Upper bounds for local cohomology modules of rings with a given Hilbert function

Dissertation zur Erlangung des Grades
Doktor der Naturwissenschaften
(Dr. rer. nat.)

dem Fachbereich
Mathematik und Informatik
der Universität GH Essen
vorgelegt von

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aus Genua, Italien

April 2000

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Tag der mündlichen Prüfung:	29. September 2000

Acknowledgments

I wish to express all my gratitude to my thesis advisor, Jürgen Herzog, whose helpful advice, deep insight into the subject, inexhaustible patience and continuous encouragement, made the accomplishment of this thesis possible.

I wish to thank Emanuela De Negri for her interest in my work as well as her constant support.

My graduate study was funded by a three year grant of the Istituto Nazionale di Alta Matematica Francesco Severi to which I am very grateful for the opportunities it afforded me.

To the friends who have been able to bear me, and particularly to Alessandro, Alex, Angelo, Bill, Bruna, Corinna, Franco, Graziella, Katia, Manuel, Markus, Paolo Pavel, Sabine: Thank you!

Contents

Preface

This thesis is concerned with the graded structure of the local cohomology modules of rings R/I, where $R = K[X_1, \ldots, X_n]$ is a polynomial ring and $I \subset R$ a graded ideal.

The importance of local cohomology as an algebraic object was first recognized by Grothendieck [G]. Since then, the local cohomology theory has become a standard tool in the theory of commutative Noetherian rings and has found algebraic treatment in many text books. Local cohomology functors are introduced as right derived functors of a certain torsion functor. Their homological properties have been thoroughly dealt with, in particular in the local and graded cases. Grothendieck's Vanishing Theorem and Local Duality Theorem, which shows the correspondence of connected sequences between the local cohomology functors and the dual of certain Ext groups, constitute the starting point of our discussion about the local cohomology modules $H^{\bullet}_{\mathfrak{m}}(R/I)$ with respect to the graded maximal ideal $\mathfrak{m} \subset R$.

The other main theme of this dissertation is lexicographic ideals. These were introduced by Macaulay [M], who proved that for any graded ideal I with a given Hilbert function there exists a unique lexicographic ideal with the same Hilbert function as I. Because of their combinatorial nature, which is derived from being a special class of a wider family of homogeneous ideals called strongly stable, lexicographic ideals have recently been the subject of enquiry by many mathematicians, cf. [AH], [AHHi], [AHHi2], [Bi], [E], [Hu], [Hu1], [P], [P1], [V] et al. It is pointed out in these works that lexicographic ideals have extremal properties, for instance that of having maximal Betti numbers in a family of graded ideals with a given Hilbert function. One of our main endeavours will be investigating the behaviour of lexicographic ideals in respect to local cohomology. In particular, we shall prove, in the same spirit of the aforementioned results, an Upper Bound Theorem as follows: Let H be a given Hilbert function (resp. f-vector) and let $\mathcal{I}$ denote the family of all homogeneous (resp. squarefree) ideals with Hilbert function (resp. f-vector) H. Let L be the lexicographic ideal of $\mathcal{I}$. Then, for each $I \in \mathcal{I}$,

$$\dim_K H^i_{\mathfrak{m}}(R/I)_j \leq \dim_K H^i_{\mathfrak{m}}(R/L)_j, \text{ for any } i, j.$$

We recall in the first chapter basic definitions and known facts about Hilbert functions, Stanley-Reisner rings, lexicographic ideals and local cohomology.

Chapter 2 is dedicated entirely to the introduction of a new functor from the category of finite multi-graded R-modules to that of multi-graded S-modules where S is a polynomial ring in one variable over R. We shall call this functor the *polar-*

ization functor and denote it ${}^{\mathrm{p}}$. In Section 2.2 we first provide the construction of the functor ${}^{\mathrm{p}}$. Secondly we show how the polarization M^{p} of any multi-graded R-module M can be determined by its minimal resolution. Then, in the main theorem of this chapter, Theorem 2.11, we prove the exactness of ${}^{\mathrm{p}}$. In Section 2.3 we discuss other properties of the polarization functor and prove some preparatory results necessary for the next chapter.

The third Chapter is devoted to the proof of the Upper Bound Theorem. In Section 3.2 we prove by way of a standard deformation argument that, given any term order $\prec$ on the monomials of R, the dimensions of the graded components of local cohomology cannot but increase in the passage from an ideal I to its initial ideal $\mathrm{in}_{\prec}(I)$ (cf. Theorem 3.3). Section 3.3 deals with the squarefree case. Combining classical and new results of the theory of Stanley-Reisner rings we are able to prove Proposition 3.8, which is the key to the Upper Bound Theorem 3.10. Section 3.4 proves the counterpart of the latter for non-squarefree ideals (cf. Theorem 3.15). The argument is based on the use of the polarization functor for monomial ideals and exploits a useful "moving to L" strategy introduced by Pardue in [P], [P1].

In Chapter 4 we prove a structure theorem for local cohomology modules of lexicographic ideals (cf. Proposition 4.6). Next, we determine the Hilbert series of such modules in terms of the Hilbert function of the ideals, in the non-squarefree case. These results follow, by duality, from Propositions 4.1 and 4.4 where we prove that the Ext groups of lexicographic ideals generated in one degree are cyclic.

Chapter 5 is divided into two sections. In the first we prove the natural generalization of Theorem 3.15 for graded submodules of free R-modules (Theorem 5.1). In the second we use Theorem 3.15 to prove an upper bound theorem for certain families of coherent sheaves on $\mathbb{P}^n_K$ with a given Euler characteristic (Theorem 5.4).

In Chapter 6 we consider another special class of strongly stable ideals, which, like lexicographic ideals generated in one degree, are determined by a single monomial called principal generator. We prove a characterization of the vanishing and structure of their Ext groups (Propositions 6.3 and 6.4), getting an analogue of Proposition 4.1.

1 Preliminaries

Throughout this work all rings are assumed to be Noetherian, commutative and with identity. Given a ring A, an A-module M, an element $t \in A$ and the closed multiplicative system $S = \{1, t, t^2, \ldots\}$, M_t denotes the localization of M at S. K will always denote a field and R the polynomial ring $K[X_1, \ldots, X_n]$ over K.

1.1 Graded modules and Hilbert function

In this section we would like to recall the main definitions and properties of graded rings, graded modules and their Hilbert function.

Definition 1.1. A ring A is said to be ($\mathbb{Z}$-)*graded* iff there exists a family of $\mathbb{Z}$-modules A_i, $i \in \mathbb{Z}$, such that $A = \oplus_{i \in \mathbb{Z}} A_i$ as a $\mathbb{Z}$-module and $A_i A_j \subset A_{i+j}$ for every $i, j \in \mathbb{Z}$.
Let A be a graded ring. An A-module M is said to be *graded* iff there exists a family of $\mathbb{Z}$-modules M_i, $i \in \mathbb{Z}$, such that $M = \oplus_{i \in \mathbb{Z}} M_i$ as a $\mathbb{Z}$-module and $A_i M_j \subset M_{i+j}$ for every $i, j \in \mathbb{Z}$.

If $x \in M_i$ we say that x is *homogeneous of degree* i and we let $\deg x \doteq i$. We also call M_i the i^{th} *homogeneous* (or *graded*) *component* of M. Notice that, from the very definition of graded module, the homogeneous components of M have an A_0-module structure. More generally, given an arbitrary abelian group $(G, +)$, a G*-graded ring* A is a ring with a family of $\mathbb{Z}$-modules A_g, for every $g \in G$, such that $A = \oplus_{g \in G} A_g$ as a $\mathbb{Z}$-module and $A_g A_h \subset A_{g+h}$ for every $g, h \in G$. The definition of a G-graded A-module is given analogously.

Example and Notation 1.2. Let $R = K[X_1, \ldots, X_n]$. If we let $\deg X_i = 1$ for $i = 1, \ldots, n$, and R_i be the set of all monomials of R of degree i, then R inherits a structure of $\mathbb{Z}$-graded ring, which is usually referred to as *standard grading.*
If $m = cX_1^{a_1} \cdot \ldots \cdot X_n^{a_n}$ is a monomial of R, the vector $a = (a_1, \ldots, a_n) \in \mathbb{N}$ is called the *multi-degree* of m. We shall also write $m = cX^a$. The polynomial ring R thus has a natural structure of $\mathbb{Z}^n$-graded ring, by letting the a^{th} homogeneous component R_a of R be the set $\{cX^a \colon c \in K\}$. In the literature this graded structure is also called *fine grading.* Note that

$$R_i = \oplus_{\substack{a \in \mathbb{Z}^n \\ |a| = i}} R_a,$$

where $|a| = \sum_{i=1}^{n} a_i$. Any of the components R_a, with $a \in \mathbb{N}^n$, is a K-vector space of dimension 1.

