Generalized Albanese and its Dual

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Abstract

A generalization of the Albanese variety to the case of a singular projective variety X over an algebraically closed field k is given in [ESV], where H. Esnault, V. Srinivas and E. Viehweg constructed a universal regular quotient of the Chow group $\mathrm{CH}_0(X)_{\deg 0}$ of 0-cycles of degree 0 modulo rational equivalence. This is a smooth connected commutative algebraic group, universal for rational maps from X to smooth commutative algebraic groups which factor through a homomorphism of groups $\mathrm{CH}_0(X)_{\deg 0} \longrightarrow G(k)$. Suppose now that in addition k is of characteristic 0. Interpreting this algebraic group as a generalized 1-motive in the sense of Laumon [L], we may ask for the dual 1-motive. The intention of these notes was to describe the functor which is represented by the dual 1-motive. This forms the main result of this work.

The notion of dual 1-motive allows to treat the problem in a more general way: We consider certain categories of rational maps from a projective variety to commutative algebraic groups (the category of rational maps factoring through $\mathrm{CH}_0(X)_{\deg 0}$ is a special case). A necessary and sufficient condition for the existence of an object of such a category satisfying the universal mapping property is given, as well as a construction of these universal objects via their dual 1-motives. In particular, this provides an independent proof of the existence and an explicit construction of the universal regular quotient for algebraically closed base field of characteristic 0.

Contents

0 Introduction

For a projective variety X over an algebraically closed field k a generalized Albanese variety $\mathrm{Alb}\,(X)$ is constructed by Esnault, Srinivas and Viehweg in [ESV] as a universal regular quotient of the Chow-group $\mathrm{CH}_0(X)_{\deg 0}$ of 0-cycles of degree 0 modulo rational equivalence. It is a smooth connected commutative algebraic group, not in general an abelian variety, if X is singular. Therefore it cannot be dualized in the same way as an abelian variety.

Laumon built up in [L] a duality theory of generalized 1-motives in characteristic 0, which are homomorphisms $[\mathcal{F} \to G]$ from a formal group $\mathcal{F}$ to an algebraic group G; and $\mathrm{Alb}\,(X)$ can be interpreted as a generalized 1-motive by setting $\mathcal{F} = 0$ and $G = \mathrm{Alb}\,(X)$. Our subject is the functor which is represented by the dual 1-motive, in the situation where the base field k is algebraically closed and of characteristic 0.

Section 1 provides some basic facts about generalized 1-motives, which are used in the rest of the paper. If G is an algebraic group which is an extension of an abelian variety A by a linear group L, then the dual 1-motive of $[0 \to G]$ is given by $[L^\vee \to A^\vee]$, where $L^\vee = \mathrm{Hom}_{\mathcal{A}b/k}\,(L, \mathbb{G}_{\mathrm{m}})$ is the Cartier-dual of L and $A^\vee = \mathrm{Pic}^0_A$ is the dual abelian variety.

Section 2 is devoted to the functor of relative divisors: For a fixed variety Y this functor assigns to an affine scheme T a family of divisors on Y, parametrized by T. The functor of relative Cartier divisors $\underline{\mathrm{Div}}_Y$ admits a natural transformation cl to the Picard functor $\underline{\mathrm{Pic}}_Y$, which maps a relative divisor to its class. Then

$$\underline{\mathrm{Div}}^0_Y := \mathrm{cl}^{-1}\,\underline{\mathrm{Pic}}^0_Y$$

is the functor of families of Cartier divisors whose associated line bundles are algebraically equivalent to the trivial bundle. If $\pi : Y \longrightarrow X$ is a finite morphism of varieties, there is a natural transformation, the push-forward π_*, between the functors of relative Weil divisors $\underline{\mathrm{WDiv}}_Y$ and $\underline{\mathrm{WDiv}}_X$. Thus we can define the functor

$$\underline{\mathrm{WDiv}}_{Y/X} := \ker\left(\pi_* : \underline{\mathrm{WDiv}}_Y \longrightarrow \underline{\mathrm{WDiv}}_X\right)$$

of Weil divisors on Y vanishing relative to X. Furthermore we introduce a functor $\underline{\mathrm{IDiv}}_Y$ of formal infinitesimal divisors, which generalizes infinitesimal deformations of the zero divisor. For Cartier divisors there exist non-trivial infinitesimal deformations, while this is not true for Weil divisors, since prime Weil divisors are always reduced. $\underline{\mathrm{IDiv}}_Y$ also admits a push-forward π_* for finite morphisms of degree 1, and we set

$$\underline{\mathrm{IDiv}}_{Y/X} := \ker\left(\pi_* : \underline{\mathrm{IDiv}}_Y \longrightarrow \underline{\mathrm{IDiv}}_X\right)$$

To a covariant functor $F : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ from the category of finitely generated k-algebras to the category of abelian groups we associate its *reduced functor* $\mathrm{Red}(F) : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ by setting

$$\mathrm{Red}(F)(R) := F(R_{\mathrm{red}})$$

where $R_{\mathrm{red}} = R/\mathrm{Nil}(R)$ is the reduced algebra of R, and we associate its *infinitesimal functor* $\mathrm{Inf}(F) : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ by setting

$$\mathrm{Inf}(F)(R) := \ker\big(F(R_{\mathrm{art}}) \longrightarrow F(k)\big)$$

where $R_{\mathrm{art}} = k + \mathrm{Nil}(R)$ is the local Artinian k-algebra of R. For example a formal group, considered as a functor on $\mathbf{Alg}/k$, is the direct product of its reduced and its infinitesimal functor.

We obtain a transformation of functors $\mathrm{weil} : \mathrm{Red}(\underline{\mathrm{Div}}_Y) \longrightarrow \underline{\mathrm{WDiv}}_Y$, which assigns to a relative Cartier divisor its associated Weil divisor (the restriction of weil to $\mathrm{Red}(\underline{\mathrm{Div}}_Y^0)$ is denoted by weil^0), and a natural transformation $\mathrm{fml} : \mathrm{Inf}(\underline{\mathrm{Div}}_Y) \longrightarrow \mathrm{IDiv}_Y$.

Section 3 states the universal factorization problem with respect to a category $\mathbf{Mr}$ of rational maps from a normal projective variety Y to connected commutative algebraic groups:

Definition 0.1 *A rational map* $(u : Y \longrightarrow \mathcal{U}) \in \mathbf{Mr}$ *is called* universal *for* $\mathbf{Mr}$ *if for all objects* $(\varphi : Y \longrightarrow G) \in \mathbf{Mr}$ *there is a unique homomorphism of algebraic groups* $h : \mathcal{U} \longrightarrow G$ *and a constant* $g \in G(k)$ *such that* $\varphi = h \circ u + g$.

We give a necessary and sufficient condition for the existence of a universal object for a category of rational maps $\mathbf{Mr}$ which contains the category $\mathbf{Mr}_0$ of morphisms from Y to abelian varieties and satisfies a certain stability condition $(\diamondsuit)$. We observe that a rational map $\varphi : Y \longrightarrow G$, where G is an extension of an abelian variety by a linear group L, induces a natural tranformation $L^\vee \longrightarrow \underline{\mathrm{Div}}_Y^0$. If $\mathcal{F}$ is a subfunctor of $\underline{\mathrm{Div}}_Y^0$ which is a formal group, denote by $\mathbf{Mr}_{\mathcal{F}}$ the category of rational maps for which the image of this induced transformation lies in $\mathcal{F}$. Then it holds

Theorem 0.2 *For a category* $\mathbf{Mr}$ *containing* $\mathbf{Mr}_0$ *and satisfying* $(\diamondsuit)$*, there exists a universal object* $\mathrm{Alb}_{\mathbf{Mr}}(Y)$ *if and only if there is a formal group* $\mathcal{F} \subset \underline{\mathrm{Div}}_Y^0$ *such that* $\mathbf{Mr}$ *is equivalent to* $\mathbf{Mr}_{\mathcal{F}}$.

The universal object $\mathrm{Alb}_{\mathcal{F}}(Y)$ of $\mathbf{Mr}_{\mathcal{F}}$ is an extension of the classical Albanese $\mathrm{Alb}(Y)$, which is the universal object of $\mathbf{Mr}_0$, by the linear group $\mathcal{F}^\vee$, the Cartier-dual of $\mathcal{F}$. The dual 1-motive of $[0 \longrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)]$ is hence given by $\big[\mathcal{F} \longrightarrow \mathrm{Pic}_Y^0\big]$, the homomorphism induced by the natural transformation $\mathrm{cl} : \underline{\mathrm{Div}}_Y^0 \longrightarrow \underline{\mathrm{Pic}}_Y^0$.

The universal regular quotient $\mathrm{Alb}\,(X)$ of a (singular) projective variety X is by definition the universal object for the category $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ of rational maps factoring through rational equivalence. More precisely, the objects of $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ are rational maps $\varphi : X \longrightarrow G$ whose associated map on pairs of k-rational points (here Z ranges over the irreducible components of X)

$$\begin{array}{rcl} \varphi^{\Pi}(k) : \bigcup_{Z\in \mathrm{Cp}(X)} Z(k)\times Z(k) & \longrightarrow & G(k) \\ (p,q) & \longmapsto & \varphi(p)-\varphi(q) \end{array}$$

factors through a homomorphism of groups $\mathrm{CH}_0(X)_{\deg 0} \longrightarrow G(k)$. Such a rational map is regular on the regular locus of X and may also be considered as a rational map from $\widetilde{X}$ to G, where $\widetilde{X} \longrightarrow X$ is the normalization of X. In particular, if X is nonsingular the universal regular quotient coincides with the classical Albanese. $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ contains $\mathbf{Mr}_0$ and satisfies $(\diamondsuit)$; therefore we have reduced the problem to find the subfuntor of $\underline{\mathrm{Div}}^0_{\widetilde{X}}$ which is represented by a formal group $\mathcal{F}$ with $\mathbf{Mr}_{\mathcal{F}}$ equivalent to $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$.

Section 4 answers the question for the formal group $\mathcal{F}$ which characterizes the category $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$. As this subfunctor of $\underline{\mathrm{Div}}^0_{\widetilde{X}}$ measures the difference between $\widetilde{X}$ and X, a natural candidate is given by the direct product $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ of a reduced and an infinitesimal functor, which are given respectively by

$$\begin{array}{rcl} \mathrm{Red}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right) & := & \left(\mathrm{weil}^0\right)^{-1}\underline{\mathrm{WDiv}}_{\widetilde{X}/X} \\ \mathrm{Inf}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right) & := & \left(\mathrm{fml}^0\right)^{-1}\underline{\mathrm{IDiv}}_{\widetilde{X}/X} \end{array}$$

The verification of the equivalence of $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ and $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}$ is done using local symbols.

This gives an independent proof (alternative to the ones in [ESV]) as well as an explicit construction of the universal regular quotient for algebraically closed base field of characteristic 0. Furthermore from the construction follows the functoriality of the universal regular quotient: Given a morphism $\sigma : V \longrightarrow X$ of projective varieties with the property that no irreducible component of $\sigma(V)$ is contained in the singular locus X_{sing} of X, and $\sigma(V)$ is a local complete intersection in a neighbourhood of $\sigma(V)\cap X_{\mathrm{sing}}$. Then σ induces a homomorphism $\mathrm{Alb}\,(\sigma) : \mathrm{Alb}\,(V) \longrightarrow \mathrm{Alb}\,(X)$ between the universal regular quotients of V and X respectively.

In the last two sections we look at the same setting from a different point of view, which allows to compute certain examples and provides a link to the results of [ESV]. Sections 5 and 6 are not necessary for the main result.

Section 5 treats the case that X is a curve C. This is an example where an explicit computation of the universal regular quotient and its dual is within easy reach: On a curve a 0-cycle is a divisor and thus $\mathrm{CH}_0(C)_{\deg 0}$ is canonically isomorphic to $\mathrm{Pic}^0 C$ and the universal regular quotient $\mathrm{Alb}\,(C)$ is just $\mathrm{CH}_0(C)_{\deg 0}$. Therefore we dualize $\mathrm{Pic}^0 C$ in the sense of 1-motives and show that the result coincides with the description given in the previous section: We inspect the extension

$$0 \longrightarrow L \longrightarrow \mathrm{Pic}^0 C \longrightarrow \mathrm{Pic}^0 \widetilde{C} \longrightarrow 0$$

where $\widetilde{C}$ is the normalization of C, $L \cong (\mathbb{G}_{\mathrm{m}})^t \times (\mathbb{G}_{\mathrm{a}})^v$, and ask what gives rise to the $\mathbb{G}_{\mathrm{m}}$- and $\mathbb{G}_{\mathrm{a}}$-parts. We introduce a curve C' lying between C and $\widetilde{C}$, which is homeomorphic to C but has only ordinary multiple points as singularities. This allows to treat the $\mathbb{G}_{\mathrm{m}}$- and $\mathbb{G}_{\mathrm{a}}$-parts separately:
We obtain extensions

$$0 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Pic}^0 C' \longrightarrow \mathrm{Pic}^0 \widetilde{C} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{V} \longrightarrow \mathrm{Pic}^0 C \longrightarrow \mathrm{Pic}^0 C' \longrightarrow 0$$

where $\mathbb{T} \cong (\mathbb{G}_{\mathrm{m}})^t$ is a torus and $\mathbb{V} \cong (\mathbb{G}_{\mathrm{a}})^v$ is a vectorial group. Then $\mathbb{T}^\vee \cong \mathbb{Z}^t$ is the étale part and $\mathbb{V}^\vee \cong \left(\widehat{\mathbb{G}_{\mathrm{a}}}\right)^v$ the infinitesimal part of the formal group $L^\vee$, the Cartier-dual of the linear group L. We show that $L^\vee$ represents $\underline{\mathrm{Div}}^0_{\widetilde{C}/C}$.

Section 6 generalizes this result to the case that the projective variety X is of higher dimension. The universal regular quotient $\mathrm{Alb}\,(X)$ is characterized by the property of being the largest algebraic group which is generated by a rational map from X factoring through $\mathrm{CH}_0(X)_{\deg 0}$. This is equivalent to the condition that $\mathrm{Alb}\,(X)$ is the largest algebraic group which is a quotient of $\mathrm{Alb}\,(C)$ for general Cartier curves C in X. Then by duality the largest linear subgroup L of $\mathrm{Alb}\,(X)$ is the largest formal group in $\underline{\mathrm{Div}}^0_{\widetilde{X}}$ which is a subgroup of $\underline{\mathrm{Div}}^0_{\widetilde{C}/C}$ for general Cartier curves C in X; and this is $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$. More generally, if $\varphi : Y \longrightarrow G$ is a rational map to an algebraic group, then for φ to have a certain property it is necessary and sufficient that the resriction $\varphi|_V$ to a subvariety has this property for all elements V of a family of subvarieties of Y satisfying certain conditions. This is shown for the properties of being an object of the category $\mathbf{Mr}_{\mathcal{F}}$, where $\mathcal{F} \subset \underline{\mathrm{Div}}^0_Y$ is a formal group, and of generating an algebraic group G.

At the end we discuss an example which illustrates some pathological properties of the universal regular quotient: While the classical Albanese of smooth projective varieties is compatible with products, for the universal regular quotient of singular projective varieties X_i it is possible that $\dim \mathrm{Alb}\,(\prod X_i) > \sum \dim \mathrm{Alb}\,(X_i)$. Moreover, if X is a smooth projective variety of dimension d and $\mathcal{L}$ a very ample line bundle on X, then for a complete intersection C of $d-1$ general divisors in the linear system $|\mathcal{L}|$ the Gysin map $\mathrm{Alb}\,(C) \longrightarrow \mathrm{Alb}\,(X)$ will be surjective. This is not true in general if X is singular (but a sufficiently high power of $\mathcal{L}$ will again have this property). We look how these phenomena fit into our picture.

The main result is summarized in the following

Theorem 0.3 *Let X be a projective variety over an algebraically closed field k of characteristic 0, and $\widetilde{X} \longrightarrow X$ its normalization. Then the universal regular quotient* $\mathrm{Alb}\,(X)$ *exists and its dual (in the sense of 1-motives) represents the functor*

$$\underline{\mathrm{Div}}^0_{\widetilde{X}/X} \longrightarrow \underline{\mathrm{Pic}}^0_{\widetilde{X}}$$

i.e. the natural transformation of functors which assigns to a relative divisor the class of its associated line bundle. $\underline{\mathrm{Pic}}^0_{\widetilde{X}}$ *is represented by an abelian variety and* $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ *by a formal group.*

The universal regular quotient for semi-abelian varieties, i.e. the universal object for rational maps to semi-abelian varieties factoring through rational equivalence (which is a quotient of our universal regular quotient), is a classical 1-motive in the sense of Deligne [D2] Définition (10.1.2). The question for the dual 1-motive of this object was already answered by Barbieri-Viale and Srinivas in [BS].

The first two sections being purely technical, it is possible (and recommended) to start reading at Section 3 and pick up definitions and facts from the preceding sections as necessary.

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0.1 Terminology

k is a fixed algebraically closed field of characteristic 0. Schemes are always locally Noetherian. A variety is a reduced scheme of finite type over k, not necessarily irreducibel. A curve is a variety of dimension 1. Algebraic groups and formal groups are always commutative and over k. We write $\mathbb{G}_{\mathrm{a}}$ for $\mathbb{G}_{\mathrm{a},k} = \operatorname{Spec} k[t]$ and $\mathbb{G}_{\mathrm{m}}$ for $\mathbb{G}_{\mathrm{m},k} = \operatorname{Spec} k\left[t, t^{-1}\right]$. The letter $\mathbb{L}$ stands for a linear algebraic group which is either $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$.

If Y is a scheme, then $y \in Y$ means that y is a point in the Zariski topological space of Y. The set of irreducible components of Y is denoted by $\mathrm{Cp}(Y)$, the set of connected components by $\mathrm{CCp}(Y)$.

If A is a ring, then K_A denotes the total quotient ring of A. If Y is a scheme, then $\mathcal{K}_Y$ denotes the sheaf of total quotient rings of $\mathcal{O}_Y$. The group of units of a ring R is denoted by R^*.

If $\sigma : Y \longrightarrow X$ is a morphism of schemes, then $\sigma^{\#} : \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ denotes the associated homomorphism of structure sheaves. If $h : A \longrightarrow B$ is a homomorphism of rings, then $h^{\dagger} : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ denotes the associated morphism of affine schemes.

We think of $\operatorname{Ext}^1(A, B)$ as the space of extensions of A by B and therefore denote it by $\operatorname{Ext}(A, B)$. When speaking of a divisor D as an element of $\operatorname{Pic}^0 X$, the class $[D] \in \operatorname{Pic}^0 X$ is meant.

The dual of an object O in its respective category is denoted by $O^{\vee}$, whereas $\widehat{O}$ is the completion of O. For example, if V is a k-vector space, then $V^{\vee} = \operatorname{Hom}_k(V, k)$ is the dual k-vector space; if G is a linear algebraic group or a formal group, then $G^{\vee} = \operatorname{Hom}_{\mathcal{A}b/k}(G, \mathbb{G}_{\mathrm{m}})$ is the Cartier-dual; if A is an abelian variety, then $A^{\vee} = \operatorname{Pic}^0 A$ is the dual abelian variety, whereas $\widehat{A} = \operatorname{Spf} \widehat{\mathcal{O}}_{A,0}$ is the completion of A w.r.t. 0_A.

1 1-Motives

The aim of this section is to summarize some foundational material about generalized 1-motives (following [L] Sections 4 and 5), as far as it is necessary for the purpose of these notes.
Throughout the whole work the base field k is algebraically closed and of characteristic 0.

1.1 Algebraic Groups and Formal Groups

At the beginning let me recall some basic facts about algebraic groups and the notion of a formal group.

1.1.1 Algebraic Groups

Definition 1.1 *An* algebraic group *(or* group-scheme*) is a commutative group-object in the category of separated schemes of finite type over k.*

Proposition 1.2 *If* $\mathrm{char}\,(k) = 0$*, an algebraic group is always smooth and equi-dimensional.*

Proof. [M] Chapter III, No. 11, p. 101. ■

Theorem 1.3 (Chevalley) *A smooth connected algebraic group G admits a* canonical decomposition

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0$$

where L is a linear algebraic group and A is an abelian variety.

Proof. [SGA3] Exposé $\mathrm{VI_A}$ and [S] Chapter III, No. 7, Proposition 11. ■

Theorem 1.4 *A linear algebraic group L splits canonically into a direct product of a torus $\mathbb{T}$ and a unipotent group $\mathbb{U}$:*

$$L = \mathbb{T} \times \mathbb{U}$$

If $\mathrm{char}\,(k) = 0$ *a unipotent group is vectorial, i.e.*

$$\mathbb{U} = \mathbb{V} := \mathrm{Spec}\,(\mathrm{Sym}\,V^\vee)$$

where V is a finite dimensional k-vector space and $V^\vee$ the dual k-vector space. A torus is a direct product of multiplicative groups and a vectorial group the *direct product of additive groups, i.e. there are natural numbers $t, v \geq 0$ such that*

$$\begin{aligned} \mathbb{T} &\cong (\mathbb{G}_{\mathrm{m}})^t \\ \mathbb{V} &\cong (\mathbb{G}_{\mathrm{a}})^v \end{aligned}$$

Proof. [SGA3] Exposé XVII, 7.2.1 and [S] Chapter III, No. 7, Proposition 12. ■

In order to classify extensions $0 \to L \to G \to A \to 0$ of an algebraic group A by a linear group L we can consider each extension G as a principal fibre bundle over the base A of fibre-type L. A principal L-fibre bundle over A is determined uniquely by a set of local sections $\{s_\alpha : U_\alpha \longrightarrow G\}$ where $\{U_\alpha\}$ is an open cover of A, since these sections give rise to local trivializations $\{\Phi_\alpha : U_\alpha \times L \longrightarrow G\}$ and corresponding transition functions $\{\Psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow L\}$. This gives a homomorphism into the space of isomorphism classes of such transition functions:

$$\mathrm{Ext}\,(A, L) \longrightarrow \mathrm{H}^1\,(A, \mathcal{L}_A)$$

where $\mathcal{L}_A$ is the sheaf of germs of regular maps from A to L. If A is an abelian variety, this map is injective (see [S] Chapter VII, No. 15, Theorem 5). A principal L-bundle G coming from an extension of algebraic groups is always translation-invariant, i.e. $G = t_a^* G$, where $t_a : x \longmapsto x + a$ is the translation by $a \in A$. This implies that one local section $s : U \longrightarrow G$ yields via translation a family of local sections $\{s_a : t_a U \longrightarrow G\}_{a \in A}$. Therefore one can recover the structure of an extension G by finding one local section.

Since a linear group L in characteristic 0 is the direct product of multiplicative and additive groups, i.e. $L \cong (\mathbb{G}_{\mathrm{m}})^t \times (\mathbb{G}_{\mathrm{a}})^v$, and therefore

$$\begin{aligned} \mathrm{Ext}\,(A, L) &\cong \mathrm{Ext}\left(A, (\mathbb{G}_{\mathrm{m}})^t \times (\mathbb{G}_{\mathrm{a}})^v\right) \\ &\cong \mathrm{Ext}\,(A, \mathbb{G}_{\mathrm{m}})^t \times \mathrm{Ext}\,(A, \mathbb{G}_{\mathrm{a}})^v \end{aligned}$$

one can find a direct sum of $\mathbb{G}_{\mathrm{m}}$- and $\mathbb{G}_{\mathrm{a}}$-bundles over A, which describe G.

1.1.2 Formal Groups

Definition 1.5 *A* formal group $\mathcal{G}$ *is a commutative group-object in the category of formal affine schemes over k.*

Proposition 1.6 *If* $\mathrm{char}\,(k) = 0$*, a formal group $\mathcal{G}$ is always formal smooth and equi-dimensional, i.e. there is a natural number $d \geq 0$ such that*

$$\mathcal{O}_{\mathcal{G},0} \cong k\,[[x_1, \ldots, x_d]]$$

Proof. [L] Section 4, (4.2). ■

Theorem 1.7 *A formal group $\mathcal{G}$ admits a canonical decomposition*

$$\mathcal{G} \cong \mathcal{G}_{\text{ét}} \times \mathcal{G}_{\text{inf}}$$

where $\mathcal{G}_{\text{ét}}$ is étale over k and $\mathcal{G}_{\text{inf}}$ is the component of the identity.

Proof. [L] (4.2.1) and [Fo] Chapter I, 6.6 and §7. ■

Definition 1.8 *A* formal group *$\mathcal{G}$ is called* étale *if its infinitesimal part $\mathcal{G}_{\text{inf}}$ is trivial, i.e. $\mathcal{G} = \mathcal{G}_{\text{ét}}$.*

A formal group *$\mathcal{G}$ is called* infinitesimal *if its étale part $\mathcal{G}_{\text{ét}}$ is trivial, i.e. $\mathcal{G} = \mathcal{G}_{\text{inf}}$.*

Theorem 1.9 *An étale formal group $\mathcal{G}_{\text{ét}}$ admits a canonical decomposition*

$$0 \longrightarrow \mathcal{G}_{\text{ét}}^{\text{tor}} \longrightarrow \mathcal{G}_{\text{ét}} \longrightarrow \mathcal{G}_{\text{ét}}^{\text{lib}} \longrightarrow 0$$

where $\mathcal{G}_{\text{ét}}^{\text{tor}}$ is the largest sub-group-scheme whose underlying k-scheme is finite and étale, and $\mathcal{G}_{\text{ét}}^{\text{lib}}(k)$ is a free abelian group of finite rank.

Proof. [L] (4.2.1) and [Fo] Chapter I, 6.6 and §7. ■

Theorem 1.10 *If* $\operatorname{char}(k) = 0$*, the Lie-functor gives an equivalence between the following categories:*

$$\{\textit{infinitesimal formal groups}/k\} \longleftrightarrow \{\textit{Lie-algebras}/k\}$$

Proof. [SGA3] VII_{B}, 3.3.2. ■

Corollary 1.11 *If* $\operatorname{char}(k) = 0$*, for an infinitesimal formal group $\mathcal{G}_{\text{inf}}$ there is a finite dimensional k-vector space V, namely $V = \operatorname{Lie}(\mathcal{G}_{\text{inf}})$, such that* $\mathcal{G}_{\text{inf}} \cong \operatorname{Spf}\left(\widehat{\operatorname{Sym} V^\vee}\right)$.

1.1.3 Sheaves of Abelian Groups

The category of algebraic groups and the category of formal groups can be considered as full subcategories of the category of sheaves of abelian groups:

Definition 1.12 *Let*

Ab	*category of abelian groups*
$\mathbf{Alg}/k$	*category of finitely generated* k*-algebras*
$\mathbf{Aff}/k$	*category of affine* k*-schemes*
$\mathbf{Sch}/k$	*category of* k*-schemes*
$\mathbf{FSch}/k$	*category of affine formal* k*-schemes*
$\mathcal{S}et/k$	*category of sheaves of sets over* $\mathbf{Aff}/k$
$\mathcal{A}b/k$	*category of sheaves of abelian groups over* $\mathbf{Aff}/k$
$\mathcal{G}a/k$	*category of algebraic groups over* k
$\mathcal{G}f/k$	*category of formal groups over* k

$\mathbf{Aff}/k$ is anti-equivalent to $\mathbf{Alg}/k$. Let $\mathbf{Aff}/k$ and $\mathbf{Alg}/k$ be equipped with the topology fppf. Interpreting a k-scheme X as a sheaf over $\mathbf{Aff}/k$ given by

$$S \longmapsto X(S) = \mathrm{Mor}_k(S, X)$$

or equivalently

$$R \longmapsto X(R) = \mathrm{Mor}_k(\mathrm{Spec}\, R, X)$$

makes $\mathbf{Sch}/k$ to a full subcategory of $\mathcal{S}et/k$ and $\mathcal{G}a/k$ to a full subcategory of $\mathcal{A}b/k$.

In the same manner $\mathbf{FSch}/k$ becomes a full subcategory of $\mathcal{S}et/k$ and $\mathcal{G}f/k$ a full subcategory of $\mathcal{A}b/k$: An affine formal k-scheme $\mathcal{Y} = \mathrm{Spf}\,\mathcal{A}$, where $\mathcal{A}$ is a complete topological k-algebra, is viewed as the sheaf over $\mathbf{Aff}/k$ given by

$$R \longmapsto \mathcal{Y}(R) = \underline{\mathrm{Spf}\,\mathcal{A}}(R) = \mathrm{Hom}_{k\text{-Alg,cont}}(\mathcal{A}, R)$$

which assigns to a finitely generated k-algebra R with discrete topology the set of continuous homomorphisms of k-algebras from $\mathcal{A}$ to R.

Then the kernel and cokernel of a homomorphism in $\mathcal{G}a/k$ or $\mathcal{G}f/k$ coincide with the ones in $\mathcal{A}b/k$, and an exact sequence $0 \to K \to G \to C \to 0$ in $\mathcal{A}b/k$, where K and C are objects of $\mathcal{G}a/k$ or $\mathcal{G}f/k$, implies that G is also an object of $\mathcal{G}a/k$ or $\mathcal{G}f/k$ respectively.

1.2 Structure of a 1-Motive

In the following by a 1-motive always a *generalized 1-motive* in the sense of Laumon [L] Définition (5.1.1) is meant:

Definition 1.13 *A* 1-motive *is a complex concentrated in degrees* -1 *and* 0 *in the category of sheaves of abelian groups of the form* $M = [\mathcal{F} \to G]$*, where* $\mathcal{F}$ *is a torsion-free formal group over* k *and* G *a connected algebraic group over* k*.*

Taking into account that each connected algebraic group G admits a canonical decomposition, i.e. is an extension of an abelian variety A by a linear group L, one can assign to each 1-motive $[\mathcal{F} \to G]$ a diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{F} & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

1.3 Cartier-Dual

Let G be an algebraic or a formal group and let $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}\left(G, \mathbb{G}_{\mathrm{m}}\right)$ be the sheaf of abelian groups over $\mathbf{Aff}/k$ associated to the functor

$$S \longmapsto \mathrm{Hom}_{\mathrm{gr}}\left(G\left(S\right), \mathbb{G}_{\mathrm{m}}\left(S\right)\right)$$

which assigns to an affine k-scheme S the set of group homomorphisms $\mathrm{Hom}_{\mathrm{gr}}\left(G\left(S\right), \mathbb{G}_{\mathrm{m}}\left(S\right)\right)$. If G is a linear algebraic group (formal group), this functor is represented by a formal group (linear algebraic group):

Definition 1.14 *If L is a linear algebraic group, the formal group which represents the sheaf $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}\left(L, \mathbb{G}_{\mathrm{m}}\right)$ is called the* Cartier-dual *of L and is denoted by $L^{\vee}$.*

If $\mathcal{F}$ is a formal group, the linear algebraic group which represents the sheaf $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}\left(\mathcal{F}, \mathbb{G}_{\mathrm{m}}\right)$ is called the Cartier-dual *of $\mathcal{F}$ and is denoted by $\mathcal{F}^{\vee}$.*

Let L be a linear algebraic group with affine algebra A, and let $\mathcal{H}$ be the dual k-vector space of A. A topology on $\mathcal{H}$ is obtained from the requirement that A is identified with the set of continuous k-linear maps from $\mathcal{H}$ to k: $A = \mathrm{Hom}_{k,\mathrm{cont}}\left(\mathcal{H}, k\right)$. In the following $\mathcal{H}\widehat{\otimes}\mathcal{H}$ denotes the completion of $\mathcal{H}\otimes\mathcal{H}$ w.r.t. the topology on $\mathcal{H}$ (see [SGA3] $\mathrm{VII_B}$, 0.3).

The group-structure $m_L : L \times L \longrightarrow L$ is given by a cogroup-structure $\Delta_A : A \longrightarrow A \otimes A$. The dualization of k-vector spaces translates Δ_A into an algebra-structure $m_{\mathcal{H}} : \mathcal{H}\widehat{\otimes}\mathcal{H} \longrightarrow \mathcal{H}$ and the algebra-structure $m_A : A \otimes A \longrightarrow A$ into a coalgebra-structure $\Delta_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H}\widehat{\otimes}\mathcal{H}$, which gives rise to a group-structure $m_{\mathcal{F}} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$ on the formal scheme $\mathcal{F} = \mathrm{Spf}\,\mathcal{H}$.

Let $\pi_L : L \longrightarrow \mathrm{Spec}\,k$ be the structural morphism of L and $\sigma_L : \mathrm{Spec}\,k \longrightarrow L$ the section of π_L which gives the neutral element of L. Then σ_L corresponds to the augmentaion $\varepsilon_A : A \longrightarrow k$, and π_L to a section $\eta_A : k \longrightarrow A$ of ε_A. Dualization of k-vector spaces translates η_A into an augmentation $\varepsilon_{\mathcal{H}} : \mathcal{H} \longrightarrow k$ and ε_A into a section $\eta_{\mathcal{H}} : k \longrightarrow \mathcal{H}$ of $\varepsilon_{\mathcal{H}}$. Then $\eta_{\mathcal{H}}$ corresponds to the structural morphism $\pi_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathrm{Spec}\,k$ and $\varepsilon_{\mathcal{H}}$ to a section $\sigma_{\mathcal{F}} : \mathrm{Spec}\,k \longrightarrow \mathcal{F}$, which yields the neutral element of $\mathcal{F}$.

Moreover the inversion morphism $i_L : L \longrightarrow L$ on L corresponds to the antipode map $S_A : A \longrightarrow A$, which dualizes to an antipode $S_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H}$ giving the inversion $i_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F}$.

In this way $\mathcal{F} = \operatorname{Spf} \mathcal{H}$ becomes a formal group, and it holds $\mathcal{F} = L^\vee$ and $L = \mathcal{F}^\vee$.

The Cartier-duality is an anti-equivalence between the category of linear algebraic groups and the category of formal goups. The functors $L \longmapsto L^\vee$ and $\mathcal{F} \longmapsto \mathcal{F}^\vee$ are quasi-inverse to each other. (see [SGA3] $\mathrm{VII_B}$, 2.2.2)

Since every linear algebraic group L is the direct product of a torus and a unipotent group, $L = \mathbb{T} \times \mathbb{V}$, the Cartier-dual $L^\vee$ is the direct product of their Cartier-duals, $L^\vee = \mathbb{T}^\vee \times \mathbb{V}^\vee$, which leads to treat the two cases separately.

1.3.1 Cartier-Dual of a Torus

Theorem 1.15 *The Cartier-dual of a torus $\mathbb{T} \cong (\mathbb{G}_{\mathrm{m}})^t$ is a lattice of the same rank:*

$$\mathbb{T}^\vee \cong \mathbb{Z}^t$$

Conversely, the Cartier-dual of a lattice $\Lambda \cong \mathbb{Z}^t$ is a torus of the same rank:

$$\Lambda^\vee \cong (\mathbb{G}_{\mathrm{m}})^t$$

Proof. Every morphism of k-schemes from $\mathbb{G}_{\mathrm{m}}$ to $\mathbb{G}_{\mathrm{m}}$ preserving the multiplicative group-structure is of the form $x \longmapsto x^\lambda$, $\lambda \in \mathbb{Z}$. Therefore

$$\operatorname{Hom}_{\mathcal{A}b/k}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}$$

and $\operatorname{Hom}_{\mathcal{A}b/k}(\mathbb{T}, \mathbb{G}_{\mathrm{m}}) \cong \operatorname{Hom}_{\mathcal{A}b/k}\left((\mathbb{G}_{\mathrm{m}})^t, \mathbb{G}_{\mathrm{m}}\right) \cong \operatorname{Hom}_{\mathcal{A}b/k}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})^t \cong \mathbb{Z}^t$.

Conversely, a homomorphism $\chi \in \operatorname{Hom}_{\mathcal{A}b/k}(\mathbb{Z}, \mathbb{G}_{\mathrm{m}})$ is determined uniquely by $\chi(1) \in \mathbb{G}_{\mathrm{m}}$, since $\mathbb{Z} = \langle 1 \rangle$ is a cyclic group. Hence

$$\operatorname{Hom}_{\mathcal{A}b/k}(\mathbb{Z}, \mathbb{G}_{\mathrm{m}}) \cong \mathbb{G}_{\mathrm{m}}$$

and $\operatorname{Hom}_{\mathcal{A}b/k}(\Lambda, \mathbb{G}_{\mathrm{m}}) \cong \operatorname{Hom}_{\mathcal{A}b/k}(\mathbb{Z}^t, \mathbb{G}_{\mathrm{m}}) \cong \operatorname{Hom}_{\mathcal{A}b/k}(\mathbb{Z}, \mathbb{G}_{\mathrm{m}})^t \cong (\mathbb{G}_{\mathrm{m}})^t$. ■

Proposition 1.16 *A lattice $\Lambda \cong \mathbb{Z}^t$ is an étale formal group with affine algebra $\prod_{\lambda \in \Lambda} k_\lambda$, where $k_\lambda = k$. The topology on $\prod_{\lambda \in \Lambda} k_\lambda$ is induced by the decreasing sequence of ideals $I_0 \supset I_1 \supset I_2 \supset \ldots$ with*

$$I_\nu = \left\{ (c_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} k_\lambda \;\middle|\; c_\lambda = 0 \textit{ for } |\lambda| \leq \nu \right\}$$

where $|\lambda| := \sum_{i=1}^{t} |l_i|$ *for* $\lambda = (l_1, \ldots, l_t) \in \mathbb{Z}^t = \Lambda$. *The* k*-algebra structure on* $\prod_{\lambda\in\Lambda} k_\lambda$ *is given by componentwise multiplication and*

$$\begin{array}{rcl} k & \longrightarrow & \prod\limits_{\lambda\in\Lambda} k_\lambda \\ c & \longmapsto & (c)_{\lambda\in\Lambda} \end{array}$$

The group structure

$$\begin{array}{rcl} \Lambda \times \Lambda & \stackrel{+}{\longrightarrow} & \Lambda \\ (\mu,\nu) & \longmapsto & \mu+\nu \end{array}$$

corresponds to the cogroup structure

$$\begin{array}{rcl} \prod\limits_{\lambda\in\Lambda} k_\lambda & \stackrel{+^{\#}}{\longrightarrow} & \left(\prod\limits_{\lambda\in\Lambda} k_\lambda\right) \widehat{\otimes} \left(\prod\limits_{\lambda\in\Lambda} k_\lambda\right) \\ \sum\limits_{\lambda} c_\lambda e_\lambda & \longmapsto & \sum\limits_{\mu,\nu} c_{\mu+\nu}\, e_\mu \,\widehat{\otimes}\, e_\nu \end{array}$$

where $e_\mu = (\delta_{\mu\nu})_{\nu\in\Lambda}, \quad \delta_{\mu\nu} = \begin{cases} 1 & \textit{for } \nu = \mu \\ 0 & \textit{for } \nu \neq \mu \end{cases}$

Proof. Set $\mathcal{F} = \mathrm{Spf}\left(\prod_{\lambda\in\Lambda} k_\lambda\right)$. I claim that the k-valued points of $\mathcal{F}$ can be identified with Λ, and the group structure on $\mathcal{F}$ coincides with the one of Λ. Indeed, according to the k-algebra structure of $\prod_{\lambda\in\Lambda} k_\lambda$ it holds

$$\begin{array}{rcl} \mathcal{F}(k) & = & \mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod\limits_{\lambda\in\Lambda} k_\lambda, k\right) \\ & = & \{\mathrm{pr}_\lambda \mid \lambda \in \Lambda\} \\ & \cong & \Lambda \end{array}$$

where $\mathrm{pr}_\lambda : \prod_{\lambda\in\Lambda} k_\lambda \longrightarrow k, \quad (c_\lambda)_{\lambda\in\Lambda} \longmapsto c_\lambda$ is the projection to the λ^{th}-component. The group structure on $\mathcal{F}(k)$ induced by $+^{\#}$

$$\begin{array}{rcl} \mathrm{Hom}_{k\text{-Alg}}\left(\prod\limits_{\lambda\in\Lambda} k_\lambda, k\right) \times \mathrm{Hom}_{k\text{-Alg}}\left(\prod\limits_{\lambda\in\Lambda} k_\lambda, k\right) & \longrightarrow & \mathrm{Hom}_{k\text{-Alg}}\left(\prod\limits_{\lambda\in\Lambda} k_\lambda, k\right) \\ (\varphi,\psi) & \longmapsto & \left(\varphi \,\widehat{\otimes}\, \psi\right) \circ +^{\#} \end{array}$$

coincides with $\Lambda \times \Lambda \stackrel{+}{\longrightarrow} \Lambda$, as one easily verifies. ∎

Lemma 1.17 *Let $\Lambda \cong \mathbb{Z}^t$ be a lattice. Let R be a finitely generated k-algebra with decomposition*

$$R = \bigoplus_{Z \in \mathrm{CCp}(R)} R_Z$$

where $\mathrm{CCp}(R)$ is the set of connected components of $\mathrm{Spec}\, R$ and $Z = \mathrm{Spec}\, R_Z$. Then the R-valued points of Λ are

$$\Lambda(R) = \bigoplus_{Z \in \mathrm{CCp}(R)} \Lambda(k)$$

Proof. Proposition 1.16 yields $\Lambda(R) = \mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod_{\lambda\in\Lambda} k_\lambda, R\right)$. In the proof of 1.16 we have seen that for $h \in \mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod_{\lambda\in\Lambda} k_\lambda, R_Z\right)$ there is a $\mu \in \Lambda$ such that $(k \longleftarrow R_Z) \circ h = \mathrm{pr}_\mu$, hence $h(k_\mu^*) \subset R_Z^*$. For $\lambda \neq \mu$ it is $e_\lambda \cdot e_\mu = 0$, where $\{e_\lambda\}_{\lambda\in\Lambda}$ is the standard basis of $\prod_{\lambda\in\Lambda} k_\lambda$. Therefore $h(e_\lambda) \cdot h(e_\mu) = h(e_\lambda \cdot e_\mu) = 0$, and since $h(e_\mu)$ is a unit, it follows that $h|_{k_\lambda} = 0$ for all $\lambda \neq \mu$. A homomorphism of k-algebras h fulfills $h \circ \left(\prod_{\lambda\in\Lambda} k_\lambda \longleftarrow k\right) = (R_Z \longleftarrow k)$, hence $h|_{k_\mu} = \mathrm{id}_{k_\mu}$, i.e. $h = \mathrm{pr}_\mu$. Thus $\mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod_{\lambda\in\Lambda} k_\lambda, R_Z\right) \cong \mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod_{\lambda\in\Lambda} k_\lambda, k\right)$ for $\mathrm{Spec}\, R_Z$ connected. With this we obtain

$$\begin{aligned}
\Lambda(R) &= \mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod_{\lambda\in\Lambda} k_\lambda, R\right)\\
&= \mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod_{\lambda\in\Lambda} k_\lambda, \bigoplus_{Z\in\mathrm{CCp}(R)} R_Z\right)\\
&= \bigoplus_{Z\in\mathrm{CCp}(R)} \mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod_{\lambda\in\Lambda} k_\lambda, R_Z\right)\\
&= \bigoplus_{Z\in\mathrm{CCp}(R)} \mathrm{Hom}_{k\text{-Alg,cont}}\left(\prod_{\lambda\in\Lambda} k_\lambda, k\right)\\
&= \bigoplus_{Z\in\mathrm{CCp}(R)} \Lambda(k)
\end{aligned}$$

■

Lemma 1.18 *Let $\mathbb{T} \cong \prod \mathbb{G}_\mathrm{m}$ be a torus. Then for each finitely generated k-algebra R the R-valued points of $\mathbb{T}$ are*

$$\mathbb{T}(R) = \prod R^*$$

In particular

$$\mathbb{G}_\mathrm{m}(R) = R^*$$

Proof. A homomorphism $h : k\left[u, u^{-1}\right] \longrightarrow R$ is uniquely determined by the value $h(u)$, and well defined if and only if $h(u) \in R^*$, since $h\left(u^{-1}\right) = h(u)^{-1}$. Thus

$$\mathbb{G}_{\mathrm{m}}(R) = \mathrm{Hom}_{k\text{-Alg}}\left(k\left[u, u^{-1}\right], R\right) = R^*$$

The statement follows now by additivity:

$$\begin{aligned}
\mathbb{T}(R) &= \mathrm{Mor}\left(\mathrm{Spec}\, R, \prod \mathbb{G}_{\mathrm{m}}\right) \\
&= \prod \mathrm{Mor}\left(\mathrm{Spec}\, R, \mathbb{G}_{\mathrm{m}}\right) \\
&= \prod \mathbb{G}_{\mathrm{m}}(R) \\
&= \prod R^*
\end{aligned}$$

■

1.3.2 Cartier-Dual of a Vectorial Group

Theorem 1.19 (Cartier) *Let V be a finite dimensional k-vector space.*

The Cartier-dual of the vectorial group $\mathbb{V} = \mathrm{Spec}\left(\mathrm{Sym}\, V^\vee\right)$ associated to V is the completion w.r.t. 0 of the vectorial group associated to the dual k-vector space $V^\vee$:

$$\mathbb{V}^\vee = \mathrm{Spf}\left(\widehat{\mathrm{Sym}\, V}\right)$$

Conversely, the Cartier-dual of the infinitesimal formal group $\mathcal{V} = \mathrm{Spf}\left(\widehat{\mathrm{Sym}\, V}\right)$ with Lie-algebra $V^\vee$ is the vectorial group associated to the dual k-vector space V:

$$\mathcal{V}^\vee = \mathrm{Spec}\left(\mathrm{Sym}\, V^\vee\right)$$

In other words: If $\{x_1, \ldots, x_v\}$ is a basis of V, $\{t_1, \ldots, t_v\}$ its dual basis and $\mathbb{V} \cong \mathrm{Spec}\, k\left[t_1, \ldots, t_v\right]$, then the formal group $\mathbb{V}^\vee \cong \mathrm{Spf}\, k\left[\left[x_1, \ldots, x_v\right]\right]$ represents the functor $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}\left(\mathbb{V}, \mathbb{G}_{\mathrm{m}}\right)$ and conversely.

