Asymptotic
and Hyperasymptotic Expansions
of Solutions of Linear Differential Equations
Near Irregular Singular Points
of Higher Rank

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0 Introduction

This thesis is concerned with the study of asymptotic expansions of linear meromorphic differential equations in the neighborhood of an irregular singular point. Such equations can be either a scalar n-th order linear homogeneous differential equation

$$y^{(n)} + g_1(z)y^{(n-1)} + \cdots + g_n(z)y = 0 \tag{0.1}$$

or a system of n first-order linear homogeneous differential equations

$$y' = G(z)\,y \tag{0.2}$$

with coefficients g_ℓ resp. G meromorphic at a certain point (which may be taken to be the origin without loss of generality). These two kinds of problems are equivalent: every n-th order scalar equation can easily be transformed into an $n \times n$ system of first order, and due to the existence of a cyclic vector, every such system can in turn be transformed into an n-th order scalar equation. We will therefore not distinguish between both problems, and refer to the integer n as to the *order* of the equation. However, all results in this thesis will be presented in the formulation for systems since this way we can benefit much from methods of Linear Algebra.

In the local theory of differential equations, there are two main kinds of solutions of (0.1) resp. (0.2):

- Formal solutions of the form

  $$\hat{y}_k(z) = e^{q_k\left(\frac{1}{z}\right)} z^{\mu_k} \sum_{\nu=1}^{n} (\log z)^{\nu} \sum_{s=0}^{\infty} y_{k\nu s} z^{s/p} \tag{0.3}$$

  where $p \in \mathbb{N}^*$, and the $q_k\left(\frac{1}{z}\right)$ are polynomials in $z^{-1/p}$. It is known that there is always a fundamental system of formal solutions of the above form. These can be computed from the the differential equation by pure algebraic methods. The formal power series occurring in (0.3) are divergent in general but of some (known) fractional Gevrey order.

- Analytic solutions in sectorial neighborhoods of 0. In every sufficiently small sector at 0, there is a fundamental system of analytic solutions which admit the formal solutions as their (Poincaré) asymptotic expansion, and which we will call *asymptotic solutions*.

When trying to numerically compute the analytical solutions, or to obtain other analytical information, one often relies on the formal solutions since these are relatively easy to compute. In this connection, three common problems arise:

(1) For given z, due to the Gevrey property of the asymptotic expansions, the remainders at first decrease and eventually grow infinitely so that there is only limited attainable accuracy of the approximation. Therefore it is natural to ask, if allowing exponentially-small functions to appear in the expansions, whether it is possible to improve the accuracy of the approximation.

(2) In addition, asymptotic expansions originating from our type of differential equations exhibit a Stokes' phenomenon: in a compound asymptotic expansion, the coefficient of the subdominant part (the Stokes multiplier function) changes rapidly at certain "singular" directions (also called "Stokes' directions" or "anti-Stokes directions" by different authors). On directions to both sides of the singular direction (which will will call "Stokes' directions" here), dominance changes from the dominant to the formerly subdominant term, and therefore the Stokes multiplier function is essential if this direction is included. Therefore one wishes to find expansions which incorporate Stokes' phenomenon, that is, are valid in sectors containing a pair of Stokes' directions. Power series expansions cannot cope with that behavior.

(3) Consider a fixed formal fundamental system (0.3) and a fundamental system of solutions $\left(y_k\right)_{k=1}^n$ with asymptotic expansion $\hat{y}_k$ on a given sector. Then any other asymptotic solution $\tilde{y}_k$ on the same sector admits a representation

$$\tilde{y}_k = \sum_{\ell=1}^{n} c_{\ell k} y_\ell . \tag{0.4}$$

The coefficients ("Stokes' multipliers") $c_{\ell k}$ in (0.4) are of particular interest in the study of the differential equation. The question now is: Can one calculate, or approximate, these multipliers from the formal solutions?

Investigations of the *first* problem have a long history. Going back to Stokes and Stieltjes, the theory of *converging factors* [Din73, Olv97] was developed, and there has been much subsequent activity in deriving re-expansions for the remainders, e.g. for the generalized exponential integral resp. the incomplete Gamma function [Olv97, Tem79, Olv91a, Dun96b] or for the confluent hypergeometric function [Olv97, Olv91b, Old92]. Some of the proofs are based on very specific properties of the functions considered and can hardly be generalized to wider classes of problems such as solutions of classes of differential equations, while others take a direct differential-equation approach. They use either modified *associated functions* in the Borel plane, or Cauchy-Heine resp. Heine-Stieltjes transforms of the original solutions. These methods have been successfully applied to solutions of differential equations:

- for second-order rank one equations [OO94, OO95a];
- for rank one equations of arbitrary order [LS94, Old98b];
- for second-order equations of arbitrary rank [MW97]; and
- for equations of arbitrary rank and order [Hoe94, HS99a].

A different approach has been taken in [Dun96c] where the author obtains representations of the remainders as solutions of a non-homogeneous differential equation and where he gives strict error bounds for the remainders.

There had been some prior results which addressed the *second* problem, but it was Berry [Ber89] who first discovered that, if the expansion is truncated near its least term, then the multiplier in the asymptotic expansion changes rapidly *but smoothly* when crossing a singular direction, and the function which describes this "switching on" is given by the complementary error function. After Berry, several authors have obtained expansions that are valid in large sectors, among others Olver [Olv91a] and Dunster [Dun96b] who considered the generalized exponential integral; Olver [Olv91b] for the confluent hypergeometric function; and Olde Daalhuis and Olver [OO94, OO95a] and Dunster [Dun96c] who considered second-order differential equations of rank one. Other results [Old98b, MW97] can also be extended to be valid in large sectors but the authors stated this without giving a proof.

Concerning the *third* problem – the calculation of Stokes' multipliers from the formal solutions –, for equations resp. systems of order two satisfactory answers have been given by Loday-Richaud [LR90] and Olde Daalhuis and Olver [OO95b]. For higher-order equations, Immink [Imm90], Hoeppner [Hoe94], and Hoeppner and Schäfke [HS99a] developed methods to calculate the leading multipliers which determine the asymptotic behavior of the coefficients of the formal solution. For equations of rank one, Lutz and Schäfke [LS97] and Olde Daalhuis [Old98b] have calculated the "difficult" multipliers as well. The authors of the first reference used conformal mappings in the Borel plane, while the the author of the second reference used hyperasymptotic expansions of higher levels. But it has been an open problem what the situation looks like in the general single-leveled, or in the multi-leveled case.

In the present thesis, an attempt is made to give answers to the three above questions in the general case of arbitrary rank and order. Briefly, these answers are affirmative for the first and partly affirmative for the second and third questions. The main results on hyperasymptotic expansions are the theorems 4.3, 4.6 and 4.8. Here we show how to obtain hyperasymptotic expansions for the multisums of the formal solutions as well as uniform estimates for the corresponding error terms. The expansions are in terms of multiple integrals which can be viewed as generalizations of the hyperterminants used in [OO95a] and [Old98b]. They are strikingly simple in form, and the coefficients at each level are a product of the coefficients of the original (Poincaré) expansion and of Stokes' multipliers.

At each level of hyperasymptotics, this is followed by an optimization procedure in which the number of terms is chosen to depend in an appropriate way upon the independent variable. This is done in Corollaries 4.4, 4.7 and 4.9. It turns out that the minimized remainder is *exponentially small* and that this exponential improvement increases from level to level so that in principle there is no limit in the attainable exponential improvement. Our expansions are valid uniformly in closed sectors between two singular directions.

In our work we have been guided by the special situation of rank one, order two which has been investigated in detail in [OO95a]. However, not all the strength and beauty of the results carries over to our general situation. For instance, the region of validity of the expansions does not necessarily increase with the level of hyperasymptotics. However we are able to show that *if the formal fundamental solution is k-summable for some k* then these regions will increase with increasing level, and thus *the hyperseries gives a smooth interpretation of Stokes' phenomenon* in this case.

As a byproduct, we have obtained hyperasymptotic expansions for the late coefficients as well, see Theorem 4.10, or (4.78). These can be used to calculate some of the Stokes multipliers. In contrast to special cases considered in the references, it is *in general not possible* using our method *to obtain all Stokes' multipliers*. However, in the single-leveled case again this is possible if considering hyperseries of high enough levels.

Our method of proof is based on a general integral representation of the remainders and the expansion coefficients coming from the *Cauchy-Heine formula*. It has been originally suggested by Y. Sibuya that this formula should give a way to express the remainders even in the very general case, and in fact it turns out to be a very powerful tool for our kind of problems.

By repeated insertion of the expansion into the integral representation we arrive at a truncated hyperasymptotic expansion, and the remainder is an expression consisting of several multiple Cauchy-Heine integrals. We have therefore derived new uniform estimates for certain Cauchy-Heine integrals that can be applied to our situation, see Theorems 5.5 and 5.6. For the proof we have used a modification of the saddle-point method for the uniform asymptotic analysis of integrals with coalescing saddle and simple pole, yielding a complementary error function term in addition to the classical Poincaré expansion. The Cauchy-Heine integrals contain two complex parameters, and the result holds uniformly with respect to both parameters in certain subsets of the complex plane. – For the multiple integrals, we need to repeat the application of such estimates, and therefore a modification (Theorem 5.6) is necessary where the parameter sets are cusps instead of sectors. For details we refer to Section 5.6.

The results obtained in the present thesis are certainly not the final answer. First, our method does not allow, in the general case, to calculate *all* Stokes' multipliers from the formal solutions. It is still an open problem whether this is possible in the general case. In view of the fact that a fundamental system of formal solutions already contains all coded information about the resurgent structure of the solutions, this question seems to be realistic. One could try to combine the present method with the method of conformal mappings proposed by Lutz and Schäfke [LS97]. Moreover, in the multi-leveled case it seems necessary to separate the different levels in order to apply our method, e.g. one could consider formal meromorphic transforms instead of formal fundamental systems. One could also imagine to consider hyperasymptotic expansions of the solutions not for very large but *for mid-size* $|z|^{-1}$ where higher order polynomials are visible before being dominated by lower-order terms.

Second, in all estimates we have exactly specified the leading term of all polynomials but we suppressed the lower-order terms. But expressions like (4.54) or (4.72) suggest that the proper setting of hyperasymptotics in the general case uses *polynomials* instead of monomials and that they could give a deeper insight into Stokes' phenomenon. However, it is not clear to the author what these polynomials look like, and how they can be calculated from the exponentials of the differential equation.

Third, we have given Big-O estimates for the remainders at all levels. But these are not strict: for direct numerical computations one could ask if it is possible to obtain strict error bounds for the remainders like in [Dun96b, Dun96c]. Furthermore, the calculation of the generalized hyperterminants still presents a problem. In [Old96, Old98a], Olde Daalhuis has given representations of the hyperterminants as convergent series of confluent hypergeometric functions. But still the coefficients of these expansions are not easy to find, and moreover our generalizations of hyperterminants are more complicated since they have polynomials figuring in the exponent.

Finally, there seems to be an intimate relation between hyperasymptotics and Écalle's theory of resurgent functions, and it would be interesting to see if it is possible to embed hyperasymptotics in this general theory – formally or, even better, in a rigorous way. But even in the single-leveled case it is not clear to the author which is the resurgent analogon of our hyperseries and how the appearance of Stokes' multipliers and of early coefficients can be explained in this theory.

The thesis is organized as follows: in **Section 1**, we will briefly sketch the main facts on asymptotic expansions, Gevrey asymptotics, and hyperasymptotics. We present the notations that we will use throughout the thesis, but the contents of this section cannot serve as an introduction into this matter. We refer to the textbooks for that. In Section 1.6.2 we define the hyperterminants which we will use in our hyperasymptotic expansions. Finally, the notion of a *cusp* is introduced which is used in Section 5.6. – In **Section 2** we note the basic asymptotic properties of solutions of meromorphic differential equations, especially the existence of asymptotic solutions and the concept of *multisummability*. – **Section 3** is devoted to the presentation of the *Cauchy-Heine formula* which we prove for slightly generalized coverings here, and to an integral representation based upon the Cauchy-Heine formula which will be the starting-point of our derivation of asymptotic expansions.

The main work of this thesis is done in **Section 4**, the derivation of hyperasymptotic expansions for multisums of the formal solutions. Separate subsections are devoted to expansions of levels 0, 1 and 2 for the solutions, and moreover a method is presented to calculate all needed Stokes multipliers from the formal solutions. The remaining **Section 5** is devoted to the uniform asymptotic analysis of a Cauchy-Heine integral which is needed for the estimation of the remainders in Section 4. The study of the Cauchy-Heine integral is quite general but also very technically involved.

I wish to thank to all the people who made this thesis possible. Especially I owe thank to my thesis advisor, Prof. Reinhard Schäfke, who introduced me to the beautiful field of Asymptotic Analysis and who suggested the topic of this thesis. During all the years with ups and downs he was not only a good teacher to me and a patient listener but also a friend.

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Finally, I wish to thank to my colleagues at the University of Essen, Prof. Schmidt, Dr. Wolf and Stephan Schultze, for many helpful discussions.

1 Asymptotic expansions

Here we review some basic concepts which will be used throughout this thesis. Throughout this section let E be a complex Banach space.

1.1 Sectors and cusps

We will denote the open and the closed disk of radius ρ centered at $z_0 \in \mathbb{C}$ by $K_\rho(z_0)$ and $\overline{K}_\rho(z_0)$, respectively. The *punctured disk* $D(\rho)$ is the set

$$D(\rho) := \{z \in \mathbb{C} : 0 < |z| < \rho\}.$$

The complex functions we will deal with, in general, have a branch point at the origin and therefore should naturally be considered on the Riemann surface $\hat{\mathbb{C}}_0$ of the logarithm instead of the complex plane.

A *direction* is a number $d \in \mathbb{R}$, and we think of it as the ray $\arg z = d$. Since we work on the Riemann surface of the logarithm, directions that differ by multiples of 2π are considered as different.

An *open* (resp. *closed*) *sector* on the Riemann surface of the logarithm is a set

$$(1.1)\qquad \begin{aligned} S_\rho(\alpha,\beta) &:= \left\{z \in \hat{\mathbb{C}}_0 : 0 < |z| < \rho,\ \alpha < \arg z < \beta\right\} \quad \text{resp.} \\ \overline{S}_\rho(\alpha,\beta) &:= \left\{z \in \hat{\mathbb{C}}_0 : 0 < |z| \le \rho,\ \alpha \le \arg z \le \beta\right\}. \end{aligned}$$

In this context sectors of opening greater than 2π are expressly allowed. However, if a sector S is of opening less than 2π one can consider the projection of S into the complex plane. We will not distinguish both objects *by the notation*. Sometimes we will use the notations $S(\alpha,\beta)$ resp. $\overline{S}(\alpha,\beta)$ for open or closed sectors with infinite radius, respectively. Note that closed sectors are not closed as subsets of $\mathbb{C}$. Sectors in this thesis, unless otherwise indicated, are always open.

Let $S = S_\rho(\alpha,\beta)$ resp. $S = \overline{S}_\rho(\alpha,\beta)$ be a sector. A *proper subsector of S* is a sector $R = S_{\tilde{\rho}}(\tilde{\alpha},\tilde{\beta})$ resp. $R = \overline{S}_{\tilde{\rho}}(\tilde{\alpha},\tilde{\beta})$ with the properties

$$\alpha < \tilde{\alpha} < \tilde{\beta} < \beta, \qquad 0 < \tilde{\rho} < \rho.$$

This property does not regard whether R or S are closed or open. For short we write $R \Subset S$.

Open / closed cusps at the origin are sets of the form

$$(1.2)\qquad \begin{aligned} \Upsilon(\theta,K,\rho) &:= \left\{z \in \hat{\mathbb{C}}_0 : \theta - \tfrac{\pi}{2} < \arg z < \theta + \tfrac{\pi}{2},\ \left|\mathrm{Im}\left(\tfrac{1}{ze^{-i\theta}}\right)\right| < K, 0 < |z| < \rho\right\} \text{ and} \\ \overline{\Upsilon}(\theta,K,\rho) &:= \left\{z \in \hat{\mathbb{C}}_0 : \theta - \tfrac{\pi}{2} < \arg z < \theta + \tfrac{\pi}{2},\ \left|\mathrm{Im}\left(\tfrac{1}{ze^{-i\theta}}\right)\right| \le K, 0 < |z| \le \rho\right\}, \end{aligned}$$

respectively. Closed cusps are not closed as subsets of $\mathbb{C}$ since they do not contain

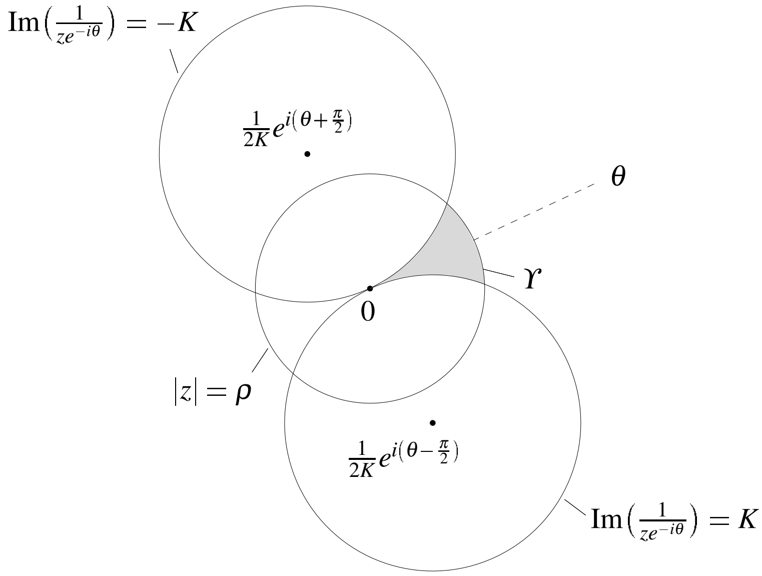

Figure 1: Shape of the cusp $\Upsilon(\theta, K, \rho)$

the origin. The essential property of cusps comes from the restriction upon $\operatorname{Im}\left(\frac{1}{ze^{-i\theta}}\right)$ which supposes that the set becomes narrower to the cuspidal point 0; more precisely, we have $\arg z - \theta = O(|z|)$ for $z \in \Upsilon(\theta, K, \rho)$, and the Big-O constant is determined by K.

Obviously, $\Upsilon(\theta_1, K_1, \rho_1) \subseteq \Upsilon(\theta_2, K_2, \rho_2)$

$$\iff \Big(\theta_1 = \theta_2, \quad K_1 \le K_2, \quad \rho_1 \le \rho_2\Big).$$

The same is true for two closed cusps. This motivates to call $\Upsilon(\theta_1, K_1, \rho_1)$ a *proper subcusp* of $\Upsilon(\theta_2, K_2, \rho_2)$ if

$$\theta_1 = \theta_2, \quad K_1 < K_2, \quad \rho_1 < \rho_2,$$

and we write $\Upsilon(\theta_1, K_1, \rho_1) \Subset \Upsilon(\theta_2, K_2, \rho_2)$. The same is defined for arbitrary combinations of open and/or closed cusps.

Cusps of this type are not usually considered in conjunction with hyperasymptotic expansions. In this thesis, too, we will obtain expansions in sectors. However, in higher levels, for repeated application of some auxiliary results they will serve as a tool to simplify the estimates.

1.2 Landau symbols

If $\emptyset \neq M$ is an arbitrary set then for two functions $f,g:\ M \to E$ define the *Landau symbols*

$$(1.3)\quad \begin{aligned} f(x) &= O\big(g(x)\big) \quad (x \in M) :\Longleftrightarrow\ \exists C>0:\ |f(x)| \le C|g(x)| \quad (x \in M),\\ f(x) &= \Omega\big(g(x)\big) \quad (x \in M) :\Longleftrightarrow\ g(x) = O\big(f(x)\big) \quad (x \in M),\\ f(x) &\asymp g(x) \qquad\quad (x \in M) :\Longleftrightarrow\ f(x) = O\big(g(x)\big) \wedge f(x) = \Omega\big(g(x)\big) \ (x \in M). \end{aligned}$$

This symbolic might be somewhat confusing since the equality sign suggests a symmetry between both sides which is not present. A clearer notation would be "$f(x) \in O\big(g(x)\big)$" and the like since the right-hand side can in fact be viewed as a set of functions. However, for convenience we will use the historic notation. We merely will sometimes use relations like $O\big(g_1(x)\big) \subseteq O\big(g_2(x)\big)$ to emphasize the non-symmetry. This last statement means that (1.3) with $g = g_1$ implies (1.3) with $g = g_2$. The estimates above respect *all* $x \in M$ and are hence automatically "uniform" with respect to $x \in M$.

If G is a region in the complex plane and z_0 is an accumulation point of G then for two functions $f,g:\ G \to E$ we define

$$\begin{aligned} f(z) = O\big(g(z)\big) \quad (G \ni z \to z_0) &:\Longleftrightarrow\ \exists U \text{ neighborhood of } z_0 \text{ such that}\\ &\qquad f(z) = O\big(g(z)\big) \quad (z \in G \cap U),\\ f(z) = \Omega\big(g(z)\big) \quad (G \ni z \to z_0) &:\Longleftrightarrow\ \exists U \text{ neighborhood of } z_0 \text{ such that}\\ &\qquad f(z) = \Omega\big(g(z)\big) \quad (z \in G \cap U),\\ f(z) = o\big(g(z)\big) \quad (G \ni z \to z_0) &:\Longleftrightarrow\ \forall \varepsilon > 0\ \exists U \text{ neighborhood of } z_0 \text{ such that}\\ &\qquad |f(z)| \le \varepsilon |g(z)| \quad (z \in G \cap U). \end{aligned}$$

If, moreover, f and g depend on an additional parameter $x \in M$ then the above estimates are said to be *uniform with respect to* $x \in M$ if the neighborhood U and the respective Big-O / Big-Omega constants can be taken to be independent from $x \in M$.

1.3 Poincaré asymptotics

Here we define a special case of the concept of an "asymptotic series" or "asymptotic expansion" due to Poincaré: the asymptotic power series. This notion can be considerably generalized; especially, one often considers asymptotic expansions of functions on sectorial neighborhoods of 0 or of quasi-functions. In our context, however, it suffices to consider asymptotic power series of functions in sectors.

We will merely give the definitions and the notation used in this thesis; for a general introduction to the matter the books of Olver [Olv97] or de Bruijn [dB67] might be helpful. Also, for an outline within the context of differential equations see [Was87, HS99b, Bal00].

Let S be an open or closed sector and $f: S \to E$ be a function. Furthermore, let $\big(f_\ell\big)_{\ell=0}^{\infty}$ be a sequence in E. If f is analytic in S and, for every $N \in \mathbb{N}$ and every closed subsector $\tilde{S}$ of S we have an estimate

$$f(z) - \sum_{\ell=0}^{N-1} f_\ell z^\ell = O\big(z^N\big)$$

then we say that f *admits the asymptotic expansion* $\hat{f}(z) := \sum_{\ell=0}^{\infty} f_\ell z^\ell$ for $z \to 0$ in S, and we write

$$f(z) \sim \hat{f}(z) \qquad (S \ni z \to 0).$$

The set of all functions admitting an asymptotic expansion in a fixed sector S is denoted by $\mathscr{A}(S,E)$ (for short: $\mathscr{A}(S)$ if $E = \mathbb{C}$), and the set of all formal power series $\sum_{\ell=0}^{\infty} f_\ell z^\ell$ will be denoted by $E[[z]]$. If an asymptotic expansion $\hat{f}$ exists then it is uniquely determined by f and S. Note that these series are purely formal and need in no way converge. However, if $\hat{f}(z)$ *does* converge on some open disk $K_\rho(0)$ then its limit function satisfies $f(z) \sim \hat{f}(z)$ $(S \ni z \to 0)$ for every sector S of radius $< \rho$. The set of all convergent power series $\sum_{\ell=0}^{\infty} f_\ell z^\ell$ will be denoted by $E\{z\}$. The sets $\mathbb{C}[[z]]$, $\mathbb{C}\{z\}$, and $\mathscr{A}(S)$ have the structure of (differential) algebras over $\mathbb{C}$, and $E[[z]]$, $E\{z\}$ and $\mathscr{A}(S,E)$ are respective vector spaces over those algebras. Define a map $J: \mathscr{A}(S,E) \to E[[z]]$ by $J(f) = \hat{f}$. The kernel $\mathscr{A}_0(S,E)$ of J is the set of analytic functions on S which admit the zero expansion $\hat{0}$ and which are called "flat" on S.

If again f depends on an additional parameter $x \in M$ then the asymptotic expansion is called *uniform with respect to* $x \in M$ if all underlying Big-O estimates are uniform w.r.to $x \in M$. Moreover, one similarly defines asymptotic power series in sectors at an arbitrary point $z_0 \in \mathbb{C}$, or at infinity. The needed translation of the above definitions is straightforward and will be omitted here.

1.4 Gevrey asymptotics

Gevrey asymptotics are a special case of Poincaré asymptotics together with some growth condition. Throughout this section let a number $s > 0$ be given.

If $f: S \to E$ is a function defined on a sector S and $\big(f_\ell\big)_{\ell=0}^{\infty}$ is a sequence in E then we say that f *admits the asymptotic expansion* $\hat{f}(z) := \sum_{\ell=0}^{\infty} f_\ell z^\ell$ *of Gevrey order s in S* if f is analytic in S and, for every closed subsector $\tilde{S}$ of S there exist numbers $C(\tilde{S}), K(\tilde{S}) > 0$ such that

$$\Big| f(z) - \sum_{\ell=0}^{N-1} f_\ell z^\ell \Big| \leq C(\tilde{S}) \cdot K(\tilde{S})^N \big(N!\big)^s |z|^N$$

uniformly w.r.to $N \in \mathbb{N}$ and $z \in \tilde{S}$. We then write

$$f(z) \sim_s \hat{f}(z) \qquad (S \ni z \to 0).$$

In this situation, the formal series $\hat{f}$ has the property

$$|f_\ell| \le C \cdot K^\ell (\ell!)^s$$

with some constants $C, K > 0$, and we call it a *Gevrey series of order s*. The set of those series is denoted by $E[[z]]_s$, and the set of all functions admitting an asymptotic expansion of Gevrey order s in S is denoted by $\mathscr{A}_s(S,E)$. For further details about Gevrey asymptotics we refer to [Bal94, Bal00, HS99b].

Formal series occurring in formal solutions of many problems of Applied Analysis (including linear and nonlinear differential equations) are known to be of some Gevrey order. However, one can go far beyond this and ”sum” formal power series solutions of meromorphic differential equations, see Section 2.2.

1.5 Exponential improvement (Superasymptotics)

Exponential improvement of an asymptotic expansion

$$f(z) \sim \hat{f}(z) \qquad (S \ni z \to 0),$$

roughly speaking, consists of *truncating, for each z, the series at or near its smallest term and then re-expanding the resulting remainder*. Though this has been done for asymptotic expansions arising from a variety of problems, including solutions of differential equations, it is hard to define what exponential improvement is like in general. We will therefore content with the above (rather vague) definition but will illustrate the procedure by means of a simple example[1].

Consider the (parameter) integral $I^{(p)}(z)$ defined by

$$I^{(p)}(z) := z \int_0^{+\infty} \frac{e^{-t} t^{p-1}}{t+z} dz$$

for $|\arg z| < \pi$, $\operatorname{Re} p > 0$, and by analytic continuation w.r.to $z \in \hat{\mathbb{C}}_0$ and $p \in \mathbb{C} \setminus (-\mathbb{N})$ elsewhere. This function is related to the generalized exponential integral $E_p(z)$ and to the confluent hypergeometric U-function by

$$I^{(p)}(z) = \Gamma(p) \cdot z e^z E_p(z) = \Gamma(p) \cdot z U(1, 2-p; z).$$

The exponential improvement of the expansion of $E_p(z)$ has been carried out in [Olv91a] and may be viewed as an example of the exponential improvement of functions defined by integrals with saddles. However, since $z^{-1} I^{(p)}(z)$ satisfies a confluent hypergeometric equation, it is at the same time an example for problems arising from second-order differential equations.

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[1] this time for $z \to \infty$

For fixed p, the function admits the well-known (Gevrey-1) asymptotic expansion

$$I^{(p)}(z) \sim \sum_{\ell=0}^{\infty}(-1)^{\ell}\Gamma(p+\ell)\,z^{-\ell}$$

as $z \to \infty$ in the sector $|\arg z| < \frac{3\pi}{2}$. From the Gevrey property of this expansion it is not hard to see that for fixed (large) $|z|$, the remainder if truncating after the first n terms

$$R^{(p)}(z,n) := I^{(p)}(z) - \sum_{\ell=0}^{n-1}(-1)^{\ell}\Gamma(p+\ell)z^{-\ell}$$

at first decreases as n increases and afterwards increases infinitely. The minimal remainder (as well as the smallest summand) is obtained approximately for $n \approx |z| - p$. Hence we couple n and z by

$$n = |z| - p + \alpha, \qquad \alpha = O(1) \tag{1.4}$$

and re-expand the remainder $R^{(p)}(z,n)$. In [Olv91a], Olver obtained the exponentially improved expansion

$$R^{(p)}(z,n) \sim e^{-|z|}z^{p}\,\frac{(-1)^{n}\sqrt{2\pi}\,e^{-i(|z|+\alpha)\theta}}{1+e^{-i\theta}}\sum_{\ell=0}^{\infty}a_{2\ell}(\theta,\alpha)|z|^{-1/2-\ell} \tag{1.5}$$

as $|z| \to \infty$, uniformly in the sector[2] $|\arg z| < \pi - \delta$. Since this expansion holds uniformly in this sector and contains as additional factor $e^{-|z|}$ it is called a *uniform, $e^{-|z|}$-improved asymptotic expansion*. Synonyms used by different authors are *"uniform, exponentially improved"* (UEI) and *"superasymptotic"* expansions, respectively.

The above re-expansion uses only elementary functions like the exponential function and powers of z. Exponentially improved expansions of this kind have been obtained more generally for solutions of differential equations [LS94, Hoe94, HS99a]. They are all in terms of elementary functions which are easy to compute. On the other hand, they have the disadvantage that the obtained expansions are not uniform near critical "singular" directions (also called "Stokes directions" or "anti-Stokes directions" by different authors) – in our example, $\arg z = \pm\pi$ – and break completely if these directions are crossed.

However, there are also exponentially improved expansions that have overcome this restriction and hold uniformly up to, and in part beyond singular directions [Tem79, Olv91b, Olv93, OO94, Dun96b, Dun96c]. These expansions have to cover Stokes' phenomenon and therefore contain non-elementary functions like the generalized exponential integral or (as a special case) the complementary error function. In the men-

[2] with the notation $\theta = \arg z$.

tioned paper [Olv91a], for the function $E_p(z)$ and with the coupling (1.4) Olver obtained a UEI expansion of this kind, too, which in our notation reads

$$R^{(p)}(z,n) \sim \\ \sim e^z\left(ze^{-\pi i}\right)^p \left[\pi i \cdot \operatorname{erfc}\left\{c(\theta)\sqrt{\tfrac{1}{2}|z|}\right\} + \sqrt{2\pi}\, e^{-\frac{1}{2}|z|c(\theta)^2} \sum_{\ell=0}^{\infty} \tilde{g}_{2\ell}(\theta,\alpha)|z|^{-1/2-\ell}\right]$$

as $|z| \to \infty$, uniformly in the sector[3] $-\pi+\delta \le \arg z \le 3\pi-\delta$. The additional term with the complementary error function describes the rapid, *but smooth*, appearance of a term $e^z\left(ze^{-\pi i}\right)^p \cdot 2\pi i$ when crossing the singular direction $\arg z = \pi$. For fixed p the exponential improvement of $e^{-|z|}$ obtained in (1.5) gradually deteriorates beyond the singular direction (together with the function e^z) and disappears completely for $\arg z = \frac{3\pi}{2}$.

1.6 Hyperasymptotics

Hyperasymptotics, roughly speaking, *is the recursive repetition of the process of exponential improvement to higher levels*. Poincaré asymptotics are considered as being level zero, and superasymptotic expansions as being level one. If truncating, for each z, the level-1 expansion at a certain stage and then re-expanding the remainder, a level-2 hyperasymptotic expansion is found, and so forth.

Though an attempt has been made to derive hyperasymptotic expansions in terms of elementary functions for solutions of certain ordinary differential equations [Old92], it turned out that only expansions involving non-elementary functions can cope with Stokes' phenomenon. Examples of such expansions are [Old93, OO95a, Old98b, MW97]. The functions involved are repeated integrals of the generalized exponential integral, in special cases, or generalizations of these called *hyperterminants*, see Section 1.6.2.

[3] with $c(\theta) = \left\{2e^{i\theta} + 2i(\theta-\pi) + 2\right\}^{1/2}$ such that $c(\theta) \sim \pi - \theta$ as $\theta \to \pi$.

1.6.1 Hyperasymptotic expansions

A level-n hyperasymptotic expansion, in its simplest form, consists of $(n+1)$ consecutive truncated expansions[4]

$$
\begin{aligned}
f(z) &= \sum_{s=0}^{N_0-1} f_s^{(0)} z^{-s} + R^{(0)}(z,N_0),\\
R^{(0)}(z,N_0) &= \sum_{s=0}^{N_1-1} f_s^{(1)} F^{(1)}(z,N_0,s) + R^{(1)}(z,N_0,N_1),\\
\vdots \qquad & \qquad\qquad\qquad \vdots\\
R^{(n-1)}(z,N_0,\dots,N_{n-1}) &= \sum_{s=0}^{N_n-1} f_s^{(n)} F^{(n)}(z,N_0,\dots,N_{n-1},s) + R^{(n)}(z,N_0,\dots,N_n)
\end{aligned}
$$

together with an estimate of $R^{(n)}(z,N_0,\dots,N_n)$. Each level-m expansion $(1 \le m \le n)$ should be exponentially improved compared with the level $(m-1)$, that is, $R^{(m)}$ should be exponentially small compared with $R^{(m-1)}$.

If, however, the originating problem is more complicated (e.g., for integrals with saddles if there are more than two saddles, or for solutions of ordinary differential equations, if the order of the equation is greater than two) then "scattering" may occur at level one and above, that is, if we start from one saddle or exponential then we obtain a *combination of expansions* in terms of hyperterminants belonging *to other* – adjacent – saddles resp. exponentials, with possibly distinct truncation points, as follows:

$$
\begin{aligned}
f_k(z) &= \sum_{s=0}^{N_0-1} f_{ks}^{(0)} z^{-s} + R_k^{(0)}(z,N_0),\\
R_k^{(0)}(z,N_0) &= \sum_{\ell\neq k}\left(\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell ks}^{(1)} F_{\ell k}^{(1)}(z,N_0,s) + R_{\ell k}^{(1)}(z,N_0,N_1)\right),\\
R_{\ell k}^{(1)}(z,N_0,N_1) &= \sum_{m\neq \ell}\left(\quad\cdots\quad \right) \qquad\qquad \text{and so forth}
\end{aligned}
$$

(cf. (4.10),(4.41),(4.59)).

It should be remarked that if choosing the right hyperterminant functions then the coefficients $f_{\ell ks}^{(1)}$ and those of higher levels are often the product of the coefficients $f_{\ell s}^{(0)}$ of the level-0 expansion and of some characteristic constants (Stokes' multipliers). This re-appearance of the early coefficients is a consequence of the resurgence of the function and one advantage over expansions in terms of elementary functions.

[4]again for $z \to \infty$

1.6.2 Hyperterminants

Here we define the hyperterminants which shall be used in Section 4. For convenience, the hyperterminants are defined suiting for the setting *at infinity* although in this thesis we will consider expansions at the origin. However, this way the integrals become slightly simpler in form and can easier be compared with the definitions given by other authors. Throughout we will use the abbreviation

$$\int_a^{[\theta]} := \int_a^{\infty e^{i\theta}} .$$

If now for all $j \in \mathbb{N}^*$, $q_j(x) \in x\mathbb{C}[x] \setminus \{0\}$, $q_j(x) = \alpha_j x^{\kappa_j} + O\big(x^{\kappa_j - 1}\big)$ with $\alpha_j \neq 0$ as well as

$$\left|\theta_j + \tfrac{1}{\kappa_j}\arg(-\alpha_j) + \tfrac{2\pi}{\kappa_j} m\right| < \tfrac{\pi}{2\kappa_j}$$

for some $m \in \mathbb{Z}$, and $N_j \in \mathbb{C}$ then we define

$$\begin{aligned}
F^{(0)}(z) &:= 1 \\
F^{(1)}\left(z;\ \begin{matrix} N_1 \\ q_1 \\ \theta_1 \end{matrix}\right) &:= \int_0^{[\theta_1]} \frac{e^{q_1(t_1)} t_1^{N_1 - 1}}{t_1 - z}\, dt_1 \\
F^{(2)}\left(z;\ \begin{matrix} N_1, & N_2 \\ q_1, & q_2 \\ \theta_1, & \theta_2 \end{matrix}\right) &:= \int_0^{[\theta_1]} \int_0^{[\theta_2]} \frac{e^{q_1(t_1) + q_2(t_2)} t_1^{N_1 - 1} t_2^{N_2 - 1}}{(t_1 - z)(t_2 - t_1)}\, dt_2\, dt_1
\end{aligned} \tag{1.6}$$

and, by induction,

$$F^{(n)}\left(z;\ \begin{matrix} N_1, & \dots, & N_n \\ q_1, & \dots, & q_n \\ \theta_1, & \dots, & \theta_n \end{matrix}\right) := \int_0^{[\theta_1]} \frac{e^{q_1(t_1)} t_1^{N_1 - 1}}{t_1 - z} F^{(n-1)}\left(t_1;\ \begin{matrix} N_2, & \dots, & N_n \\ q_2, & \dots, & q_n \\ \theta_2, & \dots, & \theta_n \end{matrix}\right) dt_1.$$

Mostly we will not need these hyperterminants for $z = 0$, and it is readily verified that the above integral $F^{(n)}$ converges for $z \neq 0$ iff

$$\operatorname{Re} \sum_{j=m}^{n} \big(N_j - 1\big) > -1 \qquad \text{for all } m = 1, \dots, n. \tag{1.7}$$

Then the integral is defined for

$$\Big\{z \in \hat{\mathbb{C}}_0 :\ \theta_1 < \arg z < \theta_1 + 2\pi\Big\}$$

and by analytic continuation elsewhere. If, in addition, we want the above hyperterminant to exist in $z=0$ then in the above condition corresponding to $m=1$, the right-hand side has to be replaced by 0.

For $n \geq 2$, a convention has to be made about the contours of integration in case that a $j \in \{2,\ldots,n\}$ exists such that $\theta_j = \theta_{j-1} \mod 2\pi$. It is understood here that in the definition above, *the inner path of t_{j-1}-integration should always be indented to the right of the point $t_{j-1} = t_j$* or more explicitly,

$$F^{(n)}\left(\begin{array}{c} \\ z; \\ \\ \end{array}\begin{array}{ccc} N_1, & \ldots, & N_n \\ q_1, & \ldots, & q_n \\ \theta_1, & \ldots, & \theta_n \end{array}\right) := \lim_{\varepsilon\searrow 0} F^{(n)}\left(\begin{array}{c} \\ z; \\ \\ \end{array}\begin{array}{ccc} N_1, & \ldots, & N_n \\ q_1, & \ldots, & q_n \\ \theta_1-\varepsilon, & \ldots, & \theta_n - n\varepsilon \end{array}\right).$$

This choice is arbitrary but consistent throughout this paper since, if translated to the situation at 0, it is in accordance with our formulation of Stokes' phenomenon in (4.6). We will need this in Section 4.

By the method of residues we obtain the following connection relation between the $F^{(n)}$:

$$\begin{aligned} F^{(n)}\left(\begin{array}{c} \\ z; \\ \\ \end{array}\begin{array}{ccc} N_1, & \ldots, & N_n \\ q_1, & \ldots, & q_n \\ \arg z+0, & \ldots, & \theta_n \end{array}\right) &- F^{(n)}\left(\begin{array}{c} \\ z; \\ \\ \end{array}\begin{array}{ccc} N_1, & \ldots, & N_n \\ q_1, & \ldots, & q_n \\ \arg z-0, & \ldots, & \theta_n \end{array}\right) \\ &= \int_{(z-)}^{[\arg z]} \frac{e^{q_1(t_1)} t_1^{N_1-1}}{t_1 - z} F^{(n-1)}\left(\begin{array}{c} \\ t_1; \\ \\ \end{array}\begin{array}{ccc} N_2, & \ldots, & N_n \\ q_2, & \ldots, & q_n \\ \theta_2, & \ldots, & \theta_n \end{array}\right) dt_1 \\ &= -2\pi i\, e^{q_1(z)} z^{N_1-1} F^{(n-1)}\left(\begin{array}{c} \\ z; \\ \\ \end{array}\begin{array}{ccc} N_2, & \ldots, & N_n \\ q_2, & \ldots, & q_n \\ \theta_2, & \ldots, & \theta_n \end{array}\right). \end{aligned} \tag{1.8}$$

This may be viewed as the Stokes' relations for hyperterminants.

Finally, another functional equation of the $F^{(n)}$ which we will need later is given by

$$zF^{(n)}\left(z;\begin{array}{ccc} N_1, & \ldots, & N_n \\ q_1, & \ldots, & q_n \\ \theta_1, & \ldots, & \theta_n \end{array}\right) - F^{(n)}\left(z;\begin{array}{ccc} N_1+1, & \ldots, & N_n \\ q_1, & \ldots, & q_n \\ \theta_1, & \ldots, & \theta_n \end{array}\right)$$

$$= \int_0^{[\theta_1]} \frac{e^{q_1(t_1)}\left(zt_1^{N_1-1} - t_1^{N_1}\right)}{t_1 - z} F^{(n-1)}\left(t_1;\begin{array}{ccc} N_2, & \ldots, & N_n \\ q_2, & \ldots, & q_n \\ \theta_2, & \ldots, & \theta_n \end{array}\right) dt_1$$

$$= -\int_0^{[\theta_1]} e^{q_1(t_1)} t_1^{N_1-1} F^{(n-1)}\left(t_1;\begin{array}{ccc} N_2, & \ldots, & N_n \\ q_2, & \ldots, & q_n \\ \theta_2, & \ldots, & \theta_n \end{array}\right) dt_1$$

$$= -F^{(n)}\left(0;\begin{array}{ccc} N_1+1, & \ldots, & N_n \\ q_1, & \ldots, & q_n \\ \theta_1, & \ldots, & \theta_n \end{array}\right). \tag{1.9}$$

2 Asymptotic solutions of meromorphic ODE

By *meromorphic ODE* we understand here either a scalar n-th order linear homogeneous ordinary differential equation

$$y^{(n)} + g_1(z)y^{(n-1)} + \cdots + g_n(z)y = 0 \tag{2.1}$$

or a system of n first-order linear homogeneous differential equations

$$y' = G(z)\,y \tag{2.2}$$

with coefficients g_ℓ resp. G meromorphic at a point $z_0 \in \mathbb{C} \cup \{\infty\}$ (and not being all identically zero). We will assume $z_0 = 0$ in the sequel.

Since every n-th order scalar equation (2.1) can be transformed into a first-order system (2.2), we will only consider systems, and all results for systems will carry over to n-th order equations, too. The inverse reduction from systems to scalar equations could also be made; however, the presentation for systems is more convenient since it highly benefits from methods of Linear Algebra.

By assumption there is an integer r such that $A(z) := z^{1+r}G(z)$ is analytic in a full neighborhood $K_{\hat\rho}(0)$ of the origin and $A_0 := \lim_{z\to 0} A(z) \neq 0$. In case $r < 0$ the origin is a regular point of (2.2); hence we assume $r \geq 0$. Therewith we rewrite the differential equation in the form

$$z^{1+r}y' = A(z)\,y, \tag{2.3}$$

and r is called the *Poincaré rank* of (2.3). In the *regular singular* case $r = 0$, the power-series part of solutions $y = \sum_{\lambda\in\Lambda}\sum_{\nu=0}^{n} z^\lambda (\log z)^\nu \sum_{s=0}^{\infty} y_{\lambda\nu s} z^s$ is known to converge in a full neighborhood of the origin and hence represents an analytic function there.

