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Der Mensch ist verurteilt, frei zu sein. Verurteilt, weil er sich nicht selbst erschaffen hat, anderweit aber dennoch frei, da er, einmal in die Welt geworfen, für alles verantwortlich ist, was er tut.

Jean-Paul Sartre

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Dedicated to my parents.

Contents

Introduction

This thesis is divided into two independent chapters which are, nevertheless, combined by the common subject of products and dual modules. The first one is on submodules of the Baer-Specker-module $P = \prod_{i<\omega} Re_i$ which are also dual modules, while the second part provides a discussion on products, reduced products, and the commutativity of products with respect to the Chase radical within the category $\mathbb{Z}$-Mod of abelian groups. Furthermore, in both parts we use combinatorial and set-theoretic ideas for the constructions and proofs. In the following we shall separately describe the contents of each chapter in more detail.

The first part is devoted to dual modules. Several authors ([12], [13], [25]) considered abelian groups H, which can be represented as dual groups $G^* = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$. The existence of such groups H is a non-trivial problem in abelian group theory. Here, we will concentrate on this problem in the context of R-modules (R a countable domain containing a multiplicatively closed subset $\mathbb{S}$ suitable for defining a linear Hausdorff topology). In fact, we will search for dual modules H within the lattice of submodules of $P = R^\omega$. Recall, given an R-module G, its dual G^* is defined as $G^* = \mathrm{Hom}_R(G, R)$. Moreover, H is called a *dual module* if $H \cong G^*$ for some G; we then also say that G is a *primal module* of H.

To be more precise, we will show that many pure submodules H of $\mathbb{D}$ are dual modules by constructing corresponding primal modules; here $\mathbb{D}$ denotes the $\mathbb{S}$-adic closure of $S = \bigoplus_{i<\omega} Re_i$ in $P = \prod_{i<\omega} Re_i$. Actually, in the end (in Section 1.5) we will construct a fully rigid system of primal modules G which will also be *essentially indecomposable*, i.e. $\mathrm{End}(G) = R \oplus \mathrm{Fin}(G)$ where

Fin(G) denotes the ideal of the endomorphism ring End(G) consisting of all endomorphisms with finite rank images. However, we will not introduce the needed techniques all at once, but, more conveniently, develop them 'step by step', respectively 'section by section'.

All the constructions make extensive use of the set-theoretic principle Martin's Axiom (MA), that is, we assume that (MA_κ) holds for all $\kappa < 2^{\aleph_0}$. Martin's Axiom is independent from ZFC which means that neither Martin's Axiom, nor its negation, is provable in ZFC. However, the countable (non-trivial !) case ($\mathrm{MA}_{\aleph_0}$) is even a consequence of ZFC. The formulation of Martin's Axiom uses partially ordered sets and families of dense subsets. Surprisingly, we can use the same partially ordered set $(\mathfrak{F}, \leq)$ throughout this chapter. Hence we already introduce and consider this partially ordered set in Section 1.1. Note, that even the used dense subsets only need to be altered slightly.

First, in Section 1.2, we consider the canonical scalar product $\phi : S \times S \to R$ $((e_i, e_j) \mapsto \delta_{ij})$ and its unique extension $\phi : \mathbb{D} \times \mathbb{D} \to \widehat{R}$. In fact, given $H \subseteq \mathbb{D}$ with $|H| = \aleph_1$, we construct $G \subseteq_* \mathbb{D}$, also of size $\aleph_1$, such that $\phi(G \times H) \subseteq R$. As a byproduct, we obtain, under the assumption of ZFC+MA, that the existence of such modules G is equivalent to the negation of the Continuum Hypothesis ($\neg$CH), i.e. $\aleph_1 < 2^{\aleph_0}$.

As mentioned before, the main objective is the construction of a primal module of a given submodule $H \subseteq_* \mathbb{D}$. This will be done in Section 1.3 using the canonical scalar product ϕ. Given pure submodules G, H of $\mathbb{D}$ with $\phi(G \times H) \subseteq R$, the mapping $H \to G^*$ defined by $h \mapsto \phi(-, h)$ is a well-defined monomorphism. The aim is to construct G in such a way that this mapping is also surjective, which then implies the desired isomorphism. For the proof it is crucial that any dual map $\varphi : G \to R$ is uniquely determined by its restriction on S, and hence $\varphi = \phi(-, h)$ for some $h \in P$ (see Lemma 1.3.1). Moreover, Martin's Axiom will be used to find solutions of infinite systems of linear equations, by considering the finite subsystems. The constructed module G will be of cardinality $2^{\aleph_0}$; we actually show that $|G|$ cannot be smaller for $G^* \cong H$.

After representing many modules as dual modules (as above), it is natural to raise the following question:
"Do there exist dual modules with prescribed endomorphism ring?"
This problem will be tackled in Section 1.4. Since we work in $\mathbb{D} \subseteq_* P$, which we assume to be separable, the smallest possible endomorphism ring of any $G \subseteq_* \mathbb{D}$ is $\operatorname{End}(G) = R \oplus \operatorname{Fin}(G)$. This is, in fact, the endomorphism ring which we will realize. Note, realization theorems have been of great interest within the last two decades of the former century (see e.g. [8], [10], [18], [19]). The construction of G basically uses the same techniques as used in the previous section. Of course, these techniques need to be refined in order to achieve the required result.

In addition, we will sharpen the main result of Section 1.4 by establishing the existence of a fully rigid system of primal modules $\{G_I : I \subseteq \omega\}$ of size $2^{\aleph_0}$, i.e.

$$\operatorname{Hom}(G_I, G_J) = \begin{cases} R \oplus \operatorname{Fin}(G_I, G_J), & \text{if } I \subseteq J \\ \operatorname{Fin}(G_I, G_J), & \text{otherwise.} \end{cases}$$

This is done in Section 1.5, again by slightly altering the techniques developed before.

In the second chapter, we consider products and reduced products of abelian groups. In particular, we investigate the behavior of the Chase radical with respect to products. Recall, that any radical $\mathfrak{R}$ is a subfunctor of the identity satisfying $\mathfrak{R}(G/\mathfrak{R}(G)) = 0$ for any group G. The Chase radical, defined by $\nu G = \bigcap\{\ker(\varphi) \mid \varphi : G \to X, X\ \aleph_1\text{-free}\}$, is a famous example for radicals in abelian group theory. It provides a criterion for testing $\aleph_1$-freeness of groups [5],[14]. Moreover, it can be characterized by $\nu G = \sum\{\nu C \mid C \subseteq G, |C| = \aleph_0, C^* = 0\}$, that means, countable subgroups with trivial dual play an important role for determining νG.

As for any radical, it is natural to ask the following question:

"What is the minimal cardinal κ such that the Chase radical ν does not commute with products with index set of size κ?"

This means, we want to find the minimal κ for ν such that $\nu \prod_{\alpha<\kappa} G_\alpha \neq \prod_{\alpha<\kappa} \nu G_\alpha$ for some family $\{G_\alpha : \alpha < \kappa\}$ of groups. Note, it is easy to see, that the minimal κ has to be bigger than $\aleph_0$ (see Lemma 2.1.5). Moreover, it is also known that, for many cardinals κ, there exist radicals $\mathfrak{R}_\kappa$ that commute with products 'up to κ', but not beyond (see [7]).

The above question has been considered before by K. Eda [11] in 1985. He showed that there is an upper bound $\kappa \leq 2^{\aleph_0}$ such that the Chase radical does not commute with direct products over κ rational groups. His proof used descending chains of types. However, due to the nature of these chains, he could not determine the exact bound when the Chase radical does not commute. The related question depends on the model of set theory, as demonstrated at the end of Section 2.2. Here we will prove (in Section 2.2), that the Chase radical does not commute with products over antichains of types of length $\aleph_1$. This finally proves that the exact bound equals $\aleph_1$. Moreover, our investigations also provide additional information on the $\aleph_1$-freeness of reduced products over rational groups. More precisely, we will show that a reduced product of rational groups is $\aleph_1$-free if and only if it is $\mathbb{Z}$-homogeneous; this property can also be characterized via conditions on the original product. As a byproduct, we also obtain an extended version of the Wald-Łoś-Lemma: If U is a countable subgroup of an $\aleph_1$-free reduced product $\prod_{\alpha<\kappa}^r R_\alpha$ of rational groups, then U can be embedded into $\prod_\kappa \mathbb{Z}$ (see Corollary 2.2.5).

Finally, in Section 2.3, we generalize the characterization from Section 2.2 to a criterion for $\aleph_1$-freeness of arbitrary reduced products.

List of Symbols

$\mathbb{N}$	all natural numbers (without zero)
$\mathbb{Z}$	all integers
$\mathbb{Q}$	all rational numbers
ω	the first infinite ordinal ($=\mathbb{N}\cup\{0\}$)
R	countable domain
Q	the quotient field (of a commutative domain R)
$\mathbb{S}$	fixed multiplicatively closed subset of R
S	$=\bigoplus_{n<\omega} Re_i$
P	$=\prod_{n<\omega} Re_i$
$\mathbb{D}$	the $\mathbb{S}$-adic closure of S in P
$\widehat{G}$	$\mathbb{S}$-adic completion of G
$\prod^{<}_{\alpha<\kappa} G_\alpha$	all elements of $\prod_{\alpha<\kappa} G_\alpha$ with support of size $<\kappa$
$\prod^{r}_{\alpha<\kappa} G_\alpha$	$=\prod_{\alpha<\kappa} G_\alpha / \prod^{<}_{\alpha<\kappa} G_\alpha$
δ_{ij}	Kronecker symbol, i.e. $\delta_{ij}\in\{0,1\}$ with $\delta_{ij}=1$ iff $i=j$
$\subseteq_*$	pure subgroup
$\subseteq_*^{\mathbb{S}}$	$\mathbb{S}$-pure subgroup
$\langle U\rangle_*^R$	the purification of U
$\langle U\rangle_*^{\mathbb{S}}$	the $\mathbb{S}$-purification of U
$\sqsubseteq$	direct summand
G^*	$=\mathrm{Hom}(G,R)$
$\mathrm{cf}(\kappa)$	cofinality of κ
$\mathrm{supp}(g)$	the support of g
$\mathrm{br}(g,h)$	the branching point of g and h
$\mathfrak{F}$	special poset defined in 1.1.10
$r\vert x$	r divides x
$\leq$	subset or order relation

1 Scalar products and dual modules in $\mathbb{D}$

Throughout this chapter, we assume that R is a countable domain with 1 containing a multiplicatively closed subset $\mathbb{S} = \{s_i \mid i < \omega\}$ such that $s_0 = 1$ is the only unit in $\mathbb{S}$ and R satisfies $\bigcap_{s\in\mathbb{S}} sR = 0$.
The main objective here is to show that, assuming Martin's Axiom, many submodules of R^ω are dual modules. This is done by constructing a corresponding primal module. A first step in this direction is to extend the canonical scalar product on the direct sum $S = R^{(\omega)}$ to larger submodules of R^ω; this will be done in the second section.
Moreover, we shall use the techniques developed in Sections 1.2 and 1.3 to construct dual modules with 'small' endomorphism rings (see Section 1.4), in fact, in Section 1.5 we will obtain a fully rigid system of such modules.
However, we begin with recalling known definitions and results and with introducing a certain partially ordered set (poset), which will be needed in all the following sections of this chapter.

1.1 Basic definitions and results

Algebraic background

First, we consider the needed algebraic concepts. Let R, $\mathbb{S} = \{s_i : i < \omega\}$ be as above. For an arbitrary R-module G, we define the $\mathbb{S}$-topology as follows:

Definition 1.1.1 *Let G be an R-module.*

(i) *Let $q_n \in \mathbb{S}$ $(n < \omega)$ be defined by $q_n = \prod_{i\leq n} s_i$.*

(ii) *The (linear)* $\mathbb{S}$-topology *on G is defined by $\{q_nG \mid n < \omega\}$ as a basis of neighborhoods of* 0.

In order to have a completion of an R-module G in the $\mathbb{S}$-topology, we need the topology to be Hausdorff, i.e. $\bigcap_{n<\omega} q_n G = 0$; in this case G is also said to be $\mathbb{S}$-*reduced*. The completion of G with respect to the $\mathbb{S}$-topology is denoted by $\widehat{G}$. In particular, our given domain R is $\mathbb{S}$-reduced by assumption and hence its completion, $\widehat{R}$, is well defined.
Moreover, we need the following notions:

Definition 1.1.2 *Let G be an R-module.*

(i) *G is called* $\mathbb{S}$-torsion-free *if $sg = 0$ ($s \in \mathbb{S}$, $g \in G$) implies $g = 0$.*

(ii) *A submodule U of G is said to be* pure *in G if $rU = rG \cap U$ for all $r \in R$. (Notation: $U \subseteq_* G$.)*

(iii) *A submodule U of G is said to be* $\mathbb{S}$-pure *in G if $sU = sG \cap U$ for all $s \in \mathbb{S}$. (Notation: $U \subseteq_*^{\mathbb{S}} G$.)*

(iv) *Let G be torsion-free. For a module $U \subseteq G$, the* $\mathbb{S}$-purification *(R-purification) of U in G is defined by $\langle U \rangle_*^{\mathbb{S}} = \{g \in G \mid sg \in U \text{ for some } s \in \mathbb{S}\}$ ($\langle U \rangle_*^{R} = \{g \in G \mid rg \in U \text{ for some } r \in R\}$).*

Recall, that torsion-freeness in general is also defined similarly, using $r \in R$ instead of $s \in \mathbb{S}$.
Note, that an $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced R-module G is always $\mathbb{S}$-pure in its completion $\widehat{G}$.
Furthermore, let Q denote the quotient field of R. Often, we shall require that $\mathbb{S}$ satisfies $\mathbb{S}^{-1}R = Q$; in this case we call $\mathbb{S}$ *full* in R.

Next we consider the cartesian product R^ω.

Definition 1.1.3

(i) *Let S be the countably infinite direct sum of copies of R, i.e.* $S := \bigoplus_{i<\omega} Re_i$.

(ii) *Let P denote the infinite cartesian product over R, i.e.* $P := \prod_{i<\omega} Re_i$.

(iii) *Let $\mathbb{D}$ be the $\mathbb{S}$-adic closure of S in P, i.e.* $\mathbb{D} = \overline{S} = \widehat{S} \cap P$.

(iv) *For an element $g \in P$ with $g = (g_i e_i)_{i<\omega}$, we define the* support *of g by* $\operatorname{supp}(g) := \{i < \omega \mid g_i \neq 0\}$.

(v) *For elements $g, h \in P$ with $g = (g_i e_i)_{i<\omega}$ and $h = (h_i e_i)_{i<\omega}$, we define the* branching point $\operatorname{br}(g, h)$ *of g and h to be the minimal $n < \omega$ such that* $g{\restriction}n = h{\restriction}n$ *but* $g_n \neq h_n$.

Note, that we identify the direct sum S with the submodule of P consisting of all elements with finite support. For an arbitrary element $g \in P$, we often use the notation $g = \sum_{i<\omega} g_i e_i$ instead of $g = (g_i e_i)_{i<\omega}$.
Next we consider separability properties of P. The definition of separability is given below; it uses finite rank submodules, where the rank of a module is the (uniquely determined!) cardinality of a maximal linearly independent subset.

Definition 1.1.4 *An R-module G is said to be* ($\mathbb{S}$-)separable *if any* ($\mathbb{S}$-)*pure submodule of G of finite rank is a free direct summand of G.*

Note that the above definition differs slightly from the usual definition of separability in case R is not a PID. Whenever it is necessary to differentiate between $\mathbb{S}$-separable and separable, we also use R-separable for the latter. Note, that $\mathbb{S}$-separability obviously implies R-separability. Moreover, if $\mathbb{S}$ is full, then an easy calculation shows that $\mathbb{S}$-purity is the same as purity (for $\mathbb{S}$-torsion-free modules) and hence the two notions of separability coincide in this case. Later on we will require that P, respectively all its submodules, are $\mathbb{S}$-separable. Note, if R is noetherian then $P = R^\omega$ is R-separable and also $\aleph_1$-free (cf. [18, Preliminaries]). Recall, that an R-module is $\aleph_1$-*free* if every countable submodule is free.

Lemma 1.1.5 *Suppose that P is R-separable. Then P is also $\mathbb{S}$-separable if and only if $\mathbb{S}$ is full in R.*

Proof. As mentioned above, if $\mathbb{S}$ is full then P is $\mathbb{S}$-separable.
Conversely, assume that P is $\mathbb{S}$-separable. So, if U is an $\mathbb{S}$-pure submodule of P of finite rank then U is a direct summand of P and hence U is pure in P. In particular, we thus have that the $\mathbb{S}$-purification and the (R-)purification of finite rank submodules of P coincide.
We now consider $q = \frac{r_1}{r_2} \in Q$. Let $b = r_1 e_1$ and $c = r_2 e_1$. Then $r_2 b = r_1 c$ and hence b is an element of the R-purification of c and so also of its $\mathbb{S}$-purification $\langle c \rangle_*$. Therefore, there are $s \in \mathbb{S}$, $r \in R$ such that $sb = rc$ and so $s r_1 e_1 = r r_2 e_1$, respectively $q = \frac{r_1}{r_2} = \frac{r}{s}$, i.e. $q \in \mathbb{S}^{-1} R$. Thus $\mathbb{S}$ is full, as required. □

We finish the algebraic part of this section with two simple results on submodules of P.

Lemma 1.1.6 *If P is ($\mathbb{S}$-)separable and U is an ($\mathbb{S}$-)pure submodule of P, then U is ($\mathbb{S}$-)separable.*

Proof. Let F be a ($\mathbb{S}$-)pure finite rank submodule of U. Then F is also ($\mathbb{S}$-)pure in P, hence $R^\omega = F \oplus D$ for some $D \subseteq R^\omega$ where F is a free R-module. Applying the modular law, implies that $U = F \oplus (D \cap U)$, i.e. U is also ($\mathbb{S}$-)separable. $\square$

Lemma 1.1.7 *Let $G \subseteq P = R^\omega$. Then there is an R-module G' with $G' \subseteq \mathbb{D}$ and $G' \cong G$.*

Proof. Let $G \subseteq P$. We define $\beta : G \longrightarrow \mathbb{D}$ by $g = \sum_{i<\omega} g_i e_i \mapsto g\beta = \sum_{i<\omega} q_i g_i e_i$. Clearly, $g\beta$ is an element of $\mathbb{D}$ for each $g \in G$. Moreover, it is straightforward to see that β is monic. Hence $G \cong G' := G\beta \subseteq \mathbb{D}$. $\square$

Set-theoretic background

Here we present all definitions which are needed for the definition of Martin's Axiom, an often used set-theoretic principle. Furthermore, we introduce a specific partially ordered set, which will be the main tool for all the later constructions of this chapter.

Definition 1.1.8 *Let $(\mathbb{P}, \leq)$ be a partially ordered set.*

(i) *Elements $p, q \in \mathbb{P}$ are called* compatible *if there is an element $r \in \mathbb{P}$ such that $r \geq p$ and $r \geq q$. Two elements $p, q \in \mathbb{P}$ are called* incompatible, *if they are not compatible.*

(ii) *A subset $A \leq \mathbb{P}$ is an* antichain *if every two elements p, q are incompatible.*

(iii) *We say, $(\mathbb{P}, \leq)$ is* directed *if every two elements $p_1, p_2 \in \mathbb{P}$ are compatible.*

(iv) *We call $(\mathbb{P}, \leq)$* σ-centered *if $\mathbb{P}$ is a countable union of directed sets.*

(v) *A subset $D \leq \mathbb{P}$ is* dense *if, for all $p \in \mathbb{P}$, there is $q \in D$ with $q \geq p$.*

We want to state Martin's Axiom using generic filters. Hence we need:

Definition 1.1.9 *Let* $(\mathbb{P}, \leq)$ *be a partially ordered set and* $F \leq \mathbb{P}$*. We say, that* F *is a* filter *if the following conditions are satisfied:*

(i) F *is non-empty;*

(ii) *if* $p \geq q$ *and* $p \in F$ *then* $q \in F$*;*

(iii) *any two elements* $p, q \in F$ *are compatible in* F*.*

Given a family $\mathcal{D}$ *of dense subsets of* $\mathbb{P}$*, a filter* $\mathbb{G}$ *is called* $\mathcal{D}$-generic *if* $\mathbb{G} \cap D \neq \emptyset$ *for all* $D \in \mathcal{D}$*.*

Next we present Martin's Axiom which was introduced by Martin and also, independently, by Rowbottom. This axiom is consistent with ZFC, i.e. neither Martin's Axiom nor its negation are provable in ZFC. Sometimes this axiom is also called 'internal forcing axiom' because of its close relation to forcing notions; for a detailed description see [29]. Here we actually state a slightly simplified version of Martin's Axiom (MA); for the original version see cf. [14, p. 164].

(MA): Let $\mathcal{D}$ be a family of less than $2^{\aleph_0}$ dense subsets of a partially ordered set $(\mathbb{P}, \leq)$ with the property, that all antichains in $\mathbb{P}$ are countable. Then there is a $\mathcal{D}$-generic filter $\mathbb{G}$ on $\mathbb{P}$.

The condition, that every antichain in $\mathbb{P}$ is at most countable (c.c.c.), can be replaced by the slightly stronger notion of σ-centered sets. We obtain *Martin's Axiom for* σ*-centered sets* (MA^{σ}), which is a proper consequence of Martin's

Axiom (cf. [4]):

($\mathbf{MA}^\sigma$): Let $\mathcal{D}$ be a family of less than $2^{\aleph_0}$ dense subsets of a σ-centered partially ordered set $(\mathbb{P}, \leq)$. Then there is a $\mathcal{D}$-generic filter $\mathbb{G}$ on $\mathbb{P}$.

It is easy to see that MA implies MA^σ. In the constructions given in the following sections we are going to use this version of Martin's Axiom, as it is more intuitive to prove σ-centered rather than the c.c.c. condition.
We finish this section with introducing the partially ordered set $\mathfrak{F}$, which is a basic tool in all our constructions.

Definition 1.1.10 *Given submodules $G \subseteq P$ and $H \subseteq P$, let $\mathfrak{F}$ consist of all quintuples $p = (l, \bar{a}, s, U, V)$ such that*

- $l \in \mathbb{N}$;
- $\bar{a} = (a_i)_{i<l}$ *is a finite sequence with* $a_i \in R$;
- $s \in \mathbb{S}$;
- U *is a finite subset of* H;
- V *is a finite subset of* G *such that* $\mathrm{br}(\bar{a}, v) < l$ *for all* $v \in V$.