Proof. We claim that the formal group $\mathrm{Spf}\left(\widehat{\mathrm{Sym}\, V}\right)$ represents the sheaf $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}\left(\mathbb{V}, \mathbb{G}_{\mathrm{m}}\right)$:

Using Lemmata 1.18, 1.21 and 1.20 we obtain

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathcal{A}b/k}\left(\mathbb{V}, \mathbb{G}_{\mathrm{m}}\right)(R) &= \mathrm{Hom}_{\mathrm{gr}}\left(\mathbb{V}(R), \mathbb{G}_{\mathrm{m}}(R)\right) \\
&= \mathrm{Hom}_{\mathrm{gr}}\left(V \otimes_k R, R^*\right) \\
&= V^\vee \otimes_k \mathrm{Hom}_{\mathrm{gr}}\left(R, R^*\right) \\
&= V^\vee \otimes_k \mathrm{Nil}(R) \\
&= \underline{\mathrm{Spf}\left(\widehat{\mathrm{Sym}\, V}\right)(R)}
\end{aligned}$$

for each finitely generated k-algebra R.

The converse direction follows from the fact that the functors $\mathbb{V} \longmapsto \mathbb{V}^\vee$ and $\mathcal{V} \longmapsto \mathcal{V}^\vee$ are quasi-inverse to each other, as mentioned above.[1] ■

Lemma 1.20 *Let V be a finite dimensional k-vector space, $\mathbb{V} = \operatorname{Spec}(\operatorname{Sym} V^\vee)$ and $\mathcal{V} = \operatorname{Spf}\left(\widehat{\operatorname{Sym} V^\vee}\right)$. Then for each finitely generated k-algebra R the R-valued points are*

$$\begin{aligned} \mathbb{V}(R) &= V \otimes_k R \\ \mathcal{V}(R) &= V \otimes_k \operatorname{Nil}(R) \end{aligned}$$

In particular

$$\begin{aligned} \mathbb{G}_{\mathrm{a}}(R) &= R \\ \widehat{\mathbb{G}_{\mathrm{a}}}(R) &= \operatorname{Nil}(R) \end{aligned}$$

Proof. A homomorphism of k-algebras $h : \operatorname{Sym} V^\vee \longrightarrow R$ or $h : \widehat{\operatorname{Sym} V^\vee} \longrightarrow R$ is uniquely determined by restriction to $V^\vee$. Let $\mathfrak{m} \subset \widehat{\operatorname{Sym} V^\vee}$ be the maximal ideal generated by $V^\vee$. According to Lemma 1.22, $h : \widehat{\operatorname{Sym} V^\vee} \longrightarrow R$ is well defined and continuous (w.r.t. the $\mathfrak{m}$-adic topology on $\widehat{\operatorname{Sym} V^\vee}$ and the discrete topology on R) if and only if there is an integer $n > 0$ such that $\mathfrak{m}^n \subset \ker(h)$. As $V^\vee$ is finite dimensional, this is equivalent to the condition that $h(t) \in R$ is nilpotent for all $t \in V^\vee$. Therefore the assignment $h \longmapsto h|_{V^\vee}$ gives isomorphisms of abelian groups

$$\begin{aligned} \operatorname{Hom}_{k\text{-Alg}}(\operatorname{Sym} V^\vee, R) &\xrightarrow{\sim} \operatorname{Hom}_k(V^\vee, R) \\ \operatorname{Hom}_{k\text{-Alg,cont}}\left(\widehat{\operatorname{Sym} V^\vee}, R\right) &\xrightarrow{\sim} \operatorname{Hom}_k(V^\vee, \operatorname{Nil}(R)) \end{aligned}$$

Then

$$\begin{aligned} \mathbb{V}(R) &= \operatorname{Hom}_{k\text{-Alg}}(\operatorname{Sym} V^\vee, R) \\ &= \operatorname{Hom}_k(V^\vee, R) \\ &= V \otimes_k R \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}(R) &= \operatorname{Hom}_{k\text{-Alg,cont}}\left(\widehat{\operatorname{Sym} V^\vee}, R\right) \\ &= \operatorname{Hom}_k(V^\vee, \operatorname{Nil}(R)) \\ &= V \otimes_k \operatorname{Nil}(R) \end{aligned}$$

■

[1] An analogue simple direct computation is not possible, as here it comes into the game that $\underline{\operatorname{Hom}}_{\mathcal{A}b/k}(\mathcal{V}, \mathbb{G}_{\mathrm{m}})$ is the sheafification of the functor $R \longmapsto \operatorname{Hom}_{\mathrm{gr}}(\mathcal{V}(R), \mathbb{G}_{\mathrm{m}}(R))$.

Lemma 1.21 *Let R be a finitely generated k-Algebra. There is an isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathrm{gr}}(R, R^*) \cong \mathrm{Nil}(R)$$

Proof. Every homomorphism which translates the additive group structure of R into the multiplicative one of R^* has to be of the form $r \longmapsto \exp(nr)$ for a certain $n \in R$. This map is well defined if and only if n is nilpotent. The assignment

$$\begin{array}{rcl} \mathrm{Nil}(R) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}}(R, R^*) \\ n & \longmapsto & \exp(n \cdot \_) \end{array}$$

gives the required isomorphism of abelian groups. ■

Lemma 1.22 *Let $\mathcal{A}$ be a topological group, G a group endowed with discrete topology and $h : \mathcal{A} \longrightarrow G$ a homomorphism of groups. Then the following conditions are equivalent:*

(i) *h is continuous*
(ii) $\ker(h)$ *is open*
(iii) $\exists\, U \ni 0_{\mathcal{A}}$ *open s.t.* $U \subset \ker(h)$

Proof. A basis for the discrete topology on G is given by the one-sets $\{g\}$, $g \in G$. In the topological group $\mathcal{A}$ the open sets are generated by the open neighbourhoods of $0_{\mathcal{A}}$ via translation.

(i)$\Longrightarrow$(ii) $\{0_G\}$ is open in G and $\ker(h) = h^{-1}\{0_G\}$ is open in $\mathcal{A}$ for h continuous.

(ii)$\Longrightarrow$(iii) Take $U = \ker(h)$.

(iii)$\Longrightarrow$(ii) $\ker(h) = \bigcup_{a \in \ker(h)} a + U$ is open.

(ii)$\Longrightarrow$(i) $h^{-1}\{g\} = a + \ker(h)$, where $a \in h^{-1}\{g\}$, is open for all $g \in G$.
■

1.4 Dual of a 1-Motive

The dual of an abelian variety A is given by $A^\vee = \mathrm{Pic}^0 A$. Unfortunately, there is no equivalent duality theory for algebraic groups in general. Instead, we dualize 1-motives; and a connected algebraic group G can be considered as a special case of a 1-motive by setting the formal group $\mathcal{F} = 0$, i.e. we assign to G the 1-motive $[0 \to G]$.

Theorem 1.23 *Let L be a linear algebraic group and A an abelian variety. There is a bijection*

$$\mathrm{Ext}(A, L) \simeq \mathrm{Hom}(L^\vee, A^\vee)$$

where $A^\vee$ is the dual abelian variety.

Proof. Since $L = \mathbb{T} \times \mathbb{V}$ and $L^\vee = \mathbb{T}^\vee \times \mathbb{V}^\vee$ we have $\operatorname{Ext}(A, L) = \operatorname{Ext}(A, \mathbb{T}) \times \operatorname{Ext}(A, \mathbb{V})$ and $\operatorname{Hom}(L^\vee, A^\vee) = \operatorname{Hom}(\mathbb{T}^\vee, A^\vee) \times \operatorname{Hom}(\mathbb{V}^\vee, A^\vee)$. Therefore it is enough to consider the two cases $L = \mathbb{T}$ and $L = \mathbb{V}$, which will be done in Lemma 1.24 and Lemma 1.25 respectively. ■

Lemma 1.24 *Let $\mathbb{T} \cong (\mathbb{G}_\mathrm{m})^t$ be a torus and A an abelian variety. There is a bijection*

$$\operatorname{Ext}(A, \mathbb{T}) \simeq \operatorname{Hom}(\mathbb{T}^\vee, A^\vee)$$

Proof. We construct a bijection $\Phi : \operatorname{Ext}(A, \mathbb{T}) \longrightarrow \operatorname{Hom}(\mathbb{T}^\vee, A^\vee)$. Given an extension G of A by $\mathbb{T}$, then let $\Phi(G) : \mathbb{T}^\vee \longrightarrow A^\vee$ be the map which assigns to $\chi \in \mathbb{T}^\vee = \operatorname{Hom}_{k\text{-Gr}}(\mathbb{T}, \mathbb{G}_\mathrm{m})$ the push-out $\chi_* G \in \operatorname{Ext}(A, \mathbb{G}_\mathrm{m}) = \operatorname{Pic}^0 A = A^\vee$ of G via χ:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{T} & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\chi} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{G}_\mathrm{m} & \longrightarrow & \chi_* G & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Conversely, consider the following map $\Psi : \operatorname{Hom}(\mathbb{T}^\vee, A^\vee) \longrightarrow \operatorname{Ext}(A, \mathbb{T})$: Given $\phi : \mathbb{T}^\vee \longrightarrow A^\vee$, and a basis $\chi_1, \ldots, \chi_t \in \mathbb{T}^\vee \cong \mathbb{Z}^t$, then let $\Psi(\phi) \cong \phi(\chi_1) \times_A \ldots \times_A \phi(\chi_t) \in \operatorname{Ext}\left(A, (\mathbb{G}_\mathrm{m})^t\right)$ be the extension obtained by taking the fibre-product of the $\mathbb{G}_\mathrm{m}$-bundles $\phi(\chi_i) \in A^\vee = \operatorname{Ext}(A, \mathbb{G}_\mathrm{m})$ and pushing out this extension $\prod_A \phi(\chi_i) \in \operatorname{Ext}\left(A, (\mathbb{G}_\mathrm{m})^t\right)$ via the isomorphism $(\mathbb{G}_\mathrm{m})^t \xrightarrow{\sim} \mathbb{T}$ corresponding to the basis $(\chi_1, \ldots, \chi_t)$ of $\mathbb{T}^\vee$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{G}_\mathrm{m})^t & \longrightarrow & \prod_A \phi(\chi_i) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\wr} & & \downarrow{\scriptstyle\wr} & & \| & & \\
0 & \longrightarrow & \mathbb{T} & \longrightarrow & \Psi(\phi) & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Then Φ and Ψ are inverse to each other, in fact they are isomorphisms of abelian groups. ■

Lemma 1.25 *Let $\mathbb{V} \cong \operatorname{Spec}(\operatorname{Sym} V^\vee) \cong (\mathbb{G}_\mathrm{a})^v$ be a vectorial algebraic group and A an abelian variety. There is a bijection*

$$\operatorname{Ext}(A, \mathbb{V}) \simeq \operatorname{Hom}(\mathbb{V}^\vee, A^\vee)$$

Proof. By Theorem 1.19 $\mathbb{V}^\vee$ is the completion of a vectorial group, hence every homomorphism $\varphi : \mathbb{V}^\vee \longrightarrow A^\vee$ factors through the completion of $A^\vee$ w.r.t. $0_{A^\vee}$, an infinitesimal formal group. Since k is of characteristic 0, the category of infinitesimal formal groups over k is equivalent to the category of Lie-algebras over k (see Theorem 1.10). This implies a bijection [2]

$$\mathrm{Hom}_{\mathcal{A}b/k}\left(\mathbb{V}^{\vee}, A^{\vee}\right) \simeq \mathrm{Hom}_k\left(\mathrm{Lie}\,\mathbb{V}^{\vee}, \mathrm{Lie}\,A^{\vee}\right)$$

Therefore it is sufficient to construct a bijection

$$\Phi : \mathrm{Ext}\,(A, \mathbb{V}) \longrightarrow \mathrm{Hom}\,(\mathrm{Lie}\,\mathbb{V}^{\vee}, \mathrm{Lie}\,A^{\vee})$$

An element $\vartheta \in \mathrm{Lie}\,\mathbb{V}^{\vee} = V^{\vee} = \mathrm{Hom}_k\,(V, k)$ induces canonically a homomorphism of algebraic groups $\mathbb{V} \longrightarrow \mathbb{G}_{\mathrm{a}}$ corresponding to the homomorphism of k-algebras $k\,[t] \longrightarrow \mathrm{Sym}\,V^{\vee}$, $t \longmapsto \vartheta$. The map $\mathrm{Ext}\,(A, \mathbb{G}_{\mathrm{a}}) \longrightarrow \mathrm{H}^1\,(A, \mathcal{O}_A)$ considered at the end of Subsubsection 1.1.1 is an isomorphism (see [S] Chapter VII, No. 17, Theorem 7). But $\mathrm{H}^1\,(A, \mathcal{O}_A)$ is the tangent-space at $0_{A^{\vee}}$ of $A^{\vee}$ (see [M] Chapter III, No. 13, Corollary 3, p. 130), which is $\mathrm{Lie}\,A^{\vee}$.

Given an extension G of A by $\mathbb{V}$, then let $\Phi\,(G) : \mathrm{Lie}\,\mathbb{V}^{\vee} \longrightarrow \mathrm{Lie}\,A^{\vee}$ be the map which assigns to $\vartheta \in \mathrm{Lie}\,\mathbb{V}^{\vee} \cong \mathrm{Hom}_{\mathcal{A}b/k}\,(\mathbb{V}, \mathbb{G}_{\mathrm{a}})$ the push-out $\vartheta_* G \in \mathrm{Ext}\,(A, \mathbb{G}_{\mathrm{a}}) = \mathrm{H}^1\,(A, \mathcal{O}_A) = \mathrm{Lie}\,A^{\vee}$ of G via ϑ:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{V} & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\vartheta} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{G}_{\mathrm{a}} & \longrightarrow & \vartheta_* G & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

On the other hand consider the following map $\Psi : \mathrm{Hom}\,(\mathrm{Lie}\,\mathbb{V}^{\vee}, \mathrm{Lie}\,A^{\vee}) \longrightarrow \mathrm{Ext}\,(A, \mathbb{V})$: Given $\phi : \mathrm{Lie}\,\mathbb{V}^{\vee} \longrightarrow \mathrm{Lie}\,A^{\vee}$, and a basis $\vartheta_1, \ldots, \vartheta_v \in \mathrm{Lie}\,\mathbb{V}^{\vee} = V^{\vee}$, then let $\Psi\,(\phi) \cong \phi\,(\vartheta_1) \times_A \ldots \times_A \phi\,(\vartheta_v) \in \mathrm{Ext}\,(A, (\mathbb{G}_{\mathrm{a}})^v)$ be the extension obtained by taking the fibre-product of the $\mathbb{G}_{\mathrm{a}}$-bundles $\phi\,(\vartheta_i) \in \mathrm{Lie}\,A^{\vee} = \mathrm{Ext}\,(A, \mathbb{G}_{\mathrm{a}})$ and pushing out this extension $\prod_A \phi\,(\vartheta_i) \in \mathrm{Ext}\,(A, (\mathbb{G}_{\mathrm{a}})^v)$ via the isomorphism $(\mathbb{G}_{\mathrm{a}})^v \xrightarrow{\sim} \mathbb{V}$ given by the cobasis to the basis $(\vartheta_1, \ldots, \vartheta_v)$ of $\mathrm{Lie}\,\mathbb{V}^{\vee}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{G}_{\mathrm{a}})^v & \longrightarrow & \prod_A \phi\,(\vartheta_i) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow\wr & & \downarrow\wr & & \| & & \\
0 & \longrightarrow & \mathbb{V} & \longrightarrow & \Psi(\phi) & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Then Φ and Ψ are inverse to each other, in fact they are isomorphisms of abelian groups. ■

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[2] For those who wish to avoid the study of [SGA3] $\mathrm{VII_B}$, 3.3.2 an ad hoc argument for $\mathrm{Hom}_{\mathcal{A}b/k}\left(\widehat{\mathbb{V}}, A\right) \cong \mathrm{Hom}_k\left(\mathrm{Lie}\,\widehat{\mathbb{V}}, \mathrm{Lie}\,A\right)$ is given in Appendix 1.4.1, p. 26.

Remark 1.26 *An other description of the bijection* $\mathrm{Ext}\,(A, L) \simeq \mathrm{Hom}\,(L^\vee, A^\vee)$ *is the following:*

Given an extension G *of an abelian variety* A *by a linear group* L

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0$$

Applying the functor $\mathrm{Hom}_{\mathcal{A}b/k}\,(\_\,, \mathbb{G}_\mathrm{m})$ *yields a long exact sequence*

$$\begin{array}{c} 0 \longrightarrow \mathrm{Hom}\,(A, \mathbb{G}_\mathrm{m}) \longrightarrow \mathrm{Hom}\,(G, \mathbb{G}_\mathrm{m}) \longrightarrow \mathrm{Hom}\,(L, \mathbb{G}_\mathrm{m}) \longrightarrow \\ \longrightarrow \mathrm{Ext}\,(A, \mathbb{G}_\mathrm{m}) \longrightarrow \mathrm{Ext}\,(G, \mathbb{G}_\mathrm{m}) \longrightarrow \mathrm{Ext}\,(L, \mathbb{G}_\mathrm{m}) \end{array}$$

Now $\mathrm{Hom}\,(A, \mathbb{G}_\mathrm{m}) = 0$ *since* A *is complete and* $\mathbb{G}_\mathrm{m}$ *is affine;* $\mathrm{Ext}\,(L, \mathbb{G}_\mathrm{m}) = 0$ *since* $L \cong (\mathbb{G}_\mathrm{m})^t \times (\mathbb{G}_\mathrm{a})^v$ *and* $\mathrm{Ext}\,(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) = 0$*,* $\mathrm{Ext}\,(\mathbb{G}_\mathrm{a}, \mathbb{G}_\mathrm{m}) = 0$*;* $\mathrm{Hom}\,(L, \mathbb{G}_\mathrm{m}) = L^\vee$ *is the Cartier-dual of* L *and* $\mathrm{Ext}\,(A, \mathbb{G}_\mathrm{m}) = \mathrm{Pic}^0 A = A^\vee$ *is the dual abelian variety. Hence we obtain a homomorphism* $L^\vee \longrightarrow A^\vee$ *with kernel* $\mathrm{Hom}\,(G, \mathbb{G}_\mathrm{m})$ *and cokernel* $\mathrm{Ext}\,(G, \mathbb{G}_\mathrm{m})$*.*

A consequence of Theorem 1.23 is the following construction, which is fundamental for the notion of a dual 1-motive:

Let $M = \left[\mathcal{F} \xrightarrow{\mu} G\right]$ be a 1-motive, and $0 \to L \to G \to A \to 0$ the canonical extension of an abelian variety A by a linear group L belonging to G. By Theorem 1.23 the composition $\bar{\mu} : \mathcal{F} \longrightarrow G \longrightarrow A$ defines an extension $0 \to \mathcal{F}^\vee \to G^{\bar{\mu}} \to A^\vee \to 0$ and the extension G defines a homomorphism of sheaves of abelian groups $\bar{\mu}^G : L^\vee \longrightarrow A^\vee$. Then the next theorem says that $\bar{\mu}^G$ factorizes through $G^{\bar{\mu}}$ and hence gives rise to a dual 1-motive $M^\vee = \left[L^\vee \xrightarrow{\mu^G} G^{\bar{\mu}}\right]$.

Theorem 1.27 *Let* $\mathcal{F}$ *be a formal group,* L *a linear group,* A *an abelian variety,* $G \in \mathrm{Ext}\,(A, L)$ *and* $\rho \in \mathrm{Hom}_{\mathcal{A}b/k}\,(\mathcal{F}, A)$*. Let* $G^\rho \in \mathrm{Ext}\,(A^\vee, \mathcal{F}^\vee)$ *be the extension induced by the homomorphism* ρ *and* $\rho^G \in \mathrm{Hom}_{\mathcal{A}b/k}\,(L^\vee, A^\vee)$ *the homomorphism induced by the extension* G*. There is a bijection*

$$\mathrm{Hom}_A\,(\mathcal{F}, G) \simeq \mathrm{Hom}_{A^\vee}\,(L^\vee, G^\rho)$$

Proof. (Taken from [L] Proposition (5.2.2).) Let E be the extension of $\mathcal{F}$ by L obtained by pulling back the extension G of A by L via ρ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\rho} & & \\ 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Let E_G^ρ be the extension of $L^\vee$ by $\mathcal{F}^\vee$ obtained by pulling back the extension G^ρ of $A^\vee$ by $\mathcal{F}^\vee$ via ρ^G:

$$\begin{CD} 0 @>>> \mathcal{F}^\vee @>>> E_G^\rho @>>> L^\vee @>>> 0 \\ @. @| @VVV @VV{\rho^G}V @. \\ 0 @>>> \mathcal{F}^\vee @>>> G^\rho @>>> A^\vee @>>> 0 \end{CD}$$

Then giving homomorphisms $\mu : \mathcal{F} \longrightarrow G$ over A
and $\mu^G : L^\vee \longrightarrow G^\rho$ over $A^\vee$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\mu} & G \\ & {\scriptstyle\rho}\searrow \quad \swarrow & \\ & A & \end{array} \qquad\qquad \begin{array}{ccc} L^\vee & \xrightarrow{\mu^G} & G^\rho \\ & {\scriptstyle\rho^G}\searrow \quad \swarrow & \\ & A^\vee & \end{array}$$

is equivalent to splittings of E and E_G^ρ respectively. Now the exact sequences

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow E_G^\rho \longrightarrow L^\vee \longrightarrow 0$$

are obtained one from the other by applying the functor $\underline{\mathrm{Hom}}_{\mathcal{A}b/k}\left(\_, \mathbb{G}_{\mathrm{m}}\right)$. Therefore one sequence is split-exact if and only if the other is. ■

Definition 1.28 *The dual 1-motive of* $M = \left[\mathcal{F} \xrightarrow{\mu} G\right]$ *with* $G \in \mathrm{Ext}\,(A, L)$ *is the 1-motive* $M^\vee = \left[L^\vee \xrightarrow{\mu^G} G^{\bar{\mu}}\right]$ *corresponding to* μ *under the bijection of Theorem 1.27.*

In terms of diagrams: Given a 1-motive

$$\begin{CD} @. @. \mathcal{F} @. @. \\ @. @. @VV{\mu}V @. @. \\ 0 @>>> L @>>> G @>>> A @>>> 0 \end{CD}$$

then the dual 1-motive looks like

$$\begin{CD} @. @. L^\vee @. @. \\ @. @. @VV{\mu^G}V @. @. \\ 0 @>>> \mathcal{F}^\vee @>>> G^{\bar{\mu}} @>>> A^\vee @>>> 0 \end{CD}$$

Corollary 1.29 *The double dual* $M^{\vee\vee}$ *of a 1-motive* M *is canonically isomorphic to* M.

1.4.1 Appendix: $\operatorname{Hom}\left(\widehat{\mathbb{V}}, A\right) = \operatorname{Hom}\left(\operatorname{Lie}\widehat{\mathbb{V}}, \operatorname{Lie} A\right)$

Lemma 1.30 *Let $\varphi : \mathcal{H} \longrightarrow \mathcal{G}$ be a homomorphism of infinitesimal formal groups. Then φ is an isomorphism if and only if $\operatorname{Lie}\varphi$ is an isomorphism.*

Proof. Let $(\mathcal{A}, \mathfrak{m})$ and $(\mathcal{B}, \mathfrak{M})$ be complete local rings such that $\mathcal{G} = \operatorname{Spf}\mathcal{A}$ and $\mathcal{H} = \operatorname{Spf}\mathcal{B}$. Since $\varphi^{\#}(\mathfrak{m}^{\nu}) \subset \mathfrak{M}^{\nu}$ the map $\varphi^{\#} : \mathcal{A} \longrightarrow \mathcal{B}$ induces a map $\left[\varphi^{\#}\right] : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{M}/\mathfrak{M}^2$. But $\operatorname{Lie}\mathcal{G} = \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ and $\operatorname{Lie}\mathcal{H} = \operatorname{Hom}_k(\mathfrak{M}/\mathfrak{M}^2, k)$, hence $\operatorname{Lie}\varphi : \operatorname{Lie}\mathcal{H} \longrightarrow \operatorname{Lie}\mathcal{G}$ is just the dual k-linear map of $\left[\varphi^{\#}\right]$:

$$\operatorname{Lie}\varphi = \left[\varphi^{\#}\right]^{\vee} : \left(\mathfrak{M}/\mathfrak{M}^2\right)^{\vee} \longrightarrow \left(\mathfrak{m}/\mathfrak{m}^2\right)^{\vee}$$

$\mathcal{A} \cong \prod \mathfrak{m}^{\nu}/\mathfrak{m}^{\nu+1}$ and $\mathcal{B} \cong \prod \mathfrak{M}^{\nu}/\mathfrak{M}^{\nu+1}$ are generated by $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{M}/\mathfrak{M}^2$ respectively, thus we have:

$$\varphi \text{ iso } \Longleftrightarrow \varphi^{\#} \text{ iso } \Longleftrightarrow \left[\varphi^{\#}\right] \text{ iso } \Longleftrightarrow \operatorname{Lie}\varphi \text{ iso}$$

■

Definition 1.31 *Let G be an algebraic group or a formal group. Then define the associated vectorial group $\mathbb{L}\mathrm{ie}\, G$ of the Lie-algebra $\operatorname{Lie} G$ to be*

$$\mathbb{L}\mathrm{ie}\, G = \operatorname{Spec}\left(\operatorname{Sym}(\operatorname{Lie} G)^{\vee}\right)$$

and the associated infinitesimal formal group $\widehat{\mathbb{L}\mathrm{ie}}\, G$ of $\operatorname{Lie} G$ to be

$$\widehat{\mathbb{L}\mathrm{ie}}\, G = \operatorname{Spf}\left(\widehat{\operatorname{Sym}(\operatorname{Lie} G)^{\vee}}\right)$$

Theorem 1.32 *Let A be an abelian variety. There is one and only one homomorphism (the* exponential map*)*

$$\eta : \widehat{\mathbb{L}\mathrm{ie}}\, A \longrightarrow A$$

with $\operatorname{Lie}\eta = \mathrm{id}_{\operatorname{Lie} A}$:

$$\begin{array}{ccc}
\widehat{\mathbb{L}\mathrm{ie}}\, A(k[\varepsilon]) & \xrightarrow{\eta(k[\varepsilon])} & A(k[\varepsilon]) \\
\| & & \cup \\
\operatorname{Lie} A & \xrightarrow{\mathrm{id}} & \operatorname{Lie} A
\end{array}$$

η induces an isomorphism

$$\widehat{\eta} : \widehat{\mathbb{L}\mathrm{ie}}\, A \xrightarrow{\sim} \widehat{A}$$

where $\widehat{A} = \operatorname{Spf}\widehat{\mathcal{O}}_{A,0}$ is the completion of A w.r.t. 0_A.

Proof. Since $0_{\widehat{\mathbb{L}\mathrm{ie}}\,A}$ is the only k-valued point of $\widehat{\mathbb{L}\mathrm{ie}}\,A$, we have $\widehat{\mathbb{L}\mathrm{ie}}\,A\,(R_{\mathrm{red}}) = \left\{0_{\widehat{\mathbb{L}\mathrm{ie}}\,A}\right\}$ for each finitely generated k-algebra R, where $R_{\mathrm{red}} = R/\operatorname{Nil}(R)$ is the reduced ring. Therefore the morphism $\eta : \widehat{\mathbb{L}\mathrm{ie}}\,A \longrightarrow A$, if it exists, is uniquely determined by its induced group-homomorphisms $\eta\,(R_{\mathrm{art}}) : \widehat{\mathbb{L}\mathrm{ie}}\,A\,(R_{\mathrm{art}}) \longrightarrow A\,(R_{\mathrm{art}})$ on the R_{art}-valued points, where $R_{\mathrm{art}} = k\,[\operatorname{Nil}(R)]$ is a local Artinian ring. Using

$$A = \operatorname{Pic}^0 A^\vee \subset \operatorname{Pic} A^\vee = \mathrm{H}^1\left(A^\vee, \mathcal{O}^*_{A^\vee}\right)$$

hence

$$\begin{array}{rcl} A\,(R_{\mathrm{art}}) & \subset & \underline{\operatorname{Pic}}\,A^\vee\,(R_{\mathrm{art}}) \\ & = & \mathrm{H}^1\left(A^\vee, \mathcal{O}^*_{A^\vee \times \operatorname{Spec} R_{\mathrm{art}}}\right) \\ & = & \mathrm{H}^1\left(A^\vee, \mathcal{O}_{A^\vee}\,[\operatorname{Nil}(R)]^*\right) \end{array}$$

and

$$\begin{array}{rcl} \widehat{\mathbb{L}\mathrm{ie}}\,A\,(R_{\mathrm{art}}) & = & \operatorname{Hom}_{k\text{-Alg,cont}}\left(\widehat{\operatorname{Sym}\,(\operatorname{Lie} A)^\vee}, k\,[\operatorname{Nil}(R)]\right) \\ & = & \operatorname{Hom}_k\left((\operatorname{Lie} A)^\vee, \operatorname{Nil}(R)\right) \\ & = & \operatorname{Lie} A \otimes_k \operatorname{Nil}(R) \\ & = & \mathrm{H}^1\left(A^\vee, \mathcal{O}_{A^\vee}\right) \otimes_k \operatorname{Nil}(R) \\ & = & \mathrm{H}^1\left(A^\vee, \mathcal{O}_{A^\vee} \otimes_k \operatorname{Nil}(R)\right) \end{array}$$

construct η by

$$\begin{array}{ccc} \widehat{\mathbb{L}\mathrm{ie}}\,A\,(R_{\mathrm{art}}) & \xrightarrow{\eta(R_{\mathrm{art}})} & A\,(R_{\mathrm{art}}) \\ \| & & \downarrow \\ \mathrm{H}^1\left(A^\vee, \mathcal{O}_{A^\vee} \otimes_k \operatorname{Nil}(R)\right) & \xrightarrow{\exp} & \mathrm{H}^1\left(A^\vee, 1 + \mathcal{O}_{A^\vee} \otimes_k \operatorname{Nil}(R)\right) \\ \left(\sum f_{\alpha n} n\right)_\alpha & \longmapsto & \left(\sum_{\nu \geq 0} \frac{1}{\nu!}\left(\sum f_{\alpha n} n\right)^\nu\right)_\alpha \end{array}$$

It is known from calculus that this map is the only one which translates the additive group-structure into a multiplicative one while lifting the identity on $\operatorname{Lie} A$.

Since $\widehat{\operatorname{Sym}\,(\operatorname{Lie} A)^\vee}$ is complete, the canonical homomorphism $\mathcal{O}_{A,0} \longrightarrow \widehat{\mathcal{O}}_{A,0}$ gives a factorization

$$\begin{array}{ccc} \widehat{\operatorname{Sym}\,(\operatorname{Lie} A)^\vee} & \xleftarrow{\eta^\#} & \mathcal{O}_{A,0} \\ & \nwarrow^{\widehat{\eta}^\#} & \downarrow \\ & & \widehat{\mathcal{O}}_{A,0} \end{array} \qquad \begin{array}{ccc} \widehat{\mathbb{L}\mathrm{ie}}\,A & \xrightarrow{\eta} & A \\ & \searrow_{\widehat{\eta}} & \uparrow \\ & & \widehat{A} \end{array}$$

By Lemma 1.30 $\widehat{\eta}$ is an isomorphism for $\operatorname{Lie}\eta = \mathrm{id}_{\operatorname{Lie} A}$ is an isomorphism. ∎

Lemma 1.33 *Let* $\mathbb{V} \cong (\mathbb{G}_{\mathrm{a}})^v$ *be a vectorial algebraic group and* A *an abelian variety. There is a bijection*

$$\operatorname{Hom}_{\mathcal{A}b/k}\left(\widehat{\mathbb{V}}, A\right) \cong \operatorname{Hom}_k\left(\operatorname{Lie}\widehat{\mathbb{V}}, \operatorname{Lie} A\right)$$

Proof. A homomorphism $\varphi : \widehat{\mathbb{V}} \longrightarrow A$ of abelian sheaves factorizes through the completion $\widehat{A}$ of A w.r.t. 0_A, hence

$$\operatorname{Hom}_{\mathcal{A}b/k}\left(\widehat{\mathbb{V}}, A\right) \cong \operatorname{Hom}_{k\text{-gf}}\left(\widehat{\mathbb{V}}, \widehat{A}\right)$$

Theorem 1.32 says that $\widehat{A} \cong \widehat{\mathbb{L}\mathrm{ie}}\, A$ as formal groups:

$$\begin{aligned}\operatorname{Hom}_{k\text{-gf}}\left(\widehat{\mathbb{V}}, \widehat{A}\right) &\cong \operatorname{Hom}_{k\text{-gf}}\left(\widehat{\mathbb{V}}, \widehat{\mathbb{L}\mathrm{ie}}\, A\right)\\ &\cong \operatorname{Hom}_{k\text{-BiAlg}}\left(\widehat{\operatorname{Sym}(\operatorname{Lie} A)^\vee}, \widehat{\operatorname{Sym}\left(\operatorname{Lie}\widehat{\mathbb{V}}\right)^\vee}\right)\end{aligned}$$

$\varphi^\# : \operatorname{Sym}(\operatorname{Lie} A)^\vee \longrightarrow \operatorname{Sym}\left(\operatorname{Lie}\widehat{\mathbb{V}}\right)^\vee$ is already determined by restriction to $(\operatorname{Lie} A)^\vee$. Since $\varphi : \widehat{\mathbb{V}} \longrightarrow \widehat{\mathbb{L}\mathrm{ie}}\, A$ preserves the additive group-structures, it has to be linear, i.e. the image $\varphi^\#(\operatorname{Lie} A)^\vee$ lies in $\left(\operatorname{Lie}\widehat{\mathbb{V}}\right)^\vee$:

$$\begin{aligned}\operatorname{Hom}_{k\text{-BiAlg}}\left(\widehat{\operatorname{Sym}(\operatorname{Lie} A)^\vee}, \widehat{\operatorname{Sym}\left(\operatorname{Lie}\widehat{\mathbb{V}}\right)^\vee}\right) &\cong \operatorname{Hom}_k\left((\operatorname{Lie} A)^\vee, \left(\operatorname{Lie}\widehat{\mathbb{V}}\right)^\vee\right)\\ &\cong \operatorname{Hom}_k\left(\operatorname{Lie}\widehat{\mathbb{V}}, \operatorname{Lie} A\right)\end{aligned}$$

which yields the assertion. ∎

2 Relative Divisors

Subject of this section is the functor of families of divisors on a scheme Y over an algebraically closed field k of characteristic 0.

2.1 Reduced and Infinitesimal Functor

First we introduce some notions on functors from the category of finitely generated k-algebras $\mathbf{Alg}/k$ to the category of abelian groups $\mathbf{Ab}$, which make it easier to deal with them.

Definition 2.1 *Let* $\mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$ *be the category of covariant functors* $F : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$*, whose morphisms are given by natural transformations between functors in* $\mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$*.*

Obviously we have

Proposition 2.2 *For a finitely generated k-algebra R let* $R_{\mathrm{red}} = R/\operatorname{Nil}(R)$ *be its reduced algebra. Then the assignment* $F \longmapsto \operatorname{Red}(F)$*, where*

$$\operatorname{Red}(F)(R) := F(R_{\mathrm{red}})$$

defines a covariant functor

$$\operatorname{Red} : \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab}) \longrightarrow \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$$

Proof. Let $h : R \longrightarrow S$ be a homomorphism of finitely generated k-algebras. Since $h(\operatorname{Nil}(R)) \subset \operatorname{Nil}(S)$ there is an induced homomorphism of reduced k-algebras $h_{\mathrm{red}} : R_{\mathrm{red}} \longrightarrow S_{\mathrm{red}}$. Then for each $F \in \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$ the homomorphism $\operatorname{Red}(F)(h) : \operatorname{Red}(F)(R) \longrightarrow \operatorname{Red}(F)(S)$ is given by $F(h_{\mathrm{red}}) : F(R_{\mathrm{red}}) \longrightarrow F(S_{\mathrm{red}})$. Thus $\operatorname{Red}(F) \in \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$.

Assume $\tau : F \longrightarrow G$ is a natural transformation between functors in $\mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$. Then for every homomorphism of finitely generated k-algebras $h : R \longrightarrow S$ the following diagram commutes:

$$\begin{array}{ccc}
F(R_{\mathrm{red}}) & \xrightarrow{\tau(R_{\mathrm{red}})} & G(R_{\mathrm{red}}) \\
{\scriptstyle F(h_{\mathrm{red}})}\downarrow & & \downarrow{\scriptstyle G(h_{\mathrm{red}})} \\
F(S_{\mathrm{red}}) & \xrightarrow{\tau(R_{\mathrm{red}})} & G(S_{\mathrm{red}})
\end{array}$$

Then $\operatorname{Red}(\tau) : \operatorname{Red}(F) \longrightarrow \operatorname{Red}(G)$ defined by $\operatorname{Red}(\tau)(R) = \tau(R_{\mathrm{red}})$ for each $R \in \mathbf{Alg}/k$ is a natural transformation of functors, i.e. Red maps $\tau \in \operatorname{Hom}(F, G)$ to $\operatorname{Red}(\tau) \in \operatorname{Hom}(\operatorname{Red}(F), \operatorname{Red}(G))$. ■

Remark 2.3 *By the same argument,* Red *is a covariant functor*

$$\mathrm{Red}: \mathbf{Fctr}\left((\mathbf{Alg}/k)^{\mathrm{op}}, \mathbf{Ab}\right) \longrightarrow \mathbf{Fctr}\left((\mathbf{Alg}/k)^{\mathrm{op}}, \mathbf{Ab}\right)$$

on the category of contravariant *functors from the category of finitely generated k-algebras to the category of abelian groups.*

Notation 2.4 *Let R be a finitely generated k-algebra and $I \subsetneqq R$ a proper ideal. Then $k[I]$ denotes the subring of R whose underlying k-vector space is $k \oplus I$ endowed with the ring-structure induced by the ring-structure of $R \supset k \oplus I$.*

Proposition 2.5 *For a finitely generated k-algebra R let $R_{\mathrm{art}} = k[\mathrm{Nil}(R)]$ and $\rho: R_{\mathrm{art}} \longrightarrow k$, $\quad \mathrm{Nil}(R) \ni n \longmapsto 0$ be the restriction. Then the assignment $F \longmapsto \mathrm{Inf}(F)$, where*

$$\mathrm{Inf}(F)(R) := \ker\Big(F(\rho): F(R_{\mathrm{art}}) \longrightarrow F(k)\Big)$$

defines a covariant functor

$$\mathrm{Inf}: \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab}) \longrightarrow \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$$

Proof. A homomorphism of k-algebras $h: R \longrightarrow S$ yields a commutative diagram

$$\begin{array}{ccc} R_{\mathrm{art}} & \xrightarrow{\rho_R} & k \\ {\scriptstyle h_{\mathrm{art}}}\downarrow & & \| \\ S_{\mathrm{art}} & \xrightarrow{\rho_S} & k \end{array}$$

Let $F \in \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$. The functoriality of F implies the commutativity of

$$\begin{array}{ccccc} \ker F(\rho_R) & \longrightarrow & F(R_{\mathrm{art}}) & \xrightarrow{F(\rho_R)} & F(k) \\ \downarrow & & {\scriptstyle F(h_{\mathrm{art}})}\downarrow & & \| \\ \ker F(\rho_S) & \longrightarrow & F(S_{\mathrm{art}}) & \xrightarrow{F(\rho_S)} & F(k) \end{array}$$

The left column gives the required homomorphism

$$\mathrm{Inf}(F)(h): \mathrm{Inf}(F)(R) \longrightarrow \mathrm{Inf}(F)(S)$$

Hence $\mathrm{Inf}(F)$ is an object of $\mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$.

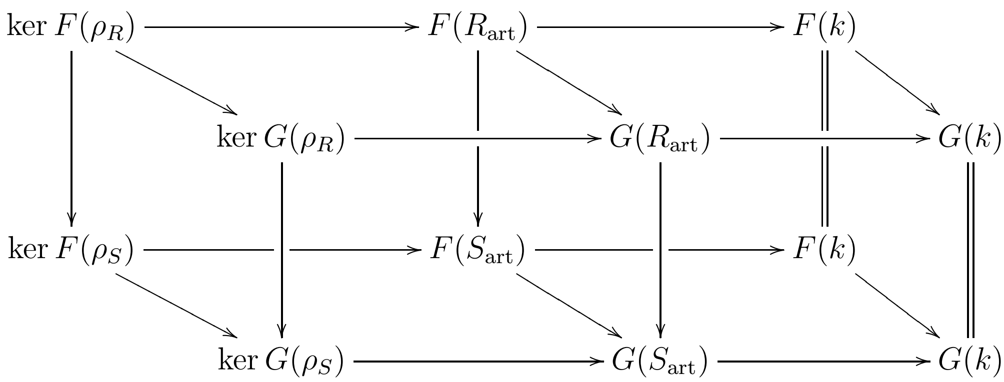
If $\tau \in \mathrm{Hom}\,(F, G)$, i.e. $\tau : F \longrightarrow G$ is a natural transformation of functors in $\mathbf{Fctr}\,(\mathbf{Alg}/k, \mathbf{Ab})$, then for each $R \in \mathbf{Alg}/k$ from the commutative diagram

$$\begin{array}{ccccc} \ker F(\rho_R) & \longrightarrow & F(R_{\mathrm{art}}) & \xrightarrow{F(\rho_R)} & F(k) \\ \downarrow & & \downarrow{\scriptstyle \tau(R_{\mathrm{art}})} & & \downarrow{\scriptstyle \tau(k)} \\ \ker G(\rho_R) & \longrightarrow & G(R_{\mathrm{art}}) & \xrightarrow{G(\rho_R)} & G(k) \end{array}$$

we obtain the homomorphism

$$\mathrm{Inf}\,(\tau)\,(R) : \mathrm{Inf}\,(F)\,(R) \longrightarrow \mathrm{Inf}\,(G)\,(R)$$

and $\mathrm{Inf}\,(\tau)$ is a natural transformation, since for each $h : R \longrightarrow S$



commutes. Thus $\mathrm{Inf}\,(\tau) \in \mathrm{Hom}\,(\mathrm{Inf}\,(F)\,, \mathrm{Inf}\,(G))$ is a morphism in the category $\mathbf{Fctr}\,(\mathbf{Alg}/k, \mathbf{Ab})$. ∎

Definition 2.6 *A functor $F : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ is called* reduced *if $F = \mathrm{Red}\,(F)$. Assume that F is covariant, then F is called* infinitesimal *if $F = \mathrm{Inf}\,(F)$.*

Definition 2.7 *A functor $F \in \mathbf{Fctr}\,(\mathbf{Alg}/k, \mathbf{Ab})$ is called* banal *if F is isomorphic to $\mathrm{Red}\,(F) \times \mathrm{Inf}\,(F)$.*

Example 2.8 *Let $\mathcal{F}$ be a formal group. Then $\mathcal{F}$ is banal by Theorem 1.7, where $\mathcal{F}_{\text{ét}} = \mathrm{Red}\,(\mathcal{F})$ and $\mathcal{F}_{\mathrm{inf}} = \mathrm{Inf}\,(\mathcal{F})$.*

Definition 2.9 *A functor $F : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$ is called* locally constant *if $F\,(R_Z) = F\,(k)$ for all finitely generated k-algebras R_Z with $\mathrm{Spec}\,R_Z$ connected, more generally*

$$F\,(R) = \bigoplus_{Z \in \mathrm{CCp}(R)} F\,(k)$$

for each finitely generated k-algebra R with decomposition $R = \bigoplus_{Z \in \mathrm{CCp}(R)} R_Z$, where $\mathrm{CCp}(R)$ denotes the set of connected components of $\mathrm{Spec}\, R$.