2.1 Existence in sectors of small opening

If (2.3) admits an *irregular singularity of Poincaré rank* $r \geq 1$, however, the situation is more complicated. First, it follows from a theorem due to Hukuhara and Turrittin that in case $r \geq 1$, equation (2.3) always admits a *formal fundamental matrix* $\hat{Y}(z)$ of the form

$$\hat{Y}(z) = \hat{F}(z) z^L e^{Q\left(\frac{1}{z}\right)} \tag{2.4}$$

with

$$\begin{cases} t = z^{1/p} \text{ with some } p \in \mathbb{N}^*, \\ Q(\frac{1}{z}) = \operatorname{diag}\big(q_1(\frac{1}{z}), \ldots, q_n(\frac{1}{z})\big) \text{ with } q_k(\frac{1}{z}) \in t^{-1}\mathbb{C}[t^{-1}], \\ L \in \mathbb{C}^{n\times n}, \\ Q(\frac{1}{z})L = LQ(\frac{1}{z}), \\ \hat{F}(z) \in GL(n; \mathbb{C}[[t]]) \end{cases}$$

(cf. [HS99b], or also [Was87]). All the quantities like the number p, the polynomials q_k and their respective degrees as well as the matrix L can be computed directly from the differential equation by pure algebraic methods (and a finite number of terms of the formal series $\hat{F}(z)$ by recurrence relations). There have been developed computer algebra packages for that; see e.g. [Pfl97a, Pfl97b, LMC]. One knows even more: the formal solutions are of some positive fractional Gevrey order which can can be determined from the differential equation as well.

Asking for *analytic solutions* in regions near $z=0$, there is a fundamental existence theorem stating that *in sectors S* at 0 of opening not exceeding $\frac{\pi}{r}$, there is an *asymptotic solution* of the matrix equation corresponding to (2.3), i.e. a solution Y of that matrix differential equation which is analytic on S and admits the asymptotic expansion

$$Y(z) \sim \hat{Y}(z) \qquad (S \ni z \to 0)$$

where $\hat{Y}$ is given by (2.4). This is a short notation for the existence of an analytic function $F:\ S \to \mathbb{C}^{n\times n}$ such that (in the sense of Poincaré asymptotics)

$$F(z) \sim \hat{F}(z) \quad (S \ni z \to 0) \qquad \text{and} \qquad Y(z) = F(z)z^L e^{Q\left(\frac{1}{z}\right)}. \tag{2.5}$$

The proof is based on the so-called *Main Asymptotic Existence Theorem* which is highly non-trivial analysis (cf. [Was87, HS99b]). Apart from being not really constructive, the above solution is not uniquely determined by $\hat{Y}$ and the sector S.

2.2 Existence in sectors of large opening (Multisummability)

There is, however, a constructive method to associate a *unique* sum to a formal solution or a formal fundamental matrix (2.4): this is the concept of *multisummability*. There have been given different definitions of multisummability by several authors: multisummability in the sense of Martinet and Ramis [MR91] which has been shown by Braaksma [Bra91] to apply to formal solutions of linear (and non-linear) meromorphic differential equations, and the multisummability in the sense of Malgrange and Ramis [MR92] which, by an isomorphism theorem due to Malgrange, is equivalent to the first definition. Nowadays multisummability is well-understood, and there have been given many independent proofs, equivalent formulations and generalizations (see e.g. [Bal00]). Since we will only use the results to construct a ”natural” system of asymptotic solutions, we will not give the very general definition but confine to *multisummability in a direction*. For further details we refer to [Bra91, B$^+$91, Bra94, BIS99] and to the textbooks [Bal94, Bal00].

We will need a few basic definitions on Borel and Laplace operators. We will follow [Bra91] to some extent. Let $k>0$, $\left(\lambda_s\right)_{s=0}^{\infty}$ a sequence in $\mathbb{R}$ with $\lambda_{s+1} > \lambda_s$ and

$$\hat{f}(z) = \sum_{s=0}^{\infty} f_s z^{\lambda_s}.$$

Then the *formal Borel transform of order k of* $\hat{f}$ is defined as

$$(\hat{\mathscr{B}}\hat{f})(z) := \sum_{s=0}^{\infty} f_s \frac{z^{\lambda_s - k}}{\Gamma(\lambda_s/k)}.$$

This definition especially applies to every $\hat{f}(z) \in \mathbb{C}[[z]]$ or $\hat{f}(z) \in \mathbb{C}[[z^{1/p}]]$.

Let $k > 0$ and $S = S(\alpha, \beta)$ be a sector. A function $f : S \to \mathbb{C}$ is said to be *of exponential growth of order* $\le k$ *in S* if there exists a constant $C > 0$ such that

$$f(z) = e^{C|z|^k} \cdot O(1) \qquad (S \ni z \to \infty). \tag{2.6}$$

If, in addition, f is analytic in S and $f(z) = O\big(z^{-k+\delta}\big)$ $(S \ni z \to 0)$ with some $\delta > 0$ then the *Laplace transform of order k of f in S* is defined by

$$\big(\mathscr{L}_{k,S} f\big)(z) := \int_0^{[\theta]} e^{-(t/z)^k} f(t)\, d(t^k) \tag{2.7}$$

which, due to the growth conditions above, converges for all z in

$$\mathscr{D}_k(S,C) := \Big\{ z \in S\big(\alpha - \tfrac{\pi}{2k}, \beta + \tfrac{\pi}{2k}\big) \,:\, \exists \theta \in (\alpha, \beta) \text{ s. th. } \mathrm{Re}\big(e^{ik\theta} z^{-k}\big) > C \Big\}. \tag{2.8}$$

For each $z \in \mathscr{D}_k(S,C)$, the direction $\theta \in (\alpha, \beta)$ in (2.7) is chosen so that $\mathrm{Re}\big(e^{ik\theta} z^{-k}\big) > C$.

Next we define Écalle's acceleration operators. The kernel functions which will appear are entire functions C_μ $(\mu > 1)$ defined by

$$C_\mu(z) := \frac{1}{2\pi i} \int_{(0+)}^{-\infty} e^{t - z t^{1/\mu}}\, dt.$$

Let now $0 < k < k'$ and κ defined by

$$\frac{1}{\kappa} = \frac{1}{k} - \frac{1}{k'}.$$

Moreover, let f be a function analytic in $S = S(\alpha, \beta)$, assume $f(z) = O\big(z^{-k+\delta}\big)$ $(S \ni z \to 0)$ with some $\delta > 0$ and that f is of exponential growth of order $\le \kappa$ in S. Then

$$\big(\mathbf{A}_{k',k,S} f\big)(z) := z^{-k'} \int_0^{[\theta]} C_{k'/k}\big((t/z)^k\big) f(t)\, d(t^k)$$

converging for $z \in \mathscr{D}_\kappa(S,C)$ where $C > 0$ is the constant in (2.6) with k replaced by κ. Therewith we define

Definition 2.1 (Multisummability). *Let $q \in \mathbb{N}^*$ and $0 < k_1 < \cdots < k_q$. With $k_{q+1} := +\infty$ let*

$$\frac{1}{\kappa_j} = \frac{1}{k_j} - \frac{1}{k_{j+1}} \qquad (j = 1, \ldots, q).$$

Let $\theta \in \mathbb{R}$ be a direction and $\varepsilon_j > \frac{\pi}{2k_j}$ be real numbers satisfying $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_q$. Furthermore let $\hat{f}(z) \in z\mathbb{C}[[z]]_{1/k_1}$.

(i) *We then have*

$$\varphi_1(z) := \big(\hat{\mathscr{B}}_{k_1} \hat{f}\big)(z) \in \mathbb{C}\{z\}.$$

Assume that φ_1 can be analytically continued on the sector $S_1 := S\big(\theta - \varepsilon_1 + \frac{\pi}{2k_1}, \theta + \varepsilon_1 - \frac{\pi}{2k_1}\big)$ and is of exponential growth of order $\leq \kappa_1$ there.

(ii) *Recursively define*

$$\varphi_j := \mathbf{A}_{k_j, k_{j-1}, S_{j-1}} \varphi_{j-1} \qquad (j = 2, \ldots, q)$$

and assume φ_j can be analytically continued on $S_j := S\big(\theta - \varepsilon_j + \frac{\pi}{2k_j}, \theta + \varepsilon_j - \frac{\pi}{2k_j}\big)$ and is of exponential growth of order $\leq \kappa_j$ there.

Then $\hat{f}$ is said to be $(k_1, \ldots, k_q)$-summable in direction θ, and its $(k_1, \ldots, k_q)$-sum is defined by

$$\mathscr{S}^{\theta}_{k_1,\ldots,k_q} \hat{f} := \mathscr{L}_{k_q, S_q} \varphi_q.$$

If all assumptions above are satisfied and $f_0 \in \mathbb{C}$ then we also say that $f_0 + \hat{f}$ is $(k_1, \ldots, k_q)$-summable in direction θ with sum

$$\mathscr{S}^{\theta}_{k_1,\ldots,k_q}\big(f_0 + \hat{f}\big) := f_0 + \mathscr{S}^{\theta}_{k_1,\ldots,k_q} \hat{f}.$$

Definition 2.2. *For $p \geq 2$ and $\hat{f}(z) \in \mathbb{C}[[z^{1/p}]]$ we also define: $\hat{f}(z)$ is called $(k_1, \ldots, k_q)$-summable in direction θ if $\hat{g}(z) := \hat{f}(z^p)$ is $(pk_1, \ldots, pk_q)$-summable in direction θ/p. Then we define*

$$\big(\mathscr{S}^{\theta}_{k_1,\ldots,k_q} \hat{f}\big)(z) := \big(\mathscr{S}^{\theta/p}_{pk_1,\ldots,pk_q} \hat{g}\big)\big(z^{1/p}\big).$$

This definition is in accordance with the properties of multisummability in the sense of the previous definition.

If there is only a single level k then there are no acceleration operators needed and multisummability reduces to the concept of *k-summability* [Ram80] which in turn, in case $k = 1$, is equivalent to ordinary Borel-Laplace summability.

There are many nice properties of the differential algebras of formal series $(k_1, \ldots, k_q)$-summable in direction d, and relations between those algebras. For further details we refer to the books [Bal94, Bal00]. We will only note one basic property which we need later: If $\hat{f}(z) \in \mathbb{C}[[z]]$ then multisummability of $\hat{f}$ in direction d is equivalent to that in direction $d+2\pi$, and we have

$$(2.9) \qquad \big(\mathscr{S}^{\theta+2\pi}_{k_1,\ldots,k_q}\hat{f}\big)(z) = \big(\mathscr{S}^{\theta}_{k_1,\ldots,k_q}\hat{f}\big)\big(ze^{-2\pi i}\big).$$

Now return to the differential equation (2.3) with formal fundamental matrix (2.4). First we introduce some notation. For $\ell, k \in \{1,\ldots,n\}$ write

$$q_\ell - q_k = q_{\ell k}.$$

For each pair (ℓ,k) such that $q_{\ell k}$ is not identically equal to zero, we can write

$$q_{\ell k}\big(\tfrac{1}{z}\big) = \alpha_{\ell k} z^{-\kappa_{\ell k}} + o\big(z^{-\kappa_{\ell k}}\big) \qquad (z \to 0)$$

with an $\alpha_{\ell k} \in \mathbb{C}^*$ and $\kappa_{\ell k}$ being a positive rational number. Denote the set of all $\kappa_{\ell k}$ by $\{h_1,\ldots,h_q\}$ arranged such that $0 < h_1 < h_2 < \cdots < h_q$. These are the *levels* of (2.3). To every nonzero $q_{\ell k}$ the *singular directions associated with* $q_{\ell k}$ are all $\theta \in \mathbb{R}$ for which

$$(2.10) \qquad \theta = \tfrac{1}{\kappa_{\ell k}} \arg(-\alpha_{\ell k}) + \tfrac{2\pi}{\kappa_{\ell k}} m$$

with some $m \in \mathbb{Z}$. In this situation, the directions $\theta \pm \frac{\pi}{2\kappa_{\ell k}}$ are called *Stokes' directions associated with* $q_{\ell k}$, and the pair

$$(2.11) \qquad \theta - \tfrac{\pi}{2\kappa_{\ell k}}, \quad \theta + \tfrac{\pi}{2\kappa_{\ell k}}$$

is called *Stokes' pair associated with* $q_{\ell k}$. Also, θ will be called *singular of level* $\kappa_{\ell k}$, and the pair (2.11) is called *Stokes' pair of level* $\kappa_{\ell k}$. A direction $\theta \in \mathbb{R}$ is said to be a *singular direction of* (2.3) if it is singular of level h_j for some $j \in \{1,\ldots,q\}$.

The multisummability of formal solutions of (2.3) is shown by the

Theorem 2.1 ([Bra91]). *Let $\theta \in \mathbb{R}$ be none of the singular directions of* (2.3) *then we can find positive numbers $\varepsilon_j > \frac{\pi}{2h_j}$ such that $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_q$ and $\hat{S}_j := S(\theta - \varepsilon_j, \theta + \varepsilon_j)$ does not contain any Stokes pair of level h_j. Then $\hat{F}$ in* (2.4) *is $(h_1,\ldots,h_q)$-summable in direction θ with sum F on $\hat{S} = S_{\hat{\rho}}(\theta - \varepsilon_q, \theta + \varepsilon_q)$, and Y defined according to* (2.5) *is a fundamental matrix solution of* (2.3).

In particular, if $k \in \{1,\ldots,n\}$ and θ is none of the singular directions of (2.3) *associated with $q_{\ell k}$ for any $\ell \neq k$ then $\hat{F}_{\cdot k}$ is $(h_{k1},\ldots,h_{k,q(k)})$-summable in direction θ with sum $F_{\cdot k}$ on $S_{\hat{\rho}}(\theta - \varepsilon_{q(k)}, \theta + \varepsilon_{q(k)})$ where $h_{k1},\ldots,h_{k,q(k)}$ are the different levels of* (2.3) *associated with $q_{\ell k}$ for some $\ell \neq k$.*

This multisum[5] is uniquely determined by $\hat{F}$ and θ. Then we have $F(z) \sim \hat{F}(z)$ $(\hat{S} \ni z \to 0)$, and we also use the short notation $\mathscr{S}^{\theta}_{h_1,\ldots,h_q}\hat{Y} := Y$.

[5] In [Bra91], Braaksma actually shows multisummability on multi-intervals and describes in detail the asymptotic sectors of F.

2.3 Stokes' phenomenon for multisums

If $\theta < \tilde{\theta}$ and there is no singular direction in the closed interval $[\theta, \tilde{\theta}]$ then $Y_2 := \mathscr{S}^{\tilde{\theta}}_{h_1,\dots,h_q}\hat{Y}$ is the analytic continuation of $Y_1 := \mathscr{S}^{\theta}_{h_1,\dots,h_q}\hat{Y}$. This needs no longer be true if there *is* a singular direction in between, since the validity of the asymptotic expansion of Y_1, just like for all solutions, stops at certain Stokes' directions. Hence the connection matrix ("Stokes matrix") V between both fundamental matrices

$$Y_2 = Y_1 V \tag{2.12}$$

is of particular interest. Multisums, in addition to their constructive definition, admit a very clear Stokes' phenomenon, and their Stokes matrices V are "Galoisian" in the sense of [LR94], i.e. come from a representation of the differential Galois group of (2.3) (cf. the remark on page 33). For multisums, one can even calculate the Stokes multipliers (cf. [Bra91, Bra94]). We will focus here on a statement concerning which of the Stokes multipliers may be nonzero:

Let $\theta < \tilde{\theta}$ be two non-singular directions of (2.3) *such that the open interval $(\theta, \tilde{\theta})$ contains* exactly one *singular direction, θ_0, say. Then the Stokes matrix V for the multisums Y_1 and Y_2 above (cf.* (2.12)*) satisfies* [Bra91][6]

$$\begin{cases} V_{kk} = 1, & \\ V_{\ell k} = 0 & \textit{for all pairs } \ell \neq k \textit{ such that } \theta_0 \\ & \textit{is no singular direction associated} \\ & \textit{with } q_{\ell k}. \end{cases} \tag{2.13}$$

[6] Again, Braaksma has shown more, namely he further factorized the Stokes matrix according to the different levels and gave relations for the non-zero multipliers.

3 A representation of the remainders and the expansion coefficients

In this section we will develop integral representations for the coefficients of the formal solutions of linear meromorphic differential equations as well as for the remainders after truncating the asymptotic expansion at the N-th term. (The result also applies to linear meromorphic difference equations, but we suppress this aspect here.) These representations are well-suited for our later analysis as they

- are integral representations and thus can be investigated by standard methods of Asymptotic Analysis,
- incorporate Stokes' phenomenon, and
- allow recursive insertion (to obtain hyperasymptotic expansions).

The main tool for this representation is the so-called Cauchy-Heine formula, a consequence of Cauchy's integral formula.

3.1 The Cauchy-Heine formula

As we have seen in Section 2, asymptotic solutions of linear ODE near irregular singular points exhibit a Stokes' phenomenon: the asymptotic behavior changes rapidly along certain lines. Thus, in general there are no solutions with prescribed asymptotic expansion in a full neighborhood of the singularity; the theory merely guarantees the existence of individual asymptotic solutions in sectors covering a full neighborhood. In this connection, it is useful to consider coverings of a punctured disk around the singularity by finitely many open sectors. The Cauchy-Heine formula deals with such a situation. But to suit Stokes' phenomenon, the sectors have to satisfy some ordering property. There are two popular concepts of such coverings in the literature: the *normal coverings* [Bal94], and the *good coverings* [Sib91a, Sib91b, HS99b].

Definition 3.1. *Let $I_\nu = (\alpha_\nu, \beta_\nu)$, $\nu = 1, \ldots, M$ be a collection of open intervals. Putting $I_0 := I_M - 2\pi$, $I_{M+1} := I_1 + 2\pi$, assume that $I_{\nu-1} \cap I_\nu \neq \emptyset$, $(\nu = 1, \ldots, M)$ and*

$$\text{(NC)} \qquad \tfrac{1}{2}(\alpha_{\nu-1} + \beta_{\nu-1}) < \tfrac{1}{2}(\alpha_\nu + \beta_\nu) \qquad (\nu = 1, \ldots, M), \quad \textit{or}$$
$$\text{(GC)} \qquad \alpha_\nu < \beta_{\nu-1} < \alpha_{\nu+1} < \beta_\nu \qquad (\nu = 1, \ldots, M).$$

Let $\rho > 0$ be arbitrarily given. Then the M sectors

$$S_\nu := S_\rho(I_\nu), \qquad (\nu = 1, \ldots, M)$$

are said to form a normal covering *(resp. a* good covering*) of the punctured disc $D(\rho)$.*

Remark. The intervals I_0 and I_{M+1} are only defined to better describe the connection between I_1 and I_M. The sector S_{M+1}, if defined the like, lies directly above S_1 (on the Riemann surface of the logarithm), as does S_M with respect to S_0. The property (NC) says that the sectors are ordered such that the bisecting directions are in ascending order. (GC), however, says that two of the sectors have a non-empty intersection if and only if they are "consecutive" sectors.

The notion of a good covering is unnecessarily strong for the formulation of the Cauchy-Heine formula below. Nothing prevents us from considering sectors with mutual intersection. The notion of a normal covering is somewhat more general. However, what we really need is another ordering property leading to a covering which we want to call a *Cauchy-Heine covering*:

Definition 3.2. *If we replace, in Definition 3.1, the property* (NC) *resp.* (GC) *by*

$$\text{(CHC)} \qquad \begin{aligned} &\exists (\theta_\nu)_{\nu=1}^M,\ \theta_\nu \in I_{\nu-1} \cap I_\nu,\ \textit{such that with } \theta_0 := \theta_M - 2\pi \\ &\qquad\qquad \textit{we have } \theta_{\nu-1} < \theta_\nu \quad (\nu = 1,\dots,M), \end{aligned}$$

then the resulting sectors S_ν are said to form a Cauchy-Heine covering *of the punctured disc $D(\rho)$.*

Lemma 3.1. *Every good covering of $D(\rho)$ is a normal covering of $D(\rho)$, and every normal covering of $D(\rho)$ is in turn a Cauchy-Heine covering of $D(\rho)$.*

Proof. The first statement is immediately clear from the definition. Let therefore $(S_\nu)_{\nu=1}^M$ be a normal covering, according to Definition 3.1. Define directions $d_\nu := \frac{1}{2}(\alpha_\nu + \beta_\nu)$ $(\nu = 0,\dots,M)$, and $\big(\theta_\nu\big)_{\nu=1}^M$ by

$$\text{(3.1)} \qquad \theta_\nu := \begin{cases} d_\nu, & d_\nu < \beta_{\nu-1}, \\ d_{\nu-1}, & d_\nu \ge \beta_{\nu-1} \wedge d_{\nu-1} > \alpha_\nu, \\ \frac{1}{2}(\alpha_\nu + \beta_{\nu-1}), & d_\nu \ge \beta_{\nu-1} \wedge d_{\nu-1} \le \alpha_\nu. \end{cases}$$

Then in all cases we obviously have $\theta_\nu \in I_{\nu-1} \cap I_\nu$ $(\nu = 1,\dots,M)$.

Now we show that $\theta_{\nu-1} \le \theta_\nu$ $(\nu = 1,\dots,M)$. If $\theta_{\nu-1}$ and θ_ν are defined by the first or the second case in (3.1), then $\theta_{\nu-1} \le \theta_\nu$ follows from the property (NC) of the normal covering. We even have $\theta_{\nu-1} < \theta_\nu$ except for the combination {first case for $\nu - 1$ and second case for ν}. Remains to consider the situation when either $\theta_{\nu-1}$ or θ_ν is defined by the third case in (3.1). E.g., for the combination {third case for $\nu - 1$ and second case for ν} we have $\beta_{\nu-2} \le d_{\nu-1}$ and hence

$$\theta_{\nu-1} = \tfrac{1}{2}(\alpha_{\nu-1} + \beta_{\nu-2}) < \beta_{\nu-2} \le d_{\nu-1} = \theta_\nu,$$

whereas for the combination {third case for both $\nu - 1$ and ν} we have $\beta_{\nu-2} \le d_{\nu-1} \le \alpha_\nu$ and hence

$$\theta_{\nu-1} = \tfrac{1}{2}(\alpha_{\nu-1} + \beta_{\nu-2}) < \beta_{\nu-2} \le d_{\nu-1} \le \alpha_\nu < \tfrac{1}{2}(\alpha_\nu + \beta_{\nu-1}) = \theta_\nu.$$

All remaining cases can be treated similarly, and we obtain $\theta_{\nu-1} < \theta_\nu$. Altogether we have the sequence of inequalities

$$\cdots \leq \theta_{\nu-1} \leq \theta_\nu \leq \theta_{\nu+1} \leq \cdots,$$

but there cannot occur two consecutive equality signs. Now, since all intervals are open, we can modify the right member of each equality by a small quantity, yielding a sequence $\left(\theta'_\nu\right)_{\nu=1}^{M}$ which satisfy (CHC). This completes the proof. $\square$

Remark. It is easy to see that any open covering $\boldsymbol{S} = \left\{S_\rho(I_\lambda),\ \lambda \in \Lambda\right\}$ of $D(\rho)$ contains a finite subcovering $(S_\nu)_{\nu=1}^{M}$ which is a Cauchy-Heine covering: First, by compactness of S^1 there is a finite open subcovering $\boldsymbol{S}'$ of $\boldsymbol{S}$. Re-write all sectors $S = S_\rho(\alpha,\beta) \in \boldsymbol{S}'$ the way that $\beta \in (0,2\pi]$. Now select $\boldsymbol{S}' \ni S_1 = S_\rho(\alpha_1,\beta_1)$ such that $(\alpha_1,\beta_1) \ni 0$ and β_1 be minimal; select $\boldsymbol{S}' \ni S_2 = S_\rho(\alpha_2,\beta_2)$ with $(\alpha_2,\beta_2) \ni \beta_1$, and so on, proceeding up to the index M when $\beta_M > \beta_1 + 2\pi$ for the first time. The resulting subcovering $(S_\nu)_{\nu=1}^{M}$ satisfies condition (CHC) — one could e.g. choose any $\theta_\nu \in \left(\max\{\alpha_\nu,\beta_{\nu-2}\},\beta_{\nu-1}\right)$.

To carry over to such a situation the power and strength of Cauchy's integral formula, one can use the Cauchy-Heine formula which will be presented below. Though it has, to my knowledge, not been stated explicitly in the literature, its idea is not new and is quite frequently used in the theory of irregular singularities, turning points, and singular perturbations [Sib81, Hoe94, FS96, HS99a, HS99b].

Theorem 3.2 (Cauchy-Heine formula). *Let $S_\nu = S_{\hat\rho}(I_\nu)$ $(\nu = 1,\dots,M)$ be a collection of open sectors forming a Cauchy-Heine covering of $D(\hat\rho)$, and let $(f_\nu)_{\nu=1}^{M}$ be functions each analytic on S_ν. Fix directions $\theta_\nu \in I_{\nu-1} \cap I_\nu$, $\theta_{\nu-1} < \theta_\nu$, and points T_ν, $\arg T_\nu = \theta_\nu$, $|T_\nu| = \rho_0 < \hat\rho$. Furthermore, assume that each f_ν is integrable in $\overline{S}_{\rho_0}(\theta_\nu,\theta_{\nu+1})$ near $z=0$:*

$$\lim_{\gamma\searrow 0}\int_\gamma^{\rho_0} \sup\Big\{\left|f_\nu(z)\right|,\ |z| = u,\ \theta_\nu \leq \arg z \leq \theta_{\nu+1}\Big\}\, du < \infty.$$

Then we have

$$(3.2)\quad f_\nu(z) = \frac{1}{2\pi i}\sum_{j=1}^{M}\int_{\overset{\frown}{T_jT_{j+1}}} \frac{f_j(w)}{w-z}\,dw + \frac{1}{2\pi i}\sum_{j=1}^{M}\int_{\overline{OT_j}} \frac{f_j(w)-f_{j-1}(w)}{w-z}\,dw$$

$$\textit{for } z \in S_{\rho_0}(\theta_\nu,\theta_{\nu+1}) \quad (\nu = 1,\dots,M).$$

(The integration has to be taken over arcs and radii, respectively. In (3.2), read $\theta_{M+1} = \theta_1 + 2\pi$, $T_{M+1} = T_1$, and $f_0 = f_M$.)

Proof. The proof is very straightforward. For $\nu \in \{1,\ldots,M\}$ and $0 \le \varepsilon < \rho_0$ define the auxiliary path $\mathscr{C}_\nu(\varepsilon)$ as the closed contour from $T_{\nu+1}$ to $\varepsilon \cdot \exp(i\theta_{\nu+1})$, then along the circular arc to $\varepsilon \cdot \exp(i\theta_\nu)$, radially to T_ν and back to $T_{\nu+1}$.

Let now $\nu \in \{1,\ldots,M\}$, $z \in S_{\rho_0}(\theta_\nu, \theta_{\nu+1})$, and choose $0 < \varepsilon < |z|$. Then Cauchy's integral formula implies

$$f_\nu(z) = \frac{1}{2\pi i} \oint_{\mathscr{C}_\nu(\varepsilon)} \frac{f_\nu(w)}{w-z}\,dw = \frac{1}{2\pi i} \oint_{\mathscr{C}_\nu(0)} \frac{f_\nu(w)}{w-z}\,dw,$$

whereas for $j \ne \nu$ we have

$$0 = \frac{1}{2\pi i} \oint_{\mathscr{C}_j(\varepsilon)} \frac{f_j(w)}{w-z}\,dw = \frac{1}{2\pi i} \oint_{\mathscr{C}_j(0)} \frac{f_j(w)}{w-z}\,dw.$$

The integration over $\mathscr{C}_j(0)$ is possible due to the integrability assumption. Taking the sum over $j \in \{1,\ldots,M\}$ and splitting up the integration paths we obtain

$$\begin{aligned} f_\nu(z) &= \frac{1}{2\pi i} \sum_{j=1}^{M} \oint_{\mathscr{C}_j(0)} \frac{f_j(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i} \sum_{j=1}^{M} \int_{\widehat{T_jT_{j+1}}} \frac{f_j(w)}{w-z}\,dw + \frac{1}{2\pi i} \sum_{j=1}^{M} \int_{\overline{OT_j}} \frac{f_j(w)}{w-z}\,dw - \frac{1}{2\pi i} \sum_{j=1}^{M} \int_{\overline{OT_{j+1}}} \frac{f_j(w)}{w-z}\,dw, \end{aligned}$$

hence Theorem 3.2 follows. □

Remark. The value of Theorem 3.2 consists of splitting up the functions f_ν into a sum of a function analytic at $z = 0$ and *independent of* ν (the first sum in (3.2)), and an individual correction function which only depends *on the differences* $f_j - f_{j-1}$ (the second sum in (3.2)). If now these differences are small, then one can approximate the f_ν's by the analytic term, and the error will be small, too. We will quantify this relation in Proposition 3.3 below.

3.2 Basic representation

For the application to asymptotic solutions of ordinary differential equations we consider systems of functions $(f_k^{(\nu)})_{k=1}^n$ analytic and bounded on sectors S_ν forming a Cauchy-Heine covering. Apart from Theorem 3.2 we essentially make use of some connection relations between the $f_k^{(\nu)}$'s originating from Stokes' phenomenon across directions θ_ν:

$$f_k^{(\nu)}(z) - f_k^{(\nu-1)}(z) = \sum_{(\ell,k)\in J_\nu} p_{\ell k}^{(\nu)}(z) f_\ell^{(\nu-1)}(z), \qquad (z \in S_{\nu-1} \cap S_\nu) \tag{3.3}$$

with some $J_\nu \in \{1,\dots,n\}^2$ and $p_{\ell k}^{(\nu)}$ analytic on $S_{\nu-1}\cap S_\nu$.

We want to show that, under some additional assumptions, the functions $f_k^{(\nu)}$ admit an asymptotic expansion $\hat{f}_k(z) = \sum_{s=0}^{\infty} f_{ks}z^s$, and we are interested in integral representations for the expansion coefficients f_{ks} as well as for the N-th remainders of the asymptotic expansion

$$R_k^{(\nu)}(z,N) := f_k^{(\nu)}(z) - \sum_{s=0}^{N-1} f_{ks}z^s. \tag{3.4}$$

To make things shorter, abbreviate the integral expressions

$$I_{\ell k}^{(j)}(z,N) = \frac{1}{2\pi i}\int\limits_{\overline{OT_j}} p_{\ell k}^{(j)}(w)w^{-N}\frac{f_\ell^{(j-1)}(w)}{w-z}\,dw, \tag{3.5}$$

$$\varepsilon_k(z,N) = \frac{1}{2\pi i}\sum_{j=1}^{M}\int\limits_{\widehat{T_jT_{j+1}}} w^{-N}\frac{f_k^{(j)}(w)}{w-z}\,dw, \tag{3.6}$$

the integration being understood over radii and arcs, respectively.

With all these ingredients, we obtain the following proposition. Note that the number and size of the sectors S_ν is still arbitrary here. Also, the origin of the functions $f_k^{(\nu)}$ is not important: one could also think of asymptotic solutions of linear meromorphic difference equations (after the transformation $z = 1/x$) or something similar.

Proposition 3.3. *Let $S_\nu = S_{\hat\rho}(I_\nu);\ \ I_\nu = (\alpha_\nu,\beta_\nu)\ (\nu = 1,\dots,M)$ be a collection of open sectors forming a Cauchy-Heine covering of $D(\hat\rho)$, let $n\in\mathbb{N}^*$, and for each ν let $(f_k^{(\nu)})_{k=1}^{n}$ be functions analytic on S_ν and bounded on every closed subsector. Fix directions $\theta_\nu\in I_{\nu-1}\cap I_\nu$, $\theta_{\nu-1}<\theta_\nu$, and points T_ν, $\arg T_\nu = \theta_\nu$, $|T_\nu| = \rho_0 < \hat\rho$. Assume that a set of connection relations* (3.3) *hold in the intersections of consecutive sectors, and moreover for all $N\in\mathbb{N}$ let $p_{\ell k}^{(\nu)}(z) = o(z^N)$ as $|z|\to 0$ on $\arg z = \theta_\nu$ whenever $(\ell,k)\in J_\nu$.*

Then there are sequences $\left(f_{ks}\right)_{s=0}^{\infty}$ $(k\in\{1,\dots,n\})$ such that $f_k^{(\nu)}$ admits the asymptotic expansion

$$f_k^{(\nu)}(z) \sim \hat{f}_k(z) := \sum_{s=0}^{\infty} f_{ks}z^s \qquad \left(S_{\rho_0}(\theta_\nu,\theta_{\nu+1})\ni z\to 0\right). \tag{3.7}$$

Moreover, the expansion coefficients f_{ks} and the remainders $R_k^{(\nu)}(z,N)$ in (3.4) *admit* *the integral representations*

$$f_{ks} = \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} I_{\ell k}^{(j)}(0,s) + \varepsilon_k(0,s), \tag{3.8}$$

$$z^{-N} R_k^{(\nu)}(z,N) = \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} I_{\ell k}^{(j)}(z,N) + \varepsilon_k(z,N) \qquad \big(z \in S_{\rho_0}(\theta_\nu, \theta_{\nu+1})\big) \tag{3.9}$$

where the $I_{\ell k}^{(j)}(z,N)$ *and* $\varepsilon_k(z,N)$ *are defined in* (3.5) *and* (3.6)*, respectively.*

Proof. For the entire proof, fix $k \in \{1,\dots,n\}$ and $N \in \mathbb{N}$.

The integrals in (3.6) are obviously well-defined for $z \in S_{\rho_0}(\theta_\nu, \theta_{\nu+1}) \cup \{0\}$, but what about the $I_{\ell k}^{(j)}(z,N)$ in (3.5)? From the assumptions we know that $\frac{f_\ell^{(j-1)}(w)}{w-z} = O(w^{-1})$ for $w \in \overline{OT_j}$ and for each of the above z. On the other hand we have $p_{\ell k}^{(j)}(w) = O(w^{N+1})$ so that we obtain that the whole integrand is bounded, hence integrable. For $z \in S_{\rho_0}(\theta_\nu, \theta_{\nu+1}) \cup \{0\}$, the integrals in (3.5) are thus well-defined, too.

As the functions $f_\ell^{(\nu)}$ are analytic on S_ν and bounded on the closed subsector $\overline{S}_{\rho_0}(\theta_\nu, \theta_{\nu+1})$, all assumptions of Theorem 3.2 (including the integrability condition) are satisfied. Therefore we can write

$$f_k^{(\nu)}(z) = \frac{1}{2\pi i} \sum_{j=1}^{M} \int_{\widehat{T_j T_{j+1}}} \frac{f_k^{(j)}(w)}{w-z}\, dw + \frac{1}{2\pi i} \sum_{j=1}^{M} \int_{\overline{OT_j}} \frac{f_k^{(j)}(w) - f_k^{(j-1)}(w)}{w-z}\, dw. \tag{3.10}$$

The usual procedure to obtain asymptotic expansions of the integrals on the right-hand side is to expand the term $\frac{1}{w-z}$ into a geometrical series. We will consider a finite part of this expansion:

$$\frac{1}{w-z} = \sum_{s=0}^{N-1} \frac{z^s}{w^{s+1}} + \frac{z^N}{w^N(w-z)}. \tag{3.11}$$

Now we can express the integrand of the second integral in (3.10) by means of (3.3) and (3.11) as to

$$\begin{aligned} \frac{f_k^{(j)}(w) - f_k^{(j-1)}(w)}{w-z} &= \sum_{(\ell,k)\in J_j} \frac{p_{\ell k}^{(j)}(w) f_\ell^{(j-1)}(w)}{w-z} \\ &= \sum_{s=0}^{N-1} z^s \sum_{(\ell,k)\in J_j} p_{\ell k}^{(j)}(w) w^{-s} \frac{f_\ell^{(j-1)}(w)}{w} + z^N \sum_{(\ell,k)\in J_j} p_{\ell k}^{(j)}(w) w^{-N} \frac{f_\ell^{(j-1)}(w)}{w-z}. \end{aligned} \tag{3.12}$$

Now, inserting (3.11) again (for the first integral) and (3.12) into (3.10), we find

$$
\begin{aligned}
f_k^{(\nu)}(z) &= \sum_{s=0}^{N-1} z^s \varepsilon_k(0,s) + z^N \varepsilon_k(z,N) \\
&\quad + \sum_{s=0}^{N-1} z^s \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} I_{\ell k}^{(j)}(0,s) + z^N \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} I_{\ell k}^{(j)}(z,N) \\
&= \sum_{s=0}^{N-1} f_{ks} z^s + z^N \underbrace{\left(\sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} I_{\ell k}^{(j)}(z,N) + \varepsilon_k(z,N) \right)}_{=:\ r_k^{(\nu)}(z,N)}
\end{aligned}
$$

for $z \in S_{\rho_0}(\theta_\nu, \theta_{\nu+1})$ and with the coefficients f_{ks} from (3.8).

Consider the closed subsector $S(\delta) := S_{\rho_0-\delta}(\theta_\nu + \delta, \theta_{\nu+1} - \delta)$ of $S_{\rho_0}(\theta_\nu, \theta_{\nu+1})$. Then for $w \in \overline{OT_j} \cup \overset{\frown}{T_j T_{j+1}}$ we have $\frac{w}{w-z} = O(1)$ uniformly w.r.to $z \in S(\delta)$. Together with $p_{\ell k}^{(j)}(w) = O(w^{N+1})$ and the boundedness of $f_\ell^{(j)}(w)$ resp. $f_\ell^{(j-1)}(w)$ we have both integrands in (3.5) and (3.6) bounded uniformly w.r.to $z \in S(\delta)$. Hence $z^N r_k^{(\nu)}(z,N) = O(z^N)$ $(z \in S(\delta))$, which shows the asymptotic expansion (3.7). If now $R_k^{(\nu)}(z,N)$ is defined according to (3.4) then we have $R_k^{(\nu)}(z,N) = z^N r_k^{(\nu)}(z,N)$ and hence (3.9). Proposition 3.3 follows. $\square$

4 Asymptotic and hyperasymptotic expansions for multisums

This section is devoted to the main topic of this thesis: the expansion of certain asymptotic solutions of linear meromorphic differential equations into hyperasymptotic series.

4.1 Notations and preliminaries

We consider linear homogeneous differential equations

$$\text{[D]} \qquad z^{1+r}y' = A(z)y$$

with

$$\begin{cases} A(z) = \sum_{s=0}^{\infty} A_s z^s \in \mathbb{C}^{n\times n}\{z\}, \\ A_0 \neq 0 \end{cases}$$

having an irregular singularity of Poincaré rank $r \geq 1$ at $z = 0$. We are interested in asymptotic and hyperasymptotic expansions of solutions of [D] in appropriate sectors at $z = 0$.

It is well-known (see Section 2.1) that there is a formal fundamental matrix $\hat{Y}(z)$ of [D] of the form

$$\hat{Y}(z) = \hat{F}(z) z^L e^{Q(\frac{1}{z})}. \tag{4.1}$$

Here,

$$\begin{cases} t = z^{1/p} \text{ with some } p \in \mathbb{N}^*, \\ Q(\frac{1}{z}) = \operatorname{diag}\big(q_1(\frac{1}{z}), \ldots, q_n(\frac{1}{z})\big) \text{ with } q_k(\frac{1}{z}) \in t^{-1}\mathbb{C}[t^{-1}], \\ L \in \mathbb{C}^{n\times n}, \\ Q(\frac{1}{z})L = LQ(\frac{1}{z}), \\ \hat{F}(z) \in GL(n;\mathbb{C}[[t]]). \end{cases}$$

Other formal fundamental matrices, e.g. like in [BJL79], §1, or in [Jur78], §3, could be considered instead.

From now on, we consider one fixed formal fundamental matrix of the form (4.1). Then all considerations from Section 2 apply, especially multisummability (cf. Theorem 2.1) in all but singular directions (2.10) of [D]. The column vectors of $\hat{Y}(z)$

$$\hat{y}_k(z) = \hat{f}_k(z) z^{\mu_k} e^{q_k(\frac{1}{z})} \qquad (k = 1, \ldots, n) \tag{4.2}$$

with $\mu_k \in \mathbb{C}$, $\hat{f}_k(z) \in \mathbb{C}^n[[t]][\log t]$, form a formal fundamental system of [D].

Define the exponentials

$$q_{\ell k}(\tfrac{1}{z}) = \alpha_{\ell k} z^{-\kappa_{\ell k}} + o\big(z^{-\kappa_{\ell k}}\big) \qquad (z \to 0)$$

of [D], the levels $\{h_1,\dots,h_q\}$ and the corresponding singular directions resp. Stokes' directions as in Section 2.2. Number all different singular directions $\theta \in \mathbb{R}$ of [D] by $\{\theta_\nu\}_{\nu\in\mathbb{Z}}$ such that

$$\theta_\nu < \theta_{\nu+1} \qquad (\nu \in \mathbb{Z}) \qquad \text{and} \qquad \theta_{-1} < 0 \le \theta_0.$$

Since $Q(\frac{1}{z})$ is a polynomial in $z^{-1/p}$, there is an $M \in \mathbb{N}$ such that

$$\theta_{\nu+M} = \theta_\nu + 2\pi p \qquad (\nu \in \mathbb{Z}).$$

Finally, for each $\nu \in \mathbb{Z}$ define a partial ordering J_ν on $\{1,\dots,n\}$ by

(4.3) $$J_\nu := \Big\{(\ell,k) \;\Big|\; \theta_\nu \text{ is a singular direction associated with } q_{\ell k}\Big\}.$$

4.2 Systems of asymptotic solutions: Generic case

Here we additionally require

(4.4) *Let* $L = \operatorname{diag}(\mu_1,\dots,\mu_n)$ *be a diagonal matrix.*

This is no substantial loss of generality, cf. Section 4.3. Then in (4.2) we have $\hat{f}_k(z) \in \mathbb{C}^n[[t]]$, which enables us to to treat (formal) vector solutions. But since the $\hat{f}_k(z)$ admit trivial formal monodromy only in the t-plane, we consider the ramification $t = z^{1/p}$ and the differential equation [D$_t$] into which [D] is carried by the change of variable $t = z^{1/p}$. It admits the formal fundamental matrix $\hat{Y}_t$ resulting from $\hat{Y}$ by that very change of variable. All essential quantities change in a very simple manner: the rank and the levels are multiplied by p while the singular directions are divided by p.

Hence, without loss of generality we will assume $p = 1$ (i.e. $t = z$) and revert to the notations of the previous section. Then we have $q_k(\frac{1}{z}) \in z^{-1}\mathbb{C}[z^{-1}]$, $\hat{f}_k(z) \in \mathbb{C}^n[[z]]$, $h_j \in \mathbb{N}^*$, and $\theta_{\nu+M} - \theta_\nu = 2\pi$. In this situation we have the following

Lemma 4.1. *Let* (4.2) *be a formal fundamental system of* [D] *with* $p = 1$, *and assume* (4.4) *holds. Then there are a* $\hat{\rho} > 0$, *open intervals* $\{I_\nu\}_{\nu\in\mathbb{Z}}$, *functions* $f_k^{(\nu)}$ *and complex numbers* $c_{\ell k}^{(\nu)}$ $\big(k,\ell = 1,\dots,n;\ \nu \in \mathbb{Z}\big)$ *such that*

- **(i)** $I_\nu = (\alpha_\nu,\beta_\nu) \supset [\theta_\nu,\theta_{\nu+1}] \qquad (\nu \in \mathbb{Z})$,
- **(ii)** *the* I_ν*'s do not contain a Stokes pair,*
- **(iii)** $I_{\nu+M} = I_\nu + 2\pi \qquad (\nu \in \mathbb{Z})$,
- **(iv)** *the sectors* $S_\nu := S_{\hat{\rho}}(I_\nu)$ $(\nu = 1,\dots,M)$ *form a Cauchy-Heine covering of* $D(\hat{\rho})$,

(v) *$f_k^{(\nu)}$ is a function analytic on S_ν and*

$$(4.5) \qquad f_k^{(\nu)}(z) \sim \hat{f}_k(z) \qquad (z \in S_\nu),$$

(vi) $f_k^{(\nu+M)}(z) = f_k^{(\nu)}(ze^{-2\pi i})$,

(vii) *for each $\nu \in \mathbb{Z}$, the functions*

$$y_k^{(\nu)}(z) := f_k^{(\nu)}(z) z^{\mu_k} e^{q_k(\frac{1}{z})} \qquad (k = 1, \ldots, n)$$

form a fundamental system of vector solutions of [D] *on S_ν,*

(viii) *for each $\nu \in \mathbb{Z}$, the $f_k^{(\nu)}$'s satisfy*

$$(4.6) \quad f_k^{(\nu)}(z) - f_k^{(\nu-1)}(z) = \sum_{(\ell,k) \in J_\nu} c_{\ell k}^{(\nu)} e^{q_{\ell k}(\frac{1}{z})} z^{\mu_{\ell k}} f_\ell^{(\nu-1)}(z) \qquad (z \in S_{\nu-1} \cap S_\nu).$$

Here, $\mu_{\ell k} = \mu_\ell - \mu_k$.