Note that any ideal I of R, which is homogeneous with respect to the standard grading of R, has the natural fine graded structure induced by that of R. If M is endowed with a structure of $\mathbb{Z}^n$-graded R-module, we say that M is *multi-graded.*

Example and Notation 1.3. Let $S \doteq K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ be a polynomial ring in $n + m$ variables and consider the grading defined by $\deg X_i \doteq (a_i, 0)$, for $i = 1, \dots, n$, and $\deg Y_j \doteq (0, b_j)$, for $j = 1, \dots, m$. An ideal I which is graded with respect to this grading is said to be *bi-homogeneous* (or *bi-graded*). For every $a, b \in \mathbb{Z}$, $I_{(a,b)}$ denotes the set of the bi-homogeneous elements of I of degree a in the variables $X_1, \dots, X_n$ and degree b in the variables $Y_1, \dots, Y_m$.

Definition 1.4. Let M and N be graded R-modules and $\varphi : M \longrightarrow N$ an R-homomorphism. The map φ is said to be *homogeneous of degree* t iff $\varphi(M_i) \subset N_{i+t}$ for every $i \in \mathbb{Z}$.

Let M an arbitrary graded A-module. For every $a \in \mathbb{Z}$, $M(a)$ denotes the graded module obtained by the assignment $M(a)_i \doteq M_{a+i}$. Note that, given a map $\varphi : M \longrightarrow N$ of degree t, there are maps $M(-t) \longrightarrow N$ and $M \longrightarrow N(t)$ of degree 0 induced by φ. This allows us to consider only maps of degree 0, which we shall refer to simply as *homogeneous.*
A submodule N of M is said to be *graded* iff N is generated by homogeneous elements of M which belong to N, i.e. $N = \oplus_{i\in\mathbb{Z}} N_i = \oplus_{i\in\mathbb{Z}} N \cap M_i$. The graded submodules of R are the homogeneous ideals.

From now on we assume A_0 to be a field, denoted by K, A to be finite over K, and M finitely generated. Thus, the homogeneous components of M are K-vector spaces of finite dimension, and it is of interest introducing the following definition.

Definition 1.5. Let M be a graded A-module. The numerical function $H(M,\ ) : \mathbb{Z} \longrightarrow \mathbb{N}$, defined by

$$H(M, n) \doteq \dim_K M_n,$$

is called the *Hilbert function* of M.

The Hilbert function of a graded module measures the dimension of all the homogeneous components of M. The same piece of information is contained in the *Hilbert series* of M, which is defined to be the formal series

$$\mathrm{Hilb}(M, t) \doteq \sum_{i\in\mathbb{Z}} H(M, i)t^i.$$

Given formal series $S_1(t)$ and $S_2(t)$, we write $S_1(t) \preceq S_2(t)$ if $S_2(t)$ is coefficient-wise greater than or equal to $S_1(t)$. The Hilbert series of a module is a rational function: If M is a non-zero module, $\mathrm{Hilb}(M, t)$ can be written as $\frac{P(t)}{Q(t)}$, where $P(t) \in \mathbb{Z}[t, t^{-1}]$

and $Q(t) \in \mathbb{Z}[t]$. If one also requires that $P(1) \neq 0$, then this presentation is unique and $Q(t) = (1-t)^d$, where d is the Krull dimension of M.
An interesting feature of the Hilbert function is that it is of *polynomial type*, i.e., for n big enough, $H(M, n)$ can be described by a polynomial, which we denote by $P_M(X)$. $P_M(X)$ is called the *Hilbert polynomial of* M. From the expression of the Hilbert series of M one can easily compute that of its Hilbert polynomial: Since $\frac{1}{1-t} = \sum_{i \in \mathbb{N}} t^i$, if $\mathrm{Hilb}(M, t) = \frac{h(t)}{(1-t)^d}$, where $h(t) = \sum_{j=-s_1}^{s_2} h_j t^j$, then

$$P_M(X) = \sum_{j=-s_1}^{s_2} h_j \binom{X+d-j-1}{d-1}.$$

Thus, if M has dimension d, $P_M(X)$ has degree $d-1$ (by convention the zero polynomial has degree -1).

Remark 1.6. If M is a non-zero finitely generated A-module of dimension d, and $\mathrm{Hilb}(M, t) = \frac{P(t)}{(1-t)^d}$, with $P(1) \neq 0$, the Hilbert polynomial of M can be expressed in the following *standard* presentation

$$\sum_{i=0}^{d-1} (-1)^{d-1-i} e_{d-1-i} \binom{X+i}{i}, \quad \text{with } e_i = \frac{P^{(i)}(1)}{i!},$$

where $P^{(i)}(t)$ denotes the i^{th} derivative of $P(t)$ for $i = 0, \ldots, d-1$.

Given a numerical function $H : \mathbb{N} \longrightarrow \mathbb{N}$, one can search for necessary and sufficient conditions for H to be *admissible*, i.e. to be the Hilbert function of some graded ring R/I. Before quoting Macaulay's well-known theorem, which provides a criterion to determine whether H is admissible or not, we recall that given a positive integer d, any $n \in \mathbb{N}$ can be written uniquely in the form

$$n = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \ldots + \binom{k(1)}{1},$$

where $k(d) > k(d-1) > \ldots > k(1) \geq 0$. In the literature the above presentation is referred to as the d^{th} *binomial expansion* (or the d^{th} *Macaulay representation*) of n. Given the d^{th} binomial expansion of n, one can define the operation

$$n^{<d>} \doteq \binom{k(d)+1}{d+1} + \binom{k(d-1)+1}{d} + \ldots + \binom{k(1)+1}{2},$$

and set $0^{<d>} \doteq 0$.

Theorem 1.7 (Macaulay). *A numerical function $H : \mathbb{N} \longrightarrow \mathbb{N}$ is admissible iff*

(i) $H(0) = 1$;

(ii) $H(n+1) \leq H(n)^{<n>}$, *for every* $n \geq 1$.

We shall discuss other properties of Hilbert functions in Section 1.3. We conclude this section by quoting one classical and well-known result of homological algebra, the graded version of Rees' Lemma, because of its many applications to be found throughout this work.

Lemma 1.8 (Rees). *Let R be a Noetherian graded ring and M, N finitely generated graded R-modules. Let x be a homogeneous element of degree d, which is R- and N-regular. If $x \in 0 : M$ then, for any $h \geq 1$, there exists a homogeneous homomorphism*

$$\mathrm{Ext}^h_R(M, N) \simeq \mathrm{Ext}^{h-1}_{R/(x)}(M, N/xN)(d).$$

Proof. See Theorem 2.2 in [R]. ▲

1.2 Stanley-Reisner rings

A Stanley-Reisner ring is a quotient of a polynomial ring $K[X_1, \ldots, X_n]$ by a "square-free" monomial ideal I. The importance of Stanley-Reisner rings arises from the fact that they can be attached to simplicial complexes so that algebraic properties of the ring can be investigated with combinatorial techniques and *vice versa*. In this section we set some notation and provide the most basic definitions related to this topic, referring the reader to [BH], [S] and [St] for more information.

Definition 1.9. Let $V = \{v_1, \ldots, v_n\}$ a finite set. A *simplicial complex* Δ on V is a family of subsets of V such that

(i) $\{v_i\} \in \Delta$ for every $i = 1, \ldots, n$;

(ii) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

V is called the *vertex set* of Δ and the elements of V are called *vertices*. Any element of Δ is called a *face*, and the faces which are maximal with respect to the partial order given by inclusion are called *facets*. The *dimension* of the face F is defined to be $\dim F \doteq |F| - 1$, where $|F|$ denotes the cardinality of F and the *dimension* of a simplicial complex is the maximum of the dimension of its facets. A simplicial complex Δ is *generated* by a set of subsets of V, let us say $\{F_1, \ldots, F_m\}$, iff it is the smallest simplicial complex which contains F_i for every i, i.e. Δ consists of all the subsets of F_i, for $i = 1, \ldots, m$. A simplicial complex which is generated by only one subset of V is called *simplex*.
Now we give the definition of Stanley-Reisner rings.

