Homological properties of monomial ideals associated to quasi-trees and lattices

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Xinxian Zheng
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Vorsitzender der Prüfungskommission: Prof. Dr. Wolfgang Lempken
Gutachter: Prof. Dr. Jürgen Herzog
Prof. Dr. Takayuki Hibi

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Contents

Introduction

Monomials are the link between Commutative Algebra and Combinatorics. Monomial algebras, also called toric rings or semigroup rings, and their presentation ideals are studied in the books of Bruns-Herzog [**2**], Hibi [**27**], Stanley [**32**], Sturmfels [**33**] and Villarreal [**39**]. In this thesis we concentrate on monomial ideals. With a simplicial complex Δ one can associate two squarefree monomial ideals: the Stanley-Reisner ideal I_Δ whose generators correspond to the non-faces of Δ, or the facet ideal $I(\Delta)$ whose generators correspond to the facets of Δ. The work of Stanley [**32**] has demonstrated that there are deep relations between the combinatorial properties of Δ and the algebraic properties of I_Δ.

Facet ideals for graphs (with no isolated vertices) have first been considered by Villarreal [**39**]. In this special case the facet ideal is called edge ideal, because its generators correspond to the edges of the graph. In his papers [**40**] and [**41**], Villarreal has shown that the edge ideal is the appropriate algebraic object attached to a graph. Among the graphs the trees are the simplest ones. Faridi generalized in [**15**] the definition of tree to simplicial complexes of any dimension, and also introduced facet ideals to study trees. However the notion quasi-tree (see Definition 2.27) introduced in [**43**] is also important for us in this thesis.

In the first two chapters of this thesis we introduce the basic notions concerning graphs, simplicial complexes, and give a characterization of pure trees which are connected in codimension 1 (Proposition 2.20).

In [**15**], Faridi showed that the facet ideal of a tree has sliding depth. We will show, in Chapter 3, that an M-sequence (introduced by Conca and Negri, see [**7**]) has sliding depth. We also prove that the facet ideals of several classes of trees are generated by M-sequences by showing that any such tree has at least one good leaf (see Definition 3.9). It is an open question whether an arbitrary tree has at least one good leaf. If this would be the case, then the facet ideal of any tree would be generated by an M-sequence.

The goal of Chapter 4 is to study the facet ideals of trees. In Section 4.1 the Koszul cycles of the facet ideal $I \subset R = K[x_1,\dots,x_n]$ of a tree are studied. By this we mean the cycles of the Koszul complex $K_\bullet(x, R/I)$ of R/I with respect to $x_1,\dots,x_n$. In Proposition 4.10 we show that the Koszul homology of the facet ideal of a tree has a K-basis with homology classes of monomial cycles as its elements. In the graph case we even show that the Koszul homology of the edge ideal of a tree is generated as a K-algebra by the homology classes of linear cycles, see Proposition 4.13. Using this fact, in Corollary 4.14, we determine the regularity and the projective dimension of the edge ideal of a 1-dimensional tree. Furthermore in Theorem 4.19 we show that for the edge ideal I of a 1-dimensional tree, the regularity of R/I is the maximal number j, for which there exist j edges which are pairwise disconnected (see Definition 4.16).

In the Section 4.2, we consider the facet ideal I of a pure tree (Definition 2.2) and describe the linear part of the resolution of R/I, see Proposition 4.23. We call a tree whose facet ideal has a linear resolution a linear tree. In Proposition 4.33 we show that a tree is a linear tree if and only if the facet ideal of this tree is a linear quotient ideal and we classify (Theorem 4.41) all linear trees of a given dimension. Moreover in Corollary 4.34, we determine the Betti numbers of the facet ideal of a linear tree.

In Proposition 4.45, we show that if there exists an order of the facets $F_1,\dots,F_m$ of Δ such that for each $i=2,\dots,m$, $F_i\setminus\bigcup_{j<i}F_j\neq\emptyset$, and there exists $j<i$ such that $|F_j\setminus F_i|=1$. Then I has the alternating sum property (see Definition 4.42). In particularly, the facet ideal of a pure quasi-tree which is connected in codimension 1 and the facet ideal of a tree (need not to be pure) which is connected in codimension 1 have the alternating sum property.

One of the fascinating results in classical graph theory is Dirac's theorem [**8**] on chordal graphs, that is, on graphs for which each cycle of G of length ≥ 4 has a chord. Dirac proved that a finite graph G is chordal if and only if G has a perfect elimination ordering on its vertices. Recall that a *perfect elimination ordering* (or a *simplicial elimination ordering*) is an ordering $v_n,\ldots,v_2,v_1$ on the vertices of G such that v_i is a simplicial vertex in the graph induced on vertices $\{v_1,\ldots,v_i\}$. Here a *simplicial vertex* in a graph is one whose neighbors form a clique. In Chapter 5, we give an algebraic proof of an equivalent form of Dirac's theorem by showing that a finite graph is chordal if and only if G is the pure 1-skeleton of a quasi-forest. Our proof of Dirac's theorem is certainly not easier than the original proof, but our algebraic approach gives new insight on the possible relation trees of a perfect ideal of codimension 2. Moreover, our version of Dirac's theorem in terms of quasi-trees allows to formulate a 'higher' Dirac theorem with applications to resolutions of powers of certain classes of monomial ideals.

In Chapter 6, we consider graded ideals in a polynomial ring over a field and ask when such an ideal has the property that all of its powers have linear resolutions.

It is known [**25**] that polymatroidal ideals have linear resolutions and that powers of polymatroidal ideals are again polymatroidal (see [**5**] and [**18**]). In particular they have again linear resolutions. In general however, powers of ideals with linear resolution need not to have linear resolutions. The first example of such an ideal was given by Terai. He showed that over a base field of characteristic $\neq 2$ the Stanley Reisner ideal $I=(abd,abf,ace,acd,aef,bde,bcf,bce,cdf,def)$ of the minimal triangulation of the projective plane has a linear resolution, while I^2 has no linear resolution. The example depends on the characteristic of the base field. If the base field has characteristic 2, then I itself has no linear resolution.

Another example, namely $I=(def,cef,cdf,cde,bef,bcd,acf,ade)$ is given by Sturmfels [**34**]. Again I has a linear resolution, while I^2 has no linear resolution. The example of Sturmfels is interesting because of two reasons: 1. it does not depend on the characteristic of the base field, and 2. it has linear quotients. Recall that an equigenerated ideal I is said to have linear quotients if there exists an order $f_1,\dots,f_m$ of the generators of I such that for all $i=1,\dots,m$ the colon ideals $(f_1,\dots,f_{i-1}):f_i$ are generated by linear forms. It is quite easy to see that such an ideal has a linear resolution (independent on the characteristic of the base field). However the example of Sturmfels also shows that powers of an ideal having linear quotients need not to have linear resolutions.

On the other hand it is known (see [**6**] and [**28**]) that the regularity of powers I^n of a graded ideal I is bounded by a linear function $an+b$, and is a linear function for large n. For ideals I whose generators are all of degree d one has the bound $\operatorname{reg}(I^n) \leq nd + \operatorname{reg}_x(\mathscr{R}(I))$, as shown by Römer [**31**]. Here $\mathscr{R}(I)$ is the Rees ring of I which is naturally bigraded, and $\operatorname{reg}_x(\mathscr{R}(I))$ is the x-regularity of $\mathscr{R}(I)$. It follows from this formula that each power of I has a linear resolution if $\operatorname{reg}_x(\mathscr{R}(I)) = 0$.

In Chapter 6, we will show (Theorem 6.16) that if $I \subset K[x_1,\ldots,x_n]$ is a monomial ideal with 2-linear resolution, then each power has a linear resolution. Our proof is based on the formula of Römer. In the second section we give a new and very short proof of his result, and remark that if there is a term order such that the initial ideal of the defining ideal P of the Rees ring $\mathscr{R}(I)$ is generated by monomials which are linear in the variables $x_1,\ldots,x_n$, then $\operatorname{reg}_x(\mathscr{R}(I)) = 0$. In Section 6.3 we view a 2-equigenerated squarefree monomial ideal as the edge ideal of a graph. By using a result of Fröberg (Theorem 5.8) and Dirac's theorem (Theorem 5.10) we define the right lexicographical term order for which the initial ideal of P is linear in the x variables. We show this in Section 6.4 and use a description of the Graver basis of the edge ring of a graph due to Oshugi and Hibi [**29**]. Based on the same ideas and using polarization we also can treat monomial ideals which are not necessarily squarefree. In the last section of Chapter 6 we extend the result of Theorem 6.16 by showing that all powers of complementary simplicial complexes of pure skeletons of a quasi-tree have linear resolutions.

In Chapter 7 we study the squarefree monomial ideals arising from lattices. One of the most influential results in the classical lattice theory is Birkhoff's fundamental structure theorem for finite distributive lattices (Theorem 7.10), which guarantees that, given a finite distributive lattice $\mathscr{L}$, there is a unique poset (partially ordered set) P such that $\mathscr{L}$ is isomorphic to the poset $\mathscr{J}(P)$ consisting of all poset ideals (including the empty set) of P, ordered by inclusion. (A poset ideal of P is a subset $I \subset P$ with the property that if $p \in I$ and $q \in P$ with $q \leq p$, then $q \in I$.) In fact, P can be chosen as the set of all join-irreducible elements of $\mathscr{L}$. Then $\mathscr{L} \simeq \mathscr{J}(P)$. (An element $p \in \mathscr{L}$ with $p \neq \hat{0}$ is called join-irreducible if there is no $q, r \in \mathscr{L}$ with $q < p$ and $r < p$ such that $p = q \vee r$.) In other words, by identifying $\mathscr{L}$ with $\mathscr{J}(P)$, if $p \in \mathscr{L}$ and $I = \{q \in P \colon q \leq p\} \in \mathscr{J}(P)$, then $p = I$.

Fix a finite distributive lattice $\mathscr{L} = \mathscr{J}(P)$. Let K be a field and $S = K[\{x_p, y_p\}_{p\in P}]$ the polynomial ring in $2|P|$ variables over K with $\deg x_p = 1$ and $\deg y_p = 1$ for all $p \in P$. We associate each element $I \subset \mathscr{J}(P) = \mathscr{L}$ with the squarefree monomial $u_I = (\prod_{p\in I} x_p)(\prod_{p\in P\setminus I} y_p) \in S$. In the paper [**19**] the monomial ideal $H_{\mathscr{L}} = (u_I)_{I\in\mathscr{L}}$ is discussed from viewpoints of both combinatorics and commutative algebra. The purpose of Chapter 7 is to introduce the squarefree monomial ideal $H_{\mathscr{L}}$ (we call it the Hibi ideal) for an arbitrary finite meet-semilattice $\mathscr{L}$ (i.e., each pair of elements in $\mathscr{L}$ has a meet) and to generalize some of the results obtained in [**19**].

Now, let $\mathscr{L}$ be an arbitrary finite meet-semilattice [**32**, p. 103] and $P \subset \mathscr{L}$ the set of join-irreducible elements of $\mathscr{L}$. For each element $q \in \mathscr{L}$ we write $\ell(q) = \{p \in P \colon p \leq q\} \subset P$. In particular $\ell(\hat{0}) = \emptyset$. Note that $\ell(q)$ is a poset ideal of P, and that $q \in \ell(q)$

if and only if q is join-irreducible. We thus obtain the map $\ell\colon \mathscr{L} \to \mathscr{B}_P$, which we call the canonical embedding of $\mathscr{L}$ into the Boolean lattice $\mathscr{B}_P$ consisting of all subsets of P ordered by inclusion. In the case of finite distributive lattices $\mathscr{L}$ with the set of join irreducible elements P, let K be a field and $S = K[\{x_p, y_p\}_{p\in P}]$ the polynomial ring in $2|P|$ variables over K with $\deg x_p = 1$ and $\deg y_p = 1$ for all $p \in P$. We associate, as in the case of a lattice, each element $q \in \mathscr{L}$ with the squarefree monomial $u_q = (\prod_{p\in\ell(q)} x_p)(\prod_{p\in P\setminus\ell(q)} y_p) \in S$ and set $H_{\mathscr{L}} = (u_q)_{q\in P} \subset S$.

In Chapter 7 the following topics on Hibi ideals $H_{\mathscr{L}}$ arising from finite meet-semilattices $\mathscr{L}$ will be studied:

- When has the Hibi ideal $H_{\mathscr{L}}$ of $\mathscr{L}$ a linear resolution? Theorem 7.30 guarantees that $H_{\mathscr{L}}$ has a linear resolution if and only if $\mathscr{L}$ is meet-distributive. (A finite meet-semilattice $\mathscr{L}$ is called meet-distributive if each interval $[x,y] = \{p \in \mathscr{L} : x \le p \le y\}$ of $\mathscr{L}$ is Boolean, where x is the meet of the lower neighbors of y in this interval. Here we call z a lower neighbor of y if y covers z.)

- How can we construct a finite multigraded free S-resolution $\mathbb{F}$ of $H_{\mathscr{L}}$? A construction of such a finite free resolution is given in Theorem 7.35 (i). Moreover, we will characterize when our resolution is minimal. In fact, it will be proved in Theorem 7.35 (ii) that our resolution is minimal if and only if $\mathscr{L}$ is meet-irredundant, i.e., for any $p \in \mathscr{L}$ and for any proper subset $S \subset N(p)$ the meet $\bigwedge\{q\colon q \in S\}$ is strictly greater than the meet $\bigwedge\{q\colon q \in N(p)\}$, where $N(p)$ is the set of lower neighbors of p in $\mathscr{L}$. In particular, if $\mathscr{L}$ is a meet-distributive meet-semilattice, then our finite free resolution is minimal (Corollary 7.36), and we describe its differential in Theorem 7.38.

- Since $H_{\mathscr{L}}$ is a squarefree monomial ideal, there is a simplicial complex Δ whose Stanley–Reisner ideal I_Δ coincides with $H_{\mathscr{L}}$. We are interested in the Alexander dual $\Delta^\vee$ of Δ. In case that $\mathscr{L}$ is a finite distributive lattice, a nice description of $\Delta^\vee$ can be obtained ([**19**, Lemma 3.1]). It seems, however, rather difficult, for an arbitrary finite meet-semilattice, to obtain an explicit description of the Alexander dual of $H_{\mathscr{L}}$. Very recently such a description has been found in [**20**]. We will consider a special meet-distributive meet-semilattice, namely, a poset ideal $\mathscr{I}$ of a finite distributive lattice. In this case a nice combinatorial description of the Alexander dual of $H_{\mathscr{I}}$ can be obtained (Theorem 7.40). Moreover, since $H_{\mathscr{I}}$ has a linear resolution, it follows that the Alexander dual of $H_{\mathscr{I}}$ is Cohen–Macaulay. The combinatorics on such Cohen–Macaulay complexes is discussed in Theorem 7.41.

- More generally, let $\mathscr{S} \subset \mathscr{L}$ be any subset of $\mathscr{L}$. The Hibi ideal $H_{\mathscr{S}}$ associated with $\mathscr{S}$ is again the monomial ideal in S generated by the monomials u_p with $p \in \mathscr{S}$, where $u_p = (\prod_{p\in\ell(p)} x_p)(\prod_{p\in P\setminus\ell(p)} y_p)$ and where $\ell(p)$ is the principal poset ideal $\{q \in P\colon q \le p\}$ in P. Let $\mathscr{S}$ be a segment of $\mathscr{L}$ (see Definition 7.44). For example, any poset ideal $\mathscr{I}$, or any poset coideal $\mathscr{J}$ of $\mathscr{L}$, as well as their intersection are segments in $\mathscr{L}$. In fact, $\mathscr{S}$ is a segment of $\mathscr{L}$ if and only if there exist poset ideal $\mathscr{I}$ and poset coideal $\mathscr{J}$ of $\mathscr{L}$ such that $\mathscr{S} = \mathscr{I} \cap \mathscr{J}$. As another generalization of [**19**, Theorem 2.4], we consider certain classes of simplicial complexes Δ (more general than those described in Theorem 7.41), and show (Theorem 7.47) that such a simplicial complex Δ is unmixed and each minimal vertex cover of Δ has cardinality n if and only if there exists a segment $\mathscr{S}$ of some distributive lattice $\mathscr{L}$ such that $H^*_{\mathscr{S}} = I(\Delta)$. Here $H^*_{\mathscr{S}}$ denotes the defining ideal of the Stanley–Reisner of the Alexander dual of Γ, where Γ is defined by the equation $H_{\mathscr{S}} = I_\Gamma$.

- We will also describe when $H_{\mathscr{I}} \cap H_{\mathscr{J}} = H_{\mathscr{I}\cap\mathscr{J}}$, and in Theorem 7.56 it is described when this ideal has a linear resolution. In the case $\mathscr{I} \cup \mathscr{J} = \mathscr{L}$ and $\mathscr{I} \cap \mathscr{J} = \emptyset$, in Theorem 7.57 we show that the ideal $H_{\mathscr{I}\cap\mathscr{J}}$ has always a linear resolution.

The results of Chapter 5, Chapter 6 and Section 7.3, 7.4, 7.5, 7.6 appear(ed) in joined papers [**21**], [**22**] and [**23**] with J. Herzog and T. Hibi.

CHAPTER 1

Basic facts on graphs

In this chapter, we will recall some notion of graphs and some basic facts of graph theories including the marriage problem.

1. Graphs and trees

A *graph* G consists of a finite set V of *vertices* and a collection E of subsets of V called *edges* where every edge of G is an unordered pair $\{v_i, v_j\}$ of distinct vertices v_i, v_j in V. We write $V(G)$ and $E(G)$ for the vertex set and edge set of G, respectively. If $e = \{v_i, v_j\}$ is an edge of G one says that the vertices v_i and v_j are *adjacent* or *connected* by e. In this case one also says that the edge e is *incident* with the vertex v_i or v_j, or the edge e has the *ends* v_i and v_j. The *degree* of a vertex v in V, denoted by $\deg(v)$, is the number of edges which is incident with v. A vertex of degree one (resp. zero) is called an *end* (resp. *isolated*) *vertex*. If all the vertices of G are isolated, G is called a *discrete graph*.

Let H and G be two graphs, H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning subgraph* is a subgraph H of G containing all the vertices of G.

Given any finite set S, we denote the cardinality of S by $|S|$. The number of edges in a graph and the degrees of its vertices are linked by a simple identity due to Euler (1736).

Proposition 1.1. *Let G be a graph with the vertex set V and edge set E. Then* $\sum_{v\in V} \deg(v) = 2|E(G)|$.

PROOF. Let $B := (b_{ve})_{v\in V, e\in E}$ be the incidence matrix of G, where

$$b_{ve} = \begin{cases} 1, & \text{if } v \text{ is an end of } e, \\ 0, & \text{otherwise.} \end{cases}$$

We compute the sum of its entries in two ways:

$$\sum_{v\in V} \deg(v) = \sum_{v\in V}\sum_{e\in E} b_{ve} = \sum_{e\in E}\sum_{v\in V} b_{ve} = \sum_{e\in E} 2 = 2|E(G)|.$$

□

One can also prove this proposition by induction on $|E(G)|$.

Corollary 1.2. *In any graph, the number of vertices of odd degree is even.*

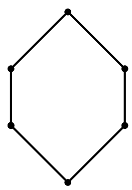
Definition 1.3. A *walk* of *length n* in a graph G is a sequence

$$w = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\},$$

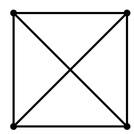
where $\{v_{i-1}, v_i\} = e_i$ is an edge for $i = 1, \ldots, n$. For the walk w we also use the simplified notation $w = \{v_0, v_1, \ldots, v_n\}$, where this set of vertices is understood to be an ordered set. If $v_0 = v_n$, the walk w is called a *closed walk*. A *path* is a walk with all its vertices distinct.

We say that G is *connected* if for every pair of vertices v_1 and v_2 there is a path from v_1 to v_2. Note that G has a vertex disjoint decomposition $G = \bigcup_{i=1}^{p} G_i$, where $G_1, \ldots, G_p$ are the maximal (with respect to inclusion) connected subgraphs of G, the G_i are called the *connected components* of G. A graph is connected if and only if it has only one connected component.

A *cycle* of *length* n is a closed walk $\{\{v_0, v_1\}, \ldots, \{v_{n-1}, v_n\}\}$ with $n \geq 3$ and the vertices $v_1, \ldots, v_n$ are distinct. A cycle is *even* (resp. *odd*) if its length is even (resp. odd). We denote by C_n the graph consisting of a cycle with n vertices. In particular, C_3 will be called a *triangle*, C_4 a *square* and so on. A *complete graph* K_n is a graph with every pair of its vertices adjacent. The following figure is an example of cycle and complete graph.



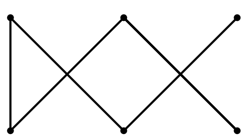
C_6



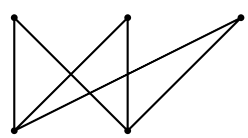
K_4

The *distance* $d(v_1, v_2)$ between two vertices v_1 and v_2 of a graph G is defined to be the minimal length of all possible paths from v_1 to v_2. If there is no path joining v_1 and v_2, then $d(v_1, v_2) = \infty$. In a complete graph, $d(v_1, v_2) = 1$ for any $v_1, v_2 \in V$.

A graph G is *bipartite* if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins V_1 with V_2, i.e., each edge of G has one end in V_1 and one end in V_2; such a partition (V_1, V_2) is called a *bipartition* of the graph G. A bipartite graph G with bipartition (V_1, V_2) is denoted by $G(V_1, V_2)$. Such a graph is *balanced* if $|V_1| = |V_2|$ and *complete* if each vertex of V_1 is joined to each vertex of V_2. For instance, the following graph G_1 is balanced and G_2 is complete.



G_1



G_2

The following characterization of bipartite graphs is due to König:

Proposition 1.4. *Let G be a graph. The following conditions are equivalent:*

(i) *G is bipartite;*
(ii) *all the cycles of G are even.*

PROOF. (i) $\Rightarrow$ (ii): Let V_1 and V_2 be a partition of $V(G)$ such that every edge of G joins V_1 and V_2. If $\{v_0, v_1, \ldots, v_n\}$ is a cycle of G, we may assume that $v_0 \in V_1$. Then

$v_1 \in V_2$, $v_2 \in V_1$, and so on. It follows that $v_i \in V_1$ if and only if i is even. Hence n must be even, i.e. $\{v_0, v_1, \ldots, v_n\}$ is an even cycle.

(ii) $\Rightarrow$ (ii): Is is enough to show that each connected component of G is bipartite. We may assume that G is connected. Let $v_0 \in V(G)$ be any vertex of G. Let $V_1 = \{v \in V(G) : d(v_0, v)$ is even $\}$ and $V_2 = V(G) \setminus X$. It follows that no two vertices of V_i are adjacent for $i = 1, 2$, otherwise G would contain an odd cycle. Hence G is bipartite. □

Definition 1.5. A graph G is called a *tree* if it is connected and has no cycle. A *forest* is a graph with all its connected components are trees.

Simply because any forest has no cycle, we have:

Corollary 1.6. *Any forest is a bipartite graph.*

2. Directed graphs

A *directed graph* or *digraph* D is a graph G in which each edge is assigned a direction, one end being designated its *tail* and the other its *head*. We call D an *orientation* of G, and write $D := \vec{G}$. An edge with tail x and head y is denoted by the ordered pair (x, y), one says that x *dominates* y and write $x \to y$. In diagrams, the direction of an edge is indicated by an arrow pointing towards the head of the edge.

Let v be a vertex of a digraph D. The *outdegree* $d_D^+(v)$ of v in D is the number of edges of D whose tail is v, the *indegree* $d_D^-(v)$ the number of edges of D whose head is v. A *sink* is a vertex of outdegree zero, a *source* a vertex of indegree zero. It follows easily from Proposition 1.1 that for any digraph D,

$$\sum_{v \in V} d^-(v) = e(D) = \sum_{v \in V} d^+(v).$$

A *directed walk* in a digraph D is a sequence $w = \{(v_0, v_1), (v_1, v_2) \ldots, (v_{n-1}, v_n)\}$. The vertices v_0 and v_n are the *tail* and the *head* of the directed walk w.

3. The marriage problem

In this section we shall state the famous marriage problem of graph theory.

Definition 1.7. A set of edges of a graph G is called a *matching* if no two of them have a vertex in common. The *matching number* of a graph G, denoted by $\beta_1(G)$, is the size of the largest matching in G. A vertex v is *saturated* by a matching M if some edge of M is incident with v. A matching which saturated all vertices of G is called *perfect*.

Note that not every graph has a perfect matching. The marriage problem theorem guarantees when a bipartite graph has a perfect matching.

Let G be a graph with vertex set V. Given a subset $U \subseteq V$, the *neighbor set* of U, denoted by $N_G(U)$ or simply $N(U)$, is defined as

$$N(U) = \{v \in V | v \text{ is adjacent to some vertex in } U\}.$$

Obviously, if C is a cycle in G with vertex set V_C, then $V_C \subseteq N(V_C)$.

Definition 1.8. Let G be a graph with vertex set V. A subset $C \subseteq V$ is called a *minimal vertex cover* of G, if the following conditions are satisfied:

(i) every edge of G is incident with one vertex in C;
(ii) there is no proper subset of C with the property (i).

If C satisfies condition (i) only, then C is called a *vertex cover* of G. The *vertex cover number* of G, denoted by $\alpha_0(G)$ is

$$\alpha_0(G) = \min\{|C| : \text{ where } C \text{ is a minimal vertex cover of } G\}.$$

For a discrete graph G, one takes the empty set as a minimal vertex cover of G. A set U of V is *independent* if no two of them are adjacent.

Remark 1.9. Let G be a graph with vertex set V. A subset U of V is a maximal independent set of G if and only if $V \setminus U$ is a minimal vertex cover of G.

Lemma 1.10. *For any graph G, we have $\beta_1(G) \leq \alpha_0(G)$.*

PROOF. Let C be a minimal vertex cover of G with $|C| = \alpha_0(G)$ and M a matching with $|M| = \beta_1(G)$. By the definition of matching, each vertex of C can cover at most one edge of M. Hence $\beta_1(G) \leq \alpha_0(G)$ as required. □

For a bipartite graph we have the following equality:

Theorem 1.11 (König). *If G is a bipartite graph, then $\beta_1(G) = \alpha_0(G)$.*

PROOF. See [**39**]. □

By using the König theorem we have

Theorem 1.12 (marriage problem). *Let G be a bipartite graph with vertex set V. Then the following conditions are equivalent:*

(i) *G has a perfect matching;*
(ii) *$|A| \leq |N(A)|$ for all independent set $A \subseteq V$.*

PROOF. Let $V = V_1 \cup V_2$ be a partition of V.

(i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i): Since V_i is an independent set for $i = 1, 2$, one has $|V_1| = |V_2|$. By König's theorem there is a matching $\{e_1, \dots, e_r\}$ with $r = \beta_1(G)$, and a minimal vertex cover C of G with r elements. Hence $e_i \cap C$ has exactly one vertex for any i and C is an independent set. Using that $V \setminus C$ is an independent set of vertices and the equality $N(V \setminus C) = C$ we get

$$|V \setminus C| \leq |N(V \setminus C)| = |C|.$$

It follows that $|V_1| = r$, thus $\{e_1, \dots, e_r\}$ is a perfect matching of G. □

4. Cohen–Macaulay graphs

In this section we introduce the edge ideal of a graph and recall some result of Cohen-Macaulay graphs.

Let R be a Noetherian local ring. A finite R-module $M \neq 0$ is a *Cohen-Macaulay module* if $\operatorname{depth} M = \dim M$. If R itself is a Cohen-Macaulay module, then it is called a *Cohen-Macaulay ring*. Let I be an ideal of R. If the quotient ring R/I is Cohen-Macaulay, then we say I is a *Cohen-Macaulay ideal*. The ideal I is said to be *height unmixed* or *unmixed* if $\operatorname{height} I = \operatorname{height} P$ for all $P \in \operatorname{Ass}_R(R/I)$.

Let G be a graph over the vertex set $[n] = \{1,\dots,n\}$ and $R = K[x_1,\dots,x_n]$ the polynomial ring in n variables over a field K.

Definition 1.13. The *edge ideal* $I(G)$ of the graph G is the ideal of R generated by the monomials $x_i x_j$ such that $\{i,j\} \in E(G)$. The *edge ring* is the subalgebra of R which is generated by the monomials $\{x_i x_j : \{i,j\} \in E(G)\}$ over K, denoted by $K[G]$, i.e.,

$$K[G] = K[\{x_i x_j : \{i,j\} \in E(G)\}].$$

If all the vertices of G are isolated, we set $I(G) = (0)$. Since G has no loop, the edge ideal $I(G)$ is a squarefree monomial ideal generated in degree 2.

Definition 1.14. A graph G is said to be *Cohen-Macaulay* if the edge ideal $I(G)$ is a Cohen-Macaulay ideal.

Remark 1.15. In general, the Cohen-Macaulay property of a graph G depend on the field K.

The next result establishes a one to one correspondence between the minimal vertex cover of a graph and the minimal primes of the corresponding edge ideal.

Proposition 1.16. *Let G be a graph over the vertex set $[n]$ and $R = K[x_1,\dots,x_n]$ a polynomial ring over a field K. If P is an ideal generated by $\{x_{i_1},\dots,x_{i_s}\}$, then the following conditions are equivalent:*

(i) *P is a minimal prime of $I(G)$;*
(ii) *$\{i_1,\dots,i_s\}$ is a minimal vertex cover of G.*

PROOF. Note that $I(G) \subset P$ if any only if $\{i_1,\dots,i_s\}$ is a vertex cover of G.

(i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Since $I(G)$ is a monomial ideal, every associated prime of $I(G)$ is an ideal generated by some variables. Since $I(G) \subset P$ and for any proper subset B of $\{i_1,\dots,i_s\}$, B is not a vertex cover of G. We have P is a minimal prime of $I(G)$. $\square$

Corollary 1.17. *Let G be a graph and $I(G)$ its edge ideal. Then $\alpha_0(G) = \operatorname{height} I(G)$.*

Definition 1.18. A graph G is *unmixed* if all minimal vertex covers of G have the same cardinality.

Since any Cohen-Macaulay ideal in a polynomial ring is unmixed, we have:

Proposition 1.19. *If G is a Cohen-Macaulay graph, then G is unmixed.*

The class of Cohen–Macaulay graphs is huge. In [**39**], Villarreal gave several constructions of Cohen–Macaulay graphs. In particularly, he gave the following effective description of Cohen–Macaulay trees and presented an interesting family of graphs containing all Cohen–Macaulay trees.

Theorem 1.20 (Villarreal). *Let T be a tree with vertex set V and edge set E. Then T is Cohen–Macaulay if and only if $|V| \le 2$ or $2 < |V| = 2r$ and there are vertices $a_1,\dots,a_r,b_1,\dots,b_r$ so that $\deg a_i = 1$, $\deg b_i \ge 2$, and $\{a_i,b_i\} \in E$ for $i = 1,\dots,r$.*

Recently, Herzog and Hibi classified all Cohen-Macaulay bipartite graphs in [**19**] by using the Alexander dual of some special simplicial complex (this kind of simplicial complexes we will discuss later in Chapter 7). Their main result is

Theorem 1.21 (Herzog-Hibi). *Let G be a bipartite graph with vertex set V and edge set E. Then G is Cohen–Macaulay if and only if after a suitable labelling of the vertices the following conditions hold:*

(i) $V = V_1 \cup V_2$ *where* $V_1 = \{x_1, \dots, x_n\}$ *and* $V_2 = \{y_1, \dots, y_n\}$*;*
(ii) $\{x_i, y_i\} \in E$ *for all* $i \in [n]$*;*
(iii) *if* $\{x_i, y_j\} \in E$*, then* $i \leq j$*;*
(iv) *if* $\{x_i, y_j\}, \{x_j, y_k\} \in E$*, with* $i < j < k$*, then* $\{x_i, y_k\} \in E$.

For the proof of this theorem, the marriage problem is needed.

Note that Theorem 1.21 implies Theorem 1.20. In fact, any tree T is a bipartite graph. By Theorem 1.21, T is Cohen–Macaulay if and only if after a suitable change of labelling of vertices, it satisfies condition (i) to (iv). Since a tree has no loop, by (iv) we have: if $\{x_i, y_j\} \in E$ with $i < j$, then $\{x_j, y_k\} \notin E$ for any $k > j$; and if $\{x_j, y_k\} \in E$, then $\{x_i, y_j\} \notin E$ for any $i < j$. Thus we can rename the vertices as follows: For all $i < j$ such that $\{x_i, y_j\} \in E$ we set $a_i = y_i$, $b_i = x_i$, $a_j = x_j$ and $b_j = y_j$. Then these vertices satisfy the conditions of Theorem 1.20.

CHAPTER 2

Simplicial complexes and quasi-trees

The main purpose of this chapter is to introduce some concepts of simplicial complexes such as Stanley–Reisner ideal and facet ideal, and define the trees and quasi-trees for simplicial complexes, both of them are generalizations of trees in graph case.

1. Stanley–Reisner ideals and facet ideals

In this section we recall the definition of simplicial complex, define the chains of a simplicial complex and study two squarefree monomial ideals (Stanley–Reisner ideal and facet ideal) associated to a simplicial complex.

Definition 2.1. A *simplicial complex* Δ over a set of vertices $V = \{v_1, \ldots, v_n\}$ is a collection of subsets of V with the property that $v_i \in \Delta$ for all i, and if $F \in \Delta$ then all the subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ, and the *dimension* of a face F is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. In particular, $\dim \emptyset = -1$. The faces of dimension 0 and 1 are called *vertices* and *edges*, respectively. The maximal faces of Δ under inclusion are called *facets*. A *nonface* of Δ is a subset W of V with $W \notin \Delta$.

We denote the simplicial complex Δ with the facets $F_1, \ldots, F_q$ by $\Delta = \langle F_1, \ldots, F_q \rangle$, and the facet set of Δ by $\mathscr{F}(\Delta)$. The simplicial complex with facets $F_1, \ldots, F_q$ is said to be generated by $F_1, \ldots, F_q$. A simplicial complex Δ generated by only one facet is called a *simplex*, note that $\emptyset$ is also a simplex. A simplicial complex Γ is called a *subcomplex* of Δ if $\mathscr{F}(\Gamma) \subseteq \mathscr{F}(\Delta)$.

The dimension of the simplicial complex Δ is the maximal dimension of its facets, that is

$$\dim \Delta = \max\{\dim F : F \in \Delta\}.$$

Definition 2.2. Let Δ be a simplicial complex of dimension d. Then Δ is called

(i) *pure*, if all of its facets have the same dimension;

(ii) *connected*, if for any two facets F and G there exists a sequence of facets $F = F_0, \ldots, F_n = G$, such that $F_i \cap F_{i+1} \neq \emptyset$ for all $i = 0, \ldots, n-1$; we call this sequence a *chain* between F and G, and n is called the *length* of this chain;

(iii) *connected in codimension 1*, if for any two facets F and G with $\dim(F) \geq \dim(G)$, there exists a chain $\mathscr{C}: F = F_0, \ldots, F_n = G$ between F and G such that $\dim(F_i \cap F_{i+1}) = \dim(F_{i+1}) - 1$ for all $i = 0, \ldots, n-1$.

The chain $\mathscr{C}$ (in Definition 2.2 (iii)) is called a *proper chain*. One can see that in a proper chain $\dim F_{i+1} \leq \dim F_i$ for $i = 0, \ldots, n-1$.

Definition 2.3. A (proper) chain $\mathscr{C}$ between F and G is called *irredundant* if no proper subsequence of $\mathscr{C}$ is a (proper) chain between F and G.

Remark 2.4. Any (proper) chain, after removing suitable facets in it, becomes an irredundant (proper) chain. In fact, let $\mathscr{C}$ be a (proper) chain between F and G. The set of (proper) subchains of $\mathscr{C}$ is with respect to inclusion a partially ordered non-empty set. The minimal elements in this set are the irredundant (proper) chains between F and G.

It is clear that an irredundant proper chain need not to be an irredundant chain. For example, $F_0 = \{a,b,c\}, F_1 = \{a,c,d\}, F_2 = \{c,d,e\}$ is an irredundant proper chain between F_0 and F_2, but it is not an irredundant chain, since F_0, F_2 is also a chain between F_0 and F_2.

Proposition 2.5. *Let $\mathscr{C}: F = F_0, F_1, \ldots, F_n = G$ be a proper chain between F and G. If $\mathscr{C}$ is irredundant, then $F_j \neq F_k$ for $j \neq k$, and $F_i \cap F_{i+1} \nsubseteq F_l \cap F_i$ for $i = 1, \ldots, n-1$, and any $l < i$.*

PROOF. Suppose there exists $k > j$ such that $F_j = F_k$, then $F_0, \ldots, F_j, F_{k+1}, \ldots, F_n$ is a proper subsequence of $\mathscr{C}$ and it is a proper chain between F and G, a contradiction.

Thus we may now assume $F_j \neq F_k$ for $j \neq k$. Suppose there exists $i \in [n-1]$, such that $F_i \cap F_{i+1} \subseteq F_l \cap F_i$ for some $l < i$. Then

$$F_l \cap F_{i+1} \supseteq (F_l \cap F_i) \cap (F_i \cap F_{i+1}) = F_i \cap F_{i+1},$$

so $\dim(F_l \cap F_{i+1}) \geq \dim(F_i \cap F_{i+1}) = \dim F_{i+1} - 1$. On the other hand, since $F_l \neq F_{i+1}$ both are facets, and $\dim F_{i+1} \leq \dim F_l$, it follows that $\dim(F_l \cap F_{i+1}) \leq \dim F_{i+1} - 1$. Hence $\dim(F_l \cap F_{i+1}) = \dim F_{i+1} - 1 = \dim(F_i \cap F_{i+1})$, together with $F_l \cap F_{i+1} \supseteq F_i \cap F_{i+1}$, we have $F_l \cap F_{i+1} = F_i \cap F_{i+1}$. Then $F_0, \ldots, F_l, F_{i+1}, \ldots, F_n$ is a proper subsequence of $\mathscr{C}$, and it is a proper chain between F and G, a contradiction. □

Let Δ be a simplicial complex over the vertex set $[n]$. Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K. For any $F = \{i_1, \ldots, i_k\} \subseteq [n]$, x_F denote the monomial $x_{i_1} \cdots x_{i_k}$ in R. There are two squarefree monomial ideals naturally associated to this simplicial complex.

Definition 2.6. The ideal $I_\Delta = (x_F : F$ is a nonface of $\Delta) \subset S$ is called the *Stanley–Reisner ideal* of Δ.

The *Stanley–Reisner ring* of Δ (with respect to the field K) is the homogeneous K-algebra $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$.

Note that I_Δ is generated by squarefree monomials. On the other hand, if $I \subset (x_1, \ldots, x_n)^2$ is any ideal which is generated by squarefree monomials, then $K[x_1, \ldots, x_n]/I \cong K[\Delta]$ for some simplicial complex Δ.

The dimension of a Stanley–Reisner ring can be easily determined.

Theorem 2.7. *Let Δ be a simplicial complex over the vertex set $[n]$, and K a field. Then*

$$I_\Delta = \bigcap_{F \in \mathscr{F}(\Delta)} P_{F^c}.$$

In particular,

$$\dim K[\Delta] = \dim \Delta + 1,$$

where P_{F^c} denotes the prime ideal generated by all x_i such that $x_i \in F^c$ and where $F^c = [n] \setminus F$.

One sees the proof for example in [**2**, Theorem 5.1.4].

Another squarefree monomial ideal associated to a simplicial complex is the so called facet ideal, introduced by S.Faridi.

Definition 2.8. The ideal $I(\Delta) = (x_F : F$ is a facet of $\Delta)$ is called the *facet ideal of* Δ.

Let G be a graph. If G contains no isolated vertex, then the facet ideal of G coincides with the edge ideal of G.

Now we introduce a very important concept related to simplicial complexes, called Alexander dual, which plays an important role in the following chapters.

Definition 2.9. Let Δ be a simplicial complex. The simplicial complex

$$\Delta^\vee = \{[n] \setminus F : F \notin \Delta\}$$

is called the *Alexander dual* of Δ.

It is easy to see that $(\Delta^\vee)^\vee = \Delta$.

Definition 2.10. Let Δ be a simplicial complex. The simplicial complex

$$\Delta^c = \langle [n] \setminus F : F \in \mathscr{F}(\Delta) \rangle.$$

is called the *complement* of Δ.

We denote $[n] \setminus F$ by F^c. As usual, we use $G(I)$ to denote the unique minimal generating system of the monomial ideal I. The following proposition gives a relation between the Alexander dual and the complimentary of a simplicial complex.

Proposition 2.11. *Let Δ be a simplicial complex. Then*

$$I_{\Delta^\vee} = I(\Delta^c).$$

PROOF. By definition, $\Delta^\vee = \langle F^c : F$ is a minimal nonface of $\Delta \rangle$. Furthermore, $x_G \in G(I_{\Delta^\vee})$ if and only if G is a minimal subset of $[n]$ such that $G \notin \Delta^\vee$. This means that G^c does not contain any minimal nonface of Δ, and for any proper subset H of G, the complement H^c contains a minimal nonface of Δ. This is equivalent to say that any subset of G^c is a face of Δ, and for any proper subset H of G, H^c is not a face of Δ. In another words, G^c is a facet of Δ. Hence $I(\Delta^c) = I_{\Delta^\vee}$, as required. □

Similar with the graph case, we have

Definition 2.12. A *vertex cover* of Δ is a set $G \subset [n]$ such that $G \cap F \neq \emptyset$ for all $F \in \mathscr{F}(\Delta)$. A vertex cover G of Δ is *minimal*, if any proper subset of G is not a vertex cover of Δ.

We denote by $\mathscr{C}(\Delta)$ the set of minimal vertex covers of Δ. If all the minimal vertex cover of Δ have the same cardinality, then we say Δ is *unmixed*.

The proof of the following proposition is similar with the proof of Proposition 1.16.

Proposition 2.13. *Let Δ be a simplicial complex over the vertex set $[n]$ and $I(\Delta)$ the facet ideal of Δ. Then an ideal $P = (x_{i_1}, \ldots, x_{i_s})$ is a minimal prime of $I(\Delta)$ if and only if $\{i_1, \ldots, i_s\}$ is a minimal vertex cover of Δ.*

For $F = \{i_1, \dots, i_k\} \subset [n]$ set $P_F = (x_{i_1}, \dots, x_{i_k})$, and let Γ be the unique simplicial complex such that $I_\Delta = I(\Gamma)$. Then

$$I_\Delta = \bigcap_{F \in \mathscr{C}(\Gamma)} P_F \quad \text{and} \quad I_{\Delta^\vee} = (x_F : F \in \mathscr{C}(\Gamma)).$$

By using Theorem 2.7 and Proposition 2.13, we have:

Corollary 2.14. *A subset F of $[n]$ is a facet of Γ if and only if F^c is a minimal vertex cover of Δ.*

2. Trees and quasi-trees

In this section we introduce the definition of higher dimensional trees and quasi-trees. Both of them are extensions of the trees in the graph case.

In [**15**] Faridi introduced the notion of trees for higher dimensional simplicial complexes. As we have seen in the graph case, a connected graph G is a tree if it has no cycle, in another word, any subgraph of G has a leaf (an edge with a free vertex). First we introduce the definition of leaf for an arbitrary simplicial complex.

Definition 2.15. Let Δ be a simplicial complex. A facet F of Δ is called a *leaf* if either F is the only facet of Δ, or there exists a facet $G \neq F$ in Δ, such that $F \cap H \subseteq F \cap G$ for any $H \in \mathscr{F}(\Delta)$, $H \neq F$. A facet G with this property is called a *branch* of F in Δ. The set of all branches of F in Δ is denoted by $\mathscr{U}_\Delta(F)$.

Let Δ be a simplicial complex with the vertex set $[n]$ and $F \in \mathscr{F}(\Delta)$. If i is a vertex of F and i does not belong to any other facets of Δ, then we call i a *free vertex* of F in Δ. It is clear that if F is a leaf of Δ, then F has at least one free vertex. But the converse is not true, even if Δ is pure.

For example, $\Delta = \langle \{1,2,3\}, \{3,4,5\}, \{5,6,7\} \rangle$ is a pure simplicial complex, the facet $\{3,4,5\}$ has a free vertex 4, but it is not a leaf of Δ.

Remark 2.16. Let Δ be a simplicial complex. It is easy to see that $F \in \mathscr{F}(\Delta)$ is a leaf of Δ if and only if $\langle F \rangle \cap \Gamma$ is a simplex, where $\Gamma = \langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ is the subcomplex of Δ.

Lemma 2.17. *Let $\mathscr{C} : F_0, \dots, F_n$ be an irredundant chain in a simplicial complex. Then $F_p \cap F_q = \emptyset$ for any $p \in \{0, \dots, n\}$ and any $q \neq p-1, p, p+1$. Furthermore, F_i is not a leaf of $\Gamma = \langle F_0, \dots, F_n \rangle$ for $i = 1, \dots, n-1$.*

PROOF. Suppose there exists $p \in \{0, \dots, n\}$ and $q > p+1$ or $q < p-1$, such that $F_p \cap F_q \neq \emptyset$. We may assume that $q > p+1$, then $F_0, \dots, F_p, F_q, \dots, F_n$ is a chain between F_0 and F_n, a contradiction.

Suppose F_j is a leaf of Γ for some $j \in \{1, \dots, n-1\}$. Since $F_j \cap F_k = \emptyset$ for any $k \neq j-1, j, j+1$, we have $F_{j-1} \cap F_j \subseteq F_j \cap F_{j+1}$ or $F_j \cap F_{j+1} \subseteq F_{j-1} \cap F_j$. We may assume that $F_{j-1} \cap F_j \subseteq F_j \cap F_{j+1}$, then $F_{j-1} \cap F_j = (F_{j-1} \cap F_j) \cap (F_j \cap F_{j+1}) \subseteq F_{j-1} \cap F_{j+1}$. On the other hand, since $\mathscr{C}$ is a chain, $F_{j-1} \cap F_j \neq \emptyset$, hence $F_{j-1} \cap F_{j+1} \neq \emptyset$. It follows that $F_0, \dots, F_{j-1}, F_{j+1}, \dots, F_n$ is a chain. This contradicts our assumption that $\mathscr{C}$ is irredundant. □

We have seen that an irredundant proper chain need not to be an irredundant chain. But as in Lemma 2.17 we also have:

Lemma 2.18. *Let $\mathscr{C}: F_0,\dots,F_n$ be an irredundant proper chain in a simplicial complex, and let $\Gamma = \langle F_0,\dots,F_n\rangle$. Then F_i is not a leaf of Γ, for $i = 1,\dots,n-1$.*

PROOF. Suppose F_i is a leaf of Γ for some $i \in [n-1]$. Then there exists an integer $k \neq i$ such that $F_i \cap F_{i+1} \subseteq F_i \cap F_k$. Since $\mathscr{C}$ is an irredundant proper chain, it follows from Proposition 2.5 that $k > i$.

For each $k \geq i+1$, we have $\dim(F_i \cap F_{i+1}) = \dim F_{i+1} - 1 \geq \dim F_k - 1 \geq \dim(F_i \cap F_k)$. It follows that $F_i \cap F_{i+1} = F_i \cap F_k$. So $F_0,\dots,F_i,F_k,\dots,F_n$ is a proper chain between F_0 and F_n, a contradiction. □

Definition 2.19 (Faridi). Let Δ be a connected simplicial complex. Then Δ is called a *tree* if every nonempty subcomplex of Δ has a leaf. A simplicial complex Δ with the property that every connected component is a tree is called a *forest*.