Proof. Let $k \in \{1, \ldots, n\}$ and $\nu \in \mathbb{Z}$ be fixed. For all $d \in (\theta_\nu, \theta_{\nu+1})$ take the multisums $\mathscr{S}^d_{h_1,\ldots,h_q}(\hat{f}_k)$ which can be glued together to an analytic function $f_k^{(\nu)}$ on (at least) the sector $S_\nu := S_{\hat{\rho}}\big(\theta_\nu - \frac{\pi}{2h_q}, \theta_{\nu+1} + \frac{\pi}{2h_q}\big)$ for small enough $\hat{\rho} > 0$ – we can even choose $\hat{\rho}$ to be the radius of convergence of $A(z)$. Then with $\alpha_\nu := \theta_\nu - \frac{\pi}{2h_q}$, $\beta_\nu := \theta_{\nu+1} + \frac{\pi}{2h_q}$ (i), (iii), (v), and (vii) are immediately clear. Since $(\theta_\nu, \theta_{\nu+1})$ does not contain a singular direction, (ii) is satisfied, too. (iv) is clear in view of $\theta_\nu \in I_{\nu-1} \cap I_\nu$. (vi) and (viii) follow from (2.9) and (2.13). This completes the proof. □

Remarks. **(i)** The Stokes matrices $V^{(\nu)}$ which describe the connection between the $(y_k^{(\nu-1)})_{k=1}^n$ and the $(y_k^{(\nu)})_{k=1}^n$ and which are defined by

$$V_{\ell k}^{(\nu)} = \begin{cases} c_{\ell k}^{(\nu)}, & (\ell, k) \in J_\nu, \\ \delta_{\ell k} & \text{else}, \end{cases}$$

are "Galoisian" in the sense of [LR94], i.e. come from a representation of the differential Galois group of [D]. In fact, since the $f_k^{(\nu)}$'s are multisums, the isomorphism $Y \mapsto YV^{(\nu)}$ of the space of solutions on $S_{\nu-1} \cap S_\nu$ can be lifted to the differential field automorphism $\mathscr{S}^{\theta_\nu +} \circ \big(\mathscr{S}^{\theta_\nu -}\big)^{-1}$ of a Picard-Vessiot extension of $K = \mathbb{C}\{z\}[z^{-1}]$ relative to [D] leaving the elements of K invariant. Hence, the $V^{(\nu)}$ are Galoisian (cf. [LR94], Theorem III.3.7).

(ii) For I_ν satisfies Lemma 4.1 (i),(ii) and (iv) and (v), it must neither be too small nor too large. From a detailed inspection of Stokes' phenomenon for multisums

(2.13) we see that, if

$$
\begin{aligned}
&\max_{\substack{(\ell,k)\in J_{\tilde\nu}\\ \tilde\nu\le\nu}}\Big(\theta_{\tilde\nu}-\frac{\pi}{2\kappa_{\ell k}}\Big) =: \hat\alpha_\nu \le \alpha_\nu < \theta_\nu,\\
&\theta_{\nu+1} < \beta_\nu \le \hat\beta_\nu := \min_{\substack{(\ell,k)\in J_{\tilde\nu}\\ \tilde\nu\ge\nu+1}}\Big(\theta_{\tilde\nu}+\frac{\pi}{2\kappa_{\ell k}}\Big).
\end{aligned}
\tag{4.7}
$$

then (i), (ii) and (v) are still satisfied for the open interval (α_ν,β_ν) (whereas $[\hat\alpha_\nu,\hat\beta_\nu]$ does contain a Stokes pair). In view of $\theta_\nu\in I_{\nu-1}\cap I_\nu$, (iv) holds, too. In general, the above properties are only guaranteed on the "maximal" sector $\hat I_\nu := (\hat\alpha_\nu,\hat\beta_\nu)$, whereas in in individual cases there might be other maximal sectors (even proper supersectors of $\hat I_\nu$) with these properties.

For each individual $k\in\{1,\dots,n\}$, however, $f_k^{(\nu)}$ can be analytically continued preserving (v) onto the larger sector $S_{\hat\rho}\big(\hat\alpha_\nu(k),\hat\beta_\nu(k)\big)$ where

$$
\begin{aligned}
\hat\alpha_\nu(k) &:= \max_{\substack{(\ell,k)\in J_{\tilde\nu}\\ \tilde\nu\le\nu}}\Big(\theta_{\tilde\nu}-\frac{\pi}{2\kappa_{\ell k}}\Big),\\
\hat\beta_\nu(k) &:= \min_{\substack{(\ell,k)\in J_{\tilde\nu}\\ \tilde\nu\ge\nu+1}}\Big(\theta_{\tilde\nu}+\frac{\pi}{2\kappa_{\ell k}}\Big).
\end{aligned}
\tag{4.8}
$$

This sector does not contain a Stokes pair associated with $q_{\ell k}$ *for that fixed* k, and we have $\hat I_\nu = \bigcap_{k=1}^{n}\big(\hat\alpha_\nu(k),\hat\beta_\nu(k)\big)$.

The $(f_k^{(\nu)})_{k=1,\dots,n}^{\nu=1,\dots,M}$ are a system of functions for which the situation of Section 3.2 applies: for $\nu=2,\dots,M$, (4.6) is a connection relation of the form (3.3) with

$$
p_{\ell k}^{(\nu)}(z) = c_{\ell k}^{(\nu)} e^{q_{\ell k}(\frac{1}{z})} z^{\mu_{\ell k}} \qquad \big((\ell,k)\in J_\nu\big),
\tag{4.9}
$$

whereas the connection relation between S_M and S_1 is given by

$$
\begin{array}{ccc}
f_k^{(1)}(z) - f_k^{(0)}(z) & = \displaystyle\sum_{(\ell,k)\in J_1} c_{\ell k}^{(1)} e^{q_{\ell k}(\frac{1}{z})} z^{\mu_{\ell k}} & f_\ell^{(0)}(z). \\
\| & & \| \\
f_k^{(M)}(ze^{2\pi i}) & & f_\ell^{(M)}(ze^{2\pi i})
\end{array}
$$

which, by identifying $ze^{2\pi i}$ with z, also has the form (3.3), (4.9) with the appropriate choice of the branch of the power $z^{\mu_{\ell k}}$. Hence we can interpret the S_ν as sectors in the complex plane rather than on the Riemann surface of the Logarithm. Moreover, for $(\ell,k)\in J_\nu$ we have by definition (4.3):

$$
\forall N\in\mathbb{N}:\quad p_{\ell k}^{(\nu)}(z) = o(z^N) \qquad \text{if } |z|\to 0 \text{ on } \arg z = \theta_\nu.
$$

Hence, with fixed points $\{T_\nu\}_{\nu=1}^{M}$, $\arg T_\nu = \theta_\nu$, $|T_\nu| = \rho_0 < \hat{\rho}$, by Proposition 3.3 the asymptotic expansions (3.7) and (4.5) must agree on $S_{\rho_0}(\theta_\nu, \theta_{\nu+1})$:

$$\hat{f}_k(z) = \sum_{s=0}^{\infty} f_{ks} z^s,$$

and for the coefficients f_{ks} and the remainders

$$R_k^{(\nu)}(z,N) := f_k^{(\nu)}(z) - \sum_{s=0}^{N-1} f_{ks} z^s \tag{4.10}$$

we have the integral representations (3.8), (3.9) with

$$I_{\ell k}^{(j)}(z,N) = \frac{c_{\ell k}^{(j)}}{2\pi i} \int_{\overline{OT_j}} e^{q_{\ell k}(\frac{1}{w})} w^{-N+\mu_{\ell k}} \frac{f_\ell^{(j-1)}(w)}{w-z}\, dw \tag{4.11}$$

and $\varepsilon_k(z,N)$ as in (3.6).

4.3 Systems of asymptotic solutions: General case

If we drop the assumption (4.4), in view of $\hat{f}_k(z) \in \mathbb{C}^n[[t]][\log t]$ it is no more practicable to treat (formal) vector solutions. However, one can deal with (formal) fundamental matrices instead.

Like in Section 4.2, without loss of generality we will assume $p = 1$ (which can be reached by the ramification $t = z^{1/p}$) and revert to the notations of Section 4.1. Then we have $q_k(\frac{1}{z}) \in z^{-1}\mathbb{C}[z^{-1}]$, $\hat{F}(z) \in GL(n;\mathbb{C}[[z]])$, $h_j \in \mathbb{N}^*$, and $\theta_{\nu+M} - \theta_\nu = 2\pi$. In this situation we have the following

Lemma 4.2. *Let* (4.1) *be a formal fundamental matrix of* [D] *with* $p = 1$. *Then there are a* $\hat{\rho} > 0$, *open intervals* $\{I_\nu\}_{\nu \in \mathbb{Z}}$, *functions* $F^{(\nu)}$ *and matrices* $C^{(\nu)} \in \mathbb{C}^{n\times n}$ $(\nu \in \mathbb{Z})$ *such that*

- **(i)** $I_\nu = (\alpha_\nu, \beta_\nu) \supset [\theta_\nu, \theta_{\nu+1}] \qquad (\nu \in \mathbb{Z})$,
- **(ii)** *the* I_ν*'s do not contain a Stokes pair,*
- **(iii)** $I_{\nu+M} = I_\nu + 2\pi \qquad (\nu \in \mathbb{Z})$,
- **(iv)** *the sectors* $S_\nu := S_{\hat{\rho}}(I_\nu)$ $(\nu = 1, \ldots, M)$ *form a Cauchy-Heine covering of* $D(\hat{\rho})$,
- **(v)** $F^{(\nu)}$ *is a function analytic on* S_ν *and*

$$F^{(\nu)}(z) \sim \hat{F}(z) \qquad (z \in S_\nu),$$

(vi) $F^{(\nu+M)}(z) = F^{(\nu)}\big(ze^{-2\pi i}\big)$,

(vii) *for each* $\nu \in \mathbb{Z}$, *the function*

$$Y^{(\nu)}(z) := F^{(\nu)}(z) z^L e^{Q(\frac{1}{z})}$$

is a fundamental matrix of [D] *on* S_ν,

(viii) *for each* $\nu \in \mathbb{Z}$, $C^{(\nu)}_{\ell k} = 0$ *for* $(\ell,k) \notin J_\nu$, *and the* $F^{(\nu)}$*'s satisfy*

(4.12)
$$F^{(\nu)}(z) - F^{(\nu-1)}(z) = F^{(\nu-1)}(z) z^L e^{Q(\frac{1}{z})} C^{(\nu)} e^{-Q(\frac{1}{z})} z^{-L} \qquad (z \in S_{\nu-1} \cap S_\nu).$$

The *Proof* is analogous to that of Lemma 4.1 with $F^{(\nu)}$ formed by the multisums $\mathscr{S}^d_{h_1,\dots,h_q}(\hat F)$ $\big(d \in (\theta_\nu, \theta_{\nu+1})\big)$. (vi) and (viii) follow from (2.9) and (2.13). □

Remark. For the Stokes matrices $V^{(\nu)} := 1 + C^{(\nu)}$ describing the connection between $Y^{(\nu-1)}$ and $Y^{(\nu)}$, and for the opening of the I_ν's, the remarks after Lemma 4.1 (page 33) apply.

We will make use of some more structure information on the formal fundamental matrix (4.1) and the corresponding fundamental matrix solutions $Y^{(\nu)}$ in Lemma 4.2. As we will see, this will lead to a strong analogy to the results of Section 4.2.

Choose the diagonal elements of $Q\big(\frac{1}{z}\big)$ to be sorted: let

$$Q\big(\tfrac{1}{z}\big) = \bigoplus_{\ell=1}^{g} \tilde q_\ell\big(\tfrac{1}{z}\big) I_\ell \qquad \text{with} \qquad \tilde q_\ell \neq \tilde q_k \text{ if } \ell \neq k,$$

and I_ℓ is the $(n_\ell \times n_\ell)$-identity matrix. Then from the commutation relation between Q and L we see that L is block-diagonal simultaneously with Q, namely

$$L = \bigoplus_{\ell=1}^{g} \tilde L_\ell \qquad \text{with} \qquad \tilde L_\ell \in \mathbb{C}^{n_\ell \times n_\ell}.$$

The individual blocks, however, may contain nilpotent parts.

Therefore arrange the Stokes matrices $C^{(\nu)}$ as block matrices compatible to Q and L as well: let

$$C^{(\nu)} = \big(\tilde C^{(\nu)}_{\ell k}\big)^g_{\ell,k=1} \qquad \text{where} \qquad \tilde C^{(\nu)}_{\ell k} \in \mathbb{C}^{n_\ell \times n_k}.$$

Finally group the columns of $F^{(\nu)}$ to corresponding column blocks setting

$$F^{(\nu)}(z) = \big(\tilde F^{(\nu)}_1(z), \dots, \tilde F^{(\nu)}_g(z)\big) \qquad \text{with} \qquad \tilde F^{(\nu)}_\ell(z) \in \mathbb{C}^{n \times n_\ell}$$

which admit the asymptotic expansions

(4.13)
$$\tilde F^{(\nu)}_\ell(z) \sim \sum_{s=0}^{\infty} \tilde F_{\ell s} z^s \qquad (z \in S_\nu).$$

Now look at Lemma 4.2 (viii). The property of the $C^{(\nu)}$ tells us that entire blocks of Stokes' multipliers vanish simultaneously:

$$\tilde{C}^{(\nu)}_{\ell k} = 0 \qquad \text{for} \qquad (\ell,k) \notin \tilde{J}_\nu, \qquad \text{where}$$

$$\tilde{J}_\nu := \left\{ (\ell,k) \;\middle|\; \theta_\nu \text{ is a singular direction associated with } \tilde{q}_{\ell k} \right\}. \tag{4.14}$$

Moreover,in terms of matrix blocks, (4.12) reads

$$\tilde{F}^{(\nu)}_k(z) - \tilde{F}^{(\nu-1)}_k(z) = \sum_{(\ell,k)\in\tilde{J}_\nu} e^{\tilde{q}_{\ell k}\left(\frac{1}{z}\right)} \tilde{F}^{(\nu-1)}_\ell(z)\, z^{\tilde{L}_\ell} \tilde{C}^{(\nu)}_{\ell k} z^{-\tilde{L}_k} \qquad (z \in S_{\nu-1} \cap S_\nu)$$

for every $k \in \{1,\dots,g\}$. This can also be viewed as a form of (3.3) (generalized to the non-commutative matrix case) where now $p^{(\nu)}_{\ell k}(z)$ is a linear operator from $\mathbb{C}^{n\times n_\ell}$ to $\mathbb{C}^{n\times n_k}$:

$$p^{(\nu)}_{\ell k}(z) A := e^{\tilde{q}_{\ell k}\left(\frac{1}{z}\right)} A\, z^{\tilde{L}_\ell} \tilde{C}^{(\nu)}_{\ell k} z^{-\tilde{L}_k}.$$

Proposition 3.3 readily carries over to this non-commutative situation, thus for the coefficients $\tilde{F}_{ks}$ in (4.13) and the remainders

$$\tilde{R}^{(\nu)}_k(z,N) := \tilde{F}^{(\nu)}_k(z) - \sum_{s=0}^{N-1} \tilde{F}_{ks} z^s \tag{4.15}$$

we find the integral representations

$$\tilde{F}_{ks} = \sum_{j=1}^{M} \sum_{(\ell,k)\in\tilde{J}_j} \tilde{I}^{(j)}_{\ell k}(0,s) + \tilde{\varepsilon}_k(0,s), \tag{4.16}$$

$$z^{-N} \tilde{R}^{(\nu)}_k(z,N) = \sum_{j=1}^{M} \sum_{(\ell,k)\in\tilde{J}_j} \tilde{I}^{(j)}_{\ell k}(z,N) + \tilde{\varepsilon}_k(z,N) \qquad \big(z \in S_{\rho_0}(\theta_\nu,\theta_{\nu+1})\big) \tag{4.17}$$

with

$$\tilde{I}^{(j)}_{\ell k}(z,N) = \frac{1}{2\pi i} \int_{\overline{OT_j}} e^{\tilde{q}_{\ell k}\left(\frac{1}{w}\right)} w^{-N} \frac{\tilde{F}^{(j-1)}_\ell(w)\, w^{\tilde{L}_\ell} \tilde{C}^{(j)}_{\ell k} w^{-\tilde{L}_k}}{w-z}\, dw, \tag{4.18}$$

$$\tilde{\varepsilon}_k(z,N) = \frac{1}{2\pi i} \sum_{j=1}^{M} \int_{\overset{\frown}{T_j T_{j+1}}} \frac{\tilde{F}^{(j)}_k(w)}{w-z} w^{-N}\, dw, \tag{4.19}$$

the integration being understood over radii and arcs, respectively. Compare (4.18), (4.19) to (4.11), (3.6)!

4.4 Level-0 expansions

In this section we will once more consider level-0 (Poincaré) expansions for the asymptotic solutions of Section 4.2 and their optimal truncation. Though this is rather easy and well-understood in the literature for many years, it will be illustrative to develop our method of proof and to show the pattern of successive levels.

The equations considered here are essentially general: of arbitrary rank and size; in addition we merely require (4.4). The confinement to this "generic" case is done for the purpose of simplicity and clarity of the presentation only and is in no way essential: see the remark on the general case at the end of this section.

Since there is no need to consider asymptotic expansions if the formal series actually converges, throughout this section we make the assumption that *the considered formal vector solution $\hat{f}_k(z)$ diverges.* This is equivalent to the existence of numbers $\ell \neq k$ and $j \in \{1,\dots,M\}$ such that $(\ell,k) \in J_j$ and $c_{\ell k}^{(j)} \neq 0$.

A first estimate can easily be derived from the Cauchy-Heine representation (3.9) and is valid between two adjacent singular directions of [D]:

Theorem 4.3. *Let the assumptions of Lemma 4.1 be satisfied: let* (4.2) *be a formal fundamental system of* [D] *with $p = 1$, assume* (4.4) *holds, and let $\hat{\rho} > 0$, $\{I_\nu\}_{\nu\in\mathbb{Z}}$, and $\{f_k^{(\nu)}\}_{\nu\in\mathbb{Z}}$ be as in Lemma 4.1. Then the N_0-th remainder $R_k^{(\nu)}(z,N_0)$ of the asymptotic expansion of $f_k^{(\nu)}$ (cf.* (4.10)*) satisfies*

(4.20)

$$R_k^{(\nu)}(z,N_0) = \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} c_{\ell k}^{(j)} \left(\frac{N_0|z|^{\kappa_{\ell k}}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{N_0/\kappa_{\ell k}} e^{O\left(N_0^{1-\frac{1}{\kappa_{\ell k}}}\right)} (1+N_0)^{\frac{-\operatorname{Re}\mu_{\ell k}+1}{\kappa_{\ell k}}} O(1)$$

uniformly for $N_0 \in \mathbb{N}$ and $z \in \overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$ for every $\rho < \hat{\rho}$.

Proof. Let $0 < \rho < \hat{\rho}$ be arbitrarily given. Choose $\rho < \rho_0 < \hat{\rho}$, and define points $\{T_0^{(\nu)}\}_{\nu=1}^{M}$ by $|T_0^{(\nu)}| = \rho_0$, $\arg T_0^{(\nu)} = \theta_\nu$. Finally let $N_0 \in \mathbb{N}$.

Putting $T_\nu := T_0^{(\nu)}$ and $N := N_0$, we have shown at the end of Section 4.2 that all assumptions of Proposition 3.3 are satisfied and hence (3.9) holds with (4.11):

$$z^{-N_0} R_k^{(\nu)}(z,N_0) = \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} I_{\ell k}^{(j)}(z,N_0) + \varepsilon_k(z,N_0)$$

$$= \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} \frac{c_{\ell k}^{(j)}}{2\pi i} \int\limits_{\overline{OT_0^{(j)}}} \frac{e^{q_{\ell k}(\frac{1}{w})} w^{-N_0+\mu_{\ell k}}}{w-z} f_\ell^{(j-1)}(w)\,dw + \varepsilon_k(z,N_0) \tag{4.21}$$

for $z \in S_{\rho_0}(\theta_\nu,\theta_{\nu+1})$. The obvious continuation to $\overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$ is done by small deformations of the integration paths: if $\arg z = \theta_\nu$ then the continuation is found by deforming the contour of w-integration of the integral for $j = \nu$ by a small circular arc *to the right of* $w = z$, and if $\arg z = \theta_{\nu+1}$ then the $(\nu+1)$-th integral is to be indented *to the left of* $w = z$.

Asymptotic estimates of each of the above integrals can be found from Theorem 5.5, as follows: Put $p(x) := q_{\ell k}(x)$, $r := \kappa_{\ell k}$, and choose $\alpha := (-\kappa_{\ell k}\alpha_{\ell k})^{1/\kappa_{\ell k}}$ such that $\arg\alpha = \theta_j$. Then $S := S_{j-1} = S_{\hat{\rho}}(\alpha_{j-1}, \beta_{j-1})$ contains the direction $\arg w = \arg\alpha$, and $F := f_\ell^{(j-1)}$ is analytic on S and bounded on every closed subsector of S. (The fact that F is a vector-valued function here should not matter, as the results of Section 5 apply to every component of F.) Furthermore, putting $\varepsilon^{-r} := N_0 + 1 - \mu_{\ell k}$ we have

$$\arg\varepsilon = O(\varepsilon^r) \subset O(\varepsilon) \qquad (N_0 \to \infty)$$

and thus we can choose Ω appropriately to ensure that $\varepsilon \in \Omega$ for large enough $N_0 \in \mathbb{R}_+$. Finally, with $T := T_0^{(j)}$ all assumptions of Theorem 5.5 are satisfied, and hence

$$\begin{aligned}
(4.22)\qquad \int_{\overline{OT_0^{(j)}}} \frac{e^{q_{\ell k}(\frac{1}{w})} w^{-N_0+\mu_{\ell k}}}{w-z} f_\ell^{(j-1)}(w)\,dw &= \int_{\mathfrak{W}} e^{p(w^{-1})} w^{-\varepsilon^{-r}} \frac{wF(w)}{w-z}\,dw \\
&= \left(e^{1/r}\alpha\varepsilon\right)^{-\varepsilon^{-r}} e^{O(\varepsilon^{-r+1})} O(1) \\
&\subseteq \left(\frac{N_0}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{N_0/\kappa_{\ell k}} e^{O\left(N_0^{1-\frac{1}{\kappa_{\ell k}}}\right)} (1+N_0)^{\frac{-\operatorname{Re}\mu_{\ell k}+1}{\kappa_{\ell k}}} O(1)
\end{aligned}$$

uniformly for $z \in \overline{K}_\rho(0)$ and (large enough) $N_0 \in \mathbb{R}_+$. The restriction upon N_0 can now be removed by directly considering the above integral for small (bounded) N_0. In addition we clearly have

$$(4.23)\qquad \varepsilon_k(z, N_0) = \sum_{j=1}^{M} \frac{1}{2\pi i} \int_{\overset{\frown}{T_0^{(j)} T_0^{(j+1)}}} w^{-N_0} \frac{f_k^{(j)}(w)}{w-z}\,dw = O\left(\rho_0^{-N_0}\right)$$

uniformly for $z \in \overline{K}_\rho(0)$ and $N_0 \in \mathbb{N}$. Theorem 4.3 follows. □

Now it is not hard to see that, whatever the choice of the truncation point N_0 is, the estimate (4.20) is not better than

$$(4.24)\qquad e^{-|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}} + O\left(|z|^{-\kappa_{\ell_0 k}+1}\right)} \cdot |z|^{\tilde{\mu}_0 - 1} O(1)$$

where

$$U^{(0)} = \bigcup_{j=1}^{M} \left\{ (\ell,k) \in J_j : \; c_{\ell k}^{(j)} \neq 0 \right\} \qquad (k \text{ fixed}), \tag{4.25}$$

$$U_0^{(0)} = \left\{ (\ell,k) \in U^{(0)} : \; |\alpha_{\ell k}| x^{\kappa_{\ell k}} = \min_{(m,k)\in U^{(0)}} |\alpha_{mk}| x^{\kappa_{mk}} \right\}, \tag{4.26}$$

$$\tilde{\mu}_0 = \min_{(\ell,k)\in U_0^{(0)}} \operatorname{Re} \mu_{\ell k} \qquad \text{and} \qquad (\ell_0,k) \in U_0^{(0)} \text{ arbitrarily.} \tag{4.27}$$

To see this, for positive real x, N_0 and $(\ell,k) \in U^{(0)}$ denote

$$\lambda_\ell(x,N_0) := \left(\frac{N_0}{e\kappa_{\ell k} |\alpha_{\ell k}| x^{\kappa_{\ell k}}} \right)^{N_0/\kappa_{\ell k}}, \qquad \lambda(x,N_0) := \max_{(\ell,k)\in U^{(0)}} \lambda_\ell(x,N_0). \tag{4.28}$$

For each ℓ, $\lambda_\ell(x,N_0)$ is a convex function with respect to N_0 with a global minimum at $N_0 = \kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}$ so that we have $\lambda_\ell(x,N_0) \geq e^{-|\alpha_{\ell k}|x^{\kappa_{\ell k}}}$. The λ_ℓ contributing to (4.20) are exactly those for which $(\ell,k) \in U^{(0)}$, and the above minimum is "worst" (i.e. maximal) if $(\ell,k) \in U_0^{(0)}$. Hence we know that $\lambda(x,N_0) \geq \lambda_{\ell_0}(x,N_0) \geq e^{-|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}}$ (Note that, though there might be several possibilities for ℓ_0, the quantities $\kappa_{\ell_0 k}$ and $|\alpha_{\ell_0 k}|$, however, are uniquely determined by (4.26)). Putting $x := |z|^{-1}$ and considering the lower-order terms, we obtain the best possible estimate (4.24).

We will now "optimize" expansion (3.7) in the following sense: we ask how large, depending on z, the truncation point N_0 must be in order for the estimate (4.20) to be best (i.e. minimal). We will show that, by an appropriate choice of $N_0 = N_0(z)$, (4.24) is an upper estimate for the remainder:

Corollary 4.4 (Optimal expansions in small sectors). *Let the assumptions of Lemma 4.1 be satisfied: let* (4.2) *be a formal fundamental system of* [D] *with* $p = 1$, *assume* (4.4) *holds, and let* $\hat{\rho} > 0$, $\{I_\nu\}_{\nu\in\mathbb{Z}}$, *and* $\{f_k^{(\nu)}\}_{\nu\in\mathbb{Z}}$ *be as in Lemma 4.1. If*

$$N_0 = \kappa_{\ell_0 k}|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}} + O(1) \tag{4.29}$$

then the remainder $R_k^{(\nu)}(z,N_0)$ *in* (4.10) *satisfies*

$$R_k^{(\nu)}(z,N_0) = e^{-|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}} + O\left(|z|^{-\kappa_{\ell_0 k}+1}\right)} \cdot |z|^{\tilde{\mu}_0 - 1} O(1)$$

uniformly for $z \in \overline{S}_\rho(\theta_\nu, \theta_{\nu+1})$ *for every* $\rho < \hat{\rho}$. *Here,* ℓ_0 *and* $\tilde{\mu}_0$ *are defined in* (4.27).

Proof. Consider the functions $\lambda_\ell(x,N_0)$ as defined in (4.28). They figure as the essential terms of the summands of the right-hand side of (4.20). To get a rough idea of the proof, let us temporarily drop the condition $N_0 \in \mathbb{N}$ and consider positive real N_0.

Then, as we have seen above, for each $(\ell,k)\in U^{(0)}$ the function $\lambda_\ell(x,N_0)$ is minimal at $N_0=\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}$. But since N_0 can only take one unique value, we choose that for $(\ell,k)\in U_0^{(0)}$, e.g. for $\ell=\ell_0$ (which means minimizing the "worst" term). By use of Lemma 4.12 (i) we can show that if we take $N_0=\kappa_{\ell_0 k}|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}$ then all other terms satisfy the same estimate: namely, if we identify $p(x)=|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}$ and $q(x)=|\alpha_{\ell k}|x^{\kappa_{\ell k}}$, then from Lemma 4.12 (i) we have

$$\lambda_\ell(x,N_0)=\left(\frac{p'(x)}{eq'(x)}\right)^{\frac{q(x)}{q'(x)}p'(x)}\le e^{-p(x)}=e^{-|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}}\qquad(x\to+\infty).$$

Reverting to $N_0\in\mathbb{N}$, we have to adapt the choice of N_0 as to

$$N_0=\kappa_{\ell_0 k}|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}+\delta_0(x)$$

with $\delta_0(x)=O(1)$. This is (4.29) if we take $x=|z|^{-1}$. Let now $(\ell,k)\in U^{(0)}$. Identifying again $p(x)=|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}$, $q(x)=|\alpha_{\ell k}|x^{\kappa_{\ell k}}$, as well as $\delta(x)=\delta_0(x)$, from Lemma 4.12 (ii) we have

$$\begin{aligned}\lambda_\ell(x,N_0)=\left(\frac{N_0}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{N_0/\kappa_{\ell k}}&=\left(\frac{xp'(x)+\delta(x)}{exq'(x)}\right)^{\frac{q(x)}{xq'(x)}(xp'(x)+\delta(x))}\\&<Ce^{-p(x)}=Ce^{-|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}}\end{aligned}\tag{4.30}$$

as $x\to+\infty$ with a suitable $C>0$, and in case $\kappa_{\ell k}>\kappa_{\ell_0 k}$ (for which Case 2 of the proof of Lemma 4.12 applies) we even have

$$\begin{aligned}\lambda_\ell(x,N_0)&=\left(\frac{xp'(x)+\delta(x)}{exq'(x)}\right)^{\frac{q(x)}{xq'(x)}(xp'(x)+\delta(x))}\\&<e^{-p(x)-C\cdot(xp'(x)+\delta(x))}=e^{-|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}-C\cdot N_0}\end{aligned}$$

as $x\to+\infty$ for every $C>0$, since both $p(x)$ and $xp'(x)$ are polynomials of degree ℓ; cf. the remark at the end of the proof of Case 2.

In this case we obtain

$$\lambda_\ell(x,N_0)\cdot e^{C\cdot N_0^{1-\frac{1}{\kappa_{\ell k}}}}\left(1+N_0\right)^{\frac{-\mathrm{Re}\,\mu_{\ell k}+1}{\kappa_{\ell k}}+\frac{\tilde\mu_0-1}{\kappa_{\ell_0 k}}}<\lambda_\ell(x,N_0)\cdot e^{C\cdot N_0}<e^{-|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}}\qquad(x\to+\infty),$$

$$\Rightarrow\quad\lambda_\ell(x,N_0)\cdot e^{O\left(N_0^{1-\frac{1}{\kappa_{\ell k}}}\right)}\left(1+N_0\right)^{\frac{-\mathrm{Re}\,\mu_{\ell k}+1}{\kappa_{\ell k}}}O(1)\subseteq e^{-|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}}x^{1-\tilde\mu_0}O(1).$$

In the case $\kappa_{\ell k}=\kappa_{\ell_0 k}$ but $(\ell,k)\notin U_0^{(0)}$, Case 1 of the proof of Lemma 4.12 applies so that we even have

$$\begin{aligned}\lambda_\ell(x,N_0)&=\left(\frac{xp'(x)+\delta(x)}{exq'(x)}\right)^{\frac{q(x)}{xq'(x)}(xp'(x)+\delta(x))}\\&<e^{-p(x)-\varepsilon q(x)}\le e^{-p(x)-\tilde\varepsilon(xp'(x)+\delta(x))}=e^{-|\alpha_{\ell_0 k}|x^{\kappa_{\ell_0 k}}-\tilde\varepsilon\cdot N_0}\end{aligned}$$

as $x \to +\infty$ for some $\varepsilon, \tilde{\varepsilon} > 0$; cf. the remark at the end of the proof of Case 1. This implies

$$\lambda_\ell(x,N_0) \cdot \big(1+N_0\big)^{\frac{-\operatorname{Re}\mu_{\ell k}+1}{\kappa_{\ell k}} + \frac{\tilde{\mu}_0-1}{\kappa_{\ell_0 k}}} < \lambda_\ell(x,N_0) \cdot e^{\tilde{\varepsilon} N_0} < e^{-|\alpha_{\ell_0 k}| x^{\kappa_{\ell_0 k}}} \qquad (x \to +\infty),$$
$$\Rightarrow \quad \lambda_\ell(x,N_0) \cdot e^{O\left(N_0^{1-\frac{1}{\kappa_{\ell k}}}\right)} \big(1+N_0\big)^{\frac{-\operatorname{Re}\mu_{\ell k}+1}{\kappa_{\ell k}}} O(1) \subseteq e^{-|\alpha_{\ell_0 k}| x^{\kappa_{\ell_0 k}} + O\left(x^{\kappa_{\ell_0 k}-1}\right)} x^{1-\tilde{\mu}_0} O(1).$$

This last estimate also holds in the case $(\ell,k) \in U_0^{(0)}$, according to (4.30) and the definition of $\tilde{\mu}_0$ in (4.27). Together with (4.4), we have estimated each summand on the right-hand side of (4.20); hence Corollary 4.4 follows. □

We ask now if the estimate in Corollary 4.4 holds outside $z \in \overline{S}_\rho(\theta_\nu, \theta_{\nu+1})$, too. Even more, we want to find the maximal sector of validity of this estimate. Therefore we make use of the Stokes' relations. But relation (4.6) is not well-suited for this purpose since it gives only the connection information about asymptotic solutions $f_k^{(\nu)}$ belonging to *consecutive* sectors, and when one crosses more than one singular direction, the Stokes' phenomenon for the functions $f_\ell^{(\nu-1)}$ has to be taken into consideration as well.

We will take another approach here which is even explicit. Since the functions $f_k^{(\nu)}$ can be continued arbitrarily throughout the Riemann surface of $\log z$ over $D(\hat{\rho})$, we consider the connection coefficients between asymptotic solutions belonging to any two sectors:

$$f_k^{(\nu)}(z) - f_k^{(\hat{\nu})}(z) = \sum_{(\ell,k)\in J_{\nu,\hat{\nu}}} c_{\ell k}^{(\nu,\hat{\nu})} e^{q_{\ell k}(\frac{1}{z})} z^{\mu_{\ell k}} f_\ell^{(\hat{\nu})}(z) \tag{4.31}$$

with some $J_{\nu,\hat{\nu}} \in \{1,\dots,n\}^2$. Now, for $\nu \in \mathbb{Z}$ and $\hat{\nu} \le \nu - 1$ let

(4.32a)
$$\alpha_{\nu,\hat{\nu}}(k) := \max\left\{ \theta_{\tilde{\nu}} - \frac{1}{\kappa_{\ell k}} \arccos\Big(\frac{|\alpha_{\ell_0 k}|}{|\alpha_{\ell k}|} \delta_{\kappa_{\ell k},\kappa_{\ell_0 k}}\Big) \in \Big(\theta_{\hat{\nu}}, \theta_{\hat{\nu}+1}\Big] \;\Big|\; (\ell,k) \in J_{\tilde{\nu}} \cap J_{\nu,\hat{\nu}} \right\},$$

and for $\hat{\nu} \ge \nu + 1$ put

(4.32b)
$$\beta_{\nu,\hat{\nu}}(k) := \min\left\{ \theta_{\tilde{\nu}} + \frac{1}{\kappa_{\ell k}} \arccos\Big(\frac{|\alpha_{\ell_0 k}|}{|\alpha_{\ell k}|} \delta_{\kappa_{\ell k},\kappa_{\ell_0 k}}\Big) \in \Big[\theta_{\hat{\nu}}, \theta_{\hat{\nu}+1}\Big) \;\Big|\; (\ell,k) \in J_{\tilde{\nu}} \cap J_{\nu,\hat{\nu}} \right\}.$$

Here, $\max\emptyset := -\infty$, $\min\emptyset := +\infty$, and $\delta_{x,y}$ denotes the Kronecker symbol. Therewith define

$$\begin{aligned} \alpha_\nu^+(k) &:= \max_{\hat{\nu} \le \nu-1} \alpha_{\nu,\hat{\nu}}(k), \\ \beta_\nu^+(k) &:= \min_{\hat{\nu} \ge \nu+1} \beta_{\nu,\hat{\nu}}(k). \end{aligned} \tag{4.33}$$

Then we have the

Proposition 4.5 (Optimal expansions in large sectors). *Let the assumptions of Lemma 4.1 be satisfied: let* (4.2) *be a formal fundamental system of* [D] *with* $p = 1$, *assume* (4.4) *holds, and let* $\hat{\rho} > 0$, $\{I_\nu\}_{\nu\in\mathbb{Z}}$, *and* $\{f_k^{(\nu)}\}_{\nu\in\mathbb{Z}}$ *be as in Lemma 4.1. Consider* $k \in \{1,\ldots,n\}$ *fixed. Among all singular directions associated with* $q_{\ell k}$ *for all* $\ell \neq k$, *let* θ_ν *and* θ_μ *be two consecutive ones. If*

$$N_0 = \kappa_{\ell_0 k}|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}} + O(1)$$

then

$$R_k^{(\nu)}(z,N_0) = e^{-|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}} + O\left(|z|^{-\kappa_{\ell_0 k}+1}\right)} \cdot |z|^{\tilde{\mu}_0 - 1} O(1)$$

uniformly w.r.to z *in closed subsectors of* $S_\nu^+(k) := S_{\hat{\rho}}\left(\alpha_\nu^+(k), \beta_{\mu-1}^+(k)\right)$. *Here,* ℓ_0 *and* $\tilde{\mu}_0$ *are defined in* (4.27), *and* $\alpha_\nu^+(k)$ *and* $\beta_{\mu-1}^+(k)$ *in* (4.33).

Proof. By construction, all $f_k^{(\tilde{\nu})}$ ($\tilde{\nu} = \nu,\ldots,\mu-1$) are analytic continuations of each other; for convenience we will still denote them by $f_k^{(\nu)}$, and the corresponding N_0-th remainders by $R_k^{(\nu)}(z,N_0)$. Furthermore the connection coefficients $c_{\ell k}^{(\tilde{\nu},\hat{\nu})}$ agree for these $\tilde{\nu}$ and arbitrary $\hat{\nu}$ - not for all k but *for the fixed k in the proposition*, as well as for the set of pairs $(\ell,k) \in J_{\tilde{\nu},\hat{\nu}}$ for that k. From (4.31) we find

$$\begin{aligned} R_k^{(\nu)}(z,N_0) &= R_k^{(\hat{\nu})}(z,N_0) + \left(R_k^{(\nu)}(z,N_0) - R_k^{(\hat{\nu})}(z,N_0)\right) \\ &= R_k^{(\hat{\nu})}(z,N_0) + \left(f_k^{(\nu)}(z) - f_k^{(\hat{\nu})}(z)\right) \\ &= R_k^{(\hat{\nu})}(z,N_0) + \sum_{(\ell,k)\in J_{\nu,\hat{\nu}}} c_{\ell k}^{(\nu,\hat{\nu})} e^{q_{\ell k}(\frac{1}{z})} z^{\mu_{\ell k}} f_\ell^{(\hat{\nu})}(z). \end{aligned} \tag{4.34}$$

Equation (4.34) holds for all values of $\arg z$ but is of particular interest in $\overline{S}_\rho(\theta_{\hat{\nu}}, \theta_{\hat{\nu}+1})$ since the term $R_k^{(\hat{\nu})}(z,N_0)$ satisfies the estimate in Corollary 4.4 there. Therefore, we have to examine the dominance relation between $\exp\left(-|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}}\right)$ and $\exp\left(\alpha_{\ell k} z^{-\kappa_{\ell k}}\right)$.

Consider the situation "to the right" of θ_ν: if $(\ell,k) \in U_0^{(0)}$ then $\exp\left(\alpha_{\ell k} z^{-\kappa_{\ell k}}\right)$ takes over dominance immediately – although smoothly – at a singular direction $\theta_{\tilde{\nu}}$ associated with $q_{\ell k}$. If $\kappa_{\ell k} > \kappa_{\ell_0 k}$ then the contribution of $\exp\left(\alpha_{\ell k} z^{-\kappa_{\ell k}}\right)$ is dominated by $\exp\left(-|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}}\right)$ up to the Stokes direction $\theta_{\tilde{\nu}} - \frac{\pi}{2\kappa_{\ell k}}$ where $\theta_{\tilde{\nu}}$ is a singular direction associated with $q_{\ell k}$. In the remaining case we have $\kappa_{\ell k} = \kappa_{\ell_0 k}$ but $|\alpha_{\ell k}| > |\alpha_{\ell_0 k}|$.

Then for $\arg z = \theta$ we find

$$\begin{aligned} & -|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}} = \operatorname{Re}\big(\alpha_{\ell k} z^{-\kappa_{\ell k}}\big) \\ \iff \quad & |\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}} = \operatorname{Re}\big(-\alpha_{\ell k} z^{-\kappa_{\ell k}}\big) \\ & \qquad = |\alpha_{\ell k}||z|^{-\kappa_{\ell k}} \cos\Big(\arg(-\alpha_{\ell k}) - \kappa_{\ell k}\theta\Big) \\ \iff \quad & \cos\Big(\kappa_{\ell k}(\theta - \theta_{\tilde\nu})\Big) = \frac{|\alpha_{\ell_0 k}|}{|\alpha_{\ell k}|} \\ \iff \quad & \kappa_{\ell k}(\theta - \theta_{\tilde\nu}) = \pm \arccos \frac{|\alpha_{\ell_0 k}|}{|\alpha_{\ell k}|} \end{aligned}$$

with some singular direction $\theta_{\tilde\nu}$ associated with $q_{\ell k}$. There is no ambiguity in choosing the right sign for the arc cosine above: if $\exp\big(\alpha_{\ell k} z^{-\kappa_{\ell k}}\big)$ takes over dominance in direction of decreasing argument then the "$-$" sign has to be taken. Altogether in all three cases dominance changes from $\exp\Big(-|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}}\Big)$ to $\exp\big(\alpha_{\ell k} z^{-\kappa_{\ell k}}\big)$ at

$$\theta = \theta_{\tilde\nu} - \frac{1}{\kappa_{\ell k}} \arccos\Big(\tfrac{|\alpha_{\ell_0 k}|}{|\alpha_{\ell k}|}\delta_{\kappa_{\ell k},\kappa_{\ell_0 k}}\Big). \tag{4.35}$$

Now assume that the estimate in the proposition does not hold at $\arg z = d \in \big(\theta_{\hat\nu}, \theta_{\hat\nu+1}\big]$ for some $\hat\nu \le \nu - 1$. Without loss of generality, $\hat\nu$ can be taken to be maximal with this property. Then from (4.34) follows that there is an ℓ with $(\ell, k) \in J_{\nu,\hat\nu}$ for which $\exp\big(\alpha_{\ell k} z^{-\kappa_{\ell k}}\big)$ dominates over $\exp\Big(-|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}}\Big)$. Let θ be the supremum of all $\arg z$ for which this dominance still holds then we clearly have $\theta \in \big(\theta_{\hat\nu}, \theta_{\hat\nu+1}\big]$. At θ, there must be a change of dominance, hence θ is of the form (4.35) with a singular direction $\theta_{\tilde\nu}$ associated with $q_{\ell k}$. Thus, $(\ell, k) \in J_{\tilde\nu}$, and from definition (4.32) we conclude that $\alpha_{\nu,\hat\nu}(k) \ge \theta \ge d$, hence z is outside of $S^+_\nu(k)$. The situation for $\hat\nu \ge \mu$ is symmetric and can be studied the same way.

The uniformity of the estimate in closed subsectors of $S^+_\nu(k)$ is clear; on the other hand, the estimate might be false on the whole sector if $\alpha_{\nu,\hat\nu}(k)$ or $\beta_{\nu,\hat\nu}(k)$ is a Stokes direction. This completes the proof. □

We can now show that if the $J_{\nu,\hat\nu}$ are chosen minimal, that is, if we have $c^{(\nu,\hat\nu)}_{\ell k} \ne 0$ in (4.31) for all $(\ell, k) \in J_{\nu,\hat\nu}$, then the according sector $S^+_\nu(k)$ in Proposition 4.5 is maximal. In fact, by (4.32), there is a pair $(\ell, k) \in J_{\nu,\hat\nu}$ for which $\alpha_{\nu,\hat\nu}(k) \in \Big(\theta_{\hat\nu}, \theta_{\hat\nu+1}\Big]$ is of the form (4.35). Hence, due to $c^{(\nu,\hat\nu)}_{\ell k} \ne 0$, there is a change of dominance from $\exp\Big(-|\alpha_{\ell_0 k}||z|^{-\kappa_{\ell_0 k}}\Big)$ to $\exp\big(\alpha_{\ell k} z^{-\kappa_{\ell k}}\big)$ at $\alpha_{\nu,\hat\nu}(k)$. The same is true for $\beta_{\nu,\hat\nu}(k)$. At these lines, the remainder can behave in different ways: if $\kappa_{\ell k} = \kappa_{\ell_0 k}$ then the exponential improvement of the remainder deteriorates smoothly while in case $\kappa_{\ell k} > \kappa_{\ell_0 k}$ it disappears dramatically at $\alpha_{\nu,\hat\nu}(k)$ resp. $\beta_{\nu,\hat\nu}(k)$.

