On minimal graded free resolutions

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Contents

Preface

Minimal graded free resolutions are an important and central topic in algebra. They are a useful tool for studying modules over finitely generated graded K-algebras. Such a resolution determines the Hilbert series, the Castelnuovo-Mumford regularity and other invariants of the module.

This thesis is concerned with the structure of minimal graded free resolutions. We relate our results to several recent trends in commutative algebra.

The first of these trends (see [**13, 22, 33, 34, 49**]) deals with relations between properties of the Stanley-Reisner ring associated to a simplicial complex and the Stanley-Reisner ring of its Alexander dual.

Another development is the investigation of the linear part of a minimal graded free resolution by Eisenbud and Schreyer in [**26**].

Several authors were interested in the problem to give lower bounds for the Betti numbers of a module. In particular, Eisenbud-Koh [**24**], Green [**31**], Herzog [**32**] and Reiner-Welker [**42**] studied the graded Betti numbers which determine the linear strand of a minimal graded free resolution.

Bigraded algebras occur naturally in many research areas of commutative algebra. A typical example of a bigraded algebra is the Rees ring of a graded ideal. In [**21**] Cutkosky, Herzog and Trung used this bigraded structure of the Rees ring to study the Castelnuovo-Mumford regularity of powers of graded ideals in a polynomial ring. Conca, Herzog, Trung and Valla dealt with diagonal subalgebras of bigraded algebras in [**20**]. Aramova, Crona and De Negri studied homological properties of bigraded K-algebras in [**3**].

This thesis is divided in 6 chapters. Chapter 1 introduces definitions, notation and gives a short survey on those facts which are relevant in the following chapters.

Recently Yanagawa [**53**] introduced the category of square-free modules over a polynomial ring $S = K[x_1, \dots, x_n]$. This concept generalizes Stanley-Reisner rings associated to simplicial complexes. In Chapter 2 we define the generalized Alexander dual for square-free S-modules. This definition is a natural extension of the well-known Alexander duality for simplicial complexes. Miller [**40**] studied Alexander duality in a more general situation. In the case of square-free S-modules his definition and ours coincide.

We extend homological theorems on Stanley-Reisner rings to square-free S-modules. Bayer, Charalambous and S. Popescu introduced in [**13**] the extremal Betti numbers, which are a refinement of the Castelnuovo-Mumford regularity and of the projective dimension of a finitely generated graded S-module. Theorem 2.2.9 states that there is a 1-1 correspondence between the extremal Betti numbers of a square-free S-module and the extremal Betti numbers of its dual. This generalizes results in [**13**]. The local cohomology of a square-free S-module is computed (see Theorem 2.3.4) and as an application we show in Corollary 2.3.5 that a square-free S-module N is Cohen-Macaulay of dimension d if and only if the dual N^* has an $(n-d)$-linear resolution. We also show that the projective dimension of N is equal to the regularity of N^*. These results extend theorems of Eagon-Reiner [**22**] and Terai [**49**]. Furthermore, we generalize a result of Herzog-Hibi [**33**] and Herzog-Reiner-Welker [**34**]. We prove that N is sequentially Cohen-Macaulay if and only if N^* is componentwise linear (see Theorem 2.4.6).

Chapter 3 is devoted to the study of the linear part of a minimal graded free resolution associated to a finitely generated graded module. Roughly speaking we obtain the linear part by deleting all entries in the matrices of the maps in the minimal graded free resolution which are not linear forms. The result is again a complex. Eisenbud and Schreyer introduced in [**26**] this construction and proved that this complex is eventually exact for finitely generated graded modules over an exterior algebra. We define the invariant lpd (the **l**inear **p**art **d**ominates) of a module as the smallest integer i such that the linear part is exact in homological degree greater than i. We show in Theorem 3.2.8 that for a finitely generated graded module M over a Koszul algebra we have $\operatorname{lpd}(M) = 0$ if and only if M is componentwise linear. Furthermore, we give in Theorem 3.3.4 a bound for the invariant lpd for certain modules over the exterior algebra.

Let S be a standard graded polynomial ring over a field K and let M be a finitely generated graded S-module. We write $\beta^S_{i,i+j}(M)$ for the graded Betti numbers of M. Assume that the initial degree of M is d, i.e. we have $M_i = 0$ for $i < d$ and $M_d \neq 0$. We are interested in the numbers $\beta^{lin}_i(M) = \beta^S_{i,i+d}(M)$ for $i \geq 0$. These numbers determine the rank of the free modules appearing in the linear strand of the minimal graded free resolution of M. Let $p = \max\{i\colon \beta^{lin}_i(M) \neq 0\}$ be the length of the linear strand. In [**32**] Herzog conjectured the following:

Let M be a k^{th}-syzygy module whose linear strand has length p, then

$$\beta^{lin}_i(M) \geq \binom{p+k}{i+k} \text{ for } i = 0, \dots, p.$$

This conjecture is motivated by a result of Green [**31**] (see also Eisenbud-Koh [**24**]) that contains the case $i = 0, k = 1$. For $k = 0$ these lower bounds were shown by Herzog [**32**]. Reiner and Welker proved them in [**42**] for $k = 1$, if M is a monomial ideal.

In Chapter 4 we prove in Theorem 4.2.13 the conjecture for $k = 1$. For $k > 1$ we get the following weaker result (Corollary 4.2.8): If $\beta^{lin}_p(M) \neq 0$ for $p > 0$ and M is a k^{th}-syzygy module, then $\beta^{lin}_{p-1}(M) \geq p + k$. We also show that the conjecture holds in full generality for finitely generated $\mathbb{Z}^n$-graded S-modules (see Theorem 4.3.4). The first three sections of this chapter are concerned with the question above.

Finally, in Section 4.4 we study a problem related to results in [**14, 36, 41**]. We fix integers $d \geq 0$ and $0 \leq k \leq \binom{n+d-1}{d}$. Let $\mathcal{B}(d,k)$ be the set of Betti sequences $\{\beta^S_{i,j}(I)\}$ where I is a graded ideal with d-linear resolution and $\beta^S_{0,d}(I) = k$. We consider the following partial order: $\{\beta^S_{i,j}(I)\} \geq \{\beta^S_{i,j}(J)\}$ if $\beta^S_{i,j}(I) \geq \beta^S_{i,j}(J)$ for all integers i, j. We show that, independent of the characteristic of the base field, there is a unique minimal and a unique maximal element in $\mathcal{B}(d, k)$ (see Corollary 4.4.8).

Chapter 5 is devoted to study homological properties of a bigraded algebra $R = S/J$ where S is a standard bigraded polynomial ring over a field K and $J \subset S$ is a bigraded ideal. First we consider the x-regularity $\operatorname{reg}_{S,x}(R)$ and the y-regularity $\operatorname{reg}_{S,y}(R)$ of R as defined by Aramova, Crona and De Negri in [**3**]. In Theorem 5.1.5 we give a homological characterization of these regularities which is similar to that in the graded case (see [**5**]). As an application we generalize a result of Trung [**50**] concerning d-sequences (Corollary 5.2.3). Moreover, we prove that $\operatorname{reg}_{S,x}(S/J) = \operatorname{reg}_{S,x}(S/\operatorname{bigin}(J))$ where $\operatorname{bigin}(J)$ is the bigeneric initial ideal of J with respect to the bigraded reverse lexicographic order induced by $y_1 > \ldots > y_m > x_1 > \ldots > x_n$ (see Theorem 5.3.6).

It was shown in [**21**] (or [**38**]) that for $j \gg 0$, $\operatorname{reg}(I^j)$ is a linear function $cj + d$ in j for a graded ideal I in a polynomial ring. In Section 5.4 we give, in case that I is equigenerated, bounds j_0 such that for $j \geq j_0$ the function is linear and give also a bound for d. Our methods can also be applied to compute $\operatorname{reg}(S^j(I))$ where $S^j(I)$ is the j^{th}-symmetric power of I.

In Section 5.5 we introduce the generalized Veronese algebra. For a bigraded K-algebra R and $\tilde{\Delta} = (s, t) \in \mathbb{N}^2$ with $(s, t) \neq (0, 0)$ we set $R_{\tilde{\Delta}} = \bigoplus_{(a,b)\in\mathbb{N}^2} R_{(as,bt)}$. In the same manner as it is done for diagonal subalgebras in [**20**], we prove in Corollary 5.5.5 that $\operatorname{reg}_{S_{\tilde{\Delta}},x}(R_{\tilde{\Delta}}) = 0$ and $\operatorname{reg}_{S_{\tilde{\Delta}},y}(R_{\tilde{\Delta}}) = 0$ if $s \gg 0$ and $t \gg 0$.

It is in general impossible to describe a minimal graded free resolution of a graded ideal in a polynomial ring explicitly. Nevertheless in some special cases there exist nice descriptions of the resolutions and in particular one gets formulas for the graded Betti numbers of the given ideal. For example Eliahou and Kervaire studied in [**27**] minimal graded free resolutions of the so-called stable ideals which, in characteristic zero, are exactly the Borel fixed ideals. In Chapter 6 we introduce two classes of ideals which generalize stable ideals. We compute the Koszul cycles of the corresponding quotient rings and obtain formulas for the Betti numbers of these ideals (see Corollary 6.1.4 and Corollary 6.2.10).

Note on references: Most of this material has been submitted, or published elsewhere. The results will appear in [**43, 44, 45, 46**]. To avoid endless citations, we do not quote single results.

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CHAPTER 1

Background

This chapter contains a brief introduction to the concepts and results used in this dissertation.

For a more detailed exposition of the contents of Sections 1.1 and 1.5 see Bruns-Herzog [**16**]. Section 1.2 is concerned with homological algebra (see Weibel's book [**52**] for a systematic treatment). The Koszul complex and the Cartan complex are explained in Sections 1.3 and 1.4. These complexes can be found in the book of Bruns-Herzog [**16**] and in the article of Aramova-Herzog [**5**]. The topic of Section 1.6 are Rees- and symmetric algebras of graded ideals. We refer to the book [**51**] of Vasconcelos where these algebras are studied in detail. In Section 1.7 we introduce aspects of the Gröbner basis theory. A good reference for this theory is Eisenbud's book [**23**]. The aim of Section 1.8 is to recall basic definitions and facts about simplicial complexes. A more detailed introduction for this material is given in Stanley's book [**48**].

We assume that the reader has fundamental knowledge about commutative algebra based on an introductory text like Matsumura [**39**].

1.1. Graded rings and modules

Throughout this work all rings are assumed to be Noetherian, commutative or skew-commutative and with identity. All considered modules are finitely generated unless otherwise stated and we fix an infinite field K.

Definition 1.1.1. Let $(G,+)$ be an arbitrary abelian group. A ring R is called a *G-graded ring* if there exists a family $\{R_g \colon g \in G\}$ of $\mathbb{Z}$-modules such that R admits a decomposition $R = \bigoplus_{g\in G} R_g$ as a $\mathbb{Z}$-module with $R_gR_h \subseteq R_{g+h}$ for all $g,h \in G$. A finitely generated R-module M is called a *G-graded R-module* if there exists a family $\{M_g \colon g \in G\}$ of $\mathbb{Z}$-modules such that M admits a decomposition $M = \bigoplus_{g\in G} M_g$ as a $\mathbb{Z}$-module with $R_gM_h \subseteq M_{g+h}$ for all $g,h \in G$.

We call $u \in M$ *homogeneous of degree g* if $u \in M_g$ for some $g \in G$ and set $\deg(u) = g$. For $g \in G$ we say that M_g is a *homogeneous component of M*. An ideal $I \subset R$ is G-graded if $I = \bigoplus_{g\in G} I_g$ with $I_g = I \cap R_g$.

Definition 1.1.2. Let R be a G-graded ring, M and N finitely generated G-graded R-modules and $\varphi : M \to N$ an R-linear map. φ is said to be *homogeneous of degree h* for some $h \in G$ if $\varphi(M_g) \subseteq N_{g+h}$ for all $g \in G$. We call φ *homogeneous* if it is homogeneous of degree 0.

For $g \in G$ the (twisted) module $M(g)$ is the G-graded module with $M(g)_h = M_{g+h}$. Note that, if $\varphi : M \to N$ is homogeneous of degree h, then the induced map $\tilde{\varphi} : M(-h) \to N$ is homogeneous.

If G equals $\mathbb{Z}$, $\mathbb{Z}^2$ or $\mathbb{Z}^n$, we say that R is a *graded, a bigraded or a* $\mathbb{Z}^n$*-graded* ring and M is a *graded, a bigraded or a* $\mathbb{Z}^n$*-graded* R-module. Let M be a bigraded module and let $u \in M$ be homogeneous with $\deg(u) = (a, b)$. In this situation we set: $\deg_x(u) = a$ and $\deg_y(u) = b$.

We observe that every bigraded module M has a natural graded structure by setting $M_i = \bigoplus_{(a,b)\in\mathbb{Z}^2,\ a+b=i} M_{(a,b)}$. Analogously every $\mathbb{Z}^n$-graded module M has a natural graded structure by setting $M_i = \bigoplus_{u\in\mathbb{Z}^n,\ |u|=i} M_u$.

Let R be a graded ring and M a finitely generated graded R-module. Then $\operatorname{indeg}(M) = \min\{d \in \mathbb{Z} : M_d \neq 0\}$ is said to be the *initial degree* of M.

Example and Definition 1.1.3. In particular the following rings will be considered in this thesis:

(i) Let $S = K[x_1, \ldots , x_n]$ be the polynomial ring in n-variables. S has a graded structure induced by $\deg(x_i) = 1$.

(ii) $S = K[x_1, \ldots , x_n]$ has also a $\mathbb{Z}^n$-graded structure by setting $\deg(x_i) = \varepsilon_i$ where ε_i denotes the i^{th}-unit vector of $\mathbb{Z}^n$.

(iii) Let $S = K[x_1, \ldots , x_n, y_1, \ldots , y_m]$ be the polynomial ring in $n+m$-variables. Then S has a bigraded structure induced by $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$.

(iv) Let $E = K\langle e_1, \ldots , e_n\rangle$ be the exterior algebra over an n-dimensional vector space V with basis $e_1, \ldots , e_n$. Then E has a graded structure induced by $\deg(e_i) = 1$.

(v) Let $\deg(e_i) = \varepsilon_i$. Then $E = K\langle e_1, \ldots , e_n\rangle$ has a $\mathbb{Z}^n$-graded structure.

The polynomial rings of (i), (ii) and (iii) will be called (*standard*) *graded, bigraded and* $\mathbb{N}^n$*-graded* polynomial rings. Similarly the exterior algebras of (iv) and (v) are said to be (*standard*) *graded and* $\mathbb{N}^n$*-graded* exterior algebras.

Definition 1.1.4. A ring R is said to be a

(i) *standard graded, bigraded or* $\mathbb{N}^n$*-graded* K*-algebra* if $R = S/I$ where S is a standard graded, bigraded or $\mathbb{N}^n$-graded polynomial ring and $I \subset S$ is a graded, bigraded or $\mathbb{Z}^n$-graded ideal.

(ii) *standard skew-commutative graded or* $\mathbb{N}^n$*-graded* K*-algebra* if $R = E/I$ where E is a standard graded or $\mathbb{N}^n$-graded exterior algebra and $I \subset E$ is a graded or $\mathbb{Z}^n$-graded ideal.

Every ring R appearing in this thesis will be of the form as in 1.1.4 and in order to simplify notation we omit the term “standard” occasionally. Let R be one of the K-algebras in 1.1.4. If R_1 has a K-basis $\mathbf{x} = x_1, \ldots , x_n$, we set $\mathfrak{m} = (x_1, \ldots , x_n)$ to be *the graded maximal ideal* of R.

Definition 1.1.5. In the framework of the theory of categories we formulate:

(i) Let R be a graded, bigraded or $\mathbb{N}^n$-graded K-algebra respectively. We denote with $\mathcal{M}_{\mathbb{Z}}(R)$, $\mathcal{M}_{bi}(R)$ and $\mathcal{M}_{\mathbb{Z}^n}(R)$ the abelian categories of finitely generated graded, bigraded or $\mathbb{Z}^n$-graded R-modules respectively.

(ii) Let R be a skew-commutative graded or $\mathbb{N}^n$-graded K-algebra respectively. We also denote with $\mathcal{M}_{\mathbb{Z}}(R)$ and $\mathcal{M}_{\mathbb{Z}^n}(R)$ the abelian categories of finitely generated graded or $\mathbb{Z}^n$-graded left and right R-modules M respectively, satisfying $ru = (-1)^{|\deg(r)||\deg(u)|}ur$ for all homogeneous elements $r \in R$ and $u \in M$.

In all cases the morphisms are the homogeneous homomorphisms.

If the reader is not familiar with categories, then just read these definitions as the collection of modules with some properties.

Let R be a graded or $\mathbb{N}^n$-graded K-algebra respectively. In the sequel we sometimes use the following notation:

$$\mathcal{M}_{\mathbb{N}}(R) = \{M \in \mathcal{M}_{\mathbb{Z}}(R) \colon M = \bigoplus_{i \in \mathbb{N}} M_i\}$$

or

$$\mathcal{M}_{\mathbb{N}^n}(R) = \{M \in \mathcal{M}_{\mathbb{Z}^n}(R) \colon M = \bigoplus_{u \in \mathbb{N}^n} M_u\}.$$

We say that $M \in \mathcal{M}_{\mathbb{N}}(R)$ is $\mathbb{N}$-graded and $M \in \mathcal{M}_{\mathbb{N}^n}(R)$ is $\mathbb{N}^n$-graded. For a bigraded K-algebra R we distinguish certain subrings of R. We define $R_x = \bigoplus_{a \in \mathbb{N}} R_{(a,0)}$ and $R_y = \bigoplus_{b \in \mathbb{N}} R_{(0,b)}$. We consider R_x and R_y as subrings of R. Observe that R_x and R_y have also the structure of a graded K-algebra. If $M \in \mathcal{M}_{bi}(R)$, then $M_{(a,*)} = \bigoplus_{b \in \mathbb{Z}} M_{(a,b)} \in \mathcal{M}_{\mathbb{Z}}(R_y)$ and $M_{(*,b)} = \bigoplus_{a \in \mathbb{Z}} M_{(a,b)} \in \mathcal{M}_{\mathbb{Z}}(R_x)$. In this situation we set $\mathfrak{m}_x \subset R_x$ and $\mathfrak{m}_y \subset R_y$ to be the graded maximal ideals of R_x and R_y.

If in addition $M \in \mathcal{M}_{bi}(R)$ is $\mathbb{Z}^n \times \mathbb{Z}^m$-graded, we write $M_{(u,v)}$ for the homogeneous component (u,v) for $u \in \mathbb{Z}^n$ and $v \in \mathbb{Z}^m$. Finally, we construct graded rings out of given rings.

Definition 1.1.6. Let R be a ring. A *filtration $\mathcal{C}$ on R* is a descending chain

$$R = C_0 \supset \ldots \supset C_i \supset \ldots$$

of ideals such that $C_i C_j \subseteq C_{i+j}$ for all i and j.

We have:

Proposition 1.1.7. *Let R be a ring, $\mathcal{C}$ a filtration on R and M a finitely generated R-module. Then*

$$\operatorname{gr}_{\mathcal{C}}(R) = \bigoplus_{i \in \mathbb{N}} C_i / C_{i+1}$$

is a standard graded ring with multiplication $(a + C_{i+1})(b + C_{j+1}) = ab + C_{i+j+1}$ for $a \in C_i$ and $b \in C_j$. Furthermore,

$$\operatorname{gr}_{\mathcal{C}}(M) = \bigoplus_{i \in \mathbb{N}} C_i M / C_{i+1} M$$

is a finitely generated $\operatorname{gr}_{\mathcal{C}}(R)$-module with scalar multiplication $(a+C_{i+1})(u+C_{k+1}M) = au + C_{k+i+1}M$ for $a \in C_i$ and $u \in C_k M$.

Observe that, if the ring R is G-graded for an abelian group G, then $\mathrm{gr}_{\mathcal{C}}(R)$ has a natural bigraded structure. For a homogeneous element $r \in C_i$ we write $[r]$ for the residue class $r + C_{i+1}$ of r. We say that $[r]$ has *the (filtration) degree* i and *the internal degree* $\deg(r)$. Analogously $\mathrm{gr}_{\mathcal{C}}(M)$ is bigraded. Note that, for $j \in \mathbb{Z}$, the twisted module $\mathrm{gr}_{\mathcal{C}}(M)(-j)$ is equal to $\bigoplus_{i\in\mathbb{N}} C_{i-j}M/C_{i+1-j}M$.

Example 1.1.8. Let R be a (commutative or skew-commutative) graded K-algebra with maximal ideal $\mathfrak{m}$. The filtration given by $C_i = \mathfrak{m}^i$ is called the $\mathfrak{m}$*-adic filtration on* R. We set $\mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{gr}_{\mathcal{C}}(R)$.

1.2. Homological algebra

We assume that the reader is familiar with homological methods in commutative algebra, but we introduce some notation and recall the basic definitions. We restrict ourself to the case that the ring is a graded K-algebra and use in this thesis that most of the results in this section can also be applied to K-algebras of the form of 1.1.4.

Definition 1.2.1. Let R be a graded K-algebra. A *(graded) complex* is a collection of finitely generated graded R-modules $\{F_i \colon i \in \mathbb{Z}\}$ and homogeneous R-linear maps $\delta_i : F_i \to F_{i-1}$ with $\mathrm{Im}(\delta_{i+1}) \subseteq \mathrm{Ker}(\delta_i)$. We write $(\mathcal{F}, \delta)$ or just $\mathcal{F}$ for a complex. We call $(\mathcal{G}, \delta^{\mathcal{G}})$ *a subcomplex of* $(\mathcal{F}, \delta^{\mathcal{F}})$ if G_i is a submodule of F_i for all integers i and $\delta^{\mathcal{G}} = \delta^{\mathcal{F}}_{|\mathcal{G}}$. To every complex we associate the *homology groups* $H_i(\mathcal{F}) = \mathrm{Ker}(\delta_i)/\mathrm{Im}(\delta_{i+1})$, which are again graded modules.

A complex $\mathcal{F}$ can also be written as

$$\ldots \to F_{i+1} \to F_i \to F_{i-1} \to \ldots$$

where the arrows represent the maps δ_i. Let $\mathcal{F}$ be a complex and $i \in \mathbb{Z}$. We call $w \in \mathrm{Ker}(\delta_i)$ a *cycle*. If in addition $w \in \mathrm{Im}(\delta_{i+1})$, then w is said to be a *boundary*. For a cycle w we denote the residue class in $H(\mathcal{F})$ with $[w]$. If needed, it is customary to write $F^i = F_{-i}$ for $i \in \mathbb{Z}$.

Remark 1.2.2. Let R be a graded K-algebra. There is a definition which is dual to that of a complex. A *(graded) cochain complex* is a collection of finitely generated graded R-modules $\{F^i \colon i \in \mathbb{Z}\}$ and homogeneous R-linear maps $\delta^i : F^i \to F^{i+1}$ with $\mathrm{Im}(\delta^i) \subseteq \mathrm{Ker}(\delta^{i+1})$. We also write $\mathcal{F}$ for the cochain complex. In the sequel we present most of the definitions and results for complexes. We leave it to the reader to write down the corresponding cochain version.

We define:

Definition 1.2.3. A complex $\mathcal{F}$ is said to be *exact at the homological degree* $i \in \mathbb{Z}$ if $H_i(\mathcal{F}) = 0$. We call $\mathcal{F}$ *exact* if $H_i(\mathcal{F}) = 0$ for all integers i.

One defines homomorphisms between complexes in the following way:

Definition 1.2.4. Let $(\mathcal{F}, \delta^{\mathcal{F}})$ and $(\mathcal{G}, \delta^{\mathcal{G}})$ be complexes. A *complex homomorphism* $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of homogeneous maps $\varphi_i : F_i \to G_i$ such that $\varphi_i \circ \delta^{\mathcal{F}}_{i+1} = \delta^{\mathcal{G}}_{i+1} \circ \varphi_{i+1}$. Moreover, φ is said to be a *monomorphism, epimorphism or isomorphism* if all φ_i are monomorphisms, epimorphisms or isomorphisms.

Note that for a complex $\mathcal{F}$ and $M \in \mathcal{M}_{\mathbb{Z}}(R)$ we have that $\mathcal{F} \otimes_R M$, $M \otimes_R \mathcal{F}$, $\operatorname{Hom}_R(\mathcal{F}, M)$ and $\operatorname{Hom}_R(M, \mathcal{F})$ are again complexes (resp. cochain complexes) with complex maps induced by $\delta \otimes_R M$, $M \otimes_R \delta$, $\operatorname{Hom}_R(\delta, M)$ and $\operatorname{Hom}_R(M, \delta)$.

Definition 1.2.5. Let R be a graded K-algebra and $M \in \mathcal{M}_{\mathbb{Z}}(R)$. A complex $\mathcal{F}$ with $F_i = 0$ for $i < 0$ is a *graded free resolution* of M if all F_i are finitely generated graded free R-modules, $H_i(\mathcal{F}) = 0$ for $i \neq 0$ and $H_0(\mathcal{F}) = M$.

A well-known result in homological algebra is:

Proposition 1.2.6. *Let R be a graded K-algebra and $M, N \in \mathcal{M}_{\mathbb{Z}}(R)$. Let $\mathcal{F}$ be a graded free resolution of M and $\mathcal{G}$ a graded free resolution of N. Then*

$$\operatorname{Tor}_i^R(M, N) \cong H_i(\mathcal{F} \otimes_R N) \cong H_i(M \otimes_R \mathcal{G})$$

and

$$\operatorname{Ext}_R^i(M, N) \cong H^i(\operatorname{Hom}_R(\mathcal{F}, N))$$

where $\operatorname{Tor}_i^R(M, N)$ and $\operatorname{Ext}_R^i(M, N)$ denote the i^{th} Tor- and Ext-groups associated to M and N.

It is possible to assign a distinguished free resolution to a finitely generated graded module by the following construction.

Construction 1.2.7. Let R be a graded K-algebra and $M \in \mathcal{M}_{\mathbb{Z}}(R)$. We choose a homogeneous minimal system of generators $g_1, \dots, g_t$ of M with $\deg(g_i) = d_i$. Define $F_0 = \bigoplus_{i=1}^t R(-d_i)$ with homogeneous basis $f_1, \dots, f_t$ and $\deg(f_i) = d_i$. The assignment $f_i \mapsto g_i$ induces a surjective homogeneous map δ_0 from F_0 to M. The kernel K_0 of δ_0 is again a finitely generated graded R-module. Choose a homogeneous minimal system of generators $g'_1, \dots, g'_{t'}$ of K_0 with $\deg(g'_i) = d'_i$. Set $F_1 = \bigoplus_{i=1}^{t'} R(-d'_i)$ with homogeneous basis $f'_1, \dots, f'_{t'}$ and $\deg(f'_i) = d'_i$. We define $\delta_1 \colon F_1 \to F_0$ by $\delta_1(f'_i) = g'_i$. By repeating this procedure one gets a graded free resolution of M.

It is easy to see that, for all $i \geq 0$, one has $\delta_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$. If we set

$$F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}^R(M)},$$

we obtain

$$\beta_{i,j}^R(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$$

which we call *the graded Betti numbers of M* for all $i, j \in \mathbb{Z}$. Moreover, $\beta_i^R(M) = \sum_j \beta_{i,j}^R(M)$ is said to be the i^{th}*-total Betti number of M*. In fact, for every other graded free resolution $\mathcal{G}$ of M with the property $\delta_{i+1}(G_{i+1}) \subseteq \mathfrak{m}G_i$ for all integers i, there exists a complex isomorphism $\varphi\colon \mathcal{F} \to \mathcal{G}$. Therefore also $\mathcal{G}$ is uniquely determined by the Betti numbers $\beta_{i,j}^R(M)$. This leads to the following definition.

Definition 1.2.8. Let R be a graded K-algebra and $M \in \mathcal{M}_{\mathbb{Z}}(R)$. The *minimal graded free resolution* $(\mathcal{F}, \delta)$ of M is the unique graded free resolution of M with $\delta_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$ for all $i \geq 0$.

Since the minimal graded free resolution is unique up to a base change, also the kernels of the maps inherit this property.

Definition 1.2.9. Let R be a graded K-algebra, $M \in \mathcal{M}_{\mathbb{Z}}(R)$ with minimal graded free resolution $(\mathcal{F}, \delta)$ and $k \geq 1$. Then $\Omega_k(M) = \operatorname{Ker}(\delta_{k-1})$ is said to be *the k^{th}-syzygy module of M.*

Note that for $i \geq k$ we have $\beta^R_{i,i+j}(M) = \beta^R_{i-k,i-k+j+k}(\Omega_k(M))$ because of trivial reasons.

Example 1.2.10. Let $S = K[x_1, x_2]$ be the graded polynomial ring in two variables. Then $K = S/\mathfrak{m}$ has the following minimal graded free resolution:

$$0 \to S(-2) \to S(-1) \oplus S(-1) \to S \to 0.$$

Therefore the only non-zero graded Betti numbers are $\beta^S_{0,0}(K) = 1$, $\beta^S_{1,1}(K) = 2$ and $\beta^S_{2,2}(K) = 1$.

We introduce several invariants to a graded module.

Definition 1.2.11. Let R be a graded K-algebra and $M \in \mathcal{M}_{\mathbb{Z}}(R)$. Then

$$\operatorname{pd}_R(M) = \sup\{i \in \mathbb{Z} \colon \beta^R_{i,j}(M) \neq 0 \text{ for some } j \in \mathbb{Z}\}$$

is said to be the *projective dimension* of M and

$$\operatorname{reg}_R(M) = \sup\{j \in \mathbb{Z} \colon \beta^R_{i,i+j}(M) \neq 0 \text{ for some } i \in \mathbb{Z}\}$$

is called the *Castelnuovo-Mumford regularity* of M.

If it is clear from the context which ring is meant, we write $\operatorname{reg}(M)$ and $\operatorname{pd}(M)$ instead of $\operatorname{reg}_R(M)$ and $\operatorname{pd}_R(M)$. We give a simple example. Let $S = K[x_1, \dots, x_n]$. It follows from Hilbert's syzygy theorem (see [**16**]) that $\operatorname{pd}_S(M) \leq n$ and $\operatorname{reg}_S(M) < \infty$ for $M \in \mathcal{M}_{\mathbb{Z}}(S)$. Bayer, Charalambous and Popescu introduced in [**13**] a refinement of the projective dimension and the regularity.

Definition 1.2.12. Let $S = K[x_1, \dots, x_n]$ be the graded polynomial ring and $M \in \mathcal{M}_{\mathbb{Z}}(S)$. A graded Betti number $\beta^S_{i,i+j}(M) \neq 0$ is said to be *extremal* if $\beta^S_{l,l+r}(M) = 0$ for all $r \geq j$ and all $l \geq i$ with $(l, r) \neq (i, j)$.

Let $\beta^S_{i_1,i_1+j_1}(M), \dots, \beta^S_{i_t,i_t+j_t}(M)$ be all extremal Betti numbers of M with $i_1 < \dots < i_t$. Then $j_1 = \operatorname{reg}_S(M)$ and $i_t = \operatorname{pd}_S(M)$.

In the case of the polynomial ring there are only finitely many non-zero Betti numbers of a module M and a nice way to present these numbers is the so-called *Betti diagram of M:*

M	0	$\cdots$	i	$\cdots$
0	$\beta^S_{0,0}(M)$	$\cdots$	$\beta^S_{i,0}(M)$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
j	$\beta^S_{0,j}(M)$	$\cdots$	$\beta^S_{i,i+j}(M)$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

The entry at the i^{th}-column and j^{th}-row is $\beta^S_{i,i+j}(M)$. Usually we write a "$-$" instead of a 0 and omit all rows and columns without any non-zero entry. Then

$\mathrm{pd}_S(M)$ is the maximal $p \in \mathbb{N}$ such that the column p has a non-zero entry. Furthermore, $\mathrm{reg}_S(M)$ is the maximal $r \in \mathbb{Z}$ such that the row r has a non-zero entry. Extremal Betti numbers correspond to upper left corners of a block of zeros.

Example 1.2.13. Let $S = K[x_1, \dots, x_6]$ and

$$I = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_3x_4, x_5x_6).$$

We used CoCoA [**18**] to compute the Betti diagram of I:

I	0	1	2	3	4	5
1	-	-	-	-	-	-
2	8	14	11	5	1	-
3	-	2	3	1	-	-
4	-	-	-	-	-	-

Thus $\mathrm{pd}(I) = 4$, $\mathrm{reg}(I) = 3$ and there are two extremal Betti numbers $\beta^S_{3,3+3}(I)$ and $\beta^S_{4,4+2}(I)$.

Sometimes we need to compare the regularity of modules in a short exact sequence (see for example [**23**, 20.19]).

Lemma 1.2.14. *Let R be a graded K-algebra and $M_1, M_2, M_3 \in \mathcal{M}_{\mathbb{Z}}(R)$. If*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence, then

(i) $\mathrm{reg}_R(M_1) \leq \sup\{\mathrm{reg}_R(M_2),\ \mathrm{reg}_R(M_3) + 1\}$.
(ii) $\mathrm{reg}_R(M_2) \leq \sup\{\mathrm{reg}_R(M_1),\ \mathrm{reg}_R(M_3)\}$.
(iii) $\mathrm{reg}_R(M_3) \leq \sup\{\mathrm{reg}_R(M_1) - 1,\ \mathrm{reg}_R(M_2)\}$.

A special class of graded K-algebras is defined as follows.

Definition 1.2.15. A graded K-algebra R is called *a Koszul algebra* if $\mathrm{reg}_R(K) = 0$.

Remark 1.2.16. For example, every graded polynomial ring is a Koszul algebra. Avramov and Eisenbud proved in [**11**] that every finitely generated graded module M over a Koszul algebra R has $\mathrm{reg}_R(M) < \infty$. Recently Avramov and Peeva [**12**] showed the converse.

Let R be a graded K-algebra and $M \in \mathcal{M}_{\mathbb{Z}}(R)$. For $d \in \mathbb{Z}$ we write $M_{\langle d \rangle}$ for the submodule of M which is generated by all homogeneous elements $u \in M$ with $\deg(u) = d$. For the following definitions see [**33**].

Definition 1.2.17. Let R be a graded K-algebra and $M \in \mathcal{M}_{\mathbb{Z}}(R)$.

(i) Let $d \in \mathbb{Z}$. Then M has a *d-linear resolution* if $\beta^R_{i,i+j}(M) = 0$ for all $i \geq 0$ and all $j \neq d$.
(ii) M is *componentwise linear* if for all integers d the module $M_{\langle d \rangle}$ has a d-linear resolution.

Example 1.2.18. By definition a graded module with a d-linear resolution has regularity d. In particular, the module is generated in degree d. For $i < d$ we have $M_{\langle i \rangle} = 0$. For $i \geq d$ the module $M_{\langle i \rangle} = \mathfrak{m}^{i-d}M$ has a linear resolution by 2.4.3. It follows that M is componentwise linear.

Obviously $\operatorname{indeg}(M) \leq \operatorname{reg}(M)$. We observe that:

Lemma 1.2.19. *Let R be a graded K-algebra and $M \in \mathcal{M}_{\mathbb{Z}}(R)$. Then* $\operatorname{indeg}(M) = \operatorname{reg}_R(M)$ *if and only if M has a linear resolution.*

Remark 1.2.20. As mentioned in the beginning of this section all these concepts can also be applied to bigraded or $\mathbb{N}^n$-graded K-algebra.

(i) Let R be a bigraded K-algebra and $M \in \mathcal{M}_{bi}(R)$. For $i, a, b \in \mathbb{Z}$ we define

$$\beta^R_{i,(a,b)}(M) = \dim_K \operatorname{Tor}^R_i(M, K)_{(a,b)}$$

as the i^{th}-bigraded Betti numbers of M in bidegree (a, b). Let $\mathcal{F}$ be the minimal bigraded free resolution of M. Then

$$F_i \cong \bigoplus_{(a,b)\in\mathbb{Z}^2} R(-(a, b))^{\beta^R_{i,(a,b)}(M)}.$$

(ii) Let R be an $\mathbb{N}^n$-graded K-algebra and $M \in \mathcal{M}_{\mathbb{Z}^n}(R)$. For $i \in \mathbb{N}$ and $u \in \mathbb{Z}^n$ we define

$$\beta^R_{i,u}(M) = \dim_K \operatorname{Tor}^R_i(M, K)_u$$

as the i^{th} $\mathbb{Z}^n$-graded Betti numbers of M in degree u. Let $\mathcal{F}$ be the minimal $\mathbb{Z}^n$-graded free resolution of M. Then

$$F_i \cong \bigoplus_{u\in\mathbb{Z}^n} R(-u)^{\beta^R_{i,u}(M)}.$$

In the case of bigraded K-algebras Aramova, Crona and De Negri introduced in [**3**] the following notion.

Definition 1.2.21. Let R be a bigraded K-algebra and $M \in \mathcal{M}_{bi}(R)$. Then

$$\operatorname{reg}_{R,x}(M) = \sup\{a \in \mathbb{Z} \colon \beta^S_{i,(a+i,b)}(M) \neq 0 \text{ for some } i, b \in \mathbb{Z}\}$$

is the *x-regularity of M* and

$$\operatorname{reg}_{R,y}(M) = \sup\{b \in \mathbb{Z} \colon \beta^S_{i,(a,b+i)}(M) \neq 0 \text{ for some } i, a \in \mathbb{Z}\}$$

is the *y-regularity of M.*

Again, for a finitely generated bigraded module over a bigraded polynomial ring these invariants are finite. If the ring is obvious from the context, we write $\operatorname{reg}_x(M)$ and $\operatorname{reg}_y(M)$ instead of $\operatorname{reg}_{R,x}(M)$ and $\operatorname{reg}_{R,y}(M)$.

1.3. The Koszul complex

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring and $M \in \mathcal{M}_{\mathbb{Z}}(S)$. Consider the graded free S-module L of rank j which is generated in degree 1, and let $\bigwedge L$ be the exterior algebra over L. Then $\bigwedge L$ inherits the structure of a bigraded S-module. If $z \in \bigwedge^i L$ and z has S-degree k, we give z the bidegree (i, k). We call i the *homological degree* (*hdeg for short*) and k *the internal degree* (*deg for short*) *of* z.

We consider maps $\mu \in L^* = \operatorname{Hom}_S(L, S)$. Note that L^* is again a graded free S-module generated in degree -1. It is well-known (see [**16**]) that μ defines a homogeneous S-homomorphism $\partial^\mu : \bigwedge L \to \bigwedge L$ of (homological) degree -1.

Recall that, if we fix a basis $e_1, \dots, e_j$ of L, then $\bigwedge^i L$ is the graded free S-module with basis consisting of all monomials $e_J = e_{j_1} \wedge \dots \wedge e_{j_i}$ with $J = \{j_1 < \dots < j_i\} \subseteq [j]$. One has

$$\partial^\mu(e_{j_1} \wedge \dots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^{\alpha(k,J)} \mu(e_{j_k}) e_{j_1} \wedge \dots \hat{e}_{j_k} \dots \wedge e_{j_i}$$

where we set $\alpha(F,G) = |\{(f,g) \colon f > g, f \in F, g \in G\}|$ for $F, G \subseteq [n]$ and where $\hat{e}_{j_k}$ indicates that e_{j_k} is omitted in the exterior product. Let $e_1^*, \dots, e_j^*$ be the basis of L^* with $e_i^*(e_i) = 1$ and $e_i^*(e_k) = 0$ for $k \neq i$. In order to simplify the notation we set $\partial^i = \partial^{e_i^*}$. Then $\partial^\mu = \sum_{k=1}^{j} \mu(e_k) \partial^k$.

Straightforward calculations yield (most of them are done in [**16**]):

Lemma 1.3.1. *Let $z, \tilde{z} \in \bigwedge L$ be bihomogeneous elements, $f \in S$ and $\mu, \nu \in L^*$.*

(i) $f \partial^\mu = \partial^{f\mu}$.
(ii) $\partial^\mu + \partial^\nu = \partial^{\mu+\nu}$.
(iii) $\partial^\mu \circ \partial^\mu = 0$.
(iv) $\partial^\mu \circ \partial^\nu = -\partial^\nu \circ \partial^\mu$.
(v) $\partial^\mu(z \wedge \tilde{z}) = \partial^\mu(z) \wedge \tilde{z} + (-1)^{\mathrm{hdeg}(z)} z \wedge \partial^\mu(\tilde{z})$.

