reducible iff its Jacobian splits as a principally polarized abelian variety, i. e. as a principally polarized abelian variety it is isomorphic to the product of two principally polarized abelian varieties.

Having done this, we further characterize splitting principally polarized abelian varieties in terms of the existence of some special endomorphisms on the abelian variety. This is what happens in the second section.

The third section contains the technique to reduce our problem to characteristic p. Knowing that the splitting of principally polarized abelian varieties is induced by some special endomorphisms, we study under which circumstances such a splitting endomorphism lifts from characteristic p to characteristic 0.

To apply these results we need to find a suitable prime p which does not divide the degree of the isogeny from the g-fold product of the elliptic curve E to the Jacobian J. That this is possible is the content of the fourth section.

B.1 Reducibility criterion for curves

Let k be a field and C/k a not necessarily smooth, projective curve. We want to be able to decide if C is geometrically reducible or not. When C/k is smooth, its Jacobian J/k is an abelian variety. The converse is not true [We57]. Nevertheless we want to deduce the reducibility of a curve via its Jacobian. For this we have to take into account an additional structure on the Jacobian, namely its canonical principal polarization.

In the beginning, k is an arbitrary field and $\overline{k}$ denotes an algebraic closure of k. All objects and morphisms should be defined over k. Let A be an abelian variety and $\widehat{A}$ its dual abelian variety. If $\mathscr{L}$ is an invertible sheaf on A, the natural identification $\mathrm{Pic}^0(A) = \widehat{A}(k)$ induces a map $A \stackrel{\varphi_{\mathscr{L}}}{\to} \widehat{A}$ via $a \mapsto t_a^*\mathscr{L} \otimes \mathscr{L}^{-1}$ where t_a is the translation-by-a map on A.

Definition B.1.1 (polarizations and principal polarizations)
An isogeny $A \stackrel{\lambda}{\to} \widehat{A}$ is called *polarization* on A if it is of the form $\varphi_{\mathscr{L}}$ over $\overline{k}$ for some ample invertible sheaf $\mathscr{L}$ on $A_{\overline{k}}$. The *degree of a polarization* is its degree as an isogeny. Polarizations of degree one are called *principal polarizations*.

Example B.1.2 Let E be an elliptic curve. Since elliptic curves have exactly one principal polarization (induced by any divisor of degree 1) we may canonically identify E with its dual $\widehat{E}$. If $\mathscr{L} = \mathcal{O}_E(D)$ where D is a divisor of degree n, then the induced map $E \stackrel{\varphi_{\mathscr{L}}}{\to} \widehat{E} \cong E$ is the multiplication-by-n map. So the polarizations on E are given by the multiplication-by-n maps with n positive.

Definition B.1.3 (polarized abelian varieties and morphisms)
An abelian variety A together with a polarization λ_A is called a *polarized abelian variety.* If λ_A is principal, (A, λ_A) is called a *principally polarized abelian variety.*

A *morphism* between two principally polarized abelian varieties (A, λ_A) and (B, λ_B) is a homomorphism $A \stackrel{f}{\to} B$ such that the diagram

$$\begin{array}{ccc} \widehat{A} & \stackrel{\widehat{f}}{\longleftarrow} & \widehat{B} \\ {\scriptstyle \lambda_A}\uparrow & & \uparrow{\scriptstyle \lambda_B} \\ A & \stackrel{f}{\longrightarrow} & B \end{array}$$

commutes where $\widehat{f}$ denotes the dual map of f.

Example B.1.4 The Jacobian J of a smooth curve C together with its canonical principal polarization λ induced by the theta divisor Θ is a principally polarized abelian variety (J, λ).

We will see that a curve is reducible iff the Jacobian *splits* as a principally polarized abelian variety.

Definition B.1.5 (split polarized abelian variety)
A polarized abelian variety (A, λ_A) *splits* if there are two positive-dimensional polarized abelian varieties (B, λ_B) and (C, λ_C) such that (A, λ_A) is isomorphic to $(B \times_k C, \lambda_B \times \lambda_C)$ as a polarized abelian variety.

Example B.1.6 If C is a reducible curve consisting of two components C_1 and C_2 intersecting in one point, then the principally polarized Jacobian (J, λ) of C is isomorphic to $(J_1 \times J_2, \lambda_1 \times \lambda_2)$ where (J_i, λ_i) denotes the Jacobian of C_i.

The converse is also true. This establishes the reducibility criterion for curves we are looking for.

Proposition B.1.7 (reducibility criterion for curves)
Let C be a curve with proper Jacobian (J, λ). Then C is reducible if and only if (J, λ) splits as a principally polarized abelian variety.

Proof. If C has a proper Jacobian then it is either smooth or it consists of smooth irreducible components C_i intersecting in a way such that they form a tree. Then from the construction of the Jacobian, we see that (J, λ) is the product of the Jacobians (J_i, λ_i) of the smooth components C_i.

It remains to show the converse that C is reducible if (J, λ) splits. Assume that $(J, \lambda) = (A_1 \times A_2, \lambda_1 \times \lambda_2)$ where (A_i, λ_i) are positive-dimensional principally polarized abelian varieties and that C is smooth. Choose an embedding $C \overset{f}{\hookrightarrow} J$ and let f_i be the composition $C \overset{f}{\hookrightarrow} J = A_1 \times A_2 \overset{p_i}{\rightarrow} A_i$ where p_i denotes the projection.

Then $f_i(C) \subset A_i$ is a 1-cycle generating A_i $\big($i. e. A_i is the smallest abelian subvariety of A_i containing $f_i(C)\big)$ because C generates its Jacobian $J = A_1 \times A_2$. Define $\tilde{C} := f_1(C) \times \{0\} + \{0\} \times f_2(C) \subset A_1 \times A_2 = J$. It follows that $\tilde{C}$ is a 1-cycle generating J.

Let $\Theta_i \subset A_i$ be a divisor inducing the polarization $A_i \overset{\lambda_i}{\rightarrow} \widehat{A_i}$. Then the divisor $\Theta := \Theta_1 \times A_2 + A_1 \times \Theta_2$ on $A_1 \times A_2 = J$ induces the polarization $J \overset{\lambda}{\rightarrow} \widehat{J}$. We compute the intersection number

$$\begin{aligned} \big(\tilde{C}.\Theta\big) &= \big(f_1(C) \times \{0\}.\Theta_1 \times A_2\big) + \big(f_1(C) \times \{0\}.A_1 \times \Theta_2\big) \\ &\quad + \big(\{0\} \times f_2(C).\Theta_1 \times A_2\big) + \big(\{0\} \times f_2(C).A_1 \times \Theta_2\big). \end{aligned}$$

The two middle terms are zero as an application of the projection formula shows. An other application of the projection formula on the remaining two terms gives us

$$\big(\tilde{C}.\Theta\big) = \big(C.\Theta_1 \times A_2\big) + \big(C.A_1 \times \Theta_2\big) = \big(C.\Theta\big) = g$$

where the last equality follows from the fact that C is the Jacobian of C.

From the Matsusaka-Ran theorem [Co84] follows that the (A_i, λ_i) are the Jacobians of the curves $f_i(C)$. But then because of the Torelli theorem the curve C is reducible with components $f_1(C)$ and $f_2(C)$. □

B.2 Splitting criterion for abelian varieties

In the previous section we saw that a curve is reducible if and only if its Jacobian splits as a polarized abelian variety. So now we want to deduce a useful criterion when a principally polarized abelian variety A splits. In particular, we want to characterize the splitting via the endomorphism ring of A. For this, as in the preceding section, we have to take into account an additional structure on the endomorphism ring which is connected to the principal polarization on A – the Rosati-involution.

Let k be an arbitrary field. Everything is supposed to be defined over k.

Definition B.2.1 (Rosati-involution)
Let (A, λ) be a principally polarized abelian variety and $\mathrm{End}_k(A)$ its endomorphism ring. The *Rosati-involution* on $\mathrm{End}_k(A)$ is the map

$$f \mapsto f^\dagger := \lambda^{-1} \circ \widehat{f} \circ \lambda.$$

So $f^\dagger$ is the endomorphism such that the diagram

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{\widehat{f}} & \widehat{A} \\ {\scriptstyle\lambda}\uparrow & & \downarrow{\scriptstyle\lambda^{-1}} \\ A & \xrightarrow{f^\dagger} & A \end{array}$$

commutes.

Remark B.2.2 (properties of the Rosati-involution)
The Rosati-involution is an anti-involution on $\mathrm{End}_k(A)$, i. e. it satisfies the relations

$$(f+g)^\dagger = f^\dagger + g^\dagger, \quad (fg)^\dagger = g^\dagger f^\dagger \quad \text{und} \quad f^{\dagger\dagger} = f.$$

where f and g are endomorphisms on A [Mi86, p.137]. Of course, the Rosati-involution depends on the chosen principal polarization on A. Since we always deal with fixed principal polarizations, no confusion will arise.

Example B.2.3 (1) Let E be an elliptic curve. If $\mathrm{End}_k(E) \cong \mathbb{Z}$, then the Rosati-involution is the identity. If $\mathrm{End}_k(E)$ is an order in an imaginary quadratic number field, then the Rosati-involution acts as the complex conjugation.

(2) Let $A = E \times_k \ldots \times_k E$ the g-fold product of an elliptic curve with $\mathrm{End}_k(E) \cong \mathbb{Z}$. There is a natural identification of $\mathrm{End}_k(A)$ with the ring $M_g(\mathbb{Z})$ of $(g \times g)$-matrices with integer coefficients by sending the identity from the j-th to the i-th component to the matrix which has a 1 at the position in the i-th row and j-th column and zeroes elsewhere.

Let λ on A be the principal polarization which is the product of the unique principal polarizations of each factor. Then the Rosati-involution on $\mathrm{End}_k(E)$ corresponds to the transposition of matrices in $M_g(\mathbb{Z})$.

We are interested in endomorphisms which are compatible with the additional structure on $\mathrm{End}_k(A)$ given by the Rosati-involution.

Definition B.2.4 (symmetric endomorphisms)
Let (A, λ) be a principally polarized abelian variety and $f \in \mathrm{End}_k(A)$ an endomorphism. We say that f is *symmetric* if it is invariant under the Rosati-involution, i. e. f fulfills $f^\dagger = f$.

Example B.2.5 As in example (B.2.3) let A be the g-fold product of an elliptic curve without complex multiplication and assume that A is equipped with the product polarization. Identifying again $\mathrm{End}_k(E)$ with $M_g(\mathbb{Z})$, the symmetric endomorphisms correspond to the symmetric matrices.

Symmetric endomorphisms are crucial for the splitting criterion.

Proposition B.2.6 (splitting criterion)
For a principally polarized abelian variety (A, λ_A) the following two statements are equivalent:

(i) (A, λ_A) *splits, i. e. (A, λ_A) is isomorphic as a principally polarized abelian variety to a product $(B \times C, \lambda_B \times \lambda_C)$ of two positive-dimensional principally polarized abelian varieties (B, λ_B) and (C, λ_C).*

(ii) (A, λ_A) *possesses a non-trivial symmetric idempotent endomorphism, i. e. it exists a map $f \in \mathrm{End}_k(A)$ different from the identity and the zero map such that the two relations $f^\dagger = f$ and $f^2 = f$ hold.*

Proof:

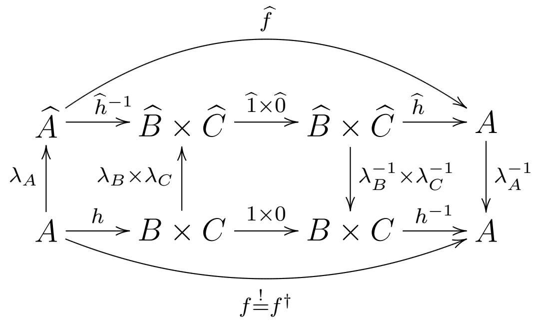
(i) $\Rightarrow$ *(ii)* Let $h : A \xrightarrow{\sim} B \times C$ be an isomorphism of principally polarized abelian varieties and define f to be the following composition of maps

$$\begin{array}{ccc} B \times C & \xrightarrow{1\times 0} & B \times C \\ {\scriptstyle h}\big\uparrow & & \big\downarrow{\scriptstyle h^{-1}} \\ A & \xrightarrow{=:f} & A. \end{array}$$

Then f is an idempotent and symmetric endomorphism of A. For the idempotence consider the commutative diagram

$$\begin{array}{ccccc} & & \overset{1\times 0}{\frown\!\!\!\longrightarrow} & & \\ B \times C & \xrightarrow{1\times 0} & B \times C & \xrightarrow{1\times 0} & B \times C \\ {\scriptstyle h}\big\uparrow & & {\scriptstyle h^{-1}}\big\downarrow \; \big\uparrow{\scriptstyle h} & & \big\downarrow{\scriptstyle h^{-1}} \\ A & \xrightarrow{f} & A & \xrightarrow{f} & A \\ & & \underset{f^2 \overset{!}{=} f}{\smile\!\!\!\longrightarrow} & & \end{array}$$

where the lower row gives us f^2. If we follow the upper way around we get f. So f and f^2 coincide and, therefor, f is idempotent. For the symmetry look at the diagram



which is commutative since dualizing endomorphisms commutes with inverting them. Again the lower row gives us f while the upper way around we obtain $f^\dagger$. So f and $f^\dagger$ are identical, telling us that f is a symmetric and idempotent endomorphism of A.

