Nonlinear elliptic-parabolic integro-differential equations with L^1-data: existence, uniqueness, asymptotics

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Contents

Chapter 1

Introduction

In this thesis we consider the history dependent initial-boundary value problem

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\kappa\big(b(v(t,x)) - b(v_0(x))\big) + \int_0^t k(t-s)\big(b(v(s,x)) - b(v_0(x))\big)\,ds\Big) \\
&\qquad = \operatorname{div}\ a(x, Dv(t,x)) + f(t,x) \quad \text{for } (t,x) \in Q := (0,T)\times\Omega, \\
&b(v)(0,\cdot) = b(v_0) \quad \text{in } \Omega, \\
&v(t,x) = 0 \quad \text{for } (t,x) \in \Gamma := (0,T)\times\partial\Omega
\end{aligned} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. We assume that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray-Lions conditions which defines a bounded continuous coercive operator from $W_0^{1,p}(\Omega)$ into its dual space for some $1 < p < \infty$. However, some of our results will also apply to (1.1) with the operator $-\operatorname{div}\ a(x, Dv)$ replaced by the operator $-\operatorname{div}\ a(x, v, Dv)$, and some will even be applicable to conservation laws with memory. Additionally, we impose the following assumptions on κ, k and b.

$\kappa \geq 0$ and $k : (0,\infty) \to \mathbb{R}$ is a nonnegative, nonincreasing function such that $k \in L^1_{\mathrm{loc}}([0,\infty))$ and $\kappa + \int_0^t k(s)\,ds > 0$ for all $t > 0$. (1.2)

$b : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, satisfying the normalization condition $b(0) = 0$. (1.3)

Note that these assumptions are quite general. As a special case, the degenerated elliptic-parabolic initial boundary value problem

$$\begin{aligned}
b(v)_t &= \operatorname{div}\ a(x, Dv) + f \quad \text{in } Q, \\
b(v)(0,\cdot) &= b(v_0) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \Gamma,
\end{aligned} \tag{1.4}$$

is included. Defining $\kappa = 0$ and $k(t) := t^{-\gamma}/\Gamma(1-\gamma)$, one easily sees that our assumptions also include the case of a fractional derivative of order $0 < \gamma < 1$ in time, i.e.

$$\begin{aligned} \frac{\partial^\gamma}{\partial t^\gamma} b(v) &= \operatorname{div}\ a(x, Dv) + f && \text{in } Q, \\ b(v)(0, \cdot) &= b(v_0) && \text{in } \Omega, \\ v &= 0 && \text{on } \Gamma. \end{aligned} \tag{1.5}$$

Here, the fractional derivative of order $0 < \gamma < 1$ is defined by

$$\begin{aligned} \frac{\partial^\gamma}{\partial t^\gamma} u(t) &:= \frac{\partial}{\partial t} \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} u(s)\, ds, \qquad t > 0, \\ \frac{\partial^\gamma}{\partial t^\gamma} u(0) &:= \frac{1}{h} \lim_{h\to 0+} \int_0^h \frac{(h-s)^{-\gamma}}{\Gamma(1-\gamma)} u(s)\, ds, \end{aligned} \tag{1.6}$$

where $u \in C([0,T]; L^1(\Omega))$ satisfying $u(0,\cdot) = 0$, c.f. [Zyg59, Chapter XII.8].
We recall that equations of the form (1.4) and (1.5) are obtained when modelling the transport of fluids in porous media. For the fractional derivative case (1.5), we refer to [Cap99, Cap00] and the references therein. Indeed, in geothermal areas the fluid may precipitate minerals in the pores of the medium, thus diminishing their size. To study such situations, Darcy's law has to be modified inducing a memory term. As shown in [Cap99], this leads to a formulation as in (1.5) for $0 < \gamma < 1$.
But also the case $\kappa > 0$ and $k \not\equiv 0$ is of particular interest. Indeed, a second application for (1.1) is the nonlinear heat flow in certain dielectric materials at very low temperatures. In this situation, finite speed of propagation of thermal disturbances has been observed experimentally. Several models to describe this phenomenon have been introduced. In particular, [GP68, Mac77] and [Nun71] introduce a model in which the constitutive relations for the internal energy and heat flux, in difference to Fourier's law, also depend on the history of the temperature and the temperature gradient, respectively. As shown in [CN81], this yields a problem of the form (1.1) under certain assumptions on the internal energy and heat flux relaxation functions.
We remark that the assumptions on κ, k in (1.2) are motivated by the fact that one wants to insure positivity of solutions. In several applications this is a physically necessary assumption. Indeed, when modelling the nonlinear heat flow in materials with memory one assumes $v(t, x)$ of problem (1.1) to denote the absolute temperature in $x \in \Omega$ at time t. Such assumptions were first introduced in [CN79] and lead to the notion of complete positivity, c.f. [CN81] and [CM88].
It is our intention to develop a solution theory for (1.1) in the state space $L^1(\Omega)$. Indeed, as the space $L^1(\Omega)$ is the natural setting for several evolution problems, such as the transport of fluids in porous media and heat conduction, we are interested in solving (1.1) for general

L^1-data, i.e. for

$$v_0 : \Omega \to \overline{\mathbb{R}} \text{ measurable with } b(v_0) \in L^1(\Omega),$$
$$f \in L^1(Q).$$

We recall that the case of the elliptic problem

$$\begin{aligned} b(v) &= \text{div } a(x, Dv) + f \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

and of the elliptic-parabolic problem (1.4) with general L^1-data has been investigated by several authors in recent years. Note that, when considering weak solutions, i.e. solutions in the sense of distributions, of (1.7) and (1.4), respectively, one encounters a problem of nonexistence of solutions for small p, i.e., for $1 < p < 2 - \frac{1}{N}$, see [BBG$^+$95, Appendix I]. Moreover, as shown in a counterexample given in [Ser64] for a linear problem, weak solutions are in general not unique. These problems of nonexistence and nonuniqueness of weak solutions carry over to the history dependent case.

In order to overcome the above mentioned problems of nonexistence and nonuniqueness of weak solutions, new notions of solutions for the elliptic and the elliptic-parabolic problem, i.e. for (1.7) and (1.4), have been introduced. One concept to guarantee uniqueness is the concept of *renormalized solutions*, which was first introduced in [DL89] for the study of the Boltzmann equation. In [BGDM93] and [Mur93], this concept was then applied to an elliptic problem, and was extended to parabolic problems in [BM97]. The second concept, which can be shown to be equivalent to the concept of renormalized solutions for the elliptic and parabolic problems, is the concept of *entropy solutions* introduced in [BBG$^+$95]. See also [AMSdLT99] for the extension to parabolic equations.

We compare the two concepts of solutions in order to develop a solution theory for the history dependent degenerated elliptic-parabolic problem (1.1) in the state space $L^1(\Omega)$. It turns out that only the notion of entropy solutions can naturally be extended to our problem (1.1) for general κ, k satisfying (1.2). This is due to the fact that the derivative in time operator in (1.1) does not satisfy a Kato equality, but only a Kato inequality. For the new notion of entropy solutions of (1.1) that we introduce, existence and uniqueness of solutions are shown under certain assumptions.

It turns out that the notion of entropy solutions of (1.1) coincides for $u = b(v)$ and $u_0 = b(v_0)$ with the notion of *generalized solutions*, see [Gri85, CGL96], of the abstract Volterra equation

$$\frac{d}{dt}\left(\kappa(u(t) - u_0) + \int_0^t k(t-s)(u(s) - u_0)\, ds\right) + Au(t) \ni f(t), \quad t \in [0, T) \tag{1.8}$$

in $L^1(\Omega)$. Here, A is an m-accretive, or m-completely accretive, possibly multivalued, operator in $L^1(\Omega)$ corresponding to b and $-\text{div } a(x, Dv)$.

Moreover, we investigate the regularity of solutions by considering generalized solutions of the abstract Volterra equation (1.8). As shown in [Gri85], the existence of strong solutions of (1.8) can be obtained for $\kappa = 0$, even if the generalized solution itself is not differentiable a.e. on the interval $(0, T)$. For $\kappa > 0$, this can not be achieved in general. Therefore, we recall that for an arbitrary m-accretive operator A in a general real Banach space X, Lipschitz continuity of the generalized solution u of (1.8) is known by [Gri85, Theorem 2] if $u_0 \in \hat{D}(A)$ and $f \in BV([0, T); X)$. Since the Radon-Nikodym property of the space X is equivalent to the differentiability of absolutely continuous functions $u : (0, T) \to X$, we can conclude that the generalized solution u of (1.8) is in fact a strong solution if X has the Radon-Nikodym property and $u_0 \in \hat{D}(A)$ and $f \in BV(0, T; X)$.
However, in spaces without the Radon-Nikodym property, such as $L^1(\Omega)$, no general regularity results are known. In the case of $\kappa > 0$, we therefore restrict our investigation to m-completely accretive operators A in a normal Banach space $X \subset L^1(\Omega)$. We show that under these assumptions regularity results can be achieved that are quite similar to those known for accretive operators in Banach spaces having the Radon-Nikodym property.
One of the most important qualitative questions in evolution problems is the one on the long-time behavior. One is interested in stability, the existence of equilibria, or periodic solutions, or those which are close-to-periodic, such as asymptotically almost periodic solutions, or weakly almost periodic solutions in the sense of Eberlein. The results obtained here extend those of [Kre92, Kre96] for mild solutions of nonlinear Cauchy problems and those of [CN81] on nonlinear Volterra equations. We give sufficient conditions such that the generalized solution of (1.8) is asymptotically almost periodic, respectively weakly almost periodic in the sense of Eberlein. Since, in applications, one is often only interested in the asymptotic behavior of the mean of the solution, we give a characterization of the almost periodic part of the generalized solution as a solution of a certain limit equation. Here, we only assume that the operator A is m-accretive. Thus, the results obtained apply to (1.1) as well as to conservation laws with memory as considered in [CGL96].
In **chapter 2** we consider the regularity of generalized solutions of the abstract Volterra equation (1.8). The main tool is an integral inequality developed in section 2.1, see proposition 2.1, allowing us to compare two generalized solutions for different data. The remainder of section 2.1 is devoted to simple applications of this integral inequality. In section 2.2 we investigate the existence of strong solutions of (1.8) in Banach spaces without the Radon-Nikodym property. Our main result is given in proposition 2.14, stating the existence of strong solutions for m-completely accretive operators in normal Banach spaces $X \subset L^1(\Omega)$ satisfying a strong convergence condition. This result is essential for the existence of weak solutions of (1.1) for regular data.
In **chapter 3** we introduce a notion of entropy solutions for the history dependent degenerated elliptic-parabolic problem (1.1), see definition 3.3, and show existence and uniqueness of solutions. Section 3.1 is concerned with the question of nonexistence and nonuniqueness

of weak solutions. In section 3.2 a Kato inequality for the time derivative operator in (1.1) is developed, see proposition 3.23 and corollary 3.24. We also give a motivation for the fact that the concept of renormalized solutions is not applicable in a straightforward way in the context of (1.1). The uniqueness of entropy solutions is shown in section 3.3, see theorem 3.26. An essential tool is Kruzhkov's method of doubling variables, which was first introduced in [Kru70]. Since we need some assumptions on the continuity of solutions, the final uniqueness result, corollary 3.31, is a consequence of the existence result for $b \equiv \mathrm{id}$, theorem 3.30, presented in section 3.4. The remainder of section 3.4 is concerned with open problems in the theory of entropy solutions, such as the strong convergence of an approximating sequence of solutions in $L^p(0,T;W_0^{1,p}(\Omega))$ and the existence of entropy solutions for general $b \not\equiv \mathrm{id}$.

Chapter 4 is concerned with the asymptotic behavior of generalized solutions of (1.8). The main estimate used for the proof of existence of asymptotically almost periodic solutions and of weakly almost periodic solutions in the sense of Eberlein is obtained in proposition 4.2 of section 4.1. It is then applied to show that generalized solutions are asymptotically almost periodic under certain assumptions, see theorem 4.10. In section 4.2 we construct a solution of a certain limit equation on the whole real line characterizing the almost periodic part of solutions of the initial value problem (1.8), see theorems 4.11 and 4.20. Section 4.3 is devoted to the study of weakly almost periodic solutions in the sense of Eberlein. Existence of such solutions is shown in theorems 4.16 and 4.19. A main tool used in this section is Grothendieck's double limit criterion for weak compactness in $(C_b(\mathbb{R}_+), \|\cdot\|_\infty)$, see [RS89, Theorem 2.1] for the Banach space valued version. We remark that the results obtained apply to (1.1) as well as to conservation laws with memory.

We conclude this introductory chapter by the investigation of two applications, in which equations of the type (1.1) naturally occur, and by stating preliminary results on the abstract Volterra equation (1.8). The applications, diffusion of fluids in porous media with memory and heat flow in materials with memory, are presented in section 1.1 and section 1.2, respectively. The main facts on the abstract Volterra equation (1.8) are given in section 1.3.

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1.1 Diffusion of fluids in porous media with memory

In this section, we present a model for the diffusion of fluids in porous media with memory introduced in [Cap99]. We first remark that the diffusion of fluids in a porous medium situated in $\Omega \subset \mathbb{R}^3$ is determined by the law of conservation of mass

$$m_t + \mathrm{div}\;\; q = h. \tag{1.9}$$

Here, $m = m(t,x)$ denotes the average mass of the fluid, and $q = q(t,x)$ is a vector field describing the mass flow of the fluid in the porous medium. Moreover, $h = h(t,x)$ denotes the external sources.
It is clear that the mass m of the fluid is the product of the porosity ϕ of the porous medium, the density ϱ of the fluid, and the saturation $S = S(p)$ of the porous medium depending on the pressure $p = p(t,x)$. Thus, we have

$$m = \phi\varrho S(p). \tag{1.10}$$

Here, for simplicity, we assume the porosity ϕ, and the density $\varrho = \varrho_0 > 0$ to be constant. Thus, we only investigate the incompressible case. Additionally, we need a constitutive relation for the flux q determined by the specific type of medium. Most authors who have studied diffusions in porous media use the classical empirical law of Darcy, stating dependence of the flux q on the gradient of the pore pressure p, i.e.

$$q = -\varrho\kappa(p)Dp, \tag{1.11}$$

where $\kappa = \kappa(p)$ denotes the permeability of the porous medium possibly depending on the pressure p.
However, some fluids may react chemically with the medium, enlarging the pores and some fluids carry solid particles that may obstruct some of the pores. In geothermal regions the fluid may precipitate minerals in the pores of the medium. In particular, steam wells used for heat extraction in such areas are often self-sealed in a relatively short time. These phenomena create permeability changes that can occur locally, see [Cap00] and the references therein.

In [Cap99], see also [Cap00], a modification of Darcy's law is suggested including the history of the pressure gradient. In particular, if one considers the case that permeability diminishes with time, the effect of the fluid pressure at the boundary on the flow of the fluid through the medium is delayed and the flow occurs as if the medium had a memory. Thus, the modified Darcy's law may be written as

$$q = -\frac{\partial^\gamma}{\partial t^\gamma}(\varrho\kappa(p)Dp). \tag{1.12}$$

Here, the fractional derivative of order γ, defined by (1.6), with $0 < \gamma < 1$ is used to model the decrease of permeability. We remark that in [Cap00] a different notion of fractional derivative is used. For the relation between these two notions we refer to [Mai96].
The law of conservation of mass together with (1.10) and the modification of Darcy's law (1.12) yield

$$(\phi S(p))_t - \frac{\partial^\gamma}{\partial t^\gamma}\,\text{div}\,(\kappa(p)Dp) = \frac{h}{\varrho}. \tag{1.13}$$

Integrating this equation over $(0,t)$ and using the initial condition $S(p(0,\cdot)) = S(p_0)$, we obtain

$$\phi S(p(t,x)) - \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,\text{div}\,(\kappa(p(s,x))Dp(s,x)) = \phi S(p_0(x)) + \int_0^t \frac{h(s,x)}{\varrho}\,ds. \tag{1.14}$$

Since the measure α on $[0,\infty)$, defined by

$$\alpha(A) := \int_A \frac{\tau^{-\gamma}}{\Gamma(1-\gamma)}\,d\tau$$

for all measurable subsets $A \subset [0,\infty)$, is completely positive (see definition A.5), we can transform (1.14) to the problem

$$\begin{aligned} \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}(\phi S(p)) - \text{div}\,(\kappa(p)Dp) &= \tilde{h} && \text{in } Q = (0,T)\times\Omega, \\ S(p)(0,\cdot) &= S(p_0) && \text{in } \Omega, \\ p &= 0 && \text{on } \Gamma = (0,T)\times\partial\Omega. \end{aligned} \tag{1.15}$$

Here, for simplicity, we assumed Dirichlet boundary conditions. Hence, for constant κ this model of diffusion of fluids in porous media yields a problem of the form (1.1).

1.2 Heat flow in materials with memory

In this section, we present a model of heat conduction in materials of fading memory type. As we will see, this model leads to a special class of history dependent initial-boundary value problems.

For simplicity, we limit our attention to the heat conduction in a homogeneous material of unit density which is situated in a bounded domain $\Omega \subset \mathbb{R}^N$. In the following, $\vartheta = \vartheta(t,x) > 0$ denotes the absolute temperature of the material at time t at $x \in \Omega$. Assuming the absence of deformation, the law of balance of energy reduces to

$$\varepsilon_t + \text{div } q = h, \tag{1.16}$$

where $\varepsilon = \varepsilon(t,x)$ is the specific internal energy, and $q = q(t,x)$ is a vector field representing the heat flux. Moreover, $h = h(t,x)$ is the external heat supply.
Equation (1.16) has to be supplemented with constitutive assumptions for the internal energy and the heat flux characterizing the particular type of material.
According to Fourier's classical theory of heat conduction, the internal energy and the heat flux are assumed to be functions of the temperature ϑ and of the temperature gradient $D\vartheta$, respectively. In particular, considering the nonlinear heat flow, the constitutive equations are given by

$$\varepsilon = b(\vartheta) \tag{1.17}$$
$$q = -a(D\vartheta). \tag{1.18}$$

Here, we assume that $a : \mathbb{R}^N \to \mathbb{R}^N$ is monotone and satisfies a coercivity assumption and has a growth bound. Moreover, we assume that b is nondecreasing. Thus, the constitutive equations (1.17), (1.18), together with the law of balance of energy, yield the elliptic-parabolic equation for the temperature ϑ

$$b(\vartheta)_t - \text{div } a(D\vartheta) = h. \tag{1.19}$$

We recall that (1.19) predicts infinite speed of propagation for thermal disturbances. Despite this prediction, Fourier's theory provides a description of heat conduction that is useful under a wide range of conditions. However, there are situations in which differences to the prediction by Fourier's law can be observed experimentally. In particular, "wavelike" pulses of heat that propagate with finite speed have been observed in certain dielectrics at very low temperatures, see [BH88] and the references therein.
There have been several attempts to overcome the problem of infinite propagation speed of thermal disturbances, and to develop a theory of heat conduction that yields finite speed of propagation. To our knowledge, the first such theory was given by Cattaneo in [Cat48], who suggested to replace the constitutive equation (1.18) by

$$\kappa(\vartheta)q_t + q = -a(D\vartheta) \tag{1.20}$$

with $\kappa > 0$.

Following the approach in [CN81], we assume that the constitutive relations for the internal energy and the heat flux are given by

$$\varepsilon(t,x) = \beta_0 b(\vartheta(t,x)) + \int_{-\infty}^{t} \beta(t-s) b(\vartheta(s,x))\, ds, \qquad t\in\mathbb{R},\ x\in\Omega, \tag{1.21}$$

$$q(t,x) = -\gamma_0 a(D\vartheta(t,x)) + \int_{-\infty}^{t} \gamma(t-s) a(D\vartheta(s,x))\, ds, \qquad t\in\mathbb{R},\ x\in\Omega. \tag{1.22}$$

Here, $\beta_0, \gamma_0 > 0$ are positive constants and $\beta, \gamma : [0,\infty) \to \mathbb{R}$ are assumed to be sufficiently smooth functions called the internal energy and the heat flux relaxation function. In the physical literature these functions are usually taken as finite linear combinations of decaying exponentials with positive coefficients. However, we consider a more general physically reasonable class of relaxation functions. Still, we require that β, γ are bounded nonnegative nonincreasing and that β and $\log\gamma$ are convex. Moreover, we assume that $\beta, \gamma \in L^1(0,\infty)$ and that

$$\gamma_0 - \int_0^\infty \gamma(\tau)\, d\tau > 0. \tag{1.23}$$

This assumption is physically reasonable. Indeed, this assumption insures "forward" heat flow at equilibrium. Consider the one dimensional case $\Omega = (0,1)$, and suppose that the temperature $\vartheta(t,x)$ converges to an equilibrium temperature $\bar\vartheta(x)$ as $t\to\infty$. For definiteness, assume that $\bar\vartheta_x(x) > 0$ for $x \in (0,1)$. Then, by the constitutive relation (1.22), we obtain

$$\lim_{t\to\infty} q(t,x) = -\left(\gamma_0 - \int_0^\infty \gamma(\tau)\, d\tau\right) a(\bar\vartheta_x) < 0.$$

Here, we used the fact that a is monotone and satisfies a coercivity assumption. Thus, condition (1.23) insures that the equilibrium flux is negative, which guarantees "forward" heat flow. Recall that even in the linear case the "backward" heat equation does not, in general, lead to well posed problems.
Additionally, we assume that

$$\beta'(t) + \frac{\gamma(0)}{\gamma_0}\beta(t) \le 0 \qquad \text{for all } t \ge 0. \tag{1.24}$$

Note that this assumption is not a severe restriction. Indeed, if we assume that $\beta(t) = \sum_{k=1}^n b_k \exp(-\beta_k t)$ with $b_k > 0$ and $0 < \beta_1 < \ldots < \beta_n$, then, since $\log(\beta)$ is convex and nonincreasing, condition (1.24) is satisfied if

$$\lim_{t\to\infty} \frac{\beta'(t)}{\beta(t)} < -\frac{\gamma(0)}{\gamma_0}.$$

Thus, in this case it is sufficient to require that $\beta_1 \ge \gamma(0)/\gamma_0$.

Without loss of generality we can assume that the material is at zero temperature up to time $t = 0$, otherwise one has to incorporate the history of the temperature up to time $t = 0$ into the forcing term h. Then the constitutive relations (1.21), (1.22) and the law of balance of energy (1.16) yield the equation

$$\big[\beta_0 b(\vartheta) + (\beta * b(\vartheta))\big]_t - \gamma_0 \operatorname{div}\ a(D\vartheta) + (\gamma * \operatorname{div}\ a(D\vartheta)) = h \quad \text{in } Q := (0,\infty)\times\Omega. \quad (1.25)$$

Here, $f * g$ denotes the convolution of two functions defined by $f * g := \int_0^t f(t-s)g(s)\, ds$. Moreover, for simplicity, we assume Dirichlet boundary conditions, i.e. $\vartheta(t,x) = 0$ on $\Gamma := (0,\infty)\times\partial\Omega$.

We show that the above assumptions on the parameters β_0, γ_0, β and γ allow us to transform (1.25) into an equation given in (1.1). Define

$$\begin{aligned} C(t) &:= \gamma_0 - \int_0^t \gamma(\tau)\, d\tau, \qquad t \ge 0, \\ G(t,x) &:= \beta_0 b(\vartheta_0(x)) + \int_0^t h(\tau,x)\, d\tau, \qquad t\ge 0,\ x\in\Omega, \end{aligned}$$

where ϑ_0 denotes the temperature at time $t = 0$. Note that

$$\gamma_0 \operatorname{div}\ a(D\vartheta) - (\gamma * (\operatorname{div}\ a(D\vartheta))) = \frac{\partial}{\partial t}\big[C * (\operatorname{div}\ a(D\vartheta))\big].$$

Thus, we can integrate equation (1.25) using the initial condition $b(\vartheta_0(x)) = b(\vartheta(0,x))$ and obtain

$$\beta_0 b(\vartheta) + (\beta * b(\vartheta)) = (C * (\operatorname{div}\ a(D\vartheta))) + G \quad \text{in } Q. \quad (1.26)$$

Defining the resolvent r_β of β to be the unique solution of the convolution equation

$$\beta_0 r_\beta + \beta * r_\beta = \beta, \qquad t \ge 0,$$

it is clear that $r_\beta \in L^1_{\text{loc}}([0,\infty))$. Thus, we can define $c : [0,\infty) \to \mathbb{R}$ by

$$c := C - (r_\beta * C).$$

By simple calculation one verifies that we can rewrite (1.26) as

$$\beta_0 b(\vartheta) = (c * (\operatorname{div}\ a(D\vartheta))) + G - r_\beta * G \quad \text{in } Q.$$

As shown in [CN81, Lemma 4.2], the assumptions imposed on β,γ_0, β and γ imply that the measure α defined by $\alpha(A) := \int_A c(\tau)\ d\tau$ for all measurable subsets $A \subset [0,\infty)$ is completely positive (see definition A.5). Moreover, there exist $\kappa = 1/\gamma_0 > 0$ and $k \in L^1(0,\infty)$ nonnegative nonincreasing such that

$$\kappa \frac{c}{\beta_0} + (k * \frac{c}{\beta_0}) = 1, \qquad t \ge 0.$$

Thus, the problem can be transformed to a problem of the form (1.1).
Finally, we refer to [BH88] for the problem of compatibility of constitutive relations (1.21), (1.22) and the above assumptions on the parameters β_0, γ_0, β and γ with the second law of thermodynamics.

1.3 Abstract Volterra equations

In this section we recall the basic results on the existence of solutions of the abstract nonlinear Volterra equation

$$\frac{d}{dt}\left(\kappa(u(t)-u_0)+\int_0^t k(t-s)(u(s)-u_0)\,ds\right)+Au(t)\ni f(t),\quad t\in[0,T) \tag{1.27}$$

in a real Banach space X. Here, we assume that κ, k satisfy (1.2) and that A is an m-accretive possibly multivalued operator in X. Moreover, in order to guarantee existence of solutions, we always assume that $u_0 \in \overline{D(A)}$ and that $f \in L^1(0,T;X)$.
To our knowledge, the first existence result for (1.27) was given in [CN78] for the case $\kappa > 0$. Using the concept of mild solutions of nonlinear Cauchy problems, see [BCP], the existence of a *generalized solution* of (1.27) is shown by a fixed point argument.
A different approach including also the case $\kappa = 0$ and, in particular, the fractional derivative case, is given in [Gri85], see also [CGL96]. Here, the idea is to construct a sequence of approximate solutions converging towards the generalized solution of (1.27) by approximating the time derivative operator by a sequence of regularizations.
Indeed, given κ, k satisfying (1.2) one can always choose a sequence $\{k_n\}_{n\in\mathbb{N}}$ of nonnegative nonincreasing functions $k_n : (0,\infty) \to \mathbb{R}$ satisfying $k_n(0+) < \infty$ for all $n \in \mathbb{N}$ such that

$$\int_0^t k_n(\tau)\,d\tau \to \kappa + \int_0^t k(\tau)\,d\tau \qquad \text{for all } t>0 \text{ as } n\to\infty. \tag{1.28}$$

Then for each $n \in \mathbb{N}$ there exists a unique strong solution u_n of (1.27) with κ, k replaced by $\kappa_n := 0$ and k_n, respectively. Here, we use the following definition of strong solutions.

Definition 1.1. Let $A \subset X \times X$ be an operator in a Banach space X, let κ, k satisfy (1.2) and $u_0 \in X$, $f \in L^1(0,T;X)$. A function $u \in L^1(0,T;X)$ is called a *strong solution* of (1.27) if the function

$$v := \kappa(u-u_0) + (k*(u-u_0))$$

is absolutely continuous and differentiable almost everywhere on $(0,T)$ satisfying $v(0) = 0$, and there exists a function $w \in L^1(0,T;X)$ such that $(u(t), w(t)) \in A$ almost everywhere for $t \in [0,T]$, and u, w satisfy the equation

$$\frac{d}{dt}\left(\kappa(u(t)-u_0)+\int_0^t k(t-s)(u(s)-u_0)\,ds\right)+w(t)=f(t),\qquad \text{a.e. for } t\in[0,T).$$

Note that if $\kappa = 0$ and $k(0+) < \infty$, then (1.27) is equivalent to

$$u(t) = J^A_{1/k(0+)}\left(\frac{1}{k(0+)}\left(f(t) + k(t)u_0 - \int_{(0,t]} u(t-s)\, dk(s)\right)\right), \qquad t \in [0,T). \quad (1.29)$$

Here, the resolvent $J^A_\lambda = (I+\lambda A)^{-1}$ of A for $\lambda > 0$ is a single valued nonexpansive mapping defined on X, since A is assumed to be m-accretive. One can show, using Banach's fixed point theorem, that the problem (1.29) admits a unique solution. In [CGL96] the convergence in $L^1(0,T;X)$ of the sequence of approximate solutions $\{u_n\}_{n\in\mathbb{N}}$ towards a generalized solution, according to the following definition, is shown.

Definition 1.2. Let $A \subset X \times X$ be an operator in a Banach space X, let κ, k satisfy (1.2) and $u_0 \in X$, $f \in L^1(0,T;X)$. For the definition of generalized solutions we distinguish two cases.

(i) For $\kappa = 0$ and $k(0+) < \infty$, a function $u \in L^1(0,T;X)$ is called a *generalized solution* of (1.27) if it is a strong solution.

(ii) If κ, k satisfy $\kappa > 0$ or $k(0+) = \infty$, then a function $u \in L^1(0,T;X)$ is called a *generalized solution* of (1.27) if for all sequences $\{k_n\}$ of functions k_n satisfying (1.2) and $k_n(0+) < \infty$ with

$$\int_0^t k_n(\tau)\, d\tau \to \kappa + \int_0^t k(\tau)\, d\tau, \qquad \text{for all } t > 0 \text{ as } n \to \infty,$$

the equation (1.27), with κ, k replaced by $\kappa_n := 0$ and k_n, respectively, admits a generalized solution u_n for all $n \in \mathbb{N}$, and $u_n \to u$ in $L^1(0,T;X)$. The sequence $\{u_n\}$ is then called a sequence of approximate solutions.

In particular, the following result is obtained in [CGL96, Theorem 1].

Theorem 1.3. *Let X be a real Banach space and assume that*

(i) κ, k *satisfy* (1.2),

(ii) $\{k_n\}_{n\in\mathbb{N}}$ *is a sequence of nonnegative nonincreasing functions* $k_n : (0,\infty) \to \mathbb{R}$ *satisfying* $k_n(0+) < \infty$ *and*

$$\int_0^t k_n(\tau)\, d\tau \to \kappa + \int_0^t k(\tau)\, d\tau, \qquad \textit{for all } t > 0 \textit{ as } n \to \infty,$$

(iii) A *is an* m*-accretive operator in* X,

(iv) $u_0 \in \overline{D(A)}$ *and* $f \in L^1(0,T;X)$,

(v) $u_n \in L^1(0,T;X)$ *is a strong solution of* (1.27) *with* κ, k *replaced by* $\kappa_n := 0$ *and* k_n, *respectively.*

Then there exists a function $u \in L^1(0,T;X)$, *independent of the choice of the sequence* $\{k_n\}_{n\in\mathbb{N}}$, *such that* $u_n \to u$ *in* $L^1(0,T;X)$. *Moreover,* u *is continuous, if either* $\kappa > 0$ *or* $k(0+) = \infty$ *and* f *is continuous. If* f *is continuous and either* $\kappa > 0$ *or* $k(0+) = \infty$, *then the convergence is uniform on* $[0,T)$.

Remark 1.4. Let the assumptions of theorem 1.3 be satisfied and let $\kappa = 1$ and $k \equiv 0$. Then the generalized solution u of (1.27) coincides with the mild solution of the abstract Cauchy problem

$$\begin{aligned} u_t + Au &\ni f, \\ u(0) &= u_0. \end{aligned}$$

For the notion of mild solutions and results on nonlinear semigroups in Banach spaces we refer to [BCP].

The second main result on generalized solutions of (1.27), which will frequently be used, is the continuous dependence of the solution on the data shown in [Gri85, Theorem 5], see also [CGL96, Theorem 4].

Theorem 1.5. *Let* X *be a real Banach space and assume that*

(i) κ, k *satisfy* (1.2),

(ii) $\{(\kappa_n, k_n)\}_{n\in\mathbb{N}}$ *is a sequence of pairs* (κ_n, k_n) *satisfying* (1.2) *for all* $n \in \mathbb{N}$ *such that*

$$\kappa_n + \int_0^t k_n(\tau)\, d\tau \to \kappa + \int_0^t k(\tau)\, d\tau, \qquad \text{for all } t > 0 \text{ as } n \to \infty,$$

(iii) $\{A_n\}_{n\in\mathbb{N}}$ *is a sequence of* m*-accretive operators in* X *converging in resolvent to an* m*-accretive operator* A *in* X, *i.e.* $J_\lambda^{A_n} x \to J_\lambda^A x$ *in* X *as* $n \to \infty$ *for all* $x \in X$ *and all* $\lambda > 0$,

(iv) $u_0, \in \overline{D(A)}$, $u_{0,n} \in \overline{D(A_n)}$ *and* $f, f_n \in L^1(0,T;X)$ *for all* $n \in \mathbb{N}$ *such that* $u_{0,n} \to u_0$ *in* X *and* $f_n \to f$ *in* $L^1(0,T;X)$ *as* $n \to \infty$.

Then the sequence $\{u_n\}_{n\in\mathbb{N}}$ *of generalized solutions* u_n *of* (1.27) *with* κ, k *replaced by* κ_n, k_n, *respectively, converges in* $L^1(0,T;X)$ *to the generalized solution* u *of* (1.27).

Chapter 2

Regularity of solutions

In the following we are going to study regularity properties of generalized solutions u of the Volterra equation

$$\frac{d}{dt}\left(\kappa(u(t)-u_0)+\int_0^t k(t-s)(u(s)-u_0)\,ds\right)+Au(t)\ni f(t), \qquad t\in[0,T), \tag{2.1}$$

where A is an m-accretive or m-completely accretive operator in a Banach space X, $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;X)$ for $0 < T < \infty$. We recall that, by the existence results of [Gri85] and [CGL96], the generalized solution u of (2.1) is an element of $L^1(0,T;X)$. Note that, for the Cauchy problem

$$\begin{aligned} \frac{d}{dt}u(t)+Au(t) &\ni f(t), \qquad t\in[0,T), \\ u(0) &= u_0, \end{aligned} \tag{2.2}$$

it is well known that mild solutions $u : [0,T) \to X$ are continuous, even if there is no further regularity assumption on the data, i.e. if only $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;X)$. In contrast to this result, generalized solutions of (2.1) do not satisfy this type of regularity for general

$\kappa \geq 0$ and $k : (0,\infty) \to \mathbb{R}$ nonnegative, nonincreasing such that $k \in L^1_{\text{loc}}([0,\infty))$ and $\kappa + \int_0^t k(s)\,ds > 0$ for all $t > 0$. (2.3)

In view of this lack of regularity, we will concentrate on sufficient conditions for boundedness and continuity of generalized solutions in the first section of this chapter. In particular, we consider uniform continuity of the generalized solution on the interval $[0,\infty)$. This will be essential for the study of the asymptotic behavior of generalized solutions.
In the second section, we will ask for sufficient conditions on the data such that the generalized solution u of (2.1) is a strong solution.
As already shown in [Gri85], the existence of strong solutions of (2.1) can be obtained for $\kappa = 0$, even if the generalized solution itself is not differentiable a.e. on the interval $(0,T)$.

For $\kappa > 0$, this can not be achieved in general. In this context we remark that this investigation covers the case of the Cauchy problem (2.2) by defining $\kappa = 1$ and $k \equiv 0$. Therefore, we recall that for an arbitrary m-accretive operator A in a general Banach space X, Lipschitz continuity of the mild solution u of the Cauchy problem (2.2) is known if $u_0 \in \hat{D}(A)$ and $f \in BV([0,T);X)$. Since the Radon-Nikodym property of the space X is equivalent to the differentiability of absolutely continuous functions $u : (0,T) \to X$, we can conclude that the mild solution u of the Cauchy problem is in fact a strong solution if X has the Radon-Nikodym property and $u_0 \in \hat{D}(A)$ and $f \in BV(0,T;X)$.
However, the natural setting for a large number of evolution problems is a space without the Radon-Nikodym property – like, for instance, L^1. In this case no general regularity results are known so far.
In the case of $\kappa > 0$, we therefore restrict ourselves to m-completely accretive operators A in a normal Banach space $X \subset L^1(\Omega)$ with a bounded domain $\Omega \subset \mathbb{R}^N$. We show that under these assumptions regularity results can be achieved that are quite similar to those known for accretive operators in Banach spaces having the Radon-Nikodym property.

2.1 Continuity of solutions

In this section, we will always assume that the constant κ and the function k satisfy the assumption (2.3), and that A is a ϕ-accretive operator in a Banach space X, with $\phi : X \to \mathbb{R}$ continuous and convex. For the concept of ϕ-accretive operators we refer to [CP78]. See also appendix B, in particular definition B.5.
Our aim is to develop sufficient conditions on the data $u_0 \in X$ and $f \in L^1(0,T;X)$ such that the generalized solution u of the Volterra equation (2.1) is a continuous function on the interval $[0,T)$. We first remark that we will at least assume that $u_0 \in \overline{D(A)}$, in order to guarantee existence of generalized solutions of (2.1) for an m-accretive operator A.
In order to be able to compare two generalized solutions, we need an integral inequality which will be used frequently in the sequel. In the case of the Cauchy problem (2.2) it is well known that an integral inequality holds, see e.g. [Bén72, Proposition 1.27]. For generalized solutions of (2.1) with an m-accretive operator A first results in this direction were obtained in [Clé80, Theorem 1] and [KKM85, Lemma 3.1].

Proposition 2.1. *Let A be a ϕ-accretive operator in a Banach space X, where $\phi : X \to \mathbb{R}$ is continuous and convex, κ, k satisfy* (2.3)*, $u_0, v_0 \in \overline{D(A)}$, $f, g \in L^1(0,T;X)$, and let u be a generalized solution of* (2.1)*, and v a generalized solution of* (2.1) *with u_0 and f replaced by v_0 and g, respectively. Then*

$$\phi(u(t) - v(t)) \leq \phi(u_0 - v_0) + \int_{[0,t]} \phi'_+[u(t-s) - v(t-s), f(t-s) - g(t-s)]\, d\alpha(s)$$

almost everywhere for $t \in [0,T)$*, whenever the functions* $\phi(u-v)$*,* $\phi'_+(u-v, f-g)$ *and* $\phi(u_n - v_n)$*,* $\phi'_+(u_n - v_n, f-g)$ *belong to* $L^1(0,T)$*. Here,* $\{u_n\}$*,* $\{v_n\}$ *are sequences of approximate solutions of* u *and* v*, respectively, for* $\kappa > 0$ *or* $k(0+) = \infty$*, and* α *is the resolvent of the first kind of the pair* (κ, k) (see proposition A.4).

Proof. In the first step we consider the case of $\kappa = 0$ and $k(0+) = \lim_{t\to 0+} k(t) < \infty$. In this case the generalized solution u is a strong solution by definition. Thus, almost everywhere for $t \in [0,T)$, we have $u(t) \in D(A)$ and

$$f(t) - k(0+)(u(t)-u_0) - \int_{(0,t]} (u(t-s)-u_0)\, dk(s) \in Au(t). \tag{2.4}$$

Obviously, (2.4) holds as well for the generalized solution v of (2.1), with u_0, f replaced by v_0 and g, respectively. Since A is ϕ-accretive, we have almost everywhere for $t \in [0,T)$

$$\begin{aligned} 0 \;\le\; & \phi'_+\big[u(t)-v(t), f(t)-g(t) - k(t)\{u(t)-v(t)-(u_0-v_0)\} \\ & \qquad + \int_{(0,t]} \{u(t)-v(t)-(u(t-s)-v(t-s))\}\, dk(s)\big] \\ \le\; & \phi'_+\big[u(t)-v(t), f(t)-g(t)\big] \\ & + \phi'_+\big[u(t)-v(t), \int_{(0,t]} \{u(t)-v(t)-(u(t-s)-v(t-s))\}\; dk(s)\big] \\ & + \phi'_+\big[u(t)-v(t), -k(t)\{u(t)-v(t)-(u_0-v_0)\}\big] \\ \le\; & \phi'_+\big[u(t)-v(t), f(t)-g(t)\big] \\ & + \int_{(0,t]} \big\{\phi(u(t)-v(t)) - \phi(u(t-s)-v(t-s))\big\}\, dk(s) \\ & - k(t)\big\{\phi(u(t)-v(t)) - \phi(u_0-v_0)\big\}. \end{aligned}$$

Here, we used the properties of the Gateaux derivative ϕ'_+ of ϕ (see proposition B.1), in particular the continuity in the second variable and the fact that k is nonincreasing. The latter implies that the measure dk is a locally finite nonpositive measure on $(0,\infty)$. Since the resolvent α of the first kind of the function k is a locally finite nonnegative measure on $[0,\infty)$, as shown in proposition A.4, the convolution of the above inequality with the measure α yields

$$\begin{aligned} \phi(u(t)-v(t)) \le \phi(u_0-v_0) & \\ + \int_{[0,t]} \phi'_+\big[u(t-s)-v(t-s), f(t-s)-g(t-s)\big]\, d\alpha(s) & \end{aligned} \tag{2.5}$$

almost everywhere for $t \in [0,T)$.

In the second step we assume that $\kappa > 0$ or that $k(0+) = \infty$. We approximate the generalized solutions u of (2.1) and v by taking a sequence $\{k_n\}_{n\in\mathbb{N}}$ of functions in $L^1_{\mathrm{loc}}([0,\infty))$

satisfying (2.3) such that

$$\int_0^t k_n(s)\,ds \to \kappa + \int_0^t k(s)\,ds \quad \text{for all } t>0 \text{ as } n\to\infty. \tag{2.6}$$

Then, for each $n \in \mathbb{N}$, let u_n be a strong solution of (2.1) with k replaced by k_n. By the definition of generalized solutions, we know that $u_n \to u$ in $L^1(0,T;X)$. Therefore, we can assume that $u_n(t) \to u(t)$ in X almost everywhere for $t \in [0,T)$. The same can be assumed to hold for the generalized solution v. As a result of the first step of the proof, the approximate solutions u_n and v_n satisfy the inequality

$$\begin{aligned}\phi(u_n(t)-v_n(t)) &\le \phi(u_0-v_0)\\ &\quad+\int_{[0,t]}\phi'_+\big[u_n(t-s)-v_n(t-s), f(t-s)-g(t-s)\big]\,d\alpha_n(s)\end{aligned} \tag{2.7}$$

almost everywhere for $t \in [0,T)$ and for all $n \in \mathbb{N}$, where α_n is the resolvent of the first kind of the function k_n.
Due to the upper semicontinuity of the Gateaux derivative ϕ'_+ of ϕ (see proposition B.1) and the fact that the sequence of measures $\{\alpha_n\}$ converges in $\mathcal{D}'([0,\infty))$ to the resolvent of the first kind α of the pair (κ,k) as $n\to\infty$ (see lemma A.10), we can apply lemma A.11 to conclude that

$$\begin{aligned}\phi(u(t)-u(t)) &\le \phi(u_0-v_0)\\ &\quad+\int_{[0,t]}\phi'_+\big[u(t-s)-v(t-s), f(t-s)-g(t-s)\big]\,d\alpha(s)\end{aligned} \tag{2.8}$$

holds almost everywhere for $t \in [0,T)$. □

Note that in special cases, such as $\phi = \|\cdot\|$, we can assume that $\phi(0) < \phi(x)$ for all $x \in X$ with $x \neq 0$. Then proposition 2.1 gives the uniqueness of generalized solutions.
We remark that for $x \in D(A)$ and $y \in Ax$ the function $v(t) := x$ for $t \ge 0$ is a strong solution, and thus, a generalized solution of

$$\frac{d}{dt}\left(\kappa(v(t)-x)+\int_0^t k(t-s)(v(s)-x)\,ds\right)+Av(t)\ni y, \qquad t\ge 0.$$

Therefore, we can conclude from proposition 2.1 that the following holds.

Corollary 2.2. *Let A be a ϕ-accretive operator in a Banach space X, κ, k satisfy* (2.3)*, $u_0 \in \overline{D(A)}$, $f \in L^1(0,T;X)$, and let u be a generalized solution of* (2.1)*. Then for all $x \in D(A)$ and $y \in Ax$*

$$\phi(u(t)-x) \le \phi(u_0-x) + \int_{[0,t]}\phi'_+\big(u(t-s)-x, f(t-s)-y\big)\,d\alpha(s) \tag{2.9}$$

almost everywhere for $t \in [0,T)$*, whenever the functions* $\phi(u-x)$*,* $\phi'_+(u-x, f-y)$ *and* $\phi(u_n - x)$*,* $\phi'_+(u_n - x, f-y)$ *belong to* $L^1(0,T)$*. Here,* $\{u_n\}$ *is a sequence of approximate solutions of* u *for* $\kappa > 0$ *or* $k(0+) = \infty$*, and* α *is the resolvent of the first kind of the pair* (κ, k) (see proposition A.4).

If we assume that A is accretive in X, then by (2.1) we have for all $(x,y) \in A$

$$\|u(t) - x\| \le \|u_0 - x\| + \int_{[0,t]} \|f(t-s) - y\| \, d\alpha(s) \quad \text{a.e. for } t \in [0,T). \tag{2.10}$$

Thus, we easily see that $f \in L^\infty(0,T;X)$ implies $u \in L^\infty(0,T;X)$ for arbitrary $u_0 \in \overline{D(A)}$. In case $\kappa > 0$, the measure α is absolutely continuous with respect to Lebesgue measure and the Radon-Nikodym derivative $a \in L^1_{\text{loc}}([0,\infty))$ of α is bounded almost everywhere by $\frac{1}{\kappa}$. Thus, in this case, $u \in L^\infty(0,T;X)$ for all $f \in L^1(0,T;X)$.
We remark that, if $u_0 \in \hat{D}(A)$, there exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset A$ such that $x_n \to u_0$ in X and $\|y_n\| \to |u_0|_A := \sup_{\lambda>0} \|A_\lambda u_0\|$ as $n \to \infty$. Here, A_λ is the Yosida approximation of the operator A, defined by $A_\lambda := \frac{1}{\lambda}(I - J_\lambda)$. By (2.10), we conclude

$$\|u(t) - u_0\| \le |u_0|_A \alpha([0,t]) + \int_{[0,t]} \|f(t-s)\| \, d\alpha(s) \quad \text{a.e. for } t \in [0,T). \tag{2.11}$$

Moreover, if $u_0 \in D(A)$ and $v_0 \in Au_0$, then the generalized solution u of (2.1) obviously satisfies

$$\|u(t) - u_0\| \le \int_{[0,t]} \|f(t-s) - v_0\| \, d\alpha(s) \quad \text{a.e. for } t \in [0,T].$$

In case $\kappa > 0$, the right hand side of the above inequality converges to 0 as $t \to 0+$.
If we consider the case of $\kappa = 0$ and $k(0+) = \infty$, and assume that $f \in L^\infty(0,T;X)$, then we might estimate the right hand side of the above inequality by

$$\int_{[0,t]} \|f(t-s) - y\| \, d\alpha(s) \le \|f - y\|_{L^\infty(0,T;X)} \alpha([0,t]) \quad \text{a.e. for } t \in [0,T].$$

We remark that in this case we have $\alpha(\{0\}) = 0$ (see lemma A.7). Therefore, it is clear that the following corollary holds.

Corollary 2.3. *Let* A *be an accretive operator in* X*,* κ*,* k *satisfy* (2.3)*,* $u_0 \in D(A)$*,* $f \in L^1(0,T;X)$*, and let* u *be the unique generalized solution of* (2.1)*. If* $\kappa > 0$ *or if* $k(0+) = \infty$*, and* $f \in L^\infty(0,T;X)$*, then* u *is continuous at* 0*.*

If we assume that A is accretive, and $u_0 \in \overline{D(A)}$, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ converging to u_0 in X as $n \to \infty$. Let $y_n \in Ax_n$ and u be a generalized solution of (2.1) with initial value u_0. Then, by proposition 2.1, we have

$$\begin{aligned} \|u(t) - u_0\| &\le \|u(t) - x_n\| + \|x_n - u_0\| \\ &\le 2\|u_0 - x_n\| + \int_{[0,t]} \|f(t-s) - y_n\| \, d\alpha(s) \end{aligned}$$

almost everywhere for $t \in [0,T)$. Thus it is obvious that the following holds.

Corollary 2.4. *Let A be an accretive operator in X, κ, k satisfy (2.3), $u_0 \in \overline{D(A)}$, $f \in L^1(0,T;X)$, and let u be the unique generalized solution of (2.1). If $\kappa > 0$, or if $k(0+) = \infty$, and $f \in L^\infty(0,T;X)$, then u is continuous at 0.*

In order to show continuity of the generalized solution at points $t_0 > 0$, we have to compare $u(t_0)$ and $u(t_0+h)$ for $|h| > 0$ small. For the Cauchy problem

$$\begin{aligned} \frac{d}{dt}u(t) + Au(t) &\ni f(t), \qquad t \in [0,T), \\ u(0) &= u_0, \end{aligned} \tag{2.12}$$

this is normally done by considering $v(t) := u(t+h)$ as a solution of the translated problem with initial value $u(h)$ and the right hand side f of (2.12) replaced by $f(\cdot+h)$. This method is not applicable for Volterra equations on $[0,\infty)$, or on intervals $[0,T)$, as these problems fail to be translation invariant in general. But we remark that translation invariance holds for Volterra equations on the real line. Therefore, we can not apply proposition 2.1 directly, in order to obtain an estimate on $\|u(t_0) - u(t_0+h)\|$, where u is the generalized solution of (2.1). But the following result will use essentially the same methods as used for the proof of proposition 2.1.

Proposition 2.5. *Let A be a ϕ-accretive operator in a Banach space X, κ, k satisfy (2.3), $u_0 \in \overline{D(A)}$, $f \in L^1(0,T;X)$, and let u be a generalized solution of (2.1). Then*

$$\begin{aligned} \phi(u(t+h) - u(t)) \le &\sup_{\tau\in[0,h]} \phi(u(\tau) - u_0) \\ &+ \int_{[0,t]} \phi'_+(u(t+h-s) - u(t-s), f(t+h-s) - f(t-s))\, d\alpha(s) \end{aligned} \tag{2.13}$$

almost everywhere for $0 < h < T$ and $t \in [0,T-h)$, whenever the functions $\phi(u(\cdot+h) - u(\cdot))$, $\phi'_+(u(\cdot+h) - u(\cdot), f(\cdot+h) - f(\cdot))$ and $\phi(u_n(\cdot+h) - u_n(\cdot))$, $\phi'_+(u_n(\cdot+h) - u_n(\cdot), f(\cdot+h) - f(\cdot))$ are in $L^1(0,T)$, where $\{u_n\}$ is a sequence of approximate solutions $\{u_n\}$ of the generalized solution u such that u is the uniform limit of the u_n on $[0,T)$ if $\kappa > 0$ or $k(0+) = \infty$. Here α is the resolvent of the first kind of the pair (κ, k) (see proposition A.4).

Proof. In the first step, we consider the case of $\kappa = 0$ and $k(0+) = \lim_{t\to 0+} k(t) < \infty$. Then the generalized solution $u \in L^1(0,T;X)$ is a strong solution of (2.1). In particular, almost everywhere for $t \in [0,T)$, we have $u(t) \in D(A)$, and

$$f(t) - k(0+)(u(t) - u_0) - \int_{(0,t]} (u(t-s) - u_0)\, dk(s) \in Au(t).$$

By the definition of ϕ-accretivity of the operator A, we have almost everywhere for $0 < h < T$ and $t \in [0, T-h)$

$$\begin{aligned}
0 \quad \leq \quad & \phi'_+\big[u(t+h)-u(t), f(t+h)-f(t)-k(0+)(u(t+h)-u(t)) \\
& \quad - \int_{(0,t+h]} (u(t+h-s)-u_0)\; dk(s) + \int_{(0,t]} (u(t-s)-u_0)\; dk(s)\big] \\
\leq \quad & \phi'_+\big[u(t+h)-u(t), f(t+h)-f(t)\big] \\
& + \phi'_+\big[u(t+h)-u(t), -k(t+h)(u(t+h)-u(t))\big] \\
& + \phi'_+\big[u(t+h)-u(t), \int_{(0,t]} \{u(t+h)-u(t) \\
& \qquad\qquad -(u(t+h-s)-u(t-s))\}\; dk(s)\big] \\
& + \phi'_+\big[u(t+h)-u(t), \int_{(t,t+h]} \{u(t+h)-u(t) \\
& \qquad\qquad -(u(t+h-s)-u_0)\}\; dk(s)\big].
\end{aligned}$$

This implies that almost everywhere for $0 < h < T$ and $t \in [0, T-h)$

$$\begin{aligned}
0 \quad \leq \quad & \phi'_+\big[u(t+h)-u(t), f(t+h)-f(t)\big] \\
& - k(t+h)\phi(u(t+h)-u(t)) \\
& + \int_{(0,t]} \{\phi(u(t+h)-u(t)) - \phi(u(t+h-s)-u(t-s))\}\; dk(s) \\
& + \int_{(t,t+h]} \{\phi(u(t+h)-u(t)) - \phi(u(t+h-s)-u_0)\}\; dk(s) \\
= \quad & \phi'_+\big[u(t+h)-u(t), f(t+h)-f(t)\big] \\
& - k(0+)\phi(u(t+h)-u(t)) - \int_{(0,t]} \phi(u(t+h-s)-u(t-s))\; dk(s) \\
& - \int_{(t,t+h]} \phi(u(t+h-s)-u_0)\; dk(s).
\end{aligned}$$

Here, we again used the properties of the Gateaux derivative ϕ'_+, in particular the continuity in the second variable and the fact that k is nonincreasing. The convolution of the above inequality with the measure α yields

$$\begin{aligned}
& \phi(u(t+h)-u(t)) \\
& \quad \leq \; -\int_{[0,t]} \int_{(t-s,t+h-s]} \phi(u(t+h-s-\sigma)-u_0)\; dk(\sigma)\; d\alpha(s) \\
& \qquad\quad + \int_{[0,t]} \phi'_+\big[u(t+h-s)-u(t-s), f(t+h-s)-f(t-s)\big]\; d\alpha(s)
\end{aligned} \tag{2.14}$$

almost everywhere for $0 < h < T$ and $t \in [0, T-h)$. This gives the assertion.

