Vector bundles of degree zero over an elliptic curve, flat bundles and Higgs bundles over a compact Kähler manifold

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Contents

Part 1

Vector bundles of degree zero over an elliptic curve

1. Introduction and Notations

Let X be a complete, connected, reduced scheme over a perfect field k. We define $\mathrm{Vect}(X)$ to be the set of isomorphism classes $[V]$ of vector bundles V over X. We can define an addition and a multiplication on $\mathrm{Vect}(X)$:

$$[V] + [V'] = [V \oplus V']$$
$$[V] \cdot [V'] = [V \otimes V'].$$

We define the ring $K(X)$ to be the Grothendieck group associated to the additive monoid $\mathrm{Vect}(X)$, endowed with the multiplication induced by the tensor product of vector bundles, i.e.

$$K(X) = \frac{\mathbb{Z}[\mathrm{Vect}(X)]}{H},$$

where H is the subgroup of $\mathbb{Z}[\mathrm{Vect}(X)]$ generated by all elements of the form $[V \oplus V'] - [V] - [V']$.
The indecomposable vector bundles over X form a free basis of $K(X)$. Since $\mathrm{H}^0(X, \mathrm{End}(V))$ is finite dimensional, the Krull-Schmidt theorem ([**2**]) holds on X. This means that a decomposition of a vector bundle into indecomposable components is unique up to isomorphism.

We want to generalize a theorem by M. Nori on finite vector bundles. A vector bundle V over X is called finite, if the collection $S(V)$ of all indecomposable components of $V^{\otimes n}$ for all integers $n \in \mathbb{Z}$ is finite, where $V^{\otimes n} := (V^{\vee})^{\otimes(-n)}$ for $n < 0$.
In the following, we denote by R(V) the $\mathbb{Q}$-subalgebra of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the set $S(V)$. If V is a finite vector bundle, the $\mathbb{Q}$-algebra $R(V)$ is of Krull dimension zero, since a finite vector bundle is integral over $\mathbb{Q}$ (see [**11**], Lemma (3.1)).

In [**11**], Nori proves the following theorem:
For every finite vector bundle V over X there exists a finite group scheme G_V and a principal G_V-bundle $\pi : P \to X$, such that $\pi^* V$ is trivial over P. In particular, the equality

$$\dim R(V) = \dim G_V \; (= 0)$$

holds.
The group scheme G_V is the group scheme associated to a Tannakian category $\mathcal{C}_V$, generated by V as subcategory of $SS(X)$, where $SS(X)$ denotes the full subcategory of the category of quasi-coherent sheaves on X, whose objects are the vector bundles that are semistable of degree zero restricted to every curve in X.
As every (arbitrary) vector bundle V over X of rank r trivializes over its associated principal $\mathrm{GL}(r)$-bundle, we can look for a group scheme

G of smallest dimension and a principal G-bundle over which the pullback of the vector bundle V is trivial. We might also compare the dimension of the group scheme to $\dim R(V)$.
If V is an object of $SS(X)$, we can also ask if it generates a Tannakian category in the same manner as described by Nori for finite bundles.
In Part 1 we consider the family of vector bundles of degree zero over an elliptic curve. In Section 2 we will prove that they trivialize over a principal G-bundle with G a group scheme of smallest possible dimension. As in the situation of Nori's theorem, this dimension turns out to be equal to the dimension of the ring $R(V)$.
In Section 3 we prove that all indecomposable vector bundles of degree zero over an elliptic curve are semistable and hence objects of $SS(X)$. We use this in Section 4 to show that the indecomposable bundles of degree zero generate Tannakian categories and that the associated group schemes are those found in Section 2. In Section 5 we construct a stable vector bundle E of degree zero over a curve of genus 2, whose ring $R(E)$ is of smaller dimension than the group scheme associated to its Tannakian category. This shows that the dimension relation found for finite bundles and the bundles treated in Section 2 is not true in general.

2. Dimension relation for vector bundles of degree zero over an elliptic curve

Let X be an elliptic curve over an algebraically closed field k of characteristic zero. We consider vector bundles of degree zero over X which can be classified according to Atiyah (see [**1**]). By $\mathcal{E}(r,0)$ we denote the set of indecomposable vector bundles of rank r and degree zero over X.

Theorem 2.1. *(Atiyah [**1**])*

1. *There exists a vector bundle $F_r \in \mathcal{E}(r,0)$, unique up to isomorphism, with $\Gamma(X,F_r) \neq 0$.*
   Moreover we have an exact sequence
   $$0 \rightarrow \mathcal{O}_X \rightarrow F_r \rightarrow F_{r-1} \rightarrow 0,$$
   and $\Gamma(X,F_r) = k$.
2. *Let $E \in \mathcal{E}(r,0)$, then $E \cong L \otimes F_r$, where L is a line bundle of degree zero, unique up to isomorphism, (and one has that $L^r \cong \det E$.)*

Lemma 2.2. *(Atiyah [**1**])*
The vector bundles F_r, $r \in \mathbb{N}$, are selfdual and fulfill the formulas

1. $F_r \otimes F_s = F_{r-s+1} \oplus F_{r-s+3} \oplus \cdots \oplus F_{(r-s)+(2s-1)}$ *for* $2 \leq s \leq r$,

(2) $F_r = S^{r-1}(F_2)$ *for* $r \geq 1$.

DEFINITION 2.3. *For any vector bundle* V *over* X, *let* $S(V)$ *be the collection of all indecomposable components of* $V^{\otimes n}$ *for all* $n \in \mathbb{Z}$, *where* $V^{\otimes n} := (V^{\vee})^{\otimes(-n)}$ *for* $n < 0$.

PROPOSITION 2.4. *The* $\mathbb{Q}$*-subalgebra* $R(F_r)$ *of* $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ *generated by* $S(F_r)$ *is* $\mathbb{Q}[x]$, *with* $x = [F_2]$, *if* r *is even, and* $x = [F_3]$, *if* r *is odd. In particular,* $R(F_r)$ *is of Krull dimension one.*

PROOF. Since all bundles F_r are selfdual, we only need to consider positive tensor powers.
For even r, the multiplication formula from the previous lemma implies by induction that there exist integers $a_i(n)$ such that

$$F_r^{\otimes n} = a_2(n)F_2 \oplus a_4(n)F_4 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}$$

for odd $n \geq 3$, and

$$F_r^{\otimes n} = a_1(n)\mathcal{O}_X \oplus a_3(n)F_3 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}$$

for even $n \geq 2$.
Therefore we obtain

$$S(F_r) = \{F_i \,|\, i = 1, 2, 3, \dots\}\,, \text{ if } r \text{ even },$$

and $S(F_r)$ generates the subring $\mathbb{Q}[F_2]$ of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, because inductively we can write every vector bundle F_i as $p(F_2)$ for some polynomial $p \in \mathbb{Z}[x]$.

For odd r, Atiyah's multiplication formula gives

$$F_r^{\otimes n} = a_1(n)\mathcal{O}_X \oplus a_3(n)F_3 \oplus \cdots \oplus a_{(r-1)n-1}(n)F_{(r-1)n-1} \oplus F_{(r-1)n+1}$$

for all $n \geq 2$. It follows that

$$S(F_r) = \{F_i \,|\, i \text{ odd }\}\,, \text{ if } r \text{ odd }.$$

For odd r, the set $S(F_r)$ generates the ring $R(F_r) = \mathbb{Q}[F_3]$, as for odd i each F_i is $p(F_3)$ for a polynomial $p \in \mathbb{Z}[x]$. □

Before we come to the next proposition we shortly recall the definition of principal G-bundles (see [**11**]).

DEFINITION 2.5. *Let* G *be an affine group scheme defined over* k. *A scheme* P, *together with a morphism* $p : P \to X$, *is called a principal* G*-bundle, if the following conditions are fulfilled:*

(1) p *is a surjective flat affine morphism,*
(2) *there is a morphism*

$$\Phi : P \times G \to P$$

defining an action of G *on* P *such that* $p \circ \Phi = p \circ p_1$, *where* p_1 *denotes the first projection,*

(3) $\Psi : P\times G \to P\times_X P$ *defined by* $\Psi = (p_1, \Phi)$ *is an isomorphism.*

REMARK 2.6. Let $E \to X$ be a vector bundle of rank r over X. The bundle E defines a principal $\mathrm{GL}(r)$-bundle $p : P \to X$ in the following way: Let $\mathcal{U} = \{U_\alpha\}$ be an open affine covering of X and let $g_{\alpha\beta}$ be the transition functions of E subordinate to this covering. Since X is separated, also the intersections $U_{\alpha\beta}$ are affine. Locally P is defined to be $U_\alpha \times \mathrm{GL}(r)$. These schemes are glued in the following manner: Let $B_{\alpha\beta}$ be a k-algebra such that $U_{\alpha\beta} = \mathrm{Spec}\ B_{\alpha\beta}$. Then we write

$$U_{\alpha\beta} \times \mathrm{GL}(r) = \mathrm{Spec}\frac{B_{\alpha\beta}[X^\alpha_{11}, \ldots, X^\alpha_{rr}, X^\alpha]}{\langle \det(X^\alpha_{ij}) \cdot X^\alpha - 1\rangle}$$

as a subscheme of $U_\alpha \times \mathrm{GL}(r)$, and

$$U_{\alpha\beta} \times \mathrm{GL}(r) = \mathrm{Spec}\frac{B_{\alpha\beta}[X^\beta_{11}, \ldots, X^\beta_{rr}, X^\beta]}{\langle \det(X^\beta_{ij}) \cdot X^\beta - 1\rangle}$$

as a subscheme of $U_\beta \times \mathrm{GL}(r)$. The glueing morphism $\phi_{\alpha\beta} : U_{\alpha\beta} \times \mathbb{G}_a \to U_{\alpha\beta} \times \mathbb{G}_a$ is the morphism corresponding to the k-algebra morphism

$$\phi^*_{\alpha\beta} : \frac{B_{\alpha\beta}[X^\alpha_{11}, \ldots, X^\alpha_{rr}, X^\alpha]}{\langle \det(X^\alpha_{ij}) \cdot X^\alpha - 1\rangle} \to \frac{B_{\alpha\beta}[X^\beta_{11}, \ldots, X^\beta_{rr}, X^\beta]}{\langle \det(X^\beta_{ij}) \cdot X^\beta - 1\rangle}$$

sending (X^α_{ij}) to $g_{\alpha\beta} \cdot (X^\beta_{ij})$, and X^α to $(\det g_{\alpha\beta})^{-1} \cdot X^\beta$. This is a well-defined algebra morphism: if a point $(P_{11}, \ldots, P_{rr}, P)$ is a zero of the polynomial $\det(X^\alpha_{ij}) \cdot X^\alpha - 1$, then also $\det(g_{\alpha\beta} \cdot (P_{ij})) \cdot \det(g^{-1}_{\alpha\beta}) \cdot P - 1$ equals zero.
The map $p : P \to X$ is locally defined to be the projection.
The pullback of E to its associated principal $\mathrm{GL}(r)$-bundle is trivial, since the transitions functions of E over P are coboundaries $g_{\alpha\beta} = (X^\alpha_{ij}) \cdot (X^\beta_{ij})^{-1}$ with $(X^\alpha_{ij}) \in \mathcal{O}_P(U_\alpha \times \mathrm{GL}(r))$ and $(X^\beta_{ij}) \in \mathcal{O}_P(U_\beta \times \mathrm{GL}(r))$. □

PROPOSITION 2.7. *There exists a principal* $\mathbb{G}_a$*-bundle* $\pi : P \to X$ *such that* $\pi^*(F_r)$ *is trivial for all* $r \geq 2$.
There is no finite group scheme G*, such that* $\pi^*(F_r)$ *trivializes over a principal* G*-bundle.*

Remark: As in the case of finite bundles we have a correspondence of dimensions

$$dim\ R(F_r) = dim\ \mathbb{G}_a\,(= 1).$$

PROOF.
By definition F_2, is an element of $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{H}^1(X, \mathcal{O}_X)$, which can be embedded into the pointed set $\mathrm{H}^1(X, \mathrm{GL}(2, \mathcal{O}_X))$. Embedding

$\mathbb{G}_a$ into $\mathrm{GL}(2,\mathbb{C})$ via $u \to \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, we can therefore define a principal $\mathbb{G}_a$-bundle in the same manner as described in the previous remark. The pullback of F_2 to this bundle is trivial.
As $F_r = S^{r-1}F_2\,, r \geq 3$, each F_r trivializes on the same principal $\mathbb{G}_a$-bundle as F_2.
Assume that there exists a finite principal G-bundle $p : Y \to X$, where F_2 is trivial. Let $u_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X)$ be a cocycle representing F_2. Then the cocycle $p^*(u_{ij})$ represents $p^*(F_2)$, and by assumption there are $s_i \in \Gamma(U_i, \mathcal{O}_Y)$ such that $p^*(u_{ij}) = s_j - s_i$. By projection formula we know, that $p_*(p^*(u_{ij})) = \deg(p) \cdot u_{ij}$, hence $u_{ij} = \deg(p)^{-1}(p_*(s_j) - p_*(s_i))$ is a coboundary, which is impossible since F_2 is non-trivial over X.
There cannot be any other F_r, $r \geq 3$, which trivializes over a finite G-bundle: Let $r \geq 3$ be the smallest r, such that F_r trivializes over a finite G-bundle $p : Y \to X$. Then there is an exact sequence

$$0 \to \mathcal{O}_Y \to \oplus^r \mathcal{O}_Y \to p^*(F_{r-1}) \to 0,$$

which implies that $p^*(F_{r-1})$ is trivial, since after a constant change of frame for $\oplus^r \mathcal{O}_Y$ we can assume that the injection maps $\mathcal{O}_Y$ identically onto the first summand and is zero on all the other summands. But $p^*(F_{r-1})$ is non-trivial by assumption, which is a contradiction. □

Remark. It follows from the previous propositions that for all $r \geq 2$ the algebra $R(F_r)$ is not only of the same dimension as the corresponding group scheme $\mathbb{G}_a$, but that it is even the corresponding Hopf algebra. The following proposition shows that this is not true in general.

Proposition 2.8. *Let $E \in \mathcal{E}(r,0)$, i.e. $E \cong L \otimes F_r$ for a line bundle of degree zero (see Theorem 2.1).*

1. *If L is not torsion, the ring $R(E)$ is isomorphic to $\mathbb{Q}[x,x^{-1}] \otimes \mathbb{Q}[y]$ and E trivializes over a principal $\mathbb{G}_m \times \mathbb{G}_a$-bundle.*
2. *If L is torsion, let $n \in \mathbb{N}$, $n \geq 1$, be the minimal number such that $L^{\otimes n} \cong \mathcal{O}_X$. If n and r are both even, the ring $R(E)$ is isomorphic to*

$$\mathbb{Q}[x]/\langle x^{n/2} - 1\rangle \otimes \mathbb{Q}[y],$$

and E trivializes over a principal $\mu_n \times \mathbb{G}_a$-bundle. There is no principal $\mu_{n/2} \times \mathbb{G}_a$-bundle over which E is trivial.
If n and r are not both even, the ring $R(E)$ is isomorphic to

$$\mathbb{Q}[x]/\langle x^n - 1\rangle \otimes \mathbb{Q}[y],$$

and E trivializes over a principal $\mu_n \times \mathbb{G}_a$-bundle.

In all the cases the trivializing principal bundle is $P_L \times_X P$, where P is the principal $\mathbb{G}_a$-bundle from Proposition 2.7 and P_L is a principal bundle, over which L is trivial.