$(ii) \Rightarrow (i)$ Let $A \xrightarrow{f} A$ be a symmetric idempotent endomorphism of A. Define $B := \mathrm{Im}(A \xrightarrow{f} A)$ and $C := \mathrm{Im}(A \xrightarrow{1-f} A)$. Then we get a homomorphism

$$B \times C = fA \times (1-f)A \xrightarrow{h} A, \quad (b,c) \mapsto b + c.$$

Since f is idempotent, the homomorphism

$$A \longrightarrow B \times C = fA \times (1-f)A, \quad a \mapsto \big(fa, (1-f)a\big)$$

is an inverse map for h and, therefore, $B \times C \xrightarrow{h} A$ is an isomorphism of abelian varieties.

Let λ_B and λ_C be the restrictions of λ_A on B and C. This makes B and C into principally polarized abelian varieties. Since f is symmetric the two diagrams

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{\widehat{f}} & \widehat{A} \\ {\scriptstyle\lambda_A}\uparrow & & \uparrow{\scriptstyle\lambda_A} \\ A & \xrightarrow{f} & A \end{array} \qquad\qquad \begin{array}{ccc} \widehat{A} & \xrightarrow{\widehat{1}-\widehat{f}} & \widehat{A} \\ {\scriptstyle\lambda_A}\uparrow & & \uparrow{\scriptstyle\lambda_A} \\ A & \xrightarrow{1-f} & A \end{array}$$

commute. In particular, B, which is the image of f, is mapped under λ_A into the image of $\widehat{A}$ under $\widehat{f}$. The same holds for C and $1-f$. But then also the diagram

$$\begin{array}{ccc} \widehat{B} \times \widehat{C} = \widehat{f}\widehat{A} \times (\widehat{1} - \widehat{f})\widehat{A} & \xleftarrow{\widehat{h}} & \widehat{A} \\ {\scriptstyle\lambda_B \times \lambda_C}\uparrow & & \uparrow{\scriptstyle\lambda_A} \\ B \times C = fA \times (1-f)A & \xrightarrow{h} & A \end{array}$$

commutes. Hence, as a principally polarized abelian variety (A, λ_A) is isomorphic to $(B \times C, \lambda_B \times \lambda_C)$ via the map h. □

B.3 Lifting endomorphisms

We want to deduce the reducibility of a certain curve C in characteristic 0 by reducing the problem to characteristic p. In the previous sections we saw that a curve is reducible iff its Jacobian splits as a principally polarized abelian variety. And that a principally polarized abelian variety splits iff it owns a non-trivial symmetric idempotent endomorphism. So if the reduction of C is reducible in characteristic p, we want to lift the induced symmetric idempotent endomorphism on the Jacobian in characteristic p to the Jacobian in characteristic 0 to deduce the reducibility of C in characteristic 0. Therefore, we study the problem of lifting endomorphisms.

Let R be an arbitrary henselian discrete valuation ring with quotient field K and residue field k. Write $S = \operatorname{Spec} R$ for the spectrum of R. We use the following notational convention. A small subscript denotes the base scheme. So a scheme X_K resp. X_k is scheme over $\operatorname{Spec} K$ resp. $\operatorname{Spec} k$. A Scheme over S is simply denoted by X instead of X_S. Then X_K is its general fiber (a K-scheme) and X_k is its special fiber (a k-scheme). Let $A \to S$ be an abelian scheme over S, so that we have the following situation.

$$\begin{array}{ccccc} A_K & \hookrightarrow & A & \hookleftarrow & A_k \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} K & \hookrightarrow & \operatorname{Spec} R & \hookleftarrow & \operatorname{Spec} k \end{array}$$

The question we study is when does an endomorphism of A_k lift to an endomorphism of A_K.

Definition B.3.1 (the lifting property)
We say that every endomorphism of A_k *lifts* if the restriction map

$$\begin{array}{ccc} \operatorname{End}_S(A) & \longrightarrow & \operatorname{End}_k(A_k) \\ f & \longmapsto & f_k := f_{|A_k} \end{array}$$

is an isomorphism.

Remark B.3.2 (the restriction map is injective)
The restriction map $\operatorname{End}_S(A) \to \operatorname{End}_k(A_k)$ is always injective. But it is not surjective in general. E. g. let E be an ordinary elliptic curve defined over a number field K. Let v be a finite place of K. Then the reduction of E at v has complex multiplication since it is defined over a finite field. So, the restriction map cannot be surjective.

Example B.3.3 If $E \to S$ is a relative elliptic curve such that E_K and E_k are both elliptic curves without complex multiplication so that the endomorphism rings $\operatorname{End}_S(E)$ and $\operatorname{End}_k(E_k)$ are isomorphic to $\mathbb{Z}$, then the restriction map is clearly an isomorphism because the endomorphisms are the multiplication-by-m maps.

The same is true for the g-fold product of the elliptic curve E since in this case the endomorphism ring is canonically isomorphic to $M_g(\mathbb{Z})$ as described in example (B.2.3) and multiplication-by-m maps lift.

We want to know if an abelian scheme $A \to S$ which is isogenous to the g-fold product of an elliptic curve $E \to S$ having the lifting property (e. g. elliptic curves like in the previous example) also has the lifting property. We will convert endomorphisms of A_k into endomorphisms of the product $E_k \times_k \ldots \times_k E_k$, lift them there and then we go back to A. For this we regard the following operators.

Definition B.3.4 (the Rosati-operator †)
Let A and B two principally polarized abelian schemes over some base scheme S. Then the *Rosati-operator* † is the map

$$\begin{array}{ccc} \mathrm{Hom}_S(A,B) & \longrightarrow & \mathrm{Hom}_S(B,A) \\ f & \longmapsto & f^\dagger := \lambda_A^{-1} \circ \widehat{f} \circ \lambda_B \end{array}$$

where $\widehat{B} \xrightarrow{\widehat{f}} \widehat{A}$ denotes the dual morphism of f.

Remember that a (principal) polarization on an abelian scheme $A \to S$ is an isogeny $A \to \widehat{A}$ which is fiber-wise a (principal) polarization [Mi86, p.149].

Remark B.3.5 (properties of the Rosati-operator $\dagger$)
The Rosati-operator is an isomorphism since $f^{\dagger\dagger} = f$ where the first operator † is the map from $\mathrm{Hom}_S(A,B)$ to $\mathrm{Hom}_S(B,A)$, and the second operator † is the map from $\mathrm{Hom}_S(B,A)$ to $\mathrm{Hom}_S(A,B)$. If $A = B$, the Rosati-operator is the Rosati-involution (B.2.1).

Definition B.3.6 (the h^*-operator)
Let A and B two principally polarized abelian schemes over some base scheme S and $A \xrightarrow{h} B$ an isogeny not necessarily compatible with the polarizations. Then the *h^*-operator* is the map

$$\begin{array}{ccc} \mathrm{End}_S(B) & \longrightarrow & \mathrm{End}_S(A) \\ f & \longmapsto & h^* f := h^\dagger \circ f \circ h \end{array}$$

where † denotes the Rosati-operator.

Remark B.3.7 (properties of the h^*-operator)
The map h^* is injective because h and $h^\dagger$ are isogenies, i. e. they are surjective with finite kernel. The map h^* is not compatible with the ring structure of the endomorphism rings unless h is an isomorphism.

Now we come to the behavior of the lifting property under isogenies. S should be as in the introduction of the section the spectrum of a henselian discrete valuation ring.

Proposition B.3.8 (the lifting property and étale isogenies)
Let A and B be two principally polarized abelian schemes over S and $A \xrightarrow{h} B$ an étale isogeny. If every endomorphism of A_k lifts, then every endomorphism of B_k lifts too.

Proof. Look at the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_S(B) & \xhookrightarrow{h^*} & \mathrm{End}_S(A) \\ \big\downarrow & & \big\downarrow \cong \\ \mathrm{End}_k(B_k) & \xhookrightarrow{h_k^*} & \mathrm{End}_k(A_k) \end{array}$$

and let $f_k \in \mathrm{End}_k(B_k)$ an endomorphism of B_k. We want to lift f_k to an endomorphism $f \in \mathrm{End}_S(B)$ so that $f_{|B_k} = f_k$. Look at the map $h_k^* f_k \in \mathrm{End}_k(A_k)$. Since A has the lifting property, the map $h_k^* f_k$ lifts to a map $A \xrightarrow{u} A$ so that $u_k = h_k^* f_k$. If we can show that $A \xrightarrow{u} A$ lies in the image of h^*, i. e. that there is map $f \in \mathrm{End}_S(B)$ with $h^* f = h^\dagger \circ f \circ h = u$, then the map f is a lifting of f_k because of the commutativity of the diagram above.

We know that $u_k = h_k^* f_k = h_k^\dagger \circ f_k \circ h_k$ factorizes through h_k so that $\mathrm{Ker}(h_k)$ is a subgroup scheme of $\mathrm{Ker}(u_k)$. Since our base S is henselian and $\mathrm{Ker}(h_k)$ étale, there is a subgroup scheme $G \subset \mathrm{Ker}(u)$ such that $G_k = \mathrm{Ker}(h_k)$. But then, being a subgroup scheme of A, the group scheme G has to coincide with $\mathrm{Ker}(h)$ since finite étale schemes over S are uniquely determined by their special fiber [Mi80, p.34]. Hence, u factorizes through h, i. e. there is a map $B \xrightarrow{g} A$ such that $u = g \circ h$ holds.

Analogously one shows that the dual $\widehat{g}$ of g factorizes through the dual $\widehat{h^\dagger}$ of $h^\dagger$. Hence, there exists an endomorphism $B \xrightarrow{f} B$ such that $g = h^\dagger \circ f$ is valid. Therefore, we get the identity $u = h^* f$ and f becomes a lifting of f_k. This implies that the abelian scheme B also has the lifting property. □

B.4 Bounding the degree of isogenies

We saw that we can lift endomorphisms for products of an elliptic curve and that this property is invariant under étale isogenies. So if J is the Jacobian of a curve C in characteristic 0 which is isogenous to the g-fold product of an elliptic curve E and splits in characteristic p, then we can deduce the splitting of J in characteristic 0 if p does not divide the degree of the isogeny $E \times \ldots \times E \to J$.

In this section we will show that we can find isogenies $E \times \ldots \times E \to J$ such that its degrees are divided by only a finite number of primes depending only on E and not on g or J. For this we need a result about Galois representations on ℓ-torsion points of E. The following theorem says that these representations are for almost all ℓ as big as possible.

Theorem B.4.1 (images of Galois representations)
Let K be a field, finitely generated over $\mathbb{Q}$, $G = Gal(\overline{K}/K)$ its absolute Galois group and E/K an elliptic curve without complex multiplication. Then for almost all ℓ the homomorphism

$$\rho_\ell : \; Gal(\overline{K}/K) \longrightarrow \mathrm{Aut}_{\mathbb{F}_\ell}\left(E[\ell](\overline{K})\right) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$$

is surjective. In particular, we have $\rho_\ell(G) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ for almost all ℓ.

Proof. We prove the theorem by induction on the transcendence degree of $K/\mathbb{Q}$. The number field case, that is the transcendence degree of $K/\mathbb{Q}$ equals zero, is a well known result of Serre [Se72].

So let K/k be an extension of transcendence degree 1 and assume that the statement of the theorem is true for k. Let v be a place of K such that E has good reduction at v and that the reduction E_v does not have complex multiplication. We denote the residue field of v by $k(v)$. It is a finite extension of k. Let $G_v \subset G$ be the decomposition group of v and $I_v \subset G_v$ the inertia group so that there is a canonical isomorphism $G_v/I_v \cong Gal\big(\overline{k(v)}/k(v)\big) =: G(v)$.

Since E has good reduction at v, using the Néron-Ogg-Shafarevich criterion [SGA7I, p.335] we conclude that $E[\ell](\overline{K})$ is an unramified G_v-module, i.e. I_v acts trivially on $E[\ell](\overline{K})$. Hence, we may regard $E[\ell](\overline{K})$ as an $G(v)$-module. In particular, by reduction mod v we get an isomorphism

$$E[\ell](\overline{K}) \xrightarrow{\cong} E_v[\ell]\big(\overline{k(v)}\big)$$

of $G(v)$-modules where E_v is the reduction of E at v. By the induction hypothesis we have $\rho_\ell(G_v) = \rho_\ell\big(G(v)\big) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ for almost all ℓ and, therefore, we also get $\rho_\ell(G) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ for almost all ℓ. □

The theorem enables us to determine the K-rational endomorphisms of $E[\ell]$ for almost all ℓ.

Corollary B.4.2 (endomorphisms on $\boldsymbol{\ell}$-torsion)
Let K be a field, finitely generated over $\mathbb{Q}$, and E/K an elliptic curve without complex multiplication. Then for almost all ℓ

(a) *$E[\ell](\overline{K})$ is an irreducible $Gal\big(\overline{K}/K\big)$-module.*

(b) *$\mathrm{End}_K(E[\ell])$ consists only of the multiplication-with-m maps for $0 \le m < \ell$.*

Proof. For almost all primes ℓ we have $\rho_\ell(G) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$. From this follows (a). For (b) observe that the K-endomorphisms of $E[\ell]$ have to lie in the center of $\rho_\ell(G) \subset \mathrm{Aut}_{\mathbb{F}_\ell}\big(E[\ell](\overline{K})\big)$ since they commute with the Galois action. But the center of $\rho_\ell(G) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ consists of the diagonal matrices with identical entries on the diagonal. These matrices correspond to the the multiplication-with-m maps independently from the chosen isomorphism $\rho_\ell(G) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$. □

This corollary gives us the main theorem of this section announced in the beginning.

