

CONTRIBUTIONS TO THE THEORY OF
NORMAL AFFINE SEMIGROUP RINGS
AND
ULRICH MODULES OF RANK ONE
OVER DETERMINANTAL RINGS

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For my wife

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INTRODUCTION

The theory of affine semigroup rings and the theory of determinantal rings are appealing and vital branches of present-day commutative algebra. In the investigation of these rings, both geometric and combinatoric aspects play an important role.

An affine semigroup ring R is a finitely generated algebra over a field K , which is isomorphic to a \mathbb{Z}^n -graded subalgebra of the ring $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of Laurent polynomials. If the unit group R^\times is equal to K^\times , we call R a positive affine semigroup ring. A fundamental result of affine semigroup rings is Hochster's theorem, which states that a normal affine semigroup ring is Cohen-Macaulay. In our thesis, we will mainly consider positive normal affine semigroup rings.

Normal affine semigroup rings occur for example in invariant theory: if K is an algebraically closed field and T is a torus group over K which acts linearly on $A = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, then the ring A^T of invariants is a normal affine semigroup ring. Therefore, some authors use the term 'toric ring' instead of 'normal affine semigroup ring'.

The general definition of a determinantal ring is rather complicated. However, in this thesis, we only consider determinantal rings of the form $K[X]/I_{r+1}(X)$, where X is an $m \times n$ -matrix of indeterminates over a field K , and $I_{r+1}(X)$ denotes the ideal of $K[X]$ which is generated by the $(r+1)$ -minors of X .

Determinantal rings are the most prominent example of algebras with straightening law. Just as normal affine semigroup rings, determinantal rings are normal Cohen-Macaulay domains.

In the first section, we study the Rees algebra of a positive normal affine semigroup ring R with respect to its graded maximal ideal \mathfrak{m} . It is obvious that $R[\mathfrak{m}t]$ is again a positive affine semigroup ring. But in general, $R[\mathfrak{m}t]$ is not normal. In fact, we show that $R[\mathfrak{m}t]$ may even fail to be Cohen-Macaulay.

The main result of the first section is a normality criterion for the Rees algebra: we prove that $R[\mathfrak{m}t]$ is normal if and only if the powers $\mathfrak{m}^i, i = 1, \dots, d-2$, with $d = \dim R$, are integrally closed in R . As a corollary, we obtain that $R[\mathfrak{m}t]$ is normal if $\dim R \leq 3$.

When proving the normality criterion, we make use of some notions from convex geometry that we learned from the preprint [BG2] of Bruns and Gubeladze. Also, the monographs [Va1] and [Va2] of Vasconcelos were valuable sources of inspiration when writing this section. A large part of this section is contained in the author's article [Wi], which will be published soon in *Manuscripta Mathematica*.

The second section is devoted to the type $r(R)$ of a simplicial normal affine semigroup ring R of dimension $d \leq 3$. The type (some authors say: Cohen-Macaulay type) is an important numerical invariant of R . It is equal to the minimal number of generators of the canonical module of R . Therefore, in a sense, it measures how far R is away from being Gorenstein.

We prove that $r(R)$ is bounded above by $r(\overline{P})$, where \overline{P} is the special fibre of an embedding $R \hookrightarrow P := K[x_1, \dots, x_d]$.

In the third section, we turn to determinantal rings. We show that the divisor class group of a determinantal ring $R = K[X]/I_{r+1}(X)$ contains two outstanding classes: the ideals which represent these classes are Ulrich modules of rank one. Although affine semigroup rings seem to have little to do with this subject, they play a crucial role in the proof of the main theorem of that section.

The results of this section appeared in the joint paper [BRW] with Bruns and Römer.

Terminology

We say that a domain is normal, if it is Noetherian and integrally closed in its field of fractions. A finitely generated graded algebra A over a field K is called standard graded, if $A_0 = K$ and $A = K[A_1]$.

The symbol \mathbb{N} denotes the set of all positive integers, and \mathbb{N}_0 denotes $\mathbb{N} \cup \{0\}$. The symbol \mathbb{Q}_+ (resp. \mathbb{R}_+) denotes the set of all nonnegative rational (resp. real) numbers. We use \subset for a proper inclusion and \subseteq to mean “contained in or equal to”.

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1. THE REES ALGEBRA OF A NORMAL AFFINE SEMIGROUP RING

The issue of this section can be summarised in the following problem: let R be a positive normal affine semigroup ring with graded maximal ideal \mathfrak{m} . Under which conditions is the Rees algebra $R[\mathfrak{m}t]$ normal or at least Cohen-Macaulay?

We have been able to achieve some results concerning this question, and we will present them here. Moreover, the section is enriched with many examples that should help the reader to get a better insight into this subject.

In the first three subsections, we introduce the basic principles of the theory of affine semigroups and affine semigroup rings. It is not our intention to give a broad overview of this topic, but rather to collect those notions and facts that will be necessary for our study.

The question whether the Rees algebra $R[\mathfrak{m}t]$ is normal can be rephrased as a question concerning the powers of \mathfrak{m} : it is known that $R[\mathfrak{m}t]$ is normal if and only if all powers of \mathfrak{m} are integrally closed in R . Therefore, the notion of the integral closure of an ideal plays a central role in our investigations. In the fourth subsection, we recall its definition and describe in particular the integral closure of a monomial ideal in an affine semigroup ring.

The fifth subsection is devoted to the associated graded ring $\text{gr}_{\mathfrak{m}}(R)$. We show that $\text{gr}_{\mathfrak{m}}(R) = R[\mathfrak{m}t] \otimes_R R/\mathfrak{m}$ is reduced, respectively a domain, if and only if the affine semigroup S satisfies certain geometric conditions.

The sixth subsection contains our main result concerning the normality of the Rees algebra: let R be a positive normal affine semigroup ring of dimension d , and assume that all powers $\mathfrak{m}^i, i = 1, \dots, d - 2$, are integrally closed in R . Then the Rees algebra $R[\mathfrak{m}t]$ is normal.

In the seventh subsection, we examine $R[\mathfrak{m}t]$ with respect to the Cohen-Macaulay property. In particular, we give an example of a positive normal affine semigroup ring R whose Rees algebra $R[\mathfrak{m}t]$ is *not* Cohen-Macaulay.

Finally, we consider the special case that the embedding dimension of R is equal to $\dim R + 1$. In this situation, the Rees algebra $R[\mathfrak{m}t]$ is always Cohen-Macaulay, and we can give an easy criterion for $R[\mathfrak{m}t]$ to be normal.

1.1. Affine semigroup rings.

An *affine semigroup* S is a finitely generated additive semigroup with neutral element that is isomorphic to a subsemigroup of \mathbb{Z}^n for some $n \in \mathbb{N}$. For instance, $(\mathbb{N}_0)^n$ is an affine semigroup for all $n \in \mathbb{N}$. For a real number $x \geq 0$, the semigroup

$$S_x = \{(a_1, a_2) \in \mathbb{Z}^2 \mid a_1, a_2 \geq 0, a_2 \geq a_1 x\}$$

is finitely generated if and only if x is rational. Thus, S_x is an affine semigroup if and only if $x \in \mathbb{Q}$.

Just like \mathbb{Z} being generated by \mathbb{N}_0 as a group, for every affine semigroup S there exists a finitely generated abelian group that contains S and is generated by S as a group. This group is unique up to canonical isomorphism and is named $\mathbb{Z}S$. Its

rank is the *dimension* of S . Moreover, $\mathbb{R}S$ denotes $\mathbb{Z}S \otimes_{\mathbb{Z}} \mathbb{R}$, and one considers S to be a subset of $\mathbb{R}S$ via the canonical map $\mathbb{Z}S \rightarrow \mathbb{Z}S \otimes_{\mathbb{Z}} \mathbb{R}$. S is said to be *positive*, if 0 is the only invertible element in S .

A subset I of an affine semigroup S is called an *ideal*, if $a + b \in I$ for all $a \in I$ and all $b \in S$. For an ideal $I \subseteq S$ and an $n \in \mathbb{N}$, the set $nI := \{\sum_{i=1}^n w_i \mid w_i \in I\}$ is again an ideal of S .

Assume that S is a positive affine semigroup. Then the set $S_+ := S \setminus \{0\}$ is an ideal of S . One sets $\text{Hilb}(S) := S_+ \setminus 2S_+$. The elements $a \in \text{Hilb}(S)$ are called the *minimal generators* of S , since they generate S and since they are contained in any set of generators of S . In particular, $\text{Hilb}(S)$ is a finite set.

Let S be an affine semigroup, and let K be a field. The K -vectorspace

$$K[S] := \bigoplus_{a \in S} Kx^a$$

becomes a K -algebra by setting $x^a \cdot x^b := x^{a+b}$ for all $a, b \in S$. It is the *affine semigroup ring* associated to S over K . Abusing language, we say that $K[S]$ is a *positive affine semigroup ring* in case S is positive. Since S is a finitely generated semigroup, $K[S]$ is a finitely generated K -algebra and thus Noetherian. Note that $K[\mathbb{Z}S]$ is isomorphic to the ring $K[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ of Laurent polynomials, where d is the dimension of S . Since $K[S]$ is a subring of $K[\mathbb{Z}S]$, one obtains in particular that $K[S]$ is an integral domain.

Assume that S is positive. A decomposition $R = \bigoplus_{n \geq 0} R_n$ of $R = K[S]$ is called an *admissible grading*, if the following conditions are fulfilled:

- (a) $R_0 = K$.
- (b) For all $n \geq 0$, R_n is a K -vectorspace that is generated by finitely many elements of the form x^a , $a \in S$.
- (c) For all $m, n \geq 0$, $R_m \cdot R_n$ is contained in R_{m+n} .

The existence of an admissible grading is guaranteed by the following

Proposition 1.1. *If S is a positive affine semigroup, then there exists a group homomorphism $\varphi : \mathbb{Z}S \rightarrow \mathbb{Z}^r$ (for some $r \in \mathbb{N}$), such that $\varphi(S) \subseteq (\mathbb{N}_0)^r$.*

For a proof, see e.g. [BH, 6.1.5]. By setting $R_n = \bigoplus_{a \in S, |\varphi(a)|=n} Kx^a$ for all $n \geq 0$, (where $|\varphi(a)|$ is the sum of the r components of $\varphi(a)$), one obtains an admissible grading.

If $R = \bigoplus_{n \geq 0} R_n$ is any admissible grading, then the maximal ideal $\bigoplus_{n > 0} R_n$ is equal to $\mathfrak{m} := \bigoplus_{a \in S_+} Kx^a$. Therefore, \mathfrak{m} is called the *graded maximal ideal* of R .

Now consider again an arbitrary affine semigroup S . If I is an ideal of S , then $\bigoplus_{a \in I} Kx^a \subseteq K[S]$ is an ideal of $K[S]$. It is called the *monomial ideal* of $K[S]$ associated to I . If $\mathfrak{a} \subseteq K[S]$ is the monomial ideal associated to I , then \mathfrak{a}^n is the monomial ideal associated to nI for all $n \in \mathbb{N}$.

The semigroup $\bar{S} := \{a \in \mathbb{Z}S \mid ma \in S \text{ for some } m \in \mathbb{N}\}$ is called the *normalization* of S . S is said to be *normal* if $S = \bar{S}$. This terminology is justified by the fundamental

Theorem 1.2. *Let S be an affine semigroup, and let $K[S]$ be the associated affine semigroup ring over a field K . Then S is normal if and only if $K[S]$ is normal.*

Trivially, S is normal in case $K[S]$ is normal. For a proof of the converse, see [BH, 6.1.4].

Corollary 1.3. *Let S be an affine semigroup, and let $R = K[S]$ be the associated affine semigroup ring over a field K . Then \overline{S} is a normal affine semigroup, and $K[\overline{S}]$ is equal to the normalization of R in its field of fractions.*

Proof. Clearly, $K[\overline{S}]$ is contained in A , where A is the normalization of R in its field of fractions. Since R is a finitely generated K -algebra, we obtain that A is finitely generated as an R -module by E. Noether's theorem on the finiteness of the integral closure (see [Ei, 13.13]). Therefore, the R -submodule $K[\overline{S}]$ of A is also finitely generated. In particular, $K[\overline{S}]$ is a finitely generated K -algebra, and thus \overline{S} is a finitely generated semigroup. The fact that \overline{S} is normal follows directly from the definition of normality. Since we have shown that \overline{S} is an affine semigroup, we may apply Theorem 1.2 and obtain that $K[\overline{S}]$ is normal. This means that $K[\overline{S}]$ is equal to A . \square

The following famous theorem is due to Hochster, see [BH, 6.3.5] for a proof.

Theorem 1.4. *Let $R = K[S]$ be an affine semigroup ring. If R is normal, then it is Cohen-Macaulay.*

Before we can describe affine semigroups in more detail, we need a few notions from convex geometry.

1.2. A short excursion into convex geometry.

Let V be a finite dimensional \mathbb{R} -vectorspace. A subset $A \subset V$ is said to be a *hyperplane* (resp. *closed halfspace*) of V , if there exists a nontrivial linear form $\tau \in \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and an $\alpha \in \mathbb{R}$, such that A is equal to

$$H(\tau; \alpha) := \{v \in V \mid \tau(v) = \alpha\} \quad (\text{resp. } H^+(\tau; \alpha) := \{v \in V \mid \tau(v) \geq \alpha\}).$$

A nonempty intersection $P = \bigcap_i H_i^+$ of finitely many halfspaces $H_i^+ = H^+(\tau_i; \alpha_i)$ of V is called a *polyhedron*. If $\alpha_i = 0$ for all i , then P is called a *cone*. If P is bounded, then it is said to be a *polytope*. The *dimension* of P is given by the dimension of its affine hull $\text{aff}(P)$, and the *boundary* of P is defined as the boundary of P in $\text{aff}(P)$.

We say that a subset A of V is *convex*, if for all $x, y \in A$ and all real numbers $\lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda)y \in A$. Since closed halfspaces are convex and since any intersection of convex sets is convex, one obtains that a polyhedron is convex. A cone fulfills a stronger condition than convexity: if P is a cone and x_1, \dots, x_r are elements in P , then $\sum_{i=1}^r \alpha_i x_i \in P$ for all $\alpha_1, \dots, \alpha_r \in \mathbb{R}_+$.

For any subset X of V , the *convex hull* $\text{conv}(X)$ is the intersection of all convex subsets of V that contain X . The following result of Carathéodory gives an alternative description of the convex hull (see e.g. [Gr, 2.3.5] for a proof).

Theorem 1.5. For any nonempty subset X of a finite dimensional \mathbb{R} -vector space V , the convex hull of X is equal to the set of all linear combinations $\sum_{i=1}^r \alpha_i x_i$, where $r \in \mathbb{N}$, $x_1, \dots, x_r \in X$, $\alpha_1, \dots, \alpha_r \in \mathbb{R}_+$, and $\sum_{i=1}^r \alpha_i = 1$.

Let P be a polyhedron. A hyperplane H of V is called a *support hyperplane* of P , if P is contained in one of the two closed halfspaces bounded by H and if the intersection $H \cap P$ is not empty. A subset F of P called a *face* of P if there is a support hyperplane H of P with $F = P \cap H$. The empty set and P are called the *improper faces* of P . Any nonempty face of P is itself a polyhedron.

A face of P that has dimension 0 (resp. $\dim P - 1$) is called a *vertex* (resp. *facet*) of P , and $\text{vert}(P)$ denotes the set of all vertices of P .

The next theorem lists some basic properties of a polyhedron and its faces.

Theorem 1.6. For a polyhedron P in V we have

- (a) The number of faces of P is finite.
- (b) If F is a face of P and F' is a face of F , then F' is also a face of P .
- (c) The boundary of P is equal to the union of all facets of P .
- (d) For a vertex v of P , the set $\text{conv}(P \setminus \{v\})$ does not contain v .
- (e) If P is a polytope, then $P = \text{conv}(\text{vert}(P))$.

It is an important fact that a polytope can also be defined as the convex hull of a finite set of points.

Theorem 1.7. Let X be a finite set in a finite dimensional \mathbb{R} -vector space V . Then $P := \text{conv}(X)$ is a polytope and $\text{vert}(P)$ is contained in X .

Proofs for Theorem 1.6 and Theorem 1.7 can be found in chapter 1 of [BG2]. Combining Theorem 1.6 (a), (e), and Theorem 1.7, one obtains that a subset P of a finite dimensional \mathbb{R} -vector space V is a polytope if and only if P is the convex hull of a finite set of points. For cones, there exists a similar characterization:

Theorem 1.8. A subset P of a finite dimensional \mathbb{R} -vector space V is a cone if and only if there exists a nonempty finite subset X of V such that P is equal to

$$\mathbb{R}_+ X = \left\{ \sum_{i=1}^r \alpha_i x_i \mid r \in \mathbb{N}, x_1, \dots, x_r \in X, \alpha_1, \dots, \alpha_r \in \mathbb{R}_+ \right\}.$$

A cone P in \mathbb{R}^n is said to be *rational* if $P = \mathbb{R}_+ X$ for some finite set $X \subset \mathbb{Q}^n$. It can be shown that P is rational if and only if P is equal to the intersection of finitely many halfspaces $H^+(\tau_i; \alpha_i)$, where $\tau_i \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^n, \mathbb{Q}) \subset \text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ and $\alpha_i \in \mathbb{Q}$ for all i . One easily proves the following

Lemma 1.9. Let v be a point in \mathbb{Q}^n and let $X = \{x_1, \dots, x_r\}$ be a subset of \mathbb{Q}^n .

- (a) If $v \in \text{conv}(X)$, then there exist $\alpha_1, \dots, \alpha_r \in \mathbb{Q}_+$ with $\sum_{i=1}^r \alpha_i = 1$ and $v = \sum_{i=1}^r \alpha_i x_i$.
- (b) If $v \in \mathbb{R}_+ X$, then there exist $\alpha_1, \dots, \alpha_r \in \mathbb{Q}_+$ with $v = \sum_{i=1}^r \alpha_i x_i$.

The next lemma is quite technical. We will apply it in the proof of our main result concerning the normality of the Rees algebra (Theorem 1.25).

Lemma 1.10. *Let P be a polytope in a finite dimensional \mathbb{R} -vector space V , that is equal to the convex hull of a finite set $X \subset V$. For every $x_0 \in P$, there exists an affinely independent subset $U \subset X$ such that $x_0 \in \text{conv}(U)$ and $x \notin \text{conv}(U)$ for all $x \in X \setminus U$.*

Proof. If $x_0 \in X$, then $U := \{x_0\}$ satisfies the required condition, and so we may assume that $x_0 \notin X$. We argue by induction on r , where r denotes the cardinality of X . Since the case $r = 1$ is trivial, we assume $r > 1$.

Let d be the dimension of P . Since P is the convex hull of X , we have $d + 1 \leq r$. If $d + 1 = r$, X is affinely independent and $U := X$ satisfies the required condition. So we consider the case $d + 1 < r$.

Assume that P has a facet F such that $X' := X \cap F$ contains $r - 1$ elements. Let y denote the element in $X \setminus X'$. Since P is equal to $\text{conv}(F \cup \{y\})$, one has $x_0 = \lambda y + (1 - \lambda)z$ for some $z \in F$ and some $\lambda \in [0, 1)$. By Theorem 1.6 (b), we have $\text{vert}(F) \subseteq \text{vert}(P)$, and by Theorem 1.7, $\text{vert}(P) \subseteq X = X' \cup \{y\}$. Since $y \notin F$, it follows that $\text{vert}(F) \subseteq X'$. Therefore, $\text{conv}(\text{vert}(F)) \subseteq \text{conv}(X') \subseteq F$. But $F = \text{conv}(\text{vert}(F))$ by Theorem 1.6 (e), and hence $F = \text{conv}(X')$.

Applying the induction hypothesis to $F = \text{conv}(X')$, we find an affinely independent subset U' of X' with $z \in \text{conv}(U')$ and $x \notin \text{conv}(U')$ for all $x \in X' \setminus U'$. We set $U = U' \cup \{y\}$ and $\Delta = \text{conv}(U)$. Since $y \notin F$, U is affinely independent. From $\Delta \cap F = \text{conv}(U')$ we see that $x \notin \Delta$ for all $x \in X \setminus U = X' \setminus U'$. Furthermore, $x_0 \in \text{conv}(\{y, z\}) \subset \Delta$.

Now assume that no facet of P contains $r - 1$ elements of X . If $\text{vert}(P)$ is equal to X , let y be any element of X . Otherwise, $\text{vert}(P)$ is a proper subset of X and we can choose y in $X \setminus \text{vert}(P)$. Since P is bounded, there is a $\lambda \in \mathbb{R}_+$ such that $z := x_0 + \lambda(x_0 - y)$ lies on the boundary of P . By Theorem 1.6 (c), z is contained in some facet F of P . Set $X' = \text{conv}(F \cup \{y\}) \cap X$, and let P' be the convex hull of X' . Since $\text{vert}(F) \subseteq \text{vert}(P) \subseteq X$ and $y \in X$, we have $\text{vert}(F) \cup \{y\} \subseteq X'$. This implies $\text{conv}(F \cup \{y\}) = \text{conv}(\text{vert}(F) \cup \{y\}) \subseteq \text{conv}(X') = P'$, and hence $P' = \text{conv}(F \cup \{y\})$. The equation $x_0 = \frac{\lambda}{1+\lambda}y + \frac{1}{1+\lambda}z$ shows that $x_0 \in P'$.

We show that X' is a proper subset of X . Then we can apply the induction hypothesis to $P' = \text{conv}(X')$ and find an affinely independent subset U of X' with $x_0 \in \text{conv}(U)$ and $x \notin \text{conv}(U)$ for all $x \in X' \setminus U$. Since $X' = P' \cap X$, we have $x \notin P' \supseteq \text{conv}(U)$ for all $x \in X \setminus X'$, and the proof would be finished.

If $\text{vert}(P) = X$, then there exists a vertex w of P with $w \notin F \cup \{y\}$, since F contains at most $r - 2$ elements of X . By Theorem 1.6 (d), $w \notin \text{conv}(F \cup \{y\})$ and therefore $w \notin X'$. If $\text{vert}(P) \neq X$, y is not a vertex of P by the choice of y . Since $\text{vert}(P) \not\subseteq F$, there must be a vertex w of P with $w \notin F \cup \{y\}$. Again, we see that $w \notin X'$. \square

The reader familiar with convex geometry will have noticed that Lemma 1.10 can be proved by choosing a triangulation Π of P such that X is equal to the vertex set of Π . However, our elementary proof shows that the assertion of the lemma can be derived directly from the theorems quoted above.

1.3. The bottom of an affine semigroup.

For an affine semigroup S , the normalization \overline{S} is obviously contained in the intersection $\mathbb{Z}S \cap \mathbb{R}_+S$ (within $\mathbb{R}S$). Conversely, noting that $\mathbb{Z}S \cap \mathbb{R}_+S \subseteq \mathbb{Q}_+S$ by Lemma 1.9 (b), one sees that every $a \in \mathbb{Z}S \cap \mathbb{R}_+S$ lies in \overline{S} . Thus, $\overline{S} = \mathbb{Z}S \cap \mathbb{R}_+S$. This observation leads to a very useful characterization of normal affine semigroups:

Theorem 1.11. *A subset S of \mathbb{Z}^n is a normal affine semigroup if and only if there exists a subgroup $G \subseteq \mathbb{Z}^n$ and a rational cone $C \subseteq \mathbb{R}^n$ such that $S = G \cap C$.*

The ‘only if’-direction has already been shown. For a proof of the ‘if’-direction, see e.g. [BH, 6.1.2].

Let S be a positive affine semigroup. We define $P(S)$ to be the polytope in $\mathbb{R}S$ that is equal to the convex hull of all $a \in \text{Hilb}(S)$. Its *bottom* $B(P(S))$ is the set

$$\{v \in P(S) \mid \lambda v \notin P(S) \text{ for all } \lambda < 1\},$$

and the intersection $B(S) := B(P(S)) \cap \text{Hilb}(S)$ is called the bottom of S . Since S is generated by $\text{Hilb}(S) \subseteq P(S)$, we have $\dim P(S) \geq \dim S - 1$.

One can show that for any positive affine semigroup of dimension $d \leq 2$, $B(S)$ is equal to $\text{Hilb}(S)$, see e.g. [BG2, 2.54]. For dimension $d \geq 3$, this is no longer true.