As a main result of this section we want to characterize when a pure tree is connected in codimension 1. For this purpose we recall the definitions of star and link of a face.

Let Δ be a simplicial complex, and H a face of Δ. Then the *star* of H is the set

$$\mathrm{st}_\Delta H = \{G \in \Delta \colon H \cup G \in \Delta\},$$

and the *link* of H is the set

$$\mathrm{lk}_\Delta H = \{G \in \Delta \colon H \cup G \in \Delta, \quad H \cap G = \emptyset\}.$$

To simplify notation we occasionally omit the index Δ in st_Δ or lk_Δ. Note that $\mathrm{lk}\, H \subset \mathrm{st}\, H$, and both are simplicial complex. Furthermore, $\mathrm{st}\, H$ is a subcomplex of Δ. Indeed one has $\mathscr{F}(\mathrm{st}\, H) = \{F \in \mathscr{F}(\Delta) \colon H \subset F\}$, and $\mathscr{F}(\mathrm{lk}\, H) = \{F \setminus H \colon F \in \mathscr{F}(\mathrm{st}\, H)\}$.

We refer the reader to [**2**] to see that these notations are crucial in the analysis of the local cohomology of a Stanley-Reisner ring.

Proposition 2.20. *Let Δ be a pure d-dimensional tree. Then the following statements are equivalent:*

(i) *for all $G \in \Delta$ with $\dim G \leq d-2$, $\mathrm{lk}\, G$ is connected;*
(ii) *Δ is connected in codimension 1.*

PROOF. (i)$\Rightarrow$(ii): Suppose Δ is not connected in codimension 1. Then there exists $F,H \in \mathscr{F}(\Delta)$ such that there is no proper chain between F and H. Since Δ is a tree, it is connected, and hence there exists a chain $F = H_0,H_1,\dots,H_q = H$ between F and H. Let $a = \min\{\dim(H_i \cap H_{i+1}) \colon i = 0,\dots,q-1\}$. Since this chain is not proper we have $0 \leq a < d-1$. We may assume that there is no other chain $F = K_0,\dots,K_p = H$ in Δ, such that $\min\{\dim(K_i \cap K_{i+1}) \colon i = 0,\dots,p-1\} > a$, otherwise we take this chain instead of $H_0,\dots,H_q$. Let $\{i_1,\dots,i_m\} \subseteq \{0,\dots,q\}$ be the subset such that $\dim(H_{i_j} \cap H_{i_j+1}) = a$, we know that $\{i_1,\dots,i_m\} \neq \emptyset$. By the choice of our chain there must exist $j \in \{1,\dots,m\}$ such that there is no chain $H_{i_j} = E_0,E_1,\dots,E_s = H_{i_j+1}$ in Δ such that $\min\{\dim(E_i \cap E_{i+1}) \colon i = 0,\dots,s-1\} > a$.

Let $G = H_{i_j} \cap H_{i_j+1}$, then $\dim G = a < d-1$. We claim that $\mathrm{lk}\, G$ is not connected. In fact, if $\mathrm{lk}\, G$ is connected, then there exists a chain $H_{i_j} = D_0,D_1,\dots,D_l = H_{i_j+1}$ in $\mathrm{st}\, G$ such that $(D_i \setminus G) \cap (D_{i+1} \setminus G) \neq \emptyset$, for any $i = 1,\dots,l-1$. This implies that $\dim(D_i \cap D_{i+1}) > a$ for any $i = 1,\dots,l-1$, a contradiction to the choice of j.

(ii)⇒(i): Suppose there exists $G \in \Delta$ with $\dim G \leq d-2$, such that $\operatorname{lk} G$ is not connected. Then there exist facets F and H in $\operatorname{st} G$ such that there is no chain between $F \setminus G$ and $H \setminus G$ in $\operatorname{lk} G$.

Since Δ is connected in codimension 1, there exists an irredundant proper chain $F = H_0, H_1, \ldots, H_r = H$ between F and H. Since $\dim G \leq d-2$, it follows that $(H_i \cap H_{i+1}) \setminus G \neq \emptyset$, $i = 0, \ldots, r-1$. Moreover not all H_i belong to $\operatorname{st} G$, because otherwise $F \setminus G = H_0 \setminus G, H_1 \setminus G, \ldots, H_r \setminus G = H \setminus G$ would be a chain between $F \setminus G$ and $H \setminus G$ in $\operatorname{lk} G$.

Let $l = \min\{j \in \{0, \ldots, r\} : H_{j+1} \notin \operatorname{st} G\}$, and let $m = \min\{j \in \{l+2, \ldots, r\} : H_j \in \operatorname{st} G\}$. Now consider the sequence of facets $H_l, \ldots, H_m$. It is an irredundant proper chain between H_l and H_m, and $H_l, H_m \in \operatorname{st} G$, $H_{l+1}, \ldots, H_{m-1} \notin \operatorname{st} G$.

Take the subcomplex $\Gamma = \langle H_l, \ldots, H_m \rangle$ of Δ. Then this subcomplex has no leaf, and so Δ is not a tree, a contradiction. Indeed, since $H_l, \ldots, H_m$ it is an irredundant proper chain, it follows from Lemma 2.18 that H_i is not a leaf of Γ for $i = l+1, \ldots, m-1$. Now consider the facet H_l, and let $H_l \cap H_{l+1} = K$. Then K is a face of Δ with dimension $d-1$. Let $\{i\} = H_l \setminus H_{l+1}$. Since $H_{l+1} \notin \operatorname{st} G$, we have $G \not\subset K$. On the other hand, since $H_l \in \operatorname{st} G$, we must have $i \in G$. Similarly we conclude that $i \in H_m$. That is to say, H_l has no free vertex in Γ, and hence H_l is not a leaf of Γ. With the same argument we can show that H_m is not a leaf of Γ. □

Corollary 2.21. *Let Δ be a pure d-dimensional tree and F a facet of Δ. If Δ is connected in codimension 1, then all the facets of $\langle F \rangle \cap \langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ are of dimension $d-1$.*

PROOF. Suppose there exists a facet F of Δ, such that $\langle F \rangle \cap \langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ is not pure of dimension $d-1$. Then there exists $H \in \mathscr{F}(\Delta)$ such that $F \cap H = G$ with $\dim G \leq d-2$ and $G \not\subset F \cap H'$ for all $H' \in \mathscr{F}(\Delta) \setminus \{F\}$.

We claim $\operatorname{lk} G$ is not connected. In fact, assume $\operatorname{lk} G$ is connected, then, since $L \in \Delta$ belongs to $\mathscr{F}(\operatorname{st} G)$ if and only if $L \setminus G \in \mathscr{F}(\operatorname{lk} G)$, there exists a sequence of facets $F = F_0, F_1, \ldots, F_r = H$ in $\operatorname{st} G$ such that $(F_i \setminus G) \cap (F_{i+1} \setminus G) \neq \emptyset$ for $i = 0, \ldots, r-1$. We may assume $F_1 \neq F$. Since $(F \setminus G) \cap (F_1 \setminus G) \neq \emptyset$, G is a proper subset of $F \cap F_1$, a contradiction.

Now Proposition 2.20 implies Δ is not connected in codimension 1, a contradiction to our hypothesis. □

Remark 2.22. Let Δ be a pure d-dimensional tree. Even if for any facet F of Δ, all the facets of $\langle F \rangle \cap \langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ are of dimension $d-1$, Δ may not be connected in codimension 1.

For example, $\Delta = \langle \{1,2,3\}, \{2,3,4\}, \{3,5,6\}, \{3,6,7\} \rangle$ is pure of dimension 2, and for any facet F of Δ, the facets of $\langle F \rangle \cap \langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ are of dimension 1, but Δ is not connected in codimension 1.

However we have:

Corollary 2.23. *Let Δ be a pure tree of dimension d and connected in codimension 1, F a facet of Δ. Then $\Gamma = \langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ is connected in codimension 1 if and only if F is a leaf of Δ.*

PROOF. Assume F is a leaf of Δ. Let G and H be any two facets in Γ. Hence $G, H \in \mathscr{F}(\Delta)$. Since Δ is connected in codimension 1, there exists an irredundant proper chain

$\mathscr{C} : G = F_0, F_1, \dots, F_l = H$ in Δ. By Lemma 2.18, F_i is not a leaf of $\langle F_0, \dots, F_l \rangle$ for all $i = 1, \dots, l-1$. Since $\mathscr{C}$ is an irredundant proper chain in Δ, F_i is not a leaf of Δ. Hence $F \neq F_i$ for all $i = 1, \dots, l-1$. Therefore $\mathscr{C}$ is a irredundant proper chain between G and H in Γ. It follows that Γ is connected in codimension 1.

Now assume Γ is connected in codimension 1. By Corollary 2.21, $\langle F \rangle \cap \Gamma$ is a pure simplicial complex of dimension $d-1$. Assume F is not a leaf of Δ. Then there exist two facets H_1 and H_2 in Γ such that $F \cap H_1 \neq F \cap H_2$ and $\dim(F \cap H_i) = d-1$, $i = 1, 2$. Let $G = H_1 \cap H_2$. Then $\dim G = d-2$. We may assume $H_1 = G \cup \{i_1, i_2\}$ and $H_2 = G \cup \{i_3, i_4\}$ and $F = G \cup \{i_1, i_4\}$, where i_j are vertices. Since Γ is a pure tree and connected in codimension 1, by Proposition 2.20, $\operatorname{lk}_\Gamma G$ is connected. Let $\{i_1, i_2\} = H_1 \setminus G = F_1 \setminus G, \dots, F_l \setminus G = H_2 \setminus G = \{i_3, i_4\}$ be an irredundant chain between $\{i_1, i_2\}$ and $\{i_3, i_4\}$ in $\operatorname{lk}_\Gamma G$. Then the subcomplex $\langle F, F_1, \dots, F_l \rangle$ of Δ has no leaf, a contradiction. Indeed, each vertex in $\langle F, F_1, \dots, F_l \rangle$ belongs to at least two facets of this subcomplex. $\square$

Another consequence of Corollary 2.21 is

Proposition 2.24. *Let Δ be a pure tree which is connected in codimension 1, and has more than one facet. Then Δ has at least two leaves.*

PROOF. Let $\dim \Delta = d$. Suppose Δ has only one leaf. Let F_1 be this leaf. Since Δ is connected and has more than one facet, there exists a facet G such that $F_1 \cap G \neq \emptyset$. Since Δ is pure it follows from Corollary 2.21 that there exists a facet F_2, such that $F_1 \cap G \subseteq F_1 \cap F_2$ and $\dim(F_1 \cap F_2) = d-1$. Let $F_2 \setminus F_1 = \{x\}$. Since F_2 is not a leaf, there exists a facet H, such that $\{x\} \subseteq F_2 \cap H$. Again by Corollary 2.21 there exists a facet F_3, such that $F_2 \cap H \subseteq F_2 \cap F_3$ and $\dim(F_2 \cap F_3) = d-1$. It is clear that $F_3 \neq F_1$. Since F_3 is not a leaf, by the same reason there exists a facet $F_4 \neq F_2$, and $\dim(F_3 \cap F_4) = d-1$, and so on. Since there are only finitely many facets, there exist integers i and j with $j < i-1$ such that $F_i = F_j$. If $F_{i-1} \cap F_i \neq F_j \cap F_{j+1}$, then the subcomplex $\langle F_j, \dots, F_{i-1} \rangle$ has no leaf. If $F_{i-1} \cap F_i = F_j \cap F_{j+1}$, then the subcomplex $\langle F_{j+1}, \dots, F_{i-1} \rangle$ has no leaf. This contradicts our assumption that Δ is a tree. $\square$

By definition, in a simplicial complex Δ which is connected in codimension 1, for any two facets F and G, there exists an irredundant proper chain between F and G. For a pure tree we even have

Proposition 2.25. *Let Δ be a pure tree and connected in codimension* 1*. Then for any two facets F and G, there exists a unique irredundant proper chain between F to G.*

PROOF. Suppose $\mathscr{C} : F = F_0, \dots, F_n = G$ and $\mathscr{C}' : F = G_0, \dots, G_m = G$ are two different irredundant proper chains between F and G. Let $l = \min\{j :$ such that $F_j \neq G_j\}$, and $k = \min\{i : i \geq j+1$ and $F_i = G_t$ for some $t\}$. Then, since $\mathscr{C}$ and $\mathscr{C}'$ both are irredundant, $t > l$ and $G_t \neq G_i$ for any $i \neq t$. Let Γ be a subcomplex of Δ, such that $F_l, \dots, F_{k-1}, G_l, \dots, G_{t-1} \in \Gamma$; if $F_{l-1} \cap F_l \neq G_{l-1} \cap G_l$, then let $F_{l-1} \in \Gamma$; if $F_{k-1} \cap F_k \neq G_{t-1} \cap G_t$, then let $F_k \in \Gamma$; and there are no other facet in Γ. By Lemma 2.18 one can easily check that Γ has no leaf, a contradiction since Δ is a tree. $\square$

According to this proposition, we give the following definition:

Definition 2.26. Let Δ be a pure tree and connected in codimension 1. For any two facets F and G, the length of the unique irredundant proper chain between F and G is called the *distance* between F and G, and denoted by $\mathrm{dist}(F,G)$.

We call $\max\{\mathrm{dist}(F,G) : F$ and G are two facets of $\Delta\}$ the *diameter* of Δ.

If Δ is a pure forest and each connected component is connected in codimension 1, then for any two facets F and G which lie in two different components, we set $\mathrm{dist}(F,G) = \infty$.

Sometimes we consider a special simplicial complex which need not to be a tree, but has some nice properties like a tree, we call it a quasi-tree.

Definition 2.27. A connected simplicial complex Δ is called a *quasi-tree*, if there exists an order $F_1,\dots,F_n$ of the facets, such that F_i is a leaf of $\langle F_1,\dots,F_i\rangle$ for all $i \in [n]$. Such an order is called a *leaf order*. A simplicial complex Δ with the property that every connected component is a quasi-tree is called a *quasi-forest*.

Note that a quasi-tree may have several different leaf orders. It is clear that a tree is a quasi-tree, since any subcomplex of a tree has a leaf. Hence for any tree there exists a leaf order of facets. But a quasi-tree need not to be a tree.

For example, $\Delta = \langle\{1,2,3\},\{2,3,4\},\{3,4,5\},\{2,4,6\}\rangle$ is a quasi-tree, but it is not a tree, because the subcomplex $\langle\{1,2,3\},\{3,4,5\},\{2,4,6\}\rangle$ has no leaf.

Remark 2.28. If the simplicial complex Δ has dimension 1, namely, Δ is a graph, then Δ is a tree if and only if Δ is a quasi-tree.

As we have seen in Proposition 2.24, any pure connected in codimension 1 tree with more than one facet has at least two leaves. In fact, in [**16**], Faridi proved that any tree with more than one facet has at least two leaves. Her proof holds also for quasi-tree case. For the convenience of the reader, we recall the proof of this property.

Proposition 2.29. *Let Δ be a quasi-tree with more than one facet. Then Δ has at least two leaves.*

PROOF. Suppose $\Delta = \langle F_1,\dots,F_m\rangle$, $m \geq 2$. We prove this proposition by induction on m.

The case $m = 2$ follows from the definition of a leaf.

Suppose $m > 2$ and F_1 is a leaf of Δ with G_1 one of its branch. Consider the subcomplex $\Delta' = \langle F_2,\dots,F_m\rangle$ of Δ. By induction hypothesis Δ' has at least two distinct leaves, say F_2 and F_3 are those two leaves. We may assume $F_2 \neq G_1$. We show that F_2 is a leaf of Δ.

Let G_2 be a branch of F_2 in Δ'. For any $i \neq 1,2$, since F_2 is a leaf of Δ', we have $F_i \cap F_2 \subseteq G_2 \cap F_2$. We need to verify this for $i = 1$. Since F_1 is a leaf of Δ with a branch G_1 and $F_2 \neq F_1$, we have $F_2 \cap F_1 \subseteq G_1 \cap F_1$. Intersecting both sides of this inclusion with F_2, we obtain

$$F_2 \cap F_1 \subseteq G_1 \cap F_1 \cap F_2 \subseteq G_1 \cap F_2 \subseteq G_2 \cap F_2,$$

where the last inclusion holds because $G_1 \neq F_2$ and F_2 is a leaf of Δ' with a branch G_2.

By the definition, $F_2 \neq F_1$ is a leaf of Δ. □

Definition 2.30. A simplicial complex Δ is called *flag*, if all minimal nonfaces of Δ consist of two elements, equivalently, I_Δ is generated by quadratic monomials.

We also consider a simplex as a flag complex. Note that if Δ has only two facets, then Δ is flag.

Lemma 2.31. *A quasi-forest is a flag complex.*

PROOF. Let Δ be a quasi-forest on $[n]$ and fix a leaf ordering of the facets $F_1,\dots,F_t$ of Δ. We work by induction on t. Let $t > 2$. Since $\Delta' = \langle F_1,\dots,F_{t-1}\rangle$ is a quasi-forest, by assumption of induction it follows that Δ' is flag. Let F_k with $k < t$ be a branch of F_t. Then Δ' consists of all faces G of Δ with $G \cap (F_t \setminus F_k) = \emptyset$.

Suppose H is a minimal nonface of Δ having at least three elements of $[n]$. Since H is a nonface, there is $p \in H$ with $p \notin F_t$. If $q \in F_t$ belongs to H, then $\{p,q\} \in \Delta$. Thus there is F_j with $j \neq t$ such that $\{p,q\} \subset F_j$. Hence $\{q\} \subset F_t \cap F_j$. Thus $q \in F_k$. Hence $H \cap (F_t \setminus F_k) = \emptyset$. This shows that H is a minimal nonface of Δ', a contradiction. $\square$

CHAPTER 3

M-sequences and good leaves

In this chapter we introduce the notion of M-sequence, and show that any ideal generated by an M-sequence has sliding depth. For several special kinds of forests Δ, we show that the facet ideal of Δ is generated by an M-sequence.

1. M-sequences and sliding depth

The notion of M-sequence is given by Conca and Negri in [**7**]. In this section we shall show that the ideal generated by an M-sequence has sliding depth.

Throughout this section $R = K[x_1,\dots,x_l]$ denotes the polynomial ring in l variables over a field K.

Given a monomial $m = \prod_{j=1}^{r} x_{i_j}^{a_j}$ in R. We say this product presentation of m is *standard*, if $i_1 < i_2 < \cdots < i_r$ and $a_1 > 0,\dots,a_r > 0$. Of course if we change the numbering of the variables the standard presentation of m also changes. In the following, unless otherwise stated, we always write the monomials in standard form.

Definition 3.1. A sequence of monomials $m_1,\dots,m_s$ in R is said to be an M-*sequence* if for all $1 \le i \le s$ there exists a numbering of the variables such that if $m_i = x_{i_1}^{a_1}\cdots x_{i_r}^{a_r}$ and whenever $x_{i_k}|m_j$ for some $1 \le k \le r$ and $i < j$, then $x_{i_k}^{a_k}\cdots x_{i_r}^{a_r}|m_j$.

Remark 3.2. In the definition of M-sequence, for each m_i the numbering of the variables may depend on i.

The following lemma follows immediately from the definition of M-sequence.

Lemma 3.3. *Let $m_1,\dots,m_s$ be an M-sequence in a set of indeterminates x. Then*

(i) *every subsequence of $m_1,\dots,m_s$ is an M-sequence;*
(ii) *if m is a monomial such that $m|m_1$, then $m_1/m, m_2,\dots,m_s$ is an M-sequence.*

Before we state the main result of this section, we recall the definition of sliding depth introduced by Herzog–Vasconcelos–Villarreal [**26**].

Definition 3.4. Let $I \subset R$ be a graded ideal. Let $\mathbb{K}$ be the Koszul complex on a homogeneous set $\mathfrak{a} = \{a_1,\dots,a_s\}$ of generators of I, and denote by $H_i(\mathbb{K})$ the homology modules of $\mathbb{K}$. We say I has *sliding depth* if

$$\operatorname{depth} H_i(\mathbb{K}) \ge l - s + i \quad \text{for all } i.$$

It can be shown that the definition does not depend on the choice of the generators.

We want to prove that any ideal generated by an M-sequence has sliding depth. For this we need the following two lemmata:

Lemma 3.5. *Let $m_1,\dots,m_s$ be homogeneous polynomials in R. If there exists $1 \le r < l$ such that $m_2,\dots,m_s \in K[x_1,\dots,x_r]$ and $m_1 \in K[x_{r+1},\dots,x_l]$, then m_1 is regular on $H_i([2,s];R)$ for all i, where $H_i([2,s];R)$ is the Koszul homology $H_i(m_2,\dots,m_s;R)$ of $m_2,\dots,m_s$.*

PROOF. We want to show m_1 is not a zero-divisor of $H_i([2,s];R)$ for all i, that is to say, $m_1 \notin \bigcup_{P\in \mathrm{Ass}(H_i([2,s];R))} P$ for all i.

Let $R_1 = K[x_1,\dots,x_r]$. Since $H_i([2,s];R) \cong H_i([2,s];R_1) \bigotimes_{R_1} R$, we have

$$\mathrm{Ass}(H_i([2,s];R)) = \{P'R \mid P' \in \mathrm{Ass}(H_i([2,s];R_1)\}.$$

Since $m_1 \in K[x_{r+1},\dots,x_l]$, $m_2,\dots,m_s \in K[x_1,\dots,x_r] = R_1$, we have $m_1 \notin P'R$ for any $P' \in \mathrm{Ass}(H_i([2,s];R_1))$, hence $m_1 \notin \bigcup_{P\in \mathrm{Ass}(H_i([2,s];R))} P$ for all i as required. □

Lemma 3.6 (depth lemma). *Let $0 \to N \to M \to L \to 0$ be a short exact sequence of graded R-modules. Then*

(i) *if* $\operatorname{depth} M < \operatorname{depth} L$, *then* $\operatorname{depth} N = \operatorname{depth} M$;
(ii) *if* $\operatorname{depth} M = \operatorname{depth} L$, *then* $\operatorname{depth} N \ge \operatorname{depth} M$;
(iii) *if* $\operatorname{depth} M > \operatorname{depth} L$, *then* $\operatorname{depth} N = \operatorname{depth} L + 1$.

This well-known lemma can be found for example in [**38**, Corollary A.6.3] or [**9**, Corollary 18.6].

In the following we assume that if $m_1,\dots,m_s$ is an M-sequence, then the variables are numbered such that $m_1 = x_{i_1}^{a_1}\cdots x_{i_n}^{a_n}$ satisfies the condition in the definition of M-sequence.

Now we show the main result of this section:

Theorem 3.7. *Let $R = K[x_1,\dots,x_l]$ be the polynomial ring in l variables over a field K, and $m_1,\dots,m_s$ monomials in R. If $m_1,\dots,m_s$ is an M-sequence, then the ideal $I = (m_1,\dots,m_s)$ has sliding depth in R.*

PROOF. We prove this theorem by induction on s.

If $s = 1$, then the Koszul complex looks like

$$\mathbb{K}\colon 0 \longrightarrow R \xrightarrow{m_1} R \longrightarrow 0,$$

which gives $H_0(\mathbb{K}) = R/m_1R$, $\operatorname{depth} R/m_1R = l-1$. And $\operatorname{depth} H_1(\mathbb{K}) = \operatorname{depth}(0:_R m_1) = \operatorname{depth} 0 = \infty$. So $I = (m_1)$ has sliding depth.

Suppose the assertion holds for any M-sequence with up to $s-1$ ($s \ge 2$) monomials. Now let $m_1,\dots,m_s$ be an M-sequence of length s, and let $m_1 = x_{i_1}^{a_1}\cdots x_{i_n}^{a_n}$.

We consider three cases:

Case 1. There exists some j with $2 \le j \le s$ such that $x_{i_1} | m_j$. Then by the definition of M-sequences, we have $m_1|m_j$. Hence $I = (m_1,\dots,m_s) = (m_1,\dots,m_{j-1},m_{j+1},\dots,m_s)$. By Lemma 3.3, $m_1,\dots,m_{j-1},m_{j+1},\dots,m_s$ is an M-sequence. Hence I has sliding depth by the induction hypothesis.

Case 2. $x_{i_k} \nmid m_j$ for all $1 \le k \le n$ and all $2 \le j \le s$. By [**2**, Corollary 1.6.13], we have the exact sequence

$$\begin{array}{l}\cdots \longrightarrow H_i(m_2,\dots,m_s;R) \xrightarrow{m_1} H_i(m_2,\dots,m_s;R) \longrightarrow H_i(m_1,\dots,m_s;R) \\ \longrightarrow H_{i-1}(m_2,\dots,m_s;R) \xrightarrow{m_1} H_{i-1}(m_2,\dots,m_s;R) \longrightarrow \cdots .\end{array}$$

By Lemma 3.5, m_1 is regular on $H_i(m_2,\dots,m_s;R)$ for all i. Hence we have

$$H_i(m_1,\dots,m_s;R)\cong H_i(m_2,\dots,m_s;R)/m_1H_i(m_2,\dots,m_s;R)\quad \text{for all } i.$$

By Lemma 3.3 and induction hypothesis, we have

$$\operatorname{depth} H_i(m_2,\dots,m_s;R)\geq l-(s-1)+i\quad \text{for all } i.$$

Hence

$$\operatorname{depth} H_i(m_1,\dots,m_s;R)=\operatorname{depth} H_i(m_2,\dots,m_s;R)-1\geq l-(s-1)+i-1=l-s+i.$$

That is to say, $I=(m_1,\dots,m_s)$ has sliding depth.

Case 3. There exists $1\leq t<n$ such that $x_{i_k}\nmid m_j$ for all $1\leq k\leq t$ and all $2\leq j\leq s$, and there exists $2\leq j\leq s$ such that $x_{i_{t+1}}|m_j$. By the definition of M-sequences, $x_{i_{t+1}}^{a_{t+1}}\cdots x_{i_n}^{a_n}|m_j$. Let $Z_1\subset\{x_1,\dots,x_l\}$ be all the indeterminates which appear in $m_2,\dots,m_s$, and $Z_2=\{x_{i_1},\dots,x_{i_t}\}$. Then $Z_1\cap Z_2=\emptyset$. Let $y_1=x_{i_{t+1}}^{a_{t+1}}\cdots x_{i_n}^{a_n}$, $y_2=x_{i_1}^{a_1}\cdots x_{i_t}^{a_t}$. Then $m_1=y_2y_1$. Let $I_1=(y_1,m_2,\dots,m_s)=(y_1,m_2,\dots,m_{j-1},m_{j+1},\dots,m_s)$. By Lemma 3.3 (i) and (ii), $y_1,m_2,\dots,m_{j-1},m_{j+1},\dots,m_s$ is an M-sequence with $s-1$ monomials. Hence the ideal I_1 has sliding depth by induction hypothesis.

We have the following short exact sequence

$$(1)\quad 0\to H_i(m_2,\dots,m_s;R)_{m_1}\to H_i(m_1,\dots,m_s;R)\to H_{i-1}(m_2,\dots,m_s;R)^{m_1}\to 0,$$

where for any R-module N and any $f\in R$, $N_f=N/fN$ and $N^f=\{m\in N:\ fm=0\}$.

Hence by using the next lemma and the depth lemma we have

$$\operatorname{depth} H_i(m_1,\dots,m_s;R)\geq l-s+i\quad \text{for all } i.$$

□

As in Lemma 3.5, to simplify the notation, in the following lemma we again write $H_i(m_2,\dots,m_s;R)$ as $H_i([2,s];R)$.

Lemma 3.8. *We use the notation as in Proposition 3.7. Then the following holds:*

(i) $\operatorname{depth} H_i([2,s];R)_{y_1}\geq l-s+i+1$, *and* $\operatorname{depth} H_i([2,s];R)^{y_1}\geq l-s+i+1$ *for all i;*

(ii) $\operatorname{depth} H_i([2,s];R)_{m_1}\geq l-s+i$, *and* $\operatorname{depth} H_i([2,s];R)^{m_1}\geq l-s+i+1$ *for all i.*

PROOF. (i) We have

$$(2)\qquad 0\longrightarrow H_i([2,s];R)_{y_1}\longrightarrow H_i(y_1,m_2\dots,m_s;R)\longrightarrow H_{i-1}([2,s];R)^{y_1}\longrightarrow 0,$$

The exact sequence:

$$0\longrightarrow H_i([2,s];R)^{y_1}\longrightarrow H_i([2,s];R)\xrightarrow{y_1} H_i([2,s];R)\longrightarrow H_i([2,s];R)_{y_1}\longrightarrow 0$$

breaks into two exact sequences

$$(3)\qquad 0\longrightarrow y_1H_i([2,s];R)\longrightarrow H_i([2,s];R)\longrightarrow H_i([2,s];R)_{y_1}\longrightarrow 0$$

and

$$(4)\qquad 0\longrightarrow H_i([2,s];R)^{y_1}\longrightarrow H_i([2,s];R)\longrightarrow y_1H_i([2,s];R)\longrightarrow 0$$

We prove (i) by induction on i. Let $i=0$. By induction hypothesis $H_0([2,s];R)=R/(m_2,\dots,m_s)$ has depth $\geq l-(s-1)$. Since $y_1|m_j$ for some $2\leq j\leq s$, we have

$$H_0([2,s];R)_{y_1}=R/(y_1,m_2,\dots,m_s)=R/(y_1,m_2,\dots,\widehat{m_j},\dots,m_s)$$

is the 0-th Koszul homology of the ideal $I_1=(y_1,m_2,\dots,m_s)=(y_1,m_2,\dots,\widehat{m_j},\dots,m_s)$. Hence $\operatorname{depth} H_0([2,s];R)_{y_1}\geq l-(s-1)$. Together with (3) and the depth lemma, we have

$$\operatorname{depth} y_1H_0([2,s];R)\geq l-(s-1).$$

Then (4) and the depth lemma imply that

$$\operatorname{depth} H_0([2,s];R)^{y_1}\geq l-(s-1).$$

Now let $i>0$. Since by induction hypothesis

$$\operatorname{depth} H_{i-1}([2,s];R)^{y_1}\geq l-s+(i-1)+1=l-s+i,$$

and since

$$\operatorname{depth} H_i(y_1,m_2\dots,m_s;R)\geq l-(s-1)+i,$$

sequence (2) and Lemma 3.6 imply that

$$\operatorname{depth} H_i([2,s];R)_{y_1}\geq l-(s-1)+i.$$

By using (3) and the fact $\operatorname{depth} H_i([2,s];R)\geq l-(s-1)+i$, we conclude that

$$\operatorname{depth} H_i([2,s];R)^{y_1}\geq l-(s-1)+i.$$

(ii) Recall that $m_1=y_2y_1$. We have the exact sequence

$$0\longrightarrow H_i([2,s];R)_{y_1}\longrightarrow H_i([2,s];R)_{m_1}\longrightarrow H_i([2,s];R)_{y_2}\longrightarrow 0. \tag{5}$$

Since $y_2\in K[Z_2]$ and $m_2,\dots,m_s\in K[Z_1]$ and $Z_1\cap Z_2=\emptyset$. By Lemma 3.5, y_2 is regular on $H_i([2,s];R)$. Hence we have

$$\operatorname{depth} H_i([2,s];R)_{y_2}=\operatorname{depth} H_i([2,s];R)-1\geq l-(s-1)+i-1=l-s+i.$$

Hence by (i), exact sequence (5) and the depth lemma we have

$$\operatorname{depth} H_i([2,s];R)_{m_1}\geq l-s+i.$$

Since $\operatorname{depth} H_i([2,s];R)^{y_1}\geq l-s+i+1$ and $m_1=y_1y_2$ we have $\operatorname{depth} H_i([2,s];R)^{m_1}\geq l-s+i+1$. $\square$

2. New classes of M-sequences

In this section, we will show that the facet ideals of some special kinds of forests are generated by M-sequences. Hence the facet ideals of these forests have sliding depth. For this, we need a new concept called "good leaf".

Let Δ be a simplicial complex. Suppose F is a leaf of Δ. It is clear that F need not to be a leaf of all subcomplexes of Δ which contains F. For example, let

$$\Delta=\{\{1,2,3\},\{1,3,4\},\{3,4,5\},\{4,6,7\}\}.$$

Then $\{3,4,5\}$ is a leaf of Δ. But it is not a leaf of the subcomplex

$$\{\{1,2,3\},\{3,4,5\},\{4,6,7\}\}.$$

Definition 3.9. Let Δ be a simplicial complex and F a facet of Δ. If F is a leaf of all subcomplexes of Δ which contain F, then we say that F is a *good leaf* of Δ.

A good leaf is obviously a leaf. On the other hand, if F is a leaf of Δ, but it is not a leaf of some subcomplex which contains F, then we say F is a *bad leaf* of Δ.

To see whether a leaf of a simplicial complex is a good leaf or not, we have the following useful fact:

Lemma 3.10. *Let Δ be a simplicial complex, F a facet of Δ. Then the following conditions are equivalent:*

(i) *F is a good leaf of Δ;*
(ii) *the set $\{F \cap H : H \in \mathscr{F}(\Delta)\}$ is totally ordered by inclusion.*

PROOF. The case that Δ is a simplex is trivial. We may assume that Δ contains at least two facets.

(i) $\Rightarrow$ (ii): Assume there exist facets H_1 and H_2 such that $F \cap H_1 \not\subseteq F \cap H_2$ and $F \cap H_2 \not\subseteq F \cap H_1$. Then F is not a leaf of $\{F, H_1, H_2\}$, this contradicts that F is a good leaf.

(ii) $\Rightarrow$ (i): Let $\Delta' = \{F, F_1, \ldots, F_s\}$ be any subcomplex of Δ which contains the facet F. Since the set $\{F \cap H : H \in \mathscr{F}(\Delta)\}$ is totally ordered, we may assume $F \cap F_1 \subseteq F \cap F_2 \subseteq \cdots \subseteq F \cap F_s$. Hence F is a leaf of Δ' with a branch F_s. By definition, F is a good leaf of Δ. □

Throughout this section we assume Δ is a simplicial complex on the vertex set $[n]$ and R is the polynomial ring $K[x_1, \ldots, x_n]$, where K is a field. Let F be a facet of Δ. As before, we write x_F for the monomial $\prod_{i \in F} x_i$ in R.

Proposition 3.11. *Let $\Delta = \langle F_1, \ldots, F_m \rangle$ be a simplicial complex. If for each $i \in [m]$, F_i is a good leaf of the subcomplex $\langle F_i, \ldots, F_m \rangle$, then $x_{F_1}, \ldots, x_{F_m}$ is an M-sequence. In particular $I(\Delta)$ has sliding depth.*

PROOF. We prove this proposition by induction on m. The case $m = 1$ is trivial. Assume $m > 1$. Let $\Delta' = \langle F_2, \ldots, F_m \rangle$. By induction hypothesis, $x_{F_2}, \ldots, x_{F_m}$ is an M-sequence. Since F_1 is a good leaf of Δ, by Lemma 3.10, there exists a permutation $i_2, \ldots, i_m$ of $2, \ldots, m$ such that $F_1 \cap F_{i_2} \supseteq \cdots \supseteq F_1 \cap F_{i_m}$.

We renumber the vertices in a way such that for any $2 \leq j \leq m$, the vertices that appear in $F_1 \cap F_{i_{j-1}} \setminus F_1 \cap F_{i_j}$ are smaller than the vertices in $F_1 \cap F_{i_j}$ (we set $F_1 = F_{i_1}$). Now we write $x_{F_1} = x_{k_1} \cdots x_{k_q}$ in standard form (with respect to this new numbering), and claim it satisfies the M-sequence condition.

If for all i, $x_{k_i} \nmid x_{F_j}$ for any $2 \leq j \leq m$, then there is nothing to prove. If there exists some $l \leq q$ and some $2 \leq j \leq m$ such that $x_{k_l} | x_{F_j}$ (i.e., $k_l \in F_1 \cap F_j$), then in our numbering, we have $k_p \in F_1 \cap F_j$ for all $l < p \leq q$. Hence $x_{k_l} \cdots x_{k_q} | x_{F_j}$. Together with the induction hypothesis that $x_{F_2}, \ldots, x_{F_m}$ is an M-sequence, we have $x_{F_1}, \ldots, x_{F_m}$ is an M-sequence. Hence the assertion holds. □

Corollary 3.12. *Let Δ be any 1-dimensional forest. The facet ideal of Δ is generated by an M-sequence.*

PROOF. Let $\Delta = \langle F_1, \ldots F_m \rangle$ such that F_i is a leaf of the subcomplex $\langle F_i, \ldots F_m \rangle$. Since Δ has dimension 1, it is obvious that F_i is a good leaf of $\langle F_i, \ldots F_m \rangle$. Hence by Proposition 3.11, the facet ideal $I = (x_{F_1}, \ldots, x_{F_m})$ is generated by the M-sequence $x_{F_1}, \ldots, x_{F_m}$. □

Now we want to show that the facet ideal of any two dimensional forest is generated by an M-sequence. We show this by showing that any two dimensional forest has at least one good leaf. For this, we need the following two lemmata.

Lemma 3.13. *Let Δ be a simplicial complex and F a bad leaf of Δ. Then there exist facets G and H such that $F\cap G \not\subseteq F\cap H$ and $F\cap H \not\subseteq F\cap G$.*

PROOF. The assertion follows immediately from Lemma 3.10. □

Let Δ be a simplicial complex and F a leaf of Δ. If $\dim F = 1$, then F is a good leaf of Δ. If $\dim F = 2$, then F is a bad leaf if and only if there exist facets F^b, G, H such that $F\cap F^b = \{i, j\}$, $F\cap G = \{i\}$ and $F\cap H = \{j\}$, where i and j are two different vertices of Δ.

Lemma 3.14. *Let Δ be a two dimensional forest, and F, F^b, G and H as above. Then at least one of $F^b\cap G$ and $F^b\cap H$ is a vertex.*

PROOF. First $i\in F^b\cap G$ and $j\in F^b\cap H$. Assume $F^b\cap G$ and $F^b\cap H$ both are edges, say $F^b\cap G = \{i,k\}$ and $F^b\cap H = \{j,l\}$. We have $\dim F^b \le 2$, since $\dim\Delta = 2$, and since F is a bad leaf, its branch F^b is of dimension 2. Hence since $i\ne j$, we must have $k = l$. It follows that $G\cap H = \{k\}$. Hence the subcomplex $\{F, G, H\}$ has no leaf. This contradicts that Δ is a forest. □

Lemma 3.15. *Let Δ be a forest, F a bad leaf of Δ, F^b a branch of F. Then F is not a good leaf of $\langle \mathscr{F}(\Delta)\setminus\{F^b\}\rangle$.*

PROOF. By Lemma 3.13, there exist facet G and H such that $F\cap G \not\subseteq F\cap H$ and $F\cap H \not\subseteq F\cap G$. Since F^b is a branch of F, we have $F\cap G\subseteq F\cap F^b$ and $F\cap H\subseteq F\cap F^b$. Hence the set $\{F\cap K \mid K\in\mathscr{F}(\Delta)\setminus\{F^b\}\}$ is not totally ordered. By Lemma 3.10, F is not a good leaf of $\langle\mathscr{F}(\Delta)\setminus\{F^b\}\rangle$. □

In the assumption of the following lemma, we need the fact that any tree with more than one facet has at least two leaves (see Proposition 2.29).

Lemma 3.16. *Let Δ be a two dimensional forest with more than one facet, F_1, F_2 two bad leaves of Δ. If F_2 is a good leaf of $\langle\mathscr{F}(\Delta)\setminus\{F_1\}\rangle$, then F_1 and F_2 have a branch in common.*

PROOF. First, the branches of F_1 can not be edges, since F_1 is a bad leaf.

We want to show there exists a branch of F_1 whose intersection with F_2 is an edge. Assume this is not the case. Let F_1^b be a branch of F_1. There are two cases to be considered. Case 1: $F_2\cap F_1^b = \emptyset$. In this case, since F_1 is a leaf, we have $F_2\cap F_1 = \emptyset$. Hence by Lemma 3.10, we have that F_2 is a bad leaf of $\langle\mathscr{F}(\Delta)\setminus\{F_1\}\rangle$. Case 2: $F_2\cap F_1^b$ is a vertex $\{i\}$. If $\{i\}\notin F_1$, then $F_2\cap F_1 = \emptyset$. As in case 1, F_2 is a bad leaf of $\langle\mathscr{F}(\Delta)\setminus\{F_1\}\rangle$. If $\{i\}\in F_1$, then $F_1\cap F_2 = F_1^b\cap F_2 = \{i\}$. Since F_2 is a bad leaf of Δ, using Lemma 3.10 and the fact $F_1\cap F_2 = F_1^b\cap F_2$, we have F_2 is not a good leaf of $\langle\mathscr{F}(\Delta)\setminus\{F_1\}\rangle$. Hence there exists a branch of F_1 which is also a branch of F_2. □

To conclude that any two dimensional forest has at least one good leaf, we also need the following two lemmata:

Lemma 3.17. *Let Δ be a forest of arbitrary dimension, F a bad leaf of Δ and $G \in \mathscr{F}(\Delta)$ a non-leaf of Δ. If G is a leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$, then G is a branch of F.*

PROOF. Suppose G is not a branch of F. Let F^b be a branch of F. Then $G \neq F^b$ and $F \cap G \subseteq F \cap F^b$. Hence $F \cap G = F \cap G \cap G \subseteq F \cap F^b \cap G \subseteq F^b \cap G$.

Since G is a leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$, there exists a facet $H \neq G, F$ such that $G \cap K \subseteq G \cap H$, for any $K \neq G, F$. In particular for $K = F^b$ we have $G \cap F^b \subseteq G \cap H$, and hence $G \cap F \subseteq G \cap H$. This implies that G is a leaf of Δ, a contradiction. □

Lemma 3.18. *Let Δ be a forest of arbitrary dimension, F a bad leaf of Δ, F^b a branch of F. Then F^b is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$.*

PROOF. Since F is a leaf of Δ and F^b is a branch of F, we have $F \cap G \subseteq F^b \cap G$ for any facet $G \neq F$ of Δ. Assume F^b is a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. Since F is a bad leaf of Δ, there exist facets H and K such that $F \cap H \nsubseteq F \cap K$ and $F \cap K \nsubseteq F \cap H$. Since by Lemma 3.10, the set $S = \{F^b \cap G : G \in \mathscr{F}(\Delta) \setminus \{F\}\}$ is totally ordered, we may assume $F^b \cap H \subseteq F^b \cap K$. Let $i \in (F \cap H) \setminus (F \cap K)$. Then $i \in F^b \cap H \subseteq F^b \cap K$. Since $i \in F$ and $i \in K$ we have $i \in F \cap K$, a contradiction. □

Corollary 3.19. *Let Δ be a forest of arbitrary dimension, F a bad leaf of Δ and $G \in \mathscr{F}(\Delta)$ a non-leaf of Δ. Then G is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$.*

Proposition 3.20. *Any two dimensional forest has at least one good leaf.*

PROOF. Suppose Δ is a two dimensional forest with m facets. We prove the assertion by induction on m.

The case $m = 1$ is trivial. Assume $m > 1$. Suppose Δ has no good leaf. Let F be a bad leaf of Δ. Remove F from Δ. By Corollary 3.19, no non-leaf of Δ becomes a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. Hence the induction hypothesis implies that there exists a bad leaf G of Δ, such that G is a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. By Lemma 3.16, F and G have a branch in common, say $F^b = \{i, j, k\}$ is this branch with $F \cap F^b = \{i, j\}$ and $G \cap F^b = \{j, k\}$. By Lemma 3.15, we have F and G are not good leaves of $\langle \mathscr{F}(\Delta) \setminus \{F^b\} \rangle$. Since Δ is a forest and F, G are leaves of Δ, we have $H \cap F^b = H \cap F$ or $H \cap F^b = H \cap G$ for any facet $H \neq F, G$ of Δ. Hence when we move F^b away from Δ, no non-leaf of Δ becomes a leaf of $\langle \mathscr{F}(\Delta) \setminus \{F^b\} \rangle$, and no bad leaf of Δ becomes a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F^b\} \rangle$. Therefore there is a forest $\langle \mathscr{F}(\Delta) \setminus \{F^b\} \rangle$ with $m-1$ facets which has no good leaf, a contradiction. □

Corollary 3.21. *Let Δ be a 2-dimensional forest. Then the facet ideal of Δ is generated by an M-sequence.*

PROOF. Using the previous proposition, there exists an order $F_1, \ldots, F_m$ of the facets of Δ such that F_i is a good leaf of the subcomplex $\langle F_i, \ldots, F_m \rangle$ for all $i \in [m]$. By Proposition 3.11, the facet ideal $I = (x_{F_1}, \ldots, x_{F_m})$ of Δ is generated by the M-sequence $x_{F_1}, \ldots, x_{F_m}$. □

In the remaining of this section, we will show that any pure tree which is connected in codimension 1 has at least one good leaf.

Lemma 3.22. *Let Δ be a pure simplicial complex of dimension d, F and F_i facets of Δ, $i = 1,2,3$. If $\dim(F \cap F_i) = d-1$ and $F \cap F_i$ are pairwise distinct, then Δ is not a tree.*

PROOF. Suppose $F = \{i_0, i_1, \ldots, i_d\}$, $F_1 = \{i, i_1, \ldots, i_d\}$, $F_2 = \{i_0, j, i_2, \ldots, i_d\}$ and $F_3 = \{i_0, i_1, k, i_3, \ldots, i_d\}$, where i_0, i, j, k are different vertices of Δ. Consider the subcomplex $\Delta' = \langle F_1, F_2, F_3 \rangle$ of Δ. Since $F_1 \cap F_2 \not\subseteq F_1 \cap F_3$ and $F_1 \cap F_3 \not\subseteq F_1 \cap F_2$, F_1 is not a leaf of Δ'. With the same argument, one sees that F_2, F_3 are not leaves of Δ'. Hence Δ is not a tree. $\square$

Lemma 3.23. *Let Δ be a d-dimensional pure tree which is connected in codimension 1. If F is a leaf with only one branch F^b, then $\langle F \rangle \cap \langle \mathscr{F}(\Delta) \setminus \{F, F^b\} \rangle$ is contained in a face of $\langle F \rangle$ with dimension $\leq d-2$.*

PROOF. Assume the assertion is not true. Since F is a leaf and F^b is the unique branch of F, there exist facets G and H such that $F \cap G \not\subseteq F \cap H$, $F \cap H \not\subseteq F \cap G$ and $(F \cap G) \cup (F \cap H)$ is not contained in any $(d-2)$-dimensional face of F. By Proposition 2.25 there exist irredundant proper chains $H = H_0, \ldots, H_m = F^b$ and $G = G_0, \ldots, G_L = F^b$ from H and G to F^b, respectively. Since $F \cap H \not\subseteq F \cap G$ and $(F \cap G) \cup (F \cap H)$ is not contained in any $(d-2)$-dimensional face of $\langle F \rangle$, we have $H_{m-1} \cap F^b \neq G_{l-1} \cap F^b$, otherwise the subcomplex $\langle H_0, \ldots, H_m \rangle$ has no leaf. But then we have four facets F^b and F, H_{m-1}, G_{l-1} satisfy the condition of Lemma 3.22, this contradicts that Δ is a tree. $\square$

Lemma 3.24. *Let Δ be a pure tree which is connected in codimension 1. If F is a bad leaf of Δ with only one branch F^b, then F^b is not a leaf of Δ.*

PROOF. Since Δ has a bad leaf, it has at least 4 facets. Let $H \neq F, F^b$ be any facet of Δ. Since Δ is connected in codimension 1 there is a proper chain between H and F. Since F^b is the unique branch of F, it must belong to this chain, and hence it is not a leaf. $\square$

Lemma 3.25. *Let Δ be a pure tree which is connected in codimension 1. If F is a bad leaf with only one branch, then any bad leaf of Δ is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$.*

PROOF. Let F^b be the branch of F with $\{i\} = F^b \setminus F$ and $K \neq F$ a bad leaf of Δ. By Lemma 3.24, $K \neq F^b$. Since F is a bad leaf of Δ, there exist facets G and H such that $F \cap G \not\subseteq F \cap H$ and $F \cap H \not\subseteq F \cap G$. It is clear that G and H can not both contain the vertex i, otherwise the subcomplex $\{F, G, H\}$ of Δ has no leaf. Assume $i \notin H$. By Lemma 3.23, $\langle F \rangle \cap \langle \mathscr{F}(\Delta) \setminus \{F, F^b\} \rangle$ is contained in a face B of $\langle F \rangle$ with $\dim B \leq d-2$. Let $H = H_0, \ldots, H_m = F^b$ and $G = G_0, \ldots, G_L = F^b$ be irredundant proper chains from H and G to F^b, respectively. There are two cases:

Case 1. F^b is a branch of K. Since F has only one branch, we have $F \cap F^b \neq K \cap F^b$. Since Δ is a tree and F has only one branch, by Lemma 3.22 we have $K \cap F^b = H_{m-1} \cap F^b = G_{l-1} \cap F^b$. Hence $\dim(F \cap K) = d-2$ and $B = F \cap K$. Therefore $K \cap H \not\subseteq K \cap G$ and $K \cap G \not\subseteq K \cap H$. This shows that K is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$.