Definition 1.10. Let Δ be a simplicial complex on the vertex set $V = \{v_1, \ldots, v_n\}$ and K a field. The *Stanley-Reisner ring* (or *face ring*) of Δ is the graded ring

$$K[\Delta] \doteq K[X_1, \ldots, X_n]/I_\Delta,$$

where I_Δ is the ideal generated by all the monomials $X_{i_1} \cdot \ldots \cdot X_{i_k}$, such that $\{v_{i_1}, \ldots, v_{i_k}\}$ is not a face of Δ. The ideal I_Δ is called the *defining ideal* of $K[\Delta]$.

Note that the previous definition can be stated more generally when K is not a field, although this will be our standard assumption.
In order to simplify the notation, we shall very often denote the vertex set $V = \{v_1, \ldots, v_n\}$ of Δ by $[n] \doteq \{1, \ldots, n\}$. A face will be accordingly denoted by $F = \{i_1, \ldots, i_k\} \subset [n]$. Moreover, we write X^F for the monomial $X_{i_1} \cdot \ldots \cdot X_{i_k}$. We say that a monomial of degree greater than or equal to 2 is *squarefree* iff all of its exponents are ≤ 1. A monomial ideal I is said to be *squarefree* iff it admits a system of generators formed by squarefree monomials. The *support* of an element $a \in \mathbb{Z}^n$ is defined to be the set

$$\operatorname{supp} a = \{i \in [n] \colon a_i \neq 0\}.$$

Since any monomial $m = X^a$ of $K[X_1, \ldots, X_n]$ is uniquely determined by its multi-degree and *vice versa*, we may define the *support* of m, denoted by $\operatorname{supp} m$, to be the support of its multi-degree a. In other words, $\operatorname{supp} m$ consists of all $i \in [n]$ such that $X_i | m$.
Observe that the defining ideal of a Stanley-Reisner ring is squarefree. On the other hand, if I is a squarefree monomial ideal, one can define Δ to be the set $\{F \subset [n] \colon X^F \notin I\}$. Clearly Δ is a simplicial complex and $I_\Delta = I$. This establishes a correspondence between simplicial complexes and squarefree ideals and, therefore, also between simplicial complexes and Stanley-Reisner rings.

Another object which can be associated to any simplicial complex Δ is its *Alexander dual*, denoted by $\overline{\Delta}$, and defined to be

$$\overline{\Delta} \doteq \{F \subset [n] \colon \overline{F} \notin \Delta\},$$

where $\overline{F}$ denotes the complement $[n] \setminus F$ of F in $[n]$. It is easy to verify that $\overline{\Delta}$ is a simplicial complex as well: If $F \in \overline{\Delta}$ and G is contained in F, then $\overline{F}$ is contained in $\overline{G}$. If $\overline{G}$ were in Δ then $\overline{F}$ should also belong to Δ, but this is not the case. Thus, $\overline{G} \notin \Delta$, i.e. $G \in \overline{\Delta}$. Clearly, $\overline{\overline{\Delta}} = \Delta$.
We let $\tilde{H}_i(\Delta, K)$ (resp. $\tilde{H}^i(\Delta, K)$) denote the i^{th} reduced simplicial homology (resp. cohomology) of Δ with values in K. The following lemma is well-known and referred to as Alexander Duality.

Lemma 1.11. *For any simplicial complex Δ on the vertex set $[n]$, one has*

$$\tilde{H}_{i-2}(\overline{\Delta}, K) \simeq \tilde{H}^{n-i-1}(\Delta, K).$$

Proof. See that of Lemma 5.5.3 in [BH]. ▲

Given a simplicial complex Δ on $[n]$ and a subset F of $[n]$, we let Δ_F denote the simplicial complex on the vertex set F defined by

$$\Delta_F \doteq \{G \subset F \colon G \in \Delta\}.$$

We now introduce another standard notion related to that of simplicial complex in the following definition.

Definition 1.12. Let Δ be a simplicial complex and F be a face of Δ. The *link* of F is the set

$$\mathrm{lk}_\Delta F \doteq \{G\colon F \cup G \in \Delta, F \cap G = \emptyset\}.$$

For any simplicial complex Δ with $\dim \Delta = d-1$, we call the d-tuple $f(\Delta) \doteq (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ, where f_i is the number of the i^{th} dimensional faces of Δ. One expects to find a strict relationship between the f-vector of a simplicial complex and the Hilbert function of its Stanley-Reisner ring. Indeed, one can show that, for every $n > 0$,

$$H(K[\Delta], n) = \sum_{i=0}^{d-1} f_i \tbinom{n-1}{i}.$$

Furthermore, in analogy to what was done in Section 1.1, one says that a vector g of $\mathbb{N}^d$ is *admissible* iff there exists a $(d-1)$-dimensional simplicial complex Δ such that g is the f-vector of Δ. The counterpart of Theorem 1.7 is the following result, attributed to Kruskal and Katona. Let $n = \sum_{i=1}^{h} \binom{k(i)}{i}$ be the h^{th} binomial expansion of n. We define $n^{(h)} \doteq \sum_{i=1}^{h} \binom{k(i)}{i+1}$.

Theorem 1.13. *A vector $g = (g_0, g_1, \ldots, g_{d-1}) \in \mathbb{N}^d$ is admissible iff*

$$0 < g_{i+1} \leq g^{(i)}, \;\; 0 \leq i \leq d-2.$$

In order to study properties of Stanley-Reisner rings and simplicial complexes, one can associate to a simplicial complex Δ a quotient of the exterior algebra E on K^n, which is strictly related to the face ring $K[\Delta]$. This approach of Kalai [K] was later followed by Aramova, Herzog and Hibi, who showed how classical theorems on Hilbert functions have an equivalent on quotients of E. We mention here just the very basics and refer the reader to [AHHi1].
Let $e_1, \ldots, e_n$ denote a basis of K^n and let us consider the standard grading on E. From the self-duality of E it follows that, if J is a graded ideal of E, then $\dim_K(E/J)_i = (0 : J)_{n-i}$ for all J.
A *monomial* of E is an element $e_F \doteq e_{i_1} \wedge \ldots \wedge e_{i_k}$, with $F = \{i_1, \ldots, i_k\}$. We let J_Δ be the ideal of E generated by all the monomials e_F such that F is a non-face of Δ. The quotient E/J_Δ is referred to as the *indicator algebra* of Δ. Observe that $f_i(\Delta) = \dim_K(E/J_\Delta)_{i+1}$. Indeed, $f_i(\Delta)$ is the number of faces of Δ of dimension i (i.e. of cardinality $i+1$). In other words $f_i(\Delta)$ counts the number of monomials of E_{i+1} which are not in J_Δ, and this number is in fact $\dim_K(E/J_\Delta)_{i+1}$.

Observe that if J is a monomial ideal of E, $0 :_E J$ is a monomial ideal generated by the monomials e_F with the property $e_{\overline{F}} \notin J$. If $e_F \in 0 :_E J$ and $e_{\overline{F}} \in J$, this would mean that $e_F \wedge e_{\overline{F}} = 0$, i.e. $F \cap \overline{F} \neq \emptyset$, which is contradictory. On the other hand, if $e_F \notin 0 :_E J$ and $e_{\overline{F}} \notin J$, then there would exist a monomial e_G of J such that $e_F \wedge e_G \neq 0$ and, accordingly, G would be a subset of $\overline{F}$, which is not possible.