We fix a graded free S-module L of rank n and K-linearly independent linear forms $l_1, \dots, l_n \in S_1$. Let $\mathbf{e} = e_1, \dots, e_n$ be a homogeneous basis of L, and $\mu \in L^*$ with $\mu(e_i) = l_i$. For $j = 1, \dots, n$ let $L(j)$ be the graded free submodule of L generated by $e_1, \dots, e_j$.

Definition 1.3.2. Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$. Then $(\mathcal{K}(j; M), \partial)$ denotes the *Koszul complex of M with respect to $l_1, \dots, l_j$* where $\mathcal{K}(j; M) = \mathcal{K}(l_1, \dots, l_j; M) = \bigwedge L(j) \otimes_S M$ and ∂ is the restriction of $\partial^\mu \otimes_S \mathrm{id}_M$ to $\bigwedge L(j) \otimes_S M$. We denote the homology of the complex $\mathcal{K}(j; M)$ with $H(j; M) = H(l_1, \dots, l_j; M)$ and call it the *Koszul homology of M with respect to $l_1, \dots, l_j$*.

If not stated otherwise, we set $l_i = x_i$ for all $i \in [n]$. The homology class of a cycle $z \in \mathcal{K}(j; M)$ is denoted with $[z]$. If the module is obvious from the context, we write $\mathcal{K}(j)$ and $H(j)$ instead of $\mathcal{K}(j; M)$ and $H(j; M)$. Observe that for $M \in \mathcal{M}_{\mathbb{N}}(S)$ we have $K_i(j; M)_{i+k} = 0$ for all $i \geq 0$ and $k < 0$.

For a homogeneous element $z \in K_i(j)$ we can write z uniquely as $z = e_k \wedge \partial^k(z) + r_z$ such that e_k divides none of the monomials of r_z.

Lemma 1.3.3. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and $z \in K_i(j)$ be a homogeneous cycle of bidegree (i, l). For all $k \in [n]$ the element $\partial^k(z)$ is a homogeneous cycle of bidegree $(i-1, l-1)$.*

Proof. Applying 1.3.1 we obtain

$$0 = \partial(z) = \partial(e_k \wedge \partial^k(z) + r_z) = x_k \partial^k(z) - e_k \wedge \partial(\partial^k(z)) + \partial(r_z).$$

We conclude that $\partial(\partial^k(z)) = 0$ and the assertion follows. $\square$

The importance of the Koszul complex is among other things based on the following result.

Proposition 1.3.4. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and $l_i = x_i$ for all $i \in [n]$. The Koszul complex $\mathcal{K}(n;S)$ is the minimal graded free resolution of $K = S/\mathfrak{m}$. Then for all $i \in \mathbb{Z}$*

$$H_i(n;M) \cong \operatorname{Tor}_i^S(K,M)$$

are isomorphic as graded K-vector spaces.

Thus we may compute the graded Betti numbers of a module with the help of the corresponding Koszul homology. Another crucial point is:

Proposition 1.3.5. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and $j \in [n-1]$. The following sequence is exact:*

$$\ldots \to H_i(j;M)(-1) \overset{l_{j+1}}{\to} H_i(j;M) \to H_i(j+1;M) \to H_{i-1}(j;M)(-1)$$

$$\overset{l_{j+1}}{\to} \ldots \to H_0(j;M)(-1) \overset{l_{j+1}}{\to} H_0(j;M) \to H_0(j+1;M) \to 0.$$

The map $H_i(j;M) \to H_i(j+1;M)$ is induced by the inclusion of the corresponding Koszul complexes and the image of a cycle $[z]$ is $[z]$. Furthermore, $H_i(j+1;M) \to H_{i-1}(j;M)(-1)$ is given by sending a cycle $[z]$ to $[\partial^{j+1}(z)]$. Finally, $H_i(j;M)(-1) \overset{l_{j+1}}{\to} H_i(j;M)$ is just the multiplication map with l_{j+1}.

Remark 1.3.6. Let S be a standard graded $\mathbb{N}^n$-graded polynomial ring. Every definition and result mentioned so far has an $\mathbb{Z}^n$-graded analogue. If $e_1, \ldots, e_n$ forms a basis of L with $\partial(e_i) = x_i$, then we set $\deg(e_i) = \varepsilon_i$ for all $i \in [n]$. Now we replace everywhere "graded" by "$\mathbb{Z}^n$-graded". In particular, for $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$, the modules $K_i(j)$ and $H_i(j)$ are $\mathbb{Z}^n$-graded. The long exact sequence of 1.3.5 is also homogeneous with respect to the $\mathbb{Z}^n$-grading.

Considering the bigraded polynomial ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ we introduce some more notation. Let $\mathcal{K}(k,l;M)$ and $H(k,l;M)$ denote the Koszul complex and the Koszul homology of M with respect to $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$. If it is obvious from the context, we write $\mathcal{K}(k,l)$ and $H(k,l)$ instead of $\mathcal{K}(k,l;M)$ and $H(k,l;M)$. Note that $\mathcal{K}(k,l;M) = \mathcal{K}(k,l;S) \otimes_S M$. Here $\mathcal{K}(k,l;S)$ is the exterior algebra of a graded free S-module with basis $e_1, \ldots, e_k, f_1, \ldots, f_l$ where $\deg(e_i) = (1,0)$ and $\deg(f_j) = (0,1)$. The differential ∂ is induced by $\partial(e_i) = x_i$ and $\partial(f_j) = y_j$. For a cycle $z \in \mathcal{K}(k,l;M)$ we write $[z]$ for the corresponding homology class in $H(k,l;M)$.

Finally, we obtain two long exact sequences from 1.3.5 relating the homology groups: For $k \in [n-1]$ and $l \in [m]$

$$\ldots \to H_i(k,l;M)(-1,0) \overset{x_{k+1}}{\to} H_i(k,l;M) \to H_i(k+1,l;M) \to H_{i-1}(k,l;M)(-1,0)$$

$$\overset{x_{k+1}}{\to} \ldots \to H_0(k,l;M)(-1,0) \overset{x_{k+1}}{\to} H_0(k,l;M) \to H_0(k+1,l;M) \to 0$$

and for $k \in [n]$ and $l \in [m-1]$

$$\ldots \to H_i(k,l;M)(0,-1) \overset{y_{l+1}}{\to} H_i(k,l;M) \to H_i(k,l+1;M) \to H_{i-1}(k,l;M)(0,-1)$$

$$\overset{y_{l+1}}{\to} \ldots \to H_0(k,l;M)(0,-1) \overset{y_{l+1}}{\to} H_0(k,l;M) \to H_0(k,l+1;M) \to 0.$$

1.4. The Cartan complex

The aim of this section is to find a complex for modules over an exterior algebra which is analogue to the Koszul complex for modules over a polynomial ring.

Let $E = K\langle e_1, \dots, e_n\rangle$ be the standard graded exterior algebra over an n-dimensional K-vector space V with basis $e_1, \dots, e_n$.

For a sequence $\mathbf{v} = v_1, \dots, v_m$ of linear forms $v_i \in E_1$ the Cartan complex $\mathcal{C}(\mathbf{v}; E)$ is defined to be the free divided power algebra $E\{x_1, \dots, x_m\}$ together with a differential δ. The free divided power algebra $E\{x_1, \dots, x_m\}$ is generated over E by the divided powers $x_i^{(j)}$ for $i = 1, \dots, m$ and $j \geq 0$, satisfying the relations $x_i^{(j)} x_i^{(k)} = \frac{(j+k)!}{j!k!} x_i^{(j+k)}$. We set $x_i^{(0)} = 1$ and $x_i^{(1)} = x_i$ for $i = 1, \dots, m$. Therefore $\mathcal{C}(\mathbf{v}; E)$ is the free E-module with basis $x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)}$ for $a \in \mathbb{N}^m$. We set $C_i(\mathbf{v}; E) = \bigoplus_{a \in \mathbb{N}^n,\ |a|=i} E x^{(a)}$. The E-linear differential δ on $\mathcal{C}(\mathbf{v}; E)$ is defined as follows: For $x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)}$ we set $\delta(x^{(a)}) = \sum_{a_i > 0} v_i x_1^{(a_1)} \cdots x_i^{(a_i - 1)} \cdots x_m^{(a_m)}$. Now $\delta \circ \delta = 0$ and $\mathcal{C}(\mathbf{v}; E)$ is indeed a complex. If not otherwise stated, we set $v_i = e_i$ for $i \in [n]$.

Definition 1.4.1. Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. We set

$$C_i(\mathbf{v}; M) = C_i(\mathbf{v}; E) \otimes_E M \text{ and } C^i(\mathbf{v}; M) = \operatorname{Hom}_E(C_i(\mathbf{v}; E), M).$$

The complexes $\{C_i(\mathbf{v}; M)\}$ and $\{C^i(\mathbf{v}; M)\}$ are called *Cartan complex* and *Cartan cocomplex* of M with respect to $\mathbf{v}$. We denote the i^{th}-homology module of these complexes with

$$H_i(\mathbf{v}; M) \text{ and } H^i(\mathbf{v}; M).$$

We call $H_i(\mathbf{v}; M)$ the i^{th}-*Cartan homology* and $H^i(\mathbf{v}; M)$ the i^{th}-*Cartan cohomology* of M with respect to $\mathbf{v}$.

The elements of $C^i(\mathbf{v}; M)$ may be identified with homogeneous polynomials $\sum m_a y^a$ in the variables $y_1, \dots, y_m$ and coefficients $m_a \in M$ where $y^a = y^{a_1} \cdots y^{a_m}$ for $a \in \mathbb{N}^m$, $a = (a_1, \dots, a_m)$. An element $m_a y^a \in C^i(\mathbf{v}; M)$ is characterized by the following property:

$$m_a y^a(x^{(b)}) = \begin{cases} m_a & b = a, \\ 0 & b \neq a. \end{cases}$$

Set $y_{\mathbf{v}} = \sum_{i=1}^n v_i y_i$, then

$$\delta^i : C^i(\mathbf{v}; M) \longrightarrow C^{i+1}(\mathbf{v}; M),\ f \mapsto y_{\mathbf{v}} f.$$

There is a natural grading of the complexes and their homology. We set

$$\deg x_i = 1,\ C_j(\mathbf{v}; M)_i = \operatorname{span}_K(m_a x^{(b)} \colon |a| + |b| = i,\ |b| = j),$$

and

$$\deg y_i = -1,\ C^j(\mathbf{v}; M)_i = \operatorname{span}_K(m_a y^b \colon |a| - |b| = i,\ |b| = j).$$

It turns out that E is injective. A reference for the next result is [**6**].

Proposition 1.4.2. *Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. Then*

(i) $M^* = \operatorname{Hom}_E(M, E) \in \mathcal{M}_{\mathbb{Z}}(E)$.

(ii) $(\ )^*$ *is an exact contravariant functor.*

We also cite another result from [**6**]: For a K-vector space W we define $W^\vee = \operatorname{Hom}_K(W, K)$.

Lemma 1.4.3. *Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. Then*

$$(M^*)_i \cong (M_{n-i})^\vee.$$

In [**5**, 4.2] the following is shown:

Proposition 1.4.4. *Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. Then for all $i \in \mathbb{N}$ there is an isomorphism of graded E-modules*

$$H_i(\mathbf{v}; M)^* \cong H^i(\mathbf{v}; M^*).$$

Cartan homology can be computed recursively (see [**5**, 4.1, 4.3]).

Proposition 1.4.5. *Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. Then for all $j \in [m-1]$ there exist exact sequences of graded E-modules*

$$\begin{aligned} \ldots \to H_i(v_1, \ldots, v_j; M) \to H_i(v_1, \ldots, v_{j+1}; M) \to H_{i-1}(v_1, \ldots, v_{j+1}; M)(-1) \\ \to H_{i-1}(v_1, \ldots, v_j; M) \to H_{i-1}(v_1, \ldots, v_{j+1}; M) \to \ldots, \end{aligned}$$

and

$$\begin{aligned} \ldots \to H^{i-1}(v_1, \ldots, v_{j+1}; M) \to H^{i-1}(v_1, \ldots, v_j; M) \to H^{i-1}(v_1, \ldots, v_{j+1}; M)(+1) \\ \to H^i(v_1, \ldots, v_{j+1}; M) \to H^i(v_1, \ldots, v_j; M) \to \ldots. \end{aligned}$$

Finally, we have:

Proposition 1.4.6. *Let $\mathbf{v} = v_1, \ldots, v_n$ be a basis of E_1. The Cartan complex $\mathcal{C}(\mathbf{v}; E)$ is the minimal graded free resolution of the residue class field K of E. In particular, for all $M \in \mathcal{M}_{\mathbb{Z}}(E)$:*

$$\operatorname{Tor}_i^E(K, M) \cong H_i(\mathbf{v}; M) \text{ and } \operatorname{Ext}_E^i(K, M) \cong H^i(\mathbf{v}; M)$$

as graded modules.

1.5. Local cohomology

Local cohomology is a useful tool in commutative algebra. In this thesis we use this theory only in a special situation where we can give a nice explicit description. Let $S = K[x_1, \ldots, x_n]$ be the standard $\mathbb{N}^n$-graded polynomial ring.

Construction 1.5.1. Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$. We define the cochain complex

$$\mathcal{L}(M)\colon 0 \to L^0 \to L^1 \to \ldots \to L^n \to 0$$

where

$$L^i = \bigoplus_{G \subseteq [n],\ |G| = i} M_{x^G}.$$

Here M_{x^G} is the localization of M at the multiplicative set $\{(x^G)^n \colon n \in \mathbb{N}\}$. The differential map d is composed of the maps

$$(-1)^{\alpha(i,G)} \mathrm{nat}\colon M_{x^G} \to M_{x^{G \cup \{i\}}} \text{ for } i \notin G.$$

This complex is called the *Čech complex* of M.

Definition 1.5.2. Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$. We define $H^i_{\mathfrak{m}}(M) = H^i(\mathcal{L}(M))$ as the i^{th}-*local cohomology module of M*.

Local cohomology can be used to compute the depth and the (Krull-) dimension of a module. The following result is due to Grothendieck.

Theorem 1.5.3. *Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$. Then:*

(i) $H^i_{\mathfrak{m}}(M) = 0$ *for* $i < \operatorname{depth}(M)$ *and* $i > \dim(M)$.

(ii) $H^{\operatorname{depth}(M)}_{\mathfrak{m}}(M) \neq 0$ *and* $H^{\dim(M)}_{\mathfrak{m}}(M) \neq 0$.

Recall that $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$ is Cohen-Macaulay if $\operatorname{depth}(M) = \dim(M)$.

Corollary 1.5.4. *Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$ and $d \in [n]$. Then M is Cohen-Macaulay of dimension d if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq d$.*

We set $\omega_S = S(-\varepsilon_1 - \ldots - \varepsilon_n)$. This module is also called the *canonical module of S*. Next we present one of the consequences of the so-called local duality theorem of Grothendieck.

Proposition 1.5.5. *Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$. For $u \in \mathbb{Z}^n$ and all integers i we have*

$$\operatorname{Ext}^i_S(M, \omega_S)_u \cong H^{n-i}_{\mathfrak{m}}(M)_{-u}$$

as a K-vector space.

At the end of this section we give a characterization of the Castelnuovo-Mumford regularity by local cohomology (see [**16**]).

Proposition 1.5.6. *Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$. Then*

$$\operatorname{reg}_S(M) = \sup\{j \in \mathbb{Z} \colon H^i_{\mathfrak{m}}(M)_{j-i} \neq 0 \text{ for some } i \in \mathbb{Z}\}.$$

1.6. The Rees algebra and the symmetric algebra of an ideal

Let $S_x = K[x_1, \ldots, x_n]$ be the standard graded polynomial ring with maximal ideal $\mathfrak{m}_x$. For studying powers of graded ideals $I = (f_1, \ldots, f_m) \subset S_x$ it is useful to consider the algebra $R(I) = S_x[f_1 t, \ldots, f_m t] \subset S_x[t]$ where t is a further indeterminate.

Definition 1.6.1. Let $I \subset S_x$ be a graded ideal. Then $R(I)$ is called the *Rees algebra of I.*

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be another polynomial ring. We define

$$\varphi : S \to R(I),\ x_i \mapsto x_i,\ y_j \mapsto f_j t,$$

and let $J = \operatorname{Ker}(\varphi)$. Then $S/J \cong R(I)$. If I is generated in one degree, we may assume that S is standard bigraded, J is a bigraded ideal and $R(I) = S/J$. In this situation $I^j \cong (S/J)_{(*,j)}(-jd)$ as S_x-modules for all $j \in \mathbb{N}$.

There exists another important construction related to ideals. As above let $I = (f_1, \ldots, f_m) \subset S_x$ be a graded ideal. Let $\mathcal{F}$ be the minimal graded free resolution of I. We may identify the map $F_1 \to F_0$ with an $m \times t$-matrix (a_{ij}) for some $t \geq 0$ and $a_{ij} \in \mathfrak{m}_x$. This matrix is also called the *relation matrix of I*. We define $J = (g_1, \ldots, g_t) \subset S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with $g_j = \sum_{i=1}^m a_{ij} y_i$ for $j = 1, \ldots, t$ and set $S(I) = S/J$.

Definition 1.6.2. Let $I \subset S_x$ be a graded ideal. Then $S(I)$ is called the *symmetric algebra of I.*

Similarly if I is generated in one degree, we may assume that S is standard bigraded, J is a bigraded ideal and $S(I) = S/J$. We also consider the finitely generated S_x-module $S^j(I) = (S/J)_{(*,j)}(-jd)$, which we call the j^{th}*-symmetric power of* I.

There exists always a surjective map from $S(I)$ to $R(I)$.

Definition 1.6.3. Let $I \subset S_x$ be a graded ideal. I is said to be of *linear type* if

$$R(I) \cong S(I).$$

Before we can give an example for ideals of linear type, we introduce a special class of ideals.

Definition 1.6.4. A sequence of elements $f_1, \dots, f_r$ in a ring is called a *d-sequence* if:

(i) $f_1, \dots, f_r$ is a minimal system of generators of the ideal $I = (f_1, \dots, f_r)$.
(ii) $(f_1, \dots, f_{i-1}) : f_i \cap I = (f_1, \dots, f_{i-1})$.

A well-known fact is (see for example [**51**]):

Proposition 1.6.5. *Let $I \subset S_x$ be a graded ideal. If I can be generated by a d-sequence, then I is of linear type.*

Hence there exists a large class of ideals which are of linear type.

Example 1.6.6. The following examples can be found in [**37**].

(i) The sequence $x_1, \dots, x_j$ is a d-sequence in S_x.
(ii) Let $K[x_{ij}]$ be the standard graded polynomial ring in $n \times n + 1$ variables. The maximal minors of the $n \times (n+1)$-matrix (x_{ij}) form a d-sequence.

1.7. Gröbner bases

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring. For $u \in \mathbb{N}^n$ we call $x^u = x_1^{u_1} \cdots x_n^{u_n}$ a *monomial in* S.

Let F be a finitely generated graded free S-module with homogeneous basis $\mathbf{f} = f_1, \dots, f_t$ and $\deg(f_i) = d_i \in \mathbb{Z}$. We also call elements $x^u e_i$ monomials (in F).

Definition 1.7.1. A (*degree refining*) *term order* on F is a total order $>$ of the monomials of F which satisfies the following conditions. For $i, j \in [t]$ and monomials $x^u, x^{u'}, x^{u''} \in S$ one has:

(i) If $\deg(x^u f_i) > \deg(x^{u'} f_j)$, then $x^u f_i > x^{u'} f_j$.
(ii) If $x^u f_i > x^{u'} f_j$, then $x^{u''} x^u f_i > x^{u''} x^{u'} f_j$.

Example 1.7.2. Usually we consider the following term orders:

(i) The (degree) lexicographic term order $>_{lex}$: Let $x^u f_i >_{lex} x^{u'} f_j$ if the first non-zero component from the left of

$$(\sum_{k=1}^{n}(u_k - u'_k) + d_i - d_j, u_1 - u'_1, \dots, u_n - u'_n, j - i)$$

is positive.

(ii) The (degree) reverse-lexicographic term order $>_{rlex}$: Let $x^u f_i >_{rlex} x^{u'} f_j$ if the first non-zero component from the left of

$$(\sum_{k=1}^{n}(u_k - u'_k) + d_i - d_j, u'_n - u_n, \dots, u'_1 - u_1, j - i)$$

is positive.

Note that if $F = S$, then in both examples $x_1 > \dots > x_n$.

Definition 1.7.3. Let $>$ be a term order on F and let $g \in F$ be a homogeneous element with unique representation $g = \sum_{u,i} \lambda_{u,i} x^u f_i$ with $\lambda_{u,i} \in K$. Then $\text{in}_>(g) = \max\{x^u f_i \colon \lambda_{u,i} \neq 0\}$ is called *the initial monomial of* g.

This notion has many nice properties. For example, let $g_1, \dots, g_t \in F$ be homogeneous elements such that $\text{in}_>(g_1), \dots, \text{in}_>(g_t)$ are K-linearly independent. Then $g_1, \dots, g_t$ are also K-linearly independent.

Now we associate to every finitely generated graded submodule M of F a module which can be useful in studying M.

Definition 1.7.4. Let $>$ be a term order on F and $M \in \mathcal{M}_{\mathbb{Z}}(S)$ a graded submodule of F. The *initial module* $\text{in}_>(M)$ *of* M with respect to $>$ is the submodule of F generated by all $\text{in}_>(g)$ with $g \in M$.

A special class of $\mathbb{Z}^n$-graded submodules of F are:

Definition 1.7.5. Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ be a graded submodule of F. M is said to be a *monomial submodule* of F if, for all homogeneous elements $f \in M$ with $f = \sum_{u,i} \lambda_{u,i} x^u f_i$ where $0 \neq \lambda_{u,i} \in K$, it follows that $x^u f_i \in M$.

We observe that $\text{in}_>(M)$ is a monomial module. In particular, monomial submodules of S are called *monomial ideals*. It is easy to see that every monomial ideal $I \subset S$ has a unique minimal system of generators $x^{u^1}, \dots, x^{u^t}$, which we denote with $G(I)$. It is well-known (see for example [**30**]) that:

Proposition 1.7.6. *Let $>$ be a term order on F and $M \in \mathcal{M}_{\mathbb{Z}}(S)$ a graded submodule of F. Then for all integers i, j one has*

$$\beta^S_{i,j}(M) \leq \beta^S_{i,j}(\text{in}_>(M)).$$

Usually we consider the reverse lexicographic term order on F. This order has some nice properties. Let R be a ring, M an R-submodule of an R-module N and I an ideal of R. Then we define the following submodule of N:

$$(M :_N I) = \{f \in N \colon If \subseteq M\}.$$

Note that $(M :_N I)$ is graded, bigraded or $\mathbb{Z}^n$-graded if R, M, N and I admit this grading. See [**23**, Chapter 15] for the following observation.

Proposition 1.7.7. *Let $>_{rlex}$ be the reverse lexicographic order on F and $M \in \mathcal{M}_{\mathbb{Z}}(S)$ a graded submodule of F. Then for all $i \in [n]$ one has:*

(i) $\text{in}_{rlex}(M + (x_n, \dots, x_{i+1})F) = \text{in}_{rlex}(M) + (x_n, \dots, x_{i+1})F$.
(ii) $(\text{in}_{rlex}(M + (x_n, \dots, x_{i+1})F) :_F x_i) = \text{in}_{rlex}(M + (x_n, \dots, x_{i+1})F :_F x_i)$.

There is another way to associate monomial modules to a given module (for details see [**23**]). We write $\mathcal{GL}(n;K)$ for the (general linear) group of $n\times n$-matrices with entries in K and non-vanishing determinant.

Construction 1.7.8. Let $>_{rlex}$ be the reverse lexicographic order on F and $M\in\mathcal{M}_{\mathbb{Z}}(S)$ a graded submodule of F. Every element $g\in\mathcal{GL}(n;K)$ induces an S-linear automorphism on F by extending

$$g(x_je_k)=\sum_{i=1}^{n}g_{ij}x_ie_k \text{ for } g=(g_{ij}).$$

There exists a non-empty open set $U\subseteq\mathcal{GL}(n;K)$ and a unique monomial S-module $\tilde{M}$ with $\tilde{M}=\mathrm{in}_{rlex}(g(M))$ for every $g\in U$. We call $\tilde{M}$ the *generic initial module of* M and denote it with $\mathrm{gin}(M)$. The elements of U are called *generic for* M.

A nice property is that $\mathrm{gin}(M)$ is *Borel-fixed,* i.e. $\mathrm{gin}(M)=b(\mathrm{gin}(M))$ for all $b\in\mathcal{B}$ where $\mathcal{B}$ is the Borel subgroup of $\mathcal{GL}(n;K)$, which is generated by all upper triangular matrices.

In the sequel we consider monomial ideals $I\subset S$. We need the following notation. For a monomial $1\neq x^u\in S$ we set

$$m(u)=m(x^u)=\max\{i\colon x_i \text{ divides } x^u\}.$$

Let $m(1)=0$.

Definition 1.7.9. Let $I\subset S$ be a monomial ideal. Then:

(i) I is called *stable* if, for all monomials $x^u\in I$ and all $i\leq m(u)$, one has $x_ix^u/x_{m(u)}\in I$.

(ii) I is called *strongly stable* if, for all monomials $x^u\in I$, all j such that $x_j|x^u$ and all $i\leq j$, one has $x_ix^u/x_j\in I$.

If $\mathrm{char}(K)=0$, then it is well-known that Borel-fixed ideals are exactly the strongly stable ideals. In particular $\mathrm{gin}(I)$ is strongly stable.

Remark 1.7.10. It suffices to show the conditions of 1.7.9 for the generators of the ideal I.

The regularity of an ideal and its generic initial ideal coincides (see [**23**]).

Proposition 1.7.11. *Let $I\subset S$ be a graded ideal. Then*

$$\mathrm{reg}_S(S/I)=\mathrm{reg}_S(S/\mathrm{gin}(I)).$$

For stable ideals there exist explicit formulas for the graded Betti numbers (see [**27**]).

Proposition 1.7.12. *Let $I\subset S$ be a stable ideal. Then for all integers i,j*

$$\beta^S_{i,i+j}(I)=\sum_{x^u\in G(I),\ |u|=j}\binom{m(x^u)-1}{i}.$$

For example, if a stable ideal I is generated in one degree, then I has a linear resolution.

It is possible to formulate the whole Gröbner basis theory for the bigraded polynomial ring with minor modifications. We recall the main facts for ideals. Let

$S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be the bigraded polynomial ring for the rest of the section.

Definition 1.7.13. A (*bidegree refining*) *term order* on S is a total order $>$ of the monomials of S which satisfies the following conditions. For monomials $x^u y^v, x^{u'} y^{v'}$, $x^{u''} y^{v''} \in S$ one has:

(i) If the first non-zero component from the left of
$$(|u| + |v| - |u'| - |v'|, |v| - |v'|, |u| - |u'|)$$
is positive, then $x^u y^v > x^{u'} y^{v'}$.

(ii) If $x^u y^v > x^{u'} y^{v'}$, then $x^{u''} y^{v''} x^u y^v > x^{u''} y^{v''} x^{u'} y^{v'}$.

Example 1.7.14. Usually we consider the (*bidegree*) *reverse-lexicographic term order* $>_{rlex}$ on S: Let $x^u y^v >_{rlex} x^{u'} y^{v'}$ if the first non-zero component from the left of
$$(|u| + |v| - |u'| - |v'|, |v| - |v'|, |u| - |u'|, u'_n - u_n, \dots, u'_1 - u_1, v'_m - v_m, \dots, v'_1 - v_1)$$
is positive.

Observe that we have $y_1 > \ldots > y_m > x_1 > \ldots > x_n$. The bigeneric initial ideal was introduced in [**3**].

Construction 1.7.15. Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be the bigraded polynomial ring, $>_{rlex}$ the reverse lexicographic order on S and $J \subset S$ a bigraded ideal. Let $G = \mathcal{GL}(n, K) \times \mathcal{GL}(m, K)$ and $g = (d_{ij}, e_{kl}) \in G$. Then g defines an S-automorphism by extending $g(x_j) = \sum_i d_{ij} x_i$ and $g(y_l) = \sum_k e_{kl} y_k$. There exists a monomial ideal $J' \subset S$ and a non-empty Zariski open set $U \subseteq G$ such that for all $g \in U$ we have $J' = \operatorname{in}(gJ)$. We call J' the *bigeneric initial ideal of* J and denote it with $\operatorname{bigin}(J)$. The elements of U are said to be *generic for* J.

We recall the following definitions from [**3**]. For a monomial $x^u y^v \in S$ we set
$$m_x(x^u y^v) = m(u), \quad m_y(x^u y^v) = m(v).$$
Similarly we define
$$m(L) = \max\{i \colon i \in L\}.$$
for $\emptyset \neq L \subseteq [n]$ and set $m(\emptyset) = 0$. Let $J \subset S$ be a monomial ideal. If $G(J) = \{z_1, \dots, z_t\}$ with $\deg(z_i) = (a_i, b_i) \in \mathbb{N}^2$, we set $m_x(J) = \max\{a_i\}$ and $m_y(J) = \max\{b_i\}$.

Definition 1.7.16. Let $J \subset S$ be a monomial ideal. Then:

(i) J is called *bistable* if, for all monomials $w \in J$, all $i \leq m_x(w)$, all $j \leq m_y(w)$, one has $x_i w / x_{m_x(w)} \in J$ and $y_j w / y_{m_y(w)} \in J$.

(ii) J is called *strongly bistable* if, for all monomials $w \in J$, all $i \leq s$ such that x_s divides w, all $j \leq t$ such that y_t divides w, one has $x_i w / x_s \in J$ and $y_j w / y_t \in J$.

If $\operatorname{char}(K) = 0$, then $\operatorname{bigin}(J)$ is strongly bistable for every bigraded ideal J.

1.8. Simplicial complexes

We introduce some objects from "combinatorial commutative algebra". This area was created by Hochster and Stanley in the mid-seventies. They used methods from commutative algebra to solve purely combinatorial problems.

Definition 1.8.1. A simplicial complex Δ (on $[n]$) is a collection of subsets of $[n]$ such that:

(i) $\{i\} \in \Delta$ for $i = 1, \dots, n$.
(ii) If $G \in \Delta$ and $F \subseteq G$, then $F \in \Delta$.

The set $[n]$ is said to be *the vertex set* of Δ. We call elements of Δ *faces* . We define $\dim(F) = |F| - 1$ as the dimension of a face $F \in \Delta$ and $\dim \Delta = \max\{\dim(F) \colon F \in \Delta\}$ as the dimension of Δ. Let $F \in \Delta$. Then F is said to be an *edge or vertex* if $\dim F = 1$ or $\dim F = 0$. The maximal faces under inclusion are called the *facets* of Δ.

Example 1.8.2. Let

$$\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Then Δ is a simplicial complex on [3]. We see that Δ has only one facet. This type of a simplicial complex is called a *simplex.*

To every simplicial complex we associate a dual simplicial complex which turns out to be useful in the forthcoming chapters. For $F \subseteq [n]$ we set $F^\vee = [n] - F$.

Definition 1.8.3. Let Δ be a simplicial complex on $[n]$. We call $\Delta^* = \{F \colon F^\vee \notin \Delta\}$ the *Alexander dual of* Δ.

Lemma 1.8.4. *Let* Δ, Γ *be simplicial complexes on* $[n]$. *Then:*

(i) Δ^* *is a simplicial complex on some vertex set.*
(ii) $\Delta^{**} = \Delta$.
(iii) *If* $\Gamma \subset \Delta$, *then* $\Delta^* \subset \Gamma^*$.

Let $S = K[x_1, \dots, x_n]$ be the $\mathbb{N}^n$-graded polynomial ring and $E = K\langle e_1, \dots, e_n\rangle$ be the $\mathbb{N}^n$-graded exterior algebra. We associate certain algebras to every simplicial complex.

Definition 1.8.5. Let Δ be a simplicial complex on $[n]$. Then

$$I_\Delta = (x_{i_1} \cdots x_{i_s} \colon \{i_1, \dots, i_s\} \notin \Delta),\ K[\Delta] = S/I_\Delta$$

are called the *Stanley-Reisner ideal* and the *Stanley-Reisner ring* of Δ. Similarly we define

$$J_\Delta = (e_{i_1} \wedge \ldots \wedge e_{i_s} \colon \{i_1, \dots, i_s\} \notin \Delta),\ K\{\Delta\} = E/J_\Delta$$

to be the *exterior Stanley-Reisner ideal* and the *exterior face ring* of Δ.

CHAPTER 2

Generalized Alexander duality and applications

We study square-free modules, which are a natural extension of the concept of Stanley-Reisner rings associated to simplicial complexes. A duality operation is introduced. We give applications generalizing well-known results about Alexander duality. In this chapter $S = K[x_1, \dots, x_n]$ denotes the standard graded (or $\mathbb{N}^n$-graded) polynomial ring and $E = K\langle e_1, \dots, e_n\rangle$ the standard graded (or $\mathbb{N}^n$-graded) exterior algebra.

2.1. Square-free modules and generalized Alexander duality

The present section is devoted to the introduction of the so-called category of square-free modules and definition of a contravariant exact functor on this category.

We need the following notation. For $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, we say that a is *square-free* if $0 \leq a_i \leq 1$ for $i = 1, \dots, n$. Sometimes a square-free vector a and $F = \operatorname{supp}(a) \subseteq [n]$ are identified. If $a \in \mathbb{N}^n$ is square-free, we set $e_a = e_{a_{j_1}} \wedge \ldots \wedge e_{a_{j_k}}$ where $\operatorname{supp}(a) = \{j_1 < \ldots < j_k\}$ and we say that e_a is a *monomial* in E. For monomials $u, v \in E$ with $\operatorname{supp}(v) \subseteq \operatorname{supp}(u)$ there exists a unique monomial $w \in E$ such that $vw = u$; we set $w = v^{-1}u$. Notice that, for monomials $u, v, w, z \in E$, the equalities below hold whenever the expressions are defined:

$$(v^{-1}u)w = v^{-1}(uw) \qquad \text{and} \qquad (z^{-1}v)(v^{-1}u) = z^{-1}u.$$

The starting point is a definition introduced by Yanagawa in [**53**].

Definition 2.1.1. Let $N \in \mathcal{M}_{\mathbb{N}^n}(S)$. The module N is called *a square-free S-module* if the multiplication maps $N_a \ni w \mapsto x_i w \in N_{a+\varepsilon_i}$ are bijective for all $a \in \mathbb{N}^n$ and all $i \in \operatorname{supp}(a)$.

For example, the Stanley-Reisner ring $K[\Delta]$ of a simplicial complex Δ is a square-free S-module. It is easy to see that, for $a \in \mathbb{N}^n$ and a square-free S-module N, we have $\dim_K N_a = \dim_K N_{\operatorname{supp}(a)}$ and N is generated by its square-free part $\{N_F \colon F \subseteq [n]\}$. Yanagawa proved in [**53**, 2.3, 2.4]:

Proposition 2.1.2. *Let $N, N' \in \mathcal{M}_{\mathbb{N}^n}(S)$ be square-free S-modules and let $\varphi \colon N \to N'$ be a homogeneous homomorphism. Then* $\operatorname{Ker}(\varphi)$ *and* $\operatorname{Coker}(\varphi)$ *are again square-free S-modules.*

This proposition has consequences on the minimal $\mathbb{N}^n$-graded free resolution of a square-free S-module.

Corollary 2.1.3. *Let $N \in \mathcal{M}_{\mathbb{N}^n}(S)$ be a square-free S-module. Then, for all integers i, the i^{th}-syzygy module $\Omega_i(N)$ in the minimal $\mathbb{N}^n$-graded free S-resolution of N is a square-free S-module.*

Remark 2.1.4. Let $N \in \mathcal{M}_{\mathbb{N}^n}(S)$ be a square-free S-module and $\mathcal{F}$ the minimal $\mathbb{N}^n$-graded free S-resolution of N. The $\mathbb{N}^n$-graded free S-module F_i is generated by elements f with square-free $\deg(f) \in \mathbb{N}^n$. We call $\mathcal{F}$ a *square-free resolution* of N. It is easy to see that N is a square-free S-module if and only if N has a square-free resolution.

This leads to the following category.

Definition 2.1.5. Let $\mathcal{SQ}(S)$ denote the abelian category of the square-free S-modules where the morphisms are the homogeneous homomorphisms.

The following construction, which is of crucial importance for this chapter, has been introduced by Aramova, Avramov and Herzog [**1**]:

Construction 2.1.6. Let $(\mathcal{F}, \theta)$ be a complex of $\mathbb{N}^n$-graded free S-modules. We assume that each F_i has a homogeneous basis B_i such that $\deg(f) \in \mathbb{N}^n$ is square-free for all $f \in B_i$.

For $a \in \mathbb{N}^n$ and $f \in B_i$ let $y^{(a)}f$ be a symbol to which we assign $\deg(y^{(a)}f) = a + \deg(f)$. Now define the $\mathbb{N}^n$-graded free E-module $G_l \in \mathcal{M}_{\mathbb{N}^n}(E)$ with basis $y^{(a)}f$ where $a \in \mathbb{N}^n$, $f \in B_i$, $\operatorname{supp}(a) \subseteq \operatorname{supp}(f)$ and $l = |a| + i$. For $f \in B_i$ and

$$\theta_i(f) = \sum_{f_j \in B_{i-1}} \lambda_j x^{b-b_j} f_j \qquad \text{with } \lambda_j \in K, \quad b = \deg(f), \quad b_j = \deg(f_j),$$

we define homomorphisms $G_l \to G_{l-1}$ of $\mathbb{N}^n$-graded E-modules by

$$\gamma_l(y^{(a)}f) = (-1)^{|b|} \sum_{k \in \operatorname{supp}(a)} y^{(a-\varepsilon_k)} f e_k,$$

$$\vartheta_l(y^{(a)}f) = (-1)^{|a|} \sum_{f_j \in B_{i-1}} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_b.$$

Set $\delta_l = \gamma_l + \vartheta_l : G_l \to G_{l-1}$. Then $(\mathcal{G}, \delta)$ is a complex of free $\mathbb{N}^n$-graded E-modules. If $(\mathcal{G}', \delta')$ is the complex obtained by a different homogeneous basis B' of $\mathcal{F}$, then $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic as complexes of $\mathbb{N}^n$-graded E-modules.

There is an important result:

Theorem 2.1.7. *Let $N \in \mathcal{SQ}(S)$ and $(\mathcal{F}, \theta)$ be the minimal $\mathbb{N}^n$-graded free S-resolution of N. Then $(\mathcal{G}, \delta)$ is the minimal $\mathbb{N}^n$-graded free E-resolution of $N_E = $* $\operatorname{Coker}(G_1 \to G_0)$.

Proof. See the proof of [**1**, 1.3]. There the theorem was shown for S/I where I is a square-free monomial ideal. The proof holds also in this more general situation. □

For example let Δ be a simplicial complex. Then $(I_\Delta)_E = J_\Delta$.

Remark 2.1.8. Sometimes we will need the following observation. We may identify a minimal homogeneous system of generators $\{f_1, \dots, f_t\}$ of N with the one of N_E. For all square-free $F \subseteq [n]$ we have $N_F \cong (N_E)_F$. If $\deg(f_i) = G_i \subseteq [n]$, then a K-basis of N_F consists of the elements $x^{H_j} f_i$ with $H_j \subseteq [n]$ and $H_j \dot\cup G_i = F$. Using the bijection above the corresponding K-basis of $(N_E)_F$ is $e_{H_j} f_i$.

Recall that N and N_E may also be considered as graded modules over S and E by defining $N_i = \bigoplus_{a\in\mathbb{N}^n, |a|=i} N_a$ and similarly N_E. By counting the ranks in the constructed resolution in 2.1.6 we get the following formulas.

Corollary 2.1.9. *Let $N \in \mathcal{SQ}(S)$. Then:*

(i) $\beta^E_{i,i+j}(N_E) = \sum_{k=0}^{i} \binom{i+j-1}{k+j-1} \beta^S_{k,k+j}(N)$ *for all $i, j \in \mathbb{N}$.*
(ii) $\beta^E_{i,a}(N_E) = \beta^S_{i,a}(N)$ *for all square-free $a \in \mathbb{N}^n$.*

It is a natural question whether the construction 2.1.6 has an inverse. This means, given a certain E-module with its minimal free E-resolution, we want to construct an S-module with a free S-resolution.