Proposition B.4.3 (bounding the degree of isogenies)
Let K be a field, finitely generated over $\mathbb{Q}$, E/K an elliptic curve without complex multiplication and $A = E \times_K \ldots \times_K E$ the g-fold product of E. Then there is a finite set $S = S(E/K)$ of primes depending only on E/K such that for every abelian variety B/K which is K-isogenous to A there is a K-isogeny between A and B whose degree has only prime divisors lying in S. In particular, the set S does not depend on g.

Proof. Let $A \xrightarrow{h} B$ an isogeny and $G \subset \mathrm{Ker}(h)$ a non-trivial irreducible subgroup scheme. So in particular, the order of G is some prime power ℓ^n. Let $S = S(E/K)$ be the set of primes such that the representations ρ_ℓ from (B.4.1) are not surjective. According to (B.4.1) this set is finite and depends only on E/K. Assume that $\ell \notin S$.

We will show that A/G is isomorphic to A so that the isogeny $A \xrightarrow{h} B$ factorizes as $A \to A/G \cong A \to B$. By continuing this process we will finally obtain a factorization $A \xrightarrow{g} A \xrightarrow{h'} B$ of h where the degree of g is only divisible by primes not in S and h' is only divisible by primes contained in S so that the proposition is proved.

We will do induction on the dimension g of $A = E \times_K \ldots \times_K E$. Let be $g = 1$ so that $A = E$ is an elliptic curve. Since we have chosen $\ell \notin S$ we conclude from (B.4.2a) that $G \cong E[\ell]$ because G is a non-trivial irreducible subgroup scheme of A of order ℓ^n for some n. But then up to isomorphism the map $A \to A/G$ is the multiplication-by-ℓ map on E. Thus $A \xrightarrow{h} B$ factorizes as $A \to A/G \to B$.

Now let $g > 1$ and assume that the statement is true for $g - 1$. Let $A \xrightarrow{p_i} E$ be the projection on the i-th factor and φ_i the composition

$$G \hookrightarrow A \xrightarrow{p_i} E, \qquad \varphi_i := p_i \circ (G \hookrightarrow A).$$

of morphisms. Without loss of generality let φ_1 and φ_2 be different from zero. Otherwise G is contained in a $g-1$-dimensional product of E and we may apply the induction hypothesis. Thus we have two isomorphisms $\varphi_i : G \xrightarrow{\sim} E[\ell]$. Both isomorphism differ only by an automorphism of $E[\ell]$. Hence, by (B.4.2b) they differ only by multiplication-by-m for some integer $0 < m < \ell$. Therefore, we have $\varphi_2 = m \cdot \varphi_1$. But then after applying a suitable automorphism of A, e. g.

$$\begin{pmatrix} 1 & & & 0 \\ -m & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \in \mathrm{End}_K(A),$$

the subgroup scheme G of A lies in $(g-1)$-dimensional factor of A. Applying the induction hypothesis we see that A/G is isomorphic to A. So this proves the proposition. □

Example B.4.4 Let $E(3) \to X(3)$ be the universal family of elliptic curves over the modular curve $X(3)$ as in (A.3.5). Let K be the function field of $X(3)$ regarded as curve defined over the number field $\mathbb{Q}(\zeta_3)$ where ζ_3 is a 3rd root of unity and let $E = E(3) \times_{X(3)} K$ be the general fiber.

According to [Ig59] the Galois representations ρ_ℓ are as big as possible for $\ell > 3$ so that the subgroup schemes $E[\ell]$ are irreducible for all $\ell > 3$. Thus we may choose $S = S(E/K) = \{2, 3\}$.

B.5 Curves over function fields

Now we come to the main result of this chapter, namely that the genus of a curve is bounded if its Jacobian is K-isogenous to the g-fold product of an elliptic curve where K is the function field of a curve defined over some number field.

We already know this result if K is the function field of a curve defined over a finite field. By reducing the above problem to this particular case, we will achieve the result.

We use again the convention that a small subscript denotes the base scheme, e. g. C_K is a curve over K. Furthermore, if we extend a scheme like C_K to a model C_S over some one-dimensional base scheme S with generic point $\operatorname{Spec} K$, then we drop the subscript and write simply C instead of C_S.

Let Y_F be a smooth, projective, geometrically connected curve defined over some number field F. Let $\mathcal{O}_F$ denote the ring of integers of F and let $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathcal{O}_F$ be any finite point of $\operatorname{Spec} \mathcal{O}_F$. We can extend $Y_F \to \operatorname{Spec} F$ to a minimal model $Y \to \operatorname{Spec} \mathcal{O}_F$, i. e. $Y \to \operatorname{Spec} \mathcal{O}_F$ is an integral, proper, regular, excellent and flat surface of finite type with general fiber Y_F together with the usual minimality property similar to the geometric case. See [Ar86] for the resolution of singularities in the arithmetic setting, and [Ch86] for the existence of arithmetic minimal models. Also the book [Li02] contains a treatment of these topics. Let $Y_{\mathbb{F}_q} \to \operatorname{Spec} \mathbb{F}_q$ be the special fiber of $Y \to \operatorname{Spec} \mathcal{O}_F$ over the point $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathcal{O}_F$. The collected data so far is presented in the two bottom rows of the following diagram.

$$\begin{array}{ccccc}
\{C_K, J_K, E_K\} & \hookrightarrow & \{C, J, E\} & \hookleftarrow & \{C_k, J_k, E_k\} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} K & \hookrightarrow & \operatorname{Spec} R & \hookleftarrow & \operatorname{Spec} k \\
\downarrow & & \downarrow & & \downarrow \\
Y_F & \hookrightarrow & Y & \hookleftarrow & Y_{\mathbb{F}_q} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} F & \hookrightarrow & \operatorname{Spec} \mathcal{O}_F & \hookleftarrow & \operatorname{Spec} \mathbb{F}_q
\end{array}$$

Now we come to the other two rows. Let K be the function field of Y_F so that $\operatorname{Spec} K \to Y_F$ is the generic point. Let k be the function field of an irreducible component of $Y_{\mathbb{F}_q}$ so that $\operatorname{Spec} k \to Y_{\mathbb{F}_q}$ is the generic point of the corresponding irreducible component. Furthermore, let R be the local ring of Y at this irreducible component. In particular, R is a discrete valuation ring (Y is regular) with generic point $\operatorname{Spec} K \to \operatorname{Spec} R$ and special point $\operatorname{Spec} k \to \operatorname{Spec} R$.

Finally, let C_K be a smooth, projective, geometrically connected curve defined over the function field K, J_K its Jacobian and E_K an elliptic curve. We may extend $C_K \to \operatorname{Spec} K$ to a minimal model $C \to \operatorname{Spec} R$ with Jacobian $J \to \operatorname{Spec} R$ and we denote the special fibers of these models by $C_k \to \operatorname{Spec} k$ and $J_k \to \operatorname{Spec} k$. Of course, also $E_K \to \operatorname{Spec} K$ extends to a (Néron) model $E \to \operatorname{Spec} R$ with special fiber $E_k \to \operatorname{Spec} k$.

Now we are ready to state and prove this chapter's main result.

Theorem B.5.1 (the genus of a curve with split Jacobian is bounded) *Let C_K be a smooth, projective, geometrically connected curve of genus g whose Jacobian J_K is K-isogenous to the g-fold product of a single non-isotrivial elliptic curve E_K. Then the genus of g is bounded, i. e. there is a constant $c = c(E_K) > 0$, depending only on E_K, such that g is smaller than c.*

Proof. We want to reduce our situation to characteristic p to apply theorem (A.4.3). Therefor we choose a finite point $\operatorname{Spec}\mathbb{F}_q \to \operatorname{Spec}\mathcal{O}_F$ of residue characteristic p such that the following properties are fulfilled.

(1) Y_F has good reduction at $\operatorname{Spec}\mathbb{F}_q$, i. e. the fiber $Y_{\mathbb{F}_q}$ is a smooth curve. This depends only on K – the function field of Y_F – and is true for almost all points of $\operatorname{Spec}\mathcal{O}_F$.

(2) $E_K \to \operatorname{Spec} K$ extends to a smooth proper model $E \to \operatorname{Spec} R$ such that $E_k \to \operatorname{Spec} k$ is a non-isotrivial elliptic curve. This is true for almost all points of $\operatorname{Spec}\mathcal{O}_F$ and depends only on E_K.

(3) There is an isogeny $E_K \times_K \ldots \times_K E_K \to J_K$ such that its degree is prime to p. Using proposition (B.4.3) we see that this is true for almost all points of $\operatorname{Spec}\mathcal{O}_F$ and depends only on E_K. Together with (2) this property will enable us to lift endomorphisms of J_k to endomorphisms of J_K with the help of proposition (B.3.8).

Since the three conditions above each hold for all but finitely many points of $\operatorname{Spec}\mathcal{O}_F$, we can find a point $\operatorname{Spec}\mathbb{F}_q \to \operatorname{Spec}\mathcal{O}_F$ fulfilling all conditions. The choice of this point depends only on E_K.

As explained in the introduction, let R be the local ring of Y at $Y_{\mathbb{F}_q}$. Extend the curve $C_K \to \operatorname{Spec} K$ to a minimal model $C \to \operatorname{Spec} R$. Its Jacobian $J \to \operatorname{Spec} R$ is equipped with a canonical principal polarization $J \xrightarrow{\lambda} \widehat{J}$ such that (J_K, λ_K) resp. (J_k, λ_k) is the principally polarized Jacobian of C_K resp. C_k. Since by assumption J_K is K-isogenous to the g-fold product of E_K, the Jacobian J_k is k-isogenous to the g-fold product of E_k (actually $J \to \operatorname{Spec} R$ is isogenous over $\operatorname{Spec} R$ to the g-fold product of $E \to \operatorname{Spec} R$).

Let $\widehat{R}$ be the completion of R and $\widehat{K}$ its quotient field. So after the base change $\operatorname{Spec}\widehat{R} \to \operatorname{Spec} R$ we get a model $C_{\widehat{R}} \to \operatorname{Spec}\widehat{R}$ with generic fiber $C_{\widehat{K}} \to \operatorname{Spec}\widehat{K}$ and special fiber $C_k \to \operatorname{Spec} k$.

We know by theorem (A.4.3) that the genus g of C_k is bounded if C_k is smooth. So for high genus g the curve C_k becomes reducible (possibly after a finite extension $\operatorname{Spec}\widehat{R}' \to \operatorname{Spec}\widehat{R}$). Hence the principally polarized Jacobian (J_k, λ_k) of C_k is a split principally polarized abelian variety by example (B.1.6) and, therefore, it owns a symmetric idempotent endomorphism because of the splitting criterion (B.2.6).

This endomorphism lifts to a symmetric idempotent endomorphism on (J_K, λ_K) using proposition (B.3.8). For this observe that the canonical polarizations on $J_{\widehat{K}}$ and J_k are just the restrictions of the canonical polarization $J_{\widehat{R}} \to \widehat{J}_{\widehat{R}}$ induced by λ so that the restriction map $\operatorname{End}_{\widehat{K}}(J_{\widehat{K}}) \to \operatorname{End}_k(J_k)$ is compatible with the Rosati-involutions on each endomorphism ring.

But, if there exists a symmetric idempotent endomorphism on $J_{\widehat{K}}$, then, using again the splitting criterion (B.2.6), we see that $J_{\widehat{K}}$ splits as a principally polarized abelian variety. So the curve $C_{\widehat{K}}$ will be reducible (at least after some finite extension $\operatorname{Spec} \widehat{L} \to \operatorname{Spec} \widehat{K}$). But this contradicts the assumption that C_K is a smooth, geometrically connected curve. Thus the genus g of C_K cannot be arbitrarily large and is, therefore, bounded. Since the bound depends on the choice of a suitable point $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathcal{O}_F$, the discussion in the beginning shows that the bound really depends only on E_K. $\square$

Chapter C

Shimura curves and the Schottky locus

We finally come to the situation where our family of curves $C \to Y$, whose Jacobian $J \to Y$ is Y-isogenous to a g-fold product of a family of elliptic curves $E \to Y$, lives over a base curve $Y/\mathbb{C}$ while the family $E \to Y$ can be defined over some number field. E. g. this is the case when $E \to Y$ is the universal family of elliptic curves over some Shimura curve. We want to show that the genus g of $C \to Y$ is still bounded and that this upper bound depends only on $E \to Y$.

This will be achieved by reducing the problem from the field of complex numbers $\mathbb{C}$ to a number field F. In the first section we show that if $E \to Y$ can be defined over some number field, then the same is true for the Jacobian $J \to Y$. We do this by studying Galois representations on the torsion points of $E \to Y$ to show that the torsion structure is quite limited.

Thereafter, we show that if the Jacobian $J \to Y$ can be defined over a number field, then somehow the corresponding curve $C \to Y$ can be, too. This is done by using fine moduli spaces of curves and algebraic varieties with level structures. There, we will see that we have to care about a subtle problem, namely that the curve $C \to Y$ will not descend to the same base as $J \to Y$ but to some covering of degree at most 2. So applying the results of the previous chapters directly, we will only be able to derive a bound for the genus g of $C \to Y$ which depends on this particular covering.

For this reason, we extend the results of the two previous chapters by allowing that the curve $C \to Y$ is defined over some covering of the base of definition of $E \to Y$. In the final section we come to the main result announced above and apply it to families of curves reaching the Arakelov bound.