In the second step, we assume that $\kappa > 0$ or that $k(0+) = \infty$. We approximate the generalized solutions u of (2.1) by taking a sequence $\{k_n\}_{n\in\mathbb{N}}$ of functions in $L^1_{\mathrm{loc}}([0,\infty))$ satisfying (2.3) such that

$$\int_0^t k_n(s)\,ds \to \kappa + \int_0^t k(s)\,ds \quad \text{for all } t > 0 \text{ as } n \to \infty. \tag{2.15}$$

For each $n \in \mathbb{N}$, let u_n be the strong solution of (2.1) with k replaced by k_n. By the definition of generalized solutions we know that $u_n \to u$ in $L^1(0,T;X)$. Therefore, we can assume that $u_n(t) \to u(t)$ almost everywhere for $t \in [0,T)$. As a result of the first step of the proof, the approximate solutions u_n satisfy the inequality

$$\begin{aligned} &\phi(u_n(t+h) - u_n(t)) \\ &\quad\le \sup_{\tau\in[0,h]} \phi(u_n(\tau) - u_0) \cdot \int_{[0,t]} (k_n(t-s) - k_n(t+h-s))\,d\alpha_n(s) \\ &\qquad + \int_{[0,t]} \phi'_+\big[u_n(t+h-s) - u_n(t-s), f(t+h-s) - f(t-s)\big]\,d\alpha_n(s) \end{aligned} \tag{2.16}$$

almost everywhere for $0 < h < T$ and $t \in [0, T-h)$ and for all $n \in \mathbb{N}$, where α_n is the resolvent of the first kind of the function k_n.
Due to the upper semicontinuity of the Gateaux derivative ϕ'_+ in the first variable (see proposition B.1), and the fact that the sequence of measures $\{\alpha_n\}$ converges in $\mathcal{D}'([0,\infty))$ to the resolvent of the first kind α of the pair (κ, k) as $n \to \infty$, we can apply lemma A.11, and we see that

$$\begin{aligned} &\phi(u(t+h) - u(t)) \\ &\quad\le \Big(\liminf_{n\to\infty} \sup_{\tau\in[0,h]} \phi(u_n(\tau) - u_0)\Big) \\ &\qquad + \int_{[0,t]} \phi'_+\big[u(t+h-s) - u(t-s), f(t+h-s) - f(t-s)\big]\,d\alpha(s) \end{aligned} \tag{2.17}$$

holds almost everywhere for $0 < h < T$ and $t \in [0, T-h)$. □

We now turn to the continuity of generalized solutions u of the Volterra equation (2.1) with an accretive operator A. For $\kappa > 0$, the following proposition has been shown in [Gri85, Theorem 1 and Theorem 2].

Proposition 2.6. *Let A be an m-accretive operator in a Banach space X, $u_0 \in \overline{D(A)}$, $f \in L^1(0,T;X)$, and let u be the unique generalized solution of* (2.1)*, with κ, k satisfying* (2.3) *and $\kappa > 0$. Then u is continuous on $[0,T)$. Moreover, if $u_0 \in \hat{D}(A)$, $f \in \mathrm{BV}([0,T);X)$, then u is Lipschitz continuous on $[0,T)$.*

For the Lipschitz continuity of u we remark that, by proposition 2.5 and lemma A.8,

$$\left\|\frac{u(t+h)-u(t)}{h}\right\| \leq \frac{1}{\kappa}\left(|u_0|_A + \|f(0+)\| + \frac{1}{h}\int_0^t \|f(\tau+h)-f(\tau)\|\,d\tau\right) \tag{2.18}$$

holds almost everywhere for $0 < h < T$ and $t \in [0, T-h)$. Here, $|x|_A := \sup_{\lambda>0}\|A_\lambda x\|$, where A_λ denotes the Yosida approximation of A.
In case $\kappa = 0$ and $k(0+) = \infty$, we will have to assume that the right-hand side f of the Volterra equation is essentially bounded in order to obtain continuity of the solution. But it is not necessary to ask for Lipschitz continuity of u, as the function

$$(0,T) \ni t \to \int_0^t k(t-s)(u(s)-u_0)\,ds$$

can be absolutely continuous and differentiable almost everywhere, even if u is not absolutely continuous.

Proposition 2.7. *Let A be an m-accretive operator in a Banach space X, $u_0 \in \overline{D(A)}$, $f \in L^\infty(0,T;X)$, and let u be the unique generalized solution of* (2.1) *with $\kappa = 0$, k satisfying* (2.3)*, $k(0+) = \infty$. If*

(i) $f \in C([0,T);X)$*, or*

(ii) $\log(k)$ *is convex on* $(0,\infty)$*,*

then u is continuous on $[0,T)$*.*

Proof. The case *(i)* has been shown in [Gri85, Theorem 1]. We therefore give only the arguments for case *(ii)*.
Let $\{u_n\}$ be a sequence of approximate solutions, i.e. there exists a sequence $\{k_n\}$ of functions satisfying (2.3), such that $k_n(0+) < \infty$, and

$$\int_0^t k_n(s)\,ds \to \int_0^t k(s)\,ds \quad \text{as } n \to \infty \text{ for all } t > 0.$$

Assume that u_n is the strong solution of (2.1) with κ, k replaced by $\kappa_n = 0$ and k_n respectively. Then, for $(x, y) \in A$ and almost everywhere for $0 < h < T$,

$$\sup_{\tau\in[0,h]} \|u_n(\tau) - u_0\| \leq 2\|u_0 - x\| + \|f-y\|_\infty \alpha_n([0,h]).$$

Choose $\varepsilon > 0$ arbitrary, and let $(x, y) \in A$ such that $\|u_0 - x\| \leq \frac{\varepsilon}{8}$. Since $\alpha([0,h]) \to 0$ as $h \to 0+$, we can choose $h_0 > 0$ such that $\alpha([0,h_0]) \leq \frac{\varepsilon}{8M}$, where $M := \|f-y\|_\infty$.

As shown in [Gri80, Theorem 3], the Radon-Nikodym derivative $a \in L^1_{\text{loc}}([0,\infty))$ of α is nonincreasing. Thus, we can choose $0 < h_1 \le \frac{h_0}{2}$ such that for all $0 < h < h_1$

$$\int_{h_0}^{t} \|f(t+h-s) - f(t-s)\| \, d\alpha(s) \le a(h_0) \int_0^{T-h} \|f(s+h) - f(s)\| \, ds \le \frac{\varepsilon}{4}.$$

Since $\alpha_n \to \alpha$ in $\mathcal{D}'([0,\infty))$, we can find $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $\alpha_n([0,h_1]) \le \alpha([0,2h_1]) + \frac{\varepsilon}{8M}$. This obviously implies by (2.17) that

$$\|u(t+h) - u(t)\| \le \varepsilon$$

almost everywhere for $0 < h < h_1$ and $t \in [0, T-h)$. □

In order to study the asymptotic behavior of solutions, we now consider the Volterra equation

$$\frac{d}{dt}\left(\kappa(u(t) - u_0) + \int_0^t k(t-s)(u(s) - u_0) \, ds\right) + Au(t) \ni f(t), \qquad t \in [0,\infty), \tag{2.19}$$

for $f \in L^1_{\text{loc}}([0,\infty); X)$ and an m-accretive operator A in a Banach space X. By the uniqueness of generalized solutions on intervals $[0,T)$, it makes sense to call a function u a generalized solution of (2.19), if u is a generalized solution of (2.1) for all $T > 0$.
The first question that arises in this context is, whether the generalized solution u is bounded. By definition and propositions 2.6 and 2.7 , it is clear that, for $f \in C([0,\infty); X)$, and $\kappa > 0$, or $k(0+) = \infty$, there can not be blow up in finite time. But still $\lim_{t\to\infty} \|u(t)\| = \infty$ is possible.

Proposition 2.8. *Let A be an m-accretive operator in a Banach space X, let $f \in L^\infty(0,\infty; X)$, $u_0 \in \overline{D(A)}$, and κ, k satisfy* (2.3) *and*

$$k(\infty) := \lim_{t\to\infty} k(t) > 0. \tag{2.20}$$

Then the unique generalized solution u of (2.19) *is bounded on* $[0,\infty)$.

Proof. If $(x,y) \in A$, then, by corollary 2.2 and by lemma A.9,

$$\|u(t) - x\| \le \|u_0 - x\| + \sup_{\tau\in[0,\infty)} \|f(\tau) - y\| \alpha([0,\infty)) < \infty \tag{2.21}$$

almost everywhere for $t \in [0,\infty)$. □

By propositions 2.6 and 2.7, we already know that the generalized solution u of (2.19) is continuous on $[0,\infty)$, whenever $\kappa > 0$ or $k(0+) = \infty$, and $f \in C([0,\infty); X)$. We now ask for uniform continuity of the generalized solution u on $[0,\infty)$. Therefore, we assume that f is uniformly continuous on $[0,\infty)$.

Proposition 2.9. *Let A be an m-accretive operator in a Banach space X, let $f \in C([0,\infty);X)$ be uniformly continuous on $[0,\infty)$, $u_0 \in \overline{D(A)}$ and κ, k satisfy* (2.3), (2.20) *and*

$$\kappa > 0 \quad or \quad k(0+) = \infty. \tag{2.22}$$

Then the unique generalized solution u of (2.19) *is uniformly continuous on* $[0,\infty)$.

Proof. Let $\varepsilon > 0$; by corollary 2.4 there exists $h_0 > 0$ such that for all $0 < h < h_0$

$$\|u(h) - u_0\| \leq \frac{\varepsilon}{2}.$$

Also, by the uniform continuity of f, we can choose h_0 small enough such that for all $0 < h < h_0$ and all $t \in [0,\infty)$

$$\|f(t+h) - f(t)\| \leq \frac{\varepsilon}{2\alpha([0,\infty))}.$$

Due to the uniform convergence of the approximate solutions, as shown in [CGL96, Theorem 1], we can apply proposition 2.5, and obtain for all $0 < h < h_0$ and all $t \in [0,\infty)$

$$\|u(t+h) - u(t)\| \leq \varepsilon.$$

□

We now define $\mathcal{G}$ to be the solution mapping

$$\begin{array}{rcl} \mathcal{G} : \overline{D(A)} \times L^1_{\text{loc}}([0,\infty);X) & \to & L^1_{\text{loc}}([0,\infty);X) \\ (u_0, f) & \mapsto & u, \end{array} \tag{2.23}$$

which maps every initial value $u_0 \in \overline{D(A)}$ and every right-hand side $f \in L^1_{\text{loc}}([0,\infty);X)$ to the generalized solution u of (2.19). By the above results, it is clear that for κ, k satisfying (2.3), (2.20) and (2.22), and for every $u_0 \in \overline{D(A)}$ the solution mapping $\mathcal{G}(u_0,\cdot)$ leaves the space $BUC([0,\infty);X)$ invariant, i.e. $\mathcal{G}(u_0,f) \in BUC([0,\infty);X)$ for all $f \in BUC([0,\infty);X)$. For the study of the asymptotic behavior in chapter 4, it will be of particular interest, whether this solution mapping $\mathcal{G}$ leaves certain subspaces of $BUC([0,\infty);X)$, such as $AAP([0,T);X)$ invariant.

2.2 Strong solutions

In this section, we will always assume that A is an m-accretive operator in a Banach space X. Recall that generalized solutions of (2.1) in the case that $\kappa = 0$ and k satisfying (2.3) with $k(0+) < \infty$ are strong solutions by definition. It is our purpose to give sufficient

conditions for u_0 and f such that generalized solutions become strong solutions for $\kappa > 0$ or $k(0+) = \infty$.
For $\kappa = 0$ and $k(0+) = \infty$, the following lemma of [Gri85, Lemma 3.4] is the main tool in order to show that generalized solutions are strong solutions under certain conditions on u_0 and f.

Lemma 2.10. *Let $k \in L^1_{\mathrm{loc}}([0,\infty))$ and $v \in BV([0,\infty);X)$, where X is a Banach space. Then the function*

$$[0,\infty) \ni t \mapsto \int_0^t k(t-s)v(s)\,ds$$

is locally absolutely continuous and differentiable almost everywhere and

$$\int_0^T \left\| \frac{d}{dt}\int_0^t k(t-s)v(s)\,ds \right\| dt \;\leq\; \int_0^T |k(s)|\,ds \Big(\|v(0+)\| + \mathrm{Var}(v;[0,T)) \Big).$$

Moreover, if $\{v_n\}$ is a sequence in $BV([0,\infty);X)$ such that for all $T > 0$

$$\sup_{n\in\mathbb{N}} \mathrm{Var}(v_n;[0,T]) < \infty,$$

and $v_n \to v$ in $L^1_{\mathrm{loc}}([0,\infty);X)$, then

$$\frac{d}{dt}\int_0^t k(t-s)v_n(s)\,ds \to \frac{d}{dt}\int_0^t k(t-s)v(s)\,ds$$

in $L^1_{\mathrm{loc}}([0,\infty);X)$ as $n \to \infty$.

We have already seen that generalized solutions are Lipschitz continuous if $u_0 \in \hat{D}(A)$, $f \in BV([0,T];X)$ and $\kappa > 0$. Thus, the following proposition of [Gri85, Theorem 2] is almost obvious.

Proposition 2.11. *Let A be an accretive operator in a Banach space X, κ, k satisfy* (2.3), *and let u be the unique generalized solution of* (2.1) *with $u_0 \in \hat{D}(A)$, $f \in BV([0,T];X)$.*

(i) If $\kappa = 0$, then u is a strong solution.

(ii) If $\kappa > 0$ and X has the Radon-Nikodym property, then u is a strong solution.

As mentioned before, many of the important spaces for applications fail to have the Radon-Nikodym property, such as for example $L^1(\Omega)$, where $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space. It is well known (see [BC91, Theorem 4.2]) that in a normal Banach space $X \subset L_0(\Omega)$ satisfying the convergence condition

$$u_n \ll u \in X \text{ for } n \in \mathbb{N} \text{ and } u_n \to u \text{ a.e.} \quad\Longrightarrow\quad \|u_n - u\|_X \to 0, \tag{2.24}$$

the semigroup $\{S(t)\}_{t\geq 0}$ generated by an m-completely accretive operator A leaves the domain $D(A)$ of the operator invariant. Here, we used the notations and definitions introduced in appendix B. Since $\{S(t)\}_{t\geq 0}$ leaves $D(A)$ invariant, all orbits $t \mapsto S(t)x$ for $x \in D(A)$ are strong solutions. Note that for all $x \in \overline{D(A)}$ the orbit $t \mapsto S(t)x$ is the unique mild solution (see remark 1.4) of the homogeneous Cauchy problem

$$\begin{aligned} u_t + Au &\ni 0 \\ u(0) &= x. \end{aligned}$$

In particular, the following regularity result holds.

Proposition 2.12. *Let $X \subset L_0(\Omega)$ be a normal Banach space, $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, A be m-completely accretive in X, and $\{S(t)\}_{t\geq 0}$ the semigroup generated by A. Then we have:*

(i) $D(A) = \left\{ u \in L_0\text{-cl}\big(D(A)\big) \cap X \;\middle|\; \exists\, v \in X : \frac{S(t)u - u}{t} \ll v \text{ for } t > 0 \text{ small} \right\}.$

(ii) $S(t)D(A) \subset D(A)$.

(iii) If $u \in D(A)$, then

$$\frac{u - S(t)u}{t} \ll v \qquad \text{for } t > 0 \text{ and } v \in Au,$$

and

$$L_0(\Omega)\text{-}\lim_{t\to 0+} \frac{S(t)u - u}{t} = -A^\circ u.$$

(iv) If, in addition, X satisfies the convergence condition (2.24)*, then*

$$X\text{-}\lim_{t\to 0+} \frac{S(t)u - u}{t} = -A^\circ u.$$

By generalizing the above result to inhomogeneous Volterra equations, two problems will arise, as we will point out in the following.
The straightforward idea to obtain regularity of generalized solutions of (2.1) would of course be to reduce the problem to the case of the homogeneous Cauchy problem. As the following proposition shows, this is practical at the point $t = 0$.

Proposition 2.13. *Let A be an m-completely accretive operator in a normal Banach space $X \subset L_0(\Omega)$, satisfying the convergence condition* (2.24)*, where $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space. Let $\kappa > 0$, k satisfy* (2.3)*, and $u_0 \in D(A)$, $f \in C([0,T); X)$. Then the unique generalized solution u of* (2.1) *is differentiable from the right at $t = 0$ and*

$$\frac{d}{dt}^{+} u(0) = \frac{1}{\kappa}(A - f(0))^\circ u_0.$$

Proof. Since A is m-completely accretive in X, one can easily verify that the operator $B = \frac{1}{\kappa}(A - f(0))$ is m-completely accretive in X as well. Then, according to proposition 2.12, the mild solution (see remark 1.4) v of the homogeneous Cauchy problem

$$\begin{aligned} \frac{d}{dt}v(t) + Bv(t) &\ni 0 \\ v(0) &= u_0 \end{aligned}$$

is strongly differentiable from the right at $t = 0$. As v is a strong solution of the Cauchy problem, it is obvious that v is as well a strong solution of the Volterra equation

$$\frac{d}{dt}\left(\kappa(v(t) - u_0) + \int_0^t k(t-s)(v(s) - u_0)\right)\, ds + Av(t) \ni g(t), \tag{2.25}$$

where, for $t \in [0, T)$,

$$g(t) := f(0) + \frac{d}{dt}\int_0^t k(t-s)(v(s) - u_0)\, ds.$$

By [CGL96, Theorem 1], v is the unique generalized solution of the Volterra equation (2.25). Since u and v are continuous, we can apply proposition 2.1 and obtain for all $0 < t < \frac{T}{2}$

$$\begin{aligned} \left\|\frac{u(t) - u_0}{t} - \frac{v(t) - u_0}{t}\right\| &\leq \frac{1}{\kappa t}\int_0^t \|f(t-s) - f(0)\|\, d\alpha(s) \\ &\quad + \frac{1}{\kappa t}\int_0^t \left\|\frac{d}{dt}\int_0^{t-s} k(t-s-\sigma)(v(\sigma) - u_0)\, d\sigma\right\| d\alpha(s) \\ &\leq \frac{1}{\kappa^2 t}\int_0^t \|f(s) - f(0)\|\, ds \\ &\quad + \frac{1}{\kappa^2}\left\|\frac{d}{dt}^+ v\right\|_{L^\infty(0,t;X)} \int_0^t k(s)\, ds. \end{aligned} \tag{2.26}$$

Here, α denotes the resolvent of the first kind of the pair (κ, k) (see proposition A.4). Moreover, by lemma A.8, the Radon-Nikodym derivative $a \in L^1(0, T)$ of α exists. As v is locally Lipschitz continuous and differentiable from the right, $\|\frac{d}{dt}^+ v\|_{L^\infty(0,T/2;X))} < \infty$. By the continuity of f, and the fact that $k \in L^1(0, \frac{T}{2})$, one can pass to the limit for $t \to 0+$ in (2.26). Thus, we conclude that u is differentiable form the right at $t = 0$, and that

$$\frac{d}{dt}^+ u(0) = \frac{d}{dt}^+ v(0) = -\frac{1}{\kappa}(A - f(0))^\circ u_0.$$

□

One might conjecture that this reduction to the homogeneous case would yield the regularity of generalized solutions even at points $t_0 > 0$. But this method turns out to be not applicable directly. Indeed, if we define for $t \in [0, t_0]$

$$g(t) := f(t) - \frac{d}{dt} \int_0^t k(t-s)(u(s) - u_0)\, ds,$$

and $B := \frac{1}{\kappa}(A - g(t_0))$, which is again m-completely accretive in X, we still can not apply proposition 2.12, as we do not know whether $u(t_0) \in D(A)$. Thus, it is unclear whether the mild solution v of the homogeneous Cauchy problem for B with initial value $u(t_0)$ is differentiable at all.

For the second problem that arises when generalizing proposition 2.12, recall that the main step in the proof of proposition 2.12 is to show that

$$\frac{S(t)u - u}{t} \ll v \tag{2.27}$$

for some $v \in X$ and $t > 0$ small. By the weak sequential compactness in $L_0(\Omega)$ of the set

$$\left\{ u \in M(\Omega) \;\middle|\; u \ll v \right\},$$

this yields

$$\frac{S(t_n)u - u}{t_n} \rightharpoonup z \qquad \text{weakly in } L_0(\Omega)$$

for some sequence $\{t_n\}$ with $t_n \to 0+$ and for some $z \in L_0(\Omega)$. Unfortunately, by proposition 2.5 we only know that for all $m > 0$

$$\begin{aligned}
&\int_\Omega \left(\frac{u(t+h) - u(t)}{h} - m \right)^+ dx \\
&\quad \le \sup_{\tau \in [0,h]} \int_\Omega \left(\frac{u(\tau) - u_0}{h} - m \right)^+ dx \\
&\qquad + \int_0^t \int_{\Omega \cap \{u(t+h-s) - u(t-s) > mh\}} \frac{f(t+h-s) - f(t-s)}{h}\, dx\, d\alpha(s) \\
&\qquad + \int_0^t \int_{\Omega \cap \{u(t+h-s) - u(t-s) = mh\}} \left(\frac{f(t+h-s) - f(t-s)}{h} \right)^+ dx\, d\alpha(s)
\end{aligned} \tag{2.28}$$

almost everywhere for $0 < h < T$ and $t \in [0, T-h)$. But as $\mathbb{R} \ni r \mapsto (r-m)^+$ fails to be sublinear, we can not estimate the last two terms of (2.28) by

$$\int_0^t \int_\Omega \left(\frac{f(t+h-s) - f(t-s)}{h} - m \right)^+ dx\, d\alpha(s).$$

However, by (2.28), we can conclude that for all $m > 0$ and almost everywhere for $0 < h < T$ and $t \in [0, T-h)$

$$\begin{aligned}\int_\Omega \left(\frac{u(t+h)-u(t)}{h} - m\right)^+ dx \;\leq\; & \sup_{\tau\in[0,h]} \int_\Omega \left(\frac{u(\tau)-u_0}{h} - m\right)^+ dx \\ & + \int_0^t \int_\Omega \left|\frac{f(t+h-s)-f(t-s)}{h}\right| dx\, d\alpha(s).\end{aligned} \tag{2.29}$$

Obviously, an analogous result can be obtained by considering

$$\phi_m^-(f) := \int_\Omega (f+m)^-$$

for $m > 0$. But this is not sufficient in order to show weak sequential compactness of the differential quotient $\frac{1}{h}(u(t+h)-u(t))$. Therefore, it will become necessary to study the Volterra equation (2.1) in an appropriate Orlicz space. For the theory of Orlicz spaces we refer to [KR61]. The following proposition is a generalization of a result for mild solutions of inhomogeneous Cauchy problems, which can be found in [Wit92, Proposition 2.4.5], to Volterra equations .

Proposition 2.14. *Let $\kappa > 0$, k satisfy* (2.3), *$(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, and let A be an m-completely accretive operator in a normal Banach space $X \subset L^1(\Omega)$ satisfying the strong convergence condition, i.e.*

$$\begin{array}{l}\{u_n\} \subset X,\ u \in L_0(\Omega),\ u_n \ll u \\ \liminf_{n\to\infty} \|u_n\|_X < \infty,\ u_n \to u \text{ a.e.}\end{array} \implies u \in X \text{ and } \|u_n - u\|_X \to 0. \tag{2.30}$$

Then, for all $u_0 \in D(A)$, and $f \in W^{1,1}(0,T;X)$, the generalized solution u of (2.1) *is locally Lipschitz continuous and differentiable a.e. on $[0,T)$ such that almost everywhere for $t \in [0,T)$*

$$-\frac{d}{dt}u(t) = \frac{1}{\kappa}\left(Au(t) - f(t) + \frac{d}{dt}\int_0^t k(t-s)(u(s)-u_0)\, ds\right)^\circ.$$

In particular, u is the unique strong solution of (2.1).

We remark that the strong convergence condition (2.30) is equivalent to

$$\begin{array}{l} X \text{ satisfies the convergence condition (2.24), and} \\ X \text{ has the Fatou property, i.e.} \\ \{u_n\}_{n\in\mathbb{N}} \subset X,\ \liminf \|u_n\|_X < \infty,\ u_n \to u \text{ a.e.} \implies u \in X,\ \|u\|_X \leq \liminf \|u_n\|_X. \end{array} \tag{2.31}$$

Simple examples of normal Banach spaces satisfying (2.31) are the Banach function spaces $L^p(\Omega)$ for $1 \leq p < \infty$ and $L_0(\Omega)$.

Proof of proposition 2.14. We assume that $u_0 \in D(A)$, $f \in W^{1,1}(0,T;X)$, and that u is the unique generalized solution of (2.1) in X. Moreover, we assume $v_0 \in Au_0$. Since u is Lipschitz continuous, by proposition 2.6, we can apply lemma 2.10 to see that the function

$$[0,T) \ni t \mapsto \int_0^t k(t-s)(u(s)-u_0)\, ds$$

is absolutely continuous and differentiable a.e., and that

$$g(t) := \frac{d}{dt}\int_0^t k(t-s)(u(s)-u_0)\, ds, \quad t \in [0,T),$$

satisfies $g \in L^1(0,T;X)$. We remark that the set $L_r \subset [0,T)$ of right Lebesgue points of g, i.e. the set of all $t \in [0,T)$, such that

$$\lim_{h\to 0+} \frac{1}{h}\int_t^{t+h} \|g(\tau)-g(t)\|\, d\tau = 0, \tag{2.32}$$

is the complement of a nullset in $[0,T)$, i.e. $\lambda([0,T)\backslash L_r) = 0$. We will now prove that u is strongly differentiable at all $t_0 \in L_r$, and that at these points u satisfies the equation. This proof will consist of several steps.

(1) In the first step, we will construct an Orlicz space $L_N(\Omega)$, where N is an N-function satisfying the Δ_2-condition (see [KR61, Definition I.4.1]), such that

$$\begin{aligned} u_0, v_0 &\in L_N(\Omega), \text{ and} \\ f, f' &\in L^1(0,T;L_N(\Omega)). \end{aligned}$$

Before proceeding with the proof, we remark that for any N-function N satisfying the Δ_2-condition, the Orlicz space $L_N(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_N := \inf\left\{k>0 \Big| \int_\Omega N\left[\left|\frac{u}{k}\right|\right] \le 1\right\}$$

is a normal Banach space.

Following the arguments in [KR61, p. 60ff], there exists an N-function N satisfying the Δ_2-condition, such that for $Q := (0,T)\times\Omega$

$$|u_0|,\ |v_0|,\ |f|,\ |f'|\ \in L_N(Q).$$

Here, we interpreted u_0, v_0 as constant functions over $(0,T)$. This implies for $\rho = u_0$ and $\rho = v_0$, respectively,

$$T\int_\Omega N\left[|\rho|\right] = \int_Q N\left[|\rho|\right] < \infty,$$

and thus $u_0, v_0 \in L_N(\Omega)$. On the other hand we have by Fubini's theorem for $\rho = f$ and $\rho = f'$, respectively,

$$\int_0^T \left| \int_\Omega N\left[|\rho|\right]\ d\mu \right| = \int_Q N\left[|\rho|\right] < \infty.$$

This implies $f, f' \in L^1(0,T; L_N(\Omega))$.

(2) Now, our purpose is to show that for $t_0 \in [0,T)$ there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ with $h_n \to 0+$ as $n \to \infty$, such that the sequence $\{\frac{1}{h_n}(u(t_0+h_n) - u(t_0))\}_{n\in\mathbb{N}}$ converges weakly in $L^1_{\text{loc}}(\Omega)$.

Note that $(u_0, v_0) \in A$, and $u_0, v_0 \in L_N(\Omega)$. Thus the restriction

$$A_Y := \left\{(u,v) \in A \;\middle|\; u, v \in L_N(\Omega)\right\}$$

is nonempty and obviously a completely accretive operator in $Y := X \cap L_N(\Omega)$. Moreover, A_Y is m-completely accretive in Y. Indeed, for $\lambda > 0$, let $y \in Y$ be arbitrary and let $x \in X$ be a solution of $(I + \lambda A)x = y$. Then, by the complete accretivity of A

$$u_0 - x \ll u_0 - x + \lambda(v_0 - \frac{1}{\lambda}(y - x)) = u_0 + \lambda v_0 - y.$$

Since $L_N(\Omega)$ is a normal Banach space, we conclude $x \in Y$.

According to [CGL96, Theorem 1], the Volterra equation (2.1) in Y admits a unique generalized solution $v \in C([0,T); Y)$. Since the embeddings

$$Y \hookrightarrow X \quad \text{and} \quad Y \hookrightarrow L_N(\Omega)$$

are continuous, $v \equiv u$, and u is as well a generalized solution of (2.1) in the space $L_N(\Omega)$. Note that the Radon-Nikodym derivative $a \in L^1(0,T)$ of the resolvent of the first kind of the pair (κ, k) exists by lemma A.8. As u is continuous by proposition 2.5, we have

$$\begin{aligned} \left\| \frac{u(t_0+h) - u(t_0)}{h} \right\|_N \leq & \sup_{\tau\in[0,h]} \left\| \frac{u(\tau) - u_0}{h} \right\|_N \\ & + \frac{1}{\kappa} \int_0^{t_0} \frac{1}{h} \int_\tau^{\tau+h} \|f'(\sigma)\|_N \, d\sigma \; a(t_0 - \tau)\, d\tau \end{aligned} \tag{2.33}$$

for $0 < h < T - t_0$. Here and in the following, a denotes the Radon-Nikodym derivative of the resolvent α of the first kind of the pair (κ, k) satisfying $a(t) \leq 1/\kappa$ a.e. $t \in [0, \infty)$ (see lemma A.8). By proposition 2.13 applied to the space $Y \hookrightarrow L_N(\Omega)$, we already know that u is strongly differentiable from the right at $t = 0$. Therefore, applying Lebesgue's dominated convergence theorem, we can pass to the limit for $h \to 0+$ in (2.33) and obtain

$$\limsup_{h\to 0+} \left\| \frac{u(t_0+h) - u(t_0)}{h} \right\|_N < \infty.$$

Thus, we have shown that the set of differential quotients is norm bounded in $L_N(\Omega)$.

Since Ω is a σ-finite measure space, we may choose an increasing sequence $\omega_k \nearrow \Omega$ of measurable subsets of Ω, satisfying $\mu(\omega_k) < \infty$ for all $k \in \mathbb{N}$. Then, we can define the injection of $L_N(\Omega)$ into the Fréchet space $\prod_{k\in\mathbb{N}} L^1(\omega_k)$ by

$$\begin{aligned} \iota : L_N(\Omega) &\hookrightarrow \prod_{k\in\mathbb{N}} L^1(\omega_k) \\ f &\rightarrow (f\mathbf{1}_{\omega_k})_{k\in\mathbb{N}}. \end{aligned}$$

By de la Vallée Poussin's theorem it is clear that $\iota(B)$ is weakly sequentially compact for all bounded subsets B of $L_N(\Omega)$. Thus, we can conclude that there exists a sequence $\{h_n\}$, with $h_n \to 0+$ as $n \to \infty$, and $(z_k)_k \in \prod_{k\in\mathbb{N}} L^1(\omega_k)$, such that

$$\frac{u(t_0+h_n)-u(t_0)}{h_n}\mathbf{1}_{\omega_k} \rightharpoonup z_k \quad \text{weakly in } L^1(\omega_k) \text{ for all } k \in \mathbb{N}. \tag{2.34}$$

Therefore, it is clear that there exists $z \in M(\Omega)$ such that $z\mathbf{1}_{\omega_k} = z_k$ for all $k \in \mathbb{N}$.

(3) Our goal is to show that $u(t_0) \in D(\overline{A})$ and $f(t_0) - g(t_0) - z \in \overline{A}u(t)$ for all $t_0 \in L_r$, where $\overline{A}$ denotes the closure of A in the space $L_0(\Omega)$. For this purpose we first show that $z \in L_0(\Omega)$.

For all $m > 0$, it is clear that, for $0 < h < T - t_0$,

$$\left\| \left(\frac{u(t_0+h)-u(t_0)}{h} - m \right)^+ \right\|_N \le \left\| \frac{u(t_0+h)-u(t_0)}{h} \right\|_N .$$

Thus, we may assume that

$$\left(\frac{u(t_0+h)-u(t_0)}{h} - m \right)^+ \mathbf{1}_{\omega_k} \rightharpoonup z_m\mathbf{1}_{\omega_k} \quad \text{weakly in } L^1(\omega_n) \text{ for all } k \in \mathbb{N},$$

for a subsequence again denoted by $\{h_n\}$ and some $z_m \in M(\Omega)$. Since the weak limit in $L^1(\Omega)$ is order preserving, it follows from

$$\frac{u(t_0+h_n)-u(t_0)}{h_n} - m \le \left(\frac{u(t_0+h_n)-u(t_0)}{h_n} - m \right)^+$$

that $z - m \le z_m$. Since $z_m \ge 0$, we conclude $(z-m)^+ \le z_m$ for all $m > 0$.

We are now going to apply (2.29). For all $k \in \mathbb{N}$ we have

$$\begin{aligned} \int_{\omega_k} (z-m)^+ &\le \int_{\omega_k} z_m \\ &= \lim_{n\to\infty} \int_{\omega_k} \left(\frac{u(t_0+h_n)-u(t_0)}{h_n} - m \right)^+ \\ &\le \int_\Omega \left(\frac{d}{dt}^+ u(0) - m \right)^+ + \frac{1}{\kappa} \int_0^t \int_\Omega |f'(\tau)| \, d\mu \, a(t-\tau) \, d\tau. \end{aligned}$$

Since, by definition, $\omega_k \nearrow \Omega$ as $k \to \infty$, we conclude that

$$\int_\Omega (z-m)^+ < \infty \qquad \text{for all } m > 0.$$

As one can apply exactly the same arguments used above for $r \mapsto (r+m)^-$ instead of $r \mapsto (r-m)^+$, we have shown that $z \in L_0(\Omega)$.

Before we proceed with the proof, we remark that the generalized solution u of the Volterra equation (2.1) is in fact a mild solution of the inhomogeneous Cauchy problem

$$\begin{aligned} \kappa \frac{d}{dt} v(t) + Av(t) &\ni f(t) - g(t), \qquad t \in [0,T), \\ v(0) &= u_0. \end{aligned} \tag{2.35}$$

Indeed, for $\lambda > 0$ the Volterra equation (2.1) with A replaced by the Yosida approximation A_λ of A admits a unique strong solution u_λ. As A_λ is m-completely accretive as well, it is clear from (2.18) that the u_λ are equi-Lipschitz continuous. Thus, we can define $g_\lambda \in L^1(0,T;X)$ by

$$g_\lambda(t) := \frac{d}{dt} \int_0^t k(t-s)(u_\lambda(s) - u_0)\, ds.$$

Since $u_\lambda \to u$ in $L^1(0,T;X)$, by the continuous dependence on the data of the solution of the Volterra equation, it is clear by lemma 2.10 that $g_\lambda \to g$ in $L^1(0,T;X)$. As u_λ is a strong solution of the Volterra equation, it is as well a mild solution of the inhomogeneous Cauchy problem (2.35), with A replaced by A_λ, and g replaced by g_λ. Due to the continuous dependence of the solution of the Cauchy problem on the data, it is now obvious that u is a mild solution of the inhomogeneous Cauchy problem (2.35).

Now, let $(\xi, \eta) \in \overline{A}$, and for $m > 0$ let $w_m \in L_0(\Omega)' = L^1(\Omega) \cap L^\infty(\Omega)$ with

$$w_m \in \partial j_m^+(u(t_0) - \xi) \qquad \text{a.e. in } \Omega,$$

where j_m^+ is the convex function on $\mathbb{R}$ defined by $j_m^+(r) := (r-m)^+$ for all $r \in \mathbb{R}$, and ∂j_m^+ denotes the subdifferential of j_m^+. It is clear that

$$\begin{aligned} w_m = 1 &\quad \text{on} \quad \{u(t_0) - \xi > m\} \\ w_m \in [0,1] &\quad \text{on} \quad \{u(t_0) - \xi = m\} \\ w_m = 0 &\quad \text{on} \quad \{u(t_0) - \xi < m\}. \end{aligned}$$

Since A is ϕ_m^+-accretive, and u is a mild solution of the inhomogeneous Cauchy problem (2.35), we can apply the integral inequality for ϕ_m^+-integral solutions (see [Bén72, Proposition 1.27])

$$
\begin{aligned}
\int_\Omega \frac{u(t_0+h_n)-u(t_0)}{h_n} w_m &\leq \frac{1}{h_n}\int_\Omega \left[(u(t_0+h_n)-\xi-m)^+ - (u(t_0)-\xi-m)^+\right] \\
&\leq \frac{1}{\kappa h_n}\int_{t_0}^{t_0+h_n} (\phi_m^+)'_+\left[u(\tau)-\xi, f(\tau)-g(\tau)-\eta\right]\, d\tau \\
&\leq \frac{1}{\kappa h_n}\int_{t_0}^{t_0+h_n}\int_\Omega \left|f(\tau)-f(t_0)-g(\tau)+g(t_0)\right|\, d\mu\, d\tau \\
&\quad + \frac{1}{\kappa h_n}\int_{t_0}^{t_0+h_n} (\phi_m^+)'_+\left[u(\tau)-\xi, f(t_0)-g(t_0)-\eta\right]\, d\tau.
\end{aligned}
\tag{2.36}
$$

As f is continuous in t, and t_0 is a right Lebesgue point of g, and $(\phi_m^+)'_+$ is upper semi-continuous, we may pass to the limit at the right-hand side of (2.36). Moreover, since by definition, $w_m \in L^\infty(\Omega)$ and $\mu(\{w_m \neq 0\}) \leq \mu(\{u(t_0)-\xi > \frac{m}{2}\}) < \infty$, we can also pass to the limit at the left-hand side of (2.36), and we obtain

$$\kappa \int_\Omega z w_m \leq (\phi_m^+)'_+\big(u(t_0)-\xi, f(t_0)-g(t_0)-\eta\big).$$

Since $w_m \in L_0(\Omega)'$ with $w_m \in \partial j_m^+(u(t_0)-\xi)$ a.e. in Ω was chosen arbitrarily, it follows that

$$(\phi_m^+)'_+\big(u(t_0)-\xi, f(t_0)-g(t_0)-\kappa z-\eta\big) \geq 0.$$

The same arguments can be applied to $j_m^-(r) := (r+m)^-$ for all $m > 0$. Therefore, for all $\lambda > 0$ and all $(\xi,\eta) \in \overline{A}$,

$$u(t_0)-\xi \ll u(t_0)-\xi+\lambda\big(f(t_0)-g(t_0)-\kappa z-\eta\big). \tag{2.37}$$

This implies $u(t_0) \in D(\overline{A})$ and

$$f(t_0)-g(t_0)-\kappa z \in \overline{A}u(t_0). \tag{2.38}$$

(4) We are now going to show that the right-hand side derivative of u exists in $L_0(\Omega)$ at all right Lebesgue points $t_0 \in L_r$ of g, and that

$$L_0(\Omega)\text{-}\frac{d}{dt}^+ u(t_0) = \frac{1}{\kappa}\left(-\overline{A}u(t_0)+f(t_0)-g(t_0)\right)^\circ.$$

To this end, we use a reduction to the homogeneous case, as we have already shown that $u(t_0) \in D(\overline{A})$. We define the operator $B \subset L_0(\Omega)\times L_0(\Omega)$ by $B := \overline{A}-f(t_0)+g(t_0)$. It is obvious that B is an m-completely accretive operator in $L_0(\Omega)$. Let v be the mild solution (see remark 1.4) of the homogeneous Cauchy problem

$$
\begin{aligned}
\kappa\frac{d}{dt}v(t)+Bv(t) &\ni 0, \qquad t \geq 0, \\
v(0) &= u(t_0),
\end{aligned}
\tag{2.39}
$$

where $t_0 \in L_r$ is a right Lebesgue point of g. Then, as shown in step (3), $u(t_0) \in D(\overline{A})$, and by proposition 2.12 applied to the operator $\frac{1}{\kappa}B$ in the space $L_0(\Omega)$, we know that v is differentiable from the right for all $t \geq 0$, and that

$$L_0(\Omega)\text{-}\lim_{t\to 0} \frac{v(t) - u(t_0)}{t} = -\frac{1}{\kappa}(Bu(t_0))^\circ. \tag{2.40}$$

In order to be able to compare v and u, we first have to shift u by t_0 and then interpret this function as a solution of a Cauchy problem. We therefore define $w(t) := u(t+t_0)$ for $t \geq 0$. As we have already mentioned in step (3), u is the mild solution of the inhomogeneous Cauchy problem (2.35) in X, and as the imbedding $X \hookrightarrow L_0(\Omega)$ is continuous, u is a mild solution of (2.35) in $L_0(\Omega)$ with A replaced by $\overline{A}$. Due to the translation invariance of Cauchy problems, w is the unique mild solution of the inhomogeneous Cauchy problem

$$\begin{aligned} \kappa\frac{d}{dt}w(t) + \overline{A}w(t) &\ni f(t+t_0) - g(t+t_0), \qquad t \in [0, T-t_0), \\ w(0) &= u(t_0). \end{aligned} \tag{2.41}$$

Since mild solutions satisfy the integral inequality, we conclude for all $0 < h < T - t_0$

$$\begin{aligned} \left\| \frac{w(h) - u(t_0)}{h} - \frac{v(h) - u(t_0)}{h} \right\|_{L_0(\Omega)} \leq\; & \frac{1}{\kappa h}\int_0^h \|f(t_0+\tau) - f(t_0)\|_{L_0(\Omega)}\, d\tau \\ & + \frac{1}{\kappa h}\int_0^h \|g(t_0+\tau) - g(t_0)\|_{L_0(\Omega)}\, d\tau. \end{aligned} \tag{2.42}$$

As f is continuous, and t_0 is a right Lebesgue point of g, this implies that u is differentiable from the right in $L_0(\Omega)$ at t_0, and that

$$L_0(\Omega)\text{-}\frac{d}{dt}^{+} u(t_0) = L_0(\Omega)\text{-}\frac{d}{dt}^{+} v(0) = -\frac{1}{\kappa}\left(\overline{A}u(t_0) - f(t_0) + g(t_0)\right)^\circ. \tag{2.43}$$

(5) The task is to show that

$$X\text{-}\lim_{h\to 0+} \frac{u(t_0+h) - u(t_0)}{h} \tag{2.44}$$

exists at all right Lebesgue points $t_0 \in L_r$. Therefore, we first note, that $u(t_0)$, $f(t_0)$, $g(t_0) \in X$. This implies, by the continuity of the embedding $X \hookrightarrow L_0(\Omega)$, that the solution v of the homogeneous Cauchy problem (2.39) in $L_0(\Omega)$ equals the mild solution of the inhomogeneous Cauchy problem in X, given by

$$\begin{aligned} \kappa\frac{d}{dt}v(t) + Av(t) &\ni f(t_0) - g(t_0), \qquad t \geq 0, \\ v(0) &= u(t_0). \end{aligned} \tag{2.45}$$

In particular, $\frac{1}{h}(v(h) - u(t_0)) \in X$ for all $h > 0$. According to the result of step (4), we already know that

$$z = L_0(\Omega)\text{-}\lim_{h \to 0+} \frac{v(h) - u(t_0)}{h} \tag{2.46}$$

exists, and that by proposition 2.12 applied to $B := A - f(t_0) + g(t_0)$

$$\frac{v(h) - u(t_0)}{h} \ll z \qquad \text{for all } h > 0 \text{ small enough.} \tag{2.47}$$

As v is the mild solution of (2.45), and $w = u(t_0 + \cdot)$ is the mild solution of (2.35), as already shown in step (4), we can apply the integral inequality in X, and obtain

$$\begin{aligned}
\left\| \frac{v(h) - u(t_0)}{h} \right\|_X &\leq \left\| \frac{v(h) - u(t_0)}{h} - \frac{w(h) - u(t_0)}{h} \right\|_X + \left\| \frac{u(t_0 + h) - u(t_0)}{h} \right\|_X \\
&\leq \frac{1}{\kappa h} \int_0^h \|f(t_0 + \tau) - f(t_0)\|_X \, d\tau + \frac{1}{\kappa h} \int_0^h \|g(t_0 + \tau) - g(t_0)\|_X \, d\tau \\
&\quad + \sup_{\tau \in [0,h]} \left\| \frac{u(\tau) - u_0}{h} \right\|_X + \frac{1}{\kappa} \int_0^t \left\| \frac{f(\tau + h) - f(\tau)}{h} \right\|_X d\tau \\
&< \infty.
\end{aligned}$$

Here, we used the fact that $f \in W^{1,1}(0, T; X)$, that u is differentiable from the right at the point $t = 0$, and that t_0 is a right Lebesgue point of g. Since convergence in $L_0(\Omega)$ implies a.e convergence of a subsequence, we can conclude by the strong convergence condition (2.30) that $z \in X$ and

$$z = X\text{-}\lim_{h \to 0+} \frac{v(h) - u(t_0)}{h}, \tag{2.48}$$

as all subsequences converge to the same limit z. Using the integral inequality, we conclude that

$$\begin{aligned}
\left\| \frac{u(t_0 + h) - u(t_0)}{h} - \frac{v(h) - u(t_0)}{h} \right\|_X &\leq \frac{1}{\kappa h} \int_0^h \|f(t_0 + \tau) - f(t_0)\|_X \, d\tau \\
&\quad \frac{1}{\kappa h} \int_0^h \|g(t_0 + \tau) - g(t_0)\|_X \, d\tau \\
&\to 0 \qquad \text{as } h \to 0+.
\end{aligned}$$

The proof will be completed by applying the following technical lemma that can be found in [Wit92, Lemma 2.4.3]. □

Lemma 2.15. *Let X be a Banach space, and $f : [0, T] \to X$ absolutely continuous and weakly differentiable from the right almost everywhere on $[0, T)$. Then f is strongly differentiable almost everywhere on $[0, T]$.*

Assuming that f is weakly differentiable a.e. in $[0,T)$, this lemma is well known. But this version, assuming only the weak differentiability from the right, can not be found in the standard literature, as far as we know. For the sake of completeness, we will now present the proof.

Proof. We first remark that, by the absolute continuity of f on $[0,T)$, the real valued function $V_f(t) := \mathrm{Var}(f,[0,t])$ is absolutely continuous on $[0,T)$. Thus, V_f is differentiable almost everywhere on $[0,T)$, and $\frac{d}{dt}V_f \in L^1(0,T)$. Moreover,

$$\limsup_{h\to 0}\left\|\frac{f(t+h)-f(t)}{h}\right\| \le \frac{d}{dt}V_f(t) \qquad \text{a.e. for } t\in[0,T). \tag{2.49}$$

Let X_0 denote the closed linear subspace generated by $f([0,T))$. Then, obviously, X_0 is separable and weakly closed. Thus, the set

$$\left\{ w\text{-}\frac{d}{dt}^{+} f(t) \;\middle|\; t\in[0,T),\ f \text{ is weakly right differentiable at } t\right\}$$

is contained in X_0. It is well known that there exists a closed separable linear subspace Y of X' which is norming for X_0, i.e.,

$$\|x_0\| = \sup_{\substack{y\in Y\\ \|y\|=1}} \langle y, x_0\rangle \qquad \text{for all } x_0\in X_0.$$

Obviously, the canonical embedding $X_0 \hookrightarrow Y'$ is an isometry.
Now, choose a countable dense set $\{y_n \,|n\in\mathbb{N}\}$ in Y. Then the scalar valued functions $[0,T)\ni t\mapsto \langle y_n, f(t)\rangle$ are absolutely continuous, and thus differentiable almost everywhere for $t\in[0,T)$. Since the union of a countable number of nullsets is again a nullset, we can find a nullset $N\subset[0,T)$ such that, for all $t\in[0,T)\setminus N$, (2.49) is satisfied, f is weakly differentiable from the right at t, and

$$\lim_{h\to 0}\left\langle y_n, \frac{f(t+h)-f(t)}{h}\right\rangle = \lim_{h\to 0+}\left\langle y_n, \frac{f(t+h)-f(t)}{h}\right\rangle = \left\langle y_n, w\text{-}\frac{d}{dt}^{+} f(t)\right\rangle \tag{2.50}$$

for all $n\in\mathbb{N}$.
Let $t\in[0,T)\setminus N$; then, for $y\in Y$ and $\varepsilon>0$ arbitrary, we can find $n\in\mathbb{N}$ such that $\|y-y_n\|\le \frac{\varepsilon}{2}\left(\frac{d}{dt}V_f(t)\right)^{-1}$. Thus,

$$\begin{aligned}
&\limsup_{h\to 0}\left|\left\langle y, \frac{f(t+h)-f(t)}{h} - w\text{-}\frac{d}{dt}^{+} f(t)\right\rangle\right| \\
\le\;& 2\|y_n-y\|\frac{d}{dt}V_f(t) \\
&+\limsup_{h\to 0}\left|\left\langle y_n, \frac{f(t+h)-f(t)}{h} - w\text{-}\frac{d}{dt}^{+} f(t)\right\rangle\right| \\
\le\;& \varepsilon.
\end{aligned}$$

Therefore, we can define $g(t) := \sigma(X_0, Y)$-$\frac{d}{dt}f(t)$ a.e. for $t \in [0,T)$. Then, g is separably valued, and $\langle y, g\rangle$ is measurable for all $y \in Y$. Since Y is a norming subspace for X_0 of X', we conclude by [DUj77, Corollary 2.1.4, p. 42] that f is measurable. By

$$\|g(t)\| \leq \limsup_{h\to 0} \left\| \frac{f(t+h) - f(t)}{h} \right\| \leq \frac{d}{dt} V_f(t) \qquad \text{a.e. for } t \in [0,T),$$

it is clear that $g \in L^1(0,T;X)$. Therefore, we can define

$$\tilde{f}(t) := f(0) + \int_0^t g(s)\, ds \qquad \text{for all } t \in [0,T).$$

Obviously, for all $y \in Y$, we have for almost all $t \in [0,T)$

$$\begin{aligned} \langle y, \tilde{f}(0)\rangle &= \langle y, f(0)\rangle, \text{ and} \\ \frac{d}{dt}\langle y, \tilde{f}(t)\rangle &= \langle y, g(t)\rangle = \lim_{h\to 0} \left\langle y, \frac{f(t+h)-f(t)}{h} \right\rangle = \frac{d}{dt}\langle y, f(t)\rangle. \end{aligned}$$

This implies $\langle y, \tilde{f}\rangle = \langle y, f\rangle$ almost everywhere on $[0,T)$. Since $f([0,T)) \subset X_0$, and Y separates points in X_0, it is clear that $\tilde{f} = f$ almost everywhere on $[0,T)$. Thus, f is strongly differentiable almost everywhere on $[0,T)$ with

$$\frac{d}{dt} f(t) = g(t) \qquad \text{a.e. for } t \in [0,T).$$

□

As mentioned before, proposition 2.14 is a generalization of the following proposition of [Wit92, Proposition 2.4.5] for inhomogeneous Cauchy problems.

Proposition 2.16. *Let A be an m-completely accretive operator in a normal Banach space $X \subset L^1(\Omega)$ satisfying the strong convergence condition* (2.30). *Then, for all $u_0 \in D(A)$, and $f \in W^{1,1}(0,T;X)$, the mild solution v of*

$$\begin{aligned} \frac{d}{dt}v(t) + Av(t) &\ni f(t), \qquad t \geq 0, \\ v(0) &= u_0, \end{aligned} \tag{2.51}$$

is locally Lipschitz continuous and differentiable a.e. on $[0,T)$, such that for almost all $t \in [0,T)$

$$-\frac{d}{dt}v(t) \in (Au(t) - f(t))^\circ .$$

In particular, v is the unique strong solution of (2.51).