Definition 2.1.10. Let $M \in \mathcal{M}_{\mathbb{N}^n}(E)$. The module M is called *a square-free E-module* if one has $M_a = 0$ for all $a \in \mathbb{N}^n$ with a is not square-free.

Example 2.1.11. We observe:

(i) The exterior face ring $K\{\Delta\}$ associated to a simplicial complex Δ is a square-free E-module.
(ii) As a direct consequence of the definition for a square-free S-module N the E-module N_E is a square-free E-module.

Observe that, for a homogeneous homomorphism $\varphi\colon M \to M'$ between two square-free E-modules, one has that $\operatorname{Ker}(\varphi)$ and $\operatorname{Coker}(\varphi)$ are again square-free E-modules. This leads to another category.

Definition 2.1.12. Let $\mathcal{SQ}(E)$ denote the abelian category of the square-free E-modules where the morphisms are the homogeneous homomorphisms.

We consider the following construction which is inverse to 2.1.6.

Construction 2.1.13. Let $M \in \mathcal{SQ}(E)$, $(\mathcal{G}, \delta)$ be the minimal $\mathbb{N}^n$-graded free E-resolution of M and B_i be a homogeneous basis of G_i for all $i \in \mathbb{N}$. We set $\tilde{B}_i = \{f \in B_i\colon \deg(f) \text{ is square-free}\}$. We define a complex $(\mathcal{F}, \theta)$ of S-modules where F_i is the $\mathbb{N}^n$-graded free S-module with homogeneous basis $\tilde{B}_i$. If $f \in \tilde{B}_i$ and

$$\delta_i(f) = \sum_{f_j \in \tilde{B}_{i-1}} f_j \lambda_j e_{b_j}^{-1} e_b \qquad \text{with } b = \deg(f),\ b_j = \deg(f_j) \text{ and } \lambda_j \in K,$$

we set

$$\theta_i(f) = \sum_{f_j \in \tilde{B}_{i-1}} f_j \lambda_j x^{b-b_j}.$$

It follows immediately from the construction that $(\mathcal{F}, \theta)$ is indeed a complex.

Hence we are able to prove:

Theorem 2.1.14. *Let $M \in \mathcal{SQ}(E)$ and let $(\mathcal{G}, \delta)$ be the minimal $\mathbb{N}^n$-graded free E-resolution of M. The constructed complex $(\mathcal{F}, \theta)$ is the minimal $\mathbb{N}^n$-graded free S-resolution of $M_S = \operatorname{Coker}(F_1 \to F_0)$ and $M_S \in \mathcal{SQ}(S)$.*

Proof. All the free S-modules F_i are square-free S-modules by definition. Thus by 2.1.2 we get that $M_S \in \mathcal{SQ}(S)$ because it is the cokernel of the homogeneous map between F_1 and F_0. Let $(\tilde{\mathcal{F}}, \tilde{\theta})$ be the minimal $\mathbb{N}^n$-graded free S-resolution of the S-module M_S. By the first construction (see 2.1.7) we get a minimal $\mathbb{N}^n$-graded free E-resolution $(\tilde{\mathcal{G}}, \tilde{\delta})$ of the E-module $(M_S)_E$. The definitions are made such that $(M_S)_E = M$. Therefore $\tilde{\mathcal{G}} \cong \mathcal{G}$ as complexes, since both complexes are minimal $\mathbb{N}^n$-graded free E-resolutions of M. If we apply the second construction to $(\tilde{\mathcal{G}}, \tilde{\delta})$, we get $(\tilde{\mathcal{F}}, \tilde{\theta})$. All in all it follows that $\tilde{\mathcal{F}} \cong \mathcal{F}$ as complexes and hence $(\mathcal{F}, \theta)$ is the minimal $\mathbb{N}^n$-graded free S-resolution of the S-module M_S. □

We get immediately:

Corollary 2.1.15. *Let $N \in \mathcal{SQ}(S)$ and $M \in \mathcal{SQ}(E)$. We write N_E for the square-free E-module defined in* 2.1.7 *and M_S for the square-free S-module defined in* 2.1.14. *Then*

$$(M_S)_E \cong M \qquad \textit{and} \qquad (N_S)_E \cong N.$$

Consider two square-free S-modules N, N' and a homogeneous homomorphism $\varphi\colon N \to N'$. Take the minimal $\mathbb{N}^n$-graded free S-resolution $(\mathcal{F}, \theta)$ of N and the minimal $\mathbb{N}^n$-graded free S-resolution $(\mathcal{F}', \theta')$ of N' with homogeneous bases $\{B_i\}$ and $\{B_i'\}$. It is well-known that φ induce a complex homomorphism $\varphi\colon \mathcal{F} \to \mathcal{F}'$. By construction 2.1.7 we get minimal $\mathbb{N}^n$-graded free E-resolutions $(\mathcal{G}, \delta)$ and $(\mathcal{G}', \delta')$ of N_E and N_E'. Let $f \in B_i$ and $\varphi_i(f) = \sum_{f_j' \in B_i'} \lambda_j x^{b-b_j'} f_j'$ where $b = \deg(f)$ and $b_j' = \deg(f_j')$. Then we define a complex homomorphism

$$\psi\colon \mathcal{G} \to \mathcal{G}', \qquad y^{(a)} f \mapsto \sum_{f_j' \in B_i'} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_b,$$

for $a \in \mathbb{N}^n$, $f \in B_i$ and $\operatorname{supp}(a) \subseteq \operatorname{supp}(f)$. Now ψ induces a homogeneous homomorphism $\psi\colon M_E \to M_E'$.

Similarly two square-free E-modules M, M' and a homogeneous homomorphism $\psi : M \to M'$ induce a homogeneous homomorphism $\varphi : M_S \to M_S'$. It turns out that these assignments define functors.

Definition 2.1.16. Let

$$F\colon \mathcal{SQ}(S) \to \mathcal{SQ}(E), \qquad N \mapsto N_E$$

and

$$G\colon \mathcal{SQ}(E) \to \mathcal{SQ}(S), \qquad M \mapsto M_S.$$

From the above discussion we observe:

Proposition 2.1.17. *F and G are additive covariant exact functors of abelian categories. In particular the categories $\mathcal{SQ}(S)$ and $\mathcal{SQ}(E)$ are equivalent.*

We give a simple example.

Example 2.1.18. Let $\Gamma \subseteq \Delta$ be simplicial complexes. Then I_Γ/I_Δ is an element of $\mathcal{SQ}(S)$ and J_Γ/J_Δ is one of $\mathcal{SQ}(E)$. We obtain

$$(I_\Gamma/I_\Delta)_E = J_\Gamma/J_\Delta \qquad \text{and} \qquad (J_\Gamma/J_\Delta)_S = I_\Gamma/I_\Delta.$$

In 1.8 we saw that there exists a duality operation on simplicial complexes. The next goal will be to extend this duality to the category of square-free modules. Recall that the functor

$$(\ )^*\colon \mathcal{M}_{\mathbb{Z}}(E) \to \mathcal{M}_{\mathbb{Z}}(E),\ M \mapsto \operatorname{Hom}_E(M, E)$$

is contravariant and exact (see 1.4.2). Moreover, we have:

Lemma 2.1.19. *Let* $\Gamma \subseteq \Delta$ *be simplicial complexes. Then*

$$(J_\Gamma/J_\Delta)^* \cong J_{\Delta^*}/J_{\Gamma^*}.$$

Proof. We observe that $(E/J_\Delta)^* = \operatorname{Hom}_E(E/J_\Delta, E) \cong (0 :_E J_\Delta) = J_{\Delta^*}$. Consider the exact sequence

$$0 \to J_\Gamma/J_\Delta \to E/J_\Delta \to E/J_\Gamma \to 0.$$

Since the functor $(\ )^*$ is contravariant and exact, we get the exact sequence

$$0 \to (E/J_\Gamma)^* \to (E/J_\Delta)^* \to (J_\Gamma/J_\Delta)^* \to 0.$$

The assertion follows. □

For example $K\{\Delta\}^* = J_{\Delta^*}$. Thus we define:

Definition 2.1.20. Let $M \in \mathcal{SQ}(E)$. We call M^* the *generalized Alexander dual* of M.

This gives us a hint how to define the generalized Alexander dual for elements in $\mathcal{SQ}(S)$.

Definition 2.1.21. Let $N \in \mathcal{SQ}(S)$. We call

$$N^* = ((N_E)^*)_S$$

the *generalized Alexander dual* of N.

Note that

$$(\ )^* : \mathcal{SQ}(S) \to \mathcal{SQ}(S), \qquad N \mapsto N^*,$$

is a contravariant exact functor.

We may also define the generalized Alexander dual without referring to the exterior algebra. Let $N \in \mathcal{SQ}(S)$ and $\mathcal{F} \to N \to 0$ be a minimal square-free resolution of N. We call $F_1 \xrightarrow{d_1} F_0 \to N$ *a square-free presentation of* N. Since $(\ )^*$ is a contravariant exact functor, the sequence $0 \to N^* \to F_0^* \xrightarrow{d_1^*} F_1^*$ is exact. Therefore the generalized Alexander dual of N is $\operatorname{Ker}(d_1^*)$. Let $P_F = (x_i \colon i \in F) \subset S$ for $F \subseteq [n]$. One obtains the following explicit description of N^*.

Remark 2.1.22. *Let* $N \in \mathcal{SQ}(S)$ *and* $F_1 \xrightarrow{d_1} F_0 \to N \to 0$ *be a square-free presentation of* N. *Furthermore, let* $F_i = \bigoplus_{G \subseteq [n]} S(-G)^{\beta^S_{i,G}(N)}$ *for* $i \in \{0,1\}$ *and*

$$d_1(f^1_{G,j}) = \sum_{H \subseteq [n]} \sum_{k=1}^{\beta^S_{0,H}(N)} \lambda^{G,j}_{H,k} x^{\deg(f^1_{G,j}) - \deg(f^0_{H,k})} f^0_{H,k}$$

where $\{f^i_{G,j} \colon G \subseteq [n],\ j \in \{1, \dots, \beta^S_{i,G}(N)\},\ \deg(f^i_{G,j}) = G\}$ *is a homogeneous basis of* F_i. *Then* N^* *is the kernel of*

$$F_0^* \xrightarrow{d_1^*} F_1^*$$

where $F_i^* = \bigoplus_{G\subseteq[n]}(S/P_G)^{\beta^S_{i,G}(N)}$,

$$d_1^*(f^{0,*}_{H,k}) = \sum_{G\subseteq[n]}\sum_{j=1}^{\beta^S_{1,G}(N)} \lambda^{G,j}_{H,k} f^{1,*}_{G,j}$$

and $\{f^{i,*}_{G,j}\colon G\subseteq[n],\ j\in\{1,\dots,\beta^S_{i,G}(N)\},\ \deg(f^{i,*}_{G,j})=0\}$ *is a homogeneous system of generators of* F_i^*.

Proof. For $i\in\{0,1\}$ it follows from 2.1.6 that

$$(F_i)_E = \bigoplus_{G\subseteq[n]}(E/(e_i\colon i\in G))(-G)^{\beta^S_{i,G}(N)}. \tag{1}$$

By 2.1.8 we may identify the homogeneous system of generators $\{f^i_{G,j}\}$ of F_i with the one of $(F_i)_E$ for $i\in\{0,1\}$. We have

$$d_{1,E}(f^1_{G,j}) = \sum_{H\subseteq[n]}\sum_{k=1}^{\beta^S_{0,H}(N)} \lambda^{G,j}_{H,k} e^{-1}_{\deg(f^0_{H,k})} e_{\deg(f^1_{G,j})} f^0_{H,k}.$$

Dualizing (1), we obtain

$$((F_i)_E)^* = \bigoplus_{G\subseteq[n]} \operatorname{Hom}_E((E/(e_i\colon i\in G))(-G),E)^{\beta^S_{i,G}(N)}.$$

Observe that we get a system of generators $\{f^{i,*}_{G,j}\}$ for $((F_i)_E)^*$ with

$$f^{i,*}_{G,j}(f^i_{H,k}) = \begin{cases} e_G & H=G \text{ and } j=k,\\ 0 & \text{else.}\end{cases}$$

Computing $d^*_{1,E}$ yields

$$d^*_{1,E}(f^{0,*}_{H,k}) = f^{0,*}_{H,k}\circ d_{1,E} = \sum_{G\subseteq[n]}\sum_{j=1}^{\beta^S_{1,G}(N)} \lambda^{G,j}_{H,k} f^{1,*}_{G,j}.$$

It follows that

$$((F_i)_E)^* \cong \bigoplus_{G\subseteq[n]}(E/(e_i\colon i\in G))^{\beta^S_{i,G}(N)}.$$

Construction 2.1.13 implies

$$F_i^* = (((F_i)_E)^*)_S = \bigoplus_{G\subseteq[n]}(S/P_G)^{\beta^S_{i,G}(N)}.$$

Again by 2.1.8 we identify the homogeneous generators of F_i^* and $((F_i)_E)^*$. Finally,

$$d_1^*(f^{0,*}_{H,k}) = (d^*_{1,E})_S(f^{0,*}_{H,k}) = \sum_{G\subseteq[n]}\sum_{j=1}^{\beta^S_{1,G}(N)} \lambda^{G,j}_{H,k} f^{1,*}_{G,j}.$$

This concludes the proof. $\square$

2.2. Extremal Betti numbers

In this section we describe the relation of the extremal Betti numbers of N and N^* for $N \in \mathcal{SQ}(S)$. First we study the behaviour of the so-called *distinguished pairs* introduced by Aramova and Herzog in [**5**] under duality. Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$ and $\mathbf{v} = v_1, \dots, v_n$ be a basis for E_1. Consider the long exact homology sequence of 1.4.5

$$\dots \to H_i(v_1, \dots, v_{j-1}; M) \to H_i(v_1, \dots, v_j; M) \to H_{i-1}(v_1, \dots, v_j; M)(-1)$$
$$\to H_{i-1}(v_1, \dots, v_{j-1}; M) \to H_{i-1}(v_1, \dots, v_j; M) \to \dots$$

To simplify the notation we set: $H_i(k; M) = H_i(v_1, \dots, v_k; M)$ for $i > 0$ and $H_0(k; M) = (0 :_{M/(v_1,\dots,v_{k-1})M} v_k)/v_k(M/(v_1, \dots, v_{k-1})M)$. Let $H_i(0; M) = 0$ for $i \geq 0$. Notice that $H_0(k; M)$ is not the 0^{th}-Cartan homology of M with respect to $v_1, \dots, v_k$. We obtain the exact sequence

$$\dots \to H_1(j-1; M) \to H_1(j; M) \to H_0(j; M)(-1) \to 0.$$

The following lemma leads to the concept of distinguished pairs [**5**, 9.5].

Lemma 2.2.1. *Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$, $1 \leq l \leq n$ and $j \in \mathbb{Z}$. The following statements are equivalent:*

(i) (1) $H_0(k; M)_j = 0$ *for $k < l$ and* $H_0(l; M)_j \neq 0$,
 (2) $H_0(k; M)_{j'} = 0$ *for all $j' > j$ and all* $k \leq l + j - j'$.
(ii) *For all $i \geq 0$*
 (1) $H_i(k; M)_{i+j} = 0$ *for $k < l$ and* $H_i(l; M)_{i+j} \neq 0$,
 (2) $H_i(k; M)_{i+j'} = 0$ *for all $j' > j$ and all* $k \leq l + j - j'$.
(iii) *Condition* (ii) *is satisfied for some i.*

Moreover, if the equivalent conditions hold, then $H_i(l; M)_{i+j} \cong H_0(l; M)_j$ for all $i \geq 0$.

Definition 2.2.2. Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. A pair of numbers (l, j) satisfying the equivalent conditions of 2.2.1 is called a *homological distinguished pair* (for M).

The author proved in his Diplom thesis [**47**] (see [**45**] for a published proof):

Theorem 2.2.3. *Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. The following statements are equivalent:*

(i) (l, j) *is a homological distinguished pair for M.*
(ii) $(l, n-j-l+1)$ *is a homological distinguished pair for M^*.*

Moreover, if the equivalent conditions hold, then $H_i(l; M)_{i+j} \cong H_i(l; M^*)_{i+n-j-l+1}$ for all integers i.

We quote the following result in [**5**]:

Proposition 2.2.4. *Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$ and $j \in \mathbb{Z}$. The formal power series $P_j^M(t) = \sum_{i \geq 0} \beta_{i,i+j}^E(M) t^i$ is the Hilbert series of a graded $K[y_1, \dots, y_n]$-module. In particular if $P_j^M(t) \neq 0$, then there exists a polynomial $Q_j(t) \in \mathbb{Z}[t]$ and an integer $d_j \in \mathbb{N}$ with $d_j \leq n$ such that*

$$P_j^M(t) = \frac{Q_j(t)}{(1-t)^{d_j}} \text{ and } e_j = Q_j(1) \neq 0.$$

Let $N \in \mathcal{SQ}(S)$. Set $k(j) = \max(\{k\colon \beta^S_{k,k+j}(N) \neq 0\} \cup \{0\})$. If $P_j^{N_E}(t) \neq 0$, then 2.1.9 yields

$$P_j^{N_E}(t) = \frac{\sum_{k=0}^{k(j)} \beta^S_{k,k+j}(N) t^k (1-t)^{k(j)-k}}{(1-t)^{k(j)+j}}. \tag{2}$$

As in [**5**, 9.2, 9.3] we conclude from this expression:

Proposition 2.2.5. *Let $N \in \mathcal{SQ}(S)$ and $j \in \mathbb{N}$ with $P_j^{N_E}(t) \neq 0$. We have $d_j(N_E) = k(j) + j$ and $e_j(N_E) = \beta^S_{k(j),k(j)+j}(N)$.*

Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. As shown in [**5**], if $P_j^M(t) \neq 0$, there exists a basis $\mathbf{v}$ of E_1 and an integer $i \gg 0$ such that $d_j(M) = n + 1 - \min\{k\colon H_i(k; M)_{i+j} \neq 0\}$ and $e_j(M) = \dim_K H_i(n - d_j(M) + 1; M)_{i+j}$. Thus if (l, j) is a homological distinguished pair for M, we have

$$d_j(M) = n + 1 - l \text{ and } e_j(M) = \dim_K H_i(l; M)_{i+j}.$$

Therefore 2.2.3 implies:

Corollary 2.2.6. *Let $M \in \mathcal{M}_{\mathbb{Z}}(E)$. If (l, j) is a homological distinguished pair for M, then $d_j(M) = d_{n-j-l+1}(M^*)$ and $e_j(M) = e_{n-j-l+1}(M^*)$.*

The definition of an extremal Betti number together with (2) imply:

Lemma 2.2.7. *Let $N \in \mathcal{SQ}(S)$. The following statements are equivalent:*

(i) *$\beta^S_{i,i+j}(N)$ is an extremal Betti number of N.*
(ii) *$i = k(j)$ and $d_{j'}(N_E) - d_j(N_E) < j' - j$ for all $j' > j$ with $P_{j'}(t) \neq 0$.*

Proof. Assume that $\beta^S_{i,i+j}(N)$ is an extremal Betti number of N. It follows that $\beta^S_{i',i'+j'}(N) = 0$ for all integers $i' \geq i$ and $j' \geq j$ with $(i', j') \neq (i, j)$. Furthermore, $\beta^S_{i,i+j}(N) \neq 0$. In particular, we obtain $k(j) = i$. For all integers $j' > j$ we get $k(j') < k(j)$. Hence by 2.2.5

$$d_{j'}(N_E) - d_j(N_E) = k(j') + j' - k(j) - j < j' - j.$$

Thus we proved (ii).

Conversely, assume that the assertion of (ii) holds. Since $i = k(j)$, we get

$$\beta^S_{i,i+j}(N) \neq 0 \text{ and } \beta^S_{i',i'+j}(N) = 0 \text{ for } i' > i.$$

Let $j' > j$. If $P_{j'}^{N_E}(t) = 0$, then

$$(*) \quad \beta^S_{i',i'+j'}(N) = 0 \text{ for } i' \geq i.$$

Otherwise we have $P_{j'}^{N_E}(t) \neq 0$. By 2.2.5 and the assumption we get that $k(j') - k(j) = d_{j'}(N_E) - d_j(N_E) - (j' - j) < 0$. Thus $(*)$ also holds in this case. It follows that $\beta^S_{i,i+j}(N)$ is an extremal Betti number of N. □

Lemma 2.2.8. *Let $N \in \mathcal{SQ}(S)$. The following statements are equivalent:*

(i) *$\beta^S_{i,i+j}(N)$ is an extremal Betti number of N.*
(ii) *$(n + 1 - i - j, j)$ is a homological distinguished pair for N_E.*

Moreover, if the equivalent conditions hold, then $\beta^S_{i,i+j}(N) = \dim_K H_0(n+1-i-j)_j$.

Proof. For all integers j and s we set $l_j^s = \inf\{k\colon H_s(k, N_E)_{s+j} \neq 0\}$. Furthermore, let $d_j(N_E) = -\infty$ if $P_j^{N_E}(t) = 0$. As observed above, there exists a basis of E_1 and an integer $s \gg 0$ such that for all integers j we have $d_j(N_E) = n + 1 - l_j^s$ and $e_j(N_E) = \dim_K H_s(l_j^s)_{s+j}$. Fix this s for the rest of the proof.

Assume that $\beta^S_{i,i+j}(N)$ is an extremal Betti number of N. By 2.2.7 this is equivalent to $i = k(j)$ and $d_{j'}(N_E) - d_j(N_E) < j' - j$ for all $j' > j$ with $P_{j'}(t) \neq 0$. Using 2.2.5 we obtain that

$$l_j^s = n + 1 - d_j(N_E) = n + 1 - i - j$$

and

$$l_{j'}^s = n + 1 - d_{j'}(N_E) > n + 1 + j - j' - d_j(N_E) = (n + 1 - j - i) + j - j'.$$

This is another way to say that $(n + 1 - i - j, j)$ is a distinguished pair for N_E.

Moreover, we get that

$$\beta^S_{i,i+j}(N) = \dim_K H_s(n + 1 - i - j; N_E)_{s+j} = \dim_K H_0(n + 1 - i - j)_j$$

where the last equality follows from 2.2.1. □

We are ready to prove the main theorem of this section.

Theorem 2.2.9. *Let $N \in \mathcal{SQ}(S)$. The following statements are equivalent:*

(i) *$\beta^S_{i,i+j}(N)$ is an extremal Betti number of N.*
(ii) *$\beta^S_{j,j+i}(N^*)$ is an extremal Betti number of N^*.*

Moreover, if the equivalent conditions hold, then $\beta^S_{i,i+j}(N) = \beta^S_{j,j+i}(N^*)$.

Proof. By 2.2.8 we get that $\beta^S_{i,i+j}(N)$ is an extremal Betti number of N if and only if $(n + 1 - i - j, j)$ is a homological distinguished pair for N_E and $\beta^S_{i,i+j}(N) = \dim_K H_0(n + 1 - i - j; N_E)_j$.

By 2.2.3 this is equivalent to the fact that $(n + 1 - i - j, i)$ is a homological distinguished pair for N_E^* and $H_0(n + 1 - i - j; N_E)_j \cong H_0(n + 1 - i - j; N_E^*)_i$.

Again by 2.2.8 this means that $\beta^S_{j,j+i}(N^*)$ is an extremal Betti number of N^* and $\beta^S_{j,j+i}(N^*) = \dim_K H_0(n + 1 - i - j; N_E^*)_i$. The assertion follows. □

In particular one has:

Corollary 2.2.10. *Let $\Gamma \subset \Delta$ be simplicial complexes on the vertex set $[n]$. Let $I_\Delta \subset I_\Gamma \subset S = K[x_1, \dots, x_n]$ be the corresponding ideals in the polynomial ring. The following statements are equivalent:*

(i) *$\beta^S_{i,i+j}(I_\Gamma/I_\Delta)$ is an extremal Betti number of I_Γ/I_Δ.*
(ii) *$\beta^S_{j,j+i}(I_{\Delta^*}/I_{\Gamma^*})$ is an extremal Betti number of $I_{\Delta^*}/I_{\Gamma^*}$.*

Moreover, if the equivalent conditions hold, then $\beta^S_{i,i+j}(I_\Gamma/I_\Delta) = \beta^S_{j,j+i}(I_{\Delta^*}/I_{\Gamma^*})$.

In the case $\Gamma = \emptyset$ this is a result in [**13**].

2.3. Local cohomology of square-free modules

In this section we want to relate homological invariants of a square-free S-module to equivalent properties of its dual module.

We need to compute the local cohomology of square-free S-modules. Let $N \in \mathcal{SQ}(S)$. Recall the following cochain complex from 1.5.1:

$$\mathcal{L}\colon 0 \to L^0 \to L^1 \to \ldots \to L^n \to 0,$$

with

$$L^i = \bigoplus_{G \subseteq [n],\ |G|=i} N_{x^G}.$$

Here N_{x^G} is the localization of N at the multiplicative set $\{(x^G)^n \colon n \in \mathbb{N}\}$ where the differential map d is composed of the maps

$$(-1)^{\alpha(i,G)}\mathrm{nat}\colon N_{x^G} \to N_{x^{G\cup\{i\}}}$$

for $i \notin G$. By 1.5.2 we have $H^i_{\mathfrak{m}}(N) = H^i(\mathcal{L})$.

In [**53**, 2.9] Yanagawa gave an explicit description of the homogeneous components of this cochain complex.

Lemma 2.3.1. *Let $N \in \mathcal{SQ}(S)$, $i \in \{0, \ldots, n\}$ and $F \subseteq [n]$. Then*

$$(L^i)_{-F} \cong \bigoplus_{F \subseteq G \subseteq [n],\ |G|=i} N_G.$$

The induced differential is given by

$$d_i(y) = \sum_{j \notin G} (-1)^{\alpha(j,G)} x_j y \quad \text{for } G \subseteq [n],\ |G| = i,\ y \in N_G.$$

In the sequel we use this description to calculate $H^i_{\mathfrak{m}}(N)_{-F}$ for $F \subseteq [n]$. The main observation is that the homogeneous components of $H^i_{\mathfrak{m}}(N)$ can be interpreted in terms of certain homogeneous components of the Cartan cohomology of N_E (see 1.4 for details and notation).

Theorem 2.3.2. *Let $N \in \mathcal{SQ}(S)$, $\mathbf{e} = e_1, \ldots, e_n$ be a basis for E_1 and $F \subseteq [n]$. We have*

$$H^i(\mathbf{e}; N_E)_F \cong H^{i+|F|}_{\mathfrak{m}}(N)_{-F}.$$

Proof. Let $N \in \mathcal{SQ}(S)$. Then $N_E \in \mathcal{SQ}(E)$. Assume that $\{f_1, \ldots, f_t\}$ is a minimal homogeneous system of generators of N with $\deg(f_j) = d_j \in \mathbb{N}^n$ square-free. We set $D_j = \operatorname{supp}\{d_j\}$. By 2.1.8 this is also a minimal homogeneous system of generators of N_E. Fix $F \subseteq [n]$ for the rest of the proof. We show that the cochain complexes of K-vector spaces $\{C^i(\mathbf{e}; N_E)_F \colon i \in \mathbb{N}\}$ and $\{(L^{i+|F|})_{-F} \colon i \in \mathbb{N}\}$ are isomorphic (as cochain complexes). This yields the theorem, because then the induced homology modules are equal.

Let $i \in \mathbb{N}$. A K-basis of $C^i(\mathbf{e}; N_E)_F$ consists of the elements $e_I f_j y^b$ where $b \in \mathbb{N}^n$ is square-free, $I \subseteq [n]$, $I \cap D_j = \emptyset$, $I + D_j - \operatorname{supp}(b) = F$ and $|b| = i$. Applying 2.3.1 it follows that a K-basis of $(L^{i+|F|})_{-F} \cong \bigoplus_{F \subseteq G \subseteq [n],\ |G|=i+|F|} N_G$ is given by $x^I f_j$ where $I \subseteq [n]$, $I \cap D_j = \emptyset$, $F \subseteq I + D_j$ and $|I| + |D_j| = i + |F|$. Observe that these K-vector spaces have the same dimension. We define a K-linear isomorphism for all $i \in \mathbb{N}$ by

$$\varphi^i : C^i(\mathbf{e}; N_E)_F \to (L^{i+|F|})_{-F}, \quad e_I f_j y^b \mapsto (-1)^{\alpha(J,D_j)} x^I f_j$$

where $J = ([n] - F) \cap I$.

We claim that $\varphi = \{\varphi^i\}$ is a cochain complex isomorphism. Since all φ^i induce a bijection between bases, it is enough to show that φ is a cochain complex homomorphism. Let δ and d denote the induced differentials on the cochain complexes $\{C^i(\mathbf{e}; N_E)_F\}$ and $\{(L^{i+|F|})_{-F}\}$. Let $e_I f_j y^b$ be an arbitrary element of the described K-basis of $C^i(\mathbf{e}; N_E)_F$. On the one hand

$$d^i \circ \varphi^i(e_I f_j y^b) = d^i((-1)^{\alpha(J,D_j)} x^I f_j)$$

$$= \sum_{i \notin I+D_j} (-1)^{\alpha(J,D_j)+\alpha(i,I+D_j)} x_i x^I f_j = \sum_{i \notin I+D_j} (-1)^{\alpha(J,D_j)+\alpha(i,I+D_j)} x^{I+\{i\}} f_j.$$

On the other hand

$$\varphi^{i+1} \circ \delta^i(e_I f_j y^b) = \varphi^{i+1}(\sum_{i\in[n]} e_i e_I f_j y^{b+\varepsilon_i}) = \varphi^{i+1}(\sum_{i \notin I+D_j} (-1)^{\alpha(i,I)} e_{I+\{i\}} f_j y^{b+\varepsilon_i}).$$

Here we used the fact that N_E is a square-free E-module, $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$. The last expression equals

$$\sum_{i \notin I+D_j} (-1)^{\alpha(i,I)+\alpha(J^i,D_j)} x^{I+\{i\}} f_j$$

where, for all $i \in [n] - I$, we set $J^i = ([n] - F) \cap (I + \{i\})$. Fix $i \in [n] - (I + D_j)$. Since $I + D_j = \mathrm{supp}(b) + F$, we obtain that $i \notin F$. Hence $J^i = J + \{i\}$. Then

$$\alpha(i, I) + \alpha(J^i, D_j) = \alpha(i, I) + \alpha(i, D_j) + \alpha(J, D_j) = \alpha(J, D_j) + \alpha(i, I + D_j).$$

It follows that $d^i \circ \varphi^i(e_I f_j y^b) = \varphi^{i+1} \circ \delta^i(e_I f_j y^b)$ and this concludes the proof. $\square$

Let $F \subseteq [n]$ and W be a K-vector space. We set $F^\vee = [n] - F$ and $W^\vee = \mathrm{Hom}_K(W, K)$.

Lemma 2.3.3. *Let $M \in \mathcal{SQ}(E)$ and $F \subseteq [n]$. We have*

$$(M^*)_F \cong (M_{F^\vee})^\vee.$$

Proof. The proof of [**5**, 3.4] can also be applied to the multigraded case. $\square$

We are able to generalize a result of Yanagawa (see [**53**, 3.4]).

Theorem 2.3.4. *Let $N \in \mathcal{SQ}(S)$ and $F \subseteq [n]$. Then*

$$\mathrm{Ext}^i_S(N, \omega_S)_F \cong H^{n-i}_{\mathfrak{m}}(N)_{-F} \cong \mathrm{Tor}^S_{|F^\vee|-i}(K, N^*)_{F^\vee}.$$

Proof. The first isomorphism is the local duality of $\mathbb{Z}^n$-graded S-modules (see 1.5.5), while the second follows from 1.4.4, 1.4.6, 2.1.9, 2.3.2 and 2.3.3 because

$$H^{n-i}_{\mathfrak{m}}(N)_{-F} \cong H^{n-i-|F|}(\mathbf{e}; N_E)_F \cong (H^{n-i-|F|}(\mathbf{e}; N_E)_F)^\vee \cong$$

$$H_{n-i-|F|}(\mathbf{e}; N_E^*)_{F^\vee} \cong \mathrm{Tor}^E_{|F^\vee|-i}(K, N_E^*)_{F^\vee} \cong \mathrm{Tor}^S_{|F^\vee|-i}(K, N^*)_{F^\vee}.$$

$\square$

The next result generalizes theorems by Eagon-Reiner [**22**] and Terai [**49**].

Corollary 2.3.5. *Let $N \in \mathcal{SQ}(S)$. Then:*

(i) $\dim(N) \geq n - \operatorname{indeg}(N^*)$.
(ii) *N is Cohen-Macaulay of* $\dim(N) = d$ *if and only if N^* has an $(n-d)$-linear resolution.*
(iii) $\operatorname{reg}(N) = \operatorname{pd}(N^*)$.

Proof. Yanagawa proved in [**53**, 2.5] that, for all integers i, we have $\operatorname{Ext}^i_S(N, \omega_S) \in \mathcal{SQ}(S)$. Hence by the definition of a square-free S-module it suffices to show that all square-free components of $\operatorname{Ext}^i_S(N, \omega_S)$ are zero to conclude that $\operatorname{Ext}^i_S(N, \omega_S) = 0$. By 1.5.5 we get that $H^i_{\mathfrak{m}}(N) = 0$ if and only if for all $F \subseteq [n]$ one has $H^i_{\mathfrak{m}}(N)_{-F} = 0$. We use this fact in the sequel.

Proof of (i): By 1.5.3 and 2.3.4 we get:

$$\begin{aligned}\dim(N) &= \max\{i\colon H^i_{\mathfrak{m}}(N) \neq 0\} = \max\{i\colon H^i_{\mathfrak{m}}(N)_{-F} \neq 0, |F| \subseteq [n]\}\\ &\geq \max\{i\colon H^i_{\mathfrak{m}}(N)_{-F} \neq 0, |F| \subseteq [n], |F| = i\}\\ &= n - \min\{i\colon H^{n-i}_{\mathfrak{m}}(N)_{-F} \neq 0, |F| \subseteq [n], |F| = n-i\}\\ &= n - \min\{i\colon \operatorname{Tor}^S_0(K, N^*)_{F^\vee} \neq 0, |F| \subseteq [n], |F^\vee| = i\} = n - \operatorname{indeg}(N^*).\end{aligned}$$

Proof of (ii): It follows from 1.5.4 that N is Cohen-Macaulay of dimension d if and only if

$$H^i_{\mathfrak{m}}(N) \begin{cases} = 0 & i \neq d, \\ \neq 0 & i = d. \end{cases}$$

Similarly to (i) by 2.3.4 this is equivalent to say that N^* has an $(n-d)$-linear resolution.

Proof of (iii): By 1.5.6 one has

$$\begin{aligned}\operatorname{reg}(N) &= \max\{j\colon H^i_{\mathfrak{m}}(N)_{j-i} \neq 0 \text{ for some } i \in \mathbb{N}\}\\ &= \max\{j\colon H^i_{\mathfrak{m}}(N)_{-F} \neq 0 \text{ for some } i \in \mathbb{N},\ F \subseteq [n],\ -|F| = j - i\}.\end{aligned}$$

Applying 2.3.4 yields:

$$\begin{aligned}\operatorname{reg}(N) &= \max\{j\colon \operatorname{Tor}^S_{|F^\vee|-n+i}(K, N^*)_{F^\vee} \neq 0 \text{ for some } i \in \mathbb{N},\ F \subseteq [n],\ -|F| = j - i\}\\ &= \max\{j\colon \operatorname{Tor}^S_j(K, N^*)_{j+n-i} \neq 0 \text{ for some } i \in \mathbb{N}\} = \operatorname{pd}(N^*).\end{aligned}$$

The last two equalities hold because a square-free S-module has only non-zero square-free Betti numbers (see 2.1.4). □

A further corollary is a formula relating the Ext-groups of N_E and N^*.

Corollary 2.3.6. *Let $N \in \mathcal{SQ}(S)$. Then*

$$\dim_K \operatorname{Ext}^i_E(K, N_E)_{n-i-j} = \sum_{k=0}^{i} \binom{i+j-1}{j+k-1} \sum_{F \subseteq [n],\ |F| = k+j} \dim_K \operatorname{Ext}^j_S(N^*, \omega_S)_{F^\vee}.$$

Proof. By 1.4.6 we have

$$\dim_K \operatorname{Ext}^i_E(K, N_E)_{n-i-j} = \dim_K H^i(\mathbf{e}; N_E)_{n-i-j} = \dim_K (H^i(\mathbf{e}; N_E)_{n-i-j})^\vee.$$

It follows from [**5**, 3.4] that

$$\dim_K (H^i(\mathbf{e}; N_E)_{n-i-j})^\vee = \dim_K H_i(\mathbf{e}; N_E)^*_{i+j}$$

By 1.4.4 and since Cartan-homology computes the Tor-groups of N^*_E and K (see 1.4.6), the last expression is equal to

$$\dim_K H_i(\mathbf{e}; N^*_E)_{i+j} = \dim_K \operatorname{Tor}^E_i(K, N^*_E)_{i+j}.$$

Applying 2.1.9 and 2.3.4 this is exactly

$$\sum_{k=0}^{i} \binom{i+j-1}{k+j-1} \dim_K \operatorname{Tor}^S_k(K, N^*)_{k+j}$$

$$= \sum_{k=0}^{i} \binom{i+j-1}{k+j-1} \sum_{F \subseteq [n],\ |F|=j+k} \dim_K \operatorname{Tor}^S_k(K, N^*)_F$$

$$= \sum_{k=0}^{i} \binom{i+j-1}{k+j-1} \sum_{F \subseteq [n],\ |F|=j+k} \dim_K \operatorname{Ext}^j_S(N^*, \omega_S)_{F^\vee}.$$

Hence we proved the corollary. □

2.4. Sequentially Cohen-Macaulay and componentwise linear modules

We study what the property to be componentwise linear for a square-free S-module means for the Alexander dual. For the following definition see for example [**48**, 2.9].

Definition 2.4.1. Let $N \in \mathcal{SQ}(S)$. Then N is said to be *sequentially Cohen-Macaulay* if it has a finite filtration $0 = N_0 \subset N_1 \subset \ldots \subset N_d = N$ of submodules $N_i \in \mathcal{SQ}(S)$ satisfying two conditions:

(i) Each quotient N_i/N_{i-1} is a Cohen-Macaulay module.
(ii) $\dim(N_1/N_0) < \dim(N_2/N_1) < \ldots < \dim(N_d/N_{d-1})$ where 'dim' denotes the Krull dimension.

Any finite filtration of N satisfying these conditions is called *a CM-filtration* of N.

We have:

Lemma 2.4.2. *Let $N \in \mathcal{SQ}(S)$ be sequentially Cohen-Macaulay and $0 = N_0 \subset N_1 \subset \ldots \subset N_d = N$ the corresponding CM-filtration of N. Then for all $i \in [d]$ one has*

$$\dim(N_i) = \dim(N_i/N_{i-1}) > \dim(N_{i-1}).$$

Proof. We prove by induction on $i \in [d]$ that $\dim(N_i) = \dim(N_i/N_{i-1}) > \dim(N_{i-1})$. The case $i = 1$ is trivial.

Let $i > 0$. By the induction hypothesis and the dimension condition of the filtration we have $\dim(N_{i-1}) = \dim(N_{i-1}/N_{i-2}) < \dim(N_i/N_{i-1})$. Therefore we deduce from the exact sequence

$$0 \to N_{i-1} \to N_i \to N_i/N_{i-1} \to 0,$$

that $\dim(N_i) = \dim(N_i/N_{i-1})$. $\square$

Analogously to [**34**] one has:

Lemma 2.4.3. *Let R be a standard graded Koszul algebra and $N \in \mathcal{M}_{\mathbb{Z}}(R)$. If N has an i-linear resolution, then $\mathfrak{m}N$ has an $(i+1)$-linear resolution.*

Proof. We have the exact sequence

$$0 \to \mathfrak{m}N \to N \to N/\mathfrak{m}N \to 0.$$

Since N has an i-linear resolution, N is generated in degree i. Thus $N/\mathfrak{m}N \cong \bigoplus K(-i)$ has an i-linear resolution because R is a Koszul algebra. Hence by 1.2.14

$$\operatorname{reg}_R(\mathfrak{m}N) \leq \max\{i,\ i+1\}.$$

One has $\operatorname{reg}_R(\mathfrak{m}N) \geq i+1$ because $\operatorname{indeg}(\mathfrak{m}N) = i+1$. It follows that $\operatorname{reg}_R(\mathfrak{m}N) = i+1$. $\square$

Lemma 2.4.4. *Let R be a standard graded K-algebra and $N,\ N',\ N'' \in \mathcal{M}_{\mathbb{Z}}(R)$. Let*

$$0 \to N' \to N \to N'' \to 0$$

be an exact sequence. If N', N'' have i-linear resolutions, then N has an i-linear resolution.