Lemma 1.14. *Let Δ be a simplicial complex and let $\overline{\Delta}$ denote its Alexander dual. Then,*

$$J_{\overline{\Delta}} = 0 :_E J_{\Delta}.$$

Proof. By definition, $J_{\overline{\Delta}}$ is generated by all the monomials e_F such that $F \notin \overline{\Delta}$. The complex $\overline{\Delta}$ is given by all the faces whose complement is not a face of Δ. This implies that e_F is a generator of $J_{\overline{\Delta}}$ iff $\overline{F}$ is a face of Δ, or, in other words, iff $e_G \nmid e_{\overline{F}}$ for any $e_G \in J_{\Delta}$. Thus, for every $e_G \in J_{\Delta}$, there exists an i such that $e_i | e_G$ and $e_i \nmid e_{\overline{F}}$, i.e. $e_i | e_G$ and $e_i | e_F$. Therefore e_F is a generator of $J_{\overline{\Delta}}$ iff $e_F \wedge e_G = 0$ for all $G \in J_{\Delta}$, and we are done. ▲

As a consequence of these results, one can find a formula to relate the f-vector of Δ and that of $\overline{\Delta}$. From the previous lemma, it follows that $\dim_K(E/J_{\Delta})_{i+1} = \dim_K(J_{\overline{\Delta}})_{n-i-1}$. Therefore, $f_i(\overline{\Delta}) = \dim_K E_{n-i-1} - \dim_K(E/J_{\Delta})_{n-i-1}$, and, consequently,

$$f_i(\overline{\Delta}) = \tbinom{n}{i+1} - f_{n-i-2}(\Delta). \tag{1.1}$$

1.3 Term orders and lexicographic ideals

The objective of this section is to recall the definitions of term order and of initial ideal along with the introduction of one of the main objects of our study: lexicographic ideals. We refer the reader to [Ei] and [St] for further details.

1.3.1 Term orders and weight functions

Making use of the notation introduced in the previous section, we call an element of R of the form $m = cX_1^{a_1} \cdot \ldots \cdot X_n^{a_n} = cX^a$, a *monomial* of R. If the scalar coefficient is equal to 1, m is said to be a *term* of R.

Definition 1.15. A total order $\prec$ on $\mathbb{N}^n$ is a term order iff

(i) 0 is the unique minimal element;

(ii) if $a \prec b$ then $a + c \prec b + c$ for all $a, b, c \in \mathbb{N}^n$.

Through the correspondence between terms of R and vectors of $\mathbb{N}^n$ one can rephrase the previous definition as follows: A total order $\prec$ on the set of terms of R is a term order iff it satisfies

(i) 1 is the least term of R;

(ii) if $m_1 \prec m_2$, then $m_1 m_3 \prec m_2 m_3$ for any m_1, m_2 and m_3 terms of R.

Example 1.16 (Lexicographical term orders).

(1) *The lexicographic term order*: $X^a \prec_{lex} X^b$ iff the first non-zero component of the vector $(b_1 - a_1, \dots, b_n - a_n)$ is positive.

(2) *The degree lexicographic term order*: $X^a \prec_{dlex} X^b$ iff the first non-zero component of the vector $(\deg X^b - \deg X^a, b_1 - a_1, \dots, b_n - a_n)$ is positive.

(3) *The degree reverse lexicographic term order*: $X^a \prec_{drevlex} X^b$ iff the last non-zero component of the vector $(b_1 - a_1, \dots, b_n - a_n, \deg X^a - \deg X^b)$ is negative.

Observe that the degree lexicographic order induced by $X_1 > \dots > X_n$ is not the same as the degree reverse lexicographic order with respect to $X_n > \dots > X_1$. Let us consider the following easy example: Let $R = K[X_1, X_2, X_3]$ and let us compare the monomials $X_1^2X_2$ and X_2^3. Clearly $X_2^3 \prec_{dlex} X_1^2X_2$. If we now order the variables the other way around, the vector of the degrees in (3) is $(0-0, 1-3, 2-0, 3-3) = (0, -2, 2, 0)$. Thus, the last non-zero component is positive, i.e. $X_1^2X_2 \prec_{drevlex} X_2^3$.

Note that, since R is Noetherian, it follows from condition (ii) that, given any term order on R, every non-empty set of terms of R has a least element.
Every polynomial $f \in R$ can be presented in a unique way as $f = \sum_i c_i m_i$. Given any term order $\prec$, every (non-zero) polynomial f has a unique *initial monomial* (or *leading monomial*), which is the greatest monomial which appears in the above presentation. The initial monomial of a polynomial f is denoted by $\mathrm{in}_\prec(f)$. Let I be any ideal of R. The *initial ideal* of I is the ideal

$$\mathrm{in}_\prec(I) \doteq \langle \mathrm{in}_\prec(f) \colon f \in I \rangle,$$

which will be denoted simply by $\mathrm{in}(I)$ anytime there is no risk of ambiguity. Observe that, if I is a monomial ideal, $\mathrm{in}(I) = I$. Note also that, since R is Noetherian, any ideal has only a finite number of possible initial ideals.
Although in the passage from an ideal I to its initial ideal $\mathrm{in}(I)$ a piece of information gets lost, there is sometimes a real advantage in working with $\mathrm{in}(I)$, since it is a monomial ideal.

Proposition 1.17 (Macaulay). *Let $I \subset R$ be any graded ideal and let $\prec$ any term order on the monomials of R. Then*

$$H(R/I, n) = H(R/\mathrm{in}_\prec(I), n), \text{ for any } n \in \mathbb{N}.$$

Proof. See for example that of Theorem 15.3 in [Ei]. ▲

Definition 1.18. Let $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$. For any polynomial $f = \sum_i c_i m_i$ we let the *initial form* $\mathrm{in}_\omega(f)$ of f be the sum of all $c_i m_i$ with the property that the inner product of ω with the multi-degree of m_i is maximal. For any ideal I we let the *initial ideal* $\mathrm{in}_\omega(I)$ of I be the ideal generated by all the initial forms of its elements.

In general $\mathrm{in}_\omega(I)$ is not a monomial ideal.

Example 1.19. Let I be the ideal of $K[X_1, X_2]$ generated by $X_1^2 + X_1^2 X_2$ and let $\omega = (1, 0)$. Evidently, $\mathrm{in}_\omega(I) = I$ which is not monomial.

Let now ω be non-negative and $\prec$ a term order on $\mathbb{N}^n$. One can define a new term order $\prec_\omega$ as follows: For any $a, b \in \mathbb{N}^n$ let

$$a \prec_\omega b \quad \text{iff} \quad \omega \cdot a < \omega \cdot b \text{ or } (\omega \cdot a = \omega \cdot b \text{ and } a \prec b),$$

where " $\cdot$ " denotes the usual inner product in $\mathbb{R}^n$. Terms orders defined as above are called *weight orders*.
The main property of weight orders we are interested in is stated in the following proposition.

Proposition 1.20. *For any term order $\prec$ and any ideal $I \subset R$, there exists a non-negative integer vector $\omega \in \mathbb{N}^n$ such that*

$$\mathrm{in}_\omega(I) = \mathrm{in}_\prec(I).$$

Proof. See Proposition 1.11 in [S]. ▲

1.3.2 Lexicographic ideals

Henceforth, if not otherwise specified, we denote the usual lexicographic term order induced by $X_1 > X_2 > \ldots > X_n$ simply with " $<$ ". Moreover, since the distinction between terms and monomials will not play any role in what follows, we shall assume, if not elsewhere explicitly stated and with some abuse of notation, all monomials to have scalar coefficient 1.

Let $[X_1, \ldots, X_n]_d$ denote the set of all monomials in the variables $X_1, \ldots, X_n$ of degree d.

Definition 1.21. A *lexicographic segment* (or *lex-segment*) $\mathcal{L}(v)$ of degree d is the subset of $[X_1, \ldots, X_n]_d$ defined by

$$\mathcal{L}(v) \doteq \{w : \ w \geq v\}.$$

A graded ideal J such that any of its homogeneous components is generated as a K-vector space by a lex-segment is called a *lexicographic ideal* (or *lex-ideal*).

In order to prove Theorem 1.7, one has to find a graded ideal I such that $\dim_K R_i - \dim_K I_i = H(i)$, for every $i \in \mathbb{N}$. One defines I to be $I = \oplus_{i \in \mathbb{N}}(\mathcal{L}_i)$, where $\mathcal{L}_i$ is the lex-segment given by the first $\dim_K R_i - H(i)$ monomials of $[X_1, \ldots, X_n]_i$. One proves that I is an ideal, which is, according to the definition, lexicographic and uniquely determined by H. By virtue of Macaulay's Theorem one may define, given any graded ideal I, the *lexicographic ideal associated to* I, denoted by I^{lex}, to be the uniquely determined lex-ideal with the same Hilbert function as I. Given any ideal I, it is easy to see that $\mathrm{in}(I)^{lex} = \mathrm{in}(I^{lex}) = I^{lex}$. If I is a monomial ideal we let $G(I)$ denote its (unique) minimal system of monomial generators. Note that if d is the greatest degree of a monomial of $G(I)$ and d' is the greatest degree of a monomial in $G(I^{lex})$ then $d \leq d'$.