Case 2. F^b is not a branch of K. If $i \notin K$, then we have $F \cap K = F^b \cap K$. Since K is a bad leaf of Δ, by Lemma 3.10, we have K is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. Now assume $i \in K$. Since K is a bad leaf of Δ, there exists a facet D such that

$$K \cap F \not\subseteq K \cap D \quad \text{and} \quad K \cap D \not\subseteq K \cap F \tag{6}$$

If $i \notin D$, then (6) also holds after replacing F by F^b. Hence K is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. Now we may assume $i \in D$. We claim there exists a facet N of Δ such that $B \subset N$ and $i \notin N$. Otherwise there exist facets F_1 and F_2 such that $i \in F_1 \cap F_2$ but $F_1 \cap F \nsubseteq F_2 \cap F$ and $F_2 \cap F \nsubseteq F_1 \cap F$. Hence the subcomplex $\{F, F_1, F_2\}$ of Δ has no leaf, a contradiction. Hence we have $K \cap F \subseteq K \cap N$. By using (6), we have $K \cap N \nsubseteq K \cap D$. On the other hand, since $i \in K \cap D$ and $i \notin K \cap N$, we have $K \cap D \nsubseteq K \cap N$. Hence K is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. □

Lemma 3.26. *Let Δ be a pure tree and connected in codimension 1, F a bad leaf with more than one branch. If G is a bad leaf of Δ, then G is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$.*

PROOF. By Lemma 3.18, we may assume G is not a branch of F. If $F \cap G = \emptyset$, then it is clear G is not a good leaf of $\Delta \setminus \{F\}$. If $F \cap G \neq \emptyset$, then let F_1, F_2 be any two branches of F with $\{i_1\} = F_1 \setminus F$, $\{i_2\} = F_2 \setminus F$ and $\{i\} = F \setminus F_i$, $i = 1, 2$. We claim i_1, i_2 can not both in G. Otherwise, F_i is not a leaf of the subcomplex $\langle F_1, F_2, G \rangle$, since it has no free vertex, for $i = 1, 2$; and G is not a leaf of $\langle F_1, F_2, G \rangle$ either, since $F_1 \cap G \nsubseteq F_2 \cap G$ and $F_2 \cap G \nsubseteq F_1 \cap G$. This contradicts that Δ is a tree. We may assume $i_1 \notin G$. Then since F is a leaf of Δ, $i \notin G$, we have $G \cap F = G \cap F_1$. Since G is a bad leaf of Δ, by Lemma 3.10, we have G is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. □

Proposition 3.27. *Any pure tree which is connected in codimension 1 has at least one good leaf.*

PROOF. Let Δ be a pure tree with m facets which is connected in codimension 1. We prove this proposition by induction on m. The case $m = 1$ is trivial. Let $m > 1$. Assume Δ has no good leaf. Let F be any leaf of Δ. There are two cases:

Case 1. F has more than one branch. By Lemma 3.17 and Lemma 3.18, any non-leaf of Δ is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. By Lemma 3.26, any bad leaf of Δ is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. Hence $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ is a tree with $m - 1$ facets having no good leaf. This contradicts our induction hypothesis.

Case 2. F has only one branch. Let F^b be the unique branch of F in Δ. By Lemma 3.17 and Lemma 3.18, any non-leaf of Δ will not become a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. By Lemma 3.24, F^b is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. And by Lemma 3.25, any bad leaf of Δ is not a good leaf of $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. Hence $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ is a tree with $m - 1$ facets having no good leaf. Again, a contradiction. □

Hence again it follows

Corollary 3.28. *Let Δ be a pure forest. If each component of Δ is connected in codimension 1, then the facet ideal of Δ is generated by an M-sequence.*

In [**7**], the following theorem is shown.

Theorem 3.29 (Conca–Negri). *Let I be a homogeneous ideal of the ring R. Suppose that there exists a monomial order $<$, such that $\mathrm{in}_<(I)$ is generated by an M-sequence $m_1, \ldots, m_s$. Then*

(i) *$\mathscr{R}(I)$ and $\mathrm{gr}_I(R) = \bigoplus_{i=0}^{\infty} I^i / I^{i+1}$ are Cohen–Macaulay;*
(ii) *if m_i is squarefree for all $i \in [s]$, then $\mathrm{gr}_I(R)$ is reduced.*

Since the facet ideal of a simplicial complex is generated by squarefree monomials, by using Theorem 3.29, Corollary 3.12, Corollary 3.21 and Corollary 3.28 we have

Corollary 3.30. *Let* Δ *be a forest and* I *the facet ideal of* Δ*. If* $\dim(\Delta) \leq 2$ *or* Δ *is pure and each component is connected in codimension* 1*, then*

(i) *I has sliding depth;*
(ii) *$\mathscr{R}(I)$ is Cohen–Macaulay;*
(iii) *$\mathrm{gr}_I(R)$ is reduced.*

In general, we believe that any forest has at least one good leaf. If this is the case, then the facet ideal I of any forest Δ is generated by an M-sequence, and hence the conclusions of Corollary 3.30 would hold for any forest.

CHAPTER 4

Facet ideals

In this chapter we study the resolutions of the facet ideals of trees. We show that the Koszul homology of the facet ideal I of a tree Δ is generated by the homology classes of monomial cycles. In case Δ is a graph, we determine the projective dimension and the regularity of I. If Δ is connected in codimension 1, then the graded Betti numbers of I satisfy an alternating sum property. We will also classify all trees whose facet ideal has a linear resolution.

1. On the Koszul cycles of the facet ideal of a tree

Let $R = K[x_1,\dots,x_n]$ be the polynomial ring in n variables over a field K and M an R-module. We denote the Koszul complex $K_\bullet(x,M)$ of M with respect to the sequence $x_1,\dots,x_n$ by $K_\bullet(M)$, and for the modules of Koszul cycles, Koszul boundaries and the Koszul homology we write $Z_\bullet(M)$, $B_\bullet(M)$ and $H_\bullet(M)$, respectively.

For this section, the following concept is important.

Definition 4.1. A graph Γ with vertex set $[n]$ and edges $\{1,i\}$ for $i = 2,\dots,n$ is called a *bouquet*. We denote this bouquet by $(1;2\dots,n)$. The vertex 1 is called the *root*, and for $i = 2,\dots,n$ the vertices i is called a *flower* and the edge $\{1,i\}$ a *stem* of this bouquet.

Let G be a graph. If a subgraph Γ of G is a bouquet, then we say Γ is a bouquet of G.

For simplicity, in the remaining of this section, any simplicial complex has the vertex set $[n]$. For a facet $F = \{i_1,\dots,i_s\}$ in Δ, as before we denote by $x_F = x_{i_1}\cdots x_{i_s}$ the monomial in R corresponding to F.

For the proof of the main result of this section which describes the Koszul cycles of certain monomial ideals, we need the following general result on the shifts in the resolution of a $\mathbb{Z}^n$-graded module: Let M be a finite $\mathbb{Z}^n$-graded R-module with minimal $\mathbb{Z}^n$-graded free resolution

$$\cdots \longrightarrow \bigoplus_{a\in\mathbb{Z}^n} R(-a)^{b_{1a}} \longrightarrow \bigoplus_{a\in\mathbb{Z}^n} R(-a)^{b_{0a}} \longrightarrow M \longrightarrow 0.$$

The numbers b_{ia} are called the multigraded Betti numbers of M.

We define the *support* of an element $a\in\mathbb{Z}^n$ to be the set $\operatorname{supp} a = \{i\colon a_i \neq 0\}$. Without ambiguity, we may set $\operatorname{supp} x^a = \operatorname{supp} a$ for any non-zero monomial. We set $\mathbb{Z}^n_+ = \{a\in\mathbb{Z}^n\colon a_i \geq 0 \text{ for all } i = 1,\dots,n\}$. Then we have

Lemma 4.2. *Let M be a torsion-free $\mathbb{Z}^n$-graded R-module, and $y_1,\dots,y_s$ a minimal homogeneous generating system of M. Suppose that* $\operatorname{supp}(\deg y_i) \subseteq \mathbb{Z}^n_+$ *and* $t \notin \operatorname{supp}(\deg y_i)$ *for $i = 1,\dots,s$. Then $t \notin \operatorname{supp} a$ for all non-zero multigraded Betti numbers b_{ia} of M.*

PROOF. We prove the assertion by induction on $\operatorname{proj}\dim(M)$. If $\operatorname{proj}\dim(M) = 0$, then the assertion is obvious. Now assume $\operatorname{proj}\dim(M) > 0$, and let

$$\mathbb{F} : \cdots \longrightarrow \bigoplus_{a\in\mathbb{Z}^n} R(-a)^{b_{1a}} \longrightarrow \bigoplus_{a\in\mathbb{Z}^n} R(-a)^{b_{0a}} \longrightarrow M \longrightarrow 0.$$

be the minimal multigraded free R resolution of M, and $\varepsilon\colon F_0 \to M$ the augmentation map.

Obviously $t \notin \operatorname{supp} a$ for all b_{0a} which are non-zero. Let $e_1,\ldots,e_s$ be a multigraded basis of F_0 with $\varepsilon(e_i) = y_i$ for $i = 1,\ldots,s$, and let $z = \sum c_i e_i$ be a homogeneous element in a minimal homogeneous set of generators of $\operatorname{Ker}(\varepsilon)$. Then $\deg z = \deg c_i + \deg e_i$ for $i = 1,\ldots,s$. By assumption we have $(\deg e_i)_t = 0$ for $i = 1,\ldots,s$. Suppose $t \in \operatorname{supp}(\deg z)$, then $(\deg c_i)_t > 0$ for all i with $c_i \neq 0$. This implies that there exist $c_i' \in R$ such that $c_i = x_t c_i'$ for all i. So we have $z = x_t \sum c_i' e_i$ and so $x_t \sum c_i' y_i = 0$. Since M is a torsion-free module, it follows that $\sum c_i' y_i = 0$, and hence $\sum c_i' e_i \in \operatorname{Ker}(\varepsilon)$. That is to say, $z \in \mathfrak{m}\operatorname{Ker}(\varepsilon)$, where $\mathfrak{m} = (x_1,\ldots,x_n)$, contradicting the assumption that z belongs to a minimal homogeneous generating system of $\operatorname{Ker}(\varepsilon)$. Therefore t does not belong to the support of any element in a minimal set of generators of $\operatorname{Ker}(\varepsilon)$. Since $\operatorname{Ker}(\varepsilon)$ is torsion free and $\operatorname{proj}\dim(\operatorname{Ker}(\varepsilon)) < \operatorname{proj}\dim(M)$, the lemma follows from our induction hypothesis. □

Let I be a monomial ideal. As usual we denote by $G(I)$ the unique minimal set of monomial generators of I.

Lemma 4.3. *Let I and J be monomial ideals in R with $G(I) = \{f_1,\ldots,f_m\}$ and $G(J) = \{f_1,\ldots,f_{m-1}\}$, and let b be the multidegree of f_m. If there exists $t \in [n]$, such that $t \in \operatorname{supp} b$, but $t \notin \operatorname{supp}(\deg f_i)$ for $i = 1,\ldots,m-1$. Then $t \in \operatorname{supp} a$ for all a with $b_{ia}(R/(J : I)(-b)) \neq 0$.*

PROOF. Let $\mathbb{F}$ be the minimal $\mathbb{Z}^n$-graded free resolution of $R/(J : I)$, then $\mathbb{F}(-b)$ is the minimal $\mathbb{Z}^n$-graded free resolution of $R/(J : I)(-b)$. Since $t \notin \operatorname{supp}(\deg f_i)$ for $1 \le i \le m-1$, t is not in the support of the elements of $G(J : I) = G(J : f_m)$. This is because $G(J : f_m)$ is a subset of $\{f_1/[f_1, f_m],\ldots, f_{m-1}/[f_{m-1}, f_m]\}$, where $[f_i, f_m]$ is the greatest common divisor of f_i and F_m. Applying Lemma 4.2 to $J : I$, we have that $t \notin \operatorname{supp} a$ for all $b_{ia}(J : I) \neq 0$. Hence $t \notin \operatorname{supp} a$ for all $b_{ia}(R/(J : I)) \neq 0$. Since $b_{ia}(R/(J : I)(-b)) = b_{i,a-b}(R/(J : I))$, we have $t \notin \operatorname{supp}(a-b)$, for any $b_{ia}(R/(J : I)(-b)) \neq 0$. But $t \in \operatorname{supp} b$, hence $t \in \operatorname{supp} a$. □

Theorem 4.4. *Let $J \subset R$ be a monomial ideal, $f \in R \setminus J$ a monomial and let $I = (J, f)$. Suppose that there exists an integer t such that x_t divides f, but x_t does not divide any $g \in G(J)$. Then for all $i > 0$ there exist short exact sequences*

$$0 \longrightarrow H_i(R/J) \longrightarrow H_i(R/I) \xrightarrow{\delta} H_{i-1}(R/(J : I)(-b)) \longrightarrow 0,$$

where b is the multidegree of f, and for each homology class $[z] \in H_{i-1}(R/(J : I)(-b))$ the homology class $[(-1)^{\deg z} z \wedge (f/x_t)e_t]$ is a preimage of $[z]$ under δ.

PROOF. From the exact sequence

$$0 \longrightarrow R/(J : I)(-b) \longrightarrow R/J \longrightarrow R/I \longrightarrow 0, \tag{7}$$

we get the long exact sequence

$$\cdots \longrightarrow H_i(R/(J:I)(-b)) \longrightarrow H_i(R/J) \longrightarrow H_i(R/I) \xrightarrow{\delta} H_{i-1}(R/(J:I)(-b)) \longrightarrow \cdots,$$

Let

$$\mathbb{F}:\cdots \longrightarrow \bigoplus_{a\in\mathbb{Z}^n} R(-a)^{b_{1a}} \longrightarrow \bigoplus_{a\in\mathbb{Z}^n} R(-a)^{b_{0a}} \longrightarrow R/J \longrightarrow 0.$$

be the minimal $\mathbb{Z}^n$-graded free resolution of R/J, then

$$\operatorname{Tor}_i^R(K,R/J)=\bigoplus K(-a)^{b_{ia}(R/J)}=H_i(R/J),$$

and

$$\operatorname{Tor}_i^R(K,R/(J:I)(-b))=\bigoplus K(-a)^{b_{ia}(R/(J:I)(-b))}=H_i(R/(J:I)(-b)).$$

From Lemma 4.2 and Lemma 4.3 we know that $t\notin \operatorname{supp} a$ for all $b_{ia}(R/J)\neq 0$, but $t\in \operatorname{supp} a$ for all $b_{ia}(R/(J:I)(-b))\neq 0$. Since $H_i(R/(J:I)(-b))\to H_i(R/J)$ is a homogeneous homomorphism, it must be the zero map. Hence we have the exact sequence as required.

To show $[(-1)^{\deg z} z\wedge (f/x_t)e_t]$ is the preimage of $[z]$, we only need to show

$$d((-1)^{\deg z} z\wedge (f/x_t)e_t)=fz \quad \text{in} \quad K_\bullet(R/J).$$

In fact, $d((-1)^{\deg z} z\wedge (f/x_t)e_t)=(-1)^{\deg z} d(z)\wedge ((f/x_t)e_t)+fz$. Now since $z\in Z_{i-1}(R/(J:I)(-b))$, it follows that $d(z)\in (J:I)K_{i-2}(R)$, and hence $x_t(f/x_t)d(z)=fd(z)\in JK_{i-2}(R)$. Since x_t does not divide any $g\in G(J)$, we have $J=J:x_t$, and so $(f/x_t)d(z)\in JK_{i-2}(R)$. Hence $d(z)\wedge (f/x_t)e_t\in JK_{i-1}(R)$. That is to say, $d(z)\wedge (f/x_t)e_t=0$ in $K_{i-1}(R/J)$. □

Corollary 4.5. *Let $L\subset R$ be a graded ideal, and $x_{i_1},\dots,x_{i_s}$ a regular sequence on R/L. If B is a K-basis of $H_\bullet(R/L)$, then $\{[z\wedge e_I]\colon [z]\in B,\quad I\subset\{i_1,\dots,i_s\}\}$ is a K-basis of $H_\bullet(R/(L+(x_{i_1},\dots,x_{i_s})))$.*

PROOF. We may assume that $s=1$. The general case is done by induction on s. Since x_{i_1} is regular on R/L, x_{i_1} does not divide any $g\in G(L)$. Therefore the result follows from Theorem 4.4. □

Corollary 4.6. *Let I, I' be monomial ideals in R with $G(I)=\{f_1,\dots,f_m,\dots,f_l\}$ and $G(I')=\{f_1,\dots,f_m\}$. If for any $i\geq m+1$ there exists a variable which divides f_i but does not divide f_j for any $j<i$. Then the map $H_i(R/I')\to H_i(R/I)$ is injective.*

PROOF. The statement follows immediately from Theorem 4.4 by induction on $l-m$. □

Corollary 4.7. *Let R, I, J be as in Theorem* 4.4. *Then we have*

$$b_{ia}(R/I)=b_{ia}(R/J)+b_{i-1,a-b}(R/(J:I)) \quad \textit{for all } i>0 \textit{ and } a\in\mathbb{Z}^n.$$

For another main result of this section, we need the following concept.

Definition 4.8. Let I be a monomial ideal of R. A cycle z of $K_\bullet(R/I)$ is called a *monomial cycle* if there exists $L\subset [n]$ and a monomial f, such that $z=fe_L$.

Even if I is a squarefree monomial ideal, $H_\bullet(R/I)$ may not be generated by homology classes of monomial cycles. For example, let

$$R = K[x_1, x_2, x_3, x_4] \quad \text{and} \quad I = (x_1x_2, x_2x_3, x_3x_4, x_4x_1).$$

Then $z = x_1e_2 \wedge e_3 \wedge e_4 + x_3e_1 \wedge e_2 \wedge e_4$ is a cycle, but z is not homologous to a monomial cycle. In fact, a boundary $b \in B_3(R/I)$ is of the form $d(fe_1 \wedge e_2 \wedge e_3 \wedge e_4)$. So z can not be a monomial cycle.

However for the facet ideal I of a forest, we have $H_\bullet(R/I)$ is generated by homology classes of monomial cycles. To prove this we need the following lemma.

Lemma 4.9. *Let Δ be a forest and I its facet ideal. If F is any facet of Δ and J is the ideal generated by $G(I) \setminus \{x_F\}$. Then the simplicial complex Δ' with facet ideal $J : I$ is again a forest.*

PROOF. Note that $\mathscr{F}(\Delta')$ is a subset of $\{G \setminus F : G \in \mathscr{F}(\Delta)\}$. Suppose Δ' is not a forest. Then there exist facets $F_1, \ldots, F_p$ of Δ, such that the subcomplex $\langle F_1 \setminus F, \ldots, F_p \setminus F\rangle$ of Δ' has no leaf. Since Δ is a forest, the subcomplex $\langle F_1, \ldots, F_p\rangle$ has a leaf F_i. Hence there exists a integer $k \in \{1, \ldots, p\}$ and $k \neq i$, such that $F_j \cap F_i \subseteq F_k \cap F_i$ for any $j \neq i$. Therefore $(F_j \cap F_i) \setminus F \subseteq (F_k \cap F_i) \setminus F$ for any $j \neq i$, and hence $(F_j \setminus F) \cap (F_i \setminus F) \subseteq (F_k \setminus F) \cap (F_i \setminus F)$ for any $j \neq i$. So $F_i \setminus F$ is a leaf of $\langle F_1 \setminus F, \ldots, F_p \setminus F\rangle$, a contradiction. $\square$

Proposition 4.10. *Let Δ be a forest and I its facet ideal. Then $H_r(R/I)$ has the K-basis*

$$\mathscr{M}_r = \{[fe_{i_1} \wedge \cdots \wedge e_{i_r}] : fe_{i_1} \wedge \cdots \wedge e_{i_r} \text{ is a monomial cycle.}\}$$

PROOF. Let $\Delta = \langle F_1 \ldots, F_m\rangle$ where $F_1 \ldots, F_m$ is a leaf order, namely, F_i is a leaf of the subcomplex $\langle F_1, \ldots, F_i\rangle$ for all $i \in [m]$. We prove the assertion by induction on m. The case $m = 1$ is trivial. Since F_m is a leaf we may assume that $x_{F_m} = hx_t$, where h is a monomial and $t \in F_m \setminus \bigcup_{j<m} F_j$. By Theorem 4.4, we have short exact sequences

$$0 \longrightarrow H_r(R/J) \longrightarrow H_r(R/I) \longrightarrow H_{r-1}(R/(J:I)(-b)) \longrightarrow 0,$$

where $J = (f_1, \ldots, f_{m-1})$ and b is the multidegree of x_{F_m}. By Lemma 4.9, $J : I$ is a facet ideal of a forest and it has at most $m-1$ facets. Again use Theorem 4.4 we have $[z]$, $[z' \wedge (f_m/x_t)e_t]$ are basis elements of $H_r(R/I)$, where $[z]$ and $[z']$ are basis elements of $H_r(R/J)$ and $H_{r-1}(R/(J:I)(-b))$, respectively. And by induction hypothesis z and z' can be chosen as monomial cycles. $\square$

Recall that the *regularity* of a finitely generated graded R-module M is defined to be

$$\operatorname{reg} M = \max\{j : \beta_{i,i+j}(M) \neq 0 \text{ for some } i\}.$$

In the following of this section, we shall determinate the regularity and projective dimension of the edge ideal of a 1-dimensional forest. For this we need:

Definition 4.11. Let I be a monomial ideal and d the least degree of its generators. A monomial cycle $z = fe_L$ in $K_\bullet(R/I)$ is called *linear* if f is a monomial of degree $d-1$.

Remark 4.12. Let G be a 1-dimensional forest with edge ideal I. Then the linear monomial cycles are of the form

$$x_l e_{l_1} \wedge \cdots \wedge e_{l_r},$$

where $\{l, l_i\}$ is an edge of G, $i = 1, \dots, r$. Hence it follows from Proposition 4.10 that the set

$$\mathscr{B}_r = \{[z(b)] \colon b = (l; l_1, \dots, l_r) \text{ is a bouquet of } G \text{ with } r \text{ flowers}\}$$

is a K-basis of $H_r(R/I)_{r+1}$, where $z(b) = x_l e_{l_1} \wedge \cdots \wedge e_{l_r}$.

Proposition 4.13. *Let G be a forest of dimension 1 and I its edge ideal. Then as a K-algebra, $H_\bullet(R/I)$ is generated by the homology classes of linear monomial cycles.*

PROOF. Let fe_L be an arbitrary monomial cycle, and let $i \in L$. Then $fx_i \in I$, and hence there exists a generator $f_1 \in G(I)$ such that $fx_i = f_1 g$. Since $f \notin I$, we conclude that x_i divides f_1. Then $f = (f_1/x_i)g$. Now let $L_1 = \{l \in L \colon (f_1/x_i)x_l \in I\}$, and $L_2 = L \setminus L_1$. Note that $i \in L_1$ and that $fe_L = (f_1/x_i)e_{L_1} \wedge ge_{L_2}$, where $(f_1/x_i)e_{L_1}$ is a linear cycle. If $g = 1$, then fe_L is a linear cycle, and if $g \neq 1$ but $L_2 = \emptyset$, then fe_L is a boundary. Thus we may assume that $g \neq 1$ and $L_2 \neq \emptyset$, and have to show that ge_{L_2} is a cycle. Then we can proceed by induction on the degree of f.

Suppose $gx_s \notin I$ for some $s \in L_2$. Since $f = (f_1/x_i)g$ we have $((f_1/x_i)g)x_s \in I$. Let $f_1/x_i = x_r$. By the choice of L_2 it follows that $x_r x_s \notin I$. Therefore there must exist x_t dividing g such that $x_t x_s \in I$. This implies $gx_s \in I$, a contradiction. □

Corollary 4.14. *Let G be a 1-dimensional forest with edge ideal I. Then*

(i) $\operatorname{reg}(R/I)$ *is the maximal number j for which there exist linear monomial cycles z_i such that $[z_1] \cdots [z_j] \neq 0$;*

(ii) $\operatorname{pd}(R/I)$ *is the maximum among the sums $\sum_{i=1}^{j} k_i$ for which there exist linear cycles $z_i \in Z_{k_i}(R/I)$ such that $[z_1] \cdots [z_j] \neq 0$.*

Proposition 4.15. *Let G be a 1-dimensional forest and let*

$$b_1 = (1; 1_1, \dots, 1_{p_1}), \dots, b_l = (l; l_1, \dots, l_{p_l})$$

be bouquets in G. Then the following statements are equivalent:

(i) $[z(b_1)] \cdots [z(b_l)] \neq 0$*;*

(ii) *the set of bouquets $b_1, \dots, b_l$ satisfies the following conditions:*

  (a) *all vertices occurring in these bouquets are pairwise distinct;*

  (b) *the roots of any two bouquets are not adjacent;*

  (c) *for any bouquet b_i there exists at least one flower which is not adjacent with the root of b_j for all $j \neq i$.*

PROOF. (i) $\Rightarrow$ (ii): It is clear that if (a) or (b) not holds, then $[z(b_1)] \cdots [z(b_l)] = 0$. Suppose there exists an i, such that each flower of b_i is adjacent with the root of some b_j. Since $d(e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_{p_i}}) = x_i e_{i_1} \wedge \cdots \wedge e_{i_{p_i}} - x_{i_1} e_i \wedge e_{i_2} \wedge \cdots \wedge e_{i_{p_i}} + \cdots + (-1)^{p_i} x_{i_{p_i}} e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_{p_i - 1}}$, we have

$$[z(b_i)] = \sum_{k=1}^{p_i} [(-1)^{k+1} x_{i_k} e_i \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_{p_i}}].$$

Since i_k has a common edge with the root of some b_j for all $k \in \{1, \dots, p_i\}$, we have $[z(b_i)][z(b_j)] = 0$, a contradiction.

(ii) $\Rightarrow$ (i): We prove the assertion by induction on l. The case $l = 1$ follows from Remark 4.12. Let G' be the subforest of G obtained as follows: If one stem of our bouquets is a leaf of G, then let $G' = G$. Otherwise let F_1 be any leaf of G, and let $G_1 = \langle \mathscr{F}(G) \setminus \{F_1\}\rangle$. Notice that G_1 is again a forest containing all our bouquets. If one stem of our bouquets is a leaf of G_1, then let $G' = G_1$. Otherwise let F_2 be any leaf of G_1, and let $G_2 = \langle \mathscr{F}(G_1) \setminus \{F_2\}\rangle$. Proceeding in this way we obtain a subforest G' of G such that

$$G = \langle \mathscr{F}(G'), F_s, \ldots, F_1\rangle,$$

where F_r is a leaf of $\langle \mathscr{F}(G'), F_s, \ldots, F_r\rangle$ for $r \in [s]$, and such that some stem of our bouquets, say $\{i, i_k\}$, is a leaf of G'. Let I' be the edge ideal of G', $\Gamma = \langle \mathscr{F}(G') \setminus \{i, i_k\}\rangle$ with edge ideal J', and let Γ' be the simplicial complex with facet ideal $J' : I'$. By Lemma 4.9, Γ' is a forest.

If $p_i > 1$, then i_k must be the free vertex of $\{i, i_k\}$ in G'. If $p_i = 1$, then $b_i = (i; i_1)$. It may be that i_1 is not the free vertex of $\{i, i_1\}$ in G'. Then we replace b_i by the bouquet $b'_i = (i_1; i)$.

Notice that the bouquets $b_1, \ldots, b_{i-1}, b'_i, b_{i+1}, \ldots, b_l$ again satisfy all conditions in (ii), and since $[b_i] = [b'_i]$ we also have $[b_1]\cdots[b_l] = [b_1]\cdots[b_{i-1}][b'_i][b_{i+1}]\cdots[b_l]$. Therefore we may as well assume that in any case the flower i_k is the free vertex of $\{i, i_k\}$ in G'.

It follows from the definition of Γ' that all the other flowers of b_i are isolated vertices of Γ'. Recall that a vertex in a simplicial complex Σ is called isolated if it is not adjacent with any other vertex in Σ.

We distinguish two cases:

Case 1: The root i of the bouquet b_i is not adjacent with any flower in the other bouquets.

In this case $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_l$ are bouquets in Γ', and this set of bouquets satisfies all conditions in (ii). By induction hypothesis, $[b_1]\cdots[b_{i-1}][b_{i+1}]\cdots[b_l] \neq 0$ in $H_\bullet(R/(J' : I'))$. Since i_m is an isolated vertices in Γ' for $m \in \{1, \ldots, p_i\}$ and $m \neq k$, by Corollary 4.5, we have $[b_1]\cdots[b_{i-1}][b_{i+1}]\cdots[b_l][e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_{p_i}}] \neq 0$ in $H_\bullet(R/(J' : I'))$. By Theorem 4.4, for any basis element $[z]$ of $H_{r-1}(R/(J' : I')(-2))$, $[z \wedge x_i e_{i_k}]$ is a basis element of $H_r(R/I')$. Since $z = z(b_1)\cdots\widehat{z(b_i)}\cdots z(b_l)(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_{p_i}})$ is a cycle in $K_\bullet(R/(J' : I')(-2))$, it follows that $0 \neq [z \wedge x_i e_{i_k}] = [b_1]\cdots[b_l]$ in $H_\bullet(R/I')$. By Corollary 4.6, we have $[b_1]\cdots[b_l] \neq 0$ in $H_\bullet(R/I)$.

Case 2. There exists an integer $j \neq i$ such that the root i of b_i has a common edge with some flower of b_j.

Let C be the set of integers having this property, and let $j \in C$. Since G is a tree, there exists only one flower of b_j which is adjacent with i, because otherwise G would have a cycle. And by the condition (c) in (ii), we have $p_j > 1$. For $j \in C$, let

$$b'_j = \begin{cases} b_j, & \text{if } j \notin C, \\ (j; j_1, \ldots, \widehat{j}_k, \ldots, j_{p_j}), & \text{if } j \in C \text{ and } \{i, j_k\} \text{ is an edge.} \end{cases}$$

Then $b'_1, \ldots, b'_{i-1}, b'_{i+1}, \ldots, b'_l$ are bouquets of Γ', and this set of bouquets satisfies all the conditions in (ii). For all $j \in C$, let $\{i, j_k\}$ be the unique common edge of the root i of b_i with the flower j_k of b_j. Then j_k is an isolated vertex of Γ'. Hence in Γ' we are in the same situation as in Case 1, and so as before the result follows by induction. □

Definition 4.16. Let G be a graph. Two edges $\{i, j\}$ and $\{k, l\}$ are called *disconnected* if

(i) $\{i,j\}\cap\{k,l\}=\emptyset$;
(ii) $\{i,k\}$, $\{i,l\}$, $\{j,k\}$, $\{j,l\}$ are not edges of G.

Corollary 4.17. *Let G be a 1-dimensional forest and $\{i_1,j_1\},\dots,\{i_m,j_m\}$ edges of G. Then the following statements are equivalent:*

(i) $[x_{i_1}e_{j_1}]\cdots[x_{i_m}e_{j_m}]\neq 0$;
(ii) *the edges $\{i_1,j_1\},\dots,\{i_m,j_m\}$ are pairwise disconnected.*

PROOF. Let $b_l=\{i_l;j_l\}$, $l=1,\dots,m$. Then b_l is a bouquet with one flower. Notice that $b_l'=\{j_l;i_l\}$ is also a bouquet with one flower of G. Since $[z(b_l)]=[z(b_l')]$, we have $[z(b_1)]\cdots[z(b_m)]\neq 0$ if and only if $[z(b_1)]\cdots[z(b_{l-1})][z(b_l')][z(b_{l+1})][z(b_m)]\neq 0$. Hence we may choose i_l or j_l as the root of b_l.

(i) $\Rightarrow$ (ii): If $[x_{i_1}e_{j_1}]\cdots[x_{i_m}e_{j_m}]\neq 0$, then all conditions in (ii) of Proposition 4.15 hold. Hence all vertices occurring in these edges are pairwise distinct, and $\{i_l,j_l\}$, $l=1,\dots,m$ are the only edges in the subgraph of G restricted to the vertices $\{i_1,\dots,i_m,j_1\dots,j_m\}$. It follows that $\{i_1,j_1\},\dots,\{i_m,j_m\}$ are pairwise disconnected.

(ii) $\Rightarrow$ (i): If $\{i_1,j_1\},\dots,\{i_m,j_m\}$ are pairwise disconnected, then the set of bouquets $b_1,\dots,b_m$ satisfies all conditions in (ii) of Proposition 4.15. Hence $[x_{i_1}e_{j_1}]\cdots[x_{i_m}e_{j_m}]\neq 0$. □

Moreover, we have

Corollary 4.18. *Let G be a* 1*-dimensional forest, and $b_1,\dots,b_l$ bouquets of G. If the set of these bouquets satisfies the condition* (ii) *of Proposition* 4.15*, then there exists one stem in each bouquet, such that these stems are pairwise disconnected.*

PROOF. We refer to the notation in the proof of Proposition 4.15. By the proof 4.15 (ii) $\Rightarrow$ (i) in each step we get a leaf $\{i,i_k\}$ in the subforest of the previous one. The arguments in the proof show that these stems are pairwise disconnected. □

By using Proposition 4.15, Corollary 4.17 and Corollary 4.18, we conclude:

Theorem 4.19. *Let G be a* 1*-dimensional forest, I its edge ideal. Then the regularity of R/I is the maximal number j, for which there exist j edges which are pairwise disconnected.*

Remark 4.20. In Theorem 4.19, the assumption that G is a forest is important. If G has a cycle, then the assertion might not be true. For example, let G be a graph with edge ideal $I=(x_1x_2,x_2x_3,x_3x_4,x_4x_5,x_5x_1)$. Then the regularity of R/I is 2, but the maximal number of the pairwise disconnected edges in G is 1.

2. Linear trees

In general, it is not easy to determine the Betti numbers of an R-module M, but for a facet ideal I of a pure tree which is connected in codimension 1, we can describe the linear part of the resolution of R/I. As before, in this section, we still assume a simplicial complex Δ has vertex set $[n]$, and let $R=K[x_1,\dots,x_n]$ be the polynomial ring in n variables over a field K.

We know that if M is a graded R-module, $z \in R$ is a homogeneous element of degree 1, and z is a non-zero divisor of M, then

$$b_{ij}(M/zM) = b_{ij}(M) + b_{i-1,j}(M(-1)) = b_{ij}(M) + b_{i-1,j-1}(M). \tag{8}$$

In fact, if $\mathbb{F}$ is a graded minimal free resolution of M, then the mapping cone of $\mathbb{F}(-1) \xrightarrow{z} \mathbb{F}$ is the minimal graded free resolution of M/zM.

Lemma 4.21. *Let L be a monomial ideal in R with $G(L) = \{g_1, \dots, g_l\}$. Suppose that* $\deg g_r = 1$ *for* $r = 1, \dots, s$, *and* $\deg g_r > 1$ *for* $r = s+1, \dots, l$. *Then* $b_{ii}(R/L) = \binom{s}{i}$.

PROOF. We may assume that $g_i = x_i$ for $i \in [s]$. Then for all $i \in [s]$, x_i does not divide any g_j for $j > s$, because $\{g_1, \dots, g_l\}$ is a minimal set of generators of L. Hence $g_1, \dots, g_s$ is a regular sequence of $R/(g_{s+1}, \dots, g_l)$. Hence the assertion follows by induction on s from (8). □

Definition 4.22. Let Δ be a d-dimensional pure and connected in codimension 1 tree and H a face of dimension $d-1$. If H is contained in at least two facets of Δ, then we call H an *adjacent face* of Δ.

Proposition 4.23. *Let Δ be a d-dimensional pure tree with m facets which is connected in codimension 1, I its facet ideal. For each adjacent face $H \in \Delta$, let $m(H) = |\{F \in \mathscr{F}(\Delta) \colon H \subset F\}|$. Then*

$$b_{i,i+d}(R/I) = \begin{cases} m, & \text{if } i = 1, \\ \sum_H \binom{m(H)}{i}, & \text{if } i \geq 2. \end{cases}$$

PROOF. Let $\Delta = \langle F_1, \dots, F_m \rangle$ such that $F_1, \dots, F_m$ is a leaf order. We prove the proposition by induction on m. The case $m = 1$ is trivial. Let $\Gamma = \langle F_1, \dots, F_{m-1} \rangle$ and J be the facet ideal of Γ. By Corollary 4.7 and Lemma 4.21, we have

$$\begin{aligned} b_{i,i+d}(R/I) &= b_{i,i+d}(R/J) + b_{i-1,i+d}(R/(J:I)(-(d+1))) \\ &= b_{i,i+d}(R/J) + b_{i-1,i-1}(R/(J:I)), \end{aligned}$$

and

$$b_{i-1,i-1}(R/(J:I)) = \binom{s}{i-1},$$

where $s = |\{F_j \colon \dim(F_j \cap F_m) = d-1, \quad j \in [m-1]\}|$, because $x_i \in J:I$ if and only if $i \in F_j \setminus F_m$ for some F_j in this set.

Let H be an adjacent face of Γ and $m'(H) = |\{F \in \mathscr{F}(\Gamma) \colon H \subset F\}|$. By our induction hypothesis

$$b_{i,i+d}(R/J) = \begin{cases} m-1, & \text{if } i = 1, \\ \sum_H \binom{m'(H)}{i}, & \text{if } i \geq 2. \end{cases}$$

Hence

$$b_{i,i+d}(R/I) = \begin{cases} m-1+1 = m, & \text{if } i = 1, \\ \sum_H \binom{m'(H)}{i} + \binom{s}{i-1} = \sum_H \binom{m(H)}{i}, & \text{if } i \geq 2. \end{cases}$$

□

For a d-dimensional pure tree Δ, we assign to each face H with dimension $d-1$ an *degree*, namely

$$\deg H = |\{F : F \text{ is a facet of } \Delta, \text{ such that } H \subset F\}|.$$

By Proposition 4.23, $b_{i,i+d}(R/I) = \sum_H \binom{\deg H}{i}$ for $i \geq 2$, where I is the facet ideal of Δ. (Notice that if H is not an adjacent face, then $\binom{\deg H}{i} = 0$ for $i \geq 2$.) If $d = 1$, then the face of dimension $d-1$ is just a vertex.

Definition 4.24. Let G be a graph. The *edge graph* of G, denoted by $L(G)$, has the vertex set equal to $E = E(G)$ with two vertices of $L(G)$ adjacent whenever the corresponding edge of G have exactly one common vertex.

For a graph G the number of edges of the edge graph is given by the following proposition. The proof of it can be found for example in [**39**].

Proposition 4.25. *Let G be a graph with vertex set $[n]$ and edge set $E(G)$. Then the number of edges of the edge graph $L(G)$ is*

$$|E(L(G))| = \sum_i \binom{\deg i}{2}$$

Remark 4.26. Let G be a 1-dimensional tree. Then by Proposition 4.23, $b_{2,3} = \sum_i \binom{\deg i}{2}$, where i runs through all the vertices of G. In [**12**], Eliahou and Villarreal proved that for any graph G, $b_{2,3} = |E(L(G))| - N_t$, where N_t is the number of triangles of G and $L(G)$ is the line graph of G. In the case G is a tree, $N_t = 0$. Hence this gives another proof of Proposition 4.25.

Lemma 4.27. *Let Δ be a d-dimensional pure tree and connected in codimension 1, V the set of faces of dimension $d-1$, and $O = \sum_{H \in V} \deg H$. Then we have $|\mathscr{F}(\Delta)| - 1 = O - |V|$.*

PROOF. The lemma follows by induction on the number of facets, observing that when we add a leaf to the tree, O will increase by $d+1$, and $|V|$ by d. □

For a d-dimensional pure tree Δ which is connected in codimension 1, let $b'_0 = |V|$, $b'_1 = O$, and $b'_i = b_{i,i+d}$ for $i \geq 2$. By using the well-known binomial formula $\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$, one sees that $\sum_i (-1)^i b'_i = 0$. Hence together with Lemma 4.27 we have

Proposition 4.28. *Let Δ be a d-dimensional pure tree with the facet ideal I. Suppose Δ is connected in codimension 1. Then*

$$1 + \sum_{i>0} (-1)^i b_{i,i+d} = 0.$$

In the next section, we will have another property on the Betti numbers of facet ideals which generalizes this proposition.

Definition 4.29. Let I be an ideal in the polynomial ring R. We say I is a *linear quotient ideal*, if for some order $f_1, \ldots, f_m$ of the elements in $G(I)$ the colon ideal $(f_1, \ldots, f_{i-1}) : f_i$ is generated by monomials of degree 1 for all $i \in [m]$.

Definition 4.30. Let I be a graded ideal of R and let

$$0 \to \bigoplus_{i=1}^{b_g} R(-d_{gi}) \to \cdots \to \bigoplus_{i=1}^{b_1} R(-d_{1i}) \to R \to R/I \to 0$$

be the minimal graded free R-resolution of R/I. We say the ideal I (or the algebra R/I) has a *pure resolution* if there are constants $d_1 < d_2 < \cdots < d_g$ such that $d_{1i} = d_1, \cdots, d_{gi} = d_g$ for all i. If in addition $d_i = d_1 + i - 1$ for $2 \le i \le g$ the resolution is said to be d_1*-linear*. In this case, we say the ideal I has a *linear* (d_1*-linear*) *R-resolution*.

For the linear quotient ideal we have the following easy but important property.

Lemma 4.31. *Let I be a linear quotient ideal in the polynomial ring R. If all the generators of I have the same degree, then R/I has a linear resolution.*

PROOF. We prove the assertion by induction on the number of minimal generators m of I.

The case $m = 1$ is trivial. Suppose $m > 1$. Let $I = (f_1, \dots, f_m)$ such that the colon ideal $(f_1, \dots, f_{i-1}) : f_i$ is generated by monomials of degree 1 for each $i \in [m]$, and let $J = (f_1, \dots, f_{m-1})$. Then the ideal $J : I$ can be generated by monomials of degree 1. By induction hypothesis, R/J has a linear resolution. Hence

$$\operatorname{Tor}_i^R(K, R/J)_j = 0 \text{ for } j \neq i + d \text{ and all } i > 0. \tag{9}$$

Since $J : I$ is generated by monomials of degree 1, we have

$$\operatorname{Tor}_i^R(K, J : I)_j = 0 \text{ for } j \neq i + 1 \text{ and all } i > 0. \tag{10}$$

From the exact sequence

$$0 \longrightarrow R/(J : I)(-d) \longrightarrow R/J \longrightarrow R/I \longrightarrow 0,$$

we have the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \operatorname{Tor}_i^R(K, R/(J : I)(-d)) &\longrightarrow \operatorname{Tor}_i^R(K, R/J) \longrightarrow \operatorname{Tor}_i^R(K, R/I) \\ \longrightarrow \operatorname{Tor}_{i-1}^R(K, R/(J : I)(-d)) &\longrightarrow \cdots, \end{aligned}$$

By using (9) and (10), this long exact sequence implies that $\operatorname{Tor}_i^R(K, R/I)_j = 0$ for $j \neq i + d$ and all $i > 0$. Hence R/I has linear resolution. □

Definition 4.32. Let Δ be a tree and I its facet ideal. If I has a linear resolution, then we call Δ a *linear tree*. If I is a linear quotient ideal and all generators of I has the same degree (i.e., Δ is pure), then we call Δ a *linear quotient tree*.

Proposition 4.33. *Let Δ be a tree, I its facet ideal.*

(i) *The following statements are equivalent:*
   (a) *Δ is a linear quotient tree;*
   (b) *Δ is a linear tree.*

(ii) *If Δ satisfies the equivalent conditions in* (i), *then Δ is pure and connected in codimension* 1.

PROOF. (i) $(a) \Rightarrow (b)$: This follows from the previous lemma.

$(b) \Rightarrow (a)$: It is clear that if Δ is not pure, then I has no linear resolution. We may assume Δ is a pure tree of dimension $d-1$. Suppose I is not a linear quotient ideal. Let $F_1, \ldots, F_m$ be a leaf order. Then $L = (x_{F_1}, \ldots, x_{F_{k-1}}) : x_{F_k}$ is not generated by monomials of degree 1 for some $k \in \{1, \ldots, m\}$, and hence $b_{1,1+j}(R/L) \neq 0$ for some $j > 1$. Let $I' = (x_{F_1}, \ldots, x_{F_k})$ and $J' = (x_{F_1}, \ldots, x_{F_{k-1}})$. By Theorem 4.4 we have the exact sequence

$$0 \longrightarrow \mathrm{Tor}_2^R(K, R/J') \longrightarrow \mathrm{Tor}_2^R(K, R/I') \longrightarrow \mathrm{Tor}_1^R(K, R/L(-d)) \longrightarrow 0,$$

which implies that $b_{2,2+j+d}(R/I') \neq 0$, so I' has no linear resolution since I' is generated in degree d. By Corollary 4.6, I has no linear resolution, a contradiction.

(ii) It is clear that Δ must be pure. Let $F_1, \ldots, F_m$ be the facets of Δ in an order such that $(x_{F_1}, \ldots, x_{F_{k-1}}) : x_{F_k}$ is generated by monomials of degree 1 for $k = 1, \ldots, m$. We prove that Δ is connected in codimension 1 by induction on m. The case $m = 1$ is trivial. Assume $m > 1$, since $(x_{F_1}, \ldots, x_{F_{m-1}})$ is a linear quotient ideal, by induction hypothesis, $\langle F_1, \ldots, F_{m-1} \rangle$ is connected in codimension 1. To show Δ is connected in codimension 1, we only need to show that for any facet F_i, with $i < m$, there exists a proper chain between F_i and F_m. Since $(x_{F_1}, \ldots, x_{F_{m-1}}) : x_{F_m}$ is generated by monomials of degree 1, we have that all the facets of $\langle F_m \rangle \cap \langle F_1, \ldots, F_{m-1} \rangle$ are of dimension $d-1$. Hence there exists an integer $j < m$ such that $\dim(F_j \cap F_m) = d-1$. Since F_i and F_j both are facets of the tree $\langle F_1, \ldots, F_{m-1} \rangle$, there exists a proper chain $F_i = F_{i_0}, \ldots, F_{i_l} = F_j$ between F_i and F_j. Hence $F_i = F_{i_0}, \ldots, F_{i_l} = F_j, F_m$ is a proper chain between F_i and F_m. □

By Proposition 4.23 and Proposition 4.33 the Betti numbers of a linear tree can now be described as follows.