Remark 2.10 *A locally constant functor is always reduced.*

Example 2.11 *The étale part $\mathcal{F}_{\text{ét}}$ of a formal group $\mathcal{F}$ is locally constant, since $\mathcal{F}_{\text{ét}}$ is discrete. For a torsion-free étale formal group this was shown in Lemma 1.17.*

Definition 2.12 *Let F be a functor $\mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$. Define the* Lie functor

$$\mathrm{Lie} : \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab}) \longrightarrow \mathbf{Ab}$$

of F by

$$\mathrm{Lie}(F) = \mathrm{Inf}(F)(k[\varepsilon])$$

Definition 2.13 *A functor $F \in \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$ is called* plain *if*

$$\mathrm{Inf}(F) \in \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Vs}/k)$$

i.e. $\mathrm{Inf}(F)$ is a functor into the category $\mathbf{Vs}/k$ of k-vector spaces, and if for every finitely generated k-algebra R there is an isomorphism of k-vector spaces

$$\mathrm{Inf}(F)(R) \cong \mathrm{Lie}(F) \otimes_k \mathrm{Nil}(R)$$

Example 2.14 *If $\mathrm{char}(k) = 0$, then each formal group $\mathcal{F}$ is plain by Corollary 1.11 and Lemma 1.20.*

Notation 2.15 *If P is a plain functor in $\mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$, then $p \,\epsilon\, P$ means that either $p \in P(k)$ or $p \in \mathrm{Lie}(P)$.*

Lemma 2.16 *Let $B, P \in \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$ be plain functors fulfilling the identity $F(\bigoplus_Z R_Z) = \bigoplus_Z F(R_Z)$ for $F = B, P$. Assume furthermore that B is banal and $\mathrm{Red}(B)$ locally constant. Then each pair (a, l) of a homomorphism of abelian groups $a : B(k) \longrightarrow F(k)$ and a k-linear map $l : \mathrm{Lie}(B) \longrightarrow \mathrm{Lie}(P)$ determines a natural transformation $\tau : B \longrightarrow P$ with $\tau(k) = a$ and $\mathrm{Lie}(\tau) = l$.*

Proof. We construct a natural transformation $\tau : B \longrightarrow P$ with the desired property by giving homomorphisms $\tau(R) : B(R) \longrightarrow P(R)$, $R \in \mathbf{Alg}/k$.

Since $B(\bigoplus_Z R_Z) = \bigoplus_Z B(R_Z)$ and $P(\bigoplus_Z R_Z) = \bigoplus_Z P(R_Z)$, the homomorphism $\tau(\bigoplus_Z R_Z)$ is given by the tuple of homomorphisms $\tau(R_Z)$:

$B(R_Z) \longrightarrow P(R_Z)$. Therefore we can reduce to k-algebras R where $\operatorname{Spec} R$ is connected. In this case

$$\begin{aligned} B(R) &= \operatorname{Red}(B)(R) \times \operatorname{Inf}(B)(R) \\ &= B(k) \times \operatorname{Inf}(B)(R) \end{aligned}$$

since $\operatorname{Red}(B)$ is locally constant. On the other hand $\iota_k : k \hookrightarrow R$ induces $P(\iota_k) : P(k) \longrightarrow P(R)$ and $\iota_{R_{\text{art}}} : k[\operatorname{Nil}(R)] = R_{\text{art}} \hookrightarrow R$ induces $P(\iota_{R_{\text{art}}}) : P(R_{\text{art}}) \longrightarrow P(R)$; and $\operatorname{Inf}(P)(R) \subset P(R_{\text{art}})$. Therefore the homomorphism $\tau(R) : B(R) \longrightarrow P(R)$ is determined by

$$\tau(k) \times \operatorname{Inf}(\tau)(R) : B(k) \times \operatorname{Inf}(B)(R) \longrightarrow P(k) \times \operatorname{Inf}(P)(R)$$

Then for τ to be a natural transformation it is sufficient that for each homomorphism of k-algebras $h : R \longrightarrow S$ the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Inf}(B)(R) & \xrightarrow{\operatorname{Inf}(\tau)(R)} & \operatorname{Inf}(P)(R) \\ \downarrow{\scriptstyle \operatorname{Inf}(B)(h)} & & \downarrow{\scriptstyle \operatorname{Inf}(P)(h)} \\ \operatorname{Inf}(B)(S) & \xrightarrow{\operatorname{Inf}(\tau)(S)} & \operatorname{Inf}(P)(S) \end{array}$$

$h : R \longrightarrow S$ induces a k-linear map $\operatorname{Nil}(h) : \operatorname{Nil}(R) \longrightarrow \operatorname{Nil}(S)$. We have

$$\begin{array}{ccc} \operatorname{Inf}(P)(R) & \xrightarrow{\sim} & \operatorname{Lie}(P) \otimes \operatorname{Nil}(R) \\ \downarrow{\scriptstyle \operatorname{Inf}(P)(h)} & & \downarrow{\scriptstyle \operatorname{id}_{\operatorname{Lie}(P)} \otimes \operatorname{Nil}(h)} \\ \operatorname{Inf}(P)(S) & \xrightarrow{\sim} & \operatorname{Lie}(P) \otimes \operatorname{Nil}(S) \end{array}$$

and the same is true for B instead of P, i.e.

$$\begin{aligned} \operatorname{Inf}(B)(h) &= \operatorname{id}_{\operatorname{Lie}(\mathcal{F})} \otimes \operatorname{Nil}(h) \\ \operatorname{Inf}(P)(h) &= \operatorname{id}_{\operatorname{Lie}(F)} \otimes \operatorname{Nil}(h) \end{aligned}$$

Then the required equality

$$\operatorname{Inf}(P)(h) \circ \operatorname{Inf}(\tau)(R) = \operatorname{Inf}(\tau)(S) \circ \operatorname{Inf}(B)(h)$$

is fulfilled if it holds

$$\operatorname{Inf}(\tau)(R) = \operatorname{Lie}(\tau) \otimes \operatorname{id}_{\operatorname{Nil}(R)}$$

for each finitely generated k-algebra R.

Thus if $a : B(k) \longrightarrow P(k)$ is a homomorphism of abelian groups and $l : \mathrm{Lie}(B) \longrightarrow \mathrm{Lie}(P)$ is a k-linear map, we obtain a natural transformation $\tau : B \longrightarrow P$ by setting

$$\begin{aligned} \tau(R) &= (P(\iota_k) \circ a) \times \left(P(\iota_{R_{\mathrm{art}}}) \circ \left(l \otimes \mathrm{id}_{\mathrm{Nil}(R)}\right)\right) : \\ &\quad B(k) \times (\mathrm{Lie}(B) \otimes_k \mathrm{Nil}(R)) = B(R) \longrightarrow P(R) \end{aligned}$$

for each finitely generated k-algebra R with $\mathrm{Spec}\, R$ connected. ∎

Example 2.17 *A formal group $\mathcal{F}$ satisfies the assumptions on B in Lemma 2.16 by Examples 2.8, 2.11 and 2.14.*

2.2 Relative Cartier Divisors

Let Y be a scheme over k (an algebraically closed field of characteristic 0).

2.2.1 Functor of Relative Cartier Divisors

A *Cartier divisor* on a k-scheme X is by definition a global section of the sheaf $\mathcal{K}_X^* / \mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of total quotient rings on X, and the star * denotes the unit groups.

$$\operatorname{Div}(X) = \Gamma\left(X, \mathcal{K}_X^* / \mathcal{O}_X^*\right)$$

is the *group of Cartier divisors.*

Notation 2.18 *If R is a finitely generated k-algebra, the scheme $Y \times_k \operatorname{Spec} R$ is often denoted by $Y \otimes R$.*

Definition 2.19 *Let $R \longrightarrow A$ be a homomorphism of finitely generated k-algebras. For $p \in \operatorname{Spec} R$ let $P(p)$ be the union of the minimal prime ideals over pA in A. The set*

$$\mathrm{S}_{A/R} = \left\{ f \in A \;\middle|\; \begin{array}{c} f \text{ not a zero divisor} \\ f \notin P(p) \quad \forall p \in \operatorname{Spec} R \end{array} \right\}$$

is a multiplicative system in A. Then the localization of A at $\mathrm{S}_{A/R}$

$$\mathrm{K}_{A/R} = \mathrm{S}_{A/R}^{-1} A$$

is called the total quotient ring of A relative to R.
Let $X \xrightarrow{\tau} T$ be a scheme over T. The sheaf $\mathcal{K}_{X/T}$ associated to the presheaf formed by the rings

$$\mathrm{K}_{\mathcal{O}_X(U)/(\tau^{-1}\mathcal{O}_T)(U)} = \mathrm{S}^{-1}_{\mathcal{O}_X(U)/(\tau^{-1}\mathcal{O}_T)(U)} \mathcal{O}_X(U)$$

for open $U \subset X$, is called the sheaf of total quotient rings of X relative to T.

Remark 2.20 *Notice that in the case $A = B \otimes_k R$ over R, where B is an integral domain, we have $P(p) = pA = B \otimes_k p$.*
Correspondingly for $X = Y \times_k \operatorname{Spec} R$ the sheaf $\mathcal{K}_{Y \otimes R/R}$ is the localization of $\mathcal{O}_{Y \otimes R}$ at functions which do not vanish on subsets of the form $Z \times \{p\}$, $p \in \operatorname{Spec} R$ and $Z \in \operatorname{Cp}(Y)$ an irreducible component of Y.

Proposition 2.21 *For a finitely generated k-algebra R let*

$$\underline{\mathrm{Div}}_Y(R) = \Gamma\left(Y \otimes R, \mathcal{K}^*_{Y\otimes R/R}\big/\mathcal{O}^*_{Y\otimes R}\right)$$

Then the assignment $R \longmapsto \underline{\mathrm{Div}}_Y(R)$ defines a covariant functor

$$\underline{\mathrm{Div}}_Y : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

from the category of finitely generated k-algebras to the category of abelian groups.

Proof. For a homomorphism $h : R \longrightarrow S$ of finitely generated k-algebras the homomorphism $\mathrm{id}_Y \otimes h : \mathcal{O}_Y \otimes_k R \longrightarrow \mathcal{O}_Y \otimes_k S$ extends by localization to $\mathcal{K}_{Y\otimes R/R} \longrightarrow \mathcal{K}_{Y\otimes S/S}$ and hence induces a homomorphism of abelian groups $\Gamma\left(\mathcal{K}^*_{Y\otimes R/R}\big/\mathcal{O}^*_{Y\otimes R}\right) \longrightarrow \Gamma\left(\mathcal{K}^*_{Y\otimes S/S}\big/\mathcal{O}^*_{Y\otimes S}\right)$ which is the required homomorphism $\underline{\mathrm{Div}}_Y(h) : \underline{\mathrm{Div}}_Y(R) \longrightarrow \underline{\mathrm{Div}}_Y(S)$. ∎

Remark 2.22 *For each finitely generated k-algebra R we have*

$$\underline{\mathrm{Div}}_Y(R) = \left\{\begin{array}{c}\textit{Cartier divisors } \mathcal{D} \textit{ on } Y \times_k \operatorname{Spec} R \\ \textit{which define Cartier divisors } \mathcal{D}_p \textit{ on } Y \times \{p\} \\ \forall p \in \operatorname{Spec} R\end{array}\right\}$$

and for a homomorphism $h : R \longrightarrow S$ in $\mathbf{Alg}/k$ the induced homomorphism $\underline{\mathrm{Div}}_Y(h) : \underline{\mathrm{Div}}_Y(R) \longrightarrow \underline{\mathrm{Div}}_Y(S)$ in $\mathbf{Ab}$ is the pull-back of Cartier divisors on $Y \times_k \operatorname{Spec} R$ to those on $Y \times_k \operatorname{Spec} S$.

Proposition 2.23 *Let R be a finitely generated k-algebra, $R_{\mathrm{art}} = k[\mathrm{Nil}(R)]$ the corresponding local Artinian ring. Then*

$$\begin{aligned}\underline{\mathrm{Div}}_Y(R_{\mathrm{art}}) &= \mathrm{Div}(Y \otimes R_{\mathrm{art}}) \\ \mathrm{Inf}(\underline{\mathrm{Div}}_Y)(R) &= \mathrm{Lie}(\underline{\mathrm{Div}}_Y) \otimes_k \mathrm{Nil}(R)\end{aligned}$$

i.e. $\underline{\mathrm{Div}}_Y \in \mathbf{Fctr}(\mathbf{Alg}/k, \mathbf{Ab})$ is a plain functor (see Definition 2.13). Explicitely

$$\begin{aligned}\underline{\mathrm{Div}}_Y(k) &= \Gamma(Y, \mathcal{K}^*_Y/\mathcal{O}^*_Y) \\ \mathrm{Lie}(\underline{\mathrm{Div}}_Y) &= \Gamma(Y, \mathcal{K}_Y/\mathcal{O}_Y)\end{aligned}$$

Proof. $R_{\mathrm{art}} = k[\mathrm{Nil}(R)]$ is a local Artinian ring with only prime ideal $\mathrm{Nil}(R) \in \operatorname{Spec} R_{\mathrm{art}}$, and it consists of zero divisors only. Therefore $\mathcal{K}^*_{Y\otimes R_{\mathrm{art}}/R_{\mathrm{art}}} = \mathcal{K}^*_{Y\otimes R_{\mathrm{art}}}$ and this implies the first equation:

$$\begin{aligned}\underline{\mathrm{Div}}_Y(R_{\mathrm{art}}) &= \Gamma\left(\mathcal{K}^*_{Y\otimes R_{\mathrm{art}}/R_{\mathrm{art}}}\big/\mathcal{O}^*_{Y\otimes R_{\mathrm{art}}}\right) \\ &= \Gamma\left(\mathcal{K}^*_{Y\otimes R_{\mathrm{art}}}\big/\mathcal{O}^*_{Y\otimes R_{\mathrm{art}}}\right) \\ &= \mathrm{Div}(Y \otimes R_{\mathrm{art}})\end{aligned}$$

In particular for $R=k$ it holds $\underline{\mathrm{Div}}_Y(k)=\Gamma\left(\mathcal{K}_Y^*/\,\mathcal{O}_Y^*\right)$.

We have $\mathcal{K}^*_{Y\otimes R_{\mathrm{art}}}=\mathcal{K}_Y\left[\mathrm{Nil}\,(R)\right]^*=\mathcal{K}_Y^*+\mathcal{K}_Y\otimes_k\mathrm{Nil}\,(R)$ and hence an exact sequence

$$1\longrightarrow\frac{1+\mathcal{K}_Y\otimes_k\mathrm{Nil}\,(R)}{1+\mathcal{O}_Y\otimes_k\mathrm{Nil}\,(R)}\longrightarrow\frac{\mathcal{K}^*_{Y\otimes R_{\mathrm{art}}}}{\mathcal{O}^*_{Y\otimes R_{\mathrm{art}}}}\longrightarrow\frac{\mathcal{K}_Y^*}{\mathcal{O}_Y^*}\longrightarrow 1$$

Now

$$\frac{1+\mathcal{K}_Y\otimes_k\mathrm{Nil}\,(R)}{1+\mathcal{O}_Y\otimes_k\mathrm{Nil}\,(R)}\cong\frac{\mathcal{K}_Y\otimes_k\mathrm{Nil}\,(R)}{\mathcal{O}_Y\otimes_k\mathrm{Nil}\,(R)}\cong\frac{\mathcal{K}_Y}{\mathcal{O}_Y}\otimes_k\mathrm{Nil}\,(R)$$

where the first isomorphism is given by $\exp^{-1}$. Applying the global section functor $\Gamma\,(Y,\_)$ yields

$$0\longrightarrow\Gamma\left(\frac{\mathcal{K}_Y}{\mathcal{O}_Y}\otimes_k\mathrm{Nil}\,(R)\right)\longrightarrow\Gamma\left(\frac{\mathcal{K}^*_{Y\otimes R_{\mathrm{art}}}}{\mathcal{O}^*_{Y\otimes R_{\mathrm{art}}}}\right)\longrightarrow\Gamma\left(\frac{\mathcal{K}_Y^*}{\mathcal{O}_Y^*}\right)$$

Here $\underline{\mathrm{Div}}_Y(R_{\mathrm{art}})=\Gamma\left(\mathcal{K}^*_{Y\otimes R_{\mathrm{art}}}/\,\mathcal{O}^*_{Y\otimes R_{\mathrm{art}}}\right)$ and $\underline{\mathrm{Div}}_Y(k)=\Gamma\left(\mathcal{K}_Y^*/\,\mathcal{O}_Y^*\right)$, therefore

$$\mathrm{Inf}\,(\underline{\mathrm{Div}}_Y)\,(R)=\Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)\otimes_k\mathrm{Nil}\,(R)$$

In particular for $R=k[\varepsilon]$, as $\mathrm{Nil}\,(k[\varepsilon])=\varepsilon k\cong k$ we have

$$\mathrm{Lie}\,(\underline{\mathrm{Div}}_Y)=\mathrm{Inf}\,(\underline{\mathrm{Div}}_Y)\,(k[\varepsilon])\cong\Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)$$

and hence

$$\mathrm{Inf}\,(\underline{\mathrm{Div}}_Y)\,(R)\cong\mathrm{Lie}\,(\underline{\mathrm{Div}}_Y)\otimes_k\mathrm{Nil}\,(R)$$

■

Definition 2.24 *If $D\in\underline{\mathrm{Div}}_Y(k)$, then* Supp (D) *denotes the locus of zeroes and poles of local sections $(f_\alpha)_\alpha\in\Gamma\left(\mathcal{K}_Y^*/\,\mathcal{O}_Y^*\right)$ representing D.*
If $\delta\in\mathrm{Lie}\,(\underline{\mathrm{Div}}_Y)$, then Supp (δ) *denotes the locus of poles of local sections $(g_\alpha)_\alpha\in\Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)$ representing δ.*

Proposition 2.25 *Let $\pi:Y\longrightarrow X$ be a morphism of k-schemes with the property that $\pi^{\#}:\mathcal{O}_X\longrightarrow\mathcal{O}_Y$ maps non zero divisors to non zero divisors. Then π induces a natural transformation of functors, the* pull-back

$$\pi^*:\underline{\mathrm{Div}}_X\longrightarrow\underline{\mathrm{Div}}_Y$$

Proof. For each finitely generated k-algebra R the transformation $\pi^*(R):\underline{\mathrm{Div}}_X(R)\longrightarrow\underline{\mathrm{Div}}_Y(R)$ is the homomorphism $\Gamma\left(\mathcal{K}^*_{X\otimes R/R}\big/\,\mathcal{O}^*_{X\otimes R}\right)\longrightarrow\Gamma\left(\mathcal{K}^*_{Y\otimes R/R}\big/\,\mathcal{O}^*_{Y\otimes R}\right)$ induced by $\pi^{\#}\otimes\mathrm{id}_R:\mathcal{O}_X\otimes_k R\longrightarrow\mathcal{O}_Y\otimes_k R$.

Let $h : R \longrightarrow S$ be a homomorphism of k-algebras and let $h^\dagger : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$, $q \longmapsto h^{-1}(q)$ be the corresponding morphism of affine schemes. Then

$$\left(\pi^{\#} \otimes \mathrm{id}_S\right) \circ (\mathrm{id}_X \otimes h) = \pi^{\#} \otimes h = (\mathrm{id}_Y \otimes h) \circ \left(\pi^{\#} \otimes \mathrm{id}_R\right)$$

and this implies a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Div}}_X(R) & \xrightarrow{\left(\pi \times \mathrm{id}_R^\dagger\right)^*} & \underline{\mathrm{Div}}_Y(R) \\ {\scriptstyle \left(\mathrm{id}_X \times h^\dagger\right)^*} \downarrow & & \downarrow {\scriptstyle \left(\mathrm{id}_Y \times h^\dagger\right)^*} \\ \underline{\mathrm{Div}}_X(S) & \xrightarrow{\left(\pi \times \mathrm{id}_S^\dagger\right)^*} & \underline{\mathrm{Div}}_Y(S) \end{array}$$

■

2.2.2 Picard Functor

The isomorphism classes of line bundles on a k-scheme X form a group $\operatorname{Pic}(X)$, the *(absolute) Picard group* of X, which is given by

$$\operatorname{Pic}(X) = \mathrm{H}^1\left(X, \mathcal{O}_X^*\right)$$

Definition 2.26 *The* (relative) Picard functor *of a k-scheme Y from the category of schemes over k to the category of abelian groups*

$$\underline{\mathrm{Pic}}_Y : \mathbf{Sch}/k \longrightarrow \mathbf{Ab}$$

is defined by

$$\underline{\mathrm{Pic}}_Y(T) = \operatorname{Pic}(Y \times_k T)/\operatorname{Pic}(T)$$

for each k-scheme T.

Simultanously we consider $\underline{\mathrm{Pic}}_Y$ as a functor from the category of finitely generated k-algebras to the category of abelian groups

$$\underline{\mathrm{Pic}}_Y : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

using the notation

$$\underline{\mathrm{Pic}}_Y(R) = \operatorname{Pic}(Y \times_k \operatorname{Spec} R)/\operatorname{Pic}(\operatorname{Spec} R)$$

for each finitely generated k-algebra R.

Remark 2.27 *For each k-scheme T we have*

$$\underline{\mathrm{Pic}}_Y(T) = \{\textit{Line bundles } \mathcal{L} \textit{ on } Y \times_k T\}/\sim_T$$

where the equivalence relation $\sim_T$ is defined by

$$\mathcal{L} \sim_T \mathcal{M} \iff \exists \textit{ line bundle } B \textit{ on } T \textit{ s.t. } \mathcal{L} \otimes \mathcal{M}^{-1} \cong \mathrm{pr}_T^* B$$

Proposition 2.28 *Let R be a finitely generated k-algebra, $R_{\text{art}} = k\left[\text{Nil}\left(R\right)\right]$. Then*

$$\begin{aligned}\underline{\text{Pic}}_Y\left(R_{\text{art}}\right) &= \text{Pic}\left(Y \otimes R_{\text{art}}\right)\\ \text{Inf}\left(\underline{\text{Pic}}_Y\right)\left(R\right) &= \text{Lie}\left(\underline{\text{Pic}}_Y\right) \otimes_k \text{Nil}\left(R\right)\end{aligned}$$

i.e. $\underline{\text{Pic}}_Y \in \mathbf{Fctr}\left(\mathbf{Alg}/k, \mathbf{Ab}\right)$ *is a plain functor (see Definition 2.13). Explicitely*

$$\begin{aligned}\underline{\text{Pic}}_Y\left(k\right) &= \text{H}^1\left(Y, \mathcal{O}_Y^*\right)\\ \text{Lie}\left(\underline{\text{Pic}}_Y\right) &= \text{H}^1\left(Y, \mathcal{O}_Y\right)\end{aligned}$$

Proof. $\text{Spec}\, R_{\text{art}}$ consists only of the point $\text{Nil}\left(R\right)$, hence every sheaf on $\text{Spec}\, R_{\text{art}}$ is flasque, which implies $\text{Pic}\left(\text{Spec}\, R_{\text{art}}\right) = \text{H}^1\left(R_{\text{art}}^*\right) = 0$. Therefore $\underline{\text{Pic}}_Y\left(R_{\text{art}}\right) = \text{Pic}\left(Y \otimes R_{\text{art}}\right) = \text{H}^1\left(\mathcal{O}_{Y\otimes R_{\text{art}}}^*\right)$. In particular for $R = k$ it holds $\underline{\text{Pic}}_Y\left(k\right) = \text{Pic}\left(Y\right) = \text{H}^1\left(\mathcal{O}_Y^*\right)$.

Furthermore we have $\mathcal{O}_{Y\otimes R_{\text{art}}}^* = \mathcal{O}_Y\left[\text{Nil}\left(R\right)\right]^* = \mathcal{O}_Y^* + \mathcal{O}_Y \otimes_k \text{Nil}\left(R\right)$ and hence an exact sequence

$$1 \longrightarrow 1 + \mathcal{O}_Y \otimes_k \text{Nil}\left(R\right) \longrightarrow \mathcal{O}_{Y\otimes R_{\text{art}}}^* \longrightarrow \mathcal{O}_Y^* \longrightarrow 1$$

Now $\exp^{-1}$ gives an isomorphism

$$1 + \mathcal{O}_Y \otimes_k \text{Nil}\left(R\right) \cong \mathcal{O}_Y \otimes_k \text{Nil}\left(R\right)$$

Taking into account that in the corresponding long exact cohomology sequence $\text{H}^0\left(\mathcal{O}_{Y\otimes R_{\text{art}}}^*\right) \longrightarrow \text{H}^0\left(\mathcal{O}_Y^*\right)$ is surjective yields

$$0 \longrightarrow \text{H}^1\left(\mathcal{O}_Y \otimes_k \text{Nil}\left(R\right)\right) \longrightarrow \text{H}^1\left(\mathcal{O}_{Y\otimes R_{\text{art}}}^*\right) \longrightarrow \text{H}^1\left(\mathcal{O}_Y^*\right)$$

Here $\underline{\text{Pic}}_Y\left(R_{\text{art}}\right) = \text{H}^1\left(\mathcal{O}_{Y\otimes R_{\text{art}}}^*\right)$ and $\underline{\text{Pic}}_Y\left(k\right) = \text{H}^1\left(\mathcal{O}_Y^*\right)$, therefore

$$\text{Inf}\left(\underline{\text{Pic}}_Y\right)\left(R\right) = \text{H}^1\left(\mathcal{O}_Y\right) \otimes_k \text{Nil}\left(R\right)$$

In particular for $R = k[\varepsilon]$, as $\text{Nil}\left(k[\varepsilon]\right) = \varepsilon k \cong k$ we have

$$\text{Lie}\left(\underline{\text{Pic}}_Y\right) = \text{Inf}\left(\underline{\text{Pic}}_Y\right)\left(k[\varepsilon]\right) \cong \text{H}^1\left(\mathcal{O}_Y\right)$$

and hence

$$\text{Inf}\left(\underline{\text{Pic}}_Y\right)\left(R\right) \cong \text{Lie}\left(\underline{\text{Pic}}_Y\right) \otimes_k \text{Nil}\left(R\right)$$

■

Theorem 2.29 *If Y is a reduced connected projective k-scheme, then the Picard functor $\underline{\text{Pic}}_Y$ is represented by a k-group-scheme Pic_Y, which is called the* Picard scheme *of Y.*

Proof. [FGA] No. 232, Theorem 2 or [BLR] Section 8.2, Theorem 2 or [K] Theorem 4.8, Theorem 4.18.1. ■

"Pic_Y represents the functor $\underline{\mathrm{Pic}}_Y$" means that $\underline{\mathrm{Pic}}_Y$ coincides with the functor of points of Pic_Y, i.e. for each k-scheme T it holds

$$\underline{\mathrm{Pic}}_Y(T) \cong \mathrm{Mor}(T, \mathrm{Pic}_Y)$$

Inserting Pic_Y for T gives the universal line bundle class $\mathcal{P} \in \underline{\mathrm{Pic}}_Y(\mathrm{Pic}_Y)$ corresponding to $\mathrm{id}_{\mathrm{Pic}_Y} \in \mathrm{Mor}(\mathrm{Pic}_Y, \mathrm{Pic}_Y)$, which is called the *Poincaré bundle.* $\mathcal{P}$ has the following universal property: Given any k-scheme T and any line bundle class $\mathcal{L} \in \underline{\mathrm{Pic}}_Y(T)$, there exists a unique k-morphism $\eta : T \longrightarrow \mathrm{Pic}_Y$ such that

$$\mathcal{L} = (\mathrm{id}_Y \times \eta)^* \mathcal{P}$$

η is called the *classifying morphism* for $\mathcal{L}$.

Remark 2.30 *If* Pic_Y *represents* $\underline{\mathrm{Pic}}_Y$*, then of course for the geometric points of* Pic_Y *it holds*

$$\mathrm{Pic}_Y(K) = \mathrm{Pic}(Y \otimes K)$$

since $\mathrm{Pic}(\mathrm{Spec}\, K) = 0$*, hence* $\underline{\mathrm{Pic}}_Y(K) = \mathrm{Pic}(Y \otimes K)$ *and on the other hand* $\underline{\mathrm{Pic}}_Y(K) = \mathrm{Mor}(\mathrm{Spec}\, K, \mathrm{Pic}_Y) = \mathrm{Pic}_Y(K)$.

The universal line bundle $\mathcal{P} \in \underline{\mathrm{Pic}}_Y(\mathrm{Pic}_Y)$ *has the property that*

$$\mathcal{P}|_{Y \times \{p\}} = p \in \mathrm{Pic}_Y$$

for each $p \in \mathrm{Pic}_Y$*. The classifying morphism* $\eta : T \longrightarrow \mathrm{Pic}_Y$ *for a line bundle* $\mathcal{L}$ *on* $Y \times_k T$ *maps*

$$\eta : t \longmapsto \mathcal{L}|_{Y \times \{t\}} \in \mathrm{Pic}_Y$$

for each $t \in T$.

Definition 2.31 *Let* M, N *be line bundles on* Y*. Then* M *is said to be* algebraically equivalent *to* N*,* $M \sim N$*, if there exists a connected* k*-scheme* C*, a line bundle* $\mathcal{L}$ *on* $Y \times_k C$ *and closed points* $p, q \in C$ *such that* $\mathcal{L}|_{Y \times \{p\}} = M$ *and* $\mathcal{L}|_{Y \times \{q\}} = N$.

Lemma 2.32 *Suppose that* $\underline{\mathrm{Pic}}_Y$ *is represented by a scheme* Pic_Y*. Let* L *be a line bundle on* Y*. Then* L *is algebraically equivalent to the trivial bundle on* Y *if and only if* L *lies in the connected component of the identity of* Pic_Y*:*

$$L \sim \mathcal{O}_Y \iff L \in \mathrm{Pic}_Y^0$$

Proof. ($\Longrightarrow$) L algebraically equivalent to $\mathcal{O}_Y$ means by definition that there is a connected k-scheme C and a line bundle $\mathcal{L}$ on $Y \times_k C$ such that for certain points $0, p \in C$ we have $\mathcal{L}|_{Y \times \{0\}} = \mathcal{O}_Y$, $\mathcal{L}|_{Y \times \{p\}} = L$. Let η be the classifying morphism for $\mathcal{L}$, i.e. $[\mathcal{L}] = (\mathrm{id}_Y \times \eta)^* \mathcal{P}$. Since C is connected $\eta(C)$ is connected, and since $\eta(0) = \mathcal{O}_Y \in \mathrm{Pic}^0_Y$ we have $\eta(C) \subset \mathrm{Pic}^0_Y$. Hence $L = \eta(p) \in \mathrm{Pic}^0_Y$.

($\Longleftarrow$) Let $L \in \mathrm{Pic}^0_Y$. Since $\mathcal{O}_Y$ is the neutral element in Pic^0_Y, the restriction to $Y \times_k \mathrm{Pic}^0_Y$ of any representative on $Y \times_k \mathrm{Pic}_Y$ of the universal line bundle $\mathcal{P} \in \underline{\mathrm{Pic}}_Y(\mathrm{Pic}_Y)$ gives a line bundle $\mathcal{P}'$ on $Y \times_k \mathrm{Pic}^0_Y$ fulfilling: Pic^0_Y is connected, $\mathcal{P}'|_{Y \times \{\mathcal{O}_Y\}} = \mathcal{O}_Y$, $\mathcal{P}'|_{Y \times \{L\}} = L$. ∎

Proposition 2.33 *Let Y be a k-scheme. Then the assignment for each k-scheme T*

$$T \longmapsto \left\{ \begin{array}{c} \text{Line bundles } \mathcal{L} \text{ on } Y \times_k T \\ \text{with } \mathcal{L}|_{Y \times \{t\}} \text{ algebraically equivalent to } \mathcal{O}_Y \\ \forall t \in T \end{array} \right\} \Big/ \sim_T$$

where

$$\mathcal{L} \sim_T \mathcal{M} \iff \exists \text{ line bundle } B \text{ on } T \text{ s.t. } \mathcal{L} \otimes \mathcal{M}^{-1} \cong \mathrm{pr}_T^* B$$

defines a contravariant functor

$$\underline{\mathrm{Pic}}^0_Y : \mathbf{Sch}/k \longrightarrow \mathbf{Ab}$$

and a covariant functor

$$\underline{\mathrm{Pic}}^0_Y : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

where we set $\underline{\mathrm{Pic}}^0_Y(R) := \underline{\mathrm{Pic}}^0_Y(\mathrm{Spec}\, R)$ for each finitely generated k-algebra R.

Suppose $\underline{\mathrm{Pic}}_Y$ is represented by a k-scheme Pic_Y. Then $\underline{\mathrm{Pic}}^0_Y$ is represented by the component of the identity Pic^0_Y.

Proof. Let $\psi : S \longrightarrow T$ be a morphism of k-schemes. Then the required homomorphism of abelian groups $\underline{\mathrm{Pic}}^0_Y(\psi) : \underline{\mathrm{Pic}}^0_Y(T) \longrightarrow \underline{\mathrm{Pic}}^0_Y(S)$ is induced by the pull-back of line bundles from $Y \times_k T$ to those on $Y \times_k S$:

Let $\mathcal{L}$ be a line bundle on $Y \times_k T$. Then for each $s \in S$ we have $(\mathrm{id}_Y \times \psi)^* \mathcal{L}|_{Y \times \{s\}} = \mathcal{L}|_{Y \times \{\psi(s)\}}$ and this is algebraically equivalent to $\mathcal{O}_Y$ if $\mathcal{L}|_{Y \times \{t\}}$ is for each $t \in T$. Furthermore if $\mathrm{pr}_S : Y \times_k S \longrightarrow S$ and $\mathrm{pr}_T : Y \times_k T \longrightarrow T$ are the projections, we have $\mathrm{pr}_T \circ (\mathrm{id}_Y \times \psi) = \psi \circ \mathrm{pr}_S$. Hence for a line bundle B on T the pull-back $(\mathrm{id}_Y \times \psi)^* \mathrm{pr}_T^* B = \mathrm{pr}_S^* \psi^* B$ is the pull-back of a line bundle $\psi^* B$ on S. Therefore $\underline{\mathrm{Pic}}^0_Y(\psi)$ is well defined.

Now assume that $\underline{\mathrm{Pic}}_Y$ is represented by Pic_Y. Let $\mathcal{L}$ be a line bundle on $Y \times_k T$ with $\mathcal{L}|_{Y\times\{t\}} \sim \mathcal{O}_Y$ for each $t \in T$. By Lemma 2.32 we have $\mathcal{L}|_{Y\times\{t\}} \in \mathrm{Pic}^0_Y$ and by Remark 2.30 this implies that the classifying morphism $\eta : T \longrightarrow \mathrm{Pic}_Y$ for $\mathcal{L}$ factorizes through Pic^0_Y. Therefore we have $\underline{\mathrm{Pic}}^0_Y \cong \mathrm{Mor}\left(\_, \mathrm{Pic}^0_Y\right)$, i.e. $\underline{\mathrm{Pic}}^0_Y$ is represented by Pic^0_Y. ■

Theorem 2.34 *Let Y be a projective and integral k-scheme. Then $\underline{\mathrm{Pic}}^0_Y$ is represented by a quasi-projective k-group-scheme Pic^0_Y. If Y is also normal, then Pic^0_Y is projective.*

Proof. Follows directly from Theorem 2.29, Proposition 2.33 and [K], Theorem 5.4. ■

Corollary 2.35 *Let Y be a normal projective variety over k. Then $\underline{\mathrm{Pic}}^0_Y$ is represented by an abelian variety Pic^0_Y.*

Proof. A normal projective variety Y over k is the disjoint union of its irreducible components, $Y = \coprod_{Z\in\mathrm{Cp}(Y)} Z$, by [Mm] Chapter 3, §9, Remark p.64. Applying Theorem 2.34 to each irreducible component Z of Y yields that $\underline{\mathrm{Pic}}^0_Y$ is represented by $\mathrm{Pic}^0_Y = \prod_{Z\in\mathrm{Cp}(Y)} \mathrm{Pic}^0_Z$, and this is a projective k-group-scheme. As $\mathrm{char}(k) = 0$, a projective k-group-scheme is an abelian variety. ■

2.2.3 Transformation $\underline{\mathrm{Div}}_Y \longrightarrow \underline{\mathrm{Pic}}_Y$

Let Y be a k-scheme and R be a finitely generated k-algebra. Consider the exact sequence $\mathrm{Seq}\,(R)$:

$$1 \longrightarrow \mathcal{O}^*_{Y\otimes R} \longrightarrow \mathcal{K}^*_{Y\otimes R} \longrightarrow \mathcal{K}^*_{Y\otimes R}/\,\mathcal{O}^*_{Y\otimes R} \longrightarrow 1$$

In the corresponding long exact sequence $\mathrm{H}^\bullet\,\mathrm{Seq}\,(R)$:

$$\longrightarrow \mathrm{H}^0\left(\mathcal{K}^*_{Y\otimes R}\right) \longrightarrow \mathrm{H}^0\left(\mathcal{K}^*_{Y\otimes R}/\,\mathcal{O}^*_{Y\otimes R}\right) \longrightarrow \mathrm{H}^1\left(\mathcal{O}^*_{Y\otimes R}\right) \longrightarrow \mathrm{H}^1\left(\mathcal{K}^*_{Y\otimes R}\right) \longrightarrow$$

the connecting homomorphism $\delta^0\,(R) : \mathrm{H}^0\left(\mathcal{K}^*_{Y\otimes R}/\,\mathcal{O}^*_{Y\otimes R}\right) \longrightarrow \mathrm{H}^1\left(\mathcal{O}^*_{Y\otimes R}\right)$ gives a natural transformation $\mathrm{Div}\,(Y \otimes R) \longrightarrow \mathrm{Pic}\,(Y \otimes R)$. Composition with the injection $\underline{\mathrm{Div}}_Y\,(R) \hookrightarrow \mathrm{Div}\,(Y \otimes R)$ and the projection $\mathrm{Pic}\,(Y \otimes R) \twoheadrightarrow \underline{\mathrm{Pic}}_Y\,(R)$ yields a natural transformation

$$\mathrm{cl} : \underline{\mathrm{Div}}_Y \longrightarrow \underline{\mathrm{Pic}}_Y$$

Since $\underline{\mathrm{Div}}_Y$ and $\underline{\mathrm{Pic}}_Y$ are plain functors (see Propositions 2.23 and 2.28), the induced transformation

$$\mathrm{Inf}\,(\mathrm{cl}) : \mathrm{Inf}\,(\underline{\mathrm{Div}}_Y) \longrightarrow \mathrm{Inf}\,(\underline{\mathrm{Pic}}_Y)$$

fulfills

$$\mathrm{Inf}\,(\mathrm{cl})\,(R) = \mathrm{Lie}\,(\mathrm{cl}) \otimes \mathrm{Nil}\,(R) : \mathrm{Lie}\,(\underline{\mathrm{Div}}_Y) \otimes \mathrm{Nil}\,(R) \longrightarrow \mathrm{Lie}\,(\underline{\mathrm{Pic}}_Y) \otimes \mathrm{Nil}\,(R)$$

where $\mathrm{Lie}\,(\underline{\mathrm{Div}}_Y) = \mathrm{H}^0\,(\mathcal{K}_Y/\,\mathcal{O}_Y)$, $\mathrm{Lie}\,(\underline{\mathrm{Pic}}_Y) = \mathrm{H}^1\,(\mathcal{O}_Y)$ (see Propositions 2.23 and 2.28) and Lie (cl) coincides with $\mathrm{Lie}\,(\delta^0)$ in the sequence $\mathrm{Lie}\,(\mathrm{H}^\bullet\,\mathrm{Seq})$.

We obtain

Proposition 2.36 *There are natural transformations*

$$\begin{aligned} \mathrm{cl} : \underline{\mathrm{Div}}_Y &\longrightarrow \underline{\mathrm{Pic}}_Y \\ \mathrm{Inf}\,(\mathrm{cl}) : \mathrm{Inf}\,(\underline{\mathrm{Div}}_Y) &\longrightarrow \mathrm{Inf}\,(\underline{\mathrm{Pic}}_Y) \end{aligned}$$

for each finitely generated k-algebra R given by

$$\begin{aligned} \mathrm{cl}\,(R) : \underline{\mathrm{Div}}_Y\,(R) &\longrightarrow \underline{\mathrm{Pic}}_Y\,(R) \\ \mathcal{D} &\longmapsto \mathcal{O}\,(\mathcal{D}) \quad \mathrm{mod}\ \mathrm{Pic}\,(\mathrm{Spec}\,R) \end{aligned}$$

and

$$\mathrm{Inf}\,(\mathrm{cl})\,(R) = \mathrm{Lie}\,(\mathrm{cl}) \otimes \mathrm{Nil}\,(R)$$

where

$$\mathrm{Lie}\,(\mathrm{cl}) : \mathrm{Lie}\,(\underline{\mathrm{Div}}_Y) \longrightarrow \mathrm{Lie}\,(\underline{\mathrm{Pic}}_Y)$$

is just the connecting homomorphism

$$\mathrm{H}^0\,(\mathcal{K}_Y/\,\mathcal{O}_Y) \longrightarrow \mathrm{H}^1\,(\mathcal{O}_Y)$$

in the long exact cohomology sequence of

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{K}_Y \longrightarrow \mathcal{K}_Y/\,\mathcal{O}_Y \longrightarrow 0$$

Definition 2.37 *Let*

$$\underline{\mathrm{Div}}^0_Y : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

be the sub-functor of $\underline{\mathrm{Div}}_Y$ defined by

$$\underline{\mathrm{Div}}^0_Y\,(R) = \mathrm{cl}^{-1}\left(\underline{\mathrm{Pic}}^0_Y\,(R)\right)$$

for each finitely generated k-algebra R.

Remark 2.38 *Let R be a finitely generated k-algebra and $\mathcal{D} \in \underline{\mathrm{Div}}_Y\,(R)$. From the definition of $\underline{\mathrm{Pic}}^0_Y$ it follows that $\mathcal{D} \in \underline{\mathrm{Div}}^0_Y\,(R)$ if and only if $\mathcal{D}_p := \underline{\mathrm{Div}}_Y\,(\rho_p)\,(\mathcal{D}) \in \underline{\mathrm{Div}}^0_Y\,(k(p))$ for all $p \in \mathrm{Spec}\,R$, where $\rho_p : R \longrightarrow k(p)$, i.e. if and only if $\mathcal{O}\,(\mathcal{D}_p) \in \underline{\mathrm{Pic}}^0_Y\,(k(p))$ for all $p \in \mathrm{Spec}\,R$.*

2.2.4 Deformations of a Divisor

Let R be a finitely generated k-algebra, $R_{\text{art}} = k\,[\text{Nil}\,(R)]$ the corresponding local Artinian ring.

The inclusion $Y \longrightarrow Y \times_k \operatorname{Spec} R_{\text{art}}$ corresponding to the augmentation $\mathcal{O}_Y \otimes_k R_{\text{art}} = \mathcal{O}_Y\,[\text{Nil}\,(R)] \longrightarrow \mathcal{O}_Y$, $\text{Nil}\,(R) \ni n \longmapsto 0$ gives via pull-back the following map of Cartier divisors:

Definition 2.39 *The* restriction from $Y \otimes R_{\text{art}}$ to Y *is the homomorphism*

$$\operatorname{res}_Y(R_{\text{art}}) = \underline{\text{Div}}_Y(\rho) : \begin{array}{rcl} \underline{\text{Div}}_Y(R_{\text{art}}) & \longrightarrow & \underline{\text{Div}}_Y(k) \\ \mathcal{D} & \longmapsto & \mathcal{D}_0 \end{array}$$

induced by $\rho : R_{\text{art}} \longrightarrow k, \quad \text{Nil}\,(R) \ni n \longmapsto 0.$

Likewise the projection $Y \times_k \operatorname{Spec} R_{\text{art}} \longrightarrow Y$ corresponding to $\mathcal{O}_Y \longrightarrow \mathcal{O}_Y \otimes R_{\text{art}}$, $f \longmapsto f \otimes 1$ yields a map of Cartier divisors:

Definition 2.40 *The* constant extension from Y to $Y \otimes R_{\text{art}}$ *is the homomorphism*

$$\operatorname{const}_Y(R_{\text{art}}) = \underline{\text{Div}}_Y(\iota) : \begin{array}{rcl} \underline{\text{Div}}_Y(k) & \longrightarrow & \underline{\text{Div}}_Y(R_{\text{art}}) \\ D & \longmapsto & D_{\text{const}} \end{array}$$

induced by $\iota : k \hookrightarrow R_{\text{art}}$.

Now we can make precise the title of this subsubsection:

Definition 2.41 *A* deformation of a Cartier divisor D on Y along R_{art} *is a Cartier divisor* $\mathcal{D}$ *on* $Y \otimes R_{\text{art}}$ *whose restriction to* Y *is* D*, i.e.* $\mathcal{D}_0 = D$.

If D *is an effective divisor on* Y*, a deformation* $\mathcal{D}$ *of* D *is called an* effective deformation *if* $\mathcal{D}$ *is an effective divisor on* $Y \otimes R_{\text{art}}$.