Proof. Consider for all $j, t \in \mathbb{Z}$ the long exact Tor-sequence

$$\ldots \to \operatorname{Tor}_j^R(K, N')_{j+t} \to \operatorname{Tor}_j^R(K, N)_{j+t} \to \operatorname{Tor}_j^R(K, N'')_{j+t} \to \ldots .$$

For $t \neq i$ the right and left hand Tor-groups vanish and therefore also the middle one. $\square$

Lemma 2.4.5. *Let R be a standard graded K-algebra and $N \in \mathcal{M}_{\mathbb{Z}}(R)$. For all integers i we have the following exact sequence:*

$$0 \to N_{\langle \operatorname{indeg}(N)\rangle \langle i\rangle} \to N_{\langle i\rangle} \to (N/N_{\langle \operatorname{indeg}(N)\rangle})_{\langle i\rangle} \to 0.$$

Proof. Without loss of generality we may assume that $\operatorname{indeg}(N) = 0$. For $i < 0$ the assertion is trivial. Let $i \geq 0$. One has

$$N_{\langle i\rangle}/N_{\langle 0\rangle\langle i\rangle} = N_{\langle i\rangle}/\mathfrak{m}^i N_{\langle 0\rangle} = (N/N_{\langle 0\rangle})_{\langle i\rangle}.$$

$\square$

We are now able to prove the main theorem of this section, which generalizes a result of Herzog-Hibi in [**33**] and Herzog-Reiner-Welker in [**34**].

Theorem 2.4.6. *Let $N \in \mathcal{SQ}(S)$. The following statements are equivalent:*

(i) *N is sequentially Cohen-Macaulay.*
(ii) *N^* is componentwise linear.*

Proof. (i)$\Rightarrow$(ii): Let N be sequentially Cohen-Macaulay with a CM-filtration $0 = N_0 \subset N_1 \subset \ldots \subset N_d = N$ of square-free submodules. By 2.4.2 we have $\dim(N_i) = \dim(N_i/N_{i-1}) > \dim(N_{i-1})$ for all $i \in [d]$. We prove (ii) by induction on d. The case $d = 0$ is trivial, and $d = 1$ was shown in 2.3.5 because N^* is componentwise linear if it has a linear resolution (see 1.2.18).

Assume $d > 1$. Consider the exact sequence

$$0 \to N_{d-1} \to N \to N/N_{d-1} \to 0.$$

Dualizing is an exact functor and we get the exact sequence

$$0 \to (N/N_{d-1})^* \to N^* \to N^*_{d-1} \to 0.$$

Let $t_d = \dim(N)$ and $t_{d-1} = \dim(N_{d-1})$. Since N/N_{d-1} is Cohen-Macaulay of dimension t_d, it follows from 2.3.5 that $(N/N_{d-1})^*$ has an $(n - t_d)$-linear resolution. This implies that $(N/N_{d-1})^*$ is componentwise linear and all generators lie in degree $(n - t_d)$ (see 1.2.18). By the induction hypothesis N^*_{d-1} is componentwise linear. 2.3.5 yields that N^*_{d-1} is generated in degrees greater or equal to $n - t_{d-1} > n - t_d$ and N^* is generated in degrees greater or equal to $n - t_d$. Therefore we get

$$(N/N_{d-1})^* = (N/N_{d-1})^*_{\langle n-t_d\rangle} = N^*_{\langle n-t_d\rangle}.$$

We have to check that $N^*_{\langle i\rangle}$ has an i-linear resolution for all integers i. By 2.4.5 the sequence

$$0 \to (N/N_{d-1})^*_{\langle i\rangle} \to N^*_{\langle i\rangle} \to (N^*_{d-1})_{\langle i\rangle} \to 0$$

is exact. Now (ii) follows from 2.4.4.

(ii)$\Rightarrow$(i): Let N^* be componentwise linear. We define $d = n - \operatorname{indeg}(N^*)$. Observe that $N^*_{\langle n-d\rangle} \in \mathcal{SQ}(S)$. Then also $N^*/N^*_{\langle n-d\rangle} \in \mathcal{SQ}(S)$. We prove by induction on $d \in \mathbb{N}$ that N is sequentially Cohen-Macaulay of dimension d. If $d = 0$, or $d > 0$ and $N^* = N^*_{\langle n-d\rangle}$, then N^* has a linear resolution and the assertion follows from 2.3.5. Assume that $d > 0$ and $N^* \neq N^*_{\langle n-d\rangle}$. The following sequence is exact

$$0 \to N^*_{\langle n-d\rangle} \to N^* \to N^*/N^*_{\langle n-d\rangle} \to 0. \tag{3}$$

We claim that $N^*/N^*_{\langle n-d\rangle}$ is componentwise linear. One has to show that, for all integers i, the module $(N^*/N^*_{\langle n-d\rangle})_{\langle i\rangle}$ has an i-linear resolution. If $(N^*/N^*_{\langle n-d\rangle})_{\langle i\rangle} = 0$, there is nothing to prove. Let $(N^*/N^*_{\langle n-d\rangle})_{\langle i\rangle} \neq 0$. By 2.4.5 we have the exact sequence

$$0 \to (N^*_{\langle n-d\rangle})_{\langle i\rangle} \to N^*_{\langle i\rangle} \to (N^*/N^*_{\langle n-d\rangle})_{\langle i\rangle} \to 0.$$

By the assumption we know that $N^*_{\langle i\rangle}$ and $(N^*_{\langle n-d\rangle})_{\langle i\rangle}$ have an i-linear resolution. It follows from 1.2.14 that $\operatorname{reg}_S((N^*/N^*_{\langle n-d\rangle})_{\langle i\rangle}) \leq i$. Since $\operatorname{indeg}((N^*/N^*_{\langle n-d\rangle})_{\langle i\rangle}) = i$, the claim follows from 1.2.19.

Let $d' = n - \operatorname{indeg}(N^*/N^*_{\langle n-d\rangle})$. We have $n - d' > n - d$, equivalently $d' < d$. Dualizing (3) we obtain the exact sequence

$$0 \to (N^*/N^*_{\langle n-d\rangle})^* \to N \to (N^*_{\langle n-d\rangle})^* \to 0.$$

By the induction hypothesis we know that the module $(N^*/N^*_{\langle n-d\rangle})^*$ is sequentially Cohen-Macaulay of dimension d'. Therefore we get a CM-filtration $0 = N_0 \subset N_1 \subset$

$\ldots \subset N_{t-1} = (N^*/N^*_{\langle n-d\rangle})^* \subset N_t = N$. Since also $N_t/N_{t-1} \cong (N^*_{\langle n-d\rangle})^*$ is Cohen-Macaulay of dimension $d > d' = \dim(N_{t-1}) = \dim(N_{t-1}/N_{t-2})$ (see 2.3.5), the module N is sequentially Cohen-Macaulay, as desired. $\square$

CHAPTER 3

The linear part of a minimal graded free resolution

We study the linear part of a minimal graded free resolution associated to a given module. For the construction of this complex and related results see Eisenbud and Schreyer [**26**]. We define the invariant lpd of a module as the smallest integer i such that the linear part is exact in homological degree greater than i. We show that for a finitely generated graded module M over a Koszul algebra we have $\operatorname{lpd}(M) = 0$ if and only if M is componentwise linear. Furthermore, we prove that for square-free modules (see Chapter 2) the invariant lpd is always finite and we give a bound for this number. The last mentioned result is joint work with Herzog.

In this chapter R is always a standard graded (commutative or skew-commutative resp.) K-algebra unless otherwise stated.

3.1. Preliminaries

The main construction associates a new complex to the graded minimal free resolution of a module. See 1.1 for details about filtered modules.

Construction 3.1.1. Let R be a standard graded commutative or skew-commutative K-algebra and $M \in \mathcal{M}_{\mathbb{Z}}(R)$. Consider the minimal graded free resolution $(\mathcal{F}, \partial)$ of M. For all integers i we have

$$\operatorname{gr}_{\mathfrak{m}}(F_i)(-i) = \bigoplus_{j \geq i} \mathfrak{m}^{j-i} F_i / \mathfrak{m}^{j+1-i} F_i \cong \operatorname{gr}_{\mathfrak{m}}(R)^{\beta_i^R(M)}(-i).$$

Note that the last isomorphism does not respect the internal degree of bihomogeneous elements of $\operatorname{gr}_{\mathfrak{m}}(F_i)(-i)$. The differential ∂ induces a bihomogeneous map

$$\partial_{i+1}^{lin} \colon \operatorname{gr}_{\mathfrak{m}}(F_{i+1})(-i-1) \to \operatorname{gr}_{\mathfrak{m}}(F_i)(-i),$$

$$w + \mathfrak{m}^{j+1-(i+1)} F_{i+1} \mapsto \partial_{i+1}(w) + \mathfrak{m}^{j+1-i} F_i \text{ for } w \in \mathfrak{m}^{j-(i+1)} F_{i+1}.$$

Since $\mathcal{F}$ is a minimal graded free resolution, it is easy to see that the maps $\{\partial_i^{lin}\}$ are well-defined and form a complex homomorphism on the complex $\{\operatorname{gr}_{\mathfrak{m}}(F_i)(-i)\}$. For $w \in \mathfrak{m}^{j-i} F_i$ let $[w] \in \mathfrak{m}^{j-i} F_i / \mathfrak{m}^{j+1-i} F_i$ be the residue class in F_i^{lin} of filtration degree j.

Definition 3.1.2. The complex constructed in 3.1.1 is called the *linear part of* $(\mathcal{F}, \partial)$. We write $(\mathcal{F}^{lin}, \partial^{lin})$ for this complex.

We are interested in the following invariant associated to a module.

Definition 3.1.3. Let R, M and $(\mathcal{F}, \partial)$ be as in 3.1.1. Define

$$\operatorname{lpd}(M) = \inf\{j \colon H_i(\mathcal{F}^{lin}) = 0 \text{ for } i \geq j+1\}.$$

If $\operatorname{lpd}(M)$ is a finite number, then we say that *the* **l***inear* **p***art* **d***ominates* from the homological degree $\operatorname{lpd}(M)$.

Example 3.1.4. The following is known:

(i) Let $S = K[x_1, \dots, x_n]$ be the graded polynomial ring and $M \in \mathcal{M}_{\mathbb{Z}}(S)$. Since $\operatorname{pd}_S(M) \leq n$, we have $\operatorname{lpd}(M) \leq n$.
(ii) Let $E = K\langle e_1, \dots, e_n\rangle$ be the graded exterior algebra and $M \in \mathcal{M}_{\mathbb{Z}}(E)$. Eisenbud and Schreyer proved in [**26**] that $\operatorname{lpd}(M) < \infty$ is always satisfied.

3.2. Componentwise linear modules over Koszul algebras

We fix a standard graded commutative Koszul algebra R with graded maximal ideal $\mathfrak{m}$. We study the question for which finitely generated graded R-modules M we have that $\operatorname{lpd}(M) = 0$. Recall that $\operatorname{indeg}(M) = \min\{d \in \mathbb{Z} \colon M_d \neq 0\}$ is the initial degree of M.

Lemma 3.2.1. *Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. If M has a linear resolution, then* $\operatorname{lpd}(M) = 0$.

Proof. Without loss of generality we may assume that $\operatorname{indeg}(M) = 0$. Let $(\mathcal{F}, \partial)$ be the minimal graded free resolution of M. It follows that $F_i = R(-i)^{\beta^R_{i,i}(M)}$ because M has a linear resolution.

We prove that $H_i(\mathcal{F}^{lin}) = 0$ for $i \geq 1$, which implies that $\operatorname{lpd}(M) = 0$. Fix an integer $i \geq 1$. We have to show that $\operatorname{Im}(\partial^{lin}_{i+1}) = \operatorname{Ker}(\partial^{lin}_i)$. Since $\operatorname{Im}(\partial^{lin}_{i+1}) \subseteq \operatorname{Ker}(\partial^{lin}_i)$, it remains to prove that $\operatorname{Im}(\partial^{lin}_{i+1}) \supseteq \operatorname{Ker}(\partial^{lin}_i)$.

Let $[w] \in F^{lin}_i$ be a homogeneous cycle of filtration degree $j \geq i$ with $w \in \mathfrak{m}^{j-i}F_i \backslash \mathfrak{m}^{j+1-i}F_i$. Since all homogeneous free generators of F_i have degree i, we may assume that w is homogeneous of degree j. We have $\partial_i(w) \in \mathfrak{m}^{j+1-(i-1)}F_{i-1}$ because $[w]$ is a cycle. All homogeneous elements in $\mathfrak{m}^{j+1-(i-1)}F_{i-1}$ have degree greater or equal to $j+1$. Hence $\partial_i(w) = 0$ because ∂_i is a homogeneous map. Thus w is a cycle in F_i and $j \geq i+1$. It follows that there exists a homogeneous element $w' \in F_{i+1}$ with $\deg(w') = j$ and $\partial_{i+1}(w') = w$ because $H_l(\mathcal{F}) = 0$ for $l \geq 1$. Since by degree reasons $w' \in \mathfrak{m}^{j-(i+1)}F_{i+1} \backslash \mathfrak{m}^{j+1-(i+1)}F_{i+1}$, we have that $\partial^{lin}_{i+1}([w']) = [w]$. This concludes the proof. $\square$

We want to extend the preceding result to componentwise linear modules over Koszul algebras. In this case it turns out that the inverse implication is also true. First we characterize the property to be componentwise linear.

Lemma 3.2.2. *Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. The following statements are equivalent:*

(i) *M is componentwise linear.*
(ii) *$M/M_{\langle \operatorname{indeg}(M)\rangle}$ is componentwise linear and $M_{\langle \operatorname{indeg}(M)\rangle}$ has an* $\operatorname{indeg}(M)$-*linear resolution.*

Proof. Without loss of generality we may assume that $\operatorname{indeg}(M) = 0$.

(i) $\Rightarrow$ (ii): Let M be componentwise linear. By definition it follows that $M_{\langle 0\rangle}$ has a linear resolution. It remains to show that for all $i \in \mathbb{Z}$ the module $(M/M_{\langle 0\rangle})_{\langle i\rangle}$ has an i-linear resolution. If $(M/M_{\langle 0\rangle})_{\langle i\rangle} = 0$, there is nothing to prove. Assume that $(M/M_{\langle 0\rangle})_{\langle i\rangle} \neq 0$. By 2.4.5 we have the exact sequence

$$0 \to M_{\langle 0\rangle\langle i\rangle} \to M_{\langle i\rangle} \to (M/M_{\langle 0\rangle})_{\langle i\rangle} \to 0. \tag{4}$$

The assumption implies that $M_{\langle i\rangle}$ has an i-linear resolution. It follows from 2.4.3 that $M_{\langle 0\rangle\langle i\rangle} = \mathfrak{m}^i M_{\langle 0\rangle}$ has also an i-linear resolution. Applying 1.2.14, we get

$$\operatorname{reg}_R((M/M_{\langle 0\rangle})_{\langle i\rangle}) \leq i.$$

Since $\operatorname{indeg}((M/M_{\langle 0\rangle})_{\langle i\rangle}) = i$, we conclude that $(M/M_{\langle 0\rangle})_{\langle i\rangle}$ has an i-linear resolution (see 1.2.19).

(ii) $\Rightarrow$ (i): Assume that $M/M_{\langle 0\rangle}$ is componentwise linear and that $M_{\langle 0\rangle}$ has a linear resolution. We have to show that for all $i \in \mathbb{Z}$ the module $M_{\langle i\rangle}$ has an i-linear resolution. For $i < 0$ we get that $M_{\langle i\rangle} = 0$ and the claim follows. Let $i \geq 0$. Again by 2.4.5 the sequence (4) is exact for all integers i. Now $M_{\langle 0\rangle\langle i\rangle} = \mathfrak{m}^i M_{\langle 0\rangle}$ has an i-linear resolution by 2.4.3. The module $(M/M_{\langle 0\rangle})_{\langle i\rangle}$ has an i-linear resolution by the assumption. Thus by 1.2.14 and 1.2.19 we obtain that $M_{\langle i\rangle}$ has an i-linear resolution. $\square$

Next we define a special subcomplex of the minimal graded free resolution of a module.

Definition 3.2.3. Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$ and let $(\mathcal{F}, \partial)$ be the minimal graded free resolution of M. We define the subcomplex $(\tilde{\mathcal{F}}, \tilde{\partial})$ of $(\mathcal{F}, \partial)$ by $\tilde{F}_i = R(-i)^{\beta^R_{i,i+\operatorname{indeg}(M)}(M)} \subseteq F_i$ and $\tilde{\partial} = \partial_{|\tilde{\mathcal{F}}}$.

Observe that by the construction of a minimal graded free resolution in 1.2 $\partial_{i+1}(\tilde{F}_{i+1}) \subseteq \tilde{F}_i$ is indeed satisfied for all $i \in \mathbb{N}$. Hence $\tilde{\partial}$ is a complex homomorphism on $\tilde{\mathcal{F}}$. Recall that for $w \in \mathfrak{m}^{j-i}F_i \backslash \mathfrak{m}^{j+1-i}F_i$ we write $[w]$ for the corresponding element in $\mathcal{F}^{lin}$ of filtration degree j. Let $\overline{w}$ be the residue class in $\mathcal{F}/\tilde{\mathcal{F}}$ and $[\overline{w}]$ the residue class in $(\mathcal{F}/\tilde{\mathcal{F}})^{lin}$. We need some technical lemmata.

Lemma 3.2.4. *Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$ such that $M_{\langle \operatorname{indeg}(M)\rangle}$ has a linear resolution and let $(\mathcal{F}, \partial)$ be the minimal graded free resolution of M. Then:*

(i) *$\tilde{\mathcal{F}}$ is the minimal graded free resolution of $M_{\langle \operatorname{indeg}(M)\rangle}$.*
(ii) *$\mathcal{F}/\tilde{\mathcal{F}}$ is the minimal graded free resolution of $M/M_{\langle \operatorname{indeg}(M)\rangle}$.*

Proof. Without loss of generality we may assume that $\operatorname{indeg}(M) = 0$.

Proof of (i): See 1.2 for the construction of a minimal graded free resolution of a module. We prove by induction on $i \in \mathbb{N}$ that we may choose $\tilde{F}_i$ to be the i^{th}-graded free module in the minimal graded free resolution $(\mathcal{G}, \partial^{\mathcal{G}})$ of $M_{\langle 0\rangle}$ and $\partial^{\mathcal{G}}_i = \tilde{\partial}_i$.

Let $i = 0$. Since

$$\dim_K(M_{\langle 0\rangle}/\mathfrak{m}M_{\langle 0\rangle}) = \dim_K((M_{\langle 0\rangle})_0) = \operatorname{rank}(\tilde{F}_0),$$

it follows that $G_0 = \tilde{F}_0$. Assume that $i > 0$. By the induction hypothesis we have that $G_j = \tilde{F}_j$ and $\partial^{\mathcal{G}}_j = \tilde{\partial}_j$ for $j < i$. Let Ω_i be the i^{th}-syzygy module of $M_{\langle 0\rangle}$, that is the kernel of $G_{i-1} \to G_{i-2}$ for $i \geq 2$, or the kernel of $G_0 \to M_{\langle 0\rangle}$ for $i = 1$. The module Ω_i is generated in degree i because $M_{\langle 0\rangle}$ has a linear resolution. Since $\operatorname{rank}(\tilde{F}_i) = \dim_K(\Omega_i)_i$, we may choose $G_i = \tilde{F}_i$ and $\partial^{\mathcal{G}}_i = \tilde{\partial}_i$.

Proof of (ii): One has the following exact sequence of complexes

$$0 \to \tilde{\mathcal{F}} \to \mathcal{F} \to \mathcal{F}/\tilde{\mathcal{F}} \to 0.$$

We obtain the associated long exact homology sequence

$$\ldots \to H_i(\mathcal{F}) \to H_i(\mathcal{F}/\tilde{\mathcal{F}}) \to H_{i-1}(\tilde{\mathcal{F}}) \to \ldots$$

This implies that $H_i(\mathcal{F}/\tilde{\mathcal{F}}) = 0$ for $i \geq 2$ because $H_i(\mathcal{F}) = H_i(\tilde{\mathcal{F}}) = 0$ for $i \geq 1$. Furthermore, we get the exact sequence

$$0 \to H_1(\mathcal{F}/\tilde{\mathcal{F}}) \to H_0(\tilde{\mathcal{F}}) \to H_0(\mathcal{F}) \to H_0(\mathcal{F}/\tilde{\mathcal{F}}) \to 0.$$

One has the following commutative diagram

$$\begin{array}{ccc} H_0(\tilde{\mathcal{F}}) & \longrightarrow & H_0(\mathcal{F}) \\ \downarrow & & \downarrow \\ M_{\langle 0 \rangle} & \longrightarrow & M \end{array}$$

where $H_0(\tilde{\mathcal{F}}) \to M_{\langle 0 \rangle}$, $H_0(\mathcal{F}) \to M$ are isomorphisms and the map $M_{\langle 0 \rangle} \to M$ is injective. Hence $H_0(\tilde{\mathcal{F}}) \to H_0(\mathcal{F})$ is injective. This implies that $H_1(\mathcal{F}/\tilde{\mathcal{F}}) = 0$ and $H_0(\mathcal{F}/\tilde{\mathcal{F}}) \cong M/M_{\langle 0 \rangle}$. Let $i \geq 1$. Since $\mathcal{F}$ is minimal, we have $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. Hence $\partial_i(F_i/\tilde{F}_i) \subseteq \mathfrak{m}F_{i-1}/\tilde{F}_{i-1}$. It follows that $\mathcal{F}/\tilde{\mathcal{F}}$ is the minimal graded free resolution of $M/M_{\langle 0 \rangle}$. □

The next step is to assume that $\operatorname{lpd}(M) = 0$.

Lemma 3.2.5. *Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$ such that* $\operatorname{lpd}(M) = 0$ *and let $(\mathcal{F}, \partial)$ be the minimal graded free resolution of M. We have:*

(i) $H_i(\tilde{\mathcal{F}}) = 0$ *for* $i \geq 1$.

(ii) *The map* $H_0(\tilde{\mathcal{F}}) \to H_0(\mathcal{F})$ *is injective.*

Proof. Without loss of generality we may assume that $\operatorname{indeg}(M) = 0$.

Proof of (i): Let $i \geq 1$ and $w \in \tilde{F}_i$ be a homogeneous cycle. We have to show that w is a boundary. Then we obtain that $H_i(\tilde{\mathcal{F}}) = 0$.

If $w \in m^{j-i}\tilde{F}_i \backslash m^{j+1-i}\tilde{F}_i$, then $\deg(w) = j$ because $\tilde{F}_i$ is generated in degree i. Since w is also a cycle in $\mathcal{F}$, it follows that $[w]$ is a cycle in $\mathcal{F}^{lin}$. The assumption $\operatorname{lpd}(M) = 0$ implies that $[w]$ is a boundary in $\mathcal{F}^{lin}$. Hence

$$w = \partial_{i+1}(w') + w''$$

where $w' \in m^{j-(i+1)}F_{i+1}$ and $w'' \in m^{j+1-i}F_i$ are homogeneous elements of degree j. We deduce that $w'' = 0$ and $w' \in \tilde{F}_{i+1}$ by degree reasons. Thus $w = \partial_{i+1}(w') = \tilde{\partial}_{i+1}(w')$.

Proof of (ii): Let $w \in \tilde{F}_0$ be a homogeneous cycle. We have to show that, if w is a boundary in $\mathcal{F}$, then it is one of $\tilde{\mathcal{F}}$. Assume that $w = \partial_1(w')$ for some homogeneous element $w' \in F_1$. If $w \in m^j\tilde{F}_0 \backslash m^{j+1}\tilde{F}_0$, then it follows that $j = \deg(w) = \deg(w')$ and that $w' \in m^{j-t-1}F_1 \backslash m^{j-(t-1)-1}F_1$ for some integer $t \geq 0$. We prove by induction on t that we find an element $w'' \in \tilde{F}_1$ with $w = \partial_1(w'')$. This concludes the proof because then $w = \tilde{\partial}_1(w'')$.

If $t = 0$, then we have already that $w' \in \tilde{F}_1$ by degree reasons. Otherwise we assume that $t > 0$. It follows that $[w']$ is a cycle in $\mathcal{F}^{lin}$. Since $\operatorname{lpd}(M) = 0$, there exist homogeneous elements $\tilde{w} \in m^{j-t-2}F_2$ and $\tilde{w'} \in m^{j-(t-1)-1}F_1$ with $w' = \partial_2(\tilde{w}) + \tilde{w'}$. We get

$$w = \partial_1(w') = \partial_1(\tilde{w'}).$$

By the induction hypothesis applied to $\tilde{w'}$ we obtain an element $w'' \in \tilde{F}_1$ such that $\partial_1(w'') = w$. □

As an application of the preceding lemmata we show:

Corollary 3.2.6. *Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$ such that $\operatorname{lpd}(M) = 0$ and let $(\mathcal{F}, \partial)$ be the minimal graded free resolution of M. Then:*

(i) *$\tilde{\mathcal{F}}$ is the minimal graded free resolution of $M_{\langle \operatorname{indeg}(M)\rangle}$.*
(ii) *$\mathcal{F}/\tilde{\mathcal{F}}$ is the minimal graded free resolution of $M/M_{\langle \operatorname{indeg}(M)\rangle}$.*

Proof. By 3.2.5 (i) we have $H_i(\tilde{\mathcal{F}}) = 0$ for $i \geq 1$. Since $\mathcal{F}$ is minimal, it follows that $\tilde{\mathcal{F}}$ is minimal. Hence $\tilde{\mathcal{F}}$ is the minimal graded free resolution of $M_{\langle \operatorname{indeg}(M)\rangle}$ because

$$M_{\langle \operatorname{indeg}(M)\rangle} \cong H_0(\mathcal{F})_{\langle \operatorname{indeg}(M)\rangle} = H_0(\tilde{\mathcal{F}})_{\langle \operatorname{indeg}(M)\rangle} = H_0(\tilde{\mathcal{F}})$$

where the second equality follows from 3.2.5. In particular $M_{\langle \operatorname{indeg}(M)\rangle}$ has a linear resolution. Applying 3.2.4 we obtain that $\mathcal{F}/\tilde{\mathcal{F}}$ is the minimal graded free resolution of $M/M_{\langle \operatorname{indeg}(M)\rangle}$. □

Analogously to 3.2.2 we characterize the invariant $\operatorname{lpd}(M)$ of a module M.

Lemma 3.2.7. *Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. The following statements are equivalent:*

(i) $\operatorname{lpd}(M) = 0$.
(ii) $\operatorname{lpd}(M/M_{\langle \operatorname{indeg}(M)\rangle}) = 0$ *and* $M_{\langle \operatorname{indeg}(M)\rangle}$ *has an* $\operatorname{indeg}(M)$*-linear resolution.*

Proof. Without loss of generality we may assume that $\operatorname{indeg}(M) = 0$. Let $(\mathcal{F}, \partial)$ be the minimal graded free resolution of M.

(i) $\Rightarrow$ (ii): Assume that $\operatorname{lpd}(M) = 0$. By 3.2.6 the complex $(\tilde{\mathcal{F}}, \tilde{\partial})$ is the minimal graded free resolution of $M_{\langle 0\rangle}$ and $(\mathcal{H}, \partial) = (\mathcal{F}/\tilde{\mathcal{F}}, \partial)$ is the minimal graded free resolution $M/M_{\langle 0\rangle}$. Thus $M_{\langle 0\rangle}$ has a linear resolution. It remains to prove that $\operatorname{lpd}(M/M_{\langle 0\rangle}) = 0$, which is equivalent to show that $H_i(\mathcal{H}^{lin}) = 0$ for $i \geq 1$. Let $i \in \mathbb{N}$. We choose a homogeneous free basis of F_i. Let $\{f_{i,l}\}$ denote the set of homogeneous free generators of F_i with $\deg(f_{i,l}) > i$ and $\{\tilde{f}_{i,l}\}$ denote the set of homogeneous free generators of F_i with $\deg(\tilde{f}_{i,l}) = i$. Then $\{\tilde{f}_{i,l}\}$ is a basis of $\tilde{F}_i$ and we may identify $\{f_{i,l}\}$ with a basis of H_i.

Assume that $i > 0$ and let $0 \neq [\overline{w}] \in H_i^{lin}$ be a cycle of filtration degree j with $\overline{w} \in \mathfrak{m}^{j-i}H_i \backslash \mathfrak{m}^{j+1-i}H_i$ where without loss of generality $w \in \mathfrak{m}^{j-i}F_i \backslash \mathfrak{m}^{j+1-i}F_i$ is a homogeneous element. It follows from the definition of H_i that $\deg(w) > j$. If $[\overline{w}]$ is a boundary, then the assertion follows.

We claim that $[w]$ is a cycle in $\mathcal{F}^{lin}$. Since $\operatorname{lpd}(M) = 0$, we know that $H_l(\mathcal{F}^{lin}) = 0$ for $l \geq 1$. Hence there exists an element $w' \in \mathfrak{m}^{j-(i+1)}F_{i+1}$ with $\partial_i^{lin}([w']) = [w]$. We conclude that $\partial_i^{lin}([\overline{w'}]) = [\overline{w}]$ in $\mathcal{H}^{lin}$. Thus it remains to prove the claim.

Since $[\overline{w}]$ is a cycle, one has

$$\partial_i(\overline{w}) \in m^{j+1-(i-1)}H_{i-1} = (m^{j+1-(i-1)}F_{i-1} + \tilde{F}_{i-1})/\tilde{F}_{i-1}.$$

Hence

$$\partial_i(w) = \sum_l g_l f_{i-1,l} + \sum_{\tilde{l}} \tilde{g}_{\tilde{l}} \tilde{f}_{i-1,\tilde{l}}$$

where $g_l, \tilde{g}_{\tilde{l}} \in R$ are homogeneous elements such that $g_l \in \mathfrak{m}^{j+1-(i-1)}$. Since $\deg(\partial_i(w)) = \deg(w) > j$ and $\deg(\tilde{f}_{i-1,\tilde{l}}) = i-1$, it follows that $\deg(\tilde{g}_{\tilde{l}}) \geq j+1-(i-1)$. Then $\tilde{g}_{\tilde{l}} \in \mathfrak{m}^{j+1-(i-1)}$ and therefore $\partial_i^{lin}([w]) = 0$ because $\partial_i(w) \in m^{j+1-(i-1)}F_{i-1}$.

(ii) $\Rightarrow$ (i): Conversely, assume that $\operatorname{lpd}(M/M_{\langle 0\rangle}) = 0$ and that $M_{\langle 0\rangle}$ has a linear resolution. It follows from 3.2.4 that $(\tilde{\mathcal{F}}, \tilde{\partial})$ is the minimal graded free resolution of $M_{\langle 0\rangle}$ and that $(\mathcal{H}, \partial) = (\mathcal{F}/\tilde{\mathcal{F}}, \partial)$ is the minimal graded free resolution $M/M_{\langle 0\rangle}$. We have to show that $H_i(\mathcal{F}^{lin}) = 0$ for $i \geq 1$.

Assume that $i \geq 1$. Let $[w]$ be a cycle in F_i^{lin} of filtration degree j such that $w \in m^{j-i}F_i \backslash \mathfrak{m}^{j+1-i}F_i$ is homogeneous and $\partial_i(w) \in \mathfrak{m}^{j+1-(i-1)}F_{i-1}$. We have to find an element $w' \in m^{j-(i+1)}F_{i+1}$ with $\partial_{i+1}(w') = w + w''$ where $w'' \in m^{j+1-i}F_i$. Then $[w]$ is a boundary in F_i^{lin} and the assertion follows. One has to consider two cases:

(1): Assume that $w \in \mathfrak{m}^{j+1-i}F_i + \tilde{F}_i$. Then $w = w' + w''$ where $w' \in \mathfrak{m}^{j+1-i}F_i$ and $w'' \in \tilde{F}_i$. Since $w \notin \mathfrak{m}^{j+1-i}F_i$, it follows that $w'' \in \mathfrak{m}^{j-i}\tilde{F}_i \backslash \mathfrak{m}^{j+1-i}\tilde{F}_i$. We conclude that $\partial_i(w'') \in \mathfrak{m}^{j+1-(i-1)}\tilde{F}_{i-1}$ because $\partial_i(\tilde{F}_i) \subset \tilde{F}_{i-1}$ and $\partial_i(w) \in m^{j+1-(i-1)}F_{i-1}$. Hence $[w'']$ is a cycle in $\tilde{F}^{lin}$ of filtration degree j. By the assumption $M_{\langle 0\rangle}$ has a linear resolution. It follows from 3.2.1 that $H_l(\tilde{F}^{lin}) = 0$ for $l \geq 1$. Hence there exist elements $\tilde{w} \in \mathfrak{m}^{j-(i+1)}\tilde{F}_{i+1}$ and $\tilde{w'} \in \mathfrak{m}^{j+1-i}\tilde{F}_i$ such that $\partial_{i+1}(\tilde{w}) = w'' + \tilde{w'}$. Then

$$\partial_{i+1}(\tilde{w}) = w - w' + \tilde{w'}$$

where $-w' + \tilde{w'} \in \mathfrak{m}^{j+1-i}F_i$. The assertion follows.

(2): Otherwise we have $w \notin \mathfrak{m}^{j+1-i}F_i + \tilde{F}_i$. Since $w \in \mathfrak{m}^{j-i}F_i$ and $[w]$ is a cycle, the element $0 \neq [\overline{w}] \in \mathfrak{m}^{j-i}H_i/\mathfrak{m}^{j+1-i}H_i$ is a cycle in $\mathcal{H}^{lin}$ of filtration degree j. By the assumption $\operatorname{lpd}(M/M_{\langle 0\rangle}) = 0$ it follows that $H_l(\mathcal{H}^{lin}) = 0$ for $l \geq 1$. Hence $[\overline{w}]$ is a boundary. This is equivalent to the fact that there exist elements $\tilde{w'} \in \mathfrak{m}^{j-(i+1)}F_{i+1}$ and $\tilde{w''} \in \mathfrak{m}^{j+1-i}F_i + \tilde{F}_i$ with $\partial_{i+1}(\tilde{w'}) = w + \tilde{w''}$. Since $\partial_i(w) = -\partial_i(\tilde{w''})$, we may apply case (1) to $\tilde{w''}$. Hence we get $w' \in m^{j-(i+1)}F_{i+1}$ and $w'' \in m^{j+1-i}F_i$ such that $\partial_{i+1}(w') = w + w''$. This concludes the proof. □

Note that by 1.2.16 one has $\operatorname{reg}_R(M) < \infty$ for $M \in \mathcal{M}_{\mathbb{Z}}(R)$ because R is a Koszul algebra. We prove the main theorem in this section. See Yanagawa [**53**] for similar results in the case that the ring is a graded polynomial ring.

Theorem 3.2.8. *Let $M \in \mathcal{M}_{\mathbb{Z}}(R)$. The following statements are equivalent:*

(i) $\operatorname{lpd}(M) = 0$.
(ii) *M is componentwise linear.*

Proof. We prove by induction on $t \in \mathbb{N}$ that for all modules $\mathcal{M}_{\mathbb{Z}}(R)$ with $t = \operatorname{reg}_R(M) - \operatorname{indeg}(M)$ the conditions (i) and (ii) are equivalent.

Let $t = 0$ and $\operatorname{reg}_R(M) = \operatorname{indeg}(M)$. Then M has a linear resolution (see 1.2.19). By 1.2.18 it follows that M is componentwise linear and by 3.2.1 $\operatorname{lpd}(M) = 0$. The assertion follows.

Let $t = \operatorname{reg}_R(M) - \operatorname{indeg}(M) > 0$. Assume that $\operatorname{lpd}(M) = 0$. By 3.2.7 this is equivalent to $\operatorname{lpd}(M/M_{\langle \operatorname{indeg}(M)\rangle}) = 0$ and $M_{\langle \operatorname{indeg}(M)\rangle}$ has a linear resolution. If $M/M_{\langle \operatorname{indeg}(M)\rangle} = 0$, then $M = M_{\langle \operatorname{indeg}(M)\rangle}$ is componentwise linear by 1.2.18. Assume that $M/M_{\langle \operatorname{indeg}(M)\rangle} \neq 0$. We observe that $\operatorname{indeg}(M/M_{\langle \operatorname{indeg}(M)\rangle}) \geq \operatorname{indeg}(M)+1$. We have the exact sequence

$$0 \to M_{\langle \operatorname{indeg}(M)\rangle} \to M \to M/M_{\langle \operatorname{indeg}(M)\rangle} \to 0.$$

By 1.2.14 one has $\operatorname{reg}_R(M/M_{\langle \operatorname{indeg}(M)\rangle}) \leq \operatorname{reg}_R(M)$. Then it follows from the induction hypothesis that $\operatorname{lpd}(M/M_{\langle \operatorname{indeg}(M)\rangle}) = 0$ if and only if $M/M_{\langle \operatorname{indeg}(M)\rangle}$ is componentwise linear. Finally, 3.2.2 implies that this is equivalent to say that M is componentwise linear. □

3.3. Square-free modules

Let $E = K\langle e_1, \dots , e_n\rangle$ be the $\mathbb{N}^n$-graded exterior algebra and $S = K[x_1, \dots , x_n]$ the $\mathbb{N}^n$-graded polynomial ring. In 2.1 we introduced the categories $\mathcal{SQ}(S)$ and $\mathcal{SQ}(E)$ of square-free S-modules and square-free E-modules. We will give bounds for the invariant lpd of square-free modules.

By 3.1.4 we have:

Corollary 3.3.1. *Let $N \in \mathcal{SQ}(S)$. Then* $\operatorname{lpd}(N) \leq \operatorname{pd}_S(N) \leq n$.

We want to give a similar bound for square-free E-modules. First we define several subcomplexes of the linear part of a minimal $\mathbb{N}^n$-graded free resolution of a square-free E-module.

Construction 3.3.2. Let $M \in \mathcal{SQ}(E)$ and let $(\mathcal{G}, \delta)$ be the minimal $\mathbb{N}^n$-graded free resolution of M. In 2.1.15 we proved that $M = (M_S)_E$. If $(\mathcal{F}, \theta)$ is the minimal $\mathbb{N}^n$-graded free resolution of M_S, then we may without loss of generality assume that $(\mathcal{G}, \delta)$ is the constructed complex in 2.1.6.

In 3.1.1 we observed that $\operatorname{gr}_{\mathfrak{m}}(E) \cong E$. Hence for all $l \in \mathbb{N}$:

$$G_l^{lin} \cong \bigoplus_{y^{(a)} f \in A_l} E y^{(a)} f$$

where $A_l = \{y^{(a)} f \colon a \in \mathbb{N}^n,\ f \in B_i,\ \operatorname{supp}(a) \subseteq \operatorname{supp}(\deg(f)) \text{ and } l = |a| + i\}$ (see 2.1.6 for notation). Note that all basis elements of G_l^{lin} have the degree l. The differential is given by

$$\delta_l^{lin} : G_l^{lin} \to G_{l-1}^{lin},$$

$$y^{(a)} f \mapsto (-1)^{|b|} \sum_{k \in \operatorname{supp}(a)} y^{(a-\varepsilon_k)} f e_k + (-1)^{|a|} \sum_{f_j \in B_{i-1} \text{ with } \deg(f_j) = b - \varepsilon_i} y^{(a)} f_j \lambda_j e_{\varepsilon_i}.$$

Let $(\mathcal{D}^{-1}, \partial^{-1}) = 0$ be the zero complex. For $j = 0, \dots , \operatorname{pd}_S(M_S)$ we define $(\mathcal{D}^j, \partial^j)$ as the subcomplex of $(\mathcal{G}^{lin}, \delta^{lin})$ given by

$$D_l^j = \bigoplus_{y^{(a)} f \in A_l,\ f \in B_i \text{ with } i \leq j} E y^{(a)} f \subseteq G_l^{lin}, \quad \partial_l^j = (\delta_l^{lin})_{|D_l^j}.$$

Observe that by the definition of δ^{lin} the map $\partial^j = \{\partial^j_l\}$ is a well defined complex homomorphism on $\mathcal{D}^j$.