Corollary 4.34. *Let Δ be a d-dimensional linear tree with m facets, I its facet ideal. Then*

$$b_i(R/I) = \begin{cases} m, & \text{if } i = 1, \\ \sum_H \binom{m(H)}{i}, & \text{if } i \geq 2, \end{cases}$$

where the sum is taken over all $(d-1)$-dimensional faces H of Δ, and $m(H) = |\{F \in \mathscr{F}(\Delta) \colon H \subset F\}|$.

Later in this section, we will classify all linear trees of a given dimension. For this, we need some preparation.

Lemma 4.35. *Let Δ be a linear tree, Γ a subcomplex of Δ which is connected in codimension 1. Then Γ is a linear tree.*

PROOF. Γ is again a pure tree. We may assume $\Gamma \neq \Delta$. We claim there exists an order of the facets $F_1, \ldots, F_l$ of $\langle \mathscr{F}(\Delta) \setminus \mathscr{F}(\Gamma) \rangle$ such that $\langle \mathscr{F}(\Gamma), F_1, \ldots, F_i \rangle$ is connected in codimension 1, $i = 1 \ldots, l$. In fact, let $F \in \mathscr{F}(\Gamma)$ and $G \in \mathscr{F}(\Delta) \setminus \mathscr{F}(\Gamma)$ be any two facets. Since Δ is connected in codimension 1, there exists a unique irredundant proper chain from F to G. Let F_1 be the first facet in this chain which does not belong to Γ. Then it is obvious that $\langle \mathscr{F}(\Gamma), F_1 \rangle$ is connected in codimension 1. The claim follows by induction on $|\mathscr{F}(\Delta) \setminus \mathscr{F}(\Gamma)|$.

By Corollary 2.23, F_i is a leaf of $\langle \mathscr{F}(\Gamma), F_1, \ldots, F_i \rangle$ for $i = 1, \ldots, l$. Let I and J be the facet ideals of Δ and Γ, respectively. By Corollary 4.6, $b_{i,i+j}(J) \leq b_{i,i+j}(I)$ for any i and

j. Since I has a linear resolution, this implies that J has a linear resolution. Hence Γ is a linear tree. $\square$

Lemma 4.36. *Let Δ be a linear tree, F and G any two facets of Δ. Let $F = F_0, \dots, F_m = G$ be the irredundant proper chain between F and G. Then the ideal $(x_{F_0}, \dots, x_{F_{l-1}}) : x_{F_l}$ is generated by monomials of degree 1 for all $l \in [m]$.*

PROOF. Since $F_0, \dots, F_m$ is an irredundant proper chain, $\langle F_0, \dots, F_i \rangle$ is a linear tree for all i, see Lemma 4.35. Assume there exists an l such that $(x_{F_0}, \dots, x_{F_{l-1}}) : x_{F_l}$ is not generated by monomials of degree 1. Since F_l is a leaf of $\langle F_0, \dots, F_l \rangle$, it follows from Theorem 4.4 that $\langle F_0, \dots, F_l \rangle$ is not a linear tree, a contradiction. $\square$

Proposition 4.37. *Let Δ be a pure d-dimensional tree and connected in codimension 1, F and G any two facets with $\dim(F \cap G) = d - k$, for some $k \in [d+1]$. Then*

(i) $\operatorname{dist}(F, G) \geq k$*;*
(ii) $\operatorname{dist}(F, G) = k$*, if Δ is a linear tree.*

PROOF. (i) is obvious. Now let Δ be a linear tree, and suppose that $\operatorname{dist}(F, G) > k$. Let $F = F_0, \dots, F_l = G$ be the irredundant proper chain between F and G, where $l > k$. Let $H = F \cap G$. By Proposition 2.20, $H \subset F_k$ for $k = 0, \dots, l$.

Let $\{i\} = F_i \setminus F_{i+1}$ for $i = 0, \dots, l-1$. We claim that $\{0, \dots, l-1\} \subset F_0$, and that the elements i are pairwise distinct.

Assume $j \notin F_0$ for some $j = 0, \dots, l-1$. Since $F_0, \dots, F_j$ is an irredundant proper chain, it follows that $F_k \cap F_{j+1}$ is a proper subset of $F_j \cap F_{j+1}$ for $k < j$. This implies that $|F_k \setminus F_{j+1}| > 1$ for all $k < j$, while $F_j \setminus F_{j+1} = \{j\}$. On the other hand, $(x_{F_0}, \dots, x_{F_j}) : x_{F_{j+1}}$ is generated by monomials of degree 1. This implies that $j \in F_k$ for all $k \leq j$. In particular, $j \in F_0$, a contradiction. Since $F_i, \dots, F_l$ is an irredundant proper chain, F_i is a leaf of $\langle F_i, \dots, F_l \rangle$ for all $i \in \{0, \dots, l-1\}$. Hence $i \notin F_k$ for all $k > i$. So the i are pairwise distinct, and $i \notin H$ for $i = 0, \dots, l-1$.

So we have $H \cup \{0, \dots, l-1\} \subseteq F_0$. Hence $|F_0| \geq d - k + 1 + l > d + 1$, a contradiction. $\square$

Definition 4.38. Let Δ be a d-dimensional pure tree and connected in codimension 1. If for any two facets F and G with $\dim(F \cap G) = d - k$, $k = 1, \dots, d+1$, we have $\operatorname{dist}(F, G) = k$, then we say Δ has the *intersection property*.

Remark 4.39. Let Δ be a d-dimensional tree with intersection property, and l the diameter of Δ. Then

(i) $l \leq d + 1$, and
(ii) for any irredundant proper chain $\mathscr{C}$ in Δ, and any face H in Γ of dimension $d - k$, where Γ is the simplicial complex generated by $\mathscr{C}$, one has that H is contained in at most $k + 1$ facets of Γ.

In fact, it is clear that for any two facets F and G of Δ, $\operatorname{dist}(F, G) \leq d + 1$. Hence $l \leq d + 1$.

Assume H is contained in more than $k + 1$ facets of Γ. Since Γ is generated by the irredundant proper chain $\mathscr{C}$, there exist two facets F and G of Γ such that $H \subseteq F \cap G$ and $\operatorname{dist}(F, G) > k$. But $\dim(F \cap G) \geq \dim H = d - k$, contradicting Proposition 4.37.

Proposition 4.40. *Let Δ be a d-dimensional linear tree, and G an adjacent face of Δ. Let $\Gamma = \langle \mathscr{F}(\Delta), F\rangle$, where $\langle F\rangle$ is a simplex of dimension d and $\langle F\rangle \cap \Delta = \langle G\rangle$. Then Γ is a linear tree.*

PROOF. By Proposition 4.33, we have $\Delta = \langle F_1,\dots,F_m\rangle$ such that $(x_{F_1},\dots,x_{F_{i-1}}) : x_{F_i}$ is generated by monomials of degree 1, $i = 1,\dots,m$. Let $F_{i_1},\dots,F_{i_l}$ be all the facets of Δ which contains G, and $\{i_j\} = F_{i_j} \setminus F$ for $j = 1,\dots,l$, where $l > 1$. We prove that $(x_{F_1},\dots,x_{F_m}) : x_F = (x_{i_1},\dots,x_{i_l})$ (which implies that Γ is also a linear tree).

It is clear that $(x_{i_1},\dots,x_{i_l}) \subseteq (x_{F_1},\dots,x_{F_m}) : x_F$. In order to prove the converse inclusion, we first notice that there exists no facet F_p of Δ, such that $F_p \cap F_{i_j} = \emptyset$ for all $j = 1,\dots,l$. Otherwise by Proposition 4.37, $\text{dist}(F_{i_j},F_p) = d+1$ for all $j = 1,\dots,l$. Since $l > 1$, this contradicts Proposition 2.25.

It remains to show that for any facet F_p of Δ we have $F_p \cap \{i_1,\dots,i_l\} \neq \emptyset$. Suppose there exists a facet F_p such that $F_p \cap \{i_1,\dots,i_l\} = \emptyset$, then $p \neq i_j$, and hence we have $F_p \cap G = F_p \cap F_{i_j} \neq \emptyset$ for $j = 1,\dots,l$. Let $\dim(F_p \cap F_{i_j}) = d-k$. Then by Proposition 4.37, $\text{dist}(F_p,F_{i_j}) = k$ for $j = 1,\dots,l$. Again, since $l > 1$, this contradicts Proposition 2.25. □

Now we are ready to show

Theorem 4.41. *Let Δ be a tree. Then the following conditions are equivalent:*

(i) *Δ is a linear tree;*
(ii) *Δ has the intersection property.*

PROOF. (i) $\Rightarrow$ (ii) follows from Proposition 4.37.

(ii) $\Rightarrow$ (i): We prove the assertion by induction on the number of facets m of Δ. The case $m = 1$ is trivial. Assume $m > 1$. Let F be a leaf of Δ. By induction hypothesis, $\langle \mathscr{F}(\Delta) \setminus \{F\}\rangle$ is a linear tree because it still satisfies the intersection property. Let $H = \langle F\rangle \cap \langle \mathscr{F}(\Delta) \setminus \{F\}\rangle$; if $|\mathscr{U}_\Delta(F)| > 1$ (see Definition 2.15), then H is an adjacent face of $\langle \mathscr{F}(\Delta) \setminus \{F\}\rangle$. Hence Δ is a linear tree by Proposition 4.40. If $|\mathscr{U}_\Delta(F)| = 1$, let $\{F'\} = \mathscr{U}_\Delta(F)$ and $\{l\} = F' \setminus F$.

We claim l is contained in every facet of $\langle \mathscr{F}(\Delta) \setminus \{F\}\rangle$. Hence, since $\langle \mathscr{F}(\Delta) \setminus \{F\}\rangle$ is a linear tree, Δ is a linear tree, too.

In order to prove the claim, consider $G \in \mathscr{F}(\Delta)$, $G \neq F$, and let $E = F \cap G$ and assume that $\dim E = d-k$. Then $G = \{i_1,\dots,i_k\} \cup E$ and $F = \{j_1,\dots,j_k\} \cup E$, where all the elements in $\{i_1,\dots,i_k,j_1,\dots,j_k\}$ are pairwise distinct. Since Δ has intersection property, Δ is pure and connected in codimension 1, and $\text{dist}(F,G) = k$. Hence there exists an irredundant proper chain $G = F_0,F_1,\dots F_k = F$ between G and F. Since F is a leaf of Δ and $\{F'\} = \mathscr{U}_\Delta(F)$, we have $F_{k-1} = F'$. Since $|F_p \setminus F_{p+1}| = 1$ for all p, we may assume $F_p = \{j_1,\dots,j_p,i_{p+1},\dots,i_k\} \cup E$ for $p = 1,\dots,k$. Hence $F_{k-1} = \{j_1,\dots,j_{k-1},i_k\} \cup E$. But on the other hand, $F_{k-1} = F' = \{j_1,\dots,j_{k-1},l\} \cup E$. Hence $l = j_k \in G$. □

3. The alternating sum property of facet ideals

In this section we show that for a special class of facet ideals I the Betti numbers have the property that $\sum_i (-1)^i b_{i,i+j}(R/I) = 0$ for all $j > d$, where d is the least degree of the generators. This class of ideals includes facet ideals of quasi-trees which are connected in codimension 1 and the facet ideals of trees (not necessary pure) which are connected in codimension 1.

Definition 4.42. Let I be a monomial ideal in the polynomial ring R with $G(I)=\{f_1\dots,f_m\}$ and $d=\min\{\deg f_i\colon\, i=1,\dots,m\}$. We say that I has the *alternating sum property*, if

$$\sum_{i\geq 1}(-1)^i b_{i,i+j}(R/I)=\begin{cases}-1, & \text{for } j=d,\\ 0, & \text{for } j>d.\end{cases}$$

To proof the main theorem of this section, we need the following fact.

Lemma 4.43. *Let I be a monomial ideal in R. Suppose $G(I)$ contains a monomial of degree 1. Then $\sum_i(-1)^i b_{i,i+j}(R/I)=0$ for all $j>1$.*

PROOF. Let $G(I)=\{m_1,\dots,m_l,x_k\}$, $J=(m_1,\dots,m_l)$. Then x_k does not divide m_j for $j=1,\dots,l$, and $J:I=J$. By Theorem 4.4 we have for $i>0$

$$\begin{aligned} b_{i,i+j}(R/I) &= b_{i,i+j}(R/J)+b_{i-1,i+j}(R/(J:I)(-1))\\ &= b_{i,i+j}(R/J)+b_{i-1,i-1+j}(R/(J:I))\\ &= b_{i,i+j}(R/J)+b_{i-1,i-1+j}(R/J). \end{aligned}$$

From this it follows that $\sum_i(-1)^i b_{i,i+j}(R/I)=0$. □

Remark 4.44. With the same arguments as in the proof of Lemma 4.43 one can show more generally: Let J be a graded ideal in R, $I=(J,f)$, where $\deg f=1$. If f is regular on R/J, then $\sum_i(-1)^i b_{i,i+j}(R/I)=0$ for all $j>1$.

Proposition 4.45. *Let Δ be a simplicial complex with facet ideal I. If there exists an order of the facets $F_1,\dots,F_m$ of Δ such that for each $i=2,\dots,m$, $F_i\setminus\bigcup_{j<i}F_j\neq\emptyset$, and there exists $j<i$ such that $|F_j\setminus F_i|=1$. Then I has the alternating sum property.*

PROOF. We prove this proposition by induction on m. The case $m=1$ is trivial. Let $d=\min\{\deg x_{F_i}\colon\, i=1,\dots,m-1\}$, $d'=\deg x_{F_m}$, and $J=(x_{F_1},\dots,x_{F_{m-1}})$. Since $|F_j\setminus F_m|=1$ for some $j<m$ it follows that $d'\geq d$, and that $G(J:I)$ contains at least one monomial of degree 1. By Lemma 4.43,

$$\sum_i(-1)^i b_{i,i+j}(R/(J:I))=0 \text{ for all } j>1. \tag{11}$$

On the other hand by Theorem 4.4, we have

$$b_{i,i+j}(R/I)=b_{i,i+j}(R/J)+b_{i-1,i+j-d'}(R/(J:I)), \tag{12}$$

for $i>0$, since $F_m\setminus\bigcup_{j<m}F_j\neq\emptyset$. By induction hypothesis J has the alternating sum property. Hence one sees that I has the alternating sum property by using (11) and (12). □

Corollary 4.46. *Let Δ be a pure quasi-tree which is connected in codimension 1. Then the facet ideal I of Δ has the alternating sum property.*

PROOF. Since Δ is a quasi-tree, there exists a leaf order of facets $F_1,\dots,F_m$. The assertion follows from Proposition 4.45 immediately. □

The next result shows that in Corollary 4.46 we can skip the assumption that Δ is pure if we assume that Δ is a tree.

Theorem 4.47. *Let Δ be a tree and connected in codimension 1. Then the facet ideal I of Δ has the alternating sum property.*

PROOF. We prove the assertion by induction on the number of facets m. The case $m = 1$ is trivial. Assume $m > 1$. Let $d = \dim \Delta$. There are two cases.

Case 1. There exists only one facet F of dimension d. Then F must be a leaf. Otherwise, there exist two facets G_1, G_2 such that $F \cap G_1 \nsubseteq F \cap G_2$ and $F \cap G_2 \nsubseteq F \cap G_1$. Since Δ is connected in codimension 1 and $\dim G_i < d$, $i = 1,2$, there exists a chain $\mathscr{C}$ between G_1 and G_2 which does not include F. Then the simplicial subcomplex Γ whose facets are the elements of $\mathscr{C}$ and F has no leaf, a contradiction.

We choose a $G \in \mathscr{U}_\Delta(F)$ of maximal dimension. Since Δ is connected in codimension 1, we have $\dim G = \dim\langle \mathscr{F}(\Delta) \setminus \{F\}\rangle$ and $\dim(F \cap G) = \dim G - 1$, i.e. $|G \setminus F| = 1$. Since F is a leaf, $\langle \mathscr{F}(\Delta) \setminus \{F\}\rangle$ is a tree with $m-1$ facets which is connected in codimension 1. By induction hypothesis there exists a leaf order of facets $F_1, \ldots, F_{m-1}$ such that for each $i = 2, \ldots, m-1$, $F_i \setminus \bigcup_{j<i} F_j \neq \emptyset$, and there exists $j < i$ such that $|F_j \setminus F_i| = 1$. Let $F = F_m$. We see that $F_1, \ldots, F_m$ satisfy the conditions of Proposition 4.45 in this order.

Case 2. There exist more than one facets of dimension d. Let $G_1, \ldots, G_s$ be all of these facets, where $s > 1$. Then for any i and j, the facets in any proper chain between G_i and G_j are all of dimension d, and hence belong to $\{G_1, \ldots, G_s\}$. Therefore $\Sigma = \langle G_1, \ldots, G_s\rangle$ is pure tree and connected in codimension 1. By Proposition 2.24 (or Proposition 2.29), $\langle G_1, \ldots, G_s\rangle$ has at least two leaves.

We claim that at least one of the leaves of Σ is a leaf of Δ. Suppose this is not the case. We take any two leaves of Σ, say G_i and G_j with free vertex i and j, respectively. Since G_i and G_j are not leaves in Δ there exist elements $F, F' \in \mathscr{F}(\Delta) \setminus \mathscr{F}(\Sigma)$ with $i \in F$ and $j \in F'$. Let $\mathscr{C}$ be a chain between F and F'. Since $\dim F < d$ and $\dim F' < d$, all elements of this chain do not belong to $\mathscr{F}(\Sigma)$. On the other hand, let $\mathscr{C}'$ be a proper chain between G_i and G_j, then all elements of the chain belong to $\mathscr{F}(\Sigma)$, because $\dim G_i = \dim G_j = d$. Then the simplicial complex generated by the elements of these two chains has no leaf, a contradiction.

We may assume that G_i is a leaf of Δ. Removing G_i from Δ yields a tree which is again connected in codimension 1, and we may proceed as in case 1. □

Corollary 4.48. *Let G be a 1-dimensional tree with edge ideal I. Then I has the alternating sum property.*

PROOF. It is clear that G is connected in codimension 1. The result follows from Theorem 4.47. □

CHAPTER 5

Dirac's theorem on chordal graphs

In this chapter we will introduce the relation tree of an monomial ideal of projective dimension 1 and give an algebraic proof of Dirac's theorem.

1. Taylor complexes and perfect modules

In this section, we recall some fundamental knowledge of commutative algebra which are needed in the following sections.

Let $S = A[x_1, \ldots, x_n]$, where A is any ring and the x_i are indeterminates. Let $u_1, \ldots, u_t$ be monomials in the x_i. The *Taylor complex* $T(u_1, \ldots u_t)$ is defined as follows. Let F_s be the free module on basis elements e_I, where I is a subset of $\{1, \ldots, t\}$ and $|I| = s$. Set

$$u_I = \text{ least common multiple of } \{u_i : i \in I\}.$$

For each pair of subsets I, J of $[n]$ such that I has s elements and J has $s-1$ elements, let $I = \{i_1, \ldots, i_s\}$ and suppose that $i_1 < \cdots < i_s$. Define:

$$c_{I,J} = \begin{cases} 0, & \text{if } J \not\subset I, \\ (-1)^k u_I/u_J, & \text{if } I = J \cup \{i_k\} \text{ for some } k. \end{cases}$$

Finally, define

$$d_s : F_s \to F_{s-1}$$

by sending e_I to $\sum_J c_{I,J}$. Let

$$T(u_1, \ldots, u_t) : 0 \longrightarrow F_t \xrightarrow{d_t} \cdots \xrightarrow{d_1} F_0.$$

One can show that $T(u_1, \ldots, u_t)$ is a free resolution of the monomial ideal $(u_1, \ldots, u_t)$. We refer the reader for example to [**9**] for more details of Taylor complex.

Let R be a Noetherian ring, and $M \neq 0$ a finite R-module. The grade of M is given by $\operatorname{grade} M = \min\{i : \operatorname{Ext}^i_R(M, R) \neq 0\}$. Since one can compute $\operatorname{Ext}^i_R(M, R)$ from a projective resolution of M, one obviously has $\operatorname{grade} M \leq \operatorname{proj\,dim} M$. Modules for which equality is attained have especially good properties.

Definition 5.1. Let R be a Noetherian ring. A non-zero finite R-module M is *perfect* if $\operatorname{proj\,dim} M = \operatorname{grade} M$. An ideal I is called *perfect* if R/I is a perfect module.

Perfect modules are 'grade unmixed':

Proposition 5.2. *Let R be a Noetherian ring, and M a perfect R-module. For a prime ideal $p \in \operatorname{Supp} M$ the following are equivalent:*

(i) $p \in \operatorname{Ass} M$;
(ii) $\operatorname{depth} R_p = \operatorname{grade} M$.

Furthermore $\operatorname{grade} P = \operatorname{grade} M$ *for all prime ideals* $P \in \operatorname{Ass} M$.

PROOF. For all finite R-module M and $P \in \operatorname{supp} M$ one has the inequalities

$$\operatorname{grade} M \le \operatorname{grade} M_P \le \operatorname{proj\,dim} M_P \le \operatorname{proj\,dim} M,$$

and moreover $\operatorname{proj\,dim} M_P + \operatorname{depth} M_P = \operatorname{depth} R_P$ by the Auslander–Buchsbaum formula. If M is perfect, then the inequalities become equations, and $\operatorname{depth} M_P = 0$ if and only if $\operatorname{depth} R_P = \operatorname{grade} M$. This shows the equivalence of (i) and (ii).

If $P \in \operatorname{Ass} M$, then $P \supset \operatorname{Ann} M$, and so $\operatorname{grade} P \ge \operatorname{grade} M$. For perfect module M the converse results from (ii) and the inequality $\operatorname{grade} P \le \operatorname{depth} R_P$. □

Since for a finite R-module M and an M-sequence $\mathbf{x}$ of length n one has $\operatorname{proj\,dim}(M/\mathbf{x}M) = \operatorname{proj\,dim} M + n$. One sees that an ideal generated by a regular sequence is perfect. Some more examples are described in the following celebrated theorem. One finds the proof of it for example in [**2**].

Theorem 5.3 (Hilbert–Burch). *Let R be a Noetherian ring, and I an ideal with a free resolution*

$$\mathbb{F}: 0 \longrightarrow R^n \xrightarrow{\varphi} R^{n+1} \longrightarrow I \longrightarrow 0.$$

Then there exists an R-regular element a such that $I = aI_n(\varphi)$. If I is projective, then $I = (a)$, and if $\operatorname{proj\,dim} I = 1$, *then $I_n(\varphi)$ is perfect of grade 2.*

Conversely, if $\varphi : R^n \to R^{n+1}$ is an R-linear may with $\operatorname{grade} I_n(\varphi) \ge 2$, *then $I = I_n(\varphi)$ has the free resolution $\mathbb{F}$.*

2. Relation trees of ideals of projective dimension 1

In this section we will define the relation tree of the monomial ideals of projective dimension 1, and describe the relation between the quasi-trees and squarefree monomial ideals of projective dimension 1.

Let Δ be a simplicial complex on $[n]$ with $\mathscr{F}(\Delta) = \{F_1, \dots, F_t\}$. We introduce the $\binom{t}{2} \times t$ matrix

$$M_\Delta = (a_k^{(i,j)})_{1 \le i < j \le t,\, 1 \le k \le t}$$

whose entries $a_k^{(i,j)} \in S$ are $a_i^{(i,j)} = x_{F_i \setminus F_j}$, $a_j^{(i,j)} = x_{F_j \setminus F_i}$, and $a_k^{(i,j)} = 0$ if $k \notin \{i, j\}$ for all $1 \le i < j \le t$ and for all $1 \le k \le t$.

Lemma 5.4. *A simplicial complex $\Delta = \langle F_1, \dots, F_t \rangle$ on $[n]$ is a quasi-tree if and only if the matrix M_Δ contains a $(t-1) \times t$ submatrix $M_\Delta^\sharp$ with the property that, for each $1 \le j \le t$, if $M_\Delta^\sharp(j)$ is the $(t-1) \times (t-1)$ submatrix of $M_\Delta^\sharp$ obtained by removing the j-th column from $M_\Delta^\sharp$, then $|\det(M_\Delta^\sharp(j))| = x_{[n]}/x_{F_j}$.*

PROOF. $\Rightarrow$: Let Δ be a quasi-tree on $[n]$ and fix a leaf ordering $F_1, \dots, F_t$ of the facets of Δ. Let $t > 1$. Let F_k with $k \ne t$ be a branch of F_t and $\Delta' = \Delta \setminus F_t$. Since Δ' is a quasi-tree, by assumption of induction, it follows that M_Δ contains a $(t-2) \times t$ submatrix M' with the property that, for each $1 \le j < t$, if $M'(j,t)$ is the $(t-2) \times (t-2)$ submatrix of M' obtained by removing the jst and t-th columns from M', then $|\det(M'(j,t))| = x_{[n] \setminus (F_t \setminus F_k)}/x_{F_j}$. Let $M_\Delta^\sharp$ denote the $(t-1) \times t$ submatrix of M_Δ obtained by adding the (k,t)-th row to M'. Since $a_t^{(k,t)} = x_{F_t \setminus F_k}$, it follows that, for each $1 \le j < t$, one has $|\det(M_\Delta^\sharp(j))| = x_{[n]}/x_{F_j}$. Moreover, since $|\det(M_\Delta^\sharp(t))| = |x_{F_k \setminus F_t} \det(M'(k,t))|$, one has $|\det(M_\Delta^\sharp(t))| = x_{[n]}/x_{F_t}$.

$\Leftarrow$: Now, suppose that the matrix M_Δ contains a $(t-1)\times t$ submatrix $M_\Delta^\sharp$ with the property that, for each $1\le j\le s$, if $M_\Delta^\sharp(j)$ is the $(t-1)\times(t-1)$ submatrix of $M_\Delta^\sharp$ obtained by removing j-th column from $M_\Delta^\sharp$, then $|\det(M_\Delta^\sharp(j))| = x_{[n]}/x_{F_j}$. Let Ω denote the subgraph on $[t]$ whose edges are those $\{i,j\}$ with $1\le i<j\le t$ such that the (i,j)-th row of M_Δ belongs to $M_\Delta^\sharp$. Then Ω contains no cycles. To see why this is true, if C is a cycle of Ω with $E(C)$ its edge set. If $\{i_0,j_0\}\in E(C)$, then in the matrix $M_\Delta^\sharp(i_0)$, the (i,j)-th rows with $\{i,j\}\in E(G)$ are linearly dependent. Thus $\det(M_\Delta^\sharp(i_0))=0$. This is impossible. Hence Ω contains no cycles. Since the number of edges of Ω is $t-1$, it follows that Ω is a tree, i.e., a connected graph without cycles. Hence there is a column of $M_\Delta^\sharp$ which contains exactly one nonzero entry. Suppose, say, that the tst column contains exactly one nonzero entry and the (k,t)-th row of M_Δ appears in $M_\Delta^\sharp$. Then, for each $1\le j<t$, the monomial $x_{F_t\setminus F_k}$ divides $|\det(M_\Delta^\sharp(j))|$. Hence $(F_t\setminus F_k)\cap F_j=\emptyset$ for all $1\le j<t$. It then follows that F_t is a leaf of Δ and F_k is a branch of F_t. Let $\Delta'=\Delta\setminus F_t$ and $M_{\Delta'}^\sharp$ the $(t-2)\times(t-1)$ submatrix of $M_{\Delta'}$ which is obtained by removing the (k,t)-th row and the tst column from $M_\Delta^\sharp$. Since Δ' is a simplicial complex on $[n]\setminus(F_t\setminus F_k)$ and since $x_{F_t\setminus F_k}(x_{[n]\setminus(F_t\setminus F_k)}/x_{F_j}) = x_{[n]}/x_{F_j}$ for each $1\le j<t$, working with induction on t, it follows that Δ' is a quasi-tree. Hence Δ is a quasi-tree. $\square$

Let I be an arbitrary monomial ideal with $G(I)=\{u_1,\ldots,u_t\}$, and let $\mathbb{T}$ be the Taylor complex associated with I. Then $T_i = S^{\binom{t}{i}}$, and the matrix A_I representing the differential $T_2\to T_1$ is a $\binom{t}{2}\times t$-matrix. To be more precise, if $T_1=\bigoplus_{i=1}^t Se_i$, then $T_2=\bigoplus_{i<j} Se_i\wedge e_j$, and $\partial(e_i\wedge e_j)=u_{ji}e_i-u_{ij}e_j$, where $u_{ij}=u_i/[u_i,u_j]$ for all $i,j\in[t]$ with $i\ne j$.

Note that for any simplicial complex Δ we have $M_\Delta = A_{I(\Delta^c)}$, because if $u_i=x_{F_i^c}$ and $u_j=x_{F_j^c}$, then $u_{ji}=x_{F_j^c\setminus F_i^c}=x_{F_i\setminus F_j}$.

Assume now that I has projective dimension 1, and that the elements of $G(I)$ have no common factor. Then I is perfect of codimension 2, that is, $\operatorname{proj\,dim} S/I = n-\dim S/I = 2$.

A subset R of the Taylor relations is called *irreducible* if R generates the first syzygy module $\operatorname{syz}_1(I)$ of I, but no proper subset of R generates $\operatorname{syz}_1(I)$. Fortunately it is known (see [**3**, Corollary 5.2]) that an irreducible subset of the Taylor relations is in fact a minimal system of generators of $\operatorname{syz}_1(I)$. In particular it follows that we can always choose a minimal free resolution

$$0 \longrightarrow S^{t-1} \xrightarrow{\varphi} S^t \longrightarrow I \longrightarrow 0$$

such that the rows of the matrix of φ correspond to Taylor relations. However the choice of an irreducible set R of Taylor relations is in general not unique.

For example, let $I=(x_4x_5x_6, x_1x_5x_6, x_1x_2x_6, x_1x_2x_5)$. Then φ can be represented by the matrix

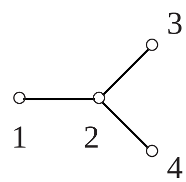
$$\begin{pmatrix} x_1 & -x_4 & 0 & 0\\ 0 & x_2 & -x_5 & 0\\ 0 & x_2 & 0 & -x_6 \end{pmatrix},$$

or by

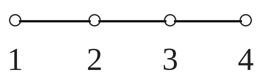
$$\begin{pmatrix} x_1 & -x_4 & 0 & 0 \\ 0 & x_2 & -x_5 & 0 \\ 0 & 0 & x_5 & -x_6 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_1 & -x_4 & 0 & 0 \\ 0 & x_2 & 0 & -x_6 \\ 0 & 0 & x_5 & -x_6 \end{pmatrix}.$$

Nevertheless for a given choice R of $t-1$ Taylor relations which generate $\operatorname{syz}_1(I)$ we can define a (1-dimensional) tree Ω as in the proof of 5.4 with $\{i,j\} \in E(\Omega)$ if $u_{ji}e_i - u_{ij}e_j \in R$ for $i < j$. We call Ω the *relation tree* of R. This relation tree was first considered in [**3**, Remark 6.3].

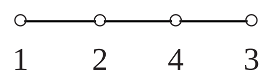
In the above example the relation tree for the first matrix is



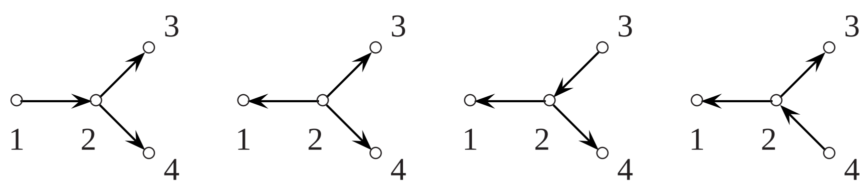
while for the other matrices it is



or



Next we want to describe how the generators u_i of I can be computed from the u_{ij} and the relation trees. To this end we introduce for each $i = 1, \dots, t$ an orientation to make Ω a directed graph which we denote Ω_i. We fix some vertex i. Let j be any other vertex of Ω. Since Ω is a tree there is a unique directed walk from i to j. This defines the orientation of the edges along this walk. The following picture explains this for the first of our relation trees in the above example.



By the Hilbert–Burch theorem one has

$$u_i = (-1)^i \det(A_i) \quad \text{for} \quad i = 1, \dots, t,$$

where the matrix A_i is obtained from the relation matrix A of I by deleting the i-th column of A. Computing $\det(A_i)$ by the determinantal expansion formula as in the proof of Lemma 5.4 one sees that

$$u_i = \prod_{(k,j)} u_{kj},$$

where the product is taken over all oriented edges (k,j) of Ω_i.

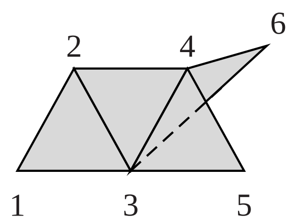
Corollary 5.5. *A simplicial complex Δ is a quasi-forest if and only if* $\operatorname{proj\,dim} I(\Delta^c) = 1$.

PROOF. Let $\mathscr{F}(\Delta) = \{F_1, \ldots, F_t\}$. By Lemma 5.4, the simplicial complex Δ is a quasi-forest if and only if M_Δ contains a $(t-1)\times t$ submatrix $M_\Delta^{\#}$ whose ideal of maximal minors is $I(\Delta^c)$. Hence, if Δ is a quasi-forest, the Hilbert–Burch theorem implies that $\operatorname{proj\,dim} I(\Delta^c) = 1$. Conversely, suppose $\operatorname{proj\,dim} I(\Delta^c) = 1$, and let A be a $(t-1)\times t$ relation matrix of this ideal consisting of Taylor relations. By the Hilbert-Burch theorem, $I(\Delta^c)$ is the ideal of maximal minors of A. Since $M_\Delta = A_{I(\Delta^c)}$, it follows that A is a submatrix of M_Δ. Hence Δ is a quasi-forest. $\square$

In our example I may be viewed as $I = I(\Delta^c)$ where the facets of Δ are

$$\{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{3,4,6\}\}.$$

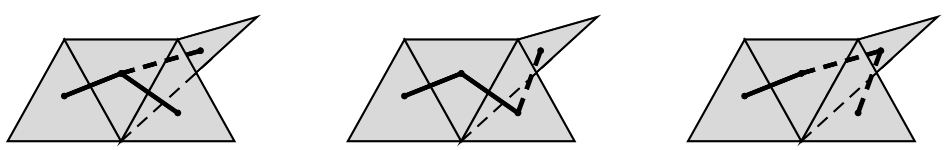
See the following picture:



This is a quasi-tree, as it should be by Corollary 5.5.

Inspecting the proof of Lemma 5.4, we see that all possible relation trees Ω of $I(\Delta^c)$ can be recovered from the quasi-forest $\Delta = \langle F_1, \ldots, F_m\rangle$ as follows: start with some leaf F_i of Δ, and let F_j be a branch of F_i. Then $\{i, j\}$ will be an edge of Ω. According to Corollary 5.11, $\langle \mathscr{F}(\Delta) \setminus \{F_i\}\rangle$ is again a quasi-forest. Then remove the leaf F_i, and continue in the same way with the remaining quasi-forest in order to find the other edges of Ω. Of course, at each step of the procedure there may be different choices. This gives us the different possible relation trees.

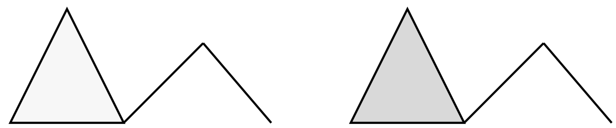
Geometrically a relation tree is obtained from a given quasi-forest by connecting the barycentric centers of the leaves and branches according to the above rules. In our example we get



3. An algebra proof of Dirac's theorem

Let G be a finite graph on $[n]$ without isolated vertices, and $E(G)$ its edge set. A *stable subset* or *clique* of G is a subset F of $[n]$ such that $\{i, j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$. We write $\Delta(G)$ for the simplicial complex on $[n]$ whose faces are the stable subsets of G. It is clear that G is the pure 1-skeleton of $\Delta(G)$, and that if Γ is a simplicial complex with $G = \Gamma(1)$, then Γ is a subcomplex of $\Delta(G)$. Hence, in a certain sense, $\Delta(G)$ is the 'largest' simplicial complex whose pure 1-skeleton is G.

The following example demonstrates this concept:



G $\Delta(G)$

Definition 5.6. A graph G is said to be *chordal* if every cycle C_n in G of length $n \geq 4$ has a chord in G. A *chord* of C_n is an edge joining two non adjacent vertices of C_n.

The simplicial complex $\Delta(i)$ whose facets are the i-dimensional faces of Δ is called the *pure i-skeleton* of Δ.

Suppose Δ is a pure $(d-1)$-dimensional simplicial complex. We then define

$$\overline{\Delta} = \langle F : F \notin \Delta, \quad |F| = d \rangle.$$

We have the following very simple

Lemma 5.7. *Let Δ be a $(d-1)$-dimensional pure simplicial complex, and let Γ be the simplicial complex such that $I_\Gamma = I(\Delta)$. Then*

$$\overline{\Delta} = \Gamma(d-1).$$

PROOF. Let $F \in \mathscr{F}(\overline{\Delta})$, then $F \notin \Delta$. Therefore $x_F \notin I(\Delta)$, and hence $x_F \notin I_\Gamma$. This means that $F \in \Gamma$. Since $|F| = d$, this implies that $F \in \Gamma(d-1)$. The converse inclusion is proved similarly. □

We recall the following result of Fröberg [**14**, Theorem 1] (see also [**39**]).

Theorem 5.8 (Fröberg). *Let G be graph. Then $I(G)$ has a linear resolution if and only if $\overline{G}$ is chordal.*

Lemma 5.9. *Let G be a graph, and Δ the simplicial complex defined by $I_\Delta = I(\overline{G})$. Then*

(i) $\Delta = \Delta(G)$*;*
(ii) $G = \Delta(1)$*;*
(iii) Δ *is a quasi-forest* $\Longleftrightarrow$ G *is chordal.*

PROOF. (i) Since the pure 1-skeleton of $\Delta(G) = G$, it follows that $I(\overline{G}) \subset I_{\Delta(G)}$. Conversely, let F be a minimal nonface of $\Delta(G)$. If $|F| > 2$, then each subset $G \subset F$ with $|G| = 2$ is an edge of G. Therefore F is a stable subset of G, and hence $F \in \Delta(G)$, a contradiction. Thus for every minimal nonface F of $\Delta(G)$ one has $|F| = 2$. This shows that $I_{\Delta(G)} = I(\overline{G})$. Therefore, $\Delta = \Delta(G)$.

(ii) follows from Lemma 5.7 (or from (i) and the remarks preceding this lemma).

(iii) The theorem of Fröberg [**14**] guarantees that the complementary graph G of $\overline{G}$ is a chordal graph if and only if $I(\overline{G}) = I_\Delta$ has a 2-linear resolution. By Theorem 6.18, $\operatorname{reg}(I_\Delta) = \operatorname{proj}\dim I_{\Delta^\vee} + 1$, and so the ideal $I(\overline{G})$ has a 2-linear resolution if and only if $\operatorname{proj}\dim I_{\Delta^\vee} = 1$. Since by Proposition 2.11, $I_{\Delta^\vee} = I(\Delta^c)$, the assertion follows from Corollary 5.5. □

As we have already seen in Lemma 2.31, a quasi-forest is a flag complex. By using this fact together with Lemma 5.9, we have:

Theorem 5.10 (Dirac). *A finite graph G on $[n]$ is a chordal graph if and only if G is the pure* 1*-skeleton of a quasi-forest on* $[n]$.

PROOF. The statements (ii) and (iii) of Lemma 5.9 imply that a chordal graph is the pure 1-skeleton of quasi-forest. Conversely, suppose that G is the pure 1-skeleton of a quasi-forest Γ. Since by Lemma 2.31, Γ is flag, the ideal I_Γ is generated by all monomials x_F with $|F| = 2$ and $F \notin \Gamma$. This shows that $I_\Gamma = I(\bar{G})$, and so $\Gamma = \Delta(G)$, by Lemma 5.9(i). Hence G is chordal by Lemma 5.9(iii). □

Corollary 5.11. *Let Δ be a quasi-forest, and F a leaf of Δ. Then $\langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$ is again a quasi-forest.*

PROOF. Let $\Delta' = \langle \mathscr{F}(\Delta) \setminus \{F\} \rangle$. Let G be the pure 1-skeleton of Δ and G' the pure 1-skeleton of Δ'. Then G' is obtained by removing all free vertices of F and all edges containing these vertices from G. Since G is chordal by Theorem 5.10, it follows that G' is also chordal. Hence again by Theorem 5.10, Δ' is a quasi-forest. □

Remark 5.12. Theorem 1 and Theorem 2 in Dirac's paper [**8**] yield the following conclusion: G is chordal if and only if $(*)$: $\Delta(G)$ can be obtained from a set of simplices $F_1, \ldots, F_m$ as follows: for each i, there exists $j < i$ such that $F_k \cap F_i \subset F_j \cap F_i$.

This does not mean that $F_1, \ldots, F_m$ is a leaf order of $\Delta(G)$. In fact, the F_i need not to be facets of $\Delta(G)$. But the above condition $(*)$ on $\Delta(G)$ is equivalent to the condition that $\Delta(G)$ is a quasi-tree.

Indeed, if Δ is any quasi-tree, then a leaf order satisfies condition $(*)$. Conversely, suppose Δ satisfies $(*)$ for the simplices $F_1, \ldots, F_m$. We show by induction on m that Δ is a quasi-tree. Hence we may assume that the simplicial complex Γ obtained from $F_1, \ldots, F_{m-1}$ is a quasi-tree, that is, there exists a leaf order $G_1, \ldots, G_r$ for Γ. (The index r may be smaller than $m-1$, since not all F_i need to be facets of Γ. In other words, $\{G_1, \ldots, G_r\}$ is a subset of $\{F_1, \ldots, F_{m-1}\}$). Now by assumption there exists $j < m$ such that $F_k \cap F_m \subset F_j \cap F_m$ for all $k < m$. This F_j is a face of some G_s. If it happens that $G_s \subset F_m$, then $G_1, \ldots, G_{s-1}, F_m, G_{s+1}, \ldots, G_r$ is the leaf order for Δ. Otherwise, F_m is a leaf of Δ, and $G_1, \ldots, G_s, F_m$ is the leaf order for Δ.

In graph theory (see for example [**42**]) Dirac's theorem is often quoted as follows: A graph G is chordal if and only if G has a perfect elimination ordering on its vertices, as explained in the introduction. It is quite clear that $\Delta(G)$ satisfies condition $(*)$ if and only if G has a perfect elimination ordering on its vertices.

We conclude this section with a sort of higher Dirac theorem.

Theorem 5.13. *Let Δ be a pure ℓ-dimensional simplicial complex over the vertex set $[n]$, and Γ its pure* 1*-skeleton. Then the following conditions are equivalent:*

(i) Δ *is the pure ℓ-skeleton of a quasi-forest;*
(ii) (a) Γ *is a chordal graph;*
  (b) Δ *is the pure ℓ-skeleton of* $\Delta(\Gamma)$.

PROOF. (ii) $\Rightarrow$ (i) follows from Lemma 5.9(iii). For the implication (i) $\Rightarrow$ (ii), suppose that Δ is the pure ℓ-skeleton of the quasi-forest Σ. Then Γ is also the pure 1-skeleton of Σ. As in the proof of Theorem 5.10 we conclude that $\Sigma = \Delta(\Gamma)$. This implies (ii)(b). Finally, by Dirac's theorem Γ is chordal. □

CHAPTER 6

The resolutions of monomial ideals related to a quasi-tree

In this chapter we consider graded ideals in a polynomial ring over a field and ask when such an ideal has the property that all of its powers have a linear resolution.

It is known [**25**] that polymatroidal ideals have linear resolutions and that powers of polymatroidal ideals are again polymatroidal (see [**5**] and [**18**]). In particular they have again linear resolutions. In general however, powers of ideals with linear resolution need not to have linear resolutions. The first example of such an ideal was given by Terai. He showed that over a base field of characteristic $\neq 2$ the Stanley Reisner ideal $I = (abd, abf, ace, adc, aef, bde, bcf, bce, cdf, def)$ of the minimal triangulation of the projective plane has a linear resolution, while I^2 has no linear resolution. The example depends on the characteristic of the base field. If the base field has characteristic 2, then I itself has no linear resolution.

Another example, namely $I = (def, cef, cdf, cde, bef, bcd, acf, ade)$ is given by Sturmfels [**34**]. Again I has a linear resolution, while I^2 has no linear resolution. The example of Sturmfels is interesting because of two reasons: 1. it does not depend on the characteristic of the base field, and 2. it is a linear quotient ideal.

Recall that an ideal $I \subset S$ is said to be equigenerated, if whose generators $f_1, \dots, f_m$ are all of same degree. As we have seen in Lemma 4.31 that a equigenerated linear quotient ideal has a linear resolution (independent on the characteristic of the base field). However the example of Sturmfels also shows that powers of a linear quotient ideal need not to be again linear quotient ideals.

In this chapter we will give some monomial ideals arriving from quasi-trees which have the property that all powers of it have linear resolutions.

1. Rees algebra and Gröbner basis

In this section we recall some fundamental concepts of Commutative algebra, which are useful in the remaining sections.

Let R be a ring and M an R-module. Given $n \geq 0$ we define

$$T^n(M) = \underbrace{M \otimes \cdots \otimes M}_{n-times} \quad \text{and} \quad T^0(M) = R.$$

Recall that the *tensor algebra* $T(M)$ of M is the non commutative graded algebra

$$T(M) = \bigoplus_{n=0}^{\infty} T^n(M),$$

where the product is induced by juxtaposition.

The *symmetric algebra* of M, denoted by $Sym_R(M)$ or simply $Sym(M)$, is defined as the quotient algebra

$$Sym(M) = T(M)/\mathfrak{I}$$

where $\mathfrak{I}$ is the two side ideal generated by the elements

$$xy - yx = x \otimes y - y \otimes x \in T^2(M)$$

with x and y running through M. Notice that $Sym(M)$ is commutative.

Since $\mathfrak{I}$ is a homogeneous elements of degree two, the symmetric algebra is graded by

$$Sym_n(M) = T^n(M)/I \cap T^n(M) \quad \text{and} \quad Sym_0(M) = R.$$

Note

$$Sym_2(M) = M \otimes M/(x \otimes y - y \otimes x),$$

with x and y running through M.

The following fact about the symmetric algebra is well known.