The composition of the two maps above gives rise to a transformation from divisors to deformations of the zero divisor:

Proposition 2.42 *Let*

$$\underline{\text{Div}}_Y^{\text{art}} : \mathbf{Art}/k \longrightarrow \mathbf{Ab}$$

be the restriction of $\underline{\text{Div}}_Y$ *to the category of local Artinian* k*-algebras.*
There is a natural transformation

$$\inf_Y : \underline{\text{Div}}_Y^{\text{art}} \longrightarrow \operatorname{Inf}\left(\underline{\text{Div}}_Y^{\text{art}}\right)$$

given by

$$\inf_Y = \mathrm{id} - \mathrm{const}_Y \circ \mathrm{res}_Y$$

i.e. for each local Artinian k-algebra R_{art}

$$\begin{array}{rrcl} \inf_Y(R_{\mathrm{art}}) : & \underline{\mathrm{Div}}_Y^{\mathrm{art}}(R_{\mathrm{art}}) & \longrightarrow & \mathrm{Inf}\left(\underline{\mathrm{Div}}_Y^{\mathrm{art}}\right)(R_{\mathrm{art}}) \\ & \mathcal{D} & \longmapsto & \mathcal{D} - (\mathcal{D}_0)_{\mathrm{const}} \end{array}$$

Proof. Straightforward. ∎

Proposition 2.43 *Let D be an effective Cartier divisor on Y. Then the assignment*

$$R_{\mathrm{art}} \longmapsto \{\textit{effective deformations of } D \textit{ along } R_{\mathrm{art}}\}$$

defines a covariant functor

$$\underline{\mathrm{Def}}_{Y,D}^{\mathrm{eff}} : \mathbf{Art}/k \longrightarrow \mathbf{Set}$$

from the category of local Artinian k-algebras to the category of sets.

Proof. $\underline{\mathrm{Def}}_{Y,D}^{\mathrm{eff}}$ is a subfunctor of the composition of $\underline{\mathrm{Div}}_Y^{\mathrm{art}}$ with the forgetful functor $\mathbf{Ab} \longrightarrow \mathbf{Set}$. It is well defined since for each homomorphism $h : R_{\mathrm{art}} \longrightarrow S_{\mathrm{art}}$ of local Artinian rings the following diagram commutes

$$\begin{array}{ccc} R_{\mathrm{art}} & \longrightarrow & S_{\mathrm{art}} \\ & \searrow \quad \swarrow & \\ & k & \end{array}$$

hence the functoriality of $\underline{\mathrm{Div}}_Y^{\mathrm{art}}$ implies a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Div}}_Y^{\mathrm{art}}(R_{\mathrm{art}}) & \xrightarrow{\underline{\mathrm{Div}}_Y^{\mathrm{art}}(h)} & \underline{\mathrm{Div}}_Y^{\mathrm{art}}(S_{\mathrm{art}}) \\ & {\scriptstyle \mathrm{res}_Y(R_{\mathrm{art}})} \searrow \quad \swarrow {\scriptstyle \mathrm{res}_Y(S_{\mathrm{art}})} & \\ & \underline{\mathrm{Div}}_Y^{\mathrm{art}}(k) & \end{array}$$

i.e. $\underline{\mathrm{Div}}_Y^{\mathrm{art}}(h)$ maps effective deformations of D along R_{art} to those along S_{art}. ∎

As $\underline{\mathrm{Div}}_Y$ is a plain functor (see Definition 2.13), it is sufficient to consider only deformations along $k[\varepsilon]$. Therefore in the following by a *deformation* always a *deformation along $k[\varepsilon]$* is meant.

Definition 2.44 *Let D be an effective Cartier divisor on Y. Then let*

$$\mathrm{Def}_Y(D) = \underline{\mathrm{Def}}^{\mathrm{eff}}_{Y,D}(k[\varepsilon])$$

be the set of effective deformations of D (along $k[\varepsilon]$).

Proposition 2.45 *Let D be an effective Cartier divisor on Y. Then* $\inf_Y(k[\varepsilon]) : \underline{\mathrm{Div}}_Y(k[\varepsilon]) \longrightarrow \mathrm{Lie}(\underline{\mathrm{Div}}_Y)$ *induces a bijection*

$$\mathrm{Def}_Y(D) \simeq \Gamma(Y, \mathcal{O}_Y(D)/\mathcal{O}_Y)$$

This gives $\mathrm{Def}_Y(D)$ *the structure of a k-vector space. In particular,* $\inf_Y(\mathrm{Def}_Y(D)) \subset \mathrm{Lie}(\underline{\mathrm{Div}}_Y)$ *is a k-linear subspace which is of finite dimension, if Y is projective.*

Proof. The restriction of the homomorphism $\inf_Y(k[\varepsilon])$ to the subset $\mathrm{Def}_Y(D)$ of the group $\underline{\mathrm{Div}}_Y(k[\varepsilon])$ is the map (of sets)

$$\begin{array}{ccccc} \mathrm{Def}_Y(D) & \longrightarrow & \mathrm{Lie}(\underline{\mathrm{Div}}_Y) & \xrightarrow{\sim} & \Gamma(\mathcal{K}_Y/\mathcal{O}_Y) \\ (f_\alpha + \varepsilon g_\alpha)_\alpha & \longmapsto & \left(1 + \varepsilon \frac{g_\alpha}{f_\alpha}\right)_\alpha & \longmapsto & \left(\frac{g_\alpha}{f_\alpha}\right)_\alpha \end{array}$$

where $f_\alpha, g_\alpha \in \mathcal{O}_Y(U_\alpha)$ for some affine open covering $(U_\alpha)_\alpha$ of Y, as $(f_\alpha + \varepsilon g_\alpha)_\alpha$ defines an effctive divisor on $Y[\varepsilon] = Y \times_k \mathrm{Spec}\, k[\varepsilon]$, and $(f_\alpha)_\alpha$ necessarily defines D. Therefore we see that the image consists of sections which have poles at most along $D \subset Y$, i.e. $\inf_Y(\mathrm{Def}_Y(D)) = \Gamma(\mathcal{O}_Y(D)/\mathcal{O}_Y)$. If $\mathcal{D}, \mathcal{D}' \in \mathrm{Def}_Y(D)$ are two effective deformations of D which have the same image under $\inf_Y(k[\varepsilon])$, then $\mathcal{D} - \mathcal{D}' \in \ker(\inf_Y(k[\varepsilon]))$. Since $(\mathcal{D})_0 = (\mathcal{D}')_0 = D$ we have $\mathcal{D} - \mathcal{D}' \in \mathrm{Lie}(\underline{\mathrm{Div}}_Y)$. But $\inf_Y(k[\varepsilon])$ is injective on $\mathrm{Lie}(\underline{\mathrm{Div}}_Y)$, hence $\mathcal{D} = \mathcal{D}'$. Thus $\mathrm{Def}_Y(D) \longrightarrow \Gamma(\mathcal{O}_Y(D)/\mathcal{O}_Y)$ is also injective. ■

Definition 2.46 *Let D be an effective Cartier divisor on Y. Then let*

$$\mathrm{Inf}_{Y,D} = \inf_Y(\mathrm{Def}_Y(D))$$

be the k-linear subspace of $\mathrm{Lie}(\underline{\mathrm{Div}}_Y)$ *given by the image of the effective deformations of D.*

2.3 Relative Weil divisors

Let Y be a scheme over k (an algebraically closed field of characteristic 0).

2.3.1 Functor of Relative Weil Divisors

A *prime cycle* on a k-scheme X is a closed reduced and irreducible subscheme. A *cycle* is a formal linear combination with integral coefficients of prime cycles. A *prime divisor* is a prime cycle of codimension 1. A *Weil divisor* is a cycle of codimension 1.

$$\mathrm{WDiv}(X) = \mathrm{Z}^1(X)$$

is the *group of Weil divisors.*

The *push-forward of cycles* is defined as follows (cf. [F] 1.4):
Let $\pi : Y \longrightarrow X$ be a proper morphism of k-schemes. Let V be a prime cycle on Y and $W = \pi(V)$ the image in X. Set

$$\deg(V/W) = \begin{cases} [\mathrm{K}_V : \mathrm{K}_W] & \text{if} \quad \dim W = \dim V \\ 0 & \text{if} \quad \dim W < \dim V \end{cases}$$

The push-forward of V under π is defined to be

$$\pi_* V = \deg(V/W)\, W$$

The push-forward of arbitrary cycles is then given by linear extension.

Definition 2.47 *The category* $\mathbf{Alg}^{\mathrm{hf}}/k$ *is defined to be the category whose objects are finitely generated* k*-algebras and whose morphisms are given by those homomorphisms of* k*-algebras* $h : A \longrightarrow B$ *such that* B *is a finite* A*-module.*

Proposition 2.48 *For a finitely generated* k*-algebra* R *let*

$$\underline{\mathrm{WDiv}}_Y(R) = \left\{ \begin{array}{c} \textit{Weil divisors } \mathcal{W} \textit{ on } Y \times_k \operatorname{Spec} R \\ \textit{s.t. } \mathcal{W}_p \textit{ is a Weil divisor on } Y \times \{p\} \\ \forall p \in \operatorname{Spec} R \end{array} \right\}$$

where $\mathcal{W}_p = \mathcal{W} \times_{\operatorname{Spec} R} \operatorname{Spec} k(p)$ *is the fibre over* $p \in \operatorname{Spec} R$.
Then the assignment $R \longmapsto \underline{\mathrm{WDiv}}_Y(R)$ *defines a contravariant functor*

$$\underline{\mathrm{WDiv}}_Y : \mathbf{Alg}^{\mathrm{hf}}/k \longrightarrow \mathbf{Ab}$$

from the category of finitely generated k*-algebras with finite homomorphisms to the category of abelian groups.*

Proof. For a finite homomorphism $h : R \longrightarrow S$ of finitely generated k-algebras with corresponding morphism $h^\dagger : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$, $q \longmapsto h^{-1}(q)$ of affine schemes the required homomorphism $\underline{\mathrm{WDiv}}_Y(h) : \underline{\mathrm{WDiv}}_Y(S) \longrightarrow \underline{\mathrm{WDiv}}_Y(R)$ of abelian groups is obtained from the push-forward of cycles $\left(\mathrm{id}_Y \times h^\dagger\right)_* : \mathrm{WDiv}\left(Y \times_k \operatorname{Spec} S\right) \longrightarrow \mathrm{WDiv}\left(Y \times_k \operatorname{Spec} R\right)$:

Let $\mathcal{W}$ be a Weil divisor on $Y \times_k \operatorname{Spec} S$. Since S is a finite R-module, for each $p \in \operatorname{Spec} R$ there are only finitely many $q \in \operatorname{Spec} S$ with $h^{-1}(q) = p$. We have $\left(\left(\mathrm{id}_Y \times h^\dagger\right)_* \mathcal{W}\right)_p = \sum_{h^{-1}(q)=p} \mathcal{W}_q$ and this is a Weil divisor on $Y \times \{p\}$ if $\mathcal{W}_q$ is one on $Y \times \{q\}$ for each $q \in \operatorname{Spec} S$. ∎

Remark 2.49 *As prime divisors are always reduced,* $\underline{\mathrm{WDiv}}_Y$ *is a reduced functor (see Definition 2.6).*

Proposition 2.50 *Let* $\pi : Y \longrightarrow X$ *be a finite morphism of* k*-schemes. Then* π *induces a natural transformation of functors, the* push-forward

$$\pi_* : \underline{\mathrm{WDiv}}_Y \longrightarrow \underline{\mathrm{WDiv}}_X$$

which is given by the push-forward of cycles (as defined at the beginning of this Subsubsection 2.3.1).

Proof. For each finitely generated k-algebra R the transformation $\pi_*(R) : \underline{\mathrm{WDiv}}_Y(R) \longrightarrow \underline{\mathrm{WDiv}}_X(R)$ is given by the push-forward of cycles $\left(\pi \times \mathrm{id}_R^\dagger\right)_* : \mathrm{WDiv}\left(Y \times_k \operatorname{Spec} R\right) \longrightarrow \mathrm{WDiv}\left(X \times_k \operatorname{Spec} R\right)$. This is well defined since for $\mathcal{W} \in \mathrm{WDiv}\left(Y \times_k \operatorname{Spec} R\right)$ and for each $p \in \operatorname{Spec} R$ we have $\left(\left(\pi \times \mathrm{id}_R^\dagger\right)_* \mathcal{W}\right)_p = \pi_* \mathcal{W}_p$, and this is a Weil divisor on $X \times \{p\}$ if $\mathcal{W}_p$ is one on $Y \times \{p\}$.

Let $h : R \longrightarrow S$ be a finite homomorphism of k-algebras, $h^\dagger : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ the corresponding morphism of affine schemes. The functoriality of Z^1 with respect to the push-forward yields $\psi_* \chi_* = (\psi\chi)_*$ for any two morphisms ψ and χ, therefore

$$\left(\pi \times \mathrm{id}_R^\dagger\right)_* \left(\mathrm{id}_Y \times h^\dagger\right)_* = \left(\pi \times h^\dagger\right)_* = \left(\mathrm{id}_X \times h^\dagger\right)_* \left(\pi \times \mathrm{id}_S\right)_*$$

which gives the commutativity of the diagram

$$\begin{array}{ccc} \underline{\mathrm{WDiv}}_Y(S) & \xrightarrow{\left(\pi \times \mathrm{id}_S^\dagger\right)_*} & \underline{\mathrm{WDiv}}_X(S) \\ {\scriptstyle \left(\mathrm{id}_Y \times h^\dagger\right)_*} \downarrow & & \downarrow {\scriptstyle \left(\mathrm{id}_X \times h^\dagger\right)_*} \\ \underline{\mathrm{WDiv}}_Y(R) & \xrightarrow{\left(\pi \times \mathrm{id}_R^\dagger\right)_*} & \underline{\mathrm{WDiv}}_X(R) \end{array}$$

∎

Proposition 2.51 *There is a transformation of functors*

$$\mathrm{weil} : \mathrm{Red}\left(\underline{\mathrm{Div}}_Y\right) \longrightarrow \underline{\mathrm{WDiv}}_Y$$

If Y is normal, then $\mathrm{weil}(R) : \mathrm{Red}\left(\underline{\mathrm{Div}}_Y\right)(R) \longrightarrow \underline{\mathrm{WDiv}}_Y(R)$ *is injective for all $R \in \mathbf{Alg}/k$.*

Proof. Let R be a finitely generated k-algebra. Then the transformation $\mathrm{weil}(R) : \underline{\mathrm{Div}}_Y(R_{\mathrm{red}}) \longrightarrow \underline{\mathrm{WDiv}}(R)$ associates to a Cartier divisor $\mathcal{D}$ on $Y \times_k \mathrm{Spec}\, R_{\mathrm{red}}$ the following Weil divisor on $Y \times_k \mathrm{Spec}\, R$ (cf. [F] 2.1):

$$\mathrm{weil}(\mathcal{D}) = \sum_{\mathrm{codim}\,\mathcal{V}=1} \mathrm{ord}_{\mathcal{V}}(\mathcal{D})\ \mathcal{V}$$

where the sum runs over all prime Weil divisors $\mathcal{V}$ in $Y \times_k \mathrm{Spec}\, R$, and $\mathrm{ord}_{\mathcal{V}}$ is the order function on $\mathcal{K}^*_{Y\times_k \mathrm{Spec}\, R} / \mathcal{O}^*_{Y\times_k \mathrm{Spec}\, R}$ defined by $\mathcal{V}$ (see [F] 1.2). Note that the definition of $\underline{\mathrm{Div}}_Y$ implies that $\mathrm{weil}(\mathcal{D})$ does not have any vertical component. Thus the transformation is well defined.

For the second assertion we observe that on a normal scheme Y the Cartier divisors are identified with the locally principal Weil divisors (see [H] Chapter II, Remark 6.11.2), i.e. the transformation $\mathrm{weil}(k) : \underline{\mathrm{Div}}_Y(k) \longrightarrow \underline{\mathrm{WDiv}}_Y(k)$ is injective. Now for an arbitrary finitely generated k-algebra R consider the diagram

$$\begin{array}{ccc} \underline{\mathrm{Div}}_Y(R_{\mathrm{red}}) & \xrightarrow{\mathrm{weil}(R)} & \underline{\mathrm{WDiv}}_X(R) \\ \downarrow & & \downarrow \\ \prod_p \underline{\mathrm{Div}}_Y(k(p)) & \xrightarrow{\prod \mathrm{weil}(k(p))} & \prod_p \underline{\mathrm{WDiv}}_X(k(p)) \end{array}$$

where the products in the bottom line range over all points $p \in \mathrm{Max}\, R = \underline{\mathrm{Spec}\, R}(k)$ in the maximal Spectrum of R. The map in the left column is the product of the homomorphisms obtained from the residue maps $R_{\mathrm{red}} \longrightarrow k(p)$ by functoriality of $\underline{\mathrm{Div}}_Y$, and the map in the right column is the product of the maps $\mathcal{W} \longmapsto \mathcal{W}_p = \mathcal{W} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, k(p)$. Then the diagram commutes by construction of the fibre-product (see [H] Chapter II, Theorem 3.3). The vertical arrows are injective and the arrow in the bottom line is also by the observation above. Hence so is the arrow in the top line. ■

Definition 2.52 *The restriction of* $\mathrm{weil} : \mathrm{Red}\left(\underline{\mathrm{Div}}_Y\right) \longrightarrow \underline{\mathrm{WDiv}}_Y$ *to* $\mathrm{Red}\left(\underline{\mathrm{Div}}^0_Y\right)$ *is denoted by*

$$\mathrm{weil}^0 : \mathrm{Red}\left(\underline{\mathrm{Div}}^0_Y\right) \longrightarrow \underline{\mathrm{WDiv}}_Y$$

2.3.2 Vanishing Weil Divisors

Definition 2.53 *A finite morphism $\pi : Y \longrightarrow X$ of k-schemes is said to be* of degree 1*, if for each irreducible component V of Y with generic point γ_V the degree of the field extension is $[k(\gamma_V) : k(\pi(\gamma_V))] = 1$, and there is exactly one irreducible component of Y lying above each irreducible component W of X, i.e. the generic fibres have cardinality $\#\,\pi^{-1}(\xi_W) = 1$, where ξ_W is the generic point of W.*

Remark 2.54 *A finite morphism $\pi : Y \longrightarrow X$ of degree 1 is characterized by the following property: The normalization $\nu : \widetilde{X} \longrightarrow X$ of X factors uniquely through $\pi : Y \longrightarrow X$, i.e. there exists a unique $\mu : \widetilde{X} \longrightarrow Y$ such that the following diagram commutes:*

$$\widetilde{X} \xrightarrow{\ \mu\ } Y \xrightarrow{\ \pi\ } X, \qquad \nu = \pi \circ \mu : \widetilde{X} \longrightarrow X$$

Proposition 2.55 *Let $\pi : Y \longrightarrow X$ be a finite morphism of k-schemes. Then*

$$\underline{\mathrm{WDiv}}_{Y/X} = \ker\left(\pi_* : \underline{\mathrm{WDiv}}_Y \longrightarrow \underline{\mathrm{WDiv}}_X\right)$$

defines a contravariant functor

$$\underline{\mathrm{WDiv}}_{Y/X} : \mathbf{Alg}^{\mathrm{hf}}/k \longrightarrow \mathbf{Ab}$$

from the category of finitely generated k-algebras with finite homomorphisms to the category of abelian groups, which is called the functor of vanishing Weil divisors on Y relative to X.

Proof. $\underline{\mathrm{WDiv}}_{Y/X}$ is a subfunctor of $\underline{\mathrm{WDiv}}_Y$. It is well defined since $\pi_* : \underline{\mathrm{WDiv}}_Y \longrightarrow \underline{\mathrm{WDiv}}_X$ is a natural transformation. ■

Proposition 2.56 *Let $\pi : Y \longrightarrow X$ be a finite morphism of k-schemes of degree 1 (see Definition 2.53). Then $\underline{\mathrm{WDiv}}_{Y/X}$ is locally constant (see Definition 2.9) and becomes also a* covariant *functor*

$$\underline{\mathrm{WDiv}}_{Y/X} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

from the category of finitely generated k-algebras to the category of abelian groups. Moreover, $\underline{\mathrm{WDiv}}_{Y/X}$ is represented by a lattice $\Lambda \cong \mathbb{Z}^t$.

Proof. A finite morphism of degree 1 is birational, i.e. there is an open dense subscheme $U \subset Y$ such that $\pi|_U : U \longrightarrow \pi(U)$ is an isomorphism. Set $S := Y \setminus U$. Then for each $W \in \underline{\mathrm{WDiv}}_{Y/X}(k)$ it holds $\mathrm{Supp}(W) \subset S$. The number of irreducible components of S is finite, in particular there are only finitely many irreducible components of codimension 1 in Y, i.e. there are only finitely many prime divisors supported on S. Therefore $\underline{\mathrm{WDiv}}_{Y/X}(k)$ is a subgroup of a free abelian group of finite rank, hence also free abelian of finite rank.

Let R be a finitely generated k-algebra with $\mathrm{Spec}\, R$ connected, let $\mathcal{W} = \mathcal{E} - \mathcal{E}' \in \underline{\mathrm{WDiv}}_{Y/X}(R)$, where $\mathcal{E}, \mathcal{E}' \in \underline{\mathrm{WDiv}}_Y(R)$ are effective divisors. There is a dense open subscheme $Q \subset \mathrm{Spec}\, R$ such that $\mathcal{W}_p \in \underline{\mathrm{WDiv}}_{Y/X}(k(p))$ for all $p \in Q$. Let $P \subset Q$ be the open subscheme where $\mathcal{E}_Q \longrightarrow Q$ is flat. The Hilbert functor $\underline{\mathrm{Hilb}}_Y$ is represented by a scheme Hilb_Y. Let $\mathcal{H} \subset Y \times_k \mathrm{Hilb}_Y$ be the universal closed subscheme. By the universal property of $(\mathrm{Hilb}_Y, \mathcal{H})$ there is a unique morphism $\eta : P \longrightarrow \mathrm{Hilb}_Y$ such that $\mathcal{E}_P = (\mathrm{id}_Y \times \eta)^* \mathcal{H}$. Then η maps $\eta : p \longmapsto \mathcal{E}_p \in \underline{\mathrm{Hilb}}_Y(k(p))$. For each closed point $p \in P$ we have $\mathcal{W}_p \in \mathrm{WDiv}_{Y/X}(k)$, thus η is a continuous map from a connected space to a discrete set, hence η is constant. Since the prime divisors of $\mathcal{E} \subset Y \times_k \mathrm{Spec}\, R$ are irreducible closed subschemes, we see that $P = \mathrm{Spec}\, R$. Applying the same argument to $\mathcal{E}'$ yields: The map $p \longmapsto \mathcal{W}_p$ is a constant map $\mathrm{Spec}\, R \longrightarrow \underline{\mathrm{WDiv}}_{Y/X}(k)$. Hence $\underline{\mathrm{WDiv}}_{Y/X}$ is locally constant of finite rank, i.e. it is represented by a lattice.

A locally constant functor always admits the trivial pull-back

$$\begin{array}{rcl} \underline{\mathrm{WDiv}}_{Y/X}(h) : \underline{\mathrm{WDiv}}_{Y/X}(R) & \longrightarrow & \underline{\mathrm{WDiv}}_{Y/X}(S) \\ W \times_k \mathrm{Spec}\, R & \longmapsto & W \times_k \mathrm{Spec}\, S \end{array}$$

for each homomorphism of finitely generated k-algebras $h : R \longrightarrow S$ with $\mathrm{Spec}\, R$ connected. Decomposing an affine scheme into its connected components yields the pull-back for arbitrary homomorphisms, which makes $\underline{\mathrm{WDiv}}_{Y/X}$ a covariant functor. ■

Proposition 2.57 *Let $\pi : Y \longrightarrow X$ be a finite morphism of k-schemes of degree 1 (see Definition 2.53). Then*

$$\underline{\mathrm{\acute{E}Div}}_{Y/X} = \mathrm{weil}^{-1}\, \underline{\mathrm{WDiv}}_{Y/X}$$

defines a covariant functor

$$\underline{\mathrm{\acute{E}Div}}_{Y/X} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

and a contravariant functor

$$\underline{\mathrm{\acute{E}Div}}_{Y/X} : \mathbf{Alg}^{\mathrm{hf}}/k \longrightarrow \mathbf{Ab}$$

Moreover, $\underline{\text{ÉDiv}}_{Y/X}$ *is locally constant (see Definition 2.9) and represented by a lattice.*

Proof. As $\underline{\text{WDiv}}_{Y/X}$ is covariant, weil : $\text{Red}\left(\underline{\text{Div}}_Y\right) \longrightarrow \underline{\text{WDiv}}_{Y/X}$ becomes a natural transformation of covariant functors. Thus $\underline{\text{ÉDiv}}_{Y/X}$ is a well defined covariant functor. Moreover, $\underline{\text{WDiv}}_{Y/X}$ is locally constant, hence $\underline{\text{ÉDiv}}_{Y/X}$ is also. Therefore $\underline{\text{Div}}_Y$ is compatible with the push-forward, which implies that $\underline{\text{ÉDiv}}_{Y/X}$ is also contravariant. Finally $\text{weil}(k)\big(\underline{\text{ÉDiv}}_{Y/X}(k)\big)$ is a subgroup of the free abelian group $\underline{\text{WDiv}}_{Y/X}(k)$ of finite rank, hence also free abelian of finite rank. Thus $\underline{\text{ÉDiv}}_{Y/X}$ is represented by a lattice. ■

Proposition 2.58 *Let* $\pi : Y \longrightarrow X$ *be a finite morphism of* k*-schemes of degree 1 (see Definition 2.53). Then*

$$\underline{\text{ÉDiv}}^0_{Y/X} = \left(\text{weil}^0\right)^{-1} \underline{\text{WDiv}}_{Y/X}$$

defines a covariant functor

$$\underline{\text{ÉDiv}}^0_{Y/X} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

and a contravariant functor

$$\underline{\text{ÉDiv}}^0_{Y/X} : \mathbf{Alg}^{\text{hf}}/k \longrightarrow \mathbf{Ab}$$

Moreover, $\underline{\text{ÉDiv}}^0_{Y/X}$ *is locally constant (see Definition 2.9) and represented by a lattice.*

Proof. Analogue to the proof of Proposition 2.57. ■

2.4 Formal Divisors

Let Y be a scheme over k (an algebraically closed field of characteristic 0).

2.4.1 Functor of Formal Infinitesimal Divisors

The functor of relative Cartier divisors $\underline{\text{Div}}_Y$ admits a pull-back, but not a push-forward of relative Cartier divisors. Supposed Y is a normal scheme, the reduced part $\text{Red}\left(\underline{\text{Div}}_Y\right)$ of $\underline{\text{Div}}_Y$ can be identified with the subfunctor of $\underline{\text{WDiv}}_Y$ consisting of locally principal relative Weil divisors, and there is a push-forward for this functor.

Therefore we are looking for a concept of infinitesimal divisors $\underline{\text{IDiv}}_Y$ which admits a push-forward and contains $\text{Inf}\left(\underline{\text{Div}}_Y\right)$ as a subfunctor under certain assumptions on the scheme Y.

Proposition 2.59 *Let $(\mathcal{A}, \mathfrak{m})$ be a Noetherian complete local ring with residue field $\kappa = \mathcal{A}/\mathfrak{m}$ of characteristic 0. Then $\mathcal{A}$ contains a system of representatives of κ which is a field.*

Proof. [Se] Chapter II, §4, Proposition 6. ■

Notation 2.60 *If $(\mathcal{A}, \mathfrak{m})$ is a Noetherian complete local ring with residue field $\kappa = \mathcal{A}/\mathfrak{m}$ of characteristic 0. Then we identify the system of representatives of κ in $\mathcal{A}$ with κ.*
If moreover R is a κ-algebra and $\mathcal{M}$ a complete $\mathcal{A}$-module, then we consider $\mathcal{M}$ as a κ-vector space and write

$$\mathcal{M}_R = \mathcal{M}\widehat{\otimes}_\kappa R$$

where $\mathcal{M}\widehat{\otimes}_\kappa R$ is the completion of $\mathcal{M} \otimes_\kappa R$ w.r.t. the $\mathfrak{m}$-topology on $\mathcal{M}$ and the discrete topology on κ (see [SGA3] VII_B, 0.5).

Proposition 2.61 *Define the set of* formal Lie divisors *on Y by*

$$\mathrm{LDiv}\,(Y) = \bigoplus_{\mathrm{ht}(\eta)=1} \mathrm{Hom}_{k(\eta),\mathrm{cont}}\,(\widehat{\mathfrak{m}}_\eta, \mathfrak{k}_\eta)$$

where the direct sum runs over all generic points η of height 1 in Y, where $\widehat{\mathfrak{m}}_\eta$ is the maximal ideal of the completion $\widehat{\mathcal{O}}_\eta$ of the local ring $\mathcal{O}_{Y,\eta}$ at η. Let $\mathcal{O}_{Y,\eta} \longrightarrow \widetilde{\mathcal{O}}_{Y,\eta}$ be the normalization of $\mathcal{O}_{Y,\eta}$, then $\mathfrak{k}_\eta = \bigoplus_{\widetilde{\eta}\to\eta} k(\widetilde{\eta})$ denotes the direct sum of all residue-fields at generic points $\widetilde{\eta} \in \mathrm{Spec}\,\widetilde{\mathcal{O}}_{Y,\eta}$ of height 1 lying over η. Endowe $\widehat{\mathcal{O}}_\eta$ with the $\widehat{\mathfrak{m}}_\eta$-adic topology, so that $\widehat{\mathfrak{m}}_\eta$ carries the induced topology, while $\mathfrak{k}_\eta$ carries the discrete topology. $\mathrm{Hom}_{k(\eta),\mathrm{cont}}$ denotes the $k(\eta)$-vector space of continuous $k(\eta)$-linear maps. For a finitely generated k-algebra R let

$$\underline{\mathrm{IDiv}}_Y\,(R) = \mathrm{LDiv}\,(Y) \otimes_k \mathrm{Nil}\,(R)$$

Then the assignment $R \longrightarrow \underline{\mathrm{IDiv}}_Y\,(R)$ defines a covariant functor

$$\underline{\mathrm{IDiv}}_Y : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

Moreover, $\underline{\mathrm{IDiv}}_Y$ is infinitesimal (see Definition 2.6) and plain (see Definition 2.13).

Proof. $\mathrm{LDiv}\,(Y)$ is well defined by Proposition 2.59 and Notation 2.60. The rest is immediate from the definitions. ■

Proposition 2.62 *Let $\pi : Y \longrightarrow X$ be a finite morphism of k-schemes of degree 1 (see Definition 2.53). Then π induces a natural transformation of functors, the* push-forward

$$\pi_* : \underline{\mathrm{IDiv}}_Y \longrightarrow \underline{\mathrm{IDiv}}_X$$

induced by the homomorphism of formal Lie divisors

$$\begin{array}{ccc} \bigoplus_{\mathrm{ht}(\eta)=1} \mathrm{Hom}_{k(\eta),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{Y,\eta}, \mathfrak{k}_{Y,\eta}\right) & \longrightarrow & \bigoplus_{\mathrm{ht}(\eta)=1} \mathrm{Hom}_{k(\pi(\eta)),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{X,\pi(\eta)}, \mathfrak{k}_{X,\pi(\eta)}\right) \\ \sum_{\mathrm{ht}(\eta)=1} h_\eta & \longmapsto & \sum_{\mathrm{ht}\,\eta=1} h_\eta \circ \widehat{\pi_\eta^\#} \end{array}$$

Proof. The homomorphism of structure sheaves $\pi^\# : \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ associated to π induces local homomorphisms of local rings $\pi_\eta^\# : \mathcal{O}_{X,\pi(\eta)} \longrightarrow \mathcal{O}_{Y,\eta}$, i.e. $\pi_\eta^\#\left(\mathfrak{m}_{X,\pi(\eta)}\right) \subset \mathfrak{m}_{Y,\eta}$, for each $\eta \in Y$. Denote by $\widehat{\pi_\eta^\#} : \widehat{\mathcal{O}}_{X,\pi(\eta)} \longrightarrow \widehat{\mathcal{O}}_{Y,\eta}$ the composition of the induced homomorphism of complete modules w.r.t. $\widehat{\mathfrak{m}}_{X,\pi(\eta)}$ with the natural homomorphism from the completion of $\mathcal{O}_{Y,\eta}$ w.r.t. $\widehat{\mathfrak{m}}_{X,\pi(\eta)}$ to the completion w.r.t. $\widehat{\mathfrak{m}}_{Y,\eta}$. The homomorphism $k(\pi(\eta)) \longrightarrow k(\eta)$ induced by $\pi_\eta^\#$ is a field extension and hence each $k(\eta)$-linear homomorphism is in particular $k(\pi(\eta))$-linear. Since $\pi : Y \longrightarrow X$ is finite of degree 1, the normalization $\widetilde{X} \longrightarrow X$ of X factors through Y, i.e. $\widetilde{X} \longrightarrow Y$ is the normalization of Y. This yields a canonical inclusion $\mathfrak{k}_{Y,\eta} \subset \mathfrak{k}_{X,\pi(\eta)}$. Thus the transformation

$$\mathrm{Lie}\,(\pi_*) : \mathrm{LDiv}\,(Y) \longrightarrow \mathrm{LDiv}\,(X)$$

is obtained from the homomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{k(\eta),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{Y,\eta}, \mathfrak{k}_{Y,\eta}\right) & \longrightarrow & \mathrm{Hom}_{k(\pi(\eta)),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{X,\pi(\eta)}, \mathfrak{k}_{X,\pi(\eta)}\right) \\ h & \longmapsto & h \circ \widehat{\pi_\eta^\#} \end{array}$$

As both functors are infinitesimal and plain, this extends canonically to a natural transformation. ■

Proposition 2.63 *Let Y be a normal k-scheme. Then there is a natural transformation of functors*

$$\mathrm{fml} : \mathrm{Inf}\,(\underline{\mathrm{Div}}_Y) \longrightarrow \underline{\mathrm{IDiv}}_Y$$

with $\mathrm{fml}\,(R) : \mathrm{Inf}\,(\underline{\mathrm{Div}}_Y)\,(R) \longrightarrow \underline{\mathrm{IDiv}}_Y\,(R)$ *injective for all* $R \in \mathbf{Alg}/k$.
If in addition $Y = C$ is a curve, then this transformation is an isomorphism of functors.

Proof. Both functors in question are infinitesimal and plain, therefore it suffices to show $\mathrm{Lie}\,(\underline{\mathrm{Div}}_Y) \hookrightarrow \mathrm{LDiv}\,(Y)$, i.e. to give an injective homomorphism

$$\Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right) \longrightarrow \bigoplus_{\mathrm{ht}(\eta)=1} \mathrm{Hom}_{k(\eta),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_\eta, k(\eta)\right)$$

We construct this homomorphism via factorization, i.e. give homomorphisms

$$\Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right) \longrightarrow \bigoplus_{\mathrm{ht}(\eta)=1} \left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)_\eta \longrightarrow \bigoplus_{\mathrm{ht}(\eta)=1} \mathrm{Hom}_{k(\eta),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_\eta, k(\eta)\right)$$

Consider the natural linear map

$$\begin{array}{ccc} \Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right) & \longrightarrow & \bigoplus_{\mathrm{ht}(\eta)=1} \left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)_\eta \\ \delta & \longmapsto & \sum_{\mathrm{ht}(\eta)=1} [\delta]_\eta \end{array}$$

This map is injective, since every non-zero $\delta \in \Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)$ determines a non-zero effective divisor by the locus of poles of its local sections. Thus if $[\delta]_\eta = 0 \in \left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)_\eta$ for all generic points η of codimension 1, then δ has support only in codimension ≥ 2 and is therefore zero. Moreover, as Y is normal, for each generic point η of height 1 the local ring $\mathcal{O}_{Y,\eta}$ is a discrete valuation ring. Hence if t_η is a local parameter of the maximal ideal $\mathfrak{m}_\eta \subset \mathcal{O}_{Y,\eta}$ it holds

$$\widehat{\mathcal{O}}_\eta = k(\eta)\,[[t_\eta]] \qquad\qquad \widehat{\mathfrak{m}}_\eta = t_\eta \cdot k(\eta)\,[[t_\eta]]$$

and

$$\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)_\eta \cong \bigcup_{\nu>0} t_\eta^{-\nu}\,\widehat{\mathcal{O}}_\eta\Big/\,\widehat{\mathcal{O}}_\eta$$

According to Lemma 2.64 below we have an isomorphism of $k(\eta)$-vector spaces

$$\begin{array}{ccc} \bigcup_{\nu>0} t_\eta^{-\nu}\,\widehat{\mathcal{O}}_\eta\Big/\,\widehat{\mathcal{O}}_\eta & \stackrel{\sim}{\longrightarrow} & \mathrm{Hom}_{k(\eta),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_\eta, k(\eta)\right) \\ \overline{f} & \longmapsto & \mathrm{Res}_\eta\left(f \cdot \mathrm{d}\_\right) \end{array}$$

where $f \in t_\eta^{-\nu}\widehat{\mathcal{O}}_\eta$ is a representative of $\overline{f}$ and $\mathrm{Res}_\eta : \Omega_{\widehat{\mathcal{O}}_\eta} \longrightarrow k(\eta)$ is the residue at η and $\mathrm{d} : \widehat{\mathcal{O}}_\eta \longrightarrow \Omega_{\widehat{\mathcal{O}}_\eta}$ the universal derivation. Then $\mathrm{Res}_\eta\left(f\ \mathrm{d}g\right)$ is independent of the choice of the representative of $\overline{f}$ in $t_\eta^{-\nu}\widehat{\mathcal{O}}_\eta$ for all $g \in \widehat{\mathfrak{m}}_\eta$.

Now assume that $Y = C$ is a curve. Then every prime ideal of height 1 is maximal and two distinct prime ideals are coprime. Using the Chinese remainder theorem one shows that the map

$$\Gamma\left(\mathcal{K}_C / \mathcal{O}_C\right) \xrightarrow{\sim} \bigoplus_{p \in C(k)} \left(\mathcal{K}_C / \mathcal{O}_C\right)_p$$

is an isomorphism. As shown above,

$$\bigoplus_{p \in C(k)} \left(\mathcal{K}_C / \mathcal{O}_C\right)_p \xrightarrow{\sim} \bigoplus_{p \in C(k)} \mathrm{Hom}_{k,\mathrm{cont}}\left(\widehat{\mathfrak{m}}_p, k\right)$$

is an isomorphism. This yields the second assertion. ■

Lemma 2.64 *Let* $(\mathcal{A}, \mathfrak{m})$ *be a complete local ring, endowed with the* $\mathfrak{m}$*-adic topology,* $\kappa = \mathcal{A}/\mathfrak{m}$ *its residue field, endowed with the discrete topology, and* $\mathcal{K} = \mathrm{Q}(\mathcal{A})$ *the quotient field of* $\mathcal{A}$. *Let* $l \in \mathrm{Hom}_\kappa(\mathfrak{m}, \kappa)$ *be a* κ*-linear map. Then the following conditions are equivalent:*

(i) l *is continuous*
(ii) $\ker(l)$ *is open*
(iii) $\ker(l) \supset \mathfrak{m}^\nu$ *for some* $\nu > 0$
(iv) $l \in \mathrm{Hom}_\kappa(\mathfrak{m}/\mathfrak{m}^\nu, \kappa)$ *for some* $\nu > 0$

If furthermore $\mathcal{A}$ *is a discrete valuation ring and* κ *of characteristic 0, this is equivalent to*

(v) $l = \mathrm{Res}\,(f \cdot \mathrm{d}\_) : g \longmapsto \mathrm{Res}\,(f \cdot \mathrm{d}g)$ *for some* $f \in t^{-\nu}\mathcal{A}/\mathcal{A}$, $\nu \geq 0$ *where* t *is a local parameter of* $\mathfrak{m}$, $\mathrm{Res} : \Omega_{\mathcal{K}/\kappa} \longrightarrow \kappa$ *the residue and* $\mathrm{d} : \mathcal{A} \longrightarrow \Omega_{\mathcal{A}/\kappa}$ *the universal derivation*

Proof. (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii) is an application of Lemma 1.22 to the $\mathfrak{m}$-adic situation.

(iii)$\Longleftrightarrow$(iv) $\mathrm{Hom}_\kappa(\mathfrak{m}/\mathfrak{m}^\nu, \kappa) = \ker\big(\mathrm{Hom}_\kappa(\mathfrak{m}, \kappa) \longrightarrow \mathrm{Hom}_\kappa(\mathfrak{m}^\nu, \kappa)\big)$

(iv)$\Longleftrightarrow$(v) If $(\mathcal{A}, \mathfrak{m})$ is a complete discrete valuation ring, κ of characteristic 0 and t a local parameter of $\mathfrak{m}$, then $\mathcal{A} \cong \kappa[[t]]$ and $\mathcal{K} \cong \kappa((t))$. Let $\overline{\kappa}$ denote the algebraic closure of κ. Then the residue over $\overline{\kappa}$ is given by

$$\begin{aligned} \mathrm{Res} : \ \Omega_{\mathcal{K}_{\overline{\kappa}}/\overline{\kappa}} & \longrightarrow \ \overline{\kappa} \\ \sum_{\nu \gg -\infty} a_\nu t^\nu \,\mathrm{d}t & \longmapsto \ a_{-1} \end{aligned}$$

and this is independent of the choice of the local parameter t (see [S] Chapter II, No. 7, Proposition 5). This implies that the image of $\Omega_{\mathcal{K}/\kappa}$ lies in κ.

$\mathrm{d} : \mathcal{A} \longrightarrow \Omega_{\mathcal{A}/\kappa}$ and $\mathrm{Res} : \Omega_{\mathcal{K}/\kappa} \longrightarrow \kappa$ are both κ-linear maps. Since $\mathrm{Res}\,(\omega) = 0$ for all $\omega \in \Omega_{\mathcal{A}/\kappa}$, the expression $\mathrm{Res}\,(f\ \mathrm{d}g)$ is well defined for $g \in \mathfrak{m}/\mathfrak{m}^{\nu+1}$ and $f \in t^{-\nu}\mathcal{A}/\mathcal{A}$.

The pairing $t^{-\nu}\mathcal{A}/\mathcal{A} \times \mathfrak{m}/\mathfrak{m}^{\nu+1} \longrightarrow \kappa, \quad (f,g) \longmapsto \mathrm{Res}\,(f\ \mathrm{d}g)$ is a perfect pairing, hence $t^{-\nu}\mathcal{A}/\mathcal{A} \xrightarrow{\sim} \mathrm{Hom}_\kappa\,(\mathfrak{m}/\mathfrak{m}^{\nu+1}, \kappa)\,, \quad f \longmapsto \mathrm{Res}\,(f\ \mathrm{d}\_)$ is an isomorphism. ■

2.4.2 Vanishing Formal Infinitesimal Divisors

Proposition 2.65 *Let $\pi : Y \longrightarrow X$ be a finite morphism of k-schemes of degree 1 (see Definition 2.53). Then*

$$\underline{\mathrm{IDiv}}_{Y/X} = \ker\left(\pi_* : \underline{\mathrm{IDiv}}_Y \longrightarrow \underline{\mathrm{IDiv}}_X\right)$$

defines a plain infinitesimal covariant functor

$$\underline{\mathrm{IDiv}}_{Y/X} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

from the category of finitely generated k-algebras to the category of abelian groups, which is called the functor of vanishing formal infinitesimal divisors on Y relative to X.

Proof. $\underline{\mathrm{IDiv}}_{Y/X}$ is a subfunctor of $\underline{\mathrm{IDiv}}_Y$, which is infinitesimal by Proposition 2.61. It is well defined since $\pi_* : \underline{\mathrm{IDiv}}_Y \longrightarrow \underline{\mathrm{IDiv}}_X$ is a natural transformation by Proposition 2.62, and both functors are plain by Proposition 2.61, hence $\underline{\mathrm{IDiv}}_{Y/X}$ is also. ■

Proposition 2.66 *Let Y be a normal k-scheme and $\pi : Y \longrightarrow X$ be a finite morphism of k-schemes of degree 1 (see Definition 2.53). Then*

$$\underline{\mathrm{IDiv}}^0_{Y/X} = \ker\left(\mathrm{Inf}\left(\underline{\mathrm{Div}}^0_Y\right) \stackrel{\mathrm{fml}}{\longrightarrow} \underline{\mathrm{IDiv}}_Y \stackrel{\pi_*}{\longrightarrow} \underline{\mathrm{IDiv}}_X\right)$$

defines a plain infinitesimal covariant functor

$$\underline{\mathrm{IDiv}}^0_{Y/X} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

from the category of finitely generated k-algebras to the category of abelian groups.