Since the proof of proposition 2.14 frequently used the fact that the generalized solution of (2.1) is as well a mild solution, one might ask whether it is not possible to apply proposition 2.16 directly. But this is in fact not possible. What we know by lemma 2.10 is that the function

$$[0,T) \ni t \mapsto \frac{d}{dt}\int_0^t k(t-s)(u(s)-u_0)\,ds$$

is an element of $L^1(0,T;X)$. To apply proposition 2.16 directly, we would have to assume that the right hand side of the Cauchy problem is in $W^{1,1}(0,T;X)$. But for general $k \in L^1(0,T)$ we have no information on the regularity of

$$\frac{d^2}{dt^2}\int_0^t k(t-s)(u(s)-u_0)\,ds.$$

Chapter 3

Entropy solutions

In this chapter, we develop an L^1-theory for the history dependent degenerated elliptic-parabolic initial value problem, given by the Volterra equation

$$\begin{aligned} \frac{\partial}{\partial t}\Big(\kappa\big(b(v(t,x)) - b(v_0(x))\big) + \int_0^t k(t-s)\big(b(v(s,x)) - b(v_0(x))\big)\,ds\Big) \\ = \text{div } a(x, Dv(t,x)) + f(t,x) \quad \text{for } (t,x) \in Q := (0,T)\times\Omega, \\ b(v)(0,\cdot) = b(v_0) \quad \text{in } \Omega, \end{aligned} \tag{3.1}$$

with Dirichlet boundary condition

$$v(t,x) = 0 \qquad \text{for } (t,x) \in \Gamma := (0,T)\times\partial\Omega. \tag{3.2}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain. We consider the above problem for L^1-data, i.e.,

$$f \in L^1(Q), \text{ and } v_0 : \Omega \to \overline{\mathbb{R}} \text{ is measurable with } b(v_0) \in L^1(\Omega). \tag{3.3}$$

We assume that the function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is Carathéodory, i.e., $a(\cdot,\xi) : \Omega \to \mathbb{R}$ is measurable for all $\xi \in \mathbb{R}^N$, and $a(x,\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous vector field for a.e. $x \in \Omega$. Moreover, we assume that a satisfies the classical Leray-Lions conditions, i.e., for some $p > 1$ and $p' := p/(p-1)$, a is monotone

$$\forall\, \xi,\zeta \in \mathbb{R}^N \text{ and a.e. } x \in \Omega : \ (a(x,\xi) - a(x,\zeta))\cdot(\xi-\zeta) \geq 0, \tag{3.4}$$

coercive

$$\exists\, \lambda > 0\ \forall\, \xi \in \mathbb{R}^N, \text{ and a.e. } x \in \Omega : \ a(x,\xi)\cdot\xi \geq \lambda|\xi|^p, \tag{3.5}$$

and satisfies a growth condition

$$\exists\, \Lambda > 0,\ j \in L^{p'}(\Omega)\ \forall\, \xi \in \mathbb{R}^N, \text{ and a.e. } x \in \Omega : \ |a(x,\xi)| \leq \Lambda(j(x) + |\xi|^{p-1}). \tag{3.6}$$

Note that we do not assume that a is strictly monotone, i.e., that

$$\forall\, \xi, \zeta \in \mathbb{R}^N, \xi \neq \zeta, \text{ and a.e. } x \in \Omega: \; (a(x,\xi) - a(x,\zeta)) \cdot (\xi - \zeta) > 0. \tag{3.7}$$

Thus, our assumptions on a are rather general. But for the existence of solutions we consider the special case of a strictly monotone a separately. It will turn out that, assuming strict monotonicity, one can obtain better convergence properties of a sequence of approximating solutions than in the general case, assuming only the monotonicity of a. Moreover, we assume that

$$\begin{array}{l}\kappa \geq 0, \text{ and } k : (0,\infty) \to \mathbb{R} \text{ is a nonnegative, nonincreasing function such that} \\ k \in L^1_{\text{loc}}([0,\infty)) \text{ and } \kappa + \int_0^t k(s)\, ds > 0 \text{ for all } t > 0,\end{array} \tag{3.8}$$

and that

$$\kappa > 0 \text{ or } k(0+) = \lim_{t\to 0+} k(t) = \infty. \tag{3.9}$$

For $b : \mathbb{R} \to \mathbb{R}$, we assume that

$$\begin{array}{l} b : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing, satisfying the normalization con-} \\ \text{dition } b(0) = 0.\end{array} \tag{3.10}$$

Thus, it may happen that b is constant on some interval. In this case, (3.1),(3.2) partially degenerates to an elliptic problem. In particular, if $b \equiv 0$, then (3.1), (3.2) is a purely elliptic problem.
The assumptions on κ, k are such that our study covers degenerate elliptic-parabolic problems

$$\begin{array}{rcll} (b(v) - b(v_0))_t &=& \text{div } a(x, Dv) + f & \text{in } Q, \\ b(v)(0,\cdot) &=& b(v_0) & \text{in } \Omega, \\ b(v) &=& 0 & \text{on } \Gamma \end{array} \tag{3.11}$$

without history dependence. Moreover, it covers the case of a fractional derivative in time. Indeed, choose $\kappa = 0$ and $k(t) := \frac{t^{-\gamma}}{\Gamma(1-\gamma)}$ for $\gamma \in (0,1)$, then the fractional derivative in time can be defined by

$$\frac{\partial^\gamma}{\partial t^\gamma} u(t) := \frac{\partial}{\partial t}\left(\int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} u(s)\, ds\right).$$

Here, in the limiting case $\gamma = 0$, i.e., $k \equiv \mathbf{1}_{[0,\infty)}$, we have the degenerated elliptic problem

$$\begin{array}{rcll} b(v) - b(v_0) &=& \text{div } a(x, Dv) + f & \text{in } Q, \\ b(v) &=& 0 & \text{on } \Gamma. \end{array} \tag{3.12}$$

In the other limiting case, i.e., $\gamma = 1$ and thus $\kappa = 1$ and $k \equiv 0$, we obtain (3.11). Thus, the degenerated elliptic-parabolic problem of fractional order $\gamma \in (0, 1)$ in time

$$\begin{aligned} \frac{\partial^\gamma}{\partial t^\gamma}(b(v) - b(v_0)) &= \operatorname{div}\ a(x, Dv) + f && \text{in } Q, \\ b(v)(0, \cdot) &= b(v_0) && \text{in } \Omega, \\ b(v) &= 0 && \text{on } \Gamma \end{aligned} \tag{3.13}$$

interpolates the degenerated elliptic and elliptic-parabolic problem, i.e. (3.12) and (3.11). There already exists a vast literature on problems of the above mentioned type. In particular, the solvability and a suitable concept of solutions in order to guarantee uniqueness of the non-degenerated elliptic problem

$$\begin{aligned} u &= \operatorname{div}\ a(x, Du) + f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.14}$$

and of the parabolic problem

$$\begin{aligned} u_t &= \operatorname{div}\ a(x, Du) + f && \text{in } Q, \\ u(0, \cdot) &= u_0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma \end{aligned} \tag{3.15}$$

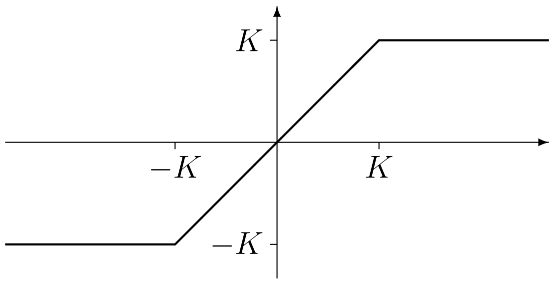
for general L^1-data, i.e., for $f \in L^1(\Omega)$ in the elliptic case, and $u_0 \in L^1(\Omega)$, and $f \in L^1(Q)$ in the parabolic case, has been investigated in recent years by several authors. It is well known, see e.g. [BBG$^+$95, Appendix I], that even for the elliptic problem (3.14) with $1 < p \leq 2 - \frac{1}{N}$ and $f \in L^1(\Omega)$ one can not expect to find a solution which solves the equation in the sense of distributions, since the gradient Du is not necessarily in $L^1(\Omega)^N$. But even if there exists a weak solution, i.e., a solution which solves the equation in the sense of distributions, this solution is in general not unique. Indeed, in [BDG97, Theorem 1.2] (see also [BP84, BG89] and [DO92]) existence of a weak solution u of (3.15) with $p \geq 2$, $u_0 = 0$ and for right hand side measures $f \in \mathcal{M}(Q)$ such that $u \in L^r(0, T; W_0^{1,q}(\Omega))$ for all pairs (r, q) satisfying

$$1 \leq q < \min\left(\frac{N(p-1)}{N-1}, p\right), \quad 1 \leq r \leq p, \quad \frac{N(p-2)+p}{r} + \frac{N}{q} > N + 1$$

is shown by approximation with regular data. But the condition $u \in L^r(0, T; W_0^{1,q}(\Omega))$, for all pairs (r, q) satisfying the above inequalities, does not imply uniqueness. A counterexample can be found in [Pri95] for the elliptic, and in [Pri97] for the parabolic problem.
In section 3.1 we show that this problem of nonexistence and nonuniqueness carries over to the history dependent problem, i.e. we show that one can not obtain weak solutions of (3.1), (3.2) for general $b(v_0) \in L^1(\Omega)$ and $f \in L^1(Q)$ if $1 < p \leq 2 - \frac{1}{N}$. Moreover, we show

by an application of a counter example given in [Ser64] that weak solutions need not be unique. In particular, we show that nonuniqueness of solutions occurs even in the linear case, i.e. for $a(x,\xi) = (a_{ij}(x))_{ij}\xi$ with $a_{ij} \in L^\infty(\Omega)$.
In order to overcome the above mentioned problems of nonexistence and nonuniqueness of weak solutions, different notions of solutions for the elliptic, the parabolic, and the degenerated elliptic-parabolic problem, i.e., (3.14), (3.15), and (3.11), were introduced. These new concepts have in common that one does not expect the solution v itself to be in a certain Sobolev space, but one introduces a class $\mathcal{T}^{1,p}$ of measurable functions $v : \Omega \to \mathbb{R}$, respectively $v : Q \to \mathbb{R}$, such that all truncations $T_K(v)$ of v are in $W^{1,p}(\Omega)$, respectively in $L^p(0,T;W^{1,p}(\Omega))$. Here, the truncation function $T_K : \mathbb{R} \to \mathbb{R}$ is given by

$$T_K(r) := \min(\max(r,-K),K) \qquad \text{for all } r \in \mathbb{R}.$$



The classes $\mathcal{T}^{1,p}(\Omega)$ and $\mathcal{T}^{1,p}(Q)$ can be defined by

$$\begin{aligned}\mathcal{T}^{1,p}(\Omega) &:= \left\{v : \Omega \to \mathbb{R} \;\middle|\; v \text{ is measurable and } T_K(v) \in W^{1,p}(\Omega) \text{ for all } K > 0\right\},\\ \mathcal{T}^{1,p}(Q) &:= \left\{v : Q \to \mathbb{R} \;\middle|\; v \text{ is measurable and } T_K(v) \in L^p(0,T;W^{1,p}(\Omega)) \text{ for all } K > 0\right\}.\end{aligned}$$

Moreover, to satisfy the boundary condition one introduces the subclasses $\mathcal{T}_0^{1,p}(\Omega)$ and $\mathcal{T}_0^{1,p}(Q)$ of $\mathcal{T}^{1,p}(\Omega)$ and $\mathcal{T}^{1,p}(Q)$, respectively, which denote those functions v such that $T_K(v) \in W_0^{1,p}(\Omega)$, respectively $T_K(v) \in L^p(0,T;W_0^{1,p}(\Omega))$. A new definition of the gradient of a function $v \in \mathcal{T}^{1,1}(\Omega)$ can be introduced. We call a measurable function $w : \Omega \to \mathbb{R}$ the gradient of $v \in \mathcal{T}^{1,1}(\Omega)$ if

$$w\mathbf{1}_{\{|v|<K\}} = DT_K(v) \qquad \text{a.e. for all } K > 0.$$

An analogous definition can be given for the gradient of a function $v \in \mathcal{T}^{1,p}(Q)$. Note that the above definition does not coincide any more with the definition of the gradient in the sense of distributions, even if both gradients exist. In fact, this new definition coincides with the definition of the so called approximate gradient in the sense of geometric measure theory, see e.g. [Fed69]. It is obvious that the above defined classes of functions in which we hope to find a solution are not linear spaces.
The two main concepts in the L^1-theory of elliptic and parabolic problems differ mainly in the way they try to guarantee uniqueness of solutions. But both concepts have in common

that one introduces an extra condition which uses a function of the solution itself as a test function in the equation (3.14), (3.15) respectively.
One concept to guarantee uniqueness is the concept of *renormalized solutions*, which was first introduced in [DL89] for the study of the Boltzmann equation. In [BGDM93] and [Mur93], this concept was then applied to the elliptic problem (3.14). See also [Rak93a, Rak93b] and [Rak94]. In [Rak94], existence and uniqueness of renormalized solutions of the elliptic problem for L^1-data according to the following definition is shown.

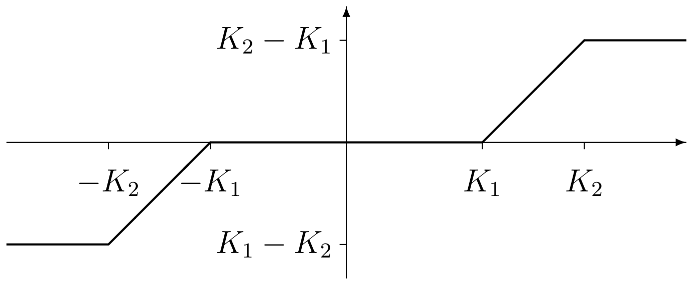
Definition 3.1. A function $u \in L^1(\Omega)$ is called a *renormalized solution* of (3.14) if $T_K(u) \in W_0^{1,p}(\Omega)$ for all $K > 0$, and

$$\int uh(u)\xi + \int_\Omega a(x, Du) \cdot D(h(u)\xi) = \int_\Omega fh(u)\xi$$

for all $h \in W^{1,\infty}(\mathbb{R})$ with compact support and all $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and, moreover,

$$\int_\Omega a(x, Du) \cdot DT_{K,K+1}(u) = \int_{\Omega\cap\{K\le|u|\le K+1\}} a(Du) \cdot Du \to 0 \qquad \text{as } K \to \infty.$$

Here, the truncation function $T_{K_1,K_2} : \mathbb{R} \to \mathbb{R}$ given by $T_{K_1,K_2} := T_{K_2} - T_{K_1}$ for $K_2 \ge K_1$ is used.



The concept of renormalized solutions was then extended to parabolic problems of the type (3.15) in [BM97]. See also [Bla93] and [DO96] for partial results in this direction. Further developments, which allow $a(x, Du)$ to be replaced by $a(t, x, u, Du)$ without any growth assumption on u can be found in [BMR99], and for the degenerated case, i.e. (3.11) with $b \not\equiv \text{id}$, we refer to [BR98], and in particular to [CW99], where uniqueness of renormalized solutions for the nonlinear degenerated elliptic-parabolic problem is shown.
The second concept, which can be shown to be equivalent to the concept of renormalized solutions for the elliptic and parabolic problems, is the concept of *entropy solutions* introduced in [BBG+95].

Definition 3.2. A function $u \in L^1(\Omega)$ is called an *entropy solution* of (3.14) if $T_K(u) \in W_0^{1,p}(\Omega)$ for all $K > 0$, and

$$\int uT_K(u - \phi) + \int_\Omega a(x, Du) \cdot DT_K(u - \phi) \le \int_\Omega fT_K(u - \phi)$$

for all $K > 0$ and all $\phi \in \mathcal{D}(\Omega)$.

The main idea of the above definition is of course adapted from the theory of conservation laws, where, in order to guarantee uniqueness, one introduces an entropy condition, see [Kru70].
The concept of entropy solutions has also been extended to parabolic problems. In [AMSdLT99], it is shown that for parabolic equations of the type (3.15) with strictly monotone a, i.e., a satisfying (3.7), the concept of entropy solutions coincides with the concept of mild solutions (see remark 1.4) of the nonlinear Cauchy problem

$$\begin{aligned} u' + Au &\ni f, \\ u(0) &= u_0 \end{aligned}$$

for $f \in L^1(0,T;L^1(\Omega))$, $u_0 \in \overline{D(A)}$ and $A \subset L^1(\Omega) \times L^1(\Omega)$ an m-completely accretive operator in $L^1(\Omega)$. Here, A can be constructed as follows, see also [AMSdLT97]. We first define the operator $A_\infty \subset L^1(\Omega) \times L^1(\Omega)$ by

$$(u,w) \in A_\infty \quad \text{if and only if} \quad \begin{aligned} & u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), w \in L^1(\Omega), \text{ and} \\ & \int_\Omega a(x,Du)\cdot D\phi = \int_\Omega w\phi \\ & \text{for all } \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{aligned} \tag{3.16}$$

The operator A_∞ is completely accretive with $\overline{D(A_\infty)} = L^1(\Omega)$ and $R(I+\lambda A_\infty) \supset L^\infty(\Omega)$ for all $\lambda > 0$. Thus, we can define $A := \overline{A_\infty}$ to be the graph-closure of A_∞. Then, A is an m-accretive operator in $L^1(\Omega)$ with $\overline{D(A)} = L^1(\Omega)$, and for all $f \in L^1(0,T;L^1(\Omega)) = L^1(Q)$, $u_0 \in L^1(\Omega)$, there exists a mild solution $u \in C([0,T];L^1(\Omega))$ of the abstract Cauchy problem (see remark 1.4) which is also the unique entropy solution of (3.15). In order to characterize the operator A, we introduce the following notation. We call a function $S : \mathbb{R} \to \mathbb{R}$ piecewise continuously differentiable, if there exist finitely many points $-\infty = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$ such that $S\big|_{(t_i,t_{i-1})} \in C^1((t_i,t_{i-1}))$ for $i = 1,\dots,m$. Moreover, $C_p^1(\mathbb{R})$ denotes the set of continuous piecewise continuously differentiable functions on $\mathbb{R}$. Letting

$$\mathcal{P} := \Big\{ S \in C_p^1(\mathbb{R}) \;\Big|\; 0 \le S' \le 1, \text{ supp } S' \text{ is compact and } S(0) = 0 \Big\}, \tag{3.17}$$

the operator A can be represented as follows.

$$(u,w) \in A \quad \text{if and only if} \quad \begin{aligned} & u,w \in L^1(\Omega), T_K(u) \in W_0^{1,p}(\Omega) \text{ for all } K > 0, \text{ and} \\ & \int_\Omega a(x,Du)\cdot DS(u-\phi) \le \int_\Omega wS(u-\phi) \\ & \text{for all } \phi \in \mathcal{D}(\Omega), S \in \mathcal{P}. \end{aligned} \tag{3.18}$$

In section 3.2, we compare the two concepts in order to develop a solution theory for L^1-data of the history dependent degenerated elliptic-parabolic problem (3.1), (3.2). It turns out that only the concept of entropy solutions can naturally be extended to this case, since the derivative in time operator in (3.1) does not satisfy a Kato equality but only a Kato inequality, see proposition 3.23 and corollary 3.24. Thus, we do not have an integration by parts formula for

$$\frac{\partial}{\partial t}\left(\kappa(u(t)-u_0)+\int_0^t k(t-s)(u(s)-u_0)\,ds\right),$$

which is an essential tool in the theory of renormalized solutions of parabolic equations. In section 3.3, we show uniqueness of entropy solutions of (3.1), (3.2) according to the following definition.

Definition 3.3. Let (3.4)-(3.6), (3.8)-(3.10) and (3.3) be satisfied. A measurable function $v : Q \to \mathbb{R}$ is called an entropy solution of (3.1), (3.2) if $b(v) \in L^1(Q)$, $T_K(v) \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $K > 0$, and

$$\begin{aligned} &-\int_Q \zeta_t(t)\Big[\kappa\int_{v_0}^{v(t)} S(r-\phi)\,db(r)+\int_0^t k_1(t-s)\int_{v_0}^{v(s)} S(r-\phi)\,db(r)\,ds\Big] \\ &+\int_Q \zeta(t)\Big[k_2(0+)(b(v(t))-b(v_0)) \\ &\qquad +\int_{(0,t]} (b(v(t-s))-b(v_0))\,dk_2(s)\Big]S(v(t)-\phi) \\ &\qquad\qquad +\int_Q \zeta(t)a(x,Dv(t))\cdot DS(v(t)-\phi) \quad\leq\quad \int_Q \zeta(t)f(t)S(v(t)-\phi) \end{aligned}$$

for all $\phi \in \mathcal{D}(\Omega)$, $\zeta \in \mathcal{D}([0,T))$ with $\zeta \geq 0$, $S \in \mathcal{P}$, where $\mathcal{P}$ is defined by (3.17), and all nonnegative nonincreasing functions $k_1, k_2 \in L^1(0,T)$, such that $k = k_1 + k_2$ and $k_2(0+) < \infty$.

The above definition is adapted from the definition of entropy solutions for fractional conservation laws as given in [CGL96, Definition 6]. A main tool in the proof of uniqueness of entropy solutions of (3.1), (3.2) will be Kruzhkov's method of doubling variables, see [Kru70].

In section 3.4, we show existence of entropy solutions for the non-degenerated problem, i.e., for (3.1), (3.2) with $b \equiv \mathrm{id}$. In particular, we show that the generalized solution of the abstract Volterra equation

$$\frac{d}{dt}\Big(\kappa(u(t)-u_0)+\int_0^t k(t-s)(u(s)-u_0)\,ds\Big)+Au(t) \ni f(t) \qquad \text{for } t \in [0,T),$$

with the operator A given by (3.18), is an entropy solution of (3.1), (3.2). To this end, we approximate the problem (3.1), (3.2) by sequences of regular data $\{u_{0,n}\}_{n\in\mathbb{N}}$, $\{f_n\}_{n\in\mathbb{N}}$. The main task in showing the existence of entropy solutions for (3.1), (3.2) is to obtain convergence of the sequence of gradients $\{Du_n\}_{n\in\mathbb{N}}$ and of $\{a(x, Du_n)\}_{n\in\mathbb{N}}$, where u_n is the solution for the regular data $u_{0,n}$, f_n.
In the strictly monotone case we can show almost everywhere convergence of the sequence $\{Du_n\}$, and thus of $\{a(x, Du_n)\}$. In the general monotone case, only weak convergence of $\{DT_K(u_n)\}$ in $L^p(Q)^N$ for all $K > 0$ can be shown by a priori estimates. Thus, one only obtains weak convergence of $\{a(x, DT_K(u_n))\}$ in $L^{p'}(Q)^N$ towards $a(x, DT_K(u))$, where u denotes the entropy solution, by a pseudo-monotonicity argument.
In the literature no result on the existence can be found for the degenerated case of the elliptic-parabolic problem (3.11), i.e., with b continuous nondecreasing and possibly constant on some interval. In particular, this problem is still open for the history dependent problem (3.1), (3.2). Remarks on the existence of renormalized solutions in the degenerated elliptic-parabolic case can be found in [BR98, BMR99] and [CW99]. We mention that the method we use in the proof of existence for the non-degenerated problem fails in the degenerated case, since there is no monotonicity in the time derivative operator any more, as shown in example 3.25.

3.1 Nonexistence and nonuniqueness of weak solutions

We assume that (3.4)-(3.6) and (3.8)-(3.10) are satisfied, and we consider the problem (3.1), (3.2) for $b \equiv \mathrm{id}$ and L^1-data $u_0 \in L^1(\Omega)$ and $f \in L^1(Q) = L^1(0, T; L^1(\Omega))$. Defining the operator $A_\infty \subset L^1(\Omega) \times L^1(\Omega)$ by (3.16), it can be shown that A_∞ is a completely accretive operator in $L^1(\Omega)$ with $R(I + \lambda A_\infty) \supset L^\infty(\Omega)$ for all $\lambda > 0$. Moreover, its closure $A := \overline{A_\infty}$ is an m-completely accretive operator in $L^1(\Omega)$; see for example [BW96, Proposition 2.4] and [AMSdLT97].
As a consequence of the existence result of [CGL96] and [Gri85], we already know that the abstract Volterra equation

$$\frac{d}{dt}\left(\kappa(u(t) - u_0) + \int_0^t k(t-s)(u(s) - u_0)\, ds\right) + Av(t) \ni f(t) \qquad t \in [0, T), \tag{3.19}$$

admits a unique generalized solution $u \in L^1(0, T; L^1(\Omega)) = L^1(Q)$. In particular, we know by the approximation result for generalized solutions that for any $k \in L^1_{\mathrm{loc}}([0, \infty))$ with $\lambda = k(0+) < \infty$ satisfying (3.8) there exists a unique strong solution u of

$$k(0+)u(t) + Au(t) \ni f(t) + k(0+)u_0 - \int_{(0,t]} (u(t-s) - u_0)\, dk(s) \quad t \in [0, T). \tag{3.20}$$

Thus, by the definition of A, $u(t) \in L^1(\Omega)$ is almost everywhere for $t \in (0,T)$ in some sense a solution of the initial value elliptic boundary value problem

$$\begin{aligned} k(0+)(u(t,x) - u_0(x)) + \int_{(0,t]} (u(t-s,x) - u_0(x))\, dk(s) & \\ - \operatorname{div}\ a(x, Du(t,x)) &= f(t,x) \quad \text{in } (0,T)\times\Omega \\ u &= 0 \qquad\quad \text{on } (0,T)\times\partial\Omega \\ u(0) &= u_0 \qquad\ \ \text{in } \Omega. \end{aligned} \tag{3.21}$$

As an example of the elliptic boundary value problem (3.21), consider

$$\begin{aligned} \lambda u - \Delta_p u &= g \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Here, we have defined $g \in L^1(\Omega)$ for $t \in (0,T)$ fixed by

$$g = f(t) + \lambda u_0 - \int_{(0,t]} (u(t-s) - u_0)\, dk(s).$$

Moreover, Δ_p denotes the p-Laplacian given by $\Delta_p = \operatorname{div}\ a(Du)$ with $a(\xi) = |\xi|^{p-2}\xi$. One easily sees that the assumptions (3.4), (3.5) and (3.6) are satisfied for this choice of a. When considering the problem (3.21), one naturally thinks of understanding the first equation in (3.21) in the sense of distributions, i.e.,

$$\int_\Omega \left(\frac{d}{dt} \int_0^t k(t-s)(u(s,x) - u_0)\, ds \right) \phi + \int_\Omega a(x, Du(t,x)) \cdot D\phi = \int_\Omega f(t,x)\phi \tag{3.22}$$

for all ϕ in the space of test function $C_c^\infty(\Omega)$ and almost everywhere for $t \in (0,T)$. Therefore we have to give a sense to the gradient Du of the solution u. Additionally, with regard to the boundary condition (3.21), the largest space in which we want the solution u to be is $L^1(0,T; W_0^{1,1}(\Omega))$. Thus, we consider the following definition of a weak solution.

Definition 3.4. A function $u \in L^1(0,T; W_0^{1,1}(\Omega))$ is called a weak solution of (3.21) for $f \in L^1((0,T)\times\Omega)$ and $u_0 \in L^1(\Omega)$, if $a(x, Du) \in L^1_{\mathrm{loc}}(Q)$, and u satisfies (3.22) for all $\phi \in C_c^\infty(\Omega)$ and almost everywhere for $t \in (0,T)$.

Now, our intention is to show that there exists a function $f \in L^1((0,T)\times\Omega)$ such that the problem (3.21) does not admit a weak solution $u \in L^1(0,T; W_0^{1,1}(\Omega))$. Therefore, we will construct a counterexample adapted from [BBG+95]. In this sense the definition 3.4 is not sufficient for the problem of existence. The main idea of showing the nonexistence of solutions will be the following nonembedding property.

Lemma 3.5. *Let Ω be an open subset of $\mathbb{R}^N$ with $N \geq 2$ then $W^{-1,\infty}(\Omega) \cap W_0^{1,N}(\Omega) \not\subset L^\infty(\Omega)$.*

Proof. We construct a function which is an element of $W^{-1,\infty}(\Omega) \cap W_0^{1,N}(\Omega)$ but which is not essentially bounded. Let $\tilde{x} \in \Omega$ and $R > 0$ such that the open ball $B_R(\tilde{x})$ with radius R centered at $\tilde{x}$ is contained in Ω. For all $x \in B_R(\tilde{x})\backslash\{\tilde{x}\}$, we define

$$u(x) := \ln\left(\ln\left(\frac{R\exp(1)}{\rho}\right)\right), \quad \text{with } \rho := |x - \tilde{x}| = \sqrt{\sum_{i=1}^{N}(x_i - \tilde{x}_i)^2},$$

and $u(x) = 0$ for all $x \in (\Omega\backslash B_R(\tilde{x})) \cup \{\tilde{x}\}$. Then, by $\lim_{x\to\tilde{x}} u(x) = \infty$, it is obvious that $u \notin L^\infty(\Omega)$. We now show that $u \in W_0^{1,N}(\Omega)$. By the simple estimate

$$\begin{aligned}
\int_\Omega |u|^N\, dx &= \int_{B_R(\tilde{x})} \ln\left(\ln\left(\frac{R\exp(1)}{\rho}\right)\right)^N dx \\
&= C_N \int_0^R \ln\left(\ln\left(\frac{R\exp(1)}{r}\right)\right)^N r^{N-1}\, dr \\
&= C_N \int_1^\infty \ln(s)^N R^N \exp\left(N(1-s)\right)\, ds \\
&\le C_N R^N \int_1^\infty s^N \exp\left(N(1-s)\right)\, ds < \infty,
\end{aligned}$$

we obtain $u \in L^N(\Omega)$. The distributional derivative of u is given by

$$\frac{\partial u}{\partial x_i} = -\frac{1}{\ln\left(\frac{R\exp(1)}{\rho}\right)} \cdot \frac{\rho}{R\exp(1)} \cdot \frac{R\exp(1)}{\rho^2} \cdot \frac{\partial \rho}{\partial x_i},$$

almost everywhere for $x \in B_R(\tilde{x})$, and $\frac{\partial u}{\partial x_i} = 0$ almost everywhere for $x \in \Omega\backslash\overline{B_R(\tilde{x})}$. Here we have $\frac{\partial \rho}{\partial x_i} = \frac{x_i - \tilde{x}_i}{\rho}$, and thus $|\frac{\partial \rho}{\partial x_i}| \le 1$ for $i = 1, ..., N$. This implies

$$\begin{aligned}
\int_\Omega |\frac{\partial u}{\partial x_i}|^N\, dx &\le \int_{B_R(\tilde{x})} \ln\left(\frac{R\exp(1)}{\rho}\right)^{-N} \frac{1}{\rho^N}\, dx \\
&= C \int_0^R \ln\left(\frac{R\exp(1)}{r}\right)^{-N} r^{N-N-1}\, dr \\
&= C \int_1^\infty s^{-N}\, ds < \infty.
\end{aligned}$$

Thus, we conclude $u \in W^{1,N}(\Omega)$. Moreover, by $\lim_{\rho\to R} u(x) = 0$ and $u(x) = 0$ almost everywhere for $x \in \Omega\backslash B_R(\tilde{x})$, we obtain $u \in W_0^{1,N}(\Omega)$.
It remains to show that $u \in W^{-1,\infty}(\Omega)$. To this end, we define the functions U_i for $i = 1, ..., N$ by

$$U_i(x) := -\frac{x_i - \tilde{x}_i}{N} \ln\left(\ln\left(\frac{R\exp(1)}{\rho}\right)\right), \quad \text{with } \rho := |x - \tilde{x}| = \sqrt{\sum_{i=1}^{N}(x_i - \tilde{x}_i)^2},$$

for $x \in B_R(\tilde{x}) \setminus \{\tilde{x}\}$, and $U_i(x) := 0$ for all $x \in (\Omega \backslash B_R(\tilde{x})) \cup \{\tilde{x}\}$. Defining

$$U_0(x) := \frac{1}{N} \cdot \frac{1}{\ln\left(\frac{R\exp(1)}{\rho}\right)}$$

for $x \in B_R(\tilde{x}) \setminus \{\tilde{x}\}$, and $U_0(x) := 0$ for all $x \in (\Omega \backslash B_R(x_0)) \cup \{\tilde{x}\}$, we easily see that $U_i \in L^\infty(\Omega)$ for $i = 0, ..., N$. Additionally, we obtain for all $v \in W_0^{1,1}(\Omega)$

$$\begin{aligned} \int_\Omega uv &= \int_{B_R(\tilde{x})} \Big[\frac{1}{N}\frac{1}{\ln\left(\frac{R\exp(1)}{\rho}\right)} - \sum_{i=1}^N \frac{(x_i - \tilde{x}_i)^2}{\rho^2 N} \frac{1}{\ln\left(\frac{R\exp(1)}{\rho}\right)} \\ &\qquad + \ln\left(\ln\left(\frac{R\exp(1)}{\rho}\right)\right)\Big]v \\ &= \int_\Omega U_0 v + \sum_{i=1}^N \int_\Omega U_i \frac{\partial v}{\partial x_i}. \end{aligned}$$

Here, we used the fact that $\lim_{\rho \to R} U_i(x) = 0$ for $i = 1, ..., N$. This implies $u \in W^{-1,\infty}(\Omega)$, which completes the proof. □

In order to show the nonexistence of weak solutions for (3.21) for certain $f \in L^1((0,T)\times\Omega)$ and $u_0 \in L^1(\Omega)$, we need to show the following duality.

Lemma 3.6. $\left(W_0^{1,1}(\Omega) + W^{-1,q'}(\Omega)\right)' = W^{-1,\infty} \cap W_0^{1,q}$ *holds for* $1 < q < \infty$ *with* $q' = \frac{q}{q-1}$.

We first note that a couple (X_1, X_2) of Banach spaces is called *compatible* if X_i is continuously embedded in a Hausdorff topological vector space X for $i = 1, 2$. For a compatible couple of Banach spaces (X_1, X_2) the spaces $X_1 + X_2$ and $X_1 \cap X_2$ are again Banach spaces, endowed with the norms

$$\|x\|_{X_1+X_2} := \inf\left\{\|x_1\|_{X_1} + \|x_2\|_{X_2} \,\Big|\, x = x_1 + x_2 \text{ with } x_1 \in X_1, x_2 \in X_2\right\},$$

respectively

$$\|x\|_{X_1 \cap X_2} := \max\left\{\|x\|_{X_1}, \|x\|_{X_2}\right\}.$$

To prove lemma 3.6, we will make use of the following abstract result on the dual space of a sum of Banach spaces, see e.g. [BS88, Exercise 3.2]

Proposition 3.7. *Let* (X_1, X_2) *be a compatible couple of Banach spaces, and let* $X_1 \cap X_2$ *be dense in* X_i *for* $i = 1, 2$. *Then* $(X_1 + X_2)' = X_1' \cap X_2'$.

Proof of lemma 3.6. By proposition 3.7, it is sufficient to show that $W_0^{1,1}(\Omega) \cap W^{-1,q'}(\Omega)$ is dense in $W_0^{1,1}(\Omega)$ and in $W^{-1,q'}(\Omega)$.

Therefore, we first note that $C_c^\infty(\Omega) \subset W_0^{1,1}(\Omega) \cap W^{-1,q'}(\Omega)$. And, by definition of $W_0^{1,1}(\Omega)$, the space $C_c^\infty(\Omega)$ is dense in $W_0^{1,1}(\Omega)$.
Take an element $F \in W^{-1,q'}(\Omega)$. Then there exist $f_i \in L^{q'}(\Omega)$ for $i = 0, ..., N$ such that for all $v \in W_0^{1,q}(\Omega)$

$$\langle F, v\rangle = \int_\Omega f_0 v + \sum_{i=1}^N \int_\Omega f_i \frac{\partial v}{\partial x_i}.$$

Choosing approximating sequences $(\phi_n^{(i)})_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ for $i = 0, ..., N$ with $\phi_n^{(i)} \to f_i$ in $L^{q'}(\Omega)$ as $n \to \infty$, we can define

$$\langle F_n, v\rangle := \int_\Omega \phi_n^{(0)} v + \sum_{i=1}^N \int_\Omega \phi_n^{(i)} \frac{\partial v}{\partial x_i}$$

for all $v \in W_0^{1,q(\Omega)}$. Then, obviously,

$$|\langle F - F_n, v\rangle| \le \max\left\{ \|f_i - \phi_n^{(i)}\|_{L^{q'}(\Omega)} \middle| i = 0, ..., N \right\} \|v\|_{W_0^{1,q}(\Omega)},$$

and thus $F_n \to F$ in $W^{-1,q'}(\Omega)$ as $n \to \infty$. □

Proposition 3.8. *Let Ω be a bounded domain in $\mathbb{R}^N$, and let $1 < p \le 2 - \frac{1}{N}$. Then there exists a function $f \in L^1((0,T) \times \Omega)$ such that the problem (3.21) with $u_0 \equiv 0$ does not admit a weak solution $u \in L^1(0,T;W_0^{1,1}(\Omega))$.*

Proof. First assume that for $f \in L^1((0,T) \times \Omega)$ and $u_0 \equiv 0$ there exists a weak solution u of (3.21), i.e. $u \in L^1(0,T;W_0^{1,1}(\Omega))$ satisfying (3.22). We show by using the growth bound (3.6) that $a(Du) \in L^{q'}((0,T) \times \Omega)$, with $q := \frac{1}{2-p}$ and $q' := \frac{q}{q-1} = \frac{1}{p-1}$. The fact that $q' = \frac{1}{p-1} < \frac{p}{p-1} = p'$, i.e., that $L^{p'}((0,T) \times \Omega) \subset L^{q'}((0,T) \times \Omega)$ holds, yields

$$\begin{aligned} \left(\int_0^T \int_\Omega |a(Du)|^{q'}\right)^{\frac{1}{q'}} &\le \Lambda \left(T^{p-1} \|j\|_{L^{q'}(\Omega)} + \left(\int_0^T \int_\Omega |Du|^{(p-1)q'}\right)^{p-1} \right) \\ &\le \Lambda \left(C\|j\|_{L^{p'}(\Omega)} + \|u\|^{p-1}_{L^1(0,T;W^{1,1}(\Omega))} \right). \end{aligned}$$

Thus, we have shown that $\mathrm{div} a(Du) \in L^{q'}(0,T;W^{-1,q'}(\Omega))$. By the definition of weak solutions this implies that $f \in L^1(0,T;W_0^{1,1}(\Omega)) + L^{q'}(0,T;W^{-1,q'}(\Omega))$.
In the second step, assume that for each $f \in L^1((0,T) \times \Omega)$ there exists a weak solution $u \in L^1(0,T;W_0^{1,1}(\Omega))$ of the elliptic boundary value problem (3.21) with $u_0 \equiv 0$. This implies by our first conclusion that

$$L^1((0,T) \times \Omega) \subset L^1(0,T;W_0^{1,1}(\Omega)) + L^{q'}(0,T;W^{-1,q'}(\Omega)).$$

Hence,

$$L^1(\Omega) \subset W_0^{1,1}(\Omega) + W^{-1,q'}(\Omega).$$

By duality and Lemma 3.6 we obtain

$$W^{-1,\infty}(\Omega) \cap W_0^{1,q}(\Omega) \subset L^\infty(\Omega).$$

But the assumption $q = \frac{1}{2-p} \leq N$ implies

$$W^{-1,\infty}(\Omega) \cap W_0^{1,N}(\Omega) \subset W^{-1,\infty}(\Omega) \cap W_0^{1,q}(\Omega) \subset L^\infty(\Omega).$$

This is a contradiction to the nonembedding shown in Lemma 3.5. Thus, there exists a function $f \in L^1((0,T) \times \Omega)$ such that the problem (3.21) with $u_0 \equiv 0$ does not admit a weak solution. □

As a consequence of the above result, we can not expect the generalized solution u of (3.20) to be a weak solution of the elliptic boundary value problem (3.21) almost everywhere for $t \in (0,T)$, even though u is a strong solution of the abstract Volterra equation (3.20). Moreover, if κ, k satisfy (3.9), we can show nonexistence of weak solutions of (3.1) with $b \equiv 0$, by using the same methods as above.

Corollary 3.9. *Let Ω be a bounded domain in $\mathbb{R}^N$, and let $1 < p \leq 2 - \frac{1}{N}$. Let $b \equiv 0$ and let κ, k be arbitrary. Then there exists $f \in L^1(Q)$ such that* (3.1) *does not admit a weak solution.*

We further show that weak solutions of Volterra equations are in general not unique. To this end, we will adapt the example given in [Ser64], see also [Pri95] and [Pri97].

Example 3.10. We consider the linear elliptic problem

$$\sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) = 0 \tag{3.23}$$

on the open unit ball $B_{\mathbb{R}^2}$ of $\mathbb{R}^2$. Here, we define the coefficients $a_{ij} \in L^\infty(B_{\mathbb{R}^2})$ for $r := \sqrt{x_1^2 + x_2^2}$ by

$$a_{ij} := \delta_{ij} + (a-1)\frac{x_i x_j}{r^2}. \tag{3.24}$$

Then, it is easy to show that the matrix $(a_{ij})_{ij}$ satisfies the coercivity assumption

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \sigma|\xi|^2 \qquad \text{for all } \xi = (\xi_i)_i \in \mathbb{R}^2 \text{ with some } \sigma > 0 \tag{3.25}$$

for all $a > 0$. Note that for $a \geq 1$ the coercivity assumption (3.25) is trivially satisfied with $\sigma = 1$ and that for $0 < a < 1$ one can obtain (3.25) for $\sigma = a > 0$. We now define the notion of weak solutions for (3.23).

Definition 3.11. A function $u \in W^{1,1}(B_{\mathbb{R}^2})$ is called a weak solution of (3.21) if

$$\int_{B_{\mathbb{R}^2}} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i}\, dx = 0 \qquad \text{for all } \varphi \in \mathcal{D}(B_{\mathbb{R}^2}) = C_c^\infty(B_{\mathbb{R}^2}).$$

Let $0 < \varepsilon < 1$, and choose $a := \dfrac{1}{\varepsilon^2}$. Defining $u : B_{\mathbb{R}^2} \to \mathbb{R}$ by

$$u(x) := x_1 r^{-(1+\varepsilon)}, \tag{3.26}$$

it is easy to see that for some constants $c_i > 0$ with $i = 0, 1, 2$

$$\int_{B_{\mathbb{R}^2}} |u|^2\, dx = c_0 \int_0^1 r^{1-2\varepsilon}\, dr, \quad \text{and} \quad \int_{B_{\mathbb{R}^2}} \left|\frac{\partial u}{\partial x_i}\right|^\beta dx = c_i \int_0^1 r^{1-\beta(1+\varepsilon)}\, dr \quad \text{for } i = 1, 2.$$

Thus, $u \in L^2(B_{\mathbb{R}^2}) \cap W^{1,\beta}(B^2_{\mathbb{R}})$ for all $\beta < \frac{2}{1+\varepsilon} < 2$. But $u \notin H^1(B_{\mathbb{R}^2})$. Moreover, since u is C^∞ for $r > 0$, it is easy to show that $u \in H^{1/2}(S^1)$, where $S^1 = \partial B_{\mathbb{R}^2}$, see also [Pri95]. By simple calculation, one obtains

$$\begin{aligned}
\int \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i}\, dx &= \lim_{\varrho \to 0+} \int_{r > \varrho} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i}\, dx \\
&= -\lim_{\varrho \to 0+} \int_{r=\rho} \varphi \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \frac{x_i}{r}\, ds \\
&= \frac{1}{\varepsilon} \lim_{\varrho \to 0+} \varrho^{-(2+\varepsilon)} \int_{r=\rho} x_1 \varphi\, ds \\
&= 0
\end{aligned}$$

for all $\varphi \in \mathcal{D}(B_{\mathbb{R}^2})$, since for all $\varphi \in \mathcal{D}(B_{\mathbb{R}^2})$ we have

$$\varrho^{-(2+\varepsilon)} \left| \int_{r=\rho} x_1 \varphi\, ds \right| \le \varrho^{1-\varepsilon} \left\| \frac{\partial \varphi}{\partial x_1} \right\|_\infty \int_0^{2\pi} \cos^2\theta\, d\theta \to 0 \qquad \text{as } \varrho \to 0+\,.$$

Thus, we have shown that u is a weak solution of (3.23). Due to $u \in H^{1/2}(S^1)$, there exists a unique variational solution $v \in H^1(B_{\mathbb{R}^2})$ of the linear elliptic boundary value problem

$$\begin{aligned}
\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial v}{\partial x_j}\right) = 0 &\qquad \text{in } B_{\mathbb{R}^2} \\
v = u &\qquad \text{on } S^1.
\end{aligned}$$

Defining $w_0 := u - v$, we conclude that the elliptic boundary value problem given by (3.23) with Dirichlet boundary condition, i.e., $u = 0$ on S^1, admits a weak solution $w_0 \in L^2(B_{\mathbb{R}^2}) \cap W^{1,\beta}(B_{\mathbb{R}^2})$ for all $\beta < \frac{2}{1+\varepsilon}$ such that $w_0 \notin H_0^1(B_{\mathbb{R}^2})$, which implies that w_0 is different from the variational solution of the same problem. Hence, weak solutions of elliptic boundary value problems are in general not unique.

We are now going to show that this nonuniqueness carries over to Volterra equations.

Example 3.12. We consider for w_0 and a_{ij} defined as above and for $0 < \gamma < 1$ the parabolic problem

$$\begin{aligned} \frac{\partial}{\partial t}\int_0^t \frac{s^{-\gamma}}{\Gamma(1-\gamma)}(w(t-s,x)-w_0(x))\,ds & \\ -\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}(t,x)\right) &= 0 \quad \text{in } (0,T)\times B_{\mathbb{R}^2}, \\ w(0,\cdot) &= w_0 \quad \text{in } B_{\mathbb{R}^2}, \\ w(t,x) &= 0 \quad \text{on } (0,T)\times S^1. \end{aligned} \tag{3.27}$$

Using the following definition of weak solutions for (3.27), it is obvious that the function $w : [0,T) \times B_{\mathbb{R}^2} \to \mathbb{R}$ defined by $w(t,x) := w_0(x)$ for all $(t,x) \in [0,T) \times B_{\mathbb{R}^2}$ is a weak solution of the above problem.

Definition 3.13. A function $w \in L^1(0,T;W_0^{1,1}(B_{\mathbb{R}^2}))$ is called a weak solution of (3.27) if for all $\varphi \in \mathcal{D}([0,T) \times B_{\mathbb{R}^2})$

$$\begin{aligned} -\int_{(0,T)\times B_{\mathbb{R}^2}}\int_0^t \frac{s^{-\gamma}}{\Gamma(1-\gamma)}(w(t-s,x)-w_0(x))\,ds\,\frac{\partial}{\partial t}\varphi(t,x)\,dx\,dt & \\ +\int_{(0,T)\times B_{\mathbb{R}^2}}\sum_{i,j} a_{ij}\frac{\partial w}{\partial x_j}(t,x)\frac{\partial \varphi}{\partial x_i}(t,x)\,dx\,dt &= 0. \end{aligned}$$

If we now define the operator $A : D(A) \to L^2(B_{\mathbb{R}^2})$ by

$$\begin{aligned} Au &:= \sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) \quad \text{for all } u \in D(A) \text{ with} \\ D(A) &:= \left\{ v \in H_0^1(B_{\mathbb{R}^2}) \;\middle|\; \sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) \in L^2(B_{\mathbb{R}^2})\right\}, \end{aligned}$$

then it is clear that A is a densely defined linear operator in $L^2(B_{\mathbb{R}^2})$ generating a C_0-semigroup. Moreover, one can show, by the same methods as used in [Paz83, Theorem 7.2.7], that A generates an analytic semigroup on $L^2(B_{\mathbb{R}^2})$. Concerning the analyticity of the semigroup generated by A, we also refer to [DL84, Exemple III.17B.3.2]. Using [Prü93, Corollary 1.2.4], we conclude that the linear Volterra equation

$$u(t) = w_0 + \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}Au(s)\,ds \tag{3.28}$$

admits an analytic resolvent $\{S(t)\}_{t\geq 0}$. Applying [Prü93, Theorem 1.2.2] to $v(\cdot) := S(\cdot)w_0$, we conclude that $v \in C([0,T);L^2(B_{\mathbb{R}^2}))$ with $v(t) \in H_0^1(B_{\mathbb{R}^2}))$ for all $t \in (0,T)$. Since, by the definition of A, v is also a weak solution of (3.27) and $v \neq w \equiv w_0 \notin H_0^1(B_{\mathbb{R}^2})$, we have shown that weak solutions of (3.27) are not unique. Therefore, the concept of weak solutions for Volterra equations is not sufficient in order to guarantee uniqueness.

We remark that the same results as above can be obtained in more than two space dimensions. This can easily be seen considering the function $u(x) := x_1(x_1^2 + x_2^2)^{-(1+\varepsilon)/2}$ on $B_{\mathbb{R}^N}$, and defining $a_{ij} = \delta_{ij}$ for the additional coefficients of the matrix $(a_{ij})_{ij}$.

3.2 A Kato inequality

This section deals with Kato inequalities for operators corresponding to the derivative in time in the Volterra equation (3.1). The min results are stated in proposition 3.23 and corollary 3.24. Thus, for $T > 0$, we consider operators of the type

$$\begin{aligned} &B : D(B) \subset L^1(0,T) \to L^1(0,T) \\ &B(u)(t) := \frac{d}{dt}\left(\kappa u(t) + \int_0^t k(t-s)u(s)\,ds\right), \\ &\text{with} \\ &D(B) := \left\{u \in L^1(0,T) \,\Big|\, t \mapsto \kappa u(t) + \int_0^t k(t-s)u(s)\,ds \in W_0^{1,1}(0,T)\right\}, \end{aligned} \tag{3.29}$$

with κ, k satisfying (3.8). Here, we define $W_0^{1,1}(0,T) := \{u \in W^{1,1}(0,T) \mid u(0) = 0\}$. The results obtained in this section are essential for the study of existence and uniqueness of entropy solutions of the history dependent problem (3.1), (3.2).
Before turning to Kato inequalities, we remark that a Kato inequality will replace the integration by parts formula which was used in the L^1-theory for degenerated elliptic-parabolic equations of the type

$$\begin{aligned} b(v)_t &= \operatorname{div}\ a(x, Du) + f && \text{in } Q := (0,T) \times \Omega, \\ v &= 0 && \text{on } (0,T) \times \partial\Omega, \\ b(v)(0,\cdot) &= b(v_0) && \text{in } \Omega. \end{aligned} \tag{3.30}$$

Here, the same assumptions as for (3.1) apply, i.e., we assume (3.4)-(3.6), (3.8)-(3.10) and (3.3). Using the definition of the set $\mathcal{P}$ given in (3.17), one can show the following integration by parts formula.

Lemma 3.14. *Let $b : \mathbb{R} \to \mathbb{R}$ be continuous and nondecreasing with $b(0) = 0$, $v \in L^p(0,T; W_0^{1,p}(\Omega))$ with $b(v) \in L^1(Q)$, $b(v)_t \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(Q)$ and $b(v)(0,\cdot) = b(v_0)$, where $v_0 : \Omega \to \overline{\mathbb{R}}$ is measurable with $b(v_0) \in L^1(\Omega)$. Then*

$$-\int_0^T \langle b(v)_t, \xi S(v-\phi)\rangle = \int_Q \xi_t \int_{v_0}^v S(r-\phi)\,db(r)\,dx\,dt$$

for all $\xi \in \mathcal{D}([0,T))$, $S \in \mathcal{P}$ *and all* $\phi \in \mathcal{D}(\Omega)$. *Moreover,*

$$-\int_0^T \langle b(v)_t, \xi h(v)\rangle = \int_Q \xi_t \int_{v_0}^v h(r)\, db(r)\, dx\, dt$$

for all $h \in W^{1,\infty}(\mathbb{R})$, $\xi \in W^{1,\infty}(Q)$ *with* $\xi(T) = 0$ *and* $h(v)\xi \in L^p(0,T;W_0^{1,p}(\Omega))$. *Here,* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $W^{-1,p'}(\Omega) + L^1(\Omega)$ *and* $W_0^{1,p}(\Omega) \cap L^\infty(Q)$.

The proof of the above lemma can be done by using exactly the same methods as in [CW99, Lemma 1.4 and Lemma 4.3], see also [AL83, Lemma 1.5] and [Ott96].
Since this integration by parts formula holds, there are two concepts of solutions of (3.30) in order to guarantee uniqueness. One concept is the concept of entropy solutions.

Definition 3.15. A measurable function $v : Q \to \mathbb{R}$ is called an *entropy solution* of (3.30) if $b(v) \in L^1(Q)$, $T_K(v) \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $K > 0$, and for all $\phi \in \mathcal{D}(\Omega)$, $S \in \mathcal{P}$, $\xi \in \mathcal{D}([0,T))$ with $\xi \geq 0$

$$-\int_Q \xi_t \int_{v_0}^v S(r-\phi)\, db(r) + \int_Q \xi\, a(x,Dv)\cdot DS(v-\phi) \leq \int_Q \xi f S(v-\phi).$$

The second concept is the concept of renormalized solutions.

Definition 3.16. A measurable function $v : Q \to \mathbb{R}$ is called a *renormalized solution* of (3.30), if $b(v) \in L^1(Q)$, $T_K(v) \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $K > 0$, for all $h \in C_c^1(\mathbb{R})$, $\xi \in \mathcal{D}([0,T) \times \Omega)$

$$-\int_Q \xi_t \int_{v_0}^v h(r)\, db(r) + \int_Q a(x,Dv)\cdot D(\xi h(v)) = \int_Q \xi f h(v), \tag{3.31}$$

and

$$\int_Q a(x,Dv)\cdot DT_{K,K+1}(v) = \int_{Q\cap\{K\leq|v|\leq K+1\}} a(x,Dv)\cdot Dv \to 0 \qquad \text{as } K \to \infty.$$

Whereas the concept of entropy solutions is only based on an inequality, the concept of renormalized solutions is based on the equality (3.31). Thus, in order to show that weak solutions $v \in L^p(0,T;W_0^{1,p}(\Omega))$ satisfying $b(v) \in L^1(Q)$ are as well renormalized solutions, one has to apply an integration by parts formula, as done in [CW99, Proposition 1.3]. Note that for (3.30) the two concepts coincide, i.e. a measurable function $v : Q \to \mathbb{R}$ is an entropy solution of (3.30) if and only if it is a renormalized solution.
The main problem that occurs when replacing the operator $\partial/\partial t$ in (3.30) by the nonlocal operator B defined by (3.29) is that we can not expect the integration by parts formula to hold in this case. Indeed, we have the following result.

Example 3.17. Take $b \equiv \mathrm{id}$, $\kappa = 0$, $k(t) := t^{-\gamma}/\Gamma(1-\gamma)$ for $0 < \gamma < 1$, and an arbitrary bounded domain $\Omega \subset \mathbb{R}^N$. We define $v_0 :\equiv 0$, and $v(t,x) := t\phi(x)$ for all $(t,x) \in Q$, where $\phi \in \mathcal{D}(\Omega)$. Since v is bounded, we may take $p \in C^\infty(\mathbb{R})$ with $0 \le p' \le 1$ such that $p(v) \equiv v$. Then, by simple calculation, we see that for certain choices of $\phi \in \mathcal{D}(\Omega)$ and $\xi \in \mathcal{D}([0,T))$ with $\phi, \xi \ge 0$

$$\begin{aligned}
&\int_Q \xi(t) \left(\frac{\partial}{\partial t} \int_0^t k(t-s)(v(s) - v_0)\, ds \right) p(v(t))\, dx\, dt \\
&\qquad = \int_\Omega \phi^2\, dx \int_0^T \xi(t) \frac{t^{2-\gamma}}{\Gamma(2-\gamma)}\, dt \\
&\qquad > \int_\Omega \phi^2\, dx \int_0^T \xi(t) \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\, dt \\
&\qquad = -\int_\Omega \phi^2\, dx \int_0^T \xi_t(t) \frac{t^{3-\gamma}}{(3-\gamma)\Gamma(3-\gamma)}\, dt \\
&\qquad = -\int_Q \xi_t(t) \int_0^t k(t-s) \int_{v_0}^{v(s)} p(r)\, dr\, ds\, dx\, dt.
\end{aligned}$$

As a consequence, one can not extend the concept of renormalized solutions to the problem given by the Volterra equation (3.1) in a straightforward way, since one can not even expect weak solutions $v \in L^p(0,T; W_0^{1,p}(\Omega))$ with $b(v) \in L^1(Q)$ to satisfy (3.31) with

$$\int_Q \xi_t \int_{v_0}^{v} h(r)\, db(r)\, dx\, dt \quad \text{replaced by} \quad \int_Q \xi_t \int_0^t k(t-s) \int_{v_0}^{v(s)} h(r)\, db(r)\, dx\, dt.$$

But the concept of entropy solutions might still be applicable if we can show an appropriate inequality. This will be done in proposition 3.23 and corollary 3.24.

To understand more clearly why we do not have an integration by parts formula for the operator B defined by (3.29), and what we can expect to hold otherwise, we now consider the above problem in an abstract setting.

It is well known that the operator $-\frac{d}{dt}$ generates the translation semigroup $(T(\tau))_{\tau \ge 0}$ on $C_0(\mathbb{R})$. Here, $C_0(\mathbb{R})$ denotes the space of continuous functions converging to 0 for $|t| \to \infty$. The translation semigroup given by $T(\tau)f = f(\cdot - \tau)$ for all $\tau \ge 0$ and all $f \in C_0(\mathbb{R})$ is strongly continuous on $C_0(\mathbb{R})$. In order to characterize semigroups and their infinitesimal generators we introduce the following notions.

Definition 3.18. Let $T : V \to V$ be a bounded linear operator on V, where $V = C_0(\mathbb{R})$ or $V = L^p(\mathbb{R})$ for some $1 \le p < \infty$.

(i) For $V = C_0(\mathbb{R})$, the operator T is called a homomorphism if T is a homomorphism of the algebra $C_0(\mathbb{R})$, i.e. T is linear and $T(fg) = (Tf)(Tg)$ for all $f, g \in C_0(\mathbb{R})$.

(ii) T is called submarkovian if $0 \leq Tf \leq 1$ holds almost everywhere in $\mathbb{R}$ for all $f \in V$ with $0 \leq f \leq 1$ a.e. in $\mathbb{R}$.

(iii) T is called positive if $0 \leq Tf$ holds almost everywhere in $\mathbb{R}$ for all $f \in V$ with $0 \leq f$ a.e. in $\mathbb{R}$.

(iv) T is called translation invariant if $T(f(\cdot - s)) = (Tf)(\cdot - s)$ for all $s \in \mathbb{R}$.

It is obvious that the translation semigroup $(T(\tau))_{\tau\geq 0}$ is a semigroup of homomorphisms, i.e. $T(\tau)$ is a homomorphism for all $\tau \geq 0$. We define

$$J_0 = \Big\{ j : \mathbb{R} \to (-\infty, \infty] \;\Big|\; j \text{ is convex and lower semicontinuous with } j(0) = 0 \Big\}. \tag{3.32}$$

For $j \in J_0$, let $D(j) = \{t \in \mathbb{R} \mid j(t) < \infty\}$. Then, according to [AB92, Théorème 13], we can fully characterize infinitesimal generators of strongly continuous semigroups of homomorphisms.