We also write $p = (l^p, \bar{a}^p, s^p, U^p, V^p)$ whenever needed. Moreover, for $p \in \mathfrak{F}$ and $x = \sum_{i<\omega} x_i e_i \in U^p$, we define $m^p_x := \sum_{l<l_p} x_l a^p_l \in R$ and put $M^p = \{m^p_x \mid x \in U^p\}$.
For $p = (l^p, \bar{a}^p, s^p, U^p, V^p)$, $q = (l^q, \bar{a}^q, s^q, U^q, V^q) \in \mathfrak{F}$ we define $p \leq q$ if and only if

- $l^p \leq l^q$;

- $\bar{a}^p = \bar{a}^q \restriction l^p$;
- $s^p | s^q$ *and* $s^p | a_l^q$ *for all* $l^p \leq l < l^q$;
- $U^p \leq U^q$;
- $V^p \leq V^q$;
- *for all* $x = \sum_{l<\omega} x_l e_l \in U^p$ *follows* $m_x^q = m_x^p = \sum_{l<l^p} x_l a_l^p$; *this is equivalent to* $\sum_{l^p \leq l < l^q} x_l a_l^q = 0$.

As mentioned before, we want to apply Martin's Axiom in the Version (MA$^\sigma$). Hence we need $\mathfrak{F}$ to be σ-centered, as we show below:

Lemma 1.1.11 *Let* $(\mathfrak{F}, \leq)$ *be the poset given in Definition 1.1.10. Then* $\mathfrak{F}$ *is* σ*-centered.*

Proof. In order to prove the assertion, we define an equivalence relation on $\mathfrak{F}$ as follows: For $p, q \in \mathfrak{F}$, we put

$$p \sim q \quad \Longleftrightarrow \quad l^p = l^q,\ \bar{a}^p = \bar{a}^q,\ s^p = s^q.$$

Note, that U^p and U^q and, similarly, V^p and V^q may differ. Clearly, the above defined relation is an equivalence relation and hence $\mathfrak{F}$ decomposes into the corresponding equivalence classes, denoted by $[p]$. Since each equivalence class is uniquely determined by $l \in \mathbb{N}$, $\bar{a} \in R^l$ and $s \in \mathbb{S}$, where $\mathbb{N}$, R^l and $\mathbb{S}$ are countable, there can only be countably many different equivalence classes. Hence it remains to show that each equivalence class is directed.

Let $p = (l, \bar{a}, s, U, V) \in \mathfrak{F}$ and $q_1, q_2 \in [p]$. It follows immediately from the definition, that $l^{q_i} = l$, $\bar{a}^{q_i} = \bar{a}$ and $s^{q_i} = s$ for $i = 1, 2$. We define $r \in \mathfrak{F}$ in the following way: Let $l^r = l$, $\bar{a}^r = \bar{a}$, $s^r = s$, $U^r = U^{q_1} \cup U^{q_2}$, and $V^r = V^{q_1} \cup V^{q_2}$.

For $v \in V^r$, we have $v \in V^{q_i}$ for some $i \in \{1, 2\}$ and hence it follows that $\mathrm{br}(\bar{a}^r, v) = \mathrm{br}(\bar{a}^{q_i}, v) < l^{q_i} = l = l^r$. Therefore r is an element of $\mathfrak{F}$ and thus obviously $r \in [p]$.

Moreover, we have $r \geq q_1$ and $r \geq q_2$ since, for all $x = \sum_{i<\omega} x_i e_i \in U^{q_1} \cap U^{q_2}$, it follows that $m_x^{q_1} = \sum_{i<l^{q_1}} x_i a_i^{q_1} = \sum_{i<l^{q_2}} x_i a_i^{q_2} = m_x^{q_2}$. This shows that $(\mathfrak{F}, \leq)$ is σ-centered. $\square$

1.2 Extensions of the scalar product on S

Let R, $\mathbb{S}$ be as described in the beginning of this chapter; for the definition of the divisor chain $\{q_n \mid n < \omega\}$ we refer the reader to Definition 1.1.1.

In this section, we shall consider submodules H, G of the cartesian product P of cardinality less than $2^{\aleph_0}$, such that the canonical scalar product on S extends to $G \times H$ in a natural way. In fact, we restrict our attention to submodules of $\mathbb{D}$, which is no loss in generality by Lemma 1.1.7.

Throughout this section we fix ϕ to be the canonical scalar product $\phi : S \times S \longrightarrow R$ defined by $(e_i, e_j) \mapsto \delta_{i,j}$ and the natural linear extension, where δ_{ij} denotes the usual Kronecker symbol. We will also use ϕ to denote the uniquely determined extension of the just defined mapping to $\phi : \mathbb{D} \times \mathbb{D} \longrightarrow \widehat{R}$. Note, however, that strictly speaking the latter is not a scalar product but just a bilinear map, since $\phi(\mathbb{D} \times \mathbb{D}) \not\subseteq R$.

Now let $H \subseteq \mathbb{D}$ be arbitrary. For the restriction $\phi\restriction(S \times H)$, we immediately deduce that $\phi(S \times H)) \subseteq R$: Let $h = \sum_{i<\omega} h_i e_i \in H$ and $d = \sum_{i<\omega} d_i e_i \in S$. Then $\phi(h, d) = \sum_{i<\omega} h_i d_i$ is an element of R since d has finite support. Therefore, for any $H \subseteq \mathbb{D}$, we may consider the scalar product $\phi : S \times H \longrightarrow R$; this will be the starting point in the construction of this section.

The aim is to enlarge S to an uncountable submodule G of $\mathbb{D}$ in such a way that $\phi : G \times H \longrightarrow \widehat{R}$ is still a scalar product, i.e. $\phi(G \times H) \subseteq R$. This will be done using Martin's Axiom.

Next, we want to formulate the step lemma, which will tell us how to construct our desired R-module G 'step-by-step'.

Note, that the step lemma below does also work for H with $|H| < 2^{\aleph_0}$ respectively for G with $|G| < 2^{\aleph_0}$, but here we want to treat the smallest possible cardinality of a submodule H of $\mathbb{D}$ such that H may not be free.

Step Lemma 1.2.1 (ZFC + ¬CH + MA) *Let $H \subseteq \mathbb{D}$ with $|H| = \aleph_1$, let G be countable with $S \subseteq G \subseteq_* \mathbb{D}$ and assume that $\phi(G \times H) \subseteq R$. Then there exists an element $a = \sum_{i<\omega} a_i e_i \in \mathbb{D} \setminus G$ such that*

$$\phi(G' \times H) \subseteq R,$$

where G' is defined by $G' = \langle G, a \rangle_*^{\mathbb{S}} \subseteq \mathbb{D}$.

Proof. First note, that we omit the upper index $\mathbb{S}$ for purity throughout this proof.

Now let G, H be given as above. In order to find the desired element a, we use the poset $\mathfrak{F}$ defined before (see Definition 1.1.10). We will apply Martin's Axiom for σ-centered set with respect to the family of dense subsets defined below. Recall, that an element $p \in \mathfrak{F}$ is of the form $(l^p, \bar{a}^p, s^p, U^p, V^p)$ with $l^p \in \mathbb{N}$, $\bar{a}^p \in R^{l^p}$, $s^p \in \mathbb{S}$, U^p, V^p finite subsets of H, G, respectively.

(i) For each $x \in H$, let $D_x^1 := \{p \in \mathfrak{F} \ : \ x \in U^p\}$.

(ii) For each $s \in \mathbb{S}$, let $D_s^2 := \{p \in \mathfrak{F} \ : \ s | s^p\}$.

(iii) For each $l_0 < \omega$, let $D_{l_0}^3 := \{p \in \mathfrak{F} \ : \ l_0 \leq l^p\}$.

(iv) For each $y \in G$, let $D_y^4 := \{p \in \mathfrak{F} \ : \ y \in V^p\}$.

We prove that these sets are, indeed, dense in $\mathfrak{F}$. Notice, that their number is less than $2^{\aleph_0}$. The sets defined in (i) will ensure that $\phi(a, h) \in R$ for all $h \in H$, those in (ii) will ensure that $a \in \mathbb{D}$, by those in (iii) we obtain infinite sequences, and those in (iv) imply $a \notin G$. In fact, we could have omitted the sets in (ii) since $S \subseteq G$, but it will be more convenient to include them.

First we show the density of the sets D_x^1 defined in (i). Let $x = \sum_{l<\omega} x_l e_l$ and take $p \in \mathfrak{F}$ arbitrary. Define q to be like p except for the subset of H; we put

$U^q = U^p \cup \{x\}$. It is then easy to see that $q \geq p$ and $q \in D^1_x$.

The density of of the sets D^2_s in (ii) follows similarly: For $s \in \mathbb{S}$ and $p \in \mathfrak{F}$, choose any $q \geq p$ with $s|s^q$, e.g. take $s^q = s^p s$.

Next, for the density of the sets $D^3_{l_0}$ in (iii), consider $l_0 < \omega$ and $p \in \mathfrak{F}$. We define q by putting $U^q = U^p$, $l^q = l^p + l_0$, $s^q = s^p$ and $V^p = V^q$ and $\bar{a}^q = (\bar{a}^p)^\frown(0, \ldots, 0)$, where $(0, \ldots, 0)$ is a zero-sequence of l_0 entries. Again, it is obvious that $q \geq p$ and $q \in D^3_{l_0}$.

It remains to prove the density of D^4_y for all $y \in G$. Take $p \in \mathfrak{F}$ and $y \in G$ arbitrary, and define $q \in \mathfrak{F}$ in the following way:

- $l^q = l^p + l$ where $|U^p| < l < \omega$;
- $\bar{a}^q \restriction l^p = \bar{a}^p$;
- $s^q = s^p$;
- $U^q = U^p$;
- $V^q = V^p \cup \{y\}$.

We still have to complete the definition of $\bar{a}^q$; the crucial point is the choice of $a^q(l)$ for all $l^p \leq l < l^q$, in such a way that

$$\mathrm{br}(\bar{a}^q, v) < l^q \text{ for all } v \in V^q,$$

$$s^p | a^q(l),$$

and

$$\sum_{l^p \leq l < l^q} x_l a^q_l = 0$$

are satisfied for all $x \in U^p$.

We note, that it is sufficient to solve a system of linear equations of the following form. For convenience, let U^p be enumerated by $U^p := \{x_i : 1 \leq i \leq n := |U^p|\}$:

$$\begin{pmatrix} x^1_{l_p} & \cdots & x^1_{l^p+l-1} \\ \vdots & \ddots & \vdots \\ x^n_{l_p} & \cdots & x^n_{l^p+l-1} \end{pmatrix} \cdot \begin{pmatrix} a^q_{l_p} \\ \vdots \\ a^q_{l^p+l-1} \end{pmatrix} = 0,$$

where $l^p + l = l^q$. Recall, that we chose l satisfying $l > n = |U^p|$. Hence there are more variables than equations and thus there is a non-trivial solution (actually, infinitely many) of this system over Q. Furthermore, since this system is homogeneous, it holds that, if $\bar{z} = (z_0, \ldots, z_{l-1})$ is a solution, then $k\bar{z}$ is also a solution for all $k \in R$. So, we can find a feasible solution $\bar{z} \in R^l$.
Now, we fix an arbitrary feasible solution $0 \neq \bar{z} = (z_0, \ldots, z_l) \in R^l$ of the above system. Let $i < l$ such that $z_i \neq 0$; such an i exists since $\bar{z} \neq 0$. Next, we choose $0 \neq k \in R$ such that $ks^p z_i \neq y_{l^p+i}$.
We define $a^q_{l^p+j} = ks^p z_j$ for all $j < l$; this finishes the definition of $\bar{a}^q$. We have to check that

$$q \geq p \textit{ and } q \in D^4_y.$$

We start with $q \geq p$: It is clear that $l^p \leq l^q$, $U^p \subseteq U^q$, $V^p \subseteq V^q$ and $\bar{a}^q \restriction l^p = \bar{a}^p$. The definition of $a^q_{l^p+j} = ks^p z_j$ ensures that $s^p | a^q_{l^p+j}$ in R for all $l^p \leq l^p + j < l^q = l^p + l$.
Moreover, $\sum_{j<l} x_{l^p+j} a^q_{l^p+j} = 0$ for all $x \in U^p$, since $ks^p \bar{z} = (a^q_{l^p+j})_{j<l}$ is a solution of the above system of linear equations. It remains to check that $q \in D^4_y$. This is readily seen since $y \in V^q$, $\mathrm{br}(\bar{a}^q, y) \leq l^p + i$ and so $q \in \mathfrak{F}$.

Finally, we are ready to apply Martin's Axiom to $\mathfrak{F}$ and to

$$\mathcal{D} = \{D^1_x \mid x \in H\} \cup \{D^2_s \mid s \in \mathbb{S}\} \cup \{D^3_{l_0} \mid l_o < \omega\} \cup \{D^4_y \mid y \in G\}.$$

Hence there exists a $\mathbb{D}$-generic filter $\mathbb{G} \subseteq \mathfrak{F}$, i.e. $\mathbb{G} \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. We define $a \in P$ by

$$a = \sum_{i<\omega} a_i e_i$$

with

$$a_i = a_i^p \text{ for } p \in \mathbb{G} \text{ with } i < l^p.$$

First note, that a is well defined since $\mathbb{G}$ is a filter and thus any two elements in $\mathbb{G}$ are compatible, i.e. for $p, q \in \mathbb{G}$ and $i < l^p, l^q$ we have $a_i^p = a_i^q$.
To see that $a \in \mathbb{D}$, let $s \in \mathbb{S}$ be arbitrary. Let $p \in D_2^s \cap \mathbb{G} \neq \emptyset$. We show that $s|a_i$ for all $i \geq l^p$. Let $i \geq l^p$ be arbitrary and $q \in \mathbb{G}$ with $l^q > i$; q exists since $\emptyset \neq D_{i+1}^3 \cap \mathbb{G}$. Since $\mathbb{G}$ is a filter there is $r \in \mathbb{G}$ such that $r \geq p$ and $r \geq q$. In particular, $l^r > i$ and $s^p|a_l^r$ for all $l^p \leq l < l^r$ by definition of '$\leq$' (see Definition 1.1.10). Therefore s^p divides $a_i^r = a_i$ and, by our choice of p, we also have $s|s^p$, i.e. $a \in \mathbb{D}$ is proven.
Next we show that $a \notin G$. Let $y \in G$ be arbitrary. Then $D_y^4 \cap \mathbb{G} \neq \emptyset$, say $p \in D_y^4 \cap \mathbb{G}$. Hence $\text{br}(\bar{a}^p, y) = m < l^p$ which implies that $a_m^p \neq y_m$. By the definition of a it follows that $a_m = a_m^p \neq y_m$. Therefore $a \neq y$ for all $y \in G$ and so $a \in \mathbb{D} \setminus G$.

It remains to show that $\phi : G' \times H \longrightarrow R$ is a well-defined scalar product, where $G' = \langle G, a \rangle_*$. For the moment we restrict our attention to $G'' = \langle G, a \rangle \subseteq G'$. Consider $c \in G''$, then c can be written as

$$c = g + ka$$

for some $g \in G$ and $k \in R$. Let $y \in H$ be arbitrary. Then

$$\phi(c, y) = \phi(g, y) + k\phi(a, y).$$

We already know, by assumption, that $\phi(g, y) \in R$. So, all we have to show is $\phi(a, y) \in R$: Since D_y^1 is dense in $\mathfrak{F}$ there is an element $p \in D_y^1 \cap \mathbb{G} \neq \emptyset$. Thus we obtain $\sum_{l<l^p} a_l^p y_l \in R$ and so it follows, by the definition of '$\leq$', that

$$\sum_{l<\omega} a_l y_l = \sum_{l<l_p} a_l^p y_l + \sum_{l_p \leq l < \omega} a_l y_l,$$

where the second summand is zero. Hence $\phi(a, y) \in R$, respectively $\phi(c, y) \in R$. We finally consider $\phi(c, y)$ for $c \in G'$ and $y \in H$. Since $c \in G' = G''_*$, there are $s \in \mathbb{S}$ and $g \in G''$ such that $sc = g$. Hence $\phi(g, y) = \phi(sc, y) = s\phi(c, y) \in s\widehat{R} \cap R = sR$, where the latter follows from the $\mathbb{S}$-purity of R in its completion $\widehat{R}$. Since R is $\mathbb{S}$-torsion-free we deduce $\phi(c, y) \in R$, i.e. $\phi(G' \times H) \subseteq R$, as required. $\square$

The following corollary is an immediate consequence of the above proof.

Corollary 1.2.2 *Let $G \subseteq \mathbb{D}$, $H \subseteq \mathbb{D}$, $\phi : G \times H \longrightarrow R$ and let $a \in \mathbb{D}$ such that $\phi(a, h) \in R$ for all $h \in H$. Then $\phi : \langle G, a \rangle_* \times H \longrightarrow R$.*

The next lemma shows that the negation of CH is necessary for our construction.

Lemma 1.2.3 (ZFC + CH) *Assume that $\mathbb{S}$ is full in R. Moreover, let $\phi : \mathbb{D} \times \mathbb{D} \longrightarrow \widehat{R}$ be as before and let $x \in \mathbb{D} \setminus S$. Then there is an element $y \in \mathbb{D}$ such that $\phi(x, y) \in \widehat{R} \setminus R$.*

Proof. First note that, by assumption, we have that R is pure in $\widehat{R}$, since purity and $\mathbb{S}$-purity coincide whenever $\mathbb{S}$ is full.

Let $x = \sum_{l<\omega} x_l e_l \in \mathbb{D} \setminus S$. Clearly, the support $I := \mathrm{supp}(x)$ is infinite. For all $\alpha \in {}^I 2 \leq {}^\omega 2$, we define $\widetilde{w_\alpha} \in \mathbb{D}$ by $\widetilde{w_\alpha} = \sum_{l<\omega} \alpha(l) t_l e_l$ where the t_l's are defined by $t_0 = s_0$ and $t_{l+1} = t_l^2 s_{l+1} \prod_{i \leq l, i \in I} x_i$ $(l < \omega)$. Moreover, let $w_\alpha := \phi(x, \widetilde{w_\alpha}) = \sum_{l<\omega} \alpha(l) t_l x_l \in \widehat{R}$ for all $\alpha \in {}^I 2$ and put

$$\mathfrak{M} = \{\widetilde{w_\alpha} : \alpha \in {}^I 2\}.$$

We also define a relation on $\mathfrak{M}$ by $\widetilde{w_\alpha} \sim \widetilde{w_\beta}$ if and only if $w_\alpha = w_\beta$; this is obviously an equivalence relation. Hence we may write $\mathfrak{M}$ as a disjoint union of the corresponding equivalence classes, i.e. $\mathfrak{M} = \dot\bigcup_{\alpha \in \mathcal{R}} N_\alpha$ for some $\mathcal{R} \subseteq {}^I 2$ where $N_\alpha = [\widetilde{w_\alpha}]$.

It is sufficient to prove that every N_α is countable since then, by the regularity of $2^{\aleph_0} = \aleph_1$ and $|\mathfrak{M}| = 2^{\aleph_0}$, we have that $|\mathcal{R}| = 2^{\aleph_0}$ and so there is $\beta \in \mathcal{R}$ with $w_\beta \in \widehat{R} \setminus R$.

It remains to prove that N_α is countable for every $\alpha \in \mathcal{R}$. In fact, we show that $|N_\alpha| = 1$ for all $\alpha \in \mathcal{R}$. So, let $\alpha \in {}^I 2$ be fixed and choose $\beta \in {}^I 2$ arbitrary with $\alpha \neq \beta$. Let $m = \operatorname{br}(\alpha, \beta)$. Note that m has to be an element of $I = \operatorname{supp}(x)$, i.e. $x_m \neq 0$. Hence

$$\begin{aligned} w_\beta - w_\alpha &= \sum_{l<\omega} (\beta(l) - \alpha(l)) t_l x_l = \sum_{m \leq l < \omega} (\beta(l) - \alpha(l)) t_l x_l \\ &= t_m \sum_{m \leq l < \omega} (\beta(l) - \alpha(l)) \frac{t_l}{t_m} x_l \in t_m \widehat{R}, \end{aligned}$$

where $\beta(m) - \alpha(m) = \pm 1$. Therefore, if we consider the above difference mod $t_{m+1}\widehat{R}$, we deduce

$$\begin{aligned} w_\beta - w_\alpha = \sum_{m \leq l < \omega} (\beta(l) - \alpha(l)) t_l x_l &\equiv (\beta(m) - \alpha(m)) t_m x_m \bmod t_{m+1}\widehat{R} \\ &\equiv \pm t_m x_m. \end{aligned}$$

We finally show that $t_{m+1} \nmid t_m x_m$ in $\widehat{R}$, respectively in R since $R \subseteq_* \widehat{R}$. Thus $w_\beta - w_\alpha \neq 0$ and hence $w_\beta \notin N_\alpha$.

Assume that $kt_{m+1} = t_m x_m$ for some $k \in R$. Then

$$t_m x_m = kt_{m+1} = kt_m^2 s_{m+1} \prod_{i \leq m, i \in I} x_i,$$

and so

$$1 = kt_m s_{m+1} \prod_{i < m, i \in I} x_i,$$

because R has no zero-divisors. This is a contradiction since s_{m+1} is not a unit in R and thus the proof is finished. □

We are now ready to formulate our main theorem of this section which, in some sense, may be considered as an algebraic characterization of the continuum hypothesis (CH) under MA.

Theorem 1.2.4 (ZFC + MA) *Assume that $\mathbb{S}$ is full in R. Then the negation of CH holds (i.e. $\aleph_1 < 2^{\aleph_0}$) if and only if, for all submodules $H \subseteq \mathbb{D}$ of cardinality $\aleph_1$, there is a submodule G with $S \subseteq G \subseteq_* \mathbb{D}$ and $|G| = \aleph_1$ such that $\phi(G \times H) \subseteq R$.*

Proof. We first assume the negation of CH and let $H \subseteq \mathbb{D}$ be given as above. We construct the desired G inductively by applying the Step Lemma 1.2.1 $\aleph_1$ times.

We begin with $G_0 = S$. As mentioned before, we have that $\phi : G_0 \times H \longrightarrow R$. Now let $\alpha < \aleph_1$ be an arbitrary ordinal and assume that G_β has already been constructed for all $\beta < \alpha$ satisfying $S \subseteq G_\beta \subseteq_* \mathbb{D}$, G_β countable, and $\phi(G_\beta \times H) \subseteq R$.

