1 Abstract

Inspired by his research in homotopy theory Emmanuel Dror Farjoun raised in 1997 the following problem:

Do uniquely transitive torsion-free abelian groups exist?

Here a torsion-free abelian group $G \neq \mathbb{Z}$ is called uniquely transitive if for any ordered pair $(a, b) \in G \times G$ of pure elements there is exactly one automorphism $\varphi \in \operatorname{Aut} G$ mapping a onto b. The group $\mathbb{Z}$ of integers has the pure elements 1 and -1 and the automorphism group $\operatorname{Aut} \mathbb{Z} = \{-1, 1\}$, thus it is uniquely transitive and therefore is excluded from the definition. Note that $\mathfrak{p}G$, the set of all pure elements in G, may be empty, if G is divisible for instance. In order to avoid this and other trivial cases we also require that G is $\aleph_1$-free, hence every countable subset of G is free.
In this thesis we will develop several fruitful new methods to attack Farjoun's problem resulting in the most extensive compilation of results concerning uniquely transitive groups up to date.
In Chapter 2 we start with basic set theoretic and algebraic tools, discuss κ-free modules and some Prediction Principles. Furthermore we deduce some general properties of uniquely transitive groups and state two Main Theorems, which will be proved in Part I and Part II.
Part I deals with the first and only existent successful construction for uniquely transitive groups given by Göbel and Shelah in 2004. They showed, that assuming ZFC for any successor cardinal $\kappa = \mu^+$ with $\mu = \mu^{\aleph_0}$ there exists an $\aleph_1$-free uniquely transitive group G of cardinality κ. Furthermore, they proved that the endomorphism ring of the constructed groups G is isomorphic to the integral group ring $\mathbb{Z}F$ over an absolute free group F of cardinality κ. We refine these arguments using new strong algebraical and combinatorial ideas to construct κ-free uniquely transitive groups G of cardinality κ with $\operatorname{End} G = \mathbb{Z}F$ in Gödel's constructible universe L for any non-reflecting cardinal κ. Here a group is called κ-free if every subgroup of cardinality less than κ is free. Also we improve the result by Göbel and Shelah to all cardinals κ with $\kappa = \kappa^{\aleph_0}$ in ZFC, particularly including limit cardinals. Thus the smallest example has size $2^{\aleph_0}$.
In Part II we present a new construction for $\aleph_1$-free uniquely transitive groups of arbitrary large cardinality κ, which leads to a family of groups very different from those in Part I: The groups G are the additive groups of rings S, which are at the same time principal ideal domains and E-rings. Here a ring R with unit is called an E-ring, if the canonical endomorphism $\varepsilon : \operatorname{End}(R^+) \to R$, $\varphi \mapsto \varphi(1)$ is a bijection. We present two versions, one for successor cardinals κ assuming Weak Diamond and one for cardinals κ with $\kappa = \kappa^{\aleph_0}$ assuming ZFC . This particularly includes small uniquely transitive groups of cardinality $\aleph_1$ assuming ZFC and $2^{\aleph_0} < 2^{\aleph_1}$.
All these results extend obviously to R-modules over arbitrary cotorsion-free principal ideal domains R.