We have chosen the different formulation (4.31) of the Stokes' phenomenon to facilitate the formulation of the results. However, the coefficients $c_{\ell k}^{(\nu,\hat{\nu})}$ are related to those of (4.6), as are the $J_{\nu,\hat{\nu}}$ to the J_ν: write (4.31) under the form

$$y_k^{(\nu)}(z) = y_k^{(\hat{\nu})}(z) + \sum_{(\ell,k)\in J_{\nu,\hat{\nu}}} c_{\ell k}^{(\nu,\hat{\nu})} y_\ell^{(\hat{\nu})}(z)$$

or, in a compact form,

$$\begin{aligned} Y^{(\nu)}(z) &= Y^{(\hat{\nu})}(z)\Big(1 + C^{(\nu,\hat{\nu})}\Big) \\ &= Y^{(\hat{\nu})}(z)\, V^{(\nu,\hat{\nu})} \end{aligned}$$

where the connection matrix $C^{(\nu,\hat{\nu})}$ is formed by the connection coefficients $c_{\ell k}^{(\nu,\hat{\nu})}$. Clearly,

$$V^{(\nu,\hat{\nu})} = \begin{cases} V^{(\hat{\nu}+1)} \cdots V^{(\nu)}, & \hat{\nu} \le \nu, \\ \big(V^{(\hat{\nu})}\big)^{-1} \cdots \big(V^{(\nu+1)}\big)^{-1}, & \hat{\nu} > \nu \end{cases}$$

with the Stokes matrices $V^{(\nu)}$ defined in the remark on page 33. Accordingly, $J_{\nu,\hat{\nu}}$ can be taken as $J_{\hat{\nu}+1} \circ \cdots \circ J_\nu$ resp. $J_{\hat{\nu}}^{-1} \circ \cdots \circ J_{\nu+1}^{-1}$ as a product of relations, but it may be smaller if there are coefficients $c_{\ell k}^{(\nu,\hat{\nu})}$ vanishing.

Remark. All results of this section can be generalized to the situation of Section 4.3. We will not go into the details here; instead, we will briefly show the procedure.

It is possible to treat each of the "blocks" $\tilde{F}_k^{(\nu)}$ at one time. Putting $T_\nu := T_0^{(\nu)}$ and $N := N_0$, the associated N_0-th remainder $\tilde{R}_k^{(\nu)}(z, N_0)$ from (4.15) satisfies the integral representation (4.17) with $\tilde{I}_{\ell k}^{(j)}(z, N_0)$ from (4.18). Since for each $\ell \neq k$ there is a real (possibly negative) constant $\tilde{\mu}_{\ell k}$ such that

$$w^{\tilde{L}_\ell} \tilde{C}_{\ell k}^{(j)} w^{-\tilde{L}_k} = O\big(w^{\tilde{\mu}_{\ell k}}\big),$$

we conclude that the term

$$\tilde{F}_\ell^{(j-1)}(w)\, w^{-\tilde{\mu}_{\ell k}} w^{\tilde{L}_\ell} \tilde{C}_{\ell k}^{(j)} w^{-\tilde{L}_k}$$

is analytic w.r.to $w \in S := S_{j-1}$ and bounded on every closed subsector. Hence, writing each of the integrals (4.18) in the form

$$\tilde{I}_{\ell k}^{(j)}(z, N_0) = \frac{1}{2\pi i} \int_{\overline{OT_0^{(j)}}} e^{\tilde{q}_{\ell k}\left(\frac{1}{w}\right)} w^{-N_0 + \tilde{\mu}_{\ell k}} \frac{\tilde{F}_\ell^{(j-1)}(w)\, w^{-\tilde{\mu}_{\ell k}} w^{\tilde{L}_\ell} \tilde{C}_{\ell k}^{(j)} w^{-\tilde{L}_k}}{w - z}\, dw,$$

we can adapt the statement and the proof of Theorem 4.3 to the level-0 remainder $\tilde{R}_k^{(\nu)}(z, N_0)$ from (4.15) to obtain:

Let the assumptions of Lemma 4.2 be satisfied: let (4.1) *be a formal fundamental matrix of* [D] *with* $p=1$ *and let* $\hat\rho>0$, $\{I_\nu\}_{\nu\in\mathbb{Z}}$, *and* $\{f_k^{(\nu)}\}_{\nu\in\mathbb{Z}}$ *be as in Lemma 4.2. Then the* N_0*-th remainder* $\tilde R_k^{(\nu)}(z,N_0)$ *of the asymptotic expansion of* $\tilde F_k^{(\nu)}$ *(cf.* (4.15)*) satisfies*

$$\tilde R_k^{(\nu)}(z,N_0)=\sum_{j=1}^{M}\sum_{(\ell,k)\in\tilde J_j}\tilde\gamma_{\ell k}^{(j)}\left(\frac{N_0|z|^{\tilde\kappa_{\ell k}}}{e\tilde\kappa_{\ell k}|\tilde\alpha_{\ell k}|}\right)^{N_0/\tilde\kappa_{\ell k}}e^{O\left(N_0^{1-\frac{1}{\tilde\kappa_{\ell k}}}\right)}\left(1+N_0\right)^{\frac{-\tilde\mu_{\ell k}+1}{\tilde\kappa_{\ell k}}}O(1).$$

uniformly for $N_0\in\mathbb{N}$ *and* $z\in\overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$ *for every* $\rho<\hat\rho$.

Here, $\tilde\kappa_{\ell k}$ and $\tilde\alpha_{\ell k}$ are the degree and the leading coefficient of $\tilde q_{\ell k}$, respectively, and $\tilde\gamma_{\ell k}^{(j)}$ is either 1 or zero according to $\tilde C_{\ell k}^{(j)}\neq 0$ or $\tilde C_{\ell k}^{(j)}=0$, respectively.

4.5 Level-1 expansions

In this section we will go beyond the results of Section 4.4 and develop level-1 expansions for the asymptotic solutions of Section 4.2 and their optimal truncation. These expansions will not be anymore in terms of elementary functions like powers, exponentials and the like, but rather in terms of level-1 "hyperterminants" (see Section 1.6.2).

Just like in Section 4.4, the restriction to the generic case is done for the purpose of simplicity and clarity of the presentation only and can readily be overcome by considering the blocks from Section 4.3.

Hyperasymptotics is meaningless if the considered formal series converges. Therefore, throughout this section we make the assumption that *the vector solution* $\hat f_k(z)$ *as well as every "adjacent"* $\hat f_\ell(z)$ *be divergent*[7][8]. This is equivalent to the existence of numbers $\ell\neq k$ and $j\in\{1,\dots,M\}$ such that $(\ell,k)\in J_j$ and $c_{\ell k}^{(j)}\neq 0$, and for each such ℓ of numbers $m\neq\ell$ and $\mu\in\{1,\dots,M\}$ such that $(m,\ell)\in J_\mu$ and $c_{m\ell}^{(\mu)}\neq 0$.

To obtain level-1 expansions, we expand the level-0 remainder $R_k^{(\nu)}(z,N_0)$ in terms of level-1 hyperterminants (cf. Section 1.6.2). The starting-point will be the integral representation (3.9). With (4.11) and (3.6), this representation reads

$$R_k^{(\nu)}(z,N_0)=\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\cdot z^{N_0}\int\limits_{\overline{OT_0^{(j)}}}\frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+\mu_{\ell k}}}{w-z}f_\ell^{(j-1)}(w)\,dw+z^{N_0}\cdot\varepsilon_k(z,N_0).$$

Inserting the asymptotic expansion $\sum_{s=0}^{\infty}f_{\ell s}w^s$ of $f_\ell^{(j-1)}(w)$ and ignoring the subdom-

[7]The term "adjacent" means one can find a $j\in\{1,\dots,M\}$ such that $(\ell,k)\in J_j$ and $c_{\ell k}^{(j)}\neq 0$.

[8]Actually the existence of *at least one* adjacent ℓ with divergence would suffice.

inant term $\varepsilon_k(z,N_0)$, we obtain

$$R_k^{(\nu)}(z,N_0) \sim \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} \frac{c_{\ell k}^{(j)}}{2\pi i} \sum_{s=0}^{\infty} f_{\ell s} z^{N_0} \int_{\overline{OT_0^{(j)}}} \frac{e^{q_{\ell k}(\frac{1}{w})} w^{-N_0+s+\mu_{\ell k}}}{w-z}\, dw. \tag{4.36}$$

This expansion is *purely formal*, and we will have to give it a precise meaning in what follows. Also, instead of the Cauchy-Heine integrals above, it is convenient to use integrals over *the whole half-line* $\arg w = \theta_j$ which are related (via the transformation $z \longleftarrow \frac{1}{z}$) to the hyperterminants $F^{(1)}$ in (1.6). If we could simply replace the incomplete integrals by the complete ones, we would have

$$\begin{aligned} R_k^{(\nu)}(z,N_0) &\sim \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} \frac{c_{\ell k}^{(j)}}{2\pi i} \sum_{s=0}^{\infty} f_{\ell s} z^{N_0} \int_0^{[\theta_j]} \frac{e^{q_{\ell k}(\frac{1}{w})} w^{-N_0+s+\mu_{\ell k}}}{w-z}\, dw \\ &= -\sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} \frac{c_{\ell k}^{(j)}}{2\pi i} \sum_{s=0}^{\infty} f_{\ell s} z^{N_0-1} F^{(1)} \begin{pmatrix} & N_0 - s - \mu_{\ell k} \\ z^{-1}; & q_{\ell k} \\ & -\theta_j \end{pmatrix}. \end{aligned} \tag{4.37}$$

But this expansion actually *does not make sense* as the hyperterminants on the right-hand side are undefined for $s \geq N_0 - \operatorname{Re}\mu_{\ell k}$, i.e. for almost all $s \in \mathbb{N}$. However, it can be made rigorous if it is truncated after a finite point N_1 *small enough compared with* N_0, and the exact remainder (including the term $\varepsilon_k(z,N_0)$ neglected above) is estimated. This is how we will proceed in the sequel.

Since the hyperterminants contributing to the right-hand side of (4.37) are of different growth for different $\ell \neq k$ and hence of different importance to the whole sum, the second-stage truncation point N_1 will be allowed to depend on the number ℓ of the vector solution $f_\ell^{(j-1)}$ (and, of course, also on k but not upon ν or j). Therefore we abbreviate

$$N_1 := \big(N_1^{(\ell)}\big)_{\ell\neq k} \in \mathbb{N}^{n-1}, \qquad \mu^{(k)} := (\mu_1,\ldots,\mu_{k-1},\mu_{k+1},\ldots,\mu_n) \in \mathbb{C}^{n-1} \tag{4.38}$$

and choose N_0, N_1 such that

$$\begin{aligned} &0 \leq N_0, \\ &0 \leq N_1 < N_0 + \operatorname{Re}\mu_k - \operatorname{Re}\mu^{(k)} + 1. \end{aligned} \tag{4.39}$$

The second line is a vector notation for a set of inequalities $0 \leq N_1^{(\ell)} < N_0 + \operatorname{Re}\mu_k - \operatorname{Re}\mu_\ell + 1$ for all $\ell \neq k$ and guarantees that for any $\ell \neq k$ in the outer sum of expansion (4.37), we have $s < N_0 - \operatorname{Re}\mu_{\ell k}$ $(0 \leq s \leq N_1^{(\ell)} - 1)$ and hence the corresponding hyperterminants in the inner sum are defined up to $s = N_1^{(\ell)} - 1$. Our result will hold uniformly in each closed subset of the form

$$\begin{aligned} &0 \le N_0,\\ &0 \le N_1 \le N_0 + \operatorname{Re}\mu_k - \operatorname{Re}\mu^{(k)} + 1 - \delta \end{aligned} \tag{4.40}$$

with a positive constant δ.

Thus we define the corresponding remainder of the level-1 expansion as to

$$\begin{aligned} R_k^{(\nu)}(z,N_0,N_1) :=& R_k^{(\nu)}(z,N_0)\\ &+\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s} z^{N_0-1} F^{(1)}\begin{pmatrix} z^{-1}; & N_0-s-\mu_{\ell k}\\ & q_{\ell k}\\ & -\theta_j\end{pmatrix}\\ =& f_k^{(\nu)}(z) - \sum_{s=0}^{N_0-1} f_{ks} z^s\\ &+\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s} z^{N_0-1} F^{(1)}\begin{pmatrix} z^{-1}; & N_0-s-\mu_{\ell k}\\ & q_{\ell k}\\ & -\theta_j\end{pmatrix} \end{aligned} \tag{4.41}$$

which is well-defined according to the considerations above.

Our goal will now be to estimate the remainder $R_k^{(\nu)}(z,N_0,N_1)$ uniformly with respect to z in some sector and without further restrictions on z, N_0, N_1. The only restriction we need is (4.39) above which, as we have seen, is indispensable for the hyperterminants to enter.

The first estimate, just as Theorem 4.3, is again valid between two adjacent singular directions of [D]:

Theorem 4.6. *Let the assumptions of Lemma 4.1 be satisfied: let* (4.2) *be a formal fundamental system of* [D] *with* $p=1$, *assume* (4.4) *holds, and let* $\hat\rho > 0$, $\{I_\nu\}_{\nu\in\mathbb{Z}}$, *and* $\{f_k^{(\nu)}\}_{\nu\in\mathbb{Z}}$ *be as in Lemma 4.1. Then the remainder* $R_k^{(\nu)}(z,N_0,N_1)$ *defined in* (4.41) *satisfies*

$$\begin{aligned} &R_k^{(\nu)}(z,N_0,N_1) =\\ &=\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu} c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} \left(\frac{N_1^{(\ell)}|z|^{\kappa_{m\ell}}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{N_1^{(\ell)}}{\kappa_{m\ell}}} \left(\frac{(N_0-N_1^{(\ell)})|z|^{\kappa_{\ell k}}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}}\times\\ &\times e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} e^{O\left((N_0-N_1^{(\ell)})^{1-\frac{1}{\kappa_{\ell k}}}\right)} \left(1+N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{m\ell}+2}{\kappa_{m\ell}}} \left(1+N_0-N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{\ell k}+2}{\kappa_{\ell k}}} O(1) \end{aligned} \tag{4.42}$$

uniformly w.r.to $N_0\in\mathbb{N}$, $N_1\in\mathbb{N}^{n-1}$ *satisfying* (4.40)[9] *for every* $\delta>0$, *and w.r.to* $z\in\overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$ *for every* $\rho<\hat\rho$.

[9]In case $N_1^{(\ell)} > N_0$ the power of $1+N_0-N_1^{(\ell)}$ in the last line of (4.42) is to be replaced by 1.

Proof. Let $0 < \rho < \hat{\rho}$ be arbitrarily given. Choose $\rho < \rho_0 < \hat{\rho}$, and define points $\{T_0^{(\nu)}\}_{\nu=1}^M$ by $|T_0^{(\nu)}| = \rho_0$, $\arg T_0^{(\nu)} = \theta_\nu$. Finally let $N_0 \in \mathbb{N}$. At the beginning of the proof of Theorem 4.3 we have shown that the remainder $R_k^{(\nu)}(z, N_0)$ of the level-0 expansion satisfies (4.21) for $z \in \overline{S}_\rho(\theta_\nu, \theta_{\nu+1})$, possibly with small deformations of the integration paths.

Now choose $\rho_0 < \rho_1 < \hat{\rho}$ and define points $\{T_1^{(\nu)}\}_{\nu=1}^M$ by $|T_1^{(\nu)}| = \rho_1$, $\arg T_1^{(\nu)} = \theta_\nu$. Fix a $\delta > 0$, and therewith consider $N_1 \in \mathbb{N}^{n-1}$ satisfying (4.40). Finally we take a $\rho_0 < \rho_{01} < \rho_1$ which we will need later. Altogether we have a sequence of inequalities

$$0 < \rho < \rho_0 < \rho_{01} < \rho_1 < \hat{\rho}. \tag{4.43}$$

Then for every $\ell \neq k$, if we put $T_j := T_1^{(j)}$ and $N := N_1^{(\ell)}$, we have shown at the end of Section 4.2 that all assumptions of Proposition 3.3 are satisfied and hence (3.9) holds with (4.11) (the integrals $I_{m\ell}^{(\mu)}$), ε_ℓ are understood to depend on the $T_1^{(\mu)}$ although this dependence is suppressed here):

$$w^{-N_1^{(\ell)}} R_\ell^{(j-1)}(w, N_1^{(\ell)}) = \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} I_{m\ell}^{(\mu)}(w, N_1^{(\ell)}) + \varepsilon_\ell(w, N_1^{(\ell)})$$

for $w \in S_{\rho_1}(\theta_{j-1}, \theta_j)$. The obvious continuation to $\overline{S}_{\rho_{01}}(\theta_{j-1}, \theta_j)$ is done by small deformations of the integration paths. Thus for the $f_\ell^{(j-1)}$ we find

$$\begin{aligned} f_\ell^{(j-1)}(w) = &\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s} w^s + \\ &+ w^{N_1^{(\ell)}} \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} \frac{c_{m\ell}^{(\mu)}}{2\pi i} \int\limits_{\overline{OT_1^{(\mu)}}} \frac{e^{q_{m\ell}(\frac{1}{t})} t^{-N_1^{(\ell)}+\mu_{m\ell}}}{t-w} f_m^{(\mu-1)}(t)\, dt + w^{N_1^{(\ell)}} \varepsilon_\ell(w, N_1^{(\ell)}). \end{aligned} \tag{4.44}$$

We want to insert this result into (4.21). Since in the formulation (4.6) of the Stokes' phenomenon we preferred "right" $(\nu - 1)$ over "left" (ν), in (4.21) we need to integrate $f_\ell^{(j-1)}$ over $\overline{OT_0^{(j)}}$, which is the "left" edge of the above sector $\overline{S}_{\rho_1}(\theta_{j-1}, \theta_j)$. Accordingly, the contour of t-integration in (4.44) has to be indented *to the left of* $t = w$ in case $\mu = j$.

This convention in mind, insertion of (4.44) into (4.21) gives

$$
\begin{aligned}
z^{-N_0}R_k^{(\nu)}(z,N_0) &= \sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\int\limits_{OT_0^{(j)}}\frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+\mu_{\ell k}}}{w-z}\sum_{s=0}^{N_1^{(\ell)}-1}f_{\ell s}w^s\,dw\\
&+\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\sum_{\mu=1}^{M}\sum_{(m,\ell)\in J_\mu}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\times\\
&\times\int\limits_{OT_0^{(j)}}\frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+\mu_{\ell k}+N_1^{(\ell)}}}{w-z}\int\limits_{OT_1^{(\mu)}}\frac{e^{q_{m\ell}(\frac{1}{t})}t^{-N_1^{(\ell)}+\mu_{m\ell}}}{t-w}f_m^{(\mu-1)}(t)\,dt\,dw\\
&+\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\int\limits_{OT_0^{(j)}}\frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+\mu_{\ell k}+N_1^{(\ell)}}}{w-z}\varepsilon_\ell(w,N_1^{(\ell)})\,dw+\varepsilon_k(z,N_0)\\
&= -\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\sum_{s=0}^{N_1^{(\ell)}-1}f_{\ell s}z^{-1}F^{(1)}\left(z^{-1};\begin{array}{c}N_0-s-\mu_{\ell k}\\ q_{\ell k}\\ -\theta_j\end{array}\right)\\
&\quad+z^{-N_0}R_k^{(\nu)}(z,N_0,N_1)
\end{aligned}
\tag{4.45}
$$

(cf. the definition (4.41)) for $z\in\overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$ and N_0, N_1 as above, where

$$
\begin{aligned}
z^{-N_0}R_k^{(\nu)}(z,N_0,N_1) &= \sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\times\\
&\times\int\limits_{OT_0^{(j)}}\int\limits_{OT_1^{(\mu)}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+\mu_{m\ell}}}{(w-z)(t-w)}f_m^{(\mu-1)}(t)\,dt\,dw\\
&\qquad+S_k^{(\nu)}(z,N_0,N_1)
\end{aligned}
\tag{4.46}
$$

and

$$
\begin{aligned}
S_k^{(\nu)}(z,N_0,N_1) &= -\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\sum_{s=0}^{N_1^{(\ell)}-1}f_{\ell s}\int\limits_{T_0^{(j)}}^{[\theta_j]}\frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+s+\mu_{\ell k}}}{w-z}\,dw\\
&+\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\int\limits_{OT_0^{(j)}}\frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}}{w-z}\varepsilon_\ell(w,N_1^{(\ell)})\,dw\\
&\qquad+\varepsilon_k(z,N_0).
\end{aligned}
\tag{4.47}
$$

The "subdominant" term $S_k^{(\nu)}(z,N_0,N_1)$ will be estimated in Lemma 4.13. We will therefore turn our attention to the double integrals in (4.46) which bring the essential contribution to the level-1 remainder $R_k^{(\nu)}(z,N_0,N_1)$.

For both the inner and the outer integral we will apply Theorem 5.6 from Section 5. We will need the two polynomials $p_1 := q_{m\ell}$ and $p_2 := q_{\ell k}$ and the derived quantities r_i, α_i, β_i $(i=1,2)$ like in Theorem 5.6 with the additional stipulation that $\arg\alpha_1 = \theta_\mu$, $\arg\alpha_2 = \theta_j$. We will also use two ε-like parameters defined by

$$\varepsilon_1^{-r_1} := N_1^{(\ell)} + 2 - \mu_{m\ell} \qquad \text{and} \qquad \varepsilon_2^{-r_2} := N_0 - N_1^{(\ell)} + 2 - \mu_{\ell k}$$

for which it is easy to see that

$$\arg\varepsilon_i = O(\varepsilon_i^{r_i}) \subset O(\varepsilon_i) \qquad (\varepsilon_i \to 0),$$

hence we can find appropriate closed cusps $\Omega_i = \overline{\Upsilon}\big(0, K(\Omega_i), \rho(\Omega_i)\big)$ such that $\varepsilon_i \in \Omega_i$ for large enough $N_1^{(\ell)} \in \mathbb{R}_+$, $N_0 - N_1^{(\ell)} \in \mathbb{R}_+$. Without loss of generality $\rho(\Omega_i)$ can be taken to satisfy $\rho(\Omega_i) < |\alpha_i|^{-1}\rho_{01}$ with ρ_{01} from (4.43). With $\hat{K} > |\alpha_i|^{-1}\big(K(\Omega_i) + \frac{1}{r_i}|\mathrm{Im}(\alpha_i\beta_i)|\big)$ consider the open cusps $G_1 = \Upsilon(\arg\alpha_1, \hat{K}, \hat{\rho})$ with $\hat{\rho}$ from (4.43) and $G_2 = \Upsilon(\arg\alpha_2, \hat{K}, \rho_{01})$.

First consider the inner integral

$$\int\limits_{OT_1^{(\mu)}} \frac{e^{q_{m\ell}(\frac{1}{t})} t^{-N_1^{(\ell)}+\mu_{m\ell}}}{t-w} f_m^{(\mu-1)}(t)\,dt.$$

In Lemma 4.1 we considered the functions $f_m^{(\nu)}$ only on the respective sectors $S_\nu = S_{\hat{\rho}}(I_\nu)$. But since it is clear that each one of these functions can be continued analytically onto sectors of arbitrary opening and since $\arg\alpha_1 \in I_{\mu-1}$ the function $F_1 := f_m^{(\mu-1)}$ is analytic and bounded on every proper subcusp of G_1. Then with $T_1 := T_1^{(\mu)}$ Theorem 5.6 applies. In case $\mu \neq j$ we use the contour of (t-) integration $\mathfrak{W}_1 = \mathfrak{W}(w)$ and (5.44) whereas in case $\mu = j$ we have to choose $\mathfrak{W}_1 = \mathfrak{W}_\ell$, according to our *"right over left"* considerations (see page 49), and apply the result in (5.43). In either case we obtain that the value of the integral depends analytically upon $w \in G_2$ and satisfies

$$\begin{aligned}
(4.48)\qquad F_2(w) &:= \int\limits_{OT_1^{(\mu)}} \frac{e^{q_{m\ell}(\frac{1}{t})} t^{-N_1^{(\ell)}+\mu_{m\ell}}}{t-w} f_m^{(\mu-1)}(t)\,dt = \int\limits_{\mathfrak{W}_1} e^{p_1(t^{-1})} t^{-\varepsilon_1^{-r_1}} \frac{t^2 F_1(t)}{t-w}\,dt \\
&= \left(e^{1/r_1}\alpha_1\varepsilon_1\right)^{-\varepsilon_1^{-r_1}} e^{O(\varepsilon_1^{-r_1+1})} O(1) \\
&\subseteq \underbrace{\left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} \left(1+N_1^{(\ell)}\right)^{\frac{-\mathrm{Re}\,\mu_{m\ell}+2}{\kappa_{m\ell}}}}_{=:\,K_{m\ell}(N_1)} O(1)
\end{aligned}$$

uniformly w.r.to w in proper subcusps of G_2 and (large enough) $N_1^{(\ell)} \in \mathbb{R}_+$. The restriction upon $N_1^{(\ell)}$ can now be removed by directly considering the above integral for small (bounded) $N_1^{(\ell)}$.

This is exactly what the function F_2 needs for the Theorem 5.6 to apply again. Taking the contour of (w-) integration $\mathfrak{W}_2 = \mathfrak{W}(z)$, (5.44) gives

$$\int\limits_{OT_0^{(j)}} \frac{e^{q_{\ell k}(\frac{1}{w})} w^{-(N_0-N_1^{(\ell)})+\mu_{\ell k}}}{w-z} \int\limits_{OT_1^{(\mu)}} \frac{e^{q_{m\ell}(\frac{1}{t})} t^{-N_1^{(\ell)}+\mu_{m\ell}}}{t-w} f_m^{(\mu-1)}(t)\, dt\, dw$$

$$= \int\limits_{\mathfrak{W}_2} e^{p_2(w^{-1})} w^{-\varepsilon_2^{-r_2}} \frac{w^2 F_2(w)}{w-z}\, dw = K_{m\ell}(N_1) \left(e^{1/r_2}\alpha_2\varepsilon_2\right)^{-\varepsilon_2^{-r_2}} e^{O(\varepsilon_2^{-r_2+1})} O(1)$$

$$\subseteq K_{m\ell}(N_1) \left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} e^{O\left((N_0-N_1^{(\ell)})^{1-\frac{1}{\kappa_{\ell k}}}\right)} \left(1+N_0-N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{\ell k}+2}{\kappa_{\ell k}}} O(1)$$

uniformly w.r.to $z \in \overline{K}_\rho(0)$, $N_1^{(\ell)} \geq 0$ and (large enough) $N_0 - N_1^{(\ell)} \in \mathbb{R}_+$. The restriction upon $N_0 - N_1^{(\ell)}$ can now be removed by directly considering the above integral for small (bounded) $N_0 - N_1^{(\ell)}$. In case $N_1^{(\ell)} > N_0$ the footnote on page 48 applies.

Observing that to every non-zero term occurring in the sum in (4.47) there is at least one non-zero double integral summand in (4.46) which dominates the former (compare the estimates in Lemma 4.13 with (4.42)), we find that $S_k^{(\nu)}(z, N_0, N_1)$ does in fact not contribute to the estimate of the whole remainder.

Taking the sum over all possible j, μ, ℓ, m and multiplying by z^{N_0}, Theorem 4.6 follows. $\square$

Now, similarly as in Section 4.4 one can see that, whatever the choice of the truncation point (N_0, N_1) is, the estimate (4.42) is not better than

$$e^{-|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} - |\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}} + O\left(|z|^{-\kappa_{m_1\ell_1}+1} + |z|^{-\kappa_{\ell_1 k}+1}\right)} \cdot |z|^{\tilde{\mu}_1 - 4} O(1) \tag{4.49}$$

where, for k fixed,

$$U^{(1)} = \bigcup_{j=1}^{M} \bigcup_{\mu=1}^{M} \left\{(m,\ell,k) :\ (\ell,k) \in J_j,\ (m,\ell) \in J_\mu,\ c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} \neq 0\right\},$$

$$U_0^{(1)} = \left\{(m,\ell,k) \in U^{(1)} :\ |\alpha_{m\ell}|x^{\kappa_{m\ell}} + |\alpha_{\ell k}|x^{\kappa_{\ell k}} = \min_{(r,s,k)\in U^{(1)}} |\alpha_{rs}|x^{\kappa_{rs}} + |\alpha_{sk}|x^{\kappa_{sk}}\right\}, \tag{4.50}$$

$$\tilde{\mu}_1 = \min_{(m,\ell,k)\in U_0^{(1)}} \operatorname{Re}\mu_{mk} \quad \text{and} \quad (m_1, \ell_1, k) \in U_0^{(1)} \text{ arbitrarily.} \tag{4.51}$$

To see this, for positive real $x, N_1^{(\ell)}, N_0 - N_1^{(\ell)}$ and $(m,\ell,k) \in U^{(1)}$ denote

$$\begin{aligned}
\lambda_{m\ell}(x,N_0,N_1) &:= \left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} \left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0 - N_1^{(\ell)}}{\kappa_{\ell k}}}, \\
\lambda(x,N_0,N_1) &:= \max_{(m,\ell,k)\in U^{(1)}} \lambda_{m\ell}(x,N_0,N_1).
\end{aligned} \tag{4.52}$$

For each pair (m,ℓ), $\lambda_{m\ell}(x,N_0,N_1)$ is the product of two convex functions of $N_1^{(\ell)}$ and $N_0 - N_1^{(\ell)}$ with global minima at $N_1^{(\ell)} = \kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}$ and $N_0 - N_1^{(\ell)} = \kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}$, respectively, so that we have $\lambda_{m\ell}(x,N_0,N_1) \geq e^{-|\alpha_{m\ell}|x^{\kappa_{m\ell}} - |\alpha_{\ell k}|x^{\kappa_{\ell k}}}$. The $\lambda_{m\ell}$ contributing to (4.42) are exactly those for which $(m,\ell,k) \in U^{(1)}$, and the above minimum is "worst" (i.e. maximal) if $(m,\ell,k) \in U_0^{(1)}$. Hence we know that $\lambda(x,N_0,N_1) \geq \lambda_{m_1\ell_1}(x,N_0,N_1) \geq e^{-|\alpha_{m_1\ell_1}|x^{\kappa_{m_1\ell_1}} - |\alpha_{\ell_1 k}|x^{\kappa_{\ell_1 k}}}$. Putting $x := |z|^{-1}$ and considering the lower-order terms, we obtain the best possible estimate (4.49).

Basically, (4.49) can also be an upper estimate for the remainder if coupling $N_0 = N_0(x)$, $N_1 = N_1(x)$ appropriately. The only difference to (4.49) is a power of z (which is of relatively little importance). Define

$$\begin{aligned}
\tilde{U}_0^{(1)} &= \left\{(\ell,k) \in U^{(0)} :\ |\alpha_{\ell k}|x^{\kappa_{\ell k}} = |\alpha_{m_1\ell_1}|x^{\kappa_{m_1\ell_1}} + |\alpha_{\ell_1 k}|x^{\kappa_{\ell_1 k}}\right\}, \\
\tilde{\tilde{\mu}}_1 &= \min\left\{\tilde{\mu}_1,\ 2 + \min_{(\ell,k)\in\tilde{U}_0^{(1)}} \operatorname{Re}\mu_{\ell k}\right\}
\end{aligned} \tag{4.53}$$

where the set $U^{(0)}$ has been defined in (4.25). Then we have

Corollary 4.7 (Optimal expansions in small sectors). *Let the assumptions of Lemma 4.1 be satisfied: let* (4.2) *be a formal fundamental system of* [D] *with* $p = 1$, *assume* (4.4) *holds, and let* $\hat{\rho} > 0$, $\{I_\nu\}_{\nu\in\mathbb{Z}}$, *and* $\{f_k^{(\nu)}\}_{\nu\in\mathbb{Z}}$ *be as in Lemma 4.1. If*

$$\begin{aligned}
N_0 &= \kappa_{m_1\ell_1}|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} + \kappa_{\ell_1 k}|\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}} + O(1), \\
N_1^{(\ell)} &= \max\left\{0, \kappa_{m_1\ell_1}|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} + \kappa_{\ell_1 k}|\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}} - \kappa_{\ell k}|\alpha_{\ell k}||z|^{-\kappa_{\ell k}}\right\} + O(1)
\end{aligned} \tag{4.54}$$

then the remainder $R_k^{(\nu)}(z,N_0,N_1)$ *in* (4.41) *satisfies*

$$R_k^{(\nu)}(z,N_0,N_1) = e^{-|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} - |\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}} + O\left(|z|^{-\kappa_{m_1\ell_1}+1} + |z|^{-\kappa_{\ell_1 k}+1}\right)} \cdot |z|^{\tilde{\tilde{\mu}}_1 - 4} O(1)$$

uniformly for $z \in \overline{S}_\rho(\theta_\nu, \theta_{\nu+1})$ *for every* $\rho < \hat{\rho}$. *Here,* m_1, ℓ_1 *and* $\tilde{\mu}_1$ *are defined in* (4.51).

Remark. Looking at (4.54) one might remark that the truncation point N_0 in the level-1 expansion is allowed to be larger than at level zero. Thus we do not perform optimization of N_0 and N_1 *one after the other*: this was the original method of Berry and

Howls [BH90, BH91] adopted in [Old92, Old93] and which only leads to limited final exponential improvement. Our strategy was inspired by a paper of Olde Daalhuis and Olver [OO95a] where it was shown in a special case that if the number of terms of the previous levels is allowed to change in subsequent levels then the remainder can become arbitrarily small.

Proof. Let $(m,\ell,k)\in U^{(1)}$. Abbreviate $x:=|z|^{-1}$ like in the proof of Corollary 4.4, and let

$$\begin{aligned}
p_1(x) &:= \max\Big\{0, |\alpha_{m_1\ell_1}|x^{\kappa_{m_1\ell_1}} + |\alpha_{\ell_1 k}|x^{\kappa_{\ell_1 k}} - |\alpha_{\ell k}|x^{\kappa_{\ell k}}\Big\},\\
q_1(x) &:= |\alpha_{m\ell}|x^{\kappa_{m\ell}},\\
p_2(x) &:= |\alpha_{m_1\ell_1}|x^{\kappa_{m_1\ell_1}} + |\alpha_{\ell_1 k}|x^{\kappa_{\ell_1 k}} - p_1(x),\\
q_2(x) &:= |\alpha_{\ell k}|x^{\kappa_{\ell k}}
\end{aligned}$$

and therewith

$$\left[\begin{aligned}
N_0 &= xp_1'(x) + xp_2'(x) + \delta_0(x), && \delta_0(x) = O(1),\\
N_1^{(\ell)} &= xp_1'(x) + \delta_1^{(\ell)}(x), && \delta_1^{(\ell)}(x) = O(1)\\
\Rightarrow\ N_0 - N_1^{(\ell)} &= xp_2'(x) + \delta_2^{(\ell)}(x), && \delta_2^{(\ell)}(x) = O(1)
\end{aligned}\right].$$

This is (4.54). Note that by (4.50),(4.51) we have $0\le p_1(x)\le q_1(x)$ $(x\to+\infty)$ and $0<p_2(x)\le q_2(x)$ $(x\to+\infty)$.

We consider now both factors of $\lambda_{m\ell}(x,N_0,N_1)$ in (4.52) individually. In the special case $p_1=0$ this essentially reduces to one factor which can be estimated by Lemma 4.12. In the case $p_1>0$, the lemma can be applied to both factors. This is done in detail in Section 4.8.3. This completes the proof. □

Remarks. **(i)** Let $N_0=N_0(x)$, $N_1=N_1(x)$ (with $x=|z|^{-1}$) be as in Corollary 4.7. Since $p_2(x)=\min\Big\{|\alpha_{m_1\ell_1}|x^{\kappa_{m_1\ell_1}}+|\alpha_{\ell_1 k}|x^{\kappa_{\ell_1 k}}, |\alpha_{\ell k}|x^{\kappa_{\ell k}}\Big\}\to+\infty$ $(x\to+\infty)$ we also have $xp_2'(x)\to+\infty$ $(x\to+\infty)$ and hence (4.40) is satisfied for large enough x. For all smaller x this condition can be reached by choice of appropriate functions $\delta_0(x)$, $\delta_1^{(\ell)}(x)$.

(ii) According to the definition of $\tilde{\tilde{\mu}}_1$, there are not only terms coming from $U_0^{(1)}$ entering the result, in contrast to the situation in level 0. The reason why the set $\tilde{U}_0^{(1)}$ shows up is not clear to the author. A possible explanation could be that the estimate (4.42) might still be sharpened for these (ℓ,k).

Like in Section 4.4, we want to ask now if the estimate in Corollary 4.7 continues to hold outside $z\in\overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$, too. In particular, since the hyperterminants in (4.41) incorporate Stokes' phenomenon, we expect that the level-1 expansion has a larger sector of validity than the level-0 expansion and allows a smooth interpretation of

Stokes' phenomenon. We will see that, in contrast to the case $n=2$, this is not true in general.

To study the behavior of $R_k^{(\nu)}(z,N_0,N_1)$ outside $z\in \overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$, we need at first a connection relation for the level-1 remainders. For simplicity, we will only consider consecutive sectors. E.g., for $z\in S_{\nu-1}\cap S_\nu$, $\arg z=\theta_\nu$ we find from (4.41)

$$\begin{aligned}
&R_k^{(\nu)}(z,N_0,N_1)-R_k^{(\nu-1)}(z,N_0,N_1)=\\
&=\left(R_k^{(\nu)}(z,N_0)+\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s}z^{N_0-1}F^{(1)}\left(z^{-1};\begin{matrix}N_0-s-\mu_{\ell k}\\ q_{\ell k}\\ -\theta_j\end{matrix}\right)\right)-\\
&\quad-\left(R_k^{(\nu-1)}(z,N_0)+\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s}z^{N_0-1}F^{(1)}\left(z^{-1};\begin{matrix}N_0-s-\mu_{\ell k}\\ q_{\ell k}\\ -\theta_j\end{matrix}\right)\right)
\end{aligned}$$

(where on the first line, $\arg z=\theta_\nu+0$ while $\arg z=\theta_\nu-0$ on the second)

$$\begin{aligned}
=&R_k^{(\nu)}(z,N_0)-R_k^{(\nu-1)}(z,N_0)+\sum_{(\ell,k)\in J_\nu}\frac{c_{\ell k}^{(\nu)}}{2\pi i}\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s}z^{N_0-1}\times\\
&\times\left[F^{(1)}\left(z^{-1};\begin{matrix}N_0-s-\mu_{\ell k}\\ q_{\ell k}\\ -\theta_\nu+0\end{matrix}\right)-F^{(1)}\left(z^{-1};\begin{matrix}N_0-s-\mu_{\ell k}\\ q_{\ell k}\\ -\theta_\nu-0\end{matrix}\right)\right].
\end{aligned}$$

Inserting the Stokes relation for hyperterminants (1.8) for $n=1$:

$$\begin{aligned}
F^{(1)}\left(z^{-1};\begin{matrix}N_0-s-\mu_{\ell k}\\ q_{\ell k}\\ -\theta_\nu+0\end{matrix}\right)-F^{(1)}\left(z^{-1};\begin{matrix}N_0-s-\mu_{\ell k}\\ q_{\ell k}\\ -\theta_\nu-0\end{matrix}\right)&\\
=-2\pi i\, e^{q_{\ell k}(z^{-1})}\left(z^{-1}\right)^{N_0-s-\mu_{\ell k}-1}&,
\end{aligned}$$

together with (4.6) this finally leads to

$$\begin{aligned}
&R_k^{(\nu)}(z,N_0,N_1)-R_k^{(\nu-1)}(z,N_0,N_1)=\\
&\qquad=f_k^{(\nu)}(z)-f_k^{(\nu-1)}(z)-\sum_{(\ell,k)\in J_\nu}c_{\ell k}^{(\nu)}\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s}z^{\mu_{\ell k}+s}e^{q_{\ell k}(z^{-1})}\\
&\qquad=\sum_{(\ell,k)\in J_\nu}c_{\ell k}^{(\nu)}e^{q_{\ell k}(z^{-1})}z^{\mu_{\ell k}}\left[f_\ell^{(\nu-1)}(z)-\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s}z^s\right]\\
&\qquad=\sum_{(\ell,k)\in J_\nu}c_{\ell k}^{(\nu)}e^{q_{\ell k}(z^{-1})}z^{\mu_{\ell k}}R_\ell^{(\nu-1)}(z,N_1^{(\ell)}).
\end{aligned}\tag{4.55}$$

By analytic continuation, this identity holds elsewhere, too. This is the level-1 analog to (4.34).

Now, let the situation of Corollary 4.7 be satisfied, and let N_0 and $N_1^{(\ell)}$ be as in the corollary. "To the right" of θ_ν – or more precisely, in $\overline{S}_\rho(\theta_{\nu-1},\theta_\nu)$ – the remainder $R_k^{(\nu-1)}(z,N_0,N_1)$ satisfies the estimate in Corollary 4.7. For the additional terms in (4.55), from Theorem 4.3 we have

$$R_\ell^{(\nu-1)}(z,N_1^{(\ell)}) = \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} c_{m\ell}^{(\mu)} \left(\frac{N_1^{(\ell)}|z|^{\kappa_{m\ell}}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} \cdot L.O.T.$$

where the term "*L.O.T.*" denotes a product of lower-order terms, and by the proof of Corollary 4.7 (Section 4.8.3) this is

$$e^{-p_1(x)} = \exp\left(\min\left\{0, -|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} - |\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}} + |\alpha_{\ell k}||z|^{-\kappa_{\ell k}}\right\}\right)$$

times lower-order terms. Hence, to find out the behavior of $R_k^{(\nu)}(z,N_0,N_1)$ in $\overline{S}_\rho(\theta_{\nu-1},\theta_\nu)$, we have to examine the dominance relation between

$$\exp\left(-|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} - |\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}}\right) \qquad \text{and}$$
$$\exp\left(\alpha_{\ell k} z^{-\kappa_{\ell k}} + \min\left\{0, -|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} - |\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}} + |\alpha_{\ell k}||z|^{-\kappa_{\ell k}}\right\}\right).$$

One can now discuss this by distinguishing several cases upon the degrees of all polynomials appearing. We will not go into the details here but only stress an important issue:

- Assume that, for all ℓ appearing in (4.55), $|\alpha_{\ell k}||z|^{-\kappa_{\ell k}}$ is small compared with $|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} + |\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}}$, that is, the above minimum is negative. Then from θ_ν on, the estimate of the optimized remainder $R_k^{(\nu)}(z,N_0,N_1)$ will deteriorate gradually *but even beyond the Stokes direction* $\theta_\nu - \frac{\pi}{2\kappa_{\ell k}}$ *it continues to be small.* Hence, level-1 optimization provides not only better accuracy but also a larger region of validity in this case.
- If, in contrast, there is an ℓ for which $|\alpha_{\ell k}||z|^{-\kappa_{\ell k}}$ is large compared with $|\alpha_{m_1\ell_1}||z|^{-\kappa_{m_1\ell_1}} + |\alpha_{\ell_1 k}||z|^{-\kappa_{\ell_1 k}}$, i.e. the above minimum is 0, then the situation is different: the term $e^{q_{\ell k}(z^{-1})}$ will take over dominance at a certain direction $\theta \in \left[\theta_\nu - \frac{\pi}{2\kappa_{\ell k}}, \theta_\nu\right)$. The sector of validity might be slightly larger than at level zero here, but *it does not contain a corresponding Stokes' direction.* In the multi-leveled case, this situation occurs quite frequently, and we cannot achieve substantial enlargement of the region of validity even at subsequent levels, cf. the discussion at the end of Section 4.6.

Another characterization of this case is that the number $N_1^{(\ell)}$ corresponding to that ℓ is zero.

Remark. Finally, just like in Section 4.4, we are able to drop (4.4), and all results of this section can be generalized to the general situation using the matrix blocks of Section 4.3. We will omit this since the procedure should be clear from the previous sections.

4.6 Higher-level expansions

The process of successive insertion, truncation, estimation and optimization can be continued to higher hyperasymptotic levels, as will be sketched in this section. Due to the generality of our presentation, there will come more and more terms like multiple sums of multiple integrals being involved, so that the derivation of the estimates becomes technically sophisticated. It is important to note, however, that the relatively simple pattern of the final estimates is being preserved through successive levels.

For the reasons mentioned above, we will outline only the step from level one to level two. The proofs of the estimates for levels three and above are to be done by induction and will be omitted here. In addition, we will confine to the generic case (that is, we require (4.4)) for the purpose of simplicity. Just like in the previous subsections, this restriction is not essential and can be overcome by considering the blocks from Section 4.3.

Finally, we assume that *the vector solution $\hat{f}_k(z)$, every adjacent $\hat{f}_\ell(z)$, and every $\hat{f}_m(z)$ adjacent to the latter ones, diverge.* Again, this can be formulated in terms of the Stokes multipliers $c_{\ell k}^{(j)}$, $c_{m\ell}^{(\mu)}$, and $c_{pm}^{(h)}$.