Example 1.12. Let S be the positive affine semigroup consisting of all elements $(a_1, a_2, a_3) \in \mathbb{Z}^3$ with $a_1, a_2, a_3 \geq 0$, $a_3 \leq 2a_1$, and $a_3 \leq 2a_2$. Then

$$B(S) = \{(1, 0, 0), (0, 1, 0), (1, 1, 2)\} \text{ and } \text{Hilb}(S) = B(S) \cup \{(1, 1, 1)\}.$$

Proof. It is easy to see that $\text{Hilb}(S) = \{(1, 0, 0), (0, 1, 0), (1, 1, 1), (1, 1, 2)\}$ and that $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 2)$ are contained in $B(S)$. The equation

$$\frac{2}{3}(1, 1, 1) = \frac{1}{3}(1, 0, 0) + \frac{1}{3}(0, 1, 0) + \frac{1}{3}(1, 1, 2) \in P(S)$$

shows that $(1, 1, 1) \notin B(S)$. □

Lemma 1.13. *Let S be a positive affine semigroup. For all $a \in S_+$ there exists a unique positive real number $\lambda \leq 1$ such that $\lambda a \in B(P(S))$.*

Proof. First, we show that 0 is not contained in $P(S)$. Otherwise, there would be $w_1, \dots, w_r \in \text{Hilb}(S)$, and rational numbers $\alpha_1, \dots, \alpha_r > 0$ with $\sum_{i=1}^r \alpha_i = 1$ and $\sum_{i=1}^r \alpha_i w_i = 0$ (see Lemma 1.9 (a)). Choosing an integer $m \gg 0$ such that $m\alpha_i \in \mathbb{N}$ for $i = 1, \dots, r$, we get $0 = \sum_{i=1}^r (m\alpha_i)w_i \in mS_+$, which is impossible, since S is positive.

Now, let $a \in S_+$. Then there are $w_1, \dots, w_r \in \text{Hilb}(S)$ and $\alpha_1, \dots, \alpha_r \in \mathbb{R}_+$ such that $a = \sum_{i=1}^r \alpha_i w_i$. For $\alpha = \sum_{i=1}^r \alpha_i$ we have $\frac{1}{\alpha}a \in P(S)$. Since $P(S)$ is compact and $0 \notin P(S)$, there exists a unique $\beta \in (0, 1]$ with $\frac{\beta}{\alpha}a \in B(P(S))$. □

Let S be a positive affine semigroup. If $H^+ = H^+(\tau; \alpha)$ is a closed halfspace of $\mathbb{R}S$ such that

$$0 \notin H^+, P(S) \subset H^+, \text{ and } \dim(P(S) \cap H(\tau; \alpha)) = \dim S - 1,$$

then $H = H(\tau; \alpha)$ is called a *bottom hyperplane* of $P(S)$, and $P(S) \cap H$ is called a *bottom face* of $P(S)$. For every bottom face F of $P(S)$, we have $F \subseteq B(P(S))$ and $F = \text{conv}(B(S) \cap F)$. If $\dim P(S) = \dim S - 1$, then $P(S)$ has only one bottom face, namely $P(S)$ itself.

Lemma 1.14. *If S be a positive affine semigroup, then $B(P(S))$ is equal to the union of all bottom faces of $P(S)$.*

Proof. It suffices to show that $B(P(S))$ is contained in the union of all bottom faces of $P(S)$. Let d be the dimension of S . Then $\dim P(S)$ equals d or $d - 1$. The case $\dim P(S) = d - 1$ is trivial, since in this case $P(S)$ itself is a bottom face. So we may assume $\dim P(S) = d$.

Let $F_i, i \in I$, be the facets of $P(S)$. For each i we have $F_i = H_i \cap P(S)$, where H_i is a support hyperplane of $P(S)$. Let $H_i^+, i \in I$, be the corresponding closed halfspaces of $\mathbb{R}S$ that contain $P(S)$. One easily deduces from Theorem 1.6(c) that $P(S)$ is equal to $\bigcap_{i \in I} H_i^+$. We set $J = \{i \in I \mid 0 \notin H_i^+\}$. Note that $i \in J$ if and only if F_i is a bottom face of $P(S)$.

Now consider an $a \in P(S)$ which does not lie in $\bigcup_{i \in J} F_i$. Then $a \in H_i^+ \setminus H_i$ for all $i \in J$. Hence there exists an $\varepsilon \in (0, 1)$ such that $(1 - \varepsilon)a \in H_i^+$ for all $i \in J$. Trivially, $(1 - \varepsilon)a \in H_i^+$ for all $i \in I \setminus J$, so that $(1 - \varepsilon)a \in \bigcap_{i \in I} H_i^+ = P(S)$. This means that $a \notin B(P(S))$. \square

Lemma 1.15. *Let S be a positive affine semigroup, and let a be an element in S_+ . Then there exists a bottom face F of $P(S)$ and a linear independent subset U of $B(S) \cap F$ such that $\text{conv}(U) \cap S = U$ and $a \in \mathbb{R}_+U$.*

Proof. By Lemma 1.13, we have $\lambda a \in B(P(S))$ for some positive real number λ . According to Lemma 1.14, λa lies on a bottom face F of $P(S)$. Let X denote the intersection $B(S) \cap F (= S \cap F)$. Since the vertices of F are contained in X , we have $F = \text{conv}(X)$. Using Lemma 1.10, we find an affinely independent subset U of X with $\lambda a \in \text{conv}(U)$ and $\text{conv}(U) \cap S = U$. Since $0 \notin \text{aff}(U)$, U is even linear independent. From $\lambda a \in \text{conv}(U)$ follows that $a \in \mathbb{R}_+U$. \square

1.4. The integral closure of an ideal.

Let \mathfrak{a} be an ideal in a ring A . An element $x \in A$ is said to be *integral over \mathfrak{a}* if there exists an $n \in \mathbb{N}$ and elements $y_1, \dots, y_n \in A$ with $y_i \in \mathfrak{a}^i$ for $i = 1, \dots, n$, and

$$x^n + y_1 x^{n-1} + \dots + y_{n-1} x + y_n = 0.$$

The *integral closure of \mathfrak{a} in A* consists of all elements in A that are integral over \mathfrak{a} , and is denoted by $\bar{\mathfrak{a}}$. Obviously, \mathfrak{a} is contained $\bar{\mathfrak{a}}$. If $\mathfrak{a} = \bar{\mathfrak{a}}$, one says that \mathfrak{a} is *integrally closed in A* . One can show that $\bar{\mathfrak{a}}$ is an ideal of A that is integrally closed in A , see e.g. [BH, 10.2.2].

Now consider an affine semigroup ring $R = K[S]$, where S is a positive normal affine semigroup. Let $I \subseteq S$ be a semigroup ideal, and let \mathfrak{a} denote the associated

monomial ideal $\bigoplus_{a \in I} Kx^a$ in R . The semigroup ideal

$$\bar{I} := \{a \in S \mid na \in nI \text{ for some } n \in \mathbb{N}\}$$

is called the integral closure of I in S . This terminology is justified by

Lemma 1.16. *The integral closure of \mathfrak{a} in R is equal to $\bigoplus_{a \in \bar{I}} Kx^a$.*

This lemma can be deduced from the fact that an affine semigroup is normal if and only if the associated semigroup ring is normal: note that $R[\mathfrak{a}t]$ is isomorphic to the semigroup ring $K[T]$, where

$$T = \{(a, 0) \mid a \in S\} \cup \{(a, n) \mid n \in \mathbb{N}, a \in nI\}.$$

Since the normalization \bar{T} of T equals $\{(a, 0) \mid a \in S\} \cup \{(a, n) \mid n \in \mathbb{N}, a \in n\bar{I}\}$, we have

$$\overline{R[\mathfrak{a}t]} = K[\bar{T}] = K \oplus \left(\bigoplus_{a \in \bar{I}} Kx^{at}\right) \oplus \left(\bigoplus_{a \in 2\bar{I}} Kx^{a^2t^2}\right) \oplus \dots,$$

and the assertion is a consequence of the following

Lemma 1.17. *Let A be a Noetherian domain, and let $\mathfrak{a} \subseteq A$ be an ideal. Then the integral closure of $A[\mathfrak{a}t]$ in $A[t]$ is equal to $\bigoplus_{n \geq 0} \bar{\mathfrak{a}}^n t^n$, where $\bar{\mathfrak{a}}^n$ denotes the integral closure of \mathfrak{a}^n in A for all $n \geq 0$. In particular, $A[\mathfrak{a}t]$ is normal if and only if A is normal and all powers of \mathfrak{a} are integrally closed in A .*

A proof of Lemma 1.17 can be found in [Ri]. The next lemma gives a geometric description of \bar{I} .

Lemma 1.18. *We have $\bar{I} = \text{conv}(I) \cap S$, where $\text{conv}(I)$ denotes the convex hull in $\mathbb{R}S (= \mathbb{Z}S \otimes_{\mathbb{Z}} \mathbb{R})$ of all points $a \in I$.*

Proof. The inclusion $\bar{I} \subseteq \text{conv}(I) \cap S$ follows immediately from the definition of \bar{I} . If on the other hand $a \in \text{conv}(I) \cap S$, then there are elements $w_1, \dots, w_r \in I$ and rational numbers $\alpha_i > 0$ ($i = 1, \dots, r$) with $a = \sum_i \alpha_i w_i$ and $\sum_i \alpha_i = 1$. Choose $n \in \mathbb{N}$ such that $n\alpha_i \in \mathbb{N}$ for $i = 1, \dots, r$. Then $na = \sum_i (n\alpha_i)w_i \in nI$ and hence $a \in \bar{I}$. \square

Finally, we quote a result that will be very useful in this section.

Lemma 1.19. *Let A be a ring, and let $\mathfrak{a} \subset A$ be an ideal such that the associated graded ring $\text{gr}_{\mathfrak{a}}(A)$ is reduced. Then all powers of \mathfrak{a} are integrally closed in A .*

Proof. Assume that $\bar{\mathfrak{a}}^n \neq \mathfrak{a}^n$ for some $n \in \mathbb{N}$. We choose $x \in \bar{\mathfrak{a}}^n \setminus \mathfrak{a}^n$ and set $m = \max\{i \geq 0 \mid x \in \mathfrak{a}^i\}$. There exists an equation

$$x^k + c_1 x^{k-1} + \dots + c_{k-1} x + c_k = 0,$$

where $c_i \in \mathfrak{a}^{ni}$ for $i = 1, \dots, k$. Since $c_i x^{k-i} \in \mathfrak{a}^{ni+m(k-i)}$ and

$$ni + m(k-i) \geq (m+1)i + m(k-i) = mk + i \geq mk + 1$$

for $i = 1, \dots, k$, we see that $x^k \in \mathfrak{a}^{mk+1}$. But this implies that the initial form x^* of x is a nilpotent element of $\text{gr}_{\mathfrak{a}}(A)$. Since $x^* \neq 0$, this contradicts the assumption that $\text{gr}_{\mathfrak{a}}(A)$ is reduced. \square

1.5. The associated graded ring of an affine semigroup ring.

Let $R = K[S]$ be a positive affine semigroup ring, and let \mathfrak{m} denote the graded maximal ideal of R . We want to study the associated graded algebra

$$\text{gr}_{\mathfrak{m}}(R) = \bigoplus_{n \geq 0} (\mathfrak{m}^n / \mathfrak{m}^{n+1}) = R[\mathfrak{m}t] \otimes_R R/\mathfrak{m}.$$

Note that for all $n \in \mathbb{N}$, the initial forms $(x^a)^*$, $a \in nS_+ \setminus (n+1)S_+$, form a K -basis of the graded component $\text{gr}_{\mathfrak{m}}(R)_n$. In general, $\text{gr}_{\mathfrak{m}}(R)$ is not an affine semigroup ring. In fact, $\text{gr}_{\mathfrak{m}}(R)$ need not even be reduced.

Proposition 1.20. *Let S be a positive affine semigroup, and let $R = K[S]$ be the associated semigroup ring over a field K . For every bottom face F of $P(S)$, let S_F denote the affine semigroup $S \cap \mathbb{R}_+F$. The following conditions are equivalent:*

- (a) *The associated graded algebra $\text{gr}_{\mathfrak{m}}(R)$ is reduced.*
- (b) *$\text{Hilb}(S_F) = B(S_F)$ for all bottom faces F of $P(S)$.*

Proof. (a) \Rightarrow (b) : Let F be a bottom face of $P(S)$, and let a be an element in $\text{Hilb}(S_F)$. One easily sees that $B(P(S_F)) = F$ and $B(S_F) = B(S) \cap F$. Since $S_F = S \cap \mathbb{R}_+F$ and $F = \text{conv}(B(S) \cap F) = \text{conv}(B(S_F))$, we have $S_F \subseteq \mathbb{R}_+B(S_F)$.

Let $\tau \in \text{Hom}_{\mathbb{R}}(\mathbb{R}S, \mathbb{R})$ be the linear form with $\tau(F) = \{1\}$, and set $m := \lfloor \tau(a) \rfloor$. Note that $H(\tau; 1)$ is a bottom hyperplane of $P(S)$. Applying Lemma 1.18, we get $a \in \overline{m(S_F)_+} \subseteq \overline{mS_+}$. By Lemma 1.19 and Lemma 1.16, condition (a) implies that $\overline{mS_+} = mS_+$, and hence $a \in mS_+$. Since $\tau(b) \geq 1$ for all elements $b \in S_+$, a cannot be contained in $(m+1)S_+$.

Assume that $\tau(a) > m$. Then there exists an $n \in \mathbb{N}$ such that $\tau(na) \geq nm + 1$. This means that $na \in (nm+1)S_+ = (nm+1)S_+$. Therefore, the n -th power of $(x^a)^* \in \text{gr}_{\mathfrak{m}}(R)_m$ is zero. This contradiction to (a) shows that $\tau(a) = m$.

Combining $a \in mS_+$ and the fact that $\tau(b) > 1$ for all $b \in S_+ \setminus B(S_F)$, we see that a can be written in the form $a = \sum_{i=1}^m w_i$, with $w_i \in B(S_F)$ for $i = 1, \dots, m$. But since $a \in \text{Hilb}(S_F)$, we must have $m = 1$ and thus $a \in B(S_F)$.

(b) \Rightarrow (a) : In order to show that $\text{gr}_{\mathfrak{m}}(R)$ is reduced, it suffices to show that no element of the form $(x^a)^*$, $a \in S_+$, is nilpotent. So let a be an arbitrary element in S_+ . By Lemma 1.15, there exists a bottom face F of $P(S)$ such that $a \in S_F$. Let $\tau \in \text{Hom}_{\mathbb{R}}(\mathbb{R}S, \mathbb{R})$ be the linear form with $\tau(F) = \{1\}$. From $\text{Hilb}(S_F) = B(S_F)$ follows that $m := \tau(a) \in \mathbb{N}$ and $a \in m(S_F)_+ \subseteq mS_+$. Since $\tau(b) \geq 1$ for all $b \in S_+$, we have $na \notin (nm+1)S_+$ for all $n \in \mathbb{N}$. This shows that $(x^a)^* \in \text{gr}_{\mathfrak{m}}(R)$ is not nilpotent. \square

Corollary 1.21. *Let S be a positive affine semigroup, and let $R = K[S]$ be the associated semigroup ring over a field K . If the associated graded algebra $\text{gr}_m(R)$ is reduced, then $\text{Hilb}(S) = B(S)$.*

Proof. If $\text{gr}_m(R)$ is reduced, then we have $\text{Hilb}(S_F) = B(S_F) \subseteq B(S)$ for all bottom faces F of $P(S)$. Since $\text{Hilb}(S)$ is contained in the union $\bigcup_F \text{Hilb}(S_F)$ (where F runs through the set of all bottom faces of $P(S)$), we obtain $\text{Hilb}(S) \subseteq B(S)$ and hence $\text{Hilb}(S) = B(S)$. \square

The following example shows that $\text{Hilb}(S) = B(S)$ does not imply that $\text{gr}_m(R)$ is reduced.

Example 1.22. Let S be the positive normal affine semigroup consisting of all elements $(a_1, \dots, a_5) \in \mathbb{Z}^5$ that satisfy

$$a_i + a_4 \geq 0, \quad 2a_i + a_5 \geq 0 \quad \text{for } i = 1, 2, 3, \quad \text{and } a_4, a_5 \geq 0,$$

and let $R = K[S]$ be the associated semigroup ring over a field K . Then $\text{Hilb}(S)$ is equal to $B(S)$, but $\text{gr}_m(R)$ is not reduced.

Proof. Set $w = (-1, -1, -1, 1, 2)$ and let e_1, \dots, e_5 be the standard basis of \mathbb{Z}^5 . We show that $\text{Hilb}(S) = \{e_1, \dots, e_5, w\}$: let $a = (a_1, \dots, a_5) \in S$ be an arbitrary element and set $r = -\min\{a_1, a_2, a_3, 0\}$. Then $r \geq 0$, $a_4 - r \geq 0$, $a_5 - 2r \geq 0$, and $a_i + r \geq 0$ for $i = 1, 2, 3$. From the equation

$$a = rw + (a_1 + r)e_1 + (a_2 + r)e_2 + (a_3 + r)e_3 + (a_4 - r)e_4 + (a_5 - 2r)e_5$$

we conclude that $\text{Hilb}(S)$ is a subset of $\{e_1, \dots, e_5, w\}$. Since, on the other hand, none of these six elements is contained in the semigroup generated by the others, we get $\text{Hilb}(S) = \{e_1, \dots, e_5, w\}$. From the equation $e_1 + e_2 + e_3 + w = e_4 + 2e_5$ follows that none of the elements in $\text{Hilb}(S)$ is contained in the cone generated by the remaining elements in $\text{Hilb}(S)$. Therefore, $\text{Hilb}(S) = B(S)$.

Now consider the element $b = (0, 0, 0, 1, 1) \in 2S_+$. Clearly, b does not lie in $3S_+$. The equation $2b = e_1 + e_2 + e_3 + e_4 + w$ shows that the square of the element $(x^b)^* \in \text{gr}_m(R)_2$ is zero. \square

Proposition 1.23. *Let S be a positive affine semigroup of dimension d , and let $R = K[S]$ be the associated semigroup ring over a field K . The following conditions are equivalent:*

- (a) *The associated graded algebra $\text{gr}_m(R)$ is isomorphic to R .*
- (b) *The associated graded algebra $\text{gr}_m(R)$ is an integral domain.*
- (c) $\dim P(S) = d - 1$.

Proof. (a) \Rightarrow (b) is trivial. (b) \Rightarrow (c) : Assume that $\dim P(S) = d$. Let w_1, \dots, w_n be the elements in $\text{Hilb}(S)$, and set $a := \sum_{i=1}^n w_i$. By Lemma 1.15, there exists a bottom face F of $P(S)$ with $a \in S_F := S \cap \mathbb{R}_+ F$. Let $\tau \in \text{Hom}_{\mathbb{R}}(\mathbb{R}S, \mathbb{R})$ be the linear form with $\tau(F) = \{1\}$. Then we have $\tau(b) \geq 1$ for all $b \in S_+$. Moreover, an element $b \in S_+$ satisfies $\tau(b) = 1$ if and only if $b \in F$.

From Proposition 1.20 we obtain $\text{Hilb}(S_F) = B(S_F)$. Hence $m := \tau(a) \in \mathbb{N}$ and $a \in m(S_F)_+ \subseteq mS_+$. Since $\dim P(S) = d$, $P(S)$ is not contained in F , and hence

$\tau(w_i) > 1$ for at least one $i \in \{1, \dots, n\}$. Therefore, $m = \sum_{i=1}^n \tau(w_i) \geq n + 1$, which means that the product of the elements $(x^{w_1})^*, \dots, (x^{w_n})^* \in \text{gr}_{\mathfrak{m}}(R)_1$ is zero. But this contradicts (b), and hence we must have $\dim P(S) = d - 1$.

(c) \Rightarrow (a) : Since $\dim P(S) = d - 1$, there exists a linear form $\tau \in \text{Hom}_{\mathbb{R}}(\mathbb{R}S, \mathbb{R})$ such that $\tau(a) = 1$ for all $a \in \text{Hilb}(S)$. By setting $\deg(x^a) = \tau(a)$ for all $a \in S$, R becomes a standard graded K -algebra, and hence $R \cong \text{gr}_{\mathfrak{m}}(R)$. \square

Corollary 1.24. *Let S be a positive affine semigroup, and let $R = K[S]$ be the associated semigroup ring over a field K . If $P(S)$ has only one bottom face, then the following conditions are equivalent:*

- (a) *The associated graded algebra $\text{gr}_{\mathfrak{m}}(R)$ is isomorphic to R .*
- (b) *The associated graded algebra $\text{gr}_{\mathfrak{m}}(R)$ is an integral domain.*
- (c) *The associated graded algebra $\text{gr}_{\mathfrak{m}}(R)$ is reduced.*
- (d) $\text{Hilb}(S) = B(S)$.

Proof. Note that $P(S)$ has only one bottom face if and only if the bottom $B(S)$ lies in a hyperplane of $\mathbb{R}S$. The implications (a) \Rightarrow (b) \Rightarrow (c) are trivial. The implication (c) \Rightarrow (d) follows from Proposition 1.20 and (d) \Rightarrow (a) follows from Proposition 1.23. \square

In general, the conditions (b) and (c) are not equivalent. For instance, consider the positive normal affine semigroup $S = \{(a_1, a_2) \in \mathbb{N}_0^2 \mid a_1 \equiv a_2 \pmod{3}\}$, and let $R = K[S]$ be the associated semigroup ring over a field K . Then we have

$$R \cong K[x^3, xy, y^3] \cong K[t_1, t_2, t_3]/(t_1 t_2 - t_3^3)$$

and

$$\text{gr}_{\mathfrak{m}}(R) \cong K[t_1, t_2, t_3]/(t_1 t_2).$$

Thus, $\text{gr}_{\mathfrak{m}}(R)$ is reduced but not an integral domain.

1.6. Normality of the Rees algebra.

Let $R = K[S]$ be a positive affine semigroup ring with graded maximal ideal \mathfrak{m} . The Rees algebra

$$R[\mathfrak{m}t] = \bigoplus_{n \geq 0} \mathfrak{m}^n t^n$$

is again a positive affine semigroup ring. In fact, $R[\mathfrak{m}t]$ is isomorphic to $K[T]$, where T is the positive affine semigroup

$$\{(a, 0) \mid a \in S\} \cup \{(a, m) \mid m \in \mathbb{N}, a \in mS_+\} \subset \mathbb{Z}S \oplus \mathbb{Z}.$$

Now assume that R is normal. By Lemma 1.17, we have

$$\overline{R[\mathfrak{m}t]} = \bigoplus_{n \geq 0} \overline{\mathfrak{m}^n t^n},$$

where $\overline{R[\mathfrak{m}t]}$ is the integral closure of $R[\mathfrak{m}t]$, and $\overline{\mathfrak{m}^n}$ is the integral closure of \mathfrak{m}^n in R . Since $R[\mathfrak{m}t]$ is a finitely generated K -algebra, $\overline{R[\mathfrak{m}t]}$ is a finitely generated $R[\mathfrak{m}t]$ -module. This means that

$$\overline{\mathfrak{m}^{n+1}} = \mathfrak{m}\overline{\mathfrak{m}^n} \text{ for all } n \gg 0.$$

For the moment, let $m(R)$ be the supremum of all $n \in \mathbb{N}$ with $\overline{\mathfrak{m}^{n+1}} \neq \mathfrak{m}\overline{\mathfrak{m}^n}$. An immediate question is: how large can $m(R)$ be? Can one give an upper bound for $m(R)$ that depends on certain numerical invariants of R ?

In the following theorem, we prove that $m(R) < d - 2$, where d is the dimension of R .

Theorem 1.25. *Let $R = K[S]$ be a positive normal affine semigroup ring of dimension d , and let \mathfrak{m} be the graded maximal ideal of R . Then $\overline{\mathfrak{m}^{n+1}} = \mathfrak{m}\overline{\mathfrak{m}^n}$ for all $n \in \mathbb{N}$ with $n \geq d - 2$.*

Proof. We may assume that $S \subseteq \mathbb{Z}^d$ and $\mathbb{Z}S = \mathbb{Z}^d$. Since the case $d = 1$ is trivial, we may assume $d \geq 2$. According to Lemma 1.16, the equation $\overline{\mathfrak{m}^{n+1}} = \mathfrak{m}\overline{\mathfrak{m}^n}$ is equivalent to the equation $\overline{(n+1)S_+} = S_+ + \overline{nS_+}$.

