Contributions to the theory of normal affine semigroup rings and Ulrich modules of rank one over determinantal rings

Abstract

In the first section, we study the Rees algebra of a positive normal affine semigroup ring R with respect to its graded maximal ideal $\mathfrak{m}$. It is obvious that $R[\mathfrak{m}t]$ is again a positive affine semigroup ring. But in general, $R[\mathfrak{m}t]$ is not normal. In fact, we show that $R[\mathfrak{m}t]$ may even fail to be Cohen-Macaulay.

The main result of the first section is a normality criterion for the Rees algebra: we prove that $R[\mathfrak{m}t]$ is normal if and only if the powers $\mathfrak{m}^i, i = 1, \dots, d-2$, with $d = \dim R$, are integrally closed in R. As a corollary, we obtain that $R[\mathfrak{m}t]$ is normal if $\dim R \leq 3$.

We also consider the special case that the embedding dimension of R is equal to $\dim R + 1$. In this situation, the Rees algebra $R[\mathfrak{m}t]$ is always Cohen-Macaulay, and we can give an easy criterion for $R[\mathfrak{m}t]$ to be normal.

The second section is devoted to the type $r(R)$ of a simplicial normal affine semigroup ring R of dimension $d \leq 3$. The type (some authors say: Cohen-Macaulay type) is an important numerical invariant of R. It is equal to the minimal number of generators of the canonical module of R. Therefore, in a sense, it measures how far R is away from being Gorenstein.

We prove that $r(R)$ is bounded above by $r(\overline{P})$, where $\overline{P}$ is the special fibre of an embedding $R \hookrightarrow P := K[x_1, \dots, x_d]$.

In the third section, we turn to determinantal rings. We show that the divisor class group of a determinantal ring $R = K[X]/I_{r+1}(X)$ contains two outstanding classes: the ideals which represent these classes are Ulrich modules of rank one.