The starting-point of our derivation of level-1 expansions was the integral representation (4.21) (resp. (3.9)) for the level-0 remainders. On the right-hand side of (4.21), we re-expanded the functions $f_\ell^{(j-1)}$ by means of (4.44). We will proceed in a similar way here: the level-1 analog to (4.21) is the representation (4.46) of the level-1 remainders. The functions $f_m^{(\mu-1)}$ will now be re-expanded by means of (4.44) again, up to a certain point $N_2^{(m\ell)}$ which is allowed to depend *upon ℓ and m*. The whole set of all those integers can be viewed as entries of a square matrix $N_2 := \left(N_2^{(m\ell)}\right)_{m\neq\ell\neq k}$. We will need additional relations between the N_0, $N_1^{(\ell)}$, and $N_2^{(m\ell)}$ later. With some $\rho_1 < \rho_2 < \hat{\rho}$ define points $\{T_2^{(\nu)}\}_{\nu=1}^{M}$ by $|T_2^{(\nu)}| = \rho_2$, $\arg T_2^{(\nu)} = \theta_\nu$, and therewith (4.44) can be re-written as to

(4.56)
$$f_m^{(\mu-1)}(t) = \sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms} t^s +$$
$$+ t^{N_2^{(m\ell)}} \sum_{h=1}^{M} \sum_{(p,m)\in J_h} \frac{c_{pm}^{(h)}}{2\pi i} \int\limits_{OT_2^{(h)}} \frac{e^{q_{pm}(\frac{1}{u})} u^{-N_2^{(m\ell)}+\mu_{pm}}}{u-t} f_p^{(h-1)}(u)\, du + t^{N_2^{(m\ell)}} \varepsilon_m(t, N_2^{(m\ell)}).$$

Inserting this into (4.46), we arrive at
(4.57)

$$z^{-N_0}R_k^{(v)}(z,N_0,N_1) = \sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms}\times$$

$$\times\int_{\overline{OT_0^{(j)}}}\int_{\overline{OT_1^{(\mu)}}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+s+\mu_{m\ell}}}{(w-z)(t-w)}\,dt\,dw$$

$$+\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{h=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\sum_{(p,m)\in J_h}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\frac{c_{pm}^{(h)}}{2\pi i}\times$$

$$\times\int_{\overline{OT_0^{(j)}}}\int_{\overline{OT_1^{(\mu)}}}\int_{\overline{OT_2^{(h)}}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})+q_{pm}(\frac{1}{u})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+N_2^{(m\ell)}+\mu_{m\ell}}u^{-N_2^{(m\ell)}+\mu_{pm}}}{(w-z)(t-w)(u-t)}\times$$

$$\times f_p^{(h-1)}(u)\,du\,dt\,dw$$

$$+\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\times$$

$$\times\int_{\overline{OT_0^{(j)}}}\int_{\overline{OT_1^{(\mu)}}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+N_2^{(m\ell)}+\mu_{m\ell}}}{(w-z)(t-w)}\varepsilon_m\big(t,N_2^{(m\ell)}\big)\,dt\,dw$$

$$+S_k^{(v)}(z,N_0,N_1).$$

Again, in case $\mu = j$ it is understood that the contour of t-integration has to be indented *to the left of* $t = w$, and in case $h = \mu$ the contour of u-integration has to be indented *to the left of* $u = t$.

If we replace the dominant double integrals on the second line of (4.57) by the more convenient ones over *the whole half-lines* $\arg w = \theta_j$, $\arg t = \theta_\mu$ then these integrals are related (via the transformation $z \longleftarrow \frac{1}{z}$) to the hyperterminants $F^{(2)}$ in (1.6). On the other hand, we have to take care about convergence of these integrals near infinity. Therefore we require

$$0 \le N_0,$$

(4.58a) $$0 \le N_1 < N_0 + \operatorname{Re}\mu_k - \operatorname{Re}\mu^{(k)}$$

as well as for every $\ell \neq k$:

(4.58b) $$0 \le N_2^{(\ell)} < N_1^{(\ell)} + \operatorname{Re}\mu_\ell - \operatorname{Re}\mu^{(\ell)} + 1$$

in the vector notation of Section 4.5. Thus we find

(4.59)

$$z^{-N_0}R_k^{(\nu)}(z,N_0,N_1) = \sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\times$$
$$\times\sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms}z^{-1}F^{(2)}\left(z^{-1};\begin{array}{cc} N_0-N_1^{(\ell)}-\mu_{\ell k}+1, & N_1^{(\ell)}-s-\mu_{m\ell} \\ q_{\ell k} & q_{m\ell} \\ -\theta_j & -\theta_\mu \end{array}\right)$$
$$+z^{-N_0}R_k^{(\nu)}(z,N_0,N_1,N_2)$$

where

(4.60)

$$z^{-N_0}R_k^{(\nu)}(z,N_0,N_1,N_2) = \sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{h=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\sum_{(p,m)\in J_h}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\frac{c_{pm}^{(h)}}{2\pi i}\times$$
$$\times\int_{OT_0^{(j)}}\int_{OT_1^{(\mu)}}\int_{OT_2^{(h)}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})+q_{pm}(\frac{1}{u})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+N_2^{(m\ell)}+\mu_{m\ell}}u^{-N_2^{(m\ell)}+\mu_{pm}}}{(w-z)(t-w)(u-t)}\times$$
$$\times f_p^{(h-1)}(u)\,du\,dt\,dw$$

$$+S_k^{(\nu)}(z,N_0,N_1,N_2)$$

and

(4.61)

$$S_k^{(\nu)}(z,N_0,N_1,N_2) = -\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms}\times$$
$$\times\int_0^{[\theta_j]}\int_{T_1^{(\mu)}}^{[\theta_\mu]}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+s+\mu_{m\ell}}}{(w-z)(t-w)}\,dt\,dw$$
$$+\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\times$$
$$\times\int_{OT_0^{(j)}}\int_{OT_1^{(\mu)}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+N_2^{(m\ell)}+\mu_{m\ell}}}{(w-z)(t-w)}\varepsilon_m\big(t,N_2^{(m\ell)}\big)\,dt\,dw+$$

$$+\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\int_{\overline{OT_0^{(j)}}}\frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}}{w-z}\varepsilon_\ell(w,N_1^{(\ell)})\,dw+\varepsilon_k(z,N_0)$$
$$+T_k^{(\nu)}(z,N_0,N_1,N_2)$$

and

$$T_k^{(\nu)}(z,N_0,N_1,N_2)=-\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j}\frac{c_{\ell k}^{(j)}}{2\pi i}\sum_{\sigma=0}^{N_1^{(\ell)}-1}f_{\ell\sigma}\int_{T_0^{(j)}}^{[\theta_j]}\frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+\sigma+\mu_{\ell k}}}{w-z}dw \tag{4.62}$$
$$-\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\sum_{s=0}^{N_2^{(m\ell)}-1}f_{ms}\times$$
$$\times\int_{T_0^{(j)}}^{[\theta_j]}\int_{\overline{OT_1^{(\mu)}}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+s+\mu_{m\ell}}}{(w-z)(t-w)}dt\,dw.$$

For the latter two equations, we have used representation (4.47) of $S_k^{(\nu)}(z,N_0,N_1)$.

In analogy to Section 4.5, we can uniformly estimate the remainder if we confine inequalities (4.58) to closed subsets of the form

$$0\le N_0,$$
$$0\le N_1\le N_0+\operatorname{Re}\mu_k-\operatorname{Re}\mu^{(k)}-\delta, \tag{4.63a}$$
$$0\le N_2^{(\ell)}\le N_1^{(\ell)}+\operatorname{Re}\mu_\ell-\operatorname{Re}\mu^{(\ell)}+1-\delta \tag{4.63b}$$

with a positive constant δ. As we will see, the "subdominant" term $S_k^{(\nu)}(z,N_0,N_1,N_2)$ is dominated by the sum over the triple integrals in (4.60). By considerations very similar to those in Section 4.5, the triple integral belonging to (j,μ,h,ℓ,m,p) is estimated by

$$\left(\frac{N_0-N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}}\left(\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{\kappa_{m\ell}}}\left(\frac{N_2^{(m\ell)}}{e\kappa_{pm}|\alpha_{pm}|}\right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}}\cdot L.O.T. \tag{4.64}$$

where the term "*L.O.T.*" denotes a product of lower-order terms such as appearing in Theorems 4.3 and 4.6, respectively. We will not write down these terms explicitly in this section since the pattern should be clear from what has been shown before.

Concerning the subdominant term $S_k^{(\nu)}(z,N_0,N_1,N_2)$, the term $T_k^{(\nu)}(z,N_0,N_1,N_2)$ is not well-suited for direct estimation since, e. g., $f_{\ell\sigma}$ is estimated by a sum of terms

$$\left(\frac{\sigma}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{\sigma}{\kappa_{m\ell}}}\cdot L.O.T.$$

(cf. (4.82)) which is too big for $\sigma \approx N_1^{(\ell)}$, compared with (4.64). We will use the integral representation (3.8) for $f_{\ell\sigma}$ instead to find

(4.65)

$$f_{\ell\sigma} = \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} \frac{c_{m\ell}^{(\mu)}}{2\pi i} \int\limits_{OT_1^{(\mu)}} e^{q_{m\ell}(\frac{1}{t})} t^{-\sigma+\mu_{m\ell}-1} f_m^{(\mu-1)}(t)\, dt + \varepsilon_\ell(0,\sigma)$$

$$= \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} \frac{c_{m\ell}^{(\mu)}}{2\pi i} \sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms} \int\limits_{OT_1^{(\mu)}} e^{q_{m\ell}(\frac{1}{t})} t^{-\sigma+s+\mu_{m\ell}-1}\, dt$$

$$+ \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} \frac{c_{m\ell}^{(\mu)}}{2\pi i} \int\limits_{OT_1^{(\mu)}} e^{q_{m\ell}(\frac{1}{t})} t^{-\sigma+N_2^{(m\ell)}+\mu_{m\ell}-1} t^{-N_2^{(m\ell)}} R_m^{(\mu-1)}\big(t, N_2^{(m\ell)}\big)\, dt + \varepsilon_\ell(0,\sigma)$$

$$= \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} \frac{c_{m\ell}^{(\mu)}}{2\pi i} \sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms} \int\limits_{OT_1^{(\mu)}} e^{q_{m\ell}(\frac{1}{t})} t^{-\sigma+s+\mu_{m\ell}-1}\, dt$$

$$+ \sum_{\mu=1}^{M} \sum_{h=1}^{M} \sum_{(m,\ell)\in J_\mu} \sum_{(p,m)\in J_h} c_{m\ell}^{(\mu)} c_{pm}^{(h)} \left(\frac{\sigma - N_2^{(m\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{\sigma-N_2^{(m\ell)}}{\kappa_{m\ell}}} \left(\frac{N_2^{(m\ell)}}{e\kappa_{pm}|\alpha_{pm}|}\right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}} \cdot L.O.T.$$

Hence we obtain

$$\begin{aligned}
T_k^{(\nu)}(z,N_0,N_1,N_2) = &-\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms}\sum_{\sigma=0}^{N_1^{(\ell)}-1}\times\\
&\times\int_{T_0^{(j)}}^{[\theta_j]}\int_{\overline{OT_1^{(\mu)}}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+\sigma+\mu_{\ell k}}t^{-\sigma+s+\mu_{m\ell}-1}}{w-z}\,dt\,dw\\
&+\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{h=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\sum_{(p,m)\in J_h}c_{\ell k}^{(j)}c_{m\ell}^{(\mu)}c_{pm}^{(h)}\sum_{\sigma=0}^{N_1^{(\ell)}-1}\times\\
&\times\rho_0^{-N_0+\sigma}\left(\frac{\sigma-N_2^{(m\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{\sigma-N_2^{(m\ell)}}{\kappa_{m\ell}}}\left(\frac{N_2^{(m\ell)}}{e\kappa_{pm}|\alpha_{pm}|}\right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}}\cdot L.O.T.\\
&-\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms}\times\\
&\times\int_{T_0^{(j)}}^{[\theta_j]}\int_{\overline{OT_1^{(\mu)}}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}t^{-N_1^{(\ell)}+s+\mu_{m\ell}}}{(w-z)(t-w)}\,dt\,dw
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\frac{c_{\ell k}^{(j)}}{2\pi i}\frac{c_{m\ell}^{(\mu)}}{2\pi i}\sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms}\times\\
&\times\int_{T_0^{(j)}}^{[\theta_j]}\int_{\overline{OT_1^{(\mu)}}}\frac{e^{q_{\ell k}(\frac{1}{w})+q_{m\ell}(\frac{1}{t})}w^{-N_0+\mu_{\ell k}}t^{s+\mu_{m\ell}}}{(w-z)(t-w)}\,dt\,dw\\
&+\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{h=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\sum_{(p,m)\in J_h}c_{\ell k}^{(j)}c_{m\ell}^{(\mu)}c_{pm}^{(h)}\times\\
&\times\rho_0^{-N_0+N_1^{(\ell)}}\left(\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{\kappa_{m\ell}}}\left(\frac{N_2^{(m\ell)}}{e\kappa_{pm}|\alpha_{pm}|}\right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}}\cdot L.O.T.,
\end{aligned}\tag{4.66}$$

by use of an analog to (3.11), and of Lemma 4.11. The first term of (4.66) can now be estimated by

$$\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{h=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\sum_{(p,m)\in J_h} c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} c_{pm}^{(h)} \times$$
$$\times\, \rho_0^{-N_0+N_1^{(\ell)}} \rho_0^{-N_1^{(\ell)}+N_2^{(m\ell)}} \left(\frac{N_2^{(m\ell)}}{e\kappa_{pm}|\alpha_{pm}|}\right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}} \cdot L.O.T.$$

by considerations similar to those in the proof of the first statement of Lemma 4.13.

The remaining terms contributing to $S_k^{(\nu)}(z,N_0,N_1,N_2)$ in (4.61) are estimated by

$$\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{h=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\sum_{(p,m)\in J_h} c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} c_{pm}^{(h)} \times$$
$$\times \left(\frac{N_0-N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} \rho_1^{-N_1^{(\ell)}+N_2^{(m\ell)}} \left(\frac{N_2^{(m\ell)}}{e\kappa_{pm}|\alpha_{pm}|}\right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}} \cdot L.O.T.,$$

$$\sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu} c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} \left(\frac{N_0-N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} \left(\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{\kappa_{m\ell}}} \rho_2^{-N_2^{(m\ell)}} \cdot L.O.T.,$$

$$\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j} c_{\ell k}^{(j)} \left(\frac{N_0-N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} \rho_1^{-N_1^{(\ell)}+N_2^{(m\ell)}} \rho_1^{-N_2^{(m\ell)}} \cdot L.O.T.$$

and

$$\rho_0^{-N_0+N_1^{(\ell)}} \rho_0^{-N_1^{(\ell)}-N_2^{(m\ell)}} \rho_0^{-N_2^{(m\ell)}} O(1),$$

respectively. Observing that to every non-zero term occurring in (4.61) there is at least one non-zero triple integral summand in (4.60) which dominates the former, we find that $S_k^{(\nu)}(z,N_0,N_1,N_2)$ does in fact not contribute to the estimate of the whole remainder.

Altogether we have shown the

Theorem 4.8. *Let the assumptions of Lemma 4.1 be satisfied: let* (4.2) *be a formal fundamental system of* [D] *with* $p=1$, *assume* (4.4) *holds, and let* $\hat{\rho}>0$, $\{I_\nu\}_{\nu\in\mathbb{Z}}$, *and* $\{f_k^{(\nu)}\}_{\nu\in\mathbb{Z}}$ *be as in Lemma 4.1. Then the remainder* $R_k^{(\nu)}(z,N_0,N_1,N_2)$ *defined in*

(4.59) *satisfies*

(4.67)

$$R_k^{(\nu)}(z,N_0,N_1,N_2) =$$
$$= \sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{h=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu}\sum_{(p,m)\in J_h} c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} c_{pm}^{(h)} \times$$
$$\times \left(\frac{(N_0-N_1^{(\ell)})|z|^{\kappa_{\ell k}}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} \left(\frac{(N_1^{(\ell)}-N_2^{(m\ell)})|z|^{\kappa_{m\ell}}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{\kappa_{m\ell}}} \left(\frac{N_2^{(m\ell)}|z|^{\kappa_{pm}}}{e\kappa_{pm}|\alpha_{pm}|}\right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}} \cdot L.O.T.$$

uniformly w.r.to N_0, N_1, N_2 *satisfying* (4.63) *for every* $\delta > 0$, *and w.r.to* $z \in \overline{S}_\rho(\theta_\nu, \theta_{\nu+1})$ *for every* $\rho < \hat{\rho}$.

Here, the term "L.O.T." denotes lower-order terms which consist of exponentials of lower-order polynomials, e.g. of $\left(N_0 - N_1^{(\ell)}\right)^{1/\kappa_{\ell k}}$ *of degree at most* $(\kappa_{\ell k}-1)$ *and so forth, times powers of* $\left(1+N_0-N_1^{(\ell)}\right)$ *etc., times a bounded function.* □

The estimate in Theorem 4.8 can now be optimized by the same method as in Sections 4.4 and 4.5. To shorten notation, for numbers $m,\ell \in \{1,\dots,n\}$, $m \neq \ell$ we introduce the function

$$u_{m\ell}(x) := |\alpha_{m\ell}| x^{\kappa_{m\ell}}$$

where x acts as replacement for $|z|^{-1}$ again. Then we consider the sets

$$U^{(2)} = \bigcup_{j=1}^{M}\bigcup_{\mu=1}^{M}\bigcup_{h=1}^{M} \Big\{(p,m,\ell,k):\ (\ell,k)\in J_j,\ (m,\ell)\in J_\mu,\ (p,m)\in J_h,\ c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} c_{pm}^{(h)} \neq 0\Big\},$$

$$\text{(4.68)}\qquad U_0^{(2)} = \Big\{(p,m,\ell,k)\in U^{(2)}:\ \left(u_{pm}+u_{m\ell}+u_{\ell k}\right)(x) = \min_{(r,s,t,k)\in U^{(2)}} \left(u_{rs}+u_{st}+u_{tk}\right)(x)\Big\},$$

choose

(4.69) $\qquad (p_2,m_2,\ell_2,k) \in U_0^{(2)}$ arbitrarily

and therewith for positive real x, $N_2^{(m\ell)}$, $N_1^{(\ell)} - N_2^{(m\ell)}$, $N_0 - N_1^{(\ell)}$ and $(p,m,\ell,k) \in U^{(2)}$ denote

(4.70)

$$\lambda_{pm\ell}(x,N_0,N_1,N_2) := \left(\frac{N_0-N_1^{(\ell)}}{exu'_{\ell k}(x)}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} \left(\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{exu'_{m\ell}(x)}\right)^{\frac{N_1^{(\ell)}-N_2^{(m\ell)}}{\kappa_{m\ell}}} \left(\frac{N_2^{(m\ell)}}{exu'_{pm}(x)}\right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}},$$
$$\lambda(x,N_0,N_1,N_2) := \max_{(p,m,\ell,k)\in U^{(2)}} \lambda_{pm\ell}(x,N_0,N_1,N_2).$$

Now since for every single (p,m,ℓ), each of the three factors is a convex function in the corresponding N-variable whose minimum is easy to determine, the global minimum of the whole function is readily verified to be $\lambda_{pm\ell}(x,N_0,N_1,N_2) \geq e^{-(u_{pm}+u_{m\ell}+u_{\ell k})(x)}$. Thus the best possible estimate is given by

$$\lambda(x,N_0,N_1,N_2) \geq e^{-(u_{p_2m_2}+u_{m_2\ell_2}+u_{\ell_2k})(x)}, \tag{4.71}$$

in analogy to the previous sections.

The estimate of the remainder $R_k^{(\nu)}(z,N_0,N_1,N_2)$ will be optimal in the sense that it is of the same order as the best possible one, if we take

$$\begin{aligned}
p_1(x) &:= \min\Big\{\big(u_{p_2m_2}+u_{m_2\ell_2}+u_{\ell_2k}\big)(x), u_{\ell k}(x)\Big\},\\
q_1(x) &:= u_{\ell k}(x),\\
p_2(x) &:= \min\left\{\max\Big\{0, \big(u_{p_2m_2}+u_{m_2\ell_2}+u_{\ell_2k}-u_{\ell k}\big)(x)\Big\}, u_{m\ell}(x)\right\},\\
q_2(x) &:= u_{m\ell}(x),\\
p_3(x) &:= \max\Big\{0, \big(u_{p_2m_2}+u_{m_2\ell_2}+u_{\ell_2k}-u_{\ell k}-u_{m\ell}\big)(x)\Big\},\\
q_3(x) &:= u_{pm}(x)
\end{aligned}$$

(observe that, by this definition, $p_j(x) \leq q_j(x)$ $(j=1,2,3)$!), and

$$\begin{aligned}
N_0 = N_0(x) &= x(p_1+p_2+p_3)'(x)+O(1)\\
&= x\big(u_{p_2m_2}+u_{m_2\ell_2}+u_{\ell_2k}\big)'(x)+O(1),\\
N_1^{(\ell)} = N_1^{(\ell)}(x) &= x(p_2+p_3)'(x)+O(1)\\
&= \max\Big\{0, x\big(u_{p_2m_2}+u_{m_2\ell_2}+u_{\ell_2k}-u_{\ell k}\big)'(x)\Big\}+O(1),\\
N_2^{(m\ell)} = N_2^{(m\ell)}(x) &= xp_3'(x)+O(1)\\
&= \max\Big\{0, x\big(u_{p_2m_2}+u_{m_2\ell_2}+u_{\ell_2k}-u_{\ell k}-u_{m\ell}\big)'(x)\Big\}+O(1)
\end{aligned}$$

where $x=|z|^{-1}$.

Then we have for the first of the factors in (4.67):

$$\begin{aligned}
N_0-N_1^{(\ell)} &= xp_1'(x)+\delta_1(x),\\
\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}} &= xq_1'(x),
\end{aligned}$$

and from Lemma 4.12 we have

$$\left(\frac{N_0-N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} = \left(\frac{p_1'(x)+\frac{\delta_1(x)}{x}}{eq_1'(x)}\right)^{\frac{q_1(x)}{q_1'(x)}\left(p_1'(x)+\frac{\delta_1(x)}{x}\right)} < Ce^{-p_1(x)} \qquad (x\to+\infty).$$

The second and the third factors are estimated by $e^{-p_2(x)}$ and $e^{-p_3(x)}$, respectively. Hence we find that each summand appearing on the right-hand side of (4.67) is estimated by $e^{-(p_1+p_2+p_3)(x)} = e^{-(u_{p_2m_2}+u_{m_2\ell_2}+u_{\ell_2k})(x)}$. This is the right-hand side of (4.71). By carefully carrying out the proof one can see that this estimate deteriorates at most by terms of lower order as in Corollaries 4.4 and 4.7 if we take the "*L.O.T.*" terms in (4.67) into consideration.

Altogether we obtain the

Corollary 4.9 (Optimal expansions in small sectors). *Let the assumptions of Lemma 4.1 be satisfied: let* (4.2) *be a formal fundamental system of* [D] *with* $p=1$, *assume* (4.4) *holds, and let* $\hat{\rho}>0$, $\{I_\nu\}_{\nu\in\mathbb{Z}}$, *and* $\{f_k^{(\nu)}\}_{\nu\in\mathbb{Z}}$ *be as in Lemma 4.1. If*
(4.72)

$$
\begin{aligned}
N_0 &= \kappa_{p_2m_2}|\alpha_{p_2m_2}||z|^{-\kappa_{p_2m_2}} + \kappa_{m_2\ell_2}|\alpha_{m_2\ell_2}||z|^{-\kappa_{m_2\ell_2}} + \kappa_{\ell_2k}|\alpha_{\ell_2k}||z|^{-\kappa_{\ell_2k}} + O(1),\\
N_1^{(\ell)} &= \max\Big\{0,\, \kappa_{p_2m_2}|\alpha_{p_2m_2}||z|^{-\kappa_{p_2m_2}} + \kappa_{m_2\ell_2}|\alpha_{m_2\ell_2}||z|^{-\kappa_{m_2\ell_2}}+\\
&\qquad + \kappa_{\ell_2k}|\alpha_{\ell_2k}||z|^{-\kappa_{\ell_2k}} - \kappa_{\ell k}|\alpha_{\ell k}||z|^{-\kappa_{\ell k}}\Big\} + O(1),\\
N_2^{(m\ell)} &= \max\Big\{0,\, \kappa_{p_2m_2}|\alpha_{p_2m_2}||z|^{-\kappa_{p_2m_2}} + \kappa_{m_2\ell_2}|\alpha_{m_2\ell_2}||z|^{-\kappa_{m_2\ell_2}}+\\
&\qquad + \kappa_{\ell_2k}|\alpha_{\ell_2k}||z|^{-\kappa_{\ell_2k}} - \kappa_{\ell k}|\alpha_{\ell k}||z|^{-\kappa_{\ell k}} - \kappa_{m\ell}|\alpha_{m\ell}||z|^{-\kappa_{m\ell}}\Big\} + O(1)
\end{aligned}
$$

then the remainder $R_k^{(\nu)}(z,N_0,N_1,N_2)$ *in* (4.59) *satisfies*

$$
R_k^{(\nu)}(z,N_0,N_1,N_2) = e^{-|\alpha_{p_2m_2}||z|^{-\kappa_{p_2m_2}} - |\alpha_{m_2\ell_2}||z|^{-\kappa_{m_2\ell_2}} - |\alpha_{\ell_2k}||z|^{-\kappa_{\ell_2k}}} \cdot L.O.T.
$$

uniformly for $z\in\overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$ *for every* $\rho<\hat{\rho}$. *Here,* p_2, m_2, ℓ_2 *are defined in* (4.69), *and the term "L.O.T." denotes lower-order terms like in Corollaries 4.4 and 4.7.* □

These are the hyperasymptotic expansions at level two. We could now, in analogy to Sections 4.4 and 4.5, try to enlarge the region of validity of this expansion. Like at level one, this is not always possible, and we will not carry out the details since the discussion at level one (page 56) already shows all important issues. For a summary, see Section 4.9.

4.7 Calculation of Stokes' multipliers

The hyperasymptotic expansions of the previous sections allow to calculate the solutions of the differential equation (in principle, to arbitrary asymptotic order). However, for this calculation we need to know the Stokes multipliers which appear in these expansions. In this section, we will sketch a method for calculating these multipliers from the formal solutions using hyperasymptotic expansions for the coefficients f_{ks} of the asymptotic expansion (3.7),(4.5). This scheme is not new: first, to obtain information about the Stokes multipliers from the formal solutions, and then to use these in higher-level expansions of the solutions. We merely derive these expansions for completeness: to show what the expansions look like, and to see that exactly those Stokes multipliers can be obtained which are needed in the hyperasymptotic expansion. We will also see that – in contrast to the case $n = 2$ or to the single-leveled case – it is not possible in general to calculate *all* multipliers using hyperasymptotic expansions, even if using high levels. This is because of the phenomenon that higher-degree exponentials may be "hidden" behind lower-level exponentials *at every level*; cf. the discussions on pages 56 and 69.

One can obtain expansions for the coefficients f_{ks} by use of the integral representation (3.8) and application of the method of the previous sections, or by taking the limit in slightly modified formulations of (4.42) and (4.67). We will choose another way here using the identity

$$f_{kN_0} = z^{-N_0} R_k^{(\nu)}(z, N_0) - z^{-N_0} R_k^{(\nu)}(z, N_0 + 1) \tag{4.73}$$

and by direct use of our results for the remainders.

At level zero, (4.73) gives an estimate

$$\sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} c_{\ell k}^{(j)} \left(\frac{N_0}{e\kappa_{\ell k}|\alpha_{\ell k}|} \right)^{N_0/\kappa_{\ell k}} e^{O\left(N_0^{1-\frac{1}{\kappa_{\ell k}}}\right)} (1+N_0)^{\frac{-\operatorname{Re}\mu_{\ell k}+1}{\kappa_{\ell k}}} O(1)$$

which may be interesting but does not give a re-expansion yet. Thus, real hyperasymptotics begins at level one. Since we will need hyperterminants at the origin (cf. Section 1.6.2), we will need a convergence condition slightly stronger than (4.39) or (4.40), namely

$$\begin{gathered} 0 \le N_0, \\ 0 \le N_1 < N_0 + \operatorname{Re}\mu_k - \operatorname{Re}\mu^{(k)} \end{gathered}$$

or, for uniformity of the result,

$$\begin{gathered} 0 \le N_0, \\ 0 \le N_1 \le N_0 + \operatorname{Re}\mu_k - \operatorname{Re}\mu^{(k)} - \delta \end{gathered} \tag{4.74}$$

with some positive constant δ. Then we have the

Theorem 4.10. *Let* (4.2) *be a formal fundamental system of* [D] *with* $p = 1$, *assume* (4.4) *holds, and let* $k \in \{1, \ldots, n\}$. *Then the coefficients* f_{ks} *in* (3.7),(4.5) *satisfy*

$$f_{kN_0} = \sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} \frac{c_{\ell k}^{(j)}}{2\pi i} \sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s} F^{(1)} \left(0; \begin{array}{c} N_0 - s - \mu_{\ell k} + 1 \\ q_{\ell k} \\ -\theta_j \end{array} \right) + r_k(N_0, N_1) \tag{4.75}$$

with

(4.76)

$$r_k(N_0,N_1) =$$

$$= \sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu} c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} \left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{N_1^{(\ell)}}{\kappa_{m\ell}}} \left(\frac{N_0-N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} \times$$

$$\times e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} e^{O\left((N_0-N_1^{(\ell)})^{1-\frac{1}{\kappa_{\ell k}}}\right)} \left(1+N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{m\ell}+2}{\kappa_{m\ell}}} \left(1+N_0-N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{\ell k}+2}{\kappa_{\ell k}}} O(1)$$

uniformly w.r.to $N_0 \in \mathbb{N}$*,* $N_1 \in \mathbb{N}^{n-1}$ *satisfying* (4.74) *for every* $\delta > 0$*.*

The *Proof* is straightforward: Equation (4.41) can be rewritten as

$$R_k^{(\nu)}(z,N_0) = -\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j} \frac{c_{\ell k}^{(j)}}{2\pi i} \sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s} z^{N_0-1} F^{(1)}\left(z^{-1};\ \begin{matrix} N_0-s-\mu_{\ell k} \\ q_{\ell k} \\ -\theta_j \end{matrix}\right) + R_k^{(\nu)}(z,N_0,N_1). \tag{4.77}$$

Inserting this and the same for $R_k^{(\nu)}(z,N_0+1)$ into (4.73), we arrive at

$$f_{kN_0} = -\sum_{j=1}^{M}\sum_{(\ell,k)\in J_j} \frac{c_{\ell k}^{(j)}}{2\pi i} \sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s} \times$$

$$\times \left[z^{-1} F^{(1)}\left(z^{-1};\ \begin{matrix} N_0-s-\mu_{\ell k} \\ q_{\ell k} \\ -\theta_j \end{matrix}\right) - F^{(1)}\left(z^{-1};\ \begin{matrix} N_0-s-\mu_{\ell k}+1 \\ q_{\ell k} \\ -\theta_j \end{matrix}\right)\right]$$

$$+ \underbrace{z^{-N_0}\left[R_k^{(\nu)}(z,N_0,N_1) - R_k^{(\nu)}(z,N_0+1,N_1)\right]}_{=:\, r_k(N_0,N_1)}.$$

The term on the second line, due to (1.9), is equal to

$$-F^{(1)}\left(0;\ \begin{matrix} N_0-s-\mu_{\ell k}+1 \\ q_{\ell k} \\ -\theta_j \end{matrix}\right),$$

and the estimate (4.76) for $r_k(N_0,N_1)$ is an immediate consequence of Theorem 4.6. This completes the proof. □

Although Theorem 4.10 holds for every N_0, $\left(N_1^{(\ell)}\right)_{\ell\neq k}$, for a given N_0 the error term will be minimal if N_0 and $N_1^{(\ell)}$ are related via (4.54). Thus, to calculate all K Stokes' multipliers appearing in the level-1 expansion of $f_k^{(\nu)}$, say, choose a number N_0 which is large compared with K, and for $\ell \neq k$ determine the numbers $N_1^{(\ell)}$ according to (4.54) (if the maximum in (4.54) is zero, the number $N_1^{(\ell)}$ corresponding to that ℓ is chosen to be zero, too). Next we can compute the coefficients f_{ks} up to $s = N_0$ from the recurrence relations, and the values for $s = N_0, N_0 - 1, \ldots, N_0 - K + 1$ yield a $K \times K$-system of linear equations for the Stokes multipliers $c_{\ell k}^{(j)}$. Basically, this process is the same as in [Old98b] and in [HS99a]. Comparing the factors in (4.75) with those in (4.41) we can see that the Stokes multipliers can be computed to almost exactly the precision required for the expansions of the remainders. We will not go into details here.

By use of (4.77) and (4.59) as well as of the estimate in Theorem 4.8, the level-2 expansion of the coefficients is easily found to be

$$
\begin{aligned}
f_{kN_0} = &\sum_{j=1}^{M} \sum_{(\ell,k)\in J_j} \frac{c_{\ell k}^{(j)}}{2\pi i} \sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s} F^{(1)} \left(0; \begin{matrix} N_0 - s - \mu_{\ell k} + 1 \\ q_{\ell k} \\ -\theta_j \end{matrix} \right) \\
&- \sum_{j=1}^{M} \sum_{\mu=1}^{M} \sum_{(\ell,k)\in J_j} \sum_{(m,\ell)\in J_\mu} \frac{c_{\ell k}^{(j)}}{2\pi i} \frac{c_{m\ell}^{(\mu)}}{2\pi i} \times \\
&\times \sum_{s=0}^{N_2^{(m\ell)}-1} f_{ms} F^{(2)} \left(0; \begin{matrix} N_0 - N_1^{(\ell)} - \mu_{\ell k} + 2, & N_1^{(\ell)} - s - \mu_{m\ell} \\ q_{\ell k} & q_{m\ell} \\ -\theta_j & -\theta_\mu \end{matrix} \right) \\
&+ r_k(N_0, N_1, N_2)
\end{aligned}
\tag{4.78}
$$

where

$$
\begin{aligned}
r_k(N_0, N_1, N_2) = &\sum_{j=1}^{M} \sum_{\mu=1}^{M} \sum_{h=1}^{M} \sum_{(\ell,k)\in J_j} \sum_{(m,\ell)\in J_\mu} \sum_{(p,m)\in J_h} c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} c_{pm}^{(h)} \times \\
&\times \left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|} \right)^{\frac{N_0 - N_1^{(\ell)}}{\kappa_{\ell k}}} \left(\frac{N_1^{(\ell)} - N_2^{(m\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|} \right)^{\frac{N_1^{(\ell)} - N_2^{(m\ell)}}{\kappa_{m\ell}}} \left(\frac{N_2^{(m\ell)}}{e\kappa_{pm}|\alpha_{pm}|} \right)^{\frac{N_2^{(m\ell)}}{\kappa_{pm}}}
\end{aligned}
$$

times lower-order terms. The multipliers from the level-1 expansion being already known, the rest of those appearing in (4.78) can be obtained from the level-2 expansion again by a system of linear equation. This procedure can be continued to arbitrary levels.

This way, we are able to calculate all Stokes' multipliers for which the corresponding number $N_1^{(\ell)}$ or $N_2^{(m\ell)}$ etc. *is not zero*.[10] Now, have a look at (4.54) and (4.72). The pattern for the following levels should be clear now, and the optimal number N_0 increases from level to level. If $q_{\ell k}$ is not of greater degree then there will be a level

[10] These are exactly the multipliers which appear in the hyperasymptotic expansion for the remainders at the corresponding level.

where the difference $N_1^{(\ell)}$ becomes *positive* and hence the Stokes multiplier $c_{\ell k}^{(j)}$ enters the expansion. If this is the case then this multiplier can be calculated. This does not necessarily occur for all multipliers, e.g. in the multi-leveled case the polynomial defining N_0 might of degree h_j at all levels while $q_{\ell k}$ is of greater degree for some ℓ. However, *in the single-leveled case for each single multiplier the above condition is finally satisfied and thus we can determine all Stokes' multipliers using hyperasymptotic expansions of high enough level.*

4.8 Completion of the proofs

4.8.1 Two auxiliary lemmas

Lemma 4.11. *Let $\alpha > 0$ and $\mu \in \mathbb{R}$ be arbitrarily given. Then*

$$\sum_{k=1}^{n} k^{\mu}\left(\frac{k}{\alpha}\right)^{k} = O\left(n^{\mu}\left(\frac{n}{\alpha}\right)^{n}\right) \qquad (n \in \mathbb{N}).$$

Proof. There are various well-known methods of Asymptotic Analysis to prove that in fact, for large enough n, the last summand dominates the sum of all others. E.g. it is known or readily verified that

$$\sum_{k=1}^{n} \left(\frac{k}{\alpha}\right)^{k} = O\left(\left(\frac{n}{\alpha}\right)^{n}\right) \qquad (n \in \mathbb{N}).$$

Using partial summation we then find

$$\sum_{k=1}^{n} k^{\mu}\left(\frac{k}{\alpha}\right)^{k} = \underbrace{n^{\mu} \underbrace{\sum_{k=1}^{n}\left(\frac{k}{\alpha}\right)^{k}}_{O\left(\left(\frac{n}{\alpha}\right)^{n}\right)}}_{O\left(n^{\mu}\left(\frac{n}{\alpha}\right)^{n}\right)} - \underbrace{\sum_{k=1}^{n} \underbrace{\left[k^{\mu}-(k-1)^{\mu}\right]}_{\substack{O((k-1)^{\mu-1})\\ \cap \\ O(n^{\mu-1})}} \underbrace{\sum_{\ell=1}^{k-1}\left(\frac{\ell}{\alpha}\right)^{\ell}}_{\substack{O\left(\left(\frac{k}{\alpha}\right)^{k}\right)\\ \cap \\ O\left(\left(\frac{n}{\alpha}\right)^{n}\right)}}}_{\subset n \cdot O(n^{\mu-1}) \cdot O\left(\left(\frac{n}{\alpha}\right)^{n}\right)} = O\left(n^{\mu}\left(\frac{n}{\alpha}\right)^{n}\right).$$

□

Lemma 4.12. *Let $p(x), q(x) \in x\mathbb{R}[x]$ be real polynomials with $0 < p(x) \leq q(x)$ $(x \to +\infty)$. Then*

(i)
$$\left(\frac{p'(x)}{eq'(x)}\right)^{\frac{q(x)}{q'(x)}p'(x)} \leq e^{-p(x)} \qquad (x \to +\infty);$$

(ii) *for every $\delta(x) = O(1)$ $(x \to +\infty)$ there is a constant $C > 0$ only depending on the Big-O constant of $\delta(x)$ such that*
$$\left(\frac{p'(x) + \frac{\delta(x)}{x}}{eq'(x)}\right)^{\frac{q(x)}{q'(x)}\left(p'(x)+\frac{\delta(x)}{x}\right)} < Ce^{-p(x)} \qquad (x \to +\infty).$$

Proof. Let $p(x) = \sum_{\mu=1}^{\ell} p_\mu x^\mu$, $p_\ell \neq 0$ and $q(x) = \sum_{\nu=1}^{m} q_\nu x^\nu$, $q_m \neq 0$, and $\delta(x) = O(1)$ $(x \to +\infty)$ be given. We consider four cases:

Case 1: Let $p(x)/q(x) \to \alpha \in (0,1)$ $(x \to +\infty)$. Then we have $\ell = m$ and
$$\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} \to \frac{p_m}{q_m} = \alpha \qquad (x \to +\infty).$$

In view of $\alpha(1 - \log\alpha) > \alpha$ we conclude
$$\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)}\left(1 - \log\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)}\right) > \frac{p(x)}{q(x)} \qquad (x \to +\infty) \tag{4.79}$$

and hence (ii) with $C = 1$ as well as, if taking $\delta(x) = 0$, (i).

(Actually, we can do even a little better in Case 1: namely, inequality (4.79) remains true if we add some small $\varepsilon > 0$ to the right-hand side. Hence, in this case Lemma 4.12 (ii) holds with $p(x)$ replaced by $p(x) + \varepsilon q(x)$ on the right-hand side.)

Case 2: If $p(x)/q(x) \to 0$ $(x \to +\infty)$ then we have $\ell < m$ and
$$\frac{p(x)}{q(x)}\bigg/\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} = \frac{xq'(x)}{q(x)} \cdot \frac{p(x)}{xp'(x) + \delta(x)} \to \frac{m}{\ell} \tag{4.80}$$

for $x \to +\infty$. On the other hand,
$$1 - \log\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} \to +\infty \qquad (x \to +\infty),$$

so that we find
$$1 - \log\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} > \frac{p(x)}{q(x)}\bigg/\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} \qquad (x \to +\infty)$$

and hence the same result as in Case 1.

(The statement of the lemma can be considerably strengthened in Case 2: if, in (4.80), we replace $p(x)$ by any polynomial $\tilde{p}(x)$ of degree $\leq \ell$ and leave all other terms – including $p'(x)$ – unchanged, then the whole term still converges to some finite real number. Thus Lemma 4.12 (ii) holds with $p(x)$ replaced by $\tilde{p}(x)$ on the right-hand side.)

Case 3: Here we suppose $p(x)/q(x) \to 1$ $(x \to +\infty)$ but $p(x) \not\equiv q(x)$. Put $h(x) := q(x) - p(x) = \sum_{\nu=1}^{n} h_\nu x^\nu$ with $n < m$. We then have on one hand

$$\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} = 1 - \frac{xh'(x) - \delta(x)}{xq'(x)} = 1 - \frac{nh_n}{mq_m} x^{n-m} + O\big(x^{n-m-1}\big),$$

$$\Rightarrow \quad \frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} \left(1 - \log \frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)}\right) = 1 + O\big(x^{n-m-1}\big)$$

as $x \to +\infty$ and on the other hand

$$\frac{p(x)}{q(x)} = 1 - \frac{h(x)}{q(x)} = 1 - \frac{h_n}{q_m} x^{n-m} + O\big(x^{n-m-1}\big) \qquad (x \to +\infty). \tag{4.81}$$

Thus the result from Case 1 follows here, too.

(We can do a little better in Case 3, too: if we replace, in (4.81), $p(x)$ by $p(x) + \varepsilon h(x)$ with some small enough $\varepsilon > 0$ then the $\frac{1}{q(x)}$-multiple of this term is still of the form $1 - \tilde{h} x^{n-m} + O\big(x^{n-m-1}\big)$ and thus less than the left-hand side of (4.79) as $x \to +\infty$. As a consequence, Lemma 4.12 (ii) remains true with $p(x)$ replaced by $p(x) + \varepsilon h(x)$ on the right-hand side.)

Case 4: If $p(x) = q(x)$ then nothing has to be shown for (i). This is the only case where the equality sign in (i) is actually needed. Concerning (ii), we see that

$$\frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} = 1 + \frac{\delta(x)}{xq'(x)} = 1 + O\big(x^{-m}\big),$$

$$\Rightarrow \quad \frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} \left(1 - \log \frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)}\right) = 1 + O\big(x^{-m}\big) \qquad (x \to +\infty).$$

Thus there is always a real (possibly negative) $\tilde{C}$ such that

$$\frac{p(x) + \tilde{C}}{q(x)} = 1 + \frac{\tilde{C}}{q(x)} < \frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)} \left(1 - \log \frac{p'(x) + \frac{\delta(x)}{x}}{q'(x)}\right) \quad (x \to +\infty).$$

Taking the exponential on both sides we obtain (ii) with $C := e^{-\tilde{C}}$. Observe that C can be greater than 1. However, this is the only case where $C > 1$ might actually be needed. □

4.8.2 Proof of Theorem 4.6

Lemma 4.13. *In the context and with the notations of Theorem 4.6, the term* $S_k^{(\nu)}(z,N_0,N_1)$ *in* (4.47) *satisfies*

$$
\begin{aligned}
S_k^{(\nu)}(z,N_0,N_1) &= \\
&= \sum_{j=1}^{M}\sum_{\mu=1}^{M}\sum_{(\ell,k)\in J_j}\sum_{(m,\ell)\in J_\mu} c_{\ell k}^{(j)} c_{m\ell}^{(\mu)} \left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{\frac{N_1^{(\ell)}}{\kappa_{m\ell}}} \rho_0^{-(N_0-N_1^{(\ell)})} \quad e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} \quad \left(1+N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{m\ell}+1}{\kappa_{m\ell}}} \quad O(1) \\
&+ \sum_{j=1}^{M}\sum_{(\ell,k)\in J_j} c_{\ell k}^{(j)} \quad \rho_1^{-N_1^{(\ell)}} \quad \left(\frac{N_0-N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} \quad e^{O\left((N_0-N_1^{(\ell)})^{1-\frac{1}{\kappa_{\ell k}}}\right)} \quad \left(1+N_0-N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{\ell k}+1}{\kappa_{\ell k}}} O(1) \\
&+ \quad \rho_0^{-N_1^{(\ell)}} \quad \rho_0^{-(N_0-N_1^{(\ell)})} \quad O(1).
\end{aligned}
$$

Proof. We will estimate all three summands of the right-hand side of (4.47) separately.