It is clear that lex-ideals are objects endowed with strong combinatorial properties. For example, one can consider the following problem: Since a lex-ideal J generated in one degree is determined by its least generator, say the monomial v, we would like to determine the multi-degree of v, and therefore v itself, from the Hilbert function of J. First we set some more notation. If J is a lex-ideal generated in one degree we write $J = (\mathcal{L}(v))$, for some monomial v of R. If J is generated in degree $d, d+1, \ldots, d+k$, we present J as the ideal generated by the monomials in $\mathcal{L}(v_d) \cup \mathcal{L}(v_{d+1}) \cup \ldots \cup \mathcal{L}(v_{d+k})$, for some $v_d, v_{d+1}, \ldots, v_{d+k} \in R$.

Proposition 1.22. *Let $J = (\mathcal{L}(v))$, with $\deg v = d$ and $v = X_1^{v_1} \cdot \ldots \cdot X_n^{v_n}$. Let $H(R/J, d) = \sum_{h=1}^{d} \binom{k(h)}{h}$ be the d^{th} binomial expansion of $H(R/J, d)$. Then,*

$$v_i = |\{k(h) \colon n - k(h) + h - 1 = i\}|,$$

for every $i = 1, \ldots, n$.

Proof. Let $\mathcal{A}(v) \doteq \{u \colon u < v\}$ and let i be the smallest integer such that $X_i | v$. It is clear that $\mathcal{A}(v) = \mathcal{A}(X_i^{-1}v)X_i \cup [X_{i+1}, \ldots, X_n]_d$, where the union is disjoint. Thus,

$$|\mathcal{A}(v)| = |\mathcal{A}(X_i^{-1}v)| + |[X_{i+1}, \ldots, X_n]_d| = |\mathcal{A}(X_i^{-1}v)| + \binom{n-i+d-1}{d}.$$

Write v as

$$v = X_{j(i)} X_{j(2)} \cdot \ldots \cdot X_{j(d)},$$

where $1 \leq j(1) \leq j(2) \leq \ldots \leq j(d)$. If we repeat the above computation, we can easily deduce that $|\mathcal{A}(v)| = \sum_{i=1}^{d} \binom{n-j(i)+d-i}{d-i+1}$, or , by substituting $h \doteq d - i + 1$,

$$|\mathcal{A}(v)| = \sum_{h=1}^{d} \binom{n-j(d-h+1)+h-1}{h},$$

where $n - j(1) + d - 1 > n - j(2) + d - 2 > \ldots > n - j(d) \geq 0$. On the other hand, since $H(R/J, d) = |\mathcal{A}(v)|$, we deduce that

$$k(h) = n - j(d - h + 1) + h - 1, \tag{1.2}$$

i.e. $j(d - h + 1) = n - k(h) + h - 1$. Thus,

$$\begin{aligned} v_i &= |\{j(d-h+1) \colon j(d-h+1) = i\}| \\ &= |\{j(d-h+1) \colon n - k(h) + h - 1 = i\}| \\ &= |\{k(h) \colon n - k(h) + h - 1 = i\}|, \end{aligned} \tag{1.3}$$

as desired. ▲

Proposition 1.23. *Let J be a lex-ideal generated in degree d. Then, for all $n \geq d$,*

$$H(R/J, n+1) = H(R/J, n)^{<n>}.$$

Proof. It is sufficient to show that $H(R/J, d+1) = H(R/J, d)^{<d>}$. If $J = (\mathcal{L}(v))$, where $v = X_{j(1)}X_{j(2)} \cdot \ldots \cdot X_{j(d)}$, with $\deg v = d$ and $j(1) \leq j(2) \leq \ldots \leq j(d)$, then the assertion is equivalent to proving that $|\mathcal{A}(vX_n)| = |\mathcal{A}(v)|^{<d>}$. Reasoning as in the previous proposition, we write $vX_n = X_{h(1)}X_{h(2)} \cdot \ldots \cdot X_{h(d)}X_{h(d+1)}$, where $j(i) = h(i)$ for $i = 1, \ldots, d$ and $h(d+1) = n$. Thus, $|\mathcal{A}(vX_n)| = \sum_{h=2}^{d+1} \binom{n-j(d-h+2)+h-1}{h}$, from which we obtain the sought after conclusion. ▲

Here we state some other results on lex-ideals which will be useful later.

Lemma 1.24. *Let J be a lex-ideal. Then $J : \mathfrak{m}^l$ is a lex-ideal.*

Proof. Observe that, since J is a monomial ideal, $J : \mathfrak{m}$ is a monomial ideal as well. $J : \mathfrak{m}$ is generated by the monomials u of R such that $u\mathfrak{m} \subseteq J$. Since J is a lex-ideal, this is equivalent to saying that $uX_n \in J$. To prove the assertion it is sufficient to show that, if v is a monomial of $J : \mathfrak{m}$ and $w > v$, then $w \in J : \mathfrak{m}$. But since $wX_n > vX_n$ and J is a lex-ideal, this is immediately seen. ▲

Given any ideal I, one can construct the ascending chain of ideals $I \subseteq I : \mathfrak{m} \subseteq I : \mathfrak{m}^2 \subseteq \ldots$. Since R is Noetherian, there is an integer k such that $I : \mathfrak{m}^{k+h} = I : \mathfrak{m}^k$ for any $h \geq 0$. Let k be the smallest one with this property. We denote $I : \mathfrak{m}^k$ by $I : \mathfrak{m}^\infty$ and call it the *saturation of I*. From the above lemma it follows that the saturation of a lex-ideal is again a lex-ideal.

In the next result the assumption that I is a monomial ideal is not restrictive.

Lemma 1.25. *Let I be a monomial ideal of R. Then $(I : \mathfrak{m}^\infty)^{lex} \subseteq (I^{lex} : \mathfrak{m}^\infty)$.*

Proof. Clearly it is enough to prove that $(I : \mathfrak{m})^{lex} \subseteq I^{lex} : \mathfrak{m}$. Observe that $(\mathfrak{m}I^{lex})_j \subseteq (\mathfrak{m}I)_j^{lex}$, since the cardinality of the set $\{X_i m \colon i = 1, \ldots, n,\ m \in I_{j-1}^{lex}\}$ is less than or equal to that of the set $\{X_i m \colon i = 1, \ldots, n,\ m \in I_{j-1}\}$. Thus it is easily seen that $H(R/I^{lex}, j) = H(R/I, j) \leq H(R/\mathfrak{m}(I : \mathfrak{m}), j) = H(R/(\mathfrak{m}(I : \mathfrak{m}))^{lex}, j) \leq H(R/\mathfrak{m}(I : \mathfrak{m})^{lex}, j)$, which implies that

$$H(I^{lex} : \mathfrak{m}, j) \geq H((I : \mathfrak{m})^{lex}, j).$$

Since, by the previous lemma, $(I^{lex} : \mathfrak{m})_j$ is a lex-segment for any j, this implies $(I : \mathfrak{m})_j^{lex} \subseteq (I^{lex} : \mathfrak{m})_j$, and we are done. ▲

Lemma 1.26. *Let J be a lex-ideal and $v_{d+k} \in G(J)$ the least monomial of highest degree in $G(J)$. Let L be the ideal generated by $\mathcal{L}(v_{d+k})$. Then $J : \mathfrak{m}^\infty = L : \mathfrak{m}^\infty$.*

Proof. Since $L \subset J$, it is clear that $L : \mathfrak{m}^\infty \subset J : \mathfrak{m}^\infty$. Conversely, if $u \in J : \mathfrak{m}^\infty$, then there exists $h \in \mathbb{N}$ such that $u\mathfrak{m}^h \in J$. Thus, $u\mathfrak{m}^h \in (\mathcal{L}(v_{d+i}))$ for some i, which implies that $u\mathfrak{m}^{h'} \in L$ for some $h' \geq h$, and so $u \in L : \mathfrak{m}^\infty$. ▲

Definition 1.27. Let I be any graded ideal and $k \in \mathbb{N}$. The *k-truncation of I*, denoted by $I_{\geq k}$, is defined as $I_{\geq k} \doteq \oplus_{j \geq k} I_j$.