Proof. Note that $\mathrm{Inf}\left(\underline{\mathrm{Div}}^0_Y\right) = \mathrm{Inf}\,(\ \underline{\mathrm{Div}}_Y)$. All functors are plain by Propositions 2.23 and 2.61, and all transformations are natural by Propositions 2.62 and 2.63. ■

Proposition 2.67 *In the situation of Proposition 2.66, if moreover Y and X are projective k-schemes, then $\underline{\mathrm{IDiv}}^0_{Y/X}$ is represented by an infinitesimal formal group.*

Proof. In characteristic 0 infinitesimal formal groups are precisely plain infinitesimal functors in $\mathbf{Fctr}\left(\mathbf{Alg}/k, \mathbf{Ab}\right)$ whose Lie-algebra is finite dimensional, according to Corollary 1.11 Lemma 1.20. Then by Proposition 2.66 it remains to show that $\operatorname{Lie}\left(\mathrm{IDiv}^0_{Y/X}\right)$ is finite dimensional:

Since a finite morphism of degree 1 is birational, the set S of generic points $\eta \in Y$ of height 1 fulfilling an inequality of local rings $(\mathcal{O}_{Y,\eta}, \mathfrak{m}_{Y,\eta}) \neq \left(\mathcal{O}_{X,\pi(\eta)}, \mathfrak{m}_{X,\pi(\eta)}\right)$ is finite. For each $\eta \in Y$ of height 1 it holds

$$\operatorname{Hom}_{k(\pi(\eta)),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{X,\pi(\eta)}, k(\eta)\right) = \operatorname{Hom}_{k(\eta),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{X,\pi(\eta)} \widehat{\otimes}_{k(\pi(\eta))} k(\eta), k(\eta)\right)$$

Denoting $\left(\widehat{\mathfrak{m}}_{X,\pi(\eta)}\right)_{k(\eta)} = \widehat{\mathfrak{m}}_{X,\pi(\eta)} \widehat{\otimes}_{k(\pi(\eta))} k(\eta)$, by Lemma 2.68 below for each $\eta \in S$ there is an integer $n_\eta > 0$ such that $\left(\widehat{\mathfrak{m}}_{Y,\eta}\right)^{n_\eta+1} \subset \left(\widehat{\mathfrak{m}}_{X,\pi(\eta)}\right)_{k(\eta)}$. Taking also into account Lemma 2.64 we obtain

$$\begin{aligned}
& \ker\left(\mathrm{LDiv}\left(Y\right) \longrightarrow \mathrm{LDiv}\left(X\right)\right) \\
= \ & \ker\left(\bigoplus_{\mathrm{ht}(\eta)=1} \operatorname{Hom}_{k(\eta),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{Y,\eta}, \mathfrak{k}_{Y,\eta}\right) \longrightarrow \bigoplus_{\mathrm{ht}(\xi)=1} \operatorname{Hom}_{k(\xi),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{X,\xi}, \mathfrak{k}_{X,\xi}\right)\right) \\
= \ & \bigoplus_{\eta \in S} \ker\left(\operatorname{Hom}_{k(\eta),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{Y,\eta}, k(\eta)\right) \longrightarrow \operatorname{Hom}_{k(\eta),\mathrm{cont}}\left(\left(\widehat{\mathfrak{m}}_{X,\pi(\eta)}\right)_{k(\eta)}, k(\eta)\right)\right) \\
= \ & \bigoplus_{\eta \in S} \operatorname{Hom}_{k(\eta)}\left(\widehat{\mathfrak{m}}_{Y,\eta} \Big/ \left(\widehat{\mathfrak{m}}_{X,\pi(\eta)}\right)_{k(\eta)}, k(\eta)\right) \\
\subset \ & \bigoplus_{\eta \in S} \operatorname{Hom}_{k(\eta)}\left(\widehat{\mathfrak{m}}_{Y,\eta} \Big/ \left(\widehat{\mathfrak{m}}_{Y,\eta}\right)^{n_\eta+1}, k(\eta)\right) \\
\cong \ & \bigoplus_{\eta \in S} t_\eta^{-n_\eta} \widehat{\mathcal{O}}_\eta / \widehat{\mathcal{O}}_\eta
\end{aligned}$$

where t_η is a local parameter of $\widehat{\mathfrak{m}}_{Y,\eta}$. Now
$\underline{\mathrm{IDiv}}^0_{Y/X} = \ker\left(\operatorname{Inf}\left(\underline{\mathrm{Div}}^0_Y\right) \xrightarrow{\mathrm{fml}} \underline{\mathrm{IDiv}}_Y \xrightarrow{\pi_*} \underline{\mathrm{IDiv}}_X\right)$, therefore

$$\begin{aligned}
\operatorname{Lie}\left(\underline{\mathrm{IDiv}}^0_{Y/X}\right) &= \operatorname{Lie}\left(\mathrm{fml}\right)^{-1}\left(\ker\left(\mathrm{LDiv}\left(Y\right) \longrightarrow \mathrm{LDiv}\left(X\right)\right)\right) \\
&\subset \operatorname{Lie}\left(\mathrm{fml}\right)^{-1}\left(\bigoplus_{\eta \in S} t_\eta^{-n_\eta} \widehat{\mathcal{O}}_\eta / \widehat{\mathcal{O}}_\eta\right) \\
&= \Gamma\left(\mathcal{O}_Y\left(\sum_{\eta \in S} n_\eta E_\eta\right) \Big/ \mathcal{O}_Y\right)
\end{aligned}$$

where E_η is the prime divisor associated to the generic point η, and for any effective Weil divisor E we denote by $\mathcal{O}_Y(E)$ the $\mathcal{O}_Y$-submodule of $\mathcal{K}_Y$

consisting of those rational functions which have poles at most along E. Since Y is projective, $\Gamma\Big(\mathcal{O}_Y \left(\sum_{\eta\in S} n_\eta E_\eta\right) \Big/ \mathcal{O}_Y\Big)$ is finite dimensional over k, thus $\mathrm{Lie}\left(\underline{\mathrm{IDiv}}^0_{Y/X}\right)$ is finite dimensional over k. ■

Lemma 2.68 *Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be Noetherian local rings of dimension 1, with residue fields $k_A = A/\mathfrak{m}_A$ and $k_B = B/\mathfrak{m}_B$. Let $h : A \longrightarrow B$ be a finite local homomorphism of degree 1 of local rings. Denote by $\widehat{\mathfrak{m}}_A$ and $\widehat{\mathfrak{m}}_B$ the maximal ideals of the completions $\widehat{A}$ and $\widehat{B}$ w.r.t. $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively. Then there is an integer $n > 0$ such that $(\widehat{\mathfrak{m}}_B)^n \subset (\widehat{\mathfrak{m}}_A)_{k_B}$.*

Proof. As $h : A \longrightarrow B$ is finite of degree 1, B is lying between A and its normalization $\widetilde{A}$: We have monomorphisms of rings $A \longrightarrow B \longrightarrow \widetilde{A}$, inducing local homomorphisms of local rings $A \longrightarrow B \longrightarrow \widetilde{A}_{\widetilde{\mathfrak{m}}}$ for all maximal primes $\widetilde{\mathfrak{m}} \in \mathrm{Spec}\,\widetilde{A}$. Let $\mathrm{Cp}\,(A)$ be the set of irreducibel components of $\mathrm{Spec}\,A$. Each $Z \in \mathrm{Cp}\,(A)$ corresponds to a minimal prime $\zeta \in \mathrm{Spec}\,A$ and the local integral domain $A_Z := A/\zeta$ is the affine algebra of Z. Then

$$\widetilde{A} = \bigoplus_{Z\in\mathrm{Cp}(A)} \widetilde{A}_Z$$

where $\widetilde{A}_Z$ is the normalization of A_Z. The normalization $A \longrightarrow \widetilde{A}$ of A is then the composition of $A \longrightarrow \bigoplus_Z A_Z$, $a \longmapsto \sum_Z [a]_\zeta$, and $\bigoplus_Z A_Z \longrightarrow \bigoplus_Z \widetilde{A}_Z$, given by the normalizations of the components $A_Z \longrightarrow \widetilde{A}_Z$.

Thus we may assume that A is a local integral domain and it is sufficient to show the assertion for its normalization $A \longrightarrow \widetilde{A}$, i.e. we suppose $B = \widetilde{A}$. As B is a discrete valuation ring, its completion is given by the formal power series ring $\widehat{B} = k_B[[t]]$ and the quotient field of its completion is the field of formal Laurant series $\mathrm{Q}\left(\widehat{B}\right) = k_B((t))$, where t is a local parameter of $\mathfrak{m}_B$. Since $\mathrm{Q}\,(A) = \mathrm{Q}\,(B)$, it holds $\{t^\nu \mid \nu \in \mathbb{Z}\} \subset \mathrm{Q}(A) \subset \mathrm{Q}\left(\widehat{A}\right)$. Hence

$$\mathrm{Q}\left(\widehat{A}\right)_{k_B} = \mathrm{Q}\left(\widehat{B}\right) = k_B((t))$$

Therefore there are coprime integers $p, q > 0$ with $t^p, t^q \in \widehat{\mathfrak{m}}_A$
(since otherwise $\left(\widehat{A}\right)_{k_B} \subset k_B[[t^r]]$ for a certain $r > 0$,
and $\mathrm{Q}\,(k_B[[t^r]]) = k_B((t^r)) \subsetneqq k_B((t))$, a contradiction).
Then by Lemma 2.69 below there exists an integer $n > 0$,
namely $n = (q-1)\,p$, such that

$$(\widehat{\mathfrak{m}}_B)^n \;=\; t^n \cdot k_B[[t]] \;\subset\; (t^p, t^q)\left(\widehat{A}\right)_{k_B} \;\subset\; (\widehat{\mathfrak{m}}_A)_{k_B}$$

■

Lemma 2.69 *Let $p, q \in \mathbb{N}$ be coprime positive integers. Then there is an integer $l \in \mathbb{N}$ such that*

$$\langle p, q\rangle_{\geq 0} \supset \mathbb{N}_{\geq l}$$

where $\langle p, q\rangle_{\geq 0} := \{np + mq \mid n, m \in \mathbb{N}\}$ *and* $\mathbb{N}_{\geq l} := \{n \in \mathbb{N} \mid n \geq l\}$.

Proof. Let $\overline{p} \in \mathbb{Z}/q\mathbb{Z}$ be the residue class of p in $\mathbb{Z}/q\mathbb{Z}$. Since p and q are coprime, the order of $\overline{p}$ in $\mathbb{Z}/q\mathbb{Z}$ is $\operatorname{ord}(\overline{p}) = q$. Hence $\overline{p}$ generates $\mathbb{Z}/q\mathbb{Z}$:

$$\langle \overline{p}\rangle = \mathbb{Z}/q\mathbb{Z}$$

Then for each $\overline{r} \in \mathbb{Z}/q\mathbb{Z}$ there is a $n \in \{0, \ldots, q-1\}$ such that $r \equiv np \mod q$. But this means that for all $r \geq (q-1)\,p$ there are $n, m \in \mathbb{N}$ such that $r = np + mq$. ■

2.5 The Functor $\underline{\mathrm{Div}}^0_{Y/X}$

The idea about $\underline{\mathrm{Div}}^0_{Y/X}$ is to define a functor which admits a natural transformation to the Picard functor $\underline{\mathrm{Pic}}^0_Y$ and measures the difference between the schemes Y and X, where $\pi : Y \longrightarrow X$ is a finite morphism of degree 1, for example the normalization of X. A natural choice is therefore the largest subfunctor of $\underline{\mathrm{Div}}^0_Y$ which lies in the kernel of the push-forward π_*.

Proposition 2.70 *Let Y be a normal k-scheme, and let $\pi : Y \longrightarrow X$ be a finite morphism of k-schemes of degree 1 (see Definition 2.53). Then there is a covariant functor*

$$\underline{\mathrm{Div}}^0_{Y/X} : \mathbf{Alg}/k \longrightarrow \mathbf{Ab}$$

from the category of finitely generated k-algebras to the category of abelian groups, defined by the following conditions:

(a) $\underline{\mathrm{Div}}^0_{Y/X}$ is a banal functor (see Definition 2.7), i.e.

$$\underline{\mathrm{Div}}^0_{Y/X} = \mathrm{Red}\left(\underline{\mathrm{Div}}^0_{Y/X}\right) \times \mathrm{Inf}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$$

(b) the reduced part of $\underline{\mathrm{Div}}^0_{Y/X}$ is given by

$$\mathrm{Red}\left(\underline{\mathrm{Div}}^0_{Y/X}\right) = \ker\left(\mathrm{Red}\left(\underline{\mathrm{Div}}^0_{Y}\right) \xrightarrow{\mathrm{weil}} \underline{\mathrm{WDiv}}_Y \xrightarrow{\pi_*} \underline{\mathrm{WDiv}}_X\right)$$

and the infinitesimal part of $\underline{\mathrm{Div}}^0_{Y/X}$ by

$$\mathrm{Inf}\left(\underline{\mathrm{Div}}^0_{Y/X}\right) = \ker\left(\mathrm{Inf}\left(\underline{\mathrm{Div}}^0_{Y}\right) \xrightarrow{\mathrm{fml}} \underline{\mathrm{IDiv}}_Y \xrightarrow{\pi_*} \underline{\mathrm{IDiv}}_X\right)$$

Proof. The assertion follows from Proposition 2.58 and Proposition 2.66. ■

Proposition 2.71 *In the situation of Proposition 2.70 above, if furthermore* Y *and* X *are projective, then* $\underline{\mathrm{Div}}^0_{Y/X}$ *is represented by a formal group.*

Proof. By Proposition 2.58 the reduced part $\mathrm{Red}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$ is represented by a torsion-free étale formal group, and by Proposition 2.67 the infinitesimal part $\mathrm{Inf}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$ is represented by an infinitesimal formal group. Thus $\underline{\mathrm{Div}}^0_{Y/X} = \mathrm{Red}\left(\underline{\mathrm{Div}}^0_{Y/X}\right) \times \mathrm{Inf}\left(\underline{\mathrm{Div}}^0_{Y/X}\right)$ is represented by a formal group. ■

3 Universal Factorization Problem

Let X be a projective variety over k, an algebraically closed field of characteristic 0. The universal factorization problem may be outlined as follows: one is looking for a "universal object" $\mathcal{U}$ and a rational map $u : X \longrightarrow \mathcal{U}$ such that for every rational map $\varphi : X \longrightarrow G$ to an algebraic group G there is a unique homomorphism $h : \mathcal{U} \longrightarrow G$ and a constant $g \in G(k)$ such that $\varphi = h \circ u + g$.

The universal object $\mathcal{U}$, if it exists, is not in general an algebraic group. But if we restrict our attention to rational maps fulfilling certain extra conditions, we may find an algebraic group solving the universal factorization problem for those rational maps. In this section a criterion for a category **Mr** of rational maps from X to algebraic groups is worked out, in which situation one can find an algebraic group $\mathrm{Alb}_{\mathbf{Mr}}(X)$ satisfying the universal mapping property for this category, and in this case a construction of $\mathrm{Alb}_{\mathbf{Mr}}(X)$ is given. The way of procedure was inspired by Serre's exposé [S3].

3.1 Categories of Rational Maps to Algebraic Groups

Let Y be a normal projective variety over k (an algebraically closed field of characteristic 0). Algebraic groups are always assumed to be connected, unless stated otherwise.

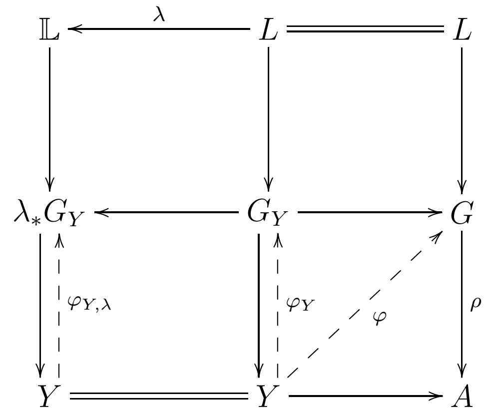
Notation 3.1 *$\mathbb{L}$ stands for one of the groups $\mathbb{G}_{\mathrm{m}}$ or $\mathbb{G}_{\mathrm{a}}$.*

Lemma 3.2 *Let P be a principal $\mathbb{L}$-bundle over Y. Then a local section $\sigma : U \subset Y \longrightarrow P$ determines uniquely an element $\mathrm{div}_{\mathbb{L}}(\sigma) \in \underline{\mathrm{Div}}_Y$ (see Notation 2.15, $\underline{\mathrm{Div}}_Y$ is a plain functor), which is a divisor on Y, if $\mathbb{L} = \mathbb{G}_{\mathrm{m}}$, or a deformation of a divisor on Y, if $\mathbb{L} = \mathbb{G}_{\mathrm{a}}$.*

Proof. For $V = \begin{cases} k & \text{if } \mathbb{L} = \mathbb{G}_{\mathrm{m}} \\ k[\varepsilon] & \text{if } \mathbb{L} = \mathbb{G}_{\mathrm{a}} \end{cases}$ let $\lambda : \mathbb{L} \longrightarrow \mathrm{Gl}(V)$ be the representation of $\mathbb{L}$ given by $l \longmapsto \begin{cases} l & \text{if } \mathbb{L} = \mathbb{G}_{\mathrm{m}} \\ 1 + \varepsilon l & \text{if } \mathbb{L} = \mathbb{G}_{\mathrm{a}} \end{cases}$. Let $E = P \times_{[\mathbb{L},\lambda]} V$ be the associated vector-bundle to P of fibre-type V. Denote by $s \in \Gamma(U, E)$ the image of σ under the map $P \longrightarrow E$, $\quad p \longmapsto \begin{cases} p & \text{if } \mathbb{L} = \mathbb{G}_{\mathrm{m}} \\ 1 + \varepsilon p & \text{if } \mathbb{L} = \mathbb{G}_{\mathrm{a}} \end{cases}$.
There is an effective divisor H, supported on $Y \setminus U$, with canonical global section $h \in \Gamma(Y, \mathcal{O}(H))$, such that $hs \in \Gamma(U, E(H))$ extends to a global section of $E(H) = E \otimes \mathcal{O}(H)$. Given local trivializations $\Phi_\alpha : E(H)|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha} \otimes_k V$, the local sections $\Phi_\alpha(hs) \in \Gamma(U_\alpha, \mathcal{O}_Y \otimes_k V)$ yield an effective divisor on Y, if $V = k$, and on $Y[\varepsilon] = Y \times_k \mathrm{Spec}\, k[\varepsilon]$, if $V = k[\varepsilon]$, called

the divisor of zeros $\mathrm{V}(hs)$ of hs. Thus $\mathrm{V}(hs)$ determines an effective divisor $D \in \underline{\mathrm{Div}}_Y(k)$ on Y, if $\mathbb{L} = \mathbb{G}_\mathrm{m}$, or a deformation $\mathcal{H} \in \underline{\mathrm{Div}}_Y(k[\varepsilon])$ of H, if $\mathbb{L} = \mathbb{G}_\mathrm{a}$. Then define $\mathrm{div}_\mathbb{L}(\sigma) = \begin{cases} D - H & \text{if } \mathbb{L} = \mathbb{G}_\mathrm{m} \\ \mathcal{H} - H_\mathrm{const} & \text{if } \mathbb{L} = \mathbb{G}_\mathrm{a} \end{cases}$, where H_const is the constant extension of H to a divisor on $Y[\varepsilon]$ (see Definition 2.40). One checks that $\mathrm{div}_\mathbb{L}(\sigma)$ is independent of the choice of H. ■

Let $\varphi : Y \longrightarrow G$ be a rational map to an algebraic group G with canonical decomposition $0 \to L \to G \xrightarrow{\rho} A \to 0$. Since a rational map to an abelian variety is defined at every normal point (see [GM] Chapter I, Theorem (1.17)), the composition $Y \xrightarrow{\varphi} G \xrightarrow{\rho} A$ extends to a morphism $\overline{\varphi} : Y \longrightarrow A$. Let $G_Y = G \times_A Y$ be the fibre-product of G and Y over A. The graph-morphism $\varphi_Y : U \subset Y \longrightarrow G \times_A Y, \quad y \longmapsto (\varphi(y), y)$ of φ is a section of the L-bundle G_Y over Y. Then for each $\lambda \in L^\vee$ (see Notation 2.15, where we consider $L^\vee$ as a plain functor on $\mathbf{Alg}/k$) the composition of φ_Y with the push-out of G_Y via λ gives a section $\varphi_{Y,\lambda} : U \subset Y \longrightarrow \lambda_* G_Y$ of the $\mathbb{L}$-bundle $\lambda_* G_Y$ over Y:

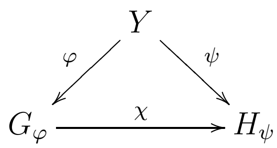


Lemma 3.2 says that the section $\varphi_{Y,\lambda}$ determines a unique divisor or deformation $\mathrm{div}_\mathbb{L}(\varphi_{Y,\lambda}) \in \begin{cases} \Gamma\left(\mathcal{K}_Y^* / \mathcal{O}_Y^*\right) & \text{if } \mathbb{L} = \mathbb{G}_\mathrm{m} \\ \Gamma\left(\mathcal{K}_{Y[\varepsilon]}^* \middle/ \mathcal{O}_{Y[\varepsilon]}^*\right) & \text{if } \mathbb{L} = \mathbb{G}_\mathrm{a} \end{cases}$. Now the bundle $\lambda_* G_Y$ comes from an extension of algebraic groups, i.e. it is the push-out of an translation-invariant bundle, hence it is a line-bundle algebraically equivalent to the trivial bundle, if $\mathbb{L} = \mathbb{G}_\mathrm{m}$, or a deformation of the trivial bundle, if $\mathbb{L} = \mathbb{G}_\mathrm{a}$. Therefore $\mathrm{div}_\mathbb{L}(\varphi_{Y,\lambda})$ is a divisor in $\underline{\mathrm{Div}}_Y^0(k)$, if $\lambda \in L(k)$, or a deformation of the zero divisor, i.e. an element of $\mathrm{Lie}\left(\underline{\mathrm{Div}}_Y^0\right)$, if $\lambda \in \mathrm{Lie}(L)$.

Proposition 3.3 *Let $G \in \operatorname{Ext}(A, L)$ be an algebraic group and $\varphi : Y \longrightarrow G$ a rational map. Then φ induces a natural transformation of functors $L^\vee \longrightarrow \underline{\mathrm{Div}}^0_Y$.*

Proof. The construction above yields a homomorphism of abelian groups $L^\vee(k) \longrightarrow \underline{\mathrm{Div}}^0_Y(k)$, $\lambda \longmapsto \operatorname{div}_{\mathbb{G}_\mathrm{m}}(\varphi_{Y,\lambda})$ and a k-linear map $\operatorname{Lie}(L) \longrightarrow \operatorname{Lie}\left(\underline{\mathrm{Div}}^0_Y\right)$, $\lambda \longmapsto \operatorname{div}_{\mathbb{G}_\mathrm{a}}(\varphi_{Y,\lambda})$. Then the assertion follows from Lemma 2.16 and Example 2.17. ■

Definition 3.4 *A* category **Mr** of rational maps from Y to algebraic groups *is defined as follows: The objects of* **Mr** *are rational maps $\varphi : Y \longrightarrow G$, where G is an algebraic group. The morphisms of* **Mr** *between two objects $\varphi : Y \longrightarrow G$ and $\psi : Y \longrightarrow H$ are given by the set of those homomorphisms of algebraic groups $\chi : G_\varphi \longrightarrow H_\psi$, where $G_\varphi = \langle \operatorname{im} \varphi \rangle$ and $H_\psi = \langle \operatorname{im} \psi \rangle$ are the subgroups of G and H generated by Y via φ and ψ respectively, such that $\chi \circ \varphi = \psi$, i.e. the following diagram commutes:*



Remark 3.5 *Definition 3.4 above implies that if* **Mr** *is a category of rational maps from Y to algebraic groups, any object $\varphi : Y \longrightarrow G$ is isomorphic to $\varphi : Y \longrightarrow \langle \operatorname{im} \varphi \rangle$, i.e. we may always replace G by the subgroup generated by Y via φ. Then a morphism χ between two objects $\varphi : Y \longrightarrow G$ and $\psi : Y \longrightarrow H$, if it exists, is uniquely determined by the condition $\chi \circ \varphi = \psi$. Therefore two categories* **Mr** *and* **Mr**$'$ *of rational maps from Y to algebraic groups are equivalent if every object of* **Mr** *is isomorphic to an object of* **Mr**$'$*:*

$$\mathbf{Mr} \sim \mathbf{Mr}' \qquad \Longleftarrow \qquad \operatorname{Ob} \mathbf{Mr} = \operatorname{Ob} \mathbf{Mr}'$$

Definition 3.6 *Let $\mathcal{F}$ be a subfunctor of $\underline{\mathrm{Div}}^0_Y$ which is a formal group. Then $\mathbf{Mr}_\mathcal{F}$ denotes the category of those rational maps $\varphi : Y \longrightarrow G$ from Y to algebraic groups for which the image of the natural transformation $L^\vee \longrightarrow \underline{\mathrm{Div}}^0_Y$ (see Proposition 3.3) lies in $\mathcal{F}$, i.e. which induce a homomorphism of formal groups $L^\vee \longrightarrow \mathcal{F}$, where $0 \to L \to G \to A \to 0$ is the canonical decomposition of G.*

In the following examples let G always be an algebraic group with canonical decomposition $0 \to L \to G \to A \to 0$.

Example 3.7 *The category* $\mathbf{Mr}_0$ *associated to the trivial formal group* 0 *is the category of morphisms from* Y *to abelian varieties:*

Let $\varphi : Y \longrightarrow G$ *be rational map to an algebraic group* G*, without loss of generality* φ *generates* G*. Then the following conditions are equivalent:*

(i) *The section* φ_Y *to the L-bundle* G_Y *over* Y *induces only a transformation to the zero-subgroup of* $\underline{\mathrm{Div}}_Y^0$

(ii) φ_Y *extends to a global section*

(iii) G_Y *is the trivial L-bundle over* Y

(iv) G *is the trivial L-bundle over* A

(v) *There exists a splitting* $\sigma : A \longrightarrow G$

(vi) $\varphi : Y \longrightarrow G$ *is isomorphic to the composition* $Y \xrightarrow{\varphi} G \longrightarrow A$ *(as objects in* $\mathbf{Mr}_0$*)*

and a rational map from Y *to an abelian variety* A *extends to a morphism* $Y \longrightarrow A$.

Example 3.8 *Let* D *be an effective divisor on* Y*, and let* $\mathcal{F}_D$ *be the formal group whose étale part is given by divisors in* $\underline{\mathrm{Div}}_Y^0(k)$ *whith support in* $\mathrm{Supp}\,(D)$ *and whose infinitesimal part is trivial. Then* $\mathbf{Mr}_{\mathcal{F}_D}$ *is the category of rational maps from* Y *to semi-abelian varieties (i.e. extensions of an abelian variety by a torus) which are regular away from* D*:*

For a rational map $\varphi : Y \longrightarrow G$ *the induced sections* $\varphi_{Y,\lambda}$ *determine divisors in* $\underline{\mathrm{Div}}_Y^0(k)$ *supported on* $\mathrm{Supp}\,(D)$ *for all* $\lambda \in L^\vee$ *if and only if* L *is a torus, i.e. it consists of several copies of* $\mathbb{G}_{\mathrm{m}}$ *only, and* φ *is regular on* $Y \setminus \mathrm{Supp}\,(D)$.

Example 3.9 *Let* $Y = C$ *be a smooth projective curve,* $\mathfrak{d} = \sum_i n_i\, p_i$ *with* $p_i \in C$*,* $n_i \geq 1$*, an effective divisor on* C*,* $\mathfrak{d}^- = \sum_i (n_i - 1)\, p_i$ *and let* v_p *be the valuation attached to the point* $p \in C$*. Let* $\mathcal{F}_\mathfrak{d}$ *be the formal group defined by*

$$\mathcal{F}_\mathfrak{d}(k) = \left\{ \sum_i l_i\, p_i \;\middle|\; \sum_i l_i = 0 \right\} \qquad\qquad \mathrm{Lie}\,(\mathcal{F}_\mathfrak{d}) = \mathrm{Inf}_{C,\mathfrak{d}^-}$$

(see Definition 2.46), i.e. the étale part of $\mathcal{F}_\mathfrak{d}$ *consists of divisors of degree* 0 *supported on* $\mathrm{Supp}\,(\mathfrak{d})$*, while the infinitesimal part of* $\mathcal{F}_\mathfrak{d}$ *is given by the image of effective deformations of* $\mathfrak{d}^-$ *in* $\mathrm{Lie}\,(\underline{\mathrm{Div}}_C)$ *(see Subsubsection 2.2.4). Then* $\mathbf{Mr}_{\mathcal{F}_\mathfrak{d}}$ *is the category of those rational maps* $\varphi : Y \longrightarrow G$ *such that for all* $f \in \mathcal{K}_C$ *it holds:*

$$v_{p_i}(1 - f) \geq n_i \quad \forall i \quad \Longrightarrow \quad \varphi\,(\mathrm{div}\,(f)) = 0$$

By constructing a singular curve defined by the modulus $\mathfrak{d}$ *(see [S] Chapter IV, No. 4), this turns out to be a special case of the next Example 3.10.*

Example 3.10 *Let X be a singular projective variety and $Y = \widetilde{X}$, where $\pi : \widetilde{X} \longrightarrow X$ is the normalization. A rational map $\varphi : X \longrightarrow G$ which is regular on the regular locus X_{reg} of X can also be considered as a rational map from Y to G. The functor $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ (see Proposition 2.58) is a formal group and $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}$ is the category of morphisms $\varphi : X_{\mathrm{reg}} \longrightarrow G$ which factor through a homomorphism of groups $\mathrm{CH}_0(X)_{\deg 0} \longrightarrow G(k)$, see Definition 4.17 (cf. [ESV] Definition 1.14 for the notion of* regular homomorphism*). This is the subject of Section 4.*

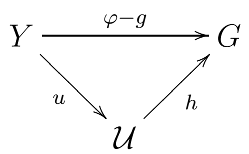
3.2 Universal Objects

Let Y be a normal projective variety over k (an algebraically closed field of characteristic 0).

3.2.1 Existence and Construction

Definition 3.11 *Let* $\mathbf{Mr}$ *be a category of rational maps from Y to algebraic groups. Then $(u : Y \longrightarrow \mathcal{U}) \in \mathbf{Mr}$ is called a* universal object for $\mathbf{Mr}$ *if it has the universal mapping property in* $\mathbf{Mr}$*:*

$$\forall\, (\varphi : Y \longrightarrow G) \in \mathbf{Mr} \quad \exists!\ \textit{homomorphism of algebraic groups}$$
$$h : \mathcal{U} \longrightarrow G \textit{ and a constant } g \in G(k) \textit{ such that } \varphi = h \circ u + g$$



For the category $\mathbf{Mr}_0$ of morphisms from Y to abelian varieties (see Example 3.7) there exists a universal object, the *Albanese mapping* to the *Albanese variety,* denoted by $\mathrm{alb} : Y \longrightarrow \mathrm{Alb}\,(Y)$. This is a classical result (see [La], [Ms], [S2]).

In the following we consider categories $\mathbf{Mr}$ of rational maps from Y to algebraic groups satisfying the following condition:

($\diamondsuit$) $(\varphi : Y \longrightarrow G) \in \mathbf{Mr}$ with $G \in \mathrm{Ext}\,(A, L)$ if and only if
$\forall \lambda \in L^{\vee}$ the induced rational map $(\varphi_\lambda : Y \longrightarrow \lambda_* G) \in \mathbf{Mr}$

Theorem 3.12 *Let* $\mathbf{Mr}$ *be a category of rational maps from Y to algebraic groups which contains* $\mathbf{Mr}_0$ *and satisfies ($\diamondsuit$). Then there exists a universal object $(u : Y \longrightarrow \mathcal{U}) \in \mathbf{Mr}$ for* $\mathbf{Mr}$ *if and only if there is a formal group $\mathcal{F}$ which is a subfunctor of $\underline{\mathrm{Div}}^0_Y$ such that* $\mathbf{Mr}$ *is equivalent to the category* $\mathbf{Mr}_{\mathcal{F}}$ *of rational maps which induce a homomorphism of formal groups to $\mathcal{F}$ (see Definition 3.6).*

Proof. ($\Longleftarrow$) Assume that **Mr** is equivalent to $\mathbf{Mr}_{\mathcal{F}}$, where $\mathcal{F}$ is a formal group in $\underline{\mathrm{Div}}^0_Y$. The first step is the construction of an algebraic group $\mathcal{U}$ and a rational map $u : Y \longrightarrow \mathcal{U}$. In a second step the universality of $u : Y \longrightarrow \mathcal{U}$ for $\mathbf{Mr}_{\mathcal{F}}$ has to be shown.

Step 1: Construction of $u : Y \longrightarrow \mathcal{U}$
Y is a normal projective variety over k, thus the functor $\underline{\mathrm{Pic}}^0_Y$ is represented by an abelian variety Pic^0_Y (see Corollary 2.35). The natural transformation $\underline{\mathrm{Div}}^0_Y \longrightarrow \underline{\mathrm{Pic}}^0_Y$ induces a 1-motive $M = \left[\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y\right]$. Let $M^\vee$ be the dual 1-motive of M. The formal group in degree -1 of $M^\vee$ is the Cartier-dual of the largest linear subgroup of Pic^0_Y, and this is zero, since an abelian variety does not contain any non-trivial linear subgroup. Then define $\mathcal{U}$ to be the algebraic group in degree 0 of $M^\vee$, i.e. $[0 \longrightarrow \mathcal{U}]$ is the dual 1-motive of $\left[\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y\right]$. The canonical decomposition $0 \to \mathcal{L} \to \mathcal{U} \to \mathcal{A} \to 0$ is the extension of $\left(\mathrm{Pic}^0_Y\right)^\vee$ by $\mathcal{F}^\vee$ induced by the homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y$ (see Theorem 1.23), where $\mathcal{L} = \mathcal{F}^\vee$ is the Cartier-dual of $\mathcal{F}$ and $\mathcal{A} = \left(\mathrm{Pic}^0_Y\right)^\vee$ is the dual abelian variety of Pic^0_Y, which is $\mathrm{Alb}\,(Y)$.

As $\mathcal{L}$ is a linear algebraic group, there is a canonical splitting $\mathcal{L} \cong \mathbb{T} \times \mathbb{V}$ of $\mathcal{L}$ into the direct product of a torus $\mathbb{T}$ of rank t and a vectorial group $\mathbb{V}$ of dimension v (see Theorem 1.4). The homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y$ is uniquely determined by the values on a basis Ω of the finite free $\mathbb{Z}$-module

$$\mathcal{F}\,(k) = \mathcal{L}^\vee\,(k) = \mathbb{T}^\vee(k) = \mathrm{Hom}_{\mathrm{gr}}\,(\mathbb{T}(k), \mathbb{G}_{\mathrm{m}}(k)) = \mathrm{Hom}_{\mathcal{A}b/k}\,(\mathbb{T}, \mathbb{G}_{\mathrm{m}})$$

and on a basis Θ of the finite dimensional k-vector space

$$\mathrm{Lie}\,(\mathcal{F}) = \mathrm{Lie}\,(\mathcal{L}^\vee) = \mathrm{Lie}\,(\mathbb{V}^\vee) = \mathrm{Hom}_k\,(\mathrm{Lie}(\mathbb{V}), k) = \mathrm{Hom}_{\mathcal{A}b/k}\,(\mathbb{V}, \mathbb{G}_{\mathrm{a}})$$

By duality, such a choice of basises corresponds to a decomposition

$$\mathcal{L} \xrightarrow{\sim} (\mathbb{G}_{\mathrm{m}})^t \times (\mathbb{G}_{\mathrm{a}})^v$$

and induces a decomposition

$$\begin{array}{ccc} \mathrm{Ext}\,(\mathcal{A}, \mathcal{L}) & \xrightarrow{\sim} & \mathrm{Ext}\,(\mathcal{A}, \mathbb{G}_{\mathrm{m}})^t \times \mathrm{Ext}\,(\mathcal{A}, \mathbb{G}_{\mathrm{a}})^v \\ \mathcal{U} & \longmapsto & \prod_{\omega\in\Omega} \omega_*\mathcal{U} \times \prod_{\vartheta\in\Theta} \vartheta_*\mathcal{U} \end{array}$$

Therefore the rational map $u : Y \longrightarrow \mathcal{U}$ is uniquely determined by the following rational maps to push-outs of $\mathcal{U}$

$$\begin{array}{ll} u_\omega : Y \longrightarrow \omega_*\mathcal{U} & \omega \in \Omega \\ u_\vartheta : Y \longrightarrow \vartheta_*\mathcal{U} & \vartheta \in \Theta \end{array}$$

whenever Ω is a basis of $\mathcal{F}(k)$ and Θ a basis of $\mathrm{Lie}(\mathcal{F})$. We have isomorphisms

$$\begin{array}{rcl} \mathrm{Ext}(\mathcal{A}, \mathbb{G}_\mathrm{m}) \simeq \mathrm{Pic}^0_{\mathcal{A}}(k) & \xrightarrow{\sim} & \mathrm{Pic}^0_Y(k) \\ P & \longmapsto & P_Y = P \times_{\mathcal{A}} Y \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Ext}(\mathcal{A}, \mathbb{G}_\mathrm{a}) \simeq \mathrm{Lie}\left(\mathrm{Pic}^0_{\mathcal{A}}\right) & \xrightarrow{\sim} & \mathrm{Lie}\left(\mathrm{Pic}^0_Y\right) \\ \tau & \longmapsto & \tau_Y = \tau \times_{\mathcal{A}} Y \end{array}$$

From the proof of Lemma 1.24 follows that $(\omega_* \mathcal{U})_Y$ is just the image of $\omega \in \mathcal{F}(k) \subset \underline{\mathrm{Div}}^0_Y(k)$ under the homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y$, which is the divisor-class $[\omega] \in \mathrm{Pic}^0_Y(k)$. Likewise from the proof of Lemma 1.25 follows that $(\vartheta_* \mathcal{U})_Y$ is the image of $\vartheta \in \mathrm{Lie}(\mathcal{F}) \subset \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Y\right)$ under the homomorphism $\mathcal{F} \longrightarrow \mathrm{Pic}^0_Y$, which is the class of deformation $[\vartheta] \in \mathrm{Lie}\left(\mathrm{Pic}^0_Y\right)$. Then define the rational map $u : Y \longrightarrow \mathcal{U}$ by the condition that for all $\omega \in \Omega$ the section

$$u_{Y,\omega} : Y \xrightarrow{u} \mathcal{U} \longrightarrow \omega_* \mathcal{U}_Y = [\omega]$$

corresponds to the divisor $\omega \in \underline{\mathrm{Div}}^0_Y(k)$, and for all $\vartheta \in \Theta$ the section

$$u_{Y,\vartheta} : Y \xrightarrow{u} \mathcal{U} \longrightarrow \vartheta_* \mathcal{U}_Y = [\vartheta]$$

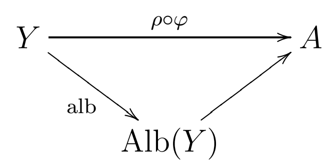
corresponds to the deformation $\vartheta \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Y\right)$, in the sense of Lemma 3.2, i.e.

$$\begin{array}{ll} \mathrm{div}_{\mathbb{G}_\mathrm{m}}(u_{Y,\omega}) = \omega & \qquad \forall \omega \in \Omega \\ \mathrm{div}_{\mathbb{G}_\mathrm{a}}(u_{Y,\vartheta}) = \vartheta & \qquad \forall \vartheta \in \Theta \end{array}$$

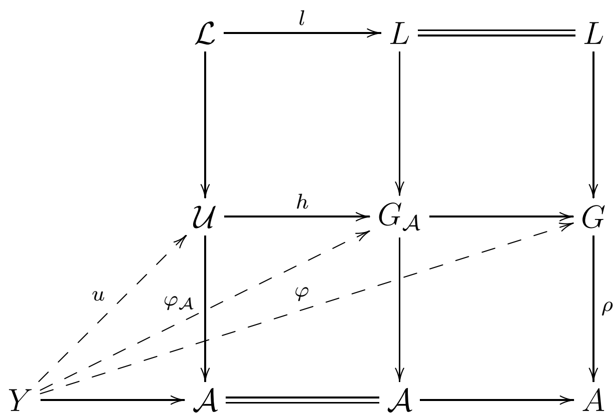
This determines u up to translation by a constant.

Step 2: Universality of $u : Y \longrightarrow \mathcal{U}$

Given a rational map $\varphi : Y \longrightarrow G$ to an algebraic group G with canonical decomposition $0 \to L \to G \xrightarrow{\rho} A \to 0$ inducing a homomorphism of formal groups $l^\vee : L^\vee \longrightarrow \mathcal{F}$, $\lambda \longmapsto \mathrm{div}_{\mathbb{L}}(\varphi_{Y,\lambda})$ for $\lambda \in L^\vee$ (see Proposition 3.3). Let $l : \mathcal{L} \longrightarrow L$ be the dual homomorphism of linear groups. The composition $Y \xrightarrow{\varphi} G \xrightarrow{\rho} A$ extends to a morphism from Y to an abelian variety. Translating φ by a constant $g \in G(k)$, if necessary, we may hence assume that $\rho \circ \varphi$ factors through $\mathcal{A} = \mathrm{Alb}(Y)$:



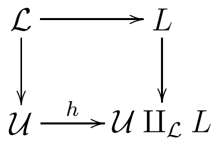
It remains to show that the following diagram commutes:



i.e. the task is to show that

$$\begin{array}{ll} \text{(a)} & G_{\mathcal{A}} = l_* \mathcal{U} \\ \text{(b)} & \varphi_{\mathcal{A}} = h \circ u \end{array}$$

where $G_{\mathcal{A}} = G \times_A \mathcal{A}$ is the fibre-product of G and $\mathcal{A}$ over A and $\varphi_{\mathcal{A}} : Y \longrightarrow G_{\mathcal{A}}$ is the unique map obtained from $(\varphi, \mathrm{alb}) : Y \longrightarrow G \times \mathcal{A}$ by the universal property of the fibre-product $G_{\mathcal{A}}$, and where h is the homomorphism obtained by the amalgamated sum



as by definition of the push-out $l_* \mathcal{U} = \mathcal{U} \amalg_{\mathcal{L}} L$.
For this purpose, by additivity of extensions, it is enough to show that for all $\lambda \in L^\vee$ it holds

$$\begin{array}{ll} \text{(a')} & \lambda_* G_{\mathcal{A}} = l^\vee(\lambda)_* \mathcal{U} \\ \text{(b')} & \varphi_{\mathcal{A},\lambda} = u_{l^\vee(\lambda)} \end{array}$$

where $l^\vee(\lambda) = \lambda \circ l$ and $l^\vee(\lambda)_* = (\lambda \circ l)_* = \lambda_* l_*$. Using the isomorphism $\mathrm{Pic}^0_{\mathcal{A}} \xrightarrow{\sim} \mathrm{Pic}^0_Y$ this is equivalent to showing that for all $\lambda \in L^\vee$ it holds

$$\begin{array}{ll} \text{(a'')} & \lambda_* G_Y = l^\vee(\lambda)_* \mathcal{U}_Y \\ \text{(b'')} & \varphi_{Y,\lambda} = u_{Y,l^\vee(\lambda)} \end{array}$$

By construction of $u : Y \longrightarrow \mathcal{U}$ we have for all $\lambda \in L^\vee$:

$$\mathrm{div}_{\mathbb{L}} \left(u_{Y,l^\vee(\lambda)} \right) = l^\vee(\lambda) = \mathrm{div}_{\mathbb{L}} \left(\varphi_{Y,\lambda} \right)$$

and hence

$$\begin{aligned} l^\vee(\lambda)_*\mathcal{U}_Y &= [l^\vee(\lambda)] \\ &= [\mathrm{div}_{\mathbb{L}}(\varphi_{Y,\lambda})] = \lambda_* G_Y \end{aligned}$$

i.e. $\varphi_{Y,\lambda}$ is isomorphic to $u_{Y,l^\vee(\lambda)}$ as objects in $\mathbf{Mr}_{\mathcal{F}}$ and $\lambda_* G_Y = l^\vee(\lambda)_*\mathcal{U}_Y$.

($\Longrightarrow$) Assume that $u : Y \longrightarrow \mathcal{U}$ is universal for $\mathbf{Mr}$. Let $0 \to \mathcal{L} \to \mathcal{U} \to \mathcal{A} \to 0$ be the canonical decomposition of $\mathcal{U}$, and let $\mathcal{F}$ be the image of the induced transformation $\mathcal{L}^\vee \longrightarrow \underline{\mathrm{Div}}_Y^0$. For $\lambda \in \mathcal{L}^\vee$ the uniqueness of the homomorphism $h_\lambda : \mathcal{U} \longrightarrow \lambda_*\mathcal{U}$ fulfilling $u_\lambda = h_\lambda \circ u$ implies that the rational maps $u_\lambda : Y \longrightarrow \lambda_*\mathcal{U}$ are non-isomorphic to each other for distinct $\lambda \in \mathcal{L}^\vee$. Hence $\mathrm{div}_{\mathbb{L}}(u_{Y,\lambda}) \neq \mathrm{div}_{\mathbb{L}}(u_{Y,\lambda'})$ for $\lambda \neq \lambda' \in \mathcal{L}^\vee$. Therefore $\mathcal{L}^\vee \longrightarrow \mathcal{F}$ is injective, hence an isomorphism.