Proposition 3.19. *Let $-A$ be the infinitesimal generator of a strongly continuous semigroup $(S(\tau))_{\tau\geq 0}$ on $C_0(\mathbb{R})$. Then the following assertions are equivalent:*

(i) $(S(\tau))_{\tau\geq 0}$ is a semigroup of homomorphisms.

(ii) $fg \in D(A)$ and $A(fg) = fAg + gAf$ for all $f, g \in D(A)$.

(iii) For all $j \in J_0$, $f \in D(A)$ with $f(\mathbb{R}) \subset D(j)$, $\mu \in D(A')$, $w \in L^1(\mathbb{R}, |\mu|)$ with $w(t) \in \partial j(f(t))$ $|\mu|$-a.e. for $t \in \mathbb{R}$, the following Kato equality *is satisfied*

$$\langle A'\mu, j \circ f \rangle = \int_{\mathbb{R}} wAf \, d\mu. \tag{3.33}$$

Here, A' denotes the adjoint of the operator A.

By the above result it is clear that we can only expect an integration by parts formula to hold if the corresponding operator generates a semigroup of homomorphisms, which is equivalent to a product rule.
As we will see, in case of Volterra equations the operator given by the derivative in time does not generate a semigroup of homomorphisms on $C_0(\mathbb{R})$ in general, but still generates a strongly continuous semigroup of translation invariant submarkovian operators on $C_0(\mathbb{R})$, respectively on $L^p(\mathbb{R})$ for $1 \leq p < \infty$.
Assume that the pair (κ, k) satisfies (3.8). Then, by [CP90, Theorem 1.6], there exists a unique Bernstein function $\varphi : (0, \infty) \to \mathbb{R}$ such that

$$\varphi(z) = z(\kappa + \hat{k}(z)) \qquad \text{for all } z > 0,$$

where $\hat{k}$ denotes the Laplace-Transform of k. Moreover, there is a one-to-one correspondence between the class of Bernstein functions and pairs (κ, k) satisfying (3.8).
According to [BF75, Proposition 9.2], for a function $\varphi : (0,\infty) \to \mathbb{R}$ to be a Bernstein function is equivalent to the condition

$$\varphi \geq 0 \text{ and } (0,\infty) \ni z \mapsto \exp(-\tau\varphi(z)) \text{ is completely monotone for all } \tau > 0.$$

Applying Bernstein's theorem, we conclude that there exists a family $(\eta_\tau)_{\tau>0}$ of nonnegative Radon measures on $\mathbb{R}_+ = [0,\infty)$ such that

$$\hat{\eta}_\tau(z) = \int_{\mathbb{R}_+} e^{-zt}\, d\eta_\tau(t) = \exp(-\tau\varphi(z)) \quad \text{for all } z > 0, \tau > 0. \tag{3.34}$$

In order to characterize the family $(\eta_\tau)_{\tau>0}$, we introduce the following notion.

Definition 3.20. A family $(\mu_\tau)_{\tau>0}$ of nonnegative bounded Radon measures on $\mathbb{R}$ is called a *convolution semigroup* if

(i) $\mu_\tau(\mathbb{R}) \leq 1$ for all $\tau > 0$,

(ii) $\mu_\tau * \mu_\sigma = \mu_{\tau+\sigma}$ for all $\tau, \sigma > 0$,

(iii) $\mu_\tau \rightharpoonup^* \delta_0$ in $(C_c(\mathbb{R}))'$ as $\tau \to 0+$.

Here, $\mu * \nu$ denotes the convolution of measures, and δ_t denotes the Dirac measure at the point $t \in \mathbb{R}$.

Extending the measure η_τ for all $\tau > 0$ by $\eta_\tau(A) = \eta_\tau(A \cap [0,\infty))$ for all measurable subsets $A \subset \mathbb{R}$, one can show that the family $(\eta_\tau)_{\tau>0}$ is a convolution semigroup, in particular the convolution $\eta_\tau * \eta_\sigma$ is well defined, since all measures are supported on $\mathbb{R}_+$. Moreover, by [BF75, Theorem 9.18], there is a one-to-one correspondence between the class of Bernstein functions and the convolution semigroups supported on $\mathbb{R}_+$, where the correspondence is given by (3.34).
Moreover, we may define for $V = C_0(\mathbb{R})$ and for $V = L^p(\mathbb{R})$ with $1 \leq p < \infty$

$$S(0)f = f \text{ and } S(\tau)f = f * \eta_\tau \text{ for all } \tau > 0, f \in V. \tag{3.35}$$

Then it is well known that the family $(S(\tau))_{\tau\geq 0}$ is a strongly continuous semigroup on $C_0(\mathbb{R})$, respectively on $L^p(\mathbb{R})$. Moreover, the operators $S(\tau)$ are submarkovian for all $\tau \geq 0$, since η_τ is a nonnegative measure on $\mathbb{R}$ satisfying $\eta_\tau(\mathbb{R}) \leq 1$ for all $\tau > 0$. Then, according to [AB92, Théorème 4], one can still show a Kato inequality.

Proposition 3.21. *(i) Let* $-A$ *be the infinitesimal generator of a strongly continuous semigroup* $(S(\tau))_{\tau\geq 0}$ *of submarkovian operators on* $L^p(\mathbb{R})$ *for* $1 \leq p < \infty$. *Then*

$$\int_{\mathbb{R}} (A'g)(j\circ f) \leq \int_{\mathbb{R}} wg(Af)$$

for all $j \in J_0$, $0 \leq g \in D(A')$, $f \in D(A)$ *with* $f(t) \in D(j)$ *a.e. for* $t \in \mathbb{R}$ *such that* $j \circ f \in L^p(\mathbb{R})$, $w \in L^\infty(\mathbb{R})$ *with* $w(t) \in \partial j(f(t))$ *a.e. for* $t \in \mathbb{R}$. *Here,* A' *denotes the adjoint of* A.
(ii) Let $-A$ *be the infinitesimal generator of a strongly continuous semigroup* $(S(\tau))_{\tau\geq 0}$ *of submarkovian operators on* $C_0(\mathbb{R})$. *Then*

$$\langle A'\mu, j\circ f\rangle \leq \int_{\mathbb{R}} w(Af)\, d\mu$$

for all $j \in J_0$, $0 \leq \mu \in D(A')$, $f \in D(A)$ *with* $f(t) \in D(j)$ *for all* $t \in \mathbb{R}$, $w \in L^1(\mathbb{R})$ *with* $w(t) \in \partial j(f(t))$ $|\mu|$*-a.e. for* $t \in \mathbb{R}$.

For the sake of completeness, we give the proof in case of a submarkovian semigroup on $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Proof. Let j, g, f, w be chosen as stated above. Then, by $w \in \partial j(f)$ a.e., we conclude for all $\tau \geq 0$

$$j\circ S(\tau)f \geq j\circ f + w(S(\tau)f - f) \qquad \text{a.e. in } \mathbb{R}.$$

Since $S(\tau)$ is submarkovian, we obtain by applying Jensen's inequality $S(\tau)(j\circ f) \geq j \circ S(\tau)f$ almost everywhere in $\mathbb{R}$ for all $\tau \geq 0$. This implies

$$-\int_{\mathbb{R}} \frac{S'(\tau)g - g}{\tau} j(f) = -\int_{\mathbb{R}} g\frac{S(\tau)j(f) - j(f)}{\tau} \leq -\int_{\mathbb{R}} gj\left(\frac{S(\tau)f - f}{\tau}\right)$$

for all $\tau \geq 0$. Noting that j is continuous on $D(j)$, and that A' is the infinitesimal generator of the dual semigroup $(S'(\tau))_{\tau\geq 0}$ in X' endowed with the weak* topology, we can take the limit for $\tau \to 0+$ in the above inequality and obtain the assertion. □

Note that, according to [BF75, Theorem 12.7], there is a one-to-one correspondence between convolution semigroups on $\mathbb{R}$ and strongly continuous contraction semigroups of translation invariant submarkovian operators on $L^p(\mathbb{R})$ for $1 \leq p < \infty$.
Having constructed a strongly continuous semigroup $(S(\tau))_{\tau\geq 0}$ on $L^p(\mathbb{R})$ for given κ, k satisfying (3.8), and already knowing that the infinitesimal generator $-B_p$ of the semigroup satisfies a Kato inequality, we refer to [CP90, Theorem 4.1] for the actual calculation of the infinitesimal generator.

Proposition 3.22. *Let κ, k satisfy (4.2), choose $k_1 \in L^1(\mathbb{R}_+)$, $k_2 \in BV(\mathbb{R}_+)$ satisfying (4.2) such that $k = k_1 + k_2$, and define for $1 \leq p < \infty$*

$$\begin{aligned} D(B_p) &:= \left\{ f \in L^p(\mathbb{R}) \;\middle|\; \kappa f + k_1 * f \in W^{1,p}(\mathbb{R}) \right\}, \text{ and for all } f \in D(B_p) \\ B_p f &:= \frac{d}{dt}\left(\kappa f(t) + \int_0^\infty k_1(t-s) f(s)\, ds \right) + k_2(0+) f(t) + \int_{(0,\infty)} f(t-s)\, dk_2(s). \end{aligned}$$

Then $-B_p$ is the infinitesimal generator of the semigroup $(S(\tau))_{\tau \geq 0}$ defined by (3.35).

Returning to our original problem (3.1) on $Q = (0,T) \times \Omega$ and omitting the calculation of the adjoint of the time derivative operator, we can still show the following Kato inequality which can be considered as a generalization of [GLS90, Exercise 20.6.30] and will serve as a replacement of the integration by parts formula.

Proposition 3.23. *Let κ, k satisfy (3.8), $b : \mathbb{R} \to \mathbb{R}$ be continuous and nondecreasing, $v : Q \to \mathbb{R}$ and $v_0 : \Omega \to \overline{\mathbb{R}}$ be measurable such that $b(v) \in BV(0,T;L^1(\Omega))$, $b(v_0) \in L^1(\Omega)$ with $b(v)(0+,\cdot) = b(v_0)$ and $\big(\kappa((b(v)) - b(v_0)) + k*(b(v) - b(v_0))\big) \in W^{1,1}(0,T;L^1(\Omega))$. Then*

$$\begin{aligned} &-\int_Q \xi_t(t) \Big(\kappa \int_{v_0}^{v(t)} S(r-\phi)\, db(r) + \int_0^t k(t-s) \int_{v_0}^{v(s)} S(r-\phi)\, db(r)\, ds \Big) \\ &\quad \leq \int_Q \frac{\partial}{\partial t} \left(\kappa(b(v(t)) - b(v_0)) + \int_0^t k(t-s)(b(v(s)) - b(v_0))\, ds \right) S(v(t) - \phi) \xi(t) \end{aligned} \tag{3.36}$$

for all $S \in \mathcal{P}$, $\phi \in \mathcal{D}(\Omega)$, $\xi \in \mathcal{D}([0,T))$ with $\xi \geq 0$. Moreover,

$$\begin{aligned} &\kappa \int_\Omega \int_{v_0}^{v(T)} S(r-\phi)\, db(r) + \int_0^T \int_\Omega k(T-s) \int_{v_0}^{v(s)} S(r-\phi)\, db(r)\, ds \\ &\quad \leq \int_0^T \int_\Omega \frac{\partial}{\partial t} \left(\kappa(b(v(t)) - b(v_0)) + \int_0^t k(t-s)(b(v(s)) - b(v_0))\, ds \right) S(v(t) - \phi) \end{aligned} \tag{3.37}$$

for all $S \in \mathcal{P}$, $\phi \in \mathcal{D}(\Omega)$.

Proof. As by assumption $u := b(v) \in BV(0,T;L^1(\Omega))$ and $u_0 := b(v_0) \in L^1(\Omega)$, we conclude by applying [Gri85, Lemma 3.4] that $k * (u - u_0) \in W^{1,1}(0,T;L^1(\Omega))$. Thus, $\kappa u \in W^{1,1}(0,T;L^1(\Omega))$.
For a maximal monotone graph β in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, we use the notation β^0 for the minimal section of β, given by $\beta^0(t) := \operatorname{sign}_0(s) \min_{s \in \beta(t)} |s|$ for all $t \in \mathbb{R}$ with $\beta(t) \neq \emptyset$. Letting $S \in \mathcal{P}$ and defining $j_x : \mathbb{R} \to (-\infty, \infty]$ by

$$j_x(r) := \int_0^r S(\cdot - \phi(x)) \circ (b^{-1})^0(s)\, ds \qquad \text{for } r \in \overline{R(b)},$$

and

$$j_x(r) := \infty \qquad \text{for } r \notin \overline{R(b)}$$

for all $x \in \Omega$, it is clear that $j_x \in J_0$ and that $w := S(v-\phi) \in \partial j_x(b(v))$ almost everywhere in Q, where ∂j_x denotes the subdifferential of j_x. Thus, we conclude for all $\tilde{r}, r \in \mathbb{R}$

$$\int_r^{\tilde{r}} S(s-\phi(x))\, db(s) = j_x(b(\tilde{r})) - j_x(b(r)) \geq \big(b(\tilde{r}) - b(r)\big) S(r-\phi(x)).$$

(1) Choosing $\phi \in \mathcal{D}(\Omega)$, $\xi \in \mathcal{D}([0,T))$ arbitrary with $\xi \geq 0$, and defining $v(t,\cdot) := v_0$ for $t < 0$, it is clear that $\zeta := \xi S(v-\phi) \in L^\infty(Q)$ and that $\zeta_\rho \in W^{1,\infty}(0,T;L^\infty(\Omega))$ for all $\rho > 0$, where ζ_ρ is given by $\zeta_\rho(t) := 1/\rho \int_t^{t+\rho} \zeta(s)\, ds$. Thus, we conclude

$$\begin{aligned}
\int_Q \kappa(b(v)-b(v_0))_t \zeta_\rho &= -\int_Q \kappa(b(v)-b(v_0))(\zeta_\rho)_t \\
&= -\frac{\kappa}{\rho}\int_Q (b(v(t))-b(v_0))(\zeta(t+\rho)-\zeta(t)) \\
&= -\frac{\kappa}{\rho}\int_Q (b(v(t-\rho))-b(v(t)))S(v(t)-\phi)\xi(t) \\
&\geq -\frac{\kappa}{\rho}\int_Q \xi(t)\int_{v(t)}^{v(t-\rho)} S(r-\phi)\, db(r) \\
&= -\int_Q \frac{\xi(t+\rho)-\xi(t)}{\rho}\kappa \int_{v_0}^{v(t)} S(r-\phi)\, db(r).
\end{aligned} \tag{3.38}$$

Since $\zeta_\rho \to \zeta$ a.e. in Q and since ζ_ρ stays uniformly bounded as $\rho \to 0+$, we can pass to the limit with $\rho \to 0+$ in the above inequality and obtain

$$-\int_Q \xi_t \kappa \int_{v_0}^{v(t)} S(r-\phi)\, db(r) \leq \int_Q \xi\kappa(b(v)-b(v_0))_t S(v-\phi).$$

Since, for $\kappa > 0$, we have $u \in W^{1,1}(0,T;L^1(\Omega))$, the above calculation stays valid for $\xi \equiv \mathbf{1}_{(-\infty,T]}$. Indeed, the boundary terms satisfy $\kappa\int_\Omega (b(v(T))-b(v_0))\zeta_\rho(T) = 0$, since $\zeta_\rho(T) = 0$, and $\kappa\int_\Omega (b(v(0))-b(v_0))\zeta_\rho(0) = 0$. But, the right hand side of estimate (3.38) gives

$$\frac{\kappa}{\rho}\int_{T-\rho}^T \int_\Omega \int_{v_0}^{v(t)} S(r-\phi)\, db(r) \to \kappa \int_\Omega \int_{v_0}^{v(T)} S(r-\phi)\, db(r) \quad \text{ as } \rho \to 0+.$$

Note that, by $|j_x(b(v(t))) - j_x(b(v(s)))| \leq \|S\|_\infty |b(v(t)) - b(v(s))|$, it is clear that $j_x(b(v)) \in BV(0,T;L^1(\Omega))$ and for $\kappa > 0$ we moreover have $j_x(b(v)) \in C([0,T];L^1(\Omega))$.
(2) In order to obtain an estimate on $(k * (u-u_0))_t$, we first assume that $k(0+) < \infty$. For $u = b(v)$, $u_0 = b(v_0)$, j_x defined as above, and $w := S(v-\phi)$, the following inequality holds almost everywhere for $x \in \Omega$:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_0^t k(t-s)(u(s)-u_0)\,ds\right)w(t)\\
&\quad = \left[k(0+)(u(t)-u_0)+\int_{(0,t]}(u(t-s)-u_0)\,dk(s)\right]w(t)\\
&\quad = k(0+)u(t)w(t)+\int_{(0,t]}(u(t-s)-u(t))\,w(t)\,dk(s)\\
&\qquad +k(t)u(t)w(t)-k(0+)u(t)w(t)-k(t)u_0w(t)\\
&\quad \geq k(t)(u(t)-u_0)w(t)+\int_{(0,t]}(j_x(u(t-s))-j_x(u(t)))\;dk(s)\\
&\quad = -k(t)\,(j_x(u(t))+(u_0-u(t))w(t))+k(0+)j_x(u(t))\\
&\qquad +\int_{(0,t]} j_x(u(t-s))\,dk(s)\\
&\quad \geq (k(0+)-k(t))j_x(u_0)+k(0+)(j_x(u(t))-j_x(u_0))\\
&\qquad +\int_{(0,t]} j_x(u(t-s))\,dk(s)\\
&\quad = \frac{\partial}{\partial t}\left(\int_0^t k(t-s)(j_x(u(s))-j_x(u_0))\,ds\right).
\end{aligned} \tag{3.39}$$

Here, we used the definition of the subdifferential, i.e., for $w \in \partial j(r)$ the inequality $j(\tilde r) \geq j(r) + y(\tilde r - r)$ holds for all $\tilde r \in \mathbb{R}$, and the fact that dk is a nonpositive measure on $(0,T]$. Multiplying the above inequality by $\xi \in \mathcal{D}([0,T))$ with $\xi \geq 0$ and integrating over Q we obtain

$$\begin{aligned}
&-\int_Q \xi_t \int_0^t k(t-s)\int_{v_0}^{v(s)} S(r-\phi)\,db(r)\,ds\\
&\qquad\qquad \leq \int_Q \xi \frac{\partial}{\partial t}\left(\int_0^t k(t-s)(b(v(s))-b(v_0))\,ds\right) S(v(t)-\phi).
\end{aligned}$$

(3) Now, assume that $k(0+)=\infty$. Then we approximate k by a sequence $\{k_n\}_{n\in\mathbb{N}} \subset L^1(0,T)$ such that each k_n is nonnegative nonincreasing with $k_n(0+) < +\infty$ and such that $k_n \to k$ in $L^1(0,T)$. Since $u \in \mathrm{BV}(0,T;L^1(\Omega))$, we have by [Gri85, Lemma 3.4]

$$\frac{\partial}{\partial t}\left(\int_0^t k_n(t-s)(b(v(s))-b(v_0))\,ds\right) \to \frac{\partial}{\partial t}\left(\int_0^t k(t-s)(b(v(s))-b(v_0))\,ds\right)$$

in $L^1(Q)$ as $n \to \infty$, and since $j(u) \in L^1(Q)$, we also conclude

$$\int_0^t k_n(t-s)\int_\Omega\int_{v_0}^{v(s)} S(r-\phi)\,db(r)\,dx\,ds \to \int_0^t k(t-s)\int_\Omega\int_{v_0}^{v(s)} S(r-\phi)\,db(r)\,dx\,ds$$

in $L^1(0,T)$ as $n \to \infty$. Applying this convergence to the result of step (2) and combining it with step (1) yields (3.36). We remark that (3.37) can be shown analogously by integrating (3.39) over $(0,T)$, since $j_x(b(v)) \in BV(0,T;L^1(\Omega))$, and thus $k * j_x(b(v)) \in W^{1,1}(0,T;L^1(\Omega))$. □

In the special case $b \equiv \mathrm{id}$, we can use the linearity to show that the derivative in time operator satisfies the following monotonicity property.

Corollary 3.24. *Let κ, k satisfy (3.8), $u, v \in BV(0,T;L^1(\Omega))$, $u_0, v_0 \in L^1(\Omega)$ with $u(0+,\cdot) = u_0$, $v(0+,\cdot) = v_0$, and let $\big(\kappa((u-u_0)+k*(u-u_0)\big)$, $\big(\kappa((v-v_0)+k*(v-v_0)\big) \in W^{1,1}(0,T;L^1(\Omega))$. Then*

$$\begin{aligned} -\int_Q \xi_t(t)\left(\kappa\int_{u_0-v_0}^{u(t)-v(t)} S(r-\phi)\,dr + \int_0^t k(t-s)\int_{u_0-v_0}^{u(s)-v(s)} S(r-\phi)\,dr\,ds\right) \\ \leq \int_Q \frac{\partial}{\partial t}\Big(\kappa(u(t)-v(t)-u_0+v_0) \qquad\qquad (3.40) \\ + \int_0^t k(t-s)(u(s)-v(s)-u_0+v_0)\,ds\Big) S(u(t)-v(t)-\phi)\xi(t) \end{aligned}$$

for all $S \in \mathcal{P}$, $\phi \in \mathcal{D}(\Omega)$, $\xi \in \mathcal{D}([0,T))$ with $\xi \geq 0$. Moreover,

$$\begin{aligned} \kappa\int_\Omega\int_{u_0-v_0}^{u(T)-v(T)} S(r-\phi)\,dr + \int_0^T\int_\Omega k(T-s)\int_{u_0-v_0}^{u(s)-v(s)} S(r-\phi)\,dr\,ds \\ \leq \int_0^T\int_\Omega \frac{\partial}{\partial t}\Big(\kappa(u(t)-v(t)-u_0+v_0) \qquad\qquad (3.41) \\ + \int_0^t k(t-s)(u(s)-v(s)-u_0+v_0)\,ds\Big) S(u(t)-v(t)-\phi) \end{aligned}$$

for all $S \in \mathcal{P}$, $\phi \in \mathcal{D}(\Omega)$.

Proof. Applying proposition 3.23 with $b \equiv \mathrm{id}$ and v, v_0 replaced by $u-v$, u_0-v_0, respectively, yields the assertion. □

Note that for $u_0 = v_0$ we can conclude by corollary 3.24 that

$$\int_0^T\int_\Omega \frac{\partial}{\partial t}\Big(\kappa(u(t)-v(t)) + \int_0^t k(t-s)(u(s)-v(s))\,ds\Big) S(u(t)-v(t)-\phi) \geq 0$$

for all $u, v \in BV(0,T;L^1(\Omega))$ with $\kappa(u-v)+k*(u-v) \in W^{1,1}(0,T;L^1(\Omega))$ and $u(0+) = v(0+)$, and all $S \in \mathcal{P}$, $\phi \in \mathcal{D}(\Omega)$. Returning to the case of an arbitrary continuous nondecreasing function $b : \mathbb{R} \to \mathbb{R}$ with $b(0) = 0$, we see that the monotonicity condition of corollary 3.24 does not hold in general. In particular, we can not expect

$$\int_0^T \frac{d}{dt}\left(\kappa(b(u(t))-b(v(t))) + \int_0^t k(t-s)(b(u(s))-b(v(s)))\,ds\right) S(u(t)-v(t))\,dt \geq 0$$

to hold for the above choices of u, v, S. Indeed, we have the following example.

Example 3.25. Take $T = 1$ and define for $\alpha > 0$ the following functions on the interval $[0, 1]$

$$\begin{aligned} k(t) &:= \exp(-\alpha t), \\ u(t) &:= \alpha t + \exp(\alpha t), \\ v(t) &:= \alpha t + 1, \end{aligned}$$

and for $x \in \mathbb{R}$ we define

$$\begin{aligned} b(x) &:= 1 - \exp(-x), \\ S(x) &:= T_K(x) \qquad \text{with } K := \max_{t\in[0,1]} |u(t) - v(t)|. \end{aligned}$$

Performing some calculations one easily sees that

$$\begin{aligned}
&\int_0^1 \frac{d}{dt}\left(\int_0^t k(t-s)(b(u(s)) - b(v(s)))\,ds\right) S(u(t)-v(t))\,dt \\
&= \int_0^1 \Big(k(0)\,(b(u(t)) - b(v(t)) \\
&\quad + \int_0^t k'(t-s)\,(b(u(s)) - b(v(s)))\;ds\Big) S(u(t)-v(t))\,dt \\
&= \int_0^1 \left(1 \cdot \exp(-\alpha t)\left(\frac{1}{e} - \exp(-\exp(\alpha t))\right)\right)(\exp(\alpha t) - 1)\;dt \\
&\quad - \int_0^1 \int_0^t \alpha \exp(-\alpha t)\left(\frac{1}{e} - \exp(-\exp(\alpha s))\right)\Big)(\exp(\alpha t) - 1)\;ds\,dt \\
&\le \frac{1}{e} + \frac{\exp(-\alpha)}{\alpha e} - \frac{1}{\alpha e} - \frac{\alpha}{2e} - \frac{\exp(-\alpha)}{e} - \frac{\exp(-\alpha)}{\alpha e} + \frac{1}{\alpha e} \\
&\quad - \int_0^1 \exp(-\exp(\alpha t))\,dt - \int_0^1\int_0^t \exp(-\exp(\alpha s))\exp(\alpha s)\alpha\,ds\,dt \\
&\quad + \int_0^1 \exp(-\alpha t)\Big(\exp(-\exp(\alpha t)) \\
&\qquad\qquad - \exp(-2\alpha t)\int_0^t \exp(-\exp(\alpha t))\exp(\alpha s)\alpha\,ds\Big)\,dt \\
&= \frac{2}{e} - \frac{\alpha}{2e} - \frac{\exp(-\alpha)}{e} - \frac{\exp(-\alpha)}{\alpha} + \frac{1}{\alpha} - \frac{\exp(-3\alpha)}{3\alpha} + \frac{1}{3\alpha} \\
&\quad + \frac{\exp(-3\alpha)}{3\alpha e} - \frac{1}{3\alpha e} - 2\int_0^1 \exp(-\exp(\alpha t))\,dt.
\end{aligned}$$

Since all terms on the right hand side of the above inequality are bounded from above, and since $-\frac{\alpha}{2e} \to -\infty$ as $\alpha \to \infty$, we can conclude that for $\alpha > 0$ sufficiently large the inequality

$$\int_0^1 \frac{d}{dt}\left(\int_0^t k(t-s)(b(u(s)) - b(v(s)))\,ds\right)\xi(u(t)-v(t))\,dt < 0$$

holds.
But even in the case of the operator $u \to \frac{d}{dt}b(u)$, we can conclude by taking the above defined functions that

$$\begin{aligned}
&\int_0^1 (b(u)-b(v))_t S(u-v) \\
&= \int_0^1 \frac{d}{dt}\left(\exp(-\alpha t)\left(\frac{1}{e}-\exp(-\exp(\alpha t))\right)\right)(\exp(\alpha t)-1)\,dt \\
&= -\exp(-\exp(\alpha)) + \frac{1}{e} - \alpha\int_0^1 \exp(-\exp(\alpha t))\exp(-\alpha t)\,dt \\
&\quad -\frac{\alpha}{e} - \frac{\exp(-\alpha t)}{e} + \frac{1}{e}.
\end{aligned}$$

Again, since all terms on the right hand side of the above equality are bounded from above and since $-\frac{\alpha}{e} \to -\infty$ as $\alpha \to \infty$, there exists an $\alpha > 0$ large enough, such that

$$\int_0^1 (b(u)-b(v))_t S(u-v) < 0.$$

Since we do not have monotonicity as in corollary 3.24 for the degenerated case, i.e. for $b \not\equiv \mathrm{id}$, we will not be able to apply the same method to show the existence of entropy solutions that we can use in the non-degenerated case. We remark that the problem of existence of entropy solutions of the degenerated equation

$$\begin{aligned}
b(v)_t - \operatorname{div} a(x,Du) &= f \quad \text{in } Q=(0,T)\times\Omega, \\
b(v)(0,\cdot) &= b(v_0) \quad \text{in } \Omega, \\
b(v) &= 0 \quad \text{on } (0,T)\times\partial\Omega
\end{aligned}$$

is also still open, if we only assume that b is continuous nondecreasing with $b(0)=0$ and with a satisfying (3.4)-(3.6).

3.3 Uniqueness of entropy solutions

As already shown in section 3.1, there is no uniqueness of weak solutions for parabolic problems of fractional order in time. In this section, we show that the concept of entropy solutions given by definition 3.3 is the right concept in order to ensure uniqueness of solutions of a general class of degenerated elliptic-parabolic problems with L^1-data.
A main tool in the proof of uniqueness of entropy solutions for the problem (3.1), (3.2) is Kruzhkov's method of doubling variables, which was first used in [Kru70] to show uniqueness of entropy solutions for scalar conservation laws. See also [CGL96] for an application of Kruzhkov's method to entropy solutions of conservation laws with memory.

Theorem 3.26. *Let* (3.4)-(3.6), (3.8)-(3.10) *be satisfied, let* $f \in L^1(Q)$ *and* $v_{0,i} : \Omega \to \overline{\mathbb{R}}$ *be measurable with* $b(v_{0,i}) \in L^1(\Omega)$ *for* $i = 1, 2$ *such that* $b(v_{0,1}) = b(v_{0,2})$ *a.e. in* Ω. *Moreover, let* $v_i : Q \to \mathbb{R}$ *be an entropy solution of* (3.1), (3.2) *with right hand side* f *and initial data* $v_{0,i}$ *such that*

$$\lim_{t\to 0+} \|b(v_1)(t,\cdot) - b(v_{0,1})\|_{L^1(\Omega)} = 0.$$

Then $b(v_1) = b(v_2)$ *a.e. in* Q.

Note that we only assume the continuity at $t = 0$ for one of the entropy solutions. Thus, the final uniqueness result, without any continuity assumption, will be a corollary of the existence result, see corollary 3.31. Indeed, by theorem 3.30 we have the existence of entropy solutions of the non-degenerated problem which are continuous at 0, whenever the generalized solution of the associated abstract Volterra equation in continuous at 0. For the proof of theorem (3.26), we need the following a priori estimate.

Lemma 3.27. *Let* (3.4)-(3.6), (3.8)-(3.10) *and* (3.3) *be satisfied. Moreover, let* $v : Q \to \mathbb{R}$ *be an entropy solution of* (3.1), (3.2). *Then*

$$\int_Q \xi |DT_{K,K+L}(v)|^p = \int_{Q\cap\{K<|v|<K+L\}} \xi |Dv|^p \to 0 \qquad \text{as } K \to \infty$$

for all $L > 0$, $\xi \in \mathcal{D}([0,T))$ *with* $\xi \geq 0$.

Proof. Using $\phi \equiv 0$, $S = T_{K,K+L} = (T_{K+L} - T_K)$ for $K, L > 0$ and $k_1 = k$, $k_2 \equiv 0$ in the definition of entropy solutions, we conclude by dividing the inequality by $L > 0$

$$-\frac{1}{L}\int_Q \xi_t(t)\Big(\kappa \int_{v_0}^{v(t)} T_{K,K+L}(r)\, db(r) + \int_0^t k(t-s) \int_{v_0}^{v(s)} T_{K,K+L}(r)\, db(r)\, ds\Big)$$
$$+ \frac{1}{L}\int_Q \xi(t) a(x, Dv(t)) \cdot DT_{K,K+L}(v(t)) \leq \frac{1}{L}\int_Q \xi(t) f(t) T_{K,K+L}(v(t))$$

for all $\xi \in \mathcal{D}([0,T))$ with $\xi \geq 0$. Applying the coercivity assumption (3.5), we obtain

$$\begin{aligned} 0 \leq \frac{\lambda}{L}\int_{Q\cap\{K<|v|<K+L\}} \xi |Dv|^p \ &\leq\ \frac{1}{L}\int_Q \xi |f| \mathbf{1}_{\{K<|v|\}} \\ &\quad + \frac{1}{L}\|\xi_t\|_\infty \int_Q \Big(\kappa\Big|\int_{v_0}^{v(t)} T_{K,K+L}(r)\, db(r)\Big| \\ &\quad + \int_0^t k(t-s)\Big|\int_{v_0}^{v(s)} T_{K,K+L}(r)\, db(r)\Big|\Big). \end{aligned} \tag{3.42}$$

Since $v : Q \to \mathbb{R}$ is measurable, $\{K < |v|\} \downarrow \emptyset$ as $K \to \infty$. Thus, by $\xi|f| \in L^1(Q)$, we conclude for the first term on the right hand side of the above inequality

$$\int_{Q\cap\{K<|v|\}} \xi |f| \to 0 \qquad \text{as } K \to \infty.$$

And for the second term we remark that, using the notation $S_K(r) := \mathrm{sign}(r)(1 - \mathbf{1}_{[-K,K]}(r))$ for all $r \in \mathbb{R}$, we get

$$
\begin{aligned}
\frac{1}{L}\left|\int_{v_0}^{v} T_{K,K+L}(r)\,db(r)\right| \le\ & \left(\int_0^v S_K(r)\,db(r) + \int_0^{v_0} S_K(r)\,db(r)\right)\mathbf{1}_{\{|v_0|,|v|\le K\}} \\
& + \left(\int_0^v S_K(r)\,db(r) + \int_0^{v_0} S_K(r)\,db(r)\right)\mathbf{1}_{\{|v_0|>K,|v|\le K\}} \\
& + \left(\int_0^v S_K(r)\,db(r) + \int_0^{v_0} S_K(r)\,db(r)\right)\mathbf{1}_{\{|v_0|\le K,|v|>K\}} \\
& + \left(\int_0^v S_K(r)\,db(r) + \int_0^{v_0} S_K(r)\,db(r)\right)\mathbf{1}_{\{|v_0|,|v|>K\}} \\
\le\ & 0 + |b(v_0) - b(\mathrm{sign}(v_0)K)|\mathbf{1}_{\{|v_0|>K,|v|\le K\}} \\
& + |b(v)|\mathbf{1}_{\{|v_0|\le K,|v|>K\}} \\
& + (|b(v)| + |b(v_0) - b(\mathrm{sign}(v_0)K)|)\mathbf{1}_{\{|v_0|,|v|>K\}} \\
\le\ & |b(v)|\mathbf{1}_{\{|v|>K\}} \\
& + |b(v_0) - b(\mathrm{sign}(v_0)K)|\mathbf{1}_{\{|v_0|>K\}} \to 0 \qquad \text{as } K \to \infty
\end{aligned}
$$

almost everywhere in Q, since $\{|v| > K\} \downarrow \emptyset$ as $K \to \infty$, and

$$
\begin{aligned}
& |b(v_0) - b(\mathrm{sign}(v_0)K)|\mathbf{1}_{\{|v_0|>K\}} \\
& \qquad \le\ |b(v_0)|\mathbf{1}_{\{K<|v_0|<\infty\}} + |b(v_0) - b(\mathrm{sign}(v_0)K)|\mathbf{1}_{\{|v_0|=\infty\}} \to 0 \qquad \text{as } K \to \infty
\end{aligned}
$$

almost everywhere in Q. Here, we used the fact that $\{K < |v_0| < \infty\} \downarrow \emptyset$ as $K \to \infty$ and $|b(v_0) - b(\mathrm{sign}(v_0)K)| \to 0$ as $K \to \infty$ in $\{|b(v_0)| < \infty, |v_0| = \infty\}$. Note that the notations $b(\infty) := \lim_{t\to\infty} b(t)$ and $b(-\infty) := \lim_{t\to-\infty} b(t)$ used above make sense, since b is nondecreasing. Moreover, since for all $K > 0$

$$
\frac{1}{L}\left|\int_{v_0}^{v} T_{K,K+L}(r)\,db(r)\right| \le |b(v)| + |b(v_0)|,
$$

we conclude by Lebesgue's dominated convergence theorem that

$$
\frac{1}{L}\left|\int_{v_0}^{v} T_{K,K+L}(r)\,db(r)\right| \to 0 \qquad \text{as } K \to \infty \text{ in } L^1(Q).
$$

Thus

$$
\kappa\left|\frac{1}{L}\int_{v_0}^{v} T_{K,K+L}(r)\,db(r)\right| + k * \left|\frac{1}{L}\int_{v_0}^{v} T_{K,K+L}(r)\,db(r)\right| \to 0.
$$

as $K \to \infty$ in $L^1(Q)$, where $f * g$ denotes the convolution of f and g, i.e., $(f * g)(t) := \int_0^t f(t-s)g(s)\,ds$. Therefore, the second term on the right hand side of (3.42) converges to 0 as $K \to \infty$. This yields the assertion. □

Using the above a priori estimate we can now prove theorem 3.26.

Proof of theorem 3.26. By the definition of entropy solutions, we already know that

$$
\begin{aligned}
&-\int_Q \zeta_s(s)\Big(\kappa\int_{v_{0,1}}^{v_1(s)} T_L(r-\phi)\,db(r)+\int_0^s k_1(s-\sigma)\int_{v_{0,1}}^{v_1(\sigma)} T_L(r-\phi)\,db(r)\,d\sigma\Big)\\
&+\int_Q \zeta(s)\Big(k_2(0+)(b(v_1(s))-b(v_{0,1}))\\
&\qquad\qquad+\int_{(0,s]}(b(v_1(s-\sigma))-b(v_{0,1}))\,dk_2(\sigma)\Big)T_L(v_1(s)-\phi)\\
&+\int_Q \zeta(s)a(x,Dv_1(s))\cdot DT_L(v_1(s)-\phi)\quad\le\quad\int_Q \zeta(s)f(s)T_L(v_1(s)-\phi)
\end{aligned}
$$

for all $L>0$, $\phi\in\mathcal{D}(\Omega)$, $\zeta\in\mathcal{D}([0,T))$ with $\zeta\ge 0$, and all nonnegative nonincreasing functions $k_1,k_2\in L^1(0,T)$, such that $k=k_1+k_2$ and $k_2(0+)<\infty$. Since by our assumption

$$T_K(v_2)(t)\in W_0^{1,p}(\Omega)=\overline{\mathcal{D}(\Omega)}^{W^{1,p}(\Omega)},$$

almost everywhere for $t\in[0,T]$, we can use the method of doubling variables in time, see also [CW99], and conclude by a simple density argument

$$
\begin{aligned}
&-\int_{Q_2}\xi_s(s,t)\Big(\kappa\int_{v_{0,1}}^{v_1(s)} T_L[r-T_K(v_2(t))]\,db(r)\\
&\qquad\qquad+\int_0^s k_1(s-\sigma)\int_{v_{0,1}}^{v_1(\sigma)} T_L[r-T_K(v_2(t))]\,db(r)\,d\sigma\Big)\\
&+\int_{Q_2}\xi(s,t)\Big(k_2(0+)(b(v_1(s))-b(v_{0,1}))\\
&\qquad\qquad+\int_{(0,s]}(b(v_1(s-\sigma))-b(v_{0,1}))\,dk_2(\sigma)\Big)T_L[v_1(s)-T_K(v_2(t))]\\
&+\int_{Q_2}\xi(s,t)a(x,Dv_1(s))\cdot DT_L[v_1(s)-T_K(v_2(t))]\\
&\qquad\qquad\qquad\qquad\le\int_{Q_2}\xi(s,t)f(s)T_L[v_1(s)-T_K(v_2(t))]
\end{aligned}
$$

for all $K,L>0$, $\xi\in\mathcal{D}([0,T)\times[0,T))$ with $\xi\ge 0$, and all nonnegative nonincreasing functions $k_1,k_2\in L^1(0,T)$, such that $k=k_1+k_2$ and $k_2(0+)<\infty$. Here, we use the notation $Q_2:=(0,T)\times(0,T)\times\Omega$.

An analogous inequality can be obtained with roles of v_1 and v_2 interchanged. Note that one still assumes that v_1 depends on (s,x) and that v_2 depends on (t,x). Adding up both inequalities, we obtain

$$
\begin{aligned}
&-\int_{Q_2}\xi_s(s,t)\Big(\kappa\int_{v_{0,1}}^{v_1(s)}T_L[r-T_K(v_2(t))]\,db(r)\\
&\qquad\qquad+\int_0^s k_1(s-\sigma)\int_{v_{0,1}}^{v_1(\sigma)}T_L[r-T_K(v_2(t))]\,db(r)\,d\sigma\Big)\\
&-\int_{Q_2}\xi_t(s,t)\Big(\kappa\int_{v_{0,2}}^{v_2(t)}T_L[r-T_K(v_1(s))]\,db(r)\\
&\qquad\qquad+\int_0^t k_1(t-\tau)\int_{v_{0,2}}^{v_2(\tau)}T_L[r-T_K(v_1(s))]\,db(r)\,d\tau\Big)\\
&+\int_{Q_2}\xi(s,t)\Big(\frac{\partial}{\partial s}\int_0^s k_2(s-\sigma)(b(v_1(\sigma))-b(v_{0,1}))\,d\sigma\Big)T_L[v_1(s)-T_K(v_2(t))]\\
&+\int_{Q_2}\xi(s,t)\Big(\frac{\partial}{\partial t}\int_0^t k_2(t-\tau)(b(v_2(\tau))-b(v_{0,2}))\,d\tau\Big)T_L[v_2(t)-T_K(v_1(s))]\\
&+\int_{Q_2}\xi(s,t)\Big(a(x,Dv_1(s))\cdot DT_L[v_1(s)-T_K(v_2(t))]\\
&\qquad\qquad+a(x,Dv_2(t))\cdot DT_L[v_2(t)-T_K(v_1(s))]\Big)\\
&\quad\le\int_{Q_2}\xi(s,t)\Big(f(s)T_L[v_1(s)-T_K(v_2(t))]+f(t)T_L[v_2(t)-T_K(v_1(s))]\Big).
\end{aligned}
\tag{3.43}
$$

In order to let K tend to infinity in the above inequality, we first consider the term

$$
\begin{aligned}
I_1^{K,L}:=\int_{Q_2}\xi(s,t)\Big(&a(x,Dv_1(s))\cdot DT_L[v_1(s)-T_K(v_2(t))]\\
&+a(x,Dv_2(t))\cdot DT_L[v_2(t)-T_K(v_1(s))]\Big).
\end{aligned}
$$

We show that $\liminf_{K\to\infty} I_1^{K,L}\ge 0$. One can split up the integral $I_1^{K,L}$ by

$$
\begin{aligned}
I_1^{K,L} \;=\;& \int_{Q_2\cap\{|v_1(s)|,|v_2(t)|<K\}}\xi(s,t)[a(x,Dv_1(s))-a(x,Dv_2(t))]\cdot DT_L[v_1(s)-v_2(t)]\\
&+\int_{Q_2\cap\{|v_2(t)|\ge K,|v_1(s)-T_K(v_2(t))|<L\}}\xi(s,t)a(x,Dv_1(s))\cdot Dv_1(s)\\
&+\int_{Q_2\cap\{|v_1(s)|\ge K,|v_2(t)-T_K(v_1(s))|<L\}}\xi(s,t)a(x,Dv_2(t))\cdot Dv_2(t)\\
&+\int_{Q_2\cap\{|v_2(t)|<K,|v_1(s)|\ge K\}}\xi(s,t)a(x,Dv_1(s))\cdot DT_L[v_1(s)-v_2(t)]\\
&+\int_{Q_2\cap\{|v_1(s)|<K,|v_2(t)|\ge K\}}\xi(s,t)a(x,Dv_2(t))\cdot DT_L[v_2(t)-v_1(s)]\\
=:\;& I_{1,1}^{K,L}+I_{1,2}^{K,L}+I_{1,3}^{K,L}+I_{1,4}^{K,L}+I_{1,5}^{K,L}.
\end{aligned}
$$

Note that the first term $I_{1,1}^{K,L}$ on the right hand side of the above equality is nonnegative by the monotonicity assumption (3.4) on a, and the second and third term $I_{1,2}^{K,L}$ and $I_{1,3}^{K,L}$ are as well nonnegative, since a satisfies the coercivity assumption (3.5). Defining $s_0 := \inf\big\{|s-T| \bigm| (s,t) \in \operatorname{supp}\xi\big\}$, $t_0 := \big\{|t-T| \bigm| (s,t) \in \operatorname{supp}\xi)\big\}$ and $\tilde{T} := T - \min(s_0, t_0) < T$, we can find $\zeta \in \mathcal{D}([0,T))$ such that $0 \le \zeta \le 1$ and $\zeta(t) = 1$ for all $t \in [0, \tilde{T}]$. Since we have chosen ζ such that $\zeta^{1/p'}(s)\zeta^{1/p}(t) = 1$ for all $(s,t) \in \operatorname{supp}\xi$, the fourth term $I_{1,4}^{K,L}$ can be estimated for all $K > L$ by applying the coercivity assumption (3.5) and Hölder's inequality by

$$\begin{aligned}
I_{1,4}^{K,L} &= \int_{Q_2\cap\{|v_2(t)|<K,|v_1(s)|\ge K,|v_1(s)-v_2(t)|<L\}} \xi(s,t)a(x,Dv_1(s))\cdot Dv_1(s)\\
&\quad - \int_{Q_2\cap\{|v_2(t)|<K,|v_1(s)|\ge K,|v_1(s)-v_2(t)|<L\}} \xi(s,t)a(x,Dv_1(s))\cdot Dv_2(t)\\
&\ge -\int_{Q_2\cap\{|v_2(t)|<K,|v_1(s)|\ge K,|v_1(s)-v_2(t)|<L\}} \xi(s,t)a(x,Dv_1(s))\cdot Dv_2(t)\\
&\ge -\|\xi\|_\infty \int_{Q_2\cap\{K\le|v_1(s)|<K+L\}\cap\{K-L<|v_2(t)|<K\}} \zeta^{1/p'}(s)\zeta^{1/p}(t)\big|a(x,Dv_1(s))\cdot Dv_2(t)\big|\\
&\ge -T\|\xi\|_\infty \left(\int_{Q\cap\{K\le|v_1|<K+L\}} \zeta|a(x,Dv_1)|^{p'}\right)^{1/p'} \left(\int_{Q\cap\{K-L<|v_2|<K\}} \zeta|Dv_2|^p\right)^{1/p}\\
&\ge -\Lambda T\|\xi\|_\infty \|\zeta^{1/p}DT_{K-L,K}(v_2)\|_{L^p(Q)^N}\Big(\|\mathbf{1}_{\{K\le|v_1|\}}j\|_{L^{p'}(Q)}\\
&\qquad\qquad + \|\zeta^{1/p}DT_{K,K+L}(v_1)\|_{L^p(Q)^N}^{p-1}\Big)
\end{aligned}$$

In the last inequality we used the growth bound (3.6). By lemma 3.27 we conclude $\liminf_{K\to\infty} I_{1,4}^{K,L} \ge 0$. Since an analogous estimate with the roles of $v_1(s,x)$ and $v_2(t,x)$ interchanged can be applied to $I_{1,5}^{K,L}$, we conclude $\liminf_{K\to\infty} I_{1,5}^{K,L} \ge 0$. This yields

$$\liminf_{K\to\infty} I_1^{K,L} \ge 0. \tag{3.44}$$

We now investigate the convergence of the remaining terms in (3.43). Note that

$$\begin{aligned}
T_L(v_1(s) - T_K(v_2(t))) &\to T_L(v_1(s) - v_2(t)) \qquad \text{and}\\
T_L(v_2(t) - T_K(v_1(s))) &\to T_L(v_2(t) - v_1(s))
\end{aligned}$$

pointwise almost everywhere in Q_2 as $K \to \infty$. Thus, we can apply Lebesgue's dominated convergence theorem, and obtain

$$\int_{Q_2} \xi(s,t)\Big(f(s)T_L[v_1(s) - T_K(v_2(t))] + f(t)T_L[v_2(t) - T_K(v_1(s))]\Big)$$
$$\to \int_{Q_2} \xi(s,t)(f(s) - f(t))T_L(v_1(s) - v_2(t))$$

as $K \to \infty$. Since $k_2(0+) < \infty$,

$$\begin{aligned}\frac{\partial}{\partial s}\left(\int_0^s k_2(s-\sigma)(b(v_1(\sigma)) - b(v_{0,1}))\right) &\in L^1(Q_2),\\ \frac{\partial}{\partial t}\left(\int_0^t k_2(t-\tau)(b(v_2(\tau)) - b(v_{0,2}))\right) &\in L^1(Q_2).\end{aligned}$$

Therefore, we may once again apply Lebesgue's dominated convergence theorem. This yields

$$\begin{aligned}&\int_{Q_2} \xi(s,t)\Big(\frac{\partial}{\partial s}\int_0^s k_2(s-\sigma)(b(v_1(\sigma)) - b(v_{0,1}))\, d\sigma\Big) T_L[v_1(s) - T_K(v_2(t))]\\ &\int_{Q_2} \xi(s,t)\Big(\frac{\partial}{\partial t}\int_0^t k_2(t-\tau)(b(v_2(\tau)) - b(v_{0,2}))\, d\tau\Big) T_L[v_2(t) - T_K(v_1(s))]\\ &\qquad \to \int_{Q_2} \xi(s,t)\Big[k_2(0+)(b(v_1(s)) - b(v_2(t))) + \int_{(0,s]} (b(v_1(s-\sigma)) - b(v_{0,1}))\, dk_2(\sigma)\\ &\qquad\qquad - \int_{(0,t]} (b(v_2(t-\tau)) - b(v_{0,2}))\, dk_2(\tau)\Big] T_L(v_1(s) - v_2(t))\end{aligned}$$

as $K \to \infty$. Moreover,

$$\int_{v_{0,1}}^{v_1(s)} T_L(r - T_K(v_2(t)))\, db(r) \to \int_{v_{0,1}}^{v_1(s)} T_L(r - v_2(t))\, db(r)$$

pointwise almost everywhere in Q_2 as $K \to \infty$. Using the fact that

$$\left|\int_{v_{0,1}}^{v_1(s)} T_L(r - T_K(v_2(t)))\, db(r)\right| \le L\left(|b(v_1(s))| + |b(v_{0,1})|\right) \in L^1(Q_2)$$

we conclude by Lebesgue's dominated convergence theorem that

$$\begin{aligned}&- \int_{Q_2} \xi_s(s,t)\Big[\kappa \int_{v_{0,1}}^{v_1(s)} T_L[r - T_K(v_2(t))]\, db(r)\\ &\qquad + \int_0^s k_1(s-\sigma) \int_{v_{0,1}}^{v_1(\sigma)} T_L[r - T_K(v_2(t))]\, db(r)\, d\sigma\Big]\\ &\qquad\qquad \to - \int_{Q_2} \xi_s(s,t)\Big[\kappa \int_{v_{0,1}}^{v_1(s)} T_L(r - v_2(t))\, db(r)\\ &\qquad\qquad\qquad + \int_0^s k_1(s-\sigma) \int_{v_{0,1}}^{v_1(\sigma)} T_L(r - v_2(t))\, db(r)\, d\sigma\Big]\end{aligned}$$

as $K \to \infty$. By the same arguments as above, we also have

$$
\begin{aligned}
-\int_{Q_2} \xi_t(s,t)\Big[\kappa \int_{v_{0,2}}^{v_2(t)} T_L[r - T_K(v_1(s))]\, db(r) \\
+ \int_0^t k_1(t-\tau) \int_{v_{0,2}}^{v_2(\tau)} T_L[r - T_K(v_1(s))]\, db(r)\, d\tau\Big] \\
\to -\int_{Q_2} \xi_t(s,t)\Big[\kappa \int_{v_{0,2}}^{v_2(t)} T_L(r - v_1(s))\, db(r) \\
+ \int_0^t k_1(t-\tau) \int_{v_{0,2}}^{v_2(\tau)} T_L(r - v_1(s))\, db(r)\, d\tau\Big].
\end{aligned}
$$

Using (3.44), we may take the limit for $K \to \infty$ in (3.43) and obtain by dividing the inequality by $L > 0$

$$
\begin{aligned}
-\int_{Q_2} \xi_s(s,t)\Big[\kappa \int_{v_{0,1}}^{v_1(s)} \frac{1}{L} T_L[r - v_2(t)]\, db(r) \\
+ \int_0^s k_1(s-\sigma) \int_{v_{0,1}}^{v_1(\sigma)} \frac{1}{L} T_L[r - v_2(t)]\, db(r)\, d\sigma\Big] \\
-\int_{Q_2} \xi_t(s,t)\Big[\kappa \int_{v_{0,2}}^{v_2(t)} \frac{1}{L} T_L[r - v_1(s)]\, db(r) \\
+ \int_0^t k_1(t-\tau) \int_{v_{0,2}}^{v_2(\tau)} \frac{1}{L} T_L[r - v_1(s)]\, db(r)\, d\tau\Big] \\
+\int_{Q_2} \xi(s,t)\Big[k_2(0+)(b(v_1(s)) - b(v_2(t))) + \int_{(0,s]} (b(v_1(s-\sigma)) - b(v_{0,1}))\, dk_2(\sigma) \\
- \int_{(0,t]} (b(v_2(t-\tau)) - b(v_{0,2}))\, dk_2(\tau)\Big] \frac{1}{L} T_L[v_1(s) - v_2(t)] \\
\le \int_{Q_2} \xi(s,t)\Big(f(s) - f(t)\Big) \frac{1}{L} T_L[v_1(s) - v_2(t)].
\end{aligned}
\tag{3.45}
$$

Now, it is our intention to let $L \to 0$ in the above inequality. First note that the term on the right hand side can always be estimated by

$$
\int_{Q_2} \xi(s,t)\Big(f(s) - f(t)\Big) \frac{1}{L} T_L[v_1(s) - v_2(t)] \le \int_0^T \int_0^T \xi(s,t) \|f(s) - f(t)\|_{L^1(\Omega)}\, ds\, dt.
$$