If α is a limit ordinal then we put $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Clearly, G_α also satisfies $\phi : G_\alpha \times H \longrightarrow R$, $S \subseteq G_\alpha \subseteq_* \mathbb{D}$ and $|G_\alpha| = \aleph_0$ (since $|\alpha| = \aleph_0$).

If $\alpha = \beta + 1$ is a successor ordinal, then we apply Step Lemma 1.2.1 to H and G_β. Hence we obtain an element $a = a_\beta \in \mathbb{D} \backslash G_\beta$ with $\phi : G_\alpha \times H \longrightarrow R$, where we put $G_\alpha = G' = \langle G_\beta, a_\beta \rangle$ and we have $G_\beta \subsetneq G_\alpha \subseteq_* \mathbb{D}$ and G_α countable. Finally, assume that all G_α's for $\alpha < \aleph_1$ have been constructed. We define

$$G = \bigcup_{\alpha < \aleph_1} G_\alpha.$$

It is clear that G has cardinality $\aleph_1$ and $\phi : G \times H \longrightarrow R$, as required.

Conversely, assume that CH holds, i.e. $\aleph_1 = 2^{\aleph_0}$. We choose $H = \mathbb{D}$ and consider $S \subseteq G \subseteq_* \mathbb{D}$ arbitrary with $|G| = \aleph_1$. Then there is an element $g \in G$ with infinite support and hence, by Lemma 1.2.3, there is an element $y \in H$ such that $\phi(g, y) \in \widehat{R} \setminus R$. This finishes the proof. $\square$

Note, that the assumption that $\mathbb{S}$ be full is only needed for proving one direction of the above statement.

1.3 On dual submodules of $\mathbb{D}$

In this section, we consider ($\mathbb{S}$-)pure submodules H of $\mathbb{D}$ of cardinality less than $2^{\aleph_0}$. In fact, we assume throughout, that $\mathbb{D}$, respectively P is separable and $\mathbb{S}$ is full in R, i.e. the two notions of purity coincide.

We shall show, under the set-theoretic assumption of Martin's Axiom (MA), that any 'admissible' (see Definition 1.3.7) $H \subseteq_* \mathbb{D}$ is a dual module. Recall, the dual of a module G is defined by $G^* = \mathrm{Hom}(G, R)$; a module H of the form $H \cong G^*$ is called *dual module* and G is called a *primal module* of H. Note, that a dual module may have many primals (see Section 1.5).

The proof will be done by constructing a module G of cardinality $2^{\aleph_0}$ such that $H \cong G^*$. For this purpose we extend the techniques developed in Section 1.2. Moreover, we will show that it is necessary for G to be of cardinality $2^{\aleph_0}$.

As before, we denote by ϕ the canonical scalar product, respectively its extension (see Section 1.2). In particular, we again obtain $\phi : S \times H \longrightarrow R$ for all $H \subseteq_* \mathbb{D}$. Moreover, we may, in fact, extend ϕ to $\mathbb{D} \times P$, respectively to $P \times \mathbb{D}$, since, for all $x = \sum_{i<\omega} x_i e_i \in \mathbb{D}$ and $b = \sum_{i<\omega} b_i e_i \in P$, the infinite sum $\sum_{i<\omega} x_i b_i$ is a well-defined element of $\widehat{R}$.

Assume, for the moment, that we already have an R-module G such that $S \subseteq G \subseteq_* \mathbb{D}$ and $\phi : G \times H \longrightarrow R$. Then the mapping defined by $h = \sum_{i<\omega} h_i e_i \mapsto \phi(-, h)$ is an embedding from H into G^* since $\varphi_h(e_i) = h_i$ for all $i < \omega$. In fact, we show that any $\varphi \in G^*$ is given by a $\phi(-, h)$ for some $h \in P = R^\omega$, as stated by the next lemma.

Lemma 1.3.1 *Let $S \subseteq G \subseteq_* \mathbb{D}$ and $\varphi \in G^*$. Then there is a uniquely determined element $b \in P$ such that $\varphi = \phi(-, b)$.*

Proof. Let $\varphi \in G^*$ be as above. We define $b = \sum_{i<\omega} b_i e_i \in R^\omega$ by $b_i = \varphi(e_i)$.

Then, for any $g = \sum_{i<\omega} g_i e_i \in G$, we have

$$\varphi(g) = \varphi(\sum_{i<\omega} g_i e_i) = \sum_{i<\omega} g_i \varphi(e_i) = \sum_{i<\omega} g_i b_i = \phi(g, b).$$

Note, that this equation is well defined, by the continuity of φ and the convergence within $\mathbb{D}$, respectively within $\widehat{R}$.

The uniqueness of b follows from $S \subseteq G$ and $\phi(e_i, b) = b_i$ for all $i < \omega$. □

In fact, the above result tells us that every $\varphi \in G^*$ is nothing else but multiplication with a certain element $b \in P$. This is an essential tool for controlling the dual maps $\varphi \in G^*$ and hence it will be crucial for our construction. As mentioned before, we assume that $\mathbb{S}$ is full in R throughout this section. Example 1.3.11 below will demonstrate the necessity of this assumption.

Before we can construct our desired module G, we need some more preliminaries which will be important for proving the density of some of the involved sets in Step Lemma 1.3.10. First, we describe the dual of $\mathbb{D}$.

Lemma 1.3.2 *There exists an isomorphism* $\alpha : S \longrightarrow \mathbb{D}^*$ *defined via* $s \mapsto \phi(-, s)$.

Proof. Let α be the mapping defined above. It is clear that α is a monomorphism.

Now consider $\varphi : \mathbb{D} \longrightarrow R$ and let $h \in P$ such that $\varphi = \phi(-, h)$, which exists by Lemma 1.3.1. If $h \in S$ then we are done. Otherwise, the support of h, $\mathrm{supp}(h)$, is infinite. Therefore, by Theorem 1.2.3, there is an element $g \in \mathbb{D}$ such that $\phi(g, h) \in \widehat{R} \setminus R$. But this implies $\varphi(g) = \phi(g, h) \in \widehat{R} \setminus R$, contradicting $\varphi(g) \in R$. It hence follows that α is an isomorphism, i.e. $\mathbb{D}^* \cong S$. □

The above result is not very surprising, since we can always embed P into $\mathbb{D}$ by $\iota : P \hookrightarrow \mathbb{D}$ defined via $(z_i e_i)_{i<\omega} \mapsto (z_i q_i e_i)_{i<\omega}$. Moreover, if, for example, R is slender then the result is actually immediate: Consider $\varphi \in \mathbb{D}^*$, respectively $\iota\varphi \in P^*$. Then, by the slenderness of R, we obtain $e_i \iota\varphi = q_i(e_i\varphi) = 0$ for almost all $i < \omega$. Hence $e_i\varphi = 0$ for almost all $i < \omega$ since, by our general assumptions, R is $\mathbb{S}$-torsion-free. Therefore $\mathbb{D}^* \cong S$. Note, it is well known that $P^* \cong S$.
Furthermore, we obtain the following corollary:

Corollary 1.3.3 *Let $H \subseteq \mathbb{D}$ with $\aleph_1 \leq |H| < 2^{\aleph_0}$ and $S \subseteq G \subseteq_* \mathbb{D}$ such that $G^* \cong H$. Then G cannot be isomorphic to $\mathbb{D}$.*
Proof. Assume $G \cong \mathbb{D}$, then $H \cong G^* \cong \mathbb{D}^* \cong S$ by Lemma 1.3.2. This implies that H is countable – a contradiction. □

In fact, this means that the R-modules G we are going to construct have to be proper submodules of $\mathbb{D}$, although they have the same size, as we will see in Lemma 1.3.12. For convenience, we introduce the following notion:

Definition 1.3.4 *Let $U \subseteq_* \mathbb{D}$. Then the R-module*

$$U^{\perp} := \{x \in S : \phi(x, y) = 0 \ \forall y \in U\} \subseteq S \cong \mathbb{D}^*$$

is the orthogonal *of U. Moreover, we define*

$$U^{\perp\perp} := \{x \in \mathbb{D} : \phi(x, y) = 0 \ \forall y \in U^{\perp}\} \subseteq \mathbb{D}$$

to be the orthogonal closure *of U. If $U = U^{\perp\perp}$, then U is said to be* orthogonally closed.

Note, that the Definition 1.3.4 coincides with the standard definition $U^{\perp} = \{\varphi \in \mathbb{D}^* : \varphi(u) = 0 \ \forall u \in U\}$, since $\mathbb{D}^* \cong S$ with $\alpha : S \longrightarrow \mathbb{D}$ via $s \mapsto \phi(-, s)$. It follows immediately from the definition that, for a given $U \subseteq_* \mathbb{D}$, we have $U \subseteq U^{\perp\perp}$. The next lemma, provides a sufficient criterion for U to be orthogonally closed.

Lemma 1.3.5 *Let $U \subseteq_* \mathbb{D}$ be a finite rank pure submodule. Then U is orthogonally closed, i.e. $U = U^{\perp\perp}$.*

PROOF. It is clear that $U \subseteq U^{\perp\perp}$.

Now, since U is of finite rank and we assume P to be $(\mathbb{S})$-separable, U is a free direct summand of $\mathbb{D}$, say $\mathbb{D} = U \oplus D$ for some $D \subseteq \mathbb{D}$. Consider $x \in \mathbb{D} \setminus U$ arbitrary, then $x = u_x + r_x$ for some $u_x \in U$ and $0 \neq r_x \in D$. It is sufficient to prove that there is $s \in \mathbb{D}$ such that $\phi(u, s) = 0$ for all $u \in U$, but $\phi(r_x, s) \neq 0$. This then implies $s \in U^{\perp}$,

$$\phi(x, s) = \phi(u_x + r_x, s) = \phi(r_x, s) \neq 0,$$

and hence $x \notin U^{\perp\perp}$.

Now, let $\theta : \mathbb{D} \longrightarrow D$ be the canonical epimorphism. Then $\theta(x) = r_x \neq 0$. Additionally, we know that $D \subseteq R^{\omega}$ and hence there is $\pi : D \longrightarrow R$ such that $\pi(\theta(x)) \neq 0$ and $\pi(\theta(u)) = 0$ for all $u \in U$. Therefore, $\theta\pi \in \mathbb{D}^*$ and hence there is $s \in R^{\omega}$ such that $\pi(\varphi(x)) = \phi(x, s)$ for all $x \in \mathbb{D}$, by Lemma 1.3.1. In fact, by Lemma 1.3.2, we obtain $s \in S$ and thus conclude $\phi(u, s) = 0$ for all $u \in U$, while

$$\phi(x, s) = \phi(r_x, s) \neq 0.$$

This finally implies $s \in U^{\perp}$ and $x \notin U^{\perp\perp}$ and so s is the desired element. $\square$

Notice, that Lemma 1.3.5 does not imply, given $U \subseteq_* \mathbb{D}$ of finite rank, that $\mathbb{D} = U \oplus U^\perp$ holds, as one may expect from functional analysis since a corresponding result is true for vector spaces. A counterexample is given by $U = \langle e_1 \rangle_*$ because then $U^\perp \subseteq S$ and so we only obtain elements of finite support, i.e. $U \oplus U^\perp \subsetneq \mathbb{D}$.

At this stage we remind the reader that the aim of the section is, given $H \subseteq_* \mathbb{D}$ with $\aleph_1 \leq |H| < 2^{\aleph_0}$, to construct a primal module $S \subseteq G \subseteq_* \mathbb{D}$ of H with $|G| = 2^{\aleph_0}$. The construction is done similar to the one in Section 1.2, i.e we will also formulate a Step Lemma according to our needs.
Before we present the step lemma itself we need a technical lemma, which provides a nice criterion for finite subsets of P to be linearly independent.

Lemma 1.3.6 *Let $M < R^\omega$ be a finite subset and let $l_0 < \omega$.*
Then $M\restriction[l_0, \omega)$ is linearly independent if and only if there exists $l_0 \leq n < \omega$ such that $M\restriction[l_0, n) := \{b\restriction[l_0, n) : b \in M\}$ is linearly independent.
Proof. First note, if $M\restriction[l_0, n)$ is linearly independent for some $n < \omega$, then this obviously implies that $M\restriction[l_0, \omega)$ is linearly independent.

Conversely, assume that $M \restriction [l_0, \omega)$ is linearly independent. We prove the assertion by induction on $m = |M|$. For $m = 1$ the claim clearly holds.
Assume now, that the assertion is true for all $k \leq m$ and let $M < R^\omega$ with $|M| = m + 1$, say $M = \{b_1, \ldots, b_{m+1}\}$. Suppose, for contradiction, that $M\restriction[l_0, n)$ is linearly dependent for all $n < \omega$. Since $\{b_1\restriction[l_0, \omega), \ldots, b_m\restriction[l_0, \omega)\} < M\restriction[l_0, \omega)$ is also linearly independent we know, by induction hypothesis, that there is $n_0 < \omega$ such that $\{b_1\restriction[l_0, n_0), \ldots, b_m\restriction[l_0, n_0)\}$ is linearly independent. Moreover, by assumption, we know that $M\restriction[l_0, n)$ is linearly dependent for all $n < \omega$. Hence, for each $n \geq n_0$, there are $c_i^n \in Q$ $(1 \leq i \leq m,\ n_0 \leq n < \omega)$

with at least one c_i^n non-zero such that

$$b_{m+1}\restriction[l_0, n) = \sum_{1\leq i\leq m} c_i^n b_i\restriction[l_0, n).$$

In fact, the coefficients are unique since we may consider all the above equations restricted to $[l_0, n_0)$ and since the elements in the right hand side of the above equation are linearly independent. Put $c_i = c_i^n$ for all $1 \leq i \leq m$ and $n \geq n_0$. We therefore obtain

$$b_{m+1}\restriction[l_0, \omega) = \sum_{1\leq i\leq m} c_i b_i\restriction[l_0, \omega),$$

contradicting the linear independence of $M\restriction[l_0, \omega)$. This finishes our proof. □

Next we consider the admissibility of $H \subseteq_* \mathbb{D}$.

Definition 1.3.7 *We call $H \subseteq \mathbb{D}$* admissible *if, for all $b \in \mathbb{D} \setminus H$ and for all $s \in S \setminus H$, we have $b + s \notin H$.*

Note, if $S \subseteq H$, then H is clearly admissible. Moreover, for the proof of our main result, it is necessary for H to be admissible, as is shown next.

Lemma 1.3.8 *Suppose $H \subseteq \mathbb{D}$ is not admissible. Then, for all $S \subseteq G \subseteq_* \mathbb{D}$, H cannot be isomorphic to G^* via the canonical mapping $h \mapsto \phi(-, h)$.*

Proof. We prove the hypothesis by contradiction. Let H, G be as above and assume $G^* \cong H$. Since H is not admissible, there is an element $b \in \mathbb{D} \setminus H$ and $s \in S \setminus H$ such that $b' := b + s \in H$. Consider now $g \in G$. Then

$$\phi(g, b) = \phi(g, b' - s) = \phi(g, b') - \phi(g, s) \in R.$$

This implies that $\phi(-, b) \in G^*$ but $b \notin H$, contradicting $G^* \cong H$ via the canonical map. □

The above lemma shows, that we need to assume that H is admissible. Moreover, we prove next, that $H \subseteq_* \mathbb{D}$ is also necessary.

Lemma 1.3.9 *Assume $H \subseteq \mathbb{D}$ non-pure. Then $G^* \not\cong H$ via the canonical homomorphism for all $S \subseteq G \subseteq_* \mathbb{D}$.*

Proof. We prove the result by contradiction. Let H be as above and assume that $G^* \cong H$ for some $S \subseteq G \subseteq_* \mathbb{D}$. Consider $b \in \langle H \rangle_* \setminus H$. Then there is $r \in R$ such that $rb \in H$. Hence, for arbitrary $g \in G$,

$$\phi(g, rb) = r\phi(g, b)$$

and so $r\phi(g, b) \in r\widehat{R} \cap R = rR$. Therefore, $\phi(g, b) \in R$ for all $g \in G$. We obtain $\phi(-, b) \in G^*$, but $b \notin H$ – a contradiction. □

We are finally ready to formulate our step lemma.

Step Lemma 1.3.10 *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$. Moreover, let $b \in R^\omega \setminus H$ and let $S \subseteq G \subseteq_* \mathbb{D}$ such that $\phi(G \times H) \subseteq R$, and $|G| < 2^{\aleph_0}$. Then there exists an element $a = \sum_{i<\omega} a_i e_i \in \mathbb{D} \setminus G$ such that*

(i) *$G' = \langle G, a \rangle_* \subseteq \mathbb{D}$ (hence $|G'| = |G| < 2^{\aleph_0}$);*

(ii) *$\phi(G' \times H) \subseteq R$;*

(iii) *$\phi(a, b) \in \widehat{R} \setminus R$.*

Proof. Let H, G, $b = (b_0, \dots, b_n, \dots)$ be as above. As in Step Lemma 1.2.1, we apply Martin's Axiom to the σ-centered poset $\mathfrak{F}$ introduced in Definition 1.1.10 and to the dense subsets defined as follows:

(i) For all $x \in H$, let $D_x^1 := \{p \in \mathfrak{F} \ : \ x \in U^p\}$.

(ii) For all $s \in \mathbb{S}$, let $D^2_s := \{p \in \mathfrak{F} \ : \ s | s^p\}$.

(iii) For all $l_0 < \omega$, let $D^3_{l_0} := \{p \in \mathfrak{F} \ : \ l_0 \leq l^p\}$.

(iv) For all $y \in G$, let $D^4_y := \{p \in \mathfrak{F} \ : \ y \in V^p\}$.

(v) For all $r \in R$, let $D^5_r := \{p \in \mathfrak{F} \ : \ \sum_{l<l^p} b_l a^p_l \not\equiv r \bmod s^p\}$.

The sets defined in (i) - (iv) are the same we used in the proof of Step Lemma 1.2.1 and hence we already know that they are dense in $\mathfrak{F}$.
So, it remains to prove the density of D^5_r for a given $r \in R$. Let $p = (l^p, \bar{a}^p, s^p, U^p, V^p) \in \mathfrak{F}$ be arbitrary. If $\sum_{l<l^p} b_l a^p_l \neq r$, then we choose $s^q \in \mathbb{S}$ such that $s^p | s^q$ and $\sum_{l<l^p} b_l a^p_l \not\equiv r \mod s^q$. Define $q \in D^5_r$ by $l^q = l^p$, $\bar{a}^q = \bar{a}^p$, s^q as above, $U^q = U^p$, $V^q = V^p$. It is easy to see that $q \geq p$.
Now assume $\sum_{l<l^p} b_l a^p_l = r$. We extend $p \in \mathfrak{F}$ such that, for an extension $q \in \mathfrak{F}$, we have $v := \sum_{l^p \leq l < l^q} b_l a^q_l \neq 0$. This then implies

$$\sum_{l<l^q} b_l a^q_l = \sum_{l<l^p} b_l a^p_l + \sum_{l^p \leq l < l^q} b_l a^q_l = r + v \neq r,$$

and so we can proceed as before.
Since we want $q \geq p$, we especially need to satisfy that $\sum_{l^p \leq l < l^q} b_l a^q_l \neq 0$ and

$$(A_{l^q}) \qquad \sum_{l^p \leq l < l^q} a^q_l u_l = 0 \ \textit{for all} \ u \in U^p. \tag{1}$$

The latter condition is a sytem of linear equations. Hence there is $l_0 < \omega$ such that the system (A_{l^q}) has a non-trivial solution for all $k = l^q$ with $l^p \leq l_0 \leq k$. For convenience, let $\mathrm{Ker}(A_k)$ be the Q-vectorspace of all solutions satisfying (A_k) with $k \geq l_0$. If there exists $l_0 \leq k < \omega$ such that $\sum_{l^p < l < l^q} b_l a^k_l \neq 0$ for some $(a^k_l)_{l^p \leq l < k} \in \mathrm{Ker}(A_k)$, we proceed as above with $l^q = k$ and hence there is an element $q \in D_r^5$ such that $q \geq p$.

Assume now, that there is no such element in $\mathrm{Ker}(A_k)$ for all $l_0 \leq k < \omega$, i.e.

$$\sum_{l^p \leq l < k} a_l^k b_l = 0$$

for any $a^k \in \ker(A_k)$. We differentiate between b being an element of $\mathbb{D}$ or not.

Case 1: Assume $b \in P \setminus \mathbb{D}$. For convenience, let $A = A_\omega$ be as above written as matrix:

$$A := \begin{pmatrix} u_1^1 & \dots & u_l^1 & \dots \\ \vdots & & & \\ u_1^{|U^p|} & \dots & x_l^{|U^p|} & \dots \end{pmatrix}.$$

Moreover, let B be the same system as A with added constraint on b:

$$B := \begin{pmatrix} u_1^1 & \dots & u_l^1 & \dots \\ \vdots & & & \\ u_1^{|U^p|} & \dots & x_l^{|U^p|} & \dots \\ b_1 & \dots & b_n & \dots \end{pmatrix}.$$

Furthermore, let g_l denote the l-th column of A and let h_l denote the l-th column of B. By the above assumption, we have for all $l_0 \leq k < \omega$ and for all $d_l \in R$ with $l \in N = [l^p, k)$, that:

$$\sum_{l \in N} d_l h_l = 0 \iff \sum_{l \in N} d_l g_l = 0. \tag{2}$$

This implies $\langle u \restriction N : u \in U^p \rangle_*{}^\perp = \langle b \restriction N, u \restriction N : u \in U^p \rangle_*{}^\perp$. We now obtain, by Lemma 1.3.5, that

$$\begin{aligned} \langle u \restriction N : u \in U^p \rangle_* &= \langle u \restriction N : u \in U^p \rangle_*{}^{\perp\perp} \\ &= \langle b \restriction N, u \restriction N : u \in U^p \rangle_*{}^{\perp\perp} = \langle b \restriction N, u \restriction N : u \in U^p \rangle_* \end{aligned}$$

and hence

$$b\restriction N \in \langle u\restriction N : u \in U^p \rangle_* .$$

Note, in this context we consider the orthogonal to be contained in $S\restriction N$.