Let $\varphi : Y \longrightarrow G$ be an object of $\mathbf{Mr}$ and $0 \to L \to G \to A \to 0$ be the canonical decomposition of G. Translating φ by a constant $g \in G(k)$, if necessary, we may assume that $\varphi : Y \longrightarrow G$ factorizes through a unique homomorphism $h : \mathcal{U} \longrightarrow G$. The restriction of h to $\mathcal{L}$ gives a homomorphism of linear groups $l : \mathcal{L} \longrightarrow L$. Then the dual homomorphism $l^\vee : L^\vee \longrightarrow \mathcal{F}$ yields a factorization of $L^\vee \longrightarrow \underline{\mathrm{Div}}_Y^0$ through $\mathcal{F}$. Thus $\mathbf{Mr}$ is a subcategory of $\mathbf{Mr}_{\mathcal{F}}$. Now the property ($\diamondsuit$) guarantees that $\mathbf{Mr}$ contains all rational maps which induce a transformation to $\mathcal{F}$, hence $\mathbf{Mr}$ is equivalent to $\mathbf{Mr}_{\mathcal{F}}$.
■

Notation 3.13 *The universal object for a category* $\mathbf{Mr}$ *of rational maps from* Y *to algebraic groups, if it exists, is denoted by* $\mathrm{alb}_{\mathbf{Mr}} : Y \longrightarrow \mathrm{Alb}_{\mathbf{Mr}}(Y)$*.*
If $\mathcal{F}$ *is a formal group in* $\underline{\mathrm{Div}}_Y^0$*, then the universal object for* $\mathbf{Mr}_{\mathcal{F}}$ *is also denoted by* $\mathrm{alb}_{\mathcal{F}} : Y \longrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)$*.*
For $\mathcal{F} = 0$ *the universal object for* $\mathbf{Mr}_0$ *is usually simply denoted by* $\mathrm{alb} : Y \longrightarrow \mathrm{Alb}(Y)$*.*

Remark 3.14 *In the proof of Theorem 3.12 we have seen that* $\mathrm{Alb}_{\mathcal{F}}(Y)$ *is an extension of the abelian variety* $\mathrm{Alb}(Y)$ *by the linear group* $\mathcal{F}^\vee$*, and the rational map* $(\mathrm{alb}_{\mathcal{F}} : Y \longrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)) \in \mathbf{Mr}_{\mathcal{F}}$ *corresponds to the identity* $\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}$*.*
More precisely, $[0 \longrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)]$ *is the dual 1-motive of* $\left[\mathcal{F} \longrightarrow \mathrm{Pic}_Y^0\right]$*.*

Example 3.15 *As mentioned above, the universal object* $\mathrm{alb} : Y \longrightarrow \mathrm{Alb}(Y)$ *for* $\mathbf{Mr}_0$ *from Example 3.7 is the* classical Albanese mapping *and* $\mathrm{Alb}(Y)$ *the* classical Albanese variety *of a normal projective variety* Y*.*

Example 3.16 *The universal object* $\mathrm{alb}_{\mathcal{F}_D} : Y \longrightarrow \mathrm{Alb}_{\mathcal{F}_D}(Y)$ *for* $\mathbf{Mr}_{\mathcal{F}_D}$ *from Example 3.8 is the* generalized Albanese of Serre *(see [S3]).*

Example 3.17 *The universal object* $\mathrm{alb}_{\mathcal{F}_{\mathfrak{d}}} : C \longrightarrow \mathrm{Alb}_{\mathcal{F}_{\mathfrak{d}}}(Y)$ *for* $\mathbf{Mr}_{\mathcal{F}_{\mathfrak{d}}}$ *from Example 3.9 is Rosenlicht's* generalized Jacobian $J_{\mathfrak{d}}$ *to the modulus* $\mathfrak{d}$ *(see [S]).*

Example 3.18 *The univeral object* $\mathrm{alb}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}} : X_{\mathrm{reg}} \longrightarrow \mathrm{Alb}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}\left(\widetilde{X}\right)$ *from Example 3.10 is the* universal regular quotient *of the Chow-group of points* $\mathrm{CH}_0(X)_{\deg 0}$ *(see [ESV]). In the following we will simply denote it by* $\mathrm{Alb}\,(X)$. *This is consistent, as in the case that* X *is normal it coincides with the classical Albanese variety.*

Remark 3.19 *Also the generalized Albanese of Serre (see Example 3.16) and the generalized Jacobian (see Example 3.17) can be interpreted as special cases of the universal regular quotient (see Example 3.18) by constructing an appropriate singular variety* X.

3.2.2 Functoriality

The Question is wether a morphism of normal projective varieties induces a homomorphism of algebraic groups between universal objects.

Proposition 3.20 *Let* $\sigma : V \longrightarrow Y$ *be a morphism of normal projective varieties. Let* $\mathbf{Vr}$ *and* $\mathbf{Yr}$ *be categories of rational maps from* V *and* Y *respectively to algebraic groups, and suppose there exist universal objects* $\mathrm{Alb}_{\mathbf{Vr}}(V)$ *and* $\mathrm{Alb}_{\mathbf{Yr}}(Y)$ *for* $\mathbf{Vr}$ *and* $\mathbf{Yr}$ *respectively. The universal property of* $\mathrm{Alb}_{\mathbf{Vr}}(V)$ *yields:*
If the composition $\mathrm{alb}_{\mathbf{Yr}} \circ \sigma : V \longrightarrow \mathrm{Alb}_{\mathbf{Yr}}(Y)$ *is an object of* $\mathbf{Vr}$, *then* σ *induces a homomorphism of algebraic groups*

$$\mathrm{Alb}^{\mathbf{Yr}}_{\mathbf{Vr}}(\sigma) : \mathrm{Alb}_{\mathbf{Vr}}(V) \longrightarrow \mathrm{Alb}_{\mathbf{Yr}}(Y)$$

More precisely, in this case we have a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\quad\sigma\quad} & Y \\ {\scriptstyle \mathrm{alb}_{\mathbf{Vr}}}\downarrow & & \downarrow{\scriptstyle \mathrm{alb}_{\mathbf{Yr}}} \\ \mathrm{Alb}_{\mathbf{Vr}}(V) & \xrightarrow{\mathrm{Alb}^{\mathbf{Yr}}_{\mathbf{Vr}}(\sigma)} & \mathrm{Alb}_{\mathbf{Yr}}(Y) \end{array}$$

Theorem 3.12 allows to give a more explicit description. But first we need some notation:

Definition 3.21 *Let* $\mathcal{F}$ *be a subfunctor of* $\underline{\mathrm{Div}}^0_Y$ *which is a formal group. The support of* $\mathcal{F} \subset \underline{\mathrm{Div}}^0_Y$ *is defined to be*

$$\mathrm{Supp}\,(\mathcal{F}) = \bigcup_{D \,\epsilon\, \mathcal{F}} \mathrm{Supp}\,(D)$$

where Supp(D) *is the support of a Cartier divisor on Y or of a deformation of the zero divisor (see Definition 2.24).*

Supp$(\mathcal{F})$ is a closed subscheme of codimension 1 in Y, since $\mathcal{F}(k)$ and Lie$(\mathcal{F})$ are both finitely generated.

Definition 3.22 *Let $\sigma : V \longrightarrow Y$ be a morphism of varieties. Then V is called* decident *to a subset $S \subset Y$, if for no irreducible component Z of V the image $\sigma(Z)$ is contained in S.*

Definition 3.23 *For a morphism $\sigma : V \longrightarrow Y$ of varieties define $\underline{\mathrm{Dec}}_{Y,V}$ to be the plain subfunctor of $\underline{\mathrm{Div}}_Y$ consisting of families of Cartier divisors to which V is decident, i.e. V decident to* Supp(D) *for all $D \in \underline{\mathrm{Dec}}_{Y,V}$.*

Proposition 3.24 *Let $\sigma : V \longrightarrow Y$ be a morphism of varieties. Then the pull-back of Cartier divisors σ^* induces a natural transformation of functors*

$$\_ \cdot V : \underline{\mathrm{Dec}}_{Y,V} \longrightarrow \underline{\mathrm{Div}}_V$$

Proof. It suffices to mention that on the subgroups $\underline{\mathrm{Dec}}_{Y,V}(k) \subset \mathcal{K}_Y^* / \mathcal{O}_Y^*$ and Lie$\left(\underline{\mathrm{Dec}}_{Y,V}\right) \subset \mathcal{K}_Y / \mathcal{O}_Y$ the pull-backs $\sigma^*(k) : \underline{\mathrm{Dec}}_{Y,V}(k) \longrightarrow \mathcal{K}_V^* / \mathcal{O}_V^*$ and Lie(σ^*) : Lie$\left(\underline{\mathrm{Dec}}_{Y,V}\right) \longrightarrow \mathcal{K}_V / \mathcal{O}_V$ are defined and extend to a natural transformation of functors $\underline{\mathrm{Dec}}_{Y,V} \longrightarrow \underline{\mathrm{Div}}_V$. ∎

Using this notation we obtain

Proposition 3.25 *Let $\sigma : V \longrightarrow Y$ be a morphism of normal projective varieties. Let $\mathcal{F} \subset \underline{\mathrm{Div}}_Y^0$ be a formal group with V decident (see Definition 3.22) to* Supp$(\mathcal{F})$.
For each formal group $\mathcal{G} \subset \underline{\mathrm{Div}}_V^0$ satisfying $\mathcal{G} \supset \mathcal{F} \cdot V$, the pull-back of relative Cartier divisors and of line bundles induces a transformation of 1-motives

$$\begin{bmatrix} \mathcal{G} \\ \downarrow \\ \underline{\mathrm{Pic}}_V^0 \end{bmatrix} \longleftarrow \begin{bmatrix} \mathcal{F} \\ \downarrow \\ \underline{\mathrm{Pic}}_Y^0 \end{bmatrix}$$

Remembering the construction of the universal objects (see Remark 3.14), dualization of 1-motives translates Proposition 3.25 into the following reformulation of Proposition 3.20:

Proposition 3.26 *Let $\sigma : V \longrightarrow Y$ be a morphism of normal projective varieties. Let $\mathcal{F} \subset \underline{\mathrm{Div}}_Y^0$ be a formal group with V decident (see Definition 3.22) to* $\mathrm{Supp}(\mathcal{F})$. *Then σ induces a homomorphism of algebraic groups*

$$\mathrm{Alb}_{\mathcal{G}}^{\mathcal{F}}(\sigma) : \mathrm{Alb}_{\mathcal{G}}(V) \longrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)$$

for each formal group $\mathcal{G} \subset \underline{\mathrm{Div}}_V^0$ satisfying $\mathcal{G} \supset \mathcal{F} \cdot V$. More precisely, we obtain a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\quad\sigma\quad} & Y \\ {\scriptstyle \mathrm{alb}_{\mathcal{G}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{alb}_{\mathcal{F}}} \\ \mathrm{Alb}_{\mathcal{G}}(V) & \xrightarrow{\mathrm{Alb}_{\mathcal{G}}^{\mathcal{F}}(\sigma)} & \mathrm{Alb}_{\mathcal{F}}(Y) \end{array}$$

4 Rational Maps Factoring through $\mathrm{CH}_0(X)_{\deg 0}$

Throughout this section let X be a projective variety over k (an algebraically closed field of characteristic 0), $\pi : Y \longrightarrow X$ its normalization, and let $U \subset Y$ be an open dense subset of Y where π is an isomorphism. U is identified with its image in X, and we suppose $U \subset X_{\mathrm{reg}}$. We consider the category $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ of morphisms $\varphi : U \longrightarrow G$ from U to algebraic groups G factoring through $\mathrm{CH}_0(X)_{\deg 0}$ (see Definition 4.17). The image of such a homomorphism being a connected algebraic subgroup of G (see [ESV] Lemma 1.15), it is no restriction to assume the algebraic group G to be connected.

The goal of this section is to show that the category $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ is equivalent to the category $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{Y/X}}$ of rational maps which induce a transformation of formal groups to $\underline{\mathrm{Div}}^0_{Y/X}$ (see Proposition 2.70).

4.1 Chow Group of Points

In this subsection the Chow group $\mathrm{CH}_0(X)_{\deg 0}$ of 0-cycles of degree 0 modulo rational equivalence is presented, quite similar as in [LW], see also [ESV], [BiS].

Definition 4.1 *A* Cartier curve *in X, relative to U, is a curve $C \subset X$ satisfying*

(a) C is pure of dimension 1
(b) no component of C is contained in $X \setminus U$
(c) if $p \in C \setminus U$, then the ideal of C in $\mathcal{O}_{X,p}$ is generated by a regular sequence

Definition 4.2 *Let C be a Cartier curve in X relative to U, $\mathrm{Cp}\,(C)$ the set of irreducible components of C and γ_Z the generic points of $Z \in \mathrm{Cp}\,(C)$. Let $\mathcal{O}_{C,\Theta}$ be the semilocal ring on C at $\Theta = (C \setminus U) \cup \{\gamma_Z \mid Z \in \mathrm{Cp}\,(C)\}$. There is a natural map on unit groups*

$$\vartheta_{C,U} : \mathcal{O}^*_{C,\Theta} \longrightarrow \bigoplus_{Z \in \mathrm{Cp}(C)} \mathrm{K}^*_Z$$

Then define

$$\mathrm{K}\,(C,U)^* = \mathrm{im}\,\vartheta_{C,U}$$

Definition 4.3 *Let C be a Cartier curve in X relative to U and $\nu : \widetilde{C} \longrightarrow C$ its normalization. For $f \in \mathrm{K}\,(C,U)^*$ and $p \in C$ let*

$$\mathrm{ord}_p\,(f) = \sum_{\widetilde{p} \to p} v_{\widetilde{p}}\left(\widetilde{f}\right)$$

where $\widetilde{f} := \nu^{\#} f \in \mathcal{K}_{\widetilde{C}}$ *and* $v_{\widetilde{p}}$ *is the discrete valuation attached to the point* $\widetilde{p} \in \widetilde{C}$ *above* $p \in C$ *(see [F] Example A.3.1).*
Define the divisor of f *to be*

$$\operatorname{div}(f)_C = \sum_{p \in C} \operatorname{ord}_p(f)\ [p]$$

Definition 4.4 *Let* $\mathrm{Z}_0(U)$ *be the group of 0-cycles on* U, *set*

$$\mathfrak{R}_0(X,U) = \left\{ (C,f) \,\middle|\, \begin{array}{c} C \text{ is a Cartier curve in } X \text{ relative to } U \\ \text{and } f \in \mathrm{K}(C,U)^* \end{array} \right\}$$

and let $\mathrm{R}_0(X,U)$ *be the subgroup of* $\mathrm{Z}_0(U)$ *generated by the elements* $\operatorname{div}(f)_C$ *with* $(C,f) \in \mathfrak{R}_0(X,U)$. *Then define*

$$\mathrm{CH}_0(X) = \mathrm{Z}_0(U) \,/\, \mathrm{R}_0(X,U)$$

Let $\mathrm{CH}_0(X)_{\deg 0}$ *be the subgroup of* $\mathrm{CH}_0(X)$ *of cycles* ζ *with* $\deg \zeta|_W = 0$ *for all irreducible components* $W \in \mathrm{Cp}(U)$ *of* U.

Remark 4.5 *The definition of* $\mathrm{CH}_0(X)$ *and* $\mathrm{CH}_0(X)_{\deg 0}$ *is independent of the choice of the dense open subscheme* $U \subset X_{\mathrm{reg}}$ *(see [ESV] Corollary 1.4).*

Remark 4.6 *Note that by our terminology a curve is always reduced, in particular a Cartier curve. In the literature, e.g. [ESV], [LW], a slightly different definition of* Cartier curve *seems to be common, which allows non-reduced Cartier curves. Actually this does not change the groups* $\mathrm{CH}_0(X)$ *and* $\mathrm{CH}_0(X)_{\deg 0}$, *see [ESV] Lemma 1.3 for more explanation.*

4.2 Local Symbols

The description of rational maps factoring through $\mathrm{CH}_0(X)_{\deg 0}$ requires the notion of a *local symbol* as in [S].

Let C be a smooth projective curve. The composition law of an unspecified algebraic group G is written additively in this subsection.

Definition 4.7 *For an effective divisor* $\mathfrak{d} = \sum n_p\, p$ *on* C *and a rational function* $f \in \mathcal{K}_C$ *define*

$$f \equiv 1 \mod \mathfrak{d} \qquad :\Longleftrightarrow \qquad v_p(1-f) \geq n_p \qquad \forall p \in \operatorname{Supp}(\mathfrak{d})$$

where v_p *is the valution attached to the point* $p \in C$.

Let $\psi : C \longrightarrow G$ be a rational map from C to an algebraic group G which is regular away from a finite subset S. The morphism $\psi : C \setminus S \longrightarrow G$ extends to a homomorphism from the group of 0-cycles $\mathrm{Z}_0(C \setminus S)$ to G by setting $\psi\left(\sum l_c c\right) := \sum l_c \psi(c)$ for $c \in C \setminus S$, $l_c \in \mathbb{Z}$, $l_c = 0$ *p.p.*

Definition 4.8 *An effective divisor $\mathfrak{d}$ on C is said to be a* modulus *for ψ if $\psi(\operatorname{div}(f)) = 0$ for all $f \in \mathcal{K}_C$ with $f \equiv 1 \mod \mathfrak{d}$.*

Theorem 4.9 *Let $\psi : C \longrightarrow G$ be a rational map from C to an algebraic group G and S the finite subset of C where ψ is not regular. Then ψ has a modulus supported on S.*

This theorem is proven in [S] Chapter III, §2, using the following concept:

Definition 4.10 *Let $\mathfrak{d}$ be an effective divisor supported on $S \subset C$ and $\psi : C \longrightarrow G$ a rational function from C to an algebraic group G, regular away from S. A* local symbol *associated to ψ and $\mathfrak{d}$ is a function*

$$(\psi, \_)_{\_} : \mathcal{K}_C^* \times C \longrightarrow G$$

which assigns to $f \in \mathcal{K}_C^*$ and $p \in C$ an element $(\psi, f)_p \in G$, satisfying the following conditions:

(a)	$(\psi, fg)_p = (\psi, f)_p + (\psi, g)_p$	
(b)	$(\psi, f)_c = v_c(f)\, \psi(c)$	*if* $c \in C \setminus S$
(c)	$(\psi, f)_s = 0$	*if* $s \in S$ *and* $f \equiv 1 \mod \mathfrak{d}$ *at* s
(d)	$\sum_{p \in C} (\psi, f)_p = 0$	

Proposition 4.11 *The rational map ψ has a modulus $\mathfrak{d}$ if and only if there exists a local symbol associated to ψ and $\mathfrak{d}$, and this symbol is then unique.*

Proof. [S] Chapter III, No. 1, Proposition 1. ■

Theorem 4.9 in combination with Proposition 4.11 states for each rational map $\psi : C \longrightarrow G$ the existence of a modulus $\mathfrak{d}$ for ψ and of a unique local symbol $(\psi, \_)_{\_}$ associated to ψ and $\mathfrak{d}$.

Let $\mathfrak{d} = \sum n_p p$ and $\mathfrak{e} = \sum m_p p$ be effective divisors, we define

$$\mathfrak{e} \geq \mathfrak{d} \qquad :\Longleftrightarrow \qquad m_p \geq n_p \qquad \forall p \in C$$

From the definitions it is clear that if $\mathfrak{d}$ is a modulus for ψ then $\mathfrak{e}$ is also for all $\mathfrak{e} \geq \mathfrak{d}$. Likewise a local symbol $(\psi, \_)_{\_}$ associated to ψ and $\mathfrak{d}$ is also associated to ψ and $\mathfrak{e}$ for all $\mathfrak{e} \geq \mathfrak{d}$.

Suppose we are given two moduli $\mathfrak{d}$ and $\mathfrak{d}'$ for ψ, and hence two local symbols $(\psi, \_)_{\_}$ and $(\psi, \_)'_{\_}$ associated to $\mathfrak{d}$ and $\mathfrak{d}'$ respectively. Then both local symbols are also associated to $\mathfrak{e} := \mathfrak{d} + \mathfrak{d}'$. The uniqueness of the local symbol associated to ψ and $\mathfrak{e}$ implies that $(\psi, \_)_{\_}$ and $(\psi, \_)'_{\_}$ coincide. It is therefore morally justified to speak about *the local symbol associated to* ψ (without mentioning a modulus).

Corollary 4.12 *For each rational map $\psi : C \longrightarrow G$ from C to an algebraic group G there exists a unique associated local symbol $(\psi, \_)_{\_} : \mathcal{K}_C^* \times C \longrightarrow G$. If $\mathfrak{d}$ is a modulus for ψ supported on S, then this local symbol is given by*

$$\begin{array}{rcll} (\psi, f)_c & = & v_c(f)\ \ \psi(c) & \forall c \in C \setminus S \\ (\psi, f)_s & = & -\sum\limits_{c \notin S} v_c(f_s)\ \ \psi(c) & \forall s \in S \end{array}$$

where $f_s \in \mathcal{K}_C^*$ is a rational function with $f_s \equiv 1 \mod \mathfrak{d}$ at z for all $z \in S \setminus s$ and $f/f_s \equiv 1 \mod \mathfrak{d}$ at s.

The above formula is shown in [S] Chapter III, No. 1, in the proof of Propostion 1.

Example 4.13 *In the case that G is the multiplicative group $\mathbb{G}_{\mathrm{m}}$, a rational map $\psi : C \longrightarrow \mathbb{G}_{\mathrm{m}}$ can be identified with a rational function in $\mathcal{K}_C$, and S is the set of zeroes and poles of ψ, i.e. $S = \mathrm{Supp}\,(\mathrm{div}\,(\psi))$. Then the local symbol associated to ψ is given by*

$$(\psi, f)_p = (-1)^{mn} \frac{\psi^m}{f^n}(p) \qquad \text{with } m = v_p(f),\, n = v_p(\psi)$$

(see [S] Chapter III, No. 4, Proposition 6)

Example 4.14 *In the case that G is the additive group $\mathbb{G}_{\mathrm{a}}$, a rational map $\psi : C \longrightarrow \mathbb{G}_{\mathrm{a}}$ can be identified with a rational function in $\mathcal{K}_C$, and S is the set of poles of ψ. Then the local symbol associated to ψ is given by*

$$(\psi, f)_p = \mathrm{Res}_p\,(\psi\ \mathrm{d}f/f)$$

(see [S] Chapter III, No. 3, Proposition 5)

Proposition 4.15 *Let $\varphi, \psi : C \longrightarrow G$ be two rational maps from C to an algebraic group G, with associated local symbols $(\varphi, \_)_{\_}$ and $(\psi, \_)_{\_}$. Then the local symbol $(\varphi + \psi, \_)_{\_}$ associated to the rational map $\varphi + \psi : C \longrightarrow G$, $c \longmapsto \varphi(c) + \psi(c)$ is given by*

$$(\varphi + \psi, f)_p = (\varphi, f)_p + (\psi, f)_p$$

Proof. Let $\mathfrak{d}_\varphi$ be a modulus for φ and $\mathfrak{d}_\psi$ one for ψ. Then both maps φ, ψ and the map $\varphi + \psi$ have $\mathfrak{d}_{\varphi+\psi} := \mathfrak{d}_\varphi + \mathfrak{d}_\psi$ as a modulus and both local symbols $(\varphi, \_)_{\_}$ and $(\psi, \_)_{\_}$ are associated to $\mathfrak{d}_{\varphi+\psi}$. Now the formula in Corollary 4.12 and the distributive law imply the assertion. ■

Lemma 4.16 *Let $\psi : C \longrightarrow G_\lambda$ be a rational map from C to an algebraic group G which is an L-bundle over an algebraic variety A, i.e. $G \in \operatorname{Ext}(A, L)$, where L is a linear group. Let $p \in C$ be a point, $U \ni p$ a neighbourhood and $\Phi : U \times L \xrightarrow{\sim} G_U$, $(u, l) \longmapsto \phi(u) + l$ a local trivialization of the induced L-bundle $G_C = G \times_A C$ over C, i.e. $\phi : U \longrightarrow G_C$ a local section. Moreover let $[\psi]_\Phi : C \longrightarrow L$, $c \longmapsto \psi(c) - \phi(c)$ be the rational map ψ considered in the local trivialization Φ. Then for each rational function $f \in \mathcal{O}^*_{C,p}$, which is a unit at p, it holds*

$$(\psi, f)_p = ([\psi]_\Phi, f)_p$$

Proof. Proposition 4.15 yields

$$\begin{aligned} ([\psi]_\Phi, f)_p &= (\psi - \phi, f)_p \\ &= (\psi, f)_p - (\phi, f)_p \end{aligned}$$

ϕ is regular at p, therefore we have $(\phi, f)_p = v_p(f) \cdot \phi(s) = 0$, since $f \in \mathcal{O}^*_{C,p}$ and hence $v_p(f) = 0$. ■

4.3 The Category $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$

Definition 4.17 $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ *is the category of rational maps from X to algebraic groups defined as follows: The objects of* $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ *are morphisms $\varphi : U \longrightarrow G$ whose associated map on pairs*

$$\begin{array}{rccc} \varphi^{\Pi} : & \bigcup_{Z \in \mathrm{Cp}(U)} Z \times Z & \longrightarrow & G \\ & (p,q) & \longmapsto & \varphi(p) - \varphi(q) \end{array}$$

applied to k-rational points of X, factors through a homomorphism of groups $\mathrm{CH}_0(X)_{\deg 0} \longrightarrow G(k)$. [3]
We refer to the objects of $\mathbf{Mr}^{\mathrm{CH}_0(x)_{\deg 0}}$ *as rational maps from X to algebraic groups* factoring through rational equivalence *or* factoring through $\mathrm{CH}_0(X)_{\deg 0}$.

Theorem 4.18 *The category* $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ *of morphisms from U to algebraic groups factoring through* $\mathrm{CH}_0(X)_{\deg 0}$ *is equivalent to the category* $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{Y/X}}$ *of rational maps from Y to algebraic groups which induce a transformation of formal groups to* $\underline{\mathrm{Div}}^0_{Y/X}$ *(see Proposition 2.70).*

Proof. First notice that a rational map from Y to an algebraic group which induces a transformation to $\underline{\mathrm{Div}}^0_{Y/X}$ is necessarily regular on U, since all $D \in \underline{\mathrm{Div}}^0_{Y/X}$ have support only on $Y \setminus U$. Then according to Definition 4.4 and Definition 3.6 the task is to show that for a morphism $\varphi : U \longrightarrow G$ from U to an algebraic group G with canonical decomposition $0 \to L \to G \to A \to 0$ the following conditions are equivalent:

(i) $\varphi\,(\mathrm{div}\,(f)_C) = 0 \qquad \forall\,(C,f) \in \mathfrak{R}_0\,(X,U)$

(ii) $\mathrm{div}_{\mathbb{L}}\,(\varphi_{Y,\lambda}) \in \underline{\mathrm{Div}}^0_{Y/X} \qquad \forall \lambda \in L^{\vee}$

where $\varphi_{Y,\lambda}$ is the induced section of the $\mathbb{L}$-bundle $\lambda_* G_Y$ over Y introduced in Subsection 3.1. The principal L-bundle G is a direct sum of $\mathbb{L}$-bundles $\lambda_* G$ over A, $\lambda \in L^{\vee}$; let $\varphi_\lambda : U \longrightarrow \lambda_* G$ be the induced morphims. Then condition (i) is equivalent to

(i') $\varphi_\lambda\,(\mathrm{div}\,(f)_C) = 0 \qquad \forall \lambda \in L^{\vee},\ \forall\,(C,f) \in \mathfrak{R}_0\,(X,U)$

Hence it comes down to show that for all $\lambda \in L^{\vee}$ the following conditions are equivalent:

(j) $\varphi_\lambda\,(\mathrm{div}\,(f)_C) = 0 \qquad \forall\,(C,f) \in \mathfrak{R}_0\,(X,U)$

(jj) $\mathrm{div}_{\mathbb{L}}\,(\varphi_{Y,\lambda}) \in \underline{\mathrm{Div}}^0_{Y/X}$

This is content of Lemma 4.19. ■

[3] A category of rational maps to algebraic groups is defined already by its objects, according to Remark 3.5.

Lemma 4.19 *Let $\varphi_\lambda : U \longrightarrow G_\lambda$ be a morphism from U to a $\mathbb{L}$-bundle G_λ over an abelian variety A, i.e. $G_\lambda \in \operatorname{Ext}(A, \mathbb{L})$. Then the following conditions are equivalent:*

(i) $\varphi_\lambda (\operatorname{div}(f)_C) = 0 \qquad \forall (C, f) \in \mathfrak{R}_0(X, U)$

(ii) $\pi_* (\operatorname{div}_{\mathbb{L}} (\varphi_{Y,\lambda})) = 0$

(iii) $\operatorname{div}_{\mathbb{L}} (\varphi_{Y,\lambda}) \in \underline{\operatorname{Div}}^0_{Y/X}$

Proof. (i)$\Longleftrightarrow$(ii) Let C be a Cartier curve in X relative to U, and let $\nu : \widetilde{C} \longrightarrow C$ be its normalization. In the case $\mathbb{L} = \mathbb{G}_{\mathrm{m}}$ Lemma 4.20 and in the case $\mathbb{L} = \mathbb{G}_{\mathrm{a}}$ Lemma 4.21 asserts that the following conditions are equivalent:

(j) $\varphi_\lambda|_C (\operatorname{div}(f)) = 0 \qquad \forall f \in \mathrm{K}(C, C \cap U)^*$

(jj) $\nu_* (\operatorname{div}_{\mathbb{L}} (\varphi_\lambda|_C)) = 0$

We have $\operatorname{div}_{\mathbb{L}} (\varphi_\lambda|_C) = \operatorname{div}_{\mathbb{L}} (\varphi_{Y,\lambda}) \cdot \widetilde{C}$, where $\_ \cdot \widetilde{C} : \underline{\operatorname{Dec}}_{Y,\widetilde{C}} \longrightarrow \underline{\operatorname{Div}}_{\widetilde{C}}$ is the pull-back of Cartier divisors from Y to $\widetilde{C}$ (see Definition 3.23, Proposition 3.24). Then condition (i) is equivalent to

(i') $(\nu_C)_* \left(\operatorname{div}_{\mathbb{L}} (\varphi_{Y,\lambda}) \cdot \widetilde{C}\right) = 0 \qquad \forall$ Cartier curves C relative to U

The equivalence of (i') and (ii) is shown in Lemma 4.23 for $\mathbb{L} = \mathbb{G}_{\mathrm{m}}$ and in Lemma 4.27 for $\mathbb{L} = \mathbb{G}_{\mathrm{a}}$.

(ii)$\Longleftrightarrow$(iii) This is just the definition of $\underline{\operatorname{Div}}^0_{Y/X}$ (see Proposition 2.70), taking into account that $\operatorname{div}_{\mathbb{L}} (\varphi_{Y,\lambda}) \in \underline{\operatorname{Div}}^0_Y$ by Proposition 3.3. ■

Lemma 4.20 *Let C be a projective curve and $\nu : \widetilde{C} \longrightarrow C$ its normalization. Let $\psi : C \longrightarrow G_\mu$ be rational map from C to a $\mathbb{G}_{\mathrm{m}}$-bundle G_μ over an abelian variety A, i.e. $G_\mu \in \operatorname{Ext}(A, \mathbb{G}_{\mathrm{m}})$. Suppose that ψ is regular on a dense open subset $U_C \subset C_{\mathrm{reg}}$, which we identify with its preimage in $\widetilde{C}$. Then the following conditions are equivalent:*

(i) $\psi (\operatorname{div}(f)) = 0 \qquad \forall f \in \mathrm{K}(C, U_C)^*$

(ii) $(f \circ \nu) (\operatorname{div}_{\mathbb{G}_{\mathrm{m}}} (\psi)) = 0 \qquad \forall f \in \mathrm{K}(C, U_C)^*$

(iii) $\nu_* (\operatorname{div}_{\mathbb{G}_{\mathrm{m}}} (\psi)) = 0$

Proof. (i)$\Longleftrightarrow$(ii) We show that for all $f \in \mathrm{K}(C, U_C)^*$ it holds

$$\psi (\operatorname{div}(f)) = (f \circ \nu) (\operatorname{div}_{\mathbb{G}_{\mathrm{m}}} (\psi))$$

Let $f \in \mathrm{K}(C, U_C)^*$. Write $\widetilde{f} := \nu^\# f = f \circ \nu$. Set $S := \widetilde{C} \setminus U_C$. For each $s \in S$ let $\Phi_s : U_s \times \mathbb{G}_{\mathrm{m}} \longrightarrow G_\mu$ be a local trivializtion of the induced $\mathbb{G}_{\mathrm{m}}$-bundle over $\widetilde{C}$ in a neighbourhood $U_s \ni s$. Notice that $v_p(\psi) := v_p\left([\psi]_{\Phi_p}\right)$ is independent of the local trivialization. Since $f \in \mathrm{K}(C, U_C)^*$ we have $f \in \mathcal{O}^*_{C,s}$ for all $s \in S$ and hence $\operatorname{div}(f) \cap S = \varnothing$. Then using Lemma 4.22, the defining properties of a local symbol from Definition 4.10, the explicit description from Example 4.13 of local symbols for rational maps to $\mathbb{G}_{\mathrm{m}}$ and Lemma 4.16 we obtain

$$\begin{aligned}
\psi(\operatorname{div}(f)) &= (\psi\circ\nu)\left(\operatorname{div}\left(\widetilde{f}\right)\right) \\
&= \prod_{c\notin S}\psi(c)^{v_c(\widetilde{f})} \\
&= \prod_{c\notin S}\left(\psi,\widetilde{f}\right)_c \\
&= \prod_{s\in S}\left(\psi,\widetilde{f}\right)_s^{-1} \\
&= \prod_{s\in S}\left([\psi]_{\Phi_s},\widetilde{f}\right)_s^{-1} \\
&= \prod_{s\in S}\left(\widetilde{f},[\psi]_{\Phi_s}\right)_s \\
&= \prod_{p\notin \operatorname{Supp}\left(\operatorname{div}\left(\widetilde{f}\right)\right)}\widetilde{f}(p)^{v_p(\psi)} \\
&= \widetilde{f}\left(\operatorname{div}_{\mathbb{G}_{\mathrm{m}}}(\psi)\right)
\end{aligned}$$

(ii)$\Longleftrightarrow$(iii) The implication (iii)$\Longrightarrow$(ii) is clear. For the converse direction first observe that the support of $\operatorname{div}_{\mathbb{G}_{\mathrm{m}}}(\psi)$ lies necessarily in $\widetilde{C}\backslash U_C$, since ψ is regular on U_C. For each $s\in C\backslash U_C$ there is a rational function $f_s\in \mathrm{K}(C,U_C)^*$ such that $f(s)=t\in\mathbb{G}_{\mathrm{m}}\setminus\{1\}$ and $f(z)=1$ for all $z\in C\setminus(U_C\cup\{s\})$ by the approximation theorem. Then $(f_s\circ\nu)(\operatorname{div}_{\mathbb{G}_{\mathrm{m}}}(\psi))=0$ if and only if $\nu_*\left(\operatorname{div}_{\mathbb{G}_{\mathrm{m}}}(\psi)|_{\nu^{-1}(s)}\right)=0$, where $\operatorname{div}_{\mathbb{G}_{\mathrm{m}}}(\psi)|_{\nu^{-1}(s)}$ is the part of $\operatorname{div}_{\mathbb{G}_{\mathrm{m}}}(\psi)$ which has support on $\nu^{-1}(s)$. As this is true for all $s\in C\setminus U_C$, it shows the implication (ii)$\Longrightarrow$(iii). ■

Lemma 4.21 *Let C be a projective curve and $\nu:\widetilde{C}\longrightarrow C$ its normalization. Let $\psi:C\longrightarrow G_\alpha$ be rational map from C to a $\mathbb{G}_{\mathrm{a}}$-bundle G_α over an abelian variety A, i.e. $G_\alpha\in\operatorname{Ext}(A,\mathbb{G}_{\mathrm{a}})$. Suppose that ψ is regular on a dense open subset $U_C\subset C_{\mathrm{reg}}$, which we identify with its preimage in $\widetilde{C}$. Then the following conditions are equivalent:*

(i)	$\psi(\operatorname{div}(f))=0$	$\forall f\in\mathrm{K}(C,U_C)^*$
(ii)	$\operatorname{Res}_q(\psi\,\mathrm{d}g)=0$	$\forall g\in\widehat{\mathcal{O}}_{C,\nu(q)},\ \forall q\in\widetilde{C}$
(iii)	$\nu_*(\operatorname{div}_{\mathbb{G}_{\mathrm{a}}}(\psi))=0$	

Proof. (i)$\Longleftrightarrow$(ii) Let $f\in\mathrm{K}(C,U_C)^*$. Write $f':=\nu^{\#}f=f\circ\nu$. Set $S:=\widetilde{C}\setminus U_C$. For each $s\in S$ let $\Phi_s:U_s\times\mathbb{G}_{\mathrm{a}}\longrightarrow G_\alpha$ be a local trivializtion of the induced $\mathbb{G}_{\mathrm{a}}$-bundle over $\widetilde{C}$ in a neighbourhood $U_s \ni s$. Notice that for each $\omega \in \Omega_{\widetilde{C}}$ which is regular at $q \in \widetilde{C}$ the expression $\mathrm{Res}_q(\psi\ \omega) := \mathrm{Res}_q\left([\psi]_{\Phi_q}\omega\right)$ is independent of the local trivialization. Then using Lemma 4.22, the defining properties of a local symbol from Definition 4.10, the explicit description from Example 4.13 of local symbols for rational maps to $\mathbb{G}_{\mathrm{a}}$ and Lemma 4.16 we obtain

$$\begin{aligned}
\psi(\mathrm{div}(f)) &= (\psi \circ \nu)(\mathrm{div}(f')) \\
&= \sum_{c \notin S} v_c(f')\ \psi(c) \\
&= \sum_{c \notin S} (\psi, f')_c \\
&= -\sum_{s \in S} (\psi, f')_s \\
&= -\sum_{s \in S} \left([\psi]_{\Phi_s}, f'\right)_s \\
&= -\sum_{s \in S} \mathrm{Res}_s(\psi\ \mathrm{d}f'/f')
\end{aligned}$$

Now $\mathrm{d}f/f = \mathrm{d}\log f$ and $\log : 1 + \widehat{\mathfrak{m}}_{C,p} \xrightarrow{\sim} \widehat{\mathfrak{m}}_{C,p}$ is an isomorphism, furthermore it holds $\mathrm{im}\left(\widehat{\mathfrak{m}}_{C,p} \xrightarrow{\mathrm{d}} \Omega_{\widehat{\mathcal{O}}_{C,p}}\right) = \mathrm{im}\left(\widehat{\mathcal{O}}_{C,p} \xrightarrow{\mathrm{d}} \Omega_{\widehat{\mathcal{O}}_{C,p}}\right)$.

Then for each $s \in S$, each $g \in \widehat{\mathcal{O}}_{C,\nu(s)}$ and each effective divisor $\mathfrak{e}$ supported on S there is a rational function $f_s \in \mathrm{K}(C, U_C)^*$ such that $\mathrm{d}\log f'_s \equiv \mathrm{d}g'$ mod $\mathfrak{e}$ at s and $f'_s \equiv 1 \mod \mathfrak{e}$ at z for all $z \in S \setminus s$, by the approximation theorem. Choosing $\mathfrak{e} = \sum_{z \in S} m_z\, z$ large enough, i.e. m_z larger than the pole order of ψ at z, yields that $\mathrm{Res}_z(\psi\ \mathrm{d}f'_s/f'_s) = 0$ for all $z \in S \setminus s$, as $\mathrm{d}f'_s/f'_s$ has a zero of order $\geq m_z - 1$ at $z \in S \setminus s$. Hence $\psi(\mathrm{div}(f_s)) = 0$ if and only if $\mathrm{Res}_s(\psi\ \mathrm{d}f'_s/f'_s) = \mathrm{Res}_s(\psi\ \mathrm{d}g') = 0$. Thus $\psi(\mathrm{div}(f)_C) = 0$ for all $f \in \mathrm{K}(C, U_C)^*$ if and only if $\mathrm{Res}_s(\psi\ \mathrm{d}g') = 0$ for all $g \in \widehat{\mathcal{O}}_{C,\nu(s)}$, $s \in S$. It remains to remark that $\mathrm{Res}_c(\psi\ \mathrm{d}h) = 0$ for all $h \in \widehat{\mathcal{O}}_{\widetilde{C},c} \supset \widehat{\mathcal{O}}_{C,\nu(c)}$, $c \in U_C$, since ψ and $\mathrm{d}h$ are both regular at c.

(ii)$\Longleftrightarrow$(iii) Let $q \in \widetilde{C}$. Then $\mathrm{Res}_q(\psi\ \mathrm{d}g') = 0$ for all $g \in \widehat{\mathcal{O}}_{C,\nu(q)}$ is equivalent to the condition that the image $[\mathrm{div}_{\mathbb{G}_{\mathrm{a}}}(\psi)]_q$ of $\mathrm{div}_{\mathbb{G}_{\mathrm{a}}}(\psi)$ in $\mathrm{Hom}_{k(q),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{\widetilde{C},q}, k(q)\right)$ vanishes on $\widehat{\mathfrak{m}}_{C,\nu(q)}$, by construction (see proof of Proposition 2.63), which says $0 = [\mathrm{div}_{\mathbb{G}_{\mathrm{a}}}(\psi)]_q \circ \widehat{\nu^\#} \in \mathrm{Hom}_{k(q),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{C,\nu(q)}, k(q)\right)$. This is true for all $q \in \widetilde{C}$ if and only if $\nu_*(\mathrm{div}_{\mathbb{G}_{\mathrm{a}}}(\psi)) = 0$ by definition of the push-forward for formal infinitesimal divisors (see Proposition 2.62). ■

Lemma 4.22 *Let C be a Cartier curve in X relative to U and $\nu : \widetilde{C} \longrightarrow C$ its normalization. If $\psi : C \cap U \longrightarrow G$ is a morphism from $C \cap U$ to an algebraic group G, then for each $f \in \mathrm{K}(C, C \cap U)^*$ it holds*

$$\psi\left(\operatorname{div}(f)_C\right) = (\psi \circ \nu)\left(\operatorname{div}(f \circ \nu)_{\widetilde{C}}\right)$$

Proof. By Definition 4.3 we have

$$\begin{aligned}
\psi\left(\operatorname{div}(f)_C\right) &= \sum_{p \in C} \operatorname{ord}_p(f)\ \psi(p) \\
&= \sum_{p \in C} \sum_{\widetilde{p} \to p} v_{\widetilde{p}}(f \circ \nu)\ \psi(p) \\
&= \sum_{\widetilde{p} \in \widetilde{C}} v_{\widetilde{p}}(f \circ \nu)\ (\psi \circ \nu)(\widetilde{p}) \\
&= (\psi \circ \nu)\left(\operatorname{div}(f \circ \nu)_{\widetilde{C}}\right)
\end{aligned}$$

■

Lemma 4.23 *For a curve C in X let $\nu_C : \widetilde{C} \longrightarrow C$ be its normalization. Then*

$$\ker\left(\pi_* \circ \operatorname{weil}_Y\right) = \bigcap_C \left(\_ \cdot \widetilde{C}\right)^{-1} \ker\left(\left(\nu_C\right)_* \circ \operatorname{weil}_{\widetilde{C}}\right)$$

where C ranges over all Cartier curves in X relative to U, $\_ \cdot \widetilde{C} : \underline{\mathrm{Dec}}_{Y,\widetilde{C}} \longrightarrow \underline{\mathrm{Div}}_{\widetilde{C}}$ is the pull-back of Cartier divisors from Y to $\widetilde{C}$ (see Definition 3.23, Proposition 3.24), and $\operatorname{weil}_Z : \underline{\mathrm{Div}}_Z(k) \longrightarrow \underline{\mathrm{WDiv}}_Z(k)$ for $Z = Y, \widetilde{C}$ is the transformation from Cartier divisors to Weil divisors (see Proposition 2.51)

Proof. The lemma asserts that for $D \in \underline{\mathrm{Div}}_Y(k)$ the following conditions are equivalent:

(j) $\pi_*\left(\operatorname{weil}_Y(D)\right) = 0$

(jj) $\left(\nu_C\right)_*\left(\operatorname{weil}_{\widetilde{C}}\left(D \cdot \widetilde{C}\right)\right) = 0$
for all Cartier curves C in X relative to U

As π is an isomorphism on U, we may assume $\operatorname{Supp}(D) \subset Y \setminus U$.

Let C be a Cartier curve in X relative to U. Since $\pi|_{C_Y}$ is of degree 1 (see Definition 2.53), the normalization $\nu : \widetilde{C} \longrightarrow C$ factors uniquely through $\pi|_{C_Y} : C_Y \longrightarrow C$, where $C_Y = \pi^{-1}C$ is the preimage of C in Y. Let $\mu : \widetilde{C} \longrightarrow C_Y$ be the unique map satisfying $\nu = \pi|_{C_Y} \circ \mu$. Let $\iota_Y : C_Y \longrightarrow Y$ be the embedding of C_Y into Y.