Some words about notations and conventions. We use again small subscripts to denote base schemes so that X_S is a scheme together with a morphism $X_S \to S$. If $S \to S_0$ is a morphism and X_S some scheme, then we will say that X_S is *defined over* S_0 if there is a scheme X_{S_0} such that $X_{S_0} \times_{S_0} S$ is S-birational to X_S. Perhaps it would be more natural to require that $X_{S_0} \times_{S_0} S$ is S-isomorphic to X_S. But in general this is not the case and for our purposes we are only concerned with schemes up to S-birationality. Also one should perhaps demand that there is a finite covering $S' \to S$, a morphism $S' \to S_0$ and a scheme X_{S_0} such that $X_S \times_S S'$ is S'-birational to $X_{S_0} \times_{S_0} S'$. But we are mostly interested in reducing a situation from the complex numbers $\mathbb{C}$ to a number field F. So we usually won't need a base extension before descending. If it is, however, the case, e. g. in (C.2.4), we will explicitly state it.

Since we will only work up to S-birationality, we further always assume that any given scheme X_S is the "minimal model" in its S-birationality class. By this convention, we will consider a scheme X_S defined over S_0 to be the scheme which arises from X_{S_0} by the base change $S \to S_0$ (in fact, it is the minimal model of $X_{S_0} \times_{S_0} S$). So, given two schemes X_S and X_{S_0} it will either be clear by assumption that X_S arises from X_{S_0} by a base change or it will turn out to be the case.

Now, being aware of the fact that there is no minimal model theory for arbitrary schemes, we should specify what we mean by a minimal model for the schemes we will encounter. If Y_k is a curve over some field k, then Y_k should be the up to isomorphism unique projective smooth curve. If $C_{Y_k} \to Y_k$ is a family of curves, then C_{Y_k} should be the regular minimal surface. And for a family of abelian varieties $A_{Y_k} \to Y_k$, i. e. a group scheme whose generic fiber is an abelian variety, A_{Y_k} should be the Néron model of its general fiber.

This choice immediately implies a slight abuse of notation, namely if $C_{Y_k} \to Y_k$ is a family of curves with Jacobian $J_{Y_k} \to Y_k$, then J_{Y_k} is not the Jacobian in the sense that it is isomorphic to $\mathrm{Pic}^0(C_{Y_k}/Y_k)$, but it is the Néron model of it. So in general, the connected component of one of J_{Y_k} will be the "real" Jacobian, but this will not do any harm.

A final word about notations. To prevent an overuse of subscripts, we will write e. g. $A_S \times \ldots \times A_S$ instead of $A_S \times_S \ldots \times_S A_S$, $\mathrm{End}_S(A)$ instead of $\mathrm{End}_S(A_S)$ and $A[N]_S$ instead of $A_S[N]$ or even $A_S[N]_S$. Also the function field of a curve Y_k will be denoted by $k(Y)$ instead of $k(Y_k)$.

C.1 Descending Jacobians

Let F be a number field and Y_F a smooth, projective, geometrically connected curve defined over F. Let further $E_{Y_F} \to Y_F$ be a non-isotrivial family of elliptic curves. After a base change $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Spec}\,F$ we get a family of elliptic curves $E_{Y_\mathbb{C}} \to Y_\mathbb{C}$.

We want to show that there exists a finite field extension F' of F, depending only on E_{Y_F}, such that any (polarized) family of abelian varieties $A_{Y_\mathbb{C}} \to Y_\mathbb{C}$ which is $Y_\mathbb{C}$-isogenous to any g-fold product of $E_{Y_\mathbb{C}}$ is already defined over $Y_{F'}$ and $Y_{F'}$-isogenous to the g-fold product of $E_{Y_{F'}}$. In particular, the number field F' will not depend on g.

This is achieved by showing that any isogeny $E_{Y_\mathbb{C}} \times \ldots \times E_{Y_\mathbb{C}} \to A_{Y_\mathbb{C}}$ has its kernel $H_{Y_\mathbb{C}}$ defined over $Y_{F'}$. So the given isogeny and $A_{Y_\mathbb{C}}$, which is the quotient of the g-fold product of $E_{Y_\mathbb{C}}$ by this isogeny's kernel, have to be defined over Y_F. Therefore, our task is to describe the finite subgroup schemes of $E_{Y_\mathbb{C}} \times \ldots \times E_{Y_\mathbb{C}}$.

Given a finite subgroup scheme $H_{Y_\mathbb{C}}$, we can find a natural number N such that $H_{Y_\mathbb{C}}$ is contained in the N-torsion subgroup scheme of $E_{Y_\mathbb{C}} \times \ldots \times E_{Y_\mathbb{C}}$ which is $E[N]_{Y_\mathbb{C}} \times \ldots \times E[N]_{Y_\mathbb{C}}$. Since every subgroup of the N-torsion points is the kernel of an endomorphism of $E[N]_{Y_\mathbb{C}} \times \ldots \times E[N]_{Y_\mathbb{C}}$, we should describe all these endomorphisms. Being a fiber product of g copies of $E[N]_{Y_\mathbb{C}}$, the endomorphisms of $E[N]_{Y_\mathbb{C}} \times \ldots \times E[N]_{Y_\mathbb{C}}$ are built up from the endomorphisms of $E[N]_{Y_\mathbb{C}}$. So we are going to study the $Y_\mathbb{C}$-rational endomorphisms of $E[N]_{Y_\mathbb{C}}$ for all numbers N.

It is enough to consider the generic fiber $E_K \to \operatorname{Spec} K$ of $E_{Y_\mathbb{C}} \to Y_\mathbb{C}$, $K = \mathbb{C}(Y)$ the function field of $Y_\mathbb{C}$, because the generic fiber's torsion structure determines the torsion structure of the whole family, and to investigate the action of the absolute Galois group $G_{\bar{K}/K} = Gal\big(\overline{K}/K\big)$ on the N-torsion subgroup schemes $E[N]_K$. If we can show that the induced representations have "large" images, then we will be able to deduce that up to a finite number of maps every endomorphism is already defined over Y_F. So after a finite extension F' of F every map will be defined.

We will study the Galois action locally, i. e. we will pass over to v-adic complete extensions K_v of K where v is a normalized discrete valuation coming from some point $y \in Y(\mathbb{C})$. Let $G_v = Gal\big(\overline{K_v}/K_v\big)$ be the absolute Galois group of K_v. Since the residue field of K_v, which is $\mathbb{C}$, is algebraically closed, G_v equals its inertia group. Thus, if $E_{Y_\mathbb{C}} \to Y_\mathbb{C}$ has good reduction in y, the Néron-Ogg-Shafarevich criterion tells us that G_v acts trivially on $E[N](\overline{K_v})$ so that we won't learn anything. Hence, we have to concentrate on points $y \in Y(\mathbb{C})$ where $E_{Y_\mathbb{C}} \to Y_\mathbb{C}$ has bad reduction.

Without loss of generality we may assume that $E_{Y_\mathbb{C}} \to Y_\mathbb{C}$ has semistable reduction everywhere, otherwise replace $Y_\mathbb{C}$ by a finite covering $Y'_\mathbb{C}$. If $y \in Y(\mathbb{C})$ now is a point of bad reduction, then the corresponding elliptic curve E_{K_v} will be a Tate curve. Hence, Tate's v-adic uniformization theorem [Si86, §14] tells us that there is an element $q \in K_v^\times$ with $v(q) > 1$ and a natural isomorphism $\overline{K_v}^\times/q^\mathbb{Z} \xrightarrow{\sim} E_{K_v}(\overline{K_v})$ of groups compatible with the action of the Galois group G_v. This description of E_{K_v} will greatly help to study the Galois action in a very explicit way.

Remember also that E_{K_v} will extend over $\operatorname{Spec} R$, where R denotes the ring of integers of K_v, to a Néron model $E \to \operatorname{Spec} R$. If k denotes the residue field of R and E_k the special fiber, then we have a map $E(K_v) \to E(k)$. We will say that a point $P \in E(\overline{K_v})$ *specializes into the connected component of one*, if $P \in E(K_v)$ and the image of P under the map $E(K_v) \to E(k)$ lies in the same connected component as the unit section. Moreover, by j_E we will denote the j-invariant of E.

Proposition C.1.1 (Galois action on torsion of Tate curves)
Let K_v be a v-adic complete field (with residue field $\mathbb{C}$) and absolute Galois group G_v, and let E_{K_v} be a Tate curve. Then for any prime power ℓ^n we can find a basis (P_1, P_2) of $E[\ell^n](\overline{K_v})$ such that for any integer n' with $\ell^{n'+1} \nmid v(j_E)$ there is an element $\sigma \in G_v$ which acts on $E[\ell^n](\overline{K_v})$ with respect to the basis (P_1, P_2) like

$$\begin{pmatrix} 1 & \ell^{n'} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}).$$

In particular, for almost all prime powers ℓ^n there is a transvection, i. e. $n' = 0$.

Furthermore, the basis (P_1, P_2) can be chosen as follows: for P_1 we may take any ℓ^n-torsion point which specializes into the connected component of one, while for P_2 we may take any other point such that (P_1, P_2) forms a basis.

Proof. We mimic and extend the proof of [Si94, V.6.1]. The assumption that the residue field is $\mathbb{C}$ is not really necessary, but this is our situation and it slightly simplifies the proof.

Let $\zeta \in \mathbb{C}^\times \subset K_v^\times$ be an ℓ^n-th root of unity, and q^{1/ℓ^n} an ℓ^n-th root of q. Then under the isomorphism $\overline{K_v}^\times/q^\mathbb{Z} \xrightarrow{\Phi} E_{K_v}(\overline{K_v})$ the ℓ^n-torsion subgroup $E[\ell^n](\overline{K_v})$ is generated by the images of ζ and q^{1/ℓ^n}.

So let us choose $P_1 = \Phi(\zeta)$ and $P_2 = \Phi(q^{1/\ell^n})$. Since the v-adic uniformization map $\overline{K_v}^\times/q^{\mathbb{Z}} \xrightarrow{\Phi} E_{K_v}(\overline{K_v})$ is compatible with the action of G_v, we see that we have $P_1^\sigma = \Phi(\zeta^\sigma) = \Phi(\zeta) = P_1$ for any $\sigma \in G_v$ because $\zeta \in K_v$. So G_v acts trivially on the point P_1.

Now we come to P_2. Let ℓ^e be the exact power of ℓ dividing $v(j_E)$, i. e. ℓ^e divides $v(j_E)$, but ℓ^{e+1} doesn't. Because of the identity

$$j_E = \frac{1}{q} + 744 + 196\,884\,q + \cdots$$

we see that $v(j_E) = -v(q)$ so that ℓ^e is also the exact power of ℓ dividing $v(q)$. Hence, $K_v(q^{1/\ell^n})/K_v$ is a Kummer extension of degree ℓ^{n-e} (resp. 1 if $e > n$). Thus for any $n' \geq e$ $\big($i. e. $\ell^{n'+1} \nmid v(j_E)\big)$ we may find a $\sigma \in G_v$ with $(q^{1/\ell^n})^\sigma = \zeta^{\ell^{n'}} \cdot q^{1/\ell^n}$ so that

$$\Phi(P_2)^\sigma = \Phi\big((q^{1/\ell^n})^\sigma\big) = \Phi\big(\zeta^{\ell^{n'}} \cdot q^{1/\ell^n}\big) = \big[\ell^{n'}\big] \cdot P_1 + P_2.$$

Hence, the action of σ with respect to the basis (P_1, P_2) is given by

$$\begin{pmatrix} 1 & \ell^{n'} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}).$$

In particular, since $\ell \nmid v(q)$ for almost all ℓ, we find for these ℓ a $\sigma \in G_v$ acting like a transvection, i. e. $n' = 0$.

Furthermore, the theory of Tate curves tells us that under the isomorphism $\overline{K_v}^\times/q^{\mathbb{Z}} \xrightarrow{\sim} E_{K_v}(\overline{K_v})$ the group of points of $E(K_v)$ specializing into the connected component of one is isomorphic to $R^\times$ where $R^\times$ denotes the units in the ring of integers of K_v [Si86, thm.14.1(b)]. So, since $\zeta \in \mathbb{C}^\times \subset R^\times$, our choice of $P_1 = \Phi(\zeta)$ will specialize into the connected component of one, while $P_2 = \Phi(q^{1/\ell^n})$ is any other point completing P_1 to a basis of the ℓ^n-torsion points of E. $\square$

We deduce several corollaries. Remember that $K = \mathbb{C}(Y)$ is the function field of our base curve $Y_{\mathbb{C}}$, and E_K is the generic fiber of the family $E_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$.

Corollary C.1.2 (Galois action on torsion of E_K)
Let E_K be a non-isotrivial elliptic curve defined over the function field K of a curve $Y_{\mathbb{C}}$. Then for any prime number ℓ there is a non-negative integer $n(\ell)$ such that for all prime powers ℓ^n there are elements σ and σ' of $G_{\bar{K}/K}$ which act like

$$\begin{pmatrix} 1 & \ell^{n(\ell)} \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 0 \\ \ell^{n(\ell)} & 1 \end{pmatrix}$$

on $E[\ell^n](\overline{K})$ with respect to a suitable basis. Moreover, for almost all ℓ, we may choose $n(\ell) = 0$.

Proof. We may assume that $E_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$ has everywhere semistable reduction and a full level-ℓ^n-structure, i. e. there is an isomorphism $(\mathbb{Z}/\ell^n\mathbb{Z})^2_{Y_{\mathbb{C}}} \to E[\ell^n]_{Y_{\mathbb{C}}}$ of group schemes. This can always be achieved after a finite base change.