Proof. Let $E \in \mathcal{E}(r,0)$ with $\Gamma(X,E)=0$. (If $\Gamma(X,E)\neq 0$, then $E \cong F_r$. This case was already treated in Propositions 2.4 and 2.7)

First we consider the case that L is not torsion. We must distinguish between odd and even r.
For odd r, Lemma 2.2 yields the following result:
For $m \in \mathbb{N}$, $m \geq 2$, the tensor power $E^{\otimes m} \cong L^{\otimes m} \otimes F_r^{\otimes m}$ has the indecomposable components $L^{\otimes m} \otimes \mathcal{O}_X$, $L^{\otimes m} \otimes F_3, \ldots, L^{\otimes m} \otimes F_{(r-1)m+1}$, the tensor power $E^{\otimes -m} \cong L^{\otimes -m} \otimes F_r^{\otimes m}$ has the indecomposable components $L^{\otimes -m} \otimes \mathcal{O}_X$, $L^{\otimes -m} \otimes F_3, \ldots, L^{\otimes -m} \otimes F_{(r-1)m+1}$.
Thus we obtain

$$S(E) = \left\{ \begin{array}{l} \mathcal{O}_X, L \otimes F_r, L^{-1} \otimes F_r, L^{\otimes \pm i} \\ L^{\otimes \pm i} \otimes F_3, L^{\otimes \pm i} \otimes F_5, \ldots, L^{\otimes \pm i} \otimes F_{(r-1)i+1},\ i \in \mathbb{N},\ i \geq 2 \end{array} \right\}.$$

The algebra $R(E)$ which is generated by $S(E)$ is the subalgebra of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by L, L^{-1} and F_3, thus

$$R(E) = \mathbb{Q}[L, L^{-1}] \otimes_{\mathbb{Z}} \mathbb{Q}[F_3].$$

For even r, a similar computation gives that

$$S(E) = \left\{ \begin{array}{l} \mathcal{O}_X, L \otimes F_r, L^{-1} \otimes F_r, \\ L^{\otimes \pm 2i}, L^{\otimes \pm 2i} \otimes F_3, \ldots, L^{\otimes \pm 2i} \otimes F_{(r-1)2i+1},\ i \in \mathbb{N} - \{0\} \\ L^{\otimes \pm (2i+1)} \otimes F_2, L^{\otimes \pm (2i+1)} \otimes F_4, \ldots, \\ \quad L^{\otimes \pm (2i+1)} \otimes F_{(r-1)(2i+1)+1},\ i \in \mathbb{N} - \{0\} \end{array} \right\}.$$

The ring $R(E)$, generated by $S(E)$, is the subring of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ that is generated by the elements $L^{\otimes 2}$, $L^{\otimes -2}$, $L^{-1} \otimes F_2$, therefore

$$R(E) = \mathbb{Q}[L^{\otimes 2}, L^{\otimes -2}] \otimes_{\mathbb{Z}} \mathbb{Q}[L^{-1} \otimes F_2].$$

If L is not a torsion bundle, it is clear that L trivializes on a principal $\mathbb{G}_m$-bundle P_L. The vector bundle $E \cong L \otimes F_2$ trivializes on the $\mathbb{G}_m \times \mathbb{G}_a$-bundle $P_L \times_X P$, where P is the principal $\mathbb{G}_a$-bundle from Proposition 2.7, where F_2 and hence all the F_r trivialize.
(Consider an open affine covering $\mathcal{U} = \{U_\alpha\}$ as in Remark 2.5, and write $U_{\alpha\beta} = \operatorname{Spec} B$. Locally $P_L \times_X P$ is $(\mathbb{G}_m \times \mathbb{G}_a) \times U_\alpha$, and the glueing morphisms are given by the algebra morphisms

$$\phi_{\alpha\beta}^* : \frac{B[X_\beta, Y_\beta]}{\langle X_\beta Y_\beta - 1 \rangle} \otimes_B B[Z_\beta] \to \frac{B[X_\alpha, Y_\alpha]}{\langle X_\alpha Y_\alpha - 1 \rangle} \otimes_B B[Z_\alpha],$$

mapping X_β to $l_{\alpha\beta} X_\alpha$, Y_β to $l_{\alpha\beta}^{-1} Y_\alpha$, and Z_β to $u_{\alpha\beta} + Z_\alpha$, where $l_{\alpha\beta}$ denotes the transition function of L and $u_{\alpha\beta}$ is the element of $B =$

$\mathcal{O}_X(U_\alpha)$ representing the transition function of F_2. This implies that $l_{\alpha\beta} = X_\beta X_\alpha^{-1}$ and $u_{\alpha\beta} = Z_\beta - Z_\alpha$, hence

$$l_{\alpha\beta}\begin{pmatrix} 1 & u_{\alpha\beta} \\ 0 & 1 \end{pmatrix} = X_\beta \begin{pmatrix} 1 & Z_\beta \\ 0 & 1 \end{pmatrix} \cdot X_\alpha^{-1} \begin{pmatrix} 1 & -Z_\alpha \\ 0 & 1 \end{pmatrix}$$

is a coboundary over P.)

Let now L be torsion and $n \in \mathbb{N}$, $n \geq 2$, the minimal number with $L^{\otimes n} \cong \mathcal{O}_X$. As the F_r are selfdual and $L^{\otimes n-1} = L^{-1}$, it suffices to consider positive tensor powers.
Again we compute the tensor powers using Lemma 2.2 to find the indecomposable components.

If r is even and n is odd, the set $S(E)$ contains the following bundles:

$$S(E) = \{L^{\otimes i} \otimes F_j \,|\, i = 0, 1, \ldots, n-1,\ j \in \mathbb{N}\}.$$

With the help of the multiplication formula for F_2 it is easy to show that all elements of $S(E)$ can be generated by L and F_2. In addition, the relation $L^{\otimes n} \cong \mathcal{O}_X$ holds. Hence we obtain

$$R(E) = \frac{\mathbb{Q}[L]}{\langle L^{\otimes n} - 1\rangle} \otimes_{\mathbb{Z}} \mathbb{Q}[F_2].$$

If r is odd and n is even or odd, the result is

$$S(E) = \{L^{\otimes i} \otimes F_j \,|\, i = 0, 1, \ldots, n-1,\ j \in \mathbb{N} \text{ odd}\}.$$

The bundles L and F_3 are in $S(E)$ and generate all elements of $S(E)$. Because of the relation $L^{\otimes n} \cong \mathcal{O}_X$, the algebra $R(E)$ is

$$R(E) = \frac{\mathbb{Q}[L]}{\langle L^{\otimes n} - 1\rangle} \otimes_{\mathbb{Z}} \mathbb{Q}[F_3].$$

If r and n are both even

$$S(E) = \{L^{\otimes 2i} \otimes F_{2j-1}, L^{\otimes 2i+1} \otimes F_{2j} \,|\, i = 0, 1, \ldots, n/2,\ j \in \mathbb{N} - \{0\}\}.$$

The algebra R(E) is generated by $L^{\otimes 2}$ and $L \otimes F_2$. The generators satisfy the relation $L^{\otimes n} \cong \mathcal{O}_X$, thus

$$R(E) = \frac{\mathbb{Q}[L^{\otimes 2}]}{\langle (L^{\otimes 2})^{\otimes m} - 1\rangle} \otimes \mathbb{Q}[L \otimes F_2],$$

where $m = n/2$.

Recall that $n \geq 2$ is the minimal number such that $L^{\otimes n} \cong \mathcal{O}_X$. Thus the bundle L trivializes on a μ_n-bundle P_L and not on a μ_m-torsor for $m < n$.
The bundle $E \cong L \otimes F_r$ then trivializes on the $\mu_n \times \mathbb{G}_a$-bundle $P_L \times_X P$, where P is again the principal $\mathbb{G}_a$-bundle from Proposition 2.6. We still have to prove that $L \otimes F_r$ does not trivialize on a $\mu_m \times \mathbb{G}_a$-bundle $P_m \times_X P$ for $m < n$. To see this we first note that any étale covering

P_m of X is again an elliptic curve. Without loss of generality we can therefore assume that $X = P_m$ and show that a bundle $L \otimes F_r$ with L non-trivial does not trivialize over P. By the construction of the projective bundle $\pi : \mathbb{P}(F_2) \to X$ associated to F_2, there is an exact sequence $0 \to \mathcal{O}_{\mathbb{P}(F_2)}(-1) \to \pi^*(F_2) \to \mathcal{O}_{\mathbb{P}(F_2)}(1) \to 0$. If the pull-back of $L \otimes F_r$ to $P = \mathbb{P}(F_2) - \{\infty\}$ is trivial, then there is $N >> 0$, such that $\mathcal{O}_{\mathbb{P}(F_2)} \hookrightarrow \pi^*(L \otimes F_r)(N\infty)$, where $\pi : \mathbb{P}(F_2) \to X$ is the projection. The projection formula and [**6**], II, (7.11) imply that $\mathcal{O}_X \hookrightarrow (L \otimes F_r) \otimes \pi_* \mathcal{O}_{\mathbb{P}(F_2)}(N\infty) = (L \otimes F_r) \otimes S^N(F_2)$, and hence that $L^{-1} \hookrightarrow F_r \otimes S^N(F_2) = F_r \otimes F_{N+1} = F_{N+2-r} \oplus F_{N+4-r} \oplus \cdots \oplus F_{N+r}$ (see Lemma 2.2). Thus L^{-1} must be a subbundle of one of the direct summands. Because of the filtration of that summand given by Proposition 1.1 there must be some $s \geq 2$, such that $L^{-1} \hookrightarrow F_s/F_{s-1} = \mathcal{O}_X$, which is a contradiction, since we assumed that L is non-trivial.

□

REMARK 2.9. The correspondence between the dimension of the "minimal" group scheme and the dimension of the ring $R(E)$ also occurs in the case of vector bundles on the projective line, as one can easily see:
Let X be the complex projective line $\mathbb{P}^1$ and $E := \mathcal{O}(a)$ a line bundle. If $a = 0$ we have $S(E) = \{\mathcal{O}\}$ and $R(E) = \mathbb{Q}$.
We define the group scheme G to be $G = \text{Spec } \mathbb{Q}$ and the trivializing torsor is simply $\mathbb{P}^1$.
If $a \neq 0$ we can easily compute that $S(E) = \{\mathcal{O}(\lambda \cdot a) | \lambda \in \mathbb{Z}\}$ and $R(E) = \mathbb{Q}[x, x^{-1}]$. We define the group scheme to be $G = \mathbb{G}_m = \text{Spec } \mathbb{Q}[x, x^{-1}]$.
The given line bundle E trivializes on a principal $\mathbb{G}_m$-bundle P_a, which depends on a.
Thus we get the correspondence of $\dim R(E)$ and $\dim G$ in the case of a line bundle on $\mathbb{P}^1$. This computation can easily be generalized to the case of vector bundles of higher rank. We illustrate this for bundles of rank two.
Let now E be a vector bundle of rank 2 on $\mathbb{P}^1$, $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$.
The case $(a, b) = (0, 0)$ is trivial. We can see at once that $S(E) = \{\mathcal{O}\}$ and therefore $R(E) = \mathbb{Q}$.
The vector bundle E trivializes on the principal Spec $\mathbb{Q}$ - bundle $\mathbb{P}^1$.
If $(a, b) \neq (0, 0)$ the computation gives that $S(\mathcal{O}(a) \oplus \mathcal{O}(b)) = S(\mathcal{O}(c))$, where $c = (a, b)$ (with $(a, 0) = a$ and $(0, b) = b$) and therefore $R(E) = \mathbb{Q}[x, x^{-1}]$. E trivializes on the principal $\mathbb{G}_m$-bundle P_c that belongs to $\mathcal{O}(c)$ as $\mathcal{O}(a) = \mathcal{O}(c)^\lambda$ and $\mathcal{O}(b) = \mathcal{O}(c)^\mu$ for appropriate integers λ and μ.

3. Semi-stability

In the following X denotes an elliptic curve over an algebraically closed field of characteristic zero, as in the previous section.

DEFINITION 3.1. *(Mumford [**9**])*
A vector bundle E over X is called semistable, if for all non-zero subbundles F of E,

$$\frac{\deg(F)}{\mathrm{rk}(F)} \leq \frac{\deg(E)}{\mathrm{rk}(E)}.$$

If the inequality is strict for all non-zero subbundles F of E, then E is called stable.

REMARK. Equivalently E is semistable if for every quotient bundle $G = E/F$,

$$\frac{\deg(G)}{\mathrm{rk}(G)} \geq \frac{\deg(E)}{\mathrm{rk}(E)}.$$

PROPOSITION 3.2. *Let*

$$0 \to E \to F \to G \to 0$$

be an exact sequence of vector bundles over X. If E and G are of degree zero and semistable, then also F is of degree zero and semistable.

PROOF. [**8**], Prop. (5.3.5)

□

PROPOSITION 3.3. *Every indecomposable vector bundle of degree zero over an elliptic curve X is semistable.*

PROOF. If L is a line bundle of degree 0, it follows directly from the definition that L is semistable. Now let $r \geq 2$ and $E \in \mathcal{E}(r, 0)$ an indecomposable vector bundle of degree zero and rank r. From Proposition 2.1 we know that there exists a line bundle L of degree 0, such that $E \cong L \otimes F_r$. Tensoring the exact sequence

$$0 \to \mathcal{O}_X \to F_r \to F_{r-1} \to 0$$

by L, we obtain an exact sequence

$$0 \to L \to L \otimes F_r \to L \otimes F_{r-1} \to 0.$$

We can assume by induction hypothesis that $L \otimes F_{r-1}$ is semistable of degree zero. It follows from Proposition 3.2 that also $L \otimes F_r$ is semistable of degree zero. □

COROLLARY 3.4. *Let E be a vector bundle of degree zero over an elliptic curve which is a subbundle or a quotient bundle of a semistable bundle of degree zero. Then E is also semistable.*

PROOF. Assume that E is a subbundle of a semistable bundle G of degree zero, and let F be a subbundle of E. Since F is also a subbundle of G, its degree must be smaller or equal to zero. Hence E is semistable. Now assume that E is a quotient bundle of a semistable bundle G of degree zero, and let E/F be a quotient of E. Since E/F is also a quotient of G, $\deg(E/F)$ is greater or equal to zero. Therefore E is semistable. $\square$

4. Tannakian category associated to a vector bundle

In the first section, we defined the category $SS(X)$ to be the full subcategory of the category of quasi-coherent sheaves, whose objects are those vector bundles which are semistable of degree zero restricted to every curve. If X is an elliptic curve, the objects of $SS(X)$ are just the semistable vector bundles of degree zero.
The category $SS(X)$ is an abelian category, as proved by Nori in [**11**], Lemma (3.6).

If E is an indecomposable vector bundle of degree zero over X, it generates a subcategory $\mathcal{C}_E$ of $SS(X)$. The category $\mathcal{C}_E$ is the full subcategory of $SS(X)$ with set of objects

$$\overline{S(E)} = \left\{ \begin{array}{l} W \cong V_2/V_1 \mid \exists\, P_i \in S(E),\, 1 \leq i \leq t, \\ \exists\, V_1, V_2 \in \text{ Obj } SS(X) \text{ such that } V_1 \subset V_2 \subset \bigoplus_{i=1}^{t} P_i \end{array} \right\},$$

with $S(E)$ as defined in Def. (2.3). Since $SS(X)$ is abelian, all objects of $\overline{S(E)}$ are objects of $SS(X)$.