At first we are going to show $\nu_*\left(\mathrm{weil}_{\widetilde{C}}\left(D\cdot\widetilde{C}\right)\right)=C\cap\pi_*\left(\mathrm{weil}_Y(D)\right)$:
We have $\nu_*\left(\mathrm{weil}_{\widetilde{C}}\left(D\cdot\widetilde{C}\right)\right)=\left(\pi|_{C_Y}\right)_*\mu_*\left(\mathrm{weil}_{\widetilde{C}}(\mu^*\iota_Y^*D)\right)$, and
$\mu_*\left(\mathrm{weil}_{\widetilde{C}}\left(\mu^*\iota_Y^*D\right)\right)=\deg\left(\widetilde{C}/C_Y\right)\mathrm{weil}_{C_Y}\left(\iota_Y^*D\right)=\mathrm{weil}_{C_Y}\left(D\cdot C_Y\right)$, see proof of [F] Proposition 2.3 (c).

If C is a complete intersection Cartier curve, Lemma 4.26 with Remark 4.25 yields $\left(\pi|_{C_Y}\right)_*\left(\mathrm{weil}_{C_Y}\left(D\cdot C_Y\right)\right)=C\cap\pi_*\left(\mathrm{weil}_Y(D)\right)$.

It holds $\pi_*\left(\mathrm{weil}_Y(D)\right)=0$ if and only if $C\cap\pi_*\left(\mathrm{weil}_Y(D)\right)=0$ for all complete intersection Cartier curves C in X relative to U, since X can be covered by such curves. For any Cartier curve C in X relative to U the support of $D\cdot C_Y$ lies in $C_Y\setminus U$ by assumption on D, and by condition (c) of Definition 4.1 each point of $\pi\left(\mathrm{Supp}\left(D\cdot C_Y\right)\right)$ has a neighbourhood in C which is a complete intersection curve. Hence it suffices to consider complete intersection Cartier curves. This gives the equivalence of (j) and (jj). ■

Definition 4.24 *(cf. [F] Definition 2.3) For a Cartier divisor D on Y and a prime cycle V decident to* Supp (D) *(see Definition 3.22) define the* intersection *of D with V to be the following Weil divisor on V:*

$$D\cap V=\mathrm{weil}_V(D\cdot V)$$

where $\_\cdot V:\underline{\mathrm{Dec}}_{Y,V}\longrightarrow\underline{\mathrm{Div}}_V$ is the pull-back of Cartier divisors from Y to V (see Definition 3.23, Proposition 3.24).
For an arbitrary cycle Z with decomposition $Z=\sum n_V V$ in prime cycles V decident to Supp (D) *set*

$$D\cap Z=\sum n_V\,\mathrm{weil}_V(D\cdot V)$$

If $C=D_1\cdot\ldots\cdot D_c$ is a complete intersection of Cartier divisors and Z a cycle with C and Z decident to each other, then form inductively

$$C\cap Z=D_1\cap\left(D_2\cap\ldots(D_c\cap Z)\right)$$

Remark 4.25 *If W is a Weil divisor in Y and C a complete intersection of Cartier divisors on Y decident to W, then $C\cap W$ is a Weil divisor on C.*

Furthermore, for a complete intersection C in Y the following diagram is commutative:

$$\begin{array}{ccc}\mathrm{Div}(Y) & \xrightarrow{\mathrm{weil}_Y} & \mathrm{WDiv}(Y)\\ {\scriptstyle \_\cdot C}\downarrow & & \downarrow{\scriptstyle C\cap\_}\\ \mathrm{Div}(C) & \xrightarrow{\mathrm{weil}_C} & \mathrm{WDiv}(C)\end{array}$$

where the vertical arrows are defined for those divisors to which C is decident.

Lemma 4.26 *Let C be a curve in X which is a complete intersection of Cartier divisors, and let $C_Y = \pi^{-1}C$ be its preimage in Y. If W is a Weil divisor in Y with C decident to W (see Definition 3.22), then*

$$\left(\pi|_{C_Y}\right)_* (C_Y \cap W) = C \cap \pi_* W$$

Proof. By induction we may assume that C is a Cartier divisor. The diagram of morphisms

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle \iota_Y}\uparrow & & \uparrow{\scriptstyle \iota} \\ C_Y & \xrightarrow{\pi|_{C_Y}} & C \end{array}$$

induces a diagram of homomorphisms

$$\begin{array}{ccc} \mathrm{WDiv}(Y) & \xrightarrow{\pi_*} & \mathrm{WDiv}(X) \\ {\scriptstyle C_Y \cap \_}\,\downarrow & & \downarrow\,{\scriptstyle C \cap \_} \\ \mathrm{WDiv}(C_Y) & \xrightarrow{\left(\pi|_{C_Y}\right)_*} & \mathrm{WDiv}(C) \end{array}$$

where the vertical arrows are defined for those Weil divisors to which C_Y or C respectively is decident. This diagram commutes by projection formula for divisors, see [F] Proposition 2.3 (c) (although the statement there is only asserted for classes of cycles, the proof is done for cycles). This yields the assertion. ■

Lemma 4.27 *For a curve C in X let $\nu_C : \widetilde{C} \longrightarrow C$ be its normalization. Then*

$$\ker\left(\pi_* \circ \mathrm{fml}_Y\right) = \bigcap_C \left(\_ \cdot \widetilde{C}\right)^{-1} \ker\left(\left(\nu_C\right)_* \circ \mathrm{fml}_{\widetilde{C}}\right)$$

where C ranges over all Cartier curves in X relative to U, $\_ \cdot \widetilde{C} : \underline{\mathrm{Dec}}_{Y,\widetilde{C}} \longrightarrow \underline{\mathrm{Div}}_{\widetilde{C}}$ is the pull-back of Cartier divisors from Y to $\widetilde{C}$ (see Definition 3.23, Proposition 3.24), and $\mathrm{fml}_Z : \mathrm{Lie}\left(\underline{\mathrm{Div}}_Z\right) \longrightarrow \mathrm{LDiv}\,(Z)$ for $Z = Y, \widetilde{C}$ is the transformation from deformations of the zero divisor to formal Lie divisors (see Propositions 2.61, 2.63)

Proof. For each Cartier curve C in X relative to U the normalization $\nu_C : \widetilde{C} \longrightarrow C$ factors uniquely through $\pi|_{C_Y} : C_Y \longrightarrow C$, where $C_Y = \pi^{-1}C$ is the preimage of C in Y. Let $\mu_C : \widetilde{C} \longrightarrow C_Y$ be the unique map satisfying $\nu_C = \pi|_{C_Y} \circ \mu_C$. When a fixed curve C is considered, we will sometimes omit the subscript "C".

Since π is an isomorphism on U, both sides of the equation in the statement are contained in $\mathrm{Lie}\left(\underline{\mathrm{Div}}_Y\right)_{Y \setminus U} = \{\delta \in \mathrm{Lie}\left(\underline{\mathrm{Div}}_Y\right) \mid \mathrm{Supp}\,(\delta) \subset Y \setminus U\}$.

By definition of the push-forward for infinitesimal divisors (see Proposition 2.62) we have to show that for $\delta \in \mathrm{Lie}\,(\underline{\mathrm{Div}}_Y)_{Y\setminus U}$ and for all generic points η of height 1 the following conditions are equivalent:

(j) $[\delta]_\eta \in \mathrm{Hom}_{k(\eta),\mathrm{cont}}\,(\widehat{\mathfrak{m}}_{Y,\eta}, k(\eta))$ vanishes on $\widehat{\mathfrak{m}}_{X,\pi(\eta)}$

(jj) $\left[\delta \cdot \widetilde{C}\right]_q \in \mathrm{Hom}_{k(q),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{\widetilde{C},q}, k(q)\right)$ vanishes on $\widehat{\mathfrak{m}}_{C,\nu(q)}$

for all Cartier curves C in X relative to U and all $q \in \mu_C^{-1}E_\eta$

where E_η is the irreducibel codimension 1 subscheme with generic point η. As $\mathrm{Supp}\,(\delta) \subset Y \setminus U$, it suffices to consider $\eta \in Y \setminus U$, hence we may assume that C_Y is decident to E_η, i.e. $\mu_C^{-1}E_\eta$ consists of a finite number of closed points in $\widetilde{C}$.

This problem is local in X, therefore the structure sheaves $\mathcal{O}_Y, \mathcal{O}_X, \mathcal{O}_C$ are thought of as affine rings. $\pi(\eta)$ generates a dense subset of $\widehat{\mathfrak{m}}_{X,\pi(\eta)}$ as $k(\pi(\eta))$-vector space, i.e. $\widehat{\mathfrak{m}}_{X,\pi(\eta)} = \overline{\mathrm{span}_{k(\pi(\eta))}\left(\mathrm{im}\left(\pi(\eta) \longrightarrow \mathfrak{m}_{X,\pi(\eta)}\right)\right)}$. Likewise the prime ideal of $\nu(q)$ in $\mathrm{Spec}\,\mathcal{O}_C$ generates a dense subset of $\widehat{\mathfrak{m}}_{C,\nu(q)}$ as $k(\nu(q))$-vector space, for a Cartier curve C in X relative to U and $q \in \mu^{-1}E_\eta \subset \widetilde{C}$. As $[\delta]_\eta$ is $k(\pi(\eta))$-linear, $\left[\delta \cdot \widetilde{C}\right]_q$ is $k(\nu(q))$-linear and both are continuous, it suffices to show the equivalence of the following conditions:

(j') $[\delta]_\eta$ vanishes on $\pi(\eta) \in \mathrm{Spec}\,\mathcal{O}_X$

(jj') $\left[\delta \cdot \widetilde{C}\right]_q$ vanishes on $\nu(q)_C := \nu(q)/\mathcal{I}_C \in \mathrm{Spec}\,\mathcal{O}_C$

for all Cartier curves C in X relative to U and all $q \in \mu_C^{-1}E_\eta$

where $\mathcal{I}_C$ is the ideal sheaf of C. If $q \in \mu_C^{-1}E_\eta$, then $\nu(q)/\mathcal{I}_C \in \mathrm{Spec}\,\mathcal{O}_C$ is an associated prime of $(\pi(\eta)+\mathcal{I}_C)/\mathcal{I}_C$ in $\mathcal{O}_C = \mathcal{O}_X/\mathcal{I}_C$. Localization at $\nu(q)_C$ on C yields $\mathfrak{m}_{C,\nu(q)} = ((\pi(\eta)+\mathcal{I}_C)/\mathcal{I}_C)_{\nu(q)_C}$.

Let $h \in \pi(\eta)$ be an arbitrary element and let $\overline{h} \in (\pi(\eta)+\mathcal{I}_C)/\mathcal{I}_C$ be its residue class, where C is a Cartier curve in X relative to U. If $[\delta]_\eta h = f \in k(\eta)$ is the image of $h \in \pi(\eta)$ under the homomorphism $[\delta]_\eta$, then for each $q \in \mu^{-1}E_\eta$ which is not a pole of f the image of $\overline{h} \in \nu(q)/\mathcal{I}_C$ under the homomorphism $\left[\delta \cdot \widetilde{C}\right]_q$ is given by $\left[\delta \cdot \widetilde{C}\right]_q \overline{h} = f(\mu(q)) \in k(\mu(q)) = k(q)$, see also Lemma 4.28.

From this the implication (j')$\Longrightarrow$(jj') is clear. For the converse direction suppose $[\delta]_\eta$ does not vanish on $\pi(\eta)$, i.e. there is a $h \in \pi(\eta)$ such that $0 \neq [\delta]_\eta h = f \in k(\eta)$. Then there exists $p \in E_\eta$ with $f(p) \neq 0, \infty$, i.e. $f \in \mathcal{O}_{E_\eta,p} \setminus \mathfrak{m}_{E_\eta,p}$, and a Cartier curve C in X relative to U such that $p \in C_Y \cap E_\eta$. As $\left[\delta \cdot \widetilde{C}\right]_q \overline{h} = f(p) \neq 0$ for $q \in \mu_C^{-1}(p)$, it follows that $\left[\delta \cdot \widetilde{C}\right]_q$ does not vanish on $\nu(q)/\mathcal{I}_C$. ■

Lemma 4.28 *Let C be a curve in X and $\nu : \widetilde{C} \longrightarrow C$ its normalization. Let $\mu : \widetilde{C} \longrightarrow C_Y$ be the unique factorization of ν through $C_Y = \pi^{-1}C$, i.e. $\nu = \pi|_{C_Y} \circ \mu$. Let η be a generic point of height 1 in Y with C_Y decident (see Definition 3.22) to its associated codimension 1 subscheme E_η. Let $\mathcal{I}_C$ be the ideal of C in an affine neighbourhood of $C \cap \pi(E_\eta)$ in X. For $q \in \nu^{-1}(C \cap \pi(E_\eta))$ and $\delta \in \mathrm{Lie}(\underline{\mathrm{Div}}_Y)$ with C_Y decident to $\mathrm{Supp}(\delta)$ we have a commutative diagram*

$$\begin{array}{ccc}
\pi(\eta) \subset \widehat{\mathfrak{m}}_{X,\pi(\eta)} & \xrightarrow{[\delta]_\eta} & k(\eta) \\
\downarrow & \searrow & \uparrow \\
 & & \mathcal{O}_{E_\eta,\mu(q)} \\
\downarrow & & \downarrow \\
\frac{\pi(\eta)+\mathcal{I}_C}{\mathcal{I}_C} \subset \widehat{\mathfrak{m}}_{C,\nu(q)} & \xrightarrow{[\delta\cdot\widetilde{C}]_q} & k(q)
\end{array}$$

Proof. Follows immediately from the construction of the homomorphisms $[\delta]_\eta \in \mathrm{Hom}_{k(\eta),\mathrm{cont}}(\widehat{\mathfrak{m}}_{Y,\eta}, k(\eta))$ and $\left[\delta \cdot \widetilde{C}\right]_q \in \mathrm{Hom}_{k(q),\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{\widetilde{C},q}, k(q)\right)$ in the proof of Proposition 2.63. ■

4.4 Universal Regular Quotient

The results obtained up to now provide the necessary foundations for a description of the universal regular quotient and its dual, which was the initial intention of this work.

4.4.1 Existence and Construction

The universal regular quotient $\mathrm{Alb}(X)$ of a (singular) projective variety X is by definition the universal object for the category $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ of morphisms from $U \subset X_{\mathrm{reg}}$ factoring through $\mathrm{CH}_0(X)_{\deg 0}$ (see Definition 4.17). In Theorem 4.18 we have seen that this category is equivalent to the category $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{Y/X}}$ of rational maps from the normalization Y of X to algebraic groups which induce a transformation to the formal group $\underline{\mathrm{Div}}^0_{Y/X}$. Now Theorem 3.12 implies the existence of a universal object $\mathrm{Alb}_{\underline{\mathrm{Div}}^0_{Y/X}}(Y)$ for this category, which was constructed (see Remark 3.14) as the dual 1-motive of $\left[\underline{\mathrm{Div}}^0_{Y/X} \longrightarrow \underline{\mathrm{Pic}}^0_Y\right]$. As $\mathrm{Alb}(X) = \mathrm{Alb}_{\underline{\mathrm{Div}}^0_{Y/X}}(Y)$, this gives the existence and an explicit construction of the universal regular quotient, as well as a description of its dual. The proof of Theorem 0.3 is thus complete.

4.4.2 Functoriality

The question coming up now in a natural way is the functoriality of the universal regular quotient, i.e. we would like to know wether a morphism $\sigma : V \longrightarrow X$ of projective varieties induces a homomorphism of algebraic groups $\mathrm{Alb}\,(V) \longrightarrow \mathrm{Alb}\,(X)$.

As the functoriality of the universal objects $\mathrm{Alb}_{\mathcal{F}}(Y)$, where Y is normal and $\mathcal{F} \subset \underline{\mathrm{Div}}_Y^0$ is a formal group, has already been treated in Subsubsection 3.2.2, we will reduce the problem to this case. Therefore it obliges to show under which assumptions the following conditions hold:

(α) A morphism of projective varieties $\sigma : V \longrightarrow X$ induces a morphism $\widetilde{\sigma} : \widetilde{V} \longrightarrow \widetilde{X}$ of their normalizations

(β) The pull-back of relative Cartier divisors maps $\underline{\mathrm{Div}}^0_{\widetilde{X}/X} \longrightarrow \underline{\mathrm{Div}}^0_{\widetilde{V}/V}$

For this purpose we introduce the following notion, analogue to Definition 4.1 (keeping the notation fixed at the beginning of this Section 4):

Definition 4.29 *A* Cartier morphism *to X, relative to U, is a morphism $\sigma : V \longrightarrow X$ of projective varieties satisfying*

(a) V is equi-dimensional

(b) V is decident to $X \setminus U$ (see Definition 3.22)

(c) if $\gamma \in \sigma(V) \setminus U$ is a generic point of $\sigma(V) \setminus U$, then the ideal of $\sigma(V)$ in $\mathcal{O}_{X,\gamma}$ is generated by a regular sequence

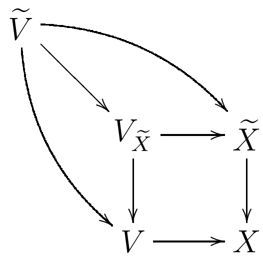
Proposition 4.30 *Let $\sigma : V \longrightarrow X$ be a Cartier morphism relative to U. Then the pull-back of relative Cartier divisors and of line bundles induces a transformation of 1-motives*

$$\begin{bmatrix} \underline{\mathrm{Div}}^0_{\widetilde{V}/V} \\ \downarrow \\ \underline{\mathrm{Pic}}^0_{\widetilde{V}} \end{bmatrix} \longleftarrow \begin{bmatrix} \underline{\mathrm{Div}}^0_{\widetilde{X}/X} \\ \downarrow \\ \underline{\mathrm{Pic}}^0_{\widetilde{X}} \end{bmatrix}$$

Proof. It suffices to verify the conditions (α) and (β) mentioned at the beginning of this Subsubsection 4.4.2.

As $\sigma(V)$ is decident to $X \setminus U$ by condition (b) of Definition 4.29, the base change $V \times_X \widetilde{X} =: V_{\widetilde{X}} \longrightarrow V$ of $\widetilde{X} \longrightarrow X$ is a morphism of degree 1 (see Definition 2.53), hence the normalization $\widetilde{V} \longrightarrow V$ factors through $V_{\widetilde{X}} \longrightarrow V$.

We obtain a commutative diagram



The morphism $\widetilde{V} \longrightarrow \widetilde{X}$ induces a pull-back of families of line bundles $\underline{\mathrm{Pic}}^0_{\widetilde{X}} \longrightarrow \underline{\mathrm{Pic}}^0_{\widetilde{V}}$ and a pull-back of relative Cartier divisors $\underline{\mathrm{Div}}^0_{\widetilde{X}/X} \longrightarrow \underline{\mathrm{Div}}^0_{\widetilde{V}}$, since V is decident to $\mathrm{Supp}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right)$. Lemmata 4.26 and 4.28 hold for $V \longrightarrow X$ instead of $C \longrightarrow X$, as one sees from the proofs. Therefore the image of $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ under pull-back $\_ \cdot \widetilde{V}$ lies actually in $\underline{\mathrm{Div}}^0_{\widetilde{V}/V}$. This gives a commutative diagram of natural transformations of functors

$$\begin{array}{ccc} \underline{\mathrm{Div}}^0_{\widetilde{V}/V} & \longleftarrow & \underline{\mathrm{Div}}^0_{\widetilde{X}/X} \\ \big\downarrow & & \big\downarrow \\ \underline{\mathrm{Pic}}^0_{\widetilde{V}} & \longleftarrow & \underline{\mathrm{Pic}}^0_{\widetilde{X}} \end{array}$$

■

Dualization of 1-motives yields the functoriality of the universal regular quotient:

Proposition 4.31 *Let $\sigma : V \longrightarrow X$ be a Cartier morphism relative to U. Then σ induces a homomorphism of algebraic groups*

$$\mathrm{Alb}(\sigma) : \mathrm{Alb}\,(V) \longrightarrow \mathrm{Alb}\,(Y)$$

More precisely, we obtain a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\sigma} & Y \\ {\scriptstyle \mathrm{alb}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{alb}} \\ \mathrm{Alb}\,(V) & \xrightarrow{\mathrm{Alb}(\sigma)} & \mathrm{Alb}\,(Y) \end{array}$$

5 Case of Curves

Let C be a projective curve. The fact that zero cycles equal divisors on C yields an identification of the universal regular quotient $\mathrm{Alb}\,(C)$ with the component of the identity $\mathrm{Pic}^0\,C$ of the Picard scheme. This offers a different point of view, independent of the previous considerations, which provides a useful tool for the explicit computation of examples.

After introducing an intermediate curve C' between C and its normalization $\widetilde{C}$ we describe the algebraic group $\mathrm{Pic}^0\,C$ as an extension of the abelian variety $\mathrm{Pic}^0\,\widetilde{C}$ by a linear group L. Then we dualize the 1-motive $\left[0 \longrightarrow \mathrm{Pic}^0\,C\right]$, taking into account that $\mathrm{Pic}^0\,C$ is the universal regular quotient $\mathrm{Alb}\,(C)$. The methods of Subsections 5.1 and 5.2 are taken from [BLR].

5.1 Normalization $\widetilde{C}$ and Largest Homeomorphic Curve C'

This section provides the construction of a curve C' lying between C and its normalization $\widetilde{C}$

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\sigma} & C' & \xrightarrow{\rho} & C \\ & & \underset{\pi}{\smile\!\!\!\longrightarrow} & & \end{array}$$

which is homeomorphic to C and has only ordinary multiple points as singularities.

Notation 5.1

$$\mathcal{O} := \mathcal{O}_C \qquad\qquad \mathcal{O}' := \rho_* \mathcal{O}_{C'} \qquad\qquad \widetilde{\mathcal{O}} := \pi_* \mathcal{O}_{\widetilde{C}}$$

Definition 5.2 *A point p of a curve C is called an* ordinary multiple point*, if it marks a transversal crossing of smooth formal local branches.*
More precisely, $p \in C$ is an ordinary m-ple point *if*

$$\widehat{\mathcal{O}}_{C,p} \cong k\,[[t_1, \ldots, t_m]] \Big/ \sum_{i \neq j} (t_i t_j)$$

Since C is reduced, the smooth locus is dense in C, hence the singular locus S is finite. The curve C' is obtained from $\widetilde{C}$ by identifying the points $\widetilde{p}_i \in \widetilde{C}$ lying over $p \in S$. A local description of C' is the following: Choose an affine neighbourhood $U = \mathrm{Spec}\,A$ of $p \in S$ with $U \cap S \setminus \{p\} = \varnothing$. Let $\widetilde{U} = \mathrm{Spec}\,\widetilde{A}$ be the inverse image of U. Then the amalgamated sum U' of

$\coprod \widetilde{p}_i \longrightarrow \widetilde{U}$ and $\coprod \widetilde{p}_i \longrightarrow p$ is (by definition) the open subset of C' lying over $U \subset C$.

$$\begin{array}{ccccc} & & U' & & \\ & \nearrow & & \nwarrow & \\ \widetilde{U} & & & & p \\ & \nwarrow & & \nearrow & \\ & & \coprod \widetilde{p}_i & & \end{array} \qquad \begin{array}{ccccc} & & A' & & \\ & \swarrow & & \searrow & \\ \widetilde{A} & & & & k(p) \\ & \searrow & & \swarrow & \\ & & \prod k\left(\widetilde{p}_i\right) & & \end{array}$$

Gluing the affine schemes $U' = \operatorname{Spec} A'$ gives C'.

By construction the map $\rho : C' \longrightarrow C$ is a bijection, and since the normalization $\pi : \widetilde{C} \longrightarrow C$ is closed, it is a homeomorphism. Moreover, the following diagram obtained from the fibre-product above (using $k(p') = k(p)$ and the Chinese remainder theorem)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}'_{p'} & \longrightarrow & A' & \longrightarrow & k\left(p'\right) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod \widetilde{\mathfrak{m}}_{\widetilde{p}_i} & \longrightarrow & \widetilde{A} & \longrightarrow & \prod k\left(\widetilde{p}_i\right) & \longrightarrow & 0 \end{array}$$

implies that the singularities p' of C' are transversal crossings of smooth formal local branches.

5.2 $\operatorname{Pic}^0 C$ as Extension of $\operatorname{Pic}^0 \widetilde{C}$ by a Linear Group L

Let $\varphi : Y \longrightarrow X$ be a morphism of projective curves, which is an isomorphism on a dense open subset $U \subset Y$. Let $\mathcal{Q}^*$ be the cokernel of the induced homomorphism $\mathcal{O}_X^* \longrightarrow \varphi_* \mathcal{O}_Y^*$, then $\mathcal{Q}^* = \prod_{x \in X} \left(\mathcal{O}_{Y,x}^* / \mathcal{O}_{X,x}^*\right)$, and $\mathcal{Q}^*$ is supported only in finitely many points, since $\mathcal{Q}_x^* = 0 \quad \forall x \in \varphi(U)$. In the corresponding long exact sequence of cohomology-groups

$$1 \to \mathrm{H}^0\left(\mathcal{O}_X^*\right) \to \mathrm{H}^0\left(\mathcal{O}_Y^*\right) \to \mathrm{H}^0\left(\mathcal{Q}^*\right) \to \mathrm{H}^1\left(\mathcal{O}_X^*\right) \to \mathrm{H}^1\left(\mathcal{O}_Y^*\right) \to \mathrm{H}^1\left(\mathcal{Q}^*\right)$$

we have $\mathrm{H}^1\left(\mathcal{Q}^*\right) = 1$ (since $\mathcal{Q}^*$ is affine), $\mathrm{H}^1\left(\mathcal{O}_Y^*\right) = \operatorname{Pic} Y$ and $\mathrm{H}^1\left(\mathcal{O}_X^*\right) = \operatorname{Pic} X$. If $\operatorname{im} \mathrm{H}^0\left(\mathcal{Q}^*\right) = \ker\left(\mathrm{H}^1\left(\mathcal{O}_X^*\right) \longrightarrow \mathrm{H}^1\left(\mathcal{O}_Y^*\right)\right)$ is connected, then it lies in the connected component of zero in $\operatorname{Pic} X$. Therefore in this case we obtain a short exact sequence of algebraic groups, referred to as $\operatorname{Pic}^0 \operatorname{Seq}(Y \to X)$:

$$1 \longrightarrow \operatorname{im} \mathrm{H}^0\left(\mathcal{Q}^*\right) \longrightarrow \operatorname{Pic}^0 X \longrightarrow \operatorname{Pic}^0 Y \longrightarrow 1$$

For a smooth connected algebraic group G there is a unique linear subgroup L such that $G/L =: A$ is an abelian variety (see Theorem 1.3 of

Chevalley) and L is the direct product of a torus and a unipotent group (see Theorem 1.4). If we set $X = C$ and $Y = \widetilde{C}$ (the normalization of C), then $\mathrm{Pic}^0 C$ is a smooth connected algebraic group, $\mathrm{Pic}^0 \widetilde{C}$ is an abelian variety and $\mathrm{im}\, \mathrm{H}^0(\mathcal{Q}^*)$ is linear, as we will see in the following. Using the intermediate curve C', we can treat the torus part and the unipotent part separately.

5.2.1 $\mathrm{Pic}^0 C'$ as Extension of $\mathrm{Pic}^0 \widetilde{C}$ by $\mathbb{T} = (\mathbb{G}_{\mathrm{m}})^t$

As mentioned in Section 5.1 the largest homeomorphic curve C' has at most ordinary multiple points as singularities, thus we can apply the following

Theorem 5.3 *Let C' be a connected curve which has only ordinary multiple points as singularities. Let $\widetilde{C}$ be the normalization of C'. Then $\mathrm{Pic}^0 C'$ is an extension of the abelian variety $\mathrm{Pic}^0 \widetilde{C}$ by a torus $\mathbb{T}$:*

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Pic}^0 C' \longrightarrow \mathrm{Pic}^0 \widetilde{C} \longrightarrow 1$$

$\mathbb{T} \cong (\mathbb{G}_{\mathrm{m}})^t$ is a torus of rank $t = 1 - \#\,\mathrm{Cp}(C') + \sum_{m \geq 1}(m-1)\# S_m$, where $\mathrm{Cp}(C')$ is the set of irreducible components of C' and S_m is the set of m-ple points (see Definition 5.2).

Proof. We specify the considerations at the beginning of this Subsection 5.2 to the case $X = C'$ and $Y = \widetilde{C}$. Let S be the singular locus of C' and $\mathrm{Cp}\left(\widetilde{C}\right)$ the set of components of $\widetilde{C}$. The long exact sequence has the form

$$1 \longrightarrow k^*_{C'} \longrightarrow \prod_{Z \in \mathrm{Cp}\left(\widetilde{C}\right)} k^*_Z \longrightarrow \prod_{p \in S} \mathbb{T}_p \longrightarrow \mathrm{Pic}\, C' \longrightarrow \prod_{Z \in \mathrm{Cp}\left(\widetilde{X}\right)} \mathrm{Pic}\, Z \longrightarrow 1$$

where $k^*_{C'} = \mathrm{H}^0(C', \mathcal{O}^*_{C'})$, $k^*_Z = \mathrm{H}^0(Z, \mathcal{O}^*_Z)$, and as each $p \in S$ is an ordinary multiple point (see Definition 5.2), we have

$$\begin{aligned} \mathbb{T}_p &= \left(\widetilde{\mathcal{O}}_p\right)^* \Big/ \left(\mathcal{O}'_p\right)^* \\ &= \frac{\left(\bigoplus_{l=1}^{m_p} k\,[[t_l]]\right)^*}{\left(k\,[[t_1, \ldots, t_{m_p}]] \Big/ \sum_{i \neq j}(t_i t_j)\right)^*} \\ &= \prod_{q \to p} k(q)^* \Big/ k(p)^* \\ &\cong (k^*)^{m_p - 1} \end{aligned}$$

For $\mathcal{Q}^* = \prod_{p\in S}\left(\left(\widetilde{\mathcal{O}}_p\right)^* \big/ \left(\mathcal{O}'_p\right)^*\right) = \prod_{p\in S}\mathbb{T}_p$ then $\operatorname{im}\mathrm{H}^0(\mathcal{Q}^*)$ is given by the torus

$$\begin{aligned}\mathbb{T} &= \operatorname{coker}\left(\prod_{Z\in\mathrm{Cp}(\widetilde{C})} k_Z^* \longrightarrow \prod_{p\in S}\mathbb{T}_p\right)\\ &= \frac{\prod_{p\in S}\mathbb{T}_p}{\prod_{Z\in\mathrm{Cp}(\widetilde{C})} k_Z^* \big/ k_{C'}^*}\\ &\cong (k^*)^t\end{aligned}$$

Since this is connected we obtain the exact sequence $\mathrm{Pic}^0\,\mathrm{Seq}\left(\widetilde{C}\to C'\right)$, where $\mathbb{T} = \operatorname{im}\mathrm{H}^0(\mathcal{Q}^*)$ and $\mathrm{Pic}^0\,\widetilde{C} = \prod_{Z\in\mathrm{Cp}(\widetilde{C})}\mathrm{Pic}^0\,Z$. ∎

5.2.2 $\mathrm{Pic}^0\,C$ as Extension of $\mathrm{Pic}^0\,C'$ by $\mathbb{V} = (\mathbb{G}_\mathrm{a})^v$

According to the previous subsection we have reduced the problem to describe $\mathrm{Pic}^0\,C$ as an extension of $\mathrm{Pic}^0\,C'$. This is described in the following

Theorem 5.4 *Let C be a projective curve, let C' be the largest homeomorphic curve between C and its normalization $\widetilde{C}$. Then* $\mathrm{Pic}^0\,C$ *is an extension of* $\mathrm{Pic}^0\,C'$ *by a vectorial group $\mathbb{V}$:*

$$1 \longrightarrow \mathbb{V} \longrightarrow \mathrm{Pic}^0\,C \longrightarrow \mathrm{Pic}^0\,C' \longrightarrow 1$$

with $\mathbb{V} = \mathrm{Spec}\left(\mathrm{Sym}\left(\mathcal{O}'/\mathcal{O}\right)^\vee\right) \cong (\mathbb{G}_\mathrm{a})^v$*, where* $(\mathcal{O}'/\mathcal{O})^\vee$ *is the dual k-vector space of $\mathcal{O}'/\mathcal{O}$ and* $v = \dim_k \mathcal{O}'/\mathcal{O}$*.*

Proof. We specify the considerations of the beginning of this Subsection 5.2 to the case $X = C$ and $Y = C'$. Since $\rho : C' \longrightarrow C$ is a homeomorphism, we have $\mathrm{H}^0((\mathcal{O}')^*) \cong \mathrm{H}^0(\mathcal{O}^*)$. Therefore the short exact sequence becomes

$$1 \longrightarrow \mathrm{H}^0(\mathcal{Q}^*) \longrightarrow \mathrm{H}^1(\mathcal{O}^*) \longrightarrow \mathrm{H}^1\left((\mathcal{O}')^*\right) \longrightarrow 1$$

Moreover, C' homeomorphic to C implies that $\mathcal{O}'/\mathcal{O} = \mathcal{J}/\mathcal{I}$ for certain proper ideals $\mathcal{J}\subset\mathcal{O}'$ and $\mathcal{I}\subset\mathcal{O}$, thus the module $\mathcal{O}'/\mathcal{O}$ carries the structure of a ring. The exponential map gives an isomorphism of vector spaces

$$\mathcal{O}'/\,\mathcal{O} \stackrel{\sim}{\longrightarrow} (\mathcal{O}')^*/\,\mathcal{O}^*$$

$\mathcal{O}'/\mathcal{O}$ and $(\mathcal{O}')^*/\,\mathcal{O}^*$ are supported only on the singular locus of C, hence are skyscraper sheaves. In particular, $\mathcal{O}'/\mathcal{O} = \mathrm{H}^0(C, \mathcal{O}'/\mathcal{O})$ and $(\mathcal{O}')^*/\,\mathcal{O}^* =$

$\mathrm{H}^0(C, \mathcal{Q}^*)$ are finite dimensional since C is projective. Taking into account the connectedness of the vectorial group $\mathbb{V}$ associated to the k-vector space $\mathcal{O}'/\mathcal{O}$ yields the exact sequence $\mathrm{Pic}^0 \,\mathrm{Seq}(C' \to C)$, where $\mathrm{im}\, \mathrm{H}^0(\mathcal{Q}^*) \cong \mathbb{V}$. ■

Definition 5.5 *Let $A \longrightarrow B$ be a ring homomorphism. We refer to an element β of the factor module B/A as* nilpotent *if there is a representative $b \in B$ of β and an integer $n > 0$ such that $b^\nu \in A$ for all $\nu \geq n$.*

The set of nilpotent elements *of B/A is denoted by* $\mathrm{Nil}(B/A)$.

Lemma 5.6 *Let $\varphi : Y \longrightarrow X$ be a finite morphism of projective curves. Then φ is a homeomorphism if and only if $\varphi_* \mathcal{O}_Y / \mathcal{O}_X$ is nilpotent.*

Proof. Since a finite morphism is a closed map, φ is a homeomorphism if and only if it is bijective.

Assume φ is bijective. Then φ is of degree 1, hence φ is an isomorphism on a dense open subscheme. Thus the cokernel $\mathcal{Q}$ of $\mathcal{O}_X \longrightarrow \varphi_* \mathcal{O}_Y$ is a skyscraper sheaf, i.e. $S := \mathrm{Supp}(\mathcal{Q})$ is finite and $\mathrm{H}^0(\mathcal{Q}) = \bigoplus_{x \in S} \mathcal{Q}_x$ is finite dimensional. For each $x \in S$ then $\mathcal{Q}_x = \mathcal{O}_{Y,x} / \mathcal{O}_{X,x} = \mathfrak{M} / \mathfrak{m}_{X,x}$ is isomorphic as k-vector space to $\bigoplus_{\nu \geq 0} \frac{\mathfrak{M}^\nu / \mathfrak{M}^{\nu+1}}{\mathfrak{m}_{X,x} \cap \mathfrak{M}^\nu / \mathfrak{m}_{X,x} \cap \mathfrak{M}^{\nu+1}}$. As this is finite dimensional over k, there is an integer $n > 0$ such that

$$\bigoplus_{\nu \geq n} \frac{\mathfrak{M}^\nu / \mathfrak{M}^{\nu+1}}{\mathfrak{m}_{X,x} \cap \mathfrak{M}^\nu / \mathfrak{m}_{X,x} \cap \mathfrak{M}^{\nu+1}} = 0$$

i.e. $\mathfrak{M}^n \subset \mathfrak{m}_{X,x}$. Hence φ is bijective if and only if $\mathcal{O}_{Y,x} / \mathcal{O}_{X,x}$ is nilpotent for every $x \in X$.

On the other hand, suppose for $x \in X$ there are several points $y_1, \ldots, y_d \in \varphi^{-1}(x)$ lying over x. Then $\mathcal{Q}_x \supset \prod_{i=1}^d k(y_i) \big/ k(x)$ is never nilpotent. ■

Conclusion 5.7 *The factor-module $\mathcal{O}'/\mathcal{O}$ is isomorphic to the submodule* $\mathrm{Nil}\left(\widetilde{\mathcal{O}}/\mathcal{O}\right)$ *of nilpotent elements in the factor-module $\widetilde{\mathcal{O}}/\mathcal{O}$.*

5.2.3 $\mathrm{Pic}^0 C$ as Extension of $\mathrm{Pic}^0 \widetilde{C}$ by $L = \mathbb{T} \times \mathbb{V}$

The main theorem of this section is now a consequence of Subsubsections 5.2.1 and 5.2.2:

Theorem 5.8 (Main) $\mathrm{Pic}^0 C$ *is an extension of the abelian variety* $\mathrm{Pic}^0 \widetilde{C}$ *by a linear group $L = \mathbb{T} \times \mathbb{V}$*

$$1 \longrightarrow L \longrightarrow \mathrm{Pic}^0 C \longrightarrow \mathrm{Pic}^0 \widetilde{C} \longrightarrow 1$$

where $\mathbb{T} \cong (\mathbb{G}_\mathrm{m})^t$ is a torus and $\mathbb{V} \cong (\mathbb{G}_\mathrm{a})^v$ is a vectorial group.

Proof. According to Theorems 5.3 and 5.4 we have a diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathbb{V} & & & & \\
 & & & & \downarrow & & & & \\
1 & \longrightarrow & L & \longrightarrow & \operatorname{Pic}^0 C & \longrightarrow & \operatorname{Pic}^0 \widetilde{C} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & \operatorname{Pic}^0 C' & \longrightarrow & \operatorname{Pic}^0 \widetilde{C} & \longrightarrow & 1 \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}$$

where L is the fibre-product of $\mathbb{T}$ and $\operatorname{Pic}^0 C$ over $\operatorname{Pic}^0 C'$. Since L is an extension of $\mathbb{T}$ by $\mathbb{V}$ and $\operatorname{Ext}(\mathbb{T}, \mathbb{V}) = 0$ in characteristc 0, we have $L = \mathbb{T} \times \mathbb{V}$.
■

5.3 Dual 1-Motive of the Universal Regular Quotient

The algebraic group $\operatorname{Pic}^0 C$ is universal for the category $\mathbf{Mr}^{\mathrm{CH}_0(C)_{\deg 0}}$ of rational maps to algebraic groups $\varphi : C \longrightarrow G$ factoring through $\mathrm{CH}_0(C)_{\deg 0}$ (see Definition 4.17). This is tautological since $\operatorname{Pic}^0 C \cong \mathrm{CH}_0(C)_{\deg 0}$ (see [LW] Proposition 1.4). The task of this subsection is to show that $\operatorname{Pic}^0 C$ coincides with the universal algebraic group $\mathrm{Alb}_{\underline{\mathrm{Div}}^0_{\widetilde{C}/C}}(C)$ for the category of rational maps from C to algebraic groups which induce a transformation to $\underline{\mathrm{Div}}^0_{\widetilde{C}/C}$.

5.3.1 Cartier-Dual of $\mathbb{T}$

The results of Subsubsection 1.3.1, especially Theorem 1.15, are used in the following

Proposition 5.9 *The Cartier-dual of* $\mathbb{T} = \ker\left(\operatorname{Pic}^0 C' \xrightarrow{\sigma^*} \operatorname{Pic}^0 \widetilde{C}\right)$ *from Proposition 5.3 is the étale formal group* $\mathbb{T}^\vee$ *given by*

$$\mathbb{T}^\vee(k) = \mathrm{WDiv}^0\left(\widetilde{C}/C'\right)$$

where $\mathrm{WDiv}\left(\widetilde{C}/C'\right) = \ker\left(\mathrm{WDiv}(\widetilde{C}) \xrightarrow{\sigma_*} \mathrm{WDiv}(C')\right)$ *is the kernel of the push-forward of Weil divisors, and*

$$\mathrm{WDiv}^0\left(\widetilde{C}/C'\right) = \mathrm{Div}^0\left(\widetilde{C}\right) \cap \mathrm{WDiv}\left(\widetilde{C}/C'\right)$$

is the subset of $\mathrm{WDiv}\left(\widetilde{C}/C'\right)$ *formed by divisors of degree 0.*

Proof. For $q \in \widetilde{C}$ we may write the Cartier-dual of the free abelian group $\mathbb{Z}q \subset \mathrm{Div}\left(\widetilde{C}\right)$ as $k(q)^*$, the pairing is given by

$$\begin{array}{rcl} \mathbb{Z}q \times k(q)^* & \longrightarrow & k^* \\ (\lambda q, t) & \longmapsto & t^\lambda \end{array}$$

Let $S \subset C'$ be the singular locus. As $\sigma|_{\sigma^{-1}C'_{\mathrm{reg}}}$ is an isomorphism on C'_{reg}, we have

$$\mathrm{WDiv}\left(\widetilde{C}/C'\right) = \prod_{p\in S} \mathrm{WDiv}\left(\sigma^{-1}p/p\right)$$

with

$$\mathrm{WDiv}\left(\sigma^{-1}p/p\right) = \ker\left(\prod_{q\to p} \mathbb{Z}q \xrightarrow{\sigma_*} \mathbb{Z}p\right)$$

Then $\mathrm{WDiv}\left(\sigma^{-1}p/p\right)$ is the Cartier-dual of

$$\mathrm{coker}\left(k(p)^* \longrightarrow \prod_{q\to p} k(q)^*\right) = \prod_{q\to p} k(q)^* \Big/ k(p)^* = \mathbb{T}_p$$

Notice that the pairing

$$\begin{array}{rcl} \mathrm{WDiv}\left(\sigma^{-1}p/p\right) \times \mathbb{T}_p & \longrightarrow & k^* \\ \left(q_i - q_j, \left[t_1, \ldots, t_{m_p}\right]_{k^*}\right) & \longmapsto & t_i \cdot t_j^{-1} \end{array}$$

is well defined. Hence $\mathrm{WDiv}\left(\widetilde{C}/C'\right)$ is the Cartier-dual of $\prod_{p\in S} \mathbb{T}_p$. Finally, denoting by $\mathrm{Cp}\left(\widetilde{C}\right)$ the set of components of $\widetilde{C}$, then

$$\mathrm{Div}^0\left(\widetilde{C}\right) \cap \mathrm{WDiv}\left(\widetilde{C}/C'\right) = \ker\left(\mathrm{WDiv}\left(\widetilde{C}/C\right) \xrightarrow{(\deg_Z)_Z} \prod_{Z\in\mathrm{Cp}(\widetilde{C})} \mathbb{Z}_Z\right)$$

which is the Cartier-dual of

$$\mathrm{coker}\left(\prod_{Z\in\mathrm{Cp}(\widetilde{C})} k_Z^* \longrightarrow \prod_{p\in S} \mathbb{T}_p\right) = \mathbb{T}$$

■

5.3.2 Cartier-Dual of $\mathbb{V}$

We are going to use the results of Subsubsection 1.3.2, especially Theorem 1.19, in the following

Proposition 5.10 *The Cartier-dual of* $\mathbb{V} = \ker\left(\operatorname{Pic}^0 C \xrightarrow{\rho^*} \operatorname{Pic}^0 C'\right)$ *from Proposition 5.4 is the infinitesimal formal group* $\mathbb{V}^\vee$ *defined by*

$$\operatorname{Lie}\mathbb{V}^\vee = \operatorname{LDiv}^0(C'/C)$$

where $\operatorname{LDiv}^0(C'/C) = \ker\left(\operatorname{LDiv}^0(C') \xrightarrow{\rho_*} \operatorname{LDiv}^0(C)\right)$ *is the kernel of the push-forward of formal Lie divisors for curves, defined as*

$$\operatorname{LDiv}^0(C) = \bigoplus_{p\in C(k)} \operatorname{Hom}_{k,\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{C,p}, k\right)$$

Proof. Let $S \subset C$ be the singular locus on C. On a curve each generic point η of height 1 is a closed point, hence $k(\eta) = k$.

$$\begin{aligned}
\operatorname{LDiv}^0(C'/C) &= \ker\left(\operatorname{LDiv}^0(C') \longrightarrow \operatorname{LDiv}^0(C)\right)\\
&= \ker\left(\bigoplus_{p'\in C'} \operatorname{Hom}_{k,\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{C',p'}, k\right) \longrightarrow \bigoplus_{p\in C} \operatorname{Hom}_{k,\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{C,p}, k\right)\right)\\
&= \bigoplus_{p\in S} \ker\left(\operatorname{Hom}_{k,\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{C',\sigma^{-1}(p)}, k\right) \longrightarrow \operatorname{Hom}_{k,\mathrm{cont}}\left(\widehat{\mathfrak{m}}_{C,p}, k\right)\right)\\
&= \bigoplus_{p\in S} \operatorname{Hom}_k\left(\mathfrak{m}_{C',\sigma^{-1}(p)}\big/\,\mathfrak{m}_{C,p}, k\right)
\end{aligned}$$

This is the dual k-vector space of

$$\begin{aligned}
\operatorname{Lie}\mathbb{V} &= \mathcal{O}'/\mathcal{O}\\
&= \bigoplus_{p\in S} \mathcal{O}'_p\big/\,\mathcal{O}_p\\
&= \bigoplus_{p\in S} \mathfrak{m}_{C',\sigma^{-1}(p)}\big/\,\mathfrak{m}_{C,p}
\end{aligned}$$

■

5.3.3 Cartier-Dual of L

Proposition 5.11 *The Cartier-dual of* $L = \ker\left(\operatorname{Pic}^0 C \xrightarrow{\pi^*} \operatorname{Pic}^0 \widetilde{C}\right)$ *from Proposition 5.8 is a formal group* $L^\vee$ *representing the functor* $\underline{\operatorname{Div}}^0_{\widetilde{C}/C}$.