Choosing a point $y \in Y(\mathbb{C})$ such that $E_{Y_\mathbb{C}}$ has bad reduction in y, we find by (C.1.1) a basis (P_1, P_2) of $E[\ell^n](\overline{K})$ such that there are elements $\sigma \in G_{\bar{K}/K}$ which act with respect to (P_1, P_2) like

$$\begin{pmatrix} 1 & \ell^{n'} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$$

for n' with $\ell^{n'+1} \nmid v(j_E)$ where v is the valuation at y. Furthermore, P_1 specializes into the connected component of one while P_2 does not. So we may find another point $y' \in Y(\mathbb{C})$ such that P_2 will specialize into the connected component of one, since having a full level-ℓ^n-structure $E_{Y_\mathbb{C}}$ is the pull-back of the universal elliptic curve $E(\ell^n) \to X(\ell^n)$ parameterizing full level-ℓ^n-structures (we may assume that $\ell^n > 2$ because if the statement is true for ℓ^n, it is also true for ℓ^{n-1}). So using again (C.1.1) we will find elements $\sigma' \in G_{\bar{K}/K}$ which act with respect to the basis (P_2, P_1) like

$$\begin{pmatrix} 1 & \ell^{n''} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$$

for n'' with $\ell^{n''+1} \nmid v'(j_E)$ where v' is the valuation in y'. Of course upper triangle matrices with respect to (P_2, P_1) will be lower triangle matrices with respect to (P_1, P_2). So choosing $n(\ell)$ such that $\ell^{n(\ell)+1} \nmid v(j_E)$ and $\ell^{n(\ell)+1} \nmid v'(j_E)$, we find two elements σ and σ' of $G_{\bar{K}/K}$ which act with respect to (P_1, P_2) like

$$\begin{pmatrix} 1 & \ell^{n(\ell)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \ell^{n(\ell)} & 1 \end{pmatrix}.$$

In particular, $n(\ell)$ does only depend on ℓ (not on ℓ^n) and for almost all ℓ, we may choose $n(\ell) = 0$ because $\ell \nmid v(j_E)$ and $\ell \nmid v'(j_E)$. □

With this knowledge, we can restrict which $Y_\mathbb{C}$-rational endomorphisms will exist on the group schemes $E[\ell^n]_{Y_\mathbb{C}}$. Consider the multiplication-by-$\ell^{n-n'}$ homomorphism $E[\ell^n] \overset{[\ell^{n-n'}]}{\longrightarrow} E[\ell^{n'}]$ for $n \geq n'$. We have a map $\mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n'}]\big) \overset{\Psi}{\to} \mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^n]\big)$ by sending an element $\varphi \in \mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n'}]\big)$ to the composition of maps

$$E[\ell^n]_{Y_\mathbb{C}} \overset{[\ell^{n-n'}]}{\longrightarrow} E[\ell^{n'}]_{Y_\mathbb{C}} \overset{\varphi}{\to} E[\ell^{n'}]_{Y_\mathbb{C}} \hookrightarrow E[\ell^n]_{Y_\mathbb{C}}.$$

We will denote the image of $\mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n'}]\big)$ under Ψ by $\ell^{n-n'} \cdot \mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n'}]\big)$ for $n \geq n'$. For $n < n'$ the set $\ell^{n-n'} \cdot \mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n'}]\big)$ is supposed to be $\mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^n]\big)$.

Corollary C.1.3 (rational endomorphisms on torsion groups)
Let $E_{Y_\mathbb{C}} \to Y_\mathbb{C}$ be a non-isotrivial family of elliptic curves. Then for any prime number ℓ there is a non-negative integer $n(\ell)$ such that for all prime powers ℓ^n the $Y_\mathbb{C}$-rational endomorphisms are given by

$$\mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^n]\big) = \langle \text{multiplication-by-}m \text{ maps} \rangle + \ell^{n-n(\ell)} \cdot \mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n(\ell)}]\big),$$

i. e. every endomorphism of $E[\ell^n]_{Y_\mathbb{C}}$ is the sum of a multiplication-by-m map and a composition of the multiplication-by-$\ell^{n-n(\ell)}$ with an endomorphism of $E[\ell^{n(\ell)}]_{Y_\mathbb{C}}$.

Moreover, for almost all ℓ we may choose $n(\ell) = 0$ so that $\mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^n]\big)$ consists only of multiplication-by-m maps.

Proof. Any endomorphism on $E[\ell^n]_{Y_\mathbb{C}}$ is invariant under the action of Galois so that it has to lie in the center of the action of Galois on $E[\ell^n]_{Y_\mathbb{C}}$. In particular, using corollary (C.1.2) an endomorphism must commute with the two matrices

$$\begin{pmatrix} 1 & \ell^{n(\ell)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \ell^{n(\ell)} & 1 \end{pmatrix}.$$

An elementary matrix calculation shows that any such endomorphism has to be represented by a matrix of the form

$$A = D + \ell^{n-n(\ell)} \cdot M \in \mathrm{M}_\ell(\mathbb{Z}/\ell^n\mathbb{Z})$$

where D is a diagonal matrix with the same entries on the diagonal, and M is some other matrix whose exact shape is unimportant. The interpretation is the following. D corresponds to a multiplication-by-m map, while $\ell^{n-n(\ell)} \cdot M$ is the composition of the multiplication-by-$\ell^{n-n(\ell)}$ map $E[\ell^n] \rightarrow E[\ell^{n(\ell)}]$ with an endomorphism of $E[\ell^{n(\ell)}]$.

Moreover, corollary (C.1.2) says that for almost all ℓ we have $n(\ell) = 0$ so that $\ell^{n-n(\ell)} \cdot \mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n(\ell)}]\big) = \{0\}$. □

This will tell us that for all N the endomorphisms of $E[N]_{Y_\mathbb{C}}$ are already defined over some base $Y_{F'}$ where F' is a number field.

Corollary C.1.4 (descending endomorphisms of torsion groups)
Let $E_{Y_F} \rightarrow Y_F$ be a non-isotrivial family of elliptic curves with base curve Y_F defined over some number field F. Then there is a finite extension F' of F such that for all natural numbers N the endomorphisms of $E[N]_{Y_\mathbb{C}}$ are defined over $Y_{F'}$, i. e. we have an isomorphism $\mathrm{End}_{Y_\mathbb{C}}\big(E[N]\big) \cong \mathrm{End}_{Y_{F'}}\big(E[N]\big)$ induced by the base change $Y_\mathbb{C} \rightarrow Y_{F'}$. In particular, the field F' depends only on E_{Y_F}.

Proof. It is enough to consider prime powers ℓ^n. By corollary (C.1.3) we have

$$\mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^n]\big) = \langle \text{multiplication-by-}m \text{ maps} \rangle + \ell^{n-n(\ell)} \cdot \mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n(\ell)}]\big).$$

The multiplication-by-m maps are clearly defined over Y_F. So $\mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^n]\big)$ is defined over some base $Y_{F'}$ if $\mathrm{End}_{Y_\mathbb{C}}\big(E[\ell^{n(\ell)}]\big)$ is defined over $Y_{F'}$. For almost all ℓ this set is trivial because $n(\ell) = 0$. For the finitely many remaining ℓ this set is not trivial but finite. So, there are at most finitely many maps which are not defined over Y_F but over $Y_\mathbb{C}$. Thus, after a suitable finite extension F' of F everything will be defined over $Y_{F'}$. Clearly, the choice of F' depends only on E_{Y_F}. □

As explained in the introduction of this section, this shows that families of abelian varieties $A_{Y_\mathbb{C}} \rightarrow Y_\mathbb{C}$, isogenous to a g-fold product of $E_{Y_\mathbb{C}}$, are defined over $Y_{F'}$, independently from g.

Proposition C.1.5 (descending isogenies and abelian varieties)
Let Y_F be a curve defined over a number field and $E_{Y_F} \rightarrow Y_F$ a non-isotrivial family of elliptic curves. Then there is a finite extension F' of F such that every $Y_\mathbb{C}$-isogeny $h_{Y_\mathbb{C}}$ from any g-fold product of $E_{Y_\mathbb{C}}$ to any family of abelian varieties $A_{Y_\mathbb{C}} \rightarrow Y_\mathbb{C}$ will descend to $Y_{F'}$, i. e. there is a family of abelian varieties $A_{Y_{F'}} \rightarrow Y_{F'}$ and an $Y_{F'}$-isogeny $h_{Y_{F'}}$ from the g-fold product of $E_{Y_{F'}}$ to $A_{Y_{F'}}$ such that $h_{Y_\mathbb{C}}$ is the extension of $h_{Y_{F'}}$ under the base change $Y_\mathbb{C} \rightarrow Y_{F'}$. In particular, F' depends only on E_{Y_F} and not on g.

Proof. Let $H_{Y_{\mathbb{C}}}$ be the kernel of $h_{Y_{\mathbb{C}}}$. For a suitable number N the group scheme $H_{Y_{\mathbb{C}}}$ is contained in $E[N]_{Y_{\mathbb{C}}} \times \ldots \times E[N]_{Y_{\mathbb{C}}}$. Since we can describe $H_{Y_{\mathbb{C}}}$ as the kernel of an endomorphism of $E[N]_{Y_{\mathbb{C}}} \times \ldots \times E[N]_{Y_{\mathbb{C}}}$, corollary (C.1.4) shows that $H_{Y_{\mathbb{C}}}$ will be defined over $Y_{F'}$, i.e. $H_{Y_{\mathbb{C}}}$ is the extension of a subgroup scheme $H_{Y_{F'}}$ of $E[N]_{Y_{F'}} \times \ldots \times E[N]_{Y_{F'}}$ with respect to the base change $Y_{\mathbb{C}} \to Y_{F'}$.

Now let $A_{Y_{F'}}$ be the quotient of the g-fold product of $E_{Y_{F'}}$ by $H_{Y_{F'}}$ and let $h_{Y_{F'}}$ be the quotient map. Clearly, $h_{Y_{\mathbb{C}}}$ and $A_{Y_{\mathbb{C}}}$ are the extensions of $h_{Y_{F'}}$ and $A_{Y_{F'}}$ under the base change $Y_{\mathbb{C}} \to Y_{F'}$. Also corollary (C.1.4) says that F' depends only on E_{Y_F}. □

With corollary (C.1.3) we can also reprove proposition (B.4.3) about bounding the degree of isogenies, but this time for the base field $\mathbb{C}$.

Proposition C.1.6 (bounding the degree of isogenies)
Let $E_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$ be a non-isotrivial family of elliptic curves. Then there is a finite set of primes $S = S(E_{Y_{\mathbb{C}}})$, depending only on $E_{Y_{\mathbb{C}}}$, such that for every family of abelian varieties $A_{Y_{\mathbb{C}}}$, which is $Y_{\mathbb{C}}$-isogenous to a g-fold product of $E_{Y_{\mathbb{C}}}$, there is a $Y_{\mathbb{C}}$-isogeny between the g-fold product of $E_{Y_{\mathbb{C}}}$ and $A_{Y_{\mathbb{C}}}$ whose degree has only prime divisors contained in S. In particular, the set S does not depend on g.

Proof. Corollary (C.1.3) tells us that for almost all primes ℓ the $Y_{\mathbb{C}}$-endomorphisms of $E[\ell]_{Y_{\mathbb{C}}}$ are just the multiplication-by-m maps. This corresponds to corollary (B.4.2b). Now verbatim the same proof as for proposition (B.4.3) works in the situation here. □

So far, we have shown in proposition (C.1.5) that families of abelian varieties $A_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$ isogenous to a g-fold product of $E_{Y_{\mathbb{C}}}$ will be defined over $Y_{F'}$. Since we want to descend families of Jacobians $J_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$, which carry a canonical polarization, we also have to take care about descending polarizations.

By definition (see [Mi86, p.149]), a polarization on a family of abelian varieties A_{Y_F} is an Y_F-isogeny $A_{Y_F} \to \widehat{A}_{Y_F}$ which induces polarization on the geometric fibers. Therefore, a polarization over $Y_{\mathbb{C}}$ is defined over Y_F if it is there defined as an isogeny. We start with polarizations on $E_{Y_{\mathbb{C}}} \times \ldots \times E_{Y_{\mathbb{C}}}$.

Lemma C.1.7 (descending polarizations on $E_{Y_{\mathbb{C}}} \times \ldots \times E_{Y_{\mathbb{C}}}$)
Let Y_F be a curve defined over a number field F and $E_{Y_F} \to Y_F$ a non-isotrivial family of elliptic curves. Then any polarization $\lambda_{Y_{\mathbb{C}}}$ on a g-fold product of $E_{Y_{\mathbb{C}}}$ will be defined over Y_F, i.e. there is a polarization λ_{Y_F} on $E_{Y_F} \times \ldots \times E_{Y_F}$ such that $\lambda_{Y_{\mathbb{C}}}$ is the extension of λ_{Y_F} under the base change $Y_{\mathbb{C}} \to Y_F$.

Proof. Let $B_{Y_F} := E_{Y_F} \times \ldots \times E_{Y_F}$ be the g-fold product of E_{Y_F} and let $\lambda_{Y_{\mathbb{C}}}$ be a polarization on $B_{Y_{\mathbb{C}}}$. Let further be $\psi_{Y_{\mathbb{C}}}$ the product polarization on $B_{Y_{\mathbb{C}}}$, i.e. $\psi_{Y_{\mathbb{C}}}$ is the g-fold product of the unique principal polarization on $E_{Y_{\mathbb{C}}}$. Clearly, $\psi_{Y_{\mathbb{C}}}$ is defined over Y_F since the unique principal polarization $E_{Y_{\mathbb{C}}} \to \widehat{E}_{Y_{\mathbb{C}}}$ is defined there. Denote this polarization by ψ_{Y_F}.