REMARK 4.1. Recall that $S(F_r) = \{F_i, i \in \mathbb{N}\}$, if r is even, and that $S(F_r) = \{F_i, i \in \mathbb{N},\ i \text{ odd }\}$, if r is odd. Therefore, for all $r \in \mathbb{N}$ the categories $\mathcal{C}_{F_r}$ coincide with $\mathcal{C}_{F_2}$: If r is even, $\mathcal{C}_{F_r}$ is defined in exactly the same way as $\mathcal{C}_{F_2}$, since $S(F_r) = S(F_2)$. If r is odd, $S(F_r)$ contains only those F_i with odd index i, but every F_s with s even is contained in the category, since it appears as subbundle of F_{s+1}. $\square$

By construction, $\mathcal{C}_E$ is abelian for every indecomposable vector bundle E on X (see [**11**], §1). We want to show that $\mathcal{C}_E$ is even a neutralized Tannakian category. We shortly explain the formalism of neutralized Tannakian categories. For details, we refer to [**13**] and [**3**].

DEFINITION 4.2. *A category $\mathcal{C}$ is called a symmetric, monoidal category, if*

(1) *$\mathcal{C}$ is endowed with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which is associative and commutative in the following sense:*
There is a functorial isomorphism

$$\{\Phi_{V_1,V_2,V_3} : V_1 \otimes (V_2 \otimes V_3) \to (V_1 \otimes V_2) \otimes V_3 \mid V_1, V_2, V_3 \in \ \mathit{Obj}\, \mathcal{C}\},$$

satisfying the pentagon axiom, and a functorial isomorphism

$$\{\Psi_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1 \,|\, V_1, V_2 \in \ Obj\, \mathcal{C}\},$$

satisfying the hexagon axiom (see [**3**], *§1).*

(2) *There is a unit object I for the tensor product, i.e. for any object V there exist functorial isomorphisms*

$$V \otimes I \to V \to I \otimes V$$

whose composition is the commutativity isomorphism.

Definition 4.3. *A tensor functor between symmetric monoidal categories is a functor between the underlying categories, which is compatible with the symmetric monoidal structures.*

Definition 4.4. *A symmetric monoidal category is called rigid, if to every object V there is assigned, in a functorial manner, an inverse object $V^\vee$ and an evaluation map $\epsilon : V \otimes V^\vee \to I$.*

Definition 4.5. *A neutralized Tannakian category over a field k is a rigid symmetric monoidal category $\mathcal{C}$, together with a fibre functor ω with values in the category k-mod of finite-dimensional k-vector spaces, such that*

(1) *Obj $\mathcal{C}$ is a set,*
(2) *$\mathcal{C}$ is an abelian category,*
(3) *there is an isomorphism $k \cong End(I)$,* *(which determines a k-vector spac e structure on Hom-sets, such that the law of composition of morphisms is k-bilinear.)*
(4) *$\omega : \mathcal{C} \to k-mod$ is a faithful, exact tensor functor, which is k-linear on Hom-sets.*

Example 4.6. *For any affine group scheme G the category G-mod of finite-dimensional representations of G, together with the forgetful functor, forms a neutralized Tannakian category.*

Proposition 4.7. *(compare* [**11**]*)*
For every $E \in \mathcal{E}(r, 0)$, $r \in \mathbb{N}$, the category $\mathcal{C}_E$ is a neutralized Tannakian category.

Proof. We already know by construction that $\mathcal{C}_E$ is abelian. As tensor product in $\mathcal{C}_E$ we use the usual tensor product of vector bundles. We must show that $\mathcal{C}_E$ is closed under the tensor product: If V, W $\in$ Obj $\mathcal{C}_E$, then there exist $V_1, V_2, W_1, W_2 \in SS(X)$, $t_i, s_j \in \mathbb{N}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and E_{t_i}, $E_{s_j} \in S(E)$, such that

$$V_1 \subset V_2 \subset \oplus_{i=1}^n E_{t_i},\ W_1 \subset W_2 \subset \oplus_{j=1}^m E_{s_j},$$

and $V \cong V_1/V_2$ and $W \cong W_1/W_2$. By definition of $S(E)$, the bundle $(\oplus_{i=1}^n E_{t_i}) \otimes (\oplus_{j=1}^m E_{s_j})$ is again a finite direct sum of elements of $S(E)$. Then

$$V_2 \otimes W_2 \subset (\oplus_{i=1}^n E_{t_i}) \otimes (\oplus_{j=1}^m E_{s_j}),$$

and $V \otimes W$ is a quotient of $V_2 \otimes W_2$, $V \otimes W = (V_2 \otimes W_2)/Q$. Since the tensor product of vector bundles of degree zero is again of degree zero, both $V \otimes W$ and $V_2 \otimes W_2$ are of degree zero. By elementary degree considerations, $\deg Q + \deg(V \otimes W) = \deg(V_2 \otimes W_2)$, it follows that also Q is of degree zero. As the tensor product of semistable bundles is semistable, $V_2 \otimes W_2$ is even semistable. By Corollary 3.5 we conclude that $V \otimes W$ and Q are semistable. But this means that $V \otimes W \in \text{Obj}\ \mathcal{C}_E$.
Because of the properties of the tensor product of vector bundles, the pentagon and the hexagon axiom are fulfilled. The trivial bundle $\mathcal{O}_X$ is an element of $S(E)$. It is a unit object for the tensor product.
Furthermore, $\mathcal{C}_E$ is rigid. For this we have to show that the category contains the dual vector bundles. This follows directly from the fact that $S(E)$ contains the duals of all elements: If $V \in \text{Obj}\ \mathcal{C}_E$ is a subbundle of a finite direct sum of bundles $E_s \in S(E)$, then the dual $V^\vee$ is a quotient of the finite direct sum of dual bundles $E_s^\vee$, which are also contained in $S(E)$, hence $V^\vee$ is an object of the category $\mathcal{C}_E$. If $V = V_1/V_2$ with $V_1 \subset V_2 \subset \oplus_{s=1}^t E_s$, dualizing the exact sequence $0 \to V_1 \to V_2 \to V \to 0$, yields that $V^\vee = \text{Ker}(V_2^\vee \to V_1^\vee)$ is an object of $\mathcal{C}_E$, since $\mathcal{C}_E$ is abelian.
We still have to define a fibre functor: Let $x \in X$ be a k-rational point. Then we define $\omega := x^*$ to be the functor, taking a vector bundle V to its fibre V_x. This functor is exact and k-linear on Hom-sets. There only remains to prove that it is also faithful. For this assume that V, W are in Obj $\mathcal{C}_E$, and that $f, g \in \text{Hom}(V, W)$, such that $\omega(f) = \omega(g)$, i.e. $f_x = g_x : V_x \to W_x$. We want to show that this implies that $f = g$. For this we consider $\text{Ker}(f - g)$. Since $\mathcal{C}_E$ is abelian, $\text{Ker}(f - g) \in \text{Obj}\ \mathcal{C}_E$. In particular, $\text{Ker}(f - g)$ is a vector bundle, hence all its fibres are of the same dimension. Because of the assumption that the fibre in x is zero-dimensional, we know that $\text{Ker}(f - g) = 0$, hence $f = g$. □

THEOREM 4.8. *(Saavedra)*
For any neutralized Tannakian category $(\mathcal{C}, \omega)$ there exists an affine group scheme G, such that $\mathcal{C}$ is equivalent to G-mod.

For a given neutralized Tannakian category $\mathcal{C}$ with fibre functor $\omega : \mathcal{C} \to k\text{-mod}$, the corresponding group scheme G can be computed in the following way, described in [**3**]:

There is an isomorphism of functors of k-algebras

$$G \overset{\sim}{\longleftrightarrow} \underline{\mathrm{Aut}}^{\otimes}(\omega),$$

where for every k-algebra R the group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ consists of all families $(\alpha(V))_{V\in\ \mathrm{Obj}\ \mathcal{C}}$, where $\alpha(V) : \omega(V) \otimes_k R \to \omega(V) \otimes_k R$ is R-linear with the properties

(1) $\alpha(I) = \mathrm{id}_R$, where I denotes the unit object,
(2) $\alpha(V_1 \otimes V_2) = \alpha(V_1) \otimes \alpha(V_2)$,
(3) for all morphisms $\phi : V \to W$ in the category $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} \omega(V) \otimes R & \xrightarrow{\alpha(V)} & \omega(V) \otimes R \\ {\scriptstyle \omega(\phi)\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \omega(\phi)\otimes \mathrm{id}} \\ \omega(W) \otimes R & \xrightarrow{\alpha(W)} & \omega(W) \otimes R \end{array}$$

For any k-algebra R, the group law in $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ is the following: If $(\alpha(V))_{V\in\ \mathrm{Obj}\ \mathcal{C}}$ and $(\beta(F))_{V\in\ \mathrm{Obj}\ \mathcal{C}}$ are two families, we define their composition to be the family

$$(\alpha(V) \circ \beta(V))_{V\in \mathrm{Obj}\ \mathcal{C}},$$

where $\alpha(V)\circ\beta(V)$ denotes the composition of morphisms of k-algebras. We must show that this family has the desired properties (1),(2) and (3). It is clear that property (1) is fulfilled, since $\alpha(I)$ and $\beta(I)$ are both the identity on R. Property (2) is fulfilled, since for $V, W \in$ Obj $\mathcal{C}$, we have that

$$\begin{aligned} \alpha(V \otimes W) \circ \beta(V \otimes W) &= (\alpha(V) \otimes \alpha(W)) \circ (\beta(V) \otimes \beta(W)) \\ &= (\alpha(V) \circ \beta(V)) \otimes (\alpha(W) \circ \beta(W)). \end{aligned}$$

Property (3) follows easily from the fact that it holds for the families $(\alpha(V))_{V\in\mathrm{Obj}\ \mathcal{C}}$ and $(\beta(F))_{V\in\mathrm{Obj}\ \mathcal{C}}$, since

$$\begin{aligned} (\omega(\phi) \otimes \mathrm{id}) \circ (\alpha(V) \otimes \beta(V)) &= ((\omega(\phi) \otimes \mathrm{id}) \circ \alpha(V)) \circ \beta(V) \\ &= (\alpha(W) \circ (\omega(\phi) \otimes \mathrm{id})) \circ \beta(V) \\ &= \alpha(W) \circ ((\omega(\phi) \otimes \mathrm{id}) \circ \beta(V)) \\ &= (\alpha(W) \circ \beta(W)) \circ (\omega(\phi) \otimes \mathrm{id}). \end{aligned}$$

We have seen in the previous proposition that $\mathcal{C}_{F_2}$ with fibre functor x^*, x a k-rational point, is a Tannakian category. Thus there exists a group scheme G, such that $\mathcal{C}_{F_2}$ is equivalent to the category G-mod, and $G \cong \underline{\mathrm{Aut}}^{\otimes}(x^*)$.

Proposition 4.9. *The group scheme corresponding to $\mathcal{C}_{F_2}$ is $\mathbb{G}_a$, i.e.*

$$\mathcal{C}_{F_2} \cong \mathbb{G}_a - mod.$$

Proof. We have to show that for every k-algebra R, $\underline{\mathrm{Aut}}^{\otimes}(x^*)(R) = \mathbb{G}_a(R)$. Let $(\alpha(V))_{V \in \text{ Obj } \mathcal{C}_{F_2}}$ be a family in $\underline{\mathrm{Aut}}^{\otimes}(x^*)(R)$. After a choice of a basis for V_x, every $\alpha(V) : V_x \otimes_k R \to V_x \otimes_k R$ can be represented as an $n \times n$-matrix with coefficients in R, where $n = \dim V_x$. Since F_2 is given by an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_2 \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

property (3) of the given family implies the existence of a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & F_{2,x} \otimes_k R & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow \mathrm{id}_R & & \downarrow \alpha(F_2) & & \downarrow \mathrm{id}_R & & \\ 0 & \longrightarrow & R & \longrightarrow & F_{2,x} \otimes_k R & \longrightarrow & R & \longrightarrow & 0. \end{array}$$

With respect to a suitable basis for $F_{2,x}$, $\alpha(F_2)$ is therefore of the form

$$\alpha(F_2) = \begin{pmatrix} 1 & \lambda(F_2) \\ 0 & 1 \end{pmatrix}$$

with $\lambda(F_2) \in R$. We will show that $\lambda(F_2)$ is the element of $\mathbb{G}_a(R)$, corresponding to the family $(\alpha(V))_{V \in \text{ Obj } \mathcal{C}_{F_2}}$.
First of all, if $(\beta(V))_{V \in \text{ Obj } \mathcal{C}_{F_2}}$ is another family with

$$\beta(F_2) = \begin{pmatrix} 1 & \mu(F_2) \\ 0 & 1 \end{pmatrix},$$

then the composition $\alpha(F_2) \circ \beta(F_2)$ is given by the matrix

$$\begin{pmatrix} 1 & \lambda(F_2) + \mu(F_2) \\ 0 & 1 \end{pmatrix},$$

thus the composition of the morphisms is compatible with the addition in $\mathbb{G}_a(R)$.
There remains to show that for every $V \in \text{Obj } \mathcal{C}_{F_2}$, $\alpha(V)$ is uniquely determined by $\alpha(F_2)$, and that every $\lambda \in R$ defines a family in $\underline{\mathrm{Aut}}^{\otimes}(x^*)(R)$.

First we remark that for $V, W \in \text{Obj } \mathcal{C}_{F_2}$,

$$\alpha(V \oplus W) = \alpha(V) \oplus \alpha(W) :$$

Property (3) of the family implies that there is a commutative diagram

$$\begin{array}{ccc} V_x \otimes R & \xrightarrow{\alpha(V)} & V_x \otimes R \\ \big\downarrow & & \big\downarrow \\ (V_x \oplus W_x) \otimes R & \xrightarrow{\alpha(V\oplus W)} & (V_x \oplus W_x) \otimes R, \end{array}$$

hence $\alpha(V \oplus W)_{|V_x} = \alpha(V)$. In the same way we show that $\alpha(V \oplus W)_{|W_x} = \alpha(W)$. From this we conclude that $\alpha(V \oplus W) = \alpha(V) \oplus \alpha(W)$.

Now we show that for every F_n, $n \geq 3$, $\alpha(F_n)$ is uniquely determined by $\alpha(F_2)$: From the multiplication formulas for the bundles F_n, see Lemma 2.2, it follows that there exist $a_i(n) \in \mathbb{Z}$, $a_i(n) \geq 0$, such that

$$F_2^{\otimes n} \cong a_1(n)\mathcal{O}_X \oplus a_2(n)F_2 \oplus \cdots \oplus a_{n-1}(n)F_{n-1} \oplus F_n.$$

Let ϕ_n denote this bundle isomorphism. From property (3) of the family it follows that

$$\begin{aligned} \alpha(\sum_{i=1}^{n-1} a_i(n)F_i \oplus F_n) \circ (x^*(\phi_n) \otimes \mathrm{id}) &= (x^*(\phi_n) \otimes \mathrm{id}) \circ \alpha(F_2^{\otimes n}) \\ &= (x^*(\phi_n) \otimes \mathrm{id}) \circ \alpha(F_2)^{\otimes n}, \end{aligned}$$

with $F_1 = \mathcal{O}_X$.
After a choice of basis for the vector space $\sum_{i=1}^{n-1} a_i(n)F_{i,x} \oplus F_{n,x}$, the matrix of $\alpha(\sum_{i=1}^{n-1} a_i(n)F_i \oplus F_n)$ can be written as a block matrix with the last block being the matrix representation of $\alpha(F_n)$ with respect to the induced basis on $F_{n,x}$. From the above formula it follows that this block is uniquely determined by $\alpha(F_2)^{\otimes n}$, and hence by $\alpha(F_2)$.