So let $n \geq d - 2$, and let a be an element in $\overline{(n+1)S_+}$. By Lemma 1.15, there exists a bottom face F of $P(S)$ and a linear independent subset U of $B(S) \cap F$, such that $U = \text{conv}(U) \cap S$ and a is contained in the rational cone $D := \mathbb{R}_+U$. Let w_1, \dots, w_r be the elements of U , and let I be the semigroup ideal of S which is generated by them. Note that $r \leq d$, since the w_i are linear independent. By Lemma 1.11, the intersection $D \cap S$ is a positive normal affine semigroup.

Since F is a bottom face of $P(S)$, there exists a linear form $\tau \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^d, \mathbb{R})$ which satisfies $\tau(b) \geq 1$ for all $b \in S_+$ and $\tau(x) = 1$ for all $x \in F$. It is clear that $\tau(b) \geq n + 1$ for all $b \in (n+1)S_+$ and hence for all $b \in \overline{(n+1)S_+}$. In particular, $\tau(a) \geq n + 1$.

Let b be any element in $D \cap S$ with $\tau(b) \geq k$ for some $k \in \mathbb{N}$. Then b can be written as $\sum_{i=1}^r \beta_i w_i$, where $\beta_1, \dots, \beta_r \in \mathbb{R}_+$ and $\sum_{i=1}^r \beta_i = \tau(b) \geq k$. Hence b is contained in the convex hull of kI in \mathbb{R}^d . By Lemma 1.18, this means that $b \in \overline{kI}$. So we have shown:

$$\text{if } b \in D \cap S \text{ and } k \in \mathbb{N}, \text{ then } \tau(b) \geq k \text{ implies } b \in \overline{kI}. \quad (\dagger)$$

According to Lemma 1.9, there exist $\alpha_1, \dots, \alpha_r \in \mathbb{Q}_+$ with $a = \sum_{i=1}^r \alpha_i w_i$. Since the w_i are linear independent, the α_i are uniquely determined. Suppose that $\alpha_i < 1$ for all $i \in \{1, \dots, r\}$. Then consider the element

$$a' := \sum_{i=1}^r w_i - a = \sum_{i=1}^r (1 - \alpha_i) w_i \in D \cap \mathbb{Z}S = D \cap S.$$

From $\tau(a') = \sum_{i=1}^r (1 - \alpha_i) > 0$ we see that $a' \in S_+$. Furthermore,

$$\tau(a') = \tau(\sum_{i=1}^r w_i) - \tau(a) = r - \tau(a) \leq d - n - 1 \leq 1.$$

Since $\tau(b) \geq 1$ for all $b \in S_+$, we conclude that $\tau(a') = 1$. This means that a' lies in $\text{conv}(U) \cap S = U$, and hence $a' = w_j$ for some $j \in \{1, \dots, r\}$. Therefore, $\alpha_j = 0$ and $\alpha_i = 1$ for all $i \neq j$, which is a contradiction to our assumption.

So we must have $\alpha_j \geq 1$ for some $j \in \{1, \dots, r\}$. Consider the element

$$a' := a - w_j = \sum_{i \neq j} \alpha_i w_i + (\alpha_j - 1)w_j \in D \cap \mathbb{Z}S = D \cap S.$$

Since $\tau(a') = \tau(a) - \tau(w_j) = \tau(a) - 1 \geq n$, we have $a' \in \overline{nI} \subseteq \overline{nS_+}$ by (†). Thus $a = w_j + a' \in S_+ + \overline{nS_+}$. \square

An immediate consequence of Lemma 1.17 and Theorem 1.25 is

Corollary 1.26. *We adopt the assumptions and notation of Theorem 1.25. Then $\overline{R[\mathfrak{m}t]}$ is generated as an $R[\mathfrak{m}t]$ -module by homogeneous elements of degree $\leq d - 2$ (with respect to the natural grading $\overline{R[\mathfrak{m}t]} = \bigoplus_{n \geq 0} \overline{\mathfrak{m}^n t^n}$).*

Since \mathfrak{m} itself is integrally closed, we obtain

Corollary 1.27. *Let R be a positive normal affine semigroup ring of dimension $d \leq 3$, and let \mathfrak{m} be the graded maximal ideal of R . Then the Rees algebra $R[\mathfrak{m}t]$ is normal.*

If $d \geq 4$, $R[\mathfrak{m}t]$ is in general not normal, as the following example shows.

Example 1.28. Let $d \geq 4$, and consider the positive normal affine semigroup S which consists of all elements $(a_1, \dots, a_d) \in \mathbb{Z}^d$ that satisfy

$$a_1, \dots, a_d \geq 0 \text{ and } a_i \equiv a_{i+1} \pmod{d-2} \text{ for } i = 1, \dots, d-1.$$

Let $R = K[S]$ be the associated semigroup ring over a field K , and let \mathfrak{m} be the graded maximal ideal of R . Then $\overline{\mathfrak{m}^i} = \mathfrak{m}^i$ for $i \leq \lceil d/2 \rceil - 1$, but the Rees algebra $R[\mathfrak{m}t]$ is not normal.

Proof. Note that

$$\text{Hilb}(S) = \{(d-2, 0, \dots, 0), \dots, (0, \dots, 0, d-2), (1, \dots, 1)\}.$$

Let $\tau : \mathbb{R}^d \rightarrow \mathbb{R}$ be the linear form which maps (a_1, \dots, a_d) to $(\sum_{i=1}^d a_i)/(d-2)$. Using Lemma 1.18, one sees that

$$\overline{iS_+} = \{a \in S \mid \tau(a) \geq i\}$$

for all $i \in \mathbb{N}$. Since

$$1 \leq \tau(a) \leq 1 + 2/(d-2)$$

for all $a \in \text{Hilb}(S)$, and

$$2j/(d-2) \leq (d-3)/(d-2) < 1$$

for $j \leq \lceil d/2 \rceil - 2$, we conclude $\overline{iS_+} = iS_+$ and thus $\overline{\mathfrak{m}^i} = \mathfrak{m}^i$ for all $i \leq \lceil d/2 \rceil - 1$.

Now consider the element

$$b = (\lceil d/2 \rceil - 1, \dots, \lceil d/2 \rceil - 1) \in S.$$

From

$$\tau(b) = \lceil d/2 \rceil - 1 + (2\lceil d/2 \rceil - 2)/(d - 2) \geq \lceil d/2 \rceil - 1 + 1 = \lceil d/2 \rceil$$

we obtain $b \in \overline{\lceil d/2 \rceil S_+}$. But since $d \geq 4$, we have $d - 2 > \lceil d/2 \rceil - 1$ and hence $b \notin \lceil d/2 \rceil S_+$. According to Lemma 1.17, this shows that $R[\mathbf{m}t]$ is not normal. \square

Lemma 1.19 yields that the Rees algebra $R[\mathbf{m}t]$ is normal in case that $\text{gr}_{\mathbf{m}}(R)$ is reduced. By Proposition 1.20, $\text{gr}_{\mathbf{m}}(R)$ is reduced if and only if $\text{Hilb}(S_F) = B(S_F)$ for all bottom faces F of $P(S)$. This leads to the question whether the condition $\text{Hilb}(S) = B(S)$ also implies the normality of $R[\mathbf{m}t]$. The next example shows that this is not the case.

Example 1.29. Let S be the positive normal affine semigroup consisting of all elements $(a_1, \dots, a_6) \in \mathbb{Z}^6$ which satisfy

$$a_i + a_5 \geq 0, 2a_i + a_6 \geq 0 \text{ for } i = 1, 2, 3, 4, \text{ and } a_5, a_6 \geq 0.$$

Then $\text{Hilb}(S) = B(S)$, but $3S_+ \neq \overline{3S_+}$.

Proof. Set $w = (-1, -1, -1, -1, 1, 2)$ and let e_1, \dots, e_6 be the standard basis of \mathbb{Z}^6 . We show that $\text{Hilb}(S) = \{e_1, \dots, e_6, w\}$: let $a = (a_1, \dots, a_6) \in S$ be an arbitrary element and set $r = -\min\{a_1, a_2, a_3, a_4, 0\}$. Then $r \geq 0$, $a_5 - r \geq 0$, $a_6 - 2r \geq 0$, and $a_i + r \geq 0$ for $i = 1, \dots, 4$. From the equation

$$a = rw + (a_1 + r)e_1 + \dots + (a_4 + r)e_4 + (a_5 - r)e_5 + (a_6 - 2r)e_6$$

we conclude that $\text{Hilb}(S)$ is a subset of $\{e_1, \dots, e_6, w\}$. Since, on the other hand, none of these seven elements is contained in the semigroup generated by the others, we get $\text{Hilb}(S) = \{e_1, \dots, e_6, w\}$. From the equation

$$e_1 + \dots + e_4 + w = e_5 + 2e_6$$

follows that none of the elements in $\text{Hilb}(S)$ is contained in the cone generated by the remaining elements in $\text{Hilb}(S)$. Therefore, $\text{Hilb}(S) = B(S)$.

Now consider $b = (0, 0, 0, 0, 1, 1)$. Of course, b does not lie in $3S_+$. But since $2b = e_1 + e_2 + e_3 + e_4 + e_5 + w \in 6S_+$, we have $b \in \overline{3S_+}$. \square

If $R = K[S]$ is standard graded with respect to some admissible grading of R , then the Rees algebra $R[\mathbf{m}t]$ is normal:

Proposition 1.30. *Let R be a positive normal affine semigroup ring with graded maximal ideal \mathbf{m} . If R is standard graded with respect to some admissible grading of R , then $R[\mathbf{m}t]$ is normal.*

Proof. If R is standard graded, then $\text{gr}_{\mathbf{m}}(R)$ is isomorphic to R and thus reduced. Now the assertion follows from Lemma 1.17 and Lemma 1.19. \square

One may ask: under which conditions is the Rees algebra of $R[\mathbf{m}t]$ normal? The next proposition will answer this question. Let T be the semigroup

$$\{(a, 0) \mid a \in S\} \cup \{(a, m) \mid m \in \mathbb{N}, a \in mS_+\} \subset \mathbb{Z}S \oplus \mathbb{Z}.$$

The semigroup ring $K[T]$ is isomorphic to $B := R[\mathbf{m}t]$. We set $T_+ = T \setminus \{0\}$ and $\mathfrak{n} = \bigoplus_{a \in T_+} Kx^a$.

Proposition 1.31. *Let $R = K[S]$ be a positive normal affine semigroup ring with graded maximal ideal \mathfrak{m} . Furthermore, let B denote the Rees algebra $R[\mathbf{m}t]$, and let \mathfrak{n} be the graded maximal ideal of B . Then B is normal if and only if its Rees algebra $B[\mathfrak{n}u]$ is normal.*

Proof. It is clear that B is normal in case that $B[\mathfrak{n}u]$ is normal. So assume that B is normal. We show that nT_+ is integrally closed in T for all $n \in \mathbb{N}$: let $n \in \mathbb{N}$ and let $(a, m) \in \overline{nT_+}$. Then there exists $p \in \mathbb{N}$ with $(pa, pm) \in pnT_+$. This implies $pa \in pnS_+$ and hence $a \in nS_+$, since B is normal. Using the next lemma, we see that $(a, m) \in nT_+$. \square

Lemma 1.32. *We have $pT_+ = \{(a, m) \in T \mid a \in pS_+\}$ for all $p \in \mathbb{N}$.*

Proof. The inclusion $pT_+ \subseteq \{(a, m) \in T \mid a \in pS_+\}$ is trivial. In order to prove the opposite inclusion, we consider an element $(a, m) \in T$ with $a \in pS_+$.

If $m \leq p$, we have $a \in mS_+ \subseteq pS_+$, and therefore we can write $a = \sum_{i=1}^p w_i$ with $w_i \in S_+$ for $i = 1, \dots, p$. From $(a, m) = \sum_{i=1}^{p-m} (w_i, 0) + \sum_{i=p-m+1}^p (w_i, 1)$ we conclude that (a, m) is contained in pT_+ .

If $m > p$, write $a = \sum_{i=1}^m w_i$ with $w_i \in S_+$ for $i = 1, \dots, m$. Furthermore, set $a' = \sum_{i=1}^{m-p} w_{p+i}$. From $(a, m) = (a', m-p) + \sum_{i=1}^p (w_i, 1)$ we conclude that (a, m) is contained in pT_+ . \square

One may ask, whether the statement of Corollary 1.27 is valid in a more general setting. For instance, let R be an arbitrary positively graded K -algebra, which is a normal domain of dimension $d \leq 3$, and let \mathfrak{m} be its graded maximal ideal. Is it generally true that the Rees algebra $R[\mathbf{m}t]$ is normal?

If $d = 1$, then R is regular and hence $\text{gr}_{\mathfrak{m}}(R)$ is a domain. By Lemma 1.19, this implies that $R[\mathbf{m}t]$ is normal. But already in dimension $d = 2$, there are examples for R , such that $R[\mathbf{m}t]$ is not normal. Before we give such an example, we state a lemma about the Rees algebra of a hypersurface ring, that we will also need in subsection 1.8.

Lemma 1.33. *Let $T = (\mathbb{N}_0)^{d+1}$, and let $P = K[y_1, \dots, y_{d+1}]$ be the polynomial ring in $d+1$ variables over a field K . Furthermore, let $f = \sum_{a \in T} \gamma_a y^a \neq 0$ be an element in $\mathfrak{m} = (y_1, \dots, y_{d+1})$, and let $R = P/(f)$. Set $m = \min\{|a| \mid \gamma_a \neq 0\}$, where $|a| = \sum_{i=1}^{d+1} a_i$ for $a = (a_1, \dots, a_{d+1}) \in T$. Let z_1, \dots, z_{d+1} be indeterminates over P , and consider the P -algebra homomorphism*

$$\pi : P[z_1, \dots, z_{d+1}] \rightarrow P[ty_1, \dots, ty_{d+1}],$$

that maps z_i to ty_i . Let h_i be a preimage of $t^i f$ with respect to π for $i = 0, \dots, m$. Then the Rees algebra $R[\mathbf{m}t]$ is isomorphic to

$$B := K[y_1, \dots, y_{d+1}, z_1, \dots, z_{d+1}] / (h_0, \dots, h_m; y_i z_j - y_j z_i, 1 \leq i < j \leq d+1)$$

Moreover,

$$B_{y_k} \cong R_{y_k}[t] \text{ for } k = 1, \dots, d+1,$$

and

$$B_{z_k} \cong K[y_k, z_1, \dots, z_{d+1}, z_k^{-1}] / (\sum_{a \in T} \gamma_a (y_k/z_k)^{|a|-m} z^a) \text{ for } k = 1, \dots, d+1.$$

Proof. Consider the sequence of surjective K -algebra homomorphisms

$$P[z_1, \dots, z_{d+1}] \xrightarrow{\pi} P[ty_1, \dots, ty_{d+1}] \xrightarrow{\varphi} R[\mathfrak{m}t] = R[t\bar{y}_1, \dots, t\bar{y}_{d+1}],$$

where φ is the natural surjection. The kernel of π is generated by the polynomials $y_i z_j - y_j z_i$, $1 \leq i < j \leq d+1$. One can show this directly or use the fact that an ideal generated by a regular sequence is an ideal of linear type, see e.g. chapter 2 of [Va1].

For a nonzero polynomial $g = \sum_{a \in T} \lambda_a y^a \in P$ we set $\delta(g) := \min\{|a| \mid \lambda_a \neq 0\}$. In particular, $\delta(f) = m$. Note that $\delta(g_1 g_2) = \delta(g_1) + \delta(g_2)$ for all $g_1, g_2 \in P \setminus \{0\}$. If $g \in P \setminus \{0\}$ and $n \in \mathbb{N}$, then $t^n g \in P[ty_1, \dots, ty_{d+1}]$ if and only if $\delta(g) \geq n$. Assume that $t^n g \neq 0$ lies in $\text{Ker } \varphi$. Then $g = f g'$ for some $g' \in P$. We define $k = \min\{m, n\}$. From

$$\delta(g') = \delta(g) - \delta(f) \geq n - m$$

one sees that $t^{n-k} g' \in P[ty_1, \dots, ty_{d+1}]$, and thus $t^n g \in (t^k f)$. So we conclude that $\text{Ker } \varphi$ is generated by the elements $t^i f$, $i = 0, \dots, m$. Combining this result with the above description of $\text{Ker } \pi$, we obtain that $R[\mathfrak{m}t]$ is isomorphic to B , as claimed.

Since $\mathfrak{m}R_{y_k} = R_{y_k}$, we have $B_{y_k} \cong (R[\mathfrak{m}t])_{y_k} = R_{y_k}[\mathfrak{m}R_{y_k}t] = R_{y_k}[t]$. To prove the statement concerning B_{z_k} , we consider the K -algebra homomorphism

$$\psi : K[y_k, z_1, \dots, z_{d+1}, z_k^{-1}] \rightarrow K[y_1, \dots, y_{d+1}, ty_1, \dots, ty_{d+1}, (ty_k)^{-1}],$$

that maps y_k to y_k and z_i to ty_i . Noting that $\psi(y_k z_k^{-1} z_i) = y_i$, one sees that ψ is surjective. Since both rings are affine domains of dimension $d+2$, ψ must be an isomorphism. For $i = 0, \dots, m$, we have

$$\begin{aligned} \psi^{-1}(t^i f) &= \psi^{-1}(t^i \sum_a \gamma_a y^a) = \psi^{-1}(\sum_a \gamma_a (t^{-1})^{|a|-i} t^{|a|} y^a) = \sum_a \gamma_a (y_k/z_k)^{|a|-i} z^a \\ &= (y_k/z_k)^{m-i} \sum_a \gamma_a (y_k/z_k)^{|a|-m} z^a = (y_k/z_k)^{m-i} \psi^{-1}(t^m f). \end{aligned}$$

Therefore, $B_{z_k} \cong K[y_1, \dots, y_{d+1}, ty_1, \dots, ty_{d+1}, (ty_k)^{-1}] / (t^i f, 0 \leq i \leq m)$ is isomorphic to $K[y_k, z_1, \dots, z_{d+1}, z_k^{-1}] / (\sum_a \gamma_a (y_k/z_k)^{|a|-m} z^a)$. \square

Now we give the example in dimension 2 promised above.

Example 1.34. Let $P = K[y_1, y_2, y_3]$ be the polynomial ring in 3 variables over a field K of characteristic zero and consider $f = y_1^2 + y_2^4 + y_3^4 \in P$. Set $R = P/(f)$ and $\mathfrak{m} = (\bar{y}_1, \bar{y}_2, \bar{y}_3) \subset R$. Then R is a normal domain of dimension 2, but its Rees algebra $R[\mathfrak{m}t]$ is not normal.

Proof. Using the Eisenstein criterion, one sees that f is irreducible in P . (Consider $g := y_1^2 + y_2^4$ in $P' := K[y_1, y_2]$, and let h be any prime factor of g in P' . Suppose that $g \in h^2 P'$. Then h divides $\partial g / \partial y_1 = 2y_1$ and hence $h = \alpha y_1$ for some $\alpha \in K^\times$. But since $g \notin y_1 P'$, we get a contradiction. Therefore, g is a squarefree polynomial in P' , and we may apply the Eisenstein criterion to $f = y_3^4 + g \in P'[y_3] = P$.)

The normality criterion of Serre states that a Noetherian ring is normal if and only if it satisfies the conditions (R_1) and (S_2) , see e.g. [Ma, 23.8]. Since R is a hypersurface ring, it satisfies (S_2) . Hence, to show that R is normal, it suffices to show that it is regular in codimension one. Since $\text{char}(K) = 0$, we may apply the Jacobian criterion (see e.g. [Ei, 11.2]) to determine the singular locus of R . It states that $\text{Sing}(R)$ is equal to $\{\mathfrak{p}R \mid \mathfrak{p} \in \text{Spec}(P), \mathfrak{p} \supseteq J := (\partial f/\partial y_1, \partial f/\partial y_2, \partial f/\partial y_3)\}$. From $J = (y_1, y_2^3, y_3^3)$ we obtain $\text{Sing}(R) = \{\mathfrak{m}\}$. Since \mathfrak{m} has height 2, R is regular in codimension one.

Now suppose that $R[\mathfrak{m}t]$ is normal. By Lemma 1.33, the ring

$$A' := K[y_2, z_1, z_2, z_3, z_2^{-1}]/(z_1^2 + (y_2/z_2)^2 z_2^4 + (y_2/z_2)^2 z_3^4)$$

is isomorphic to a localization of $R[\mathfrak{m}t]$. Note that A' is equal to $(A/(g))_{z_2}$, where

$$A = K[y_2, z_1, z_2, z_3] \text{ and } g = z_1^2 z_2^2 + y_2^2 z_2^4 + y_2^2 z_3^4.$$

Since $R[\mathfrak{m}t]$ is normal, A' is also normal and hence regular in codimension one. This means that $\text{Sing}(A/(g))$ does not contain any prime ideal $\mathfrak{q} \in \text{Spec}(A/(g))$ of height 1 with $\bar{z}_2 \notin \mathfrak{q}$.

Since the singular locus of $A/(g)$ is equal to

$$\{\bar{\mathfrak{p}} \mid \mathfrak{p} \in \text{Spec}(A), \mathfrak{p} \supseteq J := (y_2 z_2^4 + y_2 z_3^4, z_1 z_2^2, z_1^2 z_2 + 2y_2^2 z_2^3, y_2^2 z_3^3)\},$$

we see that $\mathfrak{q} := (\bar{z}_1, \bar{y}_2) \in \text{Sing}(A/(g))$. Clearly, \mathfrak{q} has height 1 in $A/(g)$ and $\bar{z}_2 \notin \mathfrak{q}$. This contradiction shows that $R[\mathfrak{m}t]$ is not normal. \square

One may also try to generalize Corollary 1.27 by replacing the maximal ideal \mathfrak{m} by an arbitrary integrally closed monomial ideal. However, there are well-known counterexamples, which show that this does not work. We present one of them:

Example 1.35. Set $I = \{a = (a_1, a_2, a_3) \in \mathbb{Z}^3 \mid a_1, a_2, a_3 \geq 0, \tau(a) \geq 60\}$, where $\tau : \mathbb{Z}^3 \rightarrow \mathbb{Z}$, $(a_1, a_2, a_3) \mapsto 20a_1 + 15a_2 + 12a_3$, and let \mathfrak{a} denote the associated ideal $\bigoplus_{a \in I} Kx^a$ in $R = K[x_1, x_2, x_3]$. Then \mathfrak{a} is integrally closed in R , but the Rees algebra $R[\mathfrak{a}t]$ is not normal.

Proof. Clearly, \mathfrak{a} is integrally closed in R (see Lemma 1.18). One easily checks that

$$w_1 = (3, 0, 0), w_2 = (0, 4, 0), \text{ and } w_3 = (0, 0, 5)$$

are the only elements in $I \cap \tau^{-1}(\{60\})$. Furthermore, $\overline{nI} = \{a \in I \mid \tau(a) \geq 60n\}$ for all $n \in \mathbb{N}$. Now consider $b = (2, 3, 3) \in I$. Since $\tau(b) = 121$, b lies in $\overline{2I}$. Assume that $b \in 2I$. Then $b \in w_i + I$ for some $i \in \{1, 2, 3\}$. But this is obviously not the case, and we get a contradiction. So $2I \neq \overline{2I}$ and thus $\mathfrak{a}^2 \neq \overline{\mathfrak{a}^2}$, which means that $R[\mathfrak{a}t]$ is not normal. \square

Nevertheless, if one replaces $d - 2$ by $d - 1$, the assertion of Theorem 1.25 can be extended to the class of all ideals in R that are associated to an integrally closed ideal in S . The following theorem is a direct consequence of a result of Vasconcelos ([Va2, 3.48]).