For each (finite) $N = [l^p, k)$ with $l_0 \le k < \omega$, let W_N be the Q-vector space given by $W_N = \langle g_l : l \in N \rangle$. Clearly, the dimension of each W_N is at most $|U^p|$ since the system (A) has only $|U^p|$ columns. Hence there is $N^* \subseteq [l_p, \omega)$ such that W_{N^*} is of maximal dimension $m = |N^*| < |U^p|$ such that $\{g_l \mid l \in N^*\}$ is a maximal independent set over Q. Now consider $N \subseteq [l^p, \omega)$ arbitrary with $N^* \subseteq N$. Then the submatrix $A_N = (g_l : l \in N)$ of A has finite column rank r and thus row rank r. Therefore there is a subset $Z \subseteq \{1, \ldots, |U^p|\}$ of size m such that

$$\{u_j\restriction N : j \in Z\}$$

is maximal independent over Q. By equation (2), we have that $b\restriction N$ is a linear combination of $\{u_j \restriction N : j \in Z\}$ and so there are elements $c_l \in Q$ such that $b \restriction N = \sum_{l \in Z} c_l u_l \restriction N$. While increasing N, the c_l's remain constant by the uniqueness of the c_l's, and hence it follows that $b \restriction [l^p, \omega) = \sum_{l \in Z} c_l u_l \restriction [l^p, \omega)$ holds. Notice, we assumed $b \in P \setminus \mathbb{D}$ and thus there is $s \in \mathbb{S}$ such that the set $L = \{i < \omega : b_i \in R \setminus sR\}$ is infinite. Now, we can choose $s' \in \mathbb{S}$ large enough such that $s'c_l \in sR$ for all $l \in Z$. If $i \in L$ is large enough, then $s'|u^i_l$ in R for all $l \in Z$ since $U^p \subseteq \mathbb{D}$. We obtain that

$$b_i = \sum_{l \in Z} c_l u^i_l = \sum_{l \in Z} s' c_l u'^i_l \in s'R$$

where $s'u'^i_l = u^i_l$ for $i \in W$ large enough. This contradicts the definition of L. Hence D^5_r is dense in $\mathfrak{F}$ in the case that $b \in R^\omega \setminus \mathbb{D}$.

Case 2: Assume now that $b \in \mathbb{D} \setminus H$. Since $H \subseteq_* \mathbb{D}$ is admissible, it follows that $b \restriction [l^p, \omega) \notin \langle U^p\restriction[l^p, \omega)\rangle_*$; note $u - u \restriction [l^p, \omega) \in S$ for all $u \in U^p$ (see

Definition 1.3.7). Since, by Definition 1.1.10, U^p is finite, there is $n_0 < \omega$ such that $b\restriction[l^p, n_0) \notin \langle u\restriction[l^p, n_0) \mid u \in U^p\rangle_*$ by Lemma 1.3.6. Without loss of generality, we may assume $n_0 \geq l_0$. Let $l^q = n_0$. Then it follows that, for all $a \in \operatorname{Ker}(A_{l^q})$ we have $a \in {\langle u\restriction[l^p, l^q) \mid u \in U^p\rangle_*}^{\perp}$. In fact, we obtain

$$\operatorname{Ker}(A_{l^q}) = {\langle u\restriction[l^p, l^q) \mid u \in U^p\rangle_*}^{\perp}.$$

Now, $\sum_{l^p<l<l^q} b_l a_l^q = 0$ for all $(a_l^q)_{l^p\leq l<l^q} \in \operatorname{Ker}(A_{l^q})$ by assumption, and hence

$$b\restriction[l^p, l^q) \in {\langle u\restriction[l^p, l^q) \mid u \in U^p\rangle_*}^{\perp\perp} = \langle u\restriction[l^p, l^q) \mid u \in U^p\rangle_*,$$

by Lemma 1.3.5. This is a contradiction to the fact that $b \restriction [l^p, l^q) \notin \langle u\restriction[l^p, l^q) \mid u \in U^p\rangle_*$. Hence there is always $q \geq p$ with $q \in D_r^5$, i.e. D_r^5 is dense in $\mathfrak{F}$, also in this case.

Finally, we are now ready to apply Martin's Axiom for σ-centered sets to $\mathfrak{F}$ and to the family

$$\mathcal{D} = \{D_x^1 \mid x \in H\} \cup \{D_s^2 \mid s \in \mathbb{S}\} \cup \{D_{l_0}^3 \mid l_0 < \omega\} \cup \{D_y^4 \mid y \in G\} \cup \{D_z^5 \mid z \in R\}$$

of dense subsets.

Therefore, there exists a generic filter $\mathbb{G}$ such that $D \cap \mathbb{G} \neq \emptyset$ for all $D \in \mathcal{D}$. As in Step Lemma 1.2.1, we define a to be $\sum_{i<\omega} a_i e_i$ with $a_i = a_i^p$ for all $p \in \mathbb{G}$ with $i < l^p$. Again, we have that a is well defined, $a \in \mathbb{D} \setminus G$ and $\phi : \langle G, a\rangle_* \times H \longrightarrow R$. So, it remains to check that $\phi(a, b) \in \widehat{R} \setminus R$.

Suppose, for contradiction, that $\phi(a, b) = \sum_{l<\omega} a_l b_l = r \in R$. Now let $p \in \mathbb{G}$ be arbitrary. Then

$$\sum_{l<\omega} a_l b_l \equiv z \bmod s^p$$

and

$$\sum_{l<\omega} a_l b_l = \sum_{l<l^p} a_l b_l + \sum_{l^p\leq l<\omega} a_l b_l.$$

Therefore we obtain

$$\sum_{l<l^p} a_l b_l \equiv r \bmod s^p,$$

since: By the definition of $a \in \mathbb{D}$ we have that, for every $l < \omega$, there is $q \in \mathbb{G}$ with $a_l = a_l^q$. If we consider a_l with $l \geq l_p$ then there is $t \in \mathbb{G}$ such that $t \geq p, q$, because $\mathbb{G}$ is directed. Hence $s^p | a_l^t$ and $a_l^t = a_l^q = a_l$ for all $l \geq l^p$, and thus $\sum_{l^p \leq l} a_l b_l \equiv 0 \bmod s^p$.
However, there is $p \in D_r^5 \cap \mathbb{G}$ and hence $\sum_{l<l^p} a_l b_l = \sum_{l<l^p} a_l^p b_l \not\equiv r \bmod s^p$, contradicting the above equation. So $\phi(a, b) = \sum_{l<\omega} a_l b_l \in \widehat{R} \setminus R$, and hence our step lemma is proven. □

Next, we give an example, showing that the above proof does not work without the assumption of $\mathbb{S}$ being full in R.

Example 1.3.11 *Let $\mathfrak{F}$, D_z^5 ($z \in R$) be as in the proof of Step Lemma 1.3.10. If we omit the condition of $\mathbb{S}$ being full, then the sets D_z^5 are not necessarily dense in $\mathfrak{F}$.*

Proof. Here we need to differentiate between $\mathbb{S}$-pure and R-pure and hence we use $\mathbb{S}$, respectively R, as upper index.
Suppose $\mathbb{S}$ is not full and consider $H \subseteq_*^{\mathbb{S}} \mathbb{D}$, $H \not\subseteq_*^R \mathbb{D}$ with $\aleph_1 \leq |H| < 2^{\aleph_0}$. Hence there is $b \in \langle H \rangle_*^R \setminus H$ with $r \in R$ such that $rb = x \in H$. Now, let $p \in \mathfrak{F}$ with $x \in U^p$ and put $z := \sum_{l<l^p} a_l^p b_l$. Obviously, $p \notin D_z^5$.
We show that, for all $q \in \mathfrak{F}$ with $q \geq p$, we have $q \notin D_m^5$, which then implies that D_z^5 is not dense.
Let $q \in \mathfrak{F}$ with $q \geq p$ be arbitrary. Then $\bar{a}^q \restriction l^p = \bar{a}^p$ and $\sum_{l^p \leq l < l^q} a_l^q u_l = 0$ for all $u \in U^p$. In particular, this is true for $u = x$, i.e.

$$\sum_{l^p \leq l < l^q} a_l^q x_l = 0.$$

So we conclude:

$$0 = \sum_{l^p \leq l < l^q} a_l^q x_l = \sum_{l^p \leq l < l^q} a_l^q r b_l = r \sum_{l^p \leq l < l^q} a_l^q b_l.$$

This implies that $\sum_{l^p \leq l < l^q} a_l^q b_l = 0$, as R is a domain, and hence

$$\sum_{l < l^q} a_l^q b_l = \sum_{l < l^p} a_l^q b_l + \sum_{l^p \leq l < l^q} a_l^q b_l = z + 0 = z.$$

So, $\sum_{l < l^q} a_l^q b_l \equiv z \mod s^q$ for all $q \geq p$, and thus $q \notin D_z^5$, as claimed. □

The next theorem will provide some information on the size of G, if $G^* \cong H$ for some $H \subseteq \mathbb{D}$ with $\aleph_1 \leq |H| < 2^{\aleph_0}$. Assume, for the moment, that $G^* \cong H$ via some $\alpha : H \longrightarrow G^*$. Since $G \subseteq \mathbb{D}$ and $H \subseteq \mathbb{D}$, we then obtain

$$(h\alpha)(g) = \left(\sum_{i<\omega} h_i(e_i\alpha) \right) \sum_{j<\omega} g_j e_j = \sum_{i,j<\omega} h_i g_j (e_i \alpha e_j).$$

Hence, any isomorphism α is given by an $\omega \times \omega$-matrix $(a_{ij})_{i,j<\omega}$, i.e. we may treat α as a bilinear map. Moreover, such an isomorphism α has infinitely many entries $a_{ij} \neq 0$ since, otherwise, we would obtain a matrix with finite rank, contradicting the cardinality of H. Also note that, every $\varphi \in G^*$ can be written as $\phi_\alpha(-, h)$ for some $h \in H$, where ϕ_α is the same map as α written as bilinear map, i.e. $\phi_\alpha(g, h) = \sum_{i,j<\omega} h_i g_j a_{ij}$. This $h \in H$ is uniquely determined, since $h \mapsto \phi_\alpha(-, h) = h\alpha$ is monic.

Theorem 1.3.12 (ZFC + MA) *Let $H \subseteq_* \mathbb{D}$ with $\aleph_1 \leq |H| < 2^{\aleph_0}$ and $S \subseteq G \subseteq_* \mathbb{D}$ with $G^* \cong H$. Then the cardinality of G has to be $2^{\aleph_0}$.*

Proof. We prove the assertion by contradiction. Let $\alpha : G^* \longrightarrow H$ be given by $(a_{ij})_{i,j<\omega}$ and let ϕ_α be the representation as bilinear map (see the discussion

above). Assume that $|G| = \kappa < 2^{\aleph_0}$. We show that there is a homomorphism $\varphi \in G^*$ such that $\varphi \neq \phi_\alpha(-, h)$ for all $h \in H$.
However, since for any $\psi \in G^*$, there is a uniquely determined element $h \in R^\omega$ with $\psi = \phi_\alpha(-, h)$, it is sufficient to find an $h \in R^\omega \setminus H$ such that $\phi_\alpha(g, h) \in R$ for all $g \in G$, i.e. $\varphi = \phi_\alpha(-, h) \in G^*$ would be the desired homomorphism.
Hence it remains to prove the existence of such an element $h \in R^\omega \setminus H$. Again, we use the poset $\mathfrak{F}$ and the dense sets from the proof of Step Lemma 1.3.10:

(i) $D_g^{1'} := \{p \in \mathfrak{F} \mid g \in U^p\}$ for all $g \in G$.

(ii) $D_h^{4'} := \{p \in \mathfrak{F} \mid h \in V^p\}$ for all $h \in H$;

First of all, notice that this time the role of G and H is swapped. Furthermore, their number is less than $2^{\aleph_0}$ and hence we may apply Martin's Axiom for σ-centered sets. Thus we obtain a filter $\mathbb{G}$ which is generic with respect to the sets in (i) and (ii). Define

$$a = \sum_{l<\omega} a_l \in R^\omega$$

via $a_l = a_l^p$ for all $p \in \mathbb{G}$ and $l < l^p$. Clearly, a is well defined since $\mathbb{G}$ is directed. By the same arguments as in Step Lemma 1.2.1 we obtain that $a \in R^\omega$, $\phi_\alpha(g, a) \in R$ for all $g \in G$ and $a \notin H$. Hence a is the desired element. □

We finish this section with constructing the desired primal module G of H. As discussed above, it is necessary to assume $\mathbb{S}$ is full in R and that $H \subseteq_* \mathbb{D}$ is admissible. Moreover, the above theorem shows that $|G| = 2^{\aleph_0}$ is also necessary.

Theorem 1.3.13 (ZFC + ¬CH + MA) *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$ and let $\mathbb{S}$ be full. Then there is a primal R-module G of H with $S \subseteq G \subseteq_* \mathbb{D}$ and $|G| = 2^{\aleph_0}$.*

PROOF. Let $H \subseteq_* \mathbb{D}$ be as above and let

$$^{\omega}R = \{\varphi_\alpha \mid \alpha < 2^{\aleph_0}\}.$$

We use the enumeration of $^{\omega}R$ to describe all candidates for homomorphisms $\varphi : G \longrightarrow R$; this is possible since every homorphism $\varphi : \mathbb{D} \longrightarrow \widehat{R}$ is uniquely determined by $\varphi\restriction S$, respectively by $\varphi(e_i)$ for all $i < \omega$.

We inductively construct G as the union of an ascending smooth chain of modules G_α of size less than $2^{\aleph_0}$ $(\alpha < 2^{\aleph_0})$.

First we put $G_0 = S$. Then $|G_0| = \aleph_0 < 2^{\aleph_0}$ and $\phi : G_0 \times H \longrightarrow R$.

Next we assume that G_β has been constructed for all $\beta < \alpha$ with $|G_\beta| < 2^{\aleph_0}$.

If α is a limit ordinal, we put $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Clearly, $|G_\alpha| < 2^{\aleph_0}$.

If $\alpha = \beta + 1$ is a successor ordinal, we differentiate between two cases:

If $\varphi_\alpha : G_\beta \longrightarrow R$ is a homomorphism such that $\varphi_\alpha \neq \phi(-, h)$ for all $h \in H$, then we apply Lemma 1.3.1 and Step Lemma 1.3.10. Hence there are an element $b = \sum_{i<\omega} b_i e_i \in R^\omega$ such that $\varphi_\alpha = \phi(-, b)$ and also an element $a_\beta \in \mathbb{D} \backslash G_\beta$ such that $\phi : G_\alpha \times H \longrightarrow R$, where $G_\beta \subsetneq G_\alpha = \langle G_\beta, a_\beta \rangle_* \subseteq_* \mathbb{D}$ and $\phi(a, b) \in \widehat{R} \setminus R$. Therefore $|G_\alpha| < 2^{\aleph_0}$ and $\mathrm{Im}(\varphi_\alpha\restriction G_\alpha) = \mathrm{Im}(\phi(-, b)\restriction G_\alpha) \nsubseteq R$.

Otherwise, we put $G_\alpha = G_\beta$.

Finally, assume that all G_α for $\alpha < 2^{\aleph_0}$ are constructed and put $G = \bigcup_{\alpha<2^{\aleph_0}} G_\alpha$. It follows immediately from the construction that G has the desired properties: $H \cong G^*$ via $h \mapsto \phi(-, h)$. □

1.4 Constructing primals with small endomorphism ring

In this section, we refine the techniques developed in the previous two sections in order to extend the achieved results. Let R, $\mathbb{S}$ be as before such that $\mathbb{S}$ is full in R and $P = R^\omega$ is separable. Given $H \subseteq_* \mathbb{D}$ admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$ we want to construct a primal module $S \subseteq G \subseteq_* \mathbb{D}$ of H with $|G| = 2^{\aleph_0}$, i.e. $G^* \cong H$ and $\mathrm{End}(G) = R \oplus \mathrm{Fin}(G)$, where $\mathrm{Fin}\, G \lhd R$ is the ideal of all endomorphisms of G with finite rank images. Since G is separable, as a pure submodule of $\mathbb{D}$, this is the smallest possible endomorphism ring. Note, a module G with such an endomorphism ring is called *essentially indecomposable.* We shall basically engage the same strategy as before. This means, we need a step lemma before we can construct our desired primal module G. In fact, here we will also need an extended step lemma which provides a whole family of possible extensions.
We begin with an important observation for endomorphisms of G. The result provides similar control of endomorphisms as Lemma 1.3.1 does for dual maps.

Lemma 1.4.1 *Let $S \subseteq G \subseteq_* \mathbb{D}$ and $\varphi \in \mathrm{End}\, G$. Then there are unique elements $b^n \in R^\omega$ for $n < \omega$ such that*

$$\varphi(x) = \sum_{n<\omega} b^n x_n$$

for all $x = \sum_{n<\omega} x_n e_n \in G$, where this sum is understood in R^ω, i.e. every endomorphism $\varphi \in \mathrm{End}\, G$ is determined by the images $b^n = \varphi(e_n)$ $(n < \omega)$.
PROOF. Let G, φ be as above and let $x \in G$ be arbitrary with $x = \sum_{n<\omega} x_n e_n$. We put $b^n = \varphi(e_n) \in G \subseteq R^\omega$ for all $n < \omega$. Then, since $G \subseteq_* \mathbb{D}$, we have

$$\varphi(x) = \varphi(\sum_{n<\omega} x_n e_n) = \sum_{n<\omega} x_n \varphi(e_n) = \sum_{n<\omega} x_n b_n$$

by continuity. $\square$

Again, we will make extensive use of the above property in the following step lemma. As we want to 'kill' all endomorphism φ, which are not of the required form $\varphi = r\,id + \sigma$ with $\sigma \in \operatorname{Fin} G$ and $r \in R$, it is sufficient to construct our module G such that, for a given sequence $\bar{b} = (b^n : n < \omega) \subseteq R^\omega$ (which represents the endomorphism), either $\langle b^n - re_n : n < \omega \rangle_*$ is a free finite rank direct summand of $\mathbb{D}$ for some $r \in R$ or there is an element $a \in G$ such that $\sum_{n<\omega} a_n b^n \notin G$.

We are now ready to formulate the preliminary step lemma. Note, that we could split this step lemma into two separate ones, one for the dual maps and one for the endomorphisms. It is here more convenient to prove the combined version below.

Step Lemma 1.4.2 (ZFC + ¬CH + MA) *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$. Moreover, let $S \subseteq G \subseteq_* \mathbb{D}$ with $|G| < 2^{\aleph_0}$, $\phi(G \times H) \subseteq R$, let $b \in R^\omega \setminus H$, and let $\bar{b} = (b^n : n < \omega) \leq \mathbb{D}$.*
Then there is an element $a = \sum_{i<\omega} a_i e_i \in \mathbb{D} \setminus G$ such that

(i) $G' = \langle G, a \rangle_* \subseteq \mathbb{D}$ *(hence $|G'| < 2^{\aleph_0}$);*

(ii) $\phi(G' \times H) \subseteq R$;

(iii) $\phi(a, b) \in \widehat{R} \setminus R$;

(iv) (a) $\sum_{i<\omega} a_i b^i \notin G'$,

(b) *or there is $t \in R$ such that $U_t = U_t(\bar{b}) := \langle b^i - te_i : i < \omega \rangle_*$ is of finite rank (hence $U_t \sqsubseteq \mathbb{D}$, by separability of $\mathbb{D}$).*

Proof. Let H, $b = \sum_{n<\omega} b_n e_n$, $\bar{b} = (b^n \mid n < \omega)$ be as above. Note, $b_n \in R$ denotes the n-th coordinate of b (lower index n), while $b^n \in \mathbb{D}$ denotes the n-th element of the given family $\bar{b}$ (upper index n). Moreover, we assume that (iv) (b) above does not hold, i.e. U_t is of infinite rank for all $t \in R$. Hence we have to prove that $\sum_{i<\omega} a_i b^i \notin G'$.

As before, we apply Martin's Axiom to the σ-centered poset $\mathfrak{F}$ introduced in Definition 1.1.10 and to the dense subsets defined as follows:

(i) For all $x \in H$, let $D^1_x := \{p \in \mathfrak{F} \ : \ x \in U^p\}$.

(ii) For all $s \in \mathbb{S}$, let $D^2_s := \{p \in \mathfrak{F} \ : \ s | s^p\}$.

(iii) For all $l_0 < \omega$, let $D^3_{l_0} := \{p \in \mathfrak{F} \ : \ l_0 \leq l^p\}$.

(iv) For all $y \in G$, let $D^4_y := \{p \in \mathfrak{F} \ : \ y \in V^p\}$.

(v) For all $r \in R$, let $D^5_z := \{p \in \mathfrak{F} \ : \ \sum_{l<l^p} b_l a^p_l \not\equiv z \bmod s^p\}$.

(vi) For all $d \in G$, $t \in R$ and $s_0 \in \mathbb{S}$, let $D^6_{dts_0} := \{p \in \mathfrak{F} : \exists\, m \in \mathbb{S}\ (m | s^p,$ $s_0 \sum_{i<l^p} a^p_i b^i - t \sum_{i<l^p} a^p_i e_i - d \not\equiv 0 \mod m\mathbb{D})\}$.

The sets defined in (i) - (v) are the same we used in the proof of Step Lemma 1.3.10 and hence we already know that they are dense in $\mathfrak{F}$, as H is admissible. So, it remains to prove the density of $D^6_{dts_0}$ for an arbitrary, but fixed choice d, t, s_0. Suppose, for contradiction, that there are $d \in G$, $t \in R$ and $s_0 \in \mathbb{S}$ such that $D^6_{dts_0}$ is not dense in $\mathfrak{F}$. Then there is $p \in \mathfrak{F}$ such that $q \notin D^6_{dts_0}$ for all $q \geq p$. For convenience, let $U^p = \{u^i = \sum_{n<\omega} u^i_n e_n : 1 \leq i \leq k\}$ and choose $l^p < l < \omega$ arbitrary. Without loss of generality, assume that U^p is linearly independent (see Definition 1.1.10). By Lemma 1.3.6, we may choose $l > k$ large enough such that $U^p \restriction [l^p, l)$ is also linearly independent, as H is admissible.