The map $\phi_{Y_{\mathbb{C}}} := \psi_{Y_{\mathbb{C}}}^{-1} \circ \lambda_{Y_{\mathbb{C}}}$ is an endomorphism of $B_{Y_{\mathbb{C}}}$. Hence, it is defined over Y_F because $\mathrm{End}_{Y_{\mathbb{C}}}(B) \cong \mathrm{End}_{Y_F}(B)$ since the endomorphisms of $E_{Y_{\mathbb{C}}}$ are just the multiplication-by-m maps (see also the discussion in example (B.2.3)). Then the isogeny $\lambda_{Y_F} := \psi_{Y_F} \circ \phi_{Y_F}$ will equal $\lambda_{Y_{\mathbb{C}}}$ after the base change $Y_{\mathbb{C}} \to Y_F$ so that $\lambda_{Y_{\mathbb{C}}}$ is defined over Y_F. □

Now we come to polarizations on families of abelian varieties isogenous to the product $E_{Y_\mathbb{C}} \times \ldots \times E_{Y_\mathbb{C}}$.

Proposition C.1.8 (descending polarizations)
In the situation of proposition (C.1.5) assume further that the family of abelian varieties $A_{Y_\mathbb{C}} \to Y_\mathbb{C}$ is equipped with a (principal) polarization $\lambda_{Y_\mathbb{C}}$. Then $\lambda_{Y_\mathbb{C}}$ is also defined over $Y_{F'}$, i. e. there is a (principal) polarization $\lambda_{Y_{F'}}$ on $A_{Y_{F'}}$ which equals $\lambda_{Y_\mathbb{C}}$ after the base change $Y_\mathbb{C} \to Y_{F'}$.

Proof. Let $\mu_{Y_\mathbb{C}}$ be the pull-back polarization of $\lambda_{Y_\mathbb{C}}$ on $B_{Y_\mathbb{C}} := E_{Y_\mathbb{C}} \times \ldots \times E_{Y_\mathbb{C}}$, i. e. $\mu_{Y_\mathbb{C}} = \widehat{h}_{Y_\mathbb{C}} \circ \lambda_{Y_\mathbb{C}} \circ h_{Y_\mathbb{C}}$ where $\widehat{h}_{Y_\mathbb{C}}$ is the dual map of $h_{Y_\mathbb{C}}$. Moreover, Mumford's characterization of pull-back polarizations [Mu70, p.331] tells us that $H_{Y_\mathbb{C}}$ – the kernel of $h_{Y_\mathbb{C}}$ – is an isotropic subgroup of $\mathrm{Ker}(\mu_{Y_\mathbb{C}})$ with respect to the Weil-pairing on $\mathrm{Ker}(\mu_{Y_\mathbb{C}})$ induced by $\mu_{Y_\mathbb{C}}$.

Now by lemma (C.1.7) $\mu_{Y_\mathbb{C}}$ descends to a polarization $\mu_{Y_{F'}}$ on $B_{Y_{F'}}$. Moreover, $H_{Y_{F'}}$ – the kernel of $h_{Y_{F'}}$ – is still an isotropic subgroup of $\mathrm{Ker}(\mu_{Y_{F'}})$ with respect to the Weil-pairing induced by $\mu_{Y_{F'}}$ because if this is false over $Y_{F'}$, it will still be false over $Y_\mathbb{C}$ where it is true. So again Mumford's characterization [Mu70, p.331] tells us that $\mu_{Y_{F'}}$ induces a polarization $\lambda_{Y_{F'}}$ on $A_{Y_{F'}}$. Clearly, $\lambda_{Y_\mathbb{C}}$ is the extension of $\lambda_{Y_{F'}}$ under the base change $Y_\mathbb{C} \to Y_{F'}$ since the polarizations are uniquely determined by the embedding $H_{Y_{F'}} \subset \mathrm{Ker}(\mu_{Y_{F'}})$ resp. $H_{Y_\mathbb{C}} \subset \mathrm{Ker}(\mu_{Y_\mathbb{C}})$. $\square$

We now have everything together to reduce principally polarized Jacobians defined over $Y_\mathbb{C}$ to a base curve $Y_{F'}$ defined over a number field $Y_{F'}$. We want to do the same for curves.

C.2 Descending curves

Let F be a number field, Y_F a curve and $C_{Y_\mathbb{C}} \to Y_\mathbb{C}$ a family of curves with Jacobian $J_{Y_\mathbb{C}} \to Y_\mathbb{C}$. We would like to show that if $J_{Y_\mathbb{C}}$ is defined over Y_F, then $C_{Y_\mathbb{C}}$ id also defined over Y_F. Therefore, regard the j-map

$$j : \mathscr{M}_g \longrightarrow \mathscr{A}_{g,1}$$

from the moduli space of curves of genus g into the moduli space of principally polarized abelian varieties of dimension g which associates to a curve C its principally polarized Jacobian (J, θ).

Unfortunately, we cannot answer questions of rationality in this setup since $\mathscr{M}_g$ and $\mathscr{A}_{g,1}$ are not fine moduli spaces. One therefore has to rigidify these two moduli problems by introducing level-N-structures. So, consider the functor

$$\mathbf{M}_g^{(N)} : \mathbf{Schemes} \longrightarrow \mathbf{Sets}$$

which associates to a scheme S the set of S-isomorphism classes of pairs (C_S, α_S) consisting of a smooth relative curve $C_S \to S$ of genus g and an S-isomorphism α_S of group schemes $(\mathbb{Z}/N\mathbb{Z})_S^{2g} \xrightarrow{\sim} J[N]_S$, where $J_S \to S$ is the relative Jacobian of $C_S \to S$. Further, consider the functor

$$\mathbf{A}_{g,1}^{(N)} : \mathbf{Schemes} \longrightarrow \mathbf{Sets}$$

which associates to a scheme S the set of S-isomorphism classes of triples $(A_S, \lambda_S, \alpha_S)$ where $A_S \to S$ is an abelian scheme, λ_S is a principal polarization $A_S \to \widehat{A}_S$ and α_S is an S-isomorphism $(\mathbb{Z}/NZ)_S^{2g} \xrightarrow{\sim} A[N]_S$ of group schemes. Additionally, we have a morphism of functors

$$\mathbf{J}^{(N)} : \mathbf{M}_g^{(N)} \longrightarrow \mathbf{A}_{g,1}^{(N)}$$

which sends the class of (C_S, α_S) to the class of $(J_S, \theta_S, \alpha_S)$ where θ_S is the canonical principal polarization on the Jacobian J_S and α_S is the same level-N-structure as in (C_S, α_S). Then we have the following theorem about the representability of $\mathbf{M}_g^{(N)}$.

Theorem C.2.1 (representability of $\mathbf{M}_g^{(N)}$)
For $N \geq 3$ the functor $\mathbf{M}_g^{(N)}$ is finely represented by a smooth scheme $\mathscr{M}_g^{(N)}$ over $\operatorname{Spec} \mathbb{Z}[1/N]$.

Proof. This is theorem 10.9 and remark (2) in [Po77, p.141-142]. □

And for the representability of $\mathbf{A}_{g,1}^{(N)}$ we have the next theorem.

Theorem C.2.2 (representability of $\mathbf{A}_{g,1}^{(N)}$)
For $N \geq 3$ the functor $\mathbf{A}_{g,1}^{(N)}$ is finely represented by a smooth scheme $\mathscr{A}_{g,1}^{(N)}$ over $\operatorname{Spec} \mathbb{Z}[1/N]$.

Proof. This is theorem 7.9 of [GIT, p.134]. □

Hence, the morphism of functors $\mathbf{J}^{(N)} : \mathbf{M}_g^{(N)} \to \mathbf{A}_{g,1}^{(N)}$ induces a morphism of schemes $j^{(N)} : \mathscr{M}_g^{(N)} \to \mathscr{A}_{g,1}^{(N)}$. Although we now have fine moduli schemes, we still cannot conclude that the family of curves $C_{Y_\mathbb{C}} \to Y_\mathbb{C}$ is defined over Y_F if $J_{Y_\mathbb{C}} \to Y_\mathbb{C}$ is defined there because the map $j^{(N)} : \mathscr{M}_g^{(N)} \to \mathscr{A}_{g,1}^{(N)}$ is not injective. The reason is that the two triples $(J_S, \theta_S, \alpha_S)$ and $(J_S, \theta_S, -\alpha_S)$ are isomorphic by the multiplication-by-(-1) map $[-1]$. Thus, they belong to the same isomorphism class. But unless $C_S \to S$ is hyperelliptic, the map $[-1]$ does not come from an automorphism of the curve C_S. Hence, (C_S, α_S) and $(C_S, -\alpha_S)$ lie in two different isomorphism classes. Therefore, the map $j^{(N)} : \mathscr{M}_g^{(N)} \to \mathscr{A}_{g,1}^{(N)}$ is generically 2-to-1.

Proposition C.2.3 (the degree of the map $j^{(N)}$)
For $g \geq 3$ the morphism of schemes

$$j^{(N)} : \mathscr{M}_g^{(N)} \to \mathscr{A}_{g,1}^{(N)}$$

is 2*-to-*1 *onto its image and ramified over the hyperelliptic locus. For $g = 2$ the map $j^{(N)}$ is* 1*-to-*1*.*

Proof. See the discussion in [OS80, p.163]. □

We may conclude the following rationality result.

Proposition C.2.4 (descending curves)
Let $N \geq 3$ be an integer and Y_F a curve defined over some number field F. Let $J_{Y_F} \to Y_F$ be a family of abelian varieties with a principal polarization θ_{Y_F} and a level-N-structure α_{Y_F}. If $C_{Y_\mathbb{C}} \to Y_\mathbb{C}$ is a family of curves whose principally polarized Jacobian is $J_{Y_\mathbb{C}} \to Y_\mathbb{C}$ together with $\lambda_{Y_\mathbb{C}}$, then there is a covering $Y'_F \to Y_F$ of degree at most 2 *such that $C_{Y'_\mathbb{C}} \to Y'_\mathbb{C}$ is defined over Y'_F.*

Proof. Let $\mathbb{C}(Y)$ be the function field of $Y_{\mathbb{C}}$. By assumption, $C_{\mathbb{C}(Y)}$ is a curve whose principally polarized Jacobian $\big(J_{\mathbb{C}(Y)}, \lambda_{\mathbb{C}(Y)}\big)$ has a level-N-structure $\alpha_{Y_{\mathbb{C}}}$. Hence, by proposition (C.2.3) $C_{\mathbb{C}(Y)}$ must be defined over a field extension $F(Y')$ of $F(Y)$ of degree at most 2, i. e. there is a curve $C_{F(Y')}$ such that $C_{F(Y')} \times_{F(Y')} \mathbb{C}(Y')$ is isomorphic to $C_{\mathbb{C}(Y)} \times_{\mathbb{C}(Y)} \mathbb{C}(Y')$. Continuing $C_{F(Y')}$ to a family of curves $C_{Y'_F} \to Y'_F$ proves the proposition. □

For curves whose Jacobian is isogenous to a g-fold product of an elliptic curve, we give a more uniform description of the descending base curve.

Proposition C.2.5 (uniformly descending curves)
Let $E_{Y_F} \to Y_F$ be a non-isotrivial family of elliptic curves over a curve Y_F defined over a number field F. Then there is a finite field extension F' of F and a curve $Y'_{F'}$ covering $Y_{F'}$, both depending only on E_{Y_F}, such that for any family of curves $C_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$, whose Jacobian $J_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$ is $Y_{\mathbb{C}}$-isogenous to a g-fold product of $E_{Y_{\mathbb{C}}}$, there is finite covering $Y''_{F'} \to Y'_{F'}$ of degree at most 2, such that $C_{Y''_{\mathbb{C}}} \to Y''_{\mathbb{C}}$ is defined over $Y''_{F'}$. In particular, $Y'_{F'}$ and the degree of $Y''_{F'} \to Y'_{F'}$ depend not on g.

Proof. Let $S = S(E_{Y_{\mathbb{C}}})$ be the set of primes from proposition (C.1.6) about bounding the degree of isogenies. Let $N \geq 3$ be an integer which is not divisible by any prime in S. (This choice of N will later ensure that the Jacobian is equipped with a level-N-structure, see below.)

After a finite extension $Y'_{F'} \to Y_F$ we may assume that $E_{Y'_{F'}}$ is equipped with a level-N-structure $\alpha_{Y'_{F'}}$. The extension $Y'_{F'} \to Y_F$ depends only on E_{Y_F}. We may also assume that the conclusion of proposition (C.1.5) about descending isogenies and abelian varieties holds. Otherwise, this will be achieved after a finite extension of F' which we call also F' (this extension will only depend on $E_{Y'_{F'}}$ and, therefore, only on E_{Y_F}.)