Lemma 3.3.3. *Let $M \in \mathcal{SQ}(E)$ and let $(\mathcal{G}, \delta)$ be the minimal $\mathbb{N}^n$-graded free resolution of M. For $j \in \{0, \dots, \mathrm{pd}_S(M_S)\}$ one has*

$$H_l(\mathcal{D}^j/\mathcal{D}^{j-1}) = 0 \text{ for } l > \mathrm{pd}_S(M_S).$$

Proof. We fix $j \in \{0, \dots, \mathrm{pd}_S(M_S)\}$ and $l \in \mathbb{N}$. Observe that D^j_l/D^{j-1}_l is the free graded E-module

$$\bigoplus_{y^{(a)} f \in A_l \text{ with } f \in B_j} E y^{(a)} f.$$

Furthermore, the induced differential on $\mathcal{D}^j/\mathcal{D}^{j-1}$ is given by

$$\partial^j_l : D^j_l/D^{j-1}_l \to D^j_{l-1}/D^{j-1}_{l-1},$$

$$y^{(a)} f \mapsto (-1)^{|b|} \sum_{k \in \mathrm{supp}(a)} y^{(a-\varepsilon_k)} f e_k \quad \text{for } y^{(a)} f \in A_l,\ f \in B_j,\ \deg(f) = b.$$

Thus we see that $\mathcal{D}^j/\mathcal{D}^{j-1}$ is a direct sum of $|B_j|$ Cartan cocomplexes which have values in E and are shifted in homological degree by j (see 1.4). These complexes have only non-trivial homology in homological degree j. Hence

$$H_l(\mathcal{D}^j/\mathcal{D}^{j-1}) = 0 \text{ for } l > \mathrm{pd}_S(M_S).$$

□

Now we are ready to prove the main theorem of this section.

Theorem 3.3.4. *Let $M \in \mathcal{SQ}(E)$ and let $(\mathcal{G}, \delta)$ be the minimal $\mathbb{N}^n$-graded free resolution of M. Then*

$$H_l(\mathcal{G}^{lin}) = 0 \text{ for } l > \mathrm{pd}_S(M_S).$$

Proof. We use the constructed complexes of 3.3.2. We show by induction on $j \in \{0, \dots, \mathrm{pd}_S(M_S)\}$ that

$$H_l(\mathcal{D}^j) = 0 \text{ for } l > \mathrm{pd}_S(M_S).$$

This yields the theorem because $\mathcal{D}^{\mathrm{pd}_S(M_S)} = \mathcal{G}^{lin}$.

Assume that $j = 0$. Since $\mathcal{D}^0 = \mathcal{D}^0/\mathcal{D}^{-1}$, the claim follows from 3.3.3. Let $j > 0$. One has the following exact sequence of complexes:

$$0 \to \mathcal{D}^{j-1} \to \mathcal{D}^j \to \mathcal{D}^j/\mathcal{D}^{j-1} \to 0.$$

We get the long exact homology sequence:

$$\dots \to H_l(\mathcal{D}^{j-1}) \to H_l(\mathcal{D}^j) \to H_l(\mathcal{D}^j/\mathcal{D}^{j-1}) \to \dots$$

By the induction hypothesis we have $H_l(\mathcal{D}^{j-1}) = 0$ for $l > \mathrm{pd}_S(M_S)$. It follows from 3.3.3 that $H_l(\mathcal{D}^j/\mathcal{D}^{j-1}) = 0$ for $l > \mathrm{pd}_S(M_S)$. Thus $H_l(\mathcal{D}^j) = 0$ for $l > \mathrm{pd}_S(M_S)$. This concludes the proof. □

Applying 3.3.4 we obtain:

Corollary 3.3.5. *Let $M \in \mathcal{SQ}(E)$. Then* $\mathrm{lpd}(M) \le \mathrm{pd}_S(M_S) \le n$.

CHAPTER 4

Bounds for Betti numbers

Let K be a field and $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring. We are interested in the following Betti numbers:

Definition 4.0.6. Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$. For $i \in \mathbb{N}$ we set $\beta_i^{lin}(M) = \beta^S_{i,i+\operatorname{indeg}(M)}(M)$. We call $\max\{i \in \mathbb{N}\colon \beta_i^{lin}(M) \neq 0\}$ the *length of the linear strand of M.*

In [**32**] Herzog conjectured the following:

Conjecture. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and M is a k^{th}-syzygy module whose linear strand has length p. Then*

$$\beta_i^{lin}(M) \geq \binom{p+k}{i+k}$$

for $i = 0, \dots, p$.

The first three sections of this chapter are devoted to prove some special cases of this conjecture.

In the last section we give upper and lower bounds for all graded Betti numbers of graded ideals with a fixed number of generators and a linear resolution.

4.1. Preliminaries

In 1.3 we introduced the Koszul complex. We recall the main facts. Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$. We fix a graded free S-module L of rank n which is generated in degree 1. We consider maps $\mu \in L^* = \operatorname{Hom}_S(L, S)$. Note that L^* is again a graded free S-module generated in degree -1. Then μ defines a homogeneous S-homomorphism $\partial^\mu : \bigwedge L \to \bigwedge L$ of (homological) degree -1.

Let $\mathbf{e} = e_1, \dots, e_n$ be a basis of L, and $\mu \in L^* = \operatorname{Hom}_S(L, S)$ with $\mu(e_i) = x_i$ for $i = 1, \dots, n$. For $j = 1, \dots, n$ let $L(j)$ be the graded free submodule of L generated by $e_1, \dots, e_j$. Then $(\mathcal{K}(j), \partial)$ is the Koszul complex of M with respect to $x_1, \dots, x_j$ where $\mathcal{K}(j) = \bigwedge L(j) \otimes_S M$ and ∂ is the restriction of $\partial^\mu \otimes_S \operatorname{id}_M$ to $\bigwedge L(j) \otimes_S M$.

We denote the homology of the complex $\mathcal{K}(j)$ with $H(j)$ and the residue class of a cycle $z \in \mathcal{K}(j)$ in $H(j)$ with $[z]$. Let $e_1^*, \dots, e_j^*$ be the basis of L^* with $e_i^*(e_i) = 1$ and $e_i^*(e_k) = 0$ for $k \neq i$. In order to simplify the notation we set $\partial^i = \partial^{e_i^*}$. For the long exact sequences of the Koszul homology and further details see 1.3.

Sometimes we perform base changes on L.

Remark 4.1.1. Let $0 \neq \mu \in L(j, \mathbf{e})^*_{-1}$ be an arbitrary element. There exists a basis $\mathbf{l} = l_1, \dots, l_n$ of L, such that $l_k = e_k$ for $k > j$, $l_k \in \sum_{l=1}^{j} K e_l$ for $k \leq j$ and $\mu = (l_j)^*$. Then $L(k, \mathbf{e}) = L(k, \mathbf{l})$ for $k \geq j$ and $\partial(l_i) = y_i$ for $i = 1, \dots, n$ is again a basis of S_1.

In this chapter we need the following:

Lemma 4.1.2. *Let $M \in \mathcal{M}_{\mathbb{N}}(S)$, $p \in \{0,\dots,n\}$ and $t \in \mathbb{N}$. Suppose that $H_p(j)_{p+l} = 0$ for $l = -1,\dots,t-1$. Then:*

(i) *$H_p(j-1)_{p+l} = 0$ for $l = -1,\dots,t-1$.*
(ii) *$H_p(j-1)_{p+t}$ is isomorphic to a submodule of $H_p(j)_{p+t}$.*
(iii) *$H_i(j)_{i+l} = 0$ for $l = -1,\dots,t-1$ and $i = p,\dots,j$.*

Proof. We prove (i) by induction on l for $l = -1,\dots,t-1$. If $l = -1$ there is nothing to show because $H_p(j-1)_{p+l} = 0$ for $l < 0$. Now let $l > -1$ and consider the exact sequence

$$\dots \to H_p(j-1)_{p+l-1} \to H_p(j-1)_{p+l} \to H_p(j)_{p+l} \to \dots .$$

Since by the induction hypothesis $H_p(j-1)_{p+l-1} = 0$ and by the assumption $H_p(j)_{p+l} = 0$, we get that $H_p(j-1)_{p+l} = 0$.

For $l = t$ the exact sequence of the Koszul homology together with (i) yields

$$0 \to H_p(j-1)_{p+t} \to H_p(j)_{p+t} \to \dots ,$$

which proves (ii).

We show by induction on $j \in [n]$ that the assertion of (iii) holds. The case $j = 1$ is trivial and for $j > 1$ and $i = p$ the assertion is true by the assumption. Now let $j > 1$, $i > p$ and consider

$$\dots \to H_i(j-1)_{i+l} \to H_i(j)_{i+l} \to H_{i-1}(j-1)_{i-1+l} \to \dots .$$

By (i) we obtain that $H_p(j-1)_{p+l} = 0$ for $l = -1,\dots,t-1$. By the induction on j we get that $H_i(j-1)_{i+l} = H_{i-1}(j-1)_{i-1+l} = 0$. Therefore $H_i(j)_{i+l} = 0$. $\square$

4.2. Lower bounds for Betti numbers of graded S-modules

The aim of this section is to give lower bounds for the Betti numbers of graded S-modules, which are related to the conjecture by Herzog. If $M \in \mathcal{M}_{\mathbb{N}}(S)$, then observe that $K_i(j)$ is generated in degrees greater or equal to i.

We introduce a partial order on the Koszul complex of M. For $0 \neq z \in K_i(j)$ we write

$$z = m_J e_J + \sum_{I \subseteq [n],\ I \neq J} m_I e_I$$

with coefficients in M, and where e_J is the lexicographic largest monomial of all e_L with $m_L \neq 0$. Recall that for $I, J \subseteq [n]$, $I = \{i_1 < \dots < i_t\}$, $J = \{j_1 < \dots < j_{t'}\}$ $e_I <_{lex} e_J$ if either $t < t'$ or $t = t'$ and there exists a number p with $i_l = j_l$ for all $l < p$ and $i_p > j_p$.

Definition 4.2.1. In the presentation above we call $\operatorname{in}(z) = m_J e_J$ the *initial term of z.*

For $I = \{i_1 < \dots < i_t\} \subseteq [n]$ we write $\partial^I = \partial^{i_1} \circ \dots \circ \partial^{i_t}$.

Lemma 4.2.2. *Let $M \in \mathcal{M}_{\mathbb{N}}(S)$, $p \in \{0,\dots,j\}$, $r \in \{0,\dots,p\}$ and $0 \neq z \in K_p(j)$ be homogeneous such that $\operatorname{in}(z) = m_J e_J$. Then for all $I \subseteq J$ with $|I| = r$ the elements $\partial^I(z)$ are K-linearly independent in $K_{p-r}(j)$. In particular, if z is a cycle, then $\{\partial^I(z) \colon I \subseteq J, |I| = r\}$ is a set of K-linearly independent cycles.*

Proof. For $I \subseteq J$ with $|I| = r$ we have $\mathrm{in}(\partial^I(z)) = m_J e_{J-I}$. Since the initial terms are pairwise disjoint, the elements $\partial^I(z)$ are K-linearly independent in $K_{p-r}(j)$.

We prove by induction on $r \in \{0, \dots, p\}$ that all $\partial^I(z)$ are cycles provided z is a cycle. The case $r = 0$ is trivial and the assertion for the case $r = 1$ follows from 1.3.3.

Assume that $r > 0$. Let $I \subseteq J$ with $|I| = r$ and choose $t \in I$. Then $\partial^{I-\{t\}}(z)$ is a cycle by the induction hypothesis. Then 1.3.3 yields that

$$\partial^I(z) = \pm\partial^t \circ \partial^{I-\{t\}}(z)$$

is a cycle. □

Lemma 4.2.3. *Let $M \in \mathcal{M}_{\mathbb{N}}(S)$, $p \in [j]$, $t \in \mathbb{N}$ and $z \in K_p(j)_{p+t}$. Assume that $H_{p-1}(j)_{p-1+l} = 0$ for $l = -1, \dots, t-1$.*

(i) *If $p < j$ and $\partial^j(z) = \partial(y)$ for some y, then there exists $\tilde{z}$ such that $\tilde{z} = z + \partial(r)$ and $\partial^j(\tilde{z}) = 0$. In particular $\tilde{z} \in K_p(j-1)$, and if z is a cycle, then $[z] = [\tilde{z}]$.*

(ii) *If $p = j$ and $\partial^j(z) = \partial(y)$ for some y, then $z = 0$. In particular if $z \neq 0$ is a cycle, then we have $0 \neq [\partial^j(z)] \in H_{j-1}(j)_{j-1+t}$.*

Proof. To prove (i) we proceed by induction on $t \in \mathbb{N}$. If $t = 0$, then $y \in K_p(j)_{p+t-1} = 0$, and so $\partial^j(z) = 0$. Thus we choose $\tilde{z} = z$.

Let $t > 0$ and assume that $\partial^j(z) = \partial(y)$. We see that $\partial^j(y)$ is a cycle because

$$0 = \partial^j(\partial^j(z)) = \partial^j(\partial(y)) = -\partial(\partial^j(y)).$$

But $\partial^j(y) \in K_{p-1}(j)_{p-1+t-1}$. Since $H_{p-1}(j)_{p-1+t-1} = 0$, we have that $\partial^j(y) = \partial(y')$ is a boundary for some y'.

By the induction hypothesis we get $\tilde{y} = y + \partial(r')$ such that $\partial^j(\tilde{y}) = 0$. Note that

$$\partial(\tilde{y}) = \partial(y) = \partial^j(z).$$

We define

$$\tilde{z} = z + \partial(e_j \wedge \tilde{y}) = z + x_j\tilde{y} - e_j \wedge \partial^j(z).$$

Then

$$\partial^j(\tilde{z}) = \partial^j(z) + x_j\partial^j(\tilde{y}) - \partial^j(e_j) \wedge \partial^j(z) + e_j \wedge \partial^j \circ \partial^j(z) = \partial^j(z) - \partial^j(z) = 0$$

and this gives (i).

If $p = j$, we see that $z = me_{[j]}$ for some $m \in M$ and therefore $\partial^j(z) \neq 0$ if and only if $z \neq 0$.

We prove (ii) by induction on $t \in \mathbb{N}$. For $t = 0$ there is nothing to show. Let $t > 0$ and assume $\partial^j(z) = \partial(y)$. By the same argument as in the proof of (i) we get $\partial^j(y) = \partial(y')$ for some y'. The induction hypothesis implies $y = 0$, and then $z = 0$. □

Lemma 4.2.4. *Let $M \in \mathcal{M}_{\mathbb{N}}(S)$, $p \in \{0, \dots, j\}$, $t \in \mathbb{N}$ and $0 \neq z \in K_p(j)_{p+t}$. Assume that $H_p(j)_{p+l} = 0$ for $l = -1, \dots, t-1$ and let $r \in [j]$. If $\partial(y) = z$ in $\mathcal{K}(j)$ for some y and $z \in K_p(j-r)_{p+t} \subseteq K_p(j)_{p+t}$, then there exists an element $\tilde{y} = y + \partial(y') \in K_{p+1}(j-r)_{p+1+t-1}$ such that $\partial(\tilde{y}) = z$.*

Proof. We prove the assertion by induction on $j - r$. For $j = r$ there is nothing to show. Let $j > r$. Since

$$0 = \partial^j(z)$$

and

$$z = \partial(y) = \partial(e_j \wedge \partial^j(y) + r) = x_j \partial^j(y) - e_j \wedge \partial(\partial^j(y)) + \partial(r),$$

we see that $\partial^j(y)$ is a cycle and therefore a boundary by the assumption that $H_p(j)_{p+t-1} = 0$. By 4.2.3 we may assume that $y \in \mathcal{K}(j-1)$. By the induction hypothesis we find the desired $\tilde{y}$ in $\mathcal{K}(j-r)$. □

Lemma 4.2.5. *Let $M \in \mathcal{M}_{\mathbb{N}}(S)$ and $t \in \mathbb{N}$. If $\beta^S_{n-1,n-1+l}(M) = 0$ for $l = -1, \dots, t-1$ and $\beta^S_{n,n+t}(M) \neq 0$, then there exists a basis* $\mathbf{e}$ *of L and a cycle $z \in K_n(n)_{n+t}$ such that*

(i) $[z] \in H_n(n)_{n+t}$ *is not zero.*
(ii) $[\partial^i(z)] \in H_{n-1}(n)_{n-1+t}$ *are K-linearly independent for $i = 1, \dots, n$.*

In particular $\beta^S_{n-1,n-1+t}(M) \geq n$.

Proof. Let $\mathbf{e}$ be an arbitrary basis of L. Since $\beta^S_{n,n+t}(M) \neq 0$, there exists a cycle $z \in K_n(n)_{n+t}$ with $0 \neq [z] \in H_n(n)_{n+t}$. Furthermore, $H_{n-1}(n)_{n-1+l} = 0$ for $l = -1, \dots, t-1$. In this situation we have $z = me_{[n]}$ for some socle element m of M and we want to show that every equation

$$0 = \sum_{i=1}^{n} \mu_i [\partial^i(z)] = [\sum_{i=1}^{n} \mu_i \partial^i(z)] \quad \text{with} \quad \mu_i \in K,$$

implies $\mu_i = 0$ for all $i \in [n]$. Assume that there is such an equation where not all μ_i are zero. After a base change of L we may assume that $\sum_{i=1}^{n} \mu_i \partial^i = \partial^n$ (see 4.1.1). We get

$$0 = [\partial^n(z)],$$

contradicting to 4.2.3 (ii). □

Theorem 4.2.6. *Let $M \in \mathcal{M}_{\mathbb{N}}(S)$, $t \in \mathbb{N}$ and $p \in [n]$. If $\beta^S_{p-1,p-1+l}(M) = 0$ for $l = -1, \dots, t-1$ and $\beta^S_{p,p+t}(M) \neq 0$, then there exists a basis* $\mathbf{e}$ *of L and a cycle $z \in K_p(n)_{p+t}$ such that*

(i) $[z] \in H_p(n)_{p+t}$ *is not zero.*
(ii) $[\partial^i(z)] \in H_{p-1}(n)_{p-1+t}$ *are K-linearly independent for $i = 1, \dots, p$.*

In particular $\beta^S_{p-1,p-1+t}(M) \geq p$.

Proof. We have $H_p(n)_{p+t} \neq 0$ because $\beta^S_{p,p+t}(M) \neq 0$. Choose $0 \neq h \in H_p(n)_{p+t}$. We prove by induction on n that we can find a basis $\mathbf{e}$ of $L(n)$ and a cycle $z \in K_p(n)_{p+t}$ representing h such that every equation

$$0 = \sum_{i=1}^{p} \mu_i [\partial^i(z)] = [\sum_{i=1}^{p} \mu_i \partial^i(z)] \quad \text{with} \quad \mu_i \in K,$$

implies $\mu_i = 0$ for all i. The cases $n = 1$ and $n > 1$, $p = n$ were shown in 4.2.5.

Let $n > 1$ and $p < n$. Assume that there is a basis $\mathbf{e}$ and such an equation for a cycle z with $[z] = h$ where not all μ_i are zero. After a base change of L we may assume that $\sum_{i=1}^{p} \mu_i \partial^i = \partial^n$. Then

$$0 = [\partial^n(z)],$$

and therefore $\partial^n(z) = \partial(y)$ for some y. By 4.2.3 we can find an element $\tilde{y}$ such that $[\tilde{y}] = [z]$ and $\tilde{y} \in K_p(n-1)_{p+t}$. Now 4.1.2 guarantees that we can apply our induction hypothesis to $\tilde{y}$ and we find a base change $\mathbf{l} = l_1, \dots, l_{n-1}$ of $e_1, \dots, e_{n-1}$, $[\tilde{z}] = [\tilde{y}]$ in $H_p(n-1)_{p+t}$ (with respect to the new basis) such that $[\partial^i(\tilde{z})] \in H_{p-1}(n-1)_{p-1+t}$ are K-linearly independent for $i = 1, \dots, p$. By 4.1.2 we have $H_i(n-1)_{i+t} \subseteq H_i(n)_{i+t}$ for $i = p-1, p$. Then $\tilde{z}$ is the desired cycle because $[\tilde{z}] = [z]$ in $H_p(n)_{p+t}$. □

Before we give our main results, we introduce some notation.

Definition 4.2.7. Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and $k \in \{0, \dots, n\}$. We define

$$d_k(M) = \min(\{j \in \mathbb{Z} \colon \beta^S_{k,k+j}(M) \neq 0\} \cup \{\operatorname{reg}_S(M)\}).$$

We are interested in the numbers $\beta_i^{k,lin}(M) = \beta^S_{i,i+d_k(M)}(M)$ for $i \geq k$. Note that $\beta_i^{0,lin}(M) = \beta_i^{lin}(M)$. If $0 \neq \Omega_k(M)$ is the k^{th}-syzygy module in the minimal graded free resolution of M, then we always have $\beta^S_{i,i+j}(M) = \beta^S_{i-k,i-k+j+k}(\Omega_k(M))$ for $i \geq k$. Therefore $\beta_i^{k,lin}(M) = \beta_{i-k}^{lin}(\Omega_k(M))$ for these i. Note that $d_0(\Omega_k(M)) = d_k(M) + k$.

Corollary 4.2.8. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and $k \in \{0, \dots, n\}$. If $\beta_p^{k,lin}(M) \neq 0$ for some $p > k$, then*

$$\beta_{p-1}^{k,lin}(M) \geq p.$$

Proof. Without loss of generality we may assume that $M \in \mathcal{M}_{\mathbb{N}}(S)$. Now apply 4.2.6. □

For the numbers $\beta_i^{lin}(M)$ and $\beta_i^{1,lin}(M)$ we get more precise results. The next result was first discovered in [**32**].

Theorem 4.2.9. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and $p \in \{0, \dots, n\}$. If $\beta_p^{lin}(M) \neq 0$, then*

$$\beta_i^{lin}(M) \geq \binom{p}{i}$$

for $i = 0, \dots, p$.

Proof. Without loss of generality we may assume that $M \in \mathcal{M}_{\mathbb{N}}(S)$. The assertion follows from 4.2.2 and the fact that there are no non-trivial boundaries in $K_i(n)_{i+d_0(M)}$. □

To prove lower bounds for $\beta_i^{1,lin}$ we use slightly different methods. We fix a basis $\mathbf{e}$ of L for the rest of this chapter. Let F be a graded free S-module and $>$ an arbitrary degree refining term order on F with $x_1 > \ldots > x_n$ (see 1.7 for details). For a homogeneous element $f \in F$ we set $\operatorname{in}_>(f)$ for the maximal monomial in a presentation of f. Note that we also defined $\operatorname{in}(z)$ for some bihomogeneous $z \in \mathcal{K}(n)$. This should not be confused with $\operatorname{in}_>(f)$.

Lemma 4.2.10. *Let $M, F \in \mathcal{M}_{\mathbb{Z}}(S)$, $M \subset F$ and let F be a free module. Assume that $0 \neq z$ is a homogeneous cycle of $K_1(n, M)$ with $z = \sum_{j \geq i} m_j e_j$ and $\mathrm{in}(z) = m_i e_i$. Then there exists an integer $j > i$ with $0 \neq \mathrm{in}_>(m_j) > \mathrm{in}_>(m_i)$. In particular m_i and m_j are K-linearly independent.*

Proof. We have
$$0 = \partial(z) = m_i x_i + \sum_{j>i} m_j x_j.$$
Hence there exists an integer $j > i$ and a monomial a_j of m_j with $\mathrm{in}_>(m_i)x_i = a_j x_j$ because all monomials have to cancel. Assume that
$$\mathrm{in}_>(m_i) \geq \mathrm{in}_>(m_j).$$
Then
$$\mathrm{in}_>(m_i)x_i > \mathrm{in}_>(m_i)x_j \geq \mathrm{in}_>(m_j)x_j \geq a_j x_j$$
is a contradiction. Therefore
$$\mathrm{in}_>(m_i) < \mathrm{in}_>(m_j).$$
□

Construction 4.2.11. Let $M, F \in \mathcal{M}_{\mathbb{Z}}(S)$, $M \subset F$ and F be a free module. Let $p \in \{0, \dots, n\}$ and $0 \neq z$ be a homogeneous cycle of $K_p(n, M)$ with $z = \sum_{J,\ |J|=p} m_J e_J$, $\mathrm{in}(z) = m_I e_I$. Assume that $I = \{1, \dots, p\}$. For $k = 0, \dots, p$ we construct inductively sets
$$J_k = \{1, \dots, p-k, j_1, \dots, j_k\}$$
with $j_k > p - k + 1$, $j_k \neq j_i$ for $i = 0, \dots, k-1$ and
$$\mathrm{in}_>(m_{J_k}) > \mathrm{in}_>(m_{J_{k-1}}) > \dots > \mathrm{in}_>(m_{J_0}).$$
Set $J_0 = I$. Assume that J_{k-1} is constructed. Then we apply 4.2.10 to
$$\partial^{\{1,\dots,p-k,j_1,\dots,j_{k-1}\}}(z) \text{ where } \mathrm{in}(\partial^{\{1,\dots,p-k,j_1,\dots,j_{k-1}\}}(z)) = m_{J_{k-1}} e_{p-k+1}$$
and find $j_k > p - k + 1$ such that $\mathrm{in}_>(m_{\{1,\dots,p-k,j_1,\dots,j_{k-1}\} \cup \{j_k\}}) > \mathrm{in}_>(m_{J_{k-1}})$. We see that $j_k \neq j_i$ for $i = 1, \dots, k-1$ because these e_{j_i} do not appear with non-zero coefficient in $\partial^{\{1,\dots,p-k,j_1,\dots,j_{k-1}\}}(z)$.

Corollary 4.2.12. *Let $M, F \in \mathcal{M}_{\mathbb{Z}}(S)$, $M \subset F$ and F be a free module. Let $p \in \{0, \dots, n\}$ and $0 \neq z$ be a homogeneous cycle of $K_p(n, M)$ with $z = \sum_{J,\ |J|=p} m_J e_J$. Then there exist $p+1$ coefficients m_J of z, which are K-linearly independent.*

Proof. This follows from 4.2.11 because the coefficients there have different leading terms. □

Theorem 4.2.13. *Let $p \geq 1$ and $M \in \mathcal{M}_{\mathbb{Z}}(S)$ such that $\beta_p^{1,lin}(M) \neq 0$. Then*
$$\beta_i^{1,lin}(M) \geq \binom{p}{i}$$
for $i = 1, \dots, p$.

Proof. Let
$$0 \to M' \to F \to M \to 0$$
be a presentation of M such that F is free and $M' = \Omega_1(M)$. We show that
$$\beta_i^{lin}(M') \geq \binom{p'+1}{i+1} \text{ for } p' = p-1.$$
Since $\beta_i^{1,lin}(M) = \beta_{i-1}^{lin}(M')$, this will prove the theorem.

After a suitable shift of the grading of M we may assume that $d_0(M') = 0$. Note that M' is a submodule of a free module and we can apply our construction 4.2.11. Since $\beta_{p'}^{lin}(M') \neq 0$, we get a homogeneous cycle $0 \neq z$ in $K_{p'}(n, M')_{p'}$. There are no boundaries except for zero in $K_i(n, M')_i$ and therefore we only have to construct enough K-linearly independent cycles in $K_i(n, M')_i$ to prove the assertion. Assume that $\operatorname{in}(z) = m_{[p']}e_{[p']}$ and construct the numbers $j_1, \dots, j_{p'}$ for z by 4.2.11.

Let $t \in \{0, \dots, p'\}$ and set $i = p' - t$. We consider the cycles $\partial^L(z)$ (see 4.2.2) with $L \in W = W_0 \dot\cup \dots \dot\cup W_t$ where
$$W_k = \{I \cup \{j_1, \dots, j_k\} \colon I \subseteq [p'-k],\ |I| = t-k\} \text{ for } k \in \{0, \dots, t\}.$$
We have
$$|W_k| = \binom{p'-k}{t-k}$$
and therefore
$$|W| = \sum_{k=0}^{t} \binom{p'-k}{t-k} = \binom{p'+1}{t} = \binom{p'+1}{p'-t+1} = \binom{p'+1}{i+1}.$$
If we show that the cycles $\partial^L(z)$ are K-linearly independent, the assertion follows.

We consider $L \in W$, $L = I_L \cup \{j_1, \dots, j_{k_L}\}$ for some $I_L \subseteq [p'-k_L]$, $|I_L| = t - k_L$. By 4.2.11 it is easy to see that
$$\operatorname{in}(\partial^L(z)) = m_{\{1,\dots,p'-k_L\} \cup \{j_1,\dots,j_{k_L}\}} e_{\{1,\dots,p'-k_L\} - I_L}.$$
It is enough to show that the initial terms of the cycles are K-linearly independent.

If cycles have different initial monomials in the e_i, there is nothing to show. Take L, L' and assume that the corresponding cycles have the same initial monomial in the e_i. We have to consider two cases. If $k_L = k_{L'}$, then $I_L = I_{L'}$ and the cycles are the same. For $k_L < k_{L'}$ the construction 4.2.11 implies
$$\operatorname{in}(m_{\{1,\dots,p'-k_L\} \cup \{j_1,\dots,j_{k_L}\}}) < \operatorname{in}(m_{\{1,\dots,p'-k_{L'}\} \cup \{j_1,\dots,j_{k_{L'}}\}}),$$
which proves the K-linearly independence. $\square$

The next corollary summarizes our results related to the conjecture of Herzog.

Corollary 4.2.14. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and $p \in \{0, \dots, n\}$. Suppose that $\beta_p^{lin}(M) \neq 0$ and M is the k^{th}-syzygy module in a minimal graded free resolution. Then:*

(i) *If $k = 0$, then $\beta_i^{lin}(M) \geq \binom{p}{i}$ for $i = 0, \dots, p$.*
(ii) *If $k = 1$, then $\beta_i^{lin}(M) \geq \binom{p+1}{i+1}$ for $i = 0, \dots, p$.*
(iii) *If $k > 1$ and $p > 0$, then $\beta_{p-1}^{lin}(M) \geq p + k$.*

Proof. Statement (i) was shown in 4.2.9. In the proof of 4.2.13 we proved in fact (ii). Finally, (iii) follows from 4.2.8 since $\beta_i^{lin}(M) = \beta_{i+k}^{k,lin}(N)$ if M is the k^{th}-syzygy module in the minimal graded free resolution of some module N. $\square$

Recall that a finitely generated graded S-module M satisfies *Serre's condition* $\mathcal{S}_k$ if
$$\operatorname{depth}(M_P) \geq \min(k, \dim S_P)$$
for all $P \in \operatorname{Spec}(S)$. We recall the Auslander-Bridger theorem [**9**]:

Lemma 4.2.15. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$. Then M is a k^{th}-syzygy module in a graded free resolution if and only if M satisfies $\mathcal{S}_k$.*

Proof. The proof is essential the same as in [**28**] where the local case is treated. $\square$

Corollary 4.2.16. *Let $M \in \mathcal{M}_{\mathbb{Z}}(S)$ and $p \in \{0, \dots, n\}$. Suppose that M satisfies $\mathcal{S}_k$ and $\beta_p^{lin}(M) \neq 0$. Then:*

(i) *If $k = 0$, then $\beta_i^{lin}(M) \geq \binom{p}{i}$ for $i = 0, \dots, p$.*
(ii) *If $k = 1$, then $\beta_i^{lin}(M) \geq \binom{p+1}{i+1}$ for $i = 0, \dots, p$.*
(iii) *If $k > 1$ and $p > 0$, then $\beta_{p-1}^{lin}(M) \geq p + k$.*

Proof. According to 4.2.15 the module M satisfies $\mathcal{S}_k$ if and only if M is a k^{th}-syzygy module in a graded free resolution $\mathcal{G} \to N \to 0$ of some graded S-module N. It is well-known (see for example [**10**]) that $\mathcal{G} = \mathcal{F} \oplus \mathcal{H}$ as graded complexes where $\mathcal{F}$ is the minimal graded free resolution of N and $\mathcal{H}$ is split exact. Then M splits as a graded module into $\Omega_k(N) \oplus W$ where W is a graded free S-module. If $p = 0$, there is nothing to show. For $p > 0$ it follows that $\beta_p^{lin}(\Omega_k(N)) \neq 0$. Then 4.2.14 applied to $\Omega_k(N)$ proves the corollary, since $\beta_{i,j}^S(M) \geq \beta_{i,j}^S(\Omega_k(N))$ for all $i, j \in \mathbb{N}$. $\square$

4.3. Lower bounds for Betti numbers of $\mathbb{Z}^n$-graded S-modules

The polynomial ring $S = K[x_1, \dots, x_n]$ is $\mathbb{Z}^n$-graded with $\deg(x_i) = \varepsilon_i$. Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$. Recall that M is naturally graded by setting $M_i = \bigoplus_{a \in \mathbb{Z}^n,\ |a| = i} M_a$. Therefore all methods from the last section can be applied in the following. Furthermore, the Koszul complex and homology are $\mathbb{Z}^n$-graded if we assign the degree ε_i to e_i. For example, if $w \in M_a$ for some $a \in \mathbb{Z}^n$, then $\deg(we_I)$ is $a + \sum_{i \in I} \varepsilon_i$; or if $z \in K_i(j)$ is homogeneous of degree a, then $\deg(\partial^I(z)) = a - \sum_{i \in I} \varepsilon_i$.

For the finer grading we want to prove results which are more precise than those in the last section. Note that 4.1.2, 4.2.3 and 4.2.4 hold for $M \in \mathcal{M}_{\mathbb{N}^n}(S)$. The proofs are verbatim the same if we replace "graded" by "$\mathbb{Z}^n$-graded". We prove now a modified version of 4.2.6.

Lemma 4.3.1. *Let $M \in \mathcal{M}_{\mathbb{N}^n}(S)$, $p \in [j]$ and $t \in \mathbb{N}$. Suppose that $H_i(j)_{i+l} = 0$ for $i = p-1, \dots, j$, $l = -1, \dots, t-1$ and let $z \in K_p(j)_{p+t}$ be a $\mathbb{Z}^n$-homogeneous cycle with $0 \neq [z] \in H_p(j)_{p+t}$. Then there exists a $\mathbb{Z}^n$-homogeneous cycle $\tilde{z}$ with:*

(i) *$[\tilde{z}] = [z] \in H_p(j)_{p+t}$.*
(ii) *$[\partial^{j_i}(\tilde{z})] \in H_{p-1}(j)_{p-1+t}$ are K-linearly independent for $i = 1, \dots, p$ and some $j_i \in [j]$.*

Proof. We prove by induction on $j \in [n]$ that we find an element $[\tilde{z}] = [z]$ and a set $\{j_1, \dots, j_p\}$ such that the cycles $[\partial^{j_i}(\tilde{z})]$ are K-linearly independent for $i = 1, \dots, p$.

The cases $j = 1$ and $j > 1, p = j$ follow from 4.2.3 (ii) because if $\deg(z) = a \in \mathbb{N}^n$, then all $\partial^k(z)$ have different degrees $a - \varepsilon_k$ and it suffices to show that these elements are not boundaries.

Let $j > 1$ and assume that $p < j$. Again it suffices to show that the cycles $\partial^{j_i}(z)$ are not boundaries for a suitable subset $\{j_1, \dots, j_p\} \subseteq [j]$. If such a set exists, then nothing is to prove. Otherwise there exists a number $k \in [j]$ with $[\partial^k(z)] = 0$ and we may assume that $k = j$. By 4.2.3 we find $\tilde{z}$ such that $[\tilde{z}] = [z]$ in $H(j)$ and $\tilde{z} \in \mathcal{K}(j-1)$. By 4.1.2 we can apply our induction hypothesis and assume that $\tilde{z}$ has the desired properties in $H(j-1)$. Again by 4.1.2 we have $H_{p-1}(j-1)_{p-1+t} \subseteq H_{p-1}(j)_{p-1+t}$ and $\tilde{z}$ is the desired element. □

We need the following simple combinatorial tool.

Construction 4.3.2. Let $p \in [n]$. Define inductively a sequence of subsets $W_i \subseteq 2^{[n]}$ for $i = 0, \dots, p$. Set

$$W_0 = \{\emptyset\}.$$

If W_{i-1} is defined, then for every set $w \in W_{i-1}$ we choose $p - i + 1$ different elements $i_1^w, \dots, i_{p-i+1}^w \in [n]$ such that $i_j^w \notin w$. Define

$$W_i = \{w \cup \{i_j^w\} \colon w \in W_{i-1} \text{ and } j = 1, \dots, p - i + 1\}.$$

Lemma 4.3.3. *Let W_i be defined as in 4.3.2. Then for $i = 0, \dots, p$ we have*

$$|W_i| \geq \binom{p}{i}.$$

Proof. We prove this by induction on $p \in [n]$. The case $p = 0$ is trivial, so let $p > 0$ and without loss of generality we may assume that $W_1 = \{\{1\}, \dots, \{p\}\}$. The set W_i is the disjoint union of the sets $W_i^1 = \{w \in W_i \colon 1 \in w\}$ and $W_i^{\hat{1}} = \{w \in W_i \colon 1 \notin w\}$. The induction hypothesis applied to W_i^1 and $W_i^{\hat{1}}$ implies

$$|W_i| = |W_i^1| + |W_i^{\hat{1}}| \geq \binom{p-1}{i-1} + \binom{p-1}{i} = \binom{p}{i}.$$

□

We prove the main theorem of this section.

Theorem 4.3.4. *Let $k \in [n]$ and $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$. If $\beta_p^{k,lin}(M) \neq 0$ for some $p \geq k$, then*

$$\beta_i^{k,lin}(M) \geq \binom{p}{i}$$

for $i = k, \dots, p$.

Proof. Without loss of generality we have that $M \in \mathcal{M}_{\mathbb{N}^n}(S)$ after a suitable regrading of M. Since $\beta_p^{k,lin}(M) \neq 0$, there exists a $\mathbb{Z}^n$-homogeneous cycle z in $K_p(n)_{p+d_k(M)}$ such that $0 \neq [z] \in H_p(n)_{p+d_k(M)}$ and $\deg(z) = a$ for some $a \in \mathbb{N}^n$. By the definition of $d_k(M)$ and 4.1.2 we have $H_i(n)_{i+l} = 0$ for $i \geq k$ and $l = -1, \dots, d_k(M) - 1$.

We will construct inductively W_i as in 4.3.2, as well as cycles z_w for each $w \in W_i$ such that $[z_w] \neq 0$, $\deg(z_w) = a - \sum_{i\in w} \varepsilon_i$, $0 \neq [\partial^{i^w_k}(z_w)]$ for $k = 1, \dots, p-i$ and suitable $i^w_k \notin w$. Furthermore, z_w is an element of the Koszul complex with respect to the variables x_i with $i \notin w$. For $i \geq k$ we take all cycles $z_w \in K_i(n)_{i+d_k(M)}$ with $w \in W_{p-i}$ which have different $\mathbb{Z}^n$-degree. They are not zero and therefore K-linearly independent in homology. By 4.3.3 there are at least $\binom{p}{p-i} = \binom{p}{i}$ of them and this concludes the proof.

Let $W_0 = \{\emptyset\}$. By 4.3.1 we can choose z in a way such that $[z_{i_j}] = [\partial^{i_j}(z)] \neq 0$ for $j = 1, \dots, p$ and some $i_j \in [n]$. Choose $z_\emptyset = z$ and $i^\emptyset_j = i_j$.

If W_{i-1} and z_w for $w \in W_{i-1}$ are constructed, then define W_i with W_{i-1} and the given i^w_k for $w \in W_{i-1}$. For $w' \in W_i$ with $w' = w \cup \{i^w_k\}$, re-choose $z_{w'} = \partial^{i^w_k}(z_w)$ by 4.3.1 in such a way that $[\partial^{i^{w'}_j}(z_{w'})] \neq 0$ for $j = 1, \dots, p-i$ and some $i^{w'}_j \in [n]$.