As an easy consequence of the previous result, we have

Corollary 1.28. *Let $J_{\geq k}$ be a truncation of J, with $k \gg 0$. Then*

$$J : \mathfrak{m}^\infty = J_{\geq k} : \mathfrak{m}^\infty.$$

In order to compute $J : \mathfrak{m}^\infty$, by the previous lemma we may assume J to be a lex-ideal generated in one degree, say $J = (\mathcal{L}(v))$, for some $v = X_1^{v_1} \cdot \ldots \cdot X_n^{v_n}$ with $v_n \geq 1$. If $v = X_n^{v_n}$ then $J : \mathfrak{m}^\infty = (1)$. Otherwise, we can define $h \doteq \max_{i \leq n-1}\{i \colon v_i \neq 0\}$.

Proposition 1.29. *Let J be a lex-ideal generated in degree d by $\mathcal{L}(v)$. If $J : \mathfrak{m}^\infty \neq (1)$, then*

$$\begin{aligned} J : \mathfrak{m}^\infty = &(X_1^{v_1+1}, X_1^{v_1} X_2^{v_2+1}, \ldots, X_1^{v_1} \cdot \ldots \cdot X_{h-2}^{v_{h-2}} X_{h-1}^{v_{h-1}+1}, \\ &X_1^{v_1} \cdot \ldots \cdot X_{h-1}^{v_{h-1}} X_h^{v_h}). \end{aligned} \tag{1.4}$$

Proof.
Let $h = 1$. In this case $v = X_1^{v_1} X_n^{v_n}$ and it is clear that $(X_1^{v_1}) \subset J : \mathfrak{m}^\infty$. Conversely, if $u \in J : \mathfrak{m}^\infty$, one has $uX_n^k \geq X_1^{v_1} X_n^{v_n+l}$, for some k, l, which implies the other inclusion.
Let $h > 1$. Let us denote by $\mathfrak{m}_j$ the ideal $(X_j, \ldots, X_n)$, for $j = 1, \ldots, n$. We observe that $J = X_1^{v_1+1} \mathfrak{m}_1^{d-v_1-1} + X_1^{v_1}(\mathcal{L}(X_1^{-v_1} v))$, where $(\mathcal{L}(X_1^{-v_1} v))$ is a lex-ideal in the variables $X_2, \ldots, X_n$ generated in degree $d - v_1$. Reasoning as above, we end up obtaining

$$J = \sum_{i=1}^{h-1} X_1^{v_1} \cdot \ldots \cdot X_{i-1}^{v_{i-1}} X_i^{v_i+1} \mathfrak{m}_i^{d-v_1-\ldots-v_i-1} + X_1^{v_1} \cdot \ldots \cdot X_h^{v_h} \mathfrak{m}_h^{v_n}.$$

Since J is a lex-ideal, $u\mathfrak{m}_j^k \in J$ implies that $u\mathfrak{m}^k \in J$ for any monomial u and "$\supseteq$" follows from this observation. The other inclusion can be deduced arguing as in the proof of the case $h = 1$. ▲

Now we recall the definition of squarefree lexicographic ideal.

Definition 1.30. A *squarefree lex-segment* of degree d is a subset $\mathcal{L}$ of the squarefree monomials of R_d with the property that if $m \in \mathcal{L}$ is a squarefree monomial, then $n \in \mathcal{L}$ for every squarefree monomial n lexicographically greater than m. If v is the least squarefree monomial of R_d belonging to $\mathcal{L}$, we shall also denote $\mathcal{L}$ by $\mathcal{L}(v)$. A squarefree monomial ideal J is a *squarefree lex-ideal* iff for every squarefree monomial $u \in J$ and all squarefree monomials v of the same degree with $v > u$, then $v \in J$.

In analogy to the proof of Macaulay's Theorem, if $I = I_\Delta$ for some simplicial complex Δ, that of the Kruskal-Katona Theorem provides a unique squarefree lexicographic ideal with the same f-vector as I_Δ. We let Δ^{lex} be the simplicial complex with defining ideal I_Δ^{lex}, i.e. we let $I_{\Delta^{lex}} \doteq I_\Delta^{lex}$. Observe that, if one defines the lexicographic order on the monomials of the external algebra E in the natural way, by virtue of the version of Kruskal-Katona Theorem provided in [AHHi1], one can also let Δ^{lex} be the simplicial complex with indicator algebra E/J_{Δ}^{lex} and the two definitions agree.

We conclude with a useful lemma on lex-ideals of E.

Lemma 1.31. *Let I be a graded ideal of E. Then*

$$(0:_E I)^{lex} = 0:_E I^{lex}.$$

Proof. Because of the uniqueness of lex-ideals with a given Hilbert function, one needs only to show that $0:_E I^{lex}$ is a lex-ideal and that $\dim_K(0:_E I)_i^{lex} = \dim_K(0:_E I^{lex})_i$ for any i. We have already observed in the previous section that, for any monomial ideal J, $0:_E J$ is the span over K of monomials e_F such that $e_{\overline{F}}$ does not belong to J. One has thus to show that, if $e_F > e_G$ with $e_{\overline{G}} \notin I^{lex}$, then $e_{\overline{F}} \notin I^{lex}$. Since $e_F > e_G$, the first element which is in F and not in G, is also the first element of $\overline{G}$ which does not belong to $\overline{F}$, and this, by definition of lex-order, implies that $e_{\overline{G}} > e_{\overline{F}}$. If $e_{\overline{F}}$ were an element of I^{lex}, then also $e_{\overline{G}}$ would be as well, which is not possible. It now remains to be shown that the two ideals have the same Hilbert function. By definition $(0:_E I)^{lex}$ has the same Hilbert function as $0:_E I$, whose i^{th} component has dimension equal to that of $(E/I)_{n-i}$. On the other hand, $\dim_K(0:_E I^{lex})_i = \dim_K(E/I^{lex})_{n-i}$ and we are done. ▲

1.4 Local cohomology

Dualizing modules and *dualizing functors* were first introduced by Grothendieck in order to investigate properties of coherent projective varieties by means of the theory of complete local rings or graded rings. Grothendieck also proved a duality theorem, which is the algebraic counterpart of Serre's projective duality theorem. Although local cohomology finds its roots in algebraic geometry, from the time Harthshorne's notes on Grothendieck's seminars appeared, it has been deeply studied also as an algebraic object, especially by means of homological algebra. This kind of approach is taken by [BH] and [BrSh], which are the main references here. Local cohomology and local cohomology functors with respect to $\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of an arbitrary ring A, can be defined for any A-module M, even if some of its properties can be understood better in case A is local or graded with unique graded maximal ideal. In the graded case, which will be the object of our study, most of the main theorems, such as the Vanishing Theorem and the Local Duality Theorem find their natural counterpart. The present section is meant to give a quick overview of some basics of the local cohomology theory, with an emphasis, in the second part, on the graded case.

1.4.1 Generalities

Let A be a ring and $\mathfrak{a}$ any ideal of A. For each A-module M we define

$$\Gamma_{\mathfrak{a}}(M) \doteq \{x \in M : x\mathfrak{a}^k = 0 \text{ for some } k \geq 0\}.$$

Clearly $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M. One can also write $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\in\mathbb{N}}(0 :_M \mathfrak{a}^n)$. Given a homomorphism $\varphi : M \longrightarrow N$, we set $\Gamma_{\mathfrak{a}}(\varphi)$ to be the restriction map $\Gamma_{\mathfrak{a}}(M) \longrightarrow \Gamma_{\mathfrak{a}}(N)$, $\Gamma_{\mathfrak{a}}(\varphi) \doteq \varphi|_{\Gamma_{\mathfrak{a}}(M)}$. Observe that $\Gamma_{\mathfrak{a}}(\varphi)$ is well-defined, since, if $x \in \Gamma_{\mathfrak{a}}(M)$ then $x\mathfrak{a}^k = 0$ for some k. Thus $\mathfrak{a}^k\Gamma_{\mathfrak{a}}(\varphi)(x) = \mathfrak{a}^k\varphi(x) = \varphi(x\mathfrak{a}^k)$, which is zero, i.e. $\Gamma_{\mathfrak{a}}(x) \in \Gamma_{\mathfrak{a}}(N)$.
Let $\mathcal{C}(A)$ denote the category of A-modules and A-linear homomorphisms.