Fact 6.1. *Let R be a ring and M an R-module. If M is free of rank n, then the symmetric algebra of M is a polynomial ring in n variables with coefficients in R.*

Let R be a ring and I an ideal generated by $f_1, \dots, f_m$. The *Rees algebra* of I, denoted by $\mathscr{R}(I)$ is the subring of $R[t]$ given by

$$\mathscr{R}(I) = R[f_1t, \dots, f_mt] \subset R[t],$$

where t is a new variable. Note

$$\mathscr{R}(I) = R \oplus It \oplus \cdots \oplus I^n t^n \oplus \cdots \subset R[t].$$

There is an epimorphism or R-algebras

$$\varphi : B = R[t_1, \dots, t_m] \longrightarrow \mathscr{R}(I) \longrightarrow 0, \quad t_i \mapsto f_i t,$$

where B is the polynomial ring in the indeterminates $t_1, \dots, t_m$ over the ring R. The kernel of φ, denoted by J, is the *presentation ideal* or *toric ideal* of $\mathscr{R}(I)$ with respect to $f_1, \dots, f_m$. Observe that $J = \bigoplus_{i=1}^{\infty} J_i$ is a graded ideal in the variables t_i, where B has the standard grading induced by $\deg t_i = 1$, $i = 1, \dots, m$.

The map

$$\psi : R^m \longrightarrow I$$

given by

$$\psi(z_1, \dots, z_m) = \sum_{i=1}^{m} z_i f_i$$

induces an R-algebra epimorphism

$$\beta : R[t_1, \dots, t_m] \longrightarrow Sym_R(I).$$

Thus the symmetric algebra of I is:

$$Sym_R(I) \simeq R[t_1, \dots, t_m]/\ker(\beta),$$

where $\ker(\beta)$ is an ideal of $R[t]$ generated by linear forms:

$$\ker(\beta) = (\{\sum_{i=1}^{m} b_i t_i : \sum_{i=1}^{m} b_i f_i = 0 \text{ and } b_i \in R\}).$$

On the other hand, the kernel of φ is generated by all forms $F(t_1,\dots,t_m)$ such that $F(f_1,\dots,f_m)=0$. In particular, one may factor φ through $Sym_R(I)$ and obtain the map

$$\alpha : Sym_R(I) \longrightarrow \mathscr{R}(I)$$

such that $\beta\circ\alpha=\varphi$. We say that I is an *ideal of linear type* if α is an isomorphism.

An important module theoretic obstruction to ideal of linear type is given by

Proposition 6.2 (Herzog-Simis-Vasconcelos). *Let I be an ideal of a ring R. If the ideal I of linear type, then for each prime ideal P containing I, I_P can be generated by* height P *elements.*

PROOF. See [**24**]. □

Let K be a field and $R=K[x_1,\dots,x_n]$ the polynomial ring in n variables over K. Let $\mathscr{M}$ be the set of all monomials in R. We say an order relation $\prec$ is a *term order* on $\mathscr{M}$ if it a *total order* which is *compatible* with the multiplication of monomials, i.e.,

- for any pair of monomials m_1 and nm_2 we have $m_1\prec m_2$ or $m_2\prec m_1$ or $m_1=m_2$;
- if $m_1\prec m_2$ and $m_2\prec m_3$ then $m_1\prec m_3$;
- $1\prec m$ for any monomial $m\neq 1$;
- if $m_1\prec m_2$ then $mm_1\prec mm_2$ for any monomial m.

For example, the lexicographical ordering, degree lexicographical ordering and the degree lexicographical ordering are term orderings.

Let $f\in R$ be a polynomial, $f\neq 0$, and suppose $\prec$ is a term order of the monomials in R. Then f can be uniquely written $f=c_1m_1+\cdots+c_lm_l$ with monomials $m_1\prec m_2\prec\cdots\prec m_l$ and $c_i\neq 0$, $i=1,\dots,l$. The *support* of f is the set $\operatorname{supp} f=\{m_1 : i=1,\dots,l\}$. The *leading monomial*, *leading term* and *leading coefficient* are defined to be $\operatorname{lm}(f)=m_1$, $\operatorname{lt}(f)=c_1m_1$ and $\operatorname{lc}(f)=c_1$, respectively. If I is a nonzero ideal in R, we define the *initial ideal* of I to be $\operatorname{in}(I)=\langle \operatorname{lm}(f) : f\in I\rangle$. The monomials which do not lie in $\operatorname{in}(I)$ are called *standard monomials*. If $I=\langle f_1,\dots,f_r\rangle$ then clearly $\operatorname{lm}(f_i)\in\operatorname{in}(I)$ for $i=1,\dots,r$. Since $\operatorname{lm}(mf)=m\operatorname{lm}(f)$ for a monomial m and a polynomial f, we have $\langle\operatorname{lm}(f_1),\dots,\operatorname{lm}(f_r)\rangle\subseteq\operatorname{in}(I)$. There might be strict inclusion. Hence we have the following

Definition 6.3. *Let $R=K[x_1,\dots,x_n]$ be a polynomial ring over the field K and I an ideal. A set $\{g_1,\dots,g_s\}$ of elements in I such that $\langle\operatorname{lm}(g_1)\dots,\operatorname{lm}(g_s)\rangle=\operatorname{in}(I)$ is called a* Gröbner basis *of I.*

Note that a Gröbner basis of an ideal is a generating set of this ideal.

Lemma 6.4. *If $\{g_1,\dots,g_s\}$ is a Gröbner basis of I, then $\langle g_1,\dots,g_s\rangle=I$.*

PROOF. $\langle g_1,\dots,g_s\rangle\subseteq I$ is clear since $g_i\in I$ for $i=1,\dots,s$. Let $f\in I$. Then $\operatorname{lm}(f)\in\langle\operatorname{lm}(g_1),\dots,\operatorname{lm}(g_s)\rangle$. Hence $\operatorname{lm}(f-mg_k)\prec\operatorname{lm}(f)$ for some $k\in[s]$ and some term $m\in R$. Since $f-mg_k\in I$, we get by recursiveness that $f\in\langle g_1,\dots,g_s\rangle$. □

The proof of the following proposition about the standard monomials is not difficult. One can also find it for example in [**13**].

Proposition 6.5. *The (images of the) standard monomials form a K-vector space basis for the residue ring $K[x_1,\dots,x_n]/I$.*

Gröbner bases of an ideal are not unique. For any Gröbner basis G of I and any element $f \in I$, $G \cup \{f\}$ is a new Gr"obner basis of I. We will introduce the concept *reduced Gröbner basis* of an ideal, which is uniquely determinant.

Definition 6.6. *Let R be a polynomial ring and $I = \langle g_1, \dots, g_s \rangle$ an ideal with $G = \{g_1, \dots, g_s\}$ a Gröbner basis. We say G is* reduced *if*

(i) $\{\mathrm{lm}(g_1), \dots, \mathrm{lm}(g_s)\}$ *constitutes a minimal set of generators for* $\mathrm{in}(I)$*;*
(ii) g_i *are monic, i.e.,* $\mathrm{lc}(g_i) = 1$, $i = 1, \dots, s$*;*
(iii) *no* $\mathrm{lm}(g_i)$ *divides any monomial in* $\mathrm{supp}\, g_j$, $i \neq j$.

Given a Gröbner basis G it is easy to construct a reduced Gröbner basis. First we pick up a subset $G' = \{g_{i_1}, \dots, g_{i_k}\}$ of G such that the condition (i) in the definition is fulfilled. Then we multiply each g_{i_k} with $\mathrm{lc}(g_{i_k})^{-1}$ so that we get monic elements. Then we take the remainder of each g_{i_k} with respect to $G' \setminus \{g_{i_k}\}$. Thus reduced Gröbner basis exist. Now suppose $\{g_1, \dots, g_s\}$ and $\{h_1, \dots, f_s\}$ are two reduced Gröbner bases of I with $\mathrm{lm}(g_i) = \mathrm{lm}(h_i)$ for $i = 1, \dots, s$. Hence $g_i - h_i \in I$. If $g_i \neq f_i$ we would have $\mathrm{lm}(g_i - h_i) \in \mathrm{in}(I)$ and the leading terms of g_i and f_i are cancelled in $g_i - h_i$. On the other hand, since $\{g_1, \dots, g_s\}$ and $\{h_1, \dots, f_s\}$ are reduced Gröbner bases, $g_i - h_i$ is a linear combination of monomials outside $\mathrm{in}(I)$, a contradiction.

Clearly, there are infinitely many term orders on $\mathscr{M}$ if $n \geq 2$. However if the ideal I is fixed, then they can be grouped into finitely many equivalence classes by the following theorem.

Theorem 6.7. *Every ideal $I \subset K[x_1, \dots, x_n]$ has only finitely many distinct initial ideals.*

PROOF. See [**33**, Theorem 1.2]. □

Theorem 6.7 permits the following definition. A finite subset $\mathscr{U} \subset I$ is called a *universal Gröbner basis* if $\mathscr{U}$ is a Gröbner basis of I respect to all term orders simultaneously.

Corollary 6.8. *Every ideal $I \subset K[x_1, \dots, x_n]$ possesses a finite universal Gröbner basis.*

PROOF. By Theorem 6.7, there exist only finitely many distinct reduced Gröbner basis of I. Their inion is again finite, and it is a universal Gröbner basis of I. □

2. The x-condition

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring, $I \subset S$ an equigenerated graded ideal. Then the Rees ring

$$\mathscr{R}(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1 t, \dots, f_m t] \subset S[t]$$

is naturally bigraded with $\deg x_i = (1,0)$ for $i = 1, \dots, n$ and $\deg f_i t = (0,1)$ for $i = 1, \dots, m$.

Let $T = S[y_1, \dots, y_m]$ be the polynomial ring over S in the variables $y_1, \dots, y_m$. We define a bigrading on T by setting $\deg x_i = (1,0)$ for $i = 1, \dots, n$, and $\deg y_j = (0,1)$ for $j = 1, \dots, m$. Then there is a natural surjective homomorphism of bigraded K-algebras $\varphi\colon T \to \mathscr{R}(I)$ with $\varphi(x_i) = x_i$ for $i = 1, \dots, n$ and $\varphi(y_j) = f_j t$ for $j = 1, \dots, m$.

Let

$$\mathbb{F}\colon 0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathscr{R}(I) \longrightarrow 0$$

be the bigraded minimal free T-resolution of $\mathscr{R}(I)$. Here $F_i = \bigoplus_j T(-a_{ij}, -b_{ij})$ for $i = 0, \ldots, p$. The *x-regularity* of $\mathscr{R}(I)$ is defined to be the number

$$\mathrm{reg}_x(\mathscr{R}(I)) = \max_{i,j}\{a_{ij} - i\}.$$

With the notation introduced one has the following result [**31**, Theorem 5.3 (i)] of Römer.

Theorem 6.9. $\mathrm{reg}(I^n) \leq nd + \mathrm{reg}_x(\mathscr{R}(I))$. *In particular, if* $\mathrm{reg}_x(\mathscr{R}(I)) = 0$, *then each power of I admits a linear resolution.*

For the reader's convenience we give a simple proof of this theorem: For all n, the exact sequence $\mathbb{F}$ gives the exact sequence of graded S-modules

$$0 \longrightarrow (F_p)_{(*,n)} \longrightarrow (F_{p-1})_{(*,n)} \cdots \longrightarrow (F_1)_{(*,n)} \longrightarrow (F_0)_{(*,n)} \longrightarrow \mathscr{R}(I)_{(*,n)} \longrightarrow 0. \tag{13}$$

We note that $\mathscr{R}(I)_{(*,n)} = I^n(-dn)$, and that $T(-a,-b)_{(*,n)}$ is isomorphic to the free S-module $\bigoplus_{|u|=n-b} S(-a)y^u$. It follows that (13) is a (possibly non-minimal) graded free S-resolution of $I^n(-dn)$. This yields at once that $\mathrm{reg}(I^n(-dn)) \leq \mathrm{reg}_x(\mathscr{R}(I))$, and thus $\mathrm{reg}(I^n) \leq nd + \mathrm{reg}_x(\mathscr{R}(I))$.

We say that I satisfies the x-condition if $\mathrm{reg}_x(\mathscr{R}(I)) = 0$.

Corollary 6.10. *Let $I \subset S$ be an equigenerated graded ideal, and let $\mathscr{R}(I) = T/P$. Then each power of I has a linear resolution if for some term order $<$ on T the defining ideal P has a Gröbner basis G whose elements are at most linear in the variables $x_1, \ldots, x_n$, that is,* $\deg_x f \leq 1$ *for all* $f \in G$.

PROOF. The hypothesis implies that $\mathrm{in}(P)$ (the initial ideal of P) is generated by monomials $u_1, \ldots, u_m$ with $\deg_x u_i \leq 1$. Let $C_\bullet$ be the Taylor resolution of $\mathrm{in}(P)$. The module C_i has the basis e_σ with $\sigma = \{j_1 < j_2 < \ldots < j_i\} \subset [m]$. Each basis element e_σ has the multidegree (a_σ, b_σ) where $x^{a_\sigma}y^{b_\sigma} = \mathrm{lcm}\{u_{j_1}, \ldots, u_{j_i}\}$. It follows that $\deg_x e_\sigma \leq i$ for all $e_\sigma \in C_i$. Since the shifts of $C_\bullet$ bound the shifts of a minimal multigraded resolution of $\mathrm{in}(P)$, we conclude that $\mathrm{reg}_x(T/\mathrm{in}(P)) = 0$. On the other hand, by semi-continuity one always has $\mathrm{reg}_x(T/P) \leq \mathrm{reg}_x(T/\mathrm{in}(P))$. □

3. Monomial ideals with 2-linear resolution

Let K be a field and $I \subset S = K[x_1, \ldots, x_n]$ be a squarefree monomial ideal generated in degree 2. We may attach to I a graph G whose vertices are the elements of $[n]$, and $\{i, j\}$ is an edge of G if and only if $x_i x_j \in I$. The ideal I is called the edge ideal of G and denoted $I(G)$. Thus the assignment $G \mapsto I(G)$ establishes a bijection between graphs which contains no isolated vertex and squarefree monomial ideals generated in degree 2.

As a consequence of Theorem 5.8 and Theorem 5.10 we obtain

Proposition 6.11. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a squarefree monomial ideal with 2-linear resolution. Then after suitable renumbering of the variables we have: if $x_i x_j \in I$ with $i \neq j$, $k > i$ and $k > j$, then either $x_i x_k$ or $x_j x_k$ belongs to I.*

PROOF. We consider I as the edge ideal of the graph G. Then by Theorem 5.8 and Theorem 5.10 the complementary graph $\overline{G}$ is the 1-skeleton of a quasi-tree Δ. Let $F_1,\dots,F_m$ be a leaf order of Δ. Let i_1 be the number of free vertices of the leaf F_m. We label the free vertices of F_m by $n, n-1,\dots,n-i_1+1$, in any order. Next F_{m-1} is a leaf of $\langle F_1,\dots,F_{m-1}\rangle$. Say, F_{m-1} has i_2 free vertices. Then we label the free vertices of F_{m-1} by $n-i_1,\dots,n-(i_1+i_2)+1$, in any order. Proceeding in this way we label all the vertices of Δ, that is, those of G, and then choose the numbering of the variables of S according to this labelling.

Suppose there exist $x_ix_j\in I$ and $k>i,j$ such that $x_ix_k\notin I$ and $x_jx_k\notin I$. Let r be the smallest number such that $\Gamma=\langle F_1,\dots,F_r\rangle$ contains the vertices $1,\dots,k$. Then k is a free vertex of F_r in Γ. Since $x_ix_k\notin I$ and $x_jx_k\notin I$, we have that $\{i,k\}$ and $\{j,k\}$ are edges of Γ, and since k is a free vertex of F_r in Γ it follows that i and j are vertices of F_r. Therefore $\{i,j\}$ is an edge of F_r and hence of Γ. However, this contradicts the assumption that $x_ix_j\in I$. □

We now consider a monomial ideal I generated in degree 2 which is not necessarily squarefree. Let $J\subset I$ be the ideal generated by all squarefree monomials in I. Then $I=(x_{i_1}^2,\dots,x_{i_k}^2,J)$.

Lemma 6.12. *Suppose I has a linear resolution. Then J has a linear resolution.*

PROOF. Polarizing (see [**2**, Lemma 4.2.16]) the ideal $I=(x_{i_1}^2,\dots,x_{i_k}^2,J)$ yields the ideal $I^*=(x_{i_1}y_1,\dots,x_{i_k}y_k,J)$ in $K[x_1,\dots,x_n,y_1,\dots,y_k]$. We consider I^* as the edge ideal of the graph G^* with the vertices $-k,\dots,-1,1,\dots,n$, where the vertices $-i$ correspond to the variables y_i and the vertices i to the variables x_i. Let G be the restriction of G^* to the vertices $1,\dots,n$. In other words, $\{i,j\}$ with $1\le i<j$ is an edge of G if and only it is an edge of G^*. Then it is clear that J is the edge ideal of G.

Assuming that I has a linear resolution implies that I^* has a linear resolution since I^* is an unobstructed deformation of I. It follows that $\overline{G^*}$ is chordal, by Theorem 5.8. Obviously the restriction of a chordal graph to a subset of the vertices is again chordal. Hence $\overline{G}$ is chordal, and so again by Theorem 5.8 we get that J has a linear resolution. □

In the situation of Lemma 6.12 let $J=I(G)$, and let Δ be the quasi-tree whose 1-skeleton is $\overline{G}$, see Theorem 5.8 and Theorem 5.10.

Proposition 6.13. *If $I=(x_{i_1}^2,\dots,x_{i_k}^2,J)$ has a linear resolution, then i_j is a free vertex of Δ for $j=1,\dots,k$, and no two of these vertices belong to the same facet.*

PROOF. We refer to the notation in the proof of Lemma 6.12. Our assumption implies that $\overline{G^*}$ is chordal. Let Δ^* the quasi-tree whose 1-skeleton is $\overline{G^*}$.

Suppose that i_j is not a free vertex of Δ. Then there exist edges $\{i_j,r\}$ and $\{i_j,s\}$ in $\overline{G}$ such that $\{r,s\}$ is not an edge in $\overline{G}$. Then $\{i_j,r\}$ and $\{i_j,s\}$ are also edges in $\overline{G^*}$, and $\{r,s\}$ is not an edge in $\overline{G^*}$. Since $x_{i_j}y_j\in I^*$, it follows that $\{i_j,-j\}$ is not an edge in G^*, and since x_ry_j and x_sy_j do not belong to I^* it follows that $\{-j,r\}$ and $\{-j,s\}$ are edges of $\overline{G^*}$. Thus $\{i_j,r\},\{r,-j\},\{-j,s\},\{s,i_j\}$ is circuit of length 4 with no chords, a contradiction.

Suppose i_j and i_l are free vertices belonging to the same facet of Δ. Then $\{i_j,i_l\}$ is an edge in $\overline{G^*}$, and we also have that $\{i_j,-l\}$, $\{i_l,-j\}$ and $\{-j,-l\}$ are edges of $\overline{G^*}$ since

$x_{i_j}y_l$, $x_{i_l}y_j$ and y_jy_l do not belong to I^*. On the other hand, $\{i_j,-j\}$ and $\{i_l,-l\}$ are not edges of $\overline{G^*}$ since $x_{i_j}y_j$ and $x_{i_l}y_l$ belong to I^*. Therefore $\{i_j,i_l\},\{i_l,-j\},\{-j,-l\},\{-l,i_j\}$ is the circuit of length 4 with no chords, a contradiction. □

Corollary 6.14. *Suppose I has a linear resolution and $x_i^2 \in I$. Then with the numbering of the variables as given in Proposition* 6.11 *the following holds: for all $j > i$ for which there exists k such that $x_kx_j \in I$, one has $x_ix_j \in I$ or $x_ix_k \in I$.*

PROOF. Suppose $x_i^2 \in I$ and there exists a $j > i$ for which there exists k such that $x_kx_j \in I$, but x_ix_j and x_ix_k both do not belong to I. Then $k \neq i$, because $x_i^2 \in I$.

If $k \neq j$, then $\{k, j\}$ is not an edge of Δ, and $\{i, j\}$, $\{i,k\}$ both are edges of Δ. This implies that i is not a free vertex of Δ, contradicting Proposition 6.13.

If $k = j$, then $x_j^2 \in I$ and j is a free vertex of Δ, by Proposition 6.13. But since $x_ix_j \notin I$ we have that $\{i, j\}$ is an edge of Δ. This implies that i and j belong to the same facet, again a contradiction to Proposition 6.13. □

4. Monomial ideals satisfying the x-condition

In the previous section we have seen that if I is a monomial ideal generated in degree 2 which has a linear resolution then it satisfies the conditions $(*)$ and $(**)$ listed in the next theorem.

Theorem 6.15. *Let $I \subset S = K[x_1,\dots,x_n]$ be an ideal which is generated by quadratic monomials and suppose that I possesses the following properties $(*)$ and $(**)$:*

$(*)$ if $x_ix_j \in I$ with $i \neq j$, $k > i$ and $k > j$, then either x_ix_k or x_jx_k belongs to I;

$(**)$ if $x_i^2 \in I$ and $j > i$ for which there is k such that $x_kx_j \in I$, then either $x_ix_j \in I$ or $x_ix_k \in I$.

Let $\mathscr{R}(I) = T/P$ be the Rees ring of I. Then there exists a lexicographic order $<_{lex}$ on T such that the reduced Gröbner basis G of the defining ideal P with respect to $<_{lex}$ consists of binomials $f \in T$ with $\deg_x f \leq 1$.

PROOF. Let Ω denote the finite graph with the vertices $1,\dots,n,n+1$ whose edge set $E(\Omega)$ consists of those edges and loops $\{i, j\}$, $1 \leq i \leq j \leq n$, with $x_ix_j \in I$ together with the edges $\{1,n+1\},\{2,n+1\},\dots,\{n,n+1\}$. Let $K[\Omega] \subset S[x_{n+1}]$ denote the edge ring of Ω studied in, e.g., [**29**] and [**30**].

Thus $K[\Omega]$ is the affine semigroup ring generated by those quadratic monomials x_ix_j, $1 \leq i \leq j \leq n+1$, with $\{i, j\} \in E(\Omega)$. Let $T = K[x_1,\dots,x_n,\{y_{\{i,j\}}\}_{\substack{1\leq i\leq n,1\leq j\leq n\\ \{i,j\}\in E(\Omega)}}]$ be the polynomial ring and define the surjective homomorphism $\pi : T \to K[\Omega]$ by setting $\pi(x_i) = x_ix_{n+1}$ and $\pi(y_{\{i,j\}}) = x_ix_j$. The *toric ideal* of $K[\Omega]$ is the kernel of π. Since the Rees ring $\mathscr{R}(I)$ is isomorphic to the edge ring $K[\Omega]$ in the obvious way, we will identify the defining ideal P of the Rees ring with the toric ideal of $K[\Omega]$.

We introduce the lexicographic order $<_{lex}$ on T induced by the ordering of the variables as follows: (i) $y_{\{i,j\}} > y_{\{p,q\}}$ if either $\min\{i, j\} < \min\{p,q\}$ or ($\min\{i, j\} = \min\{p,q\}$ and $\max\{i, j\} < \max\{p,q\}$) and (ii) $y_{\{i,j\}} > x_1 > x_2 > \cdots > x_n$ for all $y_{\{i,j\}}$. Let G denote the reduced Gröbner basis of P with respect to $<_{lex}$.

It follows (e.g., [**30**, p. 516]) that the Graver basis of P coincides with the set of all binomials f_Γ, where Γ is a primitive even closed walk in Ω. (In [**30**] a finite graph with no loop is mainly discussed. However, all results obtained there are valid for a finite graph allowing loops with the obvious modification.)

Now, let f be a binomial belonging to G and

$$\Gamma = (\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{2m}, w_1\})$$

the primitive even closed walk in Ω associated with f. In other words, with setting $y_{\{i,n+1\}} = x_i$ and $w_{2m+1} = w_1$, one has

$$f = f_\Gamma = \prod_{k=1}^{m} y_{\{w_{2k-1}, w_{2k}\}} - \prod_{k=1}^{m} y_{\{w_{2k}, w_{2k+1}\}}.$$

What we must prove is that, among the vertices $w_1, w_2, \ldots, w_{2m}$, the vertex $n+1$ appears at most one time. Let $y_{\{w_1,w_2\}}$ be the biggest variable appearing in f with respect to $<_{lex}$ with $w_1 \leq w_2$. Let $k_1, k_2, \ldots$ with $k_1 < k_2 < \cdots$ denote the integers $3 \leq k < 2m$ for which $w_k = n+1$.

Case I: Let k_1 be even. Since $\{n+1, w_1\} \in E(\Omega)$, the closed walk

$$\Gamma' = (\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{k_1-1}, w_{k_1}\}, \{w_{k_1}, w_1\})$$

is an even closed walk in Ω with $\deg_x f_{\Gamma'} = 1$. Since the initial monomial $in_{<_{lex}}(f_{\Gamma'}) = y_{\{w_1,w_2\}} y_{\{w_3,w_4\}} \cdots y_{\{w_{k_1-1}, w_{k_1}\}}$ of $f_{\Gamma'}$ divides $in_{<_{lex}}(f_\Gamma) = \prod_{k=1}^{m} y_{\{w_{2k-1}, w_{2k}\}}$, it follows that $f_\Gamma \notin G$ unless $\Gamma' = \Gamma$.

Case II: Let both k_1 and k_2 be odd. This is impossible since Γ is primitive and since the subwalk

$$\Gamma'' = (\{w_1, w_2\}, \ldots, \{w_{k_1-1}, w_{k_1}\}, \{w_{k_2}, w_{k_2+1}\}, \ldots, \{w_{2m}, w_1\})$$

of Γ is an even closed walk in Ω.

Case III: Let k_1 be odd and let k_2 be even. Let C be the odd closed walk

$$C = (\{w_{k_1}, w_{k_1+1}\}, \{w_{k_1+1}, w_{k_1+2}\}, \ldots, \{w_{k_2-1}, w_{k_2}\})$$

in Ω. Since both $\{w_2, w_{k_1}\}$ and $\{w_{k_2}, w_1\}$ are edges of Ω, the closed walk

$$\Gamma''' = (\{w_1, w_2\}, \{w_2, w_{k_1}\}, C, \{w_{k_2}, w_1\})$$

is an even closed walk in Ω and the initial monomial $in_{<_{lex}}(f_{\Gamma'''})$ of $f_{\Gamma'''}$ divides $in_{<_{lex}}(f_\Gamma)$. Thus we discuss Γ''' instead of Γ.

Since Γ''' is primitive and since C is of odd length, it follows that none of the vertices of C coincides with w_1 and that none of the vertices of C coincides with w_2.

(III – a) First, we study the case when there is $p \geq 0$ with $k_1 + p + 2 < k_2$ such that $w_{k_1+p+1} \neq w_{k_1+p+2}$. Let W and W' be the walks

$$\begin{aligned} W &= (\{w_{k_1}, w_{k_1+1}\}, \{w_{k_1+1}, w_{k_1+2}\}, \ldots, \{w_{k_1+p+1}, w_{k_1+p+2}\}), \\ W' &= (\{w_{k_2}, w_{k_2-1}\}, \{w_{k_2-1}, w_{k_2-2}\}, \ldots, \{w_{k_1+p+3}, w_{k_1+p+2}\}) \end{aligned}$$

in Ω.

(III – a – 1) Let $w_1 \neq w_2$. If either $\{w_2, w_{k_1+p+1}\}$ or $\{w_2, w_{k_1+p+2}\}$ is an edge of Ω, then it is possible to construct an even closed walk $\Gamma^\sharp$ in Ω such that $in_{<_{lex}}(f_{\Gamma^\sharp})$ divides

$in_{<_{lex}}(f_{\Gamma'''})$ and $\deg_x f_{\Gamma^\sharp} = 1$. For example, if, say, $\{w_2, w_{k_1+p+2}\} \in E(\Omega)$ and if p is even, then

$$\Gamma^\sharp = (\{w_2, w_1\}, \{w_1, w_{k_2}\}, W', \{w_{k_1+p+2}, w_2\})$$

is a desired even closed walk.

(III – a – 2) Let $w_1 \neq w_2$. Let $\{w_2, w_{k_1+p+1}\} \notin E(\Omega)$ and $\{w_2, w_{k_1+p+2}\} \notin E(\Omega)$. Since $\{w_{k_1+p+1}, w_{k_1+p+2}\}$ is an edge of Ω, by $(*)$ either $w_2 < w_{k_1+p+1}$ or $w_2 < w_{k_1+p+2}$. Let $w_2 < w_{k_1+p+2}$. Since $w_1 < w_2$ and $\{w_1, w_2\} \in E(\Omega)$, again by $(*)$ one has $\{w_1, w_{k_1+p+2}\} \in E(\Omega)$. If p is even, then consider the even closed walk

$$\Gamma^\flat = (\{w_1, w_2\}, \{w_2, w_{k_2}\}, W', \{w_{k_1+p+2}, w_1\})$$

in Ω. If p is odd, then consider the even closed walk

$$\Gamma^\flat = (\{w_1, w_2\}, \{w_2, w_{k_1}\}, W, \{w_{k_1+p+2}, w_1\})$$

in Ω. In each case, one has $\deg_x f_{\Gamma^\flat} = 1$. Since $y_{\{w_1,w_2\}} > y_{\{w_1,w_{k_1+p+2}\}}$, it follows that $in_{<_{lex}}(f_{\Gamma^\flat})$ divides $in_{<_{lex}}(f_{\Gamma'''})$.

(III – a – 3) Let $w_1 = w_2$. Since $w_1 < w_{k_1+p+1}$, by $(**)$ either $\{w_1, w_{k_1+p+1}\} \in E(\Omega)$ or $\{w_1, w_{k_1+p+2}\} \in E(\Omega)$. Thus the same technique as in (III – a – 2) can be applied.

(III – b) Second, if $C = (\{n+1, j\}, \{j, j\}, \{j, n+1\})$, then in each of the cases $w_1 < w_2 < j$, $w_1 < j < w_2$ and $w_1 = w_2 < j$, by either $(*)$ or $(**)$, one has either has $\{w_1, j\} \in E(\Omega)$ or $\{w_2, j\} \in E(\Omega)$. $\square$

As the final conclusion of our considerations we obtain

Theorem 6.16. *Let I be a monomial ideal generated in degree 2. The following conditions are equivalent:*

(i) *I has a linear resolution;*
(ii) *I has linear quotients;*
(iii) *each power of I has a linear resolution.*

PROOF. The implication (iii) $\Rightarrow$ (i) is trivial, while (ii) $\Rightarrow$ (i) is a general fact. It follows from Proposition 6.11 and Corollary 6.14 that if I has a linear resolution, then the conditions $(*)$ and $(**)$ of Theorem 6.15 are satisfied, after a suitable renumbering of the variables. Hence by Corollary 6.10 each power of I has a linear resolution.

It remains to prove (i) $\Rightarrow$ (ii): Again we may assume that the conditions $(*)$ and $(**)$ hold. Let $G(I)$ be the unique minimal set of monomial generators of I. We denote by $[u, v]$ the greatest common divisor of u and v.

We show that the following condition (q) is satisfied: the elements of $G(I)$ can be ordered such that if $u, v \in G(I)$ with $u > v$, then there exists $w > v$ such that $w/[w, v]$ is of degree 1 and $w/[w, v]$ divides $u/[u, v]$. This condition (q) then implies that I has linear quotients.

The squarefree monomials in $G(I)$ will be ordered by the lexicographical order induced by $x_n > x_{n-1} > \cdots > x_1$, and if $x_i^2 \in G(I)$ then we let $u > x_i^2 > v$, where u is the smallest squarefree monomial of the form $x_k x_i$ with $k < i$, and where v is the largest squarefree monomial less than u.

Now, for any two monomials $u, v \in G(I)$ with $u > v$ corresponding to our order, we need to show that property (q) holds. There are three cases:

Case 1: $u = x_s x_t$ and $v = x_i x_j$ both are squarefree monomials with $s < t$ and $i < j$. Since $u > v$, we have $t \geq j$. If $t = j$, take $w = u$. If $t > j$, then by $(*)$, either $x_i x_t \in G(I)$ or $x_j x_t \in G(I)$. Accordingly, let $w = x_i x_t$ or $w = x_j x_t$.

Case 2: $u = x_t^2$ and $v = x_i x_j$ with $i < j$. Since $u > v$, we have $t > j$. Hence by $(*)$, either $x_i x_t \in G(I)$ or $x_j x_t \in G(I)$. Accordingly, let $w = x_i x_t$ or $w = x_j x_t$.

Case 3: $u = x_s x_t$ with $s \leq t$ and $v = x_i^2$. If $t = i$, then $s \neq t$ and take $w = u$. If $t > i$, then by $(**)$, we have either $x_i x_t \in G(I)$ or $x_i x_s \in G(I)$. Both elements are greater than v in our order. Accordingly, let $w = x_i x_t$ or $w = x_i x_s$. Then again (q) holds. □

5. The facet ideals of the complementary of pure skeletons of quasi-trees

Let Δ be a simplicial complex, recall that the simplicial complex $\Delta(i)$ whose facets are the i-dimensional faces of Δ is called the pure i-skeleton of Δ.

Suppose Δ is a pure $(d-1)$-dimensional simplicial complex. We then define

$$\overline{\Delta} = \langle F : F \notin \Delta, \quad |F| = d \rangle.$$

Proposition 6.17. *Let Σ be a flag complex with n vertices, and let Δ and Δ' be the simplicial complexes defined by*

$$I_\Delta = I(\overline{\Sigma(\ell)}) \quad \textit{and} \quad I_{\Delta'} = I(\overline{\Sigma(1)}).$$

Then $\Delta^\vee = (\Delta')^\vee(n-\ell-2)$.

PROOF. By Proposition 2.11 we have $\Delta^\vee = (\overline{\Sigma(\ell)})^c$ and $(\Delta')^\vee = (\overline{\Sigma(1)})^c$. Since Σ is flag, any facet of $\overline{\Sigma(\ell)}$ contains a nonedge of Σ which is a facet of $\overline{\Sigma(1)}$. Therefore, any facet of $\Delta^\vee$ is a face of $(\Delta')^\vee$. It is clear that the facets of $\Delta^\vee$ are all of dimension $n-\ell-2$, so that $\Delta^\vee \subset (\Delta')^\vee(n-\ell-2)$.

On the other hand, for any $(n-\ell-2)$-dimensional face F of $(\Delta')^\vee$ its complementary set F^c contains one nonedge of Σ. Therefore, $F^c \in \overline{\Sigma(\ell)}$ and hence F is a facet of $\Delta^\vee$. □

We quote the following two results relating combinatorial or algebraic properties of a simplicial complex Δ to algebraic properties of the Alexander dual of $\Delta^\vee$.

Theorem 6.18. *Let K be a field, Δ a simplicial complex, I_Δ the Stanley–Reisner ideal and $K[\Delta]$ the Stanley–Reisner ring of Δ. Then*

(i) (Eagon–Reiner [**11**]) $K[\Delta]$ *is Cohen-Macaulay* $\Longleftrightarrow I_{\Delta^\vee}$ *has a linear resolution;*
(ii) (Terai [**37**]) $\operatorname{proj}\dim K[\Delta] = \operatorname{reg}(I_{\Delta^\vee})$*;*
(iii) Δ *is shellable* $\Longleftrightarrow I_{\Delta^\vee}$ *has linear quotients.*

For the convenience of the reader we give the easy proof of statement (iii): recall that Δ is called *shellable* if Δ is pure and there is an order $F_1,\ldots,F_m$ of the facets of Δ (called a *shelling order*), such that for all $0 < j < i$ there exists a vertex $l \in F_i \setminus F_j$ and some $k < i$ with $F_i \setminus F_k = \{l\}$, , while an ideal I is said to have *linear quotients*, if $I = (f_1,\ldots,f_m)$ and for all $i > 0$ the colon ideals $(f_1,\ldots,f_{i-1}) : f_i$ are generated by linear forms.

For a monomial ideal I we require that the f_i belong to the unique minimal set of monomial generators $G(I)$ of I. Then I has linear quotients if for all $i > 1$, and any $j < i$, there exists $k < i$ such that $f_k/[f_i, f_k]$ is a monomial of degree 1, say x_ℓ, and $x_\ell | f_j$. Here $[f_i, f_k]$ denotes the greatest common divisor of f_i and f_k.

By Proposition 2.11 one has $I_{\Delta^\vee} = (x_{F_1^c}, \ldots, x_{F_m^c})$. Hence the equivalence of the statements in (iii) are obvious.

It is well known that $K[\Delta]$ is Cohen-Macaulay for any field K, if Δ is shellable (see for instance [**2**]), and we have seen that an equigenerated linear quotient ideal has a linear resolution.

Corollary 6.19. *Let Σ be a flag complex, and let Δ and Δ' be the simplicial complexes defined by $I_\Delta = I(\overline{\Sigma(\ell)})$ and $I_{\Delta'} = I(\overline{\Sigma(1)})$. Suppose that $I_{\Delta'}$ has linear quotients, then so does I_Δ.*

PROOF. It follows from Theorem 6.18(c) that $(\Delta')^\vee$ is shellable. Since $\Delta^\vee$ is a skeleton of $(\Delta')^\vee$, the following lemma implies that $\Delta^\vee$ is shellable, too. Applying again Theorem 6.18(c), the assertion follows. □

Lemma 6.20. *Let Δ be a shellable complex with* $\dim\Delta = d-1$. *Then for each* $1 \le i < d$ *the pure i-skeleton* $\Delta(i)$ *of* Δ *is shellable.*

PROOF. Let $i < d-1$. Fix a shelling $F_1, \ldots, F_m$ of the facets of Δ. If $m = 1$, i.e., Δ is the simplex on $[n]$, then $\mathscr{F}(\Delta(i)) = \binom{[n]}{i+1}$, and $\Delta(i)$ is shellable. Let $m > 1$ and $\Delta' = \Delta \setminus \{F_m\}$. By using induction on m, we may assume that $\Delta'(i)$ is shellable. Let $V \subset [n]$ denote the set of those $b \in [n]$ such that there is $1 \le s < m$ with $\dim(F_s \cap F_m) = d-2$ and with $F_m \setminus F_s = \{b\}$. It then follows that a subset $G \in \binom{[n]}{i+1}$ belongs to $\mathscr{F}(\Delta(i)) \setminus \mathscr{F}(\Delta'(i))$ if and only if $V \subset G \subset F_m$. Hence the simplicial complex Γ with $\mathscr{F}(\Gamma) = \mathscr{F}(\Delta(i)) \setminus \mathscr{F}(\Delta'(i))$ turns out to be shellable.

Let $G_1, G_2, \ldots, G_p$ be a shelling of the facets of $\Delta'(i)$ and $G_{p+1}, \ldots, G_q$ a shelling of Γ. We claim that $G_1, G_2, \ldots, G_p, G_{p+1}, \ldots, G_q$ is a shelling of $\Delta(i)$. In fact, let $1 \le j \le p < k \le q$ and $G_j \subset F_s$ with $s < m$. Then there is $s' < m$ with $\dim(F_{s'} \cap F_m) = d-2$ such that $F_s \cap F_m \subset F_{s'} \cap F_m$. Let $F_{s'} \setminus F_m = \{a\}$ and $F_m \setminus F_{s'} = \{b\}$. Since $p < k$, one has $b \in G_k$. Let $G_{k'} = (G_k \setminus \{b\}) \cup \{a\}$ with $k' \le p$. Then $G_{k'} \cap G_k = G_k \setminus \{b\} \in \binom{[n]}{i}$. Since $b \notin F_s$, one has $b \notin G_j$. Hence $G_j \cap G_k \subset G_{k'} \cap G_k$, as desired. □

We now consider powers of facet ideals of complementary simplicial complexes of pure skeletons of quasi-forest. We first show that such ideals have linear quotients.

Theorem 6.21. *Let Δ be a quasi-tree of dimension $d-1$. Then $I = I(\overline{\Delta(\ell)})$ has linear quotients for any $\ell \le d-1$. In particular, I has a linear resolution.*

PROOF. Let $I_\Gamma = I$ and $I_{\Gamma'} = I(\overline{\Delta(1)})$. Since by the Lemma 2.31, Δ is flag we have $I_\Delta = I_{\Gamma'}$. Hence by using Theorem 6.16 $I_{\Gamma'}$ has linear quotients. By Corollary 6.19, I has linear quotients, too. □

In [**21**] a certain converse of Theorem 6.21 is shown for $\ell = 1$, namely, that if I is a monomial ideal generated in degree 2 and has linear quotients, then there exists a quasi-forest Δ such that $I = I(\overline{\Delta(1)})$. However, for $\ell > 1$, such a converse is not true: let $\Delta = \langle\{1,2,3\},\{3,4,5\},\{2,4,6\}\rangle$, and $I = I(\overline{\Delta})$. Then I has linear quotients. However, if $I = I(\overline{\Gamma(2)})$, then $\Delta = \Gamma(2)$. In particular, $\dim\Gamma \ge 2$. If $\dim\Gamma > 2$, then $\Gamma(2)$ contains at least 4 facets. But Δ has only 3 facets. Thus $\dim\Gamma = 2$, and hence $\Gamma = \Delta$. But Δ is not a quasi-forest.

The main theorem of this section is the following

Theorem 6.22. *Let Δ be a quasi-tree of dimension $d-1$. Then for any $\ell \leq d-1$, all powers of $I = I(\overline{\Delta(\ell)})$ have linear resolutions.*

To prove this main result, we need the following two lemmata.

Lemma 6.23. *Let $2 \leq \ell \leq d-1$, $I_1 = I(\overline{\Delta(1)})$ and $I_\ell = I(\overline{\Delta(\ell)})$. Then I_ℓ is generated by all squarefree monomials u of degree $\ell+1$ such that u is divided by a monomial generator of I_1.*

PROOF. Let $u = x_F$ be a squarefree monomial of degree $\ell+1$. If u is divided by a monomial generator $x_i x_j$ of I_1, then F contains the 2-element subset $\{i, j\} \notin \Delta$. Thus $F \notin \Delta$ and u is a monomial generator of I_ℓ. Conversely, suppose that u is divided by no monomial generator of I_1. Then each 2-element subset of F is a face of Δ. Since Δ is flag, it follows that F is a face of Δ. Thus $u \notin I_\ell$. □

Given integer vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we write $a \leq b$ if $a_i \leq b_i$ for all i. Let $I \subset S$ be an arbitrary monomial ideal, and $a = (a_1, \ldots, a_n)$ an integer vector with each $a_i \geq 0$. We write $I^{\leq a}$ for the monomial ideal generated by all $u = x^b \in G(I)$ with $b \leq a$. Here $x^b = x_1^{b_1} \cdots x_n^{b_n}$ if $b = (b_1, \ldots, b_n)$.

Lemma 6.24. *Let $I \subset S$ be a monomial ideal,*

$$\mathbb{F}\colon 0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I \longrightarrow 0$$

the multigraded minimal free resolution of I with $F_i = \bigoplus_j S(-q_{ij})$, and $\mathbb{G}$ the subcomplex of $\mathbb{F}$ with

$$G_i = \bigoplus_{q_{ij} \leq a} S(-q_{ij}).$$

Then $\mathbb{G}$ is a multigraded minimal free resolution of $I^{\leq a}$. In particular, if I has a linear resolution, then so does $I^{\leq a}$.

PROOF. It is clear that $H_0(\mathbb{G}) = S/I^{\leq a}$. Thus it remains to show that $\mathbb{G}$ is acyclic. We proceed by induction on the homological degree. Suppose that our claim is true up to homological degree i, and let r be a multihomogeneous element belonging to a minimal set of generator of the kernel of $\mathbb{G}_i \to \mathbb{G}_{i-1}$. Let v be the multidegree of r. It is known [**3**] that $v \leq a$.

Now r belongs to the kernel C of $\mathbb{F}_i \to \mathbb{F}_{i-1}$ as well. Let $\{c_1, \ldots, c_m\}$ be the minimal set of generators of C corresponding to the chosen basis of F_{i+1}. Then $r = \sum_i h_i c_i$ where each $h_i c_i$ has the same multidegree as r. It is then clear $h_i \neq 0$ only if the multidegree of c_i is bounded by a. Hence r belongs to the image of $G_{i+1} \to G_i$, as required. □

PROOF OF THEOREM 6.22. Let $I = I(\overline{\Delta(1)})$ and $J = I(\overline{\Delta(\ell)})$. By Lemma 6.23 it follows that $J = (I_{\langle \ell+1 \rangle})^{\leq (1,\ldots,1)}$, where for some graded ideal L, we denote by $L_{\langle j \rangle}$ the ideal generated by the elements of the j-th graded component of L. Note that $J^k = ((I^k)_{\langle k(\ell+1) \rangle})^{(k,\ldots,k)}$. By [**21**, Theorem 3.2], I^k has a linear resolution. Hence $(I^k)_{\langle k(\ell+1) \rangle}$ has a linear resolution. Then Lemma 6.24 guarantees that J^k has a linear resolution. □

CHAPTER 7

Monomial ideals arising from lattices

In this chapter we associate monomial ideals to finite lattices. The free resolution and the Alexander dual of these monomial ideals are studied.

1. Posets and lattices

In this section we give some fundamental knowledge of poset and lattice.

Definition 7.1. A *partial order* is a relation $\leq$ on a set P such that for all $x,y,z \in P$,

(i) $x \leq x$ (reflexivity);
(ii) $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry);
(iii) $x \leq y$, $y \leq z \Rightarrow x \leq z$ (transitivity).

A *partially ordered set* (*poset* for short) is a set P with a partial order $\leq$.

We use the obvious notation $x \geq y$ to mean $y \leq x$, $x < y$ to mean $x \leq y$ and $x \neq y$, and $x > y$ to mean $y < x$. We say that two elements x and y of P are *comparable* if $x \leq y$ or $y \leq x$; otherwise x and y are *incomparable*. If $x,y \in P$, then we say y *covers* x if $x < y$ and if no element $z \in P$ satisfies $x < z < y$.

A *subposet* of a poset P is a subset Q of P with the induced partial order, i.e., if $\alpha, \beta \in Q$ then $\alpha \leq \beta$ in Q if and only if $\alpha \leq \beta$ in P. A *chain* is a poset in which any two elements are comparable. A subset C of a poset P is called a *chain* if C is a chain when regarded as a subposet of P. The chain C of P is called *saturated* if there does not exist $z \in P \setminus C$ such that $x < z < y$ for some $x,y \in C$ and such that $C \cup \{z\}$ is a chain. In a finite poset, a chain $z_0 < z_1 < \cdots < z_n$ is saturated if and only if z_i covers z_{i-1} for $i \leq 1 \leq n$. The *length* $l(C)$ of a finite chain is defined by $l(C) = |C| - 1$. The *length*(or *rank*) of a finite poset P is $l(P) := \max\{l(C) : C \text{ is a chain of } P\}$. If every maximal chain of P has the same length n, then we say that P is *graded of rank n*. In this case there is a unique *rank function* $\rho : P \to [n]$ such that $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ if y covers x in P. If $\rho(x) = i$, then we say that x has *rank i*.

Definition 7.2. Let P be a poset and $x,y \in P$. If $x > y$ and for any z with $x > z$ we have $z \not> y$, then we say x *covers* y or y is a *lower neighbor* of x or x is a *upper neighbor* of x.

An element in a poset P may have more than one upper neighbor (lower neighbor) or have no upper neighbor (lower neighbor). An element in a poset P which has exactly one lower neighbor is called a *join irreducible element* of P. The set of all join irreducible elements with the induced order is a poset, called the *join irreducible subposet* of P. Conversely, an element in a poset P which has exactly one upper neighbor is called a *meet irreducible element* of P.

Let P be a poset. The poset $\widetilde{P}$ on the same set as P, such that $x \leq y$ in $\widetilde{P}$ if and only if $y \leq x$ in P is called the *dual* of P.