As we want to consider extensions q of p with $l^q = l$, it is necessary that the system

$$(A_l) \qquad \sum_{l^p \leq j < l} a_j^q u_j^i = 0 \ \ (1 \leq i \leq k)$$

has a non-trivial solution. We define

$$\ker(A_l) = \{(y_{l^p}, \ldots, y_{l-1}) \in R^{l-l^p} : \sum_{l^p \leq j < l} y_j u_j^i = 0, 1 \leq i \leq k\}.$$

Clearly, $\ker(A_l) = \{u^i \restriction [l^p, l) : 1 \leq i \leq k\}^\perp$. Using the same arguments as in the proof of Step Lemma 1.3.10, we deduce that $\ker(A_l)$ is non-trivial since $l > k$. Moreover, we abbreviate the sum

$$s_0(\sum_{1 \leq j < l^p} a_j^p b^j + \sum_{l^p \leq j < l} y_j b^j) - t(\sum_{1 \leq j < l^p} a_j^p e_j + \sum_{l^p \leq j < l} y_j e_j) - d \ \ (\in G)$$

by

$$\Xi(y_{l^p}, \ldots, y_{l-1}).$$

Next we prove

$$(y_{l^p}, \ldots, y_{l-1}) \in \ker(A_l) \quad \Longrightarrow \quad \Xi(y_{l^p}, \ldots, y_{l-1}) = 0. \tag{3}$$

Suppose not, i.e. $\Xi(y_{l^p}, \ldots, y_{l-1}) \neq 0$ for some $(y_{l^p}, \ldots, y_{l-1}) \in \ker(A_l)$. Hence there is $s \in S$ such that $\Xi(y_{l^p}, \ldots, y_{l-1}) \not\equiv 0 \bmod s\mathbb{D}$. So, $q \in \mathfrak{F}$ defined via $l^q = l$, $s^q = ss^p$, $U^q = U^p$, $V^q = V^p$ and

$$a_j^q = \begin{cases} a_j^p, \ for \ 1 \leq j < l^p \\ y_j, \ for \ l^p \leq j < l^q. \end{cases}$$

Then $q \in \mathfrak{F}$ with $q \in D^6_{dts_0}$ by assumption. Moreover, q is defined in such a way that $q \geq p$, contradicting our assumption that $q \notin D^6_{dts_0}$ for all $p \leq q$. Therefore the implication (3) holds.

Now, let $\xi^i = \sum_{n<\omega} \xi^i_n e_n = s_0 b^i - t e_i \in \mathbb{D}$ for all $l^p \leq i < \omega$. We rewrite (3) as

$$(y_{l^p}, \dots, y_{l-1}) \in \ker(A_l) \quad \Longrightarrow \quad \sum_{l^p \leq i < l} y_i \xi^i = d + t \sum_{1 \leq i < l^p} a^p_i e_i - s_0 \sum_{1 \leq i < l^p} a^p_i b^i.$$

Since $(0, \dots, 0) \in \ker(A_l)$, we obtain $d + t \sum_{1 \leq i < l^p} a^p_i e_i = s_0 \sum_{1 \leq i < l^p} a^p_i b^i$ and so it follows

$$(y_{l^p}, \dots, y_{l-1}) \in \ker(A_l) \quad \Longrightarrow \quad \sum_{l^p \leq i < l} y_i \xi^i = 0. \tag{4}$$

Note, the above 0 denotes the zero-vector in P. We consider the ξ^i's $(l^p \leq i < l)$ as the infinite rows of a matrix with finite columns $\xi_n := (\xi^i_n : l^p \leq i < l)$ for all $n < \omega$. Then each column ξ_n satisfies $\xi_n \in \ker(A_l)^\perp = \{u^i \restriction [l^p, l) : 1 \leq i \leq k\}^{\perp\perp}$ by (4). Using Lemma 1.3.5, we hence obtain $\xi_n \in \langle u^i \restriction [l^p, l) : 1 \leq i \leq k \rangle_*$ for all $n < \omega$. Hence

$$\xi_n = \sum_{1 \leq i \leq k} r^n_i u^i \restriction [l^p, l) \text{ for all } n < \omega$$

with unique (!) coefficients $r^n_i \in Q$ since $U^p \restriction [l^p, l)$ is linearly independent by assumption. Again, the coefficients remain constant while increasing l and so we can extend ξ_n to ξ'_n, where

$$\xi'_n = \sum_{1 \leq i \leq k} r^n_i u^i \restriction [l^p, \omega) \text{ for all } n < \omega.$$

From the definition of ξ'^j and $b^j = \sum_{n<\omega} b^j_n e_n$ it follows immediately that $\xi'^j_n = s_0 b^j_n - t\delta_{jn} = \sum_{1 \leq i \leq k} r^n_i c^i_n$ for all $l^p \leq n < \omega$ and thus $s_0 b^j - t e_j \restriction [l^p, \omega) \in \langle u^i \restriction [l^p, \omega) : 1 \leq i \leq k \rangle_*$. Therefore we conclude $U = \langle s_0 b^j - t e_j : j < \omega \rangle_*$ is contained in the following pure finite rank submodule of G:

$$\left\langle u^i \restriction [l^p, \omega),\ 1 \leq i \leq k;\ (s_0 b^j - t e_j) \restriction [0, l^p), j < \omega \right\rangle_* .$$

Therefore U is also of finite rank.

We also have that s_0 divides t, since otherwise $\bar{U} = \langle te_j + s_0\mathbb{D} : j < \omega \rangle_*$ is of infinite rank, which is impossible since $\bar{U}$ is an epimorphic image of U.
So, let $t' = s_0^{-1}t$. Then $U = U_{t'} = \langle b^j - t'e_j : j < \omega \rangle_*$, contradicting our assumption that $U_{t'}$ is of infinite rank. This finally implies that $D^6_{dts_0}$ is dense in $\mathfrak{F}$.

Again, we apply Martin's Axiom for σ-centered sets. As before, we obtain the existence of $a \in \mathbb{D}\backslash G$ such that $\phi(G' \times H) \subseteq R$ and $\phi(a, b) \in \widehat{R}\backslash R$, where $G' = \langle G, a \rangle_*$. Hence it remains to prove that (iv) a holds, i.e. $\sum_{i<\omega} a_i b^i \notin G'$. Note, $\sum_{i<\omega} a_i b^i$ is always a well-defined element of $\mathbb{D}$. Suppose, for contradiction, that $\sum_{i<\omega} a_i b^i \in G'$. Hence there are $s_0 \in \mathbb{S}$, $t \in R$ and $d \in G$ such that

$$s_0 \sum_{i<\omega} a_i b^i - ta - d = 0. \tag{5}$$

Since $D^6_{dts_0}$ is dense in $\mathfrak{F}$, we may choose $p \in \mathbb{G} \cap D^6_{dts_0}$. Hence we obtain

$$s_0 \sum_{i<l^p} a_i^p b^i - t \sum_{i<l^p} a_i^p e_i - d \in \mathbb{D} \setminus m\mathbb{D}$$

for some $m \in \mathbb{S}$ with $m|s^p$ since $p \in D^6_{dts_0}$. Furthermore, we know that $a_i = a_i^p$ for all $i < l^p$ from $p \in \mathbb{G}$. As the set $\mathbb{G}$ is directed, we conclude that $m|a_i$ for all $i \geq l^p$, and so $s_0 \sum_{i \geq l^p} a_i b^i \in m\mathbb{D}$, as well as $t \sum_{i \geq l^p} a_i e_i \in m\mathbb{D}$. Therefore we finally obtain that $s_0 \sum_{i<\omega} a_i b^i - ta - d \in \mathbb{D} \setminus m\mathbb{D}$, contradicting equation (5). Therefore (iv) a holds, as required. □

As mentioned before, the above step lemma is just preliminary. We extend it to the following step lemma, which provides a whole family of elements a satisfying the conclusion of Step Lemma 1.4.2. The existence of such a family is necessary for our construction, since it is possible that an endomorphism

$\varphi \notin R \oplus \mathrm{Fin}(G_\alpha)$ killed in the α-th step may 'resurrect' at a later stage in the construction. In order to control this phenomenon, we shall introduce a 'blacklist' of unwanted homomorphisms.

Step Lemma 1.4.3 (ZFC + ¬CH + MA) *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$. Moreover, let $S \subseteq G \subseteq_* \mathbb{D}$ with $|G| < 2^{\aleph_0}$, $\phi(G \times H) \subseteq R$, let $b \in R^\omega \setminus H$, and let $\bar{b} = (b^n : n < \omega) \leq \mathbb{D}$.*
Then there exists a linearly independent set $\mathcal{A} \subseteq \mathbb{D}$ of size $2^{\aleph_0}$, such that, for all $a = \sum_{i<\omega} a_i e_i \in \mathcal{A}$:

(i) $G' = \langle G, a \rangle_* \subseteq \mathbb{D}$ *(hence $|G'| < 2^{\aleph_0}$);*

(ii) $\phi(G' \times H) \subseteq R$;

(iii) $\phi(a, b) \in \widehat{R} \setminus R$;

(iv) (a) $\sum_{i<\omega} a_i b^i \notin G'$,

(b) *or there is $t \in R$ such that $U_t = U_t(\bar{b}) := \langle b^i - te_i : i < \omega \rangle_*$ is of finite rank;*

(v) $a \notin G$.

Proof. Let G, H be as above. We inductively construct modules $G_\alpha \subseteq_* \mathbb{D}$ of size $|G_\alpha| < 2^{\aleph_0}$ and elements $a^\alpha \in \mathbb{D}$ such that $a^\alpha \in G_{\alpha+1} \setminus G_\alpha$ by applying the preliminary step lemma $2^{\aleph_0}$ times.
We start with $G_0 = G$.
Now assume that we have already constructed G_β and $a^{\beta'}$ for all $\beta' + 1, \beta < \alpha$ satisfying the above properties.
If α is a limit ordinal, then put $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Clearly, $|G_\alpha| < 2^{\aleph_0}$.

If $\alpha = \beta + 1$ is a successor ordinal, then we apply Step Lemma 1.4.2 to G_β and obtain a^β and $G_\alpha = \left\langle G_\beta, a^\beta \right\rangle_*$ such that $a^\beta \notin G_\beta$ and $|G_\alpha| < 2^{\aleph_0}$.
Finally, we let

$$G^+ = \bigcup_{\alpha < 2^{\aleph_0}} G_\alpha.$$

We show that $\mathcal{A} = \{a^\alpha : \alpha < 2^{\aleph_0}\}$ is the desired family. It follows immediately from the construction that $\mathcal{A}$ is linearly independent. Moreover, by Step Lemma 1.4.2, we have that $G' = \langle G, a^\alpha \rangle_* \subseteq G_{\alpha+1}$ and hence $\phi(G' \times H) \subseteq \phi(G_{\alpha+1} \times H) \subseteq R$. It is also clear that $\phi(a^\alpha, b) \in \widehat{R} \setminus R$. Finally, if $\sum_{i<\omega} a_i^\alpha b^i \notin G'$, then also $\sum_{i<\omega} a_i^\alpha b^i \notin G_{\alpha+1}$ by construction, and hence U_t is of finite rank for some $t \in R$. This finishes the proof. □

Next we thin out the above family $\mathcal{A}$ depending on a given 'blacklist'. For an R-module G, we call a subset $\mathcal{B}$ of $\mathbb{D}$ a *blacklist* with respect to G, if $|\mathcal{B}| < 2^{\aleph_0}$ and $\mathcal{B} \cap G = \emptyset$.

Lemma 1.4.4 *Let $G \subseteq_* \mathbb{D}$ be an R-module with $|G| < 2^{\aleph_0}$ and let $\mathcal{B} \leq \mathbb{D}$ be a blacklist with respect to G. Moreover, let $\mathcal{A} \leq \mathbb{D}$ be a family of linearly independent elements of cardinality $2^{\aleph_0}$.*
Then there is a subfamily $\mathfrak{H} \leq \mathcal{A}$ of cardinality $2^{\aleph_0}$ such that $\mathcal{B} \cap \langle G, a \rangle_* = \emptyset$ for all $a \in \mathfrak{H}$.

Proof. Let G, $\mathcal{B}$, $\mathcal{A}$ be as above. Consider $b \in \mathcal{B}$, $a \in \mathcal{A}$ arbitrary with $\mathcal{B} \cap \langle G, a \rangle_* \neq \emptyset$. Then $r_b b = g + r_a a$ for some $r_b, r_a \in R$, where $r_a \neq 0$ as $b \notin G$. We conclude $a \in \langle G, b \rangle_* \subseteq \langle G, \mathcal{B} \rangle_*$.
However, $|\langle G, \mathcal{B} \rangle_*| < 2^{\aleph_0} = |\mathcal{A}|$ and hence there is $\mathfrak{H} \subseteq \mathcal{A}$ with the desired property. □

We are now ready to construct our desired module G.

Theorem 1.4.5 (ZFC + ¬CH + MA) *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq H < 2^{\aleph_0}$. Then there is $S \subseteq G \subseteq_* \mathbb{D}$ with the following properties:*

(i) *for all $x \in R^\omega \setminus H$ there is $g \in G$ such that $\phi(g, x) \in \widehat{R} \setminus R$;*

(ii) *for all $\bar{b} = (b^n : n < \omega) \leq \mathbb{D}$*

   (a) *there is $a \in G$ with $\sum_{n<\omega} a_n b^n \notin G$, or*

   (b) *there $t \in R$ such that $U_t(\bar{b}) = \langle b^n - te_n : n < \omega\rangle_*$ is of finite rank;*

(iii) *$\phi : G \times H \longrightarrow R$ is well defined and $|G| = 2^{\aleph_0}$.*

Proof. Let H be as above and enumerate $R^\omega \setminus H$, $\mathbb{D}^\omega$ by

$$R^\omega \setminus H = \{b^\alpha = (b_0^\alpha, \dots, b_n^\alpha, \dots) \mid \alpha < 2^{\aleph_0}\}$$

and

$$\mathbb{D}^\omega = \{c^\alpha = (c_0^\alpha, \dots, c_n^\alpha, \dots) \mid \alpha < 2^{\aleph_0}\}.$$

Note, that the elements in $R^\omega \setminus H$ are vectors and those in $\mathbb{D}^\omega$ are ω-sequences of vectors. Moreover, for each $t \in R$ and $\beta < 2^{\aleph_0}$, we define $U_t(c^\beta) := \left\langle c_n^\beta - te_n \mid n < \omega\right\rangle_*$ (note: $c_n^\beta \in \mathbb{D}$).

We inductively construct pure submodules $G_\alpha \subseteq \mathbb{D}$ of cardinality less than $2^{\aleph_0}$ such that $\phi(G_\alpha \times H) \subseteq R$. Furthermore, we maintain an (increasing) blacklist $\mathcal{B}$ (with respect to G) throughout the construction.

We start with $G_0 = S$ and $\mathcal{B} = \emptyset$. Then $|G_0| < 2^{\aleph_0}$ and $\phi : G_0 \times H \longrightarrow R$. Note, although the blacklist $\mathcal{B}$ will be enlarged successively, we keep calling it $\mathcal{B}$.

Now assume that G_β has been constructed for all $\beta < \alpha$ satisfying the above properties and that $\mathcal{B}$ is defined such that $G_\beta \cap \mathcal{B} = \emptyset$.

For α a limit ordinal, define $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ and let the blacklist $\mathcal{B}$ be the union of all its predecessors. Then $|G_\alpha| < 2^{\aleph_0}$ and $\mathcal{B} \cap G_\alpha = \emptyset$.

Now, let $\alpha = \beta+1$ be a successor cardinal and let $b^\beta \in R^\omega \setminus H$, $c^\beta \in \mathbb{D}^\omega$ be from the above list. We apply Step Lemma 1.4.3 to G_β, b^β, and c^β. Hence there exists a family $\mathcal{A}$ of linearly independent elements satisfying the conclusion. In particular, for all $a = \sum_{i<\omega} a_i e_i \in \mathcal{A}$, we have $\sum_{i<\omega} a_i c_i^\beta \notin \langle G_\beta, a\rangle_*$ or $U_t(c^\beta)$ is of finite rank for some $t \in R$. By Lemma 1.4.4 there is a subfamily $\mathfrak{H} \leq \mathcal{A}$ of size $2^{\aleph_0}$ such that $\langle G_\beta, a\rangle \cap \mathcal{B} = \emptyset$ for all $a \in \mathfrak{H}$.
We choose $a^\beta = a \in \mathfrak{H}$ arbitrary and put

$$G_\alpha = \left\langle G_\beta, a^\beta \right\rangle_* .$$

Moreover, we redefine $\mathcal{B} := \mathcal{B} \cup \{\sum_{i<\omega} a_i^\beta c_i^\beta\}$ if $U_t(c^\beta)$ is of infinite rank for all $t \in R$.
Finally, we define G by

$$G = \bigcup_{\alpha<2^{\aleph_0}} G_\alpha = \left\langle S, a^\alpha \mid \alpha < 2^{\aleph_0} \right\rangle_* .$$

It is clear that $|G| = 2^{\aleph_0}$ and $\phi(G \times H) \subseteq R$. Moreover, if $x \in R^\omega \setminus H$ then there is $\alpha < 2^{\aleph_0}$ such that $x = b^\alpha$ and hence $\phi(a^\alpha, x) \in \widehat{R} \setminus R$, by construction and Step Lemma 1.4.3. Also, if $\bar{b} \leq \mathbb{D}$ then $\bar{b} = c^\alpha$ for some $\alpha < 2^{\aleph_0}$. Therefore, if $U_t(c^\alpha)$ is of infinite rank for all $t \in R$, then $\sum_{i<\omega} a_i^\alpha b^i \notin G_{\alpha+1}$ and $\sum_{i<\omega} a_i^\alpha b^i \in \mathcal{B}$, by construction. This now implies that $\sum_{i<\omega} a_i^\alpha b^i \notin G_\gamma$ for all $\gamma > \alpha$ and hence $\sum_{i<\omega} a_i^\alpha b^i \notin G$. $\square$

We finish this section with proving that the above constructed module G has the desired properties. However, we first need the following observation:

Observation 1.4.6 *Let $S \subseteq G \subseteq_* \mathbb{D}$ and $S \subseteq H \subseteq_* \mathbb{D}$. Then any homomorphism $\varphi : G \longrightarrow H$ with $\varphi(S)$ of finite rank, is itself of finite rank.*

Proof. Let $B = \langle \varphi(S) \rangle_*$. Then $H = B \oplus C$ for some $C \subseteq H$, by assumption and by the separability of H. Consider the induced homomorphism

$$\bar{\varphi} : G/S \longrightarrow H/\varphi(S) \longrightarrow H/B \quad \text{defined via} \quad d + S \mapsto \varphi(d) + B.$$

Since G/S is divisible while $H/B \cong C$ is reduced, we deduce $\varphi(G) \subseteq B$. $\square$

Finally, we are ready to prove the main result of this section.

Theorem 1.4.7 *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq H < 2^{\aleph_0}$. Then there exists a primal module $S \subseteq G \subseteq_* \mathbb{D}$ of H of size $2^{\aleph_0}$ such that* $\operatorname{End} G = R \oplus \operatorname{Fin} G$.

Proof. Let H be as above and let G be the module constructed in the proof of Theorem 1.4.5. It follows immediately from the results in Section 1.3 that G is a primal module of H of the correct size, i.e. $G^* \cong H$.

Hence it remains to prove that $\operatorname{End} G = R \oplus \operatorname{Fin} G$. Consider an arbitrary $\sigma \in \operatorname{End} G$. By Lemma 1.4.1, there are elements $b^n = \sigma(e_n)$ of G $(n < \omega)$ such that $\sigma(x) = \sum_{n<\omega} x_n b^n$ for all $x = \sum_{n<\omega} x_n e_n \in G$. Hence property (ii) a of Theorem 1.4.5 is violated for all $x \in G$. Therefore there is $t \in R$ such that

$$U_t(\bar{b}) = \left\langle b^j - te_j : j < \omega \right\rangle_*$$

is of finite rank.

We show that $\varphi := \sigma - t\,id$ is of finite rank. Consider $\varphi(S)$. We obviously have that $\varphi(S) \subseteq U_t$. Hence, by Observation 1.4.6, we deduce that $\varphi(G)$ is also of finite rank. Therefore $\sigma = t\,id + \varphi \in R \oplus \mathrm{Fin}(G)$ as required. $\square$

1.5 A fully rigid system of primal modules

In this final section of our first chapter, we extend, yet again, the previous results in order to gain the existence of a fully rigid system $\mathfrak{G}$ of size $2^{\aleph_0}$ of primal modules of a given $H \subseteq_* \mathbb{D}$ as before. Recall:

Definition 1.5.1 *A family $\mathfrak{G} = \{G_I : I \in \mathcal{P}(\omega)\}$ of R-submodules of $\mathbb{D}$ is said to be a* fully rigid system*, if*

$$\mathrm{Hom}(G_I, G_J) = \begin{cases} R \oplus \mathrm{Fin}(G_I, G_J), \text{ if } I \subseteq J \\ \mathrm{Fin}(G_I, G_J), \text{ if } I \not\subseteq J. \end{cases}$$

Note, modules of a fully rigid system are pairwise non-isomorphic. Also in general, for given R-modules G, W with $\mathrm{End}(G) = R \oplus \mathrm{Fin}(G)$ and $\mathrm{End}(W) = R \oplus \mathrm{Fin}(W)$, we have that $G \cong W$ if and only if $\mathrm{Fin}(G) = \mathrm{Fin}(W)$ (cf. [14, p. 461]).

First we describe homomorphisms from G to W, where $S \subseteq G, W \subseteq_* \mathbb{D}$. The result below provides control of the homomorphisms $\varphi : G \longrightarrow W$ in a similar way as Lemma 1.3.1 does for dual maps and Lemma 1.4.1 does for endomorphisms.

Lemma 1.5.2 *Let $S \subseteq G \subseteq_* \mathbb{D}$, $S \subseteq W \subseteq_* \mathbb{D}$ and $\varphi \in \mathrm{Hom}(G, W)$. Then there is a set of unique (!) elements $\{b^n \in \mathbb{D} : n < \omega\}$ such that*

$$\varphi(x) = \sum_{n<\omega} b^n x_n$$

for all $x = \sum_{n<\omega} x_n e_n \in G$, i.e. every homomorphism $\varphi \in \mathrm{Hom}(G, W)$ is given by the images $\varphi(e_n) = b^n$ $(n < \omega)$.

Proof. Let $x \in G$ be arbitrary with $x = \sum_{n<\omega} x_n e_n$ and put $b^n = \varphi(e_n) \in W \subseteq \mathbb{D}$ for all $n < \omega$. Then, as before,

$$\varphi(x) = \varphi(\sum_{n<\omega} x_n e_n) = \sum_{n<\omega} x_n \varphi(e_n) = \sum_{n<\omega} x_n b_n$$

by continuity. $\square$

As we want to construct a fully rigid system $\mathfrak{G}$ of primal modules, we need to take care of modules G, W with $\mathrm{Hom}(G,W) = R \oplus \mathrm{Fin}(G,W)$, and also of those with $\mathrm{Hom}(G,W) = \mathrm{Fin}(G,W)$. The first case is dealt with by applying the following step lemma inductively.