Proposition 1.32. $\Gamma_{\mathfrak{a}}(\ )$ *is a left-exact and covariant functor from* $\mathcal{C}(A)$ *to itself.*

Proof. See for example that of Proposition 3.5.1 in [BH]. ▲

For any $i \in \mathbb{N}$, we call the i^{th} right derived functor of $\Gamma_{\mathfrak{a}}$ the i^{th} *local cohomology functor (with respect to* $\mathfrak{a}$*)* and denote it by $H^i_{\mathfrak{a}}(\ )$. Since $\Gamma_{\mathfrak{a}}$ is left exact, one can identify $\Gamma_{\mathfrak{a}}$ with $H^0_{\mathfrak{a}}$. Given any A-module M, we call $H^i_{\mathfrak{a}}(M)$ the i^{th} *local cohomology module of* M *(with respect to* $\mathfrak{a}$*)*. Observe that, given any short exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ there is a long exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow \Gamma_{\mathfrak{a}}(M_1) \longrightarrow \Gamma_{\mathfrak{a}}(M_2) \longrightarrow \Gamma_{\mathfrak{a}}(M_3) \longrightarrow H^1_{\mathfrak{a}}(M_1) \longrightarrow \dots \\ \dots \longrightarrow H^i_{\mathfrak{a}}(M_2) \longrightarrow H^i_{\mathfrak{a}}(M_3) \longrightarrow H^{i+1}_{\mathfrak{a}}(M_1) \longrightarrow \dots . \end{aligned}$$

To compute local cohomology modules one needs to compute the homology of the complex $\Gamma_{\mathfrak{a}}(I^\bullet)$, where $0 \longrightarrow M \longrightarrow I^\bullet$ is an injective resolution of M.
The next result illustrates some other basic properties of local cohomology.

Lemma 1.33. *Let* M *any* A*-module and let* $i > 0$.

(i) If $\mathfrak{a}$ *contains a non-zero divisor of* M*, then* $\Gamma_{\mathfrak{a}}(M) = 0$.

(ii) If $\Gamma_{\mathfrak{a}}(M) = M$ *then* $H^i_{\mathfrak{a}}(M) = 0$.

(iii) $H^i_{\mathfrak{a}}(M) \simeq H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$.

Proof. (i). If $x \in \mathfrak{a}$ is a non-zerodivisor of M and $m \in \Gamma_{\mathfrak{a}}(M)$, then $mx^k \in m\mathfrak{a}^k = 0$, for some $k \in \mathbb{N}$, and the conclusion follows.
(ii). Since $\Gamma_{\mathfrak{a}}(M) = M$, one can construct inductively an injective resolution $I^\bullet$ such that $\Gamma_{\mathfrak{a}}(I^\bullet) = I^\bullet$. The local cohomology modules $H^i_{\mathfrak{a}}(M)$ can be computed by means of such a resolution of M. Hence $H^i_{\mathfrak{a}}(M) = 0$, for all $i > 0$.
(iii) follows immediately from (ii). ▲

We recall that $\mathrm{Hom}_A(A/\mathfrak{b}, M) \simeq 0 :_M \mathfrak{b}$, for any ideal $\mathfrak{b}$ of A and A-module M. Indeed, an A-linear homomorphism $\varphi : A/\mathfrak{b} \longrightarrow M$ is determined by $\varphi([1]) = m$. We denote it by φ_m. Note that φ_m is well defined iff $m \in 0 :_M \mathfrak{b}$. Thus the assignment $\varphi_m \mapsto m$ provides the sought after isomorphism. Therefore, for any h, k with $h \leq k$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(A/\mathfrak{a}^h, M) & \longrightarrow & \mathrm{Hom}_A(A/\mathfrak{a}^k, M) \\ \downarrow & & \downarrow \\ 0 :_M \mathfrak{a}^h & \longrightarrow & 0 :_M \mathfrak{a}^k \end{array}$$

is commutative, and, since the vertical arrows are isomorphisms, there is an A-isomorphism $\varinjlim_{n \in \mathbb{N}} \mathrm{Hom}_A(A/\mathfrak{a}^n, M) \simeq \Gamma_{\mathfrak{a}}(M)$, which gives rise to a natural equivalence of functors from $\mathcal{C}(A)$ to itself

$$\Gamma_{\mathfrak{a}}(\ ) \cong \varinjlim_{n \in \mathbb{N}} \mathrm{Hom}_A(A/\mathfrak{a}^n,\ ).$$

Let B be an arbitrary ring. A family $(T^i)_{i \in \mathbb{N}}$ of covariant functors from $\mathcal{C}(A)$ to $\mathcal{C}(B)$ is said to be a *negative strongly connected sequence* iff (a) for any short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

in $\mathcal{C}(A)$ there are homomorphisms $T^i(M_3) \longrightarrow T^{i+1}(M_1)$ such that the long exact sequence

$$\ldots \longrightarrow T^i(M_2) \longrightarrow T^i(M_3) \longrightarrow T^{i+1}(M_1) \longrightarrow \ldots$$

is exact and (b) given another short exact sequence such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0 \end{array}$$

is commutative, there is a map of complexes between the corresponding long exact sequences such that all the squares commute. All the family of functors we are going to consider in this section are negative strongly connected sequences. If T and T' are two functors from $\mathcal{C}(A)$ to $\mathcal{C}(B)$ which are naturally equivalent, we write $T \cong T'$.
It is a well-known result of homological algebra that, given a left-exact additive covariant functor T from $\mathcal{C}(A)$ to $\mathcal{C}(B)$, the right derived functors of T have the following universal property: If $(T^i)_{i \in \mathbb{N}}$ is a family of functors from $\mathcal{C}(A)$ to $\mathcal{C}(B)$ such that $T^0 \cong T$ and that $T^i(I) = 0$ for any injective module I of A and for any $i \in \mathbb{N}$, then T^i is naturally equivalent to the i^{th} right derived functor of T. The same holds true, dualizing, for the left derived functors of a right-exact additive contravariant functor. Henceforth, we shall often use this characterization to re-interpret the local cohomology functors.

We recall that, since direct limit is an exact functor, taking direct limit commutes with taking homology. Hence, as a first example of application of the above characterization, observing that $\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_A(A/\mathfrak{a}^n, I) = 0$, for any $i \in \mathbb{N}$ and injective A-module I, we have

$$H^i_{\mathfrak{a}}(\ ) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_A(A/\mathfrak{a}^n,\ ).$$

We denote by $D_{\mathfrak{a}}$ the left-exact functor $\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(\mathfrak{a}^n,\ )$ and identify its i^{th} right derived functor with $\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(\mathfrak{a}^n,\ )$.

Theorem 1.34. *Let* Id *denote the identity functor from $\mathcal{C}(A)$ to itself. There are natural transformations of functors $\Gamma_{\mathfrak{a}} \longrightarrow \operatorname{Id}$, $\operatorname{Id} \longrightarrow D_{\mathfrak{a}}$ and $D_{\mathfrak{a}} \longrightarrow H^1_{\mathfrak{a}}$ such that, for any A-module M,*

(i) $\Gamma_{\mathfrak{a}}(M) \longrightarrow \operatorname{Id}(M)$ *is the inclusion map;*

(ii) the sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow D_{\mathfrak{a}}(M) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

is exact.

Furthermore, for any $i \in \mathbb{N}$, the i^{th} right derived functor of $D_{\mathfrak{a}}$ is naturally equivalent to $H^{i+1}_{\mathfrak{a}}(\ )$.