- To estimate the first term, consider the expansion coefficients $f_{\ell s}$. Theorem 4.3 gives an estimate for these coefficients, too, as follows: from

$$
z^{-s}R_\ell^{(\nu)}(z,s) = \sum_{\mu=1}^{M}\sum_{(m,\ell)\in J_\mu} c_{m\ell}^{(\mu)}\left(\frac{s}{e\kappa_{m\ell}|\alpha_{m\ell}|}\right)^{s/\kappa_{m\ell}} e^{O\left(s^{1-\frac{1}{\kappa_{m\ell}}}\right)}(1+s)^{\frac{-\operatorname{Re}\mu_{m\ell}+1}{\kappa_{m\ell}}} O(1) \tag{4.82}
$$

uniformly for $s\in\mathbb{N}$ and $z\in\overline{S}_\rho(\theta_\nu,\theta_{\nu+1})$, by taking the limit as $z\to 0$ we see that $f_{\ell s}$ satisfies exactly the same estimate. In addition, since both $q_{\ell k}(\frac{1}{w})$ and $\frac{w}{w-z}$ are bounded uniformly w.r.to $z\in\overline{K}_\rho(0)$ and $|w|\geq\rho_0$, we have

$$
\begin{aligned}
\int_{T_0^{(j)}}^{[\theta_j]} \frac{e^{q_{\ell k}(\frac{1}{w})}w^{-N_0+s+\mu_{\ell k}}}{w-z}\,dw &= \int_{|T_0^{(j)}|}^{\infty} O(1)|w|^{-N_0+s+\operatorname{Re}\mu_{\ell k}-1}O(1)\,d|w| = \int_{\rho_0}^{\infty} t^{-N_0+s+\operatorname{Re}\mu_{\ell k}-1}\,dt\cdot O(1) \\
&= \frac{1}{N_0-s-\operatorname{Re}\mu_{\ell k}}\left(\rho_0^{-N_0+s+\operatorname{Re}\mu_{\ell k}}-0\right)O(1)\subset\rho_0^{-N_0+s+\operatorname{Re}\mu_{\ell k}}O(1)\subset O\left(\rho_0^{-N_0+s}\right)
\end{aligned}
$$

which is again uniform due to (4.40)

Hence, with both above estimates we find

$$
\begin{aligned}
f_{\ell s} \int_{T_0^{(j)}}^{[\theta_j]} & \frac{e^{q_{\ell k}(\frac{1}{w})} w^{-N_0+s+\mu_{\ell k}}}{w-z}\, dw = \\
&= \rho_0^{-N_0} \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} c_{m\ell}^{(\mu)} \left(\frac{s\rho_0^{\kappa_{m\ell}}}{e\kappa_{m\ell}|\alpha_{m\ell}|} \right)^{\frac{s}{\kappa_{m\ell}}} e^{O\left(s^{1-\frac{1}{\kappa_{m\ell}}}\right)} (1+s)^{\frac{-\operatorname{Re}\mu_{m\ell}+1}{\kappa_{m\ell}}} O(1) \\
&= \rho_0^{-N_0} \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} c_{m\ell}^{(\mu)} e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} O(1) \times \\
&\qquad \times \left[\left(\frac{s\rho_0^{\kappa_{m\ell}}}{e\kappa_{m\ell}|\alpha_{m\ell}|} \right)^{\frac{s}{\kappa_{m\ell}}} (1+s)^{\frac{-\operatorname{Re}\mu_{m\ell}+1}{\kappa_{m\ell}}} \right]
\end{aligned}
$$

and, by Lemma 4.11,

$$
\begin{aligned}
\sum_{s=0}^{N_1^{(\ell)}-1} f_{\ell s} \int_{T_0^{(j)}}^{[\theta_j]} & \frac{e^{q_{\ell k}(\frac{1}{w})} w^{-N_0+s+\mu_{\ell k}}}{w-z}\, dw = \\
&= \rho_0^{-N_0} \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} c_{m\ell}^{(\mu)} e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} O(1) \times \\
&\qquad \times \left[\left(\frac{N_1^{(\ell)}\rho_0^{\kappa_{m\ell}}}{e\kappa_{m\ell}|\alpha_{m\ell}|} \right)^{N_1^{(\ell)}/\kappa_{m\ell}} \left(1+N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{m\ell}+1}{\kappa_{m\ell}}} \right] \\
&= \sum_{\mu=1}^{M} \sum_{(m,\ell)\in J_\mu} c_{m\ell}^{(\mu)} \left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|} \right)^{\frac{N_1^{(\ell)}}{\kappa_{m\ell}}} \rho_0^{-N_0+N_1^{(\ell)}} e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} \times \\
&\qquad \times \left(1+N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{m\ell}+1}{\kappa_{m\ell}}} O(1).
\end{aligned}
$$

Taking the sum over all $(\ell,k) \in J_j$ for $j = 1,\ldots,M$, the first term of the right-hand side of (4.47) is estimated by the first line of the lemma.

- Now consider the second term of (4.47). With our choice of the points $\{T_1^{(\nu)}\}_{\nu=1}^{M}$ (cf. page 49) we have

$$
\varepsilon_\ell(w, N_1^{(\ell)}) = \sum_{\mu=1}^{M} \frac{1}{2\pi i} \int_{\overset{\frown}{T_1^{(\mu)} T_1^{(\mu+1)}}} t^{-N_1^{(\ell)}} \frac{f_\ell^{(\mu)}(t)}{t-w}\, dt = \rho_1^{-N_1^{(\ell)}} \cdot O(1)
$$

uniformly for $w \in \overline{K}_{\rho_{01}}(0)$ (and is an analytic function of w there) and $N_1^{(\ell)} \in \mathbb{N}$.

Hence each of the integrals in the second line of (4.47) can be estimated using Theorem 5.5. This is identical to (4.22) except that N_0 is to be replaced by $N_0 - N_1^{(\ell)}$ and $f_\ell^{(j-1)}(w)$ by $\varepsilon_\ell(w, N_1^{(\ell)})$:

$$\int\limits_{\overline{OT_0^{(j)}}} \frac{e^{q_{\ell k}(\frac{1}{w})} w^{-N_0+N_1^{(\ell)}+\mu_{\ell k}}}{w-z} \varepsilon_\ell(w, N_1^{(\ell)})\, dw =$$

$$= \rho_1^{-N_1^{(\ell)}} \cdot \left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} e^{O\left((N_0-N_1^{(\ell)})^{1-\frac{1}{\kappa_{\ell k}}}\right)} \left(1+N_0-N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{\ell k}+1}{\kappa_{\ell k}}} O(1)$$

uniformly for $z \in \overline{K}_\rho(0)$, $N_1^{(\ell)} \in \mathbb{N}$ and (large enough) $N_0 - N_1^{(\ell)}$. The restriction upon $N_0 - N_1^{(\ell)}$ can now be removed by directly considering the above integral for small (bounded) $N_0 - N_1^{(\ell)}$. In case $N_1^{(\ell)} > N_0$ the footnote on page 48 applies.

Taking the sum over all $(\ell, k) \in J_j$ for $j = 1, \ldots, M$, the second term of the right-hand side of (4.47) is estimated by the second line of the lemma.

- The estimate of the remaining term $\varepsilon_k(z, N_0)$ is (4.23). This completes the proof. $\square$

4.8.3 Proof of Corollary 4.7

The case $p_1 = 0$. In view of the necessary condition $p(x) > 0$ $(x \to +\infty)$ in Lemma 4.12, the exceptional case $p_1 = 0$ has to be considered separately (note that $N_1^{(\ell)} = \delta_1^{(\ell)}$ is bounded here):

If $\deg p_2 < \deg q_2$ then Case 2 of the proof of Lemma 4.12 applies, and according to the remark at the end of the proof of Case 2, for every $C > 0$ we have

$$\begin{aligned} &\left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} < e^{-p_2(x) - C\cdot p_2(x)} \qquad (x \to +\infty) \\ \Rightarrow\quad &\left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} e^{O\left((N_0-N_1^{(\ell)})^{1-\frac{1}{\kappa_{\ell k}}}\right)} \subseteq e^{-p_2(x)} x^{\tilde{\alpha}} O(1) \end{aligned}$$

for arbitrary $\tilde{\alpha} \in \mathbb{R}$.

If both p_2 and q_2 are of the same degree without being equal then they must differ in the leading coefficient, hence Case 1 of the proof of Lemma 4.12 applies. With some small $\varepsilon > 0$ we obtain

$$\left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} < e^{-p_2(x)-\varepsilon p_2(x)} \qquad (x \to +\infty)$$
$$\Rightarrow \quad \left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} \subseteq e^{-p_2(x)} x^{\tilde{\alpha}} O(1)$$

for arbitrary $\tilde{\alpha} \in \mathbb{R}$.

Thus in both cases we see that

$$\begin{aligned} x^{-N_1^{(\ell)}} &\left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} e^{O\left((N_0-N_1^{(\ell)})^{1-\frac{1}{\kappa_{\ell k}}}\right)} \left(1+N_0-N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{\ell k}+2}{\kappa_{\ell k}}} O(1) \\ &\subseteq e^{-p_2(x)+O\left(p_2'(x)\right)} x^{\alpha} O(1) \\ &= e^{-p_1(x)-p_2(x)+O\left(p_1'(x)+p_2'(x)\right)} x^{\alpha} O(1) \\ &= e^{-|\alpha_{m_1\ell_1}|x^{\kappa_{m_1\ell_1}}-|\alpha_{\ell_1 k}|x^{\kappa_{\ell_1 k}}} e^{O\left(x^{\kappa_{m_1\ell_1}-1}+x^{\kappa_{\ell_1 k}-1}\right)} x^{\alpha} O(1) \end{aligned}$$

for any $\alpha \in \mathbb{R}$. With the choice $\alpha := 4 - \tilde{\tilde{\mu}}_1$, Corollary 4.7 follows in this case.

To accomplish the considerations for $p_1 = 0$ remains the case $p_1 = 0 \wedge p_2 = q_2$ which means $(\ell,k) \in \tilde{U}_0^{(1)}$. Since the corresponding Case 4 of the proof of Lemma 4.12 is sharp we only find

$$\left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} = O\left(e^{-p_2(x)}\right)$$
$$\begin{aligned} \Rightarrow \quad x^{-N_1^{(\ell)}} &\left(\frac{N_0 - N_1^{(\ell)}}{e\kappa_{\ell k}|\alpha_{\ell k}|x^{\kappa_{\ell k}}}\right)^{\frac{N_0-N_1^{(\ell)}}{\kappa_{\ell k}}} e^{O\left((N_0-N_1^{(\ell)})^{1-\frac{1}{\kappa_{\ell k}}}\right)} \left(1+N_0-N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{\ell k}+2}{\kappa_{\ell k}}} O(1) \\ &\subseteq e^{-p_2(x)+O\left(p_2'(x)\right)} x^{-N_1^{(\ell)}-\operatorname{Re}\mu_{\ell k}+2} O(1) \\ &\subseteq e^{-|\alpha_{m_1\ell_1}|x^{\kappa_{m_1\ell_1}}-|\alpha_{\ell_1 k}|x^{\kappa_{\ell_1 k}}} e^{O\left(x^{\kappa_{m_1\ell_1}-1}+x^{\kappa_{\ell_1 k}-1}\right)} x^{-\operatorname{Re}\mu_{\ell k}+2} O(1) \end{aligned}$$

Hence, according to the definition of $\tilde{\tilde{\mu}}_1$ in (4.53), Corollary 4.7 holds in this case, too.

The case $p_1 > 0$. Here we can apply Lemma 4.12 to each of the factors of $\lambda_{m\ell}(x,N_0,N_1)$ in (4.52). The case differentiation and the estimates are analogous to those of the case $p_1 = 0$. We begin by the first factor.

If $\deg p_1 < \deg q_1$ then Case 2 of the proof of Lemma 4.12 applies, and according to the remark at the end of the proof of Case 2, for every $C > 0$ we have

$$\begin{aligned} &\left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} < e^{-p_1(x)-C\cdot p_1(x)} \qquad (x \to +\infty) \\ \Rightarrow \quad &\left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} \subseteq e^{-p_1(x)} x^{\tilde{\alpha}} O(1) \end{aligned}$$

for arbitrary $\tilde{\alpha} \in \mathbb{R}$.

If both p_1 and q_1 are of the same degree but differ in the leading coefficient, or by lower-order terms, then Case 1 or Case 3 of the proof of Lemma 4.12 can be applied respectively. In both cases we obtain with some small $\varepsilon > 0$

$$\begin{aligned} &\left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} < e^{-p_1(x)-\varepsilon\left(q_1(x)-p_1(x)\right)} \qquad (x \to +\infty) \\ \Rightarrow \quad &\left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} \subseteq e^{-p_1(x)} x^{\tilde{\alpha}} O(1) \end{aligned}$$

for arbitrary $\tilde{\alpha} \in \mathbb{R}$.

Thus in all these three cases we see that

$$\begin{aligned} &\left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} \left(1+N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{m\ell}+2}{\kappa_{m\ell}}} O(1) \\ &\subseteq e^{-p_1(x)+O\left(p_1'(x)\right)} x^{\alpha} O(1) \end{aligned} \tag{4.83}$$

for any $\alpha \in \mathbb{R}$.

The last case is $p_1 = q_1$. Since the corresponding Case 4 of the proof of Lemma 4.12 is sharp we only find

$$\begin{aligned} &\left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} = O\left(e^{-p_1(x)}\right) \\ \Rightarrow \quad &\left(\frac{N_1^{(\ell)}}{e\kappa_{m\ell}|\alpha_{m\ell}|x^{\kappa_{m\ell}}}\right)^{N_1^{(\ell)}/\kappa_{m\ell}} e^{O\left((N_1^{(\ell)})^{1-\frac{1}{\kappa_{m\ell}}}\right)} \left(1+N_1^{(\ell)}\right)^{\frac{-\operatorname{Re}\mu_{m\ell}+2}{\kappa_{m\ell}}} O(1) \\ &\subseteq e^{-p_1(x)+O\left(p_1'(x)\right)} x^{-\operatorname{Re}\mu_{m\ell}+2} O(1). \end{aligned}$$

This is the same estimate as (4.83) except for the power of x.

The second factor of $\lambda_{m\ell}(x, N_0, N_1)$ in (4.52) is estimated the same way. Either of both estimates allows to choose an arbitrary power of x on the right-hand side unless the last case applies to *both* factors. This means $p_1 = q_1 \wedge p_2 = q_2$ which is equivalent to $(m, \ell, k) \in U_0^{(1)}$. In that case for the corresponding summand in (4.42) we obtain the estimate

$$e^{-p_1(x)-p_2(x)+O\left(p_1'(x)\right)+O\left(p_2'(x)\right)} x^{-\operatorname{Re}\mu_{mk}+4} O(1).$$

Observing that for nonnegative functions f,g we have $O(f)+O(g) \subseteq O\big(\max\{f,g\}\big) \subseteq O(f+g)$, the last estimate reads

$$
\begin{aligned}
&e^{-p_1(x)-p_2(x)+O\left(p_1'(x)+p_2'(x)\right)}x^{-\operatorname{Re}\mu_{mk}+4}O(1) \\
&\qquad = e^{-|\alpha_{m_1\ell_1}||x^{\kappa_{m_1\ell_1}}|-|\alpha_{\ell_1 k}||x^{\kappa_{\ell_1 k}}|}e^{O\left(x^{\kappa_{m_1\ell_1}-1}+x^{\kappa_{\ell_1 k}-1}\right)}x^{-\tilde{\mu}_1+4}O(1) \\
&\qquad \subseteq e^{-|\alpha_{m_1\ell_1}||x^{\kappa_{m_1\ell_1}}|-|\alpha_{\ell_1 k}||x^{\kappa_{\ell_1 k}}|}e^{O\left(x^{\kappa_{m_1\ell_1}-1}+x^{\kappa_{\ell_1 k}-1}\right)}x^{-\tilde{\tilde{\mu}}_1+4}O(1),
\end{aligned}
$$

according to (4.51) and (4.53). This is the desired estimate. In case $(m,\ell,k) \notin U_0^{(1)}$, the same estimate holds due to the freedom in the choice of the power as discussed above. Corollary 4.7 follows. □

4.9 Conclusions

In Sections 4.4 to 4.6 we have derived hyperasymptotic expansions for solutions of general linear differential equations at irregular singular points of arbitrary rank. The expansions are in terms of generalized hyperterminants (see Section 1.6.2) and can be viewed as generalizations of the ones derived in [OO95a, MW97, Old98b].

This is done by truncation and successive re-expansion of the remainder at each level. The Poincaré expansion truncated after N_0 terms, we add sums of $N_1^{(\ell)}$ summands with level-1 hyperterminants (with in general different $N_1^{(\ell)}$ for different ℓ), then sums of $N_2^{(m\ell)}$ level-2 hyperterminants, then $N_3^{(pm\ell)}$ level-3 hyperterminants, and so forth. The number of level-j hyperterminants taken is always less than or equal to the number of level-$(j-1)$ hyperterminants in the same expansion. This is due to the convergence condition (1.7) for these integrals. However, the number of level-j hyperterminants may vary from level to level. The expansion coefficients in the above expansions are the ones from the original (Poincaré) expansion of the respective solutions, and are prepended by a product of Stokes' multipliers.

Moreover, hyperasymptotic "scattering" occurs, that is, at successive levels the expansion comprises terms for solutions y_m which are related via (non-zero) Stokes' multipliers $c_{m\ell}^{(\mu)}$ to the terms in the preceding level. The remainder is estimated by a term which essentially is a sum of products of Gamma terms like in (4.67), one for each difference $N_0 - N_1^{(\ell)}$, $N_1^{(\ell)} - N_2^{(m\ell)}$, $N_2^{(m\ell)} - N_3^{(pm\ell)}$, and so forth.

At each level, there is limited exponential improvement. The maximal improvement at level r is given by

$$
\underbrace{|\alpha_{\ell_r k}||z|^{-\kappa_{\ell_r k}} + |\alpha_{m_r\ell_r}||z|^{-\kappa_{m_r\ell_r}} + \cdots + |\alpha_{q_r p_r}||z|^{-\kappa_{q_r p_r}}}_{r \text{ summands}}, \tag{4.84}
$$

where the tuple $(q_r, p_r, \ldots, \ell_r, k)$ is chosen from a set $U_0^{(r)}$ which, in analogy to (4.50) and (4.68), minimizes all the above sums. By appropriate choices of the number of terms, it is possible to achieve this optimal exponential improvement. The optimal truncation points, up to a $O(1)$ term, are given by

$$\kappa_{\ell_r k}|\alpha_{\ell_r k}||z|^{-\kappa_{\ell_r k}} + \kappa_{m_r \ell_r}|\alpha_{m_r \ell_r}||z|^{-\kappa_{m_r \ell_r}} + \cdots + \kappa_{q_r p_r}|\alpha_{q_r p_r}||z|^{-\kappa_{q_r p_r}}$$

for N_0, this is diminished for $\ell \neq k$ by $\kappa_{\ell k}|\alpha_{\ell k}||z|^{-\kappa_{\ell k}}$ to obtain $N_1^{(\ell)}$, and so forth, until we get a negative number from where on we replace all subsequent numbers of terms by 0. This means that in general not all Stokes' multipliers will appear in the expansion, even at high levels. However, in the *single-leveled case* (that is, if the formal solution is h-summable for some positive h, or if all polynomials $q_{\ell k}$ for all ℓ and all k are of the same degree h) all above polynomials are monomials of the same degree, and only the $|\alpha_{\ell k}|$ need to be considered. Since the number N_0 above increases from level to level, in the single-leveled case there will always be a level where this number will be major to $\kappa_{\ell k}|\alpha_{\ell k}||z|^{-\kappa_{\ell k}}$, and hence the corresponding Stokes multiplier appears at this level.

This optimization procedure is different from that of Berry and Howls [BH90, BH91] adopted in [Old92, Old93] where the authors optimized the number N_0, then, at level one, optimized the numbers $N_1^{(\ell)}$ keeping N_0 fixed, and so forth, and where only limited final exponential improvement could be achieved. Following an idea of Olde Daalhuis and Olver [OO95a], we allow the number of terms taken at some level *to change in subsequent levels*. Since the minimum in (4.84) increases from level to level there is – in theory – there is no limit in the attainable exponential improvement.

The optimally truncated expansions are uniformly valid in a sector between two singular directions. Although this might be sufficient for practical purposes, expansions which are valid in large sectors are of particular interest since they give a smooth interpretation of Stokes' phenomenon. Our optimally-truncated hyperseries do not in all cases admit this feature. Basically, this is the same problem as with the Stokes multipliers: if the numbers $N_1^{(\ell)}$ are positive for all ℓ at some level then the sector of validity is enlarged to contain Stokes' directions corresponding to the singular directions which limited the original sector. In the general multileveled case, this is not necessarily true; however, in the single-leveled case it is, and hence higher-level hyperasymptotic expansions are valid in sectors of increasing opening which contain Stokes' lines.

Finally, we have given a method to calculate the "important" Stokes multipliers from the coefficients of the formal solutions. These are exactly the same multipliers which appear in the expansion for the solutions. Once again, this is not always the complete set of Stokes' multipliers, but in the single-leveled case we are able to calculate all of them. Thus, with regarding hyperasymptotics, *the single-leveled case behaves exactly as the case of rank one and shows all the same nice properties. The multi-leveled case, in contrast, does not.*

5 Uniform asymptotic analysis of a certain Cauchy-Heine integral

5.1 Statement of the problem

In this section we will obtain a uniform asymptotic estimate of a certain Cauchy-Heine integral. The result serves as a tool for Section 4 but could be of some interest of its own.

Throughout this section let r be a positive integer, $p(x) = \sum_{\ell=1}^{r} p_\ell x^\ell \in x\mathbb{C}[x]$ a polynomial of degree r. Fix a determination of the r-th root $\alpha := (-rp_r)^{1/r}$. Let Σ be a closed sector with vertex 0, radius $\rho(\Sigma) > 0$, opening $0 < 2\delta(\Sigma) < \frac{\pi}{r}$ and bisecting direction 0; and let $S = S_{\hat{\rho}}(d_1, d_2)$ be an open sector with vertex 0, radius $\hat{\rho} > 0$ and angles $d_1 < d_2$. Let $T \in \mathbb{C}$ be fixed, and assume $T \in \alpha\Sigma \subset S$. This implies that we have $|T| =: \rho_0 < \hat{\rho}$. Let $0 < \rho < \rho_0$, and let $F :\ S \to \mathbb{C}$ be an analytic function bounded on every closed subsector of S. Then for $\varepsilon \in \Sigma$ and $z \in \overline{K}_\rho(0)$ we consider the integral

$$\mathscr{I}(z,\varepsilon) := \int\limits_{\mathfrak{W}} e^{p(w^{-1})} w^{-\varepsilon^{-r}} \frac{wF(w)}{w-z}\, dw. \tag{5.1}$$

The integration contour $\mathfrak{W}$ is a line segment from 0 to T, possibly indented by a small circular arc around z in the case $0 \neq z \in \overline{0T}$. Note that (even in the case $z = 0$) integrability is assured; see below.

This integral has the structure of a Cauchy-Heine integral in view of the denominator $w - z$ and of the contour of integration. But on the other hand, it can also be considered as a Laplace-type integral in view of the exponential term. Such integrals, in one form or another, have been the object of steady research interest as they occur e.g. as Cauchy-Heine transforms of solutions of linear meromorphic differential equations [Tem75, Tem79, Imm90, Olv91a, Olv91b, Olv93, Hoe94, OO94, OO95a, OO95b, HS99a].

First of all, since the function F is bounded on some closed sector $\alpha\Sigma' \subset S$, $\Sigma' \Supset \Sigma$, we split up the the remaining integrand: the term $\frac{w}{w-z}$ is responsible for a pole at $w = z$ (z may or may not lie inside S), and the behavior near $w = 0$ is essentially determined by $e^{\varphi(\varepsilon,w)}$ where

$$\varphi(\varepsilon, w) := p(w^{-1}) - \varepsilon^{-r} \log w.$$

Second, the essential singularity of the integrand at $w = 0$ induces different behavior on different rays when approaching the origin; expressing this behavior (for fixed ε):

$$\operatorname{Re}\varphi(\varepsilon, w) = -|p_r||w|^{-r}\cos(r \arg \tfrac{w}{\alpha}) + O(|w|^{-r+1}\log|w|), \tag{5.2}$$

we see from $\arg\frac{T}{\alpha} < \frac{\pi}{2r}$ that the above cosine is positive on the line $\overline{0T}$ and hence the integral indeed converges.

To obtain an asymptotic estimate of $\mathscr{I}(z,\varepsilon)$ we will use a modification of the saddle-point method. Let $\varepsilon \neq 0$ be arbitrarily given. A saddle point t of the function $\operatorname{Re}\varphi(\varepsilon,\cdot)$ satisfies the equation

$$\partial_2\varphi(\varepsilon,t) = 0$$

which is equivalent to a polynomial equation of order r for t^{-1} and which consequently possesses r solutions in the complex plane. We are especially interested in one of them (cf. Figure 2) which is well-suited for the deformation of the integration contour and which is given by

$$t =: t(\varepsilon) = \alpha\varepsilon\big(1+\varepsilon\gamma(\varepsilon)\big) \tag{5.3}$$

with some function γ analytic near the origin. The proof is very straightforward and identical with the one in [HS99a]. If, in addition, we put $t(0) := 0$, then $t(\varepsilon)$ is defined and depends analytically upon ε in a full neighborhood of 0. From $\alpha\Sigma' \subset S$ we know that if $\varepsilon \in \Sigma$ then we have $t(\varepsilon) \in \alpha\Sigma' \subset S$ for small enough $|\varepsilon|$.

The difficulty arises from the possibility for z to lie very *close* to $t(\varepsilon)$ or even *on* it. Some of the above results [Imm90, Hoe94, HS99a] are not uniform in the sense that z would be allowed to approach $t(\varepsilon)$. To obtain uniformly valid results one may not neglect the factor $\frac{w}{w-z}$, and the analysis will become more involved. There are various uniform results (see above), all of which have one major drawback: they are valid only under the assumption that the parameters z and ε be coupled in a certain way; moreover, most of them cover only the case $r = 1$. In this section we will overcome these restrictions, at the price of making another restriction *on the domain of valid ε only* and of obtaining slightly weaker estimates. See Section 5.5 for details.

The technique consists of splitting up, after a preliminary transformation, the integral into the sum of an integral merely involving the pole term which can be expressed in terms of the complementary error function, and a second integral whose integrand is bounded near $z = t(\varepsilon)$. This approach is due to van der Waerden [vdW51] and has been applied to the generalized exponential integral by Temme [Tem75, Tem79] and Olver [Olv91a]. A good presentation of the procedure can be found in [Jon72].

5.2 Study of the path of steepest descent

In this section we prepare the proofs of Sections 5.3 and 5.4. In the saddle-point method, though not really necessary (see [vdW51]), it is often convenient to follow the so-called *path of steepest descent*. This has the advantage of considerably simplifying the estimates. However, in the case of our integral, only a part of this contour can be used for our purposes – see the remark on page 91 for details.

For $\varepsilon \neq 0$ let $\tilde{\mathfrak{W}} = \tilde{\mathfrak{W}}(\varepsilon)$ be a regular parameterization of the path of steepest descent of $\varphi(\varepsilon,\cdot)$ through $t(\varepsilon)$. That is the contour through $t(\varepsilon)$ on which $\operatorname{Im}\varphi(\varepsilon,w)$ is constant with respect to w and $\operatorname{Re}\varphi(\varepsilon,w)$ has a relative maximum in $t(\varepsilon)$, cf. Figure 2 (b).

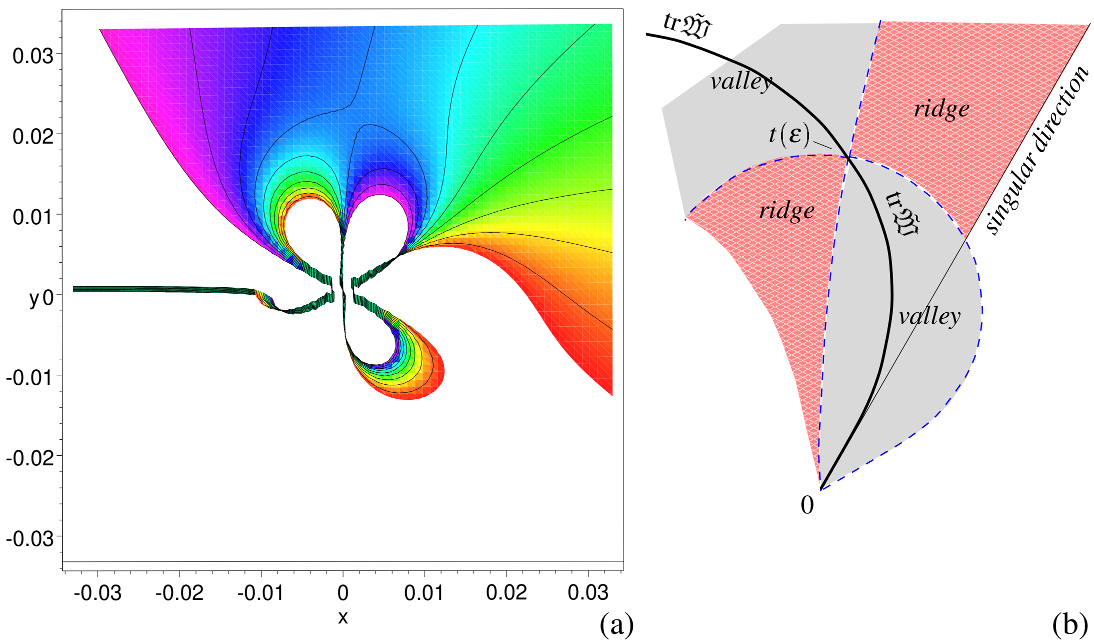

Figure 2: Shape of the function $\operatorname{Re}\varphi(\varepsilon,\cdot)$ for fixed ε – (a) Example MAPLE plot for $p(x) = 2.5x^3 - 11x^2 + 5x$ and $\varepsilon = 0.01 + 0.005i$ – (b) Orientation of the saddle, and the path $\tilde{\mathfrak{W}}(\varepsilon)$ of steepest descent

The Laplace-type nature of the integral as well as the polar part can be seen more clearly after the following change of variables: with $t = t(\varepsilon)$ from Section 5.1 let

$$\tilde{x} = \tilde{x}(\varepsilon) := -\tfrac{1}{2}t^2\partial_2^2\varphi(\varepsilon,t) \tag{5.4}$$

and a function $f(\varepsilon,\cdot):\ h \longmapsto u =: f(\varepsilon,h)$ be continuous and given by

$$\tfrac{1}{2}t^2\partial_2^2\varphi(\varepsilon,t)u^2 := \varphi(\varepsilon,(1+h)t) - \varphi(\varepsilon,t), \tag{5.5a}$$

$$u \sim h \quad (h \to 0). \tag{5.5b}$$

Except for parameter dependence, these are exactly the same transformations as in [HS99a].

Let us collect a few properties of $\tilde{\mathfrak{W}}$, $\tilde{x}$, and f:

Lemma 5.1 (Technical details). *Let the general assumptions and notations made in Section 5.1 be satisfied. Let $t = t(\varepsilon)$ be the saddle-point defined in Section 5.1, satisfying* (5.3). *Let $\tilde{\mathfrak{W}}$ be as above and $\tilde{x}$ and f defined by* (5.4) *and* (5.5)*, respectively. Then there are constants $\varepsilon_0, \delta_0 > 0$ such that:*

- **(i)** *There is a $\tau > 0$ such that the disk $K_{\tau|t|}(t)$ lies inside $\alpha\Sigma'$ for all $\varepsilon \in \Sigma$, $|\varepsilon| < \varepsilon_0$.*
- **(ii)** *For $0 < |\varepsilon| \le \varepsilon_0$ the contour $\tilde{\mathfrak{W}}$ crosses exactly one saddle (that is $t(\varepsilon)$), and $\operatorname{Re}\varphi(\varepsilon,\cdot)\circ\tilde{\mathfrak{W}}$ is strictly monotonous in both subintervals separated by $\tilde{\mathfrak{W}}^{-1}\big(t(\varepsilon)\big)$.*

(iii) $\tilde{\mathfrak{W}}(\varepsilon)$ *can be parameterized such that, in a neighborhood of* 0*, we have*

$$w = \alpha\gamma\Big(1 + O\big(\tfrac{\gamma}{\varepsilon}\log\tfrac{\gamma}{\varepsilon}\big)\Big) \qquad (r = 1),$$
$$w = \alpha\gamma\Big(1 + O\big(\tfrac{\gamma}{\varepsilon}\big)\Big) \qquad (r \geq 2)$$

where γ is a positive real parameter. The estimate is uniform with respect to $0 < |\varepsilon| \leq \varepsilon_0$ *and* $|\frac{w-t}{t}| \geq d$ *for some* $d > 0$ *(resp.* $\gamma/\varepsilon \leq K$ *for some* $K < 1$*).*

(iv) *The function $\tilde{x}$ satisfies*

$$\tilde{x}(\varepsilon) = \tfrac{r}{2}\varepsilon^{-r}\big(1 + \varepsilon\tilde{\gamma}(\varepsilon)\big) \qquad (0 < |\varepsilon| < \varepsilon_0)$$

where $\tilde{\gamma}$ can even be taken to be analytic near the origin. This suggests defining a new parameter $\tilde{\varepsilon}(\varepsilon) := \tilde{x}(\varepsilon)^{-1/r}$ $(\varepsilon \neq 0)$*;* $\tilde{\varepsilon}(0) := 0$ *which satisfies*

$$\tilde{\varepsilon}(\varepsilon) = \big(\tfrac{r}{2}\big)^{-1/r}\varepsilon\big(1 + \varepsilon\tilde{\tilde{\gamma}}(\varepsilon)\big) \qquad (0 \leq |\varepsilon| < \varepsilon_0)$$

with some $\tilde{\tilde{\gamma}}$ analytic near the origin.

(v) *If we complete definition* (5.5) *setting*

$$u := f(0,h) := h\sqrt{\frac{-\frac{1}{r}\big[(1+h)^{-r} - 1\big] - \log(1+h)}{-\frac{r}{2}h^2}},$$

then there is a function γ_1 analytic in a neighborhood of $(0,0)$ *such that for* $0 \leq |\varepsilon| < \varepsilon_0$ *and for* $0 \leq |h| < \delta_0$ *the mapping f satisfies*

$$f(\varepsilon,h) = h\big(1 + h\gamma_1(\varepsilon,h)\big)$$

(Thus (5.5b) *can indeed be satisfied).*

(vi) *For* $\varepsilon \neq 0$*, the function* $f(\varepsilon,\cdot)$ *can be continued analytically along* $\tilde{\tilde{\mathfrak{W}}} := \frac{1}{t}\tilde{\mathfrak{W}} - 1$*, and we have*

$$f(\varepsilon,\cdot):\ \operatorname{tr}\tilde{\tilde{\mathfrak{W}}} \to \tilde{\varepsilon}^{r/2}\mathbb{R} \quad \textit{bijective.}$$

Moreover, in an open neighborhood of $\operatorname{tr}\tilde{\tilde{\mathfrak{W}}}$ *the function* $f(\varepsilon,\cdot)$ *admits an inverse*

$$b(\varepsilon,\cdot) := \big(f(\varepsilon,\cdot)\big)^{-1}$$

which satisfies

(5.6) $$b(\varepsilon,u) = u\big(1 + u\gamma_2(\varepsilon,u)\big)$$

with a certain γ_2 analytic in a neighborhood of $(0,0)$*.*

(vii) *For $|x|, |y| < \delta_0$, $|\varepsilon| < \varepsilon_0$, the term*

$$g(\varepsilon, x, y) := \frac{1+x}{1+y} \cdot \frac{1}{\partial_2 f(\varepsilon, x)} \cdot \frac{1}{x-y} - \frac{1}{f(\varepsilon, x) - f(\varepsilon, y)}$$

is well-defined whenever $x \neq y$ and can be continued to an analytic function of (ε, x, y) in a full neighborhood of $(0,0,0)$.

(viii) *There is an $s > 0$ such that if $d := s \cdot \exp(\frac{r}{2} i \arg \tilde{\varepsilon})$ the point $T_1(\varepsilon, d) := \big(1 + b(\varepsilon, d)\big) t(\varepsilon) \in \mathrm{tr}\,\tilde{\mathfrak{W}}$ (cf. Figure 3) lies inside $\alpha \Sigma'$ for all $\varepsilon \in \Sigma$, $|\varepsilon| < \varepsilon_0$ and such that $\psi(x) := \operatorname{Re} \varphi(\varepsilon, T_1 x)$ $(x > 0)$ decreases monotonously for $x > 1$.*

(ix) *Let $\varepsilon \in \Sigma$, $|\varepsilon| < \varepsilon_0$. If we define $T_2(\varepsilon, d) := \frac{\rho_0}{|T_1|} T_1(\varepsilon, d)$ with T_1 as in (viii) then we have*

$$\operatorname{Re} \varphi(\varepsilon, w) < \operatorname{Re} \varphi(\varepsilon, T_1)$$

for all $w \in \overset{\frown}{T_2 T}$ (i.e. on the circular arc around 0 joining T_2 to T, see Figure 3).

(x) *Let $\varepsilon \in \Sigma$, $|\varepsilon| < \varepsilon_0$ and define $T_0(\varepsilon, d) := \big(1 + b(\varepsilon, -d)\big) t(\varepsilon) \in \mathrm{tr}\,\tilde{\mathfrak{W}}$ (cf. Figure 3. Then there are constants $\delta, \tilde{\delta} > 0$ such that for $x \in \mathrm{tr}\,\tilde{\mathfrak{W}}$, $|x| \leq \delta |t|$ we have*

$$\forall w \in K_{\tilde{\delta}|x|}(x): \quad \operatorname{Re} \varphi(\varepsilon, w) < \operatorname{Re} \varphi(\varepsilon, T_0).$$

Proof.

(i) This is immediately clear from (5.3) and from the construction of Σ'. This property is slightly stronger than the one mentioned after (5.3), but choosing ε_0 and τ small enough we can always achieve the inclusion $K_{\tau|t|}(t) \subset \alpha \Sigma'$.

(ii) First we have to calculate $\varphi\big(\varepsilon, t(\varepsilon)\big)$. But for later convenience, let us calculate $p\big((1+h)^{-1} t^{-1}\big)$ for arbitrary $h \neq -1$. Using (5.3) we have

$$\begin{aligned}
(1+h)^{-\ell} t^{-\ell} &= (\alpha \varepsilon)^{-\ell} (1+h)^{-\ell} \big(1 + \varepsilon \gamma_\ell(\varepsilon)\big), \\
p\big((1+h)^{-1} t^{-1}\big) &= \sum_{\ell=1}^{r} p_\ell \alpha^{-\ell} (1+h)^{-\ell} \big(1 + \varepsilon \gamma_\ell(\varepsilon)\big) \varepsilon^{-\ell} \\
&= \varepsilon^{-r} \sum_{\ell=0}^{r-1} p_{r-\ell} \alpha^{\ell - r} (1+h)^{\ell - r} \big(1 + \varepsilon \gamma_{r-\ell}(\varepsilon)\big) \varepsilon^{\ell} \\
&= p_r \alpha^{-r} (1+h)^{-r} \varepsilon^{-r} \big(1 + \varepsilon \tilde{\tilde{\gamma}}(\varepsilon, h)\big) \\
&= -(1+h)^{-r} \varepsilon^{-r} \Big(\frac{1}{r} + \frac{1}{r} \varepsilon \tilde{\tilde{\gamma}}(\varepsilon, h)\Big), \qquad (5.7)
\end{aligned}$$

hence

$$p(t^{-1}) = -\varepsilon^{-r}\Big(\frac{1}{r} + \frac{1}{r}\varepsilon\tilde{\tilde{\gamma}}(\varepsilon,0)\Big),$$

$$\varphi(\varepsilon,t) = -\varepsilon^{-r}\Big(\frac{1}{r} + \log\alpha + \log\varepsilon + \varepsilon\hat{\gamma}(\varepsilon)\Big). \tag{5.8}$$

To understand the global behavior of $\varphi(\varepsilon,\cdot)$ we want to choose another determination $\tilde{\alpha}$ of $(-rp_r)^{1/r}$ and consider the corresponding saddle $\tilde{t}$. Then all previous considerations apply with α replaced by $\tilde{\alpha}$, and we obtain

$$\begin{aligned}\varphi(\varepsilon,t) - \varphi(\varepsilon,\tilde{t}) &= \varepsilon^{-r}\Big(\log\frac{\tilde{\alpha}}{\alpha} + \varepsilon\hat{\hat{\gamma}}(\varepsilon)\Big),\\ &= \varepsilon^{-r}\Big(\frac{2\nu\pi i}{r} + O(\varepsilon)\Big)\end{aligned}$$

uniformly for $|\varepsilon| \le \varepsilon_0$, and

$$\operatorname{Im}\big(\varphi(\varepsilon,t) - \varphi(\varepsilon,\tilde{t})\big) = |\varepsilon|^{-r}\Big(\frac{2\nu\pi}{r} + O(\varepsilon)\Big)\sin\Big(-r\arg\varepsilon \pm \frac{\pi}{2} + O(\varepsilon)\Big),$$

but the above sine stays away uniformly from 0 for small enough ε_0 because of the assumptions on Σ. Hence on distinct saddle points the function $\operatorname{Im}(\varepsilon,\cdot)$ has distinct values.

Let now $\tilde{\mathfrak{W}}$ be as above. Assume that the directional derivative of $\operatorname{Re}\varphi(\varepsilon,\cdot)$ along $\tilde{\mathfrak{W}}$ would vanish at a certain point $a \in \operatorname{tr}\tilde{\mathfrak{W}}$. As we know that $\operatorname{Im}\varphi(\varepsilon,\cdot)$ is constant along $\tilde{\mathfrak{W}}$ we have $\partial_2\varphi(\varepsilon,a) = 0$ and hence a must be a saddle point. But a cannot be different from $t(\varepsilon)$ as in view of the considerations above $\operatorname{Im}(\varepsilon,a)$ would then differ from $\operatorname{Im}(\varepsilon,t)$. Therefore $\operatorname{Re}\varphi(\varepsilon,\cdot)\circ\tilde{\mathfrak{W}}$ has two intervals of monotony and the maximum is absolute.

(iii) From (ii) we know that along the path of steepest descent, the term

$$\varphi(\varepsilon,w) - \varphi(\varepsilon,t)$$

is real and non-positive, hence with a positive real γ put

$$\varphi(\varepsilon,w) - \varphi(\varepsilon,t) = -\tfrac{1}{r\gamma^r}. \tag{5.9}$$

Together with (5.8) this implies

$$p(w^{-1}) - \varepsilon^{-r}\log\tfrac{w}{\alpha\varepsilon} + \varepsilon^{-r}O(1) = -\tfrac{1}{r\gamma^r},$$

$$w^{-r}\Big[1 + \sum_{\ell=1}^{r-1}\tfrac{p_{r-\ell}}{p_r}w^\ell\Big] - \varepsilon^{-r}\Big(\frac{1}{p_r}\log\frac{w}{\alpha\varepsilon} + O(1)\Big) = (\alpha\gamma)^{-r}$$

uniformly with respect to $|\varepsilon| \le \varepsilon_0$. Substituting $\tilde{w} := \frac{w}{\alpha\varepsilon}$; $\tilde{\gamma} := \frac{\gamma}{\varepsilon}$ the last equation reads

$$\tilde{w}^{-r}\Big(1 + O(\tilde{w})\Big) + \big(r\log\tilde{w} + O(1)\big) = \tilde{\gamma}^{-r}.$$

If we have $|\frac{w-t}{t}| \ge d$, we see that (for small enough ε_0) $|\frac{w-\alpha\varepsilon}{\alpha\varepsilon}| \ge \tilde{d}$ and hence $\log\tilde{w} = \Omega(1)$. Thus we have

$$\tilde{w}^{-r}\Big(1 + O(\tilde{w}) + O(\tilde{w}^r\log\tilde{w})\Big) = \tilde{\gamma}^{-r}.$$

For fixed $\varepsilon \ne 0$ and small $\tilde{\gamma}$, this equation can be solved asymptotically w.r.to $\tilde{w}$ since the correspondence between $\tilde{w}$ and $\tilde{\gamma}$ is one-to-one there (cf. the monotony statements in (ii)). In the monomial case $r = 1$ we obtain

$$\begin{aligned}\tilde{w} &= \tilde{\gamma}\Big(1 + O(\tilde{\gamma}\log\tilde{\gamma})\Big),\\ w &= \alpha\gamma\Big(1 + O\big(\tfrac{\gamma}{\varepsilon}\log\tfrac{\gamma}{\varepsilon}\big)\Big)\end{aligned}$$

uniformly with respect to ε and w as above. We will be able to express the region of uniformity in terms of ε and γ by means of (iv) and (v), see below. The case $r \ge 2$ can be proved the same way, except that we have $O(\tilde{w}^r\log\tilde{w}) \subset O(\tilde{w})$ then.