Theorem 1.36. *Let $R = K[S]$ be a positive normal affine semigroup ring of dimension d . Furthermore, let I be an integrally closed ideal in S , and let \mathfrak{a} denote the associated ideal $\bigoplus_{a \in I} Kx^a$. Then $\overline{\mathfrak{a}^{n+1}} = \mathfrak{a}\overline{\mathfrak{a}^n}$ for all $n \in \mathbb{N}$ with $n \geq d - 1$.*

Proof. We quote [Va2, 3.48]: let (A, \mathfrak{n}) be a local Cohen-Macaulay ring with infinite residue field A/\mathfrak{n} and let $(\mathfrak{b}_n)_{n \geq 0}$ be a filtration of ideals in A with $\mathfrak{b}_0 = A$ and height $\mathfrak{b}_1 \geq 1$. Assume that the Rees algebra $\bigoplus_{n \geq 0} \mathfrak{b}_n t^n$ is Cohen-Macaulay and finitely generated as an $A[\mathfrak{b}_1 t]$ -module. Then $\mathfrak{b}_{n+1} = \mathfrak{b}_1 \mathfrak{b}_n$ for all $n \in \mathbb{N}$ with $n \geq \ell(\mathfrak{b}_1) - 1$, where $\ell(\mathfrak{b}_1)$ denotes the analytic spread of \mathfrak{b}_1 . (The analytic spread of \mathfrak{b}_1 is the dimension of the fiber ring $\bigoplus_{n \geq 0} \mathfrak{b}_1^n / \mathfrak{n} \mathfrak{b}_1^n$.)

In order to derive the theorem from this result, we set $A = R_{\mathfrak{m}}$, $\mathfrak{n} = \mathfrak{m}R_{\mathfrak{m}}$, and $\mathfrak{b}_n = \overline{\mathfrak{a}^n}R_{\mathfrak{m}}$ for all $n \in \mathbb{N}$. We may assume that $K \cong A/\mathfrak{n}$ is infinite. Since $\overline{R[\mathfrak{a}t]}$ is a normal semigroup ring, it is Cohen-Macaulay, and hence

$$\bigoplus_{n \geq 0} \mathfrak{b}_n t^n = \bigoplus_{n \geq 0} (\overline{\mathfrak{a}^n})_{\mathfrak{m}} t^n = (\bigoplus_{n \geq 0} \overline{\mathfrak{a}^n t^n})_{\mathfrak{m}} = (\overline{R[\mathfrak{a}t]})_{\mathfrak{m}}$$

is Cohen-Macaulay, too. The fact that the ring $\bigoplus_{n \geq 0} \mathfrak{b}_n t^n$ is a finitely generated module over $A[\mathfrak{b}_1 t]$, follows from E. Noether's theorem on the finiteness of the integral closure.

Since $\ell(\mathfrak{a}R_{\mathfrak{m}}) \leq \dim R_{\mathfrak{m}} = d$, we obtain $(\overline{\mathfrak{a}^{n+1}})_{\mathfrak{m}} = \mathfrak{a}(\overline{\mathfrak{a}^n})_{\mathfrak{m}}$ and hence $\overline{\mathfrak{a}^{n+1}} = \mathfrak{a}\overline{\mathfrak{a}^n}$ for all $n \in \mathbb{N}$ with $n \geq d - 1$. \square

We mention that in the article [RRV], the statement of Theorem 1.36 is proven for the special case that R is the polynomial ring over K .

1.7. Cohen-Macaulayness of the Rees algebra.

In the last subsection, we saw that the Rees algebra $R[\mathfrak{m}t]$ of a positive normal affine semigroup ring R is in general not normal. One may ask whether $R[\mathfrak{m}t]$ is at least always Cohen-Macaulay. (Note that for affine semigroups, the normality property is stronger than the Cohen-Macaulay property, see Theorem 1.4.) As we will see, this is not the case.

First, we show that $R[\mathfrak{m}t]$ is Cohen-Macaulay if and only if $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay.

Proposition 1.37. *Let R be a positive normal affine semigroup ring, and let \mathfrak{m} be the graded maximal ideal of R . Then the Rees algebra $R[\mathfrak{m}t]$ is Cohen-Macaulay if and only if the associated graded algebra $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay.*

Proof. Note that $\mathrm{gr}_{\mathfrak{m}}(R)$ is equal to the associated graded ring of $R_{\mathfrak{m}}$ with respect to $\mathfrak{m}R_{\mathfrak{m}}$, and that $R[\mathfrak{m}t]$ is Cohen-Macaulay if and only if $R_{\mathfrak{m}}[\mathfrak{m}t]$ is Cohen-Macaulay. So we may consider $(R_{\mathfrak{m}}, \mathfrak{m}R_{\mathfrak{m}})$ instead of (R, \mathfrak{m}) .

One implication holds in a quite general setting: let A be a local Cohen-Macaulay ring, and let $\mathfrak{a} \subset A$ be an ideal of positive height. If $A[\mathfrak{a}t]$ is Cohen-Macaulay, then so is $\mathrm{gr}_{\mathfrak{a}}(A)$. This was proved by Huneke in [Hu].

The converse holds for pseudo-rational local rings: if (A, \mathfrak{n}) is a pseudo-rational local ring, then $\mathrm{gr}_{\mathfrak{n}}(A)$ is Cohen-Macaulay if and only if $A[\mathfrak{n}t]$ is Cohen-Macaulay.

(See [LT] for the notion of pseudo-rational local rings.) This was shown by Lipman in [Li], where he proved a more general statement than the one quoted here. It remains to be shown that $R_{\mathfrak{m}}$ is pseudo-rational.

Since R is a positive normal affine semigroup ring, it can be written as the direct summand of a polynomial ring (see e.g. [BG1, Section 2] for a proof). To show that this implies the pseudo-rationality of $R_{\mathfrak{m}}$, we refer to some general facts.

In case $\text{char}(K) = 0$ we need:

(i) Let B be a finitely generated algebra over a field of characteristic 0. If B has rational singularities, then every direct summand of B has rational singularities, too. This was proved by Boutot in [Bo].

(ii) Let A be a local normal ring that is essentially of finite type over a field of characteristic 0. Then A is pseudo-rational if and only if it is a rational singularity (see e.g. [Sm, 1.14]).

In case $\text{char}(K) = p > 0$ we need:

(iii) A direct summand of a regular ring of characteristic p is F -regular (see e.g. [BH, 10.1.13]).

(iv) Let A be an excellent local ring of characteristic p . If A is F -rational (e.g. if it is F -regular), then it is pseudo-rational. This was proved by K. Smith in her paper [Sm]. \square

We mention that one can give an alternative proof to the “if”-statement of the proposition, which avoids the notion of pseudo-rationality and uses instead a tight closure argument due to Huneke, which was found by the author in [Va1, 5.1.18]. It is based on the following theorem of Goto and Shimoda ([GS, 3.1]): let (A, \mathfrak{n}) be a local Cohen-Macaulay ring of positive dimension with infinite residue field A/\mathfrak{n} . Then $A[\mathfrak{n}t]$ is Cohen-Macaulay if and only if $\text{gr}_{\mathfrak{n}}(A)$ is Cohen-Macaulay and the reduction number of A is less than $\dim A$.

(For the reader’s convenience, we briefly recall the notion of reduction number: let (A, \mathfrak{n}) be a local Noetherian ring of dimension d with infinite residue field A/\mathfrak{n} , and let $I \subseteq \mathfrak{n}$ be an ideal. An ideal $J \subseteq I$ is called a reduction ideal of I if JI^r is equal to I^{r+1} for some $r \geq 0$. It is easy to show that this condition is equivalent to the condition that the integral closure of J in R is equal to the integral closure of I in R .

One says that J is a minimal reduction of I , if J is a reduction ideal of I and if no ideal properly contained in J is a reduction ideal of I . It is well known that J is a minimal reduction of \mathfrak{n} if and only if J can be generated by d elements $f_1, \dots, f_d \in \mathfrak{n} \setminus \mathfrak{n}^2$ such that the initial forms $f_1^*, \dots, f_d^* \in \text{gr}_{\mathfrak{n}}(A)_1$ form a system of parameters of $\text{gr}_{\mathfrak{n}}(A)$. In particular, since A/\mathfrak{n} is infinite, there exist minimal reductions of \mathfrak{n} . The reduction number of A is the least integer $r \geq 0$ with the property that $J\mathfrak{n}^r = \mathfrak{n}^{r+1}$ for some minimal reduction J of \mathfrak{n} .)

We show how the tight closure proof works: assume that $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay. Let $B = \mathbb{Z}[S]$ and $B_+ = \bigoplus_{a \in S_+} \mathbb{Z}x^a$, and note that $\text{gr}_{B_+}(B)$ and $B[B_+t]$ are both finitely generated free graded \mathbb{Z} -algebras. Thus, in order to prove the Cohen-Macaulayness of $R[\mathfrak{m}t] = B \otimes_{\mathbb{Z}} K$, we may assume that K is an infinite field of characteristic $p > 0$, see Lemma 1.38 below.

Let \mathfrak{m}_0 denote the maximal ideal of $R_{\mathfrak{m}}$. We choose $y_1, \dots, y_d \in \mathfrak{m}_0 \setminus \mathfrak{m}_0^2$ such that the initial forms $y_1^*, \dots, y_d^* \in \text{gr}_{\mathfrak{m}}(R)_1$ form a system of parameters of $\text{gr}_{\mathfrak{m}}(R)$. Then $J = (y_1, \dots, y_d)$ is a minimal reduction of \mathfrak{m}_0 . We show that $J\mathfrak{m}_0^{d-1}$ is equal to \mathfrak{m}_0^d . Then it follows that the reduction number of $R_{\mathfrak{m}}$ is at most $d - 1$, and so (applying the theorem of Goto and Shimoda) we are done.

Since R is F -regular (see (iii) above), the tight closure version of the Briançon-Skoda theorem (see [BH, 10.2.6]) yields $\overline{J^d} \subseteq J$, where $\overline{J^d}$ denotes the integral closure of J^d in R . Since J is a reduction ideal of \mathfrak{m}_0 , J^d is a reduction ideal of \mathfrak{m}_0^d , which means that $\overline{J^d} = \overline{\mathfrak{m}_0^d}$. So $\mathfrak{m}_0^d \subseteq \overline{\mathfrak{m}_0^d} \subseteq J$, and hence $J \cap \mathfrak{m}_0^d = \mathfrak{m}_0^d$.

But in our situation, we have $J \cap \mathfrak{m}_0^d = J\mathfrak{m}_0^{d-1}$ by a result of Valabrega and Valla (see [VV, 2.7]): let (A, \mathfrak{n}) be a local Noetherian ring, and let $I = (f_1, \dots, f_r) \subset A$ be an ideal, where $f_1, \dots, f_r \in \mathfrak{n} \setminus \mathfrak{n}^2$. If $f_1^*, \dots, f_r^* \in \text{gr}_{\mathfrak{n}}(A)_1$ form a $\text{gr}_{\mathfrak{n}}(A)$ -regular sequence, then $\mathfrak{n}^n \cap I = \mathfrak{n}^{n-1}I$ for all $n \in \mathbb{N}$.

The next lemma is well known. Its proof is included, since the author couldn't find an appropriate reference in the literature.

Lemma 1.38. *Let $A = \bigoplus_{i \geq 0} A_i$ be a finitely generated free graded \mathbb{Z} -algebra with $A_0 = \mathbb{Z}$. The following statements are equivalent:*

- (a) *The ring $A \otimes_{\mathbb{Z}} L$ is Cohen-Macaulay for all fields L of characteristic 0.*
- (b) *There exists a field K with $\text{char}(K) = p > 0$ such that $A \otimes_{\mathbb{Z}} K$ is Cohen-Macaulay.*

Proof. Since A is a finitely generated free \mathbb{Z} -algebra, the components A_i are all finitely generated free \mathbb{Z} -modules. This implies that the Hilbert function and in particular the dimension of $A \otimes_{\mathbb{Z}} K$ is the same for all fields K . We denote this common dimension by d .

(a) \Rightarrow (b) : We choose $x_1, \dots, x_d \in A_1$ such that the elements $x_1 \otimes 1, \dots, x_d \otimes 1$ form a regular sequence in $A \otimes_{\mathbb{Z}} \mathbb{Q}$. For $i > 0$ let M_i be the i -th Koszul homology module $H_i(x_1, \dots, x_d; A)$. In particular, we have $M_i = 0$ for all $i > d$. By the generic flatness theorem (see [BH, 6.5.6]), there exists an $f \in \mathbb{Z} \setminus \{0\}$ such that $(M_i)_f$ is a free \mathbb{Z}_f -module for all $i > 0$. Since $M_i \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes for $i > 0$ (note that Koszul homology commutes with localization), we obtain $(M_i)_f = 0$ for $i > 0$. If p is a prime number that does not divide f , then $(M_i)_{(p)} = 0$ for $i > 0$, which means that $A_{(p)}$ is Cohen-Macaulay. Since p is a regular element in $A_{(p)}$, the ring $A_{(p)}/pA_{(p)} = A \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ is also Cohen-Macaulay.

(b) \Rightarrow (a) : Since $A \otimes_{\mathbb{Z}} K$ is faithfully flat over $A \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$, the latter ring is Cohen-Macaulay, too. Since p is a regular element in the unique graded maximal ideal of $A_{(p)}$, we obtain the Cohen-Macaulayness of $A_{(p)}$ and its localization $A \otimes_{\mathbb{Z}} \mathbb{Q}$. The assertion results now from the following general fact (see e.g. [BH, 2.1.10] for a proof): let B be a finitely generated Cohen-Macaulay algebra over a field L_0 . Then B is geometrically Cohen-Macaulay in the sense that $B \otimes_{L_0} L$ is Cohen-Macaulay for all field extensions L of L_0 . \square

The following example shows that the Rees algebra of a positive normal affine semigroup ring is in general not Cohen-Macaulay.

Example 1.39. Let $d \geq 4$ be an integer, and let S be the positive normal affine semigroup consisting of all elements $(a_1, \dots, a_d) \in \mathbb{Z}^d$ with $a_1, \dots, a_d \geq 0$ and

$$a_1 - a_2 \equiv a_2 - a_3 \equiv a_3 - 3a_4 \equiv a_5 \equiv \dots \equiv a_d \equiv 0 \pmod{5}.$$

Let R denote the associated semigroup ring $K[S]$ over a field K , and let \mathfrak{m} be the graded maximal ideal of R . Then the Rees algebra $R[\mathfrak{m}t]$ is *not* Cohen-Macaulay.

Proof. Let e_1, \dots, e_d denote the standard basis of \mathbb{Z}^d . One easily verifies that $\text{Hilb}(S)$ consists of the elements

$$5e_1, \dots, 5e_d, (1, 1, 1, 2, 0, \dots, 0), \text{ and } (3, 3, 3, 1, 0, \dots, 0).$$

Let $f_i = x_i^5 \in R$ for $i = 1, \dots, d$, and $J = (f_1, \dots, f_d)R_{\mathfrak{m}}$. We set $\mathfrak{m}_0 := \mathfrak{m}R_{\mathfrak{m}}$. Since $(x^b)^5 \in J^5$ for all $b \in \text{Hilb}(S)$, the initial forms $f_1^*, \dots, f_d^* \in \text{gr}_{\mathfrak{m}}(R)_1$ form a system of parameters for $\text{gr}_{\mathfrak{m}}(R)$.

Now suppose that $R[\mathfrak{m}t]$ is Cohen-Macaulay. Then $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay by Proposition 1.37, and $f_1^*, \dots, f_d^* \in \text{gr}_{\mathfrak{m}}(R)$ form a regular sequence. This implies $J \cap \mathfrak{m}_0^{n+1} = J\mathfrak{m}_0^n$ for all $n \in \mathbb{N}$ according to the theorem of Valabrega and Valla quoted in the proof of Proposition 1.37. For the element $b = (1, 1, 1, 2, 0, \dots, 0)$ we have $3b = (3, 3, 3, 1, 0, \dots, 0) + 5e_4$ and thus $(x^b)^3 \in J \cap \mathfrak{m}_0^3$. But since $(x^b)^3 \notin J\mathfrak{m}_0^2$, we get a contradiction. \square

Let $B = \bigoplus_{n \geq 0} B_n$ be a standard graded algebra over a field K . We set $d = \dim B$ and $\mathfrak{n} = \bigoplus_{n > 0} B_n$. For all integers $i \geq 0$, let $H_{\mathfrak{n}}^i(B)$ denote the i -th graded local cohomology module of B with respect to \mathfrak{n} . (We recommend Chapters 12 and 13 of [BS] as a reference for the theory of graded local cohomology modules.) It is known that each module $H_{\mathfrak{n}}^i(B)$ is an Artinian B -module, which implies that the j -th graded component $H_{\mathfrak{n}}^i(B)_j$ vanishes for $j \gg 0$.

Now assume that B is Cohen-Macaulay. By a theorem of Grothendieck, $H_{\mathfrak{n}}^d(B)$ is nonzero, but $H_{\mathfrak{n}}^i(B) = 0$ for all $i \neq d$. The integer

$$a(B) = \max\{j \in \mathbb{Z} \mid H_{\mathfrak{n}}^d(B)_j \neq 0\}$$

is called the a -invariant of B . Note that $a(B) = \max\{j \in \mathbb{Z} \mid B_j \neq 0\}$ in case that $d = 0$. The following two characterizations of the a -invariant explain why it plays an important role in the study of graded Cohen-Macaulay rings:

- (a) If ω_B denotes the *canonical module of B , then $-a(B)$ is equal to the least integer j with $(\omega_B)_j \neq 0$.
- (b) If $P_B(t) \in \mathbb{Q}[t]$ is the Hilbert polynomial of B , and $H(B, -)$ is the Hilbert function of B , then $a(B) = \max\{j \in \mathbb{Z} \mid H(B, j) \neq P_B(j)\}$.

The characterization in (a) follows directly from the graded version of the local duality theorem, see e.g. [BH, 3.6.19]. A proof for (b) can be found in [BH, 4.4.3]. If B is the polynomial ring $K[y_1, \dots, y_d]$, equipped with the usual grading, then the canonical module ω_B is equal to $B(-d)$, and hence $a(B) = -d$ by (a).

The a -invariant behaves nicely with respect to regular elements: if $f \in B$ is a homogeneous regular element of degree $r > 0$, then we have the graded exact

sequence

$$0 \longrightarrow H_{\mathfrak{n}}^{d-1}(B/(f)) \longrightarrow H_{\mathfrak{n}}^d(B)(-r) \xrightarrow{f} H_{\mathfrak{n}}^d(B) \longrightarrow 0,$$

from which one derives that $a(B/(f)) = a(B) + r$.

The following well-known lemma shows that the notion of a -invariant is closely related to the notion of reduction number.

Lemma 1.40. *Let (A, \mathfrak{m}) be a local Cohen-Macaulay ring with infinite residue field A/\mathfrak{m} , and assume that the associated graded ring $G = \text{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay. Then the reduction number of A is equal to $a(G) + d$.*

Proof. Set $d = \dim A = \dim G$, and let J be any minimal reduction of \mathfrak{m} . Then J can be generated by d elements $y_1, \dots, y_d \in \mathfrak{m} \setminus \mathfrak{m}^2$, such that the initial forms $y_1^*, \dots, y_d^* \in G_1$ form a regular sequence in G . We set

$$\overline{G} = G/(y_1^*, \dots, y_d^*) = \bigoplus_{n \geq 0} (\mathfrak{m}^n / (J\mathfrak{m}^{n-1} + \mathfrak{m}^{n+1})).$$

(Here \mathfrak{m}^{-1} denotes A .) Then

$$a(\overline{G}) = \max\{n \mid (\overline{G})_n \neq 0\} = \max\{n \mid \mathfrak{m}^n \neq J\mathfrak{m}^{n-1} + \mathfrak{m}^{n+1}\}.$$

By Nakayama's lemma, $\mathfrak{m}^n = J\mathfrak{m}^{n-1} + \mathfrak{m}^{n+1}$ holds if and only if $\mathfrak{m}^n = J\mathfrak{m}^{n-1}$, and so $\max\{n \mid \mathfrak{m}^n \neq J\mathfrak{m}^{n-1}\} = a(\overline{G}) = a(G) + d$. This proves our assertion. \square

In the proof of Proposition 1.37, we quoted the following theorem of Goto and Shimoda: let (A, \mathfrak{m}) be a local Cohen-Macaulay ring of positive dimension with infinite residue field A/\mathfrak{m} . Then $A[\mathfrak{m}t]$ is Cohen-Macaulay if and only if $\text{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay and the reduction number of A is less than $\dim A$.

By Lemma 1.40, one may replace the condition that the reduction number is less than $\dim A$ by the condition that the a -invariant of $\text{gr}_{\mathfrak{m}}(A)$ is negative. This was already mentioned by Goto and Shimoda in [GS].

Combining this observation with Corollary 1.27, we get an application for affine semigroup rings.

Proposition 1.41. *Let $R = K[S]$ be a positive normal affine semigroup ring, and let \mathfrak{m} be the graded maximal ideal of R . If $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, then its a -invariant is negative.*

Proof. If $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, then so is $R[\mathfrak{m}t]$ by Proposition 1.37. In case that K is an infinite field, we get $a(\text{gr}_{\mathfrak{m}}(R)) < 0$ from the theorem of Goto and Shimoda. If K is finite, we choose an infinite field extension L of K , and set $R' = L[S]$ and $\mathfrak{m}' = \bigoplus_{a \in S_+} Lx^a$. The Cohen-Macaulayness of $\text{gr}_{\mathfrak{m}}(R)$ implies the Cohen-Macaulayness of $\text{gr}_{\mathfrak{m}'}(R') = \text{gr}_{\mathfrak{m}}(R) \otimes_K L$. Since $\text{gr}_{\mathfrak{m}}(R)$ and $\text{gr}_{\mathfrak{m}'}(R')$ have the same Hilbert function, they also have the same a -invariant. \square

Combining Proposition 1.41 with Corollary 1.27 and Proposition 1.37, we obtain

Corollary 1.42. *Let R and \mathfrak{m} be as in Proposition 1.41 and assume $\dim R \leq 3$. Then $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay and its a -invariant is negative.*

Now consider an arbitrary positive affine semigroup S . For $n \in \mathbb{N}$, let $h_S(n)$ be the cardinality of the finite set $nS_+ \setminus (n+1)S_+$, and let $h_S(0) := 1$. If $R = K[S]$ is the associated semigroup ring over a field K and \mathfrak{m} is the graded maximal ideal of R , then $h_S(n), n \geq 0$, is equal to the Hilbert function of $G := \text{gr}_{\mathfrak{m}}(R)$.

If $P_G(t) \in \mathbb{Q}[t]$ is the Hilbert polynomial of G , then $P_G(n) = h_S(n)$ for $n \gg 0$, and

$$a(G) = \max\{n \geq 0 \mid P_G(n) \neq h_S(n)\}.$$

Therefore, by Corollary 1.42 we get

Proposition 1.43. *Let S be a positive affine semigroup of dimension d , and define $h_S(n), n \geq 0$, as above. Then there exists a unique polynomial $P_S(t) \in \mathbb{Q}[t]$ of degree $d - 1$ with $h_S(n) = P_S(n)$ for $n \gg 0$. If S is normal and $\dim S \leq 3$, then $h_S(n) = P_S(n)$ for all $n \geq 0$.*

The second statement of Proposition 1.43 is not valid for normal semigroups of arbitrary dimension. For instance, consider the positive normal affine semigroup S defined in Example 1.39, and set $d = 4$. Having the necessary endurance, one computes that

$$h_S(0) = 1, h_S(1) = 6, h_S(2) = 19, h_S(3) = 44, \text{ and } h_S(4) = 82.$$

The polynomial

$$P(t) = \frac{2}{3}t^3 + 2t^2 + \frac{7}{3}t + 1$$

satisfies $P(0) = 1, P(1) = 6, P(2) = 19$, and $P(3) = 44$. If $h_S(n) = P_S(n)$ would hold for all $n \geq 0$, then $P_S(t)$ would be equal to $P(t)$. But since

$$P(4) = 85 \neq 82 = h_S(4) = P_S(4),$$

we have $P_S(t) \neq P(t)$.

1.8. The special case of hypersurface rings.

Let $R = K[S]$ be a positive normal affine semigroup ring of dimension d . The cardinality of $\text{Hilb}(S)$ is the *embedding dimension* of R , for short $\text{emb dim } R$. We have $\text{emb dim } R \geq d$, and equality holds if and only if S is isomorphic to \mathbb{N}_0^d . In the case that $\text{emb dim } R = d + 1$, we have $R \cong K[y_1, \dots, y_{d+1}]/(f)$, where f is a polynomial in $K[y_1, \dots, y_{d+1}]$. Since R is the coordinate ring of the hypersurface $V(f)$ in \mathbb{A}_K^d , one calls R a *hypersurface ring*.