Note that since z_w has no monomial which is divided by some e_i for $i \in w$, we can use 4.2.3 and 4.2.4 to avoid these e_i in the construction of $z_{w'}$ again. By 4.1.2 the cycles $\partial^{i^{w'}_j}(z_{w'})$ are also not zero in $H(n)$. Clearly $i^{w'}_j \notin w'$ and the assertion follows. □

In the $\mathbb{Z}^n$-graded setting we prove the desired results about β^{lin}_i in full generality.

Corollary 4.3.5. *Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$ and M is the k^{th}-syzygy module in a minimal $\mathbb{Z}^n$-graded free resolution. If $\beta^{lin}_p(M) \neq 0$ for some $p \in \mathbb{N}$, then*

$$\beta^{lin}_i(M) \geq \binom{p+k}{i+k}$$

for $i = 0, \dots, p$.

Proof. This follows from 4.3.4 and the fact that

$$\beta^{lin}_i(M) = \beta^{k,lin}_{i+k}(N) \geq \binom{p+k}{i+k}$$

where M is the k^{th}-syzygy module of a $\mathbb{Z}^n$-graded S-module N. □

Analogue to 4.2.15 we get:

Lemma 4.3.6. *Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$. Then M satisfies $\mathcal{S}_k$ if and only if M is a k^{th}-syzygy module in a $\mathbb{Z}^n$-graded free resolution.*

Corollary 4.3.7. *Let $M \in \mathcal{M}_{\mathbb{Z}^n}(S)$ and M satisfies $\mathcal{S}_k$. If $\beta^{lin}_p(M) \neq 0$ for some $p \in \mathbb{N}$, then*

$$\beta^{lin}_i(M) \geq \binom{p+k}{i+k}$$

for $i = 0, \dots, p$.

Proof. The assertion follows from 4.3.5 with similar arguments as in the graded case. □

4.4. Bounds for Betti numbers of ideals with a fixed number of generators in given degree and a linear resolution

In this section we are interested in bounds for the graded Betti numbers of graded ideals of $S = K[x_1, \dots, x_n]$. We assume that the field K is infinite. Denote the lexicographic order on S with $>_{lex}$ and the reverse lexicographic order on S with $>_{rlex}$. For the notion of term orders see 1.7.

Definition 4.4.1. Let $x^u \in S$ with $|u| = d$.

(i) $R(x^u) = \{x^v \colon |v| = d,\ x^v \geq_{rlex} x^u\}$ is said to be the *revlex-segment of* x^u. For a given $d \in \mathbb{N}$ and $0 \leq k \leq \binom{n+d-1}{d}$ we define $I(d,k)$ as the unique ideal which is generated in degree d by the revlex-segment $R(x^u)$ of the monomial x^u with $|R(x^u)| = k$.

(ii) $L(x^u) = \{x^v \colon |v| = d,\ x^v \geq_{lex} x^u\}$ is said to be the *lex-segment of* x^u. For a given $d \in \mathbb{N}$ and $0 \leq k \leq \binom{n+d-1}{d}$ we define $J(d,k)$ as the unique ideal which is generated in degree d by the lex-segment $L(x^u)$ of the monomial x^u with $|L(x^u)| = k$.

Lemma 4.4.2. *Let $d \in \mathbb{N}$ and $0 \leq k \leq \binom{n+d-1}{d}$. Then:*

(i) *$I(d,k)$ is a stable ideal.*

(ii) *$J(d,k)$ is a stable ideal.*

Proof. Let $x^u \in G(I(d,k))$ and $i \leq m(u)$. Since $x_i x^u / x_{m(u)} >_{rlex} x_u$, it follows from the definition of $I(d,k)$ that $x_i x^u / x_{m(u)} \in I(d,k)$. By 1.7.10 we obtain that $I(d,k)$ is a stable ideal. Analogously $J(d,k)$ is a stable ideal. $\square$

For stable ideals there exist explicit formulas for the Betti numbers (see 1.7.12). Let $I \subset S$ be a stable ideal. Then

$$\beta^S_{i,i+j}(I) = \sum_{x^u \in G(I),\ |u|=j} \binom{m(x^u)-1}{i}. \tag{5}$$

If a stable ideal I is generated in one degree, then I has a linear resolution.

Proposition 4.4.3. *Let $I \subset S$ be a stable ideal generated in degree $d \in \mathbb{N}$ with $\beta^S_{0,d}(I) = k$. Then*

$$\beta^S_{i,i+j}(I) \geq \beta^S_{i,i+j}(I(d,k))$$

for all $i, j \in \mathbb{N}$.

Proof. Fix $d \in \mathbb{N}$ and $0 \leq k \leq \binom{n+d-1}{d}$. Let $l(I)$ be the number of monomials in I_d which are not monomials in $I(d,k)_d$. We prove the statement by induction on $l(I)$. If $l(I) = 0$, then $I = I(d,k)$ and there is nothing to show.

We assume that $l(I) > 0$. Let x^u be the smallest monomial in I_d with respect to $>_{rlex}$ which is not in $I(d,k)_d$ and x^v be the largest monomial which is in $I(d,k)_d$, but not in I_d. We define the ideal $\tilde{I}$ by $G(\tilde{I}) = (G(I) \setminus \{x^u\}) \cup \{x^v\}$. Then $l(\tilde{I}) = l(I) - 1$, and $\tilde{I}$ is also stable. Thus by the induction hypothesis

$$\beta^S_{i,i+j}(\tilde{I}) \geq \beta^S_{i,i+j}(I(d,k)).$$

Since $x^v >_{rlex} x^u$, the revlex order implies

$$m(x^v) \leq m(x^u).$$

Therefore (5) yields

$$\beta^S_{i,i+j}(I) \geq \beta^S_{i,i+j}(\tilde{I}),$$

which proves the assertion. $\square$

Let $\operatorname{gin}(I)$ denote the generic initial ideal of a graded ideal $I \subset S$ with respect to the reverse lexicographic order.

Proposition 4.4.4. *Let $d \in \mathbb{N}$ and $I \subset S$ be a graded ideal with d-linear resolution. Then $\operatorname{gin}(I)$ is stable, independent of the characteristic of K, and $\beta^S_{i,i+j}(I) = \beta^S_{i,i+j}(\operatorname{gin}(I))$ for all $i, j \in \mathbb{N}$.*

Proof. By 1.7.11 we know that $\operatorname{reg}_S(\operatorname{gin}(I)) = \operatorname{reg}_S(I)$. Therefore $\operatorname{reg}_S(\operatorname{gin}(I)) = d$ and $\operatorname{gin}(I)$ also has a d-linear resolution. [**25**, Prop. 10] implies that a Borel-fixed monomial ideal, which is generated in degree d, has regularity d if and only if it is stable. Thus we get that $\operatorname{gin}(I)$ is a stable ideal, independent of the characteristic of K. Since I has a linear resolution, we obtain by the main result in [**7**] that $\beta^S_{i,i+j}(I) = \beta^S_{i,i+j}(\operatorname{gin}(I))$ for all $i, j \in \mathbb{N}$. $\square$

Theorem 4.4.5. *Let $d \in \mathbb{N}$, $0 \leq k \leq \binom{n+d-1}{d}$ and $I \subset S$ be a graded ideal with d-linear resolution and k generators. Then*

$$\beta^S_{i,i+j}(I) \geq \beta^S_{i,i+j}(I(d,k))$$

for all $i, j \in \mathbb{N}$.

Proof. This follows from 4.4.3 and 4.4.4. $\square$

Next we show an analogue of 4.4.5 for upper bounds of Betti numbers.

Proposition 4.4.6. *Let $d \in \mathbb{N}$, $0 \leq k \leq \binom{n+d-1}{d}$ and $I \subset S$ be a graded ideal with d-linear resolution and k generators. Then*

$$\beta^S_{i,i+j}(I) \leq \beta^S_{i,i+j}(J(d,k))$$

for all $i, j \in \mathbb{N}$.

Proof. By [**41**, Thm. 31] we find an ideal L with the same Hilbert function as I and

$$\beta^S_{i,i+j}(I) \leq \beta^S_{i,i+j}(L)$$

for all $i, j \in \mathbb{N}$. Furthermore, L has the property that the set of monomials of L_j is a lex-segment for all $j \in \mathbb{N}$. We see that

$$\beta^S_{0,d}(L) = \beta^S_{0,d}(I) = k$$

because these ideals share the same Hilbert function. It follows that $J(d,k) = (L_d)$, and in particular that $G(J(d,k)) = G(L)_d$. Therefore

$$\beta^S_{i,i+d}(I) \leq \beta^S_{i,i+d}(L) = \beta^S_{i,i+d}(J(d,k)),$$

for all $i, j \in \mathbb{N}$ where the last equality follows from (5). $\square$

Definition 4.4.7. Let $d \in \mathbb{N}$, $0 \leq k \leq \binom{n+d-1}{d}$. We define $\mathcal{B}(d,k)$ as the set of Betti sequences $\{\beta^S_{i,j}(I)\}$ where $I \subset S$ is a graded ideal with d-linear resolution and $\beta^S_{0,d}(I) = k$. On $\mathcal{B}(d,k)$ we consider a partial order: We set $\{\beta^S_{i,j}(I)\} \geq \{\beta^S_{i,j}(J)\}$ if $\beta^S_{i,j}(I) \geq \beta^S_{i,j}(J)$ for all $i,j \in \mathbb{N}$.

Corollary 4.4.8. *Let $d \in \mathbb{N}$, $0 \leq k \leq \binom{n+d-1}{d}$. Then $\{\beta^S_{i,j}(I(d,k))\}$ is the unique minimal element and $\{\beta^S_{i,j}(J(d,k))\}$ is the unique maximal element of $\mathcal{B}(d,k)$.*

CHAPTER 5

Homological properties of bigraded algebras

This chapter is devoted to study homological properties of bigraded K-algebras $R = S/J$ where S is a bigraded polynomial ring and $J \subset S$ is a bigraded ideal. In the first section we give a homological characterization of the x- and y-regularity of R. As applications we reprove a result by Trung [**50**] and show that the x-regularity of S/J and $S/\mathrm{bigin}(J)$ are the same. It was shown in [**21**] (or [**38**]) that for $j \gg 0$ and a graded ideal I in a polynomial ring, $\mathrm{reg}(I^j)$ is a linear function $cj + d$ in j. In the fourth section we give, in case that I is equigenerated, bounds j_0 such that for $j \geq j_0$ the function is linear and give also a bound for d. Finally, we obtain upper bounds for the x- and y-regularity of generalized Veronese algebras.

5.1. Regularity

Let K be a field with $|K| = \infty$ and fix a bigraded K-algebra R. Then we may write $R = S/J$ where S is a bigraded polynomial ring and $J \subset S$ is a bigraded ideal. Set $\mathfrak{m}_x = (S_{(1,0)}) \subset S$, $\mathfrak{m}_y = (S_{(0,1)}) \subset S$ and $\mathfrak{m} = \mathfrak{m}_x + \mathfrak{m}_y$.

Following [**5**] (or [**50**] under the name filter regular element) we introduce:

Definition 5.1.1. We define:

(i) An element $x \in R_{(1,0)}$ is called *almost regular for* R (with respect to the x-degree) if $(0 :_R x)_{(a,*)} = 0$ for $a \gg 0$. A sequence $x_1, \dots, x_t \in R_{(1,0)}$ is called *almost regular* (with respect to the x-degree) if for all $i \in [t]$ the x_i is almost regular for $R/(x_1, \dots, x_{i-1})R$.

(ii) An element $y \in R_{(0,1)}$ is called *almost regular for* R (with respect to the y-degree) if $(0 :_R y)_{(*,b)} = 0$ for $b \gg 0$. A sequence $y_1, \dots, y_t \in R_{(0,1)}$ is *an almost regular sequence* (with respect to the y-degree) if for all $i \in [t]$ the y_i is almost regular for $R/(y_1, \dots, y_{i-1})R$.

It is well-known that, provided $|K| = \infty$, after a generic choice of coordinates we can achieve that a K-basis of $R_{(1,0)}$ is almost regular for R with respect to the x-degree and a K-basis of $R_{(0,1)}$ is almost regular for R with respect to the y-degree. For the convenience of the reader we give a proof of this fact.

Lemma 5.1.2. *We have:*

(i) *If* $\dim_K R_{(1,0)} > 0$, *then there exists an element* $x \in R_{(1,0)}$ *which is almost regular for* R *with respect to the* x*-degree.*

(ii) *If* $\dim_K R_{(0,1)} > 0$, *then there exists an element* $y \in R_{(0,1)}$ *which is almost regular for* R *with respect to the* y*-degree.*

Moreover, the property to be almost regular in (*i*) *or* (*ii*) *is a non-empty open condition.*

Proof. By symmetry it is enough to prove the statement (i). Observe that for $M \in \mathcal{M}_{bi}(S)$ every prime ideal of $\operatorname{Ass}_S(M)$ is bigraded (see [**23**, Ex. 3.5]). We claim that it is possible to choose $0 \neq x \in R_{(1,0)}$ such that for all $Q \in \operatorname{Ass}_S(0:_R x)$ one has $Q \supseteq \mathfrak{m}_x$. It follows that $\operatorname{Rad}_S(\operatorname{Ann}_S(0:_R x)) \supseteq \mathfrak{m}_x$. Hence there exists an integer i such that $\mathfrak{m}_x^i(0:_R x) = 0$ and this proves the lemma.

It remains to show the claim. If $P \supseteq \mathfrak{m}_x$ for all $P \in \operatorname{Ass}_S(R)$, then we may choose $0 \neq x \in R_{(1,0)}$ arbitrary because $\operatorname{Ass}_S(0:_R x) \subseteq \operatorname{Ass}_S(R)$. Otherwise there exists an ideal $P \in \operatorname{Ass}_S(R)$ with $P \not\supseteq \mathfrak{m}_x$. In this case we may choose $x \in R_{(1,0)}$ with

$$x \notin \bigcup_{P \in \operatorname{Ass}_S(R),\ P \not\supseteq \mathfrak{m}_x} P$$

since $|K| = \infty$. Let $Q \in \operatorname{Ass}_S(0:_R x)$ be arbitrary. Then $x \in Q$ because $x \in \operatorname{Ann}_S(0:_R x)$. We also have that $Q \in \operatorname{Ass}_S(R)$ and this implies that $Q \supseteq \mathfrak{m}_x$ by the choice of x. This gives the claim. Observe that in every case the choice of x is a non-empty open condition. □

An induction on $\dim_K R_{(1,0)}$ or $\dim_K R_{(0,1)}$ yields:

Corollary 5.1.3. *We have:*

(i) *There exists a K-basis $\mathbf{x} = x_1, \dots, x_n$ of $R_{(1,0)}$ such that $\mathbf{x}$ is almost regular for R with respect to the x-degree. Moreover, a generic K-basis of $R_{(1,0)}$ has this property.*

(ii) *There exists a K-basis $\mathbf{y} = y_1, \dots, y_m$ of $R_{(0,1)}$ such that $\mathbf{y}$ is almost regular for R with respect to the y-degree. Moreover, a generic K-basis of $R_{(0,1)}$ has this property.*

Let W be a d-dimensional K-vector space with basis $f_1, \dots, f_d$. That a generic K-basis of W satisfies a certain property $\mathcal{P}$ means the following: Every element of $\mathcal{GL}(d;K)$ induces a linear automorphism on W. Then there exists a non-empty open subset U of $\mathcal{GL}(d;K)$ such that for all $g \in U$ the K-basis $g(f_1), \dots, g(f_d)$ satisfies $\mathcal{P}$. We call this K-basis generic for W.

Thus we may always assume that $R = S/J$ where $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$, $\mathbf{x} = x_1, \dots, x_n$ is almost regular for R with respect to the x-degree and $\mathbf{y} = y_1, \dots, y_m$ is almost regular for R with respect to the y-degree. To simplify the notation we do not distinguish between the polynomial ring variables x_i or y_j and the corresponding residue classes in R.

Definition 5.1.4. Let $\mathbf{x}$ be almost regular for R with respect to the x-degree and let $\mathbf{y}$ be almost regular for R with respect to the y-degree. We define

$$s_i^x(R) = \max(\{a\colon (0:_{R/(x_1,\dots,x_{i-1})R} x_i)_{(a,*)} \neq 0\} \cup \{0\}),$$

$$s^x(R) = \max\{s_1^x(R), \dots, s_n^x(R)\}$$

and

$$s_i^y(R) = \max(\{b\colon (0:_{R/(y_1,\dots,y_{i-1})R} y_i)_{(*,b)} \neq 0\} \cup \{0\}),$$

$$s^y(R) = \max\{s_1^y(R), \dots, s_m^y(R)\}.$$

Recall that $\mathcal{K}(k,l;R)=\mathcal{K}(k,l)$ denotes the Koszul complex of R and $H(k,l;R)=H(k,l)$ denotes the Koszul homology of R with respect to $x_1,\dots,x_k$ and $y_1,\dots,y_l$ (see 1.3 for details). The following theorem characterizes the x- and y-regularity of a bigraded K-algebra over S. It is the analogue of the corresponding graded version in [**5**]. For its proof we consider

$$\tilde{H}_0(k-1,0)=(0:_{R/(x_1,\dots,x_{k-1})R}x_k)\text{ for }k\in[n]$$

and

$$\tilde{H}_0(n,k-1)=(0:_{R/(\mathfrak{m}_x+y_1,\dots,y_{k-1})R}y_k)\text{ for }k\in[m].$$

Then the beginning of the long exact Koszul sequence of the Koszul homology of R for $k\in[n]$ is modified to

$$\dots\to H_1(k-1,0)(-1,0)\xrightarrow{x_k}H_1(k-1,0)\to H_1(k,0)\to\tilde{H}_0(k-1,0)(-1,0)\to 0,$$

and for $k\in[m]$ to

$$\dots\to H_1(n,k-1)(0,-1)\xrightarrow{y_k}H_1(n,k-1)\to H_1(n,k)\to\tilde{H}_0(n,k-1)(0,-1)\to 0.$$

It is easy to see that for $k\in[n]$ and $i\geq 1$ one has $H_i(k,0)_{(a,*)}=0$ for $a\gg 0$. Similarly for $k\in[m]$ and $i\geq 1$ one has $H_i(n,k)_{(*,b)}=0$ for $b\gg 0$.

Theorem 5.1.5. *Let* $\mathbf{x}$ *be almost regular for* R *with respect to the* x*-degree and* $\mathbf{y}$ *be almost regular for* R *with respect to the* y*-degree. Then*

$$\operatorname{reg}_x(R)=s^x(R)\ \textit{and}\ \operatorname{reg}_y(R)=s^y(R).$$

Proof. By symmetry it is enough to show this theorem only for $\mathbf{x}$. Let

$$r_{(k,0)}=\max(\{a\colon H_i(k,0)_{(a+i,*)}\neq 0\text{ for }i\in[k]\}\cup\{0\})$$

for $k\in[n]$ and

$$r_{(n,k)}=\max(\{a\colon H_i(n,k)_{(a+i,*)}\neq 0\text{ for }i\in[n+k]\}\cup\{0\})$$

for $k\in[m]$. Then $r_{(n,m)}=\operatorname{reg}_x(R)$ because $H_0(n,m)=K$. We claim that:

(i) For $k\in[n]$ one has $r_{(k,0)}=\max\{s_1^x(R),\dots,s_k^x(R)\}$.
(ii) For $k\in[m]$ one has $r_{(n,k)}=\max\{s_1^x(R),\dots,s_n^x(R)\}=s^x(R)$.

This yields the theorem. We show (i) by induction on $k\in[n]$. For $k=1$ we have the following exact sequence

$$0\to H_1(1,0)\to\tilde{H}_0(0,0)(-1,0)\to 0$$

which proves this case. Let $k>1$. Since

$$\dots\to H_1(k,0)\to\tilde{H}_0(k-1,0)(-1,0)\to 0,$$

we get $r_{(k,0)}\geq s_k^x(R)$. If $r_{(k-1,0)}=0$, then $r_{(k,0)}\geq r_{(k-1,0)}$. Assume that $r_{(k-1,0)}>0$. There exists an integer i such that $H_i(k-1)_{(r_{(k-1,0)}+i,*)}\neq 0$. Then by

$$\dots\to H_{i+1}(k,0)_{(r_{(k-1,0)}+i+1,*)}\to H_i(k-1,0)_{(r_{(k-1,0)}+i,*)}$$
$$\to H_i(k-1,0)_{(r_{(k-1,0)}+i+1,*)}\to\dots$$

we have $H_{i+1}(k,0)_{(r_{(k-1,0)}+i+1,*)} \neq 0$ because $H_i(k-1,0)_{(r_{(k-1,0)}+i+1,*)} = 0$. This gives also $r_{(k,0)} \geq r_{(k-1,0)}$. On the other hand let $a > \max\{r_{(k-1,0)}, s^x_k(R)\}$. If $i \geq 2$, then by

$$\ldots \to H_i(k-1,0)_{(a+i,*)} \to H_i(k,0)_{(a+i,*)} \to H_{i-1}(k-1,0)_{(a+i-1,*)} \to \ldots$$

we get $H_i(k,0)_{(a+i,*)} = 0$ because $H_i(k-1,0)_{(a+i,*)} = H_{i-1}(k-1,0)_{(a+i-1,*)} = 0$. Similarly $H_1(k,0)_{(a+1,*)} = 0$. Therefore we obtain that $r_{(k,0)} = \max\{r_{(k-1,0)}, s^x_k(R)\} = \max\{s^x_1(R), \ldots, s^x_k(R)\}$ by the induction hypothesis.

We prove (ii) also by induction on $k \in \{0, \ldots, m\}$. The case $k = 0$ was shown in (i), so let $k > 0$. Assume that $a > s^x(R)$. For $i \geq 2$ one has

$$\ldots \to H_i(n,k-1)_{(a+i,*)} \to H_i(n,k)_{(a+i,*)} \to H_{i-1}(n,k-1)_{(a+i,*)} \to \ldots .$$

Then we get $H_i(n,k)_{(a+i,*)} = 0$ because $H_i(n,k-1)_{(a+i,*)} = H_{i-1}(n,k-1)_{(a+i,*)} = 0$. Similarly $H_1(n,k)_{(a+1,*)} = 0$ and therefore $r_{(n,k)} \leq s^x(R)$. If $s^x(R) = 0$, then $r_{(n,k)} = s^x(R)$. Assume that $0 < s^x(R) = r_{(n,k-1)}$.

There exists an integer i such that $H_i(n,k-1)_{(s^x(R)+i,*)} \neq 0$. Consider

$$\ldots \to H_i(n,k-1)_{(s^x(R)+i,*)} \xrightarrow{y_k} H_i(n,k-1)_{(s^x(R)+i,*)} \to H_i(n,k)_{(s^x(R)+i,*)} \to \ldots .$$

If $H_i(n,k)_{(s^x(R)+i,*)} = 0$, then $H_i(n,k-1)_{(s^x(R)+i,*)} = y_k H_i(n,k-1)_{(s^x(R)+i,*)}$. This is a contradiction by Nakayamas lemma because $H_i(n,k-1)_{(s^x(R)+i,*)}$ is a finitely generated S_y-module. Hence $H_i(n,k)_{(s^x(R)+i,*)} \neq 0$ and thus $r_{(n,k)} = s^x(R)$. □

5.2. d-sequences

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the bigraded polynomial ring. Observe that $S_x = K[x_1, \ldots, x_n]$ and $S_y = K[y_1, \ldots, y_m]$ are bigraded subalgebras of S. Usually we consider S_x and S_y as graded polynomial rings. Recall from 1.6 that a sequence of elements $f_1, \ldots, f_r$ in a ring is called a d-sequence, if

(i) $f_1, \ldots, f_r$ is a minimal system of generators of the ideal $I = (f_1, \ldots, f_r)$.
(ii) $(f_1, \ldots, f_{i-1}) : f_i \cap I = (f_1, \ldots, f_{i-1})$.

A result in [**50**] motivated the following proposition.

Proposition 5.2.1. *Let $J \subset S$ be a bigraded ideal and $R = S/J$. Then:*

(i) $\operatorname{reg}_x(R) = 0$ *if and only if every generic K-basis of $R_{(1,0)}$ is a d-sequence.*
(ii) $\operatorname{reg}_y(R) = 0$ *if and only if every generic K-basis of $R_{(0,1)}$ is a d-sequence.*

Proof. By symmetry we only have to prove (i). Without loss of generality the K-basis $\mathbf{x} = x_1, \ldots, x_n$ of $R_{(1,0)}$ is an almost regular sequence for R with respect to the x-degree because by 5.1.3 every generic K-basis of $R_{(1,0)}$ has this property.

By 5.1.5 one has $\operatorname{reg}_x(R) = 0$ if and only if $s^x(R) = 0$. By definition of $s^x(R)$ this is equivalent to the fact that, for all $i \in [n]$ and all $a > 0$, we have

$$\left(\frac{(x_1, \ldots, x_{i-1}) :_R x_i}{(x_1, \ldots, x_{i-1})}\right)_{(a,*)} = 0.$$

Equivalently, for all $i \in [n]$ we obtain $(x_1, \ldots, x_{i-1}) :_R x_i \cap (\mathbf{x}) = (x_1, \ldots, x_{i-1})$. This concludes the proof. □

Remark 5.2.2. If $R_{(1,0)}$ (resp. $R_{(0,1)}$) can be generated by a d-sequence (not necessarily generic), then the proof of 5.2.1 shows that $\operatorname{reg}_x(R) = 0$ (resp. $\operatorname{reg}_y(R) = 0$).

For an application we recall some facts from 1.6. Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree d. Let $R(I)$ denote the Rees algebra of I and $S(I)$ denote the symmetric algebra of I. Both algebras are bigraded and have a presentation S/J for a bigraded ideal $J \subset S$. We always assume that $R(I) = S/J$. Note that then $I^j \cong (S/J)_{(*,j)}(-jd)$ for all $j \in \mathbb{N}$. Similarly we may assume that $S(I) = S/J$ for a bigraded ideal $J \subset S$. We also consider the finitely generated S_x-module $S^j(I) = (S/J)_{(*,j)}(-jd)$, which we call the j^{th}-symmetric power of I.

For the notion of an s-sequence see [**35**]. The following results were shown in [**35**] and [**50**].

Corollary 5.2.3. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree d. Then:*

(i) *I can be generated by an s-sequence (with respect to the reverse lexicographic order) if and only if $\operatorname{reg}_y(S(I)) = 0$.*
(ii) *I can be generated by a d-sequence if and only if $\operatorname{reg}_y(R(I)) = 0$.*

Proof. In [**35**] and [**50**] it was shown that

(a) I can be generated by an s-sequence (with respect to the reverse lexicographic order) if and only if $S(I)_{(0,1)}$ can be generated by a d-sequence.
(b) I can be generated by a d-sequence if and only if $R(I)_{(0,1)}$ can be generated by a d-sequence.

Together with 5.2.1 and 5.2.2 these facts conclude the proof. □

5.3. Bigeneric initial ideals

Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be the bigraded polynomial ring. We study the relationship between the x-regularity of S/J and the x-regularity of $S/\operatorname{bigin}(J)$ where $J \subset S$ is a bigraded ideal. For notation and definitions see 1.7.

As before we set $S_x = K[x_1, \dots, x_n] \subset S$. Consider S_x as a graded polynomial ring and set $\mathbf{x} = x_1, \dots, x_n$. For the convenience of the reader we give a proof of a well-known fact.

Lemma 5.3.1. *Let $I \subset S_x$ be a stable ideal and $i \in [n]$. Then:*

(i) $(\mathbf{x})\frac{((x_n,\dots,x_{i+1})+I:_{S_x}x_i)}{(x_n,\dots,x_{i+1})+I} = 0$.
(ii) *A homogeneous K-basis of $\frac{((x_n,\dots,x_{i+1})+I:_{S_x}x_i)}{(x_n,\dots,x_{i+1})+I}$ is given by the residue classes of $x^u/x_{m(u)}$ where $x^u \in G(I)$ with $m(u) = i$.*

Proof. Let $M^i = \frac{((x_n,\dots,x_{i+1})+I:_{S_x}x_i)}{(x_n,\dots,x_{i+1})+I}$. For a monomial $x^u \in S_x$ we denote the residue class in M^i with $[x^u]$. We show:

(1) $0 \neq [x^u/x_{m(u)}] \in M^i$ for $x^u \in G(I)$ with $m(u) = i$.
(2) $0 = (\mathbf{x})[x^u/x_{m(u)}]$ in M^i for $x^u \in G(I)$ with $m(u) = i$.
(3) A system of homogeneous generators of M^i is given by all $[x^u/x_{m(u)}]$ with $x^u \in G(I)$ and $m(u) = i$.

This yields the lemma.

Proof of (1): Let $x^u \in G(I)$ with $m(u) = i$. Then $x^u/x_{m(u)} \in ((x_n, \dots, x_{i+1}) + I :_{S_x} x_i)$ and therefore $[x^u/x_{m(u)}] \in M^i$. Since $m(u) = i$ and $x^u/x_{m(u)} \notin I$, we obtain that $0 \neq [x^u/x_{m(u)}]$ in M^i.

Proof of (2): Let $j \leq i$. Since I is stable, we get that $x_j x^u/x_{m(u)} \in I$. Thus $x_j[x^u/x_{m(u)}] = 0$ in M^i. By the definition of M^i one has $x_j M^i = 0$ for $j \geq i+1$. Thus $0 = (\mathbf{x})[x^u/x_{m(u)}]$ in M^i.

Proof of (3): If $M^i = 0$, there is nothing to show. Assume that $M^i \neq 0$. Let $w \in S_x$ with $0 \neq [w]$ in M^i. Then $w \notin (x_n, \dots, x_{i+1}) + I$ and $x_i w \in (x_n, \dots, x_{i+1}) + I$. Since I is a monomial ideal, we may assume that $w = x^u$ is a monomial in S_x. We obtain that $m(u) < i+1$. We have to consider two cases:

(A) Assume that $x_i x^u = x_j x^{u'}$ for $j \geq i+1$ and for a monomial $x^{u'} \in S_x$. This is a contradiction to $m(u) < i+1$.

(B) Otherwise $x_i x^u = x^{u'} x^{u''}$ for $x^{u'} \in G(I)$ and for a monomial $x^{u''} \in S_x$. If x_i divides $x^{u''}$, then

$$x^u = x^{u'}(x^{u''}/x_i) \in I$$

and this is a contradiction to $x^u \notin I$. Hence x_i divides $x^{u'}$. Since $m(u) < i+1$, it follows that $m(u'), m(u'') < i+1$. If $|u''| \neq 0$, then

$$x^u = (x^{u'}/x_i)x^{u''} \in I,$$

because I is stable. This is again a contradiction. Hence

$$x^u = (x^{u'}/x_i) \text{ and } m(u') = i.$$

The assertion follows. ☐

We consider a bigraded K-algebra $R = S/J$ where J is a bistable ideal. We associate invariants to R.

Definition 5.3.2. *Let $J \subset S$ be a bistable ideal and $R = S/J$. For $i \in [n]$ and $j \geq 0$ we define:*

$$m_j^i(R) = \sup(\{a \colon (0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(a,j)} \neq 0\} \cup \{0\}).$$

In the next proof we need the following notation. For a bistable ideal J and $v \in \mathbb{N}^m$ we set $J_{(*,v)} = I_v y^v$ where $I_v \subset S_x$ is a monomial ideal, which is stable in S_x.

Proposition 5.3.3. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:*

(i) *For every $i \in [n]$ and for $j \geq 0$ we have $m_j^i(R) \leq \max\{m_x(J) - 1, 0\}$.*
(ii) *For every $i \in [n]$ and for $j \geq m_y(J)$ we have $m_j^i(R) = m_{m_y(J)}^i(R)$.*

Proof. If $G(J) = \{x^{u^k} y^{v^k} \colon k = 1, \dots, r\}$, then $I_v = (x^{u^k} \colon v^k \preceq v)$ for $v \in \mathbb{N}^m$. This means that for all $x^u \in G(I_v)$ one has $|u| \leq m_x(J)$. For fixed v with $|v| = j$ we have

$$(0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(*,v)} = \frac{((x_n, \dots, x_{i+1}) + I_v :_{S_x} x_i)}{(x_n, \dots, x_{i+1}) + I_v} y^v.$$

It follows from 5.3.1 that as a graded K-vector space

$$\frac{((x_n,\dots,x_{i+1})+I_v :_{S_x} x_i)}{(x_n,\dots,x_{i+1})+I_v}y^v = \bigoplus_{x^u\in G(I_v),\ m(u)=i} K(x^u/x_{m(u)})y^v$$

because I_v is stable. Thus

$$m^i_j(R) \le \max\{m_x(J)-1,0\},$$

which is (i).

To prove (ii) we replace J by $J_{(*,\ge m_y(J))}$ and may assume that J is generated in y-degree $t = m_y(J)$. Then $G(J) = \{x^{u^k}y^{v^k} : k = 1,\dots,r\}$ where $|v^k| = t$ for all $k \in [r]$. Let $|u^k|$ be maximal with $m(u^k) = i$ and define $c^i = \max\{|u^k|-1,0\}$. We show that $m^i_j(R) = c^i$ for $j \ge t$ and this gives (ii). By a similar argument as in (i) we have $m^i_{s+t}(R) \le c^i$ for $s \ge 0$. If $c^i = 0$, then $m^i_j(R) = 0$. Assume that $c^i \ne 0$.

We claim that

$$(*)\quad 0 \ne [(x^{u^k}/x_i)y^{v^k}y_n^s] \in (0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(*,s+t)} \text{ for } s \ge 0.$$

Assume this is not the case, then either

$$(x^{u^k}/x_i)y^{v^k}y_n^s = x_l x^{u'}y^{v'}$$

for some u',v' and $l \ge i+1$ which contradicts to $m(u^k) = i$, or

$$(x^{u^k}/x_i)y^{v^k}y_n^s = x^{u^{k'}}y^{v^{k'}}x^{u'}y^{v'}$$

for $x^{u^{k'}}y^{v^{k'}} \in G(J)$. It follows that $|v'| = s$. Let k_1 be the largest integer l such that $y_n^l | y^{v^{k'}}$. Then

$$(x^{u^k}/x_i)y^{v^k} = ((x^{u^{k'}}y^{v^{k'}}x^{u'})/y_n^{k_1})y^{v'}/y_n^{s-k_1} \in J,$$

because J is bistable, and this is again a contradiction. Therefore $(*)$ is true and we get $m^i_{s+t}(R) \ge c^i$ for $s \ge 0$. This concludes the proof. ☐

Remark 5.3.4. This proposition could also be formulated if the roles of $\mathbf{x} = x_1,\dots,x_n$ and $\mathbf{y} = y_1,\dots,y_m$ are exchanged.

It is easy to determine almost regular sequences for S/J where $J \subset S$ is a bistable ideal.

Corollary 5.3.5. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:*

(i) *$x_n,\dots,x_1$ is an almost regular sequence for R with respect to the x-degree.*
(ii) *$y_m,\dots,y_1$ is an almost regular sequence for R with respect to the y-degree.*

Proof. By symmetry it suffices to prove (i). Fix $i \in [n]$. Now 5.3.3 yields

$$\sup\{m^i_j(R) : j \in \mathbb{N}\} \le \max\{m_x(J)-1,0\}.$$

Then

$$(0 :_{S/((x_n,\dots,x_{i+1})+J)} x_i)_{(a,*)} = 0 \text{ for } a \ge m_x(J).$$

Thus the assertion follows. ☐

We fix the (bidegree) reverse-lexicographic term order $>_{rlex}$ on S (see 1.7.14 for the definition). Recall that for a bigraded ideal $J \subset S$ we write $\mathrm{bigin}(J)$ for the bigeneric initial ideal of J with respect to $>_{rlex}$.

Observe that for $\mathrm{char}(K) = 0$ the ideal $\mathrm{bigin}(J)$ is strongly bistable. See for example [**5**] for similar results in the graded case.

Proposition 5.3.6. *Let $J \subset S$ be a bigraded ideal. If char$(K) = 0$, then*

$$\mathrm{reg}_x(S/J) = \mathrm{reg}_x(S/\mathrm{bigin}(J)).$$

Proof. Set $\mathbf{x} = x_n, \dots , x_1$, choose $g \in G$ generic for J and let $\tilde{\mathbf{x}} = \tilde{x}_n, \dots , \tilde{x}_1$ such that $x_i = g(\tilde{x}_i)$. By 5.1.3 we may assume that the sequence $\tilde{\mathbf{x}}$ is almost regular for S/J with respect to the x-degree. Furthermore, by 5.3.5 the sequence $\mathbf{x}$ is almost regular for $S/\mathrm{bigin}(J)$ with respect to the x-degree. We have

$$(0 :_{S/((\tilde{x}_n,\dots,\tilde{x}_{i+1})+J)} \tilde{x}_i) \cong (0 :_{S/((x_n,\dots,x_{i+1})+g(J))} x_i).$$

Analogously to 1.7.7 it follows that

$$(0 :_{S/((x_n,\dots,x_{i+1})+g(J))} x_i) \cong (0 :_{S/((x_n,\dots,x_{i+1})+\mathrm{bigin}(J))} x_i).$$

By 5.1.5 we get the desired result. □

Remark 5.3.7. Observe the following:

(i) In general it is not true that

$$\mathrm{reg}_y(S/J) = \mathrm{reg}_y(S/\mathrm{bigin}(J)).$$

For example let $S = K[x_1, \dots , x_3, y_1, \dots , y_3]$ and $J = (y_2x_2 - y_1x_3, y_3x_1 - y_1x_3)$. Then the minimal bigraded free resolution of S/J is given by

$$0 \to S(-2,-2) \to S(-1,-1) \oplus S(-1,-1) \to S \to 0.$$

Therefore

$$\mathrm{reg}_x(S/J) = 0 \text{ and } \mathrm{reg}_y(S/J) = 0.$$

On the other hand $\mathrm{bigin}(J) = (y_2x_1, y_1x_1, y_1^2x_2)$ with the minimal bigraded free resolution of $S/\mathrm{bigin}(J)$

$$0 \to S(-2,-2) \oplus S(-1,-2)$$

$$\to S(-1,-1) \oplus S(-1,-1) \oplus S(-1,-2) \to S \to 0.$$

Hence

$$\mathrm{reg}_x(S/\mathrm{bigin}(J)) = 0 \text{ and } \mathrm{reg}_y(S/\mathrm{bigin}(J)) = 1.$$

(ii) It is easy to calculate the x- and the y-regularity of bistable ideals. In fact, in [**3**] (see 6.2.9 for a new proof of this result) it was shown that for a bistable ideal $J \subset S$ we have

$$\mathrm{reg}_x(J) = m_x(J) \text{ and } \mathrm{reg}_y(J) = m_y(J).$$

5.4. Regularity of powers and symmetric powers of ideals

Let S and S_x be as in 5.3. In [**21**] and [**38**] it was shown that for a graded ideal $I \subset S_x$ the function $\operatorname{reg}_{S_x}(I^j)$ is a linear function $pj + c$ for $j \gg 0$. In the case that I is generated in one degree we give an upper bound for c and find an integer j_0 for which $\operatorname{reg}_{S_x}(I^j)$ is a linear function for all $j \geq j_0$.

Theorem 5.4.1. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Let $R(I) = S/J$ for a bigraded ideal $J \subset S$. Then:*

(i) $\operatorname{reg}_{S_x}(I^j) \leq jd + \operatorname{reg}_x(R(I))$.
(ii) $\operatorname{reg}_{S_x}(I^j) = jd + c$ *for* $j \geq m_y(\operatorname{bigin}(J))$ *and some constant* $0 \leq c \leq \operatorname{reg}_x(R(I))$.