(iv) First, the second partial derivative of $\varphi(\varepsilon, w)$ with respect to w is given by

$$\partial_2^2\varphi(\varepsilon, w) = \sum_{\ell=1}^{r}\ell(\ell+1)p_\ell w^{-\ell-2} + \varepsilon^{-r}w^{-2}, \tag{5.10}$$

hence

$$w^2\partial_2^2\varphi(\varepsilon, w) = \varepsilon^{-r} + \sum_{\ell=1}^{r}\ell(\ell+1)p_\ell w^{-\ell}.$$

Insertion of (5.3) yields

$$\begin{aligned}t^2\partial_2^2\varphi(\varepsilon, t) &= \varepsilon^{-r}\Big(1 + \sum_{\ell=1}^{r}\ell(\ell+1)\alpha^{-\ell}p_\ell\varepsilon^{r-\ell}\big(1 + \varepsilon\gamma_\ell(\varepsilon)\big)\Big)\\ &= -r\varepsilon^{-r}\big(1 + \varepsilon\tilde{\gamma}(\varepsilon)\big),\end{aligned}$$

hence (iv) follows immediately from (5.4).

(v) Define $g(\varepsilon, h) := u^2$ in the context of (5.5) resp. (v). For fixed $\varepsilon \ne 0$, or for $\varepsilon = 0$, the partial analyticity of g w.r.to $|h| < 1$ is clear from the definition. If $|h| < 1$ is fixed, however, the partial analyticity of g w.r.to ε has to be shown at $\varepsilon = 0$.

According to (5.7), for $\varepsilon \ne 0$ we have

$$\begin{aligned}\varepsilon^r p\big((1+h)^{-1}t^{-1}\big) &= -\tfrac{1}{r}(1+h)^{-r}\Big(1 + \varepsilon\tilde{\tilde{\gamma}}(\varepsilon, h)\Big) \xrightarrow{\varepsilon\to 0} -\tfrac{1}{r}(1+h)^{-r},\\ \varepsilon^r p(t^{-1}) &= -\tfrac{1}{r}\Big(1 + \varepsilon\tilde{\tilde{\gamma}}(\varepsilon, 0)\Big) \qquad\qquad \xrightarrow{\varepsilon\to 0} -\tfrac{1}{r},\end{aligned}$$

hence

$$\varepsilon^r \varphi\Big(\varepsilon, (1+h)t\Big) - \varepsilon^r \varphi(\varepsilon,t) \xrightarrow{\varepsilon\to 0} -\tfrac{1}{r}\big[(1+h)^{-r}-1\big] - \log(1+h).$$

Using (iv) we conclude

$$g(\varepsilon,h) = \frac{\varepsilon^r \varphi\big(\varepsilon,(1+h)t\big) - \varepsilon^r \varphi(\varepsilon,t)}{-\varepsilon^r \tilde{x}} \xrightarrow{\varepsilon\to 0} g(0,h).$$

So we have partial analyticity w.r.to ε, too, hence analyticity by Hartog's theorem. In order for f to be analytic and of the claimed form we merely have to show that, for $|\varepsilon| < \varepsilon_0$,

$$g(\varepsilon,h) = h^2 + O\big(h^3\big).$$

For each $\varepsilon \neq 0$, we know from (5.5a) and Taylor's formula:

$$\begin{aligned} \tfrac{1}{2}t^2 \partial_2^2 \varphi(\varepsilon,t)\cdot g(\varepsilon,h) &= \varphi\big(\varepsilon,(1+h)t\big) - \varphi(\varepsilon,t) \\ &= \tfrac{1}{2}t^2 \partial_2^2 \varphi(\varepsilon,t) h^2 + O\big(h^3\big), \\ \Rightarrow \quad g(\varepsilon,h) &= h^2 + O\big(h^3\big). \end{aligned}$$

By analyticity of g this property extends to $\varepsilon = 0$, which completes the proof of (v).

(vi) The analyticity of $f(\varepsilon,\cdot)$ near the origin has been shown in (v). Outside the origin observe that $\varphi\big(\varepsilon,(1+h)t\big) - \varphi(\varepsilon,t)$ is analytic with respect to h, and since this difference is nonzero, we stay in the domain of analyticity of the square root and hence $u = f(\varepsilon,h)$ depends analytically upon h.

If $w \in \mathrm{tr}\tilde{\mathfrak{W}}$, then using the monotony from (ii), we have

$$\begin{aligned} \mathrm{Im}\big(\varphi(\varepsilon,w) - \varphi(\varepsilon,t)\big) &= 0, \qquad\qquad \text{hence} \\ \underbrace{\varphi(\varepsilon,w) - \varphi(\varepsilon,t)}_{= -\tilde{x}u^2} &= \mathrm{Re}\,\varphi(\varepsilon,w) - \mathrm{Re}\,\varphi(\varepsilon,t) \le 0. \end{aligned}$$

if $u = f(\varepsilon,h)$ as in (5.5). As a consequence, $\tilde{x}^{1/2}u \in \mathbb{R}$, thus $u \in \tilde{\varepsilon}^{r/2}\mathbb{R}$. The surjectivity follows from the behavior near $w = 0$ (cf. (5.2)) and near $w = \infty$. The injectivity in turn is clear from the monotony in (ii).

(To be exact: it is easy to see that in view of (ii) the only possibility for the other end-point of $\tilde{\tilde{\mathfrak{W}}}$ is indeed infinity. But still we would have to study the exact shape of the contour near infinity to show that the limit of $\mathrm{Re}\,\varphi(\varepsilon,w)$ is again $-\infty$. We will not go into the details here as this is irrelevant for our proof.)

Now, the partial derivative of f with respect to the second variable can be expressed by means of definitions (5.4), (5.5a) as to

$$\partial_2 f(\varepsilon,h) = -\frac{t(\varepsilon)}{2\tilde{x}(\varepsilon)} \cdot \frac{\partial_2 \varphi\big(\varepsilon,(1+h)t\big)}{f(\varepsilon,h)}.$$

Thus from the considerations in (ii) we have $\partial_2 f(\varepsilon,h) \neq 0$ for $h \in \operatorname{tr}\tilde{\tilde{\mathfrak{W}}} \setminus \{0\}$, whereas from (v) we have $\partial_2 f(\varepsilon,0) = 1 \neq 0$. Hence $f(\varepsilon,\cdot)$ is locally injective near every point $h \in \operatorname{tr}\tilde{\tilde{\mathfrak{W}}}$, and can be inverted in an open neighborhood of $\operatorname{tr}\tilde{\tilde{\mathfrak{W}}}$. From complex analysis it is known that the inverse $b(\varepsilon,\cdot)$ satisfies

$$b(\varepsilon,u) = \frac{1}{2\pi i} \oint_{\mathscr{C}(\varepsilon)} \frac{h\partial_2 f(\varepsilon,h)}{f(\varepsilon,h)-u}\,dh$$

with a suitable contour $\mathscr{C}(\varepsilon)$ circulating $b(\varepsilon,u)$ once in the positive sense.

If we choose $\mathscr{C}(\varepsilon)$ to be independent from ε ($|\varepsilon| \leq \varepsilon_0$) and differentiate under the integral sign we see that b is analytic near $(0,0)$. In the convergent Taylor expansion of b at $(0,0)$ we then have

$$b(\varepsilon,0) = 0, \qquad \partial_2 b(\varepsilon,0) = 1,$$

hence (5.6) follows.

(vii) According to (vi), as $f(\varepsilon,\cdot)$ is locally injective, we know that if δ_0 and ε_0 are small enough then we have $f(\varepsilon,x) - f(\varepsilon,y) \neq 0$ for $x \neq y$ and hence the subtrahend is defined. On the other hand, from (v) we have $\partial_2 f(\varepsilon,x) \neq 0$ near $x = 0$ and hence the minuend is defined too.

Now, by use of Taylor's formula with respect to the second variable, we have

$$f(\varepsilon,x) - f(\varepsilon,y) = (x-y)\partial_2 f(\varepsilon,x)\Big[1 + (x-y)\gamma_3(\varepsilon,x,y)\Big], \qquad \text{hence}$$

$$\frac{1}{f(\varepsilon,x)-f(\varepsilon,y)} = \frac{1}{(x-y)\partial_2 f(\varepsilon,x)} + \gamma_4(\varepsilon,x,y) \tag{5.11}$$

with some γ_3, γ_4 analytic near the origin.

Expressing for $x \neq y$

$$\frac{1+x}{1+y} \cdot \frac{1}{x-y} = \frac{1}{x-y} + \frac{1}{1+y}$$

we find

$$\frac{1+x}{1+y} \cdot \frac{1}{\partial_2 f(\varepsilon,x)} \cdot \frac{1}{x-y} = \frac{1}{(x-y)\partial_2 f(\varepsilon,x)} + \frac{1}{(1+y)\partial_2 f(\varepsilon,x)}. \tag{5.12}$$

Combining (5.11) and (5.12), the analytic continuability of g follows.

(viii) First we estimate $|T_1|$ by means of (5.3) and (5.6):

$$\begin{aligned} T_1(\varepsilon,d) &= \big(1 + b(\varepsilon,d)\big)t(\varepsilon) \\ &= \alpha\varepsilon\big(1 + O(\varepsilon)\big)\big(1 + d + O(d^2)\big), \\ \Rightarrow \quad |T_1| &= |\alpha\varepsilon|\big|1 + d + O(d^2) + O(\varepsilon)\big| \\ &= |\alpha\varepsilon|\big(1 + \operatorname{Re} d + O(d^2) + O(\varepsilon)\big), \end{aligned} \tag{5.13}$$

and observing $\arg d = \frac{r}{2}\arg\tilde{\varepsilon}$ we have

$$|T_1| = |\alpha\varepsilon|\Big(1 + |d|\cos\big(\tfrac{r}{2}\arg\varepsilon\big) + O(d^2) + O(\varepsilon)\Big) \tag{5.14}$$

uniformly with respect to $|\varepsilon| < \varepsilon_0$, $|d| \le d_0$. In view of (5.13), T_1 certainly lies inside $\alpha\Sigma'$ for small enough $\varepsilon_0, d_0 > 0$.

Consider now $\operatorname{Re}\varphi(\varepsilon, w)$ along the ray $\arg w = \arg T_1$. Putting $w = q\exp(i\arg T_1)$ we obtain

$$\operatorname{Re}\varphi(\varepsilon,w) = -|p_r|q^{-r}\cos\big(r\arg\tfrac{T_1}{\alpha}\big) + O(q^{-r+1}) - \operatorname{Re}\varepsilon^{-r}\cdot\log q + \operatorname{Im}\varepsilon^{-r}\cdot\arg T_1.$$

Solving the equation $\frac{\partial}{\partial q}\operatorname{Re}\varphi\big(\varepsilon, q\exp(i\arg T_1)\big)\Big|_{q=q_0} = 0$ asymptotically, we find

$$\begin{aligned}
&\cos\big(r\arg\tfrac{T_1}{\alpha}\big)|\alpha|^r q_0^{-r} + O(q_0^{-r+1}) = |\varepsilon|^{-r}\cos(r\arg\varepsilon)\\
\iff\quad & q_0 = q_0(\varepsilon,d) = |\alpha\varepsilon|\left(\frac{\cos\big(r\arg\frac{T_1}{\alpha}\big)}{\cos(r\arg\varepsilon)}\right)^{1/r}\big(1+O(\varepsilon)\big).
\end{aligned}$$

A lengthy calculation using the asymptotic formula (5.13) shows

$$q_0 = |\alpha\varepsilon|\Big(1 - |d|\sin\big(\tfrac{r}{2}\arg\varepsilon\big)\tan(r\arg\varepsilon) + O(d^2) + O(\varepsilon)\Big) \tag{5.15}$$

uniformly with respect to $|\varepsilon| < \varepsilon_0$, $|d| \le d_0$.

Since q_0 is the only positive solution of the above equation we see that $\operatorname{Re}\varphi\big(\varepsilon, q\exp(i\arg T_1)\big)$ decreases monotonously for $q > q_0$.

Observing $\cos\big(\frac{r}{2}\arg\varepsilon\big) \ge \cos\big(\frac{r}{2}\delta(\Sigma)\big) > \frac{1}{2}\sqrt{2} > 0$ as well as $\sin\big(\frac{r}{2}\arg\varepsilon\big)\tan(r\arg\varepsilon) \ge 0$, comparison of (5.14) and (5.15) shows the existence of an $s = |d| > 0$ such that $|T_1(\varepsilon,d)| > q_0(\varepsilon,d)$ for all $|\varepsilon| < \varepsilon_0$, and hence the desired property follows.

(ix) For the ε in question, from (5.8) we deduce

$$\begin{aligned}
&\varphi(\varepsilon,t) = -\varepsilon^{-r}\log\varepsilon - \varepsilon^{-r}\Big(\tfrac{1}{r} + \log\alpha\Big) + O\big(\varepsilon^{-r+1}\big),\\
\Rightarrow\quad & \operatorname{Re}\varphi(\varepsilon,t) = |\varepsilon|^{-r}\Big|\log|\varepsilon|\Big|\cos(r\arg\varepsilon) + O\big(\varepsilon^{-r}\big)
\end{aligned}$$

uniformly for $\varepsilon \in \Sigma$, $|\varepsilon| < \varepsilon_0$ (because $\arg\varepsilon$ remains bounded). Using (iv) we see that

$$\begin{aligned}
\operatorname{Re}\varphi(\varepsilon,T_1) &= \operatorname{Re}\varphi(\varepsilon,t) - \tilde{\varepsilon}^{-r}d^2\\
&= |\varepsilon|^{-r}\Big|\log|\varepsilon|\Big|\cos(r\arg\varepsilon) + O\big(\varepsilon^{-r}\big)
\end{aligned} \tag{5.16}$$

uniformly (because $|d|$ is bounded). On the other hand, for the w in question, we certainly have

$$\operatorname{Re}\varphi(\varepsilon,w) = \operatorname{Re} p(w^{-1}) - \operatorname{Re}\big(\varepsilon^{-r}\log w\big) = O(1) + O\big(\varepsilon^{-r}\big) = O\big(\varepsilon^{-r}\big)$$

uniformly (because $\arg w$ is bounded). Combining with (5.16) we have

$$\operatorname{Re}\varphi(\varepsilon,w) - \operatorname{Re}\varphi(\varepsilon,T_1) = -|\varepsilon|^{-r}\Big|\log|\varepsilon|\Big|\cos(r\arg\varepsilon) + O\big(\varepsilon^{-r}\big)$$

uniformly for $\varepsilon\in\Sigma$, $|\varepsilon|<\varepsilon_0$. Now observe that $\cos\big(r\arg\varepsilon\big)\geq\cos\big(r\delta(\Sigma)\big)>0$ so that we have $\operatorname{Re}\varphi(\varepsilon,w)-\operatorname{Re}\varphi(\varepsilon,T_1)<0$ for $|\varepsilon|<\varepsilon_0$ if $\varepsilon_0>0$ is chosen small enough. Hence (ix) follows.

(x) First look at the level line of $\operatorname{Re}\varphi(\varepsilon,\cdot)$ through T_0. On that curve we have

$$\operatorname{Re}\big(\varphi(\varepsilon,w)-\varphi(\varepsilon,t)\big) = \operatorname{Re}\big(\varphi(\varepsilon,T_0)-\varphi(\varepsilon,t)\big) = \varphi(\varepsilon,T_0)-\varphi(\varepsilon,t) = -\tilde{x}d^2$$

by definition. So with a positive real γ put

$$\varphi(\varepsilon,w)-\varphi(\varepsilon,t) = -\tilde{x}d^2 \pm \frac{i}{r\gamma^r}.$$

We can treat this equation now the same way as we did with equation (5.9) to see that the (two parts of the) above level line can be parameterized such that, in a neighborhood of 0, we have

$$\begin{aligned} w &= \alpha\gamma e^{\pm\frac{\pi}{2r}i}\Big(1+O\big(\tfrac{\gamma}{\varepsilon}\log\tfrac{\gamma}{\varepsilon}\big)\Big) && (r=1),\\ w &= \alpha\gamma e^{\pm\frac{\pi}{2r}i}\Big(1+O\big(\tfrac{\gamma}{\varepsilon}\big)\Big) && (r\geq 2). \end{aligned}$$

The estimate is uniform with respect to $0<|\varepsilon|\leq\varepsilon_0$ and $\left|\frac{w-t\exp(\frac{\pi}{2r}i)}{t\exp(\frac{\pi}{2r}i)}\right|$ bounded from below. So if δ and $\tilde{\delta}$ are sufficiently small then for $x\in\operatorname{tr}\tilde{\mathfrak{W}}$, $|x|\leq\delta|t|$ item (iii) is applicable, hence the disk $K_{\tilde{\delta}|x|}(x)$ lies in a sector of small opening with bisecting direction α. But on the other hand, the above level line ends up in small sectors around $\alpha\pm\frac{\pi}{2r}$ and consequently has no intersection with the disk $K_{\tilde{\delta}|x|}(x)$. Now (x) follows by a continuity argument. This completes the proof.

□

5.3 Asymptotic estimates I: The integral part

We deform now the path $\mathfrak{W}$ of integration to one which crosses the saddle at $t(\varepsilon)$. Therefore, choose a d and the corresponding $T_1(\varepsilon,d)$ which satisfy Lemma 5.1(viii), put $T_0 := T_0(\varepsilon,d) := \big(1+b(\varepsilon,-d)\big)t(\varepsilon)$, and let $\mathfrak{W}_0$ and $\mathfrak{W}_1$ be the parts of $\tilde{\mathfrak{W}}$ from

$w = 0$ up to $w = T_0$ and from $w = T_0$ to $w = T_1$, respectively, see Figure 3 (The path $\tilde{\mathfrak{W}}$ of steepest descent has been defined in Section 5.2). Define $\mathfrak{W}_2$ as the line segment from T_1 to $T_2 := \frac{\rho_0}{|T_1|}T_1$, and finally let $\mathfrak{W}_3$ be the circular arc joining T_2 to T. Now $\hat{\mathfrak{W}} := \hat{\mathfrak{W}}(\varepsilon, d)$ will be the composition $\mathfrak{W}_0\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$ of those paths. Again, this contour has to be indented by a small circular arc in the case z lies on it. According to Lemma 5.1(viii), this contour entirely lies inside $\alpha\Sigma' \subset S$.

On this contour we have $\arg w \to \arg\alpha$ as $|w| \to 0$ (see Lemma 5.1(iii)) so that near $w = 0$ the deformation is possible as we stay within the same sector of convergence of the integral (cf. (5.2)).

Remark. We cannot use more than a part of the path of steepest descent because, if ε is small, this contour twists rapidly out sideways the sector S. Therefore, $\hat{\mathfrak{W}}$ continues along a ray which lies inside $\alpha\Sigma' \subset S$.

Because this deformation possibly passes over the pole at $w = z$, we have in general

$$\mathscr{I}(z,\varepsilon) = \underbrace{\int_{\hat{\mathfrak{W}}} e^{\varphi(\varepsilon,w)}\frac{wF(w)}{w-z}\,dw}_{=:\,\hat{\mathscr{I}}(z,\varepsilon)} + \underbrace{2\pi i n(z,\varepsilon)\cdot \operatorname*{Res}_{w=z}\left[e^{\varphi(\varepsilon,w)}\frac{wF(w)}{w-z}\right]}_{=:\,r(z,\varepsilon)} \tag{5.17}$$

where $n(z,\varepsilon) \in \{0,1,-1\}$ is the (circulation) index of the closed contour $\mathfrak{W}\hat{\mathfrak{W}}^{-1}$ with respect to z (see Figure 3).

The residue term is now easily understood, so we turn our attention to the asymptotic behavior of $\hat{\mathscr{I}}(z,\varepsilon)$:

Proposition 5.2. *Let $r \in \mathbb{N}^*$, $p(x) = \sum_{\ell=1}^{r} p_\ell x^\ell \in x\mathbb{C}[x]$, $\alpha = (-rp_r)^{1/r} \neq 0$. Consider the sectors $\Sigma = \overline{S}_{\rho(\Sigma)}\big(-\delta(\Sigma),\delta(\Sigma)\big)$ with $0 < 2\delta(\Sigma) < \frac{\pi}{r}$ and $S = S_{\hat{\rho}}(d_1,d_2)$. Let $T \in \alpha\Sigma \subset S$ be fixed and $0 < \rho < \rho_0 = |T| < \hat{\rho}$. Let $F : S \to \mathbb{C}$ be an analytic function bounded on every closed subsector of S. Finally, define $\hat{\mathfrak{W}}$ as above and $\hat{\mathscr{I}}(z,\varepsilon)$ as in* (5.17)*. Then the following estimate holds:*

$$\hat{\mathscr{I}}(z,\varepsilon) = e^{\varphi(\varepsilon,t)}\cdot O(t), \qquad t = t(\varepsilon)$$

uniformly for $\varepsilon \in \Sigma$ and $z \in \overline{K}_\rho(0)$. Here, $t(\varepsilon)$ is defined in Section 5.1 and satisfies (5.3).

Proof. We have to distinguish two different principal cases: in the first case z lies outside a neighborhood of t of size $\asymp |t|$, which is the "classical" situation. In this case the main contribution comes from the value of the integrand at the saddle point. Contrarily, the second case deals with the situation that z is *close* to t, and the contribution of the pole at $w = z$ cannot be neglected anymore. Let us quantify these relations:

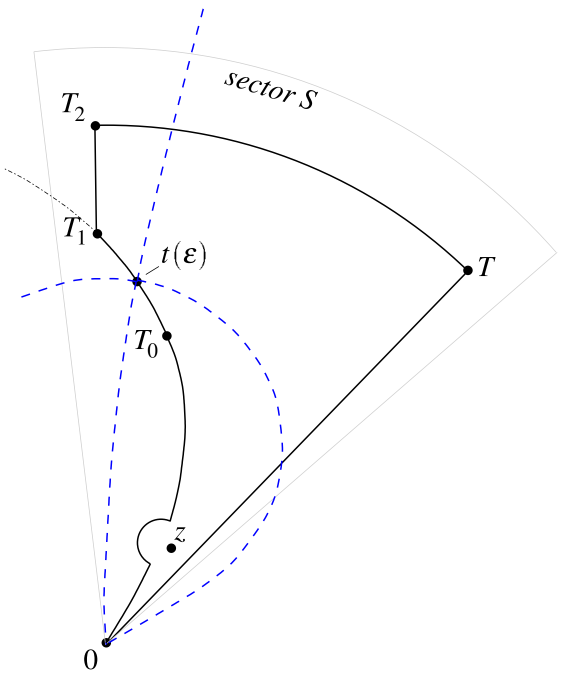

Figure 3: The path $\hat{\mathfrak{W}}(\varepsilon,d)$ of integration (ε fixed)

According to Lemma 5.1(vi) there is a $u_0>0$ such that $b(\varepsilon,u)$ is defined whenever $|u|<u_0$, $|\varepsilon|<\varepsilon_0$, and

(5.18) $$\left|\partial_2 b(\varepsilon,u)-1\right|<\frac{1}{2} \qquad (|u|<u_0,\ |\varepsilon|<\varepsilon_0).$$

Hence for every $u_1\le u_0$ we have

(5.19) $$|h|<\frac{1}{2}u_1 \implies |u|<u_1 \implies |h|<\frac{3}{2}u_1$$

if $h=b(\varepsilon,u)$. Put $u_1:=\min\{\frac{1}{3},\tau,d,u_0,\frac{2}{3}\delta_0\}$ with τ and δ_0 from Lemma 5.1, then for $|u|<u_1$ we have $|h|<\delta_0$. Thus all considerations of Lemma 5.1 are applicable, and in addition (5.19) holds. Observe that u_1,ε_0 are independent of z and ε.

Now define

(5.20) $$\sigma:=f\left(\varepsilon,\frac{z-t}{t}\right) \qquad \text{and consider}$$

Case 1: $|\sigma|\ge\frac{1}{4}u_1$; **Case 2:** $|\sigma|<\frac{1}{4}u_1$.

In general, the partition of $\hat{\mathfrak{W}}$ made at the beginning of Section 5.3 turns out to be inadequate to the situation of Cases 1 or 2, as z could lie too close to one of the partition points T_0 and T_1. Therefore, we adjust the partition made above by re-defining

$$\hat{d} := \begin{cases} \frac{1}{16}u_1 & \text{in Case 1,} \\ u_1 & \text{in Case 2,} \end{cases}$$
$$d := \hat{d} \cdot \exp\left(\tfrac{r}{2} i \arg \tilde{\varepsilon}\right).$$

Since $|d|$, after this re-definition, is not larger than before, all properties involving d remain valid. Actually we have now $d = d(z,\varepsilon)$ depend on z and ε, but it is important to note that $|d|$ is either equal to u_1 or to $\frac{1}{16}u_1$ which both are constants.

We then have

$$\left|\frac{z-t}{t}\right|, |\sigma| \geq 2\hat{d} \qquad \text{in Case 1,}$$
$$\left|\frac{z-t}{t}\right|, |\sigma| < \frac{3}{8}\hat{d} \qquad \text{in Case 2.} \tag{5.21}$$

We treat the **Case 1** first. Once more we deform the path of integration. Denote by $\hat{z} = \hat{z}(z,\varepsilon)$ the point $\hat{z} \in \operatorname{tr}\hat{\mathfrak{W}}$ which satisfies $|\hat{z}| = |z|$. Now, in case $\hat{z} \in \operatorname{tr}\mathfrak{W}_0$ or $\hat{z} \in \operatorname{tr}\mathfrak{W}_2$, depending on whether $x = \hat{z}$ satisfies the situation of Lemma 5.1(x), we indent the path $\hat{\mathfrak{W}}$ by a small circular arc around the point $\hat{z}$ with radius

$$\tau(z,\varepsilon) = \begin{cases} \tilde{\delta}|\hat{z}|, & |\hat{z}| \leq \delta|t| \quad \text{(Case 1a),} \\ \hat{\delta}|t|, & |\hat{z}| > \delta|t| \quad \text{(Case 1b)} \end{cases}$$

with a constant $\hat{\delta} > 0$ small enough to ensure $\operatorname{Re}\varphi(\varepsilon,w) < \operatorname{Re}\varphi(\varepsilon,T_0)$ $\big(w \in K_\tau(\hat{z})\big)$. The indentation will be made to the left or to the right such that $|w-z| \geq \tau$ for $w \in \operatorname{tr}\hat{\mathfrak{W}}$ and the integer n is set accordingly in equation (5.17), cf. Figure 3. The precise situation of Case 1b is not important at all, but it is crucial to know that for *small* $|z|$ (i.e. in Case 1a) a choice of a radius $\tau \asymp |z|$ is indeed possible because this way we have $\frac{w}{w-z} = O(1)$ for all $w \in \operatorname{tr}\hat{\mathfrak{W}}$ uniformly with respect to z and ε. In Case 1b we only have $\frac{w}{w-z} = O\big(t^{-1}\big)$ but this is good enough as long as the estimate does not depend on z. For simplicity the path deformed this way will still be denoted by $\hat{\mathfrak{W}}$.

Now, from Lemma 5.1(ii),(x) and the considerations above we have $\left|e^{\varphi(\varepsilon,w)}\right| \leq \left|e^{\varphi(\varepsilon,T_0)}\right|$ along $\mathfrak{W}_0$ while from Lemma 5.1(viii),(x) and the considerations above we have $\left|e^{\varphi(\varepsilon,w)}\right| \leq \left|e^{\varphi(\varepsilon,T_0)}\right|$ along $\mathfrak{W}_2$ (observe that $\left|e^{\varphi(\varepsilon,T_1)}\right| = \left|e^{\varphi(\varepsilon,T_0)}\right|$), and according to Lemma 5.1(ix) we finally have the same estimate along $\mathfrak{W}_3$.

Hence,

$$\left|\left(\int\limits_{\mathfrak{W}_0}+\int\limits_{\mathfrak{W}_2}+\int\limits_{\mathfrak{W}_3}\right)e^{\varphi(\varepsilon,w)}\frac{wF(w)}{w-z}dw\right|\le C_1\rho_0\left|e^{\varphi(\varepsilon,T_0)}\right|\cdot C_2\left|t^{-1}\right|C_3,$$

$$\Rightarrow\quad \left(\int\limits_{\mathfrak{W}_0}+\int\limits_{\mathfrak{W}_2}+\int\limits_{\mathfrak{W}_3}\right)e^{\varphi(\varepsilon,w)}\frac{wF(w)}{w-z}dw=e^{\varphi(\varepsilon,T_0)}\cdot O(t^{-1})\subset te^{\varphi(\varepsilon,t)}\cdot O\left(t^{-2}e^{-\tilde{x}d^2}\right)$$

$$\subset te^{\varphi(\varepsilon,t)}\cdot O\left(\tilde{\varepsilon}^{-2}e^{-|d|^2|\tilde{\varepsilon}|^{-r}}\right)$$

$$\subset te^{\varphi(\varepsilon,t)}\cdot O\left(\tilde{\varepsilon}^{r/2}\right)\subset te^{\varphi(\varepsilon,t)}\cdot O\left(\varepsilon^{r/2}\right)$$

uniformly with respect to (z,ε) admissible in Case 1. The dominant integral over $\mathfrak{W}_1$, however, can be transformed by

$$w=(1+h)t,\qquad h=b(\varepsilon,u) \tag{5.22}$$

and becomes

$$\int\limits_{\mathfrak{W}_1}e^{\varphi(\varepsilon,w)}\frac{wF(w)}{w-z}dw=\int\limits_{\frac{T_0-t}{t}}^{\frac{T_1-t}{t}}e^{\varphi(\varepsilon,(1+h)t)}\frac{(1+h)tF\big((1+h)t\big)}{(1+h)t-z}t\,dh$$

$$=te^{\varphi(\varepsilon,t)}\int\limits_{-d}^{d}e^{-\tilde{x}u^2}v(z,\varepsilon;u)\,du \tag{5.23}$$

$$\left[\text{where } v(z,\varepsilon;u)=\frac{(1+b(\varepsilon,u))F\big((1+b(\varepsilon,u))t\big)}{\left(1-\frac{z}{t}\right)+b(\varepsilon,u)}\partial_2 b(\varepsilon,u)\right]$$

$$=te^{\varphi(\varepsilon,t)}O\big(\tilde{x}^{-1/2}\big)\subset te^{\varphi(\varepsilon,t)}O\big(\varepsilon^{r/2}\big)$$

by a standard argument for Laplace-type integrals. The estimate holds uniformly with respect to (z,ε) admissible in Case 1 which can easily be seen from the analyticity and uniform boundedness of $v(z,\varepsilon;u)$:

We have $\left|1-\frac{z}{t}\right|\ge 2\hat{d}\wedge|u|\le\hat{d}$ ($\Rightarrow b(\varepsilon,u)\le\frac{3}{2}\hat{d}$), hence the denominator is bounded from below by $\frac{1}{2}\hat{d}$. On the other hand, from (5.18) we know that $\left|\partial_2 b(\varepsilon,u)\right|<\frac{3}{2}$ for $|u|\le\hat{d}$. Since F is required to be bounded on $\alpha\Sigma'$ the uniform boundedness is clear. Hence Proposition 5.2 follows in Case 1.

Now we come to **Case 2**. The actual difficulty consists of treating the integral over $\mathfrak{W}_1$ while the integrals over $\mathfrak{W}_0,\mathfrak{W}_2,\mathfrak{W}_3$ are very easy to estimate: from (5.21) and $|u|\ge\hat{d}$ we can see that $\frac{w}{w-z}=O(1)$ along these paths, uniformly w.r.to (z,ε), like in

Case 1. An indentation of these parts of the contour of integration is not necessary so that things become easier here. As a consequence, the estimate $\left|e^{\varphi(\varepsilon,w)}\right| \leq \left|e^{\varphi(\varepsilon,T_0)}\right|$ follows immediately from Lemma 5.1(ii),(viii),(ix). Thus, similarly to Case 1, we obtain

$$\left(\int_{\mathfrak{W}_0} + \int_{\mathfrak{W}_2} + \int_{\mathfrak{W}_3}\right) e^{\varphi(\varepsilon,w)} \frac{wF(w)}{w-z}\,dw = e^{\varphi(\varepsilon,T_0)} \cdot O(1) \subset te^{\varphi(\varepsilon,t)} \cdot O\big(\varepsilon^{r/2}\big).$$

The integral over $\mathfrak{W}_1$ can be transformed exactly as in (5.23) except that the line segment $\overline{-d,d}$ has to be indented by a small circular arc in the case that z lies *on* it – to the left or to the right, conforming to the choice of n in equation (5.17).

The function $v(z,\varepsilon;\cdot)$ admits a simple pole at $u=\sigma$. The residue at $u=\sigma$ is easily found: assume our standard variable substitutions (5.22) and consider

$$(u-\sigma)v(z,\varepsilon;u) = \underbrace{\Big(h - \frac{z-t}{t}\Big)\frac{wF(w)}{w-z}}_{\displaystyle \downarrow \atop \displaystyle \frac{z}{t}F(z)} \underbrace{\frac{u-\sigma}{h-\frac{z-t}{t}}\partial_2 b(\varepsilon,u)}_{\displaystyle \downarrow \atop \displaystyle 1}$$

for $u \to \sigma$, thus we know $\operatorname*{Res}_{u=\sigma} v(z,\varepsilon;u) = \frac{z}{t}F(z)$. Define

$$w(z,\varepsilon;u) := v(z,\varepsilon;u) - \frac{\frac{z}{t}F(z)}{u-\sigma} \tag{5.24}$$

Since we will study the integral involving $w(z,\varepsilon;\cdot)$ separately, we need to obtain estimates for this function first. Assuming our standard substitutions (5.22) and defining $y := \frac{z-t}{t}$ we find

$$\begin{aligned}
w(z,\varepsilon;u) &= v(z,\varepsilon;u) - \frac{\frac{z}{t}F(z)}{u-\sigma}\\
&= (1+h)\frac{1}{\partial_2 f(\varepsilon,h)} \cdot \frac{F\big((1+h)t\big)}{h-y} - \frac{(1+y)F\big((1+y)t\big)}{f(\varepsilon,h)-f(\varepsilon,y)},\\
\Rightarrow \quad \frac{t}{z}w(z,\varepsilon;u) &= \frac{1+h}{1+y}\cdot\frac{1}{\partial_2 f(\varepsilon,h)}\cdot\frac{F\big((1+h)t\big)}{h-y} - \frac{F\big((1+y)t\big)}{f(\varepsilon,h)-f(\varepsilon,y)}\\
&= F\big((1+h)t\big)\Big[\frac{1+h}{1+y}\cdot\frac{1}{\partial_2 f(\varepsilon,h)}\cdot\frac{1}{h-y} - \frac{1}{f(\varepsilon,h)-f(\varepsilon,y)}\Big] \qquad (5.25)\\
&\quad + \frac{F\big((1+h)t\big)-F\big((1+y)t\big)}{h-y}\cdot\frac{h-y}{f(\varepsilon,h)-f(\varepsilon,y)}.
\end{aligned}$$

Taking now another closed subsector $\Sigma'' \Supset \Sigma'$, $\alpha\Sigma'' \subset S$ and using $F(x) = O(1)$ there, by a standard argument using Cauchy's formula we conclude that $F'(x) = O(x^{-1})$ for

$x \in \alpha\Sigma'$. With this estimate we find

$$\left|\frac{F(b)-F(a)}{b-a}\right| \le C\max\{|a|^{-1},|b|^{-1}\} \qquad (a,b\in\alpha\Sigma'),$$
$$\left|\frac{F\big((1+h)t\big)-F\big((1+y)t\big)}{(h-y)t}\right| \le 2C|t|^{-1} \qquad (|h|,|y|\le\tfrac{1}{2}).$$

From the construction of d ($\frac{3}{2}u_1 \le \frac{1}{2}$) we have indeed $|h|,|y|\le\frac{1}{2}$, whereas from the construction of d ($u_1\le\tau$) and from Lemma 5.1(i) we know that $(1+y)t\in\alpha\Sigma'$ and from Lemma 5.1(viii) we know that $(1+h)t\in\alpha\Sigma'$. Hence $\frac{F((1+h)t)-F((1+y)t)}{h-y}$ is bounded uniformly for $|h|\le\frac{3}{2}\hat{d}$, $|y|\le\frac{3}{8}\hat{d}$ and ε as above and can be continued to an analytic function of ε,h,y by inserting the value $tF'\big((1+y)t\big)$ for $h=y$.

Furthermore, from the proof of Lemma 5.1(vii) we have

$$\frac{h-y}{f(\varepsilon,h)-f(\varepsilon,y)} = \frac{1}{\partial_2 f(\varepsilon,h)} + (h-y)\gamma_4(\varepsilon,h,y),$$
$$g(\varepsilon,h,y) = \frac{1}{(1+y)\partial_2 f(\varepsilon,h)} - \gamma_4(\varepsilon,h,y),$$

and observing $\partial_2 f(\varepsilon,h) = 1+O(h)$ uniformly w.r.to ε and h as above we see that all components of the right-hand side of (5.25) are bounded uniformly; hence $w(z,\varepsilon;u)$ is bounded uniformly and analytic w.r.to all variables.

As a consequence, $\int_{-d}^{d} e^{-\tilde{x}u^2} w(z,\varepsilon;u)\,du = O\big(\varepsilon^{r/2}\big)$ by the argumentation used to estimate (5.23).

Next, we express the integral over the polar part as to

$$\int_{-d}^{d} e^{-\tilde{x}u^2}\frac{du}{u-\sigma} = \int_{-\infty\tilde{\varepsilon}^{r/2}}^{\infty\tilde{\varepsilon}^{r/2}} e^{-\tilde{x}u^2}\frac{du}{u-\sigma} - \left(\int_{d}^{\infty\tilde{\varepsilon}^{r/2}} + \int_{-\infty\tilde{\varepsilon}^{r/2}}^{-d}\right) e^{-\tilde{x}u^2}\frac{du}{u-\sigma}$$

$$(5.26) \qquad = \int_{-\infty\tilde{\varepsilon}^{r/2}}^{\infty\tilde{\varepsilon}^{r/2}} e^{-\tilde{x}u^2}\frac{du}{u-\sigma} + \underbrace{O\big(e^{-\tilde{x}d^2}\big)}_{\subset O(\varepsilon^{r/2})}$$

The integral over the whole straight line can be expressed completely in terms of the complementary error function (cf. [Jon72], p.309):

$$(5.27) \qquad \int_{-\infty\tilde{\varepsilon}^{r/2}}^{\infty\tilde{\varepsilon}^{r/2}} e^{-\tilde{x}u^2}\frac{du}{u-\sigma} = \pi i e^{-\tilde{x}\sigma^2}\Big[2H\big(\mathrm{Im}(\sigma\tilde{x}^{1/2})\big) - \mathrm{erfc}(i\sigma\tilde{x}^{1/2})\Big]$$

where $H(\cdot)$ denotes the Heaviside function. It is known that

$$(5.28) \qquad 2H\big(-\mathrm{Re}\,x\big) - \mathrm{erfc}(x) = e^{-x^2}\cdot O\big(x^{-1}\big) \qquad (x\ne 0).$$

This estimate is good for *large* x but it is of no use here since σ or $x = i\sigma\tilde{x}^{1/2}$ can be *small*. However, there is another estimate which is somewhat weaker for large x but better for small arguments:

$$2H\big(-\operatorname{Re}x\big) - \operatorname{erfc}(x) = e^{-x^2}\cdot O(1) \qquad (x\in\mathbb{C}), \tag{5.29}$$

which applies to our situation. Therewith, the integral over the whole straight line in (5.26) can be estimated uniformly by $O(1)$. Observing $\frac{z}{t}F(z) = O(1)$ uniformly and combining with (5.24) and with the result for $\int_{-d}^{d} e^{-\tilde{x}u^2} w(z,\varepsilon;u)\,du$, we obtain in (5.23)

$$\int\limits_{\mathfrak{W}_1} e^{\varphi(\varepsilon,w)}\frac{wF(w)}{w-z}\,dw = te^{\varphi(\varepsilon,t)}\cdot O(1)$$

uniformly with respect to (z,ε) admissible in Case 2.

Now the result being proved for $\varepsilon\in\Sigma$, $|\varepsilon|\le\varepsilon_0$ it is clear by a continuity argument that the same result extends to $\varepsilon\in\Sigma$, which completes the proof. □

Remark. Proposition 5.2 is valid uniformly for $\varepsilon\in\Sigma$ and $z\in\overline{K}_\rho(0)$. But we can even do a little better in the "classical" Case 1: in fact, going over Case 1 of the proof of Proposition 5.2 we see that we actually have proved

$$\hat{\mathscr{I}}(z,\varepsilon) = te^{\varphi(\varepsilon,t)}\cdot O\big(\varepsilon^{r/2}\big). \tag{5.30}$$

This is the non-uniform analog to Proposition 5.2, and is in accordance with Proposition 4.1 of [HS99a] which considers the case of coupled z and ε (despite from different notations and from the fact that [HS99a] dealt with asymptotic *expansions* rather than estimates), as well as with a couple of other results covering the case $r=1$. The result in [HS99a] was proved to be uniform for $|\arg z - \arg\alpha|\ge\hat{d}$ for every $\hat{d}>0$ but could easily be extended to $\left|\frac{z-t}{t}\right|\ge\hat{d}$. Our uniform result, however, includes the case of approximation of pole and saddle-point and hence a full neighborhood of the singular direction $\arg z = \arg\alpha$. The "price" for this uniformity is the loss of a power $\varepsilon^{r/2}$.

The reason for this phenomenon is exactly the problem of estimating the integral in (5.27): while the estimate (5.28) is better for large arguments, (5.29) includes small arguments at the price of the loss of a power of x.

5.4 Asymptotic estimates II: The residue term

Remains to uniformly estimate the residue term in (5.17)

$$r(z,\varepsilon) = 2\pi i n(z,\varepsilon)\cdot zF(z)e^{\varphi(\varepsilon,z)}.$$

A universal but just as trivial estimate would be

$$r(z,\varepsilon) = n(z,\varepsilon)\cdot e^{\varphi(\varepsilon,z)}O(z). \tag{5.31}$$

But this result has some major drawbacks:

(i) The term $n(z,\varepsilon)$ is not explicit. To know whether it is actually different from zero we still have to investigate the exact shape of the region enclosed between $\operatorname{tr}\mathfrak{W}$ and $\operatorname{tr}\hat{\mathfrak{W}}$ (see Figures 3,4). But we cannot neglect it either because the result would then be too bad "far away" from $t(\varepsilon)$.

(ii) In addition, $n(z,\varepsilon)$ is discontinuous. This is inevitable because it represents exactly the difference between integrating to the "left" or to the "right" of z. The discontinuity could only be avoided if the corresponding estimate would be somewhat weakened.

(iii) This way the whole integral $\mathscr{I}(z,\varepsilon)$ in (5.17) could only be estimated by a sum of two terms involving different exponentials. This sum, although natural like the Stokes' phenomenon it represents, is not very handy in the applications except from some special cases where additional properties enter.

How good, in general, can $r(z,\varepsilon)$ be estimated? – In any case we want the result to include positive real ε with $\varepsilon < \varepsilon_0$. If now ε is real and positive and T does not lie *on* the singular direction $\arg w = \arg\alpha$ then for the two points

$$t(\varepsilon) = \alpha\varepsilon\bigl(1+O(\varepsilon)\bigr) \qquad \text{and} \qquad \hat{t}(\varepsilon) := \frac{T}{\rho_0}|\alpha|\varepsilon$$

we have $t(\varepsilon) \in \operatorname{tr}\hat{\mathfrak{W}}$, $\hat{t}(\varepsilon) \in \operatorname{tr}\mathfrak{W}$ and $\lim_{\varepsilon\to 0} \ln\frac{\hat{t}(\varepsilon)}{t(\varepsilon)} = i(\arg T - \arg\alpha)$. Hence there is an imaginary $h_0 \neq 0$ such that for the point

$$z_0 := z_0(\varepsilon) := (1+h_0)t(\varepsilon)$$

we have $n(z_0,\varepsilon) \neq 0$ for $\varepsilon < \varepsilon_0$ and

$$\begin{aligned} \varphi(\varepsilon,z_0) - \varphi(\varepsilon,t) &= -\tilde{x}f(\varepsilon,h_0)^2 = -\tilde{x}h_0^2\bigl(1+O(h_0)\bigr) \\ &= |h_0|^2\bigl(1+O(h_0)\bigr)\cdot\tilde{\varepsilon}^{-r} \asymp \varepsilon^{-r}. \end{aligned} \tag{5.32}$$

Thus $r(z_0,\varepsilon)$ could grow faster of exponential order r w.r.to ε, compared with $\hat{\mathscr{I}}(z,\varepsilon)$. This is the "worst case".