In this subsection, we consider the class of positive normal affine semigroup rings R which satisfy $\text{emb dim } R = \dim R + 1$. We show that the Rees algebra $R[\mathfrak{m}t]$ is always Cohen-Macaulay, and we give a necessary and sufficient criterion for $R[\mathfrak{m}t]$ to be normal.

First, we show that R can be written as the quotient of a polynomial ring modulo a binomial $h - g$, where g and h are coprime monomials of degree > 1 and h is squarefree.

Proposition 1.44. (a) Let R be a d -dimensional positive normal affine semigroup ring over a field K with $\text{emb dim } R = d + 1$. Then $R \cong K[y_1, \dots, y_{d+1}]/(f)$ with

$$f = \prod_{i=1}^r y_i - \prod_{i=r+1}^{d+1} y_i^{e_i},$$

where $r > 1$, e_{r+1}, \dots, e_{d+1} are nonnegative integers, and $e := \sum_{i=r+1}^{d+1} e_i > 1$.

(b) Let $f = \prod_{i=1}^r y_i - \prod_{i=r+1}^{d+1} y_i^{e_i}$ be a binomial in $K[y_1, \dots, y_{d+1}]$ that fulfills the conditions described in (a). Then $K[y_1, \dots, y_{d+1}]/(f)$ is isomorphic to a positive normal affine semigroup ring.

Proof. (a) We have $R = K[S]$, where S is a positive normal affine semigroup that is generated by $d + 1$ elements, say w_1, \dots, w_{d+1} . Let $P = K[y_1, \dots, y_{d+1}]$ be the polynomial ring over K in $d + 1$ variables, and let $\pi : P \rightarrow R$ be the map which sends y_i to x^{w_i} . The kernel of π is generated by an irreducible polynomial f . Let h be a monomial which appears in the expansion of f . Since $\pi(f) = 0$, there must be another monomial g appearing in the expansion of f such that $\pi(g) = \pi(h)$. This implies $h - g \in (f)$ and hence $f = \lambda(h - g)$ for some $\lambda \in K^\times$.

By renumbering the variables of P in a suitable way and replacing f by $\lambda^{-1}f$, we get

$$f = \prod_{i=1}^r y_i^{q_i} - \prod_{i=r+1}^{d+1} y_i^{e_i},$$

where $q_1, \dots, q_r \geq 1$, $e_{r+1}, \dots, e_{d+1} \geq 0$, $q = \sum_{i=1}^r q_i > 1$, and $e = \sum_{i=r+1}^{d+1} e_i > 1$. If $e_i \leq 1$ for $i = r + 1, \dots, d + 1$, then, after renumbering the variables again, f has the desired form.

So we consider the case that $e_j > 1$ for some $j \in \{r + 1, \dots, d + 1\}$ and show that $q_i = 1$ for $i = 1, \dots, r$. We may assume $\text{char}(K) = 0$. Then we can apply the Jacobian criterion, which states that the singular locus of R is equal to the closed subset $V(J)$ of $\text{Spec}(R)$, where J is the image of the ideal $(\partial f / \partial y_1, \dots, \partial f / \partial y_{d+1})$ in R . Note that each partial derivative $\partial f / \partial y_i$ is a monomial in P and that f is contained in $(\partial f / \partial y_1, \dots, \partial f / \partial y_{d+1})$. Assume that $q_k > 1$ for some $k \in \{1, \dots, r\}$. Then $(\partial f / \partial y_1, \dots, \partial f / \partial y_{d+1})$ is contained in $\mathfrak{p} = (y_j, y_k)$ and hence $\bar{\mathfrak{p}} \in V(J)$, where $\bar{\mathfrak{p}}$ denotes the image of \mathfrak{p} in R . Since \mathfrak{p} has height 2 in P , $\bar{\mathfrak{p}}$ has height 1 in R . But this contradicts the fact that the singular locus of a normal domain does not contain any prime ideal of height 1 (see e.g. [Ei, 11.2] for a proof).

(b) Consider the element $v = (1, \dots, 1, -e_{r+1}, \dots, -e_{d+1}) \in \mathbb{Z}^{d+1}$. Since $\mathbb{Z}^{d+1}/(v)$ is a free \mathbb{Z} -module, there exists an isomorphism $\mathbb{Z}^{d+1}/(v) \xrightarrow{\sim} \mathbb{Z}^d$. Let φ be the composition map $\mathbb{N}_0^{d+1} \xrightarrow{\text{nat}} \mathbb{Z}^{d+1}/(v) \xrightarrow{\sim} \mathbb{Z}^d$, and let S be its image. It is clear that S is a positive affine semigroup of dimension d . The induced K -algebra homomorphism $K[y_1, \dots, y_{d+1}] \rightarrow K[S]$ is surjective and f lies in its kernel. Since f is irreducible, $K[y_1, \dots, y_{d+1}]/(f)$ is a domain of dimension d . This shows that $K[y_1, \dots, y_{d+1}]/(f) \cong K[S]$.

It remains to prove that $K[S]$ is normal. For this we may assume $\text{char}(K) = 0$. The normality criterion of Serre states that a Noetherian ring is normal if and only if it satisfies the conditions (R_1) and (S_2) , see e.g. [Ma, 23.8] for a proof. Since $K[S]$ is a hypersurface ring, it satisfies (S_2) . For determining the singular locus we apply again the Jacobian criterion.

Let \mathfrak{p} be a prime ideal in $K[y_1, \dots, y_{d+1}]$ which contains the partial derivatives $\partial f/\partial y_1, \dots, \partial f/\partial y_{d+1}$. It is easy to see that \mathfrak{p} must contain at least two of the elements y_1, \dots, y_r and at least one of the elements y_{r+1}, \dots, y_{d+1} . Consequently, $\text{height } \mathfrak{p} \geq 3$, so that $\text{height } \mathfrak{p}/(f) \geq 2$. This shows that $K[S]$ is regular in codimension one, and we are done. \square

Corollary 1.45. *Under the assumptions of Proposition 1.44 (a), the Rees algebra $R[\mathfrak{m}t]$ is Cohen-Macaulay.*

Proof. The assertion follows immediately from Proposition 1.37, because $\text{gr}_{\mathfrak{m}}(R)$ is a hypersurface ring and hence Cohen-Macaulay. \square

For an arbitrary normal hypersurface ring $R = K[y_1, \dots, y_{d+1}]/(f)$, the Rees algebra $R[\mathfrak{m}t]$ need not be Cohen-Macaulay, as the following example shows.

Example 1.46. Let K be a field of characteristic zero and let

$$R = K[y_1, \dots, y_{d+1}]/(\sum_{i=1}^{d+1} y_i^{n_i}),$$

where $d \geq 2$, and $n_1, \dots, n_{d+1} \in \mathbb{N}$. Furthermore, set $\mathfrak{m} = (\bar{y}_1, \dots, \bar{y}_{d+1}) \subset R$. Then R is a normal domain, and the Rees algebra $R[\mathfrak{m}t]$ is Cohen-Macaulay if and only if $m := \min\{n_1, \dots, n_{d+1}\} \leq d$.

Proof. Just as in Example 1.34, one shows that $\sum_{i=1}^{d+1} y_i^{n_i}$ is irreducible and that R is normal. We have

$$\text{gr}_{\mathfrak{m}}(R) = K[y_1, \dots, y_{d+1}]/(\sum_{i \in I} y_i^m),$$

where I is the set of all indices $i \in \{1, \dots, d+1\}$ with $n_i = m$. In particular, $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay. Hence, applying the theorem of Goto and Shimoda mentioned below Lemma 1.40, we see that $R[\mathfrak{m}t]$ is Cohen-Macaulay if and only if $a(\text{gr}_{\mathfrak{m}}(R)) < 0$.

Since $\sum_{i \in I} y_i^m$ is a regular element of degree m in $K[y_1, \dots, y_{d+1}]$, $a(\text{gr}_{\mathfrak{m}}(R))$ is equal to $-d-1+m$, and we obtain our assertion. \square

Theorem 1.47. *We adopt the assumptions and notation of Proposition 1.44 (a). Then $R[\mathfrak{m}t]$ is normal if and only if $r \leq e+1$ or $e_i \leq 1$ for $i = r+1, \dots, d+1$.*

Proof. We may assume that $R = K[y_1, \dots, y_{d+1}]/(f)$, where f is the binomial described in Proposition 1.44 (a), and that $\text{char}(K) = 0$. Let $m = \min\{r, e\}$. Then by Lemma 1.33, we have

$$R[\mathfrak{m}t] \cong B := K[y_1, \dots, y_{d+1}, z_1, \dots, z_{d+1}]/(h_0, \dots, h_m; y_i z_j - y_j z_i, 1 \leq i < j \leq d+1),$$

where h_j ($j = 0, \dots, m$) is a preimage of $t^j f$ with respect to the K -algebra homomorphism

$$K[y_1, \dots, y_{d+1}, z_1, \dots, z_{d+1}] \rightarrow K[y_1, \dots, y_{d+1}, ty_1, \dots, ty_{d+1}], y_i \mapsto y_i, z_i \mapsto ty_i.$$

By Corollary 1.45, $R[\mathfrak{m}t]$ is Cohen-Macaulay and hence satisfies (S_2) . Therefore, $R[\mathfrak{m}t]$ is normal if and only if it is regular in codimension one. This is the case if and only if the localizations B_{y_i} and B_{z_i} ($i = 1, \dots, d+1$) are all regular in

codimension one. By Lemma 1.33, B_{y_i} is equal to the polynomial ring $R_{y_i}[t]$ and hence normal for all i . So we fix $k \in \{1, \dots, d+1\}$ and investigate

$$B_{z_k} \cong K[y_k, z_1, \dots, z_{d+1}, z_k^{-1}] / ((y_k/z_k)^{r-m} \prod_{i=1}^r z_i - (y_k/z_k)^{e-m} \prod_{i=r+1}^{d+1} z_i^{e_i}).$$

We distinguish two cases:

- (i) $r < e$.
- (ii) $r \geq e$.

In case (i) the associated graded algebra $\text{gr}_{\mathfrak{m}}(R)$ is isomorphic to

$$K[y_1, \dots, y_{d+1}] / (\prod_{i=1}^r y_i)$$

and thus reduced. By Lemma 1.17 and Lemma 1.19, this implies that B is normal.

In case (ii) we have $B_{z_k} \cong (A/(g))_{z_k}$, where $A = K[y_k, z_1, \dots, z_{d+1}]$ and

$$g = y_k^{r-e} \prod_{i=1}^r z_i - z_k^{r-e} \prod_{i=r+1}^{d+1} z_i^{e_i}.$$

Note that $\text{Sing}((A/(g))_{z_k}) = \text{Sing}(A/(g)) \cap D(z_k)$, where $D(z_k)$ denotes the open subset of $\text{Spec}(A/(g))$ consisting of all prime ideals \mathfrak{q} with $z_k \notin \mathfrak{q}$.

Assume that $r \leq e+1$ or $e_i \leq 1$ for $i = r+1, \dots, d+1$. We have to show that B_{z_k} is regular in codimension one. So let \mathfrak{p} be a prime ideal in A with

$$\mathfrak{p} \supseteq \mathfrak{a} := (\partial g / \partial y_k, \partial g / \partial z_1, \dots, \partial g / \partial z_{d+1}) \text{ and } z_k \notin \mathfrak{p}.$$

If $r \leq e+1$, then \mathfrak{p} must contain at least two of the elements y_k, z_1, \dots, z_r and at least one element of the set $\{z_{r+1}, \dots, z_{d+1}\} \setminus \{z_k\}$. (In particular, r must be less than d .) If $e_i \leq 1$ for $i = r+1, \dots, d+1$, then \mathfrak{p} must contain at least one of the elements y_k, z_1, \dots, z_r and at least two elements of the set $\{z_{r+1}, \dots, z_{d+1}\} \setminus \{z_k\}$. (In particular, r must be less than $d-1$.)

So in both cases we see that $\text{height } \mathfrak{p} \geq 3$ and thus $\text{height } \bar{\mathfrak{p}} \geq 2$, where $\bar{\mathfrak{p}}$ denotes $\mathfrak{p} \text{ mod } (g)$. This means that B_{z_k} is regular in codimension one.

Now assume that $r \geq e+2$ and $e_j \geq 2$ for some $j \in \{r+1, \dots, d+1\}$. Then $\mathfrak{p} = (y_k, z_j)$ contains \mathfrak{a} , which means that $\bar{\mathfrak{p}}$ lies in $\text{Sing}(A/(g))$. Since $\text{height } \bar{\mathfrak{p}} = 1$, we obtain that B_{z_k} is not regular in codimension one. \square

Corollary 1.48. *Let R be a positive normal affine semigroup ring, and let \mathfrak{m} be the graded maximal ideal of R . If $\text{emb dim } R \leq 5$, then the Rees algebra $R[\mathfrak{m}t]$ is normal unless $R \cong K[x_1^2, x_2^2, x_3^2, x_4^2, x_1 x_2 x_3 x_4]$.*

Proof. If $\dim R \leq 3$, then $R[\mathfrak{m}t]$ is normal by Corollary 1.27. So we may assume that R is a hypersurface ring of dimension 4. By Proposition 1.44, we have

$$R \cong K[y_1, \dots, y_5] / (\prod_{i=1}^r y_i - \prod_{i=r+1}^5 y_i^{e_i})$$

with $1 < r < 5$, $e_i \geq 0$ ($i = r+1, \dots, 5$), and $e = \sum_{i=r+1}^5 e_i > 1$. According to Theorem 1.47, $R[\mathfrak{m}t]$ is not normal if and only if $r \geq e+2$ and $e_i > 1$ for some $i \in \{r+1, \dots, 5\}$. These conditions are only satisfied if $r = 4$ and $e_5 = e = 2$, that is, if $R \cong K[x_1^2, x_2^2, x_3^2, x_4^2, x_1 x_2 x_3 x_4]$. \square

2. ON THE TYPE OF A SIMPLICIAL NORMAL AFFINE SEMIGROUP RING

Let $A = \bigoplus_{i \geq 0} A_i$ be a d -dimensional graded Cohen-Macaulay algebra over a field $K = A_0$, and let $\mathfrak{m} = \bigoplus_{i > 0} A_i$ be the graded maximal ideal of A . Then the module $\text{Ext}_A^d(A/\mathfrak{m}, A)$ is the first nontrivial object in the sequence $\text{Ext}_A^i(A/\mathfrak{m}, A), i \geq 0$. Since $\text{Ext}_A^d(A/\mathfrak{m}, A)$ is a finitely generated A -module which is annihilated by \mathfrak{m} , it is a finite dimensional vector space over K . The dimension of $\text{Ext}_A^d(A/\mathfrak{m}, A)$ over K is called the *type* of A and is denoted by $r(A)$.

The type $r(A)$ is an important invariant of the ring A . For instance, the Gorenstein property can be expressed via $r(A)$: A is Gorenstein if and only if $r(A) = 1$. This characterization is a direct consequence of the following more general result: if ω_A is the *canonical module of A , then any minimal system of homogeneous generators of ω_A consists of $r(A)$ elements (see e.g. [BH, 3.3.11] for a proof).

The ideal $\{z \in A \mid z\mathfrak{m} = 0\}$ is called the *socle* of A and is denoted by $\text{Soc}(A)$. The natural map $\text{Hom}_A(A/\mathfrak{m}, A) \rightarrow A, \varphi \mapsto \varphi(\bar{1})$, induces an isomorphism

$$\text{Hom}_A(A/\mathfrak{m}, A) \cong \text{Soc}(A).$$

Therefore, in case $d = 0$ we have $r(A) = \dim_K \text{Soc}(A)$.

In this section, we study the type of an affine semigroup ring $R = K[S]$, where S is a simplicial normal affine semigroup of dimension $d \leq 3$. For such a semigroup S , there exists an embedding $S \hookrightarrow T := \mathbb{N}_0^d$ with $S = \mathbb{Z}S \cap T$. Note that $P := K[T]$ is isomorphic to the d -dimensional polynomial ring over K . We show that $r(R)$ is bounded above by $r(P/\mathfrak{m}P)$, where \mathfrak{m} denotes the graded maximal ideal of R .

2.1. Preparations.

A d -dimensional positive affine semigroup S is said to be *simplicial*, if there are d elements $w_1, \dots, w_d \in S$ such that $\mathbb{R}_+w_1 + \dots + \mathbb{R}_+w_d = \mathbb{R}_+S$. One easily sees that any positive affine semigroup of dimension $d \leq 2$ is simplicial. For $d \geq 3$, this is no longer true.

Example 2.1. Let $d \geq 3$, and let e_1, \dots, e_d be the standard basis of \mathbb{Z}^d . The positive affine semigroup $S \subset \mathbb{Z}^d$, generated by the $d + 1$ elements

$$e_1, e_2, \dots, e_{d-1}, e_1 + e_d, e_2 + e_d,$$

is not simplicial.

Proof. Suppose that the cone \mathbb{R}_+S is equal to $\mathbb{R}_+w_1 + \dots + \mathbb{R}_+w_d$, where the w_i are elements in S . By renumbering the w_i , we may assume that $w_i = \alpha_i e_i$ with $\alpha_i > 0$ for $i = 1, \dots, d - 1$. Set $U = \mathbb{R}_+w_1 + \dots + \mathbb{R}_+w_{d-1}$. For $i = 1, 2$, we have $e_i + e_d = u_i + \beta_i w_d$, where $\beta_i > 0$ and $u_i \in U$. This implies that $w_d = \alpha_d e_d$ with $\alpha_d > 0$. But since $S \cap \mathbb{R}_+e_d = \{0\}$, we get a contradiction. \square

The following lemma gives a useful criterion for a positive affine semigroup to be simplicial, see e.g. Section 2 of [BG1] for a proof.

Lemma 2.2. *A d -dimensional positive affine semigroup S is simplicial if and only if there exists a group homomorphism $\sigma : \mathbb{Z}S \rightarrow \mathbb{Z}^d$ with $\sigma(S) = \sigma(\mathbb{Z}S) \cap (\mathbb{N}_0)^d$.*

In this sequel, we consider a d -dimensional simplicial normal affine semigroup S and its associated semigroup ring $R = K[S]$ over a field K . As usual, \mathfrak{m} denotes the graded maximal ideal of R . Because of Lemma 2.2, we may assume that

$$S \subseteq T \text{ and } S = \mathbb{Z}S \cap T,$$

where T denotes the affine semigroup \mathbb{N}_0^d . We set

$$P := K[T] = K[x_1, \dots, x_d].$$

Let e_1, \dots, e_d be the standard basis of \mathbb{Z}^d , and let $\tau_i : \mathbb{Z}^d \rightarrow \mathbb{Z}, e_j \mapsto \delta_{ij}$, denote the i -th coordinate function for $i = 1, \dots, d$. Since $\mathbb{Z}T/\mathbb{Z}S$ is a torsion group, there exist positive integers n_1, \dots, n_d such that $n_i e_i \in S$ for $i = 1, \dots, d$. We define

$$T^{\text{int}} = \{a \in T \mid \tau_i(a) > 0 \text{ for } i = 1, \dots, d\} \text{ and } S^{\text{int}} = S \cap T^{\text{int}}.$$

Note that S^{int} (resp. T^{int}) is a semigroup ideal in S (resp. T). It is known that

$$\omega_R := \bigoplus_{a \in S^{\text{int}}} x^a$$

is the *canonical module of R with respect to any admissible grading of R , see e.g. [BH, 6.3.5] for a proof. Let

$$G(S) := \{a \in S^{\text{int}} \mid a \notin S^{\text{int}} + S_+\}$$

be the minimal set of generators of the ideal S^{int} . The elements $x^a, a \in G(S)$, form a minimal system of homogeneous generators of the canonical module ω_R . Thus, the cardinality of $G(S)$ is equal to the type of R .

Note that in general, $G(S)$ is not contained in $\text{Hilb}(S)$. For example, consider $S = (\mathbb{N}_0)^d$, where $d \geq 2$. Then $w = (1, \dots, 1)$ generates S^{int} , but w does not lie in $\text{Hilb}(S)$. Nevertheless, we have the following

Proposition 2.3. *The set $G(S)$ is contained in $S_+ \setminus (d+1)S_+$.*

Proof. Assume that w_1, \dots, w_{d+1} are elements in S_+ with $w = \sum_{i=1}^{d+1} w_i \in S^{\text{int}}$. Then we can define a map $\pi : \{1, \dots, d\} \rightarrow \{1, \dots, d+1\}$ such that $\tau_i(w_{\pi(i)}) > 0$ for $i = 1, \dots, d$. If I is the image of π , then $w' = \sum_{i \in I} w_i \in S^{\text{int}}$ and $w \in w' + S_+$. This shows that $w \notin G(S)$. \square

We set

$$M(S) = \{a \in T \mid a \notin S_+ + T\} \text{ and } \overline{M}(S) = \{a \in M(S) \mid a + T_+ \subseteq S_+ + T\}.$$

Note that the residue classes of the elements $x^a, a \in M(S)$, form a K -basis of the fibre ring $\overline{P} := P/\mathfrak{m}P$. Since $\dim R = \dim P$, the ring \overline{P} is Artinian. The residue classes of the elements $x^a, a \in \overline{M}(S)$, form a K -basis of the socle of \overline{P} . Therefore, the cardinality of $\overline{M}(S)$ is equal to the type of \overline{P} .

2.2. The case of dimension 2.

If $\dim R = 2$, then the type of R can easily be computed.

Proposition 2.4. *Let S be a 2-dimensional simplicial normal affine semigroup, and let $R = K[S]$ be the associated semigroup ring over a field K . If R is regular, then $r(R) = r(\overline{P}) = 1$, where \overline{P} is the fibre ring defined in the previous subsection. If R is not regular, then $r(R) = \text{emb dim } R - 2 = r(\overline{P}) - 1$.*

Proof. Let $\text{Hilb}(S) = \{w_1, \dots, w_n\}$, where $n = \text{emb dim } R$. By renumbering the w_i , we may assume that $w_i = (\alpha_i, \beta_i)$, such that

$$0 = \alpha_1 < \alpha_2 < \dots < \alpha_n \text{ and } \beta_1 > \beta_2 > \dots > \beta_n = 0.$$

We set $v_i = (\alpha_{i+1} - 1, \beta_i - 1)$ for $i = 1, \dots, n - 1$. Assume that $v_i \in S_+ + T$ for some $i \in \{1, \dots, n - 1\}$. Then there is a $j \in \{1, \dots, n\}$ with $\alpha_{i+1} - 1 \geq \alpha_j$ and $\beta_i - 1 \geq \beta_j$. This means that $i + 1 > j$ and $i < j$, a contradiction. Therefore, all v_i must lie in $M(S)$. From $v_i + e_1 \in w_{i+1} + T$ and $v_i + e_2 \in w_i + T$ we conclude that the v_i even lie in $\overline{M}(S)$.

Let $v = (\alpha, \beta)$ be an arbitrary element in $\overline{M}(S)$. Since $v + e_1 \in S_+ + T$, there is an $i \in \{1, \dots, n\}$ such that $v + e_1 \in w_i + T$. Since $v \notin w_i + T$, we conclude that $i > 1$ and $\alpha = \alpha_i - 1$. From $v \notin w_{i-1} + T$ follows $\beta \leq \beta_{i-1} - 1$. Moreover, there exists a $j \in \{1, \dots, n\}$ with $v + e_2 \in w_j + T$. This means that $\alpha_i - 1 = \alpha \geq \alpha_j$ and $\beta_{i-1} \geq \beta + 1 \geq \beta_j$. Hence we get $i > j$ and $i - 1 \leq j$. It follows that $\beta = \beta_{i-1} - 1$ and thus $v = v_{i-1}$. So we have shown that $\overline{M}(S)$ is equal to $\{v_1, \dots, v_{n-1}\}$. In particular, $r(\overline{P}) = n - 1$.