If $V, W \in \mathcal{C}_{F_2}$ and $V \subset W$, it is obvious from the commutative diagrams given by property (3), that $\alpha(V)$ is uniquely determined by $\alpha(W)$, and that $\alpha(V/W)$ is uniquely determined by $\alpha(V)$ and $\alpha(W)$. Altogether we have shown that for all $V \in \mathcal{C}_{F_2}$, $\alpha(V)$ only depends on $\alpha(F_2)$.

If $\lambda \in R$ is arbitrary, it defines a family $(\alpha(V))$, by setting

$$\alpha(F_2) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

All the other $\alpha(V)$, $V \in \mathrm{Obj}\,\mathcal{C}_{F_2}$ are determined by $\alpha(F_2)$ in the same way as described above. $\square$

PROPOSITION 4.10. *Let $E \in \mathcal{E}(r,0)$, i.e. $E \cong L \otimes F_r$ for some line bundle L of degree zero. Then*

$$\mathcal{C}_E \cong (\mathbb{G}_m \times \mathbb{G}_a) - mod,$$

if L is not a torsion bundle, and

$$\mathcal{C}_E \cong (\mu_n \times \mathbb{G}_a) - mod,$$

if L is an n-torsion bundle with $n \geq 1$ the minimal number such that $L^{\otimes n} \cong \mathcal{O}_X$.

PROOF. Let $(\alpha(V))_{V \in \text{ Obj } \mathcal{C}_E}$ be a family in $\underline{\text{Aut}}^{\otimes}(x^*)(R)$. Because of the exact sequence

$$0 \to L \to L \otimes F_r \to L \to 0,$$

properties (2) and (3) of the family imply that we can choose a basis for $L_x \otimes F_{r,x}$, such that the matrix representation of $\alpha(L \otimes F_r)$ is of the form

$$\alpha(L \otimes F_r) = \alpha(L) \otimes \alpha(F_r) = \alpha(L) \begin{pmatrix} 1 & \lambda(F_r) \\ 0 & 1 \end{pmatrix}.$$

Since $\alpha(L^{\otimes n}) = (\alpha(L))^n$, by property (1) of the family $\alpha(L)$ is an element of $\mu_n(R)$, if L is an n-torsion bundle. Therefore $\alpha(L \otimes F_r) \in (\mu_n \times \mathbb{G}_a)(R)$, if L is a torsion bundle, and $\alpha(L \otimes F_r) \in (\mathbb{G}_m \times \mathbb{G}_a)(R)$, if L is not torsion.
As in the proof of the previous proposition, we can show that for any object V of $\mathcal{C}_E$, $\alpha(V)$ is uniquely determined by $\alpha(L \otimes F_r)$.
If $(\beta(V))_{V \in \text{ Obj } \mathcal{C}_E}$ is another family, then

$$\alpha(L \otimes F_r) \circ \beta(L \otimes F_r) = \alpha(L)\beta(L) \begin{pmatrix} 1 & \lambda(F_r) + \mu(F_r) \\ 0 & 1 \end{pmatrix},$$

where $\mu(F_r)$ denotes the element of $\mathbb{G}_a(R)$, corresponding to the family $(\beta(V))$. Therefore the composition of $\alpha(L \otimes F_r)$ and $\beta(L \otimes F_r)$ corresponds to the group law in $(\mu_n \times \mathbb{G}_a)(R)$, or $(\mathbb{G}_m \times \mathbb{G}_a)(R)$ respectively.
If (c, λ) is an arbitrary element of $(\mu_n \times \mathbb{G}_a)(R)$, or $(\mathbb{G}_m \times \mathbb{G}_a)(R)$ respectively, we define a family by setting

$$\alpha(L \otimes F_r) = c \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

□

5. Example

Let X be a curve of genus $g = 2$ over $\mathbb{C}$, and let x be a $\mathbb{C}$-rational point.

We want to show that there exists a stable bundle E over X (see Def. 3.1), generating a Tannakian category $\mathcal{C}_E$, such that the ring $R(E)$ is of smaller dimension than the group scheme associated to $\mathcal{C}_E$.

We have the Narasimhan-Seshadri correspondence ([**10**], §12):

$$\left\{ \begin{array}{c} \text{stable vector bundles} \\ \text{of degree 0 over } X \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{irreducible complex unitary} \\ \text{representations of } \pi_1(X,x) \end{array} \right\},$$

which can be extended to a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{polystable vector bundles} \\ \text{of degree 0 over } X \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{complex unitary} \\ \text{representations} \\ \text{of } \pi_1(X,x) \end{array} \right\}.$$

If E is a stable vector bundle of degree zero and ρ_E the corresponding unitary representation, both generate neutralized Tannakian categories $\mathcal{C}_E$, respectively $\mathcal{C}_{\rho_E}$, whose objects are all the subquotients of all finite direct sums of the objects $E^{\otimes m} \otimes (E^\vee)^{\otimes n}$, respectively $\rho_E^{\otimes m} \otimes (\rho_E^\vee)^{\otimes n}$, for all $n, m \in \mathbb{Z}$, $m, n \geq 0$ (see Prop. 4.7).
The objects of $\mathcal{C}_E$ are those bundles which are isomorphic to finite direct sums of elements of $S(E)$ (as defined in Def. 2.3): Let $V_1 \subset V_2 \subset \oplus_{j=1}^t E_j$ with $V_1, V_2 \in SS(X)$ and $E_j \in S(E)$. Since V_i, $i = 1, 2$, is semistable of degree zero, it has a Jordan-Hölder filtration, and the smallest subbundle in this filtration is stable of degree zero. Since all the summands E_j are stable, the smallest subbundle must be isomorphic to one of the E_j, and it must therefore split from V_i, $i = 1, 2$. Continuing this process, we see that V_i, $i = 1, 2$, must itself be isomorphic to a finite direct sum of E_j.
Hence every object of $\mathcal{C}_E$ is polystable of degree zero, and has therefore a corresponding object in $\mathcal{C}_{\rho_E}$ via the Narasimhan-Seshadri correspondence. Hence the two Tannakian categories $\mathcal{C}_E$ and $\mathcal{C}_{\rho_E}$ are equivalent, and there exists a group scheme G such that there is an equivalence of Tannakian categories

$$\mathcal{C}_E \cong \mathcal{C}_\rho \cong G - \text{mod}.$$

By [**7**], (1.2.2), we know that

$$G = \text{Zariski closure of } \rho_E(\pi_1(X,x)) \text{ in } \mathrm{GL}_n(\mathbb{C})$$

for $\rho_E : \pi_1(X,x) \to \mathrm{GL}_n(\mathbb{C})$.

In the following, we will define a special unitary representation

$$\rho : \pi_1(X,x) \to \mathrm{SL}_2(\mathbb{C}),$$

such that $\rho(\pi_1(X,x))$ is Zariski-closed in $\mathrm{SL}_2(\mathbb{C})$.
We will see that the subring $R(E)$ of $K(X)\otimes\mathbb{Q}$, generated by the stable vector bundle E corresponding to ρ, is isomorphic to the representation ring of $\mathrm{SL}_2(\mathbb{C})$ over $\mathbb{Q}$, which is one-dimensional. As $\dim \mathrm{SL}_2(\mathbb{C}) = 3$, in this case the dimensions of the ring, generated by E, and the group scheme, corresponding to E, are not equal.

The representation ρ will be defined in the following way: We find two matrices $A, B \in \mathrm{SU}_2(\mathbb{C})$ such that

$$\overline{\langle A, B\rangle}^{\mathrm{Zar}} = \mathrm{SL}_2(\mathbb{C})$$

and define ρ by setting

$$\begin{aligned} \rho(a) &:= A, \quad \rho(b) := B, \\ \rho(c) &:= B, \quad \rho(d) := A, \end{aligned}$$

where a, b, c, d are the generators of the fundamental group

$$\pi_1(X,x) = \langle a, b, c, d \,|\, [a,b][c,d] = 1\rangle.$$

This is a well-defined representation of the fundamental group as the condition

$$[\rho(a), \rho(b)][\rho(c), \rho(d)] = I$$

is fulfilled.

First Step: We define

$$A := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C}),$$

with $\lambda = e^{2\pi i\phi}$ and $\phi \in [0,1]$ irrational. Let T be the set of diagonal matrices in $\mathrm{SL}_2(\mathbb{C})$,

$$T = \{ \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \mid \mu \in \mathbb{C}^*\} \cong \mathbb{C}^*.$$

As for irrational ϕ the topological closure of $\langle\lambda\rangle$ in $\mathbb{C}^*$ is S^1, the Zariski closure must be the whole $\mathbb{C}^*$. Thus the Zariski closure of $\langle A\rangle$ in $\mathrm{SL}_2(\mathbb{C})$ is the the whole maximal torus T.

We define the second matrix B to be

$$B := C^{-1}AC,$$

where

$$C := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in \mathrm{U}_2(\mathbb{C}),$$

thus

$$B = \frac{1}{2}\begin{pmatrix} \lambda+\lambda^{-1} & i(\lambda-\lambda^{-1}) \\ -i(\lambda-\lambda^{-1}) & \lambda+\lambda^{-1} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C}),$$

As for all $n \in \mathbb{N}$

$$B^n = C^{-1}A^nC = \frac{1}{2}\begin{pmatrix} \lambda^n+\lambda^{-n} & i(\lambda^n-\lambda^{-n}) \\ -i(\lambda^n-\lambda^{-n}) & \lambda^n+\lambda^{-n} \end{pmatrix},$$

by the same arguments as above we obtain that the Zariski closure of $\langle B\rangle$ contains all matrices

$$\frac{1}{2}\begin{pmatrix} \mu+\mu^{-1} & i(\mu-\mu^{-1}) \\ -i(\mu-\mu^{-1}) & \mu+\mu^{-1} \end{pmatrix},\ \mu \in \mathbb{C}^*.$$

Second Step: We define

$$G := \text{Zariski closure of } \langle A, B\rangle$$

and claim that

$$G = \mathrm{SL}_2(\mathbb{C}).$$

To prove this, we first show that the Lie algebras g and $sl_2(\mathbb{C})$, corresponding to G and $\mathrm{SL}_2(\mathbb{C})$, coincide:
The Lie algebra $sl_2(\mathbb{C})$ consists of all traceless 2×2-matrices. The Lie algebra g contains the elements

$$a := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } b := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

as we can prove as follows. Since

$$A(t) := \begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix} \in G$$

for all $t \in (-1,1)$ and

$$A(0) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A'(t) := \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{(1+t)^{-2}} \end{pmatrix}$$

we obtain that

$$a := A'(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is an element of g. Analogously, since

$$B(t) := \frac{1}{2}\begin{pmatrix} (1+t)+(1+t)^{-1} & i((1+t)-(1+t)^{-1}) \\ -i((1+t)-(1+t)^{-1}) & (1+t)+(1+t)^{-1} \end{pmatrix} \in G$$

for all $t \in (-1,1)$ and

$$B(0) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B'(t) := \frac{1}{2}\begin{pmatrix} 1-\frac{1}{(1+t)^{-2}} & i(1+\frac{1}{(1+t)^{-2}}) \\ -i(1+\frac{1}{(1+t)^{-2}}) & 1-\frac{1}{(1+t)^{-2}} \end{pmatrix}$$

we obtain that

$$b := B'(0) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

is an element of g.
The matrices a and b generate $sl_2(\mathbb{C})$ as a Lie algebra: The matrices

$$a := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; b_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } b_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis of $sl_2(\mathbb{C})$ as vector space. As

$$b_1 = \frac{1}{4}([a,b] + 2b) \text{ and } b_2 = \frac{1}{4}([a,b] - 2b),$$

a and b generate $sl_2(\mathbb{C})$ as a Lie algebra.
Thus we have proved that $g = sl_2(\mathbb{C})$. Now let G_e be the irreducible component of G which contains the identity. As G and $\mathrm{SL}_2(\mathbb{C})$ are algebraic groups, they are smooth. Therefore

$$\dim G_e = \dim g = \dim sl_2(\mathbb{C}) = \dim \mathrm{SL}_2(\mathbb{C}).$$

Hence G_e and $\mathrm{SL}_2(\mathbb{C})$ are of the same dimension. Since G_e is Zariski closed in $\mathrm{SL}_2(\mathbb{C})$, we obtain that $G_e = \mathrm{SL}_2(\mathbb{C})$, thus $G = \mathrm{SL}_2(\mathbb{C})$.

Third Step: The representation $\rho : \pi_1(X, x) \to \mathrm{SL}_2(\mathbb{C})$ is irreducible. Denote by E the stable vector bundle corresponding to ρ via the Narasimhan-Seshadri correspondence. The Tannakian categories $\mathcal{C}_E$ and SL_2-mod are equivalent. Therefore there exist functors

$$\begin{aligned} F_1 : \mathcal{C}_E &\to \mathrm{SL}_2(\mathbb{C})\text{-mod}, \\ F_2 : \mathrm{SL}_2(\mathbb{C})\text{-mod} &\to \mathcal{C}_E, \end{aligned}$$

such that

$$F_1 F_2 \cong \mathrm{id}_{\mathrm{SL}_2(\mathbb{C})\text{-mod}}, \; F_2 F_1 \cong \mathrm{id}_{\mathcal{C}_E}.$$

This implies that there is a 1-1 correspondence of the isomorphism classes of the indecomposable objects in both categories. We can see this in the following way: Since $\mathrm{SL}(2, \mathbb{C})$ is connected and simply connected ([**12**], Ch.1, §4), the representations of $\mathrm{SL}(2, \mathbb{C})$ are in 1-1 correspondence with those of the Lie algebra $sl_2(\mathbb{C})$ ([**4**], Part II, §8.1). By [**4**], Theorem 9.19, all the objects of $sl_2(\mathbb{C})$ are completely reducible. Thus, if V is a representation of $\mathrm{SL}_2(\mathbb{C})$ and $U \subset V$ is a subrepresentation, there exists a complement U' of U in V such that

$$V = U \oplus U'.$$

Hence all the objects in $\mathrm{SL}_2(\mathbb{C})$-mod are semisimple.
The vector bundles in $\mathcal{C}_E$ are direct sums of indecomposable vector bundles and by the Krull-Schmidt-Remak theorem this decomposition

is unique up to isomorphism. Now let $V \in \mathrm{Obj}\ \mathcal{C}_E$. If $F_1(V) \in \mathrm{Obj}$ $\mathrm{SL}_2(\mathbb{C})$-mod is decomposable,

$$F_1(V) = V_1 \oplus V_2$$

we obtain that

$$V \cong F_2(F_1(V)) = F_2(V_1 \oplus V_2) \cong F_2(V_1) \oplus F_2(V_2)$$

is also decomposable. Vice versa, let $V \in \mathrm{Obj}\ \mathrm{SL}_2(\mathbb{C})$-mod. If $F_2(V) \in$ $\mathrm{Obj}\ \mathcal{C}_E$ is decomposable,

$$F_2(V) = V_1 \oplus V_2,$$

we obtain that also

$$V \cong F_1(F_2(V)) \cong F_1(V_1) \oplus F_1(V_2)$$

is decomposable.