Since $\frac{1}{L} T_L[v_1(s) - v_2(t)] \to \operatorname{sign}_0(v_1(s) - v_2(t))$ almost everywhere in Q_2 as $L \to 0$, we can apply Lebesgue's dominated convergence theorem and conclude

$$\int_{Q_2} \xi(s,t)\Big(k_2(0+)(b(v_1(s)) - b(v_2(t))) + \int_{(0,s]} (b(v_1(s-\sigma)) - b(v_{0,1}))\, dk_2(\sigma)$$
$$- \int_{(0,t]} (b(v_2(t-\tau)) - b(v_{0,2}))\, dk_2(\tau)\Big) \frac{1}{L} T_L[v_1(s) - v_2(t)]$$
$$\to \int_{Q_2} \xi(s,t)\Big(k_2(0+)(b(v_1(s)) - b(v_2(t))) + \int_{(0,s]} (b(v_1(s-\sigma)) - b(v_{0,1}))\, dk_2(\sigma)$$
$$- \int_{(0,t]} (b(v_2(t-\tau)) - b(v_{0,2}))\, dk_2(\tau)\Big) \mathrm{sign}_0(v_1(s) - v_2(t)) =: I_2$$

as $L \to 0$. Note that $(b(v_1(s)) - b(v_2(t)))\mathrm{sign}_0(v_1(s) - v_2(t)) = |b(v_1(s) - b(v_2(t))|$ almost everywhere in Q_2, since b is nondecreasing and $b(0) = 0$. Thus, using the fact that dk_2 is a nonpositive measure on $(0, \infty)$, we obtain

$$I_2 = \int_{Q_2} \xi(s,t)\Big(k_2(0+)(b(v_1(s)) - b(v_2(t))) + \int_{(0,\min(s,t)]} (b(v_1(s-\tau)) - b(v_2(t-\tau)))\, dk_2(\tau)$$
$$+ \mathbf{1}_{\{t<s\}} \int_{(t,s]} (b(v_1(s-\sigma)) - b(v_{0,1}))\, dk_2(\sigma)$$
$$- \mathbf{1}_{\{s<t\}} \int_{(s,t]} (b(v_2(t-\tau)) - b(v_{0,2}))\, dk_2(\tau)\Big) \mathrm{sign}_0(v_1(s) - v_2(t))$$
$$\geq \int_0^T \int_0^T \xi(s,t)\Big(k_2(0+)\|b(v_1(s)) - b(v_2(t))\|_{L^1(\Omega)}$$
$$+ \int_{(0,\min(s,t)]} \|b(v_1(s-\tau)) - b(v_2(t-\tau))\|_{L^1(\Omega)}\, dk_2(\tau)$$
$$+ \mathbf{1}_{\{t<s\}} \int_{(t,s]} \|b(v_1(s-\sigma)) - b(v_{0,1})\|_{L^1(\Omega)}\, dk_2(\sigma)$$
$$+ \mathbf{1}_{\{s<t\}} \int_{(s,t]} \|b(v_2(t-\tau)) - b(v_{0,2})\|_{L^1(\Omega)}\, dk_2(\tau)\Big).$$

Thus, it remains to investigate the convergence of the first two integrals in (3.45) as $L \to 0$. Therefore, we note that

$$\int_{v_{0,1}}^{v_1(s)} \frac{1}{L} T_L[r - v_2(t)]\, db(r) \to \int_{v_{0,1}}^{v_1(s)} \mathrm{sign}_0[r - v_2(t)]\, db(r)$$
$$= \big|b(v_1(s)) - b(v_2(t))\big| - \big|b(v_2(t)) - b(v_{0,1})\big|$$

almost everywhere in Q_2 as $L \to 0$. Since, moreover,

$$\int_{v_{0,1}}^{v_1(s)} \frac{1}{L} T_L[r - v_2(t)]\, db(r) \leq \big|b(v_1(s)) - b(v_2(t))\big| + \big|b(v_2(t)) - b(v_{0,1})\big| \in L^1(Q_2),$$

Lebesgue's dominated convergence theorem yields

$$
\begin{aligned}
&-\int_{Q_2} \xi_s(s,t)\Big(\kappa \int_{v_{0,1}}^{v_1(s)} \frac{1}{L} T_L[r - v_2(t)]\, db(r) \\
&\qquad\qquad + \int_0^s k_1(s-\sigma) \int_{v_{0,1}}^{v_1(\sigma)} \frac{1}{L} T_L[r - v_2(t)]\, db(r)\, d\sigma\Big) \\
&\to -\int_0^T \int_0^T \xi_s(s,t)\Big(\kappa\big(\|b(v_1(s)) - b(v_2(t))\|_{L^1(\Omega)} - \|b(v_2(t)) - b(v_{0,1})\|_{L^1(\Omega)}\big) \\
&\qquad + \int_0^s k_1(s-\sigma)\big(\|b(v_1(\sigma)) - b(v_2(t))\|_{L^1(\Omega)} - \|b(v_2(t)) - b(v_{0,1})\|_{L^1(\Omega)}\big)\, d\sigma\Big)\, ds\, dt.
\end{aligned}
$$

Note that

$$
\kappa \int_0^T \int_0^T \xi_s(s,t)\|b(v_2(t)) - b(v_{0,1})\|\, ds\, dt = -\kappa \int_0^T \xi(0,t)\|b(v_2(t)) - b(v_{0,1})\|\, dt.
$$

Using essentially the same arguments as above in order to estimate the limit for $L \to 0$ of

$$
\begin{aligned}
&-\int_{Q_2} \xi_t(s,t)\Big(\kappa \int_{v_{0,2}}^{v_2(t)} \frac{1}{L} T_L[r - v_1(s)]\, db(r) \\
&\qquad\qquad + \int_0^t k_1(t-\tau) \int_{v_{0,2}}^{v_2(\tau)} \frac{1}{L} T_L[r - v_1(s)]\, db(r)\, d\tau\Big)
\end{aligned}
$$

we conclude for $u_1 := b(v_1)$, $u_2 := b(v_2)$ and $u_0 := b(v_{0,1}) = b(v_{0,2})$ by taking the limit for $L \to 0$ in (3.45)

$$
\begin{aligned}
&-\int_0^T \int_0^T \xi_s(s,t)\Big(\kappa\|u_1(s) - u_2(t)\| + \int_0^s k_1(s-\sigma)\|u_1(\sigma) - u_2(t)\|\, d\sigma\Big)\, ds\, dt \\
&-\int_0^T \int_0^T \xi_t(s,t)\Big(\kappa\|u_1(s) - u_2(t)\| + \int_0^t k_1(t-\tau)\|u_1(s) - u_2(\tau)\|\, d\tau\Big)\, ds\, dt \\
&\quad \le \int_0^T \int_0^T \xi(s,t)\Big(F(s,t) + k_1(s)\|u_2(t) - u_0\|_{L^1(\Omega)} + k_1(t)\|u_1(s) - u_0\|_{L^1(\Omega)}\Big)\, ds\, dt \\
&\qquad + \kappa\left(\int_0^T \xi(0,t)\|u_2(t) - u_0\|_{L^1(\Omega)}\, dt + \int_0^T \xi(s,0)\|u_1(s,0) - u_0\|_{L^1(\Omega)}\, ds\right),
\end{aligned}
\tag{3.46}
$$

where

$$
\begin{aligned}
F(s,t) &:= \|f(s) - f(t)\|_{L^1(\Omega)} \\
&\quad - k_2(0+)\|u_1(s) - u_2(t)\|_{L^1(\Omega)} - \int_{(0,\min(s,t)]} \|u_1(s-\tau) - u_2(t-\tau)\|_{L^1(\Omega)}\, dk_2(\tau) \\
&\quad - \mathbf{1}_{\{t<s\}} \int_{(t,s]} \|u_1(s-\sigma) - u_0\|_{L^1(\Omega)}\, dk_2(\sigma) - \mathbf{1}_{\{s<t\}} \int_{(s,t]} \|u_2(t-\tau) - u_0\|_{L^1(\Omega)}\, dk_2(\tau).
\end{aligned}
$$

We remark that the above inequality (3.46) is equivalent to [CGL96, Equation (45)]. Thus, to complete the proof, we might follow exactly the arguments as developed in [CGL96]. But in order to omit the continuity assumption at $t = 0$ for one of the entropy solutions, we will give a somewhat different proof without applying the methods of [CGL96, Lemma 10]. This means that we will not use the existence of a nonnegative measure μ on $[0,\infty)\times[0,\infty)$ such that

$$\frac{\partial}{\partial s}\left(\kappa\mu(s,t)+\int_0^s k(s-\sigma)d\mu(\sigma,t)\right)+\frac{\partial}{\partial t}\left(\kappa\mu(s,t)+\int_0^t k(t-\tau)d\mu(s,\tau)\right)=\delta_0(s,t)$$

holds in the sense of distributions, i.e. for all $\xi\in\mathcal{D}([0,T)\times[0,T))$

$$-\int_0^T\int_0^T\left(\kappa\xi_s(s,t)+\int_0^{T-s}k(\sigma)\xi_s(\sigma+s,t)\,d\sigma\right)\,d\mu(s,t)$$
$$-\int_0^T\int_0^T\left(\kappa\xi_t(s,t)+\int_0^{T-t}k(\tau)\xi_t(s,t+\tau)\,d\tau\right)\,d\mu(s,t)=\xi(0,0).$$

We define $y(s,t):=\|u_1(s)-u_1(t)\|_{L^1(\Omega)}$ for $s,t>0$. Moreover, let $y(s,0):=\|u_1(s)-u_0\|_{L^1(\Omega)}$, and $y(0,t):=\|u_0-u_2(t)\|_{L^1(\Omega)}$ for all $s,t\geq 0$, and $y(s,t)=0$ for all $s,t\in\mathbb{R}$ such that $s<0$ or $t<0$. Since (3.46) holds for all combinations k_1, $k_2\in L^1(0,T)$ of nonnegative nonincreasing functions such that $k=k_1+k_2$ and $k_2(0+)<\infty$, we choose a sequence $\{k_{1,n}\}_{n\in\mathbb{N}}$ such that

$$k_{1,n}(t):=\max\left(0,k(t)-k\left(\frac{1}{n}\right)\right). \tag{3.47}$$

Then, obviously $k_{1,n}\to 0$ in $L^1(0,T)$ and $k_{2,n}:=k-k_{1,n}\to k$ in $L^1(0,T)$. We choose a sequence $\{\varrho_\varepsilon\}_{\varepsilon>0}$ of mollifiers on $\mathbb{R}$ such that supp $\varrho_\varepsilon\subset[0,\varepsilon]$ and $\varrho_\varepsilon\to\delta_0$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon\to 0$. Defining $\xi_\varepsilon(s,t):=\varrho_\varepsilon(s-t)\phi(t)$ for $\phi\in\mathcal{D}([0,T))$ with $\phi\geq 0$, it is easy to see that $\xi_\varepsilon\in\mathcal{D}([0,T)\times[0,T))$ with $\xi_\varepsilon\geq 0$ for $\varepsilon>0$ small enough.
By (3.46) we now conclude for all $n\in\mathbb{N}$ and all $\varepsilon>0$ small enough

$$J_1^{\varepsilon,n}+J_2^{\varepsilon,n}\leq J_3^{\varepsilon,n}+J_4^{\varepsilon,n}+J_5^{\varepsilon}+J_6^{\varepsilon} \tag{3.48}$$

where

$$\begin{aligned}J_1^{\varepsilon,n} &:= -\int_0^T\int_0^T(\xi_\varepsilon)_s(s,t)\left(\kappa y(s,t)+\int_0^s k_{1,n}(s-\sigma)y(\sigma,t)\,d\sigma\right)\,ds\,dt\\ &\quad -\int_0^T\int_0^T(\xi_\varepsilon)_t(s,t)\left(\kappa y(s,t)+\int_0^t k_{1,n}(t-\tau)y(s,\tau)\,d\tau\right)\,ds\,dt,\\ J_2^{\varepsilon,n} &:= \int_0^T\int_0^T\xi_\varepsilon(s,t)\left(k_{2,n}(0+)y(s,t)+\int_{(0,\min(s,t)]}y(s-\tau,t-\tau)\,dk_{2,n}(\tau)\right)\,ds\,dt,\end{aligned}$$

$$\begin{aligned} J_3^{\varepsilon,n} &:= -\int_0^T\int_0^T \xi_\varepsilon(s,t)\mathbf{1}_{\{t<s\}}\int_{(t,s]} y(s-\sigma,0)\;dk_{2,n}(\sigma)\;ds\;dt \\ &\quad -\int_0^T\int_0^T \xi_\varepsilon(s,t)\mathbf{1}_{\{s<t\}}\int_{(s,t]} y(0,t-\tau)\;dk_{2,n}(\tau)\;ds\;dt, \end{aligned}$$

$$\begin{aligned} J_4^{\varepsilon,n} &:= \int_0^T\int_0^T \xi_\varepsilon(s,t)\Big(k_{1,n}(s)y(0,t)+k_{1,n}(t)y(s,0)\Big)\;ds\;dt, \\ J_5^{\varepsilon} &:= \int_0^T\int_0^T \xi_\varepsilon(s,t)\|f(s)-f(t)\|_{L^1(\Omega)}\;ds\;dt, \end{aligned}$$

and

$$J_6^{\varepsilon} := \kappa\left(\int_0^T \xi_\varepsilon(0,t)y(0,t)\;dt+\int_0^T \xi_\varepsilon(s,0)y(s,0)\;ds\right).$$

It is our intention to first let $n\to\infty$ and then $\varepsilon\to 0+$ in (3.48). Since $(\xi_\varepsilon)_s(s,t) = {\varrho_\varepsilon}'(s-t)\phi(t)$ and $(\xi_\varepsilon)_t(s,t) = -{\varrho_\varepsilon}'(s-t)\phi(t)+\varrho_\varepsilon(s-t)\phi'(t)$, we conclude

$$\lim_{\varepsilon\to 0+}\lim_{n\to\infty} J_1^{\varepsilon,n} = -\lim_{\varepsilon\to 0+}\kappa\int_0^T\int_0^T \varrho_\varepsilon(s-t)\phi'(t)y(s,t)\;ds\;dt = -\kappa\int_0^T \phi'(t)y(t,t)\;dt.$$

Here, we used the fact that $k_{1,n}\to 0$ in $L^1(0,T)$ as $n\to\infty$. For $J_2^{\varepsilon,n}$ we use the transformation $(\eta,\nu)\mapsto(\eta+\frac{\nu}{2},\eta-\frac{\nu}{2})=(s,t)$ of the integral. Thus

$$\begin{aligned} J_2^{\varepsilon,n} &= \int_{-2T}^{2T}\int_{|\nu/2|}^{T-|\nu/2|} \xi_\varepsilon(\eta+\tfrac{\nu}{2},\eta-\tfrac{\nu}{2})\Big(k_{2,n}(0+)y(\eta+\tfrac{\nu}{2},\eta-\tfrac{\nu}{2}) \\ &\qquad +\int_{(0,\eta]} y(\eta+\tfrac{\nu}{2}-\tau,\eta-\tfrac{\nu}{2}-\tau)\;dk_{2,n}(\tau)\Big)\;d\eta\;d\nu \\ &= -\int_{-2T}^{2T}\int_{|\nu/2|}^{T-|\nu/2|} \varrho_\varepsilon(\nu)\phi'(\eta-\tfrac{\nu}{2})\int_0^\eta k_{2,n}(\eta-\tau)y(\tau+\tfrac{\nu}{2},\tau-\tfrac{\nu}{2})\;d\tau\;d\eta\;d\nu. \end{aligned}$$

We used the fact that, by definition, $y(s,t)=0$ for $s<0$ or $t<0$. Note, moreover, that the boundary terms are zero. Since $k_{2,n}\to k$ in $L^1(0,T)$ as $n\to\infty$, we can conclude

$$\lim_{\varepsilon\to 0+}\lim_{n\to\infty} J_2^{\varepsilon,n} = -\int_0^T \phi'(t)\int_0^t k(t-\tau)y(\tau,\tau)\;d\tau\;dt.$$

In order to estimate $J_3^{\varepsilon,n}$ we use the assumption $\lim_{t\to 0+}\|u_1(t)-u_0\|_{L^1(\Omega)}=0$ and the fact that $\xi_\varepsilon(s,t)=\varrho_\varepsilon(s-t)\phi(t)=0$ for $s<t$ and obtain

$$0\le J_3^{\varepsilon,n} \le \left(\sup_{\sigma\in[0,\varepsilon]} y(\sigma,0)\right)\int_0^T\int_t^T \xi_\varepsilon(s,t)\left(k_{2,n}(t)-k_{2,n}(s)\right)\;ds\;dt\to 0$$

as $n \to \infty$ and $\varepsilon \to 0+$. Note that

$$\lim_{n\to\infty} J_4^{\varepsilon,n} = \lim_{n\to\infty} \int_0^T \int_0^T \xi_\varepsilon(s,t)\Big(k_{1,n}(s)y(0,t) + k_{1,n}(t)y(s,0)\Big)\, ds\, dt = 0,$$

since $k_{1,n} \to 0$ in $L^1(0,T)$ as $n \to \infty$, and that

$$\lim_{\varepsilon\to 0} \int_0^T \xi_\varepsilon(s,t)\|f(s) - f(t)\|_{L^1(\Omega)} = 0$$

almost everywhere for $t \in [0,T)$. Hence, $\lim_{\varepsilon\to 0+} J_5^\varepsilon = 0$. Finally, since $\lim_{t\to 0} \|u_1(t) - u_0\|_{L^1(\Omega)} = 0$ and $\xi_\varepsilon(0,t) = \varrho_\varepsilon(0-t)\phi(t) = 0$ for $t \geq 0$, we obtain

$$\lim_{\varepsilon\to 0+} J_6^\varepsilon = \lim_{\varepsilon\to 0+} \kappa \int_0^\varepsilon \varrho_\varepsilon(s)\phi(0)y(s,0)\, ds = 0.$$

Combining the above results, we conclude by (3.48)

$$-\int_0^T \phi'(t)\left(\kappa y(t,t) + \int_0^t k(t-\tau)y(\tau,\tau)\, d\tau\right)\, dt \leq 0$$

for all $\phi \in \mathcal{D}([0,T))$ with $\phi \geq 0$. The convolution of this inequality with the completely positive measure α associated to (κ, k) and defined by (A.2) yields

$$y(t,t) \leq 0 \tag{3.49}$$

in the sense of distributions and thus almost everywhere in $(0,T)$. Therefore, we have shown that

$$b(v_1) = u_1 = u_2 = b(v_2)$$

almost everywhere in $Q = (0,T) \times \Omega$. □

3.4 Existence of entropy solutions

In this section we prove the existence of entropy solutions for the non-degenerated history dependent problem

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\kappa\big(u(t,x) - u_0(x)\big) + \int_0^t k(t-s)\big(u(s,x) - u_0(x)\big)\, ds\Big) \\
&\qquad = \quad \operatorname{div}\ a(x, Du(t,x)) + f(t,x) \quad \text{for } (t,x) \in Q := (0,T) \times \Omega, \\
&u(0,\cdot) \quad = \quad b(v_0) \quad \text{in } \Omega, \\
&u(t,x) \quad = \quad 0 \quad \text{for } (t,x) \in \Gamma := (0,T) \times \partial\Omega.
\end{aligned} \tag{3.50}$$

Here, we assume that (3.4)-(3.6) and (3.8),(3.9) hold. In particular, we show that the generalized solution of an associated abstract Volterra equation in $L^1(\Omega)$ is an entropy solution of (3.50).
To this end, define the operator $A_\infty \subset L^1(\Omega) \times L^1(\Omega)$ by

$$
\begin{aligned}
(v, w) \in A_\infty \quad :\Leftrightarrow \quad & v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), w \in L^1(\Omega) \text{ and} \\
& \int_\Omega a(x, Dv) \cdot \phi = \int_\Omega w\phi \\
& \text{for all } \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).
\end{aligned}
$$

Then A_∞ is a completely-accretive operator in $L^1(\Omega)$ with $R(I + \lambda A_\infty) \supset L^\infty(\Omega)$ for all $\lambda > 0$, and its closure $A := \overline{A_\infty}$ is an m-completely-accretive operator in $L^1(\Omega)$ which can be characterized by

$$
\begin{aligned}
(v, w) \in A \quad \Leftrightarrow \quad & v, w \in L^1(\Omega) \text{ and } T_K(v) \in W_0^{1,p}(\Omega) \text{ for all } K > 0 \text{ and} \\
& \int_\Omega a(x, Dv) \cdot DT_K(v - \phi) \leq \int_\Omega wT_K(v - \phi) \\
& \text{for all } \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), K > 0.
\end{aligned}
\tag{3.51}
$$

Note that according to the coercivity assumption (3.5) one has $0 \in A(0)$. To see this use $t\phi$ with $t \in \mathbb{R}$ and $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as a test function. Thus,

$$
t \int_\Omega a(x, tD\phi) \cdot D\phi \geq 0.
$$

Dividing the above inequality by t and letting $t \to 0+$ and $t \to 0-$ we conclude by the hemicontinuity of $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}$ with $\mathcal{A}(u) := -\operatorname{div}\ a(x, Du)$ that

$$
\int_\Omega a(x, 0) \cdot D\phi = 0
$$

for all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, i.e., that $a(x, 0) = 0$ a.e. for $x \in \Omega$.
By [CGL96], [Gri85] respectively, we already know that for all $u_0 \in \overline{D(A)} = L^1(\Omega)$ and all $f \in L^1(0, T; L^1(\Omega)) = L^1(Q)$ the abstract Volterra equation

$$
\frac{d}{dt}\big(\kappa(u(t) - u_0) + \int_0^t k(t - s)(u(s) - u_0)\, ds\big) + Au(t) \ni f(t) \tag{3.52}
$$

admits a unique generalized solution $u \in L^1(Q)$. But it is not clear in which sense this generalized solution satisfies the equation (3.50). Therefore, we are interested in an approximating sequence of solutions, which satisfy the abstract Volterra equation almost everywhere for $t \in [0, T)$ and, moreover, are weak solutions of (3.50). We define approximating sequences $\{u_{0,n}\}_{n\in\mathbb{N}} \subset D(A) \cap L^\infty(\Omega)$ and $\{f_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(0, T; L^1(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$

with $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q)$ as $n \to \infty$. Then, by proposition 2.11 and proposition 2.14, the generalized solution $u_n \in L^1(0,T,L^1(\Omega))$ of the abstract Volterra equation (3.52) with initial value $u_{0,n}$ and right hand side f_n is a strong solution for all $n \in \mathbb{N}$, i.e. the mapping

$$(0,T) \ni t \mapsto \kappa(u_n(t) - u_{0,n}) + \int_0^t k(t-s)(u_n(s) - u_{0,n})\, ds \in L^1(\Omega)$$

is absolutely continuous and differentiable almost everywhere with

$$\left(u_n(t), f_n(t) - \frac{d}{dt}\left(\kappa(u_n(t) - u_{0,n}) + \int_0^t k(t-s)(u_n(s) - u_{0,n})\, ds\right)\right) \in A$$

almost everywhere for $t \in [0,T)$. By the continuous dependence of the solution on the data, [Gri85, Theorem 5], we already know that $u_n \to u$ in $L^1(Q)$ as $n \to \infty$.
We first show that for the above choices of $u_{0,n}$ and f_n the generalized solution u_n is also a weak solution of

$$\begin{aligned} \frac{d}{dt}\Big(\kappa(u_n(t,x) - u_{0,n}(x)) + \int_0^t k(t-s)(u_n(s,x) - u_{0,n}(x))\, ds\Big) & & & \\ -\mathrm{div} a(x, Du_n(t,x)) &= f_n(t,x) & & (t,x) \in Q \\ u(0,\cdot) &= u_{0,n} & & \text{in } \Omega, \\ u(t,x) &= 0 & & \text{for } (t,x) \in \Gamma. \end{aligned} \tag{3.53}$$

To this end, we will need the following proposition which can be found in [CGL96, Proposition 5].

Proposition 3.28. *Let A be an m-accretive operator in a Banach space X, $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;X)$. We assume that κ, k satisfies* (3.8) *and that u is the generalized solution of* (3.52)*. Moreover, let Y be a Banach space such that*

(i) the set $\{x \in X \cap Y \big| \; \|x\|_Y \le 1\}$ is closed in X,

(ii) for every $x \in X \cap Y$ and every $\lambda > 0$ we have $\|(I + \lambda A)^{-1} x\|_Y \le \|x\|_Y$,

(iii) $u_0 \in X \cap Y$ and $\|f(\cdot)\|_Y \in L^1(0,T)$.

Then

$$\|u(t)\|_Y \le \|u_0\|_Y + \int_{[0,t]} \|f(t-s)\|_Y \, d\alpha(s)$$

almost everywhere for $t \in [0,T)$, where α is the completely positive measure associated to κ, k.

Note that, since Ω is a bounded domain, $L^\infty(\Omega) \subset L^1(\Omega)$. Moreover, the unit ball $B_{L^\infty(\Omega)}$ of $L^\infty(\Omega)$ is closed in $L^1(\Omega)$. Since the operator A defined by (3.51) satisfies $0 \in A(0)$, we have $\|(I+\lambda A)^{-1}f\|_\infty \le \|f\|_\infty$ for all $f \in L^\infty(\Omega)$. Here, we used the fact that the resolvent J_λ^A of A is nonexpansive in $L^\infty(\Omega)$, since A is completely accretive. Applying the above proposition, we can show that generalized solutions satisfy the equation (3.50) in the weak sense if the data are sufficiently regular.

Proposition 3.29. *Let* (3.4)-(3.6) *and* (3.8), (3.9) *be satisfied and let the m-completely accretive operator A be defined by* (3.51). *Assume that $u_0 \in D(A) \cap L^\infty(\Omega)$ and $f \in W^{1,1}(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$. Then the generalized solution u of the abstract Volterra equation* (3.52) *satisfies for a.e. $t \in [0,T)$*

$$u(t) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \qquad \frac{\partial}{\partial t}\big(\kappa(u-u_0) + k*(u-u_0)\big) \in L^1(0,T;L^1(\Omega)) = L^1(Q),$$

and, moreover,

$$\int_\Omega \frac{\partial}{\partial t}\left(\kappa(u(t)-u_0) + \int_0^t k(t-s)(u(s)-u_0)\,ds\right)\phi + \int_\Omega a(x,Du(t))\cdot D\phi = \int_\Omega f(t)\phi \tag{3.54}$$

for all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ almost everywhere for $t \in [0,T)$.

Proof. Let u be the generalized solution of (3.52) for the data $u_0 \in D(A) \cap L^\infty(\Omega)$ and $f \in W^{1,1}(0,T;L^1(\Omega)) \cap L^\infty(Q)$. We first remark that, by $W^{1,1}(0,T;L^1(\Omega)) \subset BV(0,T;L^1(\Omega))$ and proposition 2.11, u is a strong solution of (3.52) in case $\kappa = 0$. Moreover, if $\kappa > 0$, then by proposition 2.14 we also conclude that u is a strong solution. In particular

$$\frac{d}{dt}\big(\kappa(u-u_0) + k*(u-u_0)\big) \in L^1(0,T;L^1(\Omega)),$$

and $u(t) \in D(A)$ almost everywhere for $t \in [0,T)$. By the characterization of the operator A in (3.51), we obtain $T_K(u(t)) \in W_0^{1,p}(\Omega)$ for all $K > 0$ almost everywhere for $t \in [0,T)$, and

$$\int_\Omega \frac{\partial}{\partial t}\left(\kappa(u(t)-u_0) + \int_0^t k(t-s)(u(s)-u_0)\,ds\right)T_K(u-\phi) + \int_\Omega a(x,Du(t))\cdot DT_K(u-\phi) \le \int_\Omega f(t)T_K(u-\phi) \tag{3.55}$$

for all $K > 0$ and all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ almost everywhere for $t \in [0,T)$. We now use the assumptions $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(0,T;L^\infty(\Omega))$ to conclude by proposition 3.28 that $u \in L^\infty(0,T;L^\infty(\Omega)) \subset L^\infty(Q)$. Thus, taking $K := \|u\|_\infty$, we easily see that $u(t) = T_K(u(t)) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ almost everywhere for $t \in [0,T)$.

Let $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then using $\phi := u(t) - \xi$ and $K := \|u\|_\infty + \|\xi\|_\infty$ in (3.54), gives

$$\int_\Omega \frac{\partial}{\partial t}\left(\kappa(u(t)-u_0) + \int_0^t k(t-s)(u(s)-u_0)\,ds\right)\xi + \int_\Omega a(x, Du(t)) \cdot D\xi \le \int_\Omega f(t)\xi$$

almost everywhere for $t \in [0,T)$. Combining this result with the inequality one obtains when using $\phi := u(t) + \xi$ and $K := \|u\|_\infty + \|\xi\|_\infty$ in (3.54) yields the assertion. □

By the above proposition we conclude that the generalized solution u_n of the abstract Volterra equation (3.52) with data $u_{0,n} \in D(A) \cap L^\infty(\Omega)$ and $f_n \in W^{1,1}(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$ is a weak solution of (3.53).
We use this sequence of weak solutions in order to show that the generalized solution of (3.52) for $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$ is an entropy solution of (3.50). By the continuous dependence on the data, see [Gri85, Theorem 5], we already know that $u_n \to u$ in $L^1(Q)$, since we assumed $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q)$ as $n \to \infty$. Thus, the main task is to obtain convergence of the sequence $\{DT_K(u_n)\}_{n\in\mathbb{N}}$ of gradients of $T_K(u_n)$ for all $K > 0$. Therefore, we first show the boundedness of the sequence $\{DT_K(u_n)\}_n$ in $L^p(Q)^N$ by an a-priori estimate. This implies the weak convergence $DT_K(u_n) \rightharpoonup DT_K(u)$ in $L^p(Q)^N$ of a subsequence. Minty's trick then implies div $a(x, DT_K(u_n)) \rightharpoonup$ div $a(x, DT_K(u))$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Theorem 3.30. *Let* (3.4)-(3.6) *and* (3.8), (3.9) *be satisfied. Let the operator A be defined by* (3.51) *and let $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$. Then the generalized solution u of* (3.52) *is an entropy solution of* (3.50).

Proof. We choose sequences $\{u_{0,n}\}_{n\in\mathbb{N}} \subset D(A) \cap L^\infty(\Omega)$ and $\{f_n\}_{n\in\mathbb{N}} \subset W^{1,1}(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$ with $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q)$ as $n \to \infty$. Then, as shown in proposition 3.29, the generalized solution $u_n \in L^1(Q)$ of (3.52) with data $u_{0,n}$ and f_n satisfies

$$\begin{aligned}\int_\Omega \frac{\partial}{\partial t}\left(\kappa(u_n(t)-u_{0,n}) + \int_0^t k(t-s)\left(u_n(s)-u_{0,n}\right)\,ds\right)\phi \\ + \int_\Omega a(x, Du_n(t)) \cdot D\phi = \int_\Omega f(t)\phi\end{aligned} \tag{3.56}$$

for all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, $u(t) \in W_0^{1,p}(\Omega)$ almost everywhere for $t \in [0,T)$.
In the following let $K > 0$ be fixed. In order to obtain an a-priori estimate on the gradients of the approximating solutions u_n we take the truncation $T_K(u_n)$ of u_n for $K > 0$ as a test function in (3.56) and integrate this equation over $(0,T)$.

$$
\begin{aligned}
\lambda \int_Q |DT_K(u_n)|^p &\leq \int_Q a(x, Du_n) \cdot DT_K(u_n) \\
&= \int_Q f_n T_K(u_n) - \int_Q \frac{\partial}{\partial t}\Big[\kappa(u_n(t) - u_{0,n}) \\
&\qquad + \int_0^t k(t-s)\,(u_n(s) - u_{0,n})\ ds\Big] T_K(u_n(t)) \\
&\leq \int_Q f_n T_K(u_n) - \kappa \int_\Omega \int_{u_{0,n}}^{u_n(T)} T_K(r)\ dr\ dx \\
&\quad - \int_0^T k(T-s) \int_\Omega \int_{u_{0,n}}^{u_n(s)} T_K(r)\ dr\ dx\ ds \\
&\leq K\|f_n\|_{L^1(Q)} + K\left(\kappa + \|k\|_{L^1(0,T)}\right)\|u_{0,n}\|_{L^1(\Omega)}.
\end{aligned}
\tag{3.57}
$$

Here, we used the coercivity assumption (3.5) and the Kato inequality, proposition 3.23. Moreover, we used the fact that $\int_0^s T_K(r)\ dr \geq 0$ for all $s \in \mathbb{R}$. Thus, $\{DT_K(u_n)\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(Q)^N$. By Poincaré's inequality $\{T_K(u_n)\}_n$ is a bounded sequence in $L^p(0,T;W_0^{1,p}(\Omega))$, and thus admits a weakly convergent subsequence, i.e., $T_K(u_n) \rightharpoonup v_K$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ as $n \to \infty$ with some $v_K \in L^p(0,T;W_0^{1,p}(\Omega))$. The continuous dependence of the solution on the data, according to [Gri85, Theorem 5], implies $u_n \to u$ in $L^1(Q)$, since $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q)$. This implies that $T_K(u_n) \to T_K(u)$ in $L^1(Q)$. Using test functions it is easy to see that $T_K(u) = v_K$. Thus, we have $T_K(u_n) \rightharpoonup T_K(u)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ for a subsequence. Since the above argument holds for all subsequences of $\{T_K(u_n)\}_n$, we conclude $T_K(u_n) \rightharpoonup T_K(u)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ for the sequence itself. Moreover, by the growth bound (3.6)

$$
\int_Q |a(x, DT_K(u_n))|^{p'} \leq \Lambda^{p'}\left(T^{1/p'}\|j\|_{L^{p'}(\Omega)} + \|DT_K(u_n)\|_{L^p(Q)^N}^{p/p'}\right)^{p'}. \tag{3.58}
$$

Thus, the sequence $\{a(x, DT_K(u_n))\}_{n\in\mathbb{N}}$ is bounded in $L^{p'}(Q)^N$ and admits a weakly convergent subsequence in $L^{p'}(Q)^N$. Without loss of generality we assume $a(x, DT_K(u_n)) \rightharpoonup \sigma_K$ weakly in $L^{p'}(Q)^N$ with some $\sigma_K \in L^{p'}(Q)^N$.
In order to show that div $\sigma_K =$ div $a(x, DT_K(u))$, we want to apply a pseudo-monotonicity argument. Therefore, we need the following convergence result

$$
\lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty} \int_Q [a(x, DT_K(u_m)) - a(x, DT_K(u_n)] \cdot DT_L\left[T_K(u_m) - T_K(u_n)\right] = 0. \tag{3.59}
$$

To show (3.59), we use the following decomposition of the above integral:

$$
\int_Q [a(x, DT_K(u_m)) - a(x, DT_K(u_n))] \cdot DT_L\left(T_K(u_m) - T_K(u_n)\right)
$$

$$\begin{aligned}
&= \int_{Q\cap\{|u_m|<K,|u_n|<K\}} [a(x,Du_m)-a(x,Du_n)]\cdot DT_L(u_m-u_n) \\
&\quad + \int_{Q\cap\{|u_m|<K,|u_n|\ge K\}} a(x,Du_m)\cdot DT_L(u_m-\operatorname{sign}_0(u_n)K) \\
&\quad + \int_{Q\cap\{|u_m|\ge K,|u_n|<K\}} a(x,Du_n)\cdot DT_L(u_n-\operatorname{sign}_0(u_m)K) \\
&= I^{K,L}_{m,n} + J^{K,L}_{m,n} + J^{K,L}_{n,m}.
\end{aligned}$$

By the monotonicity assumption (3.4) it is obvious that

$$I^{K,L}_{m,n} \le \int_Q [a(x,Du_m)-a(x,Du_n)]\cdot DT_L\left(u_m-u_n\right).$$

Therefore we can apply (3.56) with the test function $\phi = T_L(u_m-u_n)$ for u_m and for u_n in order to compare these solutions. Adding both equalities and integrating over $(0,T)$ yields

$$\begin{aligned}
I^{K,L}_{m,n} &\le \int_Q [a(x,Du_m)-a(x,Du_n)]\cdot DT_L(u_m-u_n) \\
&\le -\int_Q \frac{\partial}{\partial t}\Big[\kappa\left(u_m(t)-u_n(t)\right) \\
&\qquad\qquad + \int_0^t k(t-s)\left(u_m(s)-u_n(s)\right)\, ds\Big] T_L(u_m(t)-u_n(t)) \\
&\quad + \int_Q k(t)(u_{0,m}-u_{0,n})T_L(u_m(t)-u_n(t)) + \int_Q (f_m-f_n)T_L(u_m-u_n) \\
&\le -\kappa\int_\Omega \int_0^{u_m(T)-u_n(T)} T_L(r)\, dr\, dx - \int_0^T k(T-s)\int_\Omega \int_0^{u_m(s)-u_n(s)} T_L(r)\, dr\, dx\, ds \\
&\quad + L(\kappa+\|k\|_{L^1(0,T)})\|u_{0,m}-u_{0,n}\|_{L^1(\Omega))} + L\|f_m-f_n\|_{L^1(Q)} \\
&\le L\left((\kappa+\|k\|_{L^1(0,T)})\|u_{0,m}-u_{0,n}\|_{L^1(\Omega)} + \|f_m-f_n\|_{L^1(Q)}\right).
\end{aligned}$$

Here, we applied the Kato inequality, corollary 3.24. By $u_{0,n}\to u_0$ in $L^1(\Omega)$ and $f_n\to f$ in $L^1(Q)$ as $n\to\infty$, we obtain

$$\lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty} I^{K,L}_{m,n} = 0.$$

We claim that

$$\lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty} J^{K,L}_{m,n} = 0. \tag{3.60}$$

The term $J_{m,n}^{K,L}$ can be rewritten in the form

$$\begin{aligned} J_{m,n}^{K,L} &= \int_{Q\cap\{|u_m|<K,K\le u_n\}} a(x,Du_m)\cdot DT_L(u_m-K)) \\ &\quad+\int_{Q\cap\{|u_m|<K,u_n\le -K\}} a(x,Du_m)\cdot DT_L(u_m+K)) \\ &= \int_{Q\cap\{|u_m|<K,K\le u_n,|u_m-K|<L\}} a(x,Du_m)\cdot Du_m \\ &\quad+\int_{Q\cap\{|u_m|<K,u_n\le -K,|u_m+K|<L\}} a(x,Du_m)\cdot Du_m. \end{aligned}$$

By the coercivity assumption (3.5) it is clear that $a(x,Du_m)\cdot Du_m\ge 0$ almost everywhere on Q. Thus, we might develop an estimate on $J_{m,n}^{K,L}$ by increasing the set of integration. Therefore, note that $|u_m|<K$ and $|u_m-K|<L$ implies that $K-L<u_m<K$. Analogously, $|u_m|<K$ and $|u_m+K|<L$ implies that $-K<u_m<-K+L$. This yields

$$\begin{aligned} &\{|u_m|<K,|u_n|\ge K,|u_m-T_K(u_n)|<L\} \\ &= \{|u_m|<K,K\le u_n,|u_m-K|<L\}\cup\{|u_m|<K,u_n\le -K,|u_m+K|<L\} \\ &\subset \{K-L<u_m<K\}\cup\{-K<u_m<-K+L\}. \end{aligned}$$

We now define for $K_2>K_1\ge 0$ the truncation function $T_{K_1,K_2}:\mathbb{R}\to\mathbb{R}$ by $T_{K_1,K_2}(r):=T_{K_2}(r)-T_{K_1}(r)$. Since the above inclusion holds, and due to the coercivity assumption (3.5), one obtains the estimate

$$J_{m,n}^{K,L}\le\int_Q a(x,Du_m)\cdot DT_{K-L,K}(u_m). \tag{3.61}$$

Again, using the fact that u_m satisfies (3.56) we can conclude for $0<L\le K$ by using the test function $T_{K-L,K}(u_m)$ and integrating the equality over $(0,T)$

$$\begin{aligned} &\int_Q a(x,Du_m)\cdot DT_{K-L,K}(u_m) \\ &= \int_Q f_m T_{K-L,K}(u_m) \\ &\quad-\int_Q \frac{\partial}{\partial t}\Big[\kappa\,(u_m(t)-u_{0,m}) \\ &\qquad\qquad+\int_0^t k(t-s)\,(u_m(s)-u_{0,m})\;dr\;ds\Big]T_{K-L,K}(u_m(t))) \\ &\le \int_Q f_m T_{K-L,K}(u_m)-\kappa\int_\Omega\int_{u_{0,m}}^{u_m(T)} T_{K-L,K}(r)\;dr\;dx \\ &\quad-\int_0^T k(T-s)\int_\Omega\int_{u_{0,m}}^{u_m(s)} T_{K-L,K}(r)\;dr\;dx\;ds \\ &\le L\|f_n\|_{L^1(Q)}+L\left(\kappa+\|k\|_{L^1(0,T)}\right)\|u_{0,m}\|_{L^1(\Omega)}. \end{aligned}$$

Here, we once again applied the Kato inequality, proposition 3.23. Inequality (3.61), and the fact that $\{u_{0,m}\}_m$ is a bounded sequence in $L^1(\Omega)$, and that $\{f_m\}_m$ is a bounded sequence in $L^1(Q)$ imply (3.60).
By an analogous estimate, we obtain

$$\lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty} J_{n,m}^{K,L} = 0.$$

Thus, combining the above results yields (3.59).
In the next step our main goal is to show that div $a(x, DT_K(u)) =$ div σ_K by using Minty's trick. Therefore, let $\phi \in L^p(0,T;W_0^{1,p}(\Omega))$ be arbitrary. Then, by (3.59), we obtain

$$\begin{aligned} 2\int_Q \sigma_K \cdot D\phi &= \lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty}\Big[\int_Q a(x, DT_K(u_m))\cdot(DT_L(T_K(u_m)-T_K(u_n))+D\phi) \\ &\quad + \int_Q a(x, DT_K(u_n))\cdot(DT_L(T_K(u_n)-T_K(u_m))+D\phi)\Big]. \end{aligned}$$

Thus,

$$\begin{aligned} 2\int_Q \sigma_K \cdot D\phi &= \lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty}\Big[\\ &\quad \int_{Q\cap\{|T_K(u_m)-T_K(u_n)|<L\}} a(x, DT_K(u_m))\cdot D\left[T_K(u_m)-T_K(u_n)+\phi\right] \\ &\quad + \int_{Q\cap\{|T_K(u_m)-T_K(u_n)|\geq L\}} a(x, DT_K(u_m))\cdot D\phi \\ &\quad + \int_{Q\cap\{|T_K(u_m)-T_K(u_n)|<L\}} a(x, DT_K(u_n))\cdot D\left[T_K(u_n)-T_K(u_m)+\phi\right] \\ &\quad + \int_{Q\cap\{|T_K(u_m)-T_K(u_n)|\geq L\}} a(x, DT_K(u_n))\cdot D\phi\Big] \\ &= \lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty}\left(\tilde{I}_{m,n}^{K,L}+\tilde{J}_{m,n}^{K,L}+\tilde{I}_{n,m}^{K,L}+\tilde{J}_{n,m}^{K,L}\right). \end{aligned} \tag{3.62}$$

For the second term $\tilde{J}_{m,n}^{K,L}$ we can apply the following estimate

$$\begin{aligned} \left|\tilde{J}_{m,n}^{K,L}\right| &= \left|\int_{Q\cap\{|T_K(u_m)-T_K(u_n)|\geq L\}} a(x, DT_K(u_m))\cdot D\phi\right| \\ &\leq \left(\int_Q |a(x, DT_K(u_m))|^{p'}\right)^{\frac{1}{p'}}\left(\int_{Q\cap\{|T_K(u_m)-T_K(u_n)|\geq L\}} |D\phi|^p\right)^{\frac{1}{p}} \\ &\leq C\left(\int_{Q\cap\{|T_K(u_m)-T_K(u_n)|\geq L\}} |D\phi|^p\right)^{\frac{1}{p}} \end{aligned}$$

Since we can assume that $T_K(u_n) \to T_K(u)$ pointwise almost everywhere on Q, and by the fact that $|D\phi|^p \in L^1(Q)$, we obtain for all $L > 0$ using Lebesgue's dominated convergence theorem

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty}\int_{Q\cap\{|T_K(u_m)-T_K(u_n)|\geq L\}} |D\phi|^p &= \lim_{m\to\infty}\lim_{n\to\infty}\int_Q \mathbf{1}_{\{|T_K(u_m)-T_K(u_n)|\geq L\}}|D\phi|^p \\
&= \lim_{m\to\infty}\int_Q \mathbf{1}_{\{|T_K(u_m)-T_K(u)|\geq L\}}|D\phi|^p \\
&= \int_Q \mathbf{1}_{\{|T_K(u)-T_K(u)|\geq L\}}|D\phi|^p \\
&= 0.
\end{aligned}$$

This implies that

$$\lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty} \tilde{J}^{K,L}_{m,n} = 0.$$

Analogously for the fourth term $\tilde{J}^{K,L}_{n,m}$ we can obtain

$$\lim_{L\to 0}\lim_{m\to\infty}\lim_{n\to\infty} \tilde{J}^{K,L}_{m,n} = 0.$$

We now investigate the convergence of the first term $\tilde{I}^{K,L}_{m,n}$ of (3.62). To this end, note that the sequence $\{a(x, DT_K(u_m))\mathbf{1}_{\{|T_K(u_m)-T_K(u_n)|<L\}}\}_{n\in\mathbb{N}}$ converges pointwise almost everywhere on Q to $a(x, DT_K(u_m))\mathbf{1}_{\{|T_K(u_m)-T_K(u)|<L\}}$ and the sequence is dominated by the $L^{p'}(Q)$-function $|a(x, DT_K(u_m))|$. By Lebesgue's dominated convergence theorem this implies

$$\begin{aligned}
&a(x, DT_K(u_m))\mathbf{1}_{\{|T_K(u_m)-T_K(u_n)|<L\}} \\
&\qquad \to \quad a(x, DT_K(u_m))\mathbf{1}_{\{|T_K(u_m)-T_K(u)|<L\}} \qquad \text{in } L^{p'}(Q)^N \text{ as } n\to\infty.
\end{aligned}$$

Since $DT_K(u_n) \rightharpoonup DT_K(u)$ weakly in $L^p(Q)^N$, we conclude

$$\begin{aligned}
&\liminf_{L\to 0}\liminf_{m\to\infty}\lim_{n\to\infty} \tilde{I}^{K,L}_{m,n} \\
&= \liminf_{L\to 0}\liminf_{m\to\infty}\lim_{n\to\infty}\int_{Q\cap\{|T_K(u_m)-T_K(u_n)|<L\}} a(x, DT_K(u_m))\cdot \\
&\qquad\qquad D\left[T_K(u_m)-T_K(u_n)+\phi\right] \\
&= \liminf_{L\to 0}\liminf_{m\to\infty}\int_{Q\cap\{|T_K(u_m)-T_K(u)|<L\}} a(x, DT_K(u_m))\cdot D\left[T_K(u_m)-T_K(u)+\phi\right] \\
&\geq \liminf_{L\to 0}\liminf_{m\to\infty}\int_{Q\cap\{|T_K(u_m)-T_K(u)|<L\}} a(x, DT_K(u)-D\phi)\cdot \\
&\qquad\qquad D\left[T_K(u_m)-T_K(u)+\phi\right] \\
&= \liminf_{L\to 0}\int_{Q\cap\{|T_K(u)-T_K(u)|<L\}} a(x, DT_K(u)-D\phi)\cdot D\left[T_K(u)-T_K(u)+\phi\right] \\
&= \int_Q a(x, DT_K(u)-D\phi)\cdot D\phi.
\end{aligned}$$

Here, we used the monotonicity assumption (3.4) and the fact that

$$\begin{aligned} &a(x, DT_K(u) + D\phi)\mathbf{1}_{\{|T_K(u_m)-T_K(u)|<L\}} \\ &\qquad \to \quad a(x, DT_K(u) + D\phi) \qquad \text{in } L^{p'}(Q)^N \text{ as } m \to \infty \end{aligned}$$

for all $L > 0$ by the same arguments as above. Moreover, we used the fact $DT_K(u_m) \rightharpoonup DT_K(u)$ weakly in $L^p(Q)^N$ as $m \to \infty$.
The limit estimate of the third term $\tilde{I}^{K,L}_{n,m}$ can not be obtained by using directly the same argument as for $\tilde{I}^{K,L}_{m,n}$, since we first have to take the limit in n as $n \to \infty$. But the roles of m and n are interchanged in $\tilde{I}^{K,L}_{n,m}$ in comparison to $\tilde{I}^{K,L}_{m,n}$. Therefore, we split up the integral term $\tilde{I}^{K,L}_{n,m}$ by

$$\begin{aligned} \tilde{I}^{K,L}_{n,m} &= \int_{Q\cap\{|T_K(u_m)-T_K(u_n)|<L\}} a(x, DT_K(u_n)) \cdot (DT_K(u_n) - DT_K(u) + D\phi) \\ &\quad + \int_{Q\cap\{|T_K(u_m)-T_K(u_n)|<L\}} a(x, DT_K(u_n)) \cdot (DT_K(u) - DT_K(u_m)) \\ &= M^{K,L}_{m,n} + N^{K,L}_{m,n}. \end{aligned}$$

By applying the monotonicity assumption (3.4) we obtain

$$\begin{aligned} &\liminf_{L\to 0} \liminf_{m\to\infty} \liminf_{n\to\infty} M^{K,L}_{m,n} \\ &\geq \liminf_{L\to 0} \liminf_{m\to\infty} \liminf_{n\to\infty} \int_{Q\cap\{|T_K(u_m)-T_K(u_n)|<L\}} a(x, DT_K(u) - D\phi) \\ &\qquad\qquad \cdot (DT_K(u_n) - DT_K(u) + D\phi) \\ &= \liminf_{L\to 0} \liminf_{m\to\infty} \int_{Q\cap\{|T_K(u_m)-T_K(u)|<L\}} a(x, DT_K(u) - D\phi) \cdot (DT_K(u) - DT_K(u) + D\phi) \\ &= \liminf_{L\to 0} \int_{Q\cap\{|T_K(u)-T_K(u)|<L\}} a(x, DT_K(u) - D\phi) \cdot D\phi \\ &= \int_Q a(x, DT_K(u) - D\phi) \cdot D\phi \end{aligned}$$

The convergence of $N^{K,L}_{m,n}$ now follows by the weak convergence $a(x, DT_K(u_n)) \rightharpoonup \sigma_K$ in $L^{p'}(Q)^N$ and $DT_K(u_m) \rightharpoonup DT_K(u)$ in $L^p(Q)^N$ and by the almost everywhere convergence of $\mathbf{1}_{\{|T_K(u_m)-T_K(u_n)|<L\}} \to \mathbf{1}_Q$ for all $L > 0$ as first $n \to \infty$ and then $m \to \infty$. This implies

$$\lim_{L\to 0} \lim_{m\to\infty} \lim_{n\to\infty} N^{K,L}_{m,n} = 0.$$

Combining the above results we obtain

$$2\int_Q \sigma_k \cdot D\phi \geq 2\int_Q a(x, DT_K(u) - D\phi) \cdot D\phi$$

for all $\phi \in L^p(0,T;W_0^{1,p}(\Omega))$. We now choose $\xi \in L^p(0,T;W_0^{1,p}(\Omega))$ arbitrary and set $\phi = r\xi$ for $r \neq 0$. By the hemicontinuity of the operator $\mathcal{A} : L^p(0,T;W_0^{1,p}(\Omega)) \to L^{p'}(0,T;W^{-1,p'}(\Omega))$ defined by $\mathcal{A}(v) := -\operatorname{div}\ a(x,Dv)$, we obtain for $r > 0$ and $r \to 0+$

$$\int_Q \sigma_k \cdot D\xi \geq \int_Q a(x, DT_K(u)) \cdot D\xi.$$

Analogously $r < 0$ and $r \to 0-$ yields

$$\int_Q \sigma_k \cdot D\xi \leq \int_Q a(x, DT_K(u)) \cdot D\xi.$$

Thus, we have shown

$$\operatorname{div}\ a(x, DT_K(u_m)) \rightharpoonup \operatorname{div}\ \sigma_K = \operatorname{div}\ a(x, DT_K(u))$$

in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ as $m \to \infty$.
We are now in the position to take the limit as $m \to \infty$ in (3.56). Therefore, take $S \in \mathcal{P}$ and $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\xi \in \mathcal{D}([0,T))$ with $\xi \geq 0$ arbitrary. Using $\xi S(u_m - \phi)$ as a test function in (3.56) and integrating this equation over $(0,T)$ yields

$$\begin{aligned}\int_Q \xi(t)\frac{\partial}{\partial t}\Big[\kappa(u_m(t) - u_{0,m}) + \int_0^t k(t-s)(u_m(s) - u_{0,m})\,ds\Big]S(u_m(t) - \phi) \\ + \int_Q \xi(t)a(x, Du_m(t)) \cdot DS(u_m(t) - \phi) = \int_Q \xi(t)f(t)S(u_m(t) - \phi).\end{aligned} \tag{3.63}$$

Take $k_1,\ k_2 \in L^1(0,T)$ arbitrary such that $k_1,\ k_2$ are nonnegative and nonincreasing satisfying $k = k_1 + k_2$ and $k_2(0+) < \infty$. Considering the first term on the left hand side of (3.63), we can apply the Kato inequality, proposition 3.23, and conclude

$$\begin{aligned}&\int_Q \xi(t)\frac{\partial}{\partial t}\Big[\kappa(u_m(t) - u_{0,m}) + \int_0^t k(t-s)(u_m(s) - u_{0,m})\,ds\Big]S(u_m(t) - \phi) \\ &\geq\ -\int_Q \xi_t(t)\Big[\kappa\int_{u_{0,m}}^{u_m(t)} S(r-\phi)\,dr + \int_0^t k_1(t-s)\int_{u_{0,m}}^{u_m(s)} S(r-\phi)\,dr\,ds\Big] \\ &\quad + \int_Q \Big[k_2(0+)(u_m(t) - u_{0,m}) + \int_{(0,t]} (u_m(t-s) - u_{0,m})\,dk_2(s)\Big]S(u_m(t) - \phi).\end{aligned}$$

Thus, assuming that $u_m \to u$ pointwise almost everywhere in Q as $m \to \infty$, we use Lebesgue's dominated convergence theorem to obtain

$$\begin{aligned}&\liminf_{m\to\infty}\int_Q \xi(t)\frac{\partial}{\partial t}\Big[\kappa(u_m(t) - u_{0,m}) + \int_0^t k(t-s)(u_m(s) - u_{0,m})\,ds\Big]S(u_m(t) - \phi) \\ &\geq\ -\int_Q \xi_t(t)\Big[\kappa\int_{u_0}^{u(t)} S(r-\phi)\,dr + \int_0^t k_1(t-s)\int_{u_0}^{u(s)} S(r-\phi)\,dr\,ds\Big] \\ &\quad + \int_Q \Big[k_2(0+)(u(t) - u_0) + \int_{(0,t]} (u(t-s) - u_0)\,dk_2(s)\Big]S(u(t) - \phi).\end{aligned}$$

Here, we used the fact that $S : \mathbb{R} \to \mathbb{R}$ is continuous and bounded. As it is our intention to take the limit in the second term on the left hand side of (3.63), we define $K := \|\phi\|_\infty + \max\{|z| \mid z \in \operatorname{supp} S\}$ and use the monotonicity assumption (3.4) to estimate this term by

$$\begin{aligned}
&\int_Q \xi a(x, Du_m) \cdot DS(u_m - \phi) \\
&= \int_Q \xi a(x, DT_K(u_m)) \cdot D[T_K(u_m) - T_K(u)]S'(u_m - \phi) \\
&\quad + \int_Q \xi a(x, DT_K(u_m)) \cdot DT_K(u)S'(u_m - \phi) \\
&\quad - \int_Q \xi a(x, DT_K(u_m)) \cdot D\phi S'(u_m - \phi) \\
&\geq \int_Q \xi a(x, DT_K(u)) \cdot D[T_K(u_m) - T_K(u)]S'(u_m - \phi) \\
&\quad + \int_Q \xi a(x, DT_K(u_m)) \cdot DT_K(u)S'(u_m - \phi) \\
&\quad - \int_Q \xi a(x, DT_K(u_m)) \cdot D\phi S'(u_m - \phi).
\end{aligned}$$

We can assume that $u_m \to u$ almost everywhere in Q as $m \to \infty$. By the piecewise continuity of S', this gives $S'(u_m - \phi) \to S'(u - \phi)$ almost everywhere in Q. Thus, by Lebesgue's dominated convergence theorem, $a(x, DT_K(u))S'(u_m - \phi) \to a(x, DT_K(u))S'(u - \phi)$ in $L^{p'}(Q)^N$ and, moreover, $DT_K(u)S'(u_m - \phi) \to DT_K(u)S'(u - \phi)$ and $D\phi S'(u_m - \phi) \to D\phi S'(u - \phi)$ in $L^p(Q)^N$ as $m \to \infty$. As we have already shown, $DT_K(u_m) \rightharpoonup DT_K(u)$ weakly in $L^p(Q)^N$ and $a(x, DT_K(u_m)) \rightharpoonup a(x, DT_K(u))$ weakly in $L^{p'}(Q)^N$. Thus, we conclude

$$\liminf_{m\to\infty} \int_Q \xi a(x, Du_m) \cdot DS(u_m - \phi) \geq \int_Q \xi a(x, Du) \cdot DS(u - \phi).$$

As the right hand side term in (3.63) converges by Lebesgue's dominated convergence theorem, we have shown

$$\begin{aligned}
-\int_Q \xi_t(t)\Big[\kappa \int_{u_0}^{u(t)} S(r - \phi)\, dr + \int_0^t k_1(t - s) \int_{u_0}^{u(s)} S(r - \phi)\, dr\, ds\Big] \\
+ \int_Q \Big[k_2(0+)(u(t) - u_0) + \int_{(0,t]} (u(t - s) - u_0)\, dk_2(s)\Big] S(u(t) - \phi) \\
+ \int_Q \xi a(x, Du) \cdot DS(u - \phi) \leq \int_Q \xi(t) f(t) S(u(t) - \phi),
\end{aligned}$$

i.e., u is an entropy solution of (3.50). □

In the case that the generalized solution of the abstract Volterra equation (3.52) is continuous at 0, we can omit the continuity assumption

$$\lim_{t\to 0+} \|u_1(t,\cdot) - u_0\|_{L^1(\Omega)} = 0$$

for one of the solutions in the uniqueness result, theorem 3.26.