Step Lemma 1.5.3 (ZFC + ¬CH + MA) *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$. Moreover, let $S \subseteq G, W \subseteq_* \mathbb{D}$ with $|G|, |W| < 2^{\aleph_0}$, $\phi(G \times H) \subseteq R$, $\phi(W \times H) \subseteq R$, and let $\bar{b} = (b^n : n < \omega) \leq \mathbb{D}$.*
Then there is an element $a = \sum_{i<\omega} a_i e_i \in \mathbb{D} \setminus G$ such that

(i) $G' = \langle G, a\rangle_* \subseteq \mathbb{D}$ *(hence $|G'| < 2^{\aleph_0}$);*

(ii) $W' = \langle W, a\rangle_* \subseteq \mathbb{D}$ *(hence $|G'| < 2^{\aleph_0}$);*

(iii) $\phi(G' \times H) \subseteq R$;

(iv) $\phi(W' \times H) \subseteq R$;

(v) (a) $\sum_{i<\omega} a_i b^i \notin W'$, *or*

(b) *there is $t \in R$ such that $U_t := \langle b^i - te_i : i < \omega\rangle_*$ is of finite rank.*

Proof. Let H, G, W, $\bar{b}$ be as above. We assume that (v) b above does not hold, i.e. U_t is of infinite rank for all $t \in R$. Hence we have to prove that $\sum_{i<\omega} a_i b^i \notin W'$.

As before, we apply Martin's Axiom to the σ-centered poset $\mathfrak{F}$ introduced in Definition 1.1.10 and to the dense subsets defined as follows:

(i) For all $x \in H$, let $D^1_x := \{p \in \mathfrak{F} \ : \ x \in U^p\}$.

(ii) For all $s \in \mathbb{S}$, let $D^2_s := \{p \in \mathfrak{F} \ : \ s|s^p\}$.

(iii) For all $l_0 < \omega$, let $D^3_{l_0} := \{p \in \mathfrak{F} \ : \ l_0 \leq l^p\}$.

(iv) For all $y \in G$, let $D^4_y := \{p \in \mathfrak{F} \ : \ y \in V^p\}$.

(v) For all $d \in W$, $t \in R$ and $s_0 \in \mathbb{S}$, let $D^6_{dts_0} := \{p \in \mathfrak{F} : \exists\, m \in \mathbb{S}\ (m|s^p,$ $s_0 \sum_{i<l^p} a^p_i b^i - t \sum_{i<l^p} a^p_i e_i - d \not\equiv 0 \mod m\mathbb{D})\}$.

The sets defined in (i) - (iv) are the same we used in the prove of Step Lemma 1.3.10 and hence we already know that they are dense in $\mathfrak{F}$. Moreover, $D^6_{dts_0}$ is defined similarly to the one in the proof of Step Lemma 1.4.2 (by replacing $d \in G$ by $d \in W$). Hence we may use the same arguments and obtain that $D^6_{dts_0}$ is also dense in $\mathfrak{F}$.

Again, we apply Martin's Axiom for σ-centered sets. As before, we obtain the existence of $a \in \mathbb{D} \setminus G$ such that $\phi(G' \times H) \subseteq R$ and $\phi(W' \times H) \subseteq R$, where $G' = \langle G, a\rangle_*$ and $W' = \langle W, a\rangle_*$ (see Corollary 1.2.2). Hence it remains to prove that (v) a holds, i.e. $\sum_{i<\omega} a_i b^i \notin W'$. However, this also follows using the same arguments as in the proof of Step Lemma 1.4.2, since we changed the definition $D^6_{dts_0}$ accordingly. □

Note, that in the above lemma both modules are extended. Next we provide the step lemma for constructing primal modules G, W with $\mathrm{Hom}(G, W) = \mathrm{Fin}(G, W)$; here we extend only G.

Step Lemma 1.5.4 (ZFC + ¬CH + MA) *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$. Moreover, let $S \subseteq G, W \subseteq_* \mathbb{D}$ with $|G|, |W| < 2^{\aleph_0}$, $\phi(G \times H) \subseteq R$, $\phi(W \times H) \subseteq R$, and let $\bar{b} = (b^n : n < \omega) \leq \mathbb{D}$.*
Then there is an element $a = \sum_{i<\omega} a_i e_i \in \mathbb{D} \setminus G$ such that

(i) $G' = \langle G, a \rangle_* \subseteq \mathbb{D}$ *(hence $|G'| < 2^{\aleph_0}$);*

(ii) $\phi(G' \times H) \subseteq R$*;*

(iii) (a) $\sum_{i<\omega} a_i b^i \notin W$*, or*

(b) $U_0 := \langle b^i : i < \omega \rangle_*$ *is of finite rank.*

Proof. The proof is basically the same as the proof of Step Lemma 1.5.3 replacing the dense subsets $D^6_{dts_0}$ by

$$D^{6'}_{ds_0} := \{p \in \mathfrak{F} : \exists\, m \in \mathbb{S}\ (m|s^p,\ s_0 \sum_{i<l^p} a_i^p b^i - d \not\equiv 0 \mod m\mathbb{D})\}$$

for all $d \in W$ and $s_0 \in \mathbb{S}$. □

As in Section 1.4, we actually need extended version of the above step lemmas, providing whole families of suitable elements. Since the proofs are exactly the same as the one of Step Lemma 1.4.3, we only state the needed results below.

Step Lemma 1.5.5 (ZFC + ¬CH + MA) *Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$. Moreover, let $S \subseteq G, W \subseteq_* \mathbb{D}$ with $|G|, |W| < 2^{\aleph_0}$, $\phi(G \times H) \subseteq R$, $\phi(W \times H) \subseteq R$, and let $\bar{b} = (b^n : n < \omega) \leq \mathbb{D}$.*
Then there is a family $\mathcal{A} \leq \mathbb{D}$ of linearly independent elements $a = \sum_{i<\omega} a_i e_i \in \mathcal{A} \setminus G$ such that, for all $a \in \mathcal{A}$,

(i) $G' = \langle G, a \rangle_* \subseteq \mathbb{D}$ *(hence $|G'| < 2^{\aleph_0}$);*

(ii) $W' = \langle W, a \rangle_* \subseteq \mathbb{D}$ *(hence* $|G'| < 2^{\aleph_0}$*);*

(iii) $\phi(G' \times H) \subseteq R$;

(iv) $\phi(W' \times H) \subseteq R$;

(v) (a) $\sum_{i<\omega} a_i b^i \notin W'$,

(b) *or there is* $t \in R$ *such that* $U_t := \langle b^i - te_i : i < \omega \rangle_*$ *is of finite rank.*

Step Lemma 1.5.6 (ZFC + ¬CH + MA) *Let* $H \subseteq_* \mathbb{D}$ *be admissible with* $\aleph_1 \leq |H| < 2^{\aleph_0}$. *Moreover, let* $S \subseteq G, W \subseteq_* \mathbb{D}$ *with* $|G|, |W| < 2^{\aleph_0}$, $\phi(G \times H) \subseteq R$, $\phi(W \times H) \subseteq R$, *and let* $\bar{b} = (b^n : n < \omega) \leq \mathbb{D}$.
Then there is a family $\mathcal{A} \leq \mathbb{D}$ *of linearly independent elements* $a = \sum_{i<\omega} a_i e_i \in \mathcal{A} \setminus G$ *such that, for all* $a \in \mathcal{A}$,

(i) $G' = \langle G, a \rangle_* \subseteq \mathbb{D}$ *(hence* $|G'| < 2^{\aleph_0}$*);*

(ii) $\phi(G' \times H) \subseteq R$;

(iii) (a) $\sum_{i<\omega} a_i b^i \notin W$,

(b) $U_0 := \langle b^i - te_i : i < \omega \rangle_*$ *is of finite rank.*

We are now ready to construct the desired fully rigid system $\mathfrak{G}$. Since the construction needs to take care of several properties at the same time, we present it separately, before we prove the main result.

Construction 1.5.7 (ZFC + ¬CH + MA) Let $H \subseteq_* \mathbb{D}$ be admissible with $\aleph_1 \leq |H| < 2^{\aleph_0}$. We construct a family $\mathfrak{G} = \{G^\delta \mid \delta < 2^{\aleph_0}\}$ of pure submodules of $\mathbb{D}$; each module is of the form $G^\delta = \bigcup_{\alpha < 2^{\aleph_0}} G^\delta_\alpha$. The construction is done inductively in the following sense: First we define G^0_0, then G^1_0, G^0_1, G^1_1, then G^2_0, G^2_1, G^0_2, G^1_2, G^2_2, and so on. One could think of this construction as extending the 'staircase' before taking the next step, rather than just taking step by step, as usual. Moreover, we maintain a family $\{\mathcal{B}^\delta \mid \delta < 2^{\aleph_0}\}$ of blacklists, one for each G^δ.

Now, let $\mathcal{P}(\omega)$ be enumerated by

$$\mathcal{P}(\omega) = \{I^\delta \mid \delta < 2^{\aleph_0}\},$$

and $R^\omega \setminus H$ by

$$R^\omega \setminus H = \{d^\alpha \mid \alpha < 2^{\aleph_0}\}.$$

Moreover, we enumerate $\mathbb{D}^\omega$ by

$$\mathbb{D}^\omega = \{f^\alpha \mid \alpha < 2^{\aleph_0}\},$$

as well as by

$$\mathbb{D}^\omega = \{h^\alpha \mid \alpha < 2^{\aleph_0}\}.$$

Note, the subsets I^δ of ω will correspond to G^δ, the d^α's are for dealing with the dual maps, the f^α's are for dealing with the endomorphisms, and the h^α's are for dealing with the homomorphisms between the different modules; of course we could just consider the endomorphisms as homomorphisms but it is more convenient to treat them separately. Moreover note, that all extensions below are obtained by applying the Step Lemma 1.4.3, 1.5.5, 1.5.6, and thus, in each case, we have that the canonical scalar product $\phi : G^\delta_\alpha \times H \longrightarrow R$ is well defined. Furthermore, by the step lemmas and by Lemma 1.4.4, we may always choose an extension in such a way that the intersection with the relevant blacklist, remains empty.

We start with putting $G^0_0 := S$ and $\mathcal{B}^\delta := \emptyset$ for all $\delta < 2^{\aleph_0}$.

Next, we consider $\alpha = \beta + 1$ and assume that G_γ^δ has been constructed for all $\gamma, \delta < \alpha$. Here we first want to define G_γ^α for $\gamma < \alpha$, and then G_α^δ for all $\delta \leq \alpha$. This will be done in several steps.

Step 1: We define inductively G_γ^α for all $\gamma \leq \beta$, by applying the Step Lemma 1.4.3 to the f^γ's and d^γ's successively

Let $G_0^\alpha = S$ and put $G_\gamma^\alpha = \bigcup_{\gamma' < \gamma} G^\alpha$ if $\gamma \leq \beta$ is a limit ordinal; in the latter case we redefine $\mathcal{B}^\alpha$ to be the union of the previous $\mathcal{B}^\alpha$'s.

If G_γ^α is defined for $\gamma < \beta$, then we apply Step Lemma 1.4.3 to $G = G_\gamma^\alpha$, $b = d^\gamma$ and $\bar{b} = f^\gamma \in \mathbb{D}\,(n < \omega)$. Hence we deduce the existence of an element $a \in \mathbb{D} \backslash G$ and a module $G_{\gamma+1}^\alpha = \left\langle G_\gamma^\alpha, a \right\rangle_*$ such that $\phi(a, d^\gamma) \in \widehat{R} \setminus R$, $\sum_{i<\omega} a_i f_i^\gamma \notin G_{\gamma+1}^\alpha$ or $U_t(f^\gamma)$ if of finite rank for some $t \in R$, and $\mathcal{B}^\alpha \cap G_{\gamma+1}^\alpha = \emptyset$. If $U_t(f^\gamma)$ if of infinite rank for all $t \in R$, we redefine $\mathcal{B}^\alpha := \mathcal{B}^\alpha \cup \{\sum_{i<\omega} a_i f_i^\gamma\}$.

Step 2: Now we deal with possible homomorphisms between G_β^α and G_β^δ for all $\delta < \alpha$. This will be done by applying Step Lemma 1.5.5 and Step Lemma 1.5.6 to all h^γ's ($\gamma < \beta$) several times, according to $I^\alpha \subseteq I^\delta$, $I^\alpha \not\subseteq I^\delta$, $I^\delta \subseteq I^\alpha$ and $I^\delta \not\subseteq I^\alpha$. For convenience, we shall denote the extended module also by G_β^γ.

Let $h^\gamma \in \mathbb{D}^\omega$ ($\gamma < \beta$) be from the above list and let $\delta < \alpha$. First we consider h^γ as candidate for a homomorphism from G_β^α to G_β^δ.

<u>Case $I^\alpha \leq I^\delta$:</u> We apply Step Lemma 1.5.5 to $\bar{b} = h^\gamma$, $G = G_\beta^\alpha$, $W = \left\langle G_\beta^{\delta'} \mid \delta' < \alpha, I^\delta \leq I^{\delta'} \right\rangle_*$. Therefore there exists a linearly independent set $\mathcal{A}$ of size $2^{\aleph_0}$ such that, for all $a \in \mathcal{A}$, $\sum_{i<\omega} a_i h_i^\gamma \notin W' = \langle W, a \rangle_*$ or $U_t(h^\gamma)$ is of finite rank for some $t \in R$. Hence we can find an element $a \in \mathcal{A}$ such that $\mathcal{B}^{\delta'} \cap \left\langle G_\beta^{\delta'}, a \right\rangle_* = \emptyset$ for all $\delta' < \alpha$ with $I^\delta \leq I^{\delta'}$, and such that $\mathcal{B}^\alpha \cap \left\langle G_\beta^\alpha, a \right\rangle_* = \emptyset$ (see Lemma 1.4.4). We put $G_\beta^\alpha := \left\langle G_\beta^\alpha, a \right\rangle_*$ and $G_\beta^{\delta'} := \left\langle G_\beta^{\delta'}, a \right\rangle_*$ for all $\delta' < \alpha$ with $I_\delta \leq I_{\delta'}$. Moreover, if $U_t(h^\gamma)$ is of infinite rank for all $t \in R$, then we put $\mathcal{B}^{\delta'} := \mathcal{B}^{\delta'} \cup \{\sum_{i<\omega} a_i h_i^\gamma\}$ for all relevant δ''s. Note, in this case we clearly have

$\sum_{a_i h_i^\gamma} \notin G_\beta^{\delta'}$.

Case $I^\alpha \not\leq I^\delta$: We apply Step Lemma 1.5.6 to $\bar{b} = h^\gamma$, $G = G_\beta^\alpha$ and $W = G_\beta^\delta$. Hence there exists an element $a \in \mathbb{D} \setminus G_\beta^\alpha$ such that $\sum_{i<\omega} a_i h_i^\gamma \notin G_\beta^\delta$ or $U_0(h^\gamma)$ is of finite rank and $\mathcal{B}^\alpha \cap \left\langle G_\beta^\alpha, a \right\rangle_* = \emptyset$. We put $G_\beta^\alpha := \left\langle G_\beta^\alpha, a \right\rangle_*$ and, if $U_0(h^\gamma)$ is of infinite rank, then we also put $\mathcal{B}^\delta := \mathcal{B}^\delta \cup \{\sum_{i<\omega} a_i h_i^\gamma\}$.

Next we consider h^γ as candidate for a homomorphism from G_β^δ to G_β^α. This is done in the same way as above, so we only state the modules, to which the step lemmas are applied.

Case $I^\delta \leq I^\alpha$: Apply Step Lemma 1.5.5 to $\bar{b} = h^\gamma$, $G = G_\beta^\delta$ and $W = \left\langle G_\beta^\alpha, G_\beta^{\delta'} : \delta' < \alpha, I^\alpha \leq I^{\delta'} \right\rangle_*$.

Case $I^\delta \not\leq I^\alpha$: Apply Step Lemma 1.5.6 to $\bar{b} = h^\gamma$, $G = G_\beta^\delta$ and $W = \left\langle G_\beta^\alpha \right\rangle_*$.

Note, since the above construction is done inductively for all h^γ ($\gamma < \beta$), we also need to consider the limit case: We put G_β^δ, $\mathcal{B}^\delta$ to be the union of all previous $G_\beta^{\delta'}$'s, respectively $\mathcal{B}^\delta$'s.

Step 3: In this step we consider f^β, d^β as candidates for endomorphisms and dual maps of G_β^δ for all $\delta \leq \alpha = \beta + 1$.
For each $\delta \leq \alpha$, we apply Step Lemma 1.4.3 to $b = d^\beta$, $\bar{b} = f^\beta$, and $G = G_\beta^\delta$. Hence there is $a \in \mathbb{D} \setminus G_\beta^\delta$ such that $\phi(a, d^\beta) \in \widehat{R} \setminus R$, $\sum_{i<\omega} a_i f_i^\beta \notin G_\alpha^\delta := \left\langle G_\beta^\delta, a \right\rangle_*$ or $U_t(f^\beta)$ is of finite rank for some $t \in R$, and $G_\alpha^\delta \cap B^\delta = \emptyset$. If $U_t(f^\beta)$ is of infinite rank for all $t \in R$, then let $\mathcal{B}^\delta := \mathcal{B}^\delta \cup \{\sum_{i<\omega} a_i f_i^\beta\}$.

Step 4: In this final step we consider h^β as candidate for homomorphisms between G_α^μ, G_α^δ for all $\mu \neq \delta \leq \alpha$; this is similar to Step 2.

Case $I^\mu \leq I^\delta$: We apply Step Lemma 1.5.5 to $\bar{b} = h^\beta$, $G = G_\alpha^\mu$ and $W = \left\langle G_\alpha^{\delta'} \mid \delta' \leq \alpha, I^\delta \leq I^{\delta'} \right\rangle_*$. As before (cf. Step 2), we deduce the existence of an element a such that, for all $\delta' \leq \alpha$ with $I^\delta \leq I^{\delta'}$, $\sum_{i<\omega} a_i h_i^\beta \notin G_\alpha^{\delta'} := \left\langle G_\alpha^{\delta'}, a\right\rangle_*$ or $U_t(h^\beta)$ is of finite rank for some $t \in R$. Moreover, we put $G_\alpha^\mu := \langle G_\alpha^\mu, a\rangle_*$ and, if $U_t(h^\beta)$ is of infinite rank for all $t \in R$, then we also put $\mathcal{B}^{\delta'} := \mathcal{B}^{\delta'} \cup \{\sum_{i<\omega} a_i h_i^\beta\}$ for all relevant δ's.

Case $I^\mu \not\leq I^\delta$: We apply Step Lemma 1.5.4 to $\bar{b} = h^\beta$, $G = G_\alpha^\mu$ and $W = G_\alpha^\delta$. Hence there is $a \in \mathbb{D} \setminus G_\alpha^\mu$ such that $\sum_{i<\omega} a_i h_i^\beta \notin G_\alpha^\delta$ or $U_0(h^\beta)$ is of finite rank, and $\mathcal{B}^\mu \cap \langle G_\alpha^\mu, a\rangle_* = \emptyset$. We put $G_\alpha^\mu := \langle G_\alpha^\mu, a\rangle_*$ and, if $U_0(h^\beta)$ is of infinite rank, then we also put $\mathcal{B}^\delta := \mathcal{B}^\delta \cup \{\sum_{i<\omega} a_i h_i^\beta\}$.

Note, the above step has to be done for each pair $(\mu, \delta) \leq (\alpha, \alpha)$.

It remains to consider a limit ordinal α, where G_β^δ has been constructed for all $\beta, \delta < \alpha$. In this case, we first put $G_\alpha^\delta = \bigcup_{\beta<\alpha} G_\beta^\delta$ for all $\delta < \alpha$. Then we define $G_\alpha^\alpha = \bigcup_{\beta<\alpha} G_\beta^\alpha$, where the G_β^α's are constructed as in Step 1. Moreover, we now redefine the G_α^δ's ($\delta \leq \alpha$) by doing Step 2 for all h^γ's with $\gamma < \alpha$.

Finally, we define $G^\delta = \bigcup_{\alpha<2^{\aleph_0}} G_\alpha^\delta$ and $\mathfrak{G} := \{G^\delta \mid \delta < 2^{\aleph_0}\}$. □

We prove that the family $\mathfrak{G}$ of R-modules constructed above has the desired properties. Note that maintaining the blacklists ensures that no homomorphism or endomorphisms can reappear.

Theorem 1.5.8 (ZFC + ¬CH + MA) *Let $H \subseteq_* \mathbb{D}$ be admissible such that $\aleph_1 \leq |H| < 2^{\aleph_0}$. Then there exists a fully rigid system $\mathfrak{G}$ of primal modules of H such that $S \subseteq G \subseteq_* \mathbb{D}$ with $|G| = 2^{\aleph_0}$ for each $G \in \mathfrak{G}$.*

Proof. Let $\mathfrak{G} = \{G^\delta \mid \delta < 2^{\aleph_0}\}$ be the family of R-modules from Construction 1.5.7 and let $\{\mathcal{B}^\delta \mid \delta < 2^{\aleph_0}\}$ be the corresponding family of (final) blacklists.

Using the same arguments as in the proof of Theorem 1.4.3, it follows immediately from the construction that, for each $\delta < 2^{\aleph_0}$,

- $S \subseteq G^\delta \subseteq_* \mathbb{D}$;
- $|G^\delta| = 2^{\aleph_0}$;
- $G^{\delta *} \cong H$; and
- $\operatorname{End} G^\delta = R \oplus \operatorname{Fin}(G^\delta)$.

(See Steps 1,3 in Construction 1.5.7.)

It remains to show that $\mathfrak{G}$ is, indeed, a fully rigid system. In order to do so, let $G_I = G^\delta$ and $\mathcal{B}_I = \mathcal{B}^\delta$ for $I = I^\delta$. In the following let $I \neq J \subseteq \omega$ be arbitrary. Moreover, let $\varphi : G_I \longrightarrow G_J$ be a homomorphism and let $\bar{b} = (b^n)_{n<\omega} \in \mathbb{D}^\omega$ be the ω-sequence describing φ, i.e. $b^n = \varphi(e_n)$ for all $n < \omega$ (cf. Lemma 1.5.2). Then there is $\gamma < 2^{\aleph_0}$ such that $h^\gamma = \bar{b}$ and thus $\bar{b}$ has been dealt with in Step 2 or Step 4 of the construction.

We have to show that $\varphi \in \operatorname{Fin}(G_I, G_J)$ or $\varphi \in R \oplus \operatorname{Fin}(G_I, G_J)$, according to $I \not\subseteq J$ or $I \subseteq J$, respectively.