Fourth Step: Next, we want to prove that there are no other indecomposable vector bundles in $\mathcal{C}_E$ than the elements of $S(E)$. First we prove that an indecomposable vector bundle in $\mathcal{C}_E$ can have no proper subbundles. If W is an indecomposable element in $\mathcal{C}_E$ and $U \subset W$ a subbundle, then $F_1(U)$ is a subrepresentation of $F_1(W)$. As the equivalence of the categories $\mathcal{C}_E$ and $\mathrm{SL}_2(\mathbb{C})$-mod supplies the 1-1 correspondence of the indecomposable objects, $F_1(W)$ must be an indecomposable representation, hence by the complete reducibility also an irreducible representation. Therefore,

$$F_1(U) = 0 \ \text{ or } \ F_1(U) = F_1(W).$$

By applying F_2 we obtain

$$U \cong F_2(F_1(U)) = 0 \ \text{ or } \ U \cong F_2(F_1(U)) = F_2(F_1(W)) \cong W.$$

Let $V \in \mathrm{Obj}\ \mathcal{C}_E$. By the definition of $\mathcal{C}_E = \overline{S(E)}$ we know that there exist semistable vector bundles $V_1 \subset V_2 \subset \oplus E_i$, where $E_i \in S(E)$ such that

$$V \cong V_2/V_1.$$

The bundles V_i, $i = 1, 2$, are direct sums of indecomposable bundles,

$$V_1 = \oplus W'_k, \ V_2 = \oplus W_j.$$

As the W_j can have no proper subbundles, all components W'_k of V_1 must be isomorphic to a component W_j of V_2. Thus $V \cong V_2/V_1$ is itself of the form

$$V = \oplus W_i$$

for indecomposable bundles W_i. But as

$$V = \oplus W_i \subset \oplus E_j,$$

where all bundles W_i and E_j are indecomposable, the same procedure yields that each W_i must be isomorphic to some E_j. Thus we have shown that V itself is a finite direct sum of elements of $S(E)$. Therefore, if V is indecomposable, it must be an element of $S(E)$.

Fifth Step: The ring $R(E)$ is isomorphic to the representation ring of $\mathrm{SL}_2(\mathbb{C})$, and thus 1-dimensional:
The ring $R(E)$ is the free group generated by the isomorphism classes of the elements of $S(E)$, the multiplication given by the tensor product of vector bundles. The representation ring of $\mathrm{SL}_2(\mathbb{C})$ is defined in a similar way with the isomorphism classes of the indecomposable representations being the generators of the free group. The isomorphism of the rings is given by assigning to an element of $S(E)$ its corresponding isomorphism class of indecomposable representations. To show that this assignment is well-defined we must prove that it is compatible with the tensor product. But this is clear by definition, as the functors F_1 and F_2, which give the assignment, are tensor functors.
We still have to prove that the representation ring of $\mathrm{SL}(2,\mathbb{C})$ is of Krull dimension 1: As already mentioned in the Third Step, the representations of $\mathrm{SL}(2,\mathbb{C})$ are in 1-1 correspondence with those of the Lie algebra $sl_2(\mathbb{C})$. But every irreducible representation of $sl_2(\mathbb{C})$ is a symmetric power of the standard representation V of $sl_2(\mathbb{C})$, and these satisfy the following formula for the tensor product: $S^n(V)\otimes S^m(V) = S^{n+m}(V) \oplus S^{n+m-2}(V) \oplus \cdots \oplus S^{n-m}(V)$ for $n \geq m$ (see [**4**], Part II, §11.1, §11.2). By the same computation as in Proposition 2.3, we conclude that the representation ring of $sl_2(\mathbb{C})$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Q}[x]$, and hence of Krull dimension 1.

Part 2

Flat and Higgs bundles over a compact Kähler manifold

6. Introduction

Let X be a compact complex Kähler manifold. There is an equivalence of categories ([**15**], Lemma 3.5)

$$\left\{\begin{array}{c}\text{Higgs bundles over } X,\\ \text{which are extensions of}\\ \text{polystable Higgs bundles}\\ \text{with vanishing Chern classes}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{complex}\\ \text{representations}\\ \text{of } \pi_1(X,x)\end{array}\right\},$$

extending the Narasimhan-Seshadri correspondence ([**10**], compare Section 5 of Part 1).
The category of representations of $\pi_1(X,x)$ is equivalent to the category of flat bundles over X. In the following sections we examine, under which conditions the isomorphism class of a flat bundle corresponds to an isomorphism class of Higgs bundles with zero Higgs field. If a Higgs bundle (E,θ) is an extension of two polystable Higgs bundles (E_1,θ_1) and (E_2,θ_2) with vanishing Chern classes, the isomorphism class of (E,θ) corresponds to a class in $\mathrm{H}^1_{\mathrm{Dol}}(X;\mathrm{Hom}(E_1,E_2))$. The underlying $\mathcal{C}^\infty$-bundles of E_1 and E_2 possess the structure of flat bundles, related to the Higgs structure via a so-called harmonic metric (see [**15**], Theorem 1). By [**15**], Lemma 2.2, the Dolbeault cohomology group is isomorphic to the de Rham cohomology group $\mathrm{H}^1_{\mathrm{DR}}(X;\mathrm{Hom}(E_1,E_2))$, giving the flat bundle extensions of E_1 and E_2. We consider the collection of those flat bundles, whose monodromy representation, extended to $\mathbb{C}\pi_1(X,x)$ by linearity, factors through J^2, where J denotes the kernel of the algebra homomorphism

$$\begin{array}{cccc}\epsilon & : & \mathbb{C}\pi_1(X,x) & \to & \mathbb{C}\\ & & \sum\alpha_\gamma\gamma & \to & \sum\alpha_\gamma.\end{array}$$

We will see that every such flat bundle (V,∇) is obtained as an extension of two trivial flat bundles $(\mathcal{A}^0_X,d)^{\oplus p}$ and $(\mathcal{A}^0_X,d)^{\oplus q}$, $p+q=\mathrm{rk}(V)$, and thus corresponds to an extension of two trivial Higgs bundles via the isomorphism of the de Rham and Dolbeault cohomology groups. We will show that (V,∇) corresponds to a class of Higgs bundles with zero Higgs field, if and only if its corresponding de Rham class in

$$\mathrm{Ext}^1_{\mathrm{DR}}((\mathcal{A}^0_X,d)^{\oplus p},(\mathcal{A}^0_X,d)^{\oplus q})=\mathrm{H}^1_{\mathrm{DR}}(X;(\mathcal{A}^0_X,d)^{\oplus pq})$$

is of type (0,1).

7. Cohomology with values in a flat or Higgs bundle

Let X be a compact complex Kähler manifold.
We will define de Rham and Dolbeault cohomology with values in flat and Higgs bundles over X, following [**15**]:
If (V,∇) is a flat $\mathcal{C}^\infty$-bundle over X, we define the de Rham cohomology

with values in (V, ∇) in the following way. Let V^∇ be the locally constant sheaf of flat sections of (V, ∇). It is resolved by the de Rham complex with coefficients in V

$$\mathcal{A}^0(V) \xrightarrow{\nabla} \mathcal{A}^1(V) \xrightarrow{\nabla} \mathcal{A}^2(V) \xrightarrow{\nabla} \cdots,$$

where $\nabla : \mathcal{A}^i_X \otimes_{\mathcal{A}^0_X} V \to \mathcal{A}^{i+1}_X \otimes_{\mathcal{A}^0_X} V$ takes $\omega \otimes v$ to $d\omega \otimes v + (-1)^i \omega \wedge \nabla(v)$. Since the sheaves of $\mathcal{C}^\infty$-forms are fine, the cohomology $\mathrm{H}^i(X; V^\nabla)$ is isomorphic to the cohomology of the complex of global sections

$$\Gamma(X, \mathcal{A}^0(V)) \xrightarrow{\nabla} \Gamma(X, \mathcal{A}^1(V)) \xrightarrow{\nabla} \Gamma(X, \mathcal{A}^2(V)) \xrightarrow{\nabla} \cdots.$$

We define $\mathrm{H}^i_{\mathrm{DR}}(X, (V, \nabla)) := \mathrm{H}^i(X; V^\nabla)$.
If (V, ∇) is the trivial flat bundle $(\mathcal{A}^0_X, d)$, this is the usual de Rham cohomology $\mathrm{H}^i_{\mathrm{DR}}(X)$ of the differentiable manifold X.

A Higgs bundle over X is a pair (E, θ), where E is a holomorphic vector bundle over X and θ is a holomorphic map

$$\theta : E \to E \otimes \Omega^1_X,$$

such that $\theta \wedge \theta = 0$. If (E, θ) is a Higgs bundle, we call the complex

$$E \xrightarrow{\theta\wedge} E \otimes \Omega^1_X \xrightarrow{\theta\wedge} E \otimes \Omega^2_X \xrightarrow{\theta\wedge} \cdots$$

the holomorphic Dolbeault complex. We define the Dolbeault cohomology groups

$$\mathrm{H}^i_{\mathrm{Dol}}(X, (E, \theta)) := \mathbb{H}^i(E \xrightarrow{\theta\wedge} E \otimes \Omega^1_X \xrightarrow{\theta\wedge} \cdots)$$

to be the hypercohomology of the Dolbeault complex.
There is an equivalent $\mathcal{C}^\infty$-description of a Higgs bundle: If (E, θ) is a Higgs bundle as defined above, it can be considered as a $\mathcal{C}^\infty$-bundle with a first order operator $D'' = \bar\partial + \theta$, where $\bar\partial$ defines the holomorphic structure of E by taking sections of E to $(0,1)$-forms with coefficients in E and by annihilating holomorphic sections. It holds that D'' is integrable, i.e. $(D'')^2 = 0$, because $\bar\partial^2 = 0$, $\theta \wedge \theta = 0$, and $\bar\partial(\theta) = 0$, as θ is holomorphic. Further it fulfills the Leibniz rule

$$D''(fe) = \bar\partial(f) \wedge e + f D''(e)$$

for f a section of $\mathcal{A}^0$ and e a section of E.
Conversely, let $D'' = \bar\partial + \theta$ be an operator on a $\mathcal{C}^\infty$-vector bundle E, so that $\bar\partial$ takes sections of E to $(0,1)$-forms with coefficients in E and θ takes sections of E to $(1,0)$-forms with coefficients in E. Such an operator defines a Higgs structure on E, if and only if it is integrable and fulfills the Leibniz rule. These conditions imply that $\bar\partial^2 = 0$,

$\theta \wedge \theta = 0$, and that θ is holomorphic. The complex of sheaves of $\mathcal{C}^\infty$-sections

$$\mathcal{A}^0(E) \xrightarrow{D''} \mathcal{A}^1(E) \xrightarrow{D''} \mathcal{A}^2(E) \cdots$$

gives a fine resolution of the holomorphic Dolbeault complex. Therefore the cohomology $\mathrm{H}^i_{\mathrm{Dol}}(X, (E, \theta))$ of the holomorphic Dolbeault complex is isomorphic to the cohomology of the complex of global sections

$$\Gamma(X, \mathcal{A}^0(E)) \xrightarrow{D''} \Gamma(X, \mathcal{A}^1(E)) \xrightarrow{D''} \Gamma(X, \mathcal{A}^2(E)) \cdots .$$

If (E, θ) is the trivial Higgs bundle $(\mathcal{O}, 0)$, this is the same as the usual Dolbeault cohomology $\mathrm{H}^i_{\mathrm{Dol}}(X) \cong \oplus_{p+q=i} \mathrm{H}^p(X, \Omega^q_X)$.

The following result can be found in [**15**], Section 1:

Proposition 7.1. (1) *Let (V, ∇) be a semisimple flat bundle. Then there is choice of a metric on V, called a harmonic metric, which induces a Higgs operator D'' on V.*

(2) *Let (E, D'') be a polystable Higgs bundle of degree zero with vanishing Chern classes. Then there is a choice of a metric on E, called a harmonic metric, which induces a flat connection D on E.*

(3) *The two constructions are inverse to each other. Thus the bundles of the above kind carry the structure of both flat and Higgs bundle.*

8. Extensions of flat and Higgs bundles

Proposition 8.1. *There is a group isomorphism*

$$\mathrm{H}^1_{\mathrm{DR}}(X; (\mathcal{A}^0, d)^{\oplus pq}) \cong \mathrm{Ext}^1_{\mathrm{DR}}((\mathcal{A}^0_X, d)^{\oplus p}, (\mathcal{A}^0_X, d)^{\oplus q}),$$

where $Ext^1_{\mathrm{DR}}((\mathcal{A}^0_X, d)^{\oplus p}, (\mathcal{A}^0_X, d)^{\oplus q})$ is the group of isomorphism classes of extensions of flat bundles of $(\mathcal{A}^0_X, d)^{\oplus p}$ by $(\mathcal{A}^0_X, d)^{\oplus q}$.

Proof. Let (V, ∇_V) be an extension of $(\mathcal{A}^0_X, d)^{\oplus p}$ by $(\mathcal{A}^0_X, d)^{\oplus q}$, and let $N := p + q$. We can choose a $\mathcal{C}^\infty$-frame $f = (f_1, \ldots, f_N)$ for V compatible with the extension such that the connection matrix $A(f)$ of ∇_V with respect to this frame is of the form

$$A(f) = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix},$$

with $\omega = (\omega_{ij})_{i=1,\ldots,q,\, j=1,\ldots,p} \in \Gamma(X, \mathcal{A}^1_X)^{\oplus pq}$. The integrability condition $dA(f) + A(f) \wedge A(f) = 0$ implies that $d\omega = 0$, i.e. all forms ω_{ij} are closed. Therefore

$$([\omega_{ij}]) \in \mathrm{H}^1_{\mathrm{DR}}(X; (\mathcal{A}^0, d)^{\oplus pq})$$

is a well-defined de Rham class. If we choose another frame $f' = (f'_1, \ldots, f'_N)$ for V, which is compatible with the extension, the resulting de Rham class is the same: The connection matrix with respect to the new frame is

$$A(f') = \begin{pmatrix} 0 & \omega' \\ 0 & 0 \end{pmatrix} = g^{-1}dg + g^{-1}A(f)g$$

where

$$g = \begin{pmatrix} E_q & \beta \\ 0 & E_p \end{pmatrix}$$

is the matrix of change of frame, with E_p and E_q denoting the identity matrices of rank p and q. This implies that

$$\omega' = d\beta + \omega,$$

hence ω' and ω define the same class in $\mathrm{H}^1_{\mathrm{DR}}(X; (\mathcal{A}^0_X, d)^{\oplus pq})$.

Conversely, let $\alpha \in \mathrm{H}^1_{\mathrm{DR}}(X; (\mathcal{A}^0_X, d)^{\oplus pq})$. We want to construct a flat bundle extension represented by this class. We define the underlying $\mathcal{C}^\infty$-bundle to be $V = (\mathcal{A}^0_X)^{\oplus q} \oplus (\mathcal{A}^0_X)^{\oplus p}$.
There are 1-forms $\omega_{ij} \in \Gamma(X, \mathcal{A}^1_X)$ such that $\alpha_{ij} = \omega_{ij}$ for all $i = 1, \ldots, q, i = 1, \ldots, p$. We define the connection matrix A of ∇_V with respect to the trivial frame to be

$$A = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}.$$

If we choose other 1-forms (ω'_{ij}) representing α, for all i and j there are $\beta_{ij} \in \Gamma(\mathcal{A}^0_X)$ such that $\omega'_{ij} = \omega_{ij} + d\beta_{ij}$. The matrix

$$A' = \begin{pmatrix} 0 & \omega' \\ 0 & 0 \end{pmatrix}$$

defines another connection ∇'_V on $V = (\mathcal{A}^0_X)^{\oplus q} \oplus (\mathcal{A}^0_X)^{\oplus p}$. It remains to show that (V, ∇_V) and (V, ∇'_V) are isomorphic extensions. We let $\beta = (\beta_{ij})$ define a mapping from $(\mathcal{A}^0_X)^{\oplus p}$ to $(\mathcal{A}^0_X)^{\oplus q}$ by mapping a section a of $(\mathcal{A}^0_X)^{\oplus p}$ to the section βa of $(\mathcal{A}^0_X)^{\oplus q}$.