Corollary 3.31. *Let* (3.4)-(3.6), (3.8), (3.9) *be satisfied and let* $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$. *Let* $\kappa > 0$ *or* $\sup_{t\in[0,\tau]} \|f(t)\|_{L^1(\Omega)} < \infty$ *for some* $0 < \tau \le T$. *Then entropy solutions of* (3.50) *are unique.*

Proof. Let u be the generalized solution of the abstract Volterra equation (3.52). Then u is an entropy solution of (3.50). Moreover, by corollary 2.4, u is continuous at 0. Thus, applying theorem (3.26), it is clear that any other entropy solution of (3.50) equals u. □

In the proof of theorem 3.30, where we could only show the weak convergence of the sequences $\{DT_K(u_n)\}_n$ in $L^p(Q)^N$ and $a(x, DT_K(u_n))$ in $L^{p'}(Q)^N$, we had to use Minty's trick to obtain div $a(x, DT_K(u_n)) \rightharpoonup$ div $a(x, DT_K(u))$ weakly in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. In the strictly monotone case, i.e., assuming that (3.7) holds, we can additionally show the convergence of the sequence of gradients $\{Du_n\}_{n\in\mathbb{N}}$ in measure. The convergence in measure implies the almost everywhere convergence of a subsequence. Using the additional assumption

$$\lim_{n\to\infty} \int_Q a(x, DT_K(u_n)) \cdot DT_K(u_n) = \int_Q a(x, DT_K(u)) \cdot DT_K(u) \qquad \text{for all } K > 0 \quad (3.64)$$

one can show the strong convergence of $T_K(u_n)$ in $L^p(0,T;W_0^{1,p}(\Omega))$. Note that by [BM97, Lemma 3.2] the condition (3.64) is satisfied in case $\kappa > 0$ and $k \equiv 0$. But in case $k \not\equiv 0$ we could only show

$$\lim_{L\to 0} \lim_{m\to\infty} \lim_{n\to\infty} \int_Q [a(x, DT_K(u_m)) - a(x, DT_K(u_n)] \cdot DT_L [T_K(u_m) - T_K(u_n)] = 0$$

for all $K > 0$, see (3.59). Therefore, it is not clear wether one has (3.64) in the general case. The strong convergence in $L^p(0,T;W_0^{1,p}(\Omega))$ is a consequence of the following well known lemma.

Lemma 3.32. *Let* g_n, $g \in L^1(E)$ *with* g_n, $g \ge 0$, *where* $(E, \mathcal{A}, \mu)$ *is a* σ*-finite measure space. If* $g_n \to g$ μ*-a.e. and*

$$\lim_{n\to\infty} \int_E g_n = \int_E g,$$

then

$$\lim_{n\to\infty} \int_E |g_n - g| = 0.$$

For the sake of completeness, we give the proof of the above lemma.

Proof. Define $h_n := \inf(g_n, g)$, then $h_n \to g$ μ-a.e. and dominated by $g \in L^1(E)$. Thus, by Lebesgue's dominated convergence theorem $h_n \to g$ in $L^1(E)$. This implies

$$\int_E |g_n - g| \le \int_E |g_n - h_n| + \int_E |h_n - g| = \int_E g_n - \int_E h_n + \int_E |h_n - g| \to 0$$

as $n \to \infty$. □

In the strictly monotone case, i.e., in case (3.7), we have the following result, which independently from theorem 3.30 implies that the generalized solution of (3.52) is an entropy solution of (3.50) in the strictly monotone case.

Theorem 3.33. *Let* (3.5)-(3.7) *and* (3.8), (3.9) *be satisfied and let the operator A be defined by* (3.51). *Moreover, let $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$ and assume that $\{u_{0,n}\}_{n\in\mathbb{N}} \subset D(A) \cap L^\infty(\Omega)$ and $\{f_n\}_{n\in\mathbb{N}} \subset W^{1,1}(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$ such that $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $f_n \to f$ in $L^1(Q)$. Assume that u_n, u are the generalized solutions to* (3.52) *for data $u_{0,n}$, f_n, and u_0, f, respectively. Then the sequence $\{Du_n\}_{n\in\mathbb{N}}$ converges in measure. If, moreover,* (3.64) *is satisfied, then $T_K(u_n) \to T_K(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ for all $K > 0$.*

Proof. First note that the approximate solutions u_n satisfy (3.56). By the proof of theorem 3.30, we already know that for fixed $K > 0$ the sequence $\{DT_K(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^p(Q)^N$, see (3.57), and that $\{a(x, DT_K(u_n))\}_{n\in\mathbb{N}}$ is bounded in $L^{p'}(Q)^N$. Moreover, we know that $DT_K(u_n) \rightharpoonup DT_K(u)$ weakly in $L^p(Q)^N$. Without loss of generality we assume that $a(x, DT_K(u_n)) \rightharpoonup \sigma_K$ in $L^{p'}(Q)^N$ for some $\sigma_K \in L^{p'}(Q)^N$.
As $u_n \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $n \in \mathbb{N}$, the sequence $\{Du_n\}_n$ is a well defined sequence in $L^p(Q)^N$. It is our intention to show that $\{Du_n\}_n$ is a Cauchy sequence in measure. Let $L > 0$, and in order to compare two solutions u_m, u_n, choose $T_L(u_n - u_m)$ as a test function in (3.56). Using the test function $T_L(u_m - u_n)$ in (3.56) with n replaced by m, adding up both equalities, and integrating over $(0,T)$ we obtain by applying the Kato inequality, corollary 3.24, and the fact that $\int_0^s T_K(r)\,dr \ge 0$ for all $s \in \mathbb{R}$,

$$\begin{aligned}
&\int_Q [a(x, Du_m) - a(x, Du_n)] \cdot DT_L(u_m - u_n) \\
&\le -\int_Q \frac{\partial}{\partial t}\Big[\kappa\,(u_m(t) - u_n(s)) + \int_0^t k(t-s)\,(u_m(s) - u_n(s))\,ds\Big] T_L(u_m(t) - u_n(t)) \\
&\quad + \int_Q k(t)(u_{0,m} - u_{0,n}) T_L(u_m(t) - u_n(t)) + \int_Q (f_m - f_n) T_L(u_m - u_n) \\
&\le -\kappa \int_\Omega \int_0^{u_m(T)-u_n(T)} T_K(r)\,dr - \int_0^T k(T-s) \int_\Omega \int_0^{u_m(s)-u_n(s)} T_K(r)\,dr\,ds \\
&\quad + L(\kappa + \|k\|_{L^1(0,T)})\|u_{0,m} - u_{0,n}\|_{L^1(\Omega))} + L\|f_m - f_n\|_{L^1(Q)}.
\end{aligned}$$

Thus,

$$\begin{aligned}&\int_Q [a(x, Du_m) - a(x, Du_n)] \cdot DT_L(u_m - u_n) \\ &\quad\le \ L\left((\kappa + \|k\|_{L^1(0,T)})\|u_{0,m} - u_{0,n}\|_{L^1(\Omega)} + \|f_m - f_n\|_{L^1(Q)}\right).\end{aligned} \tag{3.65}$$

We apply essentially the same techniques as used in [AMSdLT99] to show the convergence in measure of the sequence $\{Du_n\}$. For $A > 1$ and $r > 0$ we define the set $C(x, A, r)$ for $x \in \Omega$ by

$$C(x, A, r) := \left\{(\xi, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N \;\middle|\; |\xi| \le A, |\zeta| \le A, |\xi - \zeta| \ge r\right\} \tag{3.66}$$

As $\xi \mapsto a(x, \xi)$ is continuous for almost all $x \in \Omega$, and by the fact that the set $C(x, A, r) \subset \mathbb{R}^{N+N}$ is compact for almost all $x \in \Omega$, the function

$$c(x, A, r) := \min\left\{(a(x, \xi) - a(x, \zeta)) \cdot (\xi - \zeta) \;\middle|\; (\xi, \zeta) \in C(x, A, r)\right\}. \tag{3.67}$$

is well defined for almost all $x \in \Omega$. Also, by the strict monotonicity assumption (3.7), we obtain $c(x, A, r) > 0$ a.e. for $x \in \Omega$.

For $K > 0$ and ε, η arbitrary, we define the subset $G(m, n, A, r)$ of Q by

$$\begin{aligned}G(m, n, A, r) \ :=\ &\{|u_m - u_n| \le K^2, |u_m| \le A, |u_n| \le A, c(x, A, r) \ge K, \\ &\quad |DT_A(u_m)| \le A, |DT_A(u_n)| \le A, |Du_m - Du_n| > r\}.\end{aligned}$$

Then we obtain the following inclusion for subsets of Q

$$\begin{aligned}\{|Du_m - Du_n| > r\} \ \subset\ &\{|DT_A(u_m)| \ge A\} \cup \{|DT_A(u_n)| \ge A\} \\ &\cup\{|u_m| > A\} \cup \{|u_n| > A\} \cup \{|u_m - u_n| \ge K^2\} \\ &\cup\{c(x, A, r) \le K\} \cup G(m, n, A, r).\end{aligned}$$

We show that the measure of each of these sets is small for m, n large, K sufficiently small, and A large enough. Let $\delta > 0$ be arbitrary.

Let λ_{N+1} denote the Lebesgue measure on Q, then

$$\lambda_{N+1}\left(\{|u_m| > A\}\right) \le \int_Q \frac{|u_m|}{A} \le \frac{1}{A}\|u_m\|_{L^1(Q)}. \tag{3.68}$$

As $u_m \to u$ in $L^1(Q)$ there exists a uniform bound on $\|u_m\|_{L^1(Q)}$. Therefore we can choose A large enough such that $\lambda_{N+1}\left(\{|u_m| > A\}\right) < \delta$. The measure of the set $\{|u_n| > A\}$ can be estimated analogously.

The following argument applies to the measure of the sets $\{|DT_A(u_m)| \ge A\}$ and analogously to $\{|DT_A(u_n)| \ge A\}$. By (3.57), we conclude

$$\begin{aligned}\lambda_{N+1}\left(\{|DT_A(u_m)| \ge A\}\right) \ &\le\ \int_Q \frac{|DT_A(u_m)|^p}{A^p} \\ &\le\ \frac{A^{1-p}}{\lambda}\left(\|f_m\|_{L^1(Q)} + (\kappa + \|k\|_{L^1(0,T)})\|u_{0,m}\|_{L^1(\Omega)}\right).\end{aligned}$$

As $p > 1$ we can choose A large enough such that

$$\lambda_{N+1}(\{|DT_A(u_m)| \geq A\} \cup \{|DT_A(u_n)| \geq A\}) < \delta.$$

The measure of the set $G(m,n,A,r)$ can now be estimated using inequality (3.65).

$$\begin{aligned}
&\lambda_{N+1}\left(G(m,n,A,r)\right) \\
&\quad\leq \lambda_{N+1}\left(\{|u_m - u_n| \leq K^2, [a(x,Du_m) - a(x,Du_n)] \cdot [Du_m - Du_n] > K\}\right) \\
&\quad\leq \frac{1}{K} \int\limits_{\{|u_m-u_n|<K^2\}} [a(x,Du_m) - a(x,Du_n)] \cdot [Du_m - Du_n] \\
&\quad= \frac{1}{K} \int_Q [a(x,Du_m) - a(x,Du_n)] \cdot DT_{K^2}(u_m - u_n) \\
&\quad\leq \frac{K^2}{K} \left((\kappa + \|k\|_{L^1(0,T)})\|u_{0,m} - u_{0,n}\|_{L^1(\Omega)}) + \|f_m - f_n\|_{L^1(Q)}\right)
\end{aligned}$$

Thus we may choose $K > 0$ sufficiently small such that for all $m,\ n \in \mathbb{N}$ the following estimate holds for the above chosen $A > 1$

$$\lambda_{N+1}\left(G(m,n,A,r)\right) < \delta.$$

Finally, since A and K have already been chosen and by the fact that $\{u_m\}_m$ is a Cauchy sequence in $L^1(Q)$, there exists a $m_0 > 0$ such that for all $m,n \geq m_0$ we have

$$\lambda_{N+1}\left(\{|u_m - u_n| \geq K^2|\}\right) \leq \frac{1}{K^2} \int_Q |u_m - u_n| \leq \delta.$$

Thus, we have now shown that $\{Du_n\}_n$ is a Cauchy sequence in measure. This implies that a subsequence of $\{Du_n\}_n$ converges almost everywhere in Q. Without loss of generality, we denote this subsequence again by $\{Du_n\}_n$.

Since $DT_K(u_n) \rightharpoonup DT_K(u)$ weakly in $L^p(Q)^N$, we conclude that $Du_n \to Du$ almost everywhere in Q. By the continuity of $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ in the second component, this yields $a(x,Du_n) \to a(x,Du)$ almost everywhere in Q as $n \to \infty$.

Using the almost everywhere convergence shown above, we can give a different proof of the existence of entropy solutions of (3.50) in the strictly monotone case (3.7), which is independent from the proof of theorem 3.30. Note that the approximate solution satisfies (3.63) for all $\xi \in \mathcal{D}([0,T))$, $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and all $S \in \mathcal{P}$. For the first term on the left hand side and the right hand side term of (3.63), we use the same arguments as in the proof of theorem 3.30 to take the limit in n as $n \to \infty$. But for the second term on the left hand side of (3.63), we obtain by Fatou's lemma

$$\begin{aligned}\liminf_{n\to\infty}\int_Q \xi a(x,u_n)\cdot DS(u_n-\phi) &= \liminf_{n\to\infty}\Big[\int_Q \xi a(x,Du_n)\cdot Du_n S'(u_n-\phi) \\ &\qquad -\int_Q \xi a(x,Du_n)\cdot D\phi S'(u_n-\phi)\Big] \\ &\geq \int_Q \xi a(x,Du)\cdot DS(u-\phi).\end{aligned}$$

Thus, we have shown that u is an entropy solution of (3.50).
Now, assume that the additional assumption (3.64) holds. By the coercivity assumption, and since $a(x,DT_K(u_n))\cdot DT_K(u_n)\to a(x,DT_K(u))\cdot DT_K(u)$ almost everywhere in Q as $n\to\infty$, we can apply lemma 3.32 and obtain

$$a(x,DT_K(u_n))\cdot DT_K(u_n)\to a(x,DT_K(u))\cdot DT_K(u) \qquad \text{in } L^1(Q).$$

Thus, there exists a subsequence, again denoted by the index n, such that the above convergence holds almost everywhere in Q and dominated by some function $h\in L^1(Q)$. Due to the coercivity assumption

$$\begin{aligned}\lambda|DT_K(u_n)-DT_K(u)|^p &\leq \lambda\left(|DT_K(u_n)|+|DT_K(u)|\right)^p \\ &\leq \left([a(x,DT_K(u_n))\cdot DT_K(u_n)]^{1/p}+\lambda^{1/p}|DT_K(u)|\right)^p \\ &\leq \left(h^{1/p}+\lambda^{1/p}|DT_K(u)|\right)^p\in L^1(Q).\end{aligned}$$

We have shown that the almost everywhere convergence of $|DT_K(u_n)-DT_K(u)|\to 0$ in Q is dominated by an $L^1(Q)$-function. Thus, Lebesgue's dominated convergence theorem and Poincaré's inequality imply

$$T_K(u_n)\to T_K(u) \qquad \text{in } L^p(0,T;W_0^{1,p}(\Omega)).$$

□

We finally remark that the existence of entropy solutions in the general degenerated case, i.e., for (3.1), (3.2), is still an open problem. As we pointed out by example 3.25, we can not apply the monotonicity property of the time derivative used in the above proofs if we introduce a general continuous nondecreasing function $b:\mathbb{R}\to\mathbb{R}$.
The same problem occurs when considering the degenerated elliptic-parabolic initial boundary value problem

$$\begin{aligned} b(v)_t &= \operatorname{div}\ a(x,Dv)+f && \text{in } Q:=(0,T)\times\Omega, \\ b(v)(0,\cdot) &= b(v_0) && \text{in } \Omega, \\ v(t,x) &= 0 && \text{on } \Gamma:=(0,T)\times\partial\Omega, \end{aligned} \tag{3.69}$$

without history dependence. We recall that no existence results of entropy solutions or, equivalently, of renormalized solutions for (3.69) can be found in the literature up to now. However, the existence of renormalized solutions of (3.69) for general L^1-data, i.e,

$$v_0 : \Omega \to \overline{\mathbb{R}} \text{ measurable, with } b(v_0) \in L^1(\Omega), \qquad f \in L^1(Q),$$

can be shown by applying well known methods used in the context of parabolic equations. In the following we present the main ideas to obtain existence of renormalized solutions of (3.69).

Assume that $\{v_n\}_{n\in\mathbb{N}}$ is a sequence of weak solutions of (3.69) with data $v_{0,n} \in L^\infty(\Omega)$ and $f_n \in L^1(0,T;L^\infty(\Omega)) \cap L^{p'}(0,T;W^{-1,p'}(\Omega))$. The first problem when considering (3.69) is to obtain an almost everywhere convergent subsequence of $\{v_n\}_{n\in\mathbb{N}}$. Indeed, applying the theory of nonlinear semigroups to the Cauchy problem

$$\frac{d}{dt}u + \overline{A_b}u = f, \quad t > 0, \qquad u(0) = b(v_0)$$

with

$$\begin{aligned}(b(v), w) \in A_b \quad :\Leftrightarrow \quad & v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), w \in L^1(\Omega) \text{ and} \\ & \int_\Omega a(x, Dv) \cdot \phi = \int_\Omega w\phi \\ & \text{for all } \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\end{aligned}$$

we only obtain the convergence of the sequence $\{u_n\}$ with $u_n = b(v_n)$ in $L^1(Q)$. Since b may be constant on intervals, this does not imply the convergence of the sequence $\{v_n\}_n$. Therefore, we use the monotone vanishing perturbation method, see [Wit94]. First, consider the perturbed problem

$$\begin{aligned} b(v)_t &= \text{div } a(x, Dv) - \psi(v) + f && \text{in } Q := (0,T) \times \Omega, \\ b(v)(0, \cdot) &= b(v_0) && \text{in } \Omega, \\ v(t,x) &= 0 && \text{on } \Gamma := (0,T) \times \partial\Omega, \end{aligned} \tag{3.70}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is strictly increasing. By [BW96], one obtains the existence of weak solutions of (3.70) for bounded data, i.e., for $v_0 \in L^\infty(\Omega)$ and $f \in L^\infty(0,T;L^\infty(\Omega))$. Moreover, using $S(\psi(v))$ as a test function for $S \in \mathcal{P}$, one can show that the solution v of (3.70) satisfies

$$\|\psi(v)\|_\infty \leq C\left(\|v_0\|_\infty + \|f\|_\infty\right)$$

for some $C > 0$.

In particular, for $m, n \in \mathbb{N}$, we choose

$$\begin{aligned} v_{0,m,n} &:= \sup(\inf(v_0, n), -m), \\ f_{m,n} &:= \sup(\inf(f, n), -m), \\ \psi_{m,n}(r) &:= \frac{1}{n}r^+ - \frac{1}{m}r^-. \end{aligned}$$

If we assume that $v_{m,n}$ is the weak solution of (3.70) for the above choices of the data and perturbation, one can show

$$v_{m+1,n} \leq v_{m,n} \leq v_{m,n+1} \qquad \text{in } Q \text{ for all } m,n \in \mathbb{N}.$$

By a diagonalization argument, one can find a sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_n = v_{m(n),n}$ such that $T_K(v_n) \to T_K(v)$ in $L^1(Q)$ for all $K > 0$, and thus, $v_n \to v$ almost everywhere in Q for some subsequence. Here $v : Q \to \overline{\mathbb{R}}$ is a measurable function satisfying $\lambda_{N+1}(\{|v| = \infty\}) = 0$.
By the uniform a-priori estimate on the weak solution $T_K(v_n)$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we can also always assume that $T_K(v_n) \rightharpoonup T_K(v)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$. However, in order to pass to the limit in the nonlinearity $a(x,DT_K(v_n))$, one has to use a pseudo-monotonicity argument, and, therefore, one needs an estimate on $b(v_n)_t$ with a suitable test function. The key idea is to use a method introduced in [Lan81] of a time regularization of one of the solutions, see also [DO96].
For $w \in L^p(0,T;W_0^{1,p}(\Omega))$, we define the time regularization w_λ of w for $\lambda > 0$ and $0 \leq t \leq T$ by

$$w_\lambda(t,x) := \lambda \int_{-\infty}^{t} w(s,x)e^{-\lambda(t-s)}\, ds.$$

Here, we extend w by some $w_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ for $s < 0$. Then $w_\lambda(0) = w_0$ and $(w_\lambda)_t = \lambda(w - w_\lambda)$. We can assume that the weak solutions v_n constructed above satisfy $T_K(v_n) \in L^p(0,T;W_0^{1,p}(\Omega))$ and $b(v_n)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ for all $n \in \mathbb{N}$. By a further approximation, we may also assume that $v_{0,n} \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. Thus, the main task is to show that

$$\liminf_{\lambda\to\infty}\liminf_{n\to\infty}\liminf_{m\to\infty} \langle b(v_n)_t, (T_K(v_n) - T_K(v_m)_\lambda)h(v_n)\xi\rangle \geq 0 \tag{3.71}$$

for all $K > 0$, $h \in C_p^1(\mathbb{R})$ with compact support satisfying $h \geq 0$, and all $\xi \in \mathcal{D}([0,T))$ with $\xi \geq 0$. In the following, we use the notation $B_g(r) := \int_0^r g(s)\, db(s)$ for all $r \in \mathbb{R}$ with $g : \mathbb{R} \to \mathbb{R}$ bounded continuous. Then, applying the integration by parts formula lemma 3.14, we obtain

$$\begin{aligned}
&\langle b(v_n)_t, (T_K(v_n) - T_K(v_m)_\lambda)h(v_n)\xi\rangle \\
&= \; -\int_Q \xi_t[B_{T_Kh}(v_n) - B_{T_Kh}(v_{0,n})] + \int_Q \xi_t T_K(v_m)_\lambda[B_h(v_n) - B_h(v_{0,n})] \\
&\quad +\lambda\int_Q \xi(T_K(v_m) - T_K(v_m)_\lambda)[B_h(v_n) - B_h(T_K(v_n))] \\
&\quad +\lambda\int_Q \xi(T_K(v_m) - T_K(v_m)_\lambda)[B_h(T_K(v_n)) - B_h(T_K(v_m)_\lambda)] \\
&\quad +\lambda\int_Q \xi(T_K(v_m) - T_K(v_m)_\lambda)[B_h(T_K(v_m)_\lambda) - B_h(v_{0,n})] \\
&=: \; I_1^{m,n,\lambda} + I_2^{m,n,\lambda} + I_3^{m,n,\lambda} + I_4^{m,n,\lambda} + I_5^{m,n,\lambda}
\end{aligned}$$

Using the pointwise almost everywhere convergence of $\{v_n\}$ to v in Q and the almost everywhere convergence of $\{v_{0,n}\}$ to v_0 in Ω, we conclude

$$\liminf_{n\to\infty}\liminf_{m\to\infty} I_3^{m,n,\lambda} = \lambda \int_{Q\cap\{|v|>k\}} \xi(T_K(v) - T_K(v)_\lambda)[B_h(v) - B_h(T_K(v))] \geq 0.$$

Here, we used the fact that $B_h(v) = B_h(T_K(v))$ on $\{|v| \leq K\}$, and that $h \geq 0$ and $|T_K(v)_\lambda| \leq K$. By the monotonicity of B_h, we obtain

$$\liminf_{n\to\infty}\liminf_{m\to\infty} I_4^{m,n,\lambda} = \lambda \int_Q \xi(T_K(v) - T_K(v)_\lambda)[B_h(T_K(v)) - B_h(T_K(v)_\lambda)] \geq 0.$$

Since $T_K(v)_\lambda(0) = T_K(v_0)$, we obtain, by applying lemma 3.14 once again,

$$\begin{aligned}
&\liminf_{n\to\infty}\liminf_{m\to\infty}\langle b(v_n)_t, (T_K(v_n) - T_K(v_m)_\lambda)h(v_n)\xi\rangle \\
&\quad\geq -\int_Q \xi_t[B_{T_K h}(v) - B_{T_K h}(v_0)] + \int_Q \xi_t T_K(v)_\lambda [B_h(v) - B_h(v_0)] \\
&\qquad - \int_Q \xi_t \int_{T_K(v_0)}^{T_K(v)_\lambda} B_h(r)\, dr + \int_Q \xi_t T_K(v)_\lambda B_h(v_0) + \int_\Omega \xi(0) T_K(v_0) B_h(v_0)
\end{aligned}$$

Taking the limit for $\lambda \to \infty$ yields

$$\begin{aligned}
&\liminf_{\lambda\to\infty}\liminf_{n\to\infty}\liminf_{m\to\infty} \langle b(v_n)_t, (T_K(v_n) - T_K(v_m)_\lambda)h(v_n)\xi\rangle \\
&\quad\geq -\int_Q \xi_t \left[B_{T_K h}(v) - T_K(v)B_h(v) + \int_0^{T_K(v)} B_h(r)\, dr\right] \\
&\qquad - \int_\Omega \xi(0) \left[B_{T_K h}(v_0) - T_K(v_0)B_h(v_0) + \int_0^{T_K(v_0)} B_h(r)\, dr\right].
\end{aligned}$$

Note that

$$\begin{aligned}
&B_{T_K h}(v_0) - T_K(v_0)B_h(v_0) + \int_0^{T_K(v_0)} B_h(r)\, dr \\
&\qquad = \mathbf{1}_{\{|v_0|>K\}} \int_{T_K(v_0)}^{v_0} h(r)[T_K(r) - T_K(v_0)]\, db(r) = 0,
\end{aligned}$$

and, analogously,

$$\begin{aligned}
&B_{T_K h}(v) - T_K(v)B_h(v) + \int_0^{T_K(v)} B_h(r)\, dr \\
&\qquad = \mathbf{1}_{\{|v|>K\}} \int_{T_K(v)}^{v} h(r)[T_K(r) - T_K(v)]\, db(r) = 0.
\end{aligned}$$

This yields (3.71). Hence, by (3.70) we have

$$\limsup_{l\to\infty}\limsup_{\lambda\to\infty}\limsup_{n\to\infty}\limsup_{m\to\infty}\int_Q \xi a(x,Dv_n)\cdot D[(T_K(v_n)-T_K(v_m)_\lambda)h_l(v_n)] \geq 0,$$

where $h_l(r) := \min(\max(l+1-|r|,0),1)$. This allows to conclude, by the standard pseudo-monotonicity argument, that div $a(x,DT_K(v_n)) \rightharpoonup$ div $a(x,DT_K(v))$ weakly in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Unfortunately, it is absolutely unclear whether one can use the same method in case of the history dependent problem (3.1). In particular, one would first need a version for entropy solutions of the above estimate. Moreover, we recall that even the existence of weak solutions for bounded data is an open problem in the history dependent case of the degenerated equation.

Chapter 4

Asymptotic behavior

In the following, we study the asymptotic behavior of generalized solutions u of the nonlinear Volterra equation

$$\frac{d}{dt}\left(\kappa(u(t)-u_0)+\int_0^t k(t-s)(u(s)-u_0)\,ds\right)+Au(t)\ni f(t), \qquad t\in\mathbb{R}_+. \tag{4.1}$$

Here, A is an m-accretive operator in a Banach space X, $u_0\in\overline{D(A)}$, and $f\in L^1_{\text{loc}}(\mathbb{R}_+;X)$, with $\mathbb{R}_+ := [0,\infty)$. Again we assume that κ, k satisfy

$$\begin{aligned}&\kappa\geq 0, \text{ and } k\in L^1_{\text{loc}}(\mathbb{R}_+) \text{ is nonnegative and nonincreasing such that}\\ &\kappa+\textstyle\int_0^t k(\tau)\,d\tau>0 \text{ for all } t>0.\end{aligned} \tag{4.2}$$

Our main purpose is to investigate continuous solutions. In order to obtain sufficient regularity of solutions, we will restrict the study of the asymptotic behavior to those κ and k, for which

$$\kappa>0 \quad \text{or} \quad k(0+)=\infty. \tag{4.3}$$

In this case, the solution mapping

$$\mathcal{G}:\overline{D(A)}\times L^1_{\text{loc}}([0,\infty);X)\to L^1_{\text{loc}}([0,\infty);X) \tag{4.4}$$

introduced in (2.23) maps continuous right hand sides f of (4.1) to continuous solutions. In order to guarantee boundedness of generalized solutions of (4.1) on $[0,\infty)$ for bounded right hand sides, as shown in proposition 2.8, we will always assume that

$$k(\infty):=\lim_{t\to\infty}k(t)>0. \tag{4.5}$$

We remark that assuming (4.5) is equivalent to assuming that A is ω-accretive for $\omega=-k(\infty)<0$, i.e., to assume that $(A+\omega I)$ is accretive. Indeed, if A is ω-accretive for an

$\omega < 0$, then, as one can easily show by using approximate solutions, generalized solutions of (4.1) are as well generalized solutions of

$$\frac{d}{dt}\left(\kappa(u(t)-u_0)+\int_0^t (k(t-s)-\omega)(u(s)-u_0)\,ds\right)+Bu(t)\ni \omega u_0+f(t) \qquad t\geq 0.$$

Here, $B := (A+\omega I)$ is accretive, and $k(\infty)-\omega > 0$. On the other hand, if $k(\infty) > 0$, then one may define $\tilde{k} := k - k(\infty)$. Obviously, a generalized solution of (4.1) is a generalized solution of

$$\frac{d}{dt}\left(\kappa(u(t)-u_0)+\int_0^t \tilde{k}(t-s)(u(s)-u_0)\,ds\right)+Bu(t)\ni k(\infty)u_0+f(t) \qquad t\geq 0.$$

Here, $B := (A + k(\infty)I)$ is ω-accretive with $\omega := -k(\infty)$.
Additionally, we remark that, by proposition A.9, assumption (4.5) implies that the resolvent of the first kind α of the pair κ, k is a bounded Radon measure on $[0,\infty)$. In particular, we have

$$\alpha([0,\infty)) = \frac{1}{k(\infty)} < \infty.$$

For κ, k satisfying (4.2), (4.3) and (4.5) , the solution mapping

$$\begin{aligned} \mathcal{G} : \overline{D(A)} \times L^1_{\text{loc}}(\mathbb{R}_+;X) &\rightarrow L^1_{\text{loc}}(\mathbb{R}_+;X) \\ (u_0,f) &\mapsto u, \end{aligned} \tag{4.6}$$

which maps the data $u_0 \in \overline{D(A)}$ and $f \in L^1_{\text{loc}}(\mathbb{R}_+;X)$ to the generalized solution u of (4.1), is well defined. As we have already seen, by proposition 2.8, $\mathcal{G}(u_0,\cdot)$ leaves the space of bounded continuous functions $C_b(\mathbb{R}_+;X)$ invariant for all $u_0 \in \overline{D(A)}$, i.e., for all $f \in C_b(\mathbb{R}_+;X)$ the generalized solution of (4.1) satisfies $\mathcal{G}(u_0,f) \in C_b(\mathbb{R}_+;X)$. Moreover, by proposition 2.9, we know that $\mathcal{G}(u_0,\cdot)$ leaves the space of bounded and uniformly continuous functions $BUC(\mathbb{R}_+;X)$ invariant. In this chapter, we will concentrate on the question whether $\mathcal{G}(u_0,\cdot)$ also leaves certain subspaces of $BUC(\mathbb{R}_+;X)$ invariant for $u_0 \in \overline{D(A)}$.
It would be of particular interest, whether periodic or asymptotically periodic right hand sides of (4.1) lead to asymptotically periodic solutions. But note that the sum of two periodic functions fails to be periodic if the quotient of the two occurring periods is irrational. We therefore will concentrate on the space $AP(\mathbb{R};X)$ of almost periodic functions introduced by Bohr, which is the closed linear subspace in $(C_b(\mathbb{R}_+;X),\|\cdot\|_\infty)$ generated by the set of periodic functions on $\mathbb{R}_+$, i.e.

$$AP(\mathbb{R};X) := \overline{\text{span}}\left\{f \in C_b(\mathbb{R};X) \;\middle|\; f \text{ is periodic }\right\}.$$

In order to give a characterization of the space $AP(\mathbb{R};X)$, we introduce the notion of an orbit of a function $f \in L^1_{\text{loc}}(J;X)$. For $J \in \{\mathbb{R},\mathbb{R}_+\}$ and $f \in L^1_{\text{loc}}(J;X)$, the orbit is the set of all translates of f, i.e.

$$O(f) := \left\{ f_\tau := f(\tau + \cdot)\big|_J \;\middle|\; \tau \in J \right\}.$$

By a criterion of Bochner, a function $f \in C_b(\mathbb{R}; X)$ is almost periodic – a.p. for short, if and only if $O(f)$ is relatively compact in $(C_b(\mathbb{R}; X), \|\cdot\|_\infty)$. Hence,

$$AP(\mathbb{R}; X) = \left\{ f \in C_b(\mathbb{R}; X) \;\middle|\; O(f) \text{ is relatively compact in } (C_b(\mathbb{R}; X), \|\cdot\|_\infty) \right\}.$$

It is now canonical to define the space $AAP(\mathbb{R}_+; X)$ of asymptotically almost periodic functions – a.a.p. functions, for short – on $\mathbb{R}_+$, as in [Fré41], by

$$AAP(\mathbb{R}_+; X) := \left\{ f \in C_b(\mathbb{R}_+; X) \;\middle|\; O(f) \text{ is relatively compact in } (C_b(\mathbb{R}_+; X), \|\cdot\|_\infty) \right\}.$$

One can easily show that a.p. functions, respectively a.a.p. functions, are uniformly continuous. Thus,

$$\begin{aligned} AP(\mathbb{R}; X) &\subset BUC(\mathbb{R}; X), \qquad \text{and} \\ AAP(\mathbb{R}_+; X) &\subset BUC(\mathbb{R}_+, X). \end{aligned}$$

In section 1, we will be concerned with the problem, whether a.a.p. right hand sides in (4.1) lead to a.a.p. solutions. In this context we remark that the space $AAP(\mathbb{R}_+; X)$ has the following decomposition

$$AAP(\mathbb{R}_+; X) = AP(\mathbb{R}; X)\big|_{\mathbb{R}_+} \oplus C_0(\mathbb{R}_+; X). \tag{4.7}$$

Here, $C_0(\mathbb{R}_+; X)$ is the space of continuous functions converging to 0 as $t \to \infty$.
Thus, it will be interesting to investigate the connections between the almost periodic part of a right hand side $f \in AAP(\mathbb{R}_+; X)$ and the almost periodic part of the corresponding solution $u \in AAP(\mathbb{R}_+; X)$.
Note that results in this direction have been considered in [CN81]. They showed that under certain assumptions right hand sides $f \in AAP(\mathbb{R}_+; X)$ with constant almost periodic part, i.e. $f(t) \to f_\infty \in X$ as $t \to \infty$, lead to solutions of the same type. Moreover, the limit $u_\infty := \lim_{t\to\infty} u(t)$ can be characterized as a solution of a particular limit equation. Moreover, in the linear case, i.e. A being a linear densely defined operator in X, results of this type can be found in [PR93].
In section 2, we will show that this concept of limit equations leads to a characterization of the almost periodic part of the solution of (4.1) for general almost periodic parts of the right hand side f.
A natural extension of the space of almost periodic and asymptotically almost periodic functions was introduced by Eberlein in [Ebe49]. For $J \in \{\mathbb{R}, \mathbb{R}_+\}$, we define the space of weakly almost periodic functions in the sense of Eberlein – E.-w.a.p. for short – by

$$W(J; X) := \left\{ f \in C_b(J; X) \;\middle|\; O(f) \text{ is weakly relatively compact in } (C_b(J; X), \|\cdot\|_\infty) \right\}.$$

Note that, by [RS92, Proposition 2.1], $W(J;X)$ is a subspace of $BUC(J;X)$.
The space $W(\mathbb{R}_+;X)$ plays an important role in the ergodic theory. Indeed, by applying the ergodic theorem for bounded C_0-semigroups (see [DS57, Theorem 8.7.1 and Corollary 8.7.2]) to the translation semigroup in $BUC(\mathbb{R}_+;X)$, one easily checks that for $f \in W(\mathbb{R}_+;X)$ there exists $z \in X$ such that

$$\lim_{T\to\infty} \sup_{h\geq 0} \left\| \frac{1}{T} \int_h^{T+h} f(\tau)\, d\tau - z \right\|_X = 0.$$

Hence, each $f \in W(\mathbb{R}_+;X)$ is uniformly ergodic.
In section 3, we are going to develop sufficient conditions for the solution of (4.1) to be E.-w.a.p. It will turn out that the norm relative compactness of the range of the solution u and of the right hand side f will play an important role in the proof. We therefore introduce the space

$$WRC(J;X) := \left\{ f \in W(J;X) \;\middle|\; f \text{ has relatively compact range } \right\}.$$

We point out that one can omit the assumption on the range of the right hand side f to be relatively compact if we assume that X has a uniformly convex dual space X'.
We remark that, as a consequence of the DeLeeuw-Glicksberg theory of almost periodic functions on semigroups [DG61a, DG61b], the space $W(\mathbb{R}_+;X)$ can be decomposed as follows:

$$W(\mathbb{R}_+;X) = AP(\mathbb{R};X)\big|_{\mathbb{R}_+} \oplus W_0(\mathbb{R}_+;X). \tag{4.8}$$

Here, $W_0(J;X)$ is the space of all X-valued E.-w.a.p. functions on J, such that the constant 0 function is in the weak $(C_b(J;X), \|\cdot\|_\infty)$-closure of $O(f)$. Or, equivalently, we can define

$$W_0(J;X) := \left\{ f \in W(J;X) \;\middle|\; \exists \{\tau_n\}_{n\in\mathbb{N}} \subset J \text{ s.t. } f_{\tau_n} \rightharpoonup 0 \right\}. \tag{4.9}$$

Again, it will turn out that, under certain assumptions, the almost periodic part of a solution $u \in W(\mathbb{R}_+;X)$ of (4.1) can be characterized as solution of a corresponding limit equation.
For linear Volterra equations of the form (4.1), i.e. with a densely defined linear operator A, results of the above mentioned types can be found in [PR93]. And in the case of the nonlinear Cauchy-problem, i.e. $\kappa = 1$ and $k \equiv 0$ we refer to [Sei87],[Kre92], [RS89] and [RS90].

4.1 Asymptotic almost periodicity

In this section, we will always assume that κ, k satisfy (4.2), (4.3) and (4.5) . Moreover, u will always denote the generalized solution of the Volterra equation (4.1). We recall that u is the $L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$-limit of a sequence of approximate solutions $\{u_n\}_{n\in\mathbb{N}}$. For an m-accretive operator A, the function u_n is the unique strong solution of

$$\frac{d}{dt}\left(\int_0^t k_n(t-s)(u_n(s)-u_0)\,ds\right)+Au(t)\ni f(t). \tag{4.10}$$

Here, $\{k_n\}_{n\in\mathbb{N}}$ can be an arbitrary sequence of functions satisfying (4.2) and $k_n(0+)<\infty$ such that

$$\int_0^t k_n(s)\,ds\to\kappa+\int_0^t k(s)\,ds \qquad \text{for all } t>0 \text{ as } n\to\infty. \tag{4.11}$$

As the generalized solution does not depend on the choice of the sequence $\{u_n\}$ of approximate solutions with (4.10) and (4.11), one might choose a particular sequence $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ of approximate solutions in order to simplify calculations. Therefore, let α always denote the resolvent of the first kind of the pair (κ,k), which is by definition the unique Radon measure on $[0,\infty)$ satisfying

$$\kappa\alpha([0,t])+\int_{[0,t]}k(t-s)\alpha([0,s])\,ds=t \qquad \text{for all } t\geq 0. \tag{4.12}$$

For $\lambda>0$ the, resolvents of the measure α denoted by ϱ_λ are by definition the uniquely defined Radon measures on $[0,\infty)$ satisfying

$$\lambda\varrho_\lambda([0,t])+\int_{[0,t]}\varrho_\lambda([0,t-s])\,d\alpha(s)=\alpha([0,t]) \quad \text{for all } t\geq 0. \tag{4.13}$$

Thus, we can define

$$k_\lambda(t):=\frac{1}{\lambda}\left(1-\varrho_\lambda([0,t])\right) \qquad \text{for all } t>0. \tag{4.14}$$

Due to the results of [CN81, Proposition 2.1] and [Gri85, Lemma 3.1], and, by applying proposition A.12, we obtain the following lemma.

Lemma 4.1. *Let κ, k satisfy* (4.2) *and* (4.3)*. Let α be the resolvent of the first kind of (κ,k). Then, for all $\lambda>0$, the function k_λ defined by* (4.14) *is nonnegative nonincreasing, with $k_\lambda(0+)=\frac{1}{\lambda}$, and*

$$\int_0^t k_\lambda(s)\,ds\to\kappa+\int_0^t k(s)\,ds \quad \textit{for all } t>0 \textit{ as } \lambda\to 0+\,.$$

The resolvent of the first kind α_λ of the function k_λ defined by

$$\int_{[0,t]}k_\lambda(t-s)\,d\alpha_\lambda(s)=1, \qquad \textit{for all } t\geq 0$$

is given by $\alpha_\lambda=\lambda\delta_0+\alpha$, where δ_0 is the Dirac measure at the point 0*.*
Moreover, if k satisfies (4.5)*, then*

$$k_\lambda(\infty):=\lim_{t\to\infty}k_\lambda(t)=\frac{1}{\lambda+\alpha([0,\infty))}.$$

According to the above lemma, we will now choose sequences $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ of approximate solutions corresponding to the functions k_{λ_n} for $\lambda_n \to 0+$ as $n \to \infty$. Since our results will not depend on the choice of the particular sequence $\lambda_n \to 0+$, we will in the following statements always speak of the sequence $\lambda \to 0+$ meaning that the particular statement holds for all choices of sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \to 0+$ as $n \to \infty$.
We can now state the fundamental proposition in the study of the asymptotic behavior.

Proposition 4.2. *Let κ, k satisfy (4.2), (4.3) and (4.5) , and let A be an m-accretive operator in a Banach space X. Let $u_0 \in \overline{D(A)}$, and $f, g \in L^\infty(\mathbb{R}_+; X)$. Moreover, let u be the generalized solution of (4.1), and let v be the generalized solution of (4.1) with f replaced by g. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}\|u(t+\tau) - v(t+\sigma)\| \\ &\le \int_{[0,t]} [u(t+\tau-s) - v(t+\sigma-s), f(t+\tau-s) - g(t+\sigma-s)]_+ \, d\alpha(s) \\ &\quad + C\frac{\alpha((t,\infty))}{\alpha([0,\infty))}\end{aligned}$$

almost everywhere for $t, \tau, \sigma \in \mathbb{R}_+$.

Proof. For all $\lambda > 0$, let $u_\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ be the unique strong solution of

$$\frac{d}{dt}\left(\int_0^t k_\lambda(t-s)(u_\lambda(s) - u_0)\, ds\right) + Au(t) \ni f(t) \qquad \text{for } t \ge 0. \tag{4.15}$$

Moreover, let $v_\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}_+; X)$ be the unique strong solution of (4.15) with f replaced by g. Then, by proposition 2.8, and by the fact that $\alpha_\lambda(\mathbb{R}_+) = \lambda + \alpha(\mathbb{R}_+) < \infty$, there exists a constant $C > 0$ such that for all $0 < \lambda < 1$

$$\|u_\lambda - u_0\|_\infty + \|v_\lambda - u_0\|_\infty \le C. \tag{4.16}$$

Since, for $\lambda > 0$, u_λ and v_λ are strong solutions, we obtain almost everywhere for $t \in \mathbb{R}_+$ and for $0 < \sigma < \tau$

$$\begin{aligned}0 \le \Big[& u_\lambda(t+\tau) - v_\lambda(t+\sigma), f(t+\tau) - g(t+\sigma) \\ & -k_\lambda(0+)(u_\lambda(t+\tau) - v_\lambda(t+\sigma)) \\ & - \int_{(0,t+\tau]} (u_\lambda(t+\tau-s) - u_0)\, dk_\lambda(s) \\ & + \int_{(0,t+\sigma]} (v_\lambda(t+\sigma-s) - u_0)\, dk_\lambda(s)\Big]_+ .\end{aligned}$$

Thus,

$$\begin{aligned}
0 \;\leq\; & \big[u_\lambda(t+\tau) - v_\lambda(t+\sigma), f(t+\tau) - g(t+\sigma)\big]_+ \\
& + \big[u_\lambda(t+\tau) - v_\lambda(t+\sigma), -k_\lambda(t+\tau)(u_\lambda(t+\tau) - v_\lambda(t+\sigma))\big]_+ \\
& + \big[u_\lambda(t+\tau) - v_\lambda(t+\sigma), \\
& \qquad \int_{(0,t+\sigma]} \{u_\lambda(t+\tau) - v_\lambda(t+\sigma) - (u_\lambda(t+\tau-s) - v_\lambda(t+\sigma-s))\}\, dk_\lambda(s)\big]_+ \\
& + \big[u_\lambda(t+\tau) - v_\lambda(t+\sigma), \\
& \qquad \int_{(t+\sigma,t+\tau]} \{u_\lambda(t+\tau) - v_\lambda(t+\sigma) - (u_\lambda(t+\tau-s) - u_0)\}\, dk_\lambda(s)\big]_+ \\
\leq\; & \big[u_\lambda(t+\tau) - v_\lambda(t+\sigma), f(t+\tau) - g(t+\sigma)\big]_+ - k_\lambda(0+)\|u_\lambda(t+\tau) - v_\lambda(t+\sigma)\| \\
& - \int_{(0,t]} \|u_\lambda(t+\tau-s) - v_\lambda(t+\sigma-s)\|\, dk_\lambda(s) + C(k_\lambda(t) - k_\lambda(\infty))
\end{aligned}$$

holds almost everywhere for $t \in \mathbb{R}_+$ and for $0 < \sigma < \tau$. Here, we used the properties of the bracket $[\cdot,\cdot]_+$ (see proposition B.1), in particular the continuity in the second variable and the fact that $-dk$ is a nonnegative Radon measure on $(0,\infty)$. Note that, since k_λ is nonincreasing, we estimated $k_\lambda(t+\tau)$ and $k_\lambda(t+\sigma)$ by $k_\lambda(\infty)$. Since the resolvent of the first kind $\alpha_\lambda = \lambda\delta_0 + \alpha$ of k_λ is a nonnegative Radon measure on $[0,\infty)$, the convolution of the above inequality with α_λ yields

$$\begin{aligned}
&\|u_\lambda(t+\tau) - v_\lambda(t+\sigma)\| \\
&\quad \leq \int_{[0,t]} \big[u_\lambda(t+\tau-s) - v_\lambda(t+\sigma-s), f(t+\tau-s) - g(t+\sigma-s)\big]_+\, d\alpha_\lambda(s) \\
&\qquad + C\frac{\alpha((t,\infty))}{\lambda + \alpha(\mathbb{R}_+)}
\end{aligned} \tag{4.17}$$

almost everywhere for $t \in \mathbb{R}_+$ and for $0 < \sigma < \tau$. Here, we used the fact that, by proposition 4.1,

$$\int_{[0,t]} (k_\lambda(t-s) - k_\lambda(\infty))\, d\alpha_\lambda(s) = 1 - \frac{\lambda + \alpha([0,t])}{\lambda + \alpha(\mathbb{R}_+)} = \frac{\alpha((t,\infty))}{\lambda + \alpha(\mathbb{R}_+)} \quad \text{for all } t > 0.$$

By the existence result for generalized solutions (see [CGL96, Theorem 1] or [Gri85, Theorem 1]), we know that the sequence $\{u_\lambda\}_{\lambda>0}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$ to the generalized solution u of (4.1) as $\lambda \to 0+$, and $v_\lambda \to v$ in $L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$ as $\lambda \to 0+$. Therefore, we can assume that $u_\lambda(t) \to u(t)$ and $v_\lambda(t) \to v(t)$ almost everywhere for $t \in \mathbb{R}_+$ as $\lambda \to 0+$. Since $\{\alpha_\lambda\}_{\lambda>0}$ converges to α in $\mathcal{D}'(\mathbb{R}_+)$ as $\lambda \to 0+$, we are now in the position to apply lemma A.11. This gives the assertion, as the roles of τ and σ can be interchanged. □

In order to show the asymptotic almost periodicity of the solution of the Volterra equation

(4.1), we will have to use a different characterization of almost periodic and asymptotically almost periodic functions. We therefore introduce the notion of relatively dense sets.

Definition 4.3. Let $J \in \{\mathbb{R}, \mathbb{R}_+\}$, then a set $M \subset J$ is called *relatively dense in* J, if there exists a constant $l > 0$ such that $[t, t+l] \cap M \neq \emptyset$ for all $t \in J$.

The following characterization is due to Bochner for almost periodic functions and can be found in [AP71]. In case of asymptotic almost periodic functions the following result can be found in [Fré41] for real valued functions, and for general Banach space valued functions we refer to [RS88].

Proposition 4.4. *Let X be a Banach space.*

(i) A function $f \in C_b(\mathbb{R}; X)$ is almost periodic if and only if for all $\varepsilon > 0$ there exists a relatively dense set $p_\varepsilon \subset \mathbb{R}$ such that

$$\|f(t+\tau) - f(t)\| \leq \varepsilon \qquad \text{for all } t \in \mathbb{R} \text{ and all } \tau \in p_\varepsilon.$$

Then the elements τ of p_ε are called ε-almost periods of f.

(ii) A function $f \in BUC(\mathbb{R}_+; X)$ is asymptotically almost periodic if and only if for all $\varepsilon > 0$ there exists $T_\varepsilon > 0$ and a relatively dense set $p_\varepsilon \subset \mathbb{R}_+$ such that

$$\|f(t+\tau) - f(t)\| \leq \varepsilon \qquad \text{for all } t \geq T_\varepsilon \text{ and all } \tau \in p_\varepsilon.$$

Then the elements τ of p_ε are called asymptotic ε-almost periods of f.

Using the above characterization of asymptotically almost periodic functions, the main result of this section becomes a direct consequence of proposition 4.2.