Proof. We choose an almost regular sequence $\tilde{\mathbf{x}} = \tilde{x}_n, \dots, \tilde{x}_1$ for $R(I)$ over S with respect to the x-degree. We have that for all $j \in \mathbb{N}$ the sequence $\tilde{\mathbf{x}}$ is almost regular for I^j over S_x in the sense of [**5**] (that is $\tilde{\mathbf{x}}$ is almost regular for I^j over S_x with respect to the x-degree) because $R(I)_{(*,j)}(-dj) \cong I^j$ as graded S_x-modules and

$$(0 :_{R(I)/(\tilde{x}_n,\dots,\tilde{x}_{i+1})R(I)} \tilde{x}_i)_{(*,j)}(-dj) \cong (0 :_{I^j/(\tilde{x}_n,\dots,\tilde{x}_{i+1})I^j} \tilde{x}_i).$$

Define $m^i_j = m^i_j(S/\operatorname{bigin}(J))$. Since

$$(0 :_{R(I)/(\tilde{x}_n,\dots,\tilde{x}_{i+1})R(I)} \tilde{x}_i) \cong (0 :_{S/((x_n,\dots,x_{i+1})+\operatorname{bigin}(J))} x_i),$$

it follows that

$$jd + m^i_j = r^i_j = \max(\{l\colon (0 :_{I^j/(\tilde{x}_n,\dots,\tilde{x}_{i+1})I^j} \tilde{x}_i)_l \neq 0\} \cup \{0\}).$$

By a characterization of the regularity of graded modules in [**5**] we have

$$\operatorname{reg}_{S_x}(I^j) = \max\{jd, r^1_j, \dots, r^n_j\}.$$

Hence the assertion follows from 5.3.3, 5.3.6 and 5.3.7(ii). □

Similarly as in 5.4.1 one has:

Theorem 5.4.2. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Let $S(I) = S/J$ for a bigraded ideal $J \subset S$. Then:*

(i) $\operatorname{reg}_{S_x}(S^j(I)) \leq jd + \operatorname{reg}_x(S(I))$.
(ii) $\operatorname{reg}_{S_x}(S^j(I)) = jd + c$ *for* $j \geq m_y(\operatorname{bigin}(J))$ *and some constant* $0 \leq c \leq \operatorname{reg}_x(S(I))$.

Blum [**15**] proved the following with different methods.

Corollary 5.4.3. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree d.*

(i) *If* $\operatorname{reg}_x(R(I)) = 0$, *then* $\operatorname{reg}_{S_x}(I^j) = jd$ *for* $j \geq 1$.
(ii) *If* $\operatorname{reg}_x(S(I)) = 0$, *then* $\operatorname{reg}_{S_x}(S^j(I)) = jd$ *for* $j \geq 1$.

Proof. This follows from 5.4.1 and 5.4.2. □

Next we give a more theoretic bound for the regularity function becoming linear. Consider a bigraded K-algebra R. Let $y \in S$ be almost regular for all $\operatorname{Tor}^S_i(S/(\mathbf{x}), R)$ with respect to the y-degree. We define

$$w(R) = \max\{b\colon (0 :_{\operatorname{Tor}^S_i(S/(\mathbf{x}),R)} y)_{(*,b)} \neq 0 \text{ for some } i \in [n]\}.$$

Lemma 5.4.4. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$.*

(i) *For $j > w(R(I))$ we have $\operatorname{reg}_{S_x}(I^{j+1}) \geq \operatorname{reg}_{S_x}(I^j) + d$.*
(ii) *For $j > w(S(I))$ we have $\operatorname{reg}_{S_x}(S^{j+1}(I)) \geq \operatorname{reg}_{S_x}(S^j(I)) + d$.*

Proof. We prove the case $R = R(I)$. For $j > w(R)$ one has the exact sequence

$$0 \to \operatorname{Tor}_i^S(S/(\mathbf{x}), R)_{(*,j)} \xrightarrow{y} \operatorname{Tor}_i^S(S/(\mathbf{x}), R)_{(*,j+1)}.$$

In [**21**, 3.3] it was shown that

$$\operatorname{Tor}_i^S(S/(\mathbf{x}), R)_{(a,j)} \cong \operatorname{Tor}_i^{S_x}(K, I^j)_{a+jd}$$

and this concludes the proof. □

Lemma 5.4.5. *Let $J \subset S$ be a bigraded ideal and $R = S/J$. Then*

$$H(0, m; R)_{(*,j)} = 0 \textit{ for } j > \operatorname{reg}_y(R) + m.$$

Proof. We know that

$$H(0, m; R) \cong \operatorname{Tor}_{\bullet}^S(S/(\mathbf{y}), R) \cong H(S/(\mathbf{y}) \otimes_S \mathcal{F})$$

where $\mathcal{F}$ is the minimal bigraded free resolution of R over S. Let

$$F_i = \bigoplus S(-a, -b)^{\beta^S_{i,(a,b)}(R)}.$$

Then by the definition of the y-regularity we have $b \leq \operatorname{reg}_y(R) + m$ for all $\beta^S_{i,(a,b)}(R) \neq 0$. Thus $(S/(\mathbf{y}) \otimes_S F_i)_{(*,j)} = 0$ for $j > \operatorname{reg}_y(R) + m$. The assertion follows. □

We get the following exact sequences.

Corollary 5.4.6. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$.*

(i) *For $j > \operatorname{reg}_y(R(I)) + m$ we have the exact sequence*

$$0 \to I^{j-m}(-md) \to \bigoplus_m I^{j-m+1}(-(m-1)d) \to \dots \to \bigoplus_m I^{j-1}(-d) \to I^j \to 0.$$

(ii) *For $j > \operatorname{reg}_y(S(I)) + m$ we have the exact sequence*

$$0 \to S^{j-m}(I)(-md) \to \bigoplus_m S^{j-m+1}(I)(-(m-1)d) \to$$

$$\dots \to \bigoplus_m S^{j-1}(I)(-d) \to S^j(I) \to 0.$$

Proof. This statement follows from 5.4.5, the definition of the Koszul complex and the fact that $R(I)_{(*,j)}(-jd) \cong I^j$ or $S(I)_{(*,j)}(-jd) \cong S^j(I)$ respectively. □

Corollary 5.4.7. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Then:*

(i) *For $j \geq \max\{\operatorname{reg}_y(R(I)) + m, w(R(I)) + m\}$ we have*

$$\operatorname{reg}_{S_x}(I^{j+1}) = d + \operatorname{reg}_{S_x}(I^j).$$

(ii) *For* $j \geq \max\{\operatorname{reg}_y(S(I)) + m, w(S(I)) + m\}$ *we have*

$$\operatorname{reg}_{S_x}(S^{j+1}(I)) = d + \operatorname{reg}_{S_x}(S^j(I)).$$

Proof. We prove the corollary for $R(I)$. By 5.4.6 and by standard arguments (see 5.5.2 for the bigraded case) we get that for $j \geq \operatorname{reg}_y(R(I)) + m$

$$\operatorname{reg}_{S_x}(I^{j+1}) \leq \max\{\operatorname{reg}_{S_x}(I^{j+1-i}) + id - i + 1 \colon i \in [m]\}.$$

Since $j + 1 - i > w(R(I))$, it follows from 5.4.4 that

$$\operatorname{reg}_{S_x}(I^{j+1-i}) \leq \operatorname{reg}_{S_x}(I^{j+1-i+1}) - d \leq \ldots \leq \operatorname{reg}_{S_x}(I^{j+1}) - id.$$

Hence $\operatorname{reg}_{S_x}(I^{j+1}) = \operatorname{reg}_{S_x}(I^j) + d$. ☐

We now consider a special case where $\operatorname{reg}_{S_x}(I^j)$ can be computed precisely.

Proposition 5.4.8. *Let* $J \subset S$ *be a bigraded ideal such that* $R = S/J$ *is a complete intersection. Let* $\{z_1, \ldots, z_t\}$ *be a homogeneous minimal system of generators of* J *which is a regular sequence. Assume that* $\deg_x(z_t) \geq \ldots \geq \deg_x(z_1) > 0$ *and* $\deg_y(z_k) = 1$ *for all* $k \in [t]$*. Then for all* $j \geq t$

$$\operatorname{reg}_{S_x}(R_{(*,j+1)}) = \operatorname{reg}_{S_x}(R_{(*,j)}).$$

If in addition $\deg_x(z_k) = 1$ *for all* $k \in [t]$*, then for* $j \geq 1$

$$\operatorname{reg}_{S_x}(R_{(*,j)}) = 0.$$

Proof. The Koszul complex $\mathcal{K}(\mathbf{z})$ with respect to $\{z_1, \ldots, z_t\}$ provides a minimal bigraded free resolution of R because these elements form a regular sequence (see [**16**]). Observe that $(*, j)$ is an exact functor on complexes of bigraded modules. Note that $\mathcal{K}(\mathbf{z})_{(*,j)}$ is a complex of free S_x-modules because

$$K_i(\mathbf{z}) \cong \bigoplus_{\{j_1,\ldots,j_i\}\subseteq[t]} S(-\deg(z_{j_1}) - \ldots - \deg(z_{j_i}))$$

and

$$S(-a, -b)_{(*,j)} \cong \bigoplus_{|v|=j-b} S_x(-a)y^v \text{ as graded } S_x\text{-modules.}$$

$H_i(\mathcal{K}(\mathbf{z})_{(*,j)}) = 0$ for $i \geq 1$ and $H_0(\mathcal{K}(\mathbf{z})_{(*,j)}) = R_{(*,j)}$. Let ∂ be the differential of $\mathcal{K}(\mathbf{z})$. By the assumption that $\deg_x(z_k) > 0$ we get $\partial(K_{i+1}(\mathbf{z})) \subset (\mathbf{x})K_i(\mathbf{z})$ for all $i \in \mathbb{N}$. This implies that $\partial(K_{i+1}(\mathbf{z})_{(*,j)}) \subset (\mathbf{x})K_i(\mathbf{z})_{(*,j)}$. Hence $\mathcal{K}(\mathbf{z})_{(*,j)}$ is a minimal graded free resolution of $R_{(*,j)}$. Then we have for $j \geq t$

$$\operatorname{reg}_{S_x}(R_{(*,j)}) = \max\{\deg_x(z_t) + \ldots + \deg_x(z_{t-i+1}) - i \colon i \in [t]\}$$

and this is independent of j. If in addition $\deg_x(z_k) = 1$ for all $k \in [t]$, then we obtain

$$\operatorname{reg}_{S_x}(R_{(*,j)}) = 0$$

for $j \geq 1$. ☐

This proposition can be applied in the following situation. Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal, which is Cohen-Macaulay of codim 2. By the Hilbert-Burch theorem (see for example [**16**]) S_x/I has a minimal graded free resolution

$$0 \to \bigoplus_{i=1}^{m-1} S_x(-b_i) \xrightarrow{B} \bigoplus_{i=1}^{m} S_x(-a_i) \to S_x \to S_x/I \to 0$$

where $B = (b_{ij})$ is a $m \times (m-1)$-matrix with $b_{ij} \in \mathfrak{m}_x$ and we may assume that the ideal I is generated by the maximal minors of B. The matrix B is said to be the Hilbert-Burch matrix of I. If I is generated in degree d, then $S(I) = S/J$ where J is the bigraded ideal $(\sum_{i=1}^{m} b_{ij}y_i \colon j = 1, \dots, m-1)$.

Corollary 5.4.9. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$, which is Cohen-Macaulay of codim 2 with $m \times (m-1)$ Hilbert-Burch matrix $B = (b_{ij})$ and of linear type. Then for $j \geq m-1$:*

$$\operatorname{reg}_{S_x}(I^{j+1}) = \operatorname{reg}_{S_x}(I^j) + d.$$

If additionally $\deg_x(b_{ij}) = 1$ for $b_{ij} \neq 0$, then the equality holds for $j \geq 1$.

Proof. Since I is of linear type, we have $R(I) = S(I) = S/J$ where $J = (\sum_{i=1}^{m} b_{ij}y_i \colon j = 1, \dots, m-1)$. One knows that (Krull-) $\dim(R(I)) = n+1$ (see for example [**16**]). Since J is defined by $m-1$ equations, we conclude that $R(I)$ is a complete intersection. Now apply 5.4.8. $\square$

5.5. Bigraded Veronese algebras

The relationship between the regularity of a graded K-algebra and the regularity of its Veronese subrings were considered by several authors (see for example [**25**]). We study a more general situation. As in the sections before let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be the bigraded polynomial ring. Fix a bigraded ideal $J \subset S$ and set $R = S/J$.

Definition 5.5.1. Let $\tilde{\Delta} = (s,t) \neq (0,0)$ for $s,t \in \mathbb{N}$. We call

$$R_{\tilde{\Delta}} = \bigoplus_{(a,b)\in\mathbb{N}^2} R_{(as,bt)}$$

the *bigraded Veronese algebra of R* with respect to $\tilde{\Delta}$.

See [**20**] for similar constructions in the bigraded case. Note that $R_{\tilde{\Delta}}$ is again a bigraded K-algebra. We want to relate $\operatorname{reg}_{S_{\tilde{\Delta}},x}(R_{\tilde{\Delta}})$ and $\operatorname{reg}_{S_{\tilde{\Delta}},y}(R_{\tilde{\Delta}})$ to $\operatorname{reg}_x(R)$ and $\operatorname{reg}_y(R)$. We follow the way presented in [**20**] for the case of diagonals.

Lemma 5.5.2. *Let*

$$0 \to M_r \to \dots \to M_0 \to N \to 0$$

be an exact complex of finitely generated bigraded R-modules. Then

$$\operatorname{reg}_{R,x}(N) \leq \sup\{\operatorname{reg}_{R,x}(M_k) - k \colon 0 \leq k \leq r\}$$

and

$$\operatorname{reg}_{R,y}(N) \leq \sup\{\operatorname{reg}_{R,y}(M_k) - k \colon 0 \leq k \leq r\}.$$

Proof. We prove by induction on $r \in \mathbb{N}$ the inequality above for $\operatorname{reg}_{R,x}(N)$. The case $r = 0$ is trivial. Now let $r > 0$, and consider

$$0 \to N' \to M_0 \to N \to 0$$

where N' is the kernel of $M_0 \to N$. Then for every integer a we have the exact sequence

$$\ldots \to \operatorname{Tor}_i^R(M_0, K)_{(a+i,*)} \to \operatorname{Tor}_i^R(N, K)_{(a+i,*)} \to \operatorname{Tor}_{i-1}^R(N', K)_{(a+1+i-1,*)} \to \ldots$$

We get

$$\operatorname{reg}_{R,x}(N) \le \sup\{\operatorname{reg}_{R,x}(M_0),\ \operatorname{reg}_{R,x}(N') - 1\} \le \sup\{\operatorname{reg}_{R,x}(M_k) - k\colon 0 \le k \le r\}$$

where the last inequality follows from the induction hypothesis. Analogously we obtain the inequality for $\operatorname{reg}_{R,y}(N)$. □

Lemma 5.5.3. *Let A and B be graded K-algebras, $M \in \mathcal{M}_{\mathbb{Z}}(A)$ and $N \in \mathcal{M}_{\mathbb{Z}}(B)$. Then $M \otimes_K N \in \mathcal{M}_{\mathbb{Z}}(A \otimes_K B)$ with*

$$\operatorname{reg}_{A\otimes_K B,x}(M \otimes_K N) = \operatorname{reg}_A(M) \text{ and } \operatorname{reg}_{A\otimes_K B,y}(M \otimes_K N) = \operatorname{reg}_B(N).$$

Proof. Let $\mathcal{F}$ be the minimal graded free resolution of M over A and let $\mathcal{G}$ be the minimal graded free resolution of N over B. Then it is well-known that $H = \mathcal{F} \otimes_K \mathcal{G}$ is the minimal bigraded free resolution of $M \otimes_K N$ over $A \otimes_K B$ with $H_i = \bigoplus_{k+l=i} F_k \otimes G_l$. Since $A(-a) \otimes_K B(-b) = (A \otimes_K B)(-a, -b)$, the assertion follows. □

Theorem 5.5.4. *Let $\tilde{\Delta} = (s, t) \neq (0, 0)$ for $s, t \in \mathbb{N}$. Then*

$$\operatorname{reg}_{S_{\tilde{\Delta}},x}(R_{\tilde{\Delta}}) \le \max\{c\colon c = \lceil a/s \rceil - i, \beta^S_{i,(a,b)}(R) \neq 0 \textit{ for some } i, b \in \mathbb{N}\}$$

and

$$\operatorname{reg}_{S_{\tilde{\Delta}},y}(R_{\tilde{\Delta}}) \le \max\{c\colon c = \lceil b/t \rceil - i, \beta^S_{i,(a,b)}(R) \neq 0 \textit{ for some } i, a \in \mathbb{N}\}.$$

Proof. By symmetry it suffices to show the inequality for $\operatorname{reg}_{S_{\tilde{\Delta}},x}(R_{\tilde{\Delta}})$. Let

$$0 \to F_r \to \ldots \to F_0 \to R \to 0$$

be the minimal bigraded free resolution of R over S. Since $(\ )_{\tilde{\Delta}}$ is an exact functor, we obtain the exact complex of finitely generated $S_{\tilde{\Delta}}$-modules

$$0 \to (F_r)_{\tilde{\Delta}} \to \ldots \to (F_0)_{\tilde{\Delta}} \to R_{\tilde{\Delta}} \to 0.$$

By 5.5.2 we have

$$\operatorname{reg}_{S_{\tilde{\Delta}},x}(R_{\tilde{\Delta}}) \le \max\{\operatorname{reg}_{S_{\tilde{\Delta}},x}((F_i)_{\tilde{\Delta}}) - i\}.$$

Since

$$F_i = \bigoplus_{(a,b)\in\mathbb{N}^2} S(-a, -b)^{\beta^S_{i,(a,b)}(R)},$$

one has

$$\operatorname{reg}_{S_{\tilde{\Delta}},x}((F_i)_{\tilde{\Delta}}) = \max\{\operatorname{reg}_{S_{\tilde{\Delta}},x}(S(-a, -b)_{\tilde{\Delta}})\colon \beta^S_{i,(a,b)}(R) \neq 0\}.$$

We have to compute $\operatorname{reg}_{S_{\tilde{\Delta}},x}(S(-a, -b)_{\tilde{\Delta}})$. Let $M_0, \ldots, M_{s-1}$ be the relative Veronese modules of S_x and $N_0, \ldots, N_{t-1}$ the relative Veronese modules of S_y. That is

$M_j = \bigoplus_{k\in\mathbb{N}}(S_x)_{ks+j}$ for $j = 0, \ldots, s-1$ and $N_j = \bigoplus_{k\in\mathbb{N}}(S_y)_{kt+j}$ for $j = 0, \ldots, t-1$. Then

$$S(-a,-b)_{\tilde\Delta} = \bigoplus_{(k,l)\in\mathbb{N}^2} (S_x)_{ks-a} \otimes_K (S_y)_{lt-b} = M_i(-\lceil a/s\rceil) \otimes_K N_j(-\lceil b/t\rceil)$$

where $i \equiv -a \bmod s$ for $0 \le i \le s-1$ and $j \equiv -b \bmod t$ for $0 \le j \le t-1$.

By [**2**] the relative Veronese modules over a polynomial ring have a linear resolution over the Veronese algebra. Hence 5.5.3 yields

$$\operatorname{reg}_{S_{\tilde\Delta},x}(S(-a,-b)_{\tilde\Delta}) = \lceil a/s\rceil.$$

This concludes the proof. □

Corollary 5.5.5. *We have:*

(i) *For $s \gg 0$, $t \in \mathbb{N}$ and $\tilde\Delta = (s,t)$ one has* $\operatorname{reg}_{S_{\tilde\Delta},x}(R_{\tilde\Delta}) = 0$.

(ii) *For $t \gg 0$, $s \in \mathbb{N}$ and $\tilde\Delta = (s,t)$ one has* $\operatorname{reg}_{S_{\tilde\Delta},y}(R_{\tilde\Delta}) = 0$.

CHAPTER 6

Koszul cycles

In this chapter we compute the Koszul cycles of graded K-algebras defined by $\mathbf{a}$-stable ideals and bigraded K-algebras defined by bistable ideals. In both cases we obtain formulas for the Betti numbers.

6.1. Rings defined by bounded stable ideals

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring over a field K in n variables. In 1.7.12 we gave a formula for the graded Betti numbers of stable ideals. We extend this result to a larger class of ideals following the proof in [**4**] for the case of stable ideals.

We fix a vector $\mathbf{a} = (a_1, \dots, a_n)$ where $2 \le a_i \le \infty$. Let $I \subset S$ be a graded ideal. In this section $\mathcal{K}(k) = \mathcal{K}(x_n, \dots, x_{n-k+1}; S/I)$ and $H(k) = H(x_n, \dots, x_{n-k+1}; S/I)$ will always denote the Koszul complex and the Koszul homology of S/I with respect to the sequence $x_n, \dots, x_{n-k+1}$ (see 1.3 for details).

Definition 6.1.1. Let $I \subset S$ be a monomial ideal. I is said to be $\mathbf{a}$-*bounded* if for all $x^u \in G(I)$ and all $i \in [n]$ one has $u_i < a_i$. The ideal I is called $\mathbf{a}$-*stable* if, in addition for all $x^u \in G(I)$ and all $j \le m(u)$ with $u_j < a_j - 1$, we have $x_j x^u / x_{m(u)} \in I$.

It is easy to see that if I is $\mathbf{a}$-stable, then for all $x^u \in I$ and all $j \le m(u)$ with $u_j < a_j - 1$ we have $x_j x^u / x_{m(u)} \in I$.

Remark 6.1.2. Let $I \subset S$ be a monomial ideal with $G(I) = \{x^{u^1}, \dots, x^{u^t}\}$. Bruns and Herzog proved in [**17**, Thm. 3.1] that for all $i \ge 1$, $u \in \mathbb{N}^n$ and $0 \ne \beta^S_{i,u}(S/I)$ one has $x^u | \mathrm{lcm}(x^{u^1}, \dots, x^{u^t})$. Consequently, if I is $\mathbf{a}$-bounded, then we have $u \prec \mathbf{a}$ for these u.

Proposition 6.1.3. *Let $I \subset S$ be an $\mathbf{a}$-stable ideal. Then:*

(i) $H_0(n) \cong K$.

(ii) *For $i > 0$ a K-basis of $H_i(n)$ is given by the homology classes*

$$[x^u / x_{m(u)} e_L \wedge e_{m(u)}]$$

where $x^u \in G(I)$, $L \subseteq [n]$, $|L| = i - 1$, $m(L) < m(u)$ and $u_l < a_l - 1$ for $l \in L$.

Proof. Statement (i) is trivial, therefore assume that $i > 0$. By 6.1.2 we only have to compute $H_i(n)_v$ for $v \prec \mathbf{a}$.

In the proof we need the following fact: Let $x^u \in G(I)$, $L \subseteq \{n-k+1, \dots, n\}$, $|L| = i - 1$, $\max\{n-k, m(L)\} < m(u)$ and $u_l < a_l - 1$ for $l \in L$. It follows that

$x_l x^u / x_{m(u)} \in I$ for $l \in L$ because I is $\mathbf{a}$-stable. Hence we have

$$\partial((x^u/x_{m(u)})e_L \wedge e_{m(u)}) = \sum_{l \in L} (-1)^{\alpha(l,L)} (x_l x^u / x_{m(u)}) e_{L-\{l\}} \wedge e_{m(u)} + (-1)^{i-1} x^u e_L = 0.$$

This means that all elements $(x^u/x_{m(u)})e_L \wedge e_{m(u)}$ are cycles in $K_i(k)$.

We prove by induction on $k \in [n]$ that, for all $v \prec \mathbf{a}$ and all integers $i > 0$, the K-vector space $H_i(k)_v$ has a K-basis given by the homology classes

$$[x^u/x_{m(u)} e_L \wedge e_{m(u)}]$$

satisfying

$$(*) \quad x^u \in G(I),\ L \subseteq \{n-k+1, \dots, n\},\ |L| = i-1,\ \max\{n-k, m(L)\} < m(u),$$
$$u_l < a_l - 1 \text{ for } l \in L \text{ and } u + \sum_{l \in L} \varepsilon_l = v.$$

The case $k = n$ gives the proposition.

Let $k = 1$. We have the following exact sequence

$$0 \to H_1(1)_v \to (S/I)_{v-\varepsilon_n} \xrightarrow{x_n} (S/I)_v \to (S/(x_n + I))_v \to 0.$$

We claim that the kernel of the multiplication map with x_n has a K-basis in degree $\tilde{v} = v - \varepsilon_n$ consisting of the elements

$$(**) \quad [(x^u/x_{m(u)})] \text{ with } x^u \in G(I),\ m(u) = n \text{ and } u = v.$$

We may choose $[(x^u/x_{m(u)})e_n]$ as a preimage in $H_1(1)_v$ and the assertion follows for $k = 1$. It remains to prove the claim $(**)$. Observe that every element satisfying $(**)$ is in the kernel. On the other hand let $x^{\tilde{v}} \notin I$ with $x_n x^{\tilde{v}} \in I$. There exists $x^u \in G(I)$ such that $x_n x^{\tilde{v}} = x^u x^{u'}$ for some monomial $x^{u'}$. It follows that $\max\{m(u), m(u')\} = n$. If $m(u') = n$, then $x^{\tilde{v}} = x^u (x^{u'}/x_n) \in I$, which is a contradiction. Thus $m(u) = n$. Furthermore, we have $x^{u'} = 1$ because otherwise $x^{\tilde{v}} = (x^u/x_n)x^{u'} \in I$ since $\tilde{v} \preceq v \prec \mathbf{a}$ and I is $\mathbf{a}$-stable. This is again a contradiction. Hence $x^{\tilde{v}} = x^u/x_n$ satisfies $(**)$.

Let $k > 1$. By the induction hypothesis we have for all considered elements in $H_i(k-1)$ satisfying $(*)$ that $m(u) > n-k+1$. This implies that all the maps

$$H_i(k-1)_{v-\varepsilon_{n-k+1}} \xrightarrow{x_{n-k+1}} H_i(k-1)_v$$

are zero maps for $i \geq 1$ because $v \prec \mathbf{a}$ and I is $\mathbf{a}$-stable.

For $i = 1$ we obtain the exact sequence

$$0 \to H_1(k-1)_v \to H_1(k)_v \to (S/(x_{n-k+2}, \dots, x_n) + I)_{v-\varepsilon_{n-k+1}}$$
$$\xrightarrow{x_{n-k+1}} (S/(x_{n-k+2}, \dots, x_n) + I)_v \to (S/(x_{n-k+1}, \dots, x_n) + I)_v \to 0.$$

By the induction hypothesis the K-vector space $H_1(k-1)_v$ has a K-basis given by the elements

$$[x^u/x_{m(u)} e_{m(u)}] \text{ such that } x^u \in G(I),\ n-k+1 < m(u) \text{ and } u = v.$$

Similarly as in the case $k = 1$ the kernel of the multiplication map with x_{n-k+1} has a K-basis consisting of

$$0 \neq [x^u/x_{m(u)}] \in (S/(x_{n-k+2}, \dots, x_n) + I)_{v-\varepsilon_{n-k+1}}$$

satisfying $x^u \in G(I)$, $m(u) = n - k + 1$ and $u = v$. We may choose

$$[x^u/x_{m(u)} e_{n-k+1}]$$

as a preimage in $H_1(k)_v$ of this element. Thus we get the desired K-basis for $H_1(k)_v$.

For $i > 1$ the long exact sequence of the Koszul homology splits into short exact sequences of the following form:

$$0 \to H_i(k-1)_v \to H_i(k)_v \to H_{i-1}(k-1)_{v-\varepsilon_{n-k+1}} \to 0.$$

We may apply the induction hypothesis to $H_i(k-1)_v$ and $H_{i-1}(k-1)_{v-\varepsilon_{n-k+1}}$. A K-basis of $H_i(k-1)_v$ is given by the homology classes

$$[x^u/x_{m(u)} e_L \wedge e_{m(u)}]$$

such that $x^u \in G(I)$, $L \subseteq \{n-k+2, \dots, n\}$, $|L| = i-1$, $\max\{n-k+1, m(L)\} < m(u)$, $u_l < a_l - 1$ for $l \in L$ and $u + \sum_{l \in L} \varepsilon_l = v$. Similarly the K-vector space $H_{i-1}(k-1)_{v-\varepsilon_{n-k+1}}$ has a K-basis

$$[x^u/x_{m(u)} e_L \wedge e_{m(u)}]$$

satisfying $x^u \in G(I)$, $L \subseteq \{n-k+2, \dots, n\}$, $|L| = i-2$, $\max\{n-k+1, m(L)\} < m(u)$, $u_l < a_l - 1$ for $l \in L$ and $u + \sum_{l \in L} \varepsilon_l = v - \varepsilon_{n-k+1}$. Observe that $u_{n-k+1} < a_{n-k+1} - 1$ because $v \prec \mathbf{a}$. We may choose

$$[x^u/x_{m(u)} e_{n-k+1} \wedge e_L \wedge e_{m(u)}]$$

as a preimage in $H_i(k)_v$ of $[x^u/x_{m(u)} e_L \wedge e_{m(u)}]$. The union of all given homology classes in $H_i(k)_v$ is the desired K-basis of $H_i(k)_v$. This concludes the proof. $\square$

Let $a, b \in \mathbb{Z}$. We make the convention that $\binom{a}{0} = 1$ and $\binom{a}{b} = 0$ for $a < 0$, $b \neq 0$. Furthermore, $\binom{a}{b} = 0$ for $0 \leq a < b$ or $b < 0 \leq a$ respectively. If $x^u \in S$ with $u \prec \mathbf{a}$, then we define

$$l(u) = |\{i \colon u_i = a_i - 1,\ i < m(u)\}|.$$

The following corollary was independently discovered by Gasharov, Hibi and Peeva [**29**].

Corollary 6.1.4. *Let $I \subset S$ be an $\mathbf{a}$-stable ideal and $i, j \in \mathbb{N}$. One has, independent of the characteristic of K,*

$$\beta^S_{i,i+j}(I) = \sum_{x^u \in G(I),\ |u|=j} \binom{m(u) - 1 - l(u)}{i}.$$

As a consequence we are able to determine the regularity for $\mathbf{a}$-stable ideals.

Corollary 6.1.5. *Let $I \subset S$ be an $\mathbf{a}$-stable ideal. Then*

$$\operatorname{reg}_S(I) = \max\{|u| \colon x^u \in G(I)\}.$$

In particular, if I is generated in degree d, then I has a d-linear resolution.

Example 6.1.6. Let $S = K[x_1, \dots, x_4]$ and $I = (x_2x_4, x_2x_3, x_1x_2, x_1^2)$. We see that I is not stable because $x_2x_3 \in I$ and $x_2^2 \notin I$. Let $\mathbf{a} = (3, 2, 3, 2)$. Then I is $\mathbf{a}$-stable. By 6.1.4 we have $\beta^S_{i,i+j}(I) = 0$ for all $i, j \in \mathbb{N}$ with $j \neq 2$. For $j = 2$ we get

$$\beta^S_{i,i+2}(I) = \binom{2}{i} + 2\binom{1}{i} + \binom{0}{i}.$$

Hence the Betti diagram of I is:

I	0	1	2
1	-	-	-
2	4	4	1
3	-	-	-

6.2. Rings defined by bistable ideals

In this section we always denote the standard bigraded polynomial ring with $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$. Recall that we regard the (bigraded) polynomial rings $S_x = K[x_1, \dots, x_n]$ and $S_y = K[y_1, \dots, y_m]$ as subrings of S. In contrast to the other chapters we set $\mathfrak{m}_x = (x_1, \dots, x_n) \subset S$, $\mathfrak{m}_y = (y_1, \dots, y_m) \subset S$ and $\mathfrak{m} = \mathfrak{m}_x + \mathfrak{m}_y \subset S$. Let $J \subset S$ be a bistable ideal. We want to compute the Koszul cycles of $R = S/J$ with respect to $x_n, \dots, x_1, y_m, \dots, y_1$. For $k \in [n]$ and $l \in [m]$ let

$$\mathcal{K}(k, l) = \mathcal{K}(x_n, \dots, x_{n-k+1}, y_m, \dots, y_{m-l+1}; R)$$

and

$$H(k, l) = H(x_n, \dots, x_{n-k+1}, y_m, \dots, y_{m-l+1}; R)$$

be the Koszul complex and the Koszul homology of R with respect to the sequence $x_n, \dots, x_{n-k+1}, y_m, \dots, y_{m-l+1}$ (see 1.3 for details). Here for all integers i the modules $K_i(k, l)$ and $H_i(k, l)$ are $\mathbb{N}^n \times \mathbb{N}^m$-graded. We need the following property of bistable ideals.

Lemma 6.2.1. *Let $J \subset S$ be a bistable ideal and $x^u y^v \in J$.*

(i) *If $(x^u/x_{m(u)})y^v \notin J$, then $x^u y^v \notin \mathfrak{m}_x J$.*
(ii) *If $x^u(y^v/y_{m(v)}) \notin J$, then $x^u y^v \notin \mathfrak{m}_y J$.*

Proof. By symmetry it suffices to prove (i). Without loss of generality $|u| > 0$. Assume that $x^u y^v \in \mathfrak{m}_x J$. Then

$$x^u y^v = x^q y^r x^{q'} y^{r'} \text{ where } |q| > 0,\ x^{q'} y^{r'} \in G(J).$$

It follows that $\max\{m(q), m(q')\} = m(u)$. If $m(q) = m(u)$, then

$$(x^u/x_{m(u)})y^v = (x^q/x_{m(u)})y^r x^{q'} y^{r'} \in J$$

and this is a contradiction to the assumption that $(x^u/x_{m(u)})y^v \notin J$. If $m(q') = m(u)$, then

$$(x^u/x_{m(u)})y^v = x^q y^r (x^{q'}/x_{m(u)})y^{r'} \in J$$

because $|q| > 0$ and J is bistable. This is again a contradiction. Thus $x^u y^v \notin \mathfrak{m}_x J$. □

First we consider the Koszul homology with respect to $x_n, \dots, x_1$. Let $k \in [n]$ and $i \in [k]$. Define $\mathcal{N}(i,k)$ as the set consisting of the elements:

$$(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \in K_i(k,0)$$

with $|L| = i-1$, $\max\{m(L), n-k\} < m(u)$, $L \subseteq \{n-k+1, \dots, n\}$, $x^u y^v \in J \backslash \mathfrak{m}_x J$. Finally, let $\tilde{\mathcal{N}}(i,n) \subseteq \mathcal{N}(i,n)$ be the subset of elements where in addition $x^u y^v \in G(J)$.

Lemma 6.2.2. *Let $J \subset S$ be a bistable ideal and $R = S/J$. We have:*

(i) $H_0(n,0) = S/(\mathfrak{m}_x + J)$.
(ii) *Let $i \in [n]$. Then all elements of $\mathcal{N}(i,n)$ are cycles in $K_i(n,0)$.*
(iii) *Let $i \in [n]$, $s \in \mathbb{N}^n$ and $t \in \mathbb{N}^m$. Then $H_i(n,0)_{(s,t)}$ has a K-basis consisting of the homology classes*

$$[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]$$

where $(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \in \mathcal{N}(i,n)$ with $u + \sum_{j \in L} \varepsilon_j = s$ and $v = t$.
(iv) *The homology classes of the elements in $\tilde{\mathcal{N}}(i,n)$ form a minimal system of generators of the S_y-module $H_i(n,0)$.*

Proof. Statement (i) is trivial. It remains to prove (ii), (iii) and (iv). We show:

(1) Let $k \in [n]$ and $i \in [k]$. Then every element of $\mathcal{N}(i,k)$ is a cycle in $K_i(k,0)$.
(2) We prove by induction on $k \in [n]$, that for all $i \in [k]$, all $s \in \mathbb{N}^n$ and all $t \in \mathbb{N}^m$, $H_i(k,0)_{(s,t)}$ has a K-basis consisting of the homology classes

$$[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]$$

where $(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \in \mathcal{N}(i,k)$ with $u + \sum_{j \in L} \varepsilon_j = s$ and $v = t$.
(3) The assertion of (iv).

Then (ii) and (iii) follow from (1) and (2) for $k = n$. Finally, (3) gives (iv).

Proof of (1): Let $(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \in \mathcal{N}(i,k)$. For $l \leq m(u)$ we have $(x_l x^u/x_{m(u)})y^v \in J$ because J is bistable. Since $m(L) < m(u)$, one has

$$\partial((x^u/x_{m(u)})y^v e_L \wedge e_{m(u)})$$

$$= \sum_{l \in L} (-1)^{\alpha(l,L)} x_l (x^u/x_{m(u)}) y^v e_{L-\{l\}} \wedge e_{m(u)} + (-1)^{i-1} x_{m(u)} (x^u/x_{m(u)}) y^v e_L = 0.$$

It follows that $(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}$ is a cycle.

Proof of (2): For $k = 1$ we have the exact sequence

$$0 \to H_1(1,0)_{(s,t)} \to (S/J)_{(s-\varepsilon_n,t)} \xrightarrow{x_n} (S/J)_{(s,t)} \to (S/(x_n + J))_{(s,t)} \to 0.$$

Suppose that $[x^{u'}y^{v'}] \neq 0$ in $(S/J)_{(s-\varepsilon_n,t)}$ and $x^{u'}y^{v'}x_n = x^u y^v \in J$. It follows that $m(u) = n$ and $x^{u'}y^{v'} = (x^u/x_n)y^v \notin J$. By 6.2.1 one has $x^u y^v \notin \mathfrak{m}_x J$. Thus $x^u y^v \in J \backslash \mathfrak{m}_x J$. By (1) we may choose the homology class of $(x^u/x_n)y^v e_n \in \mathcal{N}(1,1)$ as a preimage in $H_1(1,0)_{(s,t)}$ of $[x^{u'}y^{v'}]$. Hence we proved the case $k = 1$.

Let $k > 1$. By the induction hypothesis we have for all homology classes

$$[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}] \in H_i(k-1,0)$$

where $(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \in \mathcal{N}(i,k-1)$, that $m(u) > n-k+1$. This implies that all the maps

$$H_i(k-1,0)_{(s-\varepsilon_{n-k+1},t)} \stackrel{x_{n-k+1}}{\to} H_i(k-1,0)_{(s,t)}$$

are zero maps for $i \geq 1$ because $x_{n-k+1}(x^u/x_{m(u)})y^v \in J$ since J is bistable.

Hence for $i=1$ we have the exact sequence

$$0 \to H_1(k-1,0)_{(s,t)} \to H_1(k,0)_{(s,t)} \to (S/(x_{n-k+2},\dots,x_n)+J)_{(s-\varepsilon_{n-k+1},t)}$$

$$\stackrel{x_{n-k+1}}{\to} (S/(x_{n-k+2},\dots,x_n)+J)_{(s,t)} \to (S/(x_{n-k+1},\dots,x_n)+J)_{(s,t)} \to 0.$$

By the induction hypothesis a K-basis of $H_1(k-1,0)_{(s,t)}$ is given by the elements

$$[(x^u/x_{m(u)})y^v e_{m(u)}]$$

where $(x^u/x_{m(u)})y^v e_{m(u)} \in \mathcal{N}(1,k-1)$ with $u=s$ and $v=t$. Similarly as in the case $k=1$, the kernel of the multiplication map with x_{n-k+1} has a K-basis in degree $(s-\varepsilon_{n-k+1},t)$ consisting of the elements

$$0 \neq [(x^u/x_{m(u)})y^v] \in (S/(x_{n-k+2},\dots,x_n)+J)_{(s-\varepsilon_{n-k+1},t)}$$

with $x^u y^v \in J \backslash \mathfrak{m}_x J$, $m(u)=n-k+1$, $u=s$ and $v=t$. By (1) we may choose $[(x^u/x_{m(u)})y^v e_{n-k+1}]$ as preimages in $H_1(k,0)_{(s,t)}$ of the last elements. Combining all these homology classes, we get the desired K-basis for $H_1(k,0)_{(s,t)}$.

For $i>1$ the long exact sequence of the Koszul homology splits into short exact sequences of the form:

$$0 \to H_i(k-1,0)_{(s,t)} \to H_i(k,0)_{(s,t)} \to H_{i-1}(k-1,0)_{(s-\varepsilon_{n-k+1},t)} \to 0.$$

We may apply the induction hypothesis to the Koszul homology $H_i(k-1,0)_{(s,t)}$ and $H_{i-1}(k-1,0)_{(s-\varepsilon_{n-k+1},t)}$. A K-basis of $H_i(k-1,0)_{(s,t)}$ is given by the homology classes

$$[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]$$

such that $(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \in \mathcal{N}(i,k-1)$ with $u+\sum_{j\in L}\varepsilon_j = s$ and $v=t$. A K-basis of $H_{i-1}(k-1,0)_{(s-\varepsilon_{n-k+1},t)}$ consists of the elements

$$[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]$$

where $(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \in \mathcal{N}(i-1,k-1)$ with $u+\sum_{j\in L}\varepsilon_j = s-\varepsilon_{n-k+1}$ and $v=t$. By (1) we may choose the elements

$$[(x^u/x_{m(u)})y^v e_{n-k+1} \wedge e_L \wedge e_{m(u)}]$$

as preimages in $H_i(k,0)_{(s,t)}$ of the K-basis of $H_{i-1}(k-1,0)_{(s-\varepsilon_{n-k+1},t)}$. All these homology classes together form the desired K-basis of $H_i(k,0)_{(s,t)}$. This implies (2) for $i>1$.