Proof. $\underline{\operatorname{Div}}^0_{\widetilde{C}/C}$ is the direct product of the functors

$$\operatorname{Red}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}/C}\right) = \operatorname{Red}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}}\right) \cap \underline{\operatorname{WDiv}}_{\widetilde{C}/C}$$

(see Definition 2.37 and Proposition 2.55) and

$$\operatorname{Inf}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}/C}\right) = \underline{\operatorname{IDiv}}_{\widetilde{C}/C}$$

(see Proposition 2.65), this follows from the definition (see Proposition 2.70) and the fact that on a smooth curve Cartier divisors equal Weil divisors, as well as the fact that for curves $\operatorname{Inf}\left(\underline{\operatorname{Div}}_Y\right)$ and $\underline{\operatorname{IDiv}}_Y$ are isomorphic functors (see Proposition 2.63).

As $\operatorname{Red}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}/C}\right)$ is locally constant (see Proposition 2.58), it is already defined by its k-valued points. $\widetilde{C} \longrightarrow C$ factors through C', thus $\operatorname{WDiv}\left(\widetilde{C}/C\right) = \ker\left(\operatorname{WDiv}(\widetilde{C}) \longrightarrow \operatorname{WDiv}(C') \longrightarrow \operatorname{WDiv}(C)\right)$. But C' is homeomorphic to C, hence $\operatorname{WDiv}(C'/C) = 0$. Therefore $\left(\underline{\operatorname{Div}}^0_{\widetilde{C}} \cap \underline{\operatorname{WDiv}}_{\widetilde{C}/C}\right)(k) = \operatorname{Div}^0\left(\widetilde{C}\right) \cap \operatorname{WDiv}\left(\widetilde{C}/C'\right)$. Then Proposition 5.9 implies that $\operatorname{Red}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}/C}\right) = \mathbb{T}^\vee$, i.e. is the Cartier-dual of $\mathbb{T} = \ker\left(\operatorname{Pic}^0 C' \longrightarrow \operatorname{Pic}^0 \widetilde{C}\right)$.

$\operatorname{Inf}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}/C}\right) = \underline{\operatorname{IDiv}}_{\widetilde{C}/C}$ is plain infinitesimal (see Proposition 2.65), hence it is uniquely determined by its Lie-functor. Using $\widetilde{C} \longrightarrow C' \longrightarrow C$ we have $\operatorname{LDiv}\left(\widetilde{C}/C\right) = \ker\left(\operatorname{LDiv}(\widetilde{C}) \longrightarrow \operatorname{LDiv}(C') \longrightarrow \operatorname{LDiv}(C)\right)$. By construction of C' (see Subsection 5.1) for all $\widetilde{p} \in \widetilde{C}$ the maps of maximal ideals $\mathfrak{m}'_{\sigma(\widetilde{p})} \longrightarrow \widetilde{\mathfrak{m}}_{\widetilde{p}}$ are surjective, thus $\operatorname{Hom}_{k,\mathrm{cont}}\left(\widehat{\widetilde{\mathfrak{m}}_{\widetilde{p}}}, k\right) \longrightarrow \operatorname{Hom}_{k,\mathrm{cont}}\left(\widehat{\mathfrak{m}'_{\sigma(\widetilde{p})}}, k\right)$ are injective and hence $\operatorname{LDiv}\left(\widetilde{C}/C'\right) = 0$. The image of $\operatorname{LDiv}(\widetilde{C})$ under σ_* is contained in $\operatorname{LDiv}^0(C')$. Therefore $\operatorname{Lie}\left(\underline{\operatorname{IDiv}}_{\widetilde{C}/C}\right) = \operatorname{LDiv}^0(C'/C)$. From Proposition 5.10 it follows now that $\operatorname{Inf}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}/C}\right) = \mathbb{V}^\vee$, i.e. is the Cartier-dual of $\mathbb{V} = \ker\left(\operatorname{Pic}^0 C \longrightarrow \operatorname{Pic}^0 C'\right)$.

Finally $\underline{\operatorname{Div}}^0_{\widetilde{C}/C} = \operatorname{Red}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}/C}\right) \times \operatorname{Inf}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}/C}\right)$ is the Cartier-dual of $L = \mathbb{T} \times \mathbb{V}$. ■

5.3.4 The map $\mathrm{pic}^0 : C_{\mathrm{reg}} \longrightarrow \mathrm{Pic}^0 C$

The Cartier divisors on C which correspond to a Weil divisor are given by the Weil divisors which are supported on the regular locus C_{reg}:

$$\mathrm{Div}\,(C) \cap \mathrm{WDiv}\,(C) \cong \mathrm{WDiv}\,(C_{\mathrm{reg}})$$

Fixing a base point $q_Z \in Z_{\mathrm{reg}}$ for each irreducible component Z of C, we can define the map pic^0_C :

$$\begin{array}{ccccc} C_{\mathrm{reg}} & \longrightarrow & \mathrm{Div}^0\,(C) & \longrightarrow & \mathrm{Pic}^0\,(C) \\ Z_{\mathrm{reg}} \ni p & \longmapsto & p - q_Z & \longmapsto & \mathcal{O}\,(p - q_Z) \end{array}$$

Viewing C_{reg} as an open subset of $\widetilde{C}$, the morphism $\mathrm{pic}^0_C : C_{\mathrm{reg}} \longrightarrow \mathrm{Pic}^0 C$ gives a section

$$\left(\mathrm{pic}^0_C\right)_{\widetilde{C}} : C_{\mathrm{reg}} \subset \widetilde{C} \longrightarrow \left(\mathrm{Pic}^0 C\right)_{\widetilde{C}}$$

of the principal L-bundle $\left(\mathrm{Pic}^0 C\right)_{\widetilde{C}} = \mathrm{Pic}^0 C \times_{\mathrm{Pic}^0 \widetilde{C}} \widetilde{C}$ over $\widetilde{C}$.

The structure of a translation-invariant principal L-bundle is defined by one local section (see end of Subsubsection 1.1.1). According to Proposition 5.12 below, $\mathrm{Pic}^0 C$ is determined by the section $\left(\mathrm{pic}^0_C\right)_{\widetilde{C}}$.

Proposition 5.12 *Let Y be a smooth projective variety, L a commutative linear group. Let $\mathrm{Pfb}\,(Y, L)$ denote the set of principal L-bundles over Y. Then the map*

$$\begin{array}{ccc} \mathrm{Pfb}\,(\mathrm{Alb}\,(Y)\,, L) & \longrightarrow & \mathrm{Pfb}\,(Y, L) \\ P & \longmapsto & P_Y = P \times_{\mathrm{Alb}(Y)} Y \end{array}$$

is a bijection.

Proof. Since $\mathrm{char}\,(k) = 0$ there is a decomposition $L \cong (\mathbb{G}_{\mathrm{m}})^t \times (\mathbb{G}_{\mathrm{a}})^v$. Principal L-bundles on a variety X correspond to vector-bundles whose transition functions are given by diagonal matrices, i.e. which are direct sums of line-bundles and deformations of the trivial bundle. Hence it holds additivity:

$$\mathrm{Pfb}\left(X, (\mathbb{G}_{\mathrm{m}})^t \times (\mathbb{G}_{\mathrm{a}})^v\right) \cong \mathrm{Pfb}\,(X, \mathbb{G}_{\mathrm{m}})^t \times \mathrm{Pfb}\,(X, \mathbb{G}_{\mathrm{a}})^v$$

Therefore it is sufficient to show $\mathrm{Pfb}\,(\mathrm{Alb}\,(Y)\,, \mathbb{L}) \simeq \mathrm{Pfb}\,(Y, \mathbb{L})$ for $\mathbb{L} = \mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{a}}$. Using $\mathrm{Pfb}\,(X, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{Pic}_X\,(k)$ and $\mathrm{Pfb}\,(X, \mathbb{G}_{\mathrm{a}}) \cong \mathrm{Lie}\,(\mathrm{Pic}_X)$ the assertion follows from the isomorphism $\mathrm{Pic}_{\mathrm{Alb}(Y)} \simeq \mathrm{Pic}_Y$. ■

5.3.5 Dual 1-Motive of $\mathrm{Pic}^0 C$

It is clear from construction that for all $\lambda \in L^\vee = \underline{\mathrm{Div}}^0_{\widetilde{C}/C}$ the section $\left(\mathrm{pic}^0_C\right)_{\widetilde{C},\lambda}$ defines the divisor or deformation λ, i.e.

$$\mathrm{div}_{\mathbb{L}}\left(\left(\mathrm{pic}^0_C\right)_{\widetilde{C},\lambda}\right) = \lambda$$

(see Proposition 3.3). Therefore, according to Theorem 3.12 and Remark 3.14, $\mathrm{Pic}^0 C$ coincides with the universal object $\mathrm{Alb}_{\underline{\mathrm{Div}}^0_{\widetilde{C}/C}}(C)$ for the category of rational maps $\varphi : C \longrightarrow G$ which induce a homomorphism of formal groups $M^\vee \longrightarrow \underline{\mathrm{Div}}^0_{\widetilde{C}/C}$, if M is the linear group in the canonical decomposition of the algebraic group G. Again using Remark 3.14 we obtain

Theorem 5.13 *The dual 1-motive of the universal regular quotient* $\mathrm{Alb}\,(C) = \mathrm{Pic}^0 C$ *of a projective curve* C *is given by the natural transformation of functors*

$$\underline{\mathrm{Div}}^0_{\widetilde{C}/C} \longrightarrow \mathrm{Pic}^0_{\widetilde{C}}$$

which assigns to a relative divisor $\mathcal{D} \in \underline{\mathrm{Div}}^0_{\widetilde{C}/C}(R)$ *the class of its associated line bundle* $\mathcal{O}(\mathcal{D}) \mod \mathrm{Pic}\,(\mathrm{Spec}\, R)$ *for each finitely generated* k*-algebra* R.

6 Higher Dimension

Let X be a projective variety. In this section we derive a description of the universal regular quotient $\mathrm{Alb}\,(X)$ and the dual 1-motive of $[0 \longrightarrow \mathrm{Alb}\,(X)]$ from the one for curves. This is a classical way of procedure, which is applied often in works on related topics, for example in order to show the boundedness of the dimension of the classical Albanese in [La] and of the universal regular quotient in [ESV]. Implicitly we used a similar way of thinking in the proof of the equivalence of $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ and $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}$ (see Lemma 4.19). Therefore for many questions arising in this context it is important to study the relation between a rational map $\varphi : X \longrightarrow G$ and the induced rational maps $\{\varphi|_V : V \longrightarrow G\}_{V\in T}$ on a family T of subvarieties V of X, in particular the relation between $\mathrm{Alb}\,(X)$ and $\{\mathrm{Alb}\,(V)\}_{V\in T}$.

6.1 Reducing to Subvarieties

Let Y be a normal projective variety of dimension d, and let $\mathcal{F}$ be a subfunctor of $\underline{\mathrm{Div}}^0_Y$ which is a formal group.

Let V be a subvariety of Y. Remember that V is called *decident* to a subset $S \subset Y$, if no irreducible component of V is contained in S (see Definition 3.22). $\underline{\mathrm{Dec}}_{Y,V}$ is the plain subfunctor of $\underline{\mathrm{Div}}_Y$ consisting of families of Cartier divisors to which V is decident, i.e. V decident to $\mathrm{Supp}\,(D)$ for all $D \in \underline{\mathrm{Dec}}_{Y,V}$ (see Definition 3.23). $\_ \cdot V : \underline{\mathrm{Dec}}_{Y,V} \longrightarrow \underline{\mathrm{Div}}_V$ is the pull-back of relative Cartier divisors from Y to V (see Definition 3.24).

Remark 6.1 *Let $\delta \in \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Y\right) = \Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)$ be a deformation of the zero divisor in Y. Then δ determines an effective divisor by the poles of its local sections. Hence for each generic point η of height 1 in Y, with associated discrete valuation v_η, the expression $v_\eta(\delta)$ is well defined and $v_\eta(\delta) \leq 0$. Thus we obtain a homomorphism $v_\eta : \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_Y\right) \longrightarrow \mathbb{Z}$.*

Proposition 6.2 *Let $C = D_1 \cap \ldots \cap D_{d-1}$ be a curve in Y, which is an intersection of ample divisors and decident to $\mathrm{Supp}\,(\mathcal{F})$. Let S be the set of generic points of $\mathrm{Supp}(\mathcal{F})$ and $S_{\inf}$ the corresponding set for $\mathrm{Supp}(\mathcal{F}_{\inf})$. If η is a generic point of height 1 in Y, denote by E_η the associated prime divisor. Then $(\_ \cdot C)\,|_{\mathcal{F}} : \mathcal{F} \longrightarrow \underline{\mathrm{Div}}_V$ is injective if*

(a) C intersects E_η in general points $\quad \forall \eta \in S$

(b) $\#\,(C \cap E_\eta) \cdot \#\, v_\eta(\mathrm{Lie}\,\mathcal{F}) \geq \dim_k \mathrm{im}\left(\mathrm{Lie}\,\mathcal{F} \longrightarrow \left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)_\eta\right)$ $\quad \forall \eta \in S_{\inf}$

where $\mathrm{Lie}\,\mathcal{F} \subset \Gamma\left(\mathcal{K}_Y/\,\mathcal{O}_Y\right) \longrightarrow \left(\mathcal{K}_Y/\,\mathcal{O}_Y\right)_\eta$, $\delta \longmapsto [\delta]_\eta$ is the localization at the height 1 point η.

Proof. As an ample divisor intersects each closed subscheme of codimension 1 and D_i restricted to $D_1 \cap \ldots \cap D_{i-1}$ is again ample for all $i = 2, \ldots, d-1$, it follows by induction that $C \cap \operatorname{Supp}(D) \neq \varnothing$ for all $D \in \mathcal{F}$. If C intersects $\operatorname{Supp}(D)$ in general points for each $D \in \mathcal{F}(k)$, then $(\_ \cdot C)\,|_{\mathcal{F}(k)} : \mathcal{F}(k) \longrightarrow \underline{\mathrm{Div}}_V(k)$ is injective.

For the infinitesimal part of $\mathcal{F}$ now consider the following diagram:

$$\begin{array}{ccc}
\Gamma\,(\mathcal{K}_Y/\,\mathcal{O}_Y) \supset \operatorname{Lie}\mathcal{F} & \xrightarrow{\ _\cdot C\ } & \Gamma\,(\mathcal{K}_C/\,\mathcal{O}_C) \\
\downarrow \wr & & \downarrow \wr \\
\bigoplus_{\eta\in S}\,(\mathcal{K}_Y/\,\mathcal{O}_Y)_\eta \supset \operatorname{im}(\operatorname{Lie}\mathcal{F}) & \longrightarrow & \bigoplus_{q\in C} \mathrm{K}_C/\,\mathcal{O}_{C,q}
\end{array}$$

For each $\eta \in S_{\inf}$ choose a local parameter t_η of $\mathfrak{m}_{Y,\eta}$. Since C is decident to E_η, and if C intersects E_η in general points, we may assume that each $q \in C \cap E_\eta$ is a regular closed point of C. Then the image $t_q \in \mathcal{O}_C$ of $t_\eta \in \mathcal{O}_Y$ is a local parameter of $\mathfrak{m}_{C,q}$. Choosing representatives in $\mathrm{K}_C/\,\mathcal{O}_{C,q}$ for each $q \in C$, we may consider $\operatorname{Lie}\mathcal{F}\cdot C$ as a k-linear subspace of the k-vector space

$$\bigoplus_{\eta\in S_{\inf}}\ \bigoplus_{q\in C\cap E_\eta}\ \bigoplus_{\nu\in v_\eta(\operatorname{Lie}\mathcal{F})} k\cdot t_q^\nu$$

Then the map

$$\begin{array}{rcl}
\operatorname{im}\Big(\operatorname{Lie}\mathcal{F} \longrightarrow (\mathcal{K}_Y/\,\mathcal{O}_Y)_\eta\Big) & \longrightarrow & \displaystyle\bigoplus_{q\in C\cap E_\eta}\ \bigoplus_{\nu\in v_\eta(\operatorname{Lie}\mathcal{F})} k\cdot t_q^\nu \\
f\,t_\eta^\nu & \longmapsto & \displaystyle\sum_{q\in C\cap E_\eta} f(q)\,t_q^\nu
\end{array}$$

is injective if C intersects E_η in general points and
$\dim_k \operatorname{im}\Big(\operatorname{Lie}\mathcal{F} \longrightarrow (\mathcal{K}_Y/\,\mathcal{O}_Y)_\eta\Big) \leq \#\,(C\cap E_\eta)\cdot\#\,v_\eta(\operatorname{Lie}\mathcal{F})$. ■

Lemma 6.3 *Let G be an algebraic group and $\varphi : Y \longrightarrow G$ be a rational map. Then the following conditions are equivalent:*

(i) $\varphi \in \mathbf{Mr}_\mathcal{F}$

(ii) For every subvariety V of Y decident to $\operatorname{Supp}(\mathcal{F})$ *it holds* $\varphi|_V \in \mathbf{Mr}_{\mathcal{F}\cdot V}$

(iii) There exists a family T of subvarieties of Y decident to $\operatorname{Supp}(\mathcal{F})$ *such that* *(a)* $Y \setminus \bigcup_{V\in T} V$ *is of codimension* ≥ 2 *in* Y

(b) $\varphi|_V \in \mathbf{Mr}_{\mathcal{F}\cdot V}$ *for all* $V \in T$

Proof. (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) is evident.
(iii)$\Longrightarrow$(i) Let L be the largest linear subgroup of G. By Definition 3.6 of

$\mathbf{Mr}_{\mathcal{F}}$ it comes down to show that for all $\lambda \in L^{\vee}$ (jj) implies (j) with respect to the following conditions:

(j) $\operatorname{div}_{\mathbb{L}}(\varphi_{Y,\lambda}) \in \mathcal{F}$

(jj) $\operatorname{div}_{\mathbb{L}}\left((\varphi|_V)_{V,\lambda}\right) \in \mathcal{F} \cdot V \qquad \forall V \in T$

This follows from the fact that an element $D \in \underline{\mathrm{Div}}^0_Y$ is uniquely determined on a subset whose complement is of codimension ≥ 2 in Y, and this applies to $\bigcup_{V \in T} V$ by condition (iii)(a). ∎

Definition 6.4 *For an ample line bundle $\mathcal{L}$ on Y and an integer c with $1 \leq c \leq \dim Y$ write*

$$|\mathcal{L}|^c = \mathbb{P}\left(\mathrm{H}^0(Y, \mathcal{L})\right) \times \ldots \times \mathbb{P}\left(\mathrm{H}^0(Y, \mathcal{L})\right) \qquad \text{(c copies)}$$

For a subset S of Y denote by $|\mathcal{L}|^c_S$ the open subscheme of $|\mathcal{L}|^c$ defined by c-tuples of divisors $D_1, \ldots, D_c \in |\mathcal{L}|$ such that $V = \bigcap_{i=1}^c D_i$ is decident to S. By abuse of notation we write $V \in |\mathcal{L}|^c$ instead of $(D_1, \ldots, D_c) \in |\mathcal{L}|^c$.

Definition 6.5 *If $\varphi : V \longrightarrow G$ is a rational map from a variety V to an algebraic group G, then we say (V, φ)* generates *G if $\varphi(V)$ generates G as a group.*

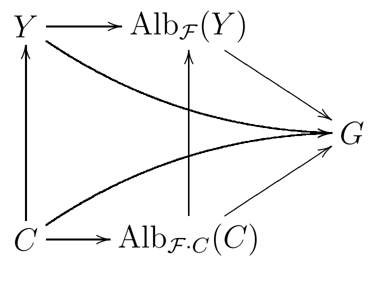
Lemma 6.6 *Let G be an algebraic group and $\varphi : Y \longrightarrow G$ be a rational map, which satisfies the equivalent conditions of Lemma 6.3. Let $\mathcal{L}$ be a very ample line bundle on Y, and let c be an integer with $1 \leq c \leq \dim Y - 1$. Then the following conditions are equivalent:*

(i) (Y, φ) generates G

(ii) There exists an integer $N \geq 1$ and a dense open subscheme $T \subset \left|\mathcal{L}^N\right|^c$ such that $(V, \varphi|_V)$ generates G for all $V \in T$

Proof. (ii)$\Longrightarrow$(i) is evident since $Y \supset V$ for all $V \in T$.
(i)$\Longrightarrow$(ii) It suffices to prove this assertion for the case that $c = \dim Y - 1$, i.e. T is a family of curves. Indeed, if the lemma is shown for curves, replacing Y by a subvariety V we can apply the implication (ii)$\Longrightarrow$(i) to each element V of a family of subvarieties of Y.

For each $C \in |\mathcal{L}|^{d-1}$ we have a commutative diagram



The task is to show that $\mathrm{Alb}_{\mathcal{F}\cdot C}(C) \longrightarrow \mathrm{Alb}_{\mathcal{F}}(Y)$ is surjective. By duality for 1-motives this is equivalent to the statement that $\mathcal{F} \longrightarrow \mathcal{F} \cdot C$ and $\mathrm{Pic}^0 Y \longrightarrow \mathrm{Pic}^0 C$ are injective homomorphisms.

For sufficiently large N the open subscheme $T \subset \left|\mathcal{L}^N\right|^{d-1}$ consisting of those C, such that the homomorphism $\_ \cdot C : \mathcal{F} \longrightarrow \underline{\mathrm{Div}}^0_C$ is injective, is dense in $\left|\mathcal{L}^N\right|^{d-1}$. This follows from Proposition 6.2.

Y and C are projective varieties over a field, therefore the groups $\mathrm{Pic}(Y)$ and $\mathrm{Pic}(C)$ may be identified with the groups $\mathrm{CaCl}(Y)$ and $\mathrm{CaCl}(C)$ of Cartier divisors modulo linear equivalence on Y and C respectively, by a result of Nakai (see [H] Chapter II, Remark 16.14.1). As C is the intersection of ample divisors, the intersection product $[D] \cdot C$ in $\mathrm{CaCl}(C)$ will be zero if and only if $[D]$ is zero in $\mathrm{CaCl}(Y)$. The injectivity on closed points implies the injectivity of the morphism of varieties $\mathrm{Pic}^0 Y \longrightarrow \mathrm{Pic}^0 C$. ■

6.2 Universal Regular Quotient and its Dual 1-Motive

The universal regular quotient $\mathrm{Alb}(X)$ is the universal object for the category $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ of rational maps from X to algebraic groups factoring through $\mathrm{CH}_0(X)_{\deg 0}$ (see Definition 4.17). The existence and a description of the universal regular quotient is now done by reduction to curves, as the case of curves has already been treated in Section 5.

The following theorem was already proven in Subsubsection 4.4.1, and accordingly the proof of it is a summary of arguments given in Section 4, but without involving *local symbols*, instead basing on the alternative description for curves from Section 5. I repeat it because we are looking at it now from a different point of view.

Theorem 6.7 *Let X be a projective variety, and $\pi : \widetilde{X} \longrightarrow X$ its normalization. Then the universal regular quotient* $\mathrm{Alb}(X)$ *of X exists and is given by the universal object* $\mathrm{Alb}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}\left(\widetilde{X}\right)$ *for the category* $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}$*, i.e. it is dual to the 1-motive*

$$\underline{\mathrm{Div}}^0_{\widetilde{X}/X} \longrightarrow \underline{\mathrm{Pic}}^0_{\widetilde{X}}$$

Proof. Identifying the regular locus X_{reg} with its preimage in $\widetilde{X}$, the category $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ can be considered as a category of rational maps from $\widetilde{X}$ to algebraic groups. We are going to show the assertion by analysing the behaviour of such a rational map φ on curves.

Let $\varphi : X \longrightarrow G$ be a rational map to an algebraic group G. Then the following conditions are equivalent:

(i) φ factors through $\mathrm{CH}_0(X)_{\deg 0}$

(ii) $\varphi|_{\pi^{-1}C}$ factors through $\mathrm{CH}_0(C)_{\deg 0}$
for all Cartier curves C in X relative to X_{reg}

i.e. $\varphi \in \mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ if and only if $\varphi|_C \in \mathbf{Mr}^{\mathrm{CH}_0(C)_{\deg 0}}$ for all Cartier curves C in X relative to X_{reg}. This follows from the fact that the relations in $\mathrm{CH}_0(X)_{\deg 0}$ are defined by Cartier curves (see Definition 4.4).

Let L be the largest linear subgroup of G. Let C be a Cartier curve in X relative to X_{reg}, and $\nu : \widetilde{C} \longrightarrow C$ its normalization. Then from Section 5 we know that the categories $\mathbf{Mr}^{\mathrm{CH}_0(C)_{\deg 0}}$ and $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{C}/C}}$ are equivalent, i.e. $\varphi|_C$ factors through $\mathrm{CH}_0(C)_{\deg 0}$ if and only if $\varphi|_C$ induces a transformation $L^\vee \longrightarrow \underline{\mathrm{Div}}^0_{\widetilde{C}/C}$. In order to prove the equivalence of the categories $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ and $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}$ it remains to show the equivalence of the following conditions:

(j) φ induces a transformation $l^\vee_{\varphi,\widetilde{X}} : L^\vee \longrightarrow \underline{\mathrm{Div}}^0_{\widetilde{X}/X}$

(jj) $\varphi|_C$ induces a transformation $l^\vee_{\varphi|_C,\widetilde{C}} : L^\vee \longrightarrow \underline{\mathrm{Div}}^0_{\widetilde{C}/C}$
for all Cartier curves C in X relative to X_{reg}

Now $l^\vee_{\varphi|_C,\widetilde{C}}(\lambda) = l^\vee_{\varphi,\widetilde{X}}(\lambda) \cdot \widetilde{C}$ for each Cartier curve C and each $\lambda \in L^\vee$, where $\_ \cdot \widetilde{C}$ is the pull-back of relative Cartier divisors from $\widetilde{X}$ to $\widetilde{C}$ (see Proposition 3.24); and $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ and $\underline{\mathrm{Div}}^0_{\widetilde{C}/C}$ are defined as kernel of the push-forward π_* and ν_* respectively (see Proposition 2.70). Therefore the equivalence of (j) and (jj) follows from Lemma 4.23 and Lemma 4.27.

The equivalence of $\mathbf{Mr}^{\mathrm{CH}_0(X)_{\deg 0}}$ and $\mathbf{Mr}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}$ implies the existence of the universal regular quotient by Theorem 3.12, as $\mathrm{Alb}\,(X) = \mathrm{Alb}_{\underline{\mathrm{Div}}^0_{\widetilde{X}/X}}\left(\widetilde{X}\right)$. The rest of the statement follows from the explicit construction of the universal objects $\mathrm{Alb}_{\mathcal{F}}\,(Y)$, see Remark 3.14. ■

6.3 Example: $X = \Gamma_\alpha \times \Gamma_\beta$

We conclude this section with the discussion of an example that was the subject of the diploma of Alexander Schwarzhaupt [Sch]. It illustrates some pathological properties: The universal regular quotient is not in general compatible with products, in this example we obtain $\dim\big(\mathrm{Alb}(\Gamma_\alpha \times \Gamma_\beta)\big) > \dim\big(\mathrm{Alb}(\Gamma_\alpha) \times \mathrm{Alb}(\Gamma_\beta)\big)$. Moreover, given a very ample line bundle $\mathcal{L}$ on the surface $X = \Gamma_\alpha \times \Gamma_\beta$ and a curve $C_N \in \left|\mathcal{L}^N\right|$ in general position, we work out a necessary and a sufficient condition on the integer N for the surjectivity of the Gysin map $\mathrm{Alb}(C_N) \longrightarrow \mathrm{Alb}(X)$.

6.3.1 The Curve Γ_α

Let $\Gamma_\alpha \subset \mathbb{P}^2_k$ be the projective curve defined by

$$\Gamma_\alpha : \qquad X^{2\alpha+1} - Y^2 Z^{2\alpha-1} = 0$$

where $X : Y : Z$ are homogeneous coordinates of $\mathbb{P}^2_k$ and $\alpha \geq 1$ is an integer. The singularities of this curve are cusps at $0 := [0:0:1]$ and $\infty := [0:1:0]$. The normalization $\widetilde{\Gamma}_\alpha$ of Γ_α is the projective line:

$$\widetilde{\Gamma}_\alpha = \mathbb{P}^1_k$$

Then $\mathrm{Alb}\left(\widetilde{\Gamma}_\alpha\right) = \mathrm{Alb}(\mathbb{P}^1_k) = 0$. Since $\mathrm{Alb}(\Gamma_\alpha)$ is an extension of $\mathrm{Alb}\left(\widetilde{\Gamma}_\alpha\right)$ by the linear group $L_{\Gamma_\alpha} = \left(\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right)^\vee$, we obtain

$$\mathrm{Alb}(\Gamma_\alpha) = L_{\Gamma_\alpha} = \left(\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right)^\vee$$

Moreover, Γ_α is homeomorphic to $\mathbb{P}^1_k$, i.e. the normalization $\widetilde{\Gamma}_\alpha$ is given by the largest homeomorphic curve Γ'_α. Then $\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}$ is an infinitesimal formal group (see Proposition 5.10), and we have

$$\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right) = \mathrm{Hom}_k\left(L_{\Gamma_\alpha}(k), k\right)$$

The k-valued points of L_{Γ_α} are given by (see Theorem 5.4)

$$\begin{aligned} L_{\Gamma_\alpha}(k) &= \mathcal{O}_{\widetilde{\Gamma}_\alpha} / \mathcal{O}_{\Gamma_\alpha} \\ &= \left(\mathcal{O}_{\widetilde{\Gamma}_\alpha} / \mathcal{O}_{\Gamma_\alpha}\right)_0 \oplus \left(\mathcal{O}_{\widetilde{\Gamma}_\alpha} / \mathcal{O}_{\Gamma_\alpha}\right)_\infty \end{aligned}$$

The dimensions are computed in [Sch] Proposition 1.5 as

$$\begin{aligned} \dim_k\left(\mathcal{O}_{\widetilde{\Gamma}_\alpha} / \mathcal{O}_{\Gamma_\alpha}\right)_0 &= \alpha \\ \dim_k\left(\mathcal{O}_{\widetilde{\Gamma}_\alpha} / \mathcal{O}_{\Gamma_\alpha}\right)_\infty &= 2\alpha(\alpha-1) \end{aligned}$$

hence

$$\begin{aligned}
\dim \operatorname{Alb}(\Gamma_\alpha) &= \dim \operatorname{Lie}\left(\underline{\operatorname{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right) \\
&= \dim_k \; \mathcal{O}_{\widetilde{\Gamma}_\alpha} / \mathcal{O}_{\Gamma_\alpha} \\
&= \alpha(2\alpha - 1)
\end{aligned}$$

Furthermore for $q \in \{0, \infty\} \subset \widetilde{\Gamma}_\alpha$ it holds

$$\begin{aligned}
\# \, v_q\left(\operatorname{Lie}\left(\underline{\operatorname{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right)\right) &= \# \left(v_q(\mathcal{O}_{\widetilde{\Gamma}_\alpha}) \setminus v_q(\mathcal{O}_{\Gamma_\alpha})\right) \\
&= \dim_k \left(\mathcal{O}_{\widetilde{\Gamma}_\alpha} / \mathcal{O}_{\Gamma_\alpha}\right)_q
\end{aligned}$$

A basis of $\operatorname{Lie}\left(\underline{\operatorname{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right)$ is given by

$$\Theta_{\Gamma_\alpha} = \left\{ t_q^\nu \;\middle|\; \nu \in v_q\left(\operatorname{Lie}\left(\underline{\operatorname{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right)\right), \; q = 0, \infty \right\}$$

where t_q is a local parameter of $\mathfrak{m}_{\widetilde{\Gamma}_\alpha, q}$.

6.3.2 The Surface $\Gamma_\alpha \times \Gamma_\beta$

Let X be the product of the cuspidal curves $\Gamma_\alpha, \Gamma_\beta$ from Subsubsection 6.3.1:

$$X = \Gamma_\alpha \times \Gamma_\beta$$

where $\alpha, \beta \geq 1$ are integers. The singular locus of X is

$$X_{\text{sing}} = (0 \times \Gamma_\beta) \cup (\infty \times \Gamma_\beta) \cup (\Gamma_\alpha \times 0) \cup (\Gamma_\alpha \times \infty)$$

The normalization $\widetilde{X}$ of X is given by

$$\widetilde{X} = \widetilde{\Gamma}_\alpha \times \widetilde{\Gamma}_\beta = \mathbb{P}^1_k \times \mathbb{P}^1_k$$

Then $\operatorname{Alb}\left(\widetilde{X}\right) = \operatorname{Alb}(\mathbb{P}^1_k) \times \operatorname{Alb}(\mathbb{P}^1_k) = 0$. Thus the universal regular quotient $\operatorname{Alb}(X)$ coincides with the linear group $L_X = \left(\underline{\operatorname{Div}}^0_{\widetilde{X}/X}\right)^\vee$:

$$\operatorname{Alb}(X) = L_X = \left(\underline{\operatorname{Div}}^0_{\widetilde{X}/X}\right)^\vee$$

and $\underline{\operatorname{Div}}^0_{\widetilde{X}/X}$ is an infinitesimal formal group (see Theorem 1.19).
If Θ_{Γ_ι} is a basis of $\operatorname{Lie}\left(\underline{\operatorname{Div}}^0_{\widetilde{\Gamma}_\iota/\Gamma_\iota}\right)$ for $\iota = \alpha, \beta$, then

$$\Theta_{\Gamma_\alpha \times \Gamma_\beta} = \left\{ \vartheta_{\Gamma_\alpha} \otimes \vartheta_{\Gamma_\beta} \;\middle|\; (\vartheta_{\Gamma_\alpha}, \vartheta_{\Gamma_\beta}) \in \left((\Theta_{\Gamma_\alpha} \cup \{1\}) \times (\Theta_{\Gamma_\beta} \cup \{1\})\right) \setminus \{(1,1)\} \right\}$$

is a basis of Lie $\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right)$. Thus the dimension of Alb (X) is

$$\begin{aligned}
& \dim \mathrm{Alb}\,(\Gamma_\alpha \times \Gamma_\beta) \\
= & \ (\dim \mathrm{Alb}\,(\Gamma_\alpha) + 1) \cdot (\dim \mathrm{Alb}\,(\Gamma_\beta) + 1) - 1 \\
= & \ \left(\dim_k \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right) + 1\right) \cdot \left(\dim_k \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\beta/\Gamma_\beta}\right) + 1\right) - 1 \\
= & \ (\alpha(2\alpha - 1) + 1) \cdot (\beta(2\beta - 1) + 1) - 1
\end{aligned}$$

The support of $\underline{\mathrm{Div}}^0_{\widetilde{X}/X}$ is the preimage of X_{sing}:

$$\mathrm{Supp}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right) = (0 \times \widetilde{\Gamma}_\beta) \cup (\infty \times \widetilde{\Gamma}_\beta) \cup (\widetilde{\Gamma}_\alpha \times 0) \cup (\widetilde{\Gamma}_\alpha \times \infty)$$

We obtain a basis of $\mathrm{im}\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right) \longrightarrow \left(\mathcal{K}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}}\right)_{\Gamma_\alpha \times q}\right)$
for $q \in \{0, \infty\} \subset \widetilde{\Gamma}_\beta$ by

$$\Theta_{\Gamma_\alpha \times q} = \left\{ \vartheta_{\Gamma_\alpha} \otimes t_q^\nu \;\middle|\; \vartheta_{\Gamma_\alpha} \in (\Theta_{\Gamma_\alpha} \cup \{1\}),\ \nu \in \upsilon_q\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\beta/\Gamma_\beta}\right)\right) \right\}$$

Now $\upsilon_q\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\beta/\Gamma_\beta}\right)\right) = \upsilon_{\Gamma_\alpha \times q}\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right)\right)$, therefore

$$\begin{aligned}
& \dim_k \mathrm{im}\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right) \longrightarrow \left(\mathcal{K}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}}\right)_{\Gamma_\alpha \times q}\right) \\
= & \ \left(\dim_k \mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right) + 1\right) \cdot \#\, \upsilon_{\Gamma_\alpha \times q}\left(\mathrm{Lie}\left(\underline{\mathrm{Div}}^0_{\widetilde{X}/X}\right)\right)
\end{aligned}$$

and analoguesly for $p \times \Gamma_\beta$, $p \in \{0, \infty\} \subset \widetilde{\Gamma}_\alpha$.

6.3.3 The Gysin map $\mathrm{Alb}\,(C_N) \longrightarrow \mathrm{Alb}\,(\Gamma_\alpha \times \Gamma_\beta)$

Consider the divisor

$$D^{k,l} = \sum_{i=1}^{k} p_i \times \Gamma_\beta + \sum_{j=1}^{l} \Gamma_\alpha \times q_j$$

The normalization of $D^{k,l}$ is isomorphic to the disjoint union of $k + l$ copies of $\mathbb{P}^1_k$. Therefore the Pic^0 group of the normalization is trivial. Then by Theorem 5.8, using the explicit formulas of Theorems 5.3 and 5.4

$$\mathrm{Pic}^0 D^{k,l} = \mathbb{T} \times \mathbb{V}$$

where $\mathbb{T} \cong (\mathbb{G}_{\mathrm{m}})^t$ is a torus of rank

$$\begin{aligned} t &= 1 - \#\operatorname{Cp}\left(D^{k,l}\right) + \#S_2 \\ &= 1 - (k+l) + k \cdot l \\ &= (k-1)\cdot(l-1) \end{aligned}$$

and $\mathbb{V} \cong (\mathbb{G}_{\mathrm{a}})^v$ is a vectorial group of dimension

$$\begin{aligned} v &= k \cdot \dim_k \ \mathcal{O}_{\widetilde{\Gamma}_\beta}\Big/ \mathcal{O}_{\Gamma_\beta} + l \cdot \dim_k \ \mathcal{O}_{\widetilde{\Gamma}_\alpha}/\,\mathcal{O}_{\Gamma_\alpha} \\ &= k \cdot \beta(2\beta - 1) + l \cdot \alpha(2\alpha - 1) \end{aligned}$$

For general $p_i \in \Gamma_\alpha$ and $q_j \in \Gamma_\beta$ the divisor $D^{2\alpha+1,2\beta+1}$ is very ample (see [Sch] Lemma 3.2). Set $\mathcal{L} = \mathcal{O}\left(D^{2\alpha+1,2\beta+1}\right)$ and let $C_N \in \left|\mathcal{L}^N\right|$ be in general position. Then the dimension of the vectorial part $\mathbb{V}_{C_N}$ of $\operatorname{Pic}^0 C_N = \operatorname{Alb}(C_N)$ is

$$\dim \mathbb{V}_{C_N} = N\big(\alpha(2\alpha-1)(2\beta+1) + \beta(2\beta-1)(2\alpha+1)\big)$$

The map $\operatorname{Alb}(C_N) \longrightarrow \operatorname{Alb}(\Gamma_\alpha \times \Gamma_\beta)$ cannot be surjective for $\dim \mathbb{V}_{C_N} < \dim \operatorname{Alb}(\Gamma_\alpha \times \Gamma_\beta)$, i.e. the simple comparison of dimensions yields:

Proposition 6.8 *The Gysin map* $\operatorname{Alb}(C_N) \longrightarrow \operatorname{Alb}(\Gamma_\alpha \times \Gamma_\beta)$ *is not surjective for*

$$N < \frac{(\alpha(2\alpha-1)+1)\cdot(\beta(2\beta-1)+1) - 1}{\alpha(2\alpha-1)(2\beta+1) + \beta(2\beta-1)(2\alpha+1)}$$

In the case $\alpha = \beta$*, this expression simplifies to*

$$N < \frac{\alpha(2\alpha-1)+2}{2(2\alpha+1)}$$

The homomorphism of vectorial groups $\mathbb{V}_{C_N} \longrightarrow \mathbb{V}_X = \operatorname{Alb}(X)$ is dual to the homomorphism of Lie algebras $\_ \cdot \widetilde{C}_N : \operatorname{Lie}\left(\underline{\operatorname{Div}}^0_{\widetilde{X}/X}\right) \longrightarrow \operatorname{Lie}\left(\underline{\operatorname{Div}}^0_{\widetilde{C}_N/C_N}\right)$, and the surjectivity of the first homomorphism is equivalent to the injectivity of the latter one. The estimation of Proposition 6.8 yields a necessary condition for surjectivity of the Gysin map, i.e. a bound for N from below. But by inspection of the map $\_ \cdot \widetilde{C}_N$ as in the proof of Proposition 6.2 one sees that this is not the greatest lower bound.

The criterion of Proposition 6.2 gives a sufficient condition for the surjectivity of the Gysin map:

$$\begin{aligned} & \#\left(\widetilde{C}_N \cap (\Gamma_\alpha \times q)\right) \cdot \# v_{\Gamma_\alpha \times q}(\operatorname{Lie}\underline{\operatorname{Div}}^0_{\widetilde{X}/X}) \\ \geq\ & \dim_k \operatorname{im}\left(\operatorname{Lie}\underline{\operatorname{Div}}^0_{\widetilde{X}/X} \longrightarrow \left(\mathcal{K}_{\widetilde{X}}/\,\mathcal{O}_{\widetilde{X}}\right)_{\Gamma_\alpha \times q}\right) \end{aligned}$$

where $q \in \{0, \infty\} \subset \widetilde{\Gamma}_\beta$, and analoguesly for $p \times \widetilde{\Gamma}_\beta$ with $p \in \{0, \infty\} \subset \widetilde{\Gamma}_\alpha$. By the formula for $\dim_k \operatorname{im}\left(\operatorname{Lie} \underline{\operatorname{Div}}^0_{\widetilde{X}/X} \longrightarrow \left(\mathcal{K}_{\widetilde{X}} / \mathcal{O}_{\widetilde{X}}\right)_{\Gamma_\alpha \times q}\right)$ at the end of Subsubsection 6.3.2, this is equivalent to

$$\#\left(\widetilde{C}_N \cap (\Gamma_\alpha \times q)\right) \geq \dim_k \operatorname{Lie}\left(\underline{\operatorname{Div}}^0_{\widetilde{\Gamma}_\alpha/\Gamma_\alpha}\right) + 1$$

Then since

$$\#\left(\widetilde{C}_N \cap (\Gamma_\alpha \times q)\right) = N(2\alpha + 1)$$

we obtain

Proposition 6.9 *The Gysin map* $\operatorname{Alb}(C_N) \longrightarrow \operatorname{Alb}(\Gamma_\alpha \times \Gamma_\beta)$ *is surjective if*

$$N \geq \frac{\alpha(2\alpha - 1) + 1}{2\alpha + 1}$$

and likewise for β *instead of* α.

In [ESV] Variant 6.4 the following sufficient condition for surjectivity of the Gysin map is given:

$$\dim_k \operatorname{im}\left(\mathrm{H}^0\left(X, \mathcal{L}^N\right) \longrightarrow \mathrm{H}^0\left(Z, \mathcal{L}^N|_Z\right)\right) \geq 2 \dim L_{C_N} + \# \operatorname{Cp}(X) + 2$$

for all $Z \in \operatorname{Cp}(X)$, where L_{C_N} is the largest linear subgroup of $\operatorname{Pic}^0 C_N$ for $C_N \in \left|\mathcal{L}^N\right|$ in general position. For $X = \Gamma_\alpha \times \Gamma_\beta$ it holds $\operatorname{Cp}(X) = \{X\}$ and $\operatorname{Pic}^0 C_N = L_{C_N} = \mathbb{V}_{C_N}$. Alexander Schwarzhaupt showed in his diploma [Sch] that in our example and for $\alpha = \beta$ this condition leads to the estimation

$$N > 2\frac{3\alpha(2\alpha - 1) - 1}{2\alpha + 1} + 1$$

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