Theorem 4.5. *Let A be an m-accretive operator in a Banach space X, let* (4.2), (4.3) *and* (4.5) *be satisfied, and let $u_0 \in \overline{D(A)}$, $f \in AAP(\mathbb{R}_+; X)$. Then the generalized solution u of* (4.1) *is asymptotically almost periodic.*

Proof. As already shown before, the generalized solution u of (4.1) exists and is an element of $BUC(\mathbb{R}_+; X)$. Now, let $\varepsilon > 0$ be arbitrary and choose $T_\varepsilon > 0$ and a relatively dense set $p_\varepsilon \subset \mathbb{R}_+$ such that

$$\|f(t+\tau) - f(t)\| \leq \frac{\varepsilon}{2\alpha(\mathbb{R}_+)} \qquad \text{for all } t \geq T_\varepsilon \text{ and all } \tau \in p_\varepsilon.$$

This is possible, since f is asymptotically almost periodic. Since, by the continuity of the measure α from above, $\alpha((t, \infty)) \to 0$ as $t \to \infty$. Thus, we choose $T > 0$ large enough such that

$$\alpha((T, \infty)) \leq \frac{\varepsilon}{4\|f\|_\infty + 2C_1}.$$

Here $C_1 := C/\alpha(\mathbb{R}_+)$, where $C > 0$ is the constant given by proposition 4.2. Again, by proposition 4.2, we conclude for all $t \geq T + T_\varepsilon$, and for all $\tau \in p_\varepsilon$

$$\begin{aligned} \|u(t+\tau) - u(t)\| &\leq \int_{[0,T]} \|f(t+\tau-s) - f(t-s)\| \, d\alpha(s) \\ &\quad + \alpha((T,\infty)) \left(2\|f\|_\infty + \frac{C}{\alpha(\mathbb{R}_+)} \right) \\ &\leq \varepsilon. \end{aligned}$$

Thus, u is asymptotically almost periodic. □

4.2 The limit equation

As we have seen in theorem 4.5, asymptotically almost periodic right hand sides f of (4.1) lead to asymptotically almost periodic solutions u. Since the space of asymptotically almost periodic functions can be decomposed by

$$AAP(\mathbb{R}_+; X) = AP(\mathbb{R}; X)\big|_{\mathbb{R}_+} \oplus C_0(\mathbb{R}_+; X),$$

there exist uniquely determined functions $u^{(\infty)}, f^{(\infty)} \in AP(\mathbb{R}; X)$ and $u^{(0)}, f^{(0)} \in C_0(\mathbb{R}_+; X)$ such that

$$f = f^{(\infty)}\big|_{\mathbb{R}_+} + f^{(0)} \quad \text{and} \quad u = u^{(\infty)}\big|_{\mathbb{R}_+} + u^{(0)}.$$

Our aim is to characterize $u^{(\infty)}$ directly by $f^{(\infty)}$. In the special case of a constant function $f^{(\infty)} \equiv f_\infty \in X$ this has already been done in [CN81]. By [CN81, Theorem 3.2], $u^{(\infty)}$ is also a constant function and $u_\infty \in X$ with $u^{(\infty)} \equiv u_\infty$ is the solution of the limit equation

$$u_\infty + \alpha(\mathbb{R}_+) A u_\infty \ni u_0 + \alpha(\mathbb{R}_+) f_\infty. \tag{4.18}$$

Assuming that A is m-accretive in X, and that (4.2), (4.3) and (4.5) hold, this limit equation has a unique solution $u_\infty \in D(A)$ given by

$$u_\infty = J^A_{\alpha(\mathbb{R}_+)} \left(u_0 + \alpha(\mathbb{R}_+) f_\infty \right).$$

Thus, one might conjecture that even in the case of nonconstant almost periodic parts $f^{(\infty)}$ of the right hand side f, the almost periodic part $u^{(\infty)}$ of the generalized solution u of (4.1) is given as a solution of a certain limit equation. Since $k \in L^1_{\text{loc}}(\mathbb{R}_+)$ in (4.1) is only assumed to be locally integrable, we choose $k_1 \in L^1(\mathbb{R}_+)$ and $k_2 \in BV(\mathbb{R}_+)$ such that

$$k(t) = k_1(t) + k_2(t) \text{ for all } t > 0 \text{ and } k_1, k_2 \text{ satisfy (4.2).} \tag{4.19}$$

Assuming that k satisfies (4.2), this choice of k_1, k_2 is always possible, since for $\delta > 0$ the functions k_1, k_2 given by

$$k_1(t) := \max(k(t) - k(\delta), 0), \qquad k_2(t) := \min(k(\delta), k(t)) \qquad \text{for } t > 0$$

obviously satisfy (4.19). Thus, for an m-accretive operator A in a Banach space X, and $u_0 \in \overline{D(A)}$, and $g \in C_b(\mathbb{R}; X)$ the equation

$$\begin{aligned} \frac{d}{dt}\Big(\kappa(u(t) - u_0) + \int_{-\infty}^{t} k_1(t-s)(u(s) - u_0)\, ds\Big) & \\ + k_2(0+)(u(t) - u_0) & \qquad t \in \mathbb{R} \qquad (4.20) \\ + \int_{(0,\infty)} (u(t-s) - u_0)\, dk_2(s) + Au(s) \;\ni\; g(t), & \end{aligned}$$

makes sense and will be called the limit equation corresponding to the problem (4.1).
Note that the limit equation (4.20) describes a problem on the whole real line $\mathbb{R}$ and is no more an initial value problem as (4.1). The fact that u_0 is still contained in (4.20) is due to our choice of the right hand side g, i.e. defining $\tilde{g} := k(\infty)u_0 + g$ we would be able to omit the term u_0 in the limit equation (4.20) by replacing g by $\tilde{g}$.
A priori it is not clear how to define the notion of solution to (4.20). But three methods of approximation of this limit equation come into view.
For the first method, one might think of approximating κ, k just as in the initial value problem on the halfline $\mathbb{R}_+$ by a sequence $\{k_n\}_{n\in\mathbb{N}}$ of nonnegative nonincreasing functions $k_n \in L^1_{\text{loc}}(\mathbb{R}_+)$ satisfying $k_n(0+) < \infty$ and $k_n(\infty) > 0$ such that

$$\int_0^t k_n(s)\, ds \to \kappa + \int_0^t k(s)\, ds \qquad \text{as } n \to \infty \text{ for all } t > 0.$$

Since $k_n(\infty) > 0$, one would be able to show that (4.20) with κ, k_1, k_2 replaced by a suitable decomposition of k_n admits a unique strong solution $u_n : \mathbb{R} \to X$ for all $n \in \mathbb{N}$. It would be interesting to know whether the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges at least in $L^1_{\text{loc}}(\mathbb{R}; X)$ and whether, as it is in the case of the initial value problem, the limit does not depend on the choice of the particular sequence $\{k_n\}$. Such a limit could then be called a *generalized solution* of (4.20). Note that in the special case $\kappa = 1$ and $k \equiv \omega > 0$ this problem is considered in [Kre]. There, it is shown that, for m-accretive operators A in X, the limit equation admits a unique solution of the above mentioned type for all right hand sides $g \in BUC(\mathbb{R}; X)$. Moreover, if the right hand side g is an element of a closed translation invariant subspace Y of $BUC(\mathbb{R}; X)$, i.e., Y satisfies $f_\tau \in Y$ for all $f \in Y$ and all $\tau \in \mathbb{R}$, and if Y is invariant under the resolvent J^A_λ of A for all $\lambda > 0$, i.e. $J^A_\lambda(Y) \subset Y$, then the solution u of (4.20) is an element of Y.

For the second method note that the Yosida approximation of an m-accretive operator A in X, given by $A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A)$ for $\lambda > 0$, is defined on all of X and is Lipschitz continuous with Lipschitz constant $\frac{2}{\lambda}$. Replacing A by A_λ in (4.20) for $\lambda > 0$, we can obtain a sequence of approximate solutions $\{u_\lambda\}_{\lambda>0}$. As in the first method, one could define a solution of (4.20) to be the $L^1_{\text{loc}}(\mathbb{R};X)$-limit for $\lambda \to 0+$ of the sequence $\{u_\lambda\}$, whenever this limit exists.

But here, we will use a different concept of generalized solutions of (4.20), which seems to be more appropriate for the investigation of the asymptotic behavior of solutions. We therefore mention that (4.20) can also be viewed as the limit of initial value problems on intervals $[-T,\infty)$ with T tending to ∞. Thus, we now consider the approximation

$$\frac{d}{dt}\left(\kappa(u^{(T)}(t)-u_0)+\int_{-T}^{t}k(t-s)(u^{(T)}(s)-u_0)\,ds\right)+Au^{(T)}(t)\ni g(t),\quad t\geq -T \tag{4.21}$$

of (4.20) with $T > 0$. But this equation is just a shifted version of the initial value problem (4.1). Indeed, assume that $u^{(T)} \in L^1_{\text{loc}}([0,\infty);X)$ is a strong solution of (4.21), then there exists $w \in L^1_{\text{loc}}(\mathbb{R}_+;X)$ such that $(u^{(T)}(t), w(t+T)) \in A$ almost everywhere for $t \in [-T,\infty)$ and

$$\frac{d}{dt}\left(\kappa(u^{(T)}(t)-u_0)+\int_{-T}^{t}k(t-s)(u^{(T)}(s)-u_0)\,ds\right)+w(t+T)\ni g(t)\quad\text{a.e. } t\in[-T,\infty).$$

Defining $v^{(T)}(t) := u^{(T)}(t-T)$ for all $t \in \mathbb{R}_+$, it is obvious that $v^{(T)} \in L^1_{\text{loc}}(\mathbb{R}_+;X)$ with $(v^{(T)}(t), w(t)) \in A$ almost everywhere for $t \in \mathbb{R}_+$ and that

$$\frac{d}{dt}\left(\kappa(v^{(T)}(t)-u_0)+\int_{0}^{t}k(t-s)(v^{(T)}(s)-u_0)\,ds\right)+w(t)\ni g(t-T), \tag{4.22}$$

almost everywhere for $t \in \mathbb{R}_+$. Thus, $v^{(T)}$ is a strong solution of (4.1) with the right hand side f replaced by $g(\cdot - T)$. Therefore, it is almost obvious how to define the notion of generalized solutions of (4.21).

Definition 4.6. A function $u^{(T)} \in L^1_{\text{loc}}([-T,\infty);X)$ is called a *generalized solution* of (4.21) for data $u_0 \in \overline{D(A)}$ and $g \in L^1_{\text{loc}}([-T,\infty);X)$ if the function $v^{(T)} \in L^1_{\text{loc}}(\mathbb{R}_+;X)$ defined by $v^{(T)}(\cdot) := u^{(T)}(\cdot - T)$ is a generalized solution of (4.1) with the right hand side f replaced by $g(\cdot - T)$.

We are now going to investigate the convergence of the net $\{u^{(T)}\}_{T>0}$ as $T \to \infty$. Using directly the net $\{u^{(T)}\}$ instead of sequences $\{u^{(T_n)}\}_{n\in\mathbb{N}}$ with $T_n \to \infty$ as $n \to \infty$, we omit possible dependence of the limit on the choice of the sequence $\{T_n\}_{n\in\mathbb{N}}$.

Theorem 4.7. *Let A be an m-accretive operator in a Banach space X, let κ, k satisfy* (4.2), (4.3), (4.5), *and let k_1, k_2 be given such that* (4.19) *holds. Moreover, assume that $u_0 \in \overline{D(A)}$ and $g \in L^\infty(\mathbb{R};X)$. Then the net $\{u^{(T)}\}_{T>0}$ of generalized solutions $u^{(T)}$ of* (4.21) *converges in $L^\infty_{\text{loc}}(\mathbb{R};X)$ as $T \to \infty$.*

Proof. We are going to show that $\{u^{(T)}\}_{T>0}$ is a Cauchy net in $L^\infty_{\text{loc}}(\mathbb{R};X)$. Here, without loss of generality we set $u^{(T)}(t) := u_0$ for $t < -T$. Defining $v^{(T)}(\cdot) := u^{(T)}(t-T)$ for all $T > 0$, it is clear, that $v^{(T)}$ is a generalized solution of (4.1) with right hand side $g(\cdot - T)\big|_{\mathbb{R}_+}$. Thus, by proposition 4.2 there exists a constant $C > 0$ such that almost everywhere for $t \in \mathbb{R}_+$ and $0 < T_1 < T_2$

$$\begin{aligned}\|v^{(T_1)}(t+T_1) - v^{(T_2)}(t+T_2)\| \\ \leq \int_{[0,t+T_1]} \|g(t+T_1-T_1-s) - g(t+T_2-T_2-s)\|\, d\alpha(s) \\ + C\frac{\alpha((t+T_1,\infty))}{\alpha(\mathbb{R}_+)}.\end{aligned}$$

For $M > 0$ and $\varepsilon > 0$ arbitrary, choose $T_0 > 0$ such that

$$\alpha((T_0 - M, \infty)) \leq \frac{\varepsilon}{C}\,\alpha(\mathbb{R}_+).$$

Then, by the above inequality, we conclude for $T_0 < T_1 < T_2$ and almost everywhere for $t \in [-M, M]$

$$\|u^{(T_1)}(t) - u^{(T_2)}(t)\| = \|v^{(T_1)}(t+T_1) - v^{(T_2)}(t+T_2)\| \leq \varepsilon.$$

Since the roles of T_1 and T_2 can be interchanged, we obtain the assertion. □

We can now give the definition of generalized solutions of the limit equation (4.20) by means of the approximate solutions $u^{(T)}$ for $T > 0$.

Definition 4.8. Let A be an m-accretive operator in a Banach space X, let κ, k satisfy (4.2), (4.3), (4.5), and let k_1, k_2 be given such that (4.19) holds. Then, a function $u^{(\infty)} \in L^\infty_{\text{loc}}(\mathbb{R};X)$ is called a *generalized solution* of the limit equation (4.20) for $u_0 \in \overline{D(A)}$ and $g \in L^\infty(\mathbb{R};X)$, if

$$u^{(T)} \to u^{(\infty)} \qquad \text{in } L^\infty_{\text{loc}}(\mathbb{R};X),$$

where $u^{(T)}$ is the generalized solution of (4.21) for $T > 0$.

Note that, since the net $\{u^{(T)}\}$ converges in $L^\infty_{\text{loc}}(\mathbb{R};X)$, we do not even know whether the generalized solution $u^{(\infty)}$ of (4.20) is essentially bounded on $\mathbb{R}$. But obviously $u^{(\infty)}$ is continuous on $\mathbb{R}$, if we assume that $g \in C_b(\mathbb{R};X)$ and that the other assumptions of theorem 4.7 are satisfied.
We are now going to state the basic properties of the generalized solution of the limit equation (4.20).

Proposition 4.9. *Let A be an m-accretive operator in a Banach space X, let κ, k satisfy* (4.2), (4.3), (4.5), *and let k_1, k_2 be given such that* (4.19) *holds. Moreover, let $u_0 \in \overline{D(A)}$ and $g \in L^\infty(\mathbb{R};X)$. Then the generalized solution $u^{(\infty)}$ of the limit equation* (4.20) *satisfies* *(i)* $u^{(\infty)} \in L^\infty(\mathbb{R}; X)$,

(ii) if $g \in BUC(\mathbb{R}; X)$, *then* $u^{(\infty)} \in BUC(\mathbb{R}; X)$.

Proof. The method of proof is almost the same as of proposition 2.8 and proposition 2.9. For almost all $t \in \mathbb{R}$, $(x, y) \in A$ and $\varepsilon > 0$, we can find $T > |t|$, such that

$$\begin{aligned} \|u^{(\infty)}(t) - x\| &\leq \|u^{(\infty)}(t) - u^{(T)}(t)\| + \|u^{(T)}(t) - x\| \\ &\leq \varepsilon + \|u_0 - x\| + \|g - y\|_\infty \alpha(\mathbb{R}_+). \end{aligned}$$

Thus, $u^{(\infty)}$ is essentially bounded. If we moreover assume that $g \in BUC(\mathbb{R}; X)$, then obviously $u^{(\infty)} \in C_b(\mathbb{R}; X)$. For the uniform continuity let $\varepsilon > 0$ be arbitrary. Then, by corollary 2.4, we can find $h_0 > 0$ such that for all $0 < h < h_0$ and all $T > 0$

$$\|u^{(T)}(h) - u_0\| \leq \frac{\varepsilon}{4}.$$

Moreover, by the uniform continuity of g, we can choose h_0 small enough such that for all $0 < h < h_0$ and for all $t \in \mathbb{R}$

$$\|g(t + h) - g(t)\| \leq \frac{\varepsilon}{4\alpha(\mathbb{R}_+)}.$$

Now, for any $t \in \mathbb{R}$ we can choose $T > |t| + h_0$ such that for all $0 < h < h_0$

$$\begin{aligned} \|u^{(\infty)}(t + h) - u^{(\infty)}(t)\| &\leq \|u^{(\infty)}(t + h) - u^{(T)}(t + h)\| + \|u^{(\infty)}(t) - u^{(T)}(t)\| \\ &\quad + \|u^{(T)}(h) - u_0\| + \int_{[0,\infty)} \|g(t + h - s) - g(t - s)\| \, d\alpha(s) \\ &\leq \varepsilon. \end{aligned}$$

□

Since we now have a meaningful definition of solutions of the limit equation, we can return to our initial question of characterizing the almost periodic part of a generalized solution of the initial value problem (4.1). Obviously, the limit equation (4.20) reduces to the limit equation (4.18) if we assume that the right hand side g of the limit equation is constant, i.e. $g \equiv f_\infty \in X$. Thus, we obtain a constant solution of the limit equation. As a generalization, we obtain the following.

Theorem 4.10. *Let A be an m-accretive operator in a Banach space X, let κ, k satisfy* (4.2), (4.3), (4.5), *and let k_1, k_2 be given such that* (4.19) *holds. Moreover, let $u_0 \in \overline{D(A)}$ and $g \in AP(\mathbb{R}; X)$. Then the generalized solution $u^{(\infty)}$ of the limit equation* (4.20) *is almost periodic.*

Proof. We already know that $u^{(\infty)}$ is bounded and uniformly continuous. In the following, let $\{u^{(T)}\}_{T>0}$ denote the net of generalized solutions $u^{(T)}$ of (4.21). Let $\varepsilon > 0$ be arbitrary, then there exists a relatively dense set $p_\varepsilon \subset \mathbb{R}$ such that

$$\|g(t+\tau) - g(t)\| \le \frac{\varepsilon}{4\alpha(\mathbb{R}_+)} \qquad \text{for all } t \in \mathbb{R} \text{ and all } \tau \in p_\varepsilon.$$

Thus, for all $t \in \mathbb{R}$ and all $\tau \in p_\varepsilon$, we can find $T > |t| + |\tau|$ such that

$$\begin{aligned}
\|u^{(\infty)}(t+\tau) - u^{(\infty)}(t)\| &\le \|u^{(\infty)}(t+\tau) - u^{(T)}(t+\tau)\| + \|u^{(\infty)}(t) - u^{(T)}(t)\| \\
&\quad + \|u^{(T)}(t+\tau) - u^{(T)}(t)\| \\
&\le \frac{\varepsilon}{2} + \int_{[0,\infty)} \|g(t+\tau-s) - g(t-s)\|\, d\alpha(s) \\
&\quad + C\frac{\alpha((t+T,\infty))}{\alpha(\mathbb{R}_+)} \\
&\le \varepsilon
\end{aligned}$$

with a constant $C > 0$ given by proposition 4.2. This implies that $u^{(\infty)}$ is almost periodic on $\mathbb{R}$. □

Moreover, by comparison of the generalized solution of the initial value problem (4.1) and the limit equation (4.20), we can characterize the almost periodic part of an asymptotically almost periodic solution of the initial value problem by means of the almost periodic part of the right hand side.

Theorem 4.11. *Let A be an m-accretive operator in a Banach space X, let κ, k satisfy* (4.2), (4.3), (4.5), *and let k_1, k_2 be given such that* (4.19) *holds. Moreover, let $u_0 \in \overline{D(A)}$, $f \in AAP(\mathbb{R}_+; X)$, and let $f^{(\infty)} \in AP(\mathbb{R}; X)$ be the almost periodic part of f.*
Then the generalized solution u of the initial value problem (4.1) *is asymptotically almost periodic, and the almost periodic part $u^{(\infty)} \in AP(\mathbb{R}; X)$ of u is the generalized solution of the limit equation* (4.20) *with the right hand side $f^{(\infty)}$.*

Proof. Let $u^{(\infty)}$ denote the generalized solution of the limit equation (4.20) with right hand side $f^{(\infty)}$. Then, by theorem 4.10 and theorem 4.5, we already know that $u^{(\infty)} \in AP(\mathbb{R}; X)$ and $u \in AAP(\mathbb{R}_+; X)$. Thus, it remains to show that $\|u^{(\infty)}(t) - u(t)\| \to 0$ as $t \to \infty$. Let $u^{(T)}$ be the generalized solution of (4.21) with right hand side $f^{(\infty)}$ for all $T > 0$ and let $\varepsilon > 0$ be arbitrary. Then, for all $t \in \mathbb{R}$, we can find $T > |t| + |\tau|$ such that

$$\begin{aligned}
\|u^{(\infty)}(t) - u(t)\| &\le \|u^{(\infty)}(t) - u^{(T)}(t)\| + \|u^{(T)}(t) - u(t)\| \\
&\le \frac{\varepsilon}{2} + \int_{[0,t]} \|f^{(\infty)}(t-s) - f(t-s)\|\, d\alpha(s) + C\frac{\alpha((t,\infty))}{\alpha(\mathbb{R}_+)}
\end{aligned}$$

with a constant $C > 0$ given by proposition 4.2. Since $\|f^{(\infty)}(t) - f(t)\| \to 0$ as $t \to \infty$, we conclude

$$\lim_{t\to\infty} \|u^{(\infty)}(t) - u(t)\| \leq \varepsilon.$$

As ε was chosen arbitrarily, we have shown that $u^{(\infty)}$ is the almost periodic part of u. $\square$

We remark that we could have introduced the concept of generalized solutions of the limiting equation by using different approximations. As we already mentioned, one can replace the operator A in the limit equation (4.20) by its Yosida approximation A_λ for $\lambda > 0$. Then, for a right hand side $f^{(\infty)} \in AP(\mathbb{R}; X)$ one can obtain a net $\{u_\lambda^{(\infty)}\}_{\lambda>0}$ of strong solutions of the limit equation. These solutions turn out to be equi-almost periodic, and $u_\lambda^{(\infty)}$ is always the almost periodic part of the generalized solution u_λ of the initial value problem (4.1) with A replaced by A_λ, where $f \in AAP(\mathbb{R}_+; X)$ such that $f^{(\infty)}$ is the almost periodic part of f. Moreover, the net $\{u_\lambda^{(\infty)}\}_{\lambda>0}$ converges uniformly on compact subsets of $\mathbb{R}$ towards the generalized solution of the limit equation given by definition 4.8. Thus, both concepts of generalized solutions of the limit equation (4.20) coincide. Since we will not further investigate the properties of generalized solutions of the limit equation, we omit the proofs of these results.
Note that, assuming that A is an m-accretive operator in a Banach space X, and that $u_0 \in \overline{D(A)}$, we have shown that the subspace $AP(\mathbb{R}; X)$ of $BUC(\mathbb{R}; X)$ is invariant under the solution mapping

$$\begin{aligned} \mathcal{G}_\infty : BUC(\mathbb{R}; X) &\to BUC(\mathbb{R}; X), \\ g &\mapsto u, \end{aligned}$$

which maps right hand sides g to the generalized solution u of the limit equation (4.20), i.e., we have shown that $\mathcal{G}_\infty(AP(\mathbb{R}; X)) \subset AP(\mathbb{R}; X)$. Moreover, in the special case of $\kappa = 1$ and $k \equiv \omega > 0$, it has been shown in [Kre] that other closed translation invariant subspaces of $BUC(\mathbb{R}; X)$ stay invariant under the solution mapping. In particular, [Kre] contains the following result.

Proposition 4.12. *Let A be an m-accretive operator in an Banach space X, let $\kappa = 1$ and $k \equiv \omega > 0$, and let Y be a closed translation invariant subspace of $BUC(\mathbb{R}; X)$ such that $J_\lambda(Y) \subset Y$ for all $\lambda > 0$. Then the solution u of (4.20) is in Y for any right hand side $g \in BUC(\mathbb{R}; X)$ and any $u_0 \in \overline{D(A)}$ with $\omega u_0 + g \in Y$.*

But whether the same result holds for Volterra equations with κ, k satisfying (4.2), (4.3) and (4.5) is still an open problem.
Finially we remark that in [Egb92] a different approach was used to show existence of solutions of (4.20) in a Hilbert space H if the operator A is a subdifferential. In particular,

let $A = \partial\varphi$ for some $\varphi \in J_0 = \{\psi : H \to [0,\infty] \mid \psi \text{ proper, convex, l.s.c, with } \psi(0) = 0\}$ and let the linear operator L be given by

$$\begin{aligned} D(L) &:= \Big\{u \in L^2(\mathbb{R}) \;\Big|\; t \mapsto \frac{\partial}{\partial t}\left(\kappa u + \int_0^\infty u(t-s)k_1(s)\,ds\right) \\ &\qquad + \tilde{k}_2(0+)u(t) + \int_{(0,\infty)} u(t-s)d\tilde{k}_2(s) \in L^2(\mathbb{R})\Big\}, \\ Lu &:= \frac{\partial}{\partial t}\left(\kappa u + \int_0^\infty u(t-s)k_1(s)\,ds\right) + \tilde{k}_2(0+)u(t) + \int_{(0,\infty)} u(t-s)d\tilde{k}_2(s), \end{aligned}$$

with $\tilde{k}_2 := k_2 - \omega$, where $\omega := k(\infty) > 0$. Then A and L can be extended naturally to m-accretive operators $\mathcal{A}$ and $\mathcal{L}$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, H)$. By [Egb92, Proposition 7.2.1] the sum $\mathcal{L} + \mathcal{A}$ of the extensions is an m-accretive operator in $\mathcal{H}$ and there exists a unique solution of

$$\omega u + (\mathcal{L} + \mathcal{A})u \ni f$$

for all $f \in \mathcal{H}$. Since $u \in D(\mathcal{L}+\mathcal{A}) = D(\mathcal{L}) \cap D(\mathcal{A})$, it turns out that u is a strong solution of (4.20) with $u_0 = 0$.

4.3 Eberlein-weak almost periodicity

In this section, we develop sufficient conditions such that the generalized solution u of the initial value problem (4.1) is Eberlein-weak almost periodic on $\mathbb{R}_+$. The main tool in this approach will be the a characterization of weak compactness in the space of bounded continuous functions. The following double limit condition is an extension of the criterion given by Grothendieck in [Gro52] for real valued functions, it can be found in [Mil80, Theorem 3] and [RS89, Theorem 2.1].

Proposition 4.13. *Let (T, τ) be a completely regular topological space and X a Banach space. Then, a subset $H \subset C_b(T; X)$ is relatively weakly compact in $(C_b(T; X), \|\cdot\|_\infty)$ if and only if*

(i) H is bounded in $C_b(T; X)$, and

(ii) for all sequences $\{h_n\}_{n\in\mathbb{N}} \subset H$, $\{t_m\}_{m\in\mathbb{N}} \subset T$, and $\{x'_m\}_{m\in\mathbb{N}} \subset \mathrm{ext}B_{X'}$ the following equality, called the double limit condition, holds

$$\lim_{m\to\infty}\lim_{n\to\infty} \langle h_n(t_m), x'_m\rangle = \lim_{n\to\infty}\lim_{m\to\infty} \langle h_n(t_m), x'_m\rangle, \tag{4.23}$$

whenever both iterated limits exist.

In order to simplify the proofs, we will make use of the following result [RS90], which enables us to verify the double limit condition only for certain sequences.

Lemma 4.14. *Let $f \in BUC(\mathbb{R}_+; X)$, where X denotes a Banach space, have relatively compact range. Let $\{t_m\}_{m\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{x'_m\}_{m\in\mathbb{N}} \subset B_{X'}$ be sequences such that $\{t_m\}_{m\in\mathbb{N}}$ or $\{\tau_n\}_{n\in\mathbb{N}}$ is bounded. Then*

$$\lim_{n\to\infty}\lim_{m\to\infty}\langle f(t_m+\tau_n), x'_m\rangle = \lim_{m\to\infty}\lim_{n\to\infty}\langle f(t_m+\tau_n), x'_m\rangle,$$

whenever both iterated limits exist.

Assuming that the right hand side of the initial value problem (4.1) is Eberlein-weak almost periodic on $\mathbb{R}_+$, we still can not describe the asymptotic behavior of the real valued function $[u(\cdot), f(\cdot)]_+$, where $u \in BUC(\mathbb{R}_+; X)$ denotes the generalized solution of (4.1), in general. But if f has relatively compact range, we can use the following lemma of [Kre92, Lemma 4.1], see also [Kre96, Lemma 4.1].

Lemma 4.15. *Let $\{f_n\}_{n\in\mathbb{N}} \subset C_b(\mathbb{R}_+; X)$ be a sequence such that for a compact set $K \subset X$ in the Banach space X*

$$\big\{f_n(t) \big|\ n \in \mathbb{N}, t \in \mathbb{R}_+\big\} \subset K \quad \text{and} \quad f_n \rightharpoonup 0 \text{ in } C_b(\mathbb{R}_+; X).$$

Then $\|f_n\| \rightharpoonup 0$ in $C_b(\mathbb{R}_+)$.

For the sake of completeness, we give the proof.

Proof. In the first step, we show that the set $\{\|f_n\| \mid n \in \mathbb{N}\}$ is weakly relatively compact in $C_b(\mathbb{R}_+)$. Therefore, by the double limit condition (4.23), we have to verify for all sequences $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, $\{t_m\}_{m\in\mathbb{N}} \subset R$ that

$$\lim_{m\to\infty}\lim_{k\to\infty}\|f_{n_k}(t_m)\| = \lim_{k\to\infty}\lim_{m\to\infty}\|f_{n_k}(t_m)\|,$$

whenever both iterated limits exist. Since $f_{n_k}(t_m) \in K$ for all $k, m \in \mathbb{N}$, we can use a diagonalization argument to obtain subsequences, again denoted by $\{n_k\}$ and $\{t_m\}$, such that the limits

$$\lim_{m\to\infty}\lim_{k\to\infty} f_{n_k}(t_m) = x \quad \text{and} \quad \lim_{k\to\infty}\lim_{m\to\infty} f_{n_k}(t_m) = y$$

exist. By proposition 4.13, we conclude $\langle x, x'\rangle = \langle y, x'\rangle$ for all $x' \in B_{X'}$ and thus $x = y$. Thus, we have shown that $\{\|f_n(\cdot)\| \bigm| n \in \mathbb{N}\}$ is weakly relatively compact in $C_b(\mathbb{R}_+)$. Since $\|f_n(t)\| \to 0$ as $n \to \infty$ for any $t \in \mathbb{R}_+$, and since the topology of pointwise convergence is Hausdorff, it coincides with the weak topology of $C_b(\mathbb{R}_+)$ on the weak closure of $\{\|f_n(\cdot)\| \bigm| n \in \mathbb{N}\}$. Thus we conclude that $\|f_n(\cdot)\| \rightharpoonup 0$ in $C_b(\mathbb{R}_+)$ as $n \to \infty$. □

Applying the characterization of weak compactness in the space $C_b(\mathbb{R}_+; X)$ given by proposition 4.13, we can show the existence of Eberlein-weak almost periodic solutions.

Theorem 4.16. *Let A be an m-accretive operator in a Banach space X, let* (4.2), (4.3) *and* (4.5) *be satisfied, and let $u_0 \in \overline{D(A)}$, $f \in WRC(\mathbb{R}_+;X)$. Then the generalized solution $u \in BUC(\mathbb{R}_+;X)$ of* (4.1) *is Eberlein-weak almost periodic on $\mathbb{R}_+$ with relatively compact range.*

Proof. In the first step, we show that the generalized solution u of (4.1) has relatively compact range. Therefore, let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$. Since $W(\mathbb{R}_+;X)$ is a subspace of $BUC(\mathbb{R}_+;X)$, f is uniformly continuous, hence u is uniformly continuous on $\mathbb{R}_+$. Thus, we may assume that $\{t_m\}_{m\in\mathbb{N}}$ is strictly increasing as $m \to \infty$. We now define

$$\tau_{m,n} := \begin{cases} t_m - t_n & \text{for } m \geq n, \\ 0 & \text{for } m < n \end{cases} \qquad \text{for all } m,n \in \mathbb{N}.$$

Then, $\tau_{m,n} \to \infty$ as $m \to \infty$ for all $n \in \mathbb{N}$ fixed. Note that, w-cl$(O(f))$ is weakly compact and contained in a closed separable linear subspace of $C_b(\mathbb{R}_+;X)$, thus the weak topology on w-cl$(O(f))$ is metrizable. Since $\{f_{\tau_{m,n}} = f(\tau_{m,n} + \cdot) \mid m,n \in \mathbb{N}\} \subset O(f)$, we can apply a diagonalization argument to obtain a common subsequence of $\{\tau_{m,n}\}_{m\in\mathbb{N}}$ for all $n \in \mathbb{N}$, without loss of generality again denoted by $\{\tau_{m,n}\}_{m\in\mathbb{N}}$, such that

$$w\text{-}\lim_{m\to\infty} f_{\tau_{m,n}} = g_n \in C_b(\mathbb{R}_+;X) \qquad \text{for all } n \in \mathbb{N}.$$

Moreover, since f has relatively compact range, we can apply lemma 4.15 and obtain

$$\|f_{\tau_{m,n}} - g_n\| \rightharpoonup 0 \quad \text{as } m \to \infty \text{ for all } n \in \mathbb{N}.$$

For $\varepsilon > 0$ given, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$C\frac{\alpha((t_n,\infty))}{\alpha(\mathbb{R}_+)} \leq \frac{\varepsilon}{3},$$

where the constant $C > 0$ is given by (4.2). For N fixed, the convolution evaluated at t_N is a bounded functional on $C_b(\mathbb{R}_+)$. Thus, there exists $M = M(N)$ such that for all $m \geq M$

$$\int_{[0,t_N]} \|f_{\tau_{m,N}}(t_N - s) - g_N(t_N - s)\|\, d\alpha(s) \leq \frac{\varepsilon}{3}.$$

For all $i,j \geq N + M$ one concludes by proposition 4.2

$$\begin{aligned} \|u(t_i) - u(t_j)\| &\leq \|u(t_N + \tau_{i,N}) - u(t_N + \tau_{j,N})\| \\ &\leq \int_{[0,t_N]} \|f_{\tau_{i,N}}(t_N - s) - g_N(t_N - s)\|\, d\alpha(s) \\ &\quad + \int_{[0,t_N]} \|f_{\tau_{j,N}}(t_N - s) - g_N(t_N - s)\|\, d\alpha(s) \\ &\quad + C\frac{\alpha((t_N,\infty))}{\alpha(\mathbb{R}_+)} \\ &\leq \varepsilon. \end{aligned}$$

This implies that $\{u(t_m)\}_{m\in\mathbb{N}}$ is a Cauchy sequence.
In the second step, we show that u is Eberlein-weak almost periodic. In order to apply proposition 4.13 and lemma 4.14, let $\{t_m\}_{m\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, and $\{x'_m\}_{m\in\mathbb{N}} \subset B_{X'}$ with $t_m \nearrow \infty$ and $\tau_n \nearrow \infty$ be given such that the iterated limits

$$\lim_{m\to\infty}\lim_{n\to\infty}\langle u(t_m+\tau_n), x'_m\rangle = \beta \quad \text{and} \quad \lim_{n\to\infty}\lim_{m\to\infty}\langle u(t_m+\tau_n), x'_m\rangle = \gamma$$

exist. We will have to show $\beta = \gamma$. Since $f \in WRC(\mathbb{R}_+; X)$, we can assume that $f_{\tau_n} \rightharpoonup g$ for some $g \in C_b(\mathbb{R}_+; X)$ as $n \to \infty$. Moreover, by lemma 4.15,

$$\|f_{\tau_n}(\cdot) - g(\cdot)\| \rightharpoonup 0 \qquad \text{in } C_b(\mathbb{R}_+) \text{ as } n \to \infty.$$

By proposition 4.2, we have for all $k, m, n \in \mathbb{N}$

$$\begin{aligned}
|\langle u(t_m+\tau_n) - u(t_m+\tau_k), x'_m\rangle| &\leq \|u(t_m+\tau_n) - u(t_m+\tau_k)\| \\
&\leq \int_{[0,t_m]} \|f_{\tau_n}(t_m - s) - g(t_m - s)\|\, d\alpha(s) \\
&\quad + \int_{[0,t_m]} \|f_{\tau_k}(t_m - s) - g(t_m - s)\|\, d\alpha(s) \\
&\quad + C\, \frac{\alpha((t_m,\infty))}{\alpha(\mathbb{R}_+)}.
\end{aligned}$$

Now, define the bounded linear functional α_m on $C_b(\mathbb{R}_+)$ for all $m \in \mathbb{N}$ by

$$\alpha_m(h) := \int_{[0,t_m]} h(t_m - s)\, d\alpha(s) \qquad \text{for all } h \in C_b(\mathbb{R}_+).$$

Then, since $V := \overline{\operatorname{span}}\{\|f_{\tau_n} - g\| \mid n \in \mathbb{N}\}$ is separable, there exists a subsequence of $\{\alpha_m\}_{m\in\mathbb{N}}$, again denoted by $\{\alpha_m\}_{m\in\mathbb{N}}$, such that $\{\alpha_m\big|_V\}_{m\in\mathbb{N}}$ is weak* convergent in V', i.e., $\alpha_m\big|_V \rightharpoonup^* \alpha_\infty$ in V' for some $\alpha_\infty \in V'$. Thus, we conclude

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} |\langle u(t_m+\tau_n) - u(t_m+\tau_k), x'_m\rangle| \\
&\qquad \leq \lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\langle \alpha_m, \|f_{\tau_n} - g\|\rangle_{V',V} + \lim_{k\to\infty}\lim_{m\to\infty}\langle \alpha_m, \|f_{\tau_k} - g\|\rangle_{V',V} \\
&\qquad\quad + \lim_{m\to\infty} C\frac{\alpha((t_m,\infty))}{\alpha(\mathbb{R}_+)} \\
&\qquad = \lim_{k\to\infty}\langle \alpha_\infty, \|f_{\tau_k} - g\|\rangle_{V',V} = 0,
\end{aligned}$$

which implies that the double limit condition holds. □

In the preceding proof, we have essentially used the relative compactness of the range of the right hand side f. This was necessary in order to apply lemma 4.15. But in special cases we may omit the assumption on the range of f if we use the following result of [Kre92, Lemma 4.6], see also [Kre96, Lemma 4.6].

Lemma 4.17. *Let K be a compact set in the dual space X' of the Banach space X. Moreover, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $C_b(\mathbb{R}_+;X)$ such that $f_n \rightharpoonup 0$ as $n \to \infty$. Then*

$$\sup_{x'\in K} |\langle f_n(\cdot), x'\rangle| \rightharpoonup 0 \quad \text{in } C_b(\mathbb{R}_+) \text{ as } n \to \infty.$$

For the sake of completeness, we present the proof.

Proof. In the first step, we show that the set $\{g_n \mid n \in \mathbb{N}\}$ is weakly relatively compact in $C_b(\mathbb{R}_+)$, where

$$g_n := \sup_{x'\in K} |\langle f_n(\cdot), x'\rangle| \qquad \text{for all } n \in \mathbb{N}.$$

Therefore, by the double limit condition (4.23), we have to verify for all sequences $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, $\{t_m\}_{m\in\mathbb{N}} \subset \mathbb{R}_+$ that $\beta = \gamma$ for

$$\lim_{m\to\infty}\lim_{k\to\infty} g_{n_k}(t_m) = \beta \qquad \text{and} \qquad \lim_{k\to\infty}\lim_{m\to\infty} g_{n_k}(t_m) = \gamma,$$

whenever both iterated limits exist. If $\{n_k\}_{k\in\mathbb{N}}$ is bounded, we can pass to a constant subsequence and obtain $\beta = \gamma$. Otherwise, we may assume that $\{n_k\}_{k\in\mathbb{N}}$ is strictly increasing. Letting $M := \sup_{n\in\mathbb{N}} \|g_n\|_\infty$, and using the compactness of K, we conclude that for given $\varepsilon > 0$ there exist $x'_1, ..., x'_N \in K$ such that

$$K \subset \bigcup_{i=1}^{N} \mathcal{U}(x'_i, \frac{\varepsilon}{M+1}).$$

By a diagonalization argument, we find subsequences of $\{t_m\}$ and $\{n_k\}$, again denoted by $\{t_m\}$ and $\{n_k\}$, respectively, such that

$$\lim_{m\to\infty}\lim_{k\to\infty} \langle f_{n_k}(t_m), x'_i\rangle = a_i \qquad \text{and} \qquad \lim_{k\to\infty}\lim_{m\to\infty} \langle f_{n_k}(t_m), x'_i\rangle = b_i$$

for all $i \in \{1, ..., N\}$. Using $f_{n_k} \rightharpoonup 0$, we conclude $a_i = b_i = 0$ for all $i \in \{1, ..., N\}$. Now, let $x' \in K$, then for some $j \in \{1, ..., N\}$

$$g_{n_k}(t_m) \le \varepsilon + |\langle f_{n_k}(t_m), x'_j\rangle|.$$

Hence, we obtain

$$0 \le \beta \le \varepsilon \qquad \text{and} \qquad 0 \le \gamma \le \varepsilon.$$

Since ε was chosen arbitrarily, we conclude $\beta = \gamma = 0$.
We show that $g_n \rightharpoonup 0$ in $C_b(\mathbb{R}_+)$ as $n \to \infty$. Since $g_n(t) \to 0$ as $n \to \infty$ for any $t \in \mathbb{R}_+$, and since the topology of pointwise convergence is Hausdorff, it coincides with the weak topology of $C_b(\mathbb{R}_+)$ on the weak closure of $\{g_n(\cdot) \mid n \in \mathbb{N}\}$. Thus we conclude that $g_n(\cdot) \rightharpoonup 0$ in $C_b(\mathbb{R}_+)$ as $n \to \infty$. □

To apply the above lemma, we will need a compact set in X'. Therefore, we will assume that X' is uniformly convex, then the duality map $\mathbf{F} : X \to 2^{X'}$, given by

$$\mathbf{F}(x) := \left\{ x' \in X' \;\middle|\; \|x\|^2 = \|x'\|^2 = \langle x, x' \rangle \right\},$$

is single-valued and continuous. We remark, that the corresponding semi-inner product, which is the right-hand Gateaux derivative of $\frac{1}{2}\|\cdot\|^2$, is given by

$$\langle x, y \rangle_+ := \lim_{\lambda \to 0+} \frac{\|x + \lambda y\|^2 - \|x\|^2}{2\lambda} = \|x\|[x, y]_+ = \sup_{x' \in \mathbf{F}(x)} \langle y, x' \rangle$$

for all $x, y \in X$. Noticing that an operator $A \subset X \times X$ is accretive if and only if

$$\langle x - \tilde{x}, y - \tilde{y} \rangle_+ \geq 0 \quad \text{for all } (x, y), (\tilde{x}, \tilde{y}) \in A,$$

we now develop a new version of proposition 4.2.

Proposition 4.18. *Let κ, k satisfy* (4.2), (4.3) *and* (4.5) *, and let A be an m-accretive operator in a Banach space X. Let $u_0 \in \overline{D(A)}$ and $f, g \in L^\infty(\mathbb{R}_+; X)$. Moreover, let u be the generalized solution of* (4.1), *and let v be the generalized solution of* (4.1) *with f replaced by g. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\|u(t+\tau) - v(t+\sigma)\|^2 \\
&\qquad \leq 2 \int_{[0,t]} \langle u(t+\tau-s) - v(t+\sigma-s), f(t+\tau-s) - g(t+\sigma-s) \rangle_+ \, d\alpha(s) \\
&\qquad\quad + C \frac{\alpha((t,\infty))}{\alpha([0,\infty))}
\end{aligned}$$

almost everywhere for $t, \tau, \sigma \in \mathbb{R}_+$.

We omit the proof of the above proposition, since it uses exactly the same arguments as the proof of proposition 4.2.
Assuming that X' is uniformly convex, we can show the existence of Eberlein-weak almost periodic solutions of (4.1) for right-hand sides f, which need not have relatively compact range.

Theorem 4.19. *Let A be an m-accretive operator in a Banach space X, with X' being uniformly convex. Let* (4.2), (4.3) *and* (4.5) *be satisfied, and let $u_0 \in \overline{D(A)}$, $f \in W(\mathbb{R}_+; X)$, and assume that the generalized solution u of* (4.1) *has relatively compact range. Then u is Eberlein-weak almost periodic.*

Proof. In order to apply proposition 4.13 and lemma 4.14, let $\{t_m\}_{m \in \mathbb{N}} \subset \mathbb{R}_+$, $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ and $\{x'_m\}_{m \in \mathbb{N}} \subset B_{X'}$ with $t_m \nearrow \infty$ and $\tau_n \nearrow \infty$ be given such that the iterated limits

$$\lim_{m \to \infty} \lim_{n \to \infty} \langle u(t_m + \tau_n), x'_m \rangle = \beta \quad \text{and} \quad \lim_{n \to \infty} \lim_{m \to \infty} \langle u(t_m + \tau_n), x'_m \rangle = \gamma$$

exist. We have to show that $\beta = \gamma$. Since the generalized solution u of (4.1) has relatively compact range, there exists a compact set $K_1 \subset X$ such that

$$\{u_{\tau_n}(t) - u_{\tau_k}(t) \mid n, k \in \mathbb{N},\ t \in \mathbb{R}_+\} \subset K_1.$$

As X' is assumed to be uniformly convex, the duality mapping $\mathbf{F}$ is single-valued and continuous, thus $K := \mathbf{F}(K_1) \subset X'$ is compact. Since $f \in W(\mathbb{R}_+; X)$, we can assume that $f_{\tau_n} \rightharpoonup g$ for some $g \in C_b(\mathbb{R}_+; X)$ as $n \to \infty$. Moreover, by lemma 4.17,

$$\sup_{x' \in K} |\langle f_{\tau_n}(\cdot) - g(\cdot), x' \rangle| \rightharpoonup 0 \qquad \text{in } C_b(\mathbb{R}_+) \text{ as } n \to \infty.$$

By proposition 4.2, we have for all $k, m, n \in \mathbb{N}$

$$\begin{aligned} |\langle u(t_m + \tau_n) - u(t_m + \tau_k), x'_m \rangle|^2 &\le \|u(t_m + \tau_n) - u(t_m + \tau_k)\|^2 \\ &\le 2 \int_{[0,t_m]} \sup_{x' \in K} |\langle f_{\tau_n}(t_m - s) - g(t_m - s), x' \rangle| \, d\alpha(s) \\ &\quad + 2 \int_{[0,t_m]} \sup_{x' \in K} |\langle f_{\tau_k}(t_m - s) - g(t_m - s), x' \rangle| \, d\alpha(s) \\ &\quad + C \, \frac{\alpha((t_m, \infty))}{\alpha(\mathbb{R}_+)}. \end{aligned}$$

Define the bounded linear functional α_m on $C_b(\mathbb{R}_+)$ for all $m \in \mathbb{N}$ by

$$\alpha_m(h) := \int_{[0,t_m]} h(t_m - s) \, d\alpha(s) \qquad \text{for all } h \in C_b(\mathbb{R}_+).$$

Then, since $V := \overline{\operatorname{span}}\{ \sup_{x' \in K} |\langle f_{\tau_n} - g, x' \rangle| \mid n \in \mathbb{N}\}$ is separable, there exists a subsequence of $\{\alpha_m\}_{m \in \mathbb{N}}$, again denoted by $\{\alpha_m\}_{m \in \mathbb{N}}$, such that $\{\alpha_m\big|_V\}_{m \in \mathbb{N}}$ is weak* convergent in V', i.e., $\alpha_m\big|_V \rightharpoonup^* \alpha_\infty$ for some $\alpha_\infty \in V'$. Thus, we conclude

$$\begin{aligned} &\lim_{k \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} |\langle u(t_m + \tau_n) - u(t_m + \tau_k), x'_m \rangle|^2 \\ &\le \lim_{k \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} 2 \langle \alpha_m, \sup_{x' \in K} |\langle f_{\tau_n} - g, x' \rangle| \rangle_{V',V} \\ &\quad + \lim_{k \to \infty} \lim_{m \to \infty} 2 \langle \alpha_m, \sup_{x' \in K} |\langle f_{\tau_k} - g, x' \rangle| \rangle_{V',V} \\ &\quad + \lim_{m \to \infty} C \frac{\alpha((t_m, \infty))}{\alpha(\mathbb{R}_+)} \\ &= \lim_{k \to \infty} 2 \langle \alpha_\infty, \sup_{x' \in K} |\langle f_{\tau_k} - g, x' \rangle| \rangle_{V',V} = 0, \end{aligned}$$

which implies that the double limit condition holds. □

Note that due to [DG61a] and [DG61b] the space of Eberlein-weak almost periodic functions on $\mathbb{R}_+$ can be decomposed by

$$W(\mathbb{R}_+, X) = AP(\mathbb{R}; X)\big|_{\mathbb{R}_+} \oplus W_0(\mathbb{R}_+, X).$$

Assuming that, for $f \in W(\mathbb{R}_+;X)$, the generalized solution $u \in BUC(\mathbb{R}_+;X)$ of (4.1) is Eberlein-weak almost periodic, there exist uniquely determined functions $u^{(\infty)}, f^{(\infty)} \in AP(\mathbb{R};X)$ and $u^{(0)}, f^{(0)} \in W_0(\mathbb{R}_+;X)$ such that

$$f = f^{(\infty)}\big|_{\mathbb{R}_+} + f^{(0)} \quad \text{and} \quad u = u^{(\infty)}\big|_{\mathbb{R}_+} + u^{(0)}.$$

As we have already seen by theorem 4.11, $u^{(\infty)}$ can directly be characterized using the limit equation, whenever f, u are asymptotically almost periodic. The same should hold for Eberlein-weak almost periodic solutions.

Theorem 4.20. *Let A be an m-accretive operator in a Banach space X, let κ, k satisfy* (4.2), (4.3), (4.5), *and let k_1, k_2 be given such that* (4.19) *holds. Moreover, let $u_0 \in \overline{D(A)}$, $f \in WRC(\mathbb{R}_+;X)$, and let $f^{(\infty)} \in AP(\mathbb{R};X)$ be the almost periodic part of f.*
Then the generalized solution u of the initial value problem (4.1) *is Eberlein-weak almost periodic and the almost periodic part $u^{(\infty)} \in AP(\mathbb{R};X)$ of u is a generalized solution of the limit equation* (4.20) *with the right hand side $f^{(\infty)}$.*

The proof of the above result will essentially use the following lemma.

Lemma 4.21. *Let $f, g \in W(\mathbb{R}_+;X)$ with almost periodic parts denoted by $f^{(\infty)}$ and $g^{(\infty)}$, respectively. Then there exists a common sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that*

$$f_{\tau_n} \rightharpoonup f^{(\infty)}\big|_{\mathbb{R}_+} \quad \textit{and} \quad g_{\tau_n} \rightharpoonup g^{(\infty)}\big|_{\mathbb{R}_+}$$

in $C_b(\mathbb{R}_+;X)$ as $n \to \infty$.

For the sake of completeness, we give the proof of the preceding lemma.

Proof. In the fist step, we show that for f, $g \in W_0(\mathbb{R}_+;X)$ there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $(f_{t_k}, g_{t_k}) \rightharpoonup (0,0)$ in $C_b(\mathbb{R}_+;X \times X)$. Since f, $g \in W_0(\mathbb{R}_+;X)$, there exist sequences $\{\tau_m\}_{m\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\sigma_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$f_{\tau_m} \rightharpoonup 0 \qquad \text{and} \qquad g_{\sigma_n} \rightharpoonup 0$$

in $C_b(\mathbb{R}_+;X)$. Note that w-cl$(O(f))$ is a weakly compact subset of a separable subspace of $C_b(\mathbb{R}_+;X)$. Thus, the weak topology on w-cl$(O(f))$ is metrizable, and by a diagonalization argument, we can construct a subsequences of $\{\tau_m\}_{m\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$, again denoted by $\{\tau_m\}_{m\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$, respectively, such that the iterated limits in $C_b(\mathbb{R}_+;X)$

$$w\text{-}\lim_{m\to\infty} w\text{-}\lim_{n\to\infty} f_{\tau_m+\sigma_n} = f_1 \quad \text{and} \quad w\text{-}\lim_{n\to\infty} w\text{-}\lim_{m\to\infty} f_{\tau_m+\sigma_n} = f_2$$

exist. Note that $f_2 = 0$, since $f_{\tau_m} \rightharpoonup 0$ in $C_b(\mathbb{R}_+;X)$. Now, by proposition 4.13, we conclude $f_1 = f_2 = 0$. As the same arguments as above apply to $\{g_{\tau_m+\sigma_n}\}_{m,n\in\mathbb{N}}$, we conclude

$$w\text{-}\lim_{m\to\infty} w\text{-}\lim_{n\to\infty} g_{\tau_m+\sigma_n} = w\text{-}\lim_{n\to\infty} w\text{-}\lim_{m\to\infty} g_{\tau_m+\sigma_n} = 0$$

for subsequences, again denoted by $\{\tau_m\}_{m\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$, respectively. Again, using a diagonalization argument in w-cl$(O(f)) \times w$-cl$(O(g))$ with the weak topology, which is metrizable, there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$(f_{t_k}, g_{t_k}) \rightharpoonup (0,0) \qquad \text{in } C_b(\mathbb{R}_+; X \times X) \text{ as } k \to \infty.$$

In the second step, we consider f, $g \in W(\mathbb{R}_+; X)$. Defining $h : \mathbb{R}_+ \to X \times X$ by $h(t) := (f(t), g(t))$, we have to show that $h \in W(\mathbb{R}_+; X \times X)$. In order to apply proposition 4.13, and using the representation of the dual of $X \times X$, we choose sequences $\{t_m\}_{m\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $\{x'_m\}_{m\in\mathbb{N}} \subset B_{X'}$ and $\{y'_m\}_{m\in\mathbb{N}} \subset B_{X'}$ such that the iterated limits

$$\begin{aligned} \lim_{m\to\infty}\lim_{n\to\infty} \left(\langle f(t_m+\tau_n), x'_m\rangle + \langle g(t_m+\tau_n), y'_m\rangle\right) &= \beta, \\ \lim_{n\to\infty}\lim_{m\to\infty} \left(\langle f(t_m+\tau_n), x'_m\rangle + \langle g(t_m+\tau_n), y'_m\rangle\right) &= \gamma \end{aligned}$$

exist. Again, by a diagonalization argument, we can take subsequences such that

$$\begin{aligned} \lim_{m\to\infty}\lim_{n\to\infty} \langle f(t_m+\tau_n), x'_m\rangle &= \lim_{n\to\infty}\lim_{m\to\infty} \langle f(t_m+\tau_n), x'_m\rangle, \\ \lim_{m\to\infty}\lim_{n\to\infty} \langle g(t_m+\tau_n), y'_m\rangle &= \lim_{n\to\infty}\lim_{m\to\infty} \langle g(t_m+\tau_n), y'_m\rangle \end{aligned}$$

all exist. Here, the equalities follow from proposition 4.13 and the fact that f, $g \in W(\mathbb{R}_+; X)$. Thus, h is Eberlein-weak almost periodic, and using the result of the first step, it is clear that the decomposition of h into an almost periodic part and a $W_0(\mathbb{R}_+; X \times X)$-function is given by

$$h = (f^{(\infty)}, g^{(\infty)}) + (f^{(0)}, g^{(0)}).$$

Here, $f^{(\infty)}, g^{(\infty)} \in AP(\mathbb{R}; X)$ and $f^{(0)}, g^{(0)} \in W_0(\mathbb{R}_+; X)$. Thus, there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$h_{t_k} \rightharpoonup (f^{(\infty)}, g^{(\infty)}) \quad \text{in } C_b(\mathbb{R}_+; X \times X) \text{ as } k \to \infty.$$

□

Using the above result, we can now give the proof of theorem 4.20.