Since the assertion concerning $r(R)$ is trivial in the case that R is regular, we may assume that R is not regular. Clearly, $\{w_2, \dots, w_{n-1}\}$ is contained in $G(S)$. Suppose that there is an element $w \in G(S)$ which does not lie in $\text{Hilb}(S)$. Then $w \in w_1 + w_n + T$ and hence $\alpha \geq \alpha_n$ and $\beta \geq \beta_1$. Since R is not regular, there exists an $i \in \{1, \dots, n\}$ with $1 < i < n$. From $\alpha > \alpha_i$ and $\beta > \beta_i$ follows that $w \in w_i + T_+$, which is impossible. So we have shown that $G(S) = \{w_2, \dots, w_{n-1}\}$ and in particular, $r(R) = n - 2$. \square

2.3. The case of dimension 3.

Now we turn to the case $d = 3$. Our goal is to show that $r(R) \leq r(\overline{P})$. For this, we need some preparation. For $i = 1, 2, 3$, we define the map

$$\pi_i : M(S) \rightarrow M(S), a \mapsto a + me_i, \text{ where } m = \sup \{n \in \mathbb{N}_0 \mid a + ne_i \in M(S)\}.$$

Note that $\tau_j(a) = \tau_j(\pi_i(a))$ for all $a \in M(S)$ and all $j \neq i$. Then we set

$$\pi := \pi_3 \circ \pi_2 \circ \pi_1.$$

Note that $\pi(a) \in \overline{M}(S)$ for all $a \in M(S)$. Furthermore, we define

$$\text{Hilb}^{\text{int}}(S) = \text{Hilb}(S) \cap S^{\text{int}} = \text{Hilb}(S) \cap G(S).$$

If a is an element in $\text{Hilb}^{\text{int}}(S)$ and $i \in \{1, 2, 3\}$, then $a - e_i \in M(S)$. Otherwise, we would have $a - e_i = b + c$ with $b \in S_+, c \in T$. Then $c + e_i = a - b \in T_+ \cap \mathbb{Z}S = S_+$ and hence $a = b + (c + e_i) \in 2S_+$, which is a contradiction. Similarly, one checks that $a - e_1 - e_2 - e_3 \in M(S)$ for all $a \in G(S)$. We set

$$G'(S) = \{a \in G(S) \setminus \text{Hilb}(S) \mid \tau_3(\pi(a - e_1 - e_2 - e_3)) \geq \tau_3(a)\}$$

and

$$\mu = \inf\{\tau_3(a) \mid a \in G'(S)\}.$$

If $G'(S) = \emptyset$, then $\mu = \infty$. Finally, we define a map $\rho : G(S) \rightarrow \overline{M}(S)$ by setting

$$\rho(a) = \begin{cases} \pi(a - e_3), & \text{if } a \in \text{Hilb}(S) \text{ and } \tau_3(a) < \mu \\ \pi(a - e_1), & \text{if } a \in \text{Hilb}(S) \text{ and } \tau_3(a) \geq \mu \\ \pi(a - e_1 - e_2 - e_3), & \text{if } a \notin \text{Hilb}(S) \end{cases}$$

We will show that ρ is injective. For this, we need several lemmas.

Lemma 2.5. *If $a \in G(S) \setminus \text{Hilb}(S)$, then $a - e_i \in S_+ + T$ for $i = 1, 2, 3$, and hence $\tau_i(\rho(a)) \neq \tau_i(a) - 1$ for at most one index $i \in \{1, 2, 3\}$.*

Proof. Since $a \in G(S) \subset S^{\text{int}}$, we have $\tau_i(a) > 0$ for $i = 1, 2, 3$. Since a does not lie in $\text{Hilb}(S)$, there exist $b, c \in S_+ \setminus S^{\text{int}}$ with $a = b + c$. We have $\tau_i(b) = 0 < \tau_i(c)$ and $\tau_j(c) = 0 < \tau_j(b)$, where i and j are distinct indices in $\{1, 2, 3\}$. If k is the remaining index in $\{1, 2, 3\}$, then $\tau_k(b) > 0$ or $\tau_k(c) > 0$, say $\tau_k(b) > 0$. Then we have $a - e_i \in b + T \subseteq S_+ + T$ and $a - e_j, a - e_k \in c + T \subseteq S_+ + T$. \square

Lemma 2.6. *Let a be an element in $\text{Hilb}^{\text{int}}(S)$. If $\tau_3(a) < \mu$, then $\tau_3(\rho(a))$ is equal to $\tau_3(a) - 1$. If $\tau_3(a) \geq \mu$, then $\tau_1(\rho(a))$ is equal to $\tau_1(a) - 1$.*

Proof. If $\tau_3(a) < \mu$, then $\rho(a) = \pi(a - e_3)$. Since $\pi(a - e_3) \notin S_+ + T$, we have $\pi(a - e_3) \notin a + T$. But $\tau_i(\pi(a - e_3)) \geq \tau_i(a)$ for $i \in \{1, 2\}$, and hence $\tau_3(\pi(a - e_3))$ must be equal to $\tau_3(a) - 1$. If $\tau_3(a) \geq \mu$ then $\rho(a) = \pi(a - e_1)$, and with the same argument as above one sees that $\tau_1(\rho(a)) = \tau_1(a) - 1$. \square

Lemma 2.7. *Let a be an element in $G'(S)$. If b and c are elements in S_+ such that $a = b + c$, then $\tau_3(b)$ and $\tau_3(c)$ are both positive. Furthermore, $\tau_i(b) = 0 < \tau_i(c)$ and $\tau_j(c) = 0 < \tau_j(b)$, where i is one of the indices 1, 2, and j is the other one.*

Proof. Assume that one of the numbers $\tau_3(b)$ and $\tau_3(c)$ is zero, say $\tau_3(b) = 0$. Then $\tau_3(c) = \tau_3(a) > 0$. Since $a \in G(S)$, c is not contained in S^{int} . Thus $\tau_i(c) = 0$ for an index $i \in \{1, 2\}$. Let j be the other index in $\{1, 2\}$.

We set $\tilde{a} := a - e_1 - e_2 - e_3$. Suppose that $\tau_j(c) < \tau_j(a)$. Then $\tilde{a} + e_3$ lies in $c + T \subseteq S_+ + T$ and thus $\tau_3(\rho(a)) = \tau_3(\pi(\tilde{a})) = \tau_3(\tilde{a}) = \tau_3(a) - 1$. But this is a contradiction to $a \in G'(S)$, and hence we must have $\tau_j(c) = \tau_j(a)$.

Combining Lemma 2.5 with the fact that $\tau_3(\rho(a)) \neq \tau_3(a) - 1$, we see that $\tilde{a} + e_j$ must lie in $S_+ + T$. This means there is an element $d \in S_+$ with $\tilde{a} \notin d + T$ and

$\tilde{a} + e_j \in d + T$. In particular, we have $d \neq a$ and $\tau_j(d) = \tau_j(a) > 0$. Since $a \in G(S)$ and $a \in (\tilde{a} + e_j) + T \subseteq d + T$, d cannot lie in S^{int} . Suppose that $\tau_i(d) = 0$. Then

$$\tau_i(d) = \tau_i(c), \tau_j(d) = \tau_j(c), \text{ and } \tau_3(d) \leq \tau_3(\tilde{a} + e_j) = \tau_3(a) - 1 = \tau_3(c) - 1.$$

Hence, $c - d \in S_+$ and $\tilde{a} + e_3 \in (c - d) + T \subseteq S_+ + T$. But then we obtain that $\tau_3(\rho(a)) = \tau_3(a) - 1$, which is a contradiction to $a \in G'(S)$. Therefore, $\tau_i(d)$ must be positive and $\tau_3(d) = 0$. Then

$$\tau_i(a - d) \leq \tau_i(a) - 1 = \tau_i(\tilde{a}),$$

$$\tau_j(a - d) = 0 \leq \tau_j(\tilde{a}),$$

$$\tau_3(a - d) = \tau_3(a) = \tau_3(\tilde{a}) + 1.$$

This implies $\tilde{a} + e_3 \in (a - d) + T \subseteq S_+ + T$, and we get again a contradiction to $a \in G'(S)$. Therefore, the case that one of the numbers $\tau_3(b), \tau_3(c)$ is zero, cannot occur. The second statement of the lemma is a direct consequence of the first. \square

Lemma 2.8. *Let a, a' be elements in $M(S)$ with $a - a' \in \mathbb{Z}S, a \notin a' + T, a' \notin a + T$, and $\tau_i(a) = \tau_i(\pi(a)) = \tau_i(\pi(a')) = \tau_i(a')$ for some $i \in \{1, 2, 3\}$. Then $\pi(a)$ is not equal to $\pi(a')$.*

Proof. Let j be the smaller one and let k be the greater one of the two indices in $\{1, 2, 3\} \setminus \{i\}$. We may assume that $\tau_j(a) > \tau_j(a')$ and $\tau_k(a) < \tau_k(a')$. By definition of π_j , there exists an element $b \in S_+$ with $\pi_j(a) \notin b + T$ and $\pi_j(a) + e_j \in b + T$. In particular, $\tau_j(b) = \tau_j(\pi_j(a)) + 1$. Consider the element $b' := b + a' - a$. From

$$\tau_i(b') = \tau_i(b) \geq 0,$$

$$\tau_j(b') \geq \tau_j(b) + \tau_j(a') - \tau_j(\pi_j(a)) = \tau_j(a') + 1 > 0,$$

$$\tau_k(b') > \tau_k(b) \geq 0,$$

we see that $b' \in \mathbb{Z}S \cap T_+ = S_+$. Combining the fact that $\pi(a) \notin S_+ + T$ with the inequalities

$$\tau_i(b') = \tau_i(b) \leq \tau_i(\pi_j(a) + e_j) = \tau_i(\pi_j(a)) = \tau_i(\pi(a)),$$

$$\tau_j(b') \leq \tau_j(b) - 1 = \tau_j(\pi_j(a)) = \tau_j(\pi(a)),$$

we obtain that $\tau_k(b') > \tau_k(\pi(a))$. Using the inequality

$$\tau_k(b) \leq \tau_k(\pi_j(a) + e_j) = \tau_k(a),$$

we get

$$\tau_k(\pi(a)) < \tau_k(b') = \tau_k(b) + \tau_k(a') - \tau_k(a) \leq \tau_k(a') \leq \tau_k(\pi(a')),$$

and hence $\pi(a) \neq \pi(a')$. \square

Theorem 2.9. *The map $\rho : G(S) \rightarrow \overline{M}(S)$ is injective.*

Proof. Let a, a' be elements in $G(S)$ with $\rho(a) = \rho(a')$. We distinguish three cases:

- (i) The elements a and a' are both contained in $\text{Hilb}(S)$.
- (ii) Only one of the elements a, a' is contained in $\text{Hilb}(S)$.
- (iii) None of the elements a, a' is contained in $\text{Hilb}(S)$.

Case (i): Lemma 2.6 states that $\tau_3(\rho(b)) = \tau_3(b) - 1 < \mu - 1$ for all elements $b \in \text{Hilb}^{\text{int}}(S)$ with $\tau_3(b) < \mu$, and we have $\tau_3(\rho(b)) \geq \tau_3(b) - 1 \geq \mu - 1$ for all $b \in \text{Hilb}^{\text{int}}(S)$ with $\tau_3(b) \geq \mu$. Hence, either $\tau_3(a), \tau_3(a') < \mu$ or $\tau_3(a), \tau_3(a') \geq \mu$. In the first case, we have

$$\tau_3(a - e_3) = \tau_3(\pi(a - e_3)) = \tau_3(\rho(a)) = \tau_3(\rho(a')) = \tau_3(\pi(a' - e_3)) = \tau_3(a' - e_3).$$

Applying Lemma 2.8, we obtain $a - e_3 = a' - e_3$, and hence $a = a'$. In the second case, we have

$$\tau_1(a - e_1) = \tau_1(\pi(a - e_1)) = \tau_1(\pi(a' - e_1)) = \tau_1(a' - e_1)$$

by Lemma 2.6. Using Lemma 2.8 again, we obtain $a = a'$.

Case (ii): We may assume that a lies in $\text{Hilb}(S)$. Then $a' \notin \text{Hilb}(S)$, and hence there exist elements $b', c' \in S_+ \setminus S^{\text{int}}$ with $a' = b' + c'$.

Suppose that $\tau_3(a) < \mu$. Then $\tau_3(\rho(a)) = \tau_3(a) - 1$ by Lemma 2.6, and hence

$$\tau_3(a') \leq \tau_3(\rho(a')) + 1 = \tau_3(\rho(a)) + 1 = \tau_3(a) < \mu.$$

By definition of μ , this means that $a' \notin G'(S)$. So we have $\tau_3(\rho(a')) = \tau_3(a') - 1$, and thus $\tau_3(a) = \tau_3(a')$. Since $a' \notin a + T$ and $a \notin a' + T$, we have $\tau_i(a') < \tau_i(a)$ and $\tau_j(a') > \tau_j(a)$, where i is one of the indices 1, 2 and j is the other one.

By Lemma 2.5, the inequality $\tau_i(a') \leq \tau_i(a) - 1 \leq \tau_i(\rho(a)) = \tau_i(\rho(a'))$ implies that $\tau_j(\rho(a')) = \tau_j(a') - 1$ and $\tau_3(\rho(a')) = \tau_3(a') - 1$. Combining

$$\tau_i(b'), \tau_i(c') \leq \tau_i(a') < \tau_i(a) \text{ and } \tau_3(b'), \tau_3(c') \leq \tau_3(a') = \tau_3(a)$$

with the fact that $a \in \text{Hilb}(S)$, we obtain $\tau_j(b'), \tau_j(c') > \tau_j(a) > 0$.

Since $b', c' \notin S^{\text{int}}$ and $\tau_i(a') = \tau_i(b') + \tau_i(c')$, $\tau_3(a') = \tau_3(b') + \tau_3(c')$ are both positive, we see that either $\tau_3(b') = 0$ or $\tau_3(c') = 0$. We may assume that $\tau_3(b') = 0$. From

$$\begin{aligned} \tau_i(\rho(a')) &\geq \tau_i(a') \geq \tau_i(b'), \\ \tau_j(\rho(a')) &= \tau_j(a') - 1 = \tau_j(b') + \tau_j(c') - 1 \geq \tau_j(b'), \\ \tau_3(\rho(a')) &\geq 0 = \tau_3(b'), \end{aligned}$$

we get $\rho(a') \in b' + T$, which is impossible. This contradiction shows that the case $\tau_3(a) < \mu$ cannot occur, and we must have $\tau_3(a) \geq \mu$.

Let d be an element in $G'(S)$ with $\tau_3(d) = \mu$. Since $d \notin \text{Hilb}(S)$, there exist $b, c \in S_+ \setminus S^{\text{int}}$ with $d = b + c$. Lemma 2.7 yields that the numbers $\tau_3(b), \tau_3(c)$ are both positive, and that $\tau_i(b) = \tau_j(c) = 0$, where i is one of the indices 1, 2, and j is the other one. Since $d \in S_+$, we have $\tau_i(c) = \tau_i(d) > 0$ and $\tau_j(b) = \tau_j(d) > 0$. One easily derives that

$$\tau_1(g) \leq \tau_1(d) \text{ and } \tau_2(g) \leq \tau_2(d) \text{ for all } g \in G'(S) \text{ with } \tau_3(g) \geq \mu. \quad (\dagger)$$

Suppose that $a' \in G'(S)$. Applying Lemma 2.5 and Lemma 2.6, we obtain

$$\begin{aligned} \tau_1(a') &= \tau_1(\rho(a')) + 1 = \tau_1(\rho(a)) + 1 = \tau_1(a), \\ \tau_2(a') &= \tau_2(\rho(a')) + 1 = \tau_2(\rho(a)) + 1 \geq \tau_2(a). \end{aligned}$$

Combining this with $a' \notin a + T$, we get $\tau_3(a') < \tau_3(a)$. By Lemma 2.7, one of the numbers $\tau_2(b'), \tau_2(c')$ is zero, say $\tau_2(b') = 0$. But then

$$\begin{aligned}\tau_1(b') &\leq \tau_1(a') = \tau_1(a), \\ \tau_2(b') &= 0 < \tau_2(a), \\ \tau_3(b') &\leq \tau_3(a') < \tau_3(a),\end{aligned}$$

and thus $a \in b' + T_+$, which is a contradiction to $a \in \text{Hilb}(S)$. Hence a' cannot lie in $G'(S)$, and we have $\tau_3(a') = \tau_3(\rho(a')) + 1 = \tau_3(\rho(a)) + 1 \geq \tau_3(a) \geq \mu$.

Suppose that the numbers $\tau_3(b'), \tau_3(c')$ are both positive. Then $\tau_k(b') = 0$ and $\tau_l(c') = 0$, where k is one of the indices 1, 2, and l is the other one. Hence

$$(a' - e_1 - e_2 - e_3) + e_l \in b' + T \subseteq S_+ + T$$

and

$$(a' - e_1 - e_2 - e_3) + e_k \in c' + T \subseteq S_+ + T.$$

Combining this with $\tau_3(\rho(a')) = \tau_3(a') - 1$, we get $\rho(a') = a' - e_1 - e_2 - e_3$. But then $a' = \rho(a) + e_1 + e_2 + e_3 = \pi(a - e_1) + e_1 + e_2 + e_3 \in a + T$, which is impossible. Therefore, one of the numbers $\tau_3(b'), \tau_3(c')$ must be zero, say $\tau_3(b') = 0$.

Now consider the element $d' := d - b'$. Applying (†) to a' , we get

$$\begin{aligned}\tau_1(d') &= \tau_1(d - a' + c') \geq \tau_1(d - a') \geq 0, \\ \tau_2(d') &= \tau_2(d - a' + c') \geq \tau_2(d - a') \geq 0, \\ \tau_3(d') &= \tau_3(d) > 0.\end{aligned}$$

This shows that $d' \in S_+$. Note that $d \in d' + T_+$.

Suppose that $\tau_i(d') = \tau_i(d)$. (Remember that i is the index in $\{1, 2\}$ for which $\tau_i(b) = 0$, and j is the index in $\{1, 2\}$ for which $\tau_j(c) = 0$.) Then $\tau_j(d) > \tau_j(d')$ since $d \in d' + T_+$. From

$$\begin{aligned}\tau_i(d' - c) &= \tau_i(d - c) = \tau_i(b) = 0, \\ \tau_j(d' - c) &= \tau_j(d') \geq 0, \\ \tau_3(d' - c) &= \tau_3(d - c) = \tau_3(b) > 0,\end{aligned}$$

we obtain $d' - c \in S_+$, and from

$$\begin{aligned}\tau_i(d) &> 0 = \tau_i(d' - c), \\ \tau_j(d) &> \tau_j(d') = \tau_j(d' - c), \\ \tau_3(d) &= \tau_3(d') > \tau_3(d' - c),\end{aligned}$$

we obtain $d \in (d' - c) + S^{\text{int}}$, which is a contradiction to $d \in G(S)$. Thus, we must have $\tau_i(d') < \tau_i(d)$. Similarly, one shows $\tau_j(d') < \tau_j(d)$.

It follows that $(d - e_1 - e_2 - e_3) + e_3 \in d' + T \subseteq S_+ + T$, and thus $\tau_3(\rho(d))$ is equal to $\tau_3(d) - 1$. But this a contradiction to $d \in G'(S)$.

So we have come to the point where we can draw the conclusion that case (ii) cannot occur.

Case (iii): From Lemma 2.5 we deduce that there is an index $i \in \{1, 2, 3\}$ with $\tau_i(a - e_1 - e_2 - e_3) = \tau_i(\pi(a - e_1 - e_2 - e_3)) = \tau_i(\pi(a' - e_1 - e_2 - e_3)) = \tau_i(a' - e_1 - e_2 - e_3)$.

By Lemma 2.8, this implies that $a - e_1 - e_2 - e_3 = a' - e_1 - e_2 - e_3$, and so we get $a = a'$. \square

As an immediate consequence of Theorem 2.9, we obtain

Theorem 2.10. *Let S be a 3-dimensional simplicial normal affine semigroup, and let $R = K[S]$ be the associated semigroup ring over a field K . Then $r(R) \leq r(\overline{P})$, where \overline{P} denotes the fibre ring defined in the beginning of this section.*

We saw that in dimension 2, the equation $r(R) = r(\overline{P})$ only holds if R is regular. In dimension 3, the case $r(R) = r(\overline{P})$ can also occur, if R is not regular.

Example 2.11. Let $T = \mathbb{N}_0^3$, and let U be the subgroup of \mathbb{Z}^3 that consists of all elements (a_1, a_2, a_3) in \mathbb{Z}^3 with $2a_1 + a_2 + a_3 \equiv 0 \pmod{3}$. Consider the simplicial normal affine semigroup $S := T \cap U$. One easily verifies that

$$G(S) = \{(1, 3, 1), (1, 2, 2), (1, 1, 3), (2, 1, 1)\} \text{ and} \\ \overline{M}(S) = \{(0, 2, 0), (0, 1, 1), (0, 0, 2), (2, 0, 0)\}.$$

In particular, $r(R) = r(\overline{P}) = 4$.

However, in general we have $r(R) < r(\overline{P})$.

Example 2.12. Assume that $n \geq d$ and let S be the simplicial normal affine semigroup consisting of all elements $(a_1, \dots, a_d) \in \mathbb{Z}^d$ with

$$a_1, \dots, a_d \geq 0 \text{ and } a_1 + \dots + a_d \equiv 0 \pmod{n}.$$

Then $r(R) = \binom{n-1}{d-1}$ and $r(\overline{P}) = \binom{n+d-2}{d-1}$. In particular, $r(R) < r(\overline{P})$ if $d > 1$.

Proof. The set $\overline{M}(S)$ consists of all $a \in T$ with $\sum_{i=1}^d a_i = n - 1$, and therefore its cardinality is equal to $\binom{n+d-2}{d-1}$. Clearly, $a \in G(S)$ for all $a \in S^{\text{int}}$ with $\sum_{i=1}^d a_i = n$. Since $n \geq d$, the converse is also true: one has $\sum_{i=1}^d a_i = n$ for all $a \in G(S)$. This means that the map $\{a \in T \mid \sum_{i=1}^d a_i = n - d\} \rightarrow G(S), a \mapsto (a_1 + 1, \dots, a_d + 1)$, is bijective. Hence, $G(S)$ contains $\binom{n-1}{d-1}$ elements. \square

Even in dimension $d > 3$, we couldn't find any example of a simplicial normal affine semigroup ring R with $r(R) > r(\overline{P})$. So we ask

Question 2.13. Is it always true that $r(R) \leq r(\overline{P})$?

Also, we would like to know whether the statement of Theorem 2.10 holds in a more general, purely algebraic setting. For instance, consider the following context: let P be the polynomial ring $K[x_1, \dots, x_d]$ over a field K , and let R be a graded K -subalgebra of P , such that

- (i) P is a finite R -module.
- (ii) There exists an R -module homomorphism $\varphi : P \rightarrow R$ with $\varphi|_R = \text{id}_R$.

By a theorem of Hochster and Eagon, conditions (i) and (ii) imply that R is a Cohen-Macaulay ring, see e.g. [BH, 6.4.5]. Let \mathfrak{m} denote the graded maximal ideal of R , and let \overline{P} be the fibre ring $\overline{P} = P/\mathfrak{m}P$. Is it true that $r(R) \leq r(\overline{P})$ (at least in case $\dim R \leq 3$)?

3. ULRICH MODULES OF RANK ONE OVER DETERMINANTAL RINGS

Let R be a standard graded Cohen-Macaulay algebra over a field K , and let M be a graded maximal Cohen-Macaulay module over R . Then $\mu(M)$, the cardinality of a minimal system of homogeneous generators of M , is bounded above by $e(M)$, the multiplicity of M . In case $\mu(M) = e(M)$, one calls M an *Ulrich module*.

In this section, we prove the existence of Ulrich modules of rank one over certain determinantal rings. To be more precise, we consider determinantal rings of the form $R_{r+1}(X) = K[X]/I_{r+1}(X)$, where X is an $m \times n$ -matrix of indeterminates over the field K , and $I_{r+1}(X)$ denotes the ideal in $K[X]$ that is generated by the $(r+1)$ -minors of X . Then we define two prime ideals in $R_{r+1}(X)$, \mathfrak{p} and \mathfrak{q} , where \mathfrak{p} (resp. \mathfrak{q}) is generated by the residue classes of the r -minors of an $r \times n$ -submatrix (resp. $m \times r$ -submatrix) of X . We show that the powers \mathfrak{p}^{m-r} and \mathfrak{q}^{n-r} are both Ulrich modules over $R_{r+1}(X)$.