Then $\mathrm{id}_{(\mathcal{A}^0_X)^{\oplus q} \oplus (\mathcal{A}^0_X)^{\oplus p}} - 0 \oplus \beta$ is an isomorphism of flat bundles from (V, ∇_V) to (V, ∇'_V), making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{A}^0_X, d)^{\oplus q} & \longrightarrow & (V, \nabla_V) & \longrightarrow & (\mathcal{A}^0_X, d)^{\oplus p} & \longrightarrow & 0 \\ & & \Vert & & \downarrow \mathrm{id}_{-0\oplus\beta} & & \Vert & & \\ 0 & \longrightarrow & (\mathcal{A}^0_X, d)^{\oplus q} & \longrightarrow & (V, \nabla'_V) & \longrightarrow & (\mathcal{A}^0_X, d)^{\oplus p} & \longrightarrow & 0. \end{array}$$

Hence the two extensions are isomorphic. $\square$

Before we prove a corresponding proposition for the extensions of $(\mathcal{O}_X, 0)^{\oplus p}$ by $(\mathcal{O}_X, 0)^{\oplus q}$ in the category of Higgs bundles, we recall Dolbeault's theorem:

THEOREM 8.2. *(Dolbeault)*
There is a group isomorphism

$$\mathrm{H}^1_{\mathrm{Dol}}(X) \cong \mathrm{H}^1(X, \mathcal{O}_X) \oplus \mathrm{H}^0(X, \Omega^1).$$

PROPOSITION 8.3. *There is a group isomorphism*

$$\mathrm{H}^1_{\mathrm{Dol}}(X)^{\oplus pq} \cong \mathrm{H}^1_{\mathrm{Dol}}(X; (\mathcal{O}, \theta = 0)^{\oplus pq}) \cong \mathrm{Ext}^1_{\mathrm{Dol}}((\mathcal{O}, 0)^{\oplus p}, (\mathcal{O}, 0)^{\oplus q})$$

where $\mathrm{Ext}^1_{\mathrm{Dol}}((\mathcal{O}, 0)^{\oplus p}, (\mathcal{O}, 0)^{\oplus q})$ *is the group of isomorphism classes of extensions of* $\mathcal{O}^{\oplus p}$ *by* $\mathcal{O}^{\oplus q}$*, both equipped with the zero Higgs field, in the category of Higgs bundles over* X*.*

PROOF. Let $[\phi]$ be a class in $\mathrm{H}^1_{\mathrm{Dol}}(X)^{\oplus pq}$. Since by Theorem 8.2,

$$\mathrm{H}^1_{\mathrm{Dol}}(X) \cong \mathrm{H}^1(X, \mathcal{O}_X) \oplus \mathrm{H}^0(X, \Omega^1),$$

there exist $\Psi \in \mathrm{Ker}\ \bar{\partial} : \Gamma(X, \mathcal{A}^{0,1}_X)^{\oplus pq} \to \Gamma(X, \mathcal{A}^{0,2}_X)^{\oplus pq}$ and $\omega \in \mathrm{H}^0(X, \Omega^1)^{\oplus pq}$, such that $[\phi] = [\Psi] + \omega$. The class $[\Psi]$ defines an isomorphism class of vector bundle extensions of $(\mathcal{O}_X)^{\oplus p}$ by $(\mathcal{O}_X)^{\oplus q}$. Let

$$0 \to (\mathcal{O}_X)^{\oplus q} \xrightarrow{\alpha} E \xrightarrow{\beta} (\mathcal{O}_X)^{\oplus p} \to 0$$

be a representative. Locally, this sequence splits.
Let s be a splitting over an open set $U \subset X$, i.e. s is an $\mathcal{O}$-linear map $s : \mathcal{O}_U \to F$, such that $\beta \circ s = \mathrm{id}$.
The splitting s induces a holomorphic frame $f = (f_1, \dots, f_N)$, with $N = p + q$, for E over U, compatible with the extension, by setting $f_i := \alpha(e_i)$, $i = 1, \dots, q$, and $f_{q+i} := s(e_i)$, $i = 1, \dots, p$. We want to define a Higgs field θ on E, which is compatible with the zero Higgs fields on $(\mathcal{O})^{\oplus p}$ and $(\mathcal{O}_X)^{\oplus q}$. With respect to the frame f we define it to be given by the matrix

$$\theta(f) = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}.$$

We must show that this defines a global holomorphic map

$$\theta : E \to \Omega^1_X \otimes E.$$

The map defined in this way will then obviously fulfill that $\theta \wedge \theta = 0$. Hence it remains to prove that θ is well-defined under the choice of another splitting $\tilde{s}$. Another splitting $\tilde{s}$ induces a frame $\tilde{f} = fg$ for E, which is given by a matrix g of the form

$$g = \begin{pmatrix} E_q & \beta \\ 0 & E_p \end{pmatrix}$$

with $\beta = (\beta_{ij})_{i=1,\ldots,q,\, j=1,\ldots,p} \in \Gamma(U, \mathcal{O}_X)^{\oplus pq}$. With $\theta(f')$ defined in the same way as $\theta(f)$, only depending on ω, we see that $g^{-1}\theta(f)g = \theta(f')$, which proves that θ is globally well-defined.
If

$$0 \to (\mathcal{O}_X)^{\oplus q} \xrightarrow{\alpha} E' \xrightarrow{\beta} (\mathcal{O}_X)^{\oplus p} \to 0$$

is another representative of $[\Psi]$, there is an isomorphism $\tau : E \to E'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{O}_X)^{\oplus q} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & (\mathcal{O}_X)^{\oplus p} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \tau} & & \| & & \\ 0 & \longrightarrow & (\mathcal{O}_X)^{\oplus q} & \longrightarrow & E' & \longrightarrow & (\mathcal{O}_X)^{\oplus p} & \longrightarrow & 0 \end{array}$$

commutes. If f' is a frame on E', induced by a splitting in the same way as above, the matrix representation of $\theta'(f')$ coincides with the matrix representation of $\theta(f)$, since this only depended on ω. Then it obviously holds that $(\tau(f) \otimes \mathrm{id}) \circ \theta(f) = \theta'(f') \circ \tau(f)$, hence τ is also an isomorphism of the Higgs bundles (E, θ) and (E', θ'). Therefore the two defined extensions of Higgs bundles are isomorphic.

Conversely, let (E, θ) be a representative of an isomorphism class of extensions in $\mathrm{Ext}^1_{\mathrm{Dol}}((\mathcal{O}_X, 0)^{\oplus p}, (\mathcal{O}_X, 0)^{\oplus q})$, i.e. there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{O}_X)^{\oplus q} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & (\mathcal{O}_X)^{\oplus p} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & (\Omega^1_X)^{\oplus q} & \longrightarrow & \Omega^1_X \otimes E & \longrightarrow & (\Omega^1_X)^{\oplus p} & \longrightarrow & 0. \end{array}$$

The exact sequence in the first row of this diagram defines an extension class $[\Psi] \in \mathrm{H}^1(X, (\mathcal{O})^{\oplus pq})$, which is independent of the chosen representative (E, θ). We have to show that the Higgs field θ determines an $\omega \in \mathrm{H}^0(X, \Omega^1_X)^{\oplus pq}$, so that we obtain a Dolbeault class

$$[\Psi] + \omega \in \mathrm{H}^1(X, \mathcal{O}_X)^{\oplus pq} \oplus \mathrm{H}^0(X, \Omega^1_X)^{\oplus pq} \cong \mathrm{H}^1_{\mathrm{Dol}}(X, (\mathcal{O}_X)^{\oplus pq}).$$

Let s be a local splitting over U for the exact sequence in the first row of the above diagram. Then $f = (f_1, \ldots, f_N)$, $N = p + q$, is a frame for E, where we set $f_i := \alpha(e_i)$, $i = 1, \ldots, q$, and $f_{q+i} := s(e_i)$, $i = 1, \ldots p$. In this frame, θ is represented by a matrix

$$\theta(f) = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix},$$

where $\omega_{ij} \in \Gamma(U, \Omega^1_X)$. Another frame $f' = (f'_1, \ldots, f'_r)$ compatible with the extension is obtained by the choice of another splitting. We have seen in the first part of the proof, that $f' = fg$, where g is a matrix of the form

$$g = \begin{pmatrix} E_q & \beta \\ 0 & E_p \end{pmatrix},$$

and we know that

$$\theta(f') = \begin{pmatrix} 0 & \omega' \\ 0 & 0 \end{pmatrix} = g^{-1}\theta(f)g,$$

as θ is a globally defined. Hence it holds that $\omega' = \omega$, and we obtain a well-defined element of $\mathrm{H}^0(X, \Omega^1)^{\oplus pq}$.

The two constructions are clearly inverse to each other, hence the two groups are isomorphic. □

COROLLARY 8.4. *The constructions in the proof of the previous theorem yield that a class in* $H^1_{\mathrm{Dol}}(X)^{\oplus pq}$ *corresponds to a Higgs bundle extension of* $(\mathcal{O}, 0)^{\oplus p}$ *by* $(\mathcal{O}, 0)^{\oplus q}$ *with Higgs field equal to zero, if and only if it is a class in* $H^1(X, \mathcal{O}_X)^{\oplus pq}$ *and is hence the same as a vector bundle extension of* $(\mathcal{O}_X)^{\oplus p}$ *by* $(\mathcal{O}_X)^{\oplus q}$.

9. Differential graded categories

We explain the formalism of differential graded categories, following Simpson [**15**], Section 3.

DEFINITION 9.1. *A differential graded category is a* $\mathbb{C}$*-linear category such that for any two objects* U *and* V*,* $\mathrm{Hom}(U, V)$ *is a differential graded algebra*

$$\oplus_{i \geq 0} \mathrm{Hom}^i(U, V),$$

i.e. it is equipped with a differential d *of degree 1 such that* $d^2 = 0$ *and such that*

$$d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$$

for the composition of two morphisms f *and* g*.*
Furthermore, we require that for every object U *the identity* 1_U *is a morphism of degree zero with* $d(1_U) = 0$*.*

An isomorphism between two objects of a differential graded category is map f *of degree zero, such that* $d(f) = 0$*, and such that there is an inverse with the same properties.*
We denote by

$$\mathrm{Ext}^i(U, V)$$

the i*-th cohomology of the Hom-complex* $\mathrm{Hom}(U, V)$*.*

DEFINITION 9.2. *An extension in a differential graded category* $\mathcal{C}$ *is a pair of morphisms*

$$M \xrightarrow{a} U \xrightarrow{b} N$$

with $a \in \mathrm{Hom}^0(M, U)$*,* $b \in \mathrm{Hom}^0(U, N)$*,* $ba = 0$*,* $d(a) = 0$ *and* $d(b) = 0$*, and such that a splitting exists, i.e. there is a pair of morphisms* g *and* h *of degree 0,*

$$M \xleftarrow{g} U \xleftarrow{h} N,$$

such that $ga = 1$*,* $bh = 1$*,* $gh = 0$*, and* $ag + hb = 1$*.*

Examples.
Now we will consider two examples of differential graded categories, which are needed in our context.
(1) Let X be a compact complex Kähler manifold, and let $\mathcal{C}_{\mathrm{DR}}$ denote the differential graded category whose objects are all flat bundles over X and with the Hom-complex

$$\mathrm{Hom}^\bullet(U, V) = (\Gamma(X, \mathcal{A}^\bullet(\mathrm{Hom}(U, V))), D),$$

where the composition of homomorphisms is given by the wedging of forms. The connection D on $\mathrm{Hom}(U, V)$ is induced by the connections D_U and D_V on U and V in the following way:
The underlying $\mathcal{C}^\infty$-bundle $\mathrm{Hom}(U, V)$ is isomorphic to $U^\vee \otimes V$. The connection $D_{U^\vee}$ on $U^\vee \cong \mathrm{Hom}(U, \mathcal{A}^0_X)$ is defined by the formula

$$D_{U^\vee}(\lambda)(u) + \lambda(D_U(u)) = d(\lambda u),$$

where $\lambda \in \mathrm{Hom}(U, \mathcal{A}^0_X)$ and $u \in U$. The connection $D_{U^\vee} \otimes D_V$ on $U^\vee \otimes V$ is defined by $(D_{U^\vee} \otimes D_V)(\lambda \otimes v) = D_{U^\vee}(\lambda) \otimes v + \lambda \otimes D_V(v)$. By $\mathcal{C}^s_{\mathrm{DR}}$ we denote the full subcategory of $\mathcal{C}_{\mathrm{DR}}$ consisting of semisimple objects.

If

$$(U, D_U) \xrightarrow{a} (E, D_E) \xrightarrow{b} (V, D_V)$$

is an extension of objects (U, D_U) and (V, D_V) in $\mathcal{C}_{\mathrm{DR}}$, then the conditions $D(a) = 0$ and $D(b) = 0$ imply that a and b map flat sections to flat sections. (Note that D denotes two different differentials, namely the one on $\mathrm{Hom}(U, E)$ and the one on $\mathrm{Hom}(E, V)$.) Together with the condition that a and b are of degree zero, we obtain that they are morphisms of flat bundles. Note that it is not required that the morphisms g and h of the splitting are in the kernel of D. Hence the splitting need not exist in the category of flat bundles, but only in the underlying category of $\mathcal{C}^\infty$- bundles.

(2) Let X be a compact complex Kähler manifold. By $\mathcal{C}_{\mathrm{Dol}}$, we denote the differential graded category whose objects are those Higgs bundles on X which are extensions of stable Higgs bundles with vanishing Chern classes. Its Hom-complex is

$$\mathrm{Hom}^\bullet(U, V) = (\Gamma(X, \mathcal{A}^\bullet(\mathrm{Hom}(U, V))), D''),$$

where the composition of morphisms is obtained by the wedging of forms. The Higgs operator D'' on $\mathrm{Hom}(U, V)$ is induced by the Higgs operators D''_U and D''_V on U and V in the following way: The Higgs bundle $(\mathrm{Hom}(U, V), D'')$ is isomorphic to $(U, D''_U)^\vee \otimes (V, D''_V)$. The underlying vector bundle of $(U, D''_U)^\vee$ is the dual vector bundle $U^\vee$. This is endowed with the Higgs operator given by the formula

$$D''_{U^\vee}(\lambda)(u) + \lambda(D''_U(u)) = \bar{\partial}(\lambda(u))$$

for λ a section of $U^\vee = \mathrm{Hom}(U, \mathcal{O})$ and u a section of U. The Higgs operator $D'' := D''_{U^\vee \otimes V}$ is then given by

$$D''(\lambda \otimes v) := D''_{U^\vee}(\lambda) \otimes v + \lambda \otimes D''_V(v)$$

for $\lambda \in U^\vee$ and $v \in V$.
$\mathcal{C}^s_{\mathrm{Dol}}$ is the full subcategory consisting of semisimple objects, i.e. polystable Higgs bundles.