Proof of theorem 4.20. Let u denote the generalized solution of the initial value problem (4.1), and let $u^{(\infty)}$ denote the solution of the limit equation (4.20) for the right hand side $f^{(\infty)}$, where $f^{(\infty)} \in AP(\mathbb{R}; X)$ is the almost periodic part of $f \in WRC(\mathbb{R}_+; X)$. Then, by theorem 4.16, we already know that $u \in WRC(\mathbb{R}_+; X)$, and, by theorem 4.10, it is clear that $u^{(\infty)} \in AP(\mathbb{R}; X)$. Moreover, let $u^{(a)} \in AP(\mathbb{R}; X)$ denote the almost periodic part of u, then we have to show that $u^{(\infty)} = u^{(a)}$. According to lemma 4.21, there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\begin{aligned} u_{\tau_n} &\rightharpoonup u^{(a)}\big|_{\mathbb{R}_+} &\quad \text{and} \quad f_{\tau_n} &\rightharpoonup f^{(\infty)}\big|_{\mathbb{R}_+}, \\ u^{(\infty)}_{\tau_n}\big|_{\mathbb{R}_+} &\rightharpoonup u^{(\infty)}\big|_{\mathbb{R}_+} &\quad \text{and} \quad f^{(\infty)}_{\tau_n}\big|_{\mathbb{R}_+} &\rightharpoonup f^{(\infty)}\big|_{\mathbb{R}_+} \end{aligned}$$

in $C_b(\mathbb{R}_+;X)$ as $n\to\infty$.
Let $t\in\mathbb{R}_+$ be fixed. For any $\varepsilon>0$, we can find $T>0$ such that by proposition 4.2

$$\begin{aligned}\|u_{\tau_n}^{(\infty)}(t)-u_{\tau_n}(t)\| &\le \varepsilon+\|u^{(T)}(t+\tau_n)-u(t+\tau_n)\| \\ &\le \varepsilon+\int_{[0,t]}\|f_{\tau_n}^{(\infty)}(t-s)-f_{\tau_n}(t-s)\|\,d\alpha(s)+\ C\frac{\alpha((t,\infty))}{\alpha(\mathbb{R}_+)}.\end{aligned} \tag{4.24}$$

Here, $u^{(T)}$ denotes the generalized solution of (4.21) with right hand side $f^{(\infty)}$. Due to the relative compactness of the range of f, we can apply lemma 4.15 to obtain $\|f_{\tau_n}^{(\infty)}-f_{\tau_n}\|\rightharpoonup 0$ in $C_b(\mathbb{R}_+)$. For $t\in\mathbb{R}_+$ fixed, α_t defined by

$$\alpha_t(h):=\int_{[0,t]}h(t-s)\,d\alpha(s)\qquad\text{for all } h\in C_b(\mathbb{R}_+)$$

is a bounded linear functional on $C_b(\mathbb{R}_+)$. Thus, noting that for $t\in\mathbb{R}_+$ fixed, the weak convergence implies

$$\big(u_{\tau_n}^{(\infty)}(t)-u_{\tau_n}(t)\big)\rightharpoonup\big(u^{(\infty)}(t)-u^{(a)}(t)\big)\qquad\text{in } X \text{ as } n\to\infty,$$

inequality (4.24) gives

$$\|u^{(\infty)}(t)-u^{(a)}(t)\|\le C\frac{\alpha((t,\infty))}{\alpha(\mathbb{R}_+)}\qquad\text{for all } t\in\mathbb{R}_+.$$

Letting $t\to\infty$ in the above inequality gives $\lim_{t\to\infty}\|u^{(\infty)}(t)-u^{(a)}(t)\|=0$. Hence, $u^{(\infty)}=u^{(a)}$, since both functions are almost periodic. □

If we do not assume that f has relatively compact range, then similarly to theorem 4.19, we have the following result.

Theorem 4.22. *Let A be an m-accretive operator in a Banach space X, with X' uniformly convex, let κ, k satisfy* (4.2), (4.3), (4.5), *and let k_1, k_2 be given such that* (4.19) *holds. Moreover, let $u_0\in\overline{D(A)}$, $f\in W(\mathbb{R}_+;X)$, and let $f^{(\infty)}\in AP(\mathbb{R};X)$ be the almost periodic part of f.*
If the generalized solution u of the initial value problem (4.1) *has relatively compact range, then u is Eberlein-weak almost periodic, and the almost periodic part $u^{(\infty)}\in AP(\mathbb{R};X)$ of u is a generalized solution of the limit equation* (4.20) *with the right hand side $f^{(\infty)}$.*

Proof. Let u denote the generalized solution of the initial value problem (4.1), and let $u^{(\infty)}$ denote the solution of the limit equation (4.20) for the right hand side $f^{(\infty)}$, where $f^{(\infty)}\in AP(\mathbb{R};X)$ is the almost periodic part of $f\in W(\mathbb{R}_+;X)$. Then, by theorem 4.16 and the relative compactness of the range of u, we already know that $u\in W(\mathbb{R}_+;X)$, and, by theorem 4.10, it is clear that $u^{(\infty)}\in AP(\mathbb{R};X)$. Moreover, let $u^{(a)}\in AP(\mathbb{R};X)$ denote the almost periodic part of u, then we have to show that $u^{(\infty)} = u^{(a)}$. According to lemma 4.21, there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\begin{array}{rclcrcl} u_{\tau_n} & \rightharpoonup & u^{(a)}\big|_{\mathbb{R}_+} & \text{and} & f_{\tau_n} & \rightharpoonup & f^{(\infty)}\big|_{\mathbb{R}_+}, \\ u^{(\infty)}_{\tau_n}\big|_{\mathbb{R}_+} & \rightharpoonup & u^{(\infty)}\big|_{\mathbb{R}_+} & \text{and} & f^{(\infty)}_{\tau_n}\big|_{\mathbb{R}_+} & \rightharpoonup & f^{(\infty)}\big|_{\mathbb{R}_+} \end{array}$$

in $C_b(\mathbb{R}_+; X)$ as $n \to \infty$.

Let $u^{(T)}$ denote the generalized solution of (4.21) with right hand side $f^{(\infty)}$ for $T > 0$. As already shown, for fixed $T > 0$, the function $u^{(T)}$ is asymptotically almost periodic, and thus has relatively compact range. Since, by assumption, $u(\mathbb{R}_+)$ is relatively compact in X, and since the duality mapping $\mathbf{F}$ is single valued and continuous, there exists a compact set $K_T \subset X'$ such that

$$\mathbf{F}(u^{(T)}_{\tau_n}(t) - u_{\tau_n}(t)) \subset K_T$$

for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}_+$. Now, let $t \in \mathbb{R}_+$ be fixed. For any $\varepsilon > 0$, we can find $T > 0$ such that by proposition 4.18

$$\begin{aligned} \|u^{(\infty)}_{\tau_n}(t) - u_{\tau_n}(t)\|^2 &\le \varepsilon + \|u^{(T)}(t+\tau_n) - u(t+\tau_n)\|^2 \\ &\le \varepsilon + 2\int_{[0,t]} \sup_{x'\in K_T} |\langle f^{(\infty)}_{\tau_n}(t-s) - f_{\tau_n}(t-s), x'\rangle|\, d\alpha(s) \\ &\quad + C\frac{\alpha((t,\infty))}{\alpha(\mathbb{R}_+)}. \end{aligned} \tag{4.25}$$

Due to the compactness of K_T in X', we can apply lemma 4.17 to obtain

$$\sup_{x'\in K_T} |\langle f^{(\infty)}_{\tau_n} - f_{\tau_n}, x'\rangle| \rightharpoonup 0 \qquad \text{in } C_b(\mathbb{R}_+).$$

For $t \in \mathbb{R}_+$ fixed, α_t defined by

$$\alpha_t(h) := \int_{[0,t]} h(t-s)\, d\alpha(s) \qquad \text{for all } h \in C_b(\mathbb{R}_+)$$

is a bounded linear functional on $C_b(\mathbb{R}_+)$. Thus, noting that for $t \in \mathbb{R}_+$ fixed, the weak convergence in $C_b(\mathbb{R}_+; X)$ implies

$$\big(u^{(\infty)}_{\tau_n}(t) - u_{\tau_n}(t)\big) \rightharpoonup \big(u^{(\infty)}(t) - u^{(a)}(t)\big) \qquad \text{in } X \text{ as } n \to \infty.$$

Therefore, inequality (4.25) gives

$$\|u^{(\infty)}(t) - u^{(a)}(t)\|^2 \le C\frac{\alpha((t,\infty))}{\alpha(\mathbb{R}_+)} \qquad \text{for all } t \in \mathbb{R}_+.$$

Letting $t \to \infty$ in the above inequality yields $\lim_{t\to\infty} \|u^{(\infty)}(t) - u^{(a)}(t)\| = 0$. Hence, $u^{(\infty)} = u^{(a)}$, since both functions are almost periodic. □

As a consequence of the above results, the following corollary is a direct consequence of the ergodic theorem for bounded C_0-semigroups, see [DS57, Theorem 8.7.1 and Corollary 8.7.2], applied to the translation semigroup in $BUC(\mathbb{R}_+; X)$.

Corollary 4.23. *Let A be an m-accretive operator in a Banach space X, let κ, k satisfy (4.2), (4.3), (4.5), and let k_1, k_2 be given such that (4.19) holds. Moreover, let $u_0 \in \overline{D(A)}$, $f \in W(\mathbb{R}_+; X)$, and let $f^{(\infty)} \in AP(\mathbb{R}; X)$ be the almost periodic part of f.*
If

(i) f has relatively compact range, or

(ii) the generalized solution u of the initial value problem (4.1) has relatively compact range, and X' is uniformly convex,

then there exists $x \in X$ such that

$$\lim_{T\to\infty} \sup_{h\geq 0} \left\| \frac{1}{T} \int_h^{T+h} u(\tau)\, d\tau - x \right\|_X = 0,$$

and

$$\lim_{T\to\infty} \sup_{h\geq 0} \left\| \frac{1}{T} \int_h^{T+h} \big(u(\tau) - u^{(\infty)}(\tau)\big)\, d\tau \right\| = 0,$$

where $u^{(\infty)} \in AP(\mathbb{R}; X)$ is the generalized solution of the limit equation (4.20) with right hand side $f^{(\infty)}$.

Thus, under the above assumptions, the generalized solution of (4.1) is uniformly ergodic.

Appendix A

Complete Positivity

In the following we will be concerned with measures on $[0,\infty)$. Therefore, let $C_c([0,\infty))$ denote the space of continuous functions on $[0,\infty)$ having compact support. We assume that $C_c([0,\infty))$ is endowed with the canonical topology, i.e. the inductive limit topology. Then, the dual space $\mathcal{M}([0,\infty)) = (C_c([0,\infty)))'$ of $C_c([0,\infty))$ is the space of Radon measures on $[0,\infty)$. All measures under consideration in the sequel will be assumed to be Radon measures. If we define $C_0([0,\infty))$ to be the space of continuous functions tending to 0 at infinity, then the dual space of $C_0([0,\infty))$ is the subset $\mathcal{M}_b([0,\infty))$ of $\mathcal{M}([0,\infty))$ consisting of bounded Radon measures.
In order to investigate the equation

$$\frac{d}{dt}\left(\kappa(u(t)-u_0)+\int_0^t k(t-s)(u(s)-u_0)\,ds\right)+Au(t)\ni f(t), \qquad t\in[0,T], \tag{A.1}$$

with an accretive operator A in a Banach space X, $u_0 \in \overline{D(A)}$, and $f \in L^1(0,T,X)$, we have to know more about the solution α of the scalar Volterra equation

$$\kappa\alpha([0,t])+\int_0^t k(t-s)\alpha([0,s])\,ds = t, \qquad t>0. \tag{A.2}$$

A Radon measure α on $[0,\infty)$ will be called *resolvent of the first kind* for the pair (κ,k) if it is a solution to (A.2).
We first note that in the case where (A.1) degenerates to the case of the Cauchy-problem, i.e. $\kappa=1$ and $k\equiv 0$, the Lebesgue measure $\alpha:=\lambda$ on $[0,\infty)$ is the unique solution of (A.2). As it is well known that mild solutions u of the Cauchy-problem satisfy the integral inequality

$$\|u(t)-x\| \le \|u_0-x\| + \int_{[0,t]} [u(t-s)-x, f(t-s)-y]_+\,d\lambda(s), \qquad t\in[0,T]$$

for all $(x,y)\in A$, we may assume that under certain assumptions on κ and k the same inequality with λ replaced by the solution α of (A.2) holds for solutions u of (A.1).

For $b \in L^1(0,T)$, we now consider the equation

$$r(t) + \int_0^t r(t-s)b(s)\, ds = b(t), \qquad t \in [0,T]. \tag{A.3}$$

Also, for a Radon measure β on the interval $[0,T)$, we consider the equation

$$\varrho([0,t]) + \int_0^t \varrho([0,t-s])\, d\beta(s)\, ds = \beta([0,t]), \qquad t \in [0,T). \tag{A.4}$$

The solutions r, respectively ϱ, will be called the *resolvent* of b, respectively β. As shown in [GLS90, Theorem 2.3.1 and Theorem 4.1.5], the following proposition holds for $T > 0$.

Proposition A.1. *For $b \in L^1(0,T)$ there exists a unique solution $r \in L^1(0,T)$ of* (A.3)*. Moreover, if β is a bounded Radon measure on $[0,T)$, then there exists a unique bounded Radon measure ϱ on $[0,T)$ such that ϱ solves* (A.4)*.*

Before we proceed finding a solution to (A.2), we need the definition of the following classes of functions (see also [BF75, p. 65]).

Definition A.2. A function $\phi \in C^\infty((0,\infty))$ is called *completely monotone* if $(-1)^n\phi^{(n)} \geq 0$ for all $n \in \mathbb{N}_0$.
A function $\psi \in C^\infty((0,\infty))$ is called a *Bernstein function* if $\psi \geq 0$ and ψ' is completely monotone.

Bernstein's theorem gives a characterization of completely monotone functions as Laplace transforms of nonnegative measures.

Theorem A.3 (Bernstein). *A function $\phi \in C^\infty((0,\infty))$ is completely monotone if and only if there exists a nonnegative Radon measure μ on $[0,\infty)$ such that*

$$\hat{\mu}(z) := \int_{[0,\infty)} e^{-zt}\, d\mu(t) = \phi(z), \qquad z \in (0,\infty). \tag{A.5}$$

The measure μ is uniquely determined by (A.5) *and*

$$\phi^{(n)}(z) = (-1)^n \int_{[0,\infty)} e^{-zt}t^n\, d\mu(t), \qquad z \in (0,\infty). \tag{A.6}$$

For the proof of theorem A.3 see [Wid41, Theorem 4.12b]. Using the above theorem, we can show the existence of solutions of (A.2) for a class of κ, k satisfying

$\kappa \geq 0$, and $k : (0,\infty) \to \mathbb{R}$ is nonnegative and nonincreasing such that $k \in L^1_{\mathrm{loc}}([0,\infty))$ and $\kappa + \int_0^t k(s)\, ds > 0$ for all $t > 0$. (A.7)

Proposition A.4. *Let κ, k satisfy* (A.7)*; then there exists a unique nonnegative Radon measure α on* $[0,\infty)$ *which solves* (A.2)*.*

Proof. By the assumptions on k, we easily see that k is Laplace transformable and that ψ, defined by $\psi(z) := z(\kappa + \hat{k}(z))$ for all $z > 0$, is a Bernstein function. Since $\psi(z) > 0$ for all $z > 0$, the function $1/\psi$ is well defined and completely monotone. Thus, according to theorem A.3, there exists a unique nonnegative Radon measure α on $[0,\infty)$ such that $\hat{\alpha} = 1/\psi$. This implies

$$\kappa\hat{\alpha}(z) + \hat{k}(z)\hat{\alpha}(z) = \frac{1}{z}, \qquad z > 0. \tag{A.8}$$

By the uniqueness of the inverse of the Laplace transformation, we obtain (A.2) and the uniqueness of α. □

As the solutions α of (A.2) exhibit certain properties, we will now introduce a special notion.

Definition A.5. A Radon measure α on $[0,\infty)$ is called *completely positive* if there exists a pair (κ, k) satisfying (A.7) such that α is a resolvent of the first kind of (κ, k), i.e. α solves (A.2).

This notion was first introduced in [CN79] and further discussed in [CN81]. The main intention in [CN79, CN81] was to develop conditions such that the solution of (A.1) stays nonnegative whenever the initial value u_0 and the right hand side f is nonnegative. This investigation obviously has a physical motivation in the theory of heat flow in materials with memory. We remark that the question of positivity of solutions can be reformulated more generally as whether for a closed convex cone $P \subset X$, for $u_0 \in P$, and $f(t) \in P$ almost everywhere the solution u stays in P almost everywhere.
As our main purpose is to investigate (A.1), we are interested in the regularity of the measure α which solves (A.2) in order obtain regularity results for the solution u of (A.1). In particular, we investigate under which conditions on κ and k the measure α has no point masses or is even absolutely continuous with respect to the Lebesgue measure λ.
As we will see, the regularity of α is mainly influenced by the behavior of $\kappa + \int_0^t k(s)\,ds$ as $t \to 0+$. Therefore we distinguish the following three cases

$$\begin{aligned} &\text{(i)} \quad \kappa = 0 \text{ and } k(0+) = \lim_{t\to 0+} k(t) < \infty, \\ &\text{(ii)} \quad \kappa = 0 \text{ and } k(0+) = \infty, \\ &\text{(iii)} \quad \kappa > 0. \end{aligned} \tag{A.9}$$

The following regularity properties of the measure α can also be found in [Prü93, Proposition 1.4.4]. We first consider the case (A.9.i).

Lemma A.6. *Let κ, k satisfy* (A.7)*, and let (A.9.i) hold. Then the resolvent of the first kind α of κ, k satisfies*

$$\alpha(\{0\}) = \frac{1}{k(0+)}, \tag{A.10}$$

and if k is continuous then α has no further point masses.

Proof. Continuing k in 0 by $k(0+)$, we obtain by differentiation of (A.2)

$$\int_{[0,t]} k(t-s)\, d\alpha(s) = 1, \qquad t > 0. \tag{A.11}$$

This implies (A.10) as $t \to 0+$.
Now, assume k is continuous on $(0,\infty)$, and let $t_0 > 0$. Then, for all $0 < t < t_0$, we obtain

$$\begin{aligned} 1 - 1 &= \int_{[0,t_0]} k(t_0 - s)\, d\alpha(s) - \int_{[0,t]} k(t-s)\, d\alpha(s) \\ &\geq -\int_{[0,t]} (k(t-s) - k(t_0 - s))\, d\alpha(s) + k(0+)\alpha(\{t_0\}). \end{aligned}$$

Since k is uniformly continuous, and the measure α is finite on $[0,t_0]$, we obtain as $t \to t_0-$ that $\alpha(\{t_0\}) = 0$. □

In the case (A.9.ii), we can even obtain more regularity of the resolvent of the first kind.

Lemma A.7. *Let κ, k satisfy* (A.7)*, and let (A.9.ii) hold. Then the resolvent of the first kind α of κ, k has no point masses on* $[0,\infty)$*. If additionally k is locally absolutely continuous on* $(0,\infty)$*, then α is absolutely continuous with respect to λ on* $[0,\infty)$*.*

Proof. By differentiation of (A.2) we obtain for $t > t_0 \geq 0$

$$1 = \alpha(\{0\})k(t) + \int_{(0,t]} k(t-s)\, d\alpha(s) \geq k(t - t_0)\alpha(\{t_0\}). \tag{A.12}$$

Let $t \to t_0+$ in this inequality. Since $k(0+) = \infty$, this implies $\alpha(\{t_0\}) = 0$.
Now, assume that k is locally absolutely continuous on $(0,\infty)$. Then k is almost everywhere differentiable and $k' \in L^1_{\mathrm{loc}}((0,\infty))$. Since

$$\int_\varepsilon^T |tk'(t)|\, dt \leq \int_0^T k(t)\, dt$$

holds for all $0 < \varepsilon < T$, the function g given by $g(t) := tk'(t)$ for all $t > 0$ is an element of $L^1_{\mathrm{loc}}([0,\infty))$. And, by the assumptions on k, the function g is Laplace transformable. We now define

$$a(t) := -\frac{1}{t} \int_{[0,t]} \int_{[0,t-s]} (t-s-\tau)k'(t-s-\tau)\, d\alpha(\tau)\, d\alpha(s), \qquad t > 0. \tag{A.13}$$

Then, by some easy calculations, one obtains for $z > 0$

$$\frac{d}{dz}\hat{a}(z) = \hat{\alpha}(z)\hat{\alpha}(z)\hat{g}(z) = \frac{d}{dz}\hat{\alpha}(z).$$

This implies $\hat{a}(z) = c + \hat{\alpha}(z)$ with a constant $c \in \mathbb{R}$. As $\alpha(\{0\}) = 0$ we have $c = 0$ and thus, by the uniqueness of the inverse of the Laplace transform, we can conclude that a is the Radon-Nikodym derivative of α. □

In the case κ, k satisfies (A.7) and (A.9,ii), the resolvent of the first kind α is not absolutely continuous with respect to the Lebesgue measure in general. To see this, set $\kappa = 0$ and define, for $t > 0$,

$$k(t) := 1 + \sum_{n=1}^{\infty} \mathbf{1}_{[0,\infty)}(\frac{1}{2^n} - t),$$

compare [Prü93, page 96].
Using the assumptions (A.9.iii), the measure α has even more regularity as it will always be absolutely continuous with respect to Lebesgue measure.

Lemma A.8. *Let κ, k satisfy* (A.7) *and (A.9,iii). Then the resolvent of the first kind α for κ, k is absolutely continuous with respect to Lebesgue measure on* $[0, \infty)$. *Additionally, the Radon-Nikodym derivative a of α satisfies $a(t) \leq \frac{1}{\kappa}$ for almost all $t \in [0, \infty)$.*

Proof. According to proposition A.1, the equation

$$r(t) + \frac{1}{\kappa}\int_0^t k(t-s)r(s)\,ds = k(t), \qquad t \geq 0$$

admits a unique solution $r \in L^1_{\text{loc}}([0, \infty))$. Define

$$a(t) := \frac{1}{\kappa} - \frac{1}{\kappa}\int_0^t r(s)\,ds, \qquad t \geq 0.$$

Then, by straightforward calculation,

$$\kappa a(t) + \int_0^t k(t-s)a(s)\,ds = 1, \qquad t \geq 0. \tag{A.14}$$

Thus, the measure μ, defined by $\mu(A) := \int_A a(s)\,ds$ for all $A \in \mathcal{B}([0, \infty))$, solves (A.2), and by uniqueness $\mu = \alpha$. Since α is a nonnegative measure, a is nonnegative almost everywhere, and from (A.14) we conclude $a(t) \leq \kappa^{-1}$ for almost all $t \geq 0$. We remark that by the definition of a it is clear that a is differentiable almost everywhere with $a' = \kappa^{-1}r$. □

In all of the above three cases we have the following result on the boundedness of the resolvent of the first kind.

Lemma A.9. *let κ, k satisfy* (A.7)*, and let α be the resolvent of the first kind corresponding to κ, k. Then*

$$\alpha([0,\infty)) < \infty \iff k(\infty) := \lim_{t\to\infty} k(t) > 0.$$

In this case, $\alpha([0,\infty)) = k(\infty)^{-1}$.

Proof. Using an Abelian theorem for the Laplace transform of α and k, we easily see

$$\alpha([0,\infty)) = \lim_{z\to 0} \hat{\alpha}(z) = \lim_{z\to 0} \frac{1}{z(\kappa + \hat{k}(z))} = \frac{1}{k(\infty)}.$$

This holds even in the case where α is not finite, or $k(\infty) = 0$. □

Now we turn to the question of continuous dependence of the resolvent of the first kind on κ, k.

Lemma A.10. *Let $\{(\kappa_n, k_n)\}_{n\in\mathbb{N}}$ be a sequence satisfying* (A.7) *and let κ, k satisfy* (A.7) *as well such that*

$$\kappa_n + \int_0^t k_n(s)\, ds \to \kappa + \int_0^t k(s)\, ds, \qquad \text{for all } t > 0. \tag{A.15}$$

Then the sequence of resolvents of the first kind $\{\alpha_n\}_{n\in\mathbb{N}}$ of κ_n, k_n converges in the sense of distributions to the resolvent α of κ, k.

Proof. As shown in the proof of proposition A.4, the functions $\psi_n(z) := z(\kappa_n + \hat{k}_n(z))$ and $\psi(z) := z(\kappa + \hat{k}(z))$ are Bernstein functions, and by assumption (A.15) we have $\psi_n(z) \to \psi(z)$ pointwise for all $z > 0$. Thus, applying [BF75, Proposition 2.9.5], we obtain the assertion. □

As we only know that the resolvents of the first kind converge in $\mathcal{D}'([0,\infty))$, we will frequently use the following lemma.

Lemma A.11. *Let $\{g_n\}$ be a sequence in $L^1_{\text{loc}}([0,\infty))$ such that $g_n \to g$ in $\mathcal{D}'([0,\infty))$ for a regular distribution $g \in L^1_{\text{loc}}([0,\infty))$. Moreover, let $\{f_n\} \subset L^1_{\text{loc}}([0,\infty))$ be such that $\limsup_{n\to\infty} f_n \le f$ for an $f \in L^1_{\text{loc}}([0,\infty))$ and $f_n \le F$ for all $n \in \mathbb{N}$ for some $F \in L^1_{\text{loc}}([0,\infty))$. Assume that for a sequence of nonnegative Radon measures $\{\alpha_n\}$, which converges in $\mathcal{D}'([0,\infty))$ to a Radon measure α the inequality*

$$g_n(t) \le \int_{[0,t]} f_n(t-s)\, d\alpha_n(s)$$

holds for almost all $t \ge 0$. Then

$$g(t) \le \int_{[0,t]} f(t-s)\, d\alpha(s), \qquad \text{a.e. for } t \in [0,\infty).$$

Proof. We extend the functions g_n, g, f_n, f to all of $\mathbb{R}$ by the value 0 for $t < 0$. We also extend the Radon measures α_n and α to Radon measures on $\mathbb{R}$ by $\alpha_n(A) := \alpha_n(A \cap [0,\infty))$, and $\alpha(A) := \alpha(A \cap [0,\infty))$ respectively, for all measurable subsets $A \subset \mathbb{R}$. For all $\phi \in \mathcal{D}(\mathbb{R})$ with $\phi \geq 0$ we easily see that

$$\int_{\mathbb{R}} \phi(t) \int_{[0,t]} f_n(t-s)\, d\alpha_n(s)\, dt = \int_{\mathbb{R}} f_n(\tau) \int_{\mathbb{R}} \phi(\tau+s)\, d\alpha_n(s)\, d\tau.$$

Since the sequence of measures converges in $\mathcal{D}'(\mathbb{R})$ we know that for all $\tau \in \mathbb{R}$

$$\int_{\mathbb{R}} \phi(\tau+s)\, d\alpha_n(s) \to \int_{\mathbb{R}} \phi(\tau+s)\, d\alpha(s), \qquad \text{as } n \to \infty.$$

Using the fact that $\phi \geq 0$ and that α_n and α are nonnegative measures we obtain

$$\limsup_{n\to\infty} \left(f_n(\tau) \int_{\mathbb{R}} \phi(\tau+s)\, d\alpha_n(s) \right) \leq f(\tau) \int_{\mathbb{R}} \phi(\tau+s)\, d\alpha(s), \text{ a.e. for } \tau \in \mathbb{R}.$$

Since the functions f_n are bounded from above by F we conclude

$$f_n(\tau) \int_{\mathbb{R}} \phi(\tau+s)\, d\alpha_n(s) \leq F(\tau) M.$$

Here, $M > 0$ is chosen such that $\|\phi\|_\infty \sup_{n\in\mathbb{N}} \alpha_n([0,K]) \leq M$, where $K > 0$ is such that $\operatorname{supp}(\phi) \subset (-\infty, K]$. Therefore, we can apply Fatou's lemma and the convergence of g_n in $\mathcal{D}'(\mathbb{R})$ to obtain

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}} g_n(t)\phi(t)\, dt &\leq \limsup_{n\to\infty} \int_{\mathbb{R}} \phi(t) \int_{\mathbb{R}} f_n(t-s)\, d\alpha_n(s)\, dt \\ &\leq \int_{\mathbb{R}} \phi(t) \int_{\mathbb{R}} f(t-s)\, d\alpha(s)\, dt. \end{aligned}$$

Defining $h(t) := \int_{\mathbb{R}} f(t-s)\, d\alpha(s)$ for all $t \in \mathbb{R}$, we have shown that

$$\int_{\mathbb{R}} \phi(t) g(t)\, dt \leq \int_{\mathbb{R}} \phi(t) h(t)\, dt \tag{A.16}$$

holds for all $\phi \in \mathcal{D}(\mathbb{R})$ with $\phi \geq 0$.
We choose $\rho \in \mathcal{D}(\mathbb{R})$ with $\rho \geq 0$ and $\int_{\mathbb{R}} \rho = 1$ and define the sequence of mollifiers $\{\rho_\varepsilon\}_{\varepsilon>0}$ by $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(\varepsilon^{-1}t)$ for all $t \in \mathbb{R}$. Choosing $\phi = \rho_\varepsilon(t - \cdot)$ in (A.16), and defining

$$\begin{aligned} g_\varepsilon(t) &:= \int_{\mathbb{R}} g(s)\rho_\varepsilon(t-s)\, ds, \\ h_\varepsilon(t) &:= \int_{\mathbb{R}} h(s)\rho_\varepsilon(t-s)\, ds \end{aligned}$$

for $t \in \mathbb{R}$, we have $g_\varepsilon \leq h_\varepsilon$ a.e. in $\mathbb{R}$. Since $g_\varepsilon \to g$ and $h_\varepsilon \to h$ in $L^1_{\text{loc}}(\mathbb{R})$, and thus almost everywhere in $\mathbb{R}$ for a subsequence, we conclude $g \leq h$ a.e. in $\mathbb{R}$. □

Finally, we specify some of the properties of the resolvents ϱ_λ for $\lambda > 0$ of a completely positive Radon measure α, i.e. the solutions of the equation

$$\lambda\varrho_\lambda([0,t]) + \int_{[0,t]} \varrho_\lambda([0,t-s])\,d\alpha(s) = \alpha([0,t]) \qquad \text{for } t \geq 0. \tag{A.17}$$

Note that, since α is defined on $[0,\infty)$, the resolvents ϱ_λ are, by proposition A.1, Radon measures on $[0,\infty)$. As shown in [Prü93, Proposition 1.4.5], the following equivalence holds for the resolvents ϱ_λ.

Proposition A.12. *A Radon measure α on $[0,\infty)$ is completely positive if and only if the resolvents $\varrho_\lambda \in \mathcal{M}([0,\infty))$ defined by* (A.17) *are nonnegative and satisfy*

$$\varrho_\lambda([0,\infty)) \leq 1.$$

Moreover, if α is finite, then

$$\varrho_\lambda([0,\infty)) = \frac{\alpha([0,\infty))}{\lambda + \alpha([0,\infty))}.$$

Appendix B

Accretivity

In this appendix we will give the definitions of accretivity, ϕ-accretivity, and complete accretivity. We will state the main facts about accretive operators, on which the study of Volterra equations relies.

In order to define accretivity of a possibly multivalued operator $A : X \to \mathcal{P}(X)$ which is identified with its graph $A = \mathrm{Graph}(A) \subset X \times X$, we will first collect some properties of convex functions on Banach spaces. Most of these properties can be found in [Gil82].

A function $\phi : X \to \mathbb{R}$ defined on a linear space X over $\mathbb{R}$ is *convex* if for all $x, y \in X$ and all $\alpha \in [0, 1]$

$$\phi(\alpha x + (1-\alpha)y) \leq \alpha\phi(x) + (1-\alpha)\phi(y).$$

Recall that a convex function $\phi : X \to \mathbb{R}$ on a Banach space X is continuous if and only if it is *locally Lipschitz continuous*, i.e. for all $a \in X$ there exists $L > 0$ and $\delta > 0$ such that for all $x_1, x_2 \in X$ with $\|x_i - a\| < \delta$ for $i = 1, 2$ we have

$$|\phi(x_1) - \phi(x_2)| \leq L\|x_1 - x_2\|.$$

Let $\phi : X \to \mathbb{R}$ be continuous and convex. Then, for all $\lambda \neq 0$, the mapping $\phi_\lambda : X \times X \to \mathbb{R}$, defined by

$$\phi_\lambda(x, y) := \frac{\phi(x + \lambda y) - \phi(x)}{\lambda},$$

is continuous. For fixed $x, y \in X$, the mapping $\lambda \to \phi_\lambda(x, y)$ is nondecreasing for $\lambda > 0$. Indeed, the function $\theta : \mathbb{R} \to \mathbb{R}$ defined by $\theta(\lambda) := \phi(x + \lambda y) - \phi(x)$ is convex. Thus, for $0 < \lambda \leq \mu$, we have

$$\frac{\theta(\lambda)}{\lambda} \leq \frac{\theta(\mu)}{\mu} + \frac{\mu - \lambda}{\lambda\mu}\theta(0) = \frac{\theta(\mu)}{\mu}.$$

Since $\phi_\lambda(x, y) = -\phi_{-\lambda}(x, -y)$, the mapping $\lambda \to \phi_\lambda(x, y)$ is nondecreasing for $\lambda < 0$. Thus, for every $x, y \in X$, we can define the *right-hand Gateaux derivative of ϕ at x in direction y* by

$$\phi'_+(x, y) := \lim_{\lambda \to 0+} \phi_\lambda(x, y) = \inf_{\lambda > 0} \phi_\lambda(x, y),$$

and the *left-hand Gateaux derivative of ϕ at x in direction y* is defined by

$$\phi'_-(x,y) := \lim_{\lambda\to 0-} \phi_\lambda(x,y) = \sup_{\lambda<0} \phi_\lambda(x,y).$$

In the following proposition, we collect some useful properties of the Gateaux derivative defined above.

Proposition B.1. *Let $\phi : X \to \mathbb{R}$ be a continuous and convex functional on a Banach space X. Then*

(i) $\phi'_+ : X \times X \to \mathbb{R}$ *is upper semicontinuous,*

(ii) for $x \in X$ fixed the mapping $X \ni y \mapsto \phi'_+(x,y)$ is a sublinear and locally Lipschitz continuous functional on X,

(iii) for all $x, y \in X$ we have $\phi'_-(x,y) \le \phi'_+(x,y)$,

(iv) for all $x, y \in X$ and all $\alpha \ge 0$

$$\phi'_+(x, \alpha(y-x)) \le \alpha(\phi(y) - \phi(x)).$$

We now consider three important examples in the context of accretivity.

Example B.2. Note that the norm $\|\cdot\|$ of a Banach space X is a convex and continuous functional on this Banach space. In this case we use the special notation

$$[x,y]_+ = \lim_{\lambda\to 0+} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

for the right-hand Gateaux derivative of the norm of X at x in direction y, and

$$[x,y]_- = \lim_{\lambda\to 0-} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

for the left-hand Gateaux derivative of the norm of X at x in direction y.
We define the duality map $\mathbf{J} : X \to 2^{X'}$ by

$$\mathbf{J}(x) := \left\{ x' \in X' \middle| \ \langle x, x' \rangle = \|x\| \text{ and } \|x'\| \le 1 \right\}.$$

Then, by the Hahn-Banach theorem, it is clear that $\mathbf{J}(x) \ne \emptyset$ for all $x \in X$. We recall that

$$[x,y]_+ = \sup_{x' \in \mathbf{J}(x)} \langle y, x' \rangle.$$

Consider the space $X = L^1(\Omega)$, where $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space. Then

$$\mathbf{J}(u) = \operatorname{sgn} u = \left\{ v \in L^\infty(\Omega) \middle| |v| \le 1 \text{ and } vu = |u| \ \mu - \text{a.e. on } \Omega \right\}.$$

Example B.3. We consider the continuous convex functional given by $\frac{1}{2}\|\cdot\|^2$, where $\|\cdot\|$ denotes the norm of the Banach space X. We denote the right-hand Gateaux derivative of this functional by

$$\langle x, y\rangle_+ := \lim_{\lambda\to 0+} \frac{\|x+\lambda y\|^2 - \|x\|^2}{2\lambda}.$$

Defining the corresponding duality mapping $\mathbf{F} : X \to 2^{X'}$ by

$$\mathbf{F}(x) := \left\{x' \in X' \middle|\ \langle x, x'\rangle = \|x\|^2 = \|x'\|^2\right\},$$

it is clear that

$$\langle x, y\rangle_+ = \sup_{x'\in\mathbf{F}(x)} \langle y, x'\rangle = \|x\|[x, y]_+ \qquad \text{for all } x, y \in X.$$

Moreover, we remark that for $x, y \in X$ the following assertions are equivalent:

(i) $[x, y]_+ \geq 0$,

(ii) $\langle x, y\rangle_+ \geq 0$,

(iii) $\|x\| \leq \|x + \lambda y\| \quad$ for all $\lambda > 0$.

Example B.4. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $M(\Omega)$ be the space of μ-a.e. equivalence classes of measurable functions on Ω with values in $\mathbb{R}$. Defining $r^+ := r \vee 0 = \max\{r, 0\}$ and $r^- := -(r \wedge 0) = -\min\{r, 0\}$, we recall that the integral

$$\int u := \int_\Omega u \, d\mu$$

is well defined on $M^+(\Omega) = \{u^+|\ u \in M(\Omega)\}$ with values in $[0, \infty]$. Apart from the classical Lebesgue spaces, as usually denoted by $L^p(\Omega)$ for $1 \leq p \leq \infty$, we define the following Banach spaces

$$\begin{aligned}
L^{1\cap\infty}(\Omega) &:= L^1(\Omega) \cap L^\infty(\Omega),\\
\text{with } \|u\|_{1\cap\infty} &:= \max\{\|u\|_1, \|u\|_\infty\},\\
L^{1+\infty}(\Omega) &:= L^1(\Omega) + L^\infty(\Omega),\\
\text{with } \|u\|_{1+\infty} &:= \inf\left\{\|u_1\|_1 + \|u_2\|_\infty \middle|\ u = u_1 + u_2,\ u_1, u_2 \in M(\Omega)\right\},\\
L_0(\Omega) &:= \left\{u \in L^{1+\infty}(\Omega)\middle|\ \int (|u| - m)^+ < \infty \text{ for all } m > 0\right\},
\end{aligned}$$

endowed with the norm $\|\cdot\|_{1+\infty}$.

For details on the following properties, we refer to [BC91]. First, the norm $\|\cdot\|_{1+\infty}$ can be written as

$$\|u\|_{1+\infty} = \inf_{m>0}(m + \int (|u| - m)^+), \qquad \text{for all } u \in L^{1+\infty}(\Omega),$$

and a sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^{1+\infty}(\Omega)$ converges to u in $L^{1+\infty}(\Omega)$ if and only if

$$\int (|u_n - u| - m)^+ \to 0 \qquad \text{for all } m > 0.$$

Moreover, if $u_n \to u$ in $L^{1+\infty}(\Omega)$, then there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ μ-a.e. on Ω.
Let $X \subset L_0(\Omega)$ be a Banach space. We define for $m > 0$ the function $\phi_m^+ : X \to \mathbb{R}$ by

$$\phi_m^+(f) := \int (f - m)^+, \qquad \text{for all } f \in X,$$

and $\phi_m^- : X \to \mathbb{R}$ by

$$\phi_m^-(f) := \int (f + m)^- = \int (-f - m)^+, \quad \text{for all } f \in X.$$

Then ϕ_m^+ and ϕ_m^- are continuous and convex functionals on X, and it is well known that their Gateaux derivatives are given by

$$\begin{aligned}
\left(\phi_m^+\right)'_+(f,g) &= \int_{\{f>m\}} g + \int_{\{f=m\}} g^+, \\
\left(\phi_m^+\right)'_-(f,g) &= \int_{\{f>m\}} g - \int_{\{f=m\}} g^-, \\
\left(\phi_m^-\right)'_+(f,g) &= -\int_{\{f<-m\}} g + \int_{\{f=-m\}} g^-, \\
\left(\phi_m^-\right)'_-(f,g) &= -\int_{\{f<-m\}} g - \int_{\{f=-m\}} g^+.
\end{aligned}$$

We have collected all results in order to give the definition of accretivity, which is adapted form [CP78]).

Definition B.5. Let X be a Banach space with norm $\|\cdot\|$, and let $\phi : X \to \mathbb{R}$ be a continuous and convex functional on X. Then an operator $A \subset X \times X$ is called ϕ-accretive in X if for all $(x,y), (\tilde{x}, \tilde{y}) \in A$

$$\phi'_+(x - \tilde{x}\ ,\ y - \tilde{y}) \geq 0.$$

The operator A is called accretive if it is $\|\cdot\|$-accretive. And if $X \subset L_0(\Omega)$, then the operator A is called completely accretive in X if A is ϕ-accretive for all $\phi = \phi_m^+$, and $\phi = \phi_m^-$ with $m > 0$.
Moreover, we call A m-accretive, respectively m-completely accretive, if $R(I + \lambda A) = X$ for all $\lambda > 0$ and A is accretive, respectively completely accretive.

We remark that an operator $A \subset X \times X$ is accretive if and only if the resolvent J_λ^A of A defined by$J_\lambda^A = (I + \lambda A)^{-1}$ is a single-valued nonexpansive mapping defined on $R(I + \lambda A)$. In order to give a characterization of completely accretive operators in $X \subset L_0(\Omega)$, we define

$$\mathbf{J}_0 = \left\{ j : \mathbb{R} \to [0, \infty] \;\middle|\; j \text{ is proper convex lower-semicontinuous, with } j(0) = 0 \right\}.$$

Then, for all $u, v \in M(\Omega)$, the following relation is well defined:

$$u \ll v \quad \text{if and only if} \quad \int j(u) \leq \int j(v) \quad \text{for all } j \in \mathbf{J}_0.$$

We remark that by [BC91, Lemma 1.3] $u \ll v$ holds for $u, v \in L_0(\Omega)$ if and only if

$$\int (u - m)^+ \leq \int (v - m)^+ \qquad \text{and} \qquad \int (u + m)^- \leq \int (v + m)^- \qquad \text{for all } m > 0.$$

Thus, an operator $A \subset X \times X$ with $X \subset L_0(\Omega)$ is completely accretive if and only if

$$u - \tilde{u} \ll u - \tilde{u} + \lambda(v - \tilde{v}) \qquad \text{for all } \lambda > 0 \text{ and all } (u, v), (\tilde{u}, \tilde{v}) \in A.$$

In some cases we will restrict our attention to the following special class of Banach spaces $X \subset L_0(\Omega)$.

Definition B.6. A Banach space $\{0\} \neq X \subset L_0(\Omega)$ is called normal if

$$u \in M(\Omega),\ v \in X,\ u \ll v \implies u \in X,\ \|u\|_X \leq \|v\|_X.$$

We refer to [BC91] for an overview on m-completely accretive operators in normal Banach spaces and to [BCP] for the concept of mild solutions of abstract Cauchy problems

$$\begin{aligned} u' + Au &\ni f \\ u(0) &= u_0 \end{aligned}$$

governed by an accretive operator A.
Moreover, we will need the following definition of a minimal section.

Definition B.7. Let C be a subset of $L_0(\Omega)$, then

$$C^\circ = \{u \in C \mid u \ll v \text{ for all } v \in C\}.$$

Moreover, let A be an operator in $L_0(\Omega)$. Then A° is the restriction of A defined by

$$A^\circ u = (Au)^\circ \qquad \text{for all } u \in D(A).$$

Remark that if $C \subset L_0(\Omega)$ is convex, then C° consists of at most one element. And that if A is an m-completely accretive operator in $X \subset L_0(\Omega)$, then A° is single-valued with $D(A^\circ) = D(A)$.

Bibliography

[AB92] W. Arendt and Ph. Bénilan. Inegalités de Kato et Semi-Groupes Sous-Markoviens. *Rev. Mat. Univ. Complutense Madr.*, 5(2, 3):279–308, 1992.

[AL83] H. W. Alt and S. Luckhaus. Quasi-linear elliptic-parabolic differential equations. *Math. Z.*, 183:311–341, 1983.

[AMSdLT97] F. Andreu, J.M. Mazón, S. Segura de León, and J. Toledo. Quasi-linear elliptic and parabolic equations in L^1 with non-linear boundary conditions. *Adv. Math. Sci. Appl.*, 7:183–213, 1997.

[AMSdLT99] F. Andreu, J.M. Mazón, S. Segura de León, and J. Toledo. Existence and uniqueness for a degenerate parabolic equation with L^1-data. *Trans. Am. Math. Soc.*, 351(1):285–306, 1999.

[AP71] L. Amerio and G. Prouse. *Almost periodic functions and functional equations.* Van Nostrand Reinhold Company, New York, 1971.

[BBG+95] Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vazquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Sc. Norm. Super. Pisa*, 22(2):241–273, 1995.

[BC91] Ph. Bénilan and M. G. Crandall. Completely accretive operators. In *Semigroup theory and evolution equations*, volume 135 of *Lect. Notes Pure Appl. Math., Marcel-Dekker*, pages 41–75, 1991.

[BCP] Ph. Bénilan, M. G. Crandall, and A. Pazy. Nonlinear evolution equations in Banach spaces. Book in preparation.

[BDG97] L. Boccardo, A. Dall'Aglio, and T. Gallouët. Nonlinear parabolic equations with measure data. *J. Funct. Anal.*, 147:237–258, 1997.

[Bén72] Ph. Bénilan. *Equation d'évolution dans un espace de Banach quelconque et applications.* Thèse d'Etat, Orsay, 1972.

[BF75] C. Berg and G. Forst. *Potential Theory on Locally Compact Abelian Groups.* Springer-Verlag, New York, 1975.

[BG89] L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87:149–169, 1989.

[BGDM93] L. Boccardo, D. Giachetti, J. I. Diaz, and F. Murat. Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms. *J. Differential Equations*, 106:215–237, 1993.

[BH88] D. Brandon and W. J. Hrusa. Construction of a class of integral models for heat flow in materials with memory. *J. Integral Equations Appl.*, 1(2):175–201, 1988.

[Bla93] D. Blanchard. Truncations and monotonicity methods for parabolic equations. *Nonlinear Anal.*, 21:725–743, 1993.

[BM97] D. Blanchard and F. Murat. Renormalized solutions of nonlinear parabolic problems with L^1-data: existence and uniqueness. *Proc. R. Soc. Edinb.*, 127A:1137–1152, 1997.

[BMR99] D. Blanchard, F. Murat, and H. Redwane. Existence et unicité de la solution renormalisée d'un problème parabolique non linéaire assez général. *C. R. Acad. Sci. Paris*, 329:575–580, 1999.

[BP84] P. Baras and M. Pierre. Problèmes paraboliques semilinéaires avec données mesures. *Appl. Ana.*, 18:111–149, 1984.

[BR98] D. Blanchard and H. Redwane. Renormalized solutions for a class of nonlinear evolution problems. *J. Math. Pures Appl.*, 77:117–151, 1998.

[BS88] C. Bennett and R. Sharpley. *Interpolation of operators.* Academic Press, Boston, 1988.

[BW96] Ph. Bénilan and P. Wittbold. On mild and weak solutions of elliptic-parabolic problems. *Adv. Differ. Equ.*, 1(6):1053–1073, 1996.

[Cap99] M. Caputo. Diffusion of fluids in porous media with memory. *Geothermics*, 28:113–130, 1999.

[Cap00] M. Caputo. Models of flux in porous media with memory. *Water Resources Res.*, 36(3):693–705, 2000.

[Cat48] C. Cattaneo. Sulla conduzione del calore. *Atti Sem. Math. Fis. Univ. Modena*, 3:83–101, 1948.

[CGL96] B. Cockburn, G. Gripenberg, and S.-O. Londen. On convergence to entropy solutions of a single conservation law. *J. Differ. Equations*, 128(1):206–251, 1996.

[Clé80] Ph. Clément. On abstract Volterra equations with kernels having a positive resolvent. *Israel J. Math.*, 36(3-4):193–200, 1980.

[CM88] Ph. Clément and E. Mitidieri. Qualitative properties of solutions of Volterra equations in Banach spaces. *Israel J. Math.*, 64(1):1–24, 1988.

[CN78] M. G. Crandall and J. A. Nohel. An abstract functional differential equation and a related nonlinear Volterra equation. *Israel J. Math.*, 29:313–328, 1978.

[CN79] Ph. Clément and J. A. Nohel. Abstract linear and nonlinear Volterra equations preserving positivity. *SIAM J. Math. Anal.*, 10:365–388, 1979.

[CN81] Ph. Clément and J. A. Nohel. Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels. *SIAM J. Math. Anal.*, 12:514–535, 1981.

[CP78] B. Calvert and C. Picard. Opérateurs accrétifs et ϕ-accrétifs dans un espace de Banach. *Hiroshima Math. J.*, 8:11–30, 1978.

[CP90] Ph. Clément and J. Prüss. Completely positive measures and Feller semigroups. *Math. Ann.*, 287:73–105, 1990.

[CW99] J. Carrillo and P. Wittbold. Uniqueness of Renormalized Solutions of Degenerate Elliptic-Parabolic Problems. *J. Differ. Equations*, 156:93–121, 1999.

[DG61a] K. DeLeeuw and I. Glicksberg. Almost periodic functions on semigroups. *Acta Math.*, 105:99–140, 1961.

[DG61b] K. DeLeeuw and I. Glicksberg. Applications of almost periodic compactifications. *Acta Math.*, 105:63–97, 1961.

[DL84] R. Dautray and J.-L. Lions. *Analyse mathématique et calcul numérique pour les sciences et les techniques.* Masson, Paris, 1984.

[DL89] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. Math.*, 130(2):321–366, 1989.

[DO92] A. Dall'Aglio and L. Orsina. Existence results for some nonlinear parabolic equations with nonregular data. *Differ. Integral Equations*, 5(6):1335–1354, 1992.

[DO96] A. Dall'Aglio and L. Orsina. Nonlinear parabolic equations with natural growth conditions and L^1-data. *Nonlinear Anal.*, 27(1):59–73, 1996.

[DS57] N. Dunford and J. Schwartz. *Linear operators.* John Wiley & Sons, New York, 1957.

[DUj77] J. Diestel and J.J. Uhl jun. *Vector measures.* American Mathematical Society, Providence, R.I., 1977.

[Ebe49] W. F. Eberlein. Abstract Ergodic Theorems and Weak Almost Periodic Functions. *Trans. Am. Math. Soc.*, 67:217–240, 1949.

[Egb92] P. Egberts. *On the sum of accretive operators.* Thesis, Technische Universiteit Delft, 1992.

[Fed69] H. Federer. *Geometric measure theory.* Springer-Verlag, Berlin, 1969.

[Fré41] M. Fréchet. Les fonctions asymptotiquement presque-périodiques. *Rev. Sci.*, 79:341–354, 1941.

[Gil82] J. R. Giles. *Convex analysis with applications in differentiation of convex functions.*, volume 58 of *Research Notes in Math.* Pitman, Boston, 1982.

[GLS90] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations.* Cambridge University Press, Cambridge, 1990.

[GP68] M. E. Gurtin and A. C. Pipkin. A general theory of heat conduction with finite wave speeds. *Arch. Rational Mech. Anal.*, 31:113–126, 1968.

[Gri80] G. Gripenberg. On Volterra equations of the first kind. *Integral Equations Oper. Theory*, 3:473–488, 1980.

[Gri85] G. Gripenberg. Volterra integro-differential equations with accretive nonlinearity. *J. Differ. Equations*, 60:57–79, 1985.

[Gro52] A. Grothendieck. Critères de compactité dans les espaces fonctionelles généraux. *Amer. J. Math.*, 74:168–186, 1952.

[KKM85] N. Kato, K. Kobayasi, and I. Miyadera. On the asymptotic behavior of solutions of evolution equations associated with nonlinear Volterra equations. *Nonlinear Anal.*, 9(5):419–430, 1985.

[KR61] M. A. Krasnosel'skij and Ya. B. Rutickij. *Convex functions and Orlicz spaces.* P. Noordhoff, Groningen, 1961.

[Kre] J. Kreulich. Bounded solutions of nonlinear Cauchy problems. to appear.

[Kre92] J. Kreulich. *Eberlein weak almost periodicity and differential equations in Banach spaces.* Dissertation, Universität Essen, 1992.

[Kre96] J. Kreulich. Eberlein-weakly almost-periodic solutions of evolution equations in Banach spaces. *Differ. Integral Equ.*, 9(5):1005–1027, 1996.

[Kru70] S. N. Kruzhkov. First-order quasilinear equations in several independent variables. *Mat. USSR-Sbornik*, 10:217–243, 1970.

[Lan81] R. Landes. On the existence of weak solutions for quasilinear parabolic initial boundary value problems. *Proc. R. Soc. Edinb.*, 89:217–237, 1981.

[Mac77] R. C. MacCamy. An integro-differential equation with application in heat flow. *Q. Appl. Math.*, 35:1–19, 1977.

[Mai96] F. Mainardi. Fractional Relaxation-Oscillation and Fractional Diffusion-Wave Phenomena. *Chaos, Solitons and Fractals*, 7(9):1461–1477, 1996.

[Mil80] P. Milnes. On vector-valued weakly almost periodic functions. *J. Lond. Math. Soc.*, 22(2):467–472, 1980.

[Mur93] F. Murat. Soluciones renormalizadas de EDP elípticas no lineales. Technical Report R 93023, Publ. Laboratoire d'Analyse Numérique, Univ. Paris 6, 1993.

[Nun71] J. W. Nunziato. On heat conduction in materials with memory. *Q. Appl. Math.*, 29:187–204, 1971.

[Ott96] F. Otto. L^1-Contraction and Uniqueness for Quasilinear Elliptic-Parabolic Equations. *J. Differ. Equations*, 131:20–38, 1996.

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, New York, 1983.

[PR93] J. Prüss and W. M. Ruess. Weak almost periodicity of convolutions. *J. Integral Equations Appl.*, 5(4):519–530, 1993.

[Pri95] A. Prignet. Remarks on the existence and uniqueness of solutions of elliptic problems with right-hand side measures. *Rend. Mat. Appl.*, 15(3):321–337, 1995.

[Pri97] A. Prignet. Existence and uniqueness of "entropy" solutions of parabolic problems with L^1 data. *Nonlinear Anal.*, 28(12):1943–1954, 1997.

[Prü93] J. Prüss. *Evolutionary Integral Equations and Applications.* Birkhaeuser Verlag, Basel, 1993.

[Rak93a] J. M. Rakotoson. Generalized solutions in a new type of sets for problems with measures as data. *Diff. Int. Equations*, 6:27–36, 1993.

[Rak93b] J. M. Rakotoson. Resolution of the critical cases for problems with L^1-data. *Asymptotic Analysis*, 6:229–246, 1993.

[Rak94] J. M. Rakotoson. Uniqueness of renormalized solutions in a T-set for the L^1-data problem and the link between various formulations. *Indiana Univ. Math. J.*, 43(2):285–293, 1994.

[RS88] W. M. Ruess and W. H. Summers. Compactness in spaces of vector valued continuous functions and asymptotic almost periodicity. *Math. Nachr.*, 135:7–33, 1988.

[RS89] W. M. Ruess and W. H. Summers. Integration of asymptotically almost periodic functions and weak asymptotic almost periodicity. *Diss. Math.*, 279:35 p., 1989.

[RS90] W. M. Ruess and W. H. Summers. Weakly almost periodic semigroups of operators. *Pacific J. Math*, 143(1):175–193, 1990.

[RS92] W. M. Ruess and W. H. Summers. Ergodic theorems for semigroups of operators. *Proc. Am. Math. Soc.*, 114(2):423–432, 1992.

[Sei87] G. Seifert. Nonlinear evolution equations with almost periodic time dependence. *SIAM J. Math. Anal.*, 18:387–392, 1987.

[Ser64] J. Serrin. Pathological solutions of elliptic differential equations. *Ann. Scuola Norm. Sup. Pisa*, 18:385–387, 1964.

[Wid41] D. V. Widder. *The Laplace Transform.* Princeton University Press, Princeton, N. J., 1941.

[Wit92] P. Wittbold. *Nonlinear evolution equations and completely accretive operators.* Diplomarbeit, Universität Essen, 1992.

[Wit94] P. Wittbold. *Absorption non linéaire.* Thèse de doctorat, Université de Franche-Comté, Besançon, 1994.

[Zyg59] A. Zygmund. *Trigonometric Series II.* Cambridge University Press, Cambridge, 1959.

Erklärung

Hiermit erkläre ich, dass ich die Arbeit selbständig und nur mit den angegebenen Hilfsmitteln angefertigt habe. Alle Stellen, die anderen Werken entnommen sind, wurden durch Angabe der Quellen kenntlich gemacht.

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