Case $I \not\subseteq J$: In this case we applied Step Lemma 1.5.6 to $h^\gamma = \bar{b}$, $G = G_I$ and $W \subseteq G_J$. Hence there is $a \in G_I$ such that $\varphi(a) = \sum_{n<\omega} a_n b^n \notin W$ or $U_0(\bar{b}) = \langle b^n \mid n < \omega \rangle_*$ is of finite rank.

Now, if $U_0(\bar{b})$ were of infinite rank then $\varphi(a) \notin W$ and also $\varphi(a) \in \mathcal{B}^J$, thus $\varphi(a) \notin G_J$ – a contradiction. Therefore $\varphi(S) \subseteq U_0(\bar{b})$ is of finite rank and so, by Observation 1.4.6, $\varphi(G_I)$ is of finite rank, i.e. $\varphi \in \operatorname{Fin}(G_I, G_J)$.

Case $I \subseteq J$: In this case we applied Step Lemma 1.5.5 to $\bar{b}$, $G \subseteq G_I$ and to some W containing a submodule of G_J. Hence there is an element $a \in G_I, G_J$ such that $\varphi(a) = \sum_{n<\omega} a_n b^n \notin \langle W, a \rangle_*$ or $U_t(\bar{b}) = \langle b^n - te_n \mid n < \omega \rangle_*$ is of finite rank for some $t \in R$. Now, if $U_t(\bar{b})$ were of infinite rank for all $t \in R$, then it would follow from the construction that $\varphi(a) \in \mathcal{B}_J$ and hence $\varphi(a) \notin G_J$ – a contradiction.

Therefore there exists $t \in R$ such that $U_t(\bar{b})$ is of finite rank. Let $\psi = \varphi - t\,id$. Then $\psi(S) \subseteq U_t(\bar{b})$ is of finite rank and thus $\psi(G_I)$ is also of finite rank. Hence $\psi \in \mathrm{Fin}(G_I, G_J)$ and so $\varphi = t\,id + \psi \in R \oplus \mathrm{Fin}(G_I, G_J)$, as required. This finishes the proof. $\square$

2 Reduced products and the Chase radical

In this chapter, we shall consider products and reduced products of abelian groups (= $\mathbb{Z}$-modules). Our main interest is in vector groups, i.e. in products of rational groups (see Section 2.2). However, we shall also discuss some generalizations to products of arbitrary torsion-free groups (see Section 2.3). In particular, we will give an example of a vector group $G = \prod_{\alpha<\aleph_1} R_\alpha$, which does not commute with the Chase radical ν, that is $\nu G \neq \prod_{\alpha<\aleph_1} \nu R_\alpha$ (see Theorem 2.2.8). This answers an open questions raised in [28].

We begin with recalling the relevant notions, the needed known results, and some general considerations on the Chase radical.

2.1 Definitions and some basic results

Throughout this chapter, all groups are abelian and torsion-free, i.e. subgroups of a vector space $\mathbb{Q}^{(\lambda)}$. So-called rational groups play an important role. Recall, a *rational group* (or rank-1 group) may be considered as a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$, and is uniquely determined by its type.

For an element a of a group G we denote by $\chi(a)$ the *characteristic* sequence of a, i.e. the sequence of the p-heights of a. The members of such a sequence are non-negative integers and the symbol ∞. Two characteristic sequences are said to be *equivalent*, if they only differ at finitely many places, and not at the places, where ∞ occurs. The corresponding equivalence classes are called *types*, denoted by t, respectively by $\mathrm{t}(a)$.

Moreover, a group G is said to be *homogeneous* if all non-zero elements have the same type; we also put $\mathrm{t}(G) = \mathrm{t}(a)$ for any $0 \neq a \in G$.

Next, we recall the definition of a reduced product and of a vector group:

Definition 2.1.1 *Let κ be a cardinal.*

(i) *For a family $\{G_\alpha : \alpha < \kappa\}$ of groups we define the* reduced product *by*

$$\prod^r_{\alpha<\kappa} G_\alpha := \prod_{\alpha<\kappa} G_\alpha / \prod^<_{\alpha<\kappa} G_\alpha,$$

where $\prod^<_{\alpha<\kappa} G_\alpha$ consists of all elements of $\prod_{\alpha<\kappa} G_\alpha$ with support of size less than κ.

(ii) *For a family $\{R_\alpha \subseteq \mathbb{Q} : \alpha < \kappa\}$ of rational groups we call the product $\prod_{\alpha<\kappa} R_\alpha$ a* vector group*, and the corresponding reduced product $\prod^r_{\alpha<\kappa} R_\alpha$ a* reduced vector group*.*

Now we define the Chase radical, which was originally introduced by Stephen Chase 1962.

Definition 2.1.2 *For a group G the* Chase radical *νG is defined by*

$$\nu G = \bigcap \{\mathrm{Ker}(\varphi) \mid \varphi : G \longrightarrow X \textit{ with } X \ \aleph_1\textit{-free}\}.$$

Note that the Chase radical is, indeed, a radical and has the nice property that it 'tests' $\aleph_1$-freeness, i.e. $\nu G = 0$ if and only if G is $\aleph_1$-free (cf. [28], [14]). Recall, that the radical properties for ν mean the following:

(i) $\nu\nu G \subseteq \nu G$,

(ii) $\nu(G/\nu G) = 0$,

(iii) $(\nu G)\sigma \subseteq \nu G'$ for every homomorphism $\sigma : G \longrightarrow G'$,

(iv) $\nu(\bigoplus_{i\in I} G_i) = \bigoplus_{i\in I} \nu G_i$ for any family $\{G_i : i \in I\}$ of groups, and

(v) $\nu(\prod_{i\in I} G_i) \subseteq \prod_{i\in I} \nu G_i$ for any family $\{G_i : i \in I\}$ of groups.

Note, that the latter inclusion above raises the question under which conditions equality holds.

For any unexplained notions we refer the reader to [15] and [16].

Next, we will state a simplified version of the *Wald-Łoś-Lemma*. For the more general version and the proof see [14, Proposition 3.4, p. 30].

Lemma 2.1.3 (Wald, Łoś) *Let $\kappa > \aleph_0$ be a cardinal and let $\{G_\alpha : \alpha < \kappa\}$ be a family of groups. Then, for every $A \subseteq \prod^r_{\alpha<\kappa} G_\alpha$ such that $|A| < \kappa$, there is a monomorphism $\gamma : A \longrightarrow \prod_{\alpha<\kappa} G_\alpha$ with $id_A = \pi \circ \gamma$, where π is the canonical epimorphism from $\prod_{\alpha<\kappa} G_\alpha$ onto $\prod^r_{\alpha<\kappa} G_\alpha$.*

Let us also recall the well-known criterion for freeness due to Pontryagin.

Lemma 2.1.4 (Pontryagin's Criterion) *A group G is $\aleph_1$-free if and only if every finite rank pure subgroup $U \subseteq G$ is free.*

PROOF. See [14, p. 98, Theorem 2.3]. □

For the Chase radical and for an arbitrary countable family of groups, the following result holds; the proof uses an old result of Balcerzyk, respectively a more specific result due to Hulanicki (cf. [15, Corollary 42.2]).

Lemma 2.1.5 *Let $\{G_\alpha : \alpha < \omega\}$ be a countable family of groups. Then $\nu\left(\prod_{\alpha<\omega} G_\alpha\right) = \prod_{\alpha<\omega} \nu G_\alpha$.*

PROOF. See [9, Section 5]. □

The above result raises the natural question, if the Chase radical ν commutes with uncountable products, i.e. products with uncountable index set. This question was answered negatively by K. Eda [11]. He showed that there is a family of groups of size κ with $\aleph_0 < \kappa \leq 2^{\aleph_0}$ such that the Chase radical does not commute with this family. However, his proof does not provide a satisfying answer in ZFC, since he uses descending chains of types with infimum $\mathrm{t}_0 = \mathrm{t}(\mathbb{Z})$; note, the minimal length for the existence of such a chain is undecidable. Later, in Section 2, we will generalize Eda's result to antichains and prove (in ZFC) that there is a family of groups of cardinality $\aleph_1$ such that the Chase radical does not commute.

Next we show, however, that the Chase radical commutes with arbitrary products over a fixed countable group C.

Lemma 2.1.6 *Let κ be a cardinal and let C be a countable group. Then $\nu \prod_\kappa C = \prod_\kappa \nu C$.*

Proof. Let κ, C be as above. Since we already know from the previous lemma that ν commutes with countable products, it is obviously enough to consider κ with $\mathrm{cf}(\kappa) > \aleph_0$. Suppose, for a contradiction, that $\nu(\prod_{\alpha<\kappa} C) \subsetneq \prod_{\alpha<\kappa} \nu C$. Then there is

$$0 \neq c = (c_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa} \nu C \setminus \nu(\prod_{\alpha<\kappa} C).$$

We now define $I_q := \{\alpha < \kappa \mid c_\alpha = q\}$ for all $q \in C$. This implies $c \in \prod_{q\in C} \prod_{\alpha\in I_q} \nu C$. Since C is countable it follows that

$$c \notin \nu(\prod_{q\in C} \prod_{\alpha\in I_q} C) = \prod_{q\in C} \nu(\prod_{\alpha\in I_q} C)$$

and hence there is $l \in C$ with $|I_l| > \aleph_0$ such that $c \restriction I_l \notin \nu(\prod_{\alpha\in I_l} C)$. It is clear that $c_\alpha = l$ for all $\alpha \in I_l$. It thus follows that there is a homomorphism $\varphi : \prod_{\alpha\in I_l} C \longrightarrow X$ with X $\aleph_1$-free and $(c\restriction I_l)\varphi \neq 0$. Therefore we may define

$$\nabla : C \longrightarrow \prod_{\alpha\in I_l} C$$

$$\text{via} \quad c \mapsto (c, \dots, c, \dots).$$

Then $\nabla\varphi : C \longrightarrow X$ and $l\nabla\varphi = c \restriction I_l\varphi \neq 0$. We obtain that $l \notin \nu C$, but $c\restriction I_l \in \prod_{\alpha\in I_l} \nu C$ and so it is immediate that $l \in \nu C$ – a contradiction. Therefore the product really commutes. □

Recall, that the Chase radical for a group G is determined by its countable subgroups C with trivial dual, i.e.

$$\nu G = \sum\{\nu C \mid C \subseteq G, |C| = \aleph_0, C^* = 0\},$$

as proved in [11].

We finish this section with presenting some details of Eda's proof. First we need:

Lemma 2.1.7 (K. Eda [11]) *There exists a descending chain of types* $\{t_\alpha : \alpha < \kappa\}$ *for some cardinal* κ *with* $\aleph_0 < \kappa \leq 2^{\aleph_0}$ *such that, for every countable group* C *with* $\mathrm{Hom}(C, \mathbb{Z}) = 0$*, there is* $\beta < \kappa$ *such that* $\mathrm{Hom}(C, R_{t_\beta}) = 0$*, where* R_{t_β} *is a rational group of type* t_β*.*

PROOF. See [11, Theorem 5] □

Theorem 2.1.8 (K. Eda [11]) *There is a cardinal* $\kappa > \aleph_0$ *and a group* $G := \prod_{\alpha<\kappa} G_\alpha$ *with* $\nu G \neq G$*, but* $\nu G_\alpha = G_\alpha$ *for all* $\alpha < \kappa$*. Hence the Chase radical does not commute with uncountable products.*

PROOF. Let $\{t_\alpha : \alpha < \kappa\}$ be as in Lemma 2.1.7 and R_{t_α} the corresponding rational groups. Moreover, let

$$G := \prod_{\alpha<\kappa} R_{t_\alpha}.$$

Since $t_\alpha > t_0$ which implies that $\mathbb{Z} \subsetneq R_{t_\alpha}$, it follows that $\mathrm{Hom}(R_{t_\alpha}, \mathbb{Z}) = 0$ and hence $\nu R_{t_\alpha} = R_{t_\alpha}$ for all $\alpha < \kappa$. Now, let $a = (a_i)_{i<\kappa} \in G$ be arbitrary with $a_i \neq 0$ for all $i < \kappa$.

The element a cannot be contained in any countable subgroup $U \subseteq G$ with trivial dual. Otherwise, by Lemma 2.1.7, there is an $\alpha < \kappa$ with the property that $\mathrm{Hom}(U, R_{t_\alpha}) = 0$. This is obviously a contradiction, since $a \in U$ and hence the canonical projection $\pi_\alpha : G \longrightarrow R_{t_\alpha}$ implies that $a\pi_\alpha \neq 0$.

Therefore $a \notin \nu G$ but $a \in G$ and thus we conclude $\nu \left(\prod_{\alpha<\kappa} R_{t_\alpha}\right) \subsetneq \prod \nu R_{t_\alpha} = \prod_{\alpha<\kappa} R_{t_\alpha}$, as required. □

2.2 Reduced products of rational groups

As mentioned before, we here investigate vector groups, respectively reduced vector groups. In particular, we shall characterize those reduced vector groups, which are $\aleph_1$-free. For this purpose, we introduce the following: Given a cardinal κ, we say that the vector group $V = \prod_{\alpha<\kappa} R_\alpha$ satisfies the *S_κ-property* if and only if, for all $0 \neq x \in V$ with $|\operatorname{supp}(x)| = \kappa$, we have $\mathrm{t}(x) = \mathrm{t}(\mathbb{Z})$.
We begin with showing that the S_κ-property of V is sufficient for the reduced vector group $V^r = \prod_{\alpha<\kappa}^r R_\alpha$ to be $\mathbb{Z}$-homogeneous, i.e. homogeneous of type $\mathrm{t}(\mathbb{Z})$.

Lemma 2.2.1 *Let κ be a regular cardinal and let V, V^r be as above. If V has the S_κ-property, then V^r is $\mathbb{Z}$-homogeneous.*

Proof. For $0 \neq [g] \in V^r$ there is, by Lemma 2.1.3, a monomorphism $\varphi : \langle [g] \rangle_* \subseteq V^r \longrightarrow V$ with $[g] = [g]\varphi\pi$ where π is the canoncial projection. Let us now consider $[g]\varphi$. Since $[g]\varphi\pi = [g]$ it follows that $[g]$ is the coset of $[g]\varphi$. Hence $|\operatorname{supp}([g]\varphi)| = \kappa$ because otherwise $[g] = 0$. Therefore $\mathrm{t}([g]\varphi) = \mathrm{t}(\mathbb{Z})$ since V has the S_κ-property by assumption. It now follows that $\mathrm{t}([g]) \leq \mathrm{t}([g]\varphi) = \mathrm{t}(\mathbb{Z})$ and hence $\mathrm{t}([g]) = \mathrm{t}(\mathbb{Z})$; thus we are done. □

It is easy to see that, whenever the reduced vector group V^r is $\mathbb{Z}$-homogeneous, then the vector group V satisfies the S_κ-property. Hence these properties are equivalent:

Corollary 2.2.2 *Let κ be a regular cardinal, V a vector group and V^r the corresponding reduced vector group. Then V has the S_κ-property if and only if V^r is $\mathbb{Z}$-homogeneous.*

The next lemma provides an important tool for characterizing $\aleph_1$-free reduced vector groups. In our context, we use the expression 'almost all' to describe a property which is true for all but less than κ elements. Moreover, for an element $[g] \in V^r$ and $E \leq \kappa$, we say that $[g]\restriction E$ is *constant* if $g\restriction E$ is almost constant, i.e. $g\restriction E \in V$ is constant for all but $< \kappa$ elements.

Lemma 2.2.3 *Let κ be a regular cardinal and let V, V^r be as above. Moreover, let $g^1, \ldots, g^n \in V$.*
Then there is a countable family of pairwise disjoint sets $\{E_j \mid j \in J \leq \omega\}$ such that

- $|\kappa \setminus \bigcup_{j\in J} E_j| < \kappa$;
- $|E_j| = \kappa$ *for all* $j \in J$;
- $[g^i]\restriction E_j$ *is constant for all* $1 \leq i \leq n$ *and* $j \in J$.

Proof. First we define sets $G_i^q := \{\alpha < \kappa : g_\alpha^i = q\}$ for all $q \in \mathbb{Q}$ and $1 \leq i \leq n$, i.e. G_i^q collects all coordinates α of g^i which are equal to q.
Next let

$$F := \{G_1^{q_1} \cap \cdots \cap G_n^{q_n} \mid (q_1, \ldots, q_n) \in \mathbb{Q}^n\}.$$

Obviously, F is countable, say $F = \{G_j \mid j \in J\}$ for some $J \leq \omega$. It follows immediately that $[g^i]\restriction G_j$ is constant for all $1 \leq i \leq n$, $j \in J$.
We claim $\kappa = \bigcup_{j\in J} G_j$:
For all $1 \leq i \leq n$ and for each $\alpha < \kappa$, we have $\alpha \in G_i^{g_\alpha^i}$ and hence $\alpha \in G_1^{g_\alpha^1} \cap \cdots \cap G_n^{g_\alpha^n}$. Therefore $\kappa = \bigcup_{j\in J} G_j$, as claimed.
Now let $F' = \{G_j \mid j \in J, |G_j| < \kappa\}$. Then $|\bigcup_{G\in F'} G| < \kappa$ and hence we have that the family $F'' := F \setminus F'$ satisfies the property $|\kappa \setminus \bigcup_{G\in F''} G| < \kappa$. Since κ is regular by assumption, we obtain that F'' is non-empty. Obviously, F''

is also countable and hence we may enumerate F'' by $F'' = \{E_j \mid j \in J'\}$ for some $J' \leq \omega$. Finally note, that the E_j's in F'' are pairwise disjoint since the g^i_α's can only take one value at a time. □

Next we use the above lemma to characterize $\aleph_1$-free reduced vector groups.

Theorem 2.2.4 *Let κ be a regular cardinal, $V = \prod_{\alpha<\kappa} R_\alpha$ be a vector group and let V^r be the corresponding reduced vector group.*
Then the following are equivalent:

(i) *V^r is $\aleph_1$-free;*

(ii) *V satisfies the S_κ-property;*

(iii) *V^r is $\mathbb{Z}$-homogeneous.*

Proof. First note, that the equivalence of (ii) and (iii) has already been established (see Corollary 2.2.2). Moreover, it is clear that $\aleph_1$-freeness implies $\mathbb{Z}$-homogeneity. Hence it remains to prove that (iii) implies (i).
So, assume that V^r is $\mathbb{Z}$-homogeneous. We show that V^r is $\aleph_1$-free by applying Pontryagin's Criterion (see Lemma 2.1.4). Therefore we need to prove that every (pure) finite rank subgroup C of V^r is free. In order to do so, let $[g_1], \ldots, [g_n] \in V^r$ and put $C := \langle [g_1], \ldots, [g_n] \rangle_*$. By Lemma 2.2.3, there is a family

$$\mathcal{F} := \{E_j \mid j \in I \leq \omega\}$$

of subsets of κ such that $|\kappa \setminus \bigcup_{j\in I} E_j| < \kappa$, $|E_j| = \kappa$ for all $j \in I$, and $[g_i]\restriction E_j$ is constant for all $1 \leq i \leq n$ and $j \in I$.

Since the $[g_i]\restriction E_j$'s are constant, there is $(r_1,\ldots,r_n)\in\mathbb{Q}^n$ with $g^i_j = r_i$ for almost all $j\in E_j$. Now, let F be the subgroup of V^r given by

$$F = \{[v]\in V^r \mid [v]\restriction E_j \textit{ is constant for all } j\in I\}.$$

Clearly, $[g^i]\in F$ for each $1\le i\le n$. We show that F is a pure, $\aleph_1$-free subgroup of V^r. Then we are done, since $C\subseteq F$ and so C has to be free.

First we prove that F is $\aleph_1$-free. For all $j\in I$, let $[e^j]$ be defined in the following way :

$$e^j_l := \begin{cases} 1, & \textit{if } l\in E_j \\ 0, & \textit{if } l\notin E_j. \end{cases}$$

Because V^r is $\mathbb{Z}$-homogeneous and $|E_j| = \kappa$, it follows that $\mathrm{t}(e^j) = \mathrm{t}([e^j]) = \mathrm{t}(\mathbb{Z})$ for all $j\in I$. Hence we can find $l_j\in\mathbb{Z}$ such that $\chi([\frac{e_j}{l_j}]) = (0,\ldots,0,\ldots)$ for all $j\in I$. Let us now define the following homomorphism:

$$\varphi : F \longrightarrow \mathbb{Z}^I,\quad [f]\mapsto z$$

$$\textit{via}\quad z_j = l_j k_j \quad\Longleftrightarrow\quad [f]\restriction E_j = k_j[e_j].$$

It is easy to check that φ is well defined: If $[f]=[f']$ then $|\,\mathrm{supp}(f-f')| < \kappa$ and hence, if $f_i = r$ for some $r\in\mathbb{Q}$ and for almost all $i\in E_j$, then also $f'_i = r$ for almost all $i\in E_j$. It remains to show that $l_jk_j\in\mathbb{Z}$. We know that $k_j\in\mathbb{Q}$ and $\mathrm{t}(e_j) = \mathrm{t}(\mathbb{Z})$. Furthermore, $[f]\restriction E_j = k_j[e_j]$ and $[\frac{e_j}{l_j}]$ is not divisible in V^r by any integer. Therefore

$$\left\langle\left[\frac{e_j}{l_j}\right]\right\rangle = \{[v]\in V \mid [v]\restriction E_j \textit{ is constant and } \mathrm{supp}([v])\le E_j\}$$

and thus it follows that there is $k'_j\in\mathbb{Z}$ such that $k'_j([\frac{e_j}{l_j}]) = [f]\restriction E_j = k_j[e_j]$. Hence $k_jl_j\in\mathbb{Z}$ as required, i.e. φ is, indeed, well defined.

Moreover, φ is obviously onto and φ is also monic since $|\kappa \setminus \bigcup_{j\in I} E_j| < \kappa$. Therefore $F \cong \mathbb{Z}^I$ is $\aleph_1$-free.

Now we prove that $F \subseteq_* V^r$. So, let $[v] \in V$ with $k[v] \in F$ for some $k \in \mathbb{Z}\setminus\{0\}$. It follows that $k[v]\restriction E_j$ is constant for all $j \in I$, and hence $[v]\restriction E_j$ has to be constant, too. Thus we have $[v] \in F$ and so we are done. □

The corollary below is some kind of a generalization of the Wald-Łoś-Lemma 2.1.3 in the following sense: It does not only provide a criterion for countable $U \subseteq \prod_\kappa^r \mathbb{Z}$ to be embeddable into $\prod_\omega \mathbb{Z}$, but also for countable subgroups of more general reduced vector groups.