An important class of posets is known as lattice. If x and y belongs to a poset P, then an *upper bound* of x and y is an element $z \in P$ satisfying $z \geq x$ and $z \geq y$. A *least upper bound* of x and y is an upper bound of x and y such that every upper bound w of x and y satisfies $w \geq z$. If a least upper bound of x and y exists, then it is clearly unique and is denoted by $x \vee y$ (read 'x join y'). Dually one can define the greatest lower bound $x \wedge y$ (read 'x meet y').

Definition 7.3. A *lattice* is a poset $\mathscr{L}$ for which each pair of elements has a least upper bound and a greatest lower bound.

One sees immediately from the definition that in a lattice $\mathscr{L}$, there is a unique element u satisfies that $u \geq x$ for any $x \in \mathscr{L}$, call this u the *maximum* of $\mathscr{L}$, denoted by $\hat{1}$; and there is a unique element v satisfies $v \leq x$ for any $x \in L$, call this v the *minimum* of $\mathscr{L}$, denoted by $\hat{0}$. In a lattice every element except $\hat{1}$ (resp. $\hat{0}$) has a upper neighbor (resp. lower neighbor).

It is easy to check that in a lattice $\mathscr{L}$, one has:

(i) the operations $\vee$ and $\wedge$ are associative, commutative, and idempotent (i.e., $x \wedge x = x \vee x = x$);
(ii) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ (absorption laws);
(iii) $x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \leq y$.

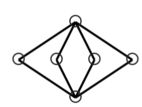
In checking whether a finite poset is a lattice, it is sometimes easy to see that meets, say, exist, but the existence of joins is not so clear. Thus the criterion of the next proposition ([**32**, Proposition 3.3]) can be useful. If every pair of elements of a poset P has a meet (respectively, join), we say that P is a *meet-semilattice* (respectively, *join-semilattice*).

Proposition 7.4. *Let $\mathscr{L}$ be a finite meet-semilattice with $\hat{1}$. Then $\mathscr{L}$ is a lattice. Dually a finite join- semilattice with $\hat{0}$ is a lattice.*

Examples 7.5. (i) Let $n \in \mathbb{N}+$. The set $[n]$ with its usual order forms an n-element lattice with the special property that any two elements are comparable. This lattice is denoted by $\mathbf{n}$. Of course $\mathbf{n}$ and $[n]$ coincide as sets, but we use the notation $\mathbf{n}$ to emphasize the order structure.

(ii) Let $n \in \mathbb{N}$. We can make the set $2^{[n]}$ of all subsets of $[n]$ into a poset $\mathscr{B}_n$ by defining $S \leq T$ in $\mathscr{B}_n$ if $S \subseteq T$ as sets. One says that $\mathscr{B}_n$ consists of the subsets of $[n]$ ordered by inclusion. B_n is a lattice with $\hat{0} = \emptyset$ and $\hat{1} = [n]$, and called the *Boolean lattice of rank n*.

(iii) The following figure is the Hasse diagram of a lattice with 6 elements.



A subposet P of a lattice $\mathscr{L}$ is called a *sublattice* of $\mathscr{L}$ if both $\alpha \wedge \beta$ and $\alpha \vee \beta$ in $\mathscr{L}$ are contained in P for all $\alpha, \beta \in P$.

Definition 7.6. Let $\mathscr{L}$ be a finite lattice. If $\mathscr{L}$ satisfies either of the following two conditions, then we call $\mathscr{L}$ a *finite upper semimodular lattice*.

(i) $\mathscr{L}$ is graded, and the rank function ρ of $\mathscr{L}$ satisfies $\rho(x)+\rho(y) \geq \rho(x\wedge y)+\rho(x\vee y)$ for all $x,y \in \mathscr{L}$.
(ii) If x and y both cover $x\wedge y$, then $x\vee y$ covers both x and y.

The conditions (i) and (ii) in the previous definition are equivalent. One sees the proof, for example, in [**32**, Proposition 3.3.2].

A finite lattice $\mathscr{L}$ whose dual $\widetilde{\mathscr{L}}$ is upper semimodular is called *lower semimodular*. A finite lattice is both upper and lower semimodular is called a *modular lattice*. The following fact follows immediately from the definition of the semimodular lattice.

Fact 7.7. *A finite lattice $\mathscr{L}$ is modular if and only if it is graded and its rank function ρ satisfies*

$$\rho(x)+\rho(y) = \rho(x\wedge y)+\rho(x\vee y) \quad \textit{for all } x,y \in \mathscr{L}.$$

The most important class of lattice from the combinatorial point of view are the distributive lattice.

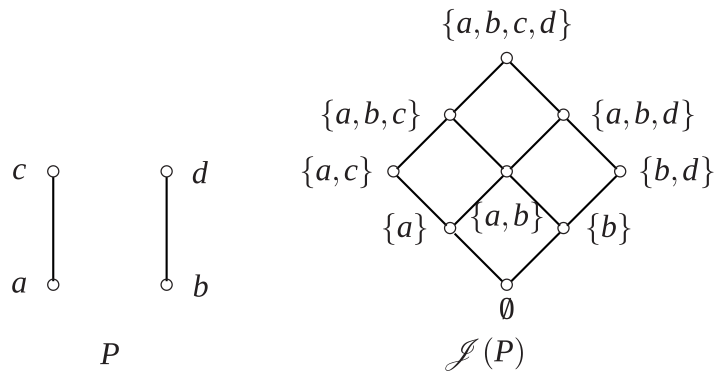
Definition 7.8. A lattice $\mathscr{L}$ is called *distributive* if for all $\alpha,\beta,\gamma \in \mathscr{L}$, we have

(i) $\alpha\wedge(\beta\vee\gamma) = (\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$;
(ii) $\alpha\vee(\beta\wedge\gamma) = (\alpha\vee\beta)\wedge(\alpha\vee\gamma)$.

Remark 7.9. (i) One can prove that either of these two laws implies the other.
(ii) In Example 7.5, (i) and (ii) are distributive lattices, but (iii) is not a distributive lattice.
(iii) Any sublattice of a distributive lattice is distributive. In fact, let P be a sublattice of a distributive lattice $\mathscr{L}$, then by the definition of sublattice, one has both $\alpha\wedge\beta$ and $\alpha\vee\beta$ in $\mathscr{L}$ are contained in P for any $\alpha,\beta \in P$. Hence P satisfies the distributive laws.
(iv) A finite distributive lattice is a finite upper semimodular lattice.

A *poset ideal* of a poset P is a poset I in P such that if $\alpha \in I$ and $\beta < \alpha$, then $\beta \in I$. The maximal elements in I are called the *generators* of I, denoted by $G(I)$. Similarly, a *poset coideal* of P is a poset J in P such that if $\alpha \in I$ and $\beta > \alpha$, then $\beta \in I$. The minimal elements in J are called the *cogenerators* of J, denoted by $G(J)$.

Let P be an arbitrary finite poset. We write $\mathscr{J}(P)$ for the poset which consists of all poset ideals of P ordered by inclusion. For example



Suppose the poset P is an antichain, i.e., any two elements of P are incomparable. Then $\mathscr{J}(P)$ is a Boolean lattice consists of all subset of P ordered by inclusion. We write this Boolean lattice as $\mathscr{B}_P$.

Since the union $I \cup J$ and the intersection $I \cap J$ of poset ideals I and J of P are also poset ideals of P, the poset $\mathscr{J}(P)$ is in fact a lattice. Furthermore, it follows from the well-known distributivity of set union and intersection that the lattice $\mathscr{J}(P)$ is a distributive lattice. Moreover, Birkhoff's fundamental theorem for finite distributive lattice guarantees that the converse of this is also true up to isomorphism.

We say that a poset P is *isomorphic* to a poset Q if there exists a bijection $\theta : P \to Q$ such that $\alpha \leq \beta$ in P if and only if $\theta(a) \leq \theta(b)$ in Q.

Theorem 7.10 (Birkhoff). *Let $\mathscr{L}$ be a finite distributive lattice. Then there exists a unique (up to isomorphism) poset P such that $\mathscr{L}$ is isomorphic to $\mathscr{J}(P)$.*

One finds the proof, for example, in [**32**, Theorem 3.4.1] . In fact, P is the subposet of join-irreducible elements of $\mathscr{L}$.

Definition 7.11. A finite meet-semilattice $\mathscr{L}$ is called *meet-distributive* if each interval $[x,y] = \{p \in \mathscr{L} : x \leq p \leq y\}$ of $\mathscr{L}$ such that x is the meet of the lower neighbors of y in this interval is Boolean.

The following combinatorial characterization of meet-distributive lattices are discussed in the survey article [**10**]. A finite meet-semilattice $\mathscr{L}$ is called *graded* if for each $p \in \mathscr{L}$ all of its maximal chains have the same length.

Theorem 7.12. *For a finite lattice $\mathscr{L}$ the following conditions are equivalent:*

(i) *$\mathscr{L}$ is meet-distributive;*
(ii) *$\mathscr{L}$ is graded and* $\deg \hat{1} = \operatorname{rank} \hat{1}$*;*
(iii) *$\mathscr{L}$ is graded and* $\deg \hat{p} = \operatorname{rank} \hat{p}$ *for all $p \in \mathscr{L}$;*
(iv) *each element in $\mathscr{L}$ is a unique minimal join of join-irreducible elements;*
(v) *$\mathscr{L}$ is lower semimodular, and any upper semimodular sublattice is distributive.*

2. Standard labelling and Hibi ideals

Let $\mathscr{L}$ be a finite lattice and P the join irreducible subposet of $\mathscr{L}$. In this section we introduce a standard labelling ℓ of $\mathscr{L}$ which associates to each element of this lattice a subset of P. We use $H_{\mathscr{L}}$ to denote the ideal generated by the monomials corresponding to these subsets, and show that for an upper semimodular lattice $\mathscr{L}$, the ideal $H_{\mathscr{L}}$ has a linear resolution if and only if $\mathscr{L}$ is a distributive lattice.

Definition 7.13. Let $\mathscr{L}$ be a finite lattice. A map ℓ from $\mathscr{L}$ to a Boolean lattice $\mathscr{B}$ is called a *labelling* of $\mathscr{L}$ if it is an injective and order preserving map. Sometimes we also call a labelling ℓ an *embedding* of $\mathscr{L}$ into $\mathscr{B}$.

Thus if ℓ is a labelling of $\mathscr{L}$, then the subset $\ell(\mathscr{L}) \subset \mathscr{B}$ with the induced order is a subposet of $\mathscr{B}$ which is isomorphic to $\mathscr{L}$.

As we will see in Corollary 7.16 (i) and (ii) the map ℓ in the following definition is a labelling.

Definition 7.14. Let $\mathscr{L}$ be any finite lattice and P the join irreducible subposet of $\mathscr{L}$. We call the map ℓ from $\mathscr{L}$ to $\mathscr{B}_P$ a *standard labelling* if for any $q \in \mathscr{L}$, we set $\ell(q) = \{p : p \leq q \text{ and } p \in P\}$. In particular, $\ell(\hat{0}) = \emptyset$.

For each element $q \in \mathscr{L}$ we call the cardinality of $\ell(q)$ the *degree* of q, i.e., $\deg q = |\ell(q)|$, where ℓ is the standard labelling. In particular, $\deg \hat{0} = 0$ and $\deg \hat{1} = |P|$.

Lemma 7.15. *Let $\mathscr{L}$ be a finite lattice and P the join irreducible subposet of $\mathscr{L}$. Then any element $\hat{0} \neq q \in \mathscr{L}$ is the join of some elements of P.*

PROOF. We prove this assertion by induction on the degree d of q. The case $d = 1$ is clear. Assume $d > 1$. If $q \in P$, then q is the join of all $r \in P$ with $r \leq q$. If $q \notin P$, then q is the join of all its lower neighbors. For each lower neighbor s of q, from the definition of ℓ we have $\deg s = |\ell(s)| < |\ell(q)| = \deg q$. By induction hypothesis, s is the join of some elements of P. Since the operation $\vee$ is associative, we have q is the join of some elements of P. $\square$

Corollary 7.16. *Let $\mathscr{L}$ be a finite lattice, ℓ the standard labelling and $s,t \in \mathscr{L}$ any two elements. We have*

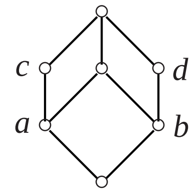
(i) $s = t$ *if and only if* $\ell(s) = \ell(t)$;
(ii) $s \leq t$ *if and only if* $\ell(s) \subseteq \ell(t)$;
(iii) $\ell(s) \cap \ell(t) = \ell(s \wedge t)$;
(iv) $\deg s + \deg t \leq \deg(s \wedge t) + \deg(s \vee t)$.

PROOF. (i) $\Rightarrow$ is clear. Assume $\ell(s) = \ell(t)$. By the previous lemma, $s = \vee_i p_i$ where $p_i \in P$ and $t = \vee_j p_j$ where $p_j \in P$. Since $\ell(s) = \{p_i : s = \vee_i p_i\}$, $\ell(t) = \{p_j : t = \vee_j p_j\}$ and $\ell(s) = \ell(t)$, we have $\vee_i p_i = \vee_j p_j$, i.e., $s = t$.

(ii) and (iii) are clear by the definition of ℓ and the previous lemma.

(iv) Since $s < s \vee t$ and $t < s \vee t$, by (ii) we have $\ell(s) \subset \ell(s \vee t)$ and $\ell(t) \subset \ell(s \vee t)$. Hence $\ell(s) \cup \ell(t) \subseteq \ell(s \vee t)$. So together with (iii) we have $|\ell(s \vee t)| \geq |\ell(s) \cup \ell(t)| = |\ell(s)| + |\ell(t)| - |\ell(s) \cap \ell(t)| = |\ell(s)| + |\ell(t)| - |\ell(s \wedge t)|$. Hence $\deg s + \deg t - \deg(s \wedge t) \leq \deg(s \vee t)$ as required. $\square$

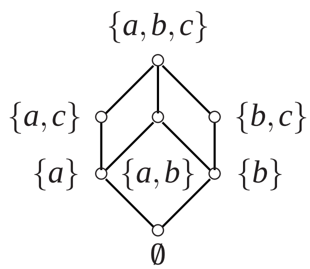
For a finite lattice $\mathscr{L}$, however, there may exist different labellings. For example, for the following lattice $\mathscr{L}$



$\mathscr{L}$

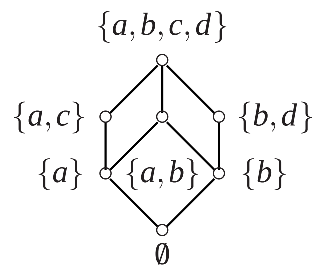
Figure (A)

there exist two labellings ℓ_0 and ℓ as in (I) and (II), such that $\ell_0(\mathscr{L})$ is a subposet of the Boolean lattices $\mathscr{B}_{\{a,b,c\}}$ and $\ell(\mathscr{L})$ is a subposet of $\mathscr{B}_{\{a,b,c,d\}}$. The labelling ℓ is a standard labelling, but ℓ_0 is not.



$\ell_0(\mathscr{L})$

(I)



$\ell(\mathscr{L})$

(II)

Even though a labelling of a lattice $\mathscr{L}$ yields an embedding of $\mathscr{L}$ into a Boolean lattice, the image need not to be a sublattice of the Boolean lattice. In fact, we have:

Lemma 7.17. *Let $\mathscr{L}$ be a finite lattice, P the join irreducible subposet of $\mathscr{L}$, and ℓ the standard labelling of $\mathscr{L}$. Then $\mathscr{L}$ is distributive if and only if $\ell(\mathscr{L})$ is a sublattice of $\mathscr{B}_P$.*

PROOF. $\Leftarrow$: See Remark 7.9(iii).

$\Rightarrow$: If $\mathscr{L}$ is a distributive lattice, then by Birkhoff's theorem, $\mathscr{L} \cong \mathscr{J}(P) = \ell(\mathscr{L})$. Since in $\mathscr{J}(P)$, the join and the meet of any two elements are in fact union and intersection of two sets, and the same holds in the Boolean lattice $\mathscr{B}_P$. We have for any two elements I and J in $\ell(\mathscr{L})$, $I \wedge J = I \cup J$ and $I \vee J = I \cap J$ in $\mathscr{B}_P$ are contained in $\ell(\mathscr{L})$. Hence $\ell(\mathscr{L})$ is a sublattice of $\mathscr{B}_P$. □

Definition 7.18. Let $\mathscr{L}$ be any finite lattice, P the join irreducible subposet of $\mathscr{L}$. We call the finite poset $\mathscr{J}(P)$ the *distributive closure* of $\mathscr{L}$, and denote by $\hat{\mathscr{L}}$.

With the notation introduced, let $S = K[\{x_p, y_p\}_{p\in P}]$ be the polynomial ring in $2|P|$ variables over a field K. We associate each element q of $\mathscr{L}$ with the squarefree monomial

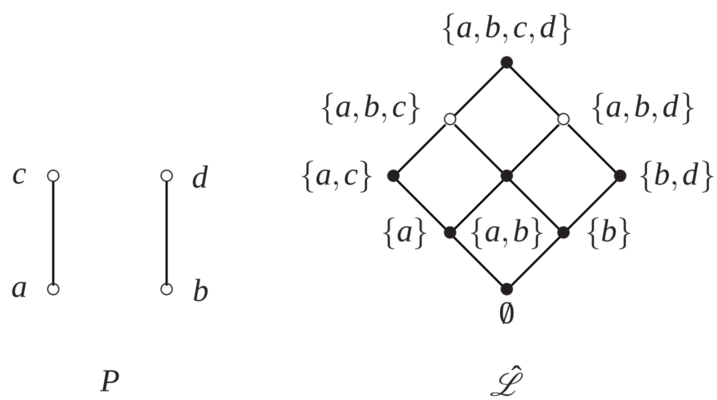
$$u_q = (\prod_{p\in\ell(q)} x_p)(\prod_{p\in P\setminus\ell(q)} y_p)$$

in S. In particular, $u_{\hat{1}} = \prod_{p\in P} x_p$ and $u_{\hat{0}} = \prod_{p\in P} y_p$. Note that $\deg u_q = |P|$ for all $q \in \mathscr{L}$.

Definition 7.19. Let $\mathscr{L}$ be any finite lattice. The ideal $H_{\mathscr{L}} = (\{u_q\}_{q\in\mathscr{L}})$ in S is called the *Hibi ideal* of $\mathscr{L}$.

Since $\hat{\mathscr{L}} = \mathscr{J}(P)$, each element of $\hat{\mathscr{L}}$ is a subset of P. There exist a nature map $\hat{\ell}$ from $\hat{\mathscr{L}}$ to $\mathscr{B}_P$. Let τ be the map from $\mathscr{L}$ to $\hat{\mathscr{L}}$, which maps each element $q \in \mathscr{L}$ to the set $\{p : p \in P \text{ and } p \leq q \text{ in } \mathscr{L}\}$. This τ is well defined since $\{p : p \in P \text{ and } p \leq q \text{ in } \mathscr{L}\}$ is a poset ideal of P. By the definition of ℓ, we have $\ell = \hat{\ell} \circ \tau$.

We recall the notation H_P introduced in [**19**]. There the authors associate a squarefree monomial ideal $H_P = (\{u_I\}_{I\in\mathscr{J}(P)})$ to a poset P, where $u_I = (\prod_{p\in I} x_p)(\prod_{p\in P\setminus I} y_p)$. If P is the join irreducible subposet of a finite lattice $\mathscr{L}$, we identify the Hibi ideal $H_{\hat{\mathscr{L}}}$ of $\hat{\mathscr{L}}$ with $H_{\mathscr{J}(P)}$. For example, the lattice $\mathscr{L}$ in Figure (A) has the join irreducible poset P and the distributive closure $\hat{\mathscr{L}} = \mathscr{J}(P)$ as following:



The embedded subposet $\mathscr{L}$ is indicated by the bullet vertices.

The Hibi ideal of $\mathscr{L}$ is

$$H_{\mathscr{L}} = (y_ay_by_cy_d, x_ay_by_cy_d, x_ax_by_cy_d, x_by_ay_cy_d, x_ax_cy_by_d, x_bx_dy_ay_c, x_ax_bx_cx_d).$$

It is a subset of $H_{\hat{\mathscr{L}}}$. In general, we have:

Remark 7.20. *Let $\mathscr{L}$ be a finite lattice. Then $H_{\mathscr{L}} \subseteq H_{\hat{\mathscr{L}}}$.*

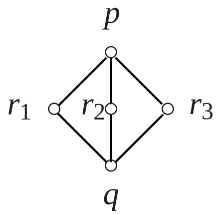
Proposition 7.21. *Let $\mathscr{L}$ be any finite lattice. Then $\mathscr{L}$ is distributive if and only if $\mathscr{L} \cong \hat{\mathscr{L}}$. Hence $\mathscr{L}$ is distributive if and only if $H_{\mathscr{L}} = H_{\hat{\mathscr{L}}}$.*

PROOF. $\Leftarrow$ is clear, since $\hat{\mathscr{L}} = \mathscr{J}(P)$ is distributive, where P is the join irreducible subposet of $\mathscr{L}$.

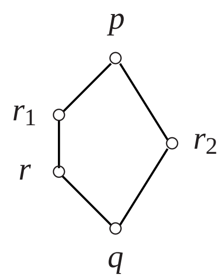
$\Rightarrow$: Assume $\mathscr{L}$ is a distributive lattice. By the proof of Birkhoff's theorem, we have $\mathscr{L} \cong \mathscr{J}(P)$, where P is the join irreducible subposet of $\mathscr{L}$. By definition $\hat{\mathscr{L}} = \mathscr{J}(P)$, hence $\mathscr{L} \cong \hat{\mathscr{L}}$. □

The following lemma is well known in lattice theory, we refer the reader to [**1**] to see the proof.

Lemma 7.22. *A finite lattice $\mathscr{L}$ is distributive if and only if $\mathscr{L}$ does not contain the following two lattice as sublattice*



$\mathscr{L}_a$



$\mathscr{L}_b$

Furthermore, we have the following fact, see e.g., in [**35**].

Fact 7.23. *Let $\mathscr{L}$ be a finite lattice. Then*

(i) *$\mathscr{L}$ is a modular lattice if and only if $\mathscr{L}$ does not contain a sublattice isomorphic to the lattice $\mathscr{L}_b$ in the previous lemma;*
(ii) *$\mathscr{L}$ is a distributive lattice if and only if $\mathscr{L}$ is a modular lattice and does not contain a sublattice isomorphic to the lattice $\mathscr{L}_a$ in the previous lemma.*

As in Definition 7.14, the degree of an element q in a finite lattice $\mathscr{L}$ is $\deg q = |\ell(q)|$, where ℓ is a standard labelling of $\mathscr{L}$. For an upper semimodular lattice, since it is graded, it has a unique rank function $\rho : \mathscr{L} \to [n]$, where n is the rank of $\mathscr{L}$. The following lemma gives a description of an upper semimodular lattice to be distributive by using the degree and rank function:

Lemma 7.24. *Let $\mathscr{L}$ be an upper semimodular lattice and ρ the unique rank function. Then the following statements are equivalent:*

(i) *$\mathscr{L}$ is a distributive lattice;*
(ii) *$\rho(c) = \deg c$ for all $c \in \mathscr{L}$, i.e., for any $c, d \in \mathscr{L}$ where d is a lower neighbor of c, we have $\ell(d) \subset \ell(c)$ and $|\ell(c)| - |\ell(d)| = 1$.*

PROOF. (i) $\Rightarrow$ (ii): By Lemma 7.17, $\ell(\mathscr{L})$ is a sublattice of the Boolean lattice $\mathscr{B}_P$, where P is the join irreducible subposet of $\mathscr{L}$ and ℓ is the standard labelling. It is clear that for any element c in $\mathscr{B}_P$ we have $\rho(c) = \deg c$. Since $\mathscr{L}$ is isomorphic to the sublattice $\ell(\mathscr{L})$ of $\mathscr{B}_P$, the equality also holds for any element in $\mathscr{L}$.

(ii) $\Rightarrow$ (i): Suppose $\mathscr{L}$ is not distributive. Then there are two cases:

Case 1. $\mathscr{L}$ is a modular lattice. Then by Fact 7.23, it contains $\mathscr{L}_a$ (as in Lemma 7.22) as a sublattice. Hence we have $r_1 \wedge r_2 \wedge r_3 = r_i \wedge r_j = q$ and $r_1 \vee r_2 \vee r_3 = r_i \vee r_j = p$ in $\mathscr{L}$ for any $i, j \in \{1,2,3\}$ and $i \neq j$. By Fact 7.7,

$$\rho(r_1) + \rho(r_2) = \rho(r_1 \wedge r_2) + \rho(r_1 \vee r_2) = \rho(q) + \rho(p).$$

By the assumption $\rho(c) = \deg c$ for all $c \in \mathscr{L}$, we have

$$\deg p = \deg r_1 + \deg r_2 - \deg q. \quad (\sharp)$$

On the other hand, since $r_i < p$ for $i = 1,2,3$, by Corollary 7.16 (ii) we have $\ell(p) \supseteq \ell(r_1) \cup \ell(r_2) \cup \ell(r_3)$. Hence by using also Corollary 7.16(iii), we have

$$\begin{aligned} \deg p &= \ell(p) \geq |\ell(r_1) \cup \ell(r_2) \cup \ell(r_3)| \\ &= |\ell(r_1)| + |\ell(r_2)| + |\ell(r_3)| - |\ell(r_1) \cap \ell(r_2)| - |\ell(r_2) \cap \ell(r_3)| \\ &\quad -|\ell(r_3) \cap \ell(r_1)| + |\ell(r_1) \cap \ell(r_2) \cap \ell(r_3)| \\ &= \deg r_1 + \deg r_2 + \deg r_3 - 2\deg q. \end{aligned}$$

Since $r_3 > q$, again by Corollary 7.16 (ii), we have $\deg r_3 > \deg q$. Hence $\deg p > \deg r_1 + \deg r_2 - \deg q$, this contradicts ($\sharp$).

Case 2. $\mathscr{L}$ is not a modular lattice. Since $\mathscr{L}$ is an upper semimodular lattice, by Fact 7.7 there exist elements $c, d \in \mathscr{L}$, such that $\rho(c) + \rho(d) > \rho(c \wedge d) + \rho(c \vee d)$. Again by our assumption $\rho(c) = \deg c$ for all $c \in \mathscr{L}$, we have $\deg c + \deg d > \deg(c \wedge d) + \deg(c \vee d)$. But this contradicts that ℓ is a standard labelling (see Corollary 7.16(iv)).

Hence we conclude that $\mathscr{L}$ is a distributive lattice. □

Now we state the main result of this section.

Theorem 7.25. *Let $\mathscr{L}$ be an upper semimodular lattice and ℓ the standard labelling. Then the following statements are equivalent:*

(i) *$\mathscr{L}$ is a distributive lattice.*
(ii) *The Hibi ideal $H_{\mathscr{L}}$ of $\mathscr{L}$ has a linear resolution.*

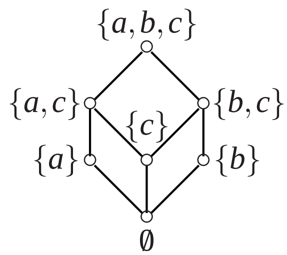
PROOF. (i) $\Rightarrow$ (ii): If $\mathscr{L}$ is a distributive lattice, then by Proposition 7.21, we have $H_{\mathscr{L}} = H_{\hat{\mathscr{L}}} = H_P$, where P is the join irreducible subposet of $\mathscr{L}$. By [**19**, Corollary 1.3], it has a linear resolution.

(ii) $\Rightarrow$ (i): Suppose $\mathscr{L}$ is not a distributive lattice. By Lemma 7.24, there exists an element c and one of its lower neighbor d, such that $|\ell(c)| - |\ell(d)| \geq 2$. Now we restrict $H_{\mathscr{L}}$ on those x_p, where $p \in P \setminus \ell(c)$, and those y_q, where $q \in \ell(d)$, we get the ideal $J = (u_c, u_d)$. Here, by restrict a squarefree ideal I on a valuable x, we mean the ideal generate by those generators of I which do not divide x. Since $H_{\mathscr{L}}$ has a linear resolution, by the previous restrict lemma, J has a linear resolution too. But it is obvious that $J = (u_c, u_d)$ has no linear resolution, since $|\ell(c)| - |\ell(d)| \geq 2$. A contradiction. □

Remark 7.26. We see from the previous theorem that for any finite lattice $\mathscr{L}$, if the Hibi ideal has no linear resolution, then $\mathscr{L}$ is not a distributive lattice. For an upper semimodular lattice, Theorem 7.25 holds only for standard labelling. As we have seen the lattice $\mathscr{L}$ in Figure (A) is an upper semimodular and the ideal

$$H_0 = (y_a y_b y_c, x_a y_b y_c, x_a x_b y_c, x_b y_a y_c, x_a x_c y_b, x_b x_c y_a, x_a x_b x_c)$$

has a linear resolution, but $\mathscr{L}$ is not distributive, since ℓ_0 is not a standard labelling. But without the assumption that $\mathscr{L}$ is an upper semimodular lattice, even if the Hibi ideal has a linear resolution, $\mathscr{L}$ might not be distributive. The lattice $\mathscr{L}$ with the standard labelling in the following figure is a counter example. The Hibi ideal $H_{\mathscr{L}}$ of $\mathscr{L}$ has a linear resolution, but $\mathscr{L}$ is not distributive.



$\ell(\mathscr{L})$

However, this lattice $\mathscr{L}$ is a lower semimodular lattice. Let $\widetilde{\mathscr{L}}$ be the dual of $\mathscr{L}$. Then $\widetilde{\mathscr{L}}$ is an upper semimodular lattice. For each element q of $\widetilde{\mathscr{L}}$, let $\ell_0(q) = P \setminus \ell(q)$, where P is the join irreducible subposet of $\mathscr{L}$. Then the ideal

$$H_0 = (y_a y_b y_c, x_a y_b y_c, x_a x_b y_c, x_b y_a y_c, x_a x_c y_b, x_b x_c y_a, x_a x_b x_c)$$

has a linear resolution. But $\widetilde{\mathscr{L}}$ is not distributive, since ℓ_0 is not a standard labelling of $\widetilde{\mathscr{L}}$ (as in Figure I). In fact, P is not the irreducible subposet of $\widetilde{\mathscr{L}}$.

3. Algebraic characterizations of meet-distributive meet-semilattices

In this section we will introduce the Hibi ideal of a meet-semilattice, and give some Algebraic characterizations of meet-distributive meet-semilattices. This is a generalization of Theorem 7.25.

Let $\mathscr{L}$ be an arbitrary finite meet-semilattice, and $P \subset \mathscr{L}$ the set of join-irreducible elements of $\mathscr{L}$.

As in the previous section, to each element $p \in \mathscr{L}$ we associate the subset $\ell(p) = \{q \in P \colon q \leq p\}$ of P. Note that $p \in \ell(p)$ if and only if p is join irreducible. In any case, $\ell(p)$ is a poset ideal of P.

Remark 7.27. Let $I \subset P$. Then the following conditions are equivalent:

(i) I is a poset ideal (coideal) in P;
(ii) $P \setminus I$ is a poset coideal (ideal) of P;
(iii) $\widetilde{I}$ is a poset coideal (ideal) of $\widetilde{P}$.

As a consequence of Remark 7.27 we have the following

Lemma 7.28. $\widetilde{\mathscr{L}} \cong \mathscr{J}(\widetilde{P})$.

PROOF. Let $q \in \widetilde{\mathscr{L}}$. Since the underlying set of $\widetilde{\mathscr{L}}$ is the same as that of $\mathscr{L}$, we may apply $\ell \colon \mathscr{L} \to \mathscr{J}(P)$ to q. Then $\widetilde{\mathscr{L}} \to \mathscr{J}(\widetilde{P})$, $q \mapsto P \setminus \ell(q)$ is the desired isomorphism. $\square$

Lemma 7.29. *Let $\mathscr{L}$ be a lattice. Then* $\text{height}(H_{\mathscr{L}}) = 2$.

PROOF. It is clear that $H_{\mathscr{L}} \subset (x_p, y_p)$ for any $p \in P$, i.e., $\text{height}(H_{\mathscr{L}}) \leq 2$. While on the other hand $u_{\hat{0}} = \prod_{p \in P} y_p$ and $u_{\hat{1}} = \prod_{p \in P} x_p$ both belong to $H_{\mathscr{L}}$ and have no common factor, $\text{height}(H_{\mathscr{L}}) \geq 2$. Hence $\text{height}(H_{\mathscr{L}}) = 2$. $\square$

Let I be a monomial ideal with the (unique) minimal set $G(I) = \{u_1, \ldots, u_m\}$ of monomial generators. If I is squarefree, then I has linear quotients if and only if for each i and each $j < i$ there exists $k < i$ such that $u_k/[u_k, u_i]$ is a variable and divides u_j. Here $[u, v]$ denotes the greatest common divisor of u and v.

In Section 7.1 we defined the rank of an element in a graded poset. More generally, let $\mathscr{L}$ be any poset and $p \in \mathscr{L}$. The *rank* of p is the maximal length of chains descending from p.

We now come to our algebraic characterization of meet-distributive meet-semilattices.

Theorem 7.30. *Let $\mathscr{L}$ be an arbitrary finite meet-semilattice. The following conditions are equivalent:*

(i) *$\mathscr{L}$ is meet-distributive;*
(ii) *$H_{\mathscr{L}}$ has linear quotients;*
(iii) *$H_{\mathscr{L}}$ has a linear resolution;*
(iv) *$H_{\mathscr{L}}$ has linear relations.*

PROOF. (i) $\Rightarrow$ (ii): We fix a linear order $\prec$ on $\mathscr{L}$ which extends the partial order given by the degree. We put $u_r < u_q$ if $r \prec q$. For any $u_q \in H_{\mathscr{L}}$ and any $u_r < u_q$, let t be a lower neighbor of q in the interval $[r \wedge q, q]$. Then $u_t/[u_t, u_q] = y_p$, where $\{p\} = \ell(q) \setminus \ell(t)$. We claim that y_p divides u_r. If not, then x_p divides u_r and so $p \in \ell(r) \cap \ell(q) = \ell(r \wedge q)$. Thus $p \in \ell(t)$, since $r \wedge q \leq t$.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (i): Suppose $\mathscr{L}$ is not meet-distributive. Then equivalence of (iii) and (i) of Theorem 7.12 (which is also valid if $\mathscr{L}$ is a meet-distributive meet-semilattice) there exist $p, q \in \mathscr{L}$ such that q is lower neighbor of p and $\deg p - \deg q > 1$. The ideal (u_p, u_q) is generated by precisely those monomials in $G(H_{\mathscr{L}})$ which are not divided by x_r for all $r \in P \setminus \ell(p)$, and are not divided by all y_s for all $s \in \ell(q)$. Since we assume that $H_{\mathscr{L}}$ has linear relations, Lemma 6.24 implies that (u_p, u_q) has linear relations contradicting the fact that $\deg p - \deg q > 1$. □

As a corollary, we obtain the result of Theorem 7.25:

Corollary 7.31. *Let $\mathscr{L}$ be a finite upper semimodular lattice. Then the following conditions are equivalent:*

(i) *$H_{\mathscr{L}}$ has a linear resolution;*
(ii) *$\mathscr{L}$ is distributive.*

PROOF. The assertion follows from Theorem 7.12(v) and Theorem 7.30. □

Recall some facts in the Section 2.1. For $F = \{i_1, \ldots, i_k\} \subset [n]$ set $P_F = (x_{i_1}, \ldots, x_{i_k})$, and let Γ be the unique simplicial complex such that $I_\Delta = I(\Gamma)$. Then

$$I_\Delta = \bigcap_{F \in \mathscr{C}(\Gamma)} P_F \quad \text{and} \quad I_{\Delta^\vee} = (x_F : F \in \mathscr{C}(\Gamma)). \tag{14}$$

Set $F^c = [n] \setminus F$, and

$$\Delta^c = \langle F^c \colon F \in \mathscr{F}(\Delta) \rangle.$$

Then

$$I_{\Delta^\vee} = I(\Delta^c). \tag{15}$$

Theorem 7.30 together with Theorem 6.18 yields

Corollary 7.32. *Let $\mathscr{L}$ be an arbitrary finite meet-semilattice, and let $\Delta_{\mathscr{L}}$ be the simplicial complex whose Stanley–Reisner ideal is $H_{\mathscr{L}}$. The following conditions are equivalent:*

(i) *$(\Delta_{\mathscr{L}})^\vee$ is shellable;*
(ii) *$(\Delta_{\mathscr{L}})^\vee$ is Cohen–Macaulay;*
(iii) *$\mathscr{L}$ is meet-distributive.*

Proposition 7.33. *Let $\mathscr{L}$ be a finite lattice, P the set of join irreducible elements of $\mathscr{L}$. Then*

(i) *the minimal prime ideals of height 2 of $H_{\mathscr{L}}$ are (x_p, y_q) where $p,q \in P$ and $p \leq q$;*
(ii) *$H_{\mathscr{L}}$ has only height 2 minimal prime ideals if and only if $\mathscr{L}$ is distributive.*

PROOF. Let $\hat{\mathscr{L}}$ be the distributive closure of $\mathscr{L}$. Then ℓ induces an injective order preserving map $\ell\colon \mathscr{L} \to \hat{\mathscr{L}}$. Thus $H_{\mathscr{L}} \subset H_{\hat{\mathscr{L}}}$, and equality holds if and only if $\mathscr{L}$ is distributive. This follows from Birkhoff's theorem (see in Section 7.1).

(i) The minimal prime ideals of $H_{\hat{\mathscr{L}}}$ are precisely the ideals (x_p, y_q) where $p,q \in P$ and $p \leq q$, see [**19**]. Of course these are also minimal prime ideals of $H_{\mathscr{L}}$. We claim that there are no other minimal prime ideals of height 2 of $H_{\mathscr{L}}$. Indeed, any such ideal must contain some x_p and some y_q, since $\prod_{p\in P} x_p$ and $\prod_{p\in P} y_p$ belong to $H_{\mathscr{L}}$. Suppose $p \not\leq q$, then u_q is not contained in (x_p, y_q).

(ii) It remains to show that if $\mathscr{L}$ is not distributive, then there exists a minimal prime ideal of $H_{\mathscr{L}}$ of height > 2. In fact, the proof of (i) shows that if such a minimal prime ideal does not exist, then $H_{\mathscr{L}} = H_{\hat{\mathscr{L}}}$. Therefore $\mathscr{L} = \hat{\mathscr{L}}$, and hence $\mathscr{L}$ is distributive. □

Proposition 7.33 together with (14) implies

Corollary 7.34. *A finite lattice $\mathscr{L}$ is distributive if and only if $(\Delta_{\mathscr{L}})^\vee$ is flag.*

4. A free resolution of $H_{\mathscr{L}}$

The main theorem of the present section is the following:

Theorem 7.35. *Let $\mathscr{L}$ be finite meet-semilattice.*

(i) *There exists a finite multigraded free S-resolution $\mathbb{F}$ of $H_{\mathscr{L}}$ such that for each $i \geq 0$, the free module F_i has a basis with basis elements*

$$b(p;S)$$

where $p \in \mathscr{L}$ and S is a subset of the set of lower neighbors $N(p)$ of p with $|S| = i$.

The multidegree of $b(p;S)$ is the least common multiple of u_p and all monomials u_q with $q \in S$.

(ii) *The following conditions are equivalent:*

(a) *the resolution constructed in* (i) *is minimal;*

(b) *for any* $p \in \mathscr{L}$ *and any proper subset* $S \subset N(p)$ *the meet* $\bigwedge\{q\colon q \in S\}$ *is strictly greater than the meet* $\bigwedge\{q\colon q \in N(p)\}$.

We call a finite meet-semilattice satisfying condition (ii)(b) *meet-irredundant*.

PROOF OF THEOREM 7.35. (i) The resolution will be built by an iterated mapping cone construction. As in the proof of Theorem 7.30 we fix a linear order $\prec$ on $\mathscr{L}$ which extends the partial order given by the degree. For any p in $\mathscr{L}$ we construct inductively a complex $\mathbb{F}(p)$ which is a multigraded free S-resolution of the ideal $H_{\mathscr{L}}(p)$ generated by all $u_q \in H_{\mathscr{L}}$ with $q \preceq p$. Then $\mathbb{F}(q)$ is the desired resolution, where $q \in \mathscr{L}$ is the maximal element with respect to $\prec$.

The complex $\mathbb{F}(\hat{0})$ is defined as $F_i(\hat{0}) = 0$ for $i > 0$, and $F_0(\hat{0}) = S$. This complex together with the augmentation map $\varepsilon\colon S \to H_{\mathscr{L}}(\hat{0})$, $1 \mapsto u_{\hat{0}}$ is a free resolution of $H_{\mathscr{L}}(\hat{0})$.

Now let $p \in \mathscr{L}$, $p \neq \hat{0}$, and let $q \in \mathscr{L}$, $q \prec p$ be the element preceding p. Then $H_{\mathscr{L}}(p) = (H_{\mathscr{L}}(q), u_p)$, and hence we get an exact sequence of multigraded S-modules

$$0 \longrightarrow (S/L)(-\operatorname{multideg} u_p) \longrightarrow S/H_{\mathscr{L}}(q) \longrightarrow S/H_{\mathscr{L}}(p) \longrightarrow 0,$$

where L is the colon ideal $H_{\mathscr{L}}(q) : u_p$. As in the proof of 7.30 one shows that

$$L = (\{u_t/[u_t, u_p]\}_{t \in N(p)}).$$

Let $\mathbb{T}$ be the Taylor complex associated with the monomials $u_t/[u_t, u_p]$, $t \in N(p)$. Then $\mathbb{T}$ is a multigraded free resolution of S/L with $T_0 = S$, $T_1 = \bigoplus_{t \in N(p)} Se_t$ and $T_i = \bigwedge^i T_0$ for $i \geq 1$. Thus T_i has a basis whose elements are $e_{t_1} \wedge e_{t_2} \wedge \ldots \wedge e_{t_i}$ with $t_1 < t_2 < \cdots < t_i$. The multidegree of $e_{t_1} \wedge e_{t_2} \wedge \cdots \wedge e_{t_i}$ is the least common multiple of the elements $u_{t_j}/[u_{t_j}, u_p]$, $j = 1, \ldots, i$.

The shifted complex

$$\mathbb{T}(-\operatorname{multideg} u_p)$$

is a multigraded free resolution of $(S/L)(-\operatorname{multideg} u_p)$. We denote the basis element of $T_i(-\operatorname{multideg} u_p)$ which corresponds to $e_{t_1} \wedge e_{t_2} \wedge \cdots \wedge e_{t_i}$ by $b(p; \{t_1, \ldots, t_i\})$. Then $\operatorname{multideg} b(p; t_1, \ldots, t_i) = \operatorname{multideg} u_p + \operatorname{multideg} e_{t_1} \wedge e_{t_2} \wedge \cdots \wedge e_{t_i}$, and hence it is the least common multiple of $u_p, u_{t_1}, \ldots, u_{t_i}$.

The monomorphism $(S/L)(-\operatorname{multideg} u_p) \to S/H_{\mathscr{L}}(q)$ induces a comparison map

$$\alpha\colon \mathbb{T}(-\operatorname{multideg} u_p) \longrightarrow \mathbb{F}(q)$$

of multigraded complexes. We let $\mathbb{F}(p)$ be the mapping cone of α. Then $\mathbb{F}(p)$ is a multigraded free S-resolution of $H_{\mathscr{L}}(p)$, and has the desired multigraded basis.

(ii) (a) $\Rightarrow$ (b): Let $p \in \mathscr{L}$ with $|N(p)| > 1$, and let $S \subset N(p)$ be a subset. By the definition of the differential ∂ of $\mathbb{F}$ we have

$$\partial b(p; S) = \sum_{q \in S} \pm v_q b(p; S \setminus \{q\}) + \cdots$$

where $v_q = \operatorname{multideg} b(p; S)/\operatorname{multideg} b(p; S \setminus \{q\})$. Therefore the resolution can be minimal only if the multidegree of $b(p; S \setminus \{q\})$ is a proper divisor the multidegree of $b(p; S)$ for all q in S.

By (i)

$$\operatorname{multideg} b(p; S) = x_A y_B \quad \text{and} \quad \operatorname{multideg} b(p; S \setminus \{q\}) = x_A y_C,$$

where $A = \ell(p)$, $B = \ell(p)^c \cup \bigcup_{r\in S} \ell(r)^c$ and $C = \ell(p)^c \cup \bigcup_{r\in S, r\neq q} \ell(r)^c$. Here, for any subset $F \subset P$, we set $F^c = P \setminus F$.

It follows that $v_q = 1$ if and only if $\bigcap_{r\in S} \ell(r) = \bigcap_{r\in S, r\neq q} \ell(r)$. By Lemma 7.16(iii) this is equivalent to say that

$$\bigwedge\{r\colon r\in S\} = \bigwedge\{r\colon r\in S, r\neq q\}. \tag{16}$$

Hence if the resolution is minimal, then we do not have equality in (16) for any $S \subset N(p)$ and any $q \in S$. In particular, for $S = N(p)$ we obtain the desired result.

(b) $\Rightarrow$ (a): Let $b(p;S)$ and $b(q;T)$ be two basis elements with $|T| = |S| - 1$. It suffices to show that in the following three cases the coefficient of $b(q;T)$ in $\partial b(p;S)$ is either 0 or a monomial $\neq 1$:

- $p = q$ and $T \not\subset S$;
- $q < p$;
- $q \not< p$.

In the first case we show that multideg $b(p;T)$ does not divide multideg $b(p;S)$. Otherwise we would have that $\bigcup_{r\in T} \ell(r)^c \subseteq \bigcup_{r\in S} \ell(r)^c$. This would imply that $\bigcap_{r\in S} \ell(r) \subseteq \bigcap_{r\in T} \ell(r)$, which in turn would imply that $\bigwedge\{r\colon r\in S\} \leq \bigwedge\{r\colon r\in T\}$. But then $\bigwedge\{r\colon r\in N(p)\} = \bigwedge\{r\colon r\in N(p)\setminus(T\setminus S)\}$, a contradiction.

In the second case we have multideg $b(p;S) = x_{\ell(p)} y_A$ and multideg $b(q;T) = x_{\ell(q)} y_B$ for some A and B. If multideg $b(q;T)$ does not divide multideg $b(p;S)$ then the coefficient of $b(q;T)$ is 0. Otherwise it is $x_{\ell(p)\setminus\ell(q)} y_{A\setminus B}$. Since $q < p$ this coefficient is not 1.

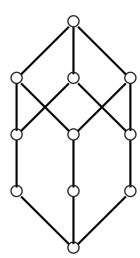
In the last case $\ell(q) \not\subseteq \ell(p)$, and so multideg $b(q;T)$ does not divide multideg $b(p;S)$. Hence the coefficient of $b(q;T)$ is 0. $\square$

Corollary 7.36. *If $\mathscr{L}$ is a meet-distributive meet-semilattice, then the finite multigraded free S-resolution given in Theorem* 7.35 *is minimal.*

PROOF. By definition meet-distributive meet-semilattices have the property that for any element $p \in \mathscr{L}$ the interval $[\bigwedge\{q\colon q\in N(p)\}, p]$ is a Boolean lattice (of rank $|N(p)|$). This implies condition (ii)(b) in Theorem 7.35. $\square$

Note that condition (ii)(b) in Theorem 7.35 is satisfied for any meet-semilattice $\mathscr{L}$ for which $|N(p)| \leq 2$ for all $p \in \mathscr{L}$. Other examples can easily be constructed, as follows: let $\mathscr{L}$ be a meet-semilattice satisfying the condition (ii)(b), and let $p, q \in \mathscr{L}$ such that $q \in N(p)$. Let $\mathscr{L}'$ be the meet-semilattice adding a new element r with $q < r < p$. Then this new meet-semilattice again satisfies (ii)(b).