Now let $C_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$ be a curve whose Jacobian $J_{Y_{\mathbb{C}}} \to Y_{\mathbb{C}}$ is $Y_{\mathbb{C}}$-isogenous to a g-fold product of $E_{Y_{\mathbb{C}}}$. We may assume by proposition (C.1.6) that there is an isogeny $h_{Y_{\mathbb{C}}}$ from $E_{Y_{\mathbb{C}}} \times \ldots \times E_{Y_{\mathbb{C}}}$ to $J_{Y_{\mathbb{C}}}$ whose degree has only prime divisors in S. Thus, the level-N-structure $\alpha_{Y'_{\mathbb{C}}}$ will be mapped under $h_{Y'_{\mathbb{C}}}$ injectively into $J_{Y'_{\mathbb{C}}}$ so that $J_{Y'_{\mathbb{C}}}$ itself is equipped with a level-N-structure which we call $\beta_{Y'_{\mathbb{C}}}$.

By proposition (C.1.5) respectively (C.1.8) and the choice of $Y'_{F'}$ we see that $J_{Y'_{\mathbb{C}}}$ together with its principal polarization $\theta_{Y'_{\mathbb{C}}}$ are defined over $Y'_{F'}$. Also the level-N-structure $\beta_{Y_{\mathbb{C}}}$ on $J_{Y'_{\mathbb{C}}}$ is defined over $Y'_{F'}$ since it is the image of the level-N-structure $\alpha_{Y'_{F'}}$ of $E_{Y'_{F'}} \times \ldots \times E_{Y'_{F'}}$ under $h_{Y'_{F'}}$. So, the triple $(J_{Y'_{\mathbb{C}}}, \theta_{Y'_{\mathbb{C}}}, \beta_{Y'_{\mathbb{C}}})$ is defined over $Y'_{F'}$. Hence, by proposition (C.2.4) the curve $C_{Y'_{\mathbb{C}}}$ is defined over a covering $Y''_{F'} \to Y'_{F'}$ of degree at most 2. Also, as mentioned above, the choice of the base $Y'_{F'}$ depends only on E_{Y_F}. □

We can now reduce the situation from the curve $Y_{\mathbb{C}}$ defined over the complex numbers to the curve $Y'_{F'}$ defined over some number field F'. Applying theorem (B.5.1) we can derive a bound for the genus of $C_{Y_{\mathbb{C}}}$. But since our curve $C_{Y_{\mathbb{C}}}$ will only be defined over a covering $Y''_{F'} \to Y'_{F'}$ of degree at most 2, the bound will depend on this covering $Y''_{F'} \to Y'_{F'}$. To make the bound independent of the covering, we will extend the results of the previous two chapters by taking coverings of bounded degrees into account.

C.3 Uniform boundedness in positive characteristic

Fix some finite field $\mathbb{F}_q$ and let Y be a smooth, projective, geometrically connected curve defined over $\mathbb{F}_q$. Let further $E_Y \to Y$ be a family of elliptic curves. In section (A.4) we showed that any family of curves $C_Y \to Y$ of genus g whose Jacobian $J_Y \to Y$ is Y-isogenous to the g-fold product of E_Y has its genus bounded by some constant $c = c(E_Y)$ depending only on E_Y.

Now we want to extend this result by assuming that the family of curves is not defined over Y but over some covering $Y' \to Y$ of degree d. We will show that the genus g is bounded by some constant $\mathscr{C} = \mathscr{C}(E_Y, d)$ depending only on E_Y and d. In particular, $\mathscr{C}$ is independent of the particular choice of the covering $Y' \to Y$. We therefor extend the two results (A.4.2) and (A.4.3) to this case.

Proposition C.3.1 (uniform lower bound for δ)
Let $E_Y \to Y$ be a non-isotrivial family of elliptic curves and $Y' \to Y$ some finite covering of degree d. Let $C_{Y'} \to Y'$ be a family of curves of genus g whose Jacobian $J_{Y'} \to Y'$ is Y'-isogenous to the g-fold product of $E_{Y'}$. Then there is a constant $c = c(E_Y, d) > 0$, depending only on E_Y and d such that

$$c \cdot \frac{\log g}{\log \log g} \cdot g \leq \delta$$

where δ is the number of singularities in the geometric fibers of $C_{Y'} \to Y'$. In particular, δ is not linearly bounded above by g and the lower bound does not depend on the particular choice of the covering $Y' \to Y$.

Proof. Recall the proof of (A.4.2). We extended the base field $\mathbb{F}_q$ such that we may assume, using the Sato-Tate-conjecture, that $\#\{y \in Y(\mathbb{F}_{q^n}) \,|\, 0 \leq \Theta(y) < \frac{\pi}{2}\} > \frac{1}{4} q^n$ where $\Theta(y)$ was the Frobenius angle of the fiber above y of $E_Y \to Y$. Then we counted the number of singularities, namely that there is a constant $c(E_Y)$ such that $C_Y \to Y$ has at least $c(E_Y) \cdot \frac{\log g}{\log \log g}$ singularities in its geometric fibers. For this we used only properties of E_Y, the family of curves $C_Y \to Y$ was never involved in the counting.

Now assume that we have a covering $Y' \to Y$ of degree d. Then any $\mathbb{F}_{q^n}$-rational point of Y has at least one $\mathbb{F}_{q^{rn}}$-rational preimage with $r \leq d$. So applying the Sato-Tate-conjecture on $E_{Y'}$, which is the extension of E_Y with respect to the base change $Y' \to Y$, we see that

$$\# \left\{ y \in Y'(\mathbb{F}_{q^{dn}}) \,\middle|\, 0 \leq \Theta(y) < \frac{\pi}{2} \right\} \geq \# \left\{ y \in Y(\mathbb{F}_{q^n}) \,\middle|\, 0 \leq \Theta(y) < \frac{\pi}{2d} \right\} > \varepsilon q^n$$

where $\varepsilon > 0$ is some constant depending only on d. Now verbatim the same counting as in (A.4.2) gives us a constant $c = c(E_Y, d)$, depending only on E_Y and d, such that $C_{Y'} \to Y'$ has at least $c \cdot \frac{\log g}{\log \log g}$ singularities. $\square$

As in (A.4.3) we can now derive a bound for the genus g. We will denote the function field of Y by K and the function field of Y' by K'.

Theorem C.3.2 (uniform bound for the genus)
Let E_K be a non-isotrivial elliptic curve and $C_{K'}$ a smooth, projective, geometrically connected curve of genus g defined over a field extension K' of K of degree at most d. Assume that the Jacobian $J_{K'}$ of $C_{K'}$ is K' isogenous to the g-fold product of $E_{K'}$. Then the genus of $C_{K'}$ is bounded, i. e. there is a constant $\mathscr{C} = \mathscr{C}(E_K, d) > 0$, depending only on E_K and d, such that g is smaller than $\mathscr{C}$.

Proof. This is essentially the same proof as in (A.4.3). Without loss of generality we may assume that E_K has everywhere semistable reduction and we extend $C_{K'}$ to a semistable family of curves $C_{Y'}{\rightarrow}Y'$. Furthermore, (C.3.1) gives the lower bound $c_0(E_K, d) \cdot \frac{\log g}{\log\log g} \cdot g$ for the number of singularities δ of $C_{Y'}{\rightarrow}Y'$ where $c_0(E_K, d)$ is the minimum of the constants $c(E_Y, 1), \ldots, c(E_Y, d)$ of proposition (C.3.1). This constant depends only on E_Y and d.

An upper bound for δ is given by $12h(E_{Y'}) \cdot g$, see (A.3.10). Since $E_{Y'}$ is the extension of E_Y with respect to the base change $Y'{\rightarrow}Y$, it follows that

$$h(E_{Y'}) = d(Y' : Y) \cdot h(E_Y) \leq d \cdot h(E_Y)$$

where $d(Y' : Y)$ is the degree of $Y'{\rightarrow}Y$. In fact, the first equality holds because E_Y is semistable and since the height of $E_{Y'}$ resp. E_Y depends only on the connected component of one $E^0_{Y'}$ resp. E^0_Y. The semistability assumption ensures that $E^0_{Y'} \cong E^0_Y \times_Y Y'$.

So, for the number of singularities δ we have the inequalities

$$c_0(E_Y, d) \cdot \frac{\log g}{\log\log g} \cdot g \leq \delta \leq 12 \cdot d \cdot h(E_Y) \cdot g$$

which implies a contradiction for large g. Hence, the genus g is bounded by a constant $\mathscr{C}$ which depends only on E_K, thus on E_K, and d. □

As in chapter B this will imply the uniform boundedness of the genus in the case that the base curve Y is defined over some number field. We show this in the following section.

C.4 Uniform boundedness in characteristic zero

In this section we extend the result of theorem (B.5.1) by allowing the curve C to be defined over some field extension K' of K of degree at most d instead of K itself. We show that the genus is still bounded by some constant $\mathscr{C} = \mathscr{C}(E_K, d)$ depending only on E_K and d. As in (B.5.1) we will prove this by reduction to characteristic p.

So let F be a number field with ring of integers $\mathcal{O}_F$ and let Y_F be a smooth, projective, geometrically connected curve over F. So $Y_F{\rightarrow}\operatorname{Spec} F$ extends to a minimal arithmetic surface $Y{\rightarrow}\operatorname{Spec}\mathcal{O}_F$. Remember our explanations in section B.5. If $\operatorname{Spec}\mathbb{F}_q{\rightarrow}\operatorname{Spec}\mathcal{O}_F$ is some finite point, then $Y_{\mathbb{F}_q}$ denotes the fiber of $Y{\rightarrow}\operatorname{Spec}\mathcal{O}_F$ over this point. K is supposed to be the function field of Y_F while k is the function field of some irreducible component of $Y_{\mathbb{F}_q}$. R was the local ring of Y at this irreducible component so that $\operatorname{Spec} K$ and $\operatorname{Spec} k$ are the generic and special point of $\operatorname{Spec} R$. We further extend given abelian varieties J_K and E_K to Néron models $J{\rightarrow}\operatorname{Spec} R$ and $E{\rightarrow}\operatorname{Spec} E$ with special fibers J_k and E_k. This is summarized in the diagram on page 40.

Additionally, we now assume that there is a second curve Y'_F covering Y_F. We regard the same structures as above related to Y'_F. More precisely, $Y' \to \operatorname{Spec} \mathcal{O}_F$ is the minimal arithmetic surface corresponding to $Y'_F \to \operatorname{Spec} F$ and $Y'_{\mathbb{F}_q}$ is its fiber over the point $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathcal{O}_F$. Let K' be the function field of Y'_F and let k' be the function field of an irreducible component of $Y'_{\mathbb{F}_q}$. We denote the local ring of $Y' \to \operatorname{Spec} \mathcal{O}_F$ at this irreducible component by R'. So we have the following diagram analogous to the diagram on page 40.

$$
\begin{array}{ccccc}
\{C_{K'}, J_{K'}, E_{K'}\} & \hookrightarrow & \{C', J', E'\} & \hookleftarrow & \{C_{k'}, J_{k'}, E_{k'}\} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} K' & \hookrightarrow & \operatorname{Spec} R' & \hookleftarrow & \operatorname{Spec} k' \\
\downarrow & & \downarrow & & \downarrow \\
Y'_F & \hookrightarrow & Y' & \hookleftarrow & Y'_{\mathbb{F}_q} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} F & \hookrightarrow & \operatorname{Spec} \mathcal{O}_F & \hookleftarrow & \operatorname{Spec} \mathbb{F}_q
\end{array}
$$

Given a curve $C_{K'}$ or abelian varieties $J_{K'}$ resp. $E_{K'}$, we denote the corresponding minimal models over $\operatorname{Spec} R'$ by C', J' or E' and their special fibers by $C_{k'}, J_{k'}$ or $E_{k'}$. The morphism $Y'_F \to Y_F$ induces an $\mathcal{O}_F$-rational map $Y' \to Y$. Assume that the irreducible component of $Y'_{\mathbb{F}_q}$ corresponding to k' maps onto the irreducible component of $Y_{\mathbb{F}_q}$ corresponding to k. Then we have a morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ that maps $\operatorname{Spec} K'$ to $\operatorname{Spec} K$ and $\operatorname{Spec} k'$ to $\operatorname{Spec} k$.

We have the following extension of theorem (B.5.1).

Theorem C.4.1 (uniform bound for the genus)
Let E_K be a non-isotrivial elliptic curve and J_K an abelian variety which is K-isogenous to a g-fold product of E_K. Let K' be a field extension of K of degree at most d such that $J_{K'}$ becomes the Jacobian of a smooth, projective, geometrically connected curve $C_{K'}$. Then the genus of $C_{K'}$ is bounded, i.e. there is a constant $\mathscr{C} = \mathscr{C}(E_K, d)$, depending only on E_K and d, such that g is smaller than $\mathscr{C}$.

Proof. We reduce the situation to characteristic p. In (B.5.1) we choosed a finite point $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathcal{O}_F$ of residue characteristic p such that the following properties are fulfilled.

(1) Y_F has good reduction at $\operatorname{Spec} \mathbb{F}_q$, i.e. the fiber $Y_{\mathbb{F}_q}$ is a smooth curve. This depends only on K – the function field of Y_F – and is true for almost all points of $\operatorname{Spec} \mathcal{O}_F$. (We actually demand this property only to slightly simplify the situation.)

(2) $E_K \to \operatorname{Spec} K$ extends to a smooth proper model $E \to \operatorname{Spec} R$ such that $E_k \to \operatorname{Spec} k$ is a non-isotrivial elliptic curve. This is true for almost all points of $\operatorname{Spec} \mathcal{O}_F$ and depends only on E_K.

(3) There is an isogeny $E_K \times \ldots \times E_K \to J_K$ such that its degree is prime to p. Using proposition (B.4.3) we see that this is true for almost all points of $\operatorname{Spec} \mathcal{O}_F$ and depends only on E_K.