Proof of (3): First observe the following. Let $w \in \mathfrak{m}_y H_i(n,0)$ be an $\mathbb{N}^n \times \mathbb{N}^m$-homogeneous element. By (iii) we may write

$$w = \sum_{(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \in \mathcal{N}(i,n),\ j\in[m]} \lambda_{u,v,L,j}\, y_j [(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]$$

for some $\lambda_{u,v,L,j} \in K$. If $x^u y_j y^v \in J\backslash \mathfrak{m}_x J$, then $(x^u/x_{m(u)})y_j y^v e_L \wedge e_{m(u)}$ is again an element of $\mathcal{N}(i,n)$.

Otherwise we have $x^u y_j y^v \in \mathfrak{m}_x J$. By 6.2.1 we get that $x^u/x_{m(u)} y_j y^v \in J$. Then $(x^u/x_{m(u)}) y_j y^v e_L \wedge e_{m(u)} = 0$ in $H_i(n,0)$ because the coefficients of this Koszul complex are taken in R.

Hence we get an expression

$$w = \sum_{(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)} \in \mathcal{N}(i,n),\ x^u y^v \notin G(J)} \lambda_{u,v,L} [(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)}]$$

for some $\lambda_{u,v,L} \in K$. We will use this fact later in the proof.

It follows from (iii) that all homology classes of elements of $\tilde{\mathcal{N}}(i,n)$ form a system of generators of the S_y-module $H_i(n,0)$. It remains to show that the residue classes are K-linearly independent in $H_i(n,0)/\mathfrak{m}_y H_i(n,0)$.

Assume that

$$\sum_{(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)} \in \tilde{\mathcal{N}}(i,n)} \lambda_{u,v,L} [(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)}] \in \mathfrak{m}_y H_i(n,0)$$

where $\lambda_{u,v,L} \in K$. Without loss of generality all considered homology classes have the same $\mathbb{N}^n \times \mathbb{N}^m$-degree. By the preceding remark we get

$$\sum_{(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)} \in \tilde{\mathcal{N}}(i,n)} \lambda_{u,v,L} [(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)}]$$

$$= \sum_{(x^{\tilde{u}}/x_{m(\tilde{u})}) y^{\tilde{v}} e_{\tilde{L}} \wedge e_{m(\tilde{u})} \in \mathcal{N}(i,n),\ x^{\tilde{u}} y^{\tilde{v}} \notin G(J)} \lambda_{\tilde{u},\tilde{v},\tilde{L}} [(x^{\tilde{u}}/x_{m(\tilde{u})}) y^{\tilde{v}} e_{\tilde{L}} \wedge e_{m(\tilde{u})}]$$

for $\lambda_{\tilde{u},\tilde{v},\tilde{L}} \in K$. Since all elements in the second sum satisfy $x^{\tilde{u}} y^{\tilde{v}} \notin G(J)$, they are different from the homology classes of the first sum. It follows from (iii) that

$$\sum_{(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)} \in \tilde{\mathcal{N}}(i,n)} \lambda_{u,v,L} [(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)}] = 0 \text{ in } H_i(n,0).$$

Then again by (iii) all $\lambda_{u,v,L} = 0$. The assertion follows. □

Observe that each $H_i(n,0)$ is a finitely generated $\mathbb{N}^n \times \mathbb{N}^m$-graded S_y-module. Let

$$K_{i,j}(k) = K_j(y_m, \dots, y_{m-k+1}; H_i(n,0))$$

and

$$H_{i,j}(k) = H_j(y_m, \dots, y_{m-k+1}; H_i(n,0))$$

be the j^{th}-module of the Koszul complex and the j^{th}-Koszul homology of $H_i(n,0)$ with respect to $y_m, \dots, y_{m-k+1}$. We set $H_{i,j}(k) = 0$ for $i < 0$. Note that $H_{i,0}(k-1) = H_i(n,0)/(y_{m-k+2}, \dots, y_m) H_i(n,0)$.

Furthermore, we define for $k \in [m]$

$$\tilde{H}_{i,0}(k-1) = (0 :_{H_{i,0}(k-1)} y_{m-k+1}) \subseteq H_{i,0}(k-1).$$

For $c \in H_i(n,0)$ we write $[c]$ for the homology class in $H_{i,0}(k-1)$. For example, if by 6.2.2 we have $[(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)}] \in H_i(n,0)$, then $[[(x^u/x_{m(u)}) y^v e_L \wedge e_{m(u)}]]$ is the corresponding homology class in $H_{i,0}(k-1)$. In the next proposition we describe the modules $\tilde{H}_{i,0}(k-1)$ explicitly.

Proposition 6.2.3. *Let $J \subset S$ be a bistable ideal, $R = S/J$ and $i \in [n]$:*

(i) *For $k \in [m]$ we have $\mathfrak{m}\tilde{H}_{i,0}(k-1) = 0$.*

(ii) *For $k \in [m]$ a K-basis of $\tilde{H}_{i,0}(k-1)$ is given by the homology classes:*

(a) $[[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]]$ *where* $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u y^v \in G(J)$, $m-k+1 < m(v)$,

(b) $[[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]]$ *where* $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $m-k+1 = m(v)$, $x^u(y^v/y_{m(v)}) \in J \backslash \mathfrak{m}_x J$, $(x^u/x_{m(u)})y^v \in J \backslash \mathfrak{m}_y J$.

Proof. Fix $i \in [n]$ and $k \in [m]$. To prove this proposition, we show that the elements of type (a) and (b)

(1) belong to the socle of $H_{i,0}(k-1)$,

(2) are K-linearly independent,

(3) form a system of generators of $\tilde{H}_{i,0}(k-1)$.

From (1) it follows that the elements of type (a) and (b) lie in $\tilde{H}_{i,0}(k-1)$. Then (1), (2) and (3) imply (i) and (ii).

Proof of (1): Let $[c] \in H_{i,0}(k-1)$ with $c \in H_i(n,0)$ is of type (a) or (b). We have to show that $\mathfrak{m}_x[c] = 0$ and $\mathfrak{m}_y[c] = 0$. Since $\mathfrak{m}_x H_{i,0}(k-1) = 0$ and $y_l H_{i,0}(k-1) = 0$ for $l \geq m-k+2$, it remains to prove that for $l \leq m-k+1$ one has $y_l[c] = 0$.

Fix $l \leq m-k+1$. Let

$$c = [(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]$$

where $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u y^v \in G(J)$, $m-k+1 < m(v)$. This means $[c]$ is of type (a). We have to show that

$$y_l c \in (y_{m-k+2}, \dots, y_m)H_i(n,0).$$

If $(x^u/x_{m(u)})(y_l y^v/y_{m(v)}) \in J$, then

$$y_l c = y_{m(v)}[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}] = 0 \text{ in } H_i(n,0).$$

Otherwise one has $(x^u/x_{m(u)})(y_l y^v/y_{m(v)}) \notin J$. But $x^u(y_l y^v/y_{m(v)}) \in J$ because $x^u y^v \in J$ and J is bistable. Thus by 6.2.1 the element $x^u(y_l y^v/y_{m(v)}) \notin \mathfrak{m}_x J$. It follows that

$$[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}]$$

is of the form as described in 6.2.2. Hence

$$y_l c = y_{m(v)}[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}] \in (y_{m-k+2}, \dots, y_m)H_i(n,0)$$

because $m(v) > m-k+1$.

Now consider an element $[c]$ of type (b) with

$$c = [(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]$$

where

$$L \subseteq [n],\ |L| = i-1,\ m(L) < m(u),\ m-k+1 = m(v),\ x^u(y^v/y_{m(v)}) \in J \backslash \mathfrak{m}_x J,$$

$$(x^u/x_{m(u)})y^v \in J \backslash \mathfrak{m}_y J.$$

Since J is bistable, we have $(x^u/x_{m(u)})(y_l y^v/y_{m(v)}) \in J$. Therefore

$$y_l c = 0 \text{ in } H_i(n,0).$$

Proof of (2): Assume that

$$0 = \sum_{u,v,L} \lambda^{(a)}_{u,v,L}[[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]] + \sum_{u,v,L} \lambda^{(b)}_{u,v,L}[[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]]$$

where $\lambda^{(a)}_{u,v,L}, \lambda^{(b)}_{u,v,L} \in K$, the first sum is taken over elements of type (a) and the second sum of those elements of type (b). We may assume that all terms have the same $\mathbb{N}^n \times \mathbb{N}^m$-degree. It follows that

$$(*) \quad \sum_{u,v,L} \lambda^{(a)}_{u,v,L}[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}] + \sum_{u,v,L} \lambda^{(b)}_{u,v,L}[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]$$

$$\in (y_{m-k+2}, \dots, y_m)H_i(n,0).$$

Observe that by 6.2.2 the non-zero terms in the summation can not cancel each other. If the sum is zero in $H_i(n,0)$, then we get that all $\lambda^{(a)}_{u,v,L} = \lambda^{(b)}_{u,v,L} = 0$. Assume that the sum is not zero. For all homology classes of type (b) we have

$$\operatorname{supp}(\deg_y([(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}])) \subseteq \{1, \dots, m-k+1\}.$$

Thus all $\lambda^{(b)}_{u,v,L} = 0$ because for elements $0 \neq c \in (y_{m-k+2}, \dots, y_m)H_i(n,0)$ one has $\operatorname{supp}(\deg_y(c)) \cap \{m-k+2, \dots, m\} \neq \emptyset$. By 6.2.2 all elements of type (a) are part of a minimal system of generators of $H_i(n,0)$. We get that the K-linear combination

$$0 \neq \sum_{u,v,L} \lambda^{(a)}_{u,v,L}[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}] \notin \mathfrak{m}_y H_i(n,0).$$

This is a contradiction to $(*)$. Therefore the sum is zero and also all $\lambda^{(a)}_{u,v,L} = 0$.

Proof of (3): Let $[c] \in \tilde{H}_{i,0}(k-1)$ be an arbitrary $\mathbb{N}^n \times \mathbb{N}^m$-homogeneous element with $c \in H_i(n,0)$ and $y_{m-k+1}c \in (y_{m-k+2}, \dots, y_m)H_i(n,0)$. We may write

$$c = \sum_{u,v,L} \lambda_{u,v,L}[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]$$

where $0 \neq \lambda_{u,v,L} \in K$ and the summation is taken over the K-basis elements in 6.2.2(iii), which are of degree $\deg(c)$. We claim that for all v with $m(v) \geq m-k+2$, we may assume that $x^u y^v \in G(J)$. Otherwise let $x^u y^v \notin G(J)$. By 6.2.2 one has $x^u y^v \in J \backslash \mathfrak{m}_x J$. We may write $x^u y^v = x^u y^{v'} y^{v''}$ where $v', v'' \in \mathbb{N}^m$, $|v''| > 0$, $x^u y^{v'} \in J$. If $m(v'') \geq m(v')$, then $m(v'') > m-k+1$ and therefore

$$y_{m(v'')}[(x^u/x_{m(u)})y^{v'}(y^{v''}/y_{m(v'')})e_L \wedge e_{m(u)}] \in (y_{m-k+2}, \dots, y_m)H_i(n,0),$$

which is zero in $H_i(n,0)/(y_{m-k+2}, \dots, y_m)H_i(n,0)$. We subtract this term from c and get an element c' with $[c'] = [c]$ in $H_{i,0}(k-1)$. Hence without loss of generality we may replace c by c'. If $m(v'') < m(v')$, then $m(v') > m-k+1$. Since J is bistable, we have

$$x^u(y^{v'}/y_{m(v')})y^{v''} \in J.$$

It follows that

$$y_{m(v')}[(x^u/x_{m(u)})(y^{v'}/y_{m(v')})y^{v''}e_L \wedge e_{m(u)}] \in (y_{m-k+2},\dots,y_m)H_i(n,0),$$

which is zero in $H_i(n,0)/(y_{m-k+2},\dots,y_m)H_i(n,0)$. We also subtract this term from c.

Since the elements $[[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]]$ with $m(v) \geq m-k+2$ and $x^u y^v \in G(J)$ are of type (a), we may assume that

$$c = \sum_{u,v,L} \lambda_{u,v,L}[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}] \in H_i(n,0)$$

with $m(v) \leq m-k+1$ for all v. One has

$$\sum_{u,v,L} \lambda_{u,v,L}[(x^u/x_{m(u)})y_{m-k+1}y^v e_L \wedge e_{m(u)}] = y_{m-k+1}c \in (y_{m-k+2},\dots,y_m)H_i(n,0).$$

Since

$$\operatorname{supp}(\deg_y([(x^u/x_{m(u)})y_{m-k+1}y^v e_L \wedge e_{m(u)}])) \subseteq \{1,\dots,m-k+1\},$$

we get

$$(**) \quad \sum_{u,v,L} \lambda_{u,v,L}[(x^u/x_{m(u)})y_{m-k+1}y^v e_L \wedge e_{m(u)}] = 0.$$

Assume that there exist u,v such that $(x^u/x_{m(u)})y_{m-k+1}y^v \notin J$. By 6.2.1 it follows that $x^u y_{m-k+1}y^v \in J \backslash \mathfrak{m}_x J$. Therefore

$$[(x^u/x_{m(u)})y_{m-k+1}y^v e_L \wedge e_{m(u)}]$$

belongs to the K-basis given in 6.2.2. Then $(**)$ leads to a contradiction. Thus we obtain $(x^u/x_{m(u)})y_{m-k+1}y^v \in J$ for all u,v.

Without loss of generality we may therefore assume that

$$c = [(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}] = [(x^u/x_{m(u)})(y_{m-k+1}y^v/y_{m-k+1})e_L \wedge e_{m(u)}]$$

where $x^u y^v \in J\backslash \mathfrak{m}_x J$, $(x^u/x_{m(u)})y_{m-k+1}y^v \in J$ and $m(v) \leq m-k+1$.

We have $m(v+\varepsilon_{m-k+1}) = m-k+1$. By 6.2.1 we get $(x^u/x_{m(u)})y_{m-k+1}y^v \in J\backslash \mathfrak{m}_y J$ because $(x^u/x_{m(u)})y^v \notin J$. Hence $[c]$ is of type (b). This concludes the proof. $\square$

In the sequel we need the following observation.

Remark 6.2.4. Let $J \subset S$ be a bistable ideal and $R = S/J$. Let $x^u y^v \in J\backslash \mathfrak{m}_x J$, $L \subseteq [n]$, $m(L) < m(u)$, $|L| = i-1$ and $l \leq m(v)$. It follows that $x^u(y_l y^v/y_{m(v)}) \in J$ because J is bistable. Either we have that $x^u(y_l y^v/y_{m(v)}) \in J\backslash \mathfrak{m}_x J$ and then $(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)} \in \mathcal{N}(i,n)$. In particular this element is a cycle in $K_i(n,0)$ and we may consider the corresponding homology class in $H_i(n,0)$. Otherwise $x^u(y_l y^v/y_{m(v)}) \in \mathfrak{m}_x J$. It follows from 6.2.1 that $(x^u/x_{m(u)})(y_l y^v/y_{m(v)}) \in J$. Then $(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)} = 0$ in $K_i(n,0)$ because the coefficients in this Koszul complex are taken in R. Again we may write $[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}] = 0$ to be the corresponding homology class in $H_i(n,0)$.

Let $i \in [n]$, $k \in [m]$ and $j \in [k]$. Define $\mathcal{M}_{(a)}(i,j,k)$ as the set of elements:

$$[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]f_F +$$
$$(-1)^j \sum_{l \in F} (-1)^{\alpha(l,F)}[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{F-\{l\}} \wedge f_{m(v)} \in K_{i,j}(k)$$

where $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u y^v \in G(J)$, $\max\{m-k+1, m(F)\} < m(v)$, $|F| = j$ and $F \subseteq \{m-k+1, \dots, m\}$.

Let $\mathcal{M}_{(b)}(i,j,k)$ be the set of elements:

$$[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_F \wedge f_{m(v)} \in K_{i,j}(k)$$

where $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u(y^v/y_{m(v)}) \in J \backslash \mathfrak{m}_x J$, $(x^u/x_{m(u)})y^v \in J \backslash \mathfrak{m}_y J$, $\max\{m-k, m(F)\} < m(v)$, $|F| = j-1$ and $F \subseteq \{m-k+1, \dots, m\}$.

Lemma 6.2.5. *Let $J \subset S$ be a bistable ideal, $R = S/J$, $i \in [n]$, $k \in [m]$ and $j \in [k]$.*

(i) *All elements of $\mathcal{M}_{(a)}(i,j,k)$ and $\mathcal{M}_{(b)}(i,j,k)$ are cycles in $K_{i,j}(k)$.*
(ii) *The homology classes of the elements of $\mathcal{M}_{(a)}(i,j,k)$ and $\mathcal{M}_{(b)}(i,j,k)$ lie in the socle of $H_{i,j}(k)$.*

Proof. Let

$$c = [(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]f_F +$$
$$(-1)^j \sum_{l \in F} (-1)^{\alpha(l,F)}[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{F-\{l\}} \wedge f_{m(v)} \in \mathcal{M}_{(a)}(i,j,k).$$

Then

$$\partial(c) = \sum_{l \in F} (-1)^{\alpha(l,F)}[(x^u/x_{m(u)})(y_l y^v)e_L \wedge e_{m(u)}]f_{F-\{l\}} +$$
$$(-1)^j \partial(\sum_{l \in F} (-1)^{\alpha(l,F)}[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{F-\{l\}}) \wedge f_{m(v)}$$
$$-y_{m(v)} \sum_{l \in F} (-1)^{\alpha(l,F)}[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{F-\{l\}} = 0.$$

Hence we proved (i) for all elements of $\mathcal{M}_{(a)}(i,j,k)$. For (ii) it suffices to show that the homology classes of these elements are annihilated by $y_{\tilde{l}}$ for all $\tilde{l} < m-k+1$ because $\mathfrak{m}_x H_{i,j}(k) = 0$ and $y_{\tilde{l}} H_{i,j}(k) = 0$ for $\tilde{l} \geq m-k+1$. Fix $\tilde{l} < m-k+1$. Observe that $m(v) > m-k+1$. By 6.2.4 we have

$$[(x^u/x_{m(u)})(y_{\tilde{l}} y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_F \wedge f_{m(v)} \in K_{i,j}(k).$$

Hence

$$y_{\tilde{l}} c = \partial((-1)^j [(x^u/x_{m(u)})(y_{\tilde{l}} y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_F \wedge f_{m(v)})$$

and therefore

$$[y_{\tilde{l}} c] = 0 \text{ in } H_{i,j}(k).$$

Thus the assertion of (ii) follows for all elements of $\mathcal{M}_{(a)}(i,j,k)$.

Consider $c = [(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_F \wedge f_{m(v)} \in \mathcal{M}_{(b)}(i,j,k)$. Observe that $m(v) > m-k$. Let $l \leq m(v)$. We have

$$y_l[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}] = 0 \text{ in } H_i(n,0)$$

because $(x^u/x_{m(u)})(y_l y^v/y_{m(v)}) \in J$ since J is bistable. This implies that c is a cycle because $m(F) < m(v)$ and thus

$$\partial(c) = (-1)^{j-1}[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]f_F +$$

$$\sum_{l\in F}(-1)^{\alpha(l,F)} y_l[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{F-\{l\}} \wedge f_{m(v)} = 0.$$

Since $\mathfrak{m}_x H_{i,j}(k) = 0$ and $y_{\tilde{l}} H_{i,j}(k) = 0$ for $\tilde{l} \geq m-k+1$, it follows similar that $\mathfrak{m}[c] = 0$ in $H_{i,j}(k)$. This concludes the proof. □

Next we study the homology groups $H_{i,j}(k)$ for $i \geq 1$.

Proposition 6.2.6. *Let $J \subset S$ be a bistable ideal, $R = S/J$ and $i \in [n]$:*

(i) *$H_{i,0}(m)$ has a K-basis consisting of the homology classes*

$$[[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]]$$

where $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$ and $x^u y^v \in G(J)$.

(ii) *For $k \in [m]$ and $j \in [k]$ we have $\mathfrak{m}H_{i,j}(k) = 0$.*

(iii) *For $k \in [m]$ and $j \in [k]$ a K-basis of $H_{i,j}(k)$ is given by the homology classes of the elements of $\mathcal{M}_{(a)}(i,j,k)$ and $\mathcal{M}_{(b)}(i,j,k)$.*

Proof. Fix $i \in [n]$ for the rest of the proof. Statement (i) follows from 6.2.2 (iv) and the fact that $H_{i,0}(m) = H_i(n,0)/\mathfrak{m}_y H_i(n,0)$.

We prove by induction on $k \in [m]$ that for all $j \in [k]$ the assertion of (iii) holds. Finally, (ii) follows from this and 6.2.5 (ii).

For $k = 1$ we have the exact sequence

$$0 \to H_{i,1}(1) \to \tilde{H}_{i,0}(0)(0, -\varepsilon_m) \to 0.$$

By 6.2.3 and 6.2.5 (i) preimages in $H_{i,1}(1)$ of a K-basis of $\tilde{H}_{i,0}(0)$ are:

$$[[(x^u/x_{m(u)})(y^v/y_m)e_L \wedge e_{m(u)}]f_m]$$

where $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u(y^v/y_{m(v)}) \in J\backslash\mathfrak{m}_x J$, $(x^u/x_{m(u)})y^v \in J\backslash\mathfrak{m}_y J$, $m = m(v)$. Hence the assertion follows.

Let $k > 1$. By 6.2.5 (ii) and our induction hypothesis we have $\mathfrak{m}H_{i,j}(k-1) = 0$. It follows that for $j \geq 1$ the multiplication maps

$$H_{i,j}(k-1)(0, -\varepsilon_{m-k+1}) \stackrel{y_{m-k+1}}{\to} H_{i,j}(k-1)$$

are zero maps.

For $j = 1$ we get the exact sequence

$$0 \to H_{i,1}(k-1) \to H_{i,1}(k) \to \tilde{H}_{i,0}(k-1)(0, -\varepsilon_{m-k+1}) \to 0.$$

By the induction hypothesis the homology classes of the elements of $\mathcal{M}_{(a)}(i,1,k-1)$ and $\mathcal{M}_{(b)}(i,1,k-1)$ are a K-basis of $H_{i,1}(k-1)$. By 6.2.3 and 6.2.5 preimages in $H_{i,1}(k)$ of a K-basis of $\tilde{H}_{i,0}(k-1)$ are:

(A) $[[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]f_{m-k+1} -$
$[(x^u/x_{m(u)})(y_{m-k+1}y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{m(v)}]$ where $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u y^v \in G(J)$, $m-k+1 < m(v)$.

(B) $[[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{m-k+1}]$ where $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u(y^v/y_{m(v)}) \in J\backslash \mathfrak{m}_x J$, $(x^u/x_{m(u)})y^v \in J\backslash \mathfrak{m}_y J$, $m-k+1 = m(v)$.

We see that the homology classes of the elements of $\mathcal{M}_{(a)}(i,1,k)$ and $\mathcal{M}_{(b)}(i,1,k)$ are a K-basis for $H_{i,1}(k)$.

Let $j > 1$. The long exact sequence of the Koszul homology splits in short exact sequences of the following form:

$$0 \to H_{i,j}(k-1) \to H_{i,j}(k) \to H_{i,j-1}(k-1)(0, -\varepsilon_{m-k+1}) \to 0.$$

By the induction hypothesis a K-basis of $H_{i,j}(k-1)$ is given by the homology classes of the elements of $\mathcal{M}_{(a)}(i,j,k-1)$ and $\mathcal{M}_{(b)}(i,j,k-1)$. Analogously a K-basis of $H_{i,j-1}(k-1)$ is given by the homology classes of the elements of $\mathcal{M}_{(a)}(i,j-1,k-1)$ and $\mathcal{M}_{(b)}(i,j-1,k-1)$.

Applying 6.2.5 we may choose the following preimages in $H_{i,j}(k)$ of a K-basis of $H_{i,j-1}(k-1)$:

(A) $[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]f_{\tilde{F}}+$ $(-1)^j \sum_{l \in \tilde{F}} (-1)^{\alpha(l,\tilde{F})}[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{\tilde{F}-\{l\}} \wedge f_{m(v)}$ where $\tilde{F} = F \dot{\cup} \{m-k+1\}$, $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u y^v \in G(J)$, $\max\{m-k+1, m(\tilde{F})\} < m(v)$, $|F| = j-1$, $F \subseteq \{m-k+2, \ldots, m\}$.

(B) $[[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{\tilde{F}} \wedge f_{m(v)}]$ where $\tilde{F} = F \dot{\cup} \{m-k+1\}$, $L \subseteq [n]$, $|L| = i-1$, $m(L) < m(u)$, $x^u(y^v/y_{m(v)}) \in J\backslash \mathfrak{m}_x J$, $(x^u/x_{m(u)})y^v \in J\backslash \mathfrak{m}_y J$, $\max\{m-k, m(\tilde{F})\} < m(v)$, $|F| = j-2$, $F \subseteq \{m-k+2, \ldots, m\}$.

We conclude that the homology classes of the elements of $\mathcal{M}_{(a)}(i,j,k)$ and of $\mathcal{M}_{(b)}(i,j,k)$ form a K-basis for $H_{i,j}(k)$. Thus we proved the proposition. □

In the following we describe the homology groups $H_{0,j}(k)$.

Proposition 6.2.7. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:*

(i) $H_{0,0}(m) = K$.
(ii) *For $k \in [m]$ we have* $\mathfrak{m}\tilde{H}_{0,0}(k-1) = 0$.
(iii) *For $k \in [m]$ a K-basis of $\tilde{H}_{0,0}(k-1) = 0$ is given by:*

$$[(y^v/y_{m(v)})]$$

where $m-k+1 = m(v)$ and $y^v \in G(J)$.
(iv) *For $k \in [m]$ and $j \in [k]$ we have* $\mathfrak{m}H_{0,j}(k) = 0$.
(v) *For $k \in [m]$ and $j \in [k]$ a K-basis of $H_{0,j}(k)$ is given by:*

$$[(y^v/y_{m(v)})f_F \wedge f_{m(v)}]$$

where $F \subseteq \{m-k+1, \ldots, m\}$, $|F| = j-1$, $\max\{m-k, m(F)\} < m(v)$ and $y^v \in G(J)$.

Proof. This follows from [**4**] and the fact that $S/(\mathfrak{m}_x + J) \cong S_y/I$ where I is a stable ideal with minimal generators $y^v \in G(J)$. □

We are ready to prove the main theorem of this section.

Theorem 6.2.8. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then for $0 \le i \le n+m$ we have*

$$\bigoplus_{j=0}^{m} H_{i-j,j}(m) \cong H_i(n,m)$$

as graded K-vector spaces.

Proof. At the beginning we define homogeneous S-module homomorphisms from $H_{i,j}(k)$ to $H_{i+j}(n,k)$. These maps will be used to prove the theorem. For $i,j \in \mathbb{N}$ we distinguish the cases:

(1) $i > 0$ and $j > 0$.
(2) $i > 0$ and $j = 0$.
(3) $i = 0$ and $j > 0$.
(4) $i = 0$ and $j = 0$.

Case (1): Let $i \in [n]$, $k \in [m]$ and $j \in [k]$. We define the map

$$\varphi_{i,j,k} \colon H_{i,j}(k) \to H_{i+j}(n,k)$$

which is induced by

$$\begin{aligned}\varphi_{i,j,k}([[(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)}]f_F +\\ (-1)^j \sum_{l\in F} (-1)^{\alpha(l,F)}[(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_{F-\{l\}} \wedge f_{m(v)}])\\ = [(x^u/x_{m(u)})y^v e_L \wedge e_{m(u)} \wedge f_F +\\ (-1)^j \sum_{l\in F} (-1)^{\alpha(l,F)}(x^u/x_{m(u)})(y_l y^v/y_{m(v)})e_L \wedge e_{m(u)} \wedge f_{F-\{l\}} \wedge f_{m(v)}]\end{aligned}$$

and

$$\begin{aligned}&\varphi_{i,j,k}([[(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)}]f_F \wedge f_{m(v)}])\\ &= [(x^u/x_{m(u)})(y^v/y_{m(v)})e_L \wedge e_{m(u)} \wedge f_F \wedge f_{m(v)}].\end{aligned}$$

Observe that the expressions on the right hand sides are well defined homology classes in $H_i(n,k)$. As in the proof of 6.2.5 one checks, that the image of these maps belong to the socle of $H_i(n,k)$.

Case (2): For all $i \in [n]$ and $k \in [m]$ there exists a natural homogeneous S-module homomorphism from $H_i(n,0)$ to $H_i(n,k)$ induced by the inclusion of Koszul complexes. This map factors through

$$H_{i,0}(k) = H_i(n,0)/(y_{m-k+1},\dots,y_m)H_i(n,0)$$

because

$$(y_{m-k+1},\dots,y_m)H_i(n,k) = 0.$$

Thus we obtain

$$\varphi_{i,0,k} \colon H_{i,0}(k) \to H_i(n,k)$$

with

$$\varphi_{i,0,k}([[(x^u/x_{m(u)})e_L \wedge e_{m(u)}]]) = [(x^u/x_{m(u)})e_L \wedge e_{m(u)}]$$

for $[(x^u/x_{m(u)})e_L \wedge e_{m(u)}] \in H_i(n,0)$ as described in 6.2.2.

Case (3): Let $k \in [m]$, $j \in [k]$ and $[(y^v/y_{m(v)})f_F \wedge f_{m(v)}]$ be an element as described in 6.2.7. Similarly $[(y^v/y_{m(v)})f_F \wedge f_{m(v)}]$ belongs to the socle of $H_j(n,k)$. Then we define a homogeneous map

$$\varphi_{0,j,k} \colon H_{0,j}(k) \to H_j(n,k)$$

by

$$\varphi_{0,j,k}([(y^v/y_{m(v)})f_F \wedge f_{m(v)}]) = [(y^v/y_{m(v)})f_F \wedge f_{m(v)}].$$

Case (4): Note that for $k \in [m]$ one has

$$(*) \quad H_0(n,k) = S/(\mathfrak{m}_x + (y_m, \dots, y_{m-k+1}) + J) = H_{0,0}(k).$$

Set $\varphi_{0,0,k} = \mathrm{id}_{H_0(n,k)}$.

All these $\varphi_{i,j,k}$ are said to be the natural maps from $H_{i,j}(k)$ to $H_{i+j}(k)$. By $(*)$ the assertion of the theorem follows for $i = 0$. Hence we may assume that $i \geq 1$.

We prove by induction on $k \in [m]$ that for all $i \in [n+k]$

$$\bigoplus_{j=0}^{k} H_{i-j,j}(k) \cong H_i(n,k)$$

where the isomorphism is given by the direct sum of the $\varphi_{i-j,j,k}$. The case $k = m$ will prove the theorem.

Let $k = 1$. One has the long exact sequence

$$\dots \to H_i(n,0)(0,-\varepsilon_m) \xrightarrow{y_m} H_i(n,0) \to H_i(n,1)$$

$$\to H_{i-1}(n,0)(0,-\varepsilon_m) \xrightarrow{y_m} H_{i-1}(n,0) \to \dots,$$

and therefore the exact sequence

$$0 \to H_{i,0}(1) \to H_i(n,1) \to \tilde{H}_{i-1,0}(0)(0,-\varepsilon_m) \to 0.$$

We have $\tilde{H}_{i-1,0}(0)(0,-\varepsilon_m) \cong H_{i-1,1}(1)$. Thus we get the exact sequence

$$0 \to H_{i,0}(1) \to H_i(n,1) \to H_{i-1,1}(1) \to 0.$$

Denote the map between $H_i(n,1)$ and $H_{i-1,1}(1)$ with α. We see that $\alpha \circ \varphi_{i-1,1,1} = \mathrm{id}_{H_{i-1,1}(1)}$. Hence this sequence splits and this implies the case $k = 1$.

Let $k > 1$. We have the long exact sequence

$$(**) \quad \dots \to H_i(n,k-1)(0,-\varepsilon_{m-k+1}) \xrightarrow{y_{m-k+1}} H_i(n,k-1) \to H_i(n,k)$$

$$\to H_{i-1}(n,k-1)(0,-\varepsilon_{m-k+1}) \xrightarrow{y_{m-k+1}} H_{i-1}(n,k-1) \to \dots .$$

By the induction hypothesis it follows that for $i \geq 1$

$$\bigoplus_{j=0}^{k-1} H_{i-j,j}(k-1) \cong H_i(n,k-1)$$

and the isomorphism is given by the direct sum of the $\varphi_{i-j,j,k-1}$. We also know that $H_0(n,k-1) = H_{0,0}(k-1)$. By 6.2.6 and 6.2.7 one has $\mathfrak{m}H_{i-j,j}(k-1) = 0$ for $j \geq 1$.

Therefore by calculating the kernel and cokernel of the multiplication maps with y_{m-k+1} in $(**)$, we obtain the exact sequence

$$0 \to H_{i,0}(k) \oplus \bigoplus_{j=1}^{k-1} H_{i-j,j}(k-1) \to H_i(n,k)$$

$$\to \tilde{H}_{i-1,0}(k-1)(0,-\varepsilon_{m-k+1}) \oplus \bigoplus_{j=1}^{k-1} H_{i-1-j,j}(k-1)(0,-\varepsilon_{m-k+1}) \to 0.$$

Let $\varphi = \bigoplus_{j=0}^{k} \varphi_{i-j,j,k}$. We have the following commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H_{i,0}(k) \oplus \bigoplus_{j=1}^{k-1} H_{i-j,j}(k-1) & = & H_{i,0}(k) \oplus \bigoplus_{j=1}^{k-1} H_{i-j,j}(k-1) \\
\downarrow & & \downarrow \\
H_{i,0}(k) \oplus \bigoplus_{j=1}^{k-1} H_{i-j,j}(k) \oplus H_{i-k,k}(k) & \xrightarrow{\varphi} & H_i(n,k) \\
\downarrow & & \downarrow \\
C & = & C \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where

$$C = \tilde{H}_{i-1,0}(k-1)(0,-\varepsilon_{m-k+1}) \oplus \bigoplus_{j=1}^{k-1} H_{i-1-j,j}(k-1)(0,-\varepsilon_{m-k+1}).$$

This implies that φ is an isomorphism and thus $\bigoplus_{j=0}^{k} H_{i-j,j}(k) \cong H_i(n,k)$. $\square$

The following corollary was first discovered in [**3**].

Corollary 6.2.9. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:*

(i) $\operatorname{reg}_x(R) = \max\{m_x(J) - 1, 0\}$.
(ii) $\operatorname{reg}_y(R) = \max\{m_y(J) - 1, 0\}$.

Proof. By symmetry it suffices to prove (i). Observe that for $x^u y^v \in J \backslash \mathfrak{m}_x J$ one has $|u| \le m_x(J)$. Then (i) follows from 6.2.6, 6.2.7 and 6.2.8. $\square$

Finally, we compute all total Betti numbers of a bigraded algebra R which is defined by a bistable ideal. Let $a, b \in \mathbb{Z}$. Recall the convention that $\binom{a}{0} = 1$ and $\binom{a}{b} = 0$ for $a < 0$, $b \neq 0$. Furthermore, $\binom{a}{b} = 0$ for $0 \le a < b$ or $b < 0 \le a$ respectively.

Corollary 6.2.10. *Let $J \subset S$ be a bistable ideal and $R = S/J$. For $1 \leq i \leq n+m$ one has, independent of the characteristic of K,*

$$\beta_i^S(R) = \sum_{\{(u,v):\, x^u y^v \in G(J)\}} \sum_{j=0}^{m-1} \binom{m(u)-1}{i-j-1}\binom{m(v)-1}{j} +$$

$$\sum_{\{(u,v):\, x^u(y^v/y_{m(v)}) \in J \setminus \mathfrak{m}_x J,\ (x^u/x_{m(u)})y^v \in J\setminus \mathfrak{m}_y J\}} \sum_{j=1}^{m} \binom{m(u)-1}{i-j-1}\binom{m(v)-1}{j-1}.$$

Proof. The formula follows from 6.2.6, 6.2.7 and 6.2.8. $\square$

Example 6.2.11. Let $S = K[x_1, x_2, y_1]$ and

$$J = (x_1^3, x_1^2x_2, x_1x_2^2, x_1^2y_1, x_1x_2y_1, x_2^2y_1).$$

To apply 6.2.10 we have to find those monomials $x^u y^v \in S$ with $x^u(y^v/y_{m(v)}) \in J\setminus\mathfrak{m}_x J$ and $(x^u/x_{m(u)})y^v \in J\setminus\mathfrak{m}_y J$. We see that this condition is only satisfied for the monomials

$$x_1x_2^2y_1,\ \ x_1^2x_2y_1 \text{ and } x_1^3y_1.$$

We get

$$\beta_i^S(S/J) = \left[4\binom{1}{i-1} + 2\binom{0}{i-1}\right] + \left[2\binom{1}{i-2} + \binom{0}{i-2}\right].$$

Hence we obtain

$$\beta_1^S(S/J) = 6, \quad \beta_2^S(S/J) = 7, \quad \beta_3^S(S/J) = 2 \text{ and } \beta_i^S(S/J) = 0 \text{ for } i \geq 3.$$

Index of symbols

symbol	definition	page
$m^i_j(R)$	$\sup(\{a\colon (0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(a,j)} \neq 0\} \cup \{0\})$	74
$m_x(J)$	$\max\{a_i\colon \deg(z_i) = (a_i, b_i), z_i \in G(J)\}$	29
$\mathfrak{m}_x$	graded maximal ideal of R_x	15
$m_y(J)$	$\max\{b_i\colon \deg(z_i) = (a_i, b_i), z_i \in G(J)\}$	29
$\mathfrak{m}_y$	graded maximal ideal of R_y	15
$\Omega_k(M)$	k^{th}-syzygy module of a f.g. graded module M	18
ω_R	canonical module of a ring R	
$\mathrm{pd}_R(M)$	projective dimension of a module M	18
$R(x^u)$	revlex-segment of a monomial x^u	65
$R(I)$	Rees algebra of an ideal I	25
$\mathrm{reg}_R(M)$	regularity of a f.g. graded module M	18
$\mathrm{reg}_{R,x}(M)$	x-regularity of a f.g. bigraded module M	20
$\mathrm{reg}_{R,y}(M)$	y-regularity of a f.g. bigraded module M	20
S	a polynomial ring	14
$\mathrm{socle}(M)$	socle of a module M	
$\mathrm{Rad}(I)$	radical ideal of an ideal I	
$\mathrm{rank}(F)$	rank of a free module F	
$R_{\tilde{\Delta}}$	bigraded Veronese algebra of a bigraded K-algebra R	80
R_x	$\bigoplus_{a\in\mathbb{N}} R_{(a,0)}$ for a bigraded K-algebra R	15
R_y	$\bigoplus_{b\in\mathbb{N}} R_{(0,b)}$ for a bigraded K-algebra R	15
$\mathrm{Spec}(M)$	spectrum of a module M	
$\mathrm{supp}(a)$	$\{i\colon a_i \neq 0\}$ for $a \in \mathbb{N}^n$	
$S(I)$	symmetric algebra of an ideal I	25
$\mathcal{SQ}(E)$	category of square-free E-modules	33
$\mathcal{SQ}(S)$	category of square-free S-modules	32
$\mathrm{Tor}^R_i(M, N)$	i^{th} Tor-module of M and N	17
$\|u\|$	$u_1 + \ldots + u_n$ for $u \in \mathbb{Z}^n$	
$W^\vee$	$\mathrm{Hom}_K(W, K)$ where W is a K-vector space	41
$\mathbf{x}$	$x_1, \dots , x_n$	14
x^u	$x_1^{u_1} \cdots x_n^{u_n}$ for $u \in \mathbb{N}^n$	26
$\mathbb{Z}$	integers	

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