If

$$(U, D''_U) \xrightarrow{a} (E, D''_E) \xrightarrow{b} (V, D''_V)$$

is an extension of objects (U, D''_U) and (V, D''_V) in $\mathcal{C}_{\mathrm{Dol}}$, then the conditions $D''(a) = 0$ and $D''(b) = 0$ imply that a and b map holomorphic sections to holomorphic sections. (Note that D'' denotes two different differentials, namely the one on $\mathrm{Hom}(U, E)$ and the one on $\mathrm{Hom}(E, V)$.) Together with the condition that a and b are of degree zero, we obtain that they are morphisms of Higgs bundles. Note that it is not required that the morphisms g and h of the splitting are in the kernel of D''. Hence the splitting need not exist in the category of Higgs bundles, but only in the underlying category of $\mathcal{C}^\infty$- bundles. □

Next we want to construct the completion of a differential graded category (see [**15**] and [**5**]). Before we do this, we have to introduce the notion of a complete category. Given an extension in a differential graded category $\mathcal{C}$,

$$M \underset{g}{\overset{a}{\rightleftarrows}} U \underset{h}{\overset{b}{\rightleftarrows}} N,$$

we define $\delta \in \mathrm{Hom}^1(N, M)$ by $\delta := g \circ d(h)$. Since $d(\delta) = 0$, δ defines a class $[\delta] \in \mathrm{Ext}^1(N, M)$.
$\mathcal{C}$ is called a *complete category*, if every element of $\mathrm{Ext}^1(N, M)$ comes from an extension in that way.

First we construct a new differential graded category $\bar{\mathcal{C}}$, whose objects are pairs (U, η) with U an object of $\mathcal{C}$ and $\eta \in \mathrm{Hom}^1(U, U)$ an endomorphism of U of degree one, satisfying $d(\eta) + \eta^2 = 0$. If (U, η)

and (V, ξ) are two objects of $\bar{\mathcal{C}}$, we define the differential graded algebra $\mathrm{Hom}((U, \eta), (V, \xi))$ to be $\mathrm{Hom}(U, V)$ with the same grading, but we introduce a new differential, namely

$$\hat{d}(f) = d(f) + \xi f - (-1)^{\deg(f)} f \eta.$$

The category $\mathcal{C}$ can be embedded into $\bar{\mathcal{C}}$ by mapping an object U to $(U, 0)$.
Then we define the *completion* $\hat{\mathcal{C}}$ of $\mathcal{C}$ to be the full subcategory of $\bar{\mathcal{C}}$, whose objects are successive extensions of objects of $\mathcal{C}$.

LEMMA 9.3. *The completion $\hat{\mathcal{C}}$ is a complete category.*

PROOF. If (U, η) and (V, ξ) are objects of $\hat{\mathcal{C}}$, and if

$$[\delta] \in \mathrm{Ext}^1((V, \xi), (U, \eta)) = \mathrm{H}^1(X; \mathrm{Hom}(V, U)),$$

then the corresponding extension can be constructed in the following way, described by Simpson in [**15**], Lemma (3.1):
Let $\delta \in \mathrm{Hom}^1(V, U)$ be a representative of the extension class. Then the extension corresponding to $[\delta]$ is

$$(U, \eta) \to (U \oplus V, \eta \oplus (\xi + \delta)) \to (V, \xi).$$

We have to prove that this is independent of the chosen representative. Let $\tilde{\delta}$ be another representative of the same class, hence there is a $\beta \in \mathrm{Hom}^0(V, U)$ such that $\tilde{\delta} = \delta + \hat{d}\beta$. Then the extension corresponding to $\tilde{\delta}$ is

$$(U, \eta) \to (U \oplus V, \eta \oplus (\xi + \tilde{\delta})) \to (V, \xi).$$

We have to show that the two extensions are in the same extension class. But this is clear since $1_{U \oplus V} - 0 \oplus \beta$ is an isomorphism of $U \oplus V$ in the category $\bar{\mathcal{C}}$ (fulfilling $\hat{d}(1_{U \oplus V} - 0 \oplus \beta) = 0$), such that the diagram

$$\begin{array}{ccccc} (U, \eta) & \longrightarrow & (U \oplus V, \eta \oplus (\xi + \delta)) & \longrightarrow & (V, \xi) \\ \| & & \downarrow{\scriptstyle 1_{U \oplus V} - 0 \oplus \beta} & & \| \\ (U, \eta) & \longrightarrow & (U \oplus V, \eta \oplus (\xi + \tilde{\delta})) & \longrightarrow & (V, \xi) \end{array}$$

commutes. □

Coming back to our examples, we can form the completions $\hat{\mathcal{C}}^s_{\mathrm{DR}}$ and $\hat{\mathcal{C}}^s_{\mathrm{Dol}}$. There is the following lemma:

LEMMA 9.4. *(Simpson, [**15**])*
There are equivalences of differential graded categories

$$\hat{\mathcal{C}}^s_{\mathrm{DR}} \cong \mathcal{C}_{\mathrm{DR}} \text{ and } \hat{\mathcal{C}}^s_{\mathrm{Dol}} \cong \mathcal{C}_{\mathrm{Dol}}.$$

Remark.
In the Section 11, we will need a concrete description of the functors

$$\hat{\mathcal{C}}^s_{\mathrm{DR}} \to \mathcal{C}_{\mathrm{DR}} \text{ and } \hat{\mathcal{C}}^s_{\mathrm{Dol}} \to \mathcal{C}_{\mathrm{Dol}},$$

which can be found in [**15**], Section 3:
Let a pair $((U, D), \eta)$ be an object of $\hat{\mathcal{C}}^s_{\mathrm{DR}}$, i.e. (U, D) is a semisimple flat bundle and η is 1-form with values in $\mathrm{End}(U)$ satisfying $D(\eta) + \eta^2 = 0$. The corresponding (in general not semisimple) object of $\mathcal{C}_{\mathrm{DR}}$ is $(U, D + \eta)$.
Analogously, an object $((U, D''), \eta)$ of $\hat{\mathcal{C}}^s_{\mathrm{Dol}}$ is mapped to $(U, D'' + \eta)$.

10. Unipotent representations

Let G be a group.
The homomorphism of G to the trivial group induces the algebra homomorphism

$$\begin{array}{rcll} \epsilon & : & \mathbb{C}G & \to & \mathbb{C} \\ & & \sum \alpha_g g & \to & \sum \alpha_g \end{array}$$

The kernel

$$J := \mathrm{Ker}\ (\epsilon)$$

is called *augmentation ideal.* It is spanned by $\{g - 1 \,|\, g \in G\}$.

Definition 10.1. *Let V be a complex vector space. A representation*

$$\rho : G \to Aut(V)$$

is called unipotent, if for all $g \in G$ the automorphism $\rho(g)$ is unipotent (i.e. $\rho(g) = 1 + n_g$ with n_g a nilpotent endomorphism of V), or equivalently, if for all $g \in G$ all eigenvalues of $\rho(g)$ are equal to 1.

Every representation $\rho : G \to \mathrm{Aut}(V)$ can be extended by linearity to an algebra homomorphism

$$\bar{\rho} : \mathbb{C}G \to \mathrm{End}(V).$$

Proposition 10.2. *The following are equivalent:*

(1) *ρ is unipotent.*
(2) *$\bar{\rho}$ induces a homomorphism*

$$\bar{\bar{\rho}} : \mathbb{C}G/J^n \to End(V),$$

where $n := \dim V$.

For the proof of the proposition we need Kolchin's theorem (e.g. in [**14**], Part I, Chapter V)

Theorem 10.3. *(Kolchin)*
Let V be a finite-dimensional $\mathbb{C}$-vector space, and G a subgroup of $Aut(V)$, such that every element $g \in G$ is unipotent (i.e. $g = 1 + n$, with $n \in End(V)$ nilpotent).
Then there exists a basis for V in which all elements $g \in G$ are represented simultaneously by triangular matrices, (hence by triangular matrices with 1's on the diagonal, since the eigenvalues are all 1 by hypothesis.)

Proof of the proposition:
(2)⇒(1): If $\bar{\bar{\rho}}$ is well-defined, every element of J^n is mapped to zero by $\bar{\rho}$.
For every $g \in G$, $g - 1$ is in J, thus $(g-1)^n \in J^n$.
Therefore

$$0 = \bar{\rho}((g-1)^n) = (\rho(g) - 1)^n,$$

and we obtain that $\rho(g)$ is unipotent for every $g \in G$, which means that ρ is unipotent.

(1)⇒(2): The assertion follows from Kolchin's theorem:
We assume that $\rho : G \to \mathrm{Aut}(V)$ is unipotent. Then $\rho(G)$ is a subgroup of $\mathrm{Aut}(V)$, whose elements are all unipotent. Applying Kolchin's theorem to $\rho(G)$, we know that there is a basis for V, in which all elements of $\rho(G)$ are simultaneously represented by upper-triangular matrices with 1's on the diagonal.
Thus ρ is of the form

$$\begin{array}{cccc} \rho & : & G & \to & \mathrm{GL}(n,\mathbb{C}) \\ & & g & \to & A_g \end{array},$$

where $n = \dim V$ and $A_g = E_n + N_g$ with the identity matrix E_n and an upper-triangular matrix N_g with 0's on the diagonal.
Therefore we obtain for arbitrary $g_1, \ldots, g_n \in G$ that

$$\begin{array}{rcl} \bar{\rho}(\Pi_{i=1}^n (g_i - 1)) & = & \Pi_{i=1}^n (\rho(g_i) - E_n) \\ & = & \Pi_{i=1}^n N_{g_i} \\ & = & 0, \end{array}$$

which means that $\bar{\rho}$ induces a well-defined homomorphism

$$\bar{\bar{\rho}} : \mathbb{C}G/J^n \to \mathrm{End}(V).$$

□

11. Extensions of trivial bundles

In the following, we denote by 1 the trivial bundle in the category $\mathcal{C}_{\mathrm{DR}}$, i.e. $1 = (\mathcal{A}^0, d)$, or the trivial bundle in the category $\mathcal{C}_{\mathrm{Dol}}$, i.e.

$1 = (\mathcal{A}^0_X, \bar{\partial})$.
We consider extensions of $1^{\oplus p}$ by $1^{\oplus q}$ in these categories. Since by Lemma 9.3 these are complete categories, the considered extensions are classified by the extension groups

$$\mathrm{Ext}^1_{\mathrm{Dol/DR}}(1^{\oplus p}, 1^{\oplus q}) = \mathrm{H}^1_{\mathrm{Dol/DR}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q})).$$

Considering the coordinates of $\mathrm{Hom}(1^{\oplus p}, 1^{\oplus q})$, by renumbering the coordinates we obtain an isomorphism

$$\mathrm{Hom}(1^{\oplus p}, 1^{\oplus q}) \cong 1^{\oplus pq},$$

which induces an isomorphism of cohomology groups

$$\mathrm{H}^1_{\mathrm{Dol/DR}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q})) \cong \mathrm{H}^1_{\mathrm{Dol/DR}}(X; 1^{\oplus pq}) \cong \mathrm{H}^1_{\mathrm{Dol/DR}}(X)^{\oplus pq}.$$

This isomorphism is obtained by just renumbering the coordinates of representatives.

Proposition 11.1. *Let $p, q \in \mathbb{N} - \{0\}$ and (E, D'') be a Higgs bundle in $\mathcal{C}_{\mathrm{Dol}}$, which is an extension of $1^{\oplus p}$ by $1^{\oplus q}$. Then (E, D'') is a Higgs bundle with Higgs field equal to zero, if and only if its corresponding class $[\delta] \in Ext^1_{\mathrm{Dol}}(1^{\oplus p}, 1^{\oplus q}) = H^1_{\mathrm{Dol}}(X; Hom(1^{\oplus p}, 1^{\oplus q}))$ is of type (0,1), i.e. $[\delta] \in H^1(X, \mathcal{O}_X)^{\oplus pq}$, via the identification*

$$\mathrm{H}^1_{\mathrm{Dol}}(X; Hom(1^{\oplus p}, 1^{\oplus q})) \cong \mathrm{H}^1_{\mathrm{Dol}}(X)^{\oplus pq} \cong \mathrm{H}^1(X, \mathcal{O}_X)^{\oplus pq} \oplus \mathrm{H}^0(X; \Omega^1_X)^{\oplus pq}.$$

Proof. The given Higgs bundle extension corresponds to a class

$$[\delta] \in \mathrm{H}^1_{\mathrm{Dol}}(X, \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q})).$$

Using the remark at the end of Section 9, we obtain that the corresponding extension in $\hat{\mathcal{C}}_{\mathrm{Dol}}$ is given by

$$(1^{\oplus q} \oplus 1^{\oplus p}, 0 \oplus (0 + \delta)).$$

This maps to the Higgs bundle (in $\mathcal{C}_{\mathrm{Dol}}$)

$$((\mathcal{A}^0_X)^{\oplus q} \oplus (\mathcal{A}^0_X)^{\oplus p}), D'' = \bar{\partial}^{\oplus (q+p)} + \begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix}).$$

It suffices to look at the Higgs field of this bundle, since it is in the same extension class as (E, D'') and since a Higgs bundle has Higgs field equal to zero if and only if every other representative of the same extension class does.
Now δ can be split into

$$\delta = \delta^{1,0} + \delta^{0,1} \in \Gamma(X, \mathcal{A}^{1,0}(\mathrm{Hom}(1^{\oplus p}, 1^{\oplus q}))) \oplus \Gamma(X, \mathcal{A}^{0,1}(\mathrm{Hom}(1^{\oplus p}, 1^{\oplus q}))).$$

Then $\delta^{0,1}$ defines the holomorphic structure on $(\mathcal{A}^0_X)^{\oplus (p+q)}$ together with $\bar{\partial}^{\oplus (q+p)}$, whereas the Higgs field is defined by $\delta^{1,0}$ (compare to the proof of Proposition 8.3).
Therefore, the Higgs field is zero if and only if $[\delta]$ is of type (0,1). □

Since by definition $\mathrm{Ext}^1_{\mathrm{Dol}}(1^{\oplus p}, 1^{\oplus q}) = \mathrm{H}^1_{\mathrm{Dol}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q}))$ and $\mathrm{Ext}^1_{\mathrm{DR}}(1^{\oplus p}, 1^{\oplus q}) = \mathrm{H}^1_{\mathrm{DR}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q}))$, it follows from the following proposition that there is an isomorphism of de Rham and Dolbeault extension classes of semisimple flat and Higgs bundles.

Proposition 11.2. *Let E be a harmonic bundles, i.e. E is a $\mathcal{C}^\infty$-bundle, which carries the structure of both flat and Higgs bundle related by a harmonic metric in the sense of Proposition 7.1.*
Then there is a natural isomorphism of cohomology groups

$$\mathrm{H}^1_{\mathrm{DR}}(X; E) \cong \mathrm{H}^1_{\mathrm{Dol}}(X; E),$$

and the isomorphism is obtained by considering harmonic representatives of the cohomology classes.

Proof:
This follows from [**15**], Lemma (2.2).

Remark.
In the following, we restrict ourselves to extensions of $1^{\oplus p}$ by $1^{\oplus q}$ and use the above isomorphism of cohomology groups only in the special case

$$\mathrm{H}^1_{\mathrm{DR}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q})) \cong \mathrm{H}^1_{\mathrm{Dol}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q})),$$

where it can also be concluded from classical Hodge theory via the isomorphism $\mathrm{Hom}(1^{\oplus p}, 1^{\oplus q}) \cong 1^{\oplus pq}$. □

Using the isomorphism of the de Rham and Dolbeault extension classes, we want to express Proposition 11.1 in terms of the monodromy representations of flat bundles:

Definition 11.3. *A flat bundle is called unipotent, if its monodromy representation ρ is unipotent.*

The unipotent flat bundles are exactly those, which are successive extensions of $(\mathcal{A}^0_X, d)$ by itself, as follows from the following proposition:

Proposition 11.4. *There is an equivalence of categories between the category of unipotent representations of $\pi_1(X, x)$ and the category of flat bundles, which are successive extensions of $(\mathcal{A}^0_X, d)$.*
The correspondence is the usual Riemann-Hilbert correspondence.