The most difficult part was to prove that \mathfrak{p}^{m-r} and \mathfrak{q}^{n-r} are Cohen-Macaulay modules over $R_{r+1}(X)$. In fact, the author tried for a considerable amount of time to solve this problem, but remained unsuccessful. He then received the decisive hint from Prof. W. Bruns, who explained a clever method that would possibly lead to the goal. Bruns suggested to use a certain deformation argument, that transfers the problem into the realm of affine semigroup rings and monomial ideals. Since the condition of Cohen-Macaulayness should be easier to handle in this context, it seemed to be a feasible way to achieve the desired result.

Indeed, using Bruns' advice, the author was finally able to complete this part of the proof.

3.1. Ulrich modules.

Let R be a standard graded algebra over a field K , and let $M \neq 0$ be a finitely generated graded R -module of dimension d . Furthermore, let $P_M(t) \in \mathbb{Q}[t]$ be the Hilbert polynomial of M . Then $P_M(t)$ has degree $d-1$, and $P_M(n)$ is equal to $\dim_K M_n$ for all $n \gg 0$. Recall that the multiplicity of M is defined as

$$e(M) = \begin{cases} (d-1)! \cdot [\text{leading coefficient of } P_M(t)], & \text{if } d > 0, \\ \text{length of } M, & \text{if } d = 0. \end{cases}$$

It is well-known that $e(M)$ is a positive integer, see e.g. section 4.1 of [BH]. For explicit computations, the following result is very useful.

Proposition 3.1. *If $y \in R$ is a homogeneous M -regular element of degree one, then $e(M)$ is equal to $e(M/yM)$.*

Proof. We set $\overline{M} = M/yM$. Since y is M -regular, we have $\dim(M/yM) = d-1$. If $d = 1$, then there exists an $n_0 \in \mathbb{Z}$ such that $e(M) = \dim_K M_n = \dim_K M_{n_0}$ for all

$n \geq n_0$. Noting that $\dim_K(\overline{M})_n = \dim_K M_n - \dim_K M_{n-1}$ for all $n \in \mathbb{Z}$, we obtain

$$e(\overline{M}) = \sum_{n \in \mathbb{Z}} \dim_K(\overline{M})_n = \sum_{n \in \mathbb{Z}} (\dim_K M_n - \dim_K M_{n-1}) = \dim_K M_{n_0} = e(M).$$

Now assume $d > 1$. From the exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{y} M \longrightarrow \overline{M} \longrightarrow 0$$

we obtain $P_{\overline{M}}(t) = P_M(t) - P_{M(-1)}(t) = P_M(t) - P_M(t-1)$. Since

$$P_M(t) - P_M(t-1) = (d-1) \left(\frac{e(M)}{(d-1)!} t^{d-2} \right) + \text{terms of lower degree},$$

we conclude that $e(\overline{M}) = e(M)$. \square

Let \mathfrak{m} denote the graded maximal ideal $\bigoplus_{n>0} R_n$. Then $\dim M_{\mathfrak{m}} = \dim M$, and depth $M_{\mathfrak{m}}$ is equal to the maximal length of an M -regular sequence contained in \mathfrak{m} . In case K is infinite, even R_1 contains an M -regular sequence of length depth $M_{\mathfrak{m}}$. (See section 1.5 of [BH] for proofs to these statements.) We simply write depth M instead of depth $M_{\mathfrak{m}}$. It is well-known that M is a Cohen-Macaulay module over R (which by definition means that $M_{\mathfrak{p}}$ is a Cohen-Macaulay module over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec}(R)$) if and only depth $M = \dim M$. In case depth $M = \dim R$, M is called a *maximal* Cohen-Macaulay module.

Note that every minimal system of homogeneous generators of M has the same cardinality, namely $\dim_K(M/\mathfrak{m}M) = \mu(M_{\mathfrak{m}})$. We set $\mu(M) := \mu(M_{\mathfrak{m}})$.

From Theorem 3.1, one derives the following two standard results concerning the multiplicity.

Proposition 3.2. *If R is a standard graded Cohen-Macaulay algebra over a field K , then $e(R) \geq \text{emb dim } R - \dim R + 1$.*

Proof. For any field extension L of K , the ring $R \otimes_K L$ is a standard graded Cohen-Macaulay algebra over L , and has the same dimension, embedding dimension, and multiplicity as R . Hence we may assume that K is infinite. Then we can choose elements y_1, \dots, y_d in R_1 which form a regular system of parameters of R . Let R' denote the residue class ring $R/(y_1, \dots, y_d)$. We have $e(R) = e(R') = \ell(R') = \dim_K R' \geq \dim_K R'_0 + \dim_K R'_1 = 1 + \text{emb dim } R' = 1 + \text{emb dim } R - \dim R$. \square

If $e(R) = \text{emb dim } R - \dim R + 1$, one says that R has *minimal multiplicity*.

Proposition 3.3. *Let R be a standard graded algebra over a field K , and let M be a Cohen-Macaulay module over R . Then $\mu(M) \leq e(M)$.*

Proof. Just as in the proof of Proposition 3.2, we may assume that K is infinite. Then there exists an M -regular sequence y_1, \dots, y_n in R_1 such that the module $M' := M/(y_1, \dots, y_n)M$ has finite length. If \mathfrak{m} denotes the graded maximal ideal of R , we have $e(M) = e(M') = \ell(M') \geq \ell(M/\mathfrak{m}M) = \mu(M)$. \square

If (A, \mathfrak{n}) is a local Noetherian ring and M a finitely generated A -module, then its multiplicity $e(M)$ is defined to be the multiplicity of the $\text{gr}_{\mathfrak{n}}(A)$ -module

$$\text{gr}_{\mathfrak{n}}(M) = \bigoplus_{n \geq 0} (\mathfrak{n}^n M / \mathfrak{n}^{n+1} M).$$

We remark that Proposition 3.2 and Proposition 3.3 remain valid, if one replaces ‘standard graded algebra over a field K ’ by ‘local Noetherian ring’ (see sections 4.6 and 4.7 of [BH]).

Let R be again a standard graded K -algebra, and set $d = \dim R$. Assume that R is a domain, and let S be the set of all homogeneous elements in $R \setminus \{0\}$. Let M be a finitely generated graded R -module. Since any nonzero element in $S^{-1}R$ is invertible, there exist homogeneous elements $z_1, \dots, z_r \in S^{-1}M$ which form an $S^{-1}R$ -basis of $S^{-1}M$. The number r is called the *rank* of M over R .

Now assume that M is torsionfree over R . Then $\dim M = d$. We choose $f \in S$ such that fM is contained in $F := Rz_1 + \dots + Rz_r$, and set $N := F/fM$. Note that $e(F) = e(R)r$, because z_1, \dots, z_r is an R -basis of F . Since M is torsionfree, fM is (up to a shift) isomorphic to M , and thus $e(fM) = e(M)$. The Hilbert polynomial $P_{fM}(t)$ is equal to $P_F(t) - P_N(t)$. Since $S^{-1}N = 0$, we have $\dim N < d$. Therefore, the leading coefficients of the polynomials $P_{fM}(t)$ and $P_F(t)$ coincide, and we get $e(M) = e(fM) = e(F) = e(R)r$.

This observation leads to

Corollary 3.4. *Let R and M be as in Proposition 3.3. Assume furthermore that R is a domain and that $\dim M = \dim R$. Then $\mu(M) \leq e(R)r$, where r is the rank of M over R . In particular, $\mu(M) \leq e(R)$ in case that M has rank one over R .*

Proof. Since $\dim M = \dim R$, M is torsionfree over R , and hence $e(M) = e(R)r$. Now the assertion follows from Proposition 3.3. \square

Let R be a standard graded Cohen-Macaulay algebra over a field K . A graded maximal Cohen-Macaulay module M over R is called an *Ulrich module*, if

$$e(M) = \mu(M).$$

(In case that R is a local Cohen-Macaulay ring, the definition is analogous.) The name ‘Ulrich module’ honors the mathematician Bernd Ulrich. In his paper [U], Ulrich has posed the following question: let R be a local Cohen-Macaulay ring with positive dimension and infinite residue field. Does there exist a Cohen-Macaulay module M over R that has positive rank and satisfies $\mu(M) = e(R) \text{rank}(M)$? (Here ‘positive rank’ means that $M \otimes_R Q$ is a nonzero free Q -module, where Q denotes the total ring of fractions of R .)

Until today, one is far way from having an answer to Ulrich’s question. In fact, during the last twenty years, only a few results on the existence of Ulrich modules could be achieved. The following list contains a selection.

Let R be a standard graded Cohen-Macaulay algebra over a field K , and let \mathfrak{m} be the graded maximal ideal of R .

- (a) If $\dim R = 0$, then $R/\mathfrak{m} = K$ is an Ulrich module over R .
- (b) If $\dim R = 1$, then $\mathfrak{m}^{e(R)-1}$ is an Ulrich module over R .
- (c) If R is a 2-dimensional domain and K is infinite, then R admits an Ulrich module of rank 2.
- (d) If $\dim R > 0$ and R has minimal multiplicity, then the i -th syzygy module of $R/\mathfrak{m} = K$ is an Ulrich module for all $i \geq d$.
- (e) If $R = P/(f_1, \dots, f_r)$, where f_1, \dots, f_r are homogeneous polynomials that form a regular sequence in the polynomial ring $P = K[x_1, \dots, x_n]$, then R possesses an Ulrich module.
- (f) If R is the d -th Veronese subring of the polynomial ring $K[x_1, \dots, x_n]$ over a field K of characteristic zero, then there exists an Ulrich module of rank $d^{\binom{n+1}{2}}$ over R .

Proofs for (a), (b), (c), and (d) can be found in [BHU]. For (e), see [HUB], and for (e), see [ESW].

If all generators of M have the same degree, then the condition of being an Ulrich module can be expressed via the graded Betti numbers of M .

Theorem 3.5. *Let $R = P/I$, where $P = K[x_1, \dots, x_n]$ is the polynomial ring over an infinite field K , and $I \subset P$ is a graded ideal. Let M be a graded maximal Cohen-Macaulay module over R , and assume that all generators of M have the same degree, say m . If*

$$\dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} P(-j)^{b_{ij}} \longrightarrow \dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} P(-j)^{b_{1j}} \longrightarrow \bigoplus_{j \in \mathbb{Z}} P(-j)^{b_{0j}} \longrightarrow M \longrightarrow 0$$

denotes the minimal graded resolution of M over P , then M is an Ulrich module over R if and only if the resolution is linear, that is, if $b_{ij} = 0$ whenever $j \neq m + i$.

For a proof, see [BHU, 1.5].

3.2. Determinantal rings.

Let $X_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, be indeterminates over a field K , and write X for the $m \times n$ -matrix (X_{ij}) . Let $K[X]$ denote the polynomial ring

$$K[X_{ij}, 1 \leq i \leq m, 1 \leq j \leq n].$$

A *minor* of X is the determinant of a nonempty quadratic submatrix of X . We write $\Delta(X)$ for the set of all minors of X . A *monomial on $\Delta(X)$* is a product of elements $\delta_1, \dots, \delta_k \in \Delta(X)$, where $k \geq 0$. By convention, the empty product is equal to 1.

Now fix an integer t with $1 \leq t \leq \min\{m, n\}$, and let a_1, \dots, a_t and b_1, \dots, b_t be two sequences of integers satisfying

$$1 \leq a_1 < \dots < a_t \leq m \text{ and } 1 \leq b_1 < \dots < b_t \leq n.$$

The determinant of the $t \times t$ -submatrix $(X_{a_i b_j})_{1 \leq i, j \leq t}$ of X is denoted by

$$[a_1, \dots, a_t \mid b_1, \dots, b_t].$$

One calls $[a_1, \dots, a_t \mid b_1, \dots, b_t]$ a t -minor of X . On the set $\Delta(X)$, one defines the following partial order:

$$[a_1, \dots, a_t \mid b_1, \dots, b_t] \preceq [c_1, \dots, c_u \mid d_1, \dots, d_u]$$

if and only if

$$t \geq u \text{ and } a_i \leq c_i, b_i \leq d_i \text{ for } i = 1, \dots, u.$$

Note that $\Delta(X)$ is not totally ordered with respect to \preceq unless $m = 1$ or $n = 1$. However, $\Delta(X)$ possesses a greatest element and a smallest element with respect to \preceq , namely $[m, n]$ resp. $[1, \dots, \min\{m, n\} \mid 1, \dots, \min\{m, n\}]$.

If $\delta_1, \dots, \delta_k$ are minors of X with $\delta_1 \preceq \dots \preceq \delta_k$, then the product $\delta_1 \cdots \delta_k$ is called a *standard monomial on $\Delta(X)$* . Note that the element 1 is also a standard monomial on $\Delta(X)$. By a famous theorem of Doubilet, Rota, and Stein (see e.g. [BH, 7.2.7]), the standard monomials on $\Delta(X)$ form a K -basis of $K[X]$. Moreover, $K[X]$ is a graded *algebra with straightening law (for short: ASL) on $\Delta(X)$* . See section 4 of [BV] or [BH, 7.1] for this notion.

For any element $\delta \in \Delta(X)$, the set

$$\{\xi \in \Delta(X) \mid \delta \not\preceq \xi\}$$

is called the *ideal cogenerated by δ in $\Delta(X)$* . The complementary set

$$\{\xi \in \Delta(X) \mid \delta \preceq \xi\}$$

is denoted by $\Delta(X; \delta)$. One defines monomials and standard monomials on $\Delta(X; \delta)$ in an obvious way. The ideal in $K[X]$ which is generated by all elements $\xi \in \Delta(X)$ with $\delta \not\preceq \xi$, is denoted by $I(X; \delta)$, and the residue class ring

$$R(X; \delta) := K[X]/I(X; \delta)$$

is called a *determinantal ring*. The residue classes of the elements X_{ij} in $R(X; \delta)$ are denoted by x_{ij} .

Using some basic facts on ASLs (see e.g. [BV, 5.A]), one obtains the following generalization of the theorem mentioned above:

Theorem 3.6. *Let X be an $m \times n$ -matrix of indeterminates over a field K , and let δ be a minor of X . Then the residue classes of the standard monomials on $\Delta(X; \delta)$ form a K -basis of the determinantal ring $R(X; \delta)$.*

In case that $\delta = [1, \dots, r \mid 1, \dots, r]$ for some r with $1 \leq r \leq \min\{m, n\}$, the ideal cogenerated by δ in $\Delta(X)$ consists of all t -minors of X with $t \geq r + 1$. In this case, one writes $\Delta_{r+1}(X)$ for $\Delta(X; \delta)$, $I_{r+1}(X)$ for $I(X; \delta)$, and $R_{r+1}(X)$ for $R(X; \delta)$. Note that $R_{r+1}(X)$ is equal to $K[X]$, if $r = \min\{m, n\}$. Using Laplace's expansion formula, one sees that $I_{r+1}(X)$ is generated by the $(r + 1)$ -minors of X .

In the sequel, we will consider determinantal rings of the form $R_{r+1}(X)$. The algebraic structure of these rings is well understood. It is known that $R_{r+1}(X)$ is a normal Cohen-Macaulay domain of dimension $(m + n - r)r$, and its divisor class group $\text{Cl}(R_{r+1}(X))$ is isomorphic to \mathbb{Z} . For proofs, see e.g. [BH, 7.3].

3.3. The existence of Ulrich modules of rank one.

In their paper [BHU], Brennan, Herzog, and Ulrich proved the following

Theorem 3.7. *Let $P = K[x_1, \dots, x_s]$ be the polynomial ring over a field K , let m, n be positive integers with $m \leq n$, and let $A = (f_{ij})$ be a matrix whose entries $f_{ij} \in P$ are homogeneous polynomials of degree one. Let I denote the ideal in P which is generated by the m -minors of A , and assume that I is a prime ideal of height $n - m + 1$. Then the residue class ring $R = P/I$ possesses an Ulrich module of rank one.*

As an immediate consequence, we obtain

Corollary 3.8. *Let X be an $m \times n$ -matrix of indeterminates over a field K . For $r + 1 = \min\{m, n\}$, the determinantal ring $R_{r+1}(X) = K[X]/I_{r+1}(X)$ possesses an Ulrich module of rank one.*

Proof. We may assume that $r + 1 = m \leq n$. As mentioned in the last subsection, $R_{r+1}(X)$ is an integral domain of dimension $(m + n - r)r = (n + 1)(m - 1)$. Thus, $I_{r+1}(X)$ is a prime ideal, and

$$\begin{aligned} \text{height } I_{r+1}(X) &= \dim K[X] - \dim R_{r+1}(X) \\ &= mn - (n + 1)(m - 1) \\ &= n - m + 1. \end{aligned}$$

Now, the assertion follows from Theorem 3.7. □

Our aim is to show that *any* determinantal ring of the form $R_{r+1}(X)$ possesses an Ulrich module of rank one.

The first question that has to be solved is where to look for such a module. So assume for the moment that M is an Ulrich module of rank one over $R_{r+1}(X)$. Then M is a torsionfree $R_{r+1}(X)$ -module of rank one, and hence it is isomorphic to a fractionary ideal I of $R_{r+1}(X)$. Moreover, since $R_{r+1}(X)$ is normal and M is Cohen-Macaulay, M is reflexive (see e.g. [BH, 1.4.1]), so that I must be a divisorial fractionary ideal.

This shows that in our search for Ulrich modules of rank one, we have to consider only divisorial fractionary ideals of $R_{r+1}(X)$. In fact, it suffices to look at a family

$I_n, n \in \mathbb{Z}$, of divisorial fractionary ideals of $R_{r+1}(X)$, which form a system of representatives for the divisor class group. Fortunately, there exists such a family, which may be described in a very easy and elegant way. Let \mathfrak{p} be the image of the ideal

$$([1, \dots, r \mid b_1, \dots, b_r]_{1 \leq b_1 < \dots < b_r \leq n})$$

in $R_{r+1}(X)$, and let \mathfrak{q} be the image of the ideal

$$([a_1, \dots, a_r \mid 1, \dots, r]_{1 \leq a_1 < \dots < a_r \leq m})$$

in $R_{r+1}(X)$. Note that

$$\mathfrak{p} = I(X; \delta) / I_{r+1}(X) \text{ for } \delta = [1, \dots, r-1, r+1 \mid 1, \dots, r],$$

and

$$\mathfrak{q} = I(X; \delta) / I_{r+1}(X) \text{ for } \delta = [1, \dots, r \mid 1, \dots, r-1, r+1].$$

The ideals \mathfrak{p} and \mathfrak{q} are prime ideals of height one and hence divisorial. The divisor classes $[\mathfrak{p}]$ and $[\mathfrak{q}]$ are inverse to each other, and each of them generates the divisor class group $\text{Cl}(R_{r+1}(X))$. (For proofs, see [BH, 7.3.5].) Moreover, the symbolic powers of \mathfrak{p} and \mathfrak{q} coincide with the ordinary ones, that is, $\mathfrak{p}^{(t)} = \mathfrak{p}^t$ and $\mathfrak{q}^{(t)} = \mathfrak{q}^t$ for all $t \geq 1$ (see [BV, 9.18]). Hence, the ideals $\mathfrak{p}^t, \mathfrak{q}^t, t \geq 1$, form a system of representatives for the nontrivial classes of $\text{Cl}(R_{r+1}(X))$. So, in order to prove the existence of an Ulrich module of rank one over $R_{r+1}(X)$, we have to show that there exists an integer $t \geq 1$ such that

$$R_{r+1}(X) / \mathfrak{p}^t \text{ is Cohen-Macaulay and } \mu(\mathfrak{p}^t) = e(R_{r+1}(X))$$

or

$$R_{r+1}(X) / \mathfrak{q}^t \text{ is Cohen-Macaulay and } \mu(\mathfrak{q}^t) = e(R_{r+1}(X)).$$

Here we used the fact that if $I \subset R_{r+1}(X)$ is a graded ideal of height one, then I is a (maximal) Cohen-Macaulay module over $R_{r+1}(X)$ if and only if the residue class ring $R_{r+1}(X) / I$ is Cohen-Macaulay.

First, we determine the minimal number of generators for the powers of \mathfrak{p} and \mathfrak{q} .

Lemma 3.9. *For any integer $t \geq 1$ the number $\mu(\mathfrak{p}^t)$ is equal to the determinant of the matrix*

$$\left[\binom{t+n-j}{n-i} \right]_{1 \leq i, j \leq r}$$

and the number $\mu(\mathfrak{q}^t)$ is equal to the determinant of the matrix

$$\left[\binom{t+m-j}{m-i} \right]_{1 \leq i, j \leq r}.$$

Proof. By symmetry, it is enough to prove the assertion for \mathfrak{p}^t . Since \mathfrak{p} is generated by elements of degree r in $R_{r+1}(X)$, the number $\mu(\mathfrak{p}^t)$ is equal to the dimension of the homogeneous component $(\mathfrak{p}^t)_{tr}$ as a K -vector space.

We consider the $r \times n$ -submatrix X' of X that consists of the first r rows of X . Let $G(X')$ be the graded K -subalgebra of $K[X']$ generated by the r -minors of X' .

Since $G(X')$ is the coordinate ring of the Grassmannian variety of r -dimensional subspaces of K^n (see [BV, 1.2]), it has been studied extensively. In particular, the Hilbert series of $G(X')$ is well known: for all $t \geq 0$, one has

$$\dim_K G(X')_{tr} = \left[\binom{t+n-j}{n-i} \right]_{1 \leq i, j \leq r},$$

see e.g. [Gh, Theorem 6].

Therefore, it remains to show that $\dim_K G(X')_{tr} = \dim_K (\mathfrak{p}^t)_{tr}$. So consider the composition of natural maps $G(X') \hookrightarrow K[X'] \hookrightarrow K[X] \rightarrow R_{r+1}(X)$. Clearly, $G(X')_{tr}$ is mapped surjectively to $(\mathfrak{p}^t)_{tr}$. It is known that $G(X')_{tr}$ is generated as a K -vectorspace by all standard monomials $\delta_1 \cdots \delta_t$ on $\Delta(X')$ whose factors $\delta_1 \cdots \delta_t$ are r -minors of X' (see [BV, 9.3]). By Theorem 3.6, the images of these elements are K -linear independent in $R_{r+1}(X)$, and hence we obtain our assertion. \square

Next we show:

Lemma 3.10. *The multiplicity of $R_{r+1}(X)$ coincides with $\mu(\mathfrak{p}^{m-r})$ and $\mu(\mathfrak{q}^{n-r})$.*

Proof. The multiplicity of $R_{r+1}(X)$ is known to be the determinant of the matrix

$$B = \left[\binom{m+n-i-j}{n-j} \right]_{1 \leq i, j \leq r},$$

see e.g. [HT]. By Lemma 3.9, $\mu(\mathfrak{p}^{m-r})$ is equal to the determinant of the matrix

$$A = \left[\binom{m+n-r-j}{n-i} \right]_{1 \leq i, j \leq r}.$$

Using the binomial identity $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$, one can transform A into the transpose of B by elementary row operations, which do not affect the determinant. This proves $\mu(\mathfrak{p}^{m-r}) = e(R_{r+1}(X))$. The equation $\mu(\mathfrak{q}^{n-r}) = e(R_{r+1}(X))$ can be obtained analogously. \square

In the proof of Lemma 3.9 we saw that \mathfrak{p}^t is generated by the residue classes of all standard monomials on $\Delta(X)$ which have the form $\delta_1 \cdots \delta_t$, where $\delta_1, \dots, \delta_t$ are r -minors of an $r \times n$ -submatrix of X . From this description one sees that $\mu(\mathfrak{p}^t)$ is strictly increasing as a function of t , except in the trivial case $r = \min\{m, n\}$. The same holds for $\mu(\mathfrak{q}^t)$.

Hence, the only candidates for an Ulrich module of rank one over $R_{r+1}(X)$ are \mathfrak{p}^{m-r} and \mathfrak{q}^{n-r} . The decisive question is now: are $R_{r+1}(X)/\mathfrak{p}^{m-r}$ and $R_{r+1}(X)/\mathfrak{q}^{n-r}$ Cohen-Macaulay? Theorem 3.12 will give an affirmative answer to this question. Moreover, we will show that $R_{r+1}(X)/\mathfrak{p}^t$ (resp. $R_{r+1}(X)/\mathfrak{q}^t$) is Cohen-Macaulay if and only if $t \leq m-r$ (resp. $t \leq n-r$).