Under certain assumptions, however, it is possible to obtain better and nevertheless uniform results. In the first situation we do not impose any restrictions on z or on the polynomial p, but in order to avoid a behavior like in (5.32), we reduce the region of admissible ε:

Proposition 5.3. *Let* $r \in \mathbb{N}^*$, $p(x) = \sum_{\ell=1}^{r} p_\ell x^\ell \in x\mathbb{C}[x]$, $\alpha = (-rp_r)^{1/r} \neq 0$. *Consider the sectors* $\Sigma = \overline{S}_{\rho(\Sigma)}\bigl(-\delta(\Sigma),\delta(\Sigma)\bigr)$ *with* $0 < 2\delta(\Sigma) < \frac{\pi}{r}$ *and* $S = S_{\hat{\rho}}(d_1,d_2) \supset \alpha\Sigma$ *and a bounded region* $\Omega \subset \Sigma$ *with the property*

$$\arg\varepsilon = O(\varepsilon) \qquad (\varepsilon \in \Omega).$$

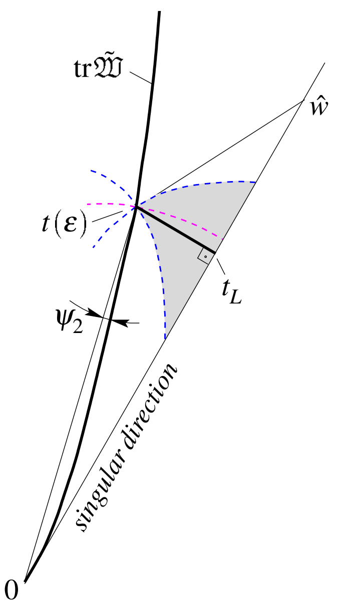

Figure 4: Behavior of $\operatorname{Re}\varphi(\varepsilon,\cdot)$ near the saddle point and shaded region of "critical" z (ε is fixed)

Let T with $\arg T = \arg\alpha$ be fixed and $0 < \rho < \rho_0 = |T| < \hat{\rho}$. Let $F : S \to \mathbb{C}$ be an analytic function bounded on every closed subsector of S. Finally, define $\hat{\mathfrak{W}}$ and $r(z,\varepsilon)$ as in Section 5.3. Then the following estimate holds:

$$r(z,\varepsilon) = e^{\varphi(\varepsilon,t)+O(\varepsilon^{-r+2})} \cdot O(z), \qquad t = t(\varepsilon)$$

uniformly for $\varepsilon \in \Omega$ and $z \in \overline{K}_\rho(0)$. Here, $t(\varepsilon)$ is defined in Section 5.1 and satisfies (5.3).

Remark. In the case of "rank" one or two ($r \in \{1,2\}$), Proposition 5.3 reads

$$r(z,\varepsilon) = e^{\varphi(\varepsilon,t)} \cdot O(z), \qquad t = t(\varepsilon).$$

This is more or less what one would expect in the case $r = 1$: the integral $\mathscr{I}(z,\varepsilon)$ in (5.1) satisfies an estimate with the leading term $e^{\varphi(\varepsilon,t)}$, just like the integral $\hat{\mathscr{I}}(z,\varepsilon)$ in (5.17). This is in accordance with many of the earlier results on uniform asymptotics. However, it is a new result that the same is true for $r = 2$, too.

Proof. Let $\varepsilon \in \Sigma$ be arbitrarily given then the assumptions of Lemma 5.1 and Proposition 5.2 are satisfied. In view of (5.5a) the function $\operatorname{Re}\varphi(\varepsilon,\cdot)$ has a simple saddle at $t(\varepsilon)$. For z near $\operatorname{tr}\tilde{\mathfrak{W}}$ we have $\operatorname{Re}\varphi(\varepsilon,z) \le \operatorname{Re}\varphi(\varepsilon,t)$, thus Proposition 5.3 is obviously true for these z. Remains to consider among the z satisfying $n(z,\varepsilon) \neq 0$ only those "critical" z which satisfy $\operatorname{Re}\varphi(\varepsilon,z) > \operatorname{Re}\varphi(\varepsilon,t)$ and which lie between the level lines of $\operatorname{Re}\varphi(\varepsilon,\cdot)$ through $t(\varepsilon)$ (marked shaded in Figure 4).

To obtain estimates for $\frac{z-t}{t}$ we introduce the auxiliary lines $\overline{Ot}$ and $\overline{tt_L}$ (the latter being perpendicular to $\overline{OT}$) and the tangent to the level line of $\operatorname{Re}\varphi(\varepsilon,\cdot)$ at $w=t$ which, as we shall see by trigonometric arguments below, intersects $\overline{OT}$ at a certain point $\hat{w}$ that yields an upper bound for the "critical" z. Define three angles $\psi_1 := |\sphericalangle(t_L,O,t)|$, $\psi_3 := |\sphericalangle(t_L,t,\hat{w})|$, and ψ_2 as the angle between $\overline{Ot}$ and $\operatorname{tr}\tilde{\mathfrak{W}}$ in $t(\varepsilon)$ (see Fig. 4).

It is immediate from (5.3) that

$$\psi_1 = \arg\varepsilon + O(\varepsilon) \subset O(\varepsilon).$$

On the other hand, from Lemma 5.1(iv),(v),(vi) it is not hard to see that the angle ψ_2 satisfies

$$\psi_2 = \frac{r}{2}\arg\varepsilon + O\big(\varepsilon^{1/2}\big) \subset O\big(\varepsilon^{1/2}\big),$$

for $r \geq 2$ even little better. Hence we conclude

$$\begin{aligned}\psi_3 &= \psi_2 + \frac{3\pi}{4} - \Big(\frac{\pi}{2} - \psi_1\Big)\\ &= \frac{\pi}{4} + O\big(\varepsilon^{1/2}\big),\\ \hat{w} - t &= \big(t_L - t\big)O(1)\\ &= t\cdot O(\varepsilon),\\ h &= \frac{w-t}{t} = O(\varepsilon)\end{aligned}$$

for $\varepsilon \in \Omega$, $|\varepsilon| \leq \varepsilon_0$ with small enough ε_0 and for all "critical" w. Thus we also have $u = f(\varepsilon,h) = O(\varepsilon)$ for these h. It follows

$$\varphi(\varepsilon,w) - \varphi(\varepsilon,t) = -\tilde{\varepsilon}^{-r}u^2 = O\big(\varepsilon^{-r+2}\big).$$

The ε with $|\varepsilon| > \varepsilon_0$ can now be included by a continuity argument, which proves Proposition 5.3. $\square$

Remark. Proposition 5.3 can be modified in different ways. If e.g. we would require $\arg\varepsilon = O\big(\varepsilon^{1/2}\big)$ instead of $O(\varepsilon)$ for $\varepsilon \in \Omega$ then in the result we would lose one power of ε in the exponential. Moreover it is not really necessary that $\arg T = \arg\alpha$. If these arguments differ, $\arg\varepsilon$ has to vary, too: it is easy to see that a condition of the kind $\arg\frac{\varepsilon}{T/\alpha} = O(\varepsilon)$ leads to the same result as in Proposition 5.3.

We can do even better, namely in the monomial case $p(x) = -\frac{1}{r}(\alpha x)^r$. Like in Proposition 5.3, we have to consider subregions of the sector Σ, but if we do so we can even achieve

$$e^{\varphi(\varepsilon,z)} = O\Big(e^{\varphi(\varepsilon,t)}\Big), \qquad t = t(\varepsilon)$$

for the "critical" z. This is the "best case":

Proposition 5.4. *Let $r \in \mathbb{N}^*$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $p(x) = -\frac{1}{r}(\alpha x)^r$. Consider the sectors $\Sigma = \overline{S}_{\rho(\Sigma)}\big(-\delta(\Sigma), \delta(\Sigma)\big)$ with $0 < 2\delta(\Sigma) < \frac{\pi}{r}$ and $S = S_{\hat\rho}(d_1, d_2) \supset \alpha\Sigma$ and a bounded region $\Omega \subset \Sigma$ with the property*

$$\arg\varepsilon = O\big(\varepsilon^{r/2}\big) \qquad (\varepsilon \in \Omega).$$

Let T with $\arg T = \arg\alpha$ be fixed and $0 < \rho < \rho_0 = |T| < \hat\rho$. Let $F: S \to \mathbb{C}$ be an analytic function bounded on every closed subsector of S. Finally, define $\hat{\mathfrak{W}}$ and $r(z,\varepsilon)$ as in Section 5.3. Then the following estimate holds:

$$r(z,\varepsilon) = e^{\varphi(\varepsilon,t)} \cdot O(z), \qquad t = t(\varepsilon)$$

uniformly for $\varepsilon \in \Omega$ and $z \in \overline{K}_\rho(0)$. Here, $t(\varepsilon)$ is defined in Section 5.1 and satisfies (5.3).

Proof. Going over part of the analysis of Sections 5.1 and 5.2 it is easy to see that $t(\varepsilon)$ and $\tilde\varepsilon(\varepsilon)$ are exact multiples of ε:

$$t(\varepsilon) = \alpha\varepsilon, \qquad \tilde\varepsilon(\varepsilon) = \big(\tfrac{2}{r}\big)^{1/r}\varepsilon.$$

Repeating the steps of the proof of Proposition 5.3 we find

$$\begin{aligned} \psi_1 &= \arg\varepsilon, & \hat w - t &= t \cdot O\big(\varepsilon^{r/2}\big),\\ \psi_2 &= \frac{r}{2}\arg\varepsilon + O\big(\varepsilon^{r/2}\big), & \varphi(\varepsilon, w) - \varphi(\varepsilon, t) &= O(1) \end{aligned}$$

for small enough $|\varepsilon|$ and "critical" w. Proposition 5.4 now follows by the same argumentation used for the proof of Proposition 5.3. □

5.5 Main asymptotic result

In this section we assemble the results from Sections 5.3 and 5.4.

Theorem 5.5. *Let $r \in \mathbb{N}^*$, $p(x) = \sum_{\ell=1}^r p_\ell x^\ell \in x\mathbb{C}[x]$, $\alpha = (-rp_r)^{1/r} \neq 0$. Consider the sector $S = S_{\hat\rho}(d_1, d_2)$ with $d_1 < \arg\alpha < d_2$ and a bounded region $\Omega \subset \mathbb{C}$ with the property*

$$\arg\varepsilon = O(\varepsilon) \qquad (\varepsilon \in \Omega).$$

Let T with $\arg T = \arg\alpha$ be fixed and $0 < \rho < \rho_0 = |T| < \hat\rho$. Define a path $\mathfrak{W}$ to be the line segment from 0 to T, possibly indented by a small circular arc around z in the case $0 \neq z \in \overline{0T}$. Let $F: S \to \mathbb{C}$ be an analytic function bounded on every closed subsector of S. Then in (5.1) *we have*

$$\int_{\mathfrak{W}} e^{p(w^{-1})} w^{-\varepsilon^{-r}} \frac{wF(w)}{w-z}\, dw = e^{\varphi(\varepsilon,t) + O(\varepsilon^{-r+2})} \cdot O(1), \qquad t = t(\varepsilon) \tag{5.33}$$

uniformly for $\varepsilon \in \Omega$ *and* $z \in \overline{K}_\rho(0)$. *Especially we can estimate*

$$\int_{\mathfrak{W}} e^{p(w^{-1})} w^{-\varepsilon^{-r}} \frac{wF(w)}{w-z}\,dw = \left(e^{1/r}\alpha\varepsilon\right)^{-\varepsilon^{-r}} \cdot e^{O(\varepsilon^{-r+1})} \cdot O(1) \tag{5.34}$$

uniformly for these (z, ε). *Here,* $t(\varepsilon)$ *is defined in Section 5.1 and satisfies* (5.3).

Remark. If Ω satisfies $\arg\varepsilon = O(\varepsilon^{r/2})$ $(\varepsilon \in \Omega)$ and p is a monomial then by Proposition 5.4, the estimate (5.33) still holds with the term $O(\varepsilon^{-r+2})$ canceled.

Proof of the theorem. In view of the assumption on Ω there is an $\varepsilon_0 > 0$ such that $\Omega \cap D(\varepsilon_0)$ is part of a closed sector Σ with opening $< \frac{\pi}{r}$, bisecting direction 0 for which $\alpha\Sigma \subset S$ holds. Therewith we can deform the path of integration to the contour $\hat{\mathfrak{W}}$ constructed at the beginning of Section 5.3 and split up the integral (5.1) into the integral part $\hat{\mathscr{I}}(z, \varepsilon)$ and the residue term $r(z, \varepsilon)$ conforming to (5.17). Then the Propositions 5.2 and 5.3 are applicable, and (5.33) follows immediately for these ε. Inserting (5.8) into (5.33) yields (5.34) for these ε.

Both estimates again extend to $\varepsilon \in \Omega$ by continuity. Theorem 5.5 follows. □

Let us compare the result of Theorem 5.5 with earlier results. To my knowledge, only Hoeppner and Schäfke [HS99a] have considered the case $r > 1$, so let us take a simple example where $r = 1$:

$$\mathscr{I}(z, \varepsilon) = \int_0^{-1} e^{w^{-1}} w^{-\varepsilon^{-1}} \frac{wF(w)}{w-z}\,dw; \qquad \alpha := e^{\pi i}. \tag{5.35}$$

By the substitutions $\tilde{z} = z^{-1}$, $N = \varepsilon^{-1} - 1$, $t = e^{\pi i} w^{-1}$, $u(t) = F(w)$ this integral is the $e^{-\pi i N}$-multiple of the Stieltjes-type integral

$$\tilde{z} \int_1^{+\infty} \frac{e^{-t} t^{N-1}}{t + \tilde{z}} u(t)\,dt$$

which appears in most of the references, rather than the integral (5.35). The restriction upon the case $r = 1$ presents no substantial loss of generality as this case already shows all important features of the general situation.

Following the terminology of Section 5.1, the function

$$\varphi(\varepsilon, w) = w^{-1} - \varepsilon^{-1} \log w$$

has a saddle point at $t(\varepsilon) = \alpha\varepsilon$ (for short we write $t(\varepsilon) = -\varepsilon$).

We have to distinguish two principal kinds of asymptotic results regarding (5.35): first, the results that are uniform only in sectors $|\arg z| \leq \pi - \delta$ which we will call *non-uniform* (w.r.to $\delta \geq 0$, that is, in a full neighborhood of the singular directions $\arg z = \pm\pi$). In contrast, we will denote the results for $|\arg z| \leq \pi$ as being *uniform*.

Speaking about uniform asymptotics, nothing has been said so far in the literature about the case of independent z and ε. Theorem 5.5 states that if ε is restricted to Ω, then $\mathscr{I}(z,\varepsilon) = e^{\varphi(\varepsilon,-\varepsilon)} \cdot O(1)$. All of the earlier investigations have made the assumption $\varepsilon^{-1} = \beta|z|^{-1} + O(1)$ with a positive constant β. In this special situation, Olde Daalhuis and Olver have obtained the estimate $e^{\varphi(\varepsilon,-\varepsilon)} \cdot O(\varepsilon^{1/2})$ (Equation (4.6) in [OO95a]). If this coupling is optimized – that is, if $\varepsilon^{-1} = |z|^{-1} + O(1)$ –, then this estimate can be improved by an additional factor $\varepsilon^{1/2}$ (cf. the case $m = 0$ in Equation (2.16) of [OO94], and Equation (4.11) in [OO95a]).

On the other hand, in the case $|\arg z| \le \pi - \delta$ we have seen in (5.30) that $\mathscr{I}(z,\varepsilon) = e^{\varphi(\varepsilon,-\varepsilon)} \cdot O(\varepsilon^{3/2})$ (observe that $n(z,\varepsilon) = 0$ for small enough $\varepsilon \in \Omega$). This is the very general case of non-uniform asymptotics for (5.35), without any coupling. It may be surprising, however, that a coupling of z and ε by $\varepsilon^{-1} = \beta|z|^{-1} + O(1)$ or even by $\varepsilon^{-1} = |z|^{-1} + O(1)$ does not lead to sharper estimates: the former case has been studied by Hoeppner and Schäfke (proof of Proposition 4.1 of [HS99a]), the latter by Olde Daalhuis and Olver (case $m = 0$ in Equation (5.6) of [OO94]).

Altogether we obtain the following overview:

	Uniform estimates	Non-uniform estimates
No coupling	$O(1)$	$O(\varepsilon^{3/2})$
General coupling	$O(\varepsilon^{1/2})$	
Optimal coupling	$O(\varepsilon)$	

Table 1: Asymptotic estimates of the integral (5.35) under different assumptions: factors after $e^{\varphi(\varepsilon,-\varepsilon)}$

5.6 A variant on cusps

In the previous subsections, a function F entered the integral (5.1) which was supposed to be analytic on some open sector S containing a singular direction of p and bounded on proper subsectors. This is perfect in applications where F is e.g. an analytic function admitting an asymptotic expansion in an open sector, like in the situation in Section 4.4.

However, in Section 4.5 and successive levels of hyperasymptotics, we have to estimate integrals in whose integrand again integrals of the kind of (5.1) appear in place of the function F. Though in Theorem 5.5, we have estimated such integrals uniformly in a whole neighborhood of the origin, they are not an analytic function of the parameter: by definition of $\mathfrak{W}$, they have a discontinuity along $\overline{OT}$.

To obtain results which can be applied recursively, we have to overcome this restriction, i.e. to consider analytic continuations of $\mathscr{I}(z,\varepsilon)$. Therefore, in (5.1) think of $\mathfrak{W}$ as passing $w=z$ always to the "left" ($\mathfrak{W}=\mathfrak{W}_\ell$) or always to the "right" ($\mathfrak{W}=\mathfrak{W}_r$). Then we still have the decomposition (5.17) with the appropriate new $n(z,\varepsilon)$.

The integral part $\hat{\mathscr{I}}(z,\varepsilon)$ has been estimated uniformly in Proposition 5.2, hence remains the residue term $r(z,\varepsilon)$. For fixed ε, the set of "critical" z, however, can be larger than in Proposition 5.3 and in Figure 4. See Figure 5. We can obtain the same estimate as in Proposition 5.3 if for those critical z we still have

$$z-t=O(\varepsilon^2).$$

Since in a sector, in general, we only have $z-t=O(\varepsilon)$, we will confine our considerations to smaller regions: cusped domains with cuspidal point 0 (for short: cusps, see (1.2)). They are such that $\arg\frac{z}{\alpha}=O(z)$ is satisfied for these z. This property is well-suited to obtain the desired estimates, see (xii) on page 109 for details.

Thus, we modify the situation of Section 5.1 as follows: Throughout this section let r be a positive integer and $p(x)=\sum_{\ell=1}^{r}p_\ell x^\ell\in x\mathbb{C}[x]$ a polynomial of degree r. Fix a determination of the r-th root $\alpha:=(-rp_r)^{1/r}$, and denote $\beta:=\frac{(r-1)p_{r-1}}{rp_r}$. Let $\Omega=\overline{\Upsilon}\big(0,K(\Omega),\rho(\Omega)\big)$ be a closed cusp at 0 with bisecting direction 0, and let $G=\Upsilon(\arg\alpha,\hat{K},\hat{\rho})$ be an open cusp at 0. Both cusped domains are related: G must be large enough with respect to Ω, namely $\hat{\rho}>|\alpha|\rho(\Omega)$ and $\hat{K}>|\alpha|^{-1}\big(K(\Omega)+\frac{1}{r}|\mathrm{Im}(\alpha\beta)|\big)$. This is to guarantee that the saddle point $t(\varepsilon)$ lies in G for small enough $\varepsilon\in\Omega$, see (i) below. Let T with $\arg T=\arg\alpha$ be fixed, and assume $T\in\alpha\Omega\subset G$. This implies that we have $|T|=:\rho_0<\hat{\rho}$. Let $0<\rho<\rho_0$, and let $F:\ G\to\mathbb{C}$ be an analytic function bounded on every proper subcusp of G. Then for $\varepsilon\in\Omega$ and $z\in\overline{K}_\rho(0)$ we consider the integral

$$\mathscr{I}(z,\varepsilon):=\int\limits_{\mathfrak{W}}e^{p(w^{-1})}w^{-\varepsilon^{-r}}\frac{w^2F(w)}{w-z}\,dw \tag{5.36}$$

where $\mathfrak{W}\in\{\mathfrak{W}_\ell,\mathfrak{W}_r,\mathfrak{W}(z)\}$ is a contour in G from 0 to T, either passing $w=z$ to the "left" ($\mathfrak{W}=\mathfrak{W}_\ell$) or to the "right" ($\mathfrak{W}=\mathfrak{W}_r$), or following the line segment $\overline{OT}$ and being indented by a small circular arc around z in the case $0\neq z\in\overline{0T}$ ($\mathfrak{W}=\mathfrak{W}(z)$). Accordingly, we consider the regions G_ℓ, G_r and $\overline{K}_\rho(0)$ where $G_\ell:=G\cup\overline{S}_{\hat{\rho}}(\arg\alpha-\pi,\arg\alpha)$ and $G_r:=G\cup\overline{S}_{\hat{\rho}}(\arg\alpha,\arg\alpha+\pi)$ are "right" and "left" extensions of G, respectively.

This is not simply a special case of the situation of Section 5.1 as the domain G is considerably smaller than any sector S was. So for this modified situation, we have to adapt the proofs of the previous subsections. Most of the considerations can be left unchanged while we have to calculate some quantities to higher order.

(i) E.g., the saddle point $t(\varepsilon)$ of the function $\operatorname{Re}\varphi(\varepsilon,\cdot)$ satisfies

(5.37) $$t(\varepsilon) = \alpha\varepsilon\Big(1 + \tfrac{\alpha\beta}{r}\varepsilon + \varepsilon^2\gamma(\varepsilon)\Big)$$

with some function γ analytic near the origin. If $w \in K_{\delta|t|^2}(t)$ then

$$\operatorname{Im}\tfrac{\alpha}{w} = \operatorname{Im}\tfrac{1}{\varepsilon} - \tfrac{1}{r}\operatorname{Im}(\alpha\beta) - \operatorname{Im}(\alpha\delta\mu) + O(t)$$

with some $|\mu| \le 1$, hence from the assumption on $\hat{K}$ we know that for any closed subcusp $\alpha\Omega' \subset G$, $\Omega' \Supset \Omega$ there are small enough $\delta, \varepsilon_0 > 0$ such that $K_{\delta|t|^2}(t) \subset \alpha\Omega' \subset G$ for $\varepsilon \in \Omega$, $|\varepsilon| \le \varepsilon_0$.

(ii) Other objects like $\tilde{\mathfrak{W}}$, $\tilde{x}$, $\tilde{\varepsilon}$, and f do not change, and the statements of Lemma 5.1(ii)–(vii) can be left unchanged.

(iii) We deform the path $\mathfrak{W}$ of integration to one which crosses the saddle at $t(\varepsilon)$. With $d := \delta|\alpha||\varepsilon|\exp\big(\tfrac{r}{2}i\arg\tilde{\varepsilon}\big)$ define $T_0 := \big(1 + b(\varepsilon,-d)\big)t$ and $T_1 := \big(1 + b(\varepsilon,d)\big)t$, respectively. For small enough ε_0 we then have $T_0, T_1 \in \alpha\Omega'$ for $|\varepsilon| \le \varepsilon_0$. Let $\mathfrak{W}_0$ be a regular parameterization of the curve $\operatorname{Im}\frac{\alpha}{w} = \operatorname{Im}\frac{\alpha}{T_0}$ from 0 to T_0, $\mathfrak{W}_1$ the part of the path $\tilde{\mathfrak{W}}$ of steepest descent from T_0 to T_1, $\mathfrak{W}_2$ the line segment from T_1 to $T_2 := \frac{\rho}{|T_1|}T_1$, and $\mathfrak{W}_3$ be the circular arc joining T_2 to T. $\hat{\mathfrak{W}} := \hat{\mathfrak{W}}(\varepsilon,d)$ will be the composition $\mathfrak{W}_0\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$ of those paths. This contour has to be indented by a small circular arc in the case z lies on it. Now we have

(5.38) $$\mathscr{I}(z,\varepsilon) = \underbrace{\int_{\hat{\mathfrak{W}}} e^{\varphi(\varepsilon,w)}\frac{w^2F(w)}{w-z}\,dw}_{=:\,\hat{\mathscr{I}}(z,\varepsilon)} + \underbrace{2\pi i n(z,\varepsilon)\cdot\operatorname*{Res}_{w=z}\left[e^{\varphi(\varepsilon,w)}\frac{w^2F(w)}{w-z}\right]}_{=:\,r(z,\varepsilon)}.$$

(iv) First we want to estimate $\hat{\mathscr{I}}(z,\varepsilon)$. Let δ be as above, let δ_0 be as in Lemma 5.1(v), and choose u_0 such that (5.18), (5.19) are satisfied. Without loss of generality we may choose δ as small as to satisfy

$$\delta \le \tfrac{1}{\rho(\Omega)}\min\{\tfrac{2}{3}\delta_0, u_0\}.$$

Then for $|u| \le \delta|t|$ we have $|h| \le \frac{3}{2}\delta|t| \le \delta_0$. With σ from (5.20) consider
Case 1: $|\sigma| \ge \frac{1}{4}\delta|t|$; **Case 2:** $|\sigma| < \frac{1}{4}\delta|t|$.

Again, re-define

$$\hat{\delta} := \begin{cases} \frac{1}{16}\delta & \text{in Case 1,} \\ \delta & \text{in Case 2,} \end{cases}$$
$$d := \hat{\delta}|t|\exp\big(\tfrac{r}{2}i\arg\tilde{\varepsilon}\big)$$

and all subsequent quantities (T_0, T_1, $\hat{\mathfrak{W}}$) with δ replaced by $\hat{\delta}$. For convenience, we will again write δ in place of $\hat{\delta}$. We then have

$$\begin{aligned} &\left|\frac{z-t}{t}\right|, |\sigma| \geq 2\delta|t| && \text{in Case 1,} \\ &\left|\frac{z-t}{t}\right|, |\sigma| < \frac{3}{8}\delta|t| && \text{in Case 2.} \end{aligned} \tag{5.39}$$

(v) Consider the **Case 1** first. In case $\hat{z} \in \mathrm{tr}\,\mathfrak{W}_0$ or $\hat{z} \in \mathrm{tr}\,\mathfrak{W}_2$, we again deform the path of integration by a small circular arc around $\hat{z}$ with radius

$$\tau(z) = \tilde{\delta}|\hat{z}|^2$$

with a constant $\tilde{\delta} > 0$ small enough to ensure $\overline{K}_\tau(\hat{z}) \subset \alpha\Omega'$ and $\operatorname{Re}\varphi(\varepsilon, w) \leq \operatorname{Re}\varphi(\varepsilon, T_0)$ for $w \in \overline{K}_\tau(\hat{z})$. The indentation will be made to the left or to the right such that $|w - z| \geq \tau$ for $w \in \mathrm{tr}\,\hat{\mathfrak{W}}$ and the integer n is set accordingly in equation (5.38). For simplicity the path deformed this way will still be denoted by $\hat{\mathfrak{W}}$. We then have $F(w) = O(1)$ and $\frac{w^2}{w-z} = O(1)$ for $w \in \mathrm{tr}\,\hat{\mathfrak{W}}$, uniformly with respect to $|\varepsilon| \leq \varepsilon_0$, $z \in \overline{K}_\rho(0)$ in Case 1.

(vi) From $T_1 := \big(1 + b(\varepsilon, d)\big)t$ we have $\arg(T_1/\alpha) = O(\varepsilon)$ as well as

$$|T_1| = |\alpha\varepsilon|\Big[1 + \big(\tfrac{1}{r}\operatorname{Re}(\alpha\beta) + \delta|\alpha|\big)|\varepsilon| + O(\varepsilon^2)\Big]. \tag{5.40}$$

Therewith we can investigate the function $\operatorname{Re}\varphi(\varepsilon, \cdot)$ along $\mathfrak{W}_2$, similarly to Lemma 5.1(viii). We merely have to calculate the solution q_0 up to the second order to obtain

$$q_0\big[1 - \tfrac{\gamma}{r}q_0 + O(q_0^2)\big] = |\alpha\varepsilon|\left(\frac{\cos\big(r\arg\frac{T_1}{\alpha}\big)}{\cos(r\arg\varepsilon)}\right)^{1/r} \tag{5.41}$$

with $\gamma = |\beta|\cos\arg(\beta T_1) + |\beta|\sin\arg(\beta T_1)\tan\big(r\arg\frac{T_1}{\alpha}\big) = |\beta|\cos\arg(\alpha\beta) + O(\varepsilon)$. Inserting into (5.41), a lengthy calculation shows

$$q_0 = |\alpha\varepsilon|\Big[1 + \tfrac{1}{r}\operatorname{Re}(\alpha\beta)|\varepsilon| + O(\varepsilon^2)\Big]$$

which is smaller than $|T_1|$ for small enough ε_0, see (5.40). Since q_0 is the only positive solution of the equation $\frac{\partial}{\partial q}\operatorname{Re}\varphi\big(\varepsilon, q\exp(i\arg T_1)\big)\big|_{q=q_0} = 0$ we see that $\operatorname{Re}\varphi\big(\varepsilon, q\exp(i\arg T_1)\big)$ decreases monotonously for $q > q_0$.

(vii) Similarly we can prove that $\operatorname{Re}\varphi(\varepsilon, w) \leq \operatorname{Re}\varphi(\varepsilon, T_0)$ for $w \in \mathrm{tr}\,\mathfrak{W}_0$. Apart from the circular arc around $\hat{z}$ we have $w^{-1} = T_0^{-1} + \alpha^{-1}q$ with a $q \geq 0$. Insertion gives

$$\tfrac{\partial}{\partial q}\varphi(\varepsilon, w) = \Big(p'(T_0^{-1})\alpha^{-1} + \varepsilon^{-r}\alpha^{-1}T_0\Big) + \Big(p''(T_0^{-1})\alpha^{-2} - \varepsilon^{-r}\alpha^{-2}T_0^2\Big)q + O(q^2).$$

Solving the equation $\operatorname{Re}\frac{\partial}{\partial q}\varphi(\varepsilon,w)\big|_{q=q_0}=0$ asymptotically, we find

$$q_0\big(1+O(q_0)\big)=-\frac{\operatorname{Re}\big(p'(T_0^{-1})\alpha^{-1}+\varepsilon^{-r}\alpha^{-1}T_0\big)}{\operatorname{Re}\big(p''(T_0^{-1})\alpha^{-2}-\varepsilon^{-r}\alpha^{-2}T_0^2\big)}=-\delta|\alpha|\big(1+O(\varepsilon)\big)$$
$$\Rightarrow\quad q_0=\Big(-\delta|\alpha|+O(\delta^2)\Big)+O(\varepsilon)$$

uniformly w.r.to $|\varepsilon|\le\varepsilon_0$. Thus for small enough $\delta,\varepsilon_0>0$ we have $q_0<0$, and consequently there is no positive solution q of the above equation. Thus if $w^{-1}=T_0^{-1}+\alpha^{-1}q$ we know that $\operatorname{Re}\varphi(\varepsilon,w)$ decreases monotonously w.r.to $q\ge 0$.

(viii) Then on one hand we find

$$\Bigg(\int\limits_{\mathfrak{W}_0}+\int\limits_{\mathfrak{W}_2}+\int\limits_{\mathfrak{W}_3}\Bigg)e^{\varphi(\varepsilon,w)}\frac{w^2F(w)}{w-z}\,dw=e^{\varphi(\varepsilon,T_0)}\cdot O(1)$$
$$=e^{\varphi(\varepsilon,t)}\cdot O\Big(e^{-\delta^2|\alpha|^2|\varepsilon|^2|\tilde\varepsilon|^{-r}}\Big)\subset e^{\varphi(\varepsilon,t)}\cdot\begin{cases}O(1), & r=1,2\\ O(\varepsilon^{r/2+1}), & r\ge 3\end{cases}$$

and on the other hand

(5.42)
$$\int\limits_{\mathfrak{W}_1}e^{\varphi(\varepsilon,w)}\frac{w^2F(w)}{w-z}\,dw=te^{\varphi(\varepsilon,t)}\int\limits_{-d}^{d}e^{-\tilde{x}u^2}\tilde{v}(z,\varepsilon;u)\,du$$
$$\left[\begin{array}{l}\text{where } \tilde v(z,\varepsilon;u)=t\cdot\big(1+b(\varepsilon,u)\big)v(z,\varepsilon;u)\\ \text{is analytic and bounded uniformly in Case 1}\end{array}\right]$$
$$=|\alpha||\varepsilon|te^{\frac{r}{2}i\arg\tilde\varepsilon}e^{\varphi(\varepsilon,t)}\int\limits_{-\delta}^{\delta}e^{-|\alpha|^2|\varepsilon|^2|\tilde\varepsilon|^{-r}s^2}\,\tilde v\Big(z,\varepsilon;|\alpha||\varepsilon|\exp(\tfrac{r}{2}i\arg\tilde\varepsilon)s\Big)\,ds$$
$$=e^{\varphi(\varepsilon,t)}\cdot\begin{cases}O(\varepsilon^2), & r=1,2\\ O(\varepsilon^{r/2+1}), & r\ge 3.\end{cases}$$

So in summary we have

$$\hat{\mathscr{I}}(z,\varepsilon)=e^{\varphi(\varepsilon,t)}\cdot\begin{cases}O(1), & r=1,2\\ O(\varepsilon^{r/2+1}), & r\ge 3\end{cases}$$

in Case 1.

(ix) Consider the **Case 2** now. An indentation of the paths $\mathfrak{W}_0$ or $\mathfrak{W}_2$ is not necessary here, and similarly to Case 1 we obtain $\frac{w^2}{w-z}=O(1)$, $\big|e^{\varphi(\varepsilon,w)}\big|\le\big|e^{\varphi(\varepsilon,T_0)}\big|$ for

$w \in \operatorname{tr}\mathfrak{W}_0 \cup \operatorname{tr}\mathfrak{W}_2 \cup \operatorname{tr}\mathfrak{W}_3$.

$$\Rightarrow \quad \left(\int_{\mathfrak{W}_0} + \int_{\mathfrak{W}_2} + \int_{\mathfrak{W}_3} \right) e^{\varphi(\varepsilon,w)} \frac{w^2 F(w)}{w-z}\,dw = e^{\varphi(\varepsilon,t)} \cdot \begin{cases} O(1), & r=1,2 \\ O(\varepsilon^{r/2+1}), & r \geq 3. \end{cases}$$

(x) The integral over $\mathfrak{W}_1$ can be transformed exactly as in (5.42) except that the line segment $\overline{-d,d}$ has to be indented by a small circular arc in the case that z lies *on* it – to the left or to the right, conforming to the choice of n in equation (5.38). The function $\tilde{v}(z,\varepsilon;\cdot)$ admits a simple pole at $u=\sigma$, and the residue at $u=\sigma$ is easily found to be $\frac{z^2}{t}F(z)$. We then have

$$\tilde{v}(z,\varepsilon;u) = \frac{\frac{z^2}{t}F(z)}{u-\sigma} + \tilde{w}(z,\varepsilon;u)$$

where

$$\tilde{w}(z,\varepsilon;u) = zF(z) \cdot \frac{b(\varepsilon,u)-b(\varepsilon,\sigma)}{u-\sigma} + t \cdot \big(1+b(\varepsilon,u)\big) w(z,\varepsilon;u),$$

the difference quotient being replaced by $\partial_2 b(\varepsilon,\sigma)$ for $u=\sigma$. Going over the Case 2 of the proof of Proposition 5.2, we find that $w(z,\varepsilon;u)$ is analytic w.r.to $\varepsilon \in \Omega$, $|\varepsilon| \leq \varepsilon_0$, $|u| \leq \delta|t|$, $\left|\frac{z-t}{t}\right| \leq \frac{3}{8}\delta|t|$, and is in the class $O(t^{-1})$ with uniformity w.r.to all variables. (Note that, in contrast to the situation of Proposition 5.2, the boundedness of F in proper subcusps of G implies only $F'(x) = O(x^{-2})$ in proper subcusps.) Hence, $\tilde{w}(z,\varepsilon;u)$ is bounded uniformly and analytic w.r.to all variables.

As a consequence, $\displaystyle\int_{-d}^{d} e^{-\tilde{x}u^2} \tilde{w}(z,\varepsilon;u)\,du = e^{\varphi(\varepsilon,t)} \cdot \begin{cases} O(\varepsilon), & r=1,2 \\ O(\varepsilon^{r/2}), & r \geq 3. \end{cases}$

(xi) Furthermore, by a straightforward application of the Laplace method, we have

$$\begin{aligned} \int_{-d}^{d} e^{-\tilde{x}u^2} \frac{du}{u-\sigma} &= \int_{-\infty\tilde{\varepsilon}^{r/2}}^{\infty\tilde{\varepsilon}^{r/2}} e^{-\tilde{x}u^2} \frac{du}{u-\sigma} - \left(\int_{-\infty\tilde{\varepsilon}^{r/2}}^{-d} + \int_{d}^{\infty\tilde{\varepsilon}^{r/2}} \right) e^{-\tilde{x}u^2} \frac{du}{u-\sigma} \\ &= O(1) - \begin{cases} O(\varepsilon^{r-2}), & r=1,2 \\ O(1), & r \geq 3 \end{cases} \end{aligned}$$

(see (5.27),(5.29) for the first summand).

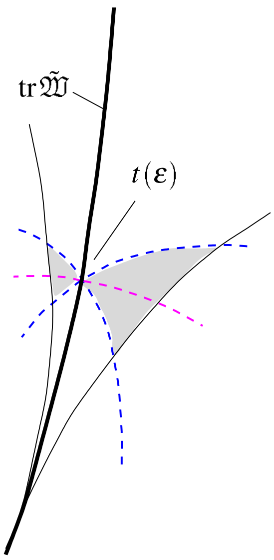

Figure 5: Region of ”critical” z (ε is fixed)

Altogether we obtain

$$\begin{aligned}\int_{\mathfrak{W}_1} e^{\varphi(\varepsilon,w)}\frac{w^2F(w)}{w-z}\,dw &= z^2F(z)e^{\varphi(\varepsilon,t)}\int_{-d}^{d} e^{-\tilde{x}u^2}\frac{du}{u-\sigma} + te^{\varphi(\varepsilon,t)}\int_{-d}^{d} e^{-\tilde{x}u^2}\tilde{w}(z,\varepsilon;u)\,du\\ &= e^{\varphi(\varepsilon,t)}\left\{\begin{matrix} O(\varepsilon^r), & r=1,2\\ O(\varepsilon^2), & r\geq 3\end{matrix}\right\} + e^{\varphi(\varepsilon,t)}\begin{cases} O(\varepsilon^2), & r=1,2\\ O(\varepsilon^{r/2+1}), & r\geq 3\end{cases}\\ &= e^{\varphi(\varepsilon,t)}\begin{cases} O(\varepsilon^r), & r=1,2\\ O(\varepsilon^2), & r\geq 3,\end{cases}\end{aligned}$$

hence

$$\hat{\mathscr{I}}(z,\varepsilon) = e^{\varphi(\varepsilon,t)}\cdot\begin{cases} O(1), & r=1,2\\ O(\varepsilon^2), & r\geq 3\end{cases}$$

in Case 2. So in both Cases 1 and 2 we have

$$\hat{\mathscr{I}}(z,\varepsilon) = e^{\varphi(\varepsilon,t)}\cdot\begin{cases} O(1), & r=1,2\\ O(\varepsilon^2), & r\geq 3\end{cases}$$

uniformly with respect to $\varepsilon \in \Omega$ and $z \in \overline{K}_\rho(0)$.

(xii) Finally we want to estimate $r(z,\varepsilon) = 2\pi i n(z,\varepsilon)\cdot z^2F(z)e^{\varphi(\varepsilon,z)}$. Let $G' \Subset G$ be a proper subcusp of G, $z \in G'$. We can then modify the proof of Proposition 5.3.

The region of "critical" z becomes different here because of $z \in G'$ and because $\mathfrak{W}$ can be a contour to the "left" as well as to the "right" of z, cf. Figure 5. As a consequence, the utter point $\hat{w}$ of the proof of Proposition 5.3 has to be replaced with the intersecting point of the tangent to the level line of $\operatorname{Re}\varphi(\varepsilon,\cdot)$ at $w = t$ and one of the edges of G'. It may not be surprising (although the proof is technically involved) that for small enough ε_0, this intersection indeed exists and that[11] $\psi_1 = |\sphericalangle(\hat{w}, O, t)| = O(\varepsilon)$. The rest of the proof of Proposition 5.3 can be left unchanged to see that

$$\varphi(\varepsilon, z) - \varphi(\varepsilon, t) = O\big(\varepsilon^{-r+2}\big)$$

for $\varepsilon \in \Omega$ and "critical" $z \in G'$, thus

$$r(z, \varepsilon) = e^{\varphi(\varepsilon,t)+O(\varepsilon^{-r+2})} \cdot O(z^2)$$

uniformly with respect to $\varepsilon \in \Omega$, $z \in G'$.

We thus have proved the

Theorem 5.6. *Let $r \in \mathbb{N}^*$, $p(x) = \sum_{\ell=1}^{r} p_\ell x^\ell \in x\mathbb{C}[x]$, $\alpha = (-rp_r)^{1/r} \neq 0$, $\beta = \frac{(r-1)p_{r-1}}{rp_r}$. Consider the cusps $G = \Upsilon(\arg\alpha, \hat{K}, \hat{\rho})$ and $\Omega = \overline{\Upsilon}\big(0, K(\Omega), \rho(\Omega)\big)$ with the relations $\hat{\rho} > |\alpha|\rho(\Omega)$ and $\hat{K} > |\alpha|^{-1}\big(K(\Omega) + \frac{1}{r}|\operatorname{Im}(\alpha\beta)|\big)$. Let T with $\arg T = \arg\alpha$ be fixed and $0 < \rho < \rho_0 = |T| < |\alpha|\rho(\Omega)$. Let $F :\ G \to \mathbb{C}$ be an analytic function bounded on every proper subcusp of G.*

(i) *Consider a contour $\mathfrak{W}_\ell$ in G from 0 to T, passing $w = z$ to the "left", and a region $G_\ell := \Upsilon(\arg\alpha, \hat{K}, \rho) \cup \overline{S}_\rho(\arg\alpha - \pi, \arg\alpha)$. Then in* (5.36) *we have*

$$\text{(5.43)} \quad \int\limits_{\mathfrak{W}_\ell} e^{p(w^{-1})} w^{-\varepsilon^{-r}} \frac{w^2 F(w)}{w - z}\, dw = e^{\varphi(\varepsilon,t)+O(\varepsilon^{-r+2})} \cdot O(1), \qquad t = t(\varepsilon)$$

uniformly with respect to $\varepsilon \in \Omega$ and $z \in G'_\ell = G' \cup \overline{S}_\rho(\arg\alpha - \pi, \arg\alpha)$ for every $G' \Subset \Upsilon(\arg\alpha, \hat{K}, \rho)$. The same is true if we replace $\mathfrak{W}_\ell$ by a contour $\mathfrak{W}_r$ passing $w = z$ to the "right", and G_ℓ by $G_r := \Upsilon(\arg\alpha, \hat{K}, \rho) \cup \overline{S}_\rho(\arg\alpha, \arg\alpha + \pi)$.

(ii) *Consider a path $\mathfrak{W}(z)$ which is the line segment from 0 to T, possibly indented by a small circular arc around z in the case $0 \neq z \in \overline{OT}$. Then in* (5.36) *we have*
(5.44)

$$\int\limits_{\mathfrak{W}(z)} e^{p(w^{-1})} w^{-\varepsilon^{-r}} \frac{w^2 F(w)}{w - z}\, dw = e^{\varphi(\varepsilon,t)+O(\varepsilon^{-r+2})} \cdot O(1), \qquad t = t(\varepsilon)$$

uniformly with respect to $\varepsilon \in \Omega$ and $z \in \overline{K}_\rho(0)$.

Here, $t(\varepsilon)$ is defined in Section 5.1 and satisfies (5.37). □

[11] In fact, we have $\psi_1 \asymp |\varepsilon|$ even on the ray $\arg\varepsilon = 0$. A better result comparable to Proposition 5.4 does not seem possible unless G is made considerably smaller.

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