Proof. Let $\rho : \pi_1(X, x) \to \mathrm{Aut}(V)$ be a unipotent representation of the fundamental group, where V is an n-dimensional $\mathbb{C}$-vector space. We prove by induction over n that it corresponds to a flat bundle, which is a successive extension of $(\mathcal{A}^0_X, d)$ by itself.
If $n = 1$, ρ must be the trivial representation, as it is unipotent. Therefore it corresponds to $(\mathcal{A}^0_X, d)$.

For $n \geq 2$, due to Kolchin's theorem (Theorem 10.3), we can choose a basis for V, such that all elements of $\rho(\pi_1(X,x))$ are upper-triangular matrices with only 1's on the diagonal. Thus there exists a one-dimensional sub-vector space V_1 of V, such that $\rho_1 := \rho_{|V_1}$ is trivial, and thus corresponds to the trivial flat bundle $(\mathcal{A}^0_X, d)$. The quotient representation $(V,\rho)/(V_1,\rho_1)$ is of rank $n-1$ and again unipotent. By induction hypothesis it corresponds to an extension of $(\mathcal{A}^0_X, d)$ by itself. Conversely, if (U,∇) is a flat bundle, which is a successive extension of $(\mathcal{A}^0_X, d)$ by itself, we obtain that the corresponding monodromy representation ρ_U is an extension of trivial representations, since the Riemann-Hilbert correspondence respects exact sequences. Hence ρ_U is unipotent. $\square$

Let us consider a flat bundle (V,∇_V), which is a successive extension of trivial bundles. The monodromy representation of (V,∇_V),

$$\rho : \pi_1(X,x) \to \mathrm{Aut}(V_x)$$

is unipotent. We have seen in the previous section that this implies that there exists an $m \in \mathbb{N}$, $1 \leq m \leq N$, $N = \dim V_x$, such that

$$\bar{\rho} : \mathbb{C}\pi_1(X,x)/J^m \to \mathrm{End}(V_x)$$

is well-defined.
We have also seen, that we can choose a basis for V_x, so that all matrices in the image of ρ are upper triangular (with only ones on the diagonal.)
We write $A_\gamma := \rho(\gamma) = E_N + N_\gamma$, where E_N is the identity matrix of rank N and N_γ is upper-triangular and nilpotent with zeros on the diagonal.
Furthermore we choose a global $\mathcal{C}^\infty$-frame for V respecting the successive extension, i.e. with respect to this frame

$$\nabla_V = d + A,$$

with a matrix of 1-forms

$$A = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \cdots & \omega_{1N} \\ 0 & 0 & \omega_{23} & \dots & \omega_{2N} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & \omega_{N-1,N} \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

The associated transport is

$$T = E_N + \int A + \int AA + \cdots + \int \underbrace{A \cdots A}_{N \ times}.$$

We assume that the chosen basis for the fibre V_x is the one induced by this frame. Then the transport is connected with the monodromy representation in the following way:

$$A_\gamma = T(\gamma).$$

From now on, let us assume that $m = 2$. Then we know that for all $\gamma_1, \gamma_2 \in \pi_1(X, x)$

$$(A_{\gamma_1} - E_N)(A_{\gamma_2} - E_N) = 0.$$

This implies that

$$\begin{aligned} \rho(\gamma_1\gamma_2) &= A_{\gamma_1\gamma_2} = A_{\gamma_1}A_{\gamma_2} \\ &= ((A_{\gamma_1} - E_N) + E_N)((A_{\gamma_2} - E_N) + E_N) \\ &= (A_{\gamma_1} - E_N)(A_{\gamma_2} - E_N) + (A_{\gamma_1} - E_N) + (A_{\gamma_2} - E_N) + E_N \\ &= A_{\gamma_1} + A_{\gamma_2} - E_N \\ &= E_N + N_{\gamma_1} + N_{\gamma_2}, \end{aligned}$$

and hence that $N_{\gamma_1\gamma_2} = N_{\gamma_1} + N_{\gamma_2}$.

There is a direct sum decomposition

$$\mathbb{C}\pi_1(X, x)/J^2 = \mathbb{C} \oplus J/J^2 = \mathbb{C} \oplus \mathrm{H}_1(X, \mathbb{C}),$$

where the group isomorphism $\mathrm{H}_1(X, \mathbb{C}) \to J/J^2$ is induced by the map

$$\begin{array}{ccc} \pi_1(X, x) & \longrightarrow & J/J^2 \\ \gamma & \to & \overline{\gamma - e}. \end{array}$$

As $\bar{\rho}$ maps $c \in \mathbb{C}$ to cE_N, the restriction of $\bar{\rho}$ to $\mathrm{H}_1(X, \mathbb{C})$ already carries the complete information about $\bar{\rho}$. Because of the above formula for the N_γ, we obtain a well-defined group homomorphism

$$\begin{array}{cccc} \bar{\rho}: & \mathrm{H}_1(X, \mathbb{C}) & \longrightarrow & \mathrm{M}(N, \mathbb{C}) \\ & [\gamma] & \to & N_\gamma \end{array}$$

corresponding to the extension.

Writing $N_\gamma = (n_\gamma^{ij})$, we obtain group homomorphisms

$$\begin{array}{cccc} \bar{\rho}_{ij}: & \mathrm{H}_1(X, \mathbb{C}) & \longrightarrow & \mathbb{C} \\ & [\gamma] & \to & n_\gamma^{ij}, \end{array}$$

for $1 \leq i < j \leq N$. (For $1 \leq j \leq i \leq N$, $\bar{\rho}_{ij}$ is constantly zero and need not be considered.)

Since $\mathrm{Hom}(\mathrm{H}_1(X, \mathbb{C}), \mathbb{C}) \cong \mathrm{H}^1(X, \mathbb{C}) \cong \mathrm{H}^1_{\mathrm{DR}}(X)$, every $\bar{\rho}_{ij}$ defines a de Rham class

$$\bar{\rho}_{ij} \in \mathrm{H}^1_{\mathrm{DR}}(X).$$

From this it follows that for every $1 \le i < j \le N$ there is a closed 1-form ω'_{ij} such that $\bar\rho_{ij} = [\omega'_{ij}]$. The values of the homomorphism $\bar\rho_{ij}$ are obtained by integration over ω'_{ij},

$$\bar\rho_{ij}([\gamma]) = \int_\gamma \omega'_{ij} \ \text{ for all } [\gamma] \in \mathrm{H}_1(X,\mathbb{C}).$$

Considering the coordinate homomorphisms

$$\bar\rho_{ij}: \ \mathbb{C}\pi_1(X,x)/J^2 \longrightarrow \mathbb{C}$$

of $\bar\rho : \mathbb{C}\pi_1(X,x)/J^2 \to \mathrm{M}(N,\mathbb{C})$, not restricted to the first homology group, there is another way of attaching a closed 1-form to each $\bar\rho_{ij}$, involving Chen's theorem:

THEOREM 11.5. *(Chen)*
For each $s \ge 0$, denote by $H^0(B_s(X),x)$ the set of iterated integrals of length $\le s$, which are homotopy functionals restricted to loops in X based at x. The integration map

$$\mathrm{H}^0(B_s(X),x) \longrightarrow \mathit{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(X,x)/J^{s+1},\mathbb{C})$$

is an isomorphism.

Applying this theorem to $\bar\rho_{ij} : \mathbb{C}\pi_1(X,x)/J^2 \longrightarrow \mathbb{C}$, we obtain an iterated integral I_{ij} of length ≤ 1 such that

$$\bar\rho_{ij}(\gamma) = \langle I_{ij},\gamma\rangle \text{ for all } \gamma \in \pi_1(X,x).$$

Since I_{ij} is a homotopy functional restricted to loops based at x, it must be of the form

$$I_{ij} = c + \int \omega'_{ij},$$

with $c \in \mathbb{C}$ and a closed 1-form ω'_{ij}. As $\bar\rho_{ij}(e) = 0$, it follows that $c = 0$, hence $\bar\rho_{ij}(\gamma) = \int_\gamma \omega'_{ij}$ for all $\gamma \in \pi_1(X,x)$. It is clear, that the 1-form, attached to $\bar\rho_{ij}$ in this way, coincides with the previous one up to addition of an exact form, and hence defines the same de Rham class.

THEOREM 11.6. (1) *Let $(E,D'') \in \mathrm{Ext}^1_{\mathrm{Dol}}(1^{\oplus p},1^{\oplus q})$ be a Higgs bundle extension, and let (V,D) be an extension of trivial flat bundles, corresponding to the given Higgs bundle extension via the isomorphism*

$$\mathrm{H}^1_{\mathrm{DR}}(X;\mathrm{Hom}(1^{\oplus p},1^{\oplus q})) \cong \mathrm{H}^1_{\mathrm{Dol}}(X;\mathrm{Hom}(1^{\oplus p},1^{\oplus q})).$$

Then the monodromy representation ρ of (V,D) factors through J^2, and (E,D'') has Higgs field equal to zero, if and only if the pq classes $\bar\rho_{ij}$, $1 \le i \le q$, $q+1 \le j \le q+p$, in $\mathrm{H}^1_{\mathrm{DR}}(X) \cong \mathrm{H}^1_{\mathrm{Dol}}(X)$, induced by $\bar\rho$ in the way described above are of type (0,1). This is independent of the choice of frame for V, for any frame that is compatible with the

extension.
(2) *Conversely, if* (V, D) *is any flat bundle of rank* N*, whose monodromy representation factors through* J^2*, there exists an* $n \in \mathbb{N}$*,* $1 \leq n \leq N-1$*, such that*

$$(V, D) \in \mathrm{Ext}^1_{\mathrm{DR}}(1^{\oplus n}, 1^{\oplus(N-n)}),$$

and hence there is a Higgs pair $(E, D'') \in \mathrm{Ext}^1_{\mathrm{Dol}}(1^{\oplus n}, 1^{\oplus(N-n)})$ *such that the isomorphism classes of* (V, D) *and* (E, D'') *correspond via the isomorphism of the cohomology groups.*

Proof. (1) With respect to a frame f compatible with the extension, D is of the form

$$D = d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$$

with $\omega = (\omega_{ij})$ a $(q \times p)$-matrix of global 1-forms. The monodromy representation

$$\rho : \pi_1(X, x) \to \mathrm{GL}(p+q, \mathbb{C})$$

with respect to the basis for V_x, which is induced by the chosen frame, is upper-triangular: for every $\gamma \in \pi_1(X, x)$

$$\rho(\gamma) = \begin{pmatrix} E_q & M_\gamma \\ 0 & E_p \end{pmatrix}$$

with a $(q \times p)$-matrix M_γ with coefficients in $\mathbb{C}$. Hence ρ factors through J^2, since for all γ_1, γ_2

$$(\rho(\gamma_1) - E_{p+q})(\rho(\gamma_2) - E_{p+q}) = 0.$$

In the considerations preceding the proposition, we have seen that ρ, factoring through J^2, induces a well-defined group homomorphism

$$\bar{\rho} : \mathrm{H}_1(X, \mathbb{C}) \to \mathrm{M}(p+q, \mathbb{C}).$$

We only consider the coordinates $\bar{\rho}_{ij}$, $1 \leq i \leq q$, $q+1 \leq j \leq q+p$, since all the others are constantly zero. The $\bar{\rho}_{ij}$ are independent of the chosen frame: If f' is any other frame, compatible with the extension, the induced basis for V_x is obtained by a base change with a matrix of the form

$$g = \begin{pmatrix} E_q & h \\ 0 & E_p \end{pmatrix}.$$

With respect to the new basis, the matrix representation of $\rho(\gamma)$ coincides with the previous one. In this way we obtain pq classes $\bar{\rho}_{ij} \in \mathrm{H}^1(X, \mathbb{C})$, which are the coordinates of $[\omega] \in \mathrm{H}^1_{\mathrm{DR}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q}))$. If all coordinates of ω are harmonic 1-forms, ω is a representative for the de Rham and Dolbeault cohomology class,

$$[\omega] \in \mathrm{H}^1_{\mathrm{DR}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q})) \cong \mathrm{H}^1_{\mathrm{Dol}}(X; \mathrm{Hom}(1^{\oplus p}, 1^{\oplus q})),$$

and the type consideration follows from Proposition 11.1.
Otherwise we find $\phi_{ij} \in \Gamma(X, \mathcal{A}^0_X)$ such that

$$\tilde{\omega}_{ij} := \omega_{ij} + d\phi_{ij}$$

are harmonic 1-forms for all $1 \leq i \leq q$, $q+1 \leq j \leq q+p$. Now let $\tilde{f} = fg$ be a new frame for V with

$$g = \begin{pmatrix} E_q & \phi \\ 0 & E_p \end{pmatrix},$$

where ϕ is the matrix of 1-forms $\phi = (\phi_{ij})$. With respect to the frame $\tilde{f}$ for V the connection matrix of D is

$$g^{-1}dg + g^{-1} \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & \tilde{\omega} \\ 0 & 0 \end{pmatrix}.$$

Therefore we have obtained a representative $\tilde{\omega}$ for the de Rham and Dolbeault cohomology class, whose coordinates are harmonic 1-forms, and we can again apply Proposition 11.1 for the type consideration.

(2) By Proposition 10.2. the monodromy representation of (V, D) is unipotent, hence it follows from Proposition 11.4 that (V, D) is a successive extension of trivial bundles. Let

$$\rho : \pi_1(X, x) \longrightarrow \mathrm{Aut}(V_x)$$

be the monodromy representation of (V, D), and let γ_i, $i \in I$ be a (finite) set of generators of $\pi_1(X, x)$. We define for all $i \in I$

$$\eta_i := \rho(\gamma_i) - \mathrm{id}_{V_x} \in \mathrm{End}(V_x).$$

Since ρ factors through J^2, the η_i have the following property: For all $i, j \in I$, $\eta_i \circ \eta_j = (\rho(\gamma_i) - \mathrm{id}) \circ (\rho(\gamma_j) - \mathrm{id}) = 0$, hence for all $i, j \in I$, $\mathrm{im}(\eta_j) \subset \ker(\eta_i)$. We define

$$W := \cap_{i \in I} \ker(\eta_i).$$

If $\eta_j = 0$ for all $j \in I$, then ρ is trivial and there is nothing to prove. If there is a j, such that $\eta_j \neq 0$, then $n := \dim W \geq 1$, since $\mathrm{im}(\eta_j) \subset W$. Thus there exists a basis $\{w_1, \ldots, w_n, v_{n+1}, \ldots, v_N\}$ for V, such that $\{w_1, \ldots, w_n\}$ is a basis for W. With respect to such a basis, all η_i are represented by matrices

$$N_i = \begin{pmatrix} 0 & B_i \\ 0 & 0 \end{pmatrix},$$

with B_i an $n \times (N-n)$-matrix for all $i \in I$. Hence for all i, $\rho(\gamma_i)$ has the matrix representation

$$\rho(\gamma_i) = \begin{pmatrix} E_n & B_i \\ 0 & E_{N-n} \end{pmatrix}.$$

Therefore, ρ is an extension of trivial representations

$$\rho \in \mathrm{Ext}^1(1^{\oplus n}, 1^{\oplus(N-n)}),$$

where 1 denotes the trivial representation $\pi_1(X, x) \to \mathbb{C}^*$. Because of the identification of flat bundles with their monodromy representation (see Proposition 11.4), the assertion follows. □

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