An example of such a meet-semilattice is



$\mathscr{L}$

Observe that $\mathscr{L}$ is neither upper nor lower semimodular. The resolution of $H_{\mathscr{L}}$ is

$$0 \longrightarrow S(-12) \longrightarrow S^6(-10) \longrightarrow S^9(-8) \oplus S^6(-7) \longrightarrow S^{11}(-6) \longrightarrow H_{\mathscr{L}} \longrightarrow 0.$$

We close this section with discussing the regularity of $H_{\mathscr{L}}$.

Corollary 7.37. *Let $\mathscr{L}$ be a finite meet-semilattice, P the set of join irreducible elements of $\mathscr{L}$. Then*

(i) $\operatorname{reg}(H_{\mathscr{L}}) \leq |P| + \max_{\substack{p\in\mathscr{L} \\ S\subset N(p)}} \{\deg p - \deg \bigwedge\{q\colon q \in S\} - |S|\}$;

(ii) *if $\mathscr{L}$ satisfies condition (ii)(b) in Theorem 7.35, then*

$$\operatorname{reg}(H_{\mathscr{L}}) = |P| + \max_{p\in\mathscr{L}} \{\deg p - \deg \bigwedge\{q\colon q \in N(p)\} - |N(p)|\}.$$

PROOF. Since $\mathbb{F}$ is a possibly non-minimal free resolution of $H_{\mathscr{L}}$ it follows that

$$\operatorname{reg} H_{\mathscr{L}} \leq \max\{\deg b(p;S) - |S|\}$$

where the maximum is taken over all basis elements in the resolution.

By our computation in the proof of Theorem 7.35 one has

$$\deg b(p;S) - |S| = |P| + \deg p - \deg \bigwedge\{q\colon q \in S\} - |S|.$$

This implies assertion (i).

If $\mathscr{L}$ satisfies the condition (ii)(b) in Theorem 7.35, then our resolution is minimal and hence we have equality in formula (i). Moreover, if $S' \subset S \subset N(p)$ with $|S| = |S'| + 1$, then

$$\deg \bigwedge\{q\colon q \in S\} - \deg \bigwedge\{q\colon q \in S'\} \geq 1.$$

Hence

$$\bigwedge\{q\colon q \in S\} - \bigwedge\{q\colon q \in N(p)\} \geq |N(p)| - |S|.$$

□

5. The resolution of $H_{\mathscr{L}}$ for a meet-distributive meet-semilattice

In this section we describe the differential ∂ in the graded minimal free resolution $\mathbb{F}$ of $H_{\mathscr{L}}$ when $\mathscr{L}$ is a meet-distributive meet-semilattice.

As we have seen in the previous section, a basis of F_i is given by the basis elements

$$b(p;S),$$

where $p \in \mathscr{L}$ and $S \subset N(p)$ with $|S| = i$. Thus it amounts to describe $\partial(b(p;S))$ for each such basis element. To this end we introduce some notation:

Let $\mathscr{L}$ be any meet-distributive meet-semilattice, and P the set of join-irreducible elements of $\mathscr{L}$. We extend the partial order on P to a total order $<$.

For a subset $T \subset P$ and $q \in P$ we set

$$\lambda(q;T) = |\{r \in T : r < q\}|.$$

For each $q \in N(p)$, we have $|\ell(p) \setminus \ell(q)| = 1$. We denote the unique element in $\ell(p) \setminus \ell(q)$ by $p \setminus q$. Furthermore, for any subset $S \subset N(p)$ we set $p \setminus S = \{p \setminus q\colon q \in S\}$. Let $p \in \mathscr{L}$ and $S \subset \mathscr{L}$. We use $p \vee S$ and $p \wedge S$ to denote the set $\{p \vee s : s \in S\}$ and $\{p \wedge s : s \in S\}$, respectively. With the notation introduced we now have

Theorem 7.38. *For each $p \in \mathscr{L}$ and each $S \subset N(p)$, one has*

$$\partial(b(p;S)) = \sum_{q\in S}(-1)^{\lambda(p\setminus q;p\setminus S)}(y_{p\setminus q}b(p;S\setminus\{q\}) - x_{p\setminus q}b(q;q\wedge(S\setminus\{q\})).$$

Before we give the proof of the theorem we first note that $q\wedge(S\setminus\{q\})\subset N(q)$ for all $q\in S$. This is the case because by assumption $\mathscr{L}$ is meet-distributive, so that for any two distinct lower neighbors q_1 and q_2 of p, the element $q_1\wedge q_2$ is a lower neighbor of q_1 and q_2.

We also note that the differential defined in Theorem 7.38 is multi-homogeneous. To see this, recall that $\operatorname{multideg}(b(p;S))$ is the least common multiple of u_p and all u_q with $q\in S$. Since $u_q = y_{p\setminus q}u_p/x_{p\setminus q}$, we have $\operatorname{multideg}(b(p;S\setminus\{q\})) = \operatorname{multideg}(b(p;S))/y_{p\setminus q}$, and $\operatorname{multideg}(b(q;q\wedge(S\setminus\{q\}))) = \operatorname{multideg}(b(p;S))/x_{p\setminus q}$. This shows that ∂ is indeed multi-homogeneous.

PROOF OF 7.38. We use the linear order $\prec$ on $\mathscr{L}$ introduced in the proof of Theorem 7.35, and show by induction on $p\in\mathscr{L}$ that the differential ∂ is given on the free resolution $\mathbb{F}(p)$ of $H_{\mathscr{L}}(p)$ by the iterated mapping cone construction as described in Theorem 7.35.

Recall that for $p\in\mathscr{L}$ there is an exact sequence of multigraded S-modules

$$0\longrightarrow (S/L)(-\operatorname{multideg}u_p)\longrightarrow S/H_{\mathscr{L}}(q)\longrightarrow S/H_{\mathscr{L}}(p)\longrightarrow 0,$$

where $q\prec p$ is the element in $\mathscr{L}$ preceding p, and where L is the colon ideal

$$H_{\mathscr{L}}(q):u_p = (\{u_t/[u_t,u_p]\}_{t\in N(p)}) = (y_{p\setminus t}\colon t\in N(p)).$$

By induction hypothesis, the differential on $\mathbb{F}(q)$ is obtained by iterated mapping cones from exact sequences as before.

Let $\mathbb{C} = \mathbb{T}(-\operatorname{multideg}u_p)$ be the shifted Taylor complex associated with the sequence $y_{p\setminus t}$, $t\in N(P)$, where the order of the sequence is given by the order of the elements $p\setminus t$ in P. For a subset $S\in N(p)$, $S=\{t_1,\dots,t_i\}$ with $p\setminus t_1 < p\setminus t_2 < \cdots < p\setminus t_i$, we denote the element $e_{t_1}\wedge e_{t_2}\wedge\cdots\wedge e_{t_i}\in T_i$ by $b(p;S)$.

Let $\alpha\colon \mathbb{C}\to\mathbb{F}(q)$ be a complex homomorphism extending the map

$$(S/L)(-\operatorname{multideg}u_p)\longrightarrow S/H_{\mathscr{L}}(q).$$

Then the differential given by the mapping cone is defined as follows:

$$\partial_i = (\partial_i^{\mathbb{T}} + (-1)^i\alpha_i, \partial_{i+1}^{\mathbb{F}(q)})\quad \text{for all}\quad i.$$

Comparing this equation with the definition of ∂ in the theorem it remains to show that for each $S\subset N(p)$ we have:

(i) $\partial^{\mathbb{T}}(b(p;S)) = \sum_{q\in S}(-1)^{\lambda(p\setminus q;p\setminus S)}y_{p\setminus q}b(p;S\setminus\{q\})$, and
(ii) α can be chosen such that

$$(-1)^i\alpha_i(b(p;S)) = -\sum_{q\in S}(-1)^{\lambda(p\setminus q;p\setminus S)}x_{p\setminus q}b(q;q\wedge(S\setminus\{q\}).$$

Equation (i) is obvious, because this is exactly how the differential in the Taylor complex is defined.

We conclude the proof of the theorem by showing that if α is defined as in (ii), then $\alpha\colon \mathbb{C} \to \mathbb{F}(q)$ is a complex homomorphism. This amounts to show that

$$\partial_i^{\mathbb{F}(q)} \circ \alpha_i = \alpha_{i-1} \circ \partial_i^{\mathbb{T}}.$$

To see this we choose $b(p;S) \in T_i$. Then

$$(17)\quad (\partial_i^{\mathbb{F}(q)} \circ \alpha_i)(b(p;S)) \;=\; (-1)^{i+1} \sum_{q \in S} (-1)^{\lambda(p\setminus q; p\setminus S)} x_{p\setminus q} \partial_i^{\mathbb{F}(q)}(b(q; q \wedge (S \setminus \{q\}))).$$

By our induction hypothesis we have that

$$\begin{aligned} \partial_i^{\mathbb{F}(q)}(b(q;q\wedge(S\setminus\{q\}))) \;=\; & \sum_{q' \in S\setminus\{q\}} (-1)^{\lambda(p\setminus q'; (p\setminus S)\setminus\{p\setminus q\})} \big(y_{p\setminus q'} b(q; q\wedge(S\setminus\{q,q'\})) \\ & - \; x_{p\setminus q'} b(q\wedge q'; q' \wedge [(q\wedge(S\setminus\{q\})\setminus\{q\wedge q'\})])\big). \end{aligned}$$

Here we used that $q \setminus q\wedge q' = p \setminus q'$.

Substituting this in equation (17) we get

$$(18)\quad (\partial_i^{\mathbb{F}(q)} \circ \alpha_i)(b(p;S)) =$$
$$(-1)^{i+1} \sum_{q,q' \in S, q \neq q'} (-1)^{(\lambda(p\setminus q; p\setminus S) + \lambda(p\setminus q'; (p\setminus S)\setminus\{p\setminus q\})} x_{p\setminus q} y_{p\setminus q'} b(q; q\wedge(S\setminus\{q,q'\}).$$

On the other hand

$$(19)\quad (\alpha_{i-1} \circ \partial_i^{\mathbb{T}})(b(p;S)) = \sum_{q\in S} (-1)^{\lambda(p\setminus q; p\setminus S)} y_{p\setminus q} \alpha_{i-1}(b(p; S\setminus\{q\}))$$
$$= (-1)^{i+1} \sum_{q,q' \in S, q\neq q'} (-1)^{(\lambda(p\setminus q; p\setminus S) + \lambda(p\setminus q'; (p\setminus S)\setminus\{p\setminus q\})))} y_{p\setminus q} x_{p\setminus q'} b(q'; q' \wedge (S\setminus\{q,q'\}).$$

Here we used that $q \setminus q\wedge q' = p\setminus q'$.

It follows that the right hand sides of the equations (18) and (19) coincide after exchanging q and q'. This concludes the proof. □

In fact this resolution is a cellular resolution in the sense of Bayer and Sturmfels [**4**], the cells being cubes. Each basis element $b(p;S)$ can be identified with the interval $[q,p]$ where q is the meet of all elements in S. Since $\mathscr{L}$ is meet-distributive, this interval is a Boolean lattice, and hence may be identified with a cube.

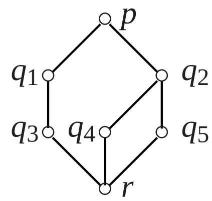
It would be desirable to have also an explicit description of the differentials for the resolution of $H_{\mathscr{L}}$ when $\mathscr{L}$ is a meet-irredundant meet-semilattice. Quite generally, according to the iterated mapping cone construction described in Theorem 7.35, the differentials in the resolution of $H_{\mathscr{L}}$ for a meet-irredundant meet-semilattice are of the form

$$\partial(b(p;S)) = \sum_{q\in S} (-1)^{\lambda(p\setminus q; p\setminus S)} y_{p\setminus q} b(p; S\setminus\{q\}) + \sum_{t\in[r,p], t\neq p} c_t b(t; S_t),$$

where

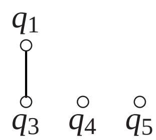
(i) r is the meet of all elements in S,
(ii) $c_t = \lambda_t v_t$ with $\lambda_t \in K$ and v_t the monomial whose multidegree is multideg$(b(p;S))-$ multideg$(b(t;S_t))$,
(iii) S_t is a set of lower neighbors of t in the interval $[r,p]$ with $|S_t| = |S|-1$.

For example consider the following meet irredundant meet-semilattice



$\mathscr{L}$

whose poset of join irreducible elements is



P

It is easy to see that in this case there are two, equally natural choices, to define $\partial(b(p;\{q_1,q_2\})$, namely:

$$\begin{aligned}\partial(b(p;\{q_1,q_2\}) = & -y_1y_3b(p;\{q_1\}) + y_4y_5b(p;\{q_2\}) - x_4x_5y_3b(q_1;\{q_3\}) - x_1x_4x_5b(q_3;\{r\}) \\ & + x_1x_3y_4b(q_2;\{q_4\}) + x_1x_3x_5b(q_4;\{r\}),\end{aligned}$$

or,

$$\begin{aligned}\partial(b(p;\{q_1,q_2\}) = & -y_1y_3b(p;\{q_1\}) + y_4y_5b(p;\{q_2\}) - x_4x_5y_3b(q_1;\{q_3\}) - x_1x_4x_5b(q_3;\{r\}) \\ & + x_1x_3y_5b(q_2;\{q_5\}) + x_1x_3x_4b(q_5;\{r\}).\end{aligned}$$

Here we wrote for simplicity x_i and y_i instead of x_{q_i} and y_{q_i}, respectively.

6. On the Alexander dual of $H_{\mathscr{I}}$

Let $\mathscr{L}$ be a finite distributive lattice and $\mathscr{I}$ a poset ideal of $\mathscr{L}$. Since the ideal Hibi ideal $H_{\mathscr{I}}$ is a squarefree monomial ideal, there exists a simplicial complex Δ such that $I_\Delta = H_{\mathscr{I}}$. In this section we study Stanlay–Reisne ideal $I_{\Delta^\vee}$ of the Alexander dual of Δ, sometimes we also call it the Alexander dual of the Hibi ideal $H_{\mathscr{I}}$. In particular, we know the Alexander dual of $H_{\mathscr{L}}$, which is studied in [**19**]. There the authors showed that a bipartite graph G with bipartition (V,V'), $|V| = |V'|$ is Cohen–Macaulay if and only if there exists a finite distributive lattice $\mathscr{L}$ such that the Alexander dual of $H_{\mathscr{L}}$ coincides with the edge ideal of G.

For the convenience we introduce the following notation: let I be a squarefree monomial ideal. Then $I = I_\Delta$ for some simplicial complex Δ, and we write I^* for $I_{\Delta^\vee}$. Here, as before, $\Delta^\vee$ is the Alexander dual of the simplicial complex Δ and $I_{\Delta^\vee}$ is the Stanley–Reisner ideal of $\Delta^\vee$.

Let $\mathscr{L}$ be a finite distributive lattice. In particular $\mathscr{L}$ is a poset and we may consider a poset ideal $\mathscr{I} \subset \mathscr{L}$. Note that any poset ideal $\mathscr{I}$ of $\mathscr{L}$ is a (special) meet-semilattice.

Let $p \in \mathscr{L}$, then the poset ideal

$$\mathscr{I}_p = \{q \in \mathscr{L} : q \ngeq p\}$$

is called 1-*cogenerated*. It is clear that for any poset ideal $\mathscr{I}$ we have

$$\mathscr{I} = \bigcap_{p\in\mathscr{L}\setminus\mathscr{I}} \mathscr{I}_p.$$

We set $H_{\mathscr{I}} = (\{u_q : q \in \mathscr{I}\})$. Then

Lemma 7.39. *Let $\mathscr{L}$ be a finite distributive lattice and $\mathscr{I} \in \mathscr{L}$ a poset ideal. Then*

$$H_{\mathscr{I}} = \bigcap_{q\in\mathscr{L}\setminus\mathscr{I}} H_{\mathscr{I}_q} \quad \textit{and} \quad H^*_{\mathscr{I}} = \sum_{q\in\mathscr{L}\setminus\mathscr{I}} H^*_{\mathscr{I}_q}.$$

PROOF. In order to prove the first equation, it suffices to show that if $\mathscr{I}'$ and $\mathscr{I}''$ are two poset ideals in $\mathscr{L}$, and $\mathscr{I} = \mathscr{I}' \cap \mathscr{I}''$, then $H_{\mathscr{I}} = H'_{\mathscr{I}} \cap H''_{\mathscr{I}}$. It is clear that $H_{\mathscr{I}} \subseteq H_{\mathscr{I}'} \cap H_{\mathscr{I}''}$. Since $H_{\mathscr{I}'}$ and $H''_{\mathscr{I}}$ both are monomial ideals, $H_{\mathscr{I}}$ is a monomial ideal. Let $m \in H'_{\mathscr{I}} \cap H''_{\mathscr{I}}$ be a monomial. Then there exist $p \in \mathscr{I}'$ and $q \in \mathscr{I}''$ such that $u_p|m$ and $u_q|m$. Let $t = p\wedge q$. Since $\mathscr{L}$ is distributive, we have $u_t = x_{\ell(p)\cap\ell(q)}y_{P\setminus(\ell(p)\cap\ell(q))} = x_{\ell(p)\cap\ell(q)}y_{(P\setminus\ell(p))\cup(P\setminus\ell(q))}$; hence $u_t|m$. Since $t \le p$ and $t \le q$, it follows that $t \in \mathscr{I}' \cap \mathscr{I}'' = \mathscr{I}$. Therefore, $m \in H_{\mathscr{I}}$.

Let P be a monomial prime ideal. Then $\bigcap_{q\in\mathscr{L}\setminus\mathscr{I}} H_{\mathscr{I}_q} \subset P$ if and only if $H_{\mathscr{I}_q} \subset P$ for some q. Hence the assertion follows from (14). □

Theorem 7.40. *Let $\mathscr{L}$ be a finite distributive lattice, $P \subset \mathscr{L}$ the poset of join irreducible elements of $\mathscr{L}$, and $\mathscr{I} \subset \mathscr{L}$ a poset ideal of $\mathscr{L}$. Then*

$$H^*_{\mathscr{I}} = (H^*_{\mathscr{L}}, \{\prod_{r\in G(\ell(q))} y_r : q \in \mathscr{L}\setminus\mathscr{I}\}),$$

where $G(\ell(q))$ is the set of generators of the poset ideal $\ell(q) \subset P$.

PROOF. By using Lemma 7.39 it suffices to prove the theorem for a 1-cogenerated poset ideal $\mathscr{I}_p$. In this case what we must prove is

$$H^*_{\mathscr{I}_p} = (H^*_{\mathscr{L}}, \{\prod_{r\in G(\ell(q))} y_r : q \ge p\}).$$

Let x_Ay_B be a squarefree monomial with $A, B \subset P$. Then $x_Ay_B \in H^*_{\mathscr{I}_p}$ if and only if $A\cap\ell(r) \ne \emptyset$, or $B\cap\ell(r)^c \ne \emptyset$ for all $r \not\ge p$.

Let $T = (H^*_{\mathscr{L}}, \{\prod_{r\in G(\ell(q))} y_r : q \ge p\})$. We first show that $T \subset H^*_{\mathscr{I}_p}$. Since $H_{\mathscr{I}_p} \subset H_{\mathscr{L}}$ it follows that $H^*_{\mathscr{L}} \subset H^*_{\mathscr{I}_p}$. Moreover, suppose that for some $q \ge p$ the monomial $\prod_{r\in G(\ell(q))} y_r$ does not belong to $H^*_{\mathscr{I}_p}$. Then there exists $t \not\ge p$ such that $G(\ell(q))\cap\ell(t)^c = \emptyset$, equivalently $G(\ell(q)) \subset \ell(t)$. Hence $\ell(q) \subset \ell(t)$. However, since $q \ge p$, we have $\ell(p) \subset \ell(q)$, so that $\ell(p) \subset \ell(q)$, a contradiction.

It remains to show that $H^*_{\mathscr{I}_p} \subset T$.

Suppose $B = \emptyset$. Then $A\cap\ell(\hat{0}) = \emptyset$ since $\ell(\hat{0}) = \emptyset$ and also $B\cap\ell(\hat{0})^c = \emptyset$, a contradiction.

Suppose $A = \emptyset$. Let $\Delta^\vee$ denote the simplicial complex whose Stanley–Reisner ideal is equal to $H^*_{\mathscr{I}_p}$ and $\Delta^\vee_y$ the restriction of $\Delta^\vee$ over the vertex set $\{y_t : t \in P\}$. Then the facets of $\Delta^\vee_y$ are $\{y_t : t \in \mathscr{I}\}$, where $\mathscr{I}$ is a maximal poset ideal of P which does not contain $\ell(p)$. Such a poset ideal is of the form $P\setminus\{t \in P : t \ge h\}$ with $h \in G(\ell(p))$. If y_B belongs to $H^*_{\mathscr{I}_p}$, then B is contained in no facet of $\Delta_y^\vee$. Hence, for each $h \in G(\ell(p))$, there is $h' \in P$ with $h' \geq h$ such that $h' \in B$. Let $\mathscr{I}_0$ denote the poset ideal of P consisting of all $t \in P$ with $t \leq h'$ for some $h \in G(\ell(p))$. Let $q \in \mathscr{L}$ with $\ell(q) = \mathscr{I}_0$. It then follows that $\prod_{r \in G(\ell(q))} y_r$ divides y_B.

Finally we consider the case that $A \neq \emptyset$, and $y_B \notin H^*_{\mathscr{I}_p}$. We will show that in this case $x_A y_B \in H^*_{\mathscr{L}}$. In fact, since $y_B \notin H^*_{\mathscr{I}_p}$, there exists $r \not\geq p$ such that $B \cap \ell(r)^c = \emptyset$, equivalently $B \subset \ell(r)$. Let $(B) \subset P$ be the poset ideal generated by B. Then there exists $t \in \mathscr{L}$ such that $\ell(t) = (B)$. Since $\ell(t) = (B) \subset \ell(r)$ it follows that $t \leq r$, and hence $t \in \mathscr{I}_p$.

Suppose $x_A y_B \notin H^*_{\mathscr{L}}$, then $a \not\leq b$ for all $a \in A$ and $b \in B$. This implies that $A \cap (B) = A \cap \ell(t) = \emptyset$. This is a contradiction because also $B \cap \ell(t)^c = \emptyset$. □

Recall from [**19**, Theorem 3.4] that if G is a Cohen–Macaulay bipartite graph over the vertex set $V \cup V'$ with $V \cap V' = \emptyset$ and $|V| = |V'|$, then there exists a partial order $<$ on V such that the distributive lattice $\mathscr{J}(P)$ with $P = (V, <)$ satisfies $H^*_{\mathscr{J}(P)} = I(G)$. We write $\mathscr{L}(G)$ for the distributive lattice $\mathscr{J}(P)$.

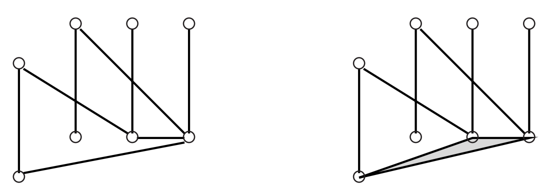
Theorem 7.41. *Let Δ be a simplicial complex over the vertex set $V \cup V'$ with $V \cap V' = \emptyset$ and $|V| = |V'|$. Suppose that*

(i) *there is no $F \in \mathscr{F}(\Delta)$ with $F \subset V$,*
(ii) *$G = \{F \in \mathscr{F}(\Delta) \colon F \cap V \neq \emptyset, \quad F \cap V' \neq \emptyset\}$ is a Cohen–Macaulay bipartite graph with no isolated vertex.*

Then the following conditions are equivalent:

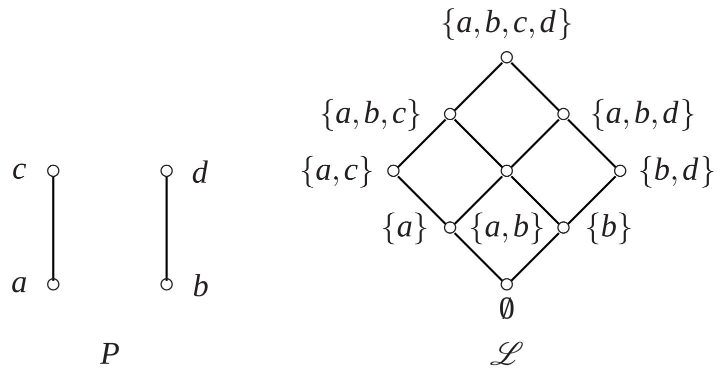
(a) *$S/I(\Delta)$ is Cohen–Macaulay;*
(b) *The simplicial complex Γ with $I_\Gamma = I(\Delta)$ is pure;*
(c) *There exists a poset ideal $\mathscr{I} \subset \mathscr{L}(G)$ containing all join-irreducible elements of $\mathscr{L}(G)$ such that $H^*_{\mathscr{I}} = I(\Delta)$.*

The following pictures show examples of simplicial complexes satisfying the conditions (i) and (ii) of Theorem 7.41.



Δ Δ'

The module $S/I(\Delta)$ is Cohen–Macaulay, while the module $S/I(\Delta')$ is not Cohen-Macaulay. In fact, the distributive lattice $\mathscr{L}$ and its poset P of join irreducible elements corresponding to the bipartite graph in Δ and Δ' is in both cases



The simplicial complex Δ corresponds to the ideal

$$\mathscr{I} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,d\}, \{a,b,c\}\}.$$

Since all poset ideals of $\mathscr{L}$ are generated by at most two elements, it follows from Theorem 7.40 the simplicial complex Δ' cannot correspond to any poset ideal in $\mathscr{L}$. Therefore, by Theorem 7.41 it cannot be Cohen-Macaulay.

PROOF OF THEOREM 7.41. Since every Cohen–Macaulay simplicial complex is pure, one has (a) $\Rightarrow$ (b). Moreover, since Theorem 7.30 guarantees that $H_{\mathscr{I}}$ has a linear resolution, it follows from Theorem 6.18(i) that (c) $\Rightarrow$ (a).

We now prove that (b) $\Rightarrow$ (c). Let $V = \{x_1, \dots, x_n\}$ and $V' = \{y_1, \dots, y_n\}$. Since Γ is pure and since V is a facet of Γ, it follows that each facet of Γ is a facet of Γ_0, where Γ_0 is a simplicial complex on $V \cup V'$ with $I_{\Gamma_0} = I(G)$. In other words, each minimal nonface of $\Gamma^\vee$ is a minimal nonface of $\Gamma_0^\vee$. Thus we may regards that the minimal set $\mathscr{I}^\flat$ of monomial generators of $I_{\Gamma^\vee}$ is a subset of $\mathscr{L}(G)$. Now, what we must prove is that $\mathscr{I}^\flat$ is a poset ideal of $\mathscr{L}(G) = \mathscr{J}(P)$, where $P = (V, <)$ is the poset consisting of all join-irreducible elements of $\mathscr{L}(G)$. Suppose, on the contrary, that $\mathscr{I}^\flat$ is not a poset ideal, and choose two elements δ and ξ of $\mathscr{L}(G)$ with $\delta \in \mathscr{I}^\flat$ and $\xi \notin \mathscr{I}^\flat$ such that δ covers ξ in $\mathscr{L}(G)$. To simplify the notation, we will assume that $\delta = \{x_1, \dots, x_k\}$ and $\xi = \{x_1, \dots, x_{k-1}\}$. Thus $\{y_1, \dots, y_k, x_{k+1}, \dots, x_n\}$ is a facet of Γ and $\{y_1, \dots, y_{k-1}, x_k, x_{k+1}, \dots, x_n\}$ is not a facet of Γ. Thus there is a monomial generator u of $I(\Delta)$ which divides $y_1 \cdots y_{k-1} x_k x_{k+1} \cdots x_n$. However, since $\{y_1, \dots, y_{k-1}, x_{k+1}, \dots, x_n\}$ is a face of Γ, it follows that the variable x_k must appear in the support of u. Hence $u = x_k y_j$ with $1 \leq j \leq k-1$. Then [**?**, Theorem 3.4] says that $x_k < x_j$ in P. This is impossible, since ξ is a poset ideal of $\mathscr{L}(G)$. Consequently, it turns out that $\mathscr{I}^\flat$ is a poset ideal of $\mathscr{L}(G)$.

Finally, in case that $\mathscr{I}^\flat$ does not contain of a join-irreducible element x_i of $\mathscr{L}(G)$, the vertex y_i belongs to all facets of Γ. This is impossible, since G possesses no isolated vertex. This completes the proof of (b) $\Rightarrow$ (c). □

Corollary 7.42. *Let Δ be a simplicial complex over the vertex set $V = \{v_1, \dots, v_n\}$, and let $W = \{w_1, \dots, w_n\}$ be a vertex set with $W \cap V = \emptyset$. Let Γ be the simplicial complex over the vertex set $V \cup W$ whose facets are those of Δ and all the edges $\{v_i, w_i\}$ for $i = 1, \dots, n$. Then the facet ideal of Γ is Cohen-Macaulay.*

PROOF. Our work is to show that the simplicial complex Σ with $I_\Sigma = I(\Gamma)$ is pure. Let $F = \{v_i\colon i \in A\} \cup \{w_j\colon j \in B\}$ be a face of Σ, then $A \cap B = \emptyset$. If $A \cup B \neq [n]$, then $F \cup \{w_i\colon i \in [n] \setminus (A \cup B)\}$ is a face of Σ. Thus all facets of Σ have the cardinality n. Hence Σ is pure, as desired. □

The results of Theorem 7.30 and Theorem 7.40 can be extended as follows. Let $\mathscr{L}$ be a finite distributive lattice, and let $\mathscr{I} \subset \mathscr{L}$ be a poset ideal, and $\mathscr{J}$ a poset coideal in $\mathscr{L}$. Then $H_\mathscr{I}$ and $H_\mathscr{J}$ have linear resolutions. We know this for $H_\mathscr{I}$ by Theorem 7.30 and for $H_\mathscr{J}$ it follows by the same theorem using the fact that the dual of $\mathscr{L}$ is again a distributive lattice. What can be said about $H_\mathscr{I} \cap H_\mathscr{J}$? One might expect that this ideal has again a linear resolution. However this is not the case. For example, consider the Boolean lattice $\mathscr{B}_3$ of rank 3, and let $\mathscr{I} = \mathscr{B}_3 \setminus \{\hat{1}\}$ and $\mathscr{J} = \mathscr{B}_3 \setminus \{\hat{0}\}$. Then $H_\mathscr{I} \cap H_\mathscr{J}$ does not have a linear resolution.

However in the positive direction we have

Proposition 7.43. *Let $\mathscr{I}$ be a poset ideal and $\mathscr{J}$ a poset coideal in $\mathscr{L}$. Then*

(i) $\operatorname{rank} \mathscr{L} \leq \operatorname{reg}(H_\mathscr{I} \cap H_\mathscr{J}) \leq \operatorname{rank} \mathscr{L} + 1$, *if* $\mathscr{L} = \mathscr{I} \cup \mathscr{J}$*;*

(ii) $(H_\mathscr{I} \cap H_\mathscr{J})^* = (H^*_\mathscr{L}, \{\prod_{r \in G(\ell(q))} y_r\colon q \in \mathscr{L} \setminus \mathscr{I}\}, \{\prod_{r \in G(\ell(q)^c)} x_r\colon q \in \mathscr{L} \setminus \mathscr{J}\})$.

PROOF. (i) Consider the long exact Tor-sequence

$$\cdots \to \operatorname{Tor}_{i+1}(K, H_\mathscr{I} + H_\mathscr{J}) \to \operatorname{Tor}_i(K, H_\mathscr{I} \cap H_\mathscr{J}) \to \operatorname{Tor}_i(K, H_\mathscr{I}) \oplus \operatorname{Tor}_i(K, H_\mathscr{J}) \to \cdots$$

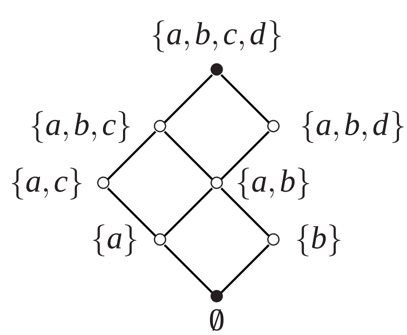
arising from the short exact sequence

$$0 \longrightarrow H_\mathscr{I} \cap H_\mathscr{J} \longrightarrow H_\mathscr{I} \oplus H_\mathscr{J} \longrightarrow H_\mathscr{I} + H_\mathscr{J} \longrightarrow 0.$$

Since $\mathscr{L} = \mathscr{I} \cup \mathscr{J}$, we have $H_\mathscr{L} = H_\mathscr{I} + H_\mathscr{J}$. Hence the ideals $H_\mathscr{I}$, $H_\mathscr{J}$ and $H_\mathscr{I} + H_\mathscr{J}$ have linear resolutions by Theorem 7.30. It follows that $\operatorname{Tor}_i(K, H_\mathscr{I})_j = \operatorname{Tor}_i(K, H_\mathscr{J})_j = 0$ for $j \neq i + \operatorname{rank} \mathscr{L}$, and $\operatorname{Tor}_{i+1}(K, H_\mathscr{I} + H_\mathscr{J})_j = 0$ for $j \neq i + 1 + \operatorname{rank} \mathscr{L}$. Thus the assertion follows from the long exact Tor-sequence.

(ii) Since $(H_\mathscr{I} \cap H_\mathscr{J})^* = H^*_\mathscr{I} + H^*_\mathscr{J}$, the claim follows from Theorem 7.40. □

Consider the following example.



$\mathscr{L}$

Here we take in $\mathscr{L}$ the poset ideal

$$\mathscr{I} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}\},$$

and the poset coideal

$$\mathscr{J} = \{\{a\},\{b\},\{a,b\},\{a,c\},\{a,b,c\},\{a,b,d\},\{a,b,c,d\}\}.$$

Then $H_{\mathscr{I}} \cap H_{\mathscr{J}} = (avwx, buwx, acvx, abwx, abcx, abdw)$. Thus this intersection is generated by all generators of $H_{\mathscr{L}}$ except $u_{\hat{0}}$ and $u_{\hat{1}}$, as indicated in the picture. The resolution of $H_{\mathscr{I}} \cap H_{\mathscr{J}}$ is linear, namely

$$0 \longrightarrow S(-6) \longrightarrow S(-5)^6 \longrightarrow S(-4)^6 \longrightarrow H_{\mathscr{I}} \cap H_{\mathscr{J}} \longrightarrow 0.$$

Quite generally it would be interesting to know when $H_{\mathscr{I}} \cap H_{\mathscr{J}} = H_{\mathscr{I}\cap\mathscr{J}}$, and when an ideal of the form $H_{\mathscr{I}\cap\mathscr{J}}$ has a linear resolution. Of particular interest are the following cases:

(i) $H = (\{u_p\}_{p\in\mathscr{L}\setminus\{\hat{0},\hat{1}\}})$;
(ii) $H = (\{u_p \colon r \leq \operatorname{rank} p \leq s\})$ for some r and s with $0 < r \leq s < \operatorname{rank}\mathscr{L}$.

These two cases we will discuss in the following section.

7. A class of unmixed simplicial complexes

In this section we will give a new class of unmixed simplicial complexes related to the segments of distributive lattices.

A simplicial complex Δ on the vertex set $[n]$ is *Cohen–Macaulay* over a field K, if the Stanley–Reisner ideal I_Δ of Δ is a Cohen–Macaulay ideal, while for a graph G, we say G is Cohen–Macaulay, if the edge ideal $I(G)$ of G is a Cohen–Macaulay ideal.

Recall that a graph G is bipartite if its vertex set V can be partitioned into disjoint subsets V_1 and V_2 such that every edge $\{v_1, v_2\}$ of G satisfies $v_1 \in V_1$ and $v_2 \in V_2$. Let G be a bipartite graph with no isolated vertex on the vertex set $V \cup V'$, where $V \cap V' = \emptyset$ and $|V| = |V'|$. In [**19**, Theorem 2.4], the authors showed that a bipartite graph G is a Cohen–Macaulay if and only if $I(G) = H^*_{\mathscr{L}}$ for some distributive lattice $\mathscr{L}$. In the previous section we considered more generalize simplicial complexes Δ on the vertex set $V \cup V'$ with $V \cap V' = \emptyset$ and $|V| = |V'|$, such that

(1) there is no $F \in \mathscr{F}(\Delta)$ with $F \subset V$, and
(2) $G = \{F \in \mathscr{F}(\Delta) \colon F \cap V \neq \emptyset, \quad F \cap V' \neq \emptyset\}$ is a Cohen–Macaulay bipartite graph with no isolated vertex,

and showed when the facet ideal $I(\Delta)$ of Δ is Cohen–Macaulay, see Theorem 7.41.

In this section we will prove another generalization of Theorem 2.4 in [**19**]. For this we need some preparation.

The poset ideals and poset coideals of lattices are special subsets of lattices. Now we introduce a more general class of subsets of lattices:

Definition 7.44. Let $\mathscr{L}$ be a lattice. A subset $\mathscr{S}$ of $\mathscr{L}$ is called a *segment* of $\mathscr{L}$, if for all $p, q \in \mathscr{S}$ with $p \leq q$, we have $[p,q] \subseteq \mathscr{S}$.

It is clear that any poset ideal and any poset coideal of a lattice $\mathscr{L}$ are segments of $\mathscr{L}$. Furthermore, we have

Lemma 7.45. *Let $\mathscr{L}$ be a lattice, $\mathscr{S}$ a subset of $\mathscr{L}$. Then the following statements are equivalent:*

(i) *$\mathscr{S}$ is a segment of $\mathscr{L}$;*
(ii) *$\mathscr{S}$ is the intersection of a poset ideal and a poset coideal of $\mathscr{L}$.*

PROOF. (i) $\Rightarrow$ (ii): Let $\mathscr{I} = \{r \in \mathscr{L} :$ there exists an element $s \in \mathscr{S}$ such that $r \leq s\}$ and $\mathscr{J} = \{r \in \mathscr{L} :$ there exists an element $s \in \mathscr{S}$ such that $r \geq s\}$. Then $\mathscr{I}$ is a poset ideal of $\mathscr{L}$ and $\mathscr{J}$ is a poset coideal of $\mathscr{L}$. For any $s \in \mathscr{S}$, we have $s \in \mathscr{I} \cap \mathscr{J}$. This implies $\mathscr{S} \subseteq \mathscr{I} \cap \mathscr{J}$. Now let r be an arbitrary element in $\mathscr{I} \cap \mathscr{J}$. Then there exist $p, q \in \mathscr{S}$ such that $p \leq r \leq q$, i.e., $r \in [p, q]$. Since $\mathscr{S}$ is a segment, we have $r \in \mathscr{S}$. Hence $\mathscr{I} \cap \mathscr{J} \subseteq \mathscr{S}$.

(ii) $\Rightarrow$ (i): Assume $\mathscr{S} = \mathscr{I} \cap \mathscr{J}$, where $\mathscr{I}$ is a poset ideal of $\mathscr{L}$ and $\mathscr{J}$ is a poset coideal of $\mathscr{L}$. Let $r \in [p, q]$ with $p, q \in \mathscr{S}$ and $p \leq q$. Since $q \in \mathscr{I}$ and $r \leq q$, we have $r \in \mathscr{I}$. Since $p \in \mathscr{J}$ and $r \geq p$, we have $r \in \mathscr{J}$. Hence $r \in \mathscr{I} \cap \mathscr{J} = \mathscr{S}$. This implies that $\mathscr{S}$ is a segment of $\mathscr{L}$. $\square$

Remark 7.46. Let $\mathscr{S}$ be a segment of a lattice $\mathscr{L}$. The poset ideal $\mathscr{I}$ and poset coideal $\mathscr{J}$ with the property $\mathscr{S} = \mathscr{I} \cap \mathscr{J}$ are not uniquely determined. The poset ideal $\mathscr{I}$ and poset coideal $\mathscr{J}$ in the proof (i) $\Rightarrow$ (ii) of Lemma 7.45 are the minimal one with this property.

Let G be a Cohen–Macaulay bipartite graph on the vertex set $V \cup V'$ with $V \cap V' = \emptyset$ and $|V| = |V'| = n$, and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the polynomial ring over a field K. Recall from [**19**, Theorem 2.4] that the vertices $V = \{x_1, \ldots, x_n\}$ and $V' = \{y_1, \ldots, y_n\}$ can be labelled such that there exists a partial order $<$ on V with the property that $\{x_i, y_j\}$ is an edge of G if and only if $x_i \leq x_j$. Moreover it is shown that for $P = (V, <)$ the distributive lattice $\mathscr{J}(P)$ satisfies $H^*_{\mathscr{J}(P)} = I(G)$. We denote this lattice by $\mathscr{L}(G)$. As a generalization of this result we have:

Theorem 7.47. *Let Δ be a simplicial complex on the vertex set $V \cup V'$ with $V \cap V' = \emptyset$ and $|V| = |V'|$. Suppose that $G = \{F \in \mathscr{F}(\Delta) \colon F \cap V \neq \emptyset, \quad F \cap V' \neq \emptyset\}$ is a Cohen–Macaulay bipartite graph with no isolated vertex. Then the following conditions are equivalent:*
(i) *Δ is unmixed, and all minimal vertex covers of Δ have cardinality $|V|$;*
(ii) *there exists a lattice segment $\mathscr{S} \subseteq \mathscr{L}(G)$ such that $H^*_{\mathscr{S}} = I(\Delta)$.*

PROOF. (i) $\Rightarrow$ (ii): Let Γ be the (unique) simplicial complex defined by the equation $I_\Gamma = I(\Delta)$. Since Δ is unmixed, we have Γ is pure. Let $V = \{x_1, \ldots, x_n\}$ and $V' = \{y_1, \ldots, y_n\}$ with the labelling as described before this theorem. Since Δ is a complex with $2n$ vertices and the minimal vertex cover of Δ has cardinality n, it follows from Corollary 2.14, that $|F| = n$ for each $F \in \mathscr{F}(\Gamma)$.

Let Γ_0 be the simplicial complex on $V \cup V'$ with $I_{\Gamma_0} = I(G)$. Then any minimal vertex cover of Δ is a minimal vertex cover of G. Indeed, a minimal vertex cover C of Δ is also a vertex cover of G, and it has cardinality n, by assumption. On the other hand, since G contains all the edges $\{x_i, y_i\}$, each vertex cover of G has at least cardinality n. Hence C is a minimal vertex cover of G.

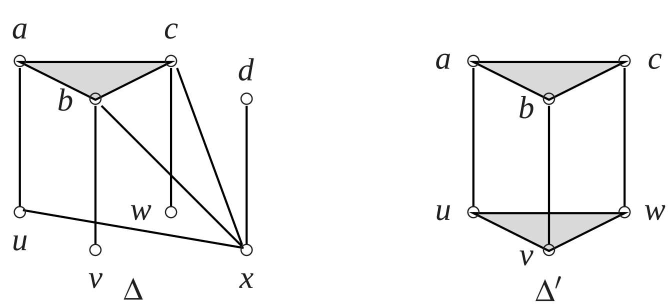
It follows that each facet of Γ is a facet of Γ_0. In other words, each minimal nonface of $\Gamma^\vee$ is a minimal nonface of $\Gamma_0^\vee$. Therefore, $G(I_{\Gamma^\vee}) \subset G(I_{\Gamma_0^\vee}) = G(H_{\mathscr{L}(G)})$. That is, there exists a subset $\mathscr{S} \neq \emptyset$ of $\mathscr{L}(G)$, such that $G(I_{\Gamma^\vee}) = \{u_s : s \in \mathscr{S}\}$, and this implies that $I(\Delta) = H^*_{\mathscr{S}}$.

Now, what we must prove is that for any $p,q \in \mathscr{S}$ with $p \leq q$ one has $[p,q] \subseteq \mathscr{S}$. Suppose, on the contrary, there exist two elements δ and ξ of $\mathscr{L}(G)$ with $\xi < \delta$, and $\gamma \in \mathscr{L}(G)$ such that $\gamma \in [\xi,\delta]$ but $\gamma \notin \mathscr{S}$.

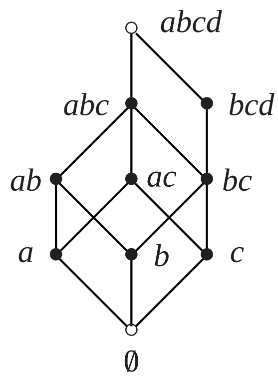
Recall that the elements of $\mathscr{L}(G)$ are poset ideals of $P = (V,<)$. To simplify the notation, we will assume that $\xi = \{x_1,\ldots,x_l\}$, $\gamma = \{x_1,\ldots,x_r\}$ and $\delta = \{x_1,\ldots,x_k\}$ with $l < r < k$. Since $\xi = \{x_1,\ldots,x_l\} \in \mathscr{S}$, we have $x_1\cdots x_l y_{l+1}\cdots y_n \in G(H_{\mathscr{S}})$. Thus $\{x_1,\ldots,x_l,y_{l+1},\ldots,y_n\}$ is a minimal vertex cover of Δ. It follows from Corollary 2.14 that $\{y_1,\ldots,y_l,x_{l+1},\ldots,x_n\} \in \mathscr{F}(\Gamma)$. By the same reason we have $\{y_1,\ldots,y_k,x_{k+1},\ldots,x_n\} \in \mathscr{F}(\Gamma)$, but $\{y_1,\ldots,y_r,x_{r+1},\ldots,x_n\} \notin \mathscr{F}(\Gamma)$. Hence there exists a monomial generator u of $I_\Gamma = I(\Delta)$ such that u does not divide $y_1\cdots y_k x_{k+1}\cdots x_n$ and $y_1\cdots y_l x_{l+1}\cdots x_n$, but divides $y_1\cdots y_r x_{r+1}\cdots x_n$. Hence there exists an i with $r < i \leq k$, such that $x_i \mid u$ and a j with $l < j \leq r$ such that $y_j \mid u$. By our assumption, $u = x_i y_j$. By our labelling of the vertices it follows that $x_i < x_j$ in P. Since $j \leq r$, we have that $x_j \in \gamma$. Since γ is a poset ideal it follows that also $x_i \in \gamma$. This is impossible, since $i > r$.

(ii) $\Rightarrow$ (i): Since all generators of $H_{\mathscr{S}}$ are of same degree and since $H^*_{\mathscr{S}} = I(\Delta)$, it follows that Δ is unmixed. $\square$

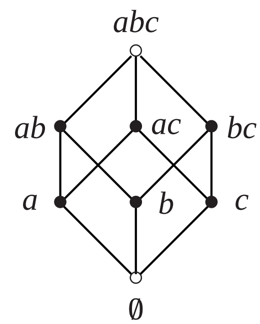
The following two simplicial complexes are unmixed and satisfy the assumption in Theorem 7.47 with $V_\Delta = \{a,b,c,d\}$, $V'_\Delta = \{u,v,w,x\}$, and $V_{\Delta'} = \{a,b,c\}$, $V'_{\Delta'} = \{u,v,w\}$.



The segments $\mathscr{S}$ and $\mathscr{S}'$ such that $I(\Delta) = H^*_{\mathscr{S}}$ and $I(\Delta') = H^*_{\mathscr{S}'}$ are given in the next figures. For the simplicity, sets in these figures are written as monomials. For example $abcd$ stands for $\{a,b,c,d\}$. The elements of the segments are indicated by the bullet vertices.