Corollary 2.2.5 *Let κ be a regular cardinal and let $V^r = \prod_{\alpha<\kappa}^r R_\alpha$ be a $\mathbb{Z}$-homogeneous, reduced vector group.*
Then, for all $U \subseteq V^r$ with $|U| = \aleph_0$, there is a monomorphism $\alpha : U \longrightarrow \prod_\omega \mathbb{Z}$.
Proof. The result follows from the proof of Theorem 2.2.4. □

Before we can proceed with presenting an example of a product of rational groups which does not commute with the Chase radical ν, we need the following lemmas. The first one characterizes the Chase radical νG of a group G as the minimal group such that $G/\nu G$ is $\aleph_1$-free, while the second one provides a criterion for the Chase radical ν to commute with products depending on the corresponding reduced product.

Lemma 2.2.6 (S. U. Chase [5]) *Let G be a group and let $U \subseteq G$ be a subgroup of G such that G/U is $\aleph_1$-free. Then $\nu G \subseteq U$, i.e. νG is minimal among all subgroups U of G with $\aleph_1$-free quotient G/U.*

PROOF. Let G, U be as above and let $\pi_U : G \longrightarrow G/U$ denote the canonical epimorphism. Since G/U is $\aleph_1$-free, we obtain that

$$\nu G = \bigcap \{\mathrm{Ker}(\varphi) \mid \varphi : G \longrightarrow X \textit{ with } X\ \aleph_1\textit{-free}\} \subseteq \mathrm{Ker}\, \pi_U = U.$$

Hence νG is minimal with respect to the desired property. □

Lemma 2.2.7 *Let κ be a regular cardinal and let $V^r = \prod_{\alpha<\kappa}^r G_\alpha$ be an $\aleph_1$-free reduced product. Moreover, suppose that ν commutes with products of size λ for all $\lambda < \kappa$, and that $\nu G_\alpha = G_\alpha$ for all $\alpha < \kappa$.*
Then $\nu(\prod_{\alpha<\kappa} G_\alpha) \subsetneq \prod_{\alpha<\kappa} \nu G_\alpha$.
PROOF. Let κ, V^r be as above. By assumption, ν commutes with products of size λ for all $\lambda < \kappa$ and hence we obtain, for all subfamilies $\{G_\alpha : \alpha < \lambda\}$, that $\nu(\prod_{\alpha<\lambda} G_\alpha) = \prod_{\alpha<\lambda} \nu G_\alpha = \prod_{\alpha<\lambda} G_\alpha$. Therefore we deduce

$$\nu \left(\prod\nolimits_{\alpha<\kappa}^{<} G_\alpha\right) = \prod\nolimits_{\alpha<\kappa}^{<} G_\alpha.$$

This implies

$$\prod\nolimits_{\alpha<\kappa}^{<} G_\alpha \subseteq \nu \left(\prod\nolimits_{\alpha<\kappa} G_\alpha\right).$$

On the other hand, $V^r = \prod_{\alpha<\kappa} G_\alpha / \prod_{\alpha<\kappa}^{<} G_\alpha$ is $\aleph_1$-free and hence, by Lemma 2.2.6, we have

$$\prod\nolimits_{\alpha<\kappa}^{<} G_\alpha \supseteq \nu \left(\prod\nolimits_{\alpha<\kappa} G_\alpha\right),$$

and thus equality holds. We finally conclude $\nu(\prod_{\alpha<\kappa} G_\alpha) \neq \prod_{\alpha<\kappa} \nu G_\alpha = \prod_{\alpha<\kappa} G_\alpha$ as required. □

We are now ready to prove that the Chase radical ν does not commute with arbitrary products of size $\aleph_1$.

Theorem 2.2.8 *There is a vector group $V = \prod_{\alpha<\aleph_1} R_\alpha$ such that the Chase radical ν does not commute with V, i.e.*

$$\nu(\prod_{\alpha<\aleph_1} R_\alpha) \neq \prod_{\alpha<\aleph_1} \nu R_\alpha.$$

Proof. First we show that there exists an antichain $\{R_\alpha : \alpha < \aleph_1\}$ of rational groups such that $\mathrm{t}(R_\alpha) \wedge \mathrm{t}(R_\beta) = \mathrm{t}(\mathbb{Z})$ for all $\alpha \neq \beta < \aleph_1$, but $\mathrm{t}(R_\alpha) \neq \mathrm{t}(\mathbb{Z})$ for each $\alpha < \aleph_1$.

As is well known, there exists a family of almost disjoint subsets A_α of the set of primes of size $\aleph_1$, in fact, of size $2^{\aleph_0}$.

Now define

$$R_\alpha := \left\langle \frac{1}{p} : p \in A_\alpha \right\rangle.$$

Since $A_\alpha \cap A_\beta$ is finite for any $\alpha \neq \beta$, it follows immediately, that the family $\{R_\alpha : \alpha < \aleph_1\}$ satisfies the desired properties.

By the definition of the Chase radical and since $\mathrm{t}(R_\alpha) \neq \mathrm{t}(\mathbb{Z})$, we have that $\nu R_\alpha = R_\alpha$ for all $\alpha < \aleph_1$. Moreover, it is easy to see that $V = \prod_{\alpha<\aleph_1} R_\alpha$ satisfies the $S_{\aleph_1}$-property and hence V^r is $\aleph_1$-free, by Theorem 2.2.4. Finally, we apply Lemma 2.2.7 and deduce

$$\nu\left(\prod_{\alpha<\aleph_1} R_\alpha\right) \neq \prod_{\alpha<\aleph_1} R_\alpha = \prod_{\alpha<\aleph_1} \nu R_\alpha.$$

This finishes the proof. □

As mentioned in Section 2.1, Eda has already shown that there exists a product of rational groups such that the Chase radical does not commute with this product. We want to finish this section with two examples, demonstrating that the property for a reduced vector group to be $\mathbb{Z}$-homogeneous may depend on the underlying set theory.

Example 2.2.9 Under the assumption of $ZFC + MA$, every strictly descending chain $C = \{t_\alpha : \alpha < \kappa\} \leq T$ of cofinality $\mathrm{cf}(C) = \kappa < 2^{\aleph_0}$ has a lower bound S with $t_\alpha > S > t(\mathbb{Z})$ for all $\alpha < \kappa$ (for a proof [22], [28]).
Therefore the vector group $V = \prod_{\alpha<\kappa} R_{t_\alpha}$ with $\mathtt{t}(R_{t_\alpha}) = t_\alpha$ does not satisfy the S_κ-property and hence the corresponding reduced vector group V^r is not $\mathbb{Z}$-homogeneous.

On the other hand, we have:

Example 2.2.10 Under the assumption of $ZFC + CH$, every strictly descending chain $C = \{t_\alpha : \alpha < \kappa\} \leq T$ of cofinality $\mathrm{cf}(C) = \kappa > \aleph_0$ can be extended in such a way that $\inf C = \mathtt{t}(\mathbb{Z})$. In particular, there is a chain C of cofinality $\aleph_1 = 2^{\aleph_0}$ such that the reduced vector group $V^r = \prod^r_{\alpha<\kappa} R_{t_\alpha}$ is $\mathbb{Z}$-homogeneous.

2.3 Reduced products of arbitrary groups

In this final section, we investigate possible generalizations of Theorem 2.2.4. In fact, we shall prove two results which can be considered as generalizations of this theorem.

However, first we give an example which demonstrates that the theorem cannot be generalized to arbitrary groups in the canonical way.

Example 2.3.1 Let C be a $\mathbb{Z}$-homogeneous, indecomposable group of rank 2 (for the existence of such a group see [16, Theorem 88.4]). Moreover, let κ be regular, let $V := \prod_\kappa C$ and let V^r be the corresponding reduced product. Then it is easily seen, using Lemma 2.1.3, that V^r is $\mathbb{Z}$-homogeneous: For any $[x] \in V^r$, there is an embedding $\varphi : \langle [x] \rangle_* \hookrightarrow \prod_\kappa C$ and hence $\mathrm{t}([x]) \leq \mathrm{t}([x]\varphi) = \mathrm{t}(\mathbb{Z})$, i.e. $\mathrm{t}([x]) = \mathrm{t}(\mathbb{Z})$.

However, V^r is not $\aleph_1$-free, as proven below in two different ways. Note, if Theorem 2.2.4 were true for arbitrary groups, then V^r would be $\aleph_1$-free.

First consider the homorphism

$$\nabla : C \longrightarrow V^r$$

$$\textit{defined via} \;\; c \mapsto [(c, \ldots, c, \ldots)].$$

It is easy to see that ∇ is injective. Since C is indecomposable of rank 2, it cannot be free and hence V^r cannot be $\aleph_1$-free.

Alternative: Since C is indecomposable of rank 2, it follows immediately that $\mathrm{Hom}(C, \mathbb{Z}) = 0$ and hence $\nu C = C$. Therefore, by Theorem 2.1.6, we have $\prod_\kappa C = \prod_\kappa \nu C = \nu \prod_\kappa C$. In particular, $\prod_\kappa^< C \subsetneq \nu \prod_\kappa C$. However, $\nu \prod_\kappa C$ is the minimal group such that $\prod_\kappa C / \nu \prod_\kappa C$ is $\aleph_1$-free by Lemma 2.2.6, and thus it follows that $V = \prod_\kappa^r C$ cannot be $\aleph_1$-free.

Next we present a straightforward generalization of Theorem 2.2.4. In fact, we consider groups which can be embedded in a vector group.

Theorem 2.3.2 *Let κ be a regular cardinal and let $\{G_\alpha \mid \alpha < \kappa\}$ be a family of groups such that each G_α is embeddable in a vector group V_α over less than κ rational groups. If $\prod_{\alpha<\kappa} V_\alpha$ has the S_κ-property, then the reduced product $\prod^r_{\alpha<\kappa} G_\alpha$ is $\aleph_1$-free.*

Proof. Let G_α, V_α be as above, say

$$\iota_\alpha : G_\alpha \longrightarrow V_\alpha = \prod_{\lambda<\theta_\alpha} R^\alpha_\lambda,$$

for some $\theta_\alpha < \kappa$. Moreover, put $\varphi = \prod_{\alpha<\kappa} \iota_\alpha$, i.e.

$$\varphi : \prod\nolimits_{\alpha<\kappa} G_\alpha \longrightarrow \prod\nolimits_{\alpha<\kappa} V_\alpha = \prod\nolimits_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda$$

is defined in the obvious way. By assumption and by Theorem 2.2.4, we have that $\prod^r_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda$ is $\aleph_1$-free.

Next let

$$\tilde{\varphi} : \prod\nolimits^r_{\alpha<\kappa} G_\alpha \longrightarrow \prod\nolimits^r_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda$$

$$\text{be defined via } \ [x] \mapsto x\varphi\pi$$

with $\pi : \prod_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda \longrightarrow \prod^r_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda$ the canonical projection. We show that $\tilde{\varphi}$ is a well-defined embedding, which then implies the desired $\aleph_1$-freeness of $\prod_{\alpha<\kappa} G_\alpha$, since subgroups of $\aleph_1$-free groups are also $\aleph_1$-free.

First, we prove

$$(\prod\nolimits^<_{\alpha<\kappa} G_\alpha)\varphi = \prod\nolimits^<_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda \cap (\prod\nolimits_{\alpha<\kappa} G_\alpha)\varphi.$$

Let $x \in \prod^<_{\alpha<\kappa} G_\alpha$. Then $x\varphi \in (\prod_{\alpha<\kappa} G_\alpha)\varphi$ and $|\operatorname{supp}(x\varphi)| < \kappa$, which implies that $x\varphi \in \prod^<_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda \cap (\prod_{\alpha<\kappa} G_\alpha)\varphi$.

For the converse inclusion, let $y \in \prod^{<}_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda \cap (\prod_{\alpha<\kappa} G_\alpha)\varphi$. Then there is $x \in \prod_{\alpha<\kappa} G_\alpha$ such that $x\varphi = y$. Moreover, we know that $|\operatorname{supp}(y)| < \kappa$ and, as φ is monic, we obtain $|\operatorname{supp}(x)| \leq |\operatorname{supp}(y)| < \kappa$. Therefore, $y = x\varphi \in (\prod^{<}_{\alpha<\kappa} G_\alpha)\varphi$ and so the above equality is proven.
Now it is clear, that the induced homorphism $\tilde{\varphi}$ is well defined.
It remains to prove that $\tilde{\varphi}$ is monic. In order to do so, let $[x] \in \prod^{r}_{\alpha<\kappa} G_\alpha$ and assume $x\varphi\pi = 0$, i.e. $x\varphi \in \prod^{<}_{\alpha<\kappa,\lambda<\theta_\alpha} R^\alpha_\lambda$. Moreover, $x\varphi \in (\prod_{\alpha<\kappa} G_\alpha)\varphi$ which implies that $x\varphi \in (\prod^{<}_{\alpha<\kappa} G_\alpha)\varphi$ and thus $[x] = 0$. So the proof is finished. $\square$

Before we prove a more sophisticated generalization of Theorem 2.2.4, we generalize the main tool Lemma 2.2.3, that is, we give a criterion for checking whether a given reduced product of a family of groups $\{G_\beta \mid \beta < \kappa\}$ is $\aleph_1$-free.

Lemma 2.3.3 *Let κ, λ be cardinals with κ regular and let $\{G_\alpha \subseteq \mathbb{Q}^{(\lambda)} \mid \alpha < \kappa\}$ be a family of torsion-free groups. Moreover, let V^r denote the reduced product $V^r = \prod^r_{\alpha<\kappa} G_\alpha$ and let $\{[g^\beta] : \beta < \eta\} \subseteq V^r$ be a subset of η elements for some η with $\lambda^\eta < \kappa$.*
Then there exist a cardinal $\tau \leq \lambda^\eta$ and a family of pairwise disjoint sets $\mathcal{F} := \{E_n \mid n < \tau\}$ such that

- $|\kappa \setminus \bigcup_{n<\tau} E_n| < \kappa$;
- $|E_n| = \kappa$ *for all* $n < \tau$;
- $[g^\beta]\restriction E_j$ *is constant for all* $\beta < \eta$, *and* $j < \tau$.

Proof. Let $\{[g^\beta] : \beta < \eta\} \subseteq V^r$ be the given subset. We define sets $H_{r,\beta}$ by

$$H_{r,\beta} := \{\alpha < \kappa : g^\beta_\alpha = r\}$$

for all $r \in \mathbb{Q}^{(\lambda)}$ and $\beta < \eta$. Note that the lower indices identify the coordinates. Furthermore, let

$$F := \{\bigcap_{\beta<\eta} H_{r_\beta,\beta} \mid (r_\beta)_{\beta<\eta} \in (\mathbb{Q}^{(\lambda)})^\eta\}.$$

It is easy to see that F is of cardinality $\theta \leq \lambda^\eta < \kappa$. So, let us enumerate those sets with θ, i.e. $F = \{H_\alpha \mid \alpha < \theta\}$. If we now consider the $[g^\beta]$'s, it is immediate that $[g^\beta] \restriction H_j$ is constant for all $\beta < \eta$, $j < \theta$. We claim that $\kappa = \bigcup_{j<\theta} H_j$: Take $\xi < \kappa$. Then $\xi \in H_{g^\beta_\xi,\beta}$ for all $\beta < \eta$ and hence

$$\xi \in \bigcap_{\beta<\eta} H_{g^\beta_\xi,\beta}.$$

Since this holds for all $\xi < \kappa$, it follows that $\kappa = \bigcup_{j\in I} H_j$.
If we now define $F' = \{H_j \mid j < \theta, |H_j| < \kappa\}$, then $|\bigcup_{H\in F'} H| < \kappa$. Hence, if we omit those H_j's with $|H_j| < \kappa$, we obtain the family of sets $F'' := F \setminus F'$ with the property that $|\kappa \setminus \bigcup_{H\in F''} H| < \kappa$. Clearly, F'' satisfies the required properties (cf. proof of Lemma 2.2.3). □

We are now ready to characterize those families of groups, for which the reduced product is $\aleph_1$-free. Note that, the necessary and sufficient condition resembles the S_κ-property for rational groups.

Theorem 2.3.4 *Let $\lambda < \kappa = \mathrm{cf}(\kappa)$ be cardinals and let $V^r = \prod^r_{\alpha<\kappa} G_\alpha$ be the reduced product of $G_\alpha \subseteq \mathbb{Q}^{(\lambda)}$ $(\alpha < \kappa)$.*
Then V^r is $\aleph_1$-free if and only if, for all $I \leq \kappa$ with $|I| = \kappa$, the intersection $\bigcap_{\alpha\in I} G_\alpha$ is $\aleph_1$-free.
PROOF. First we prove that V^r is $\aleph_1$-free provided that $\bigcap_{\alpha\in I} G_\alpha$ is $\aleph_1$-free for all $I \leq \kappa$ with $|I| = \kappa$. By Pontryagin's Criterion 2.1.4, it is sufficient to show, that every finite rank pure subgroup of V^r is free.

Let $[g_1], \ldots, [g_n] \in V^r$ and define $C := \langle [g_1], \ldots, [g_n] \rangle_*$. By Lemma 2.3.3, there is a family

$$\mathcal{F} = \{E_j \mid j < \tau \leq \lambda\}$$

with $|\kappa \setminus \bigcup_{j<\tau} E_j| < \kappa$, $|E_j| = \kappa$ for all $j < \tau$ and $[g_j]\restriction E_j$ is constant for all $1 \leq i \leq n$, $j < \tau$. Hence there is $(r_1, \ldots, r_n) \in (\mathbb{Q}^{(\lambda)})^n$ with $g_i(l) = r_i$ for all $l \in E_j$ with $j < \tau$ and $1 \leq i \leq n$. We now consider

$$F := \{[v] \in V^r \mid [v]\restriction E_j \textit{ is constant for all } j < \tau\}.$$

Obviously, it is sufficient to prove that $F \subseteq_* V^r$ and F is $\aleph_1$-free, since then $C \subseteq_* V^r$ and $C \subseteq F$ implies that C is free.

Let $[v] \in V^r$ such that $k[v] \in F$ with $k \in \mathbb{Z} \setminus \{0\}$. Then $k[v]\restriction E_j$ is constant for all $j < \tau$ and hence it is clear that $[v]\restriction E_j$ is constant for all $j < \tau$. This implies that $[v] \in F$, i.e. $F \subseteq_* V^r$.

It remains to show that F is $\aleph_1$-free. In order to do so, we define the map

$$\varphi : F \longrightarrow \prod_{j<\tau} \left(\bigcap_{\alpha \in E_j} G_\alpha \right)$$

$$\textit{via} \quad [f] \mapsto z \textit{ with } z_j = g_j \in \bigcap_{\alpha \in E_j} G_\alpha \Leftrightarrow [f]\restriction E_j = g_j.$$

It is easy to check that φ is well defined (cf. proof of Theorem 2.2.4).

Moreover, the injectivity of φ again follows from $|\kappa \setminus \bigcup_{j<\tau} E_j| < \kappa$. Since a product of $\aleph_1$-free groups is clearly also $\aleph_1$-free, we obtain F is $\aleph_1$-free and hence $C \subseteq_* F$ is free.

Conversely, assume that V^r is $\aleph_1$-free. We need to show that $\bigcap_{\alpha \in I} G_\alpha$ is $\aleph_1$-free for all $I \leq \kappa$ with $|I| = \kappa$. Suppose not. Then there is $I \subseteq \kappa$ with $|I| = \kappa$ such that $K := \bigcap_{\alpha \in I} G_\alpha$ is not $\aleph_1$-free. Hence K contains a non-free subgroup of finite rank, that means, there are $h_1, \ldots, h_n \in K$ such that $\langle h_1, \ldots, h_n \rangle_*$ is not free. Define $[g_i] \in V^r$ $(1 \le i \le n)$ in the following way:

$$g_i \restriction I = h_i \ \text{ and } \ g_i \restriction (\kappa \setminus I) = 0 \ \textit{ for all } \ 1 \le i \le n.$$

This now allows us to define

$$\psi : \langle [g_1], \ldots, [g_n] \rangle_* \longrightarrow \langle h_1, \ldots, h_n \rangle_* \subseteq_* K$$

with $\psi(g_i) = h_i$ for all $1 \le i \le n$ and $\psi(x)$, which is obviously an isomorphism. Therefore, $\langle [g_1], \ldots, [g_n] \rangle_* \cong \langle h_1, \ldots, h_n \rangle_*$ is not free, contradicting the $\aleph_1$-freeness of V^r. □

We finish this section, respectively the chapter, with a special case of the above theorem, which is interesting on its own right.

Corollary 2.3.5 *Let $\kappa > 2^\lambda$ be cardinals and $V^r = \prod^r_{\alpha<\kappa} G_\alpha$ with $G_\alpha \subseteq \mathbb{Q}^{(\lambda)}$. Then V^r is $\aleph_1$-free if and only if G_α is $\aleph_1$-free for almost all $\alpha < \kappa$.*

Proof. Recall first, that in this context 'for almost all' means for all but less than κ.

Assume that there is $I \subseteq \kappa$ with $|I| = \kappa$ such that G_α is not $\aleph_1$-free for all $\alpha \in I$. Since there are at most $2^\lambda < \kappa$ subgroups of $\mathbb{Q}^{(\lambda)}$, there is $I' \subseteq I$ with $|I'| = \kappa$ such that G_α is constant for all $\alpha \in I'$. Hence

$$G_\alpha = \bigcap_{\alpha \in I'} G_\alpha$$

is not $\aleph_1$-free and thus, by Theorem 2.3.4, V^r cannot be $\aleph_1$-free.

Conversely, let V^r be $\aleph_1$-free. Then $\bigcap_{\alpha \in I} G_\alpha$ is $\aleph_1$-free for all $I \subseteq \kappa$ with $|I| = \kappa$, by Theorem 2.3.4. For each $\alpha < \kappa$, define $I_\alpha = \{\beta < \kappa \mid G_\beta = G_\alpha\}$

and let

$$I' := \bigcup_{\alpha<\kappa, |I_\alpha|<\kappa} I_\alpha.$$

Then $|I'| < \kappa$ since only $2^\lambda < \kappa$ different $G_\alpha \subseteq \mathbb{Q}^{(\lambda)}$ may exist. Hence, for all $\alpha \in \kappa \setminus I'$ holds that

$$G_\alpha = \bigcap_{\beta \in I_\alpha} G_\beta$$

is $\aleph_1$-free, by hypothesis and since $|G_\alpha| = \kappa$ in this case. □

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