Since the three conditions above each hold for all but finitely many points of $\operatorname{Spec}\mathcal{O}_F$, we can find a point $\operatorname{Spec}\mathbb{F}_q \to \operatorname{Spec}\mathcal{O}_F$ fulfilling all conditions. The choice of this point depends only on E_K.

Now let k' be the function field of an irreducible component of $Y'_{\mathbb{F}_q}$ which maps onto $Y_{\mathbb{F}_q}$. We always have such components since the finite map $Y'_F \to Y_F$ induces a finite map $J(Y_F) \to J(Y'_F)$ between the Jacobians . This map extends to a finite map between the Néron models over $\operatorname{Spec}\mathcal{O}_F$ of the Jacobians $J(Y_F)$ and $J(Y'_F)$ and restricts over $\operatorname{Spec}\mathbb{F}_q \to \mathcal{O}_F$ to a finite map $J(Y_{\mathbb{F}_q}) \to J(Y'_{\mathbb{F}_q})$ between the Jacobians of $Y_{\mathbb{F}_q}$ and $Y'_{\mathbb{F}_q}$. Because this map is finite, there has to be an irreducible component of $Y'_{\mathbb{F}_q}$ which covers $Y_{\mathbb{F}_q}$. Then the following properties are fulfilled.

(1) k' is a field extension of k of degree at most d. We already discussed this fact above. This follows because the degree of $Y'_F \to Y_F$ is at most d. So also the degree of the covering corresponding to the field extension $\operatorname{Spec} k' \to \operatorname{Spec} k$ is at most d.

(2) $E_{K'}$ extends to a smooth proper model $E' \to \operatorname{Spec} R'$ such that $E_{k'} \to \operatorname{Spec} k'$ is a non-isotrivial elliptic curve. This follows because $E' \to \operatorname{Spec} k'$ is just the extension of $E \to \operatorname{Spec} R$ with respect to the base change $\operatorname{Spec} R' \to \operatorname{Spec} R$ and the assumption is true for $E \to \operatorname{Spec} R$.

(3) There is an isogeny $E_{k'} \times \ldots \times E_{K'} \to J_{K'}$ such that its degree is prime to p. Just take the extension of $E_K \times \ldots \times E_K \to J_K$ with respect to the base change $\operatorname{Spec} K' \to \operatorname{Spec} K$. Together with (2) this property will enable us to lift endomorphisms of $J_{k'}$ to endomorphisms of $J_{K'}$ with the help of proposition (B.3.8).

So $C_{k'}$ is a curve defined over a field extension k' of k of degree at most d and whose Jacobian $J_{k'}$ is k'-isogenous to the g-fold product of $E_{k'}$ where $E_{k'}$ is an elliptic curve defined over k. Hence, by theorem (C.3.2) for g larger than $\mathscr{C}(E_{k'}, d)$, where $\mathscr{C}(E_{k'}, d)$ is the constant from theorem (C.3.2), the curve becomes geometrically singular resp. reducible.

Now the same line of arguments as in theorem (B.5.1) shows that C_K is not smooth. Briefly, since $C_{k'}$ is reducible (at least after some finite base extension), its Jacobian $J_{k'}$ splits as a principally polarized abelian variety. So $J_{k'}$ owns a symmetric idempotent endomorphism which lifts to a symmetric idempotent endomorphism of $J_{K'}$. So $J_{K'}$ splits implying that $C_{K'}$ is reducible. But this contradicts the smoothness and geometrically connectedness of $C_{K'}$.

Hence, the genus of $C_{K'}$ is bounded by the constant $\mathscr{C}(E_k, d)$. Since E_k depends only on E_K because the choice of the point $\operatorname{Spec}\mathbb{F}_q \to \operatorname{Spec}\mathcal{O}_F$ does, we see that the genus is bounded by some constant $\mathscr{C}(E_K, d)$ depending only on E_K and d. □

If we now combine this theorem with the results of the first two sections about descending Jacobians and curves, then we can derive that the family of curves $C \to Y$ may be defined over the field of complex numbers $\mathbb{C}$ while the family of elliptic curves $E \to Y$ should be still defined over a number field. We show this in the following section.

C.5 Families of Jacobians reaching the Arakelov bound

We come to our main result. Let F be a number field and Y_F a smooth, projective, geometrically connected curve over F. Remember that a family of curves $C \to Y$ should have a smooth, projective, geometrically connected fiber.

Theorem C.5.1 (the genus of a curve with split Jacobian is bounded) *Let $E \to Y_F$ be a non-isotrivial family of elliptic curves and $C \to Y_{\mathbb{C}}$ a family of curves of genus g whose Jacobian is $Y_{\mathbb{C}}$-isogenous to the g-fold product of $E_{Y_{\mathbb{C}}}$. Then the genus g of the fibers of $C \to Y_{\mathbb{C}}$ is bounded, i. e. there is a constant $\mathscr{C} = \mathscr{C}(E_{Y_F})$, depending only on E_{Y_F}, such that g is smaller than $\mathscr{C}$.*

Proof. We may assume that there is a covering $Y'_F \to Y_F$ of degree at most 2 such that $C_{Y'_{\mathbb{C}}} \to Y'_{\mathbb{C}}$ is defined over Y'_F. Using proposition (C.2.5) about uniformly descending curves this can always be achieved after replacing F and Y_F by finite extensions depending only on $E \to Y_F$.

Theorem (C.4.1) then implies that the genus of the general fiber of $C_{Y'_{\mathbb{C}}} \to Y'_{\mathbb{C}}$ and, hence, of $C \to Y_F$ is bounded by a constant $\mathscr{C} = \mathscr{C}(E_K, 2)$ depending only on the general fiber of $E \to Y_F$. In particular, $\mathscr{C}$ depends only on $E \to Y_F$. □

We now come to families of curves and Jacobians reaching the Arakelov bound. We briefly repeat the explanation in [VZ04] what this means. The base field of what follows is $\mathbb{C}$ – the field of complex numbers.

Let $C \to Y$ be a semistable, non-isotrivial family of curves and $J \xrightarrow{f} Y$ its (compactified) family of Jacobians. Let $U \subset Y$ be the smooth locus of $J \to Y$, i. e. the restriction of $J \to Y$ over U is an abelian scheme $J_0 \to U$ while the fibers over the set $S = Y - U$ are all singular. Consider the weight 1 variation of Hodge structures $R^1 f_* \mathbb{Z}_{J_0}$ and let F be the non-flat part of the Higgs bundle (E, θ) given by taking the graded sheaf of the Deligne extension of $R^1 f_* \mathbb{Z}_{J_0} \otimes \mathcal{O}_U$ to Y which carries a Hodge filtration. Then the Arakelov inequality for families of abelian varieties [JZ02] says that

$$2 \cdot \deg(F^{1,0}) \leq g_0 \cdot \big(2q - 2 + \#S\big)$$

where q denotes the genus of the base curve Y and g_0 is the rank of $F^{1,0}$. We say that the family of Jacobians $J \to Y$ *reaches the Arakelov bound* if the above inequality becomes an equality. Viehweg and Zuo showed in [VZ04] that this property is equivalent to the maximality of the Higgs field for F, i. e. the map $\theta|_{F^{1,0}} : F^{1,0} \to F^{0,1} \otimes \Omega^1_Y(\log S)$ is an isomorphism.

Moreover, they show that if $S \neq \emptyset$, then there is an étale covering $Y' \to Y$ such that $J_{Y'} \to Y'$ is Y'-isogenous to a product $B \times_{\mathbb{C}} E \times_{Y'} \ldots \times_{Y'} E$ where $B/\mathbb{C}$ is an abelian variety of dimension $g - g_0$ and $E \to Y'$ is a modular family of elliptic curves. Modular means that the smooth locus U' of $E \to Y'$ is the quotient $\Gamma \backslash \mathbb{H}$ of the upper half-plane $\mathbb{H}$ by a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of finite index and $E \to Y'$ is over U' the quotient of $\mathbb{H} \times \mathbb{C}$ by the semi-direct product of Γ and $\mathbb{Z}^2$.

By theorem (C.5.1) the g-fold product of $E \to Y'$ can not be a Jacobian for g bigger than a constant depending only on $E \to Y'$. We want to make the bound more uniform, i. e. it should depend only on some numerical data of $E \to Y'$. In the right hand side of the Arakelov inequality, there are three numerical invariants involved, namely g_0 which is the dimension of the non-constant part of the Jacobian $J \to Y$, q which is the genus of the base curve Y and $\#S$ the cardinality of the singular locus of $J \to Y$. Since we will consider completely decomposable Jacobians, we have no constant part so that $g_0 = g$, the genus of the fibers of $C \to Y$. So there are two numerical invariants left – the genus of the base curve q and the cardinality of the singular locus S. To get a bound depending only on these two data, we need a finiteness result for modular families of elliptic curves.

Proposition C.5.2 (finiteness of modular families of elliptic curves)
Fix two integers q and s. Then there are only finitely many semistable modular families of elliptic curves $E \to Y$ defined over a base curve Y of genus at most q and smooth outside a set $S \subset Y$ of cardinality at most s.

Proof. Let $E \to Y$ be a modular family as in the proposition and let $Y \xrightarrow{j_E} \mathbb{P}^1_{\mathbb{C}}$ be the j-map corresponding to the family $E \to Y$. Because of the semistability of $E \to Y$, an application of the ABC-conjecture for function fields yields

$$\deg(j_E) \leq 6 \cdot (2q - 2 + s) =: d.$$

So, in particular, the degree of the j-map is absolutely bounded by d. Therefore, the modular family of elliptic curves $E \to Y$ is given by a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of index at most d.

Since $\mathrm{SL}_2(\mathbb{Z})$ is finitely generated, there are only finitely many subgroups Γ in $\mathrm{SL}_2(\mathbb{Z})$ of index at most d. Thus, we have only finitely many semistable modular families of elliptic curves $E \to Y$ over a curve of genus at most q and smooth outside a set of cardinality at most s. $\square$

Example C.5.3 Let $q = 0$ so that $Y = \mathbb{P}^1_{\mathbb{C}}$. Beauville [Be81] showed that for $s \leq 3$ there are no non-isotrivial semistable families of elliptic curves at all. For $s = 4$ Beauville showed in [Be82] that there are six non-isotrivial semistable families of elliptic curves, all modular, corresponding to the congruence subgroups $\Gamma(3)$, $\Gamma_1(4) \cap \Gamma(2)$, $\Gamma_1(5)$, $\Gamma_1(6)$, $\Gamma_0(8) \cap \Gamma_1(4)$ and $\Gamma_0(9) \cap \Gamma_1(3)$.

So, if the family of Jacobians is Y-isogenous to the g-fold product of a modular family of elliptic curves, then we get the following result from theorem (C.5.1).

Corollary C.5.4 (uniform bound for modular families)
Fix two integers q and s. Then there is a constant $\mathscr{C} = \mathscr{C}(q, s)$ such that for any semistable family of curves $C \to Y$, which is defined over a base curve Y of genus at most q and whose family of Jacobians $J \to Y$ is smooth outside a set $S \subset Y$ of cardinality at most s and Y-isogenous to the g-fold product of a modular family of elliptic curves $E \to Y$, the genus g of the fibers of $C \to Y$ is bounded above by $\mathscr{C}$. In particular, $\mathscr{C}$ depends only on q and s.

Proof. By proposition (C.5.2) there are only finitely many semistable modular families of elliptic curves $E \to Y$ over a curve of genus at most q and smooth outside a set of cardinality at most s. Because of the modularity, each one can be defined over some number field [De79].

So theorem (C.5.1) gives for each $E \to Y$ a bound $\mathscr{C}(E_Y)$ such that for g larger than $\mathscr{C}(E_Y)$ the g-fold product of $E \to Y$ is not Y-isogenous to a Jacobian. Thus, taking $\mathscr{C} = \mathscr{C}(q,s)$ to be the maximum of these finitely many numbers $\mathscr{C}(E_Y)$ proves the corollary. □

Consider again the case $Y = \mathbb{P}^1_{\mathbb{C}}$, and let $C \to Y$ be a family of curves whose Jacobian $J \to Y$ reaches the Arakelov bound. Then the Arakelov (in)equality implies that $\#S = 4$ and $J \to Y$ is Y-isogenous to a product of a constant abelian variety with a product of a modular family of elliptic curves $E \to Y$ because the only étale covers of $\mathbb{P}^1_{\mathbb{C}}$ are automorphisms of itself. Such a family of curves exist for $g = 2$ [VZ04, ex.7.1] and, as explained in the introduction, such families conjecturally do not exist for high genus g. We derive this result, assuming that there is no constant part, as a corollary from theorem (C.5.1).

Corollary C.5.5 (curves over $\mathbb{P}^1_{\mathbb{C}}$ and the Arakelov bound)
There is a natural number $\mathscr{C}$ such that for any family of curves $C \to \mathbb{P}^1_{\mathbb{C}}$, whose Jacobian $J \to \mathbb{P}^1_{\mathbb{C}}$ has no constant part and reaches the Arakelov bound, the genus of the fibers of $C \to \mathbb{P}^1_{\mathbb{C}}$ is bounded by $\mathscr{C}$.

Proof. Choose $\mathscr{C}$ to be the constant $\mathscr{C}(0,4)$ from corollary (C.5.4). □

In particular, rational Shimura curves parameterizing a family of high-dimensional abelian varieties reaching the Arakelov bound and without constant part do not intersect the open Schottky locus.

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