$\mathscr{S}$



$\mathscr{S}'$

Note that $I(\Delta)$ is Cohen–Macaulay, while the facet ideal $I(\Delta')$ is not. This is because the ideal $H_{\mathscr{S}} = \{avwx, buwx, cuvx, abwx, acvx, bcux, abcx, bcdu\}$ has a linear resolution, while the ideal $H_{\mathscr{S}'} = \{avw, buw, cuv, abw, acv, bcu\}$ has no linear resolution. It is therefore of interest to know for which kind of segments $\mathscr{S}$ of a finite distributive lattice $\mathscr{L}$, the ideal $H_{\mathscr{S}}$ has a linear resolution.

8. Lattice segments and poset ideals

We use the notation as in the previous sections. We have already seen that $\mathscr{S} = \mathscr{I} \cap \mathscr{J}$ where $\mathscr{I}$ is a poset ideal and $\mathscr{J}$ a poset coideal in $\mathscr{L}$. In case $H_{\mathscr{S}} = H_{\mathscr{I}} \cap H_{\mathscr{J}}$, necessary and sufficient conditions for $H_{\mathscr{S}}$ to have a linear resolution will be given in this section. We will also discuss when $H_{\mathscr{S}} = H_{\mathscr{I}} \cap H_{\mathscr{J}}$.

Let $p \in \mathscr{L}$, as before, $N(p)$ denote the set of lower neighbors. We set $M(p)$ for the set of upper neighbors of p. For any subset $T \subseteq M(p)$ we set $T \setminus p = \{\ell(t) \setminus \ell(p) : t \in T\}$.

Let P be the set of join-irreducible elements of $\mathscr{L}$, and $<$ a total order on $\mathscr{L}$ which extends the partial order on P. For a subset $T \subset P$ and $q \in P$, as before, we set

$$\lambda(q;T) = |\{r \in T : r < q\}|.$$

For each element $q \in N(p)$, we have $|\ell(p) \setminus \ell(q)| = 1$. We denote the unique element in $\ell(p) \setminus \ell(q)$ by $p \setminus q$. Let $p \in \mathscr{L}$ and $S \subset \mathscr{L}$. We use $p \vee S$ ($p \wedge S$) to denote the set $\{p \vee s : s \in S\}$ ($\{p \wedge s : s \in S\}$). The following theorem is shown in Section 7.4 and Section 7.5.

Theorem 7.48. *Let $\mathscr{L}$ be finite meet-semilattice.*

(i) *There exists a finite multigraded free S-resolution $\mathbb{F}$ of $H_{\mathscr{L}}$ such that for each $i \geq 0$, the free module $\mathbb{F}_i$ has a basis with basis elements*

$$b(p;S)$$

where $p \in \mathscr{L}$ and $S \subseteq N(p)$ with $|S| = i$. The multidegree of $b(p;S)$ is the least common multiple of u_p and all monomials u_q with $q \in S$.

(ii) *The following conditions are equivalent:*
 (a) *the resolution constructed in* (i) *is minimal;*

(b) *for any* $p \in \mathscr{L}$ *and for any proper subset* $S \subset N(p)$ *the meet* $\bigwedge\{q\colon q \in S\}$ *is strictly greater than the meet* $\bigwedge\{q\colon q \in N(p)\}$.

(iii) *If* $\mathbb{F}$ *is minimal, then the differential* ∂ *in* $\mathbb{F}$ *is as follows: for each* $p \in \mathscr{L}$ *and each* $S \subset N(p)$*, one has*

$$\partial(b(p;S)) = \sum_{q\in S} (-1)^{\lambda(p\setminus q;p\setminus S)} (y_{p\setminus q} b(p;S\setminus\{q\}) - x_{p\setminus q} b(q;q\wedge(S\setminus\{q\})).$$

Corollary 7.49. *Let* $\mathscr{L}$ *be a finite distributive lattice and* $\mathscr{I}$ *a poset ideal (coideal) of* $\mathscr{L}$. *Then the minimal free resolution of* $H_{\mathscr{I}}$ *is linear.*

PROOF. Let $\mathscr{I}$ be a poset ideal of $\mathscr{L}$. Then $\mathscr{I}$ is a meet-semilattice, and has property (ii)(b) of Theorem 7.48. Hence the free resolution of $H_{\mathscr{I}}$ as described in Theorem 7.48(i) is minimal. For any $p \in \mathscr{I}$ and any $S \subseteq N(p)$, the total degree of $b(p;S)$ equals $\operatorname{rank}\mathscr{L} + |S|$. This shows that the resolution of $H_{\mathscr{I}}$ is linear.

Now assume that $\mathscr{I}$ is a poset coideal. Then by Remark 7.27, $\widetilde{\mathscr{I}}$ is a poset ideal in $\widetilde{\mathscr{L}}$. Therefore $H_{\widetilde{\mathscr{I}}}$ has a linear resolution by the first part of the proof. By Lemma 7.28 (and its proof) the canonical labelling $\widetilde{\ell}$ of $\widetilde{\mathscr{L}}$ is given by $\widetilde{\ell}(p) = P\setminus \ell(p)$ for all $p \in \widetilde{\mathscr{L}}$. It follows that $H_{\widetilde{\mathscr{I}}}$ is generated by the monomials $\widetilde{u}_p = x_{\widetilde{\ell}(p)} y_{P\setminus\widetilde{\ell}(p)} = x_{P\setminus\ell(p)} y_{\ell(p)}$. Now we apply the following involution

$$\sigma\colon K[\{x_p,y_p\}_{p\in P}] \to K[\{x_p,y_p\}_{p\in P}], \quad x_p \mapsto y_p \quad \text{and} \quad y_p \mapsto x_p, \tag{20}$$

and we obtain $\sigma(H_{\widetilde{\mathscr{I}}}) = H_{\mathscr{I}}$. This shows that $H_{\mathscr{I}}$ has a linear resolution, too. □

As we have already seen that for a poset ideal $\mathscr{I}$ and a poset coideal $\mathscr{J}$ of a finite distributive lattice $\mathscr{L}$, the ideal $H_{\mathscr{I}}$ and $H_{\mathscr{J}}$ both have linear resolutions, one might except that if we write $\mathscr{S} = \mathscr{I}\cap\mathscr{J}$ for some poset ideal $\mathscr{I}$ and some poset coideal $\mathscr{J}$ of $\mathscr{L}$ and if $H_{\mathscr{S}} = H_{\mathscr{I}} \cap H_{\mathscr{J}}$, then the ideal $H_{\mathscr{S}}$ has a linear resolution. However there are two questions arising: (1) when $H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}}\cap H_{\mathscr{J}}$ and (2) whether $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ has a linear resolution. In general, the intersection of two ideals with linear resolutions need not to have a linear resolution, even for the special ideals $H_{\mathscr{I}}$ and $H_{\mathscr{J}}$. For example, consider the Boolean lattice $\mathscr{B}_3$ of rank 3, and let $\mathscr{I} = \mathscr{B}_3\setminus\{\hat{1}\}$ and $\mathscr{J} = \mathscr{B}_3\setminus\{\hat{0}\}$. Then $H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}}\cap H_{\mathscr{J}}$, but it has no linear resolution.

To see when $H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}}\cap H_{\mathscr{J}}$ and when $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ has a linear resolution, we need some preparation.

Let $\mathscr{I}$ be a poset ideal of a finite distributive lattice $\mathscr{L}$ and let $\mathbb{F}$ be the minimal free resolution of $H_{\mathscr{L}}$ and $\mathbb{P}$ the minimal free resolution of $H_{\mathscr{I}}$. Then by Theorem 7.48 one sees that $\mathbb{P}$ is a subcomplex of $\mathbb{F}$. More precisely we have

Lemma 7.50. *For each* $i \geq 0$ *there exists an injective map* $\operatorname{Tor}_i(K,H_{\mathscr{I}}) \to \operatorname{Tor}_i(K,H_{\mathscr{L}})$ *which maps the basis elements* $b(p;S)$ *of* $\operatorname{Tor}_i(K,H_{\mathscr{I}})$ *to the corresponding basis elements of* $\operatorname{Tor}_i(K,H_{\mathscr{L}})$.

For convenience, in this lemma and the remaining of this section the basis elements in a free resolution and corresponding basis in the Tor-groups are denoted by the same symbol.

Let $\mathscr{J}$ be a poset coideal of $\mathscr{L}$. Then $\widetilde{\mathscr{J}}$ is a poset ideal of $\widetilde{\mathscr{L}}$. Let $\widetilde{\mathbb{F}}$ be the minimal multigraded free resolution of $H_{\widetilde{\mathscr{L}}}$ and $\widetilde{\mathbb{T}}$ the minimal multigraded free resolution of $H_{\widetilde{\mathscr{J}}}$, as described in Theorem 7.48. Then $\widetilde{\mathbb{T}}$ is a subcomplex of $\widetilde{\mathbb{F}}$, and the injective map $\mathrm{Tor}_i(K, H_{\widetilde{\mathscr{J}}}) \to \mathrm{Tor}_i(K, H_{\widetilde{\mathscr{L}}})$ is as described in Lemma 7.50. Since $\sigma(H_{\widetilde{\mathscr{L}}}) = H_{\mathscr{L}}$, we have $\sigma(\widetilde{\mathbb{F}})$ is a minimal multigraded free resolution of $H_{\mathscr{L}}$. Since $\mathbb{F}$ is also a minimal multigraded free resolution of $H_{\mathscr{L}}$, it is natural to ask what is the isomorphic chain map from $\widetilde{\mathbb{F}}$ to $\mathbb{F}$.

To answer this question we need the following two lemmata:

Lemma 7.51. *Let $\widetilde{\mathscr{L}}$ be the dual of the distributive lattice $\mathscr{L}$ and $\widetilde{\mathbb{F}}$ the minimal multigraded free resolution of $H_{\widetilde{\mathscr{L}}}$. Then*

(i) *for each $i \geq 0$, the free module $\widetilde{F_i}$ has a basis with basis elements $\widetilde{b}(r;T)$ with $r \in \mathscr{L}$, $T \subseteq M(r)$ in $\mathscr{L}$ and $|T| = i$. The multidegree of $\widetilde{b}(r;T)$ is the least common multiple of $\widetilde{u}_r$ and all monomials $\widetilde{u}_s$ with $s \in T$;*

(ii) *the differential $\widetilde{\partial}$ in $\widetilde{\mathbb{F}}$ is as follows: for each $r \in \mathscr{L}$ and each $T \subseteq M(r)$, one has*

$$\widetilde{\partial}(\widetilde{b}(r;T)) = \sum_{s \in T} (-1)^{\lambda(s \setminus r; T \setminus r)} (y_{s \setminus r} \widetilde{b}(r; T \setminus \{s\}) - x_{s \setminus r} \widetilde{b}(s; s \vee (T \setminus \{s\}))).$$

PROOF. We may assume that $\widetilde{\mathbb{F}}$ is a minimal free resolution of $H_{\widetilde{\mathscr{L}}}$ as described in Theorem 7.48. Therefore $\widetilde{\mathbb{F}}$ has a basis $\widetilde{b}(r;T)$ where $r \in \widetilde{\mathscr{L}}$ and T is a subset of lower neighbors of r in $\widetilde{\mathscr{L}}$. Moreover, we have

$$\widetilde{\partial}(\widetilde{b}(r;T)) = \sum_{s \in T} (-1)^{\lambda(\widetilde{r \setminus s}; \widetilde{r \setminus T})} (y_{\widetilde{\ell}(r) \setminus \widetilde{\ell}(s)} \widetilde{b}(r; T \setminus \{s\}) - x_{\widetilde{\ell}(r) \setminus \widetilde{\ell}(s)} \widetilde{b}(s; s \wedge (T \setminus \{s\})),$$

where $\widetilde{r \setminus s}$ denote the unique element in $\widetilde{\ell}(r) \setminus \widetilde{\ell}(s)$ and $\widetilde{r \setminus T}$ the set $\{\widetilde{\ell}(r) \setminus \widetilde{\ell}(s) : s \in T\}$.

Notice that for any element $r \in \widetilde{\mathscr{L}}$ (hence $r \in \mathscr{L}$, too), a lower (upper) neighbor of r in $\widetilde{\mathscr{L}}$ is just a upper (lower) neighbor of r in $\mathscr{L}$, and for any two element p and q in $\widetilde{\mathscr{L}}$, the meet (join) of p and q in $\widetilde{\mathscr{L}}$ is just the join (meet) of them in $\mathscr{L}$.

Let $r \in \widetilde{\mathscr{L}}$ and s a lower neighbor of r in $\widetilde{\mathscr{L}}$. We have

$$\widetilde{\ell}(r) \setminus \widetilde{\ell}(s) = (P \setminus \ell(r)) \setminus (P \setminus \ell(s)) = s \setminus r$$

and

$$\lambda(\widetilde{r \setminus s}; \widetilde{r \setminus T}) = \lambda(\widetilde{\ell}(r) \setminus \widetilde{\ell}(s); \{\widetilde{\ell}(r) \setminus \widetilde{\ell}(s) : s \in T\}) = \lambda(s \setminus r; \{s \setminus r : s \in T\}) = \lambda(s \setminus r; T \setminus r).$$

Thus we obtain the desired formula. □

Let S be any subset of $\mathscr{L}$. We write $\vee S$ ($\wedge S$) for the element $\vee\{s : s \in S\}$ ($\wedge\{s : s \in S\}$).

Lemma 7.52. *Let $\mathscr{L}$ be a finite distributive lattice, $p \in \mathscr{L}$ and $S \subseteq N(p)$ with $|S| = i$. Let $r = \wedge\{q : q \in S\}$, and T the set of all upper neighbors of r in the interval $[r, p]$. Then*

(i) *$|T| = i$ and $\vee T = p$;*

(ii) $\mathrm{lcm}(u_p, \{u_q\}_{q \in S}) = \mathrm{lcm}(u_r, \{u_s\}_{s \in T})$*;*

(iii) *for any $r' \neq r$ and $T' \subseteq M(r')$, one has* $\mathrm{lcm}(u_{r'}, \{u_{s'}\}_{s' \in T'}) \neq \mathrm{lcm}(u_p, \{u_q\}_{q \in S})$.

PROOF. (i) Since $\mathscr{L}$ is a distributive lattice, the interval $[r,p]$ is a Boolean lattice. Hence $|T|=|S|=i$ and $\vee T=p$.

(ii) The monomial associated to p is $u_p=x_{\ell(p)}y_{P\setminus\ell(p)}$, where P is the set of join irreducible elements of $\mathscr{L}$. Let $q\in N(p)$. Then $u_q=x_{\ell(p)\setminus(p\setminus q)}y_{(P\setminus\ell(p))\cup(p\setminus q)}$. Hence

$$\operatorname{lcm}(u_p,\{u_q\}_{q\in S})=x_{\ell(p)}y_{(P\setminus\ell(p))\cup(\bigcup_{q\in S}(p\setminus q))}.$$

On the other hand, $\ell(r)=\ell(p)\setminus(\bigcup_{q\in S}(p\setminus q))$, $u_r=x_{\ell(p)\setminus(\bigcup_{q\in S}(p\setminus q))}y_{P\setminus(\ell(p)\setminus\bigcup_{q\in S}(p\setminus q))}$. Since $\ell(p)=\ell(r)\cup(\bigcup_{s\in T}\ell(s))$, we have

$$\operatorname{lcm}(u_r,\{u_s\}_{s\in T})=x_{\ell(p)}y_{P\setminus(\ell(p)\setminus(\bigcup_{q\in S}(p\setminus q)))}.$$

Since $\ell(p)\subseteq P$ and $\bigcup_{p\in S}(p\setminus q)\subseteq P$, we have

$$(P\setminus\ell(p))\cup(\bigcup_{q\in S}(p\setminus q))=P\setminus(\ell(p)\setminus\bigcup_{q\in S}(p\setminus q)).$$

Hence (ii) follows.

(iii) As in the proof of (ii) we see that the y-part of $\operatorname{lcm}(u_{r'},\{u_{s'}\}_{s'\in T'})$ equals the y-part of $u_{r'}$. Since for any $r'\neq r$, we have $\ell(r')\neq\ell(r)$. The assertion follows from (ii). □

We fix some notation. For each element $r\in\mathscr{L}$ and $T\subseteq M(r)$, we write r^T for the join of all elements in T, and T_r the set of all lower neighbors of r^T in the interval $[r,r^T]$.

The polynomial ring S viewed as a S-module via the involution $\sigma: S\to S$ is denoted by ${}^{\sigma}S$. Let $\widetilde{\mathbb{F}}$ be the minimal free resolution of the ideal $H_{\widetilde{\mathscr{L}}}$ with basis elements $\widetilde{b}(r;T)$ as described in Lemma 7.51. Then $\widetilde{\mathbb{F}}\bigotimes_S {}^{\sigma}S$ with basis elements $\widetilde{b}(r;T)\bigotimes_S 1$ is a minimal free resolution of $\sigma(H_{\widetilde{\mathscr{L}}})=H_{\mathscr{L}}$. We denote the complex $\widetilde{\mathbb{F}}\bigotimes_S {}^{\sigma}S$ by $\sigma(\widetilde{\mathbb{F}})$ and the basis elements $\widetilde{b}(r;T)\bigotimes_S 1$ by $\sigma(\widetilde{b}(r;T))$.

Proposition 7.53. *Let $\mathscr{L}$ be a finite distributive lattice, and let $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ be the minimal multigraded free resolutions of $H_{\mathscr{L}}$ and $H_{\widetilde{\mathscr{L}}}$, respectively. Then the map $\pi:\sigma(\widetilde{\mathbb{F}})\to\mathbb{F}$ with $\pi(\sigma(\widetilde{b}(r;T)))=-b(r^T;T_r)$ is an isomorphism of complexes.*

PROOF. Let $|T|=i$. By Lemma 7.51, we have

$$
\begin{aligned}
(21)\quad & \pi_{i-1}(\widetilde{\partial}_i(\sigma(\widetilde{b}(r;T))))\\
= \;& \pi_{i-1}(\sum_{s\in T}(-1)^{\lambda(s\setminus r;T\setminus r)}(y_{s\setminus r}\widetilde{b}(r;T\setminus\{s\})-x_{s\setminus r}\widetilde{b}(s;s\vee(T\setminus\{s\}))))\\
= \;& \sum_{s\in T}(-1)^{\lambda(s\setminus r;T\setminus r)+(i-1)+1}(y_{s\setminus r}b(s^{s\vee(T\setminus\{s\})};(s\vee(T\setminus\{s\}))_s)-x_{s\setminus r}b(r^{T\setminus\{s\}};(T\setminus\{s\})_r))\\
= \;& \sum_{s\in T}(-1)^{\lambda(s\setminus r;T\setminus r)+i}(y_{s\setminus r}b(s^{s\vee(T\setminus\{s\})};(s\vee(T\setminus\{s\}))_s)-x_{s\setminus r}b(r^{T\setminus\{s\}};(T\setminus\{s\})_r)).
\end{aligned}
$$

Let $s\in T$ and $q=\vee\{s'\in T: s'\neq s\}$. Since $\mathscr{L}$ is a distributive lattice, the interval $[r,r^T]$ is a Boolean lattice. Hence $q\in T_r$, $r^T\setminus q=s\setminus r$ and $r^T\setminus T_r=T\setminus r$. Furthermore, we have

$$(s\vee(T\setminus\{s\}))_s=T_r\setminus q$$

and

$$r^{T\setminus\{s\}}=q,\quad (T\setminus\{s\})_r=q\wedge(T_r\setminus\{q\}).$$

These facts together with Theorem 7.48 yields

$$
\begin{aligned}
(22)\qquad & \partial_i(\pi_i(\sigma(\widetilde{b}(r;T)))) = \partial_i((-1)^i b(r^T;T_r)) \\
= \quad & \sum_{q\in T_r}(-1)^{\lambda(r^T\setminus q;T_r\setminus r_T)+i}(y_{r^T\setminus q}b(r^T;T_r\setminus q) - x_{r^T\setminus q}b(q;q\wedge(T_r\setminus\{q\})) \\
= \quad & \sum_{s\in T}(-1)^{\lambda(s\setminus r;T\setminus r)+i}(y_{s\setminus r}b(s^{s\vee(T\setminus\{s\})};(s\vee(T\setminus\{s\}))_s) - x_{s\setminus r}b(r^{T\setminus\{s\}};(T\setminus\{s\})_r)).
\end{aligned}
$$

Hence (21) and (22) imply that π is an isomorphism of complexes. □

Let $\widetilde{\mathbb{T}}$ and $\widetilde{\mathbb{F}}$ be the minimal free resolutions of $H_{\widetilde{\mathscr{J}}}$ and $H_{\widetilde{\mathscr{L}}}$ as described in Theorem 7.48, and let $\iota : \widetilde{\mathbb{T}} \to \widetilde{\mathbb{F}}$ be the injective complex homomorphism which maps the basis elements $\widetilde{b}(r;T)$ of $\mathbb{T}$ to the corresponding basis elements of $\mathbb{F}$. Then we have the following sequence of complex homomorphisms:

$$
\widetilde{\mathbb{T}} \xrightarrow{\ \iota\ } \widetilde{\mathbb{F}} \xrightarrow{\ \sigma\ } \sigma(\widetilde{\mathbb{F}}) \xrightarrow{\ \pi\ } \mathbb{F}.
$$

Let ψ be the map from $\mathrm{Tor}(K,H_{\widetilde{\mathscr{J}}})$ to $\mathrm{Tor}(K,H_{\mathscr{L}})$ induced by $\pi\circ\sigma\circ\iota$.

As a consequence of the previous proposition, we now have:

Corollary 7.54. *For each $i\geq 0$ the map $\psi_i : \mathrm{Tor}_i(K,H_{\widetilde{\mathscr{J}}}) \to \mathrm{Tor}_i(K,H_{\mathscr{L}})$ is injective and maps the basis elements $\widetilde{b}(r;T)$ of $\mathrm{Tor}_i(K,H_{\widetilde{\mathscr{J}}})$ to the basis elements $(-1)^{|T|}b(r^T;T_r)$ of $\mathrm{Tor}_i(K,H_{\mathscr{L}})$.*

Now we are ready to present one of the main results of this section:

Theorem 7.55. *Let $\mathscr{L}$ be a finite distributive lattice, $\mathscr{I}$ a poset ideal and $\mathscr{J}$ a poset coideal of $\mathscr{L}$ such that $\mathscr{I}\cup\mathscr{J}=\mathscr{L}$. Then $H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}}\cap H_{\mathscr{J}}$ if and only if for each pair $p,q\in\mathscr{L}$ with $q\in N(p)$, either $p\in\mathscr{I}$ or $q\in\mathscr{J}$.*

PROOF. We may assume that $|\mathscr{L}|>1$, because otherwise the assertions are trivial.

Notice that $H_{\mathscr{I}\cap\mathscr{J}} \subseteq H_{\mathscr{I}}\cap H_{\mathscr{J}}$ holds always, and $H_{\mathscr{I}}\cap H_{\mathscr{J}} \subseteq H_{\mathscr{I}\cap\mathscr{J}}$ if and only if all generators of $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ have degree r, where r is the rank of $\mathscr{L}$.

Consider the long exact Tor-sequence

$$
\begin{aligned}
&\longrightarrow \mathrm{Tor}_1(K,H_{\mathscr{I}})\oplus\mathrm{Tor}_1(K,H_{\mathscr{J}}) \xrightarrow{\ \beta_1\ } \mathrm{Tor}_1(K,H_{\mathscr{I}}+H_{\mathscr{J}}) \xrightarrow{\ \alpha_1\ } \mathrm{Tor}_0(K,H_{\mathscr{I}}\cap H_{\mathscr{J}}) \\
&\longrightarrow \mathrm{Tor}_0(K,H_{\mathscr{I}})\oplus\mathrm{Tor}_0(K,H_{\mathscr{J}}) \longrightarrow \mathrm{Tor}_0(K,H_{\mathscr{I}}+H_{\mathscr{J}}) \longrightarrow 0
\end{aligned}
$$

arising from the short exact sequence

$$
0\longrightarrow H_{\mathscr{I}}\cap H_{\mathscr{J}} \longrightarrow H_{\mathscr{I}}\oplus H_{\mathscr{J}} \longrightarrow H_{\mathscr{I}}+H_{\mathscr{J}} \longrightarrow 0.
$$

Since $\mathscr{L}=\mathscr{I}\cup\mathscr{J}$, it follows that $H_{\mathscr{L}} = H_{\mathscr{I}}+H_{\mathscr{J}}$, and since $H_{\mathscr{L}}$ has an r-linear resolution we have $\mathrm{Tor}_i(K,H_{\mathscr{L}})_{i+j}=0$ if $j\neq r$, and $\mathrm{Tor}_1(K,H_{\mathscr{L}})_{1+r}\neq 0$. Thus we see that all generators of $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ have degree r if and only if α_1 is a zero map, i.e, β_1 is a surjective map.

By Lemma 7.50 we have that the K-vector space $\beta_1(\mathrm{Tor}_1(K,H_{\mathscr{I}}))$ is spanned by the elements $b(p;\{q\})$ with $p\in\mathscr{I}$ and $q\in N(p)$, and by Corollary 7.54 we have that

$\beta_1(\mathrm{Tor}_1(K,H_{\mathscr{J}}))$ is spanned by the elements $b(q^{\{p\}};\{p\}_q) = b(p;\{q\})$ with $q \in \mathscr{J}$ and $p \in M(q)$. It follows that the image of β_1 is spanned by the subset

$$B' = \{b(p;\{q\})\colon\ p \in \mathscr{I} \text{ and } q \in N(p), \text{ or } q \in \mathscr{J} \text{ and } q \in N(p)\}$$

of the basis

$$B = \{b(p;\{q\})\colon\ p \in \mathscr{L} \text{ and } q \in N(p)\}$$

of $\mathrm{Tor}_1(K,H_{\mathscr{L}})$. Therefore, β_1 is surjective if and only if $B' = B$. This implies the assertion. □

It is clear that if $\mathscr{I} \cap \mathscr{J} \neq \emptyset$ and $H_{\mathscr{I}\cap\mathscr{J}} \neq H_{\mathscr{I}} \cap H_{\mathscr{J}}$, then not all generators of $H_{\mathscr{I}} \cap H_{\mathscr{J}}$ have the same degree. Therefore in this case, the ideal $H_{\mathscr{I}} \cap H_{\mathscr{J}}$ has no linear resolution. However in case $\mathscr{I} \cap \mathscr{J} \neq \emptyset$ and $H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}} \cap H_{\mathscr{J}}$ we have

Theorem 7.56. *Let $\mathscr{L}$ be a finite distributive lattice, $\mathscr{I}$ a poset ideal and $\mathscr{J}$ a poset coideal of $\mathscr{L}$ such that $\mathscr{I} \cup \mathscr{J} = \mathscr{L}$. If $H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}} \cap H_{\mathscr{J}}$, then the following statements are equivalent:*

(i) *$H_{\mathscr{I}} \cap H_{\mathscr{J}}$ has a linear resolution;*
(ii) *for each element $p \in \mathscr{L}$, either $p \in \mathscr{I}$ or $\wedge N(p) \in \mathscr{J}$;*
(iii) *for each element $r \in \mathscr{L}$, either $r \in \mathscr{J}$ or $\vee M(r) \in \mathscr{I}$.*

PROOF. We may assume that $|\mathscr{L}| > 1$.

(ii) $\Rightarrow$ (iii): Assume there exists some element $r \in \mathscr{L}$ such that $r \notin \mathscr{J}$ and $p = \vee M(r)$ does not belong to $\mathscr{I}$. Since $\mathscr{L}$ is a distributive lattice, the interval $[r,p]$ is a Boolean lattice. Hence $\wedge N(p) = r$. Therefore we have $p \notin \mathscr{I}$ and $r = \wedge N(p) \notin \mathscr{J}$, a contradiction.

By the same argument, one sees that (iii) implies (ii).

Now, we prove that the conditions (i) and (ii) are equivalent. Consider the long exact Tor-sequence

$$\cdots \to \mathrm{Tor}_{i+1}(K,H_{\mathscr{I}}) \oplus \mathrm{Tor}_{i+1}(K,H_{\mathscr{J}}) \xrightarrow{\beta_{i+1}} \mathrm{Tor}_{i+1}(K,H_{\mathscr{L}}) \xrightarrow{\alpha_{i+1}} \mathrm{Tor}_i(K,H_{\mathscr{I}} \cap H_{\mathscr{J}})$$
$$\longrightarrow \mathrm{Tor}_i(K,H_{\mathscr{I}}) \oplus \mathrm{Tor}_i(K,H_{\mathscr{J}}) \longrightarrow \mathrm{Tor}_i(K,H_{\mathscr{L}}) \longrightarrow \cdots$$

arising from the short exact sequence

$$0 \longrightarrow H_{\mathscr{I}} \cap H_{\mathscr{J}} \longrightarrow H_{\mathscr{I}} \oplus H_{\mathscr{J}} \longrightarrow H_{\mathscr{L}} \longrightarrow 0.$$

Here we used that $\mathscr{I} \cup \mathscr{J} = \mathscr{L}$, so that $H_{\mathscr{I}} + H_{\mathscr{J}} = H_{\mathscr{L}}$. Let $r = \operatorname{rank} \mathscr{L}$. Since the ideal $H_{\mathscr{I}} \cap H_{\mathscr{J}} = H_{\mathscr{I}\cap\mathscr{J}}$ is generated in degree r, it has a linear resolution if and only if

$$\mathrm{Tor}_{i+1}(K,H_{\mathscr{L}})_j \xrightarrow{\alpha_{i+1,j}} \mathrm{Tor}_i(K,H_{\mathscr{I}} \cap H_{\mathscr{J}})_j$$

is the zero map for all $j \neq i+r$ and all $i \geq 0$, since the ideals $H_{\mathscr{I}}$ and $H_{\mathscr{J}}$ have r-linear resolutions. Since the ideal $H_{\mathscr{L}}$ has an r-linear resolution, the map $\alpha_{i+1,j} = 0$ for $j \neq i+1+r$. Hence $H_{\mathscr{I}} \cap H_{\mathscr{J}}$ has a linear resolution if and only if $\alpha_{i+1,i+1+r} = 0$ for all $i \geq 0$, and this is the case if and only if $\beta_{i+1,i+1+r}$ is surjective for all $i \geq 0$.

We argue as in the proof of Theorem 7.55. The set

$$B_{i+1} = \{b(p;S)\colon\ p \in \mathscr{L} \text{ and } S \subset N(p),\ |S| = i+1\}$$

is a K-basis of $\operatorname{Tor}_{i+1}(K,H_{\mathscr{L}})_{i+1+r}$. Using Lemma 7.50 and Corollary 7.54 we see that the set

$$\begin{aligned} B'_{i+1} &= \{b(p;S)\colon p\in\mathscr{I} \text{ and } S\subset N(p),\, |S|=i+1\} \\ &\cup \{b(r^T;T_r)\colon r\in\mathscr{J} \text{ and } T\subset M(r),\, |T|=i+1\} \end{aligned}$$

spans the image of $\beta_{i+1,i+1+r}$. Thus $\beta_{i+1,i+1+r}$ is surjective if and only if $B'_{i+1}=B_{i+1}$ for all $i>0$. Note that $B'_{i+1}\subset B_{i+1}$. Suppose condition (ii) holds, and let $b(p;S)\in B_{i+1}$. If $p\in\mathscr{I}$, then $b(p;S)\in B'_{i+1}$. If $p\notin\mathscr{I}$, then $p\in\mathscr{J}$. Let $r=\wedge S$. It follows from condition (ii) that $r\in\mathscr{J}$. Let T be the set of upper neighbors of r in the interval $[r,p]$. Then $p=r^T$ and $S=T_r$, and hence $b(p;S)=b(r^T;T_r)$ belongs to B'_{i+1}.

Conversely assume that $B'_{i+1}=B_{i+1}$. In particular, for all $p\in\mathscr{L}$ we have $b(p;N(p))\in B'_{|N(p)|}$. So either $p\in\mathscr{I}$ or there is some $r\in\mathscr{J}$ and $T\subset M(r)$ such that $p=r^T$ and $T_r=N(p)$. Since $\wedge T_r=r$, it follows that $\wedge N(p)=r$ which is in $\mathscr{J}$. □

Up to now, we always assume that $\mathscr{I}\cup\mathscr{J}=\mathscr{L}$ and $\mathscr{I}\cap\mathscr{J}\neq\emptyset$. Now we consider the case $\mathscr{I}\cup\mathscr{J}=\mathscr{L}$ and $\mathscr{I}\cap\mathscr{J}=\emptyset$. In this case, $H_{\mathscr{I}\cap\mathscr{J}}\neq H_{\mathscr{I}}\cap H_{\mathscr{J}}$. However, we have:

Theorem 7.57. *Let $\mathscr{L}$ be a finite distributive lattice, $\mathscr{I}$ a poset ideal and $\mathscr{J}$ a poset coideal of $\mathscr{L}$. If $\mathscr{I}\cup\mathscr{J}=\mathscr{L}$ and $\mathscr{I}\cap\mathscr{J}=\emptyset$, then the ideal $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ has a linear resolution.*

This theorem follows immediately from the following two lemmata.

Lemma 7.58. *Under the assumption of Proposition 7.57, the ideal $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ is generated by the monomials* $\operatorname{lcm}(u_p,u_q)$ *with $p\in\mathscr{J}$, $q\in\mathscr{I}$ and $q\in N(p)$. In particular, all generators of $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ are of degree* $\operatorname{rank}\mathscr{L}+1$.

PROOF. Since $\mathscr{I}\cap\mathscr{J}=\emptyset$, all generators of $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ have degree greater than the rank of $\mathscr{L}$. Let $H=\langle\operatorname{lcm}(u_p,u_q): p\in\mathscr{J}, q\in\mathscr{I} \text{ and } q\in N(p)\rangle$. It is clear that $H\subseteq H_{\mathscr{I}}\cap H_{\mathscr{J}}$. Since $H_{\mathscr{I}}$ and $H_{\mathscr{J}}$ both are monomial ideals, the intersection $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ is again a monomial ideal. Let m be any monomial in $H_{\mathscr{I}}\cap H_{\mathscr{J}}$. Then there exist $r\in\mathscr{I}$ and $s\in\mathscr{J}$, such that $u_r|m$ and $u_s|m$. Let C be any chain between $r\wedge s$ and $r\vee s$. Since $\mathscr{I}$ is a poset ideal and $\mathscr{J}$ is a poset coideal, and $r\in\mathscr{I}$, $s\in\mathscr{J}$, we have $r\wedge s\in\mathscr{I}$ and $r\vee s\in\mathscr{J}$. Hence there exist $p,q\in C$ such that $q\in\mathscr{I}$, $p\in\mathscr{J}$ and q is a lower neighbor of p. We claim $\operatorname{lcm}(u_p,u_q)|m$. To see this, we write $m=\prod_{i=1}^n x^{a_i}y^{b_i}$ as m_xm_y where $m_x=\prod_{i=1}^n x^{a_i}$ and $m_y=\prod_{i=1}^n y^{b_i}$, and as before we write $u_t=x_{\ell(t)}y_{P\setminus\ell(t)}$, where P is the set of join irreducible elements of $\mathscr{L}$. Since $\ell(p)\subseteq\ell(r\vee s)$, we have $x_{\ell(p)}|x_{\ell(r\vee s)}$. Since $u_r|m$ and $u_s|m$, we have $x_{\ell(r\vee s)}|m$ and hence $x_{\ell(p)}|m_x$. By the same argument we see that $y_{P\setminus\ell(q)}|m_y$. Hence $\operatorname{lcm}(u_p,u_q)=x_{\ell(p)}y_{P\setminus\ell(q)}$ divides m. □

Lemma 7.59. *Let R be a polynomial ring over a field K, I and J ideals in R. Suppose I, J and $I+J$ have d-linear resolutions. If all elements of $G(I\cap J)$ have degree $d+1$, then the ideal $I\cap J$ has a $(d+1)$-linear resolution.*

PROOF. Consider the long exact Tor-sequence

$$\begin{array}{ccccccc}\cdots \to \mathrm{Tor}_{i+1}(K,I)\oplus \mathrm{Tor}_{i+1}(K,J) & \xrightarrow{\beta_{i+1}} & \mathrm{Tor}_{i+1}(K,I+J) & \xrightarrow{\alpha_{i+1}} & \mathrm{Tor}_i(K,I\cap J)\\ \longrightarrow \mathrm{Tor}_i(K,I)\oplus \mathrm{Tor}_i(K,J) & \longrightarrow & \mathrm{Tor}_i(K,I+J) & \longrightarrow & \cdots .\end{array}$$

Since I, J and $I+J$ have d-linear resolutions. It follows that $\mathrm{Tor}_i(K,I)_j=\mathrm{Tor}_i(K,J)_j=0$ for any $j\neq i+d$ and $\mathrm{Tor}_{i+1}(K,I+J)_j=0$ for any $j\neq i+1+d$. Hence $\mathrm{Tor}_i(K,I\cap J)_j=0$ for $j<i+d$ or $j>i+1+d$. Since $I\cap J$ is generated in degree $d+1$, we have $\mathrm{Tor}_i(K,I\cap J)_j=0$ for $j=i+d$. Therefore $\mathrm{Tor}_i(K,I\cap J)_j=0$ for any $j\neq i+1+d$. Hence $I\cap J$ has a $(d+1)$-linear resolution. □

In the remaining of this section we discuss some special classes of segments of a finite distributive lattice $\mathscr{L}$ to which our results apply and where some additional information can be obtained . As we have already seen, for any segment $\mathscr{S}$ there exist a poset ideal $\mathscr{I}$ and a poset coideal $\mathscr{J}$ such that $\mathscr{S}=\mathscr{I}\cap\mathscr{J}$. Now Let $\mathscr{L}$ be a finite distributive lattice of rank r. We consider a special class of segments $\mathscr{S}$ of $\mathscr{L}$ which consisting of all elements p in $\mathscr{L}$ such that $i\leq \operatorname{rank} p\leq j$ for some i and j with $0\leq i\leq j\leq r$. We denote it by $\mathscr{L}_{i,j}$.

Lemma 7.60. *Let $\mathscr{L}$ be a finite distributive lattice of rank r, and let $\mathscr{L}_{i,j}$ be a segment of $\mathscr{L}$. Then there exists a poset ideal $\mathscr{I}$ and a poset coideal $\mathscr{J}$ of $\mathscr{L}$ such that $H_{\mathscr{L}_{i,j}}=H_{\mathscr{I}\cap\mathscr{J}}=H_{\mathscr{I}}\cap H_{\mathscr{J}}$.*

PROOF. Let $\mathscr{I}=\{p\in\mathscr{L}:\operatorname{rank} p\leq j\}$, and $\mathscr{J}=\{p\in\mathscr{L}:\operatorname{rank} p\geq i\}$. Then $\mathscr{I}$ is a poset ideal, $\mathscr{J}$ is a poset coideal of $\mathscr{L}$, and $\mathscr{L}_{i,j}=\mathscr{I}\cap\mathscr{J}$. It remains to show that $H_{\mathscr{I}\cap\mathscr{J}}=H_{\mathscr{I}}\cap H_{\mathscr{J}}$. Let $p,q\in\mathscr{L}$ and $q\in N(p)$. If $p\notin\mathscr{I}$, then $\operatorname{rank} p>j$. Hence $\operatorname{rank} q=\operatorname{rank} p-1\geq j\geq i$, i.e., $q\in\mathscr{J}$. The assertion follows from Proposition 7.55. □

With the assumptions and notation of the previous lemma, the ideal $H_{\mathscr{L}_{i,j}}$ has a linear resolution if and only if $H_{\mathscr{I}}\cap H_{\mathscr{J}}$ has a linear resolution.

Corollary 7.61. *Let $\mathscr{L}\neq\{\hat{0},\hat{1}\}$ be a finite distributive lattice and $\mathscr{S}=\mathscr{L}\setminus\{\hat{0},\hat{1}\}$. Then*

(i) *$H_{\mathscr{S}}$ has a linear resolution if and only if $\mathscr{L}$ is not a Boolean lattice;*

(ii) *in case $\mathscr{L}$ is the Boolean lattice $\mathscr{B}_r$, the ideal $H_{\mathscr{S}}$ has the following minimal free resolution:*

$$\mathbb{T}\colon 0\longrightarrow T_{r-1}\longrightarrow\cdots T_1\longrightarrow T_0\longrightarrow H_{\mathscr{S}}\longrightarrow 0.$$

with $T_i=S^{\binom{r}{i}(2^{r-i}-2)}(-r-i)$ for $i=0,\dots,r-2$ and $T_{r-1}=S(-2r)$.

PROOF. (i) Let $\mathscr{I}=\{p\in\mathscr{L}:\operatorname{rank} p\leq\operatorname{rank}\mathscr{L}-1\}$, and $\mathscr{J}=\{p\in\mathscr{L}:\operatorname{rank} p\geq 1\}$. Then by Lemma 7.60, $\mathscr{I}$ and $\mathscr{J}$ are the poset ideal and poset coideal of $\mathscr{L}$ such that $H_{\mathscr{S}}=H_{\mathscr{I}}\cap H_{\mathscr{J}}$. The distributive lattice $\mathscr{L}$ is a Boolean lattice if and only if the meet of all lower neighbors of $\hat{1}$ is $\hat{0}$. Since $\hat{1}$ is the only element which is not in $\mathscr{I}$, and $\hat{0}$ is the only element which is not in $\mathscr{J}$, by using the Theorem 7.56, we have $H_{\mathscr{S}}$ has a linear resolution if and only if $\mathscr{L}$ is not a Boolean lattice.

(ii) Choose $\mathscr{I}$ and $\mathscr{J}$ as in the proof of (i). Hence $H_{\mathscr{S}} = H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}} \cap H_{\mathscr{J}}$. We consider long exact Tor-sequence as in the proof of Theorem 7.56. Notice that $(\hat{1}, N(\hat{1}))$ is the only pair with the form (p,S) with $p \in \mathscr{B}_r$ and $S \subset N(p)$ such that $p \notin \mathscr{I}$ and $\wedge S \notin \mathscr{J}$. It follows that the map β_i is surjective for $i < r$. Hence for all $i < r-1$ we have the exact sequence

$$0 \longrightarrow \mathrm{Tor}_i(K, H_{\mathscr{S}}) \longrightarrow \mathrm{Tor}_i(K, H_{\mathscr{I}}) \oplus \mathrm{Tor}_i(K, H_{\mathscr{J}}) \longrightarrow \mathrm{Tor}_i(K, H_{\mathscr{B}_r}) \longrightarrow 0, \tag{23}$$

and so

$$b_i(H_{\mathscr{S}}) = b_i(H_{\mathscr{I}}) + b_i(H_{\mathscr{J}}) - b_i(H_{\mathscr{B}_r}) \quad \text{for} \quad i < r-1,$$

where $b_i(I)$ is the i-th Betti number of the ideal I. By using Theorem 7.48, Corollary 7.54 and the combinatorial fact that each Boolean lattice $\mathscr{B}_r$ contains $\binom{r}{i}2^{r-i}$ Boolean sublattices $\mathscr{B}_i$, we have $b_i(H_{\mathscr{I}}) = b_i(H_{\mathscr{J}}) = \binom{r}{i}2^{r-i} - \binom{r}{i}$ and $b_i(H_{\mathscr{L}}) = \binom{r}{i}2^{r-i}$. Hence

$$b_i(H_{\mathscr{S}}) = 2(\binom{r}{i}2^{r-i} - \binom{r}{i}) - \binom{r}{i}2^{r-i} = \binom{r}{i}(2^{r-i} - 2),$$

for any $i < r-1$. It also follows from (23) that the resolution of $H_{\mathscr{S}}$ is linear up to homological degree $r-2$.

Now let $i = r-1$. Since $\mathrm{Tor}_r(K, H_{\mathscr{I}}) = \mathrm{Tor}_r(K, H_{\mathscr{J}}) = 0$, we get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Tor}_r(K, H_{\mathscr{B}_r}) \to \quad & \mathrm{Tor}_{r-1}(K, H_{\mathscr{S}}) \quad \to \mathrm{Tor}_{r-1}(K, H_{\mathscr{I}}) \oplus \mathrm{Tor}_{r-1}(K, H_{\mathscr{J}}) \\ \longrightarrow \quad & \mathrm{Tor}_{r-1}(K, H_{\mathscr{B}_r}) \to 0. \end{aligned}$$

Since $\dim_K \mathrm{Tor}_{r-1}(K, H_{\mathscr{I}}) \oplus \mathrm{Tor}_{r-1}(K, H_{\mathscr{J}}) = \dim_K \mathrm{Tor}_{r-1}(K, H_{\mathscr{J}}) = 2r$, it follows that $\mathrm{Tor}_{r-1}(K, H_{\mathscr{I}}) \oplus \mathrm{Tor}_{r-1}(K, H_{\mathscr{J}}) \to \mathrm{Tor}_{r-1}(K, H_{\mathscr{B}_r})$ is an isomorphism. Hence

$$\mathrm{Tor}_{r-1}(K, H_{\mathscr{S}}) \cong \mathrm{Tor}_r(K, H_{\mathscr{B}_r}) \cong K(-2r),$$

as desired. □

Using Lemma 7.60 and Theorem7.56, we have the following two facts:

Corollary 7.62. *Let $\mathscr{L}$ be a finite distributive lattice, and i an integer. If $|\mathscr{L}_{i,i}| > 1$, then the ideal $H_{\mathscr{L}_{i,i}}$ has no linear resolution.*

PROOF. Let $\mathscr{I} = \{p \in \mathscr{L} : \operatorname{rank} p \le i\}$ and $\mathscr{J} = \{p \in \mathscr{L} : \operatorname{rank} p \ge i\}$. Then by Lemma 7.60 we have $H_{\mathscr{L}_{i,i}} = H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}} \cap H_{\mathscr{J}}$. Since $\mathscr{L}$ is distributive, there exist elements u and v in $\mathscr{L}_{i,i}$ such that $\operatorname{rank}(u \vee v) = i+1$. Hence $\operatorname{rank}(\wedge N(u \vee v)) < \operatorname{rank} u = i$. Therefore we have $u \vee v \notin \mathscr{I}$ and $\wedge N(u \vee v) \notin \mathscr{J}$. By Theorem 7.56, $H_{\mathscr{L}_{i,i}}$ has no linear resolution. □

Corollary 7.63. *If $\mathscr{L}$ is a finite planar distributive lattice, then the ideal $H_{\mathscr{L}_{i,j}}$ has a linear resolution, if $i < j$.*

PROOF. Let $\mathscr{I} = \{p \in \mathscr{L} : \operatorname{rank} p \le j\}$ and $\mathscr{J} = \{p \in \mathscr{L} : \operatorname{rank} p \ge i\}$. Then $H_{\mathscr{L}_{i,j}} = H_{\mathscr{I}\cap\mathscr{J}} = H_{\mathscr{I}} \cap H_{\mathscr{J}}$. Since $\mathscr{L}$ is a planar distributive lattice, each element p in $\mathscr{L}$ has at most two lower neighbors. Hence $\operatorname{rank} \wedge N(p) \ge p-2$. Thus if $p \notin \mathscr{I}$, then $\operatorname{rank} p > j$. Therefore since $i < j$, we have $\operatorname{rank} q \ge \operatorname{rank} p - 2 \ge i$, i.e., $q \in \mathscr{J}$. By Theorem 7.56, $H_{\mathscr{L}_{i,j}}$ has a linear resolution. □

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