But since we are not able to compute the depth of $R_{r+1}(X)/\mathfrak{p}^t$ (or $R_{r+1}(X)/\mathfrak{q}^t$) directly, we have to make a detour: using the concept of initial algebras, we will transfer the problem to the realm of affine semigroups and solve it there. Thus, before presenting the theorem, we give a short outline of initial algebras in which the required facts are summarized.

Let A denote the polynomial ring $K[x_1, \dots, x_n]$ over a field K , and let $<_\tau$ be a monomial order on A . (See e.g. [Ei, 15.2] for the notion of monomial orders.) Every nonzero polynomial $f \in A$ has a unique presentation $f = \alpha_1 m_1 + \dots + \alpha_r m_r$, where $\alpha_1, \dots, \alpha_r \in K^\times$ and m_1, \dots, m_r are monomials with $m_1 <_\tau \dots <_\tau m_r$. The term $\alpha_r m_r$ is called the *initial term of f* with respect to τ and is denoted by $\text{in}_\tau(f)$.

For all $f, g \neq 0$, one has

$$\text{in}_\tau(fg) = \text{in}_\tau(f) \text{in}_\tau(g) \text{ and } \text{in}_\tau(f + g) \leq_\tau \max\{\text{in}_\tau(f), \text{in}_\tau(g)\}.$$

In case that $\text{in}_\tau(f) \neq \text{in}_\tau(g)$, one has $\text{in}_\tau(f + g) = \max\{\text{in}_\tau(f), \text{in}_\tau(g)\}$. If $V \neq 0$ is a K -subspace of A , then $\text{in}_\tau(V)$ denotes the K -subspace of A generated by the elements $\text{in}_\tau(f), f \in V$. In case that V is a graded K -subspace of A , $\text{in}_\tau(V)$ is also a graded K -subspace of A and one has $\dim_K V_i = \dim_K (\text{in}_\tau(V))_i$ for all $i \geq 0$, see [BC, 3.4].

Assume that B is a K -subalgebra of A and I is an ideal of B . Then $\text{in}_\tau(B)$ is a K -subalgebra of A and $\text{in}_\tau(I)$ is an ideal of $\text{in}_\tau(B)$. But even if B is finitely generated over K , $\text{in}_\tau(B)$ need not be finitely generated over K .

The following result will be important for us.

Theorem 3.11. *Let B be a K -subalgebra of A , and let I be an ideal of B . Assume that $\text{in}_\tau(B)$ is finitely generated over K . If $\text{in}_\tau(B)/\text{in}_\tau(I)$ is Cohen-Macaulay, then B/I is Cohen-Macaulay, too.*

The proof of Theorem 3.11 is based on a deformation argument. One constructs a $K[t]$ -graded algebra C , such that C is flat over $k[t]$, $C/(t) \cong \text{in}_\tau(B)/\text{in}_\tau(I)$, and $C/(t - \alpha) \cong B/I$ for $\alpha \in K^\times$. For details, see [BC, 3.16].

After this excursion, we come to the promised

Theorem 3.12. *Let r be an integer with $1 \leq r < \min\{m, n\}$. Then $R_{r+1}(X)/\mathfrak{p}^t$ (resp. $R_{r+1}(X)/\mathfrak{q}^t$) is Cohen-Macaulay if and only if $t \leq m - r$ (resp. $t \leq n - r$).*

Proof. By symmetry, we only have to prove the statement concerning $R_{r+1}(X)/\mathfrak{p}^t$. As already mentioned, $\mu(\mathfrak{p}^t) > \mu(\mathfrak{p}^{m-r}) = e(R_{r+1}(X))$ for $t > m - r$. Therefore, Corollary 3.4 implies that $R_{r+1}(X)/\mathfrak{p}^t$ cannot be Cohen-Macaulay for $t > m - r$.

It remains to show that $R_{r+1}(X)/\mathfrak{p}^t$ is Cohen-Macaulay for $t \leq m - r$. Before we do so, let us first sketch the strategy:

(1) Embed the determinantal ring $R_{r+1}(X)$ into the polynomial ring $K[Y, Z]$, where Y is an $m \times r$ -matrix of indeterminates and Z is an $r \times n$ -matrix of indeterminates:

$$\varphi : R_{r+1}(X) \hookrightarrow K[Y, Z], \quad x_{ij} \mapsto (YZ)_{ij}.$$

(Note that the product matrix YZ is an $m \times n$ matrix.)

(2) Define a ‘suitable’ monomial order $<_\tau$ on $K[Y, Z]$. Set $D = \text{in}_\tau(\varphi(R_{r+1}(X)))$ and $\mathfrak{a}_t = \text{in}_\tau(\varphi(\mathfrak{p}^t))$ for $t \geq 1$. Show that $D = K[E]$, where $E \subseteq \mathbb{Z}^{mr} \oplus \mathbb{Z}^{rn}$ is a positive normal affine semigroup. Furthermore, show that \mathfrak{a}_t is a monomial ideal that is associated to a semigroup ideal E_t of E .

(3) Now assume that $t \leq m - r$. Prove that D/\mathfrak{a}_t is Cohen-Macaulay by showing that \mathfrak{a}_t is a conic ideal of D .

Finally, Theorem 3.11 yields that $R_{r+1}(X)/\mathfrak{p}^t$ is Cohen-Macaulay.

So much for the strategy. The real work starts now:

(1) Let Y be an $m \times r$ -matrix of indeterminates Y_{ij} over K , and let Z be an $r \times n$ -matrix of indeterminates Z_{kl} over K . Furthermore, let $K[Y, Z]$ denote the polynomial ring

$$K[Y_{ij}, Z_{kl}, 1 \leq i \leq m, 1 \leq j, k \leq r, 1 \leq l \leq n].$$

We consider the K -algebra homomorphism

$$\Phi : K[X] \rightarrow K[Y, Z], \quad X_{ij} \mapsto (YZ)_{ij},$$

where $(YZ)_{ij}$ denotes the (i, j) -th element of the product matrix YZ . Since YZ has rank r , Φ induces a K -algebra homomorphism

$$\varphi : R_{r+1}(X) \rightarrow K[Y, Z], \quad x_{ij} \mapsto (YZ)_{ij}.$$

The map φ is known to be an embedding, see e.g. [BV, 7.2].

(2) We introduce a monomial order $<_\tau$ on $K[Y, Z]$. For simplicity, we write $<$ instead of $<_\tau$ and $\text{in}(-)$ instead of $\text{in}_\tau(-)$. First, we order the variables of $K[Y, Z]$ in the following way:

$$Y_{m1} > Y_{m-1,1} > \cdots > Y_{11} > Y_{m2} > \cdots > Y_{1r} > Z_{1n} > \cdots > Z_{11} > Z_{2n} > \cdots > Z_{r1}.$$

In words: list the entries of Y column by column from bottom to top, starting with the first column, and then list the entries of Z row by row from right to left, starting with the first row. The announced monomial order $<$ is obtained by extending this order of the variables to the induced reverse lexicographic order on $K[Y, Z]$. (For the definition of the reverse lexicographic order, see [Ei, 15.2].) We set

$$D = \text{in}(\varphi(R_{r+1}(X))) \text{ and } \mathfrak{a}_t = \text{in}(\varphi(\mathfrak{p}^t)) \text{ for } t \geq 1.$$

Since the residue classes of the standard monomials on $\Delta_{r+1}(X)$ form a K -basis of $R_{r+1}(X)$, we want to compute the initial terms $\text{in}(\Phi(\delta)), \delta \in \Delta_{r+1}(X)$.

So choose an integer t with $1 \leq t \leq r$, and let $\delta = [a_1, \dots, a_t \mid b_1, \dots, b_t]$ be any t -minor of X . Then $\delta = \det(X')$, where X' denotes the $t \times t$ -matrix $(X_{a_i, b_j})_{1 \leq i, j \leq t}$. Note that Φ maps X' to $Y'Z'$, where Y' is the $t \times r$ -submatrix $(Y_{a_i, j})_{1 \leq i \leq t, 1 \leq j \leq r}$ of Y , and Z' is the $r \times t$ -submatrix $(Z_{i, b_j})_{1 \leq i \leq r, 1 \leq j \leq t}$ of Z . Let Y'' (resp. Z'') be the $t \times t$ -matrix that consists of the first t columns of Y' (resp. first t rows of Z').

We want to show that $\text{in}(\det(Y''))$ is equal to the product of the entries on the main diagonal of Y'' , that is, $\text{in}(\det(Y'')) = Y_{a_1, 1} \cdots Y_{a_t, t}$. For this, we have to prove that $Y_{a_1, 1} \cdots Y_{a_t, t} > Y_{a_{\pi(1)}, 1} \cdots Y_{a_{\pi(t)}, t}$ for every permutation $\pi \in \mathfrak{S}_t$ with $\pi \neq \text{id}$.

So let $\pi \in \mathfrak{S}_t$ be a nontrivial permutation, and set $k := \max\{i \mid \pi(i) \neq i\}$. Then $\pi(k) < k$, and hence $a_{\pi(k)} < a_k$. Recalling the order of the variables introduced above, we obtain

$$Y_{a_i, i} > Y_{a_{\pi(i)}, i} \text{ for } i = 1, \dots, k, \text{ and } Y_{a_i, i} = Y_{a_{\pi(i)}, i} \text{ for } i = k + 1, \dots, t.$$

Therefore, $Y_{a_{1,1}} \cdots Y_{a_{t,t}} > Y_{a_{\pi(1),1}} \cdots Y_{a_{\pi(t),t}}$ follows from the definition of the degree reverse lexicographic order. So we get

$$\text{in}(\det(Y'')) = \prod_{i=1}^t Y_{a_i,i},$$

as desired. Analogously, one shows that

$$\text{in}(\det(Z'')) = \prod_{i=1}^t Z_{i,b_i}.$$

We have

$$\begin{aligned} \Phi(\delta) &= \det(Y'Z') \\ &= \sum_{\pi \in \mathfrak{S}_t} \prod_{i=1}^t (Y'Z')_{i,\pi(i)} \\ &= \sum_{\pi \in \mathfrak{S}_t} \prod_{i=1}^t \left(\sum_{j=1}^r Y_{a_i,j} Z_{j,b_{\pi(i)}} \right) \\ &= \left(\sum_{\pi \in \mathfrak{S}_t} \prod_{i=1}^t \left(\sum_{j=1}^t Y_{a_i,j} Z_{j,b_{\pi(i)}} \right) \right) + f \\ &= \left(\sum_{\pi \in \mathfrak{S}_t} \prod_{i=1}^t (Y''Z'')_{i,\pi(i)} \right) + f \\ &= \det(Y''Z'') + f \\ &= \det(Y'') \det(Z'') + f, \end{aligned}$$

where $f \in K[Y, Z]$ is a polynomial with the property that all monomials occurring in the expansion of f contain at least one variable Z_{ij} with $i > t$. This means that $\text{in}(f) < \prod_{i=1}^t Y_{a_i,i} Z_{i,b_i} = \text{in}(\det(Y'')) \text{in}(\det(Z'')) = \text{in}(\det(Y'') \det(Z''))$, and so we get

$$\text{in}(\Phi(\delta)) = \prod_{i=1}^t Y_{a_i,i} Z_{i,b_i}. \quad (*)$$

Now assume that ζ is a standard monomial on $\Delta_{r+1}(X)$. Then ζ is equal to a product of elements $\delta_i = [a_{i1}, \dots, a_{it_i} \mid b_{i1}, \dots, b_{it_i}] \in \Delta_{r+1}(X)$, $i = 1, \dots, k$, such that $\delta_1 \preceq \dots \preceq \delta_k$. From the definition of \preceq , it follows that $r \geq t_1 \geq \dots \geq t_k \geq 1$ and $a_{ij} \leq a_{lj}$ for all $i, l \in \{1, \dots, k\}$ with $i < l$ and all $j \in \{1, \dots, t_l\}$. From (*) one deduces that

$$\text{in}(\Phi(\zeta)) = \prod_{i=1}^k \prod_{j=1}^{t_i} Y_{a_{ij},j} Z_{j,b_{ij}}. \quad (**)$$

Note that ζ can be recovered from the monomial $g := \text{in}(\Phi(\zeta))$. One has

$$\begin{aligned} t_1 &= \max\{j \mid Y_{ij} \text{ divides } g \text{ for some } i \in \{1, \dots, m\}\} \\ &= \max\{i \mid Z_{ij} \text{ divides } g \text{ for some } j \in \{1, \dots, n\}\}, \\ a_{1j} &= \min\{l \mid Y_{lj} \text{ divides } g\}, j = 1, \dots, t_1, \\ b_{1j} &= \min\{l \mid Z_{jl} \text{ divides } g\}, j = 1, \dots, t_1. \end{aligned}$$

Thus, one obtains δ_1 . Set $g' = g / (\prod_{i=1}^{t_1} Y_{a_{1j},j} Z_{j,b_{1j}})$. If $g' = 1$, then ζ is equal to δ_1 . Otherwise, one has

$$\begin{aligned} t_2 &= \max\{j \mid Y_{ij} \text{ divides } g' \text{ for some } i \in \{1, \dots, m\}\} \\ &= \max\{i \mid Z_{ij} \text{ divides } g' \text{ for some } j \in \{1, \dots, n\}\}, \\ a_{2j} &= \min\{l \mid Y_{lj} \text{ divides } g'\}, j = 1, \dots, t_2, \\ b_{2j} &= \min\{l \mid Z_{jl} \text{ divides } g'\}, j = 1, \dots, t_2. \end{aligned}$$

So one obtains δ_2 . Continuing this algorithm, one clearly gets all remaining factors of ζ . This means, the map

$$\text{in}(\Phi(-)) : \{\text{standard monomials on } \Delta_{r+1}(X)\} \rightarrow \{\text{monomials in } K[Y, Z]\}$$

is injective. Since the residue classes of the standard monomials on $\Delta_{r+1}(X)$ form a K -basis of $R_{r+1}(X)$, and since

$$\text{in}(\varphi(R_{r+1}(X)_i)) = D_{2i} \text{ and } \dim_K R_{r+1}(X)_i = \dim_K D_{2i} \text{ for all } i \geq 0,$$

we conclude that $\text{in}(\Phi(-))$ maps the standard monomials on $\Delta_{r+1}(X)$ to a K -basis of D .

Having come so far, the rest of (2) is not difficult anymore. It is clear that $D = K[E]$, where E is a subsemigroup of $(\mathbb{Z}^{mr}) \oplus (\mathbb{Z}^{rn})$. Using (**), one verifies that E is equal to the set of all elements $[(c_{ij}), (d_{uv})] \in (\mathbb{Z}^{mr}) \oplus (\mathbb{Z}^{rn})$ satisfying the following linear equations and inequalities:

$$\begin{aligned} c_{ij} &= d_{uv} = 0, & j > i, \quad u > v, \\ c_{ij}, d_{uv} &\geq 0, & i \geq j, \quad v \geq u, \\ \sum_{i=j-1}^{k-1} c_{i,j-1} - \sum_{i=j}^k c_{ij} &\geq 0, & j = 2, \dots, r, \quad k = j, \dots, m, \\ \sum_{t=u-1}^{w-1} d_{u-1,t} - \sum_{t=u}^w d_{ut} &\geq 0, & u = 2, \dots, r, \quad w = u, \dots, n, \\ \sum_{i=1}^m c_{ij} - \sum_{v=1}^n d_{jv} &= 0, & j = 1, \dots, r. \end{aligned}$$

Let E_t be the subset of E consisting of all vectors in $(\mathbb{Z}^{mr}) \oplus (\mathbb{Z}^{rn})$ which appear as exponent vectors of the elements in \mathfrak{a}_t . Using (**) again, one sees that

$$\begin{aligned} E_t &= \{[(c_{ij}), (d_{uv})] \in E \mid c_{ii} \geq t, i = 1, \dots, r\} \\ &= \{[(c_{ij}), (d_{uv})] \in E \mid c_{rr} \geq t\}. \end{aligned}$$

(3) First, we recall the definition of a conic ideal. Let S be a positive normal affine semigroup, and let $C = \mathbb{R}_+ S$ be the cone generated by S in $\mathbb{R}S$. A monomial ideal I of $R = K[S]$ is called *conic*, if there exists an element $b \in \mathbb{R}S$ such that

$$I = \bigoplus_{a \in T_b} Kx^a, \text{ where } T_b = \mathbb{Z}S \cap (b + C).$$

If I is conic, then R/I is Cohen-Macaulay, see [BG1, 3.3].

So let us show that \mathfrak{a}_t is a conic ideal in D for $t \leq m - r$. For this, we have to find an element $w_t \in \mathbb{R}E$ such that $E_t = \mathbb{Z}E \cap (w_t + \mathbb{R}_+E)$. Note that $\mathbb{R}E$ is the set of all vectors $[(c_{ij}), (d_{uv})] \in (\mathbb{R}^{mr}) \oplus (\mathbb{R}^{rn})$ that satisfy the equations

$$\begin{aligned} c_{ij} = d_{uv} = 0, & & j > i, \quad u > v, \\ \sum_{i=1}^m c_{ij} - \sum_{v=1}^n d_{jv} = 0, & & j = 1, \dots, r. \end{aligned}$$

Furthermore, $\mathbb{Z}E = \mathbb{R}E \cap ((\mathbb{Z}^{mr}) \oplus (\mathbb{Z}^{rn}))$. We choose a positive real number $\varepsilon < 1$ and define $w_t = [(c_{ij}), (d_{uv})]$ by setting

$$c_{ij} = \begin{cases} t - \varepsilon, & \text{for } i = j, \\ -(t - \varepsilon)/(m - r), & \text{for } j < i \leq m - r + j, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$d_{uv} = 0 \quad \text{for all } u, v.$$

It is clear that $w_t \in \mathbb{R}E$. Since $-(t - \varepsilon)/(m - r) > -1$ (this is the point, where we need $t \leq m - r$!) we have $\mathbb{Z}E \cap (w_t + \mathbb{R}_+E) = E_t$. So \mathfrak{a}_t is indeed a conic ideal. \square

Combining Lemma 3.10 and Theorem 3.12, we obtain

Theorem 3.13. *The powers \mathfrak{p}^{m-r} and \mathfrak{q}^{n-r} are Ulrich modules of rank one over $R_{r+1}(X)$.*

Once one has shown that \mathfrak{p}^{m-r} is an Ulrich module over $R_{r+1}(X)$, it follows by symmetry that \mathfrak{q}^{n-r} is an Ulrich module, too. However, there exists also another, more abstract reason why \mathfrak{q}^{n-r} must be an Ulrich module, provided that \mathfrak{p}^{m-r} is an Ulrich module. It involves the canonical module of $R_{r+1}(X)$, and we want to explain it briefly.

Assume that $m \geq n$. Then the power \mathfrak{p}^{m-n} is (isomorphic to) the *canonical module $\omega_{R_{r+1}(X)}$ of $R_{r+1}(X)$, see e.g. [BV, 8.8]. Note that the functor

$$D(-) := \text{Hom}_{R_{r+1}(X)}(-, \omega_{R_{r+1}(X)})$$

satisfies $D(D(M)) \cong M$ for any graded maximal Cohen-Macaulay module M over $R_{r+1}(X)$ (see e.g. [BH, 3.3.10] for a proof). The following calculation shows that $D(\mathfrak{p}^{m-r}) \cong \mathfrak{q}^{n-r}$ (and hence $D(\mathfrak{q}^{n-r}) \cong \mathfrak{p}^{m-r}$). Denoting $\text{Hom}_{R_{r+1}(X)}(-, R_{r+1}(X))$ by $(-)^*$, we have

$$\begin{aligned} D(\mathfrak{p}^{m-r}) &\cong \text{Hom}_{R_{r+1}(X)}(\mathfrak{p}^{m-r}, \mathfrak{p}^{m-n}) \\ &\cong \text{Hom}_{R_{r+1}(X)}(\mathfrak{p}^{m-r}, (\mathfrak{q}^{m-n})^*) \\ &\cong (\mathfrak{p}^{m-r} \otimes_{R_{r+1}(X)} \mathfrak{q}^{m-n})^* \\ &\cong (\mathfrak{p}^{m-r} \otimes_{R_{r+1}(X)} \mathfrak{q}^{m-n})^{***}. \end{aligned}$$

The last isomorphism follows from the general fact that $\text{Hom}_A(M, A)$ is reflexive whenever M is a finitely generated module over a domain A . Since

$$(\mathfrak{p}^{m-r} \otimes_{R_{r+1}(X)} \mathfrak{q}^{m-n})^{**} \cong ((\mathfrak{p}^{m-r} \mathfrak{q}^{m-n})^{-1})^{-1}$$

and

$$[((\mathfrak{p}^{m-r} \mathfrak{q}^{m-n})^{-1})^{-1}] = [\mathfrak{p}^{m-r}] + [\mathfrak{q}^{m-n}] = (m-r)[\mathfrak{p}] - (m-n)[\mathfrak{p}] = [\mathfrak{p}^{n-r}]$$

in $\text{Cl}(R_{r+1}(X))$, we obtain

$$(\mathfrak{p}^{m-r} \otimes_{R_{r+1}(X)} \mathfrak{q}^{m-n})^{***} \cong (\mathfrak{p}^{n-r})^* \cong \mathfrak{q}^{n-r}.$$

Thus we have shown:

$$\text{Hom}_{R_{r+1}(X)}(\mathfrak{p}^{m-r}, \omega_{R_{r+1}(X)}) \cong \mathfrak{q}^{n-r} \text{ and } \text{Hom}_{R_{r+1}(X)}(\mathfrak{q}^{n-r}, \omega_{R_{r+1}(X)}) \cong \mathfrak{p}^{m-r}.$$

It remains to show that $\text{Hom}_{R_{r+1}(X)}(-, \omega_{R_{r+1}(X)})$ maps the category of Ulrich modules over $R_{r+1}(X)$ into itself.

Proposition 3.14. *Let R be a standard graded Cohen-Macaulay algebra over a field K and let ω_R be the *canonical module of R . If M is an Ulrich module over R , then $\text{Hom}_R(M, \omega_R)$ is also an Ulrich module over R .*

Proof. Let M be an Ulrich module over R . We set $d = \dim R$. Since M is a maximal Cohen-Macaulay module, $\text{Hom}_R(M, \omega_R)$ is a maximal Cohen-Macaulay module, too (see e.g. [BH, 3.3.10]). Therefore, we only have to show that

$$\mu(\text{Hom}_R(M, \omega_R)) = e(\text{Hom}_R(M, \omega_R)).$$

For this, we may assume that K is an infinite field. Then there exist elements $x_1, \dots, x_d \in S_1$, which form a system of parameters of R . These elements form a regular sequence on each of the three R -modules R, M , and $\text{Hom}_R(M, \omega_R)$. Set $\bar{R} = R/(x_1, \dots, x_d)$ and $\bar{M} = M \otimes_R \bar{R}$. Then

$$\text{Hom}_R(M, \omega_R) \otimes_R \bar{R} = \text{Hom}_{\bar{R}}(\bar{M}, \omega_R \otimes_R \bar{R}),$$

see e.g. [BH, 3.3.3]. From $n := \mu(\bar{M}) = \mu(M) = e(M) = e(\bar{M}) = \ell(\bar{M})$ we deduce that \bar{M} is isomorphic to $\bigoplus_{i=1}^n K(-a_i)$, where $a_1, \dots, a_n \in \mathbb{Z}$. Since $\omega_R \otimes_R \bar{R}$ is the *canonical module of \bar{R} , one has $\text{Hom}_{\bar{R}}(K, \omega_R \otimes_R \bar{R}) \cong K(-b)$ for some $b \in \mathbb{Z}$, and hence $\text{Hom}_{\bar{R}}(\bar{M}, \omega_R \otimes_R \bar{R}) \cong \bigoplus_{i=1}^n K(a_i - b)$. Thus we get

$$\begin{aligned} \mu(\text{Hom}_R(M, \omega_R)) &= \mu(\text{Hom}_R(M, \omega_R) \otimes_R \bar{R}) \\ &= \mu(\bigoplus_{i=1}^n K(a_i - b)) = n \\ &= e(\bigoplus_{i=1}^n K(a_i - b)) \\ &= e(\text{Hom}_R(M, \omega_R) \otimes_R \bar{R}) \\ &= e(\text{Hom}_R(M, \omega_R)). \end{aligned}$$

□

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