Proof. The short exact sequence of A-modules

$$0 \longrightarrow \mathfrak{a}^n \longrightarrow A \longrightarrow A/\mathfrak{a}^n \longrightarrow 0$$

gives rise to the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(A/\mathfrak{a}^n, M) \longrightarrow M \longrightarrow \operatorname{Hom}_A(\mathfrak{a}^n, M) \longrightarrow \operatorname{Ext}^1_A(A/\mathfrak{a}^n, M) \longrightarrow 0$$

and to isomorphisms $\operatorname{Ext}^i_A(\mathfrak{a}^n, M) \longrightarrow \operatorname{Ext}^{i+1}_A(A/\mathfrak{a}^n, M)$, for all $i > 0$. By virtue of the natural equivalences of functors we have shown before the theorem, taking direct limits, we obtain, for any A-module M, a short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow D_{\mathfrak{a}}(M) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

and isomorphisms $\varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_A(\mathfrak{a}^n, M) \longrightarrow \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^{i+1}(A/\mathfrak{a}^n, M)$ for all $i > 0$. It remains to be shown that this construction is functorial, but this is easy. ▲

Let $\varphi : A \longrightarrow B$ a homomorphism of rings. We recall, without proof, the following fact: If φ is flat then $I \otimes_A B$ is $\Gamma_{\mathfrak{a}B}$-acyclic for any injective A-module I (i.e. $H^i_{\mathfrak{a}B}(I \otimes_A B) = 0$ for any $i > 0$). Let M be an A-module. As we have already observed, the i^{th} local cohomology modules of M can be computed as the i^{th} homology of the complex $\Gamma_{\mathfrak{a}}(I^\bullet)$, where $I^\bullet$ is an injective resolution of M. On the other hand, as we have just mentioned, $I^\bullet \otimes_A B$ is a $\Gamma_{\mathfrak{a}B}$-acyclic (not necessarily injective) resolution of $M \otimes_A B$. Therefore $H^i_{\mathfrak{a}B}(M \otimes_A B) \simeq H^i(\Gamma_{\mathfrak{a}B}(I^\bullet \otimes_A B))$. Observe now that, since $\Gamma_{\mathfrak{a}}(\ ) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_A(A/\mathfrak{a}^n, \ )$, $A/\mathfrak{a}^n$ is finitely generated and B is A-flat,

$$\begin{aligned}\Gamma_{\mathfrak{a}B}(I^\bullet \otimes_A B) &\simeq \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_B(B/\mathfrak{a}^n B, I^\bullet \otimes_A B) \\ &\simeq \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_A(A/\mathfrak{a}^n A, I^\bullet) \otimes_A B \simeq \Gamma_{\mathfrak{a}}(I^\bullet) \otimes_A B.\end{aligned}$$

Thus, we can conclude that, for any $i \in \mathbb{N}$, the functors $H^i_{\mathfrak{a}}(\ ) \otimes_A B$ and $H^i_{\mathfrak{a}B}(\ \otimes_A B)$ are naturally equivalent. This fact is known as Flat Base Change Theorem.

Now we state the well-known Vanishing Theorem for local cohomology.

Theorem 1.35 (Grothendieck). *Let M be an A-module and $\mathfrak{a}$ an arbitrary ideal of A. Let t and d denote the depth and the dimension of M respectively. Then*

(i) $H^i_{\mathfrak{a}}(M) = 0$ *for* $i > d$.

(ii) If, in addition, A is local with maximal ideal $\mathfrak{m}$ and $M \neq 0$ is finitely generated, then $H^i_{\mathfrak{m}}(M) = 0$, for $i < t$, and $H^i_{\mathfrak{m}}(M) \neq 0$, for $i = t, d$.

We recall now some standard notions such as that of Gorenstein ring, injective hull and canonical module, in order to state Grothendieck's Local Duality Theorem by means of Matlis duality.

A local ring A is a *Gorenstein ring* iff its injective dimension is finite. A ring A is a *Gorenstein ring* iff its localization at every maximal ideal is Gorenstein. Given a ring A and A-modules M, N such that $N \subseteq M$, M is said to be an *essential extension* of N iff for any $0 \neq U$ submodule of N one has $U \cap M \neq 0$. An injective A-module $E \supseteq M$ which is an essential extension of M is called the *injective hull of M*. The injective hull of a module is characterized by the property of being the unique minimal injective module, up to isomorphisms, in which M can be embedded. We recall that, if $(A, \mathfrak{m}, K)$ is local, M is finitely generated and $x_1, \ldots, x_m$ is a maximal M-sequence, then the *type of M* is the dimension as a K-vector space of $\mathrm{Soc}(M/(x_1, \ldots, x_m)M) \doteq (0 :_{M/(x_1,\ldots,x_m)M} \mathfrak{m})$. If A is local, one can show that A is Gorenstein iff A is a Cohen-Macaulay ring of type 1.

Now let $(A, \mathfrak{m}, K)$ be Cohen-Macaulay. A Cohen-Macaulay A-module M of type 1, such that the injective dimension of M is finite and $\dim M = \dim R$ is said to be a *canonical module of A*, denoted by ω_A. For a Cohen-Macaulay local ring A, the property of being Gorenstein can be characterized by the property of having a canonical module ω_A which is isomorphic to the ring itself.

Let us denote by $\mathcal{A}(A)$ (resp. $\mathcal{F}(A)$) the full subcategory of Artinian (resp. finitely generated) A-modules. Let E be the injective hull of K and $D \doteq \mathrm{Hom}_A(\ , E)$ be the *dualizing functor*. For any A-module M we let $M^\vee \doteq \mathrm{Hom}_A(M, E)$ be the *Matlis dual of M*.

Theorem 1.36. *Suppose that $(A, \mathfrak{m}, K)$ is a complete local ring and let $N \in \mathcal{A}(A)$ and $M \in \mathcal{F}(A)$. Then*

(i) $D(E) \simeq A$ and $D(A) \simeq E$;

(ii) $D(N) \in \mathcal{F}(A)$ and $D(M) \in \mathcal{A}(A)$;

(iii) $D(D(N)) \simeq N$ and $D(D(M)) \simeq M$.

Finally, we state Grothendieck's Local Duality Theorem. Observe that, if we denote by $\hat{M}$ the ($\mathfrak{m}$-adic) completion of M, by the Flat Base Change Theorem one has $H^i_{\mathfrak{m}}(M) \simeq H^i_{\hat{\mathfrak{m}}}(\hat{M})$, since $\varphi : A \longrightarrow \hat{A}$ is (faithfully) flat. This explains why in the next theorem the hypothesis of completeness is not required.

Theorem 1.37. *Let $(A, \mathfrak{m}, K)$ be a Cohen-Macaulay local ring of dimension d which is a homomorphic image of a Gorenstein ring. Then, for all finitely generated A-modules M and all $i \in \mathbb{N}$, there are natural isomorphisms*

$$H^i_{\mathfrak{m}}(M) \simeq D(\mathrm{Ext}^{d-i}_A(M, \omega_A)).$$

In other words, the i^{th} local cohomology module of M is the Matlis dual of $\mathrm{Ext}^{d-i}_A(M, \omega_A)$. In particular, for any finitely generated A-module M and all $i \in \mathbb{N}$, $H^i_{\mathfrak{m}}(M)$ is Artinian.

1.4.2 The graded case

Let ${}^*\mathcal{C}(A)$ denote the category of graded modules and homogeneous homomorphisms on a graded ring A, and M a graded A-module. Since ${}^*\mathcal{C}(A)$ has enough projectives and injectives (i.e. each module is the homomorphic image of a graded free module and can be embedded in a graded injective module in this category), one can develop the same definitions and argumentations in this category as those we introduced before. For any graded ideal $\mathfrak{a} \subseteq A$, $\Gamma_{\mathfrak{a}}$ can be viewed as a functor from ${}^*\mathcal{C}(A)$ to itself. We denote it by ${}^*\Gamma_{\mathfrak{a}}$ and its right derived functors by ${}^*\mathrm{H}^{\bullet}_{\mathfrak{a}}$.
In fact $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M and therefore inherits a natural structure of graded module. The local cohomology modules of M can be computed as the homology of the complex $\Gamma_{\mathfrak{a}}(I^{\bullet})$, where $I^{\bullet}$ is a resolution of M by graded injective A-modules in $\mathcal{C}(A)$, and in addition, inherit a natural graded structure. On the other hand, local cohomology modules can be re-interpreted as the direct limit of $\mathrm{Ext}^{\bullet}_A(A/\mathfrak{a}^n, M)$, which are endowed with a graded structure as well. *A priori* these gradings might be different from that which descend from the standard theory of homological algebra in the category ${}^*\mathcal{C}(A)$. We sketch now some of the ideas of the proof that these different approaches are coherent and recover a graded version of the results we stated in the previous section.

Let B a graded ring. A covariant functor $T : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$ is said to have the **restriction property r.p* iff (a) for any graded A-module M, $T(M)$ is graded as a B-module and (b) given a homogeneous homomorphism $\varphi : M \longrightarrow N$ of graded

A-modules, $T(\varphi)$ is a homogeneous homomorphism of graded B-modules. In other words T has r.p iff the restriction of T to ${}^*\mathcal{C}(A)$, denoted by $T|_*$, is a functor between ${}^*\mathcal{C}(A)$ and ${}^*C(B)$.
A negative strongly connected sequence of covariant functors from $\mathcal{C}(A)$ to $\mathcal{C}(B)$ has the **restriction property r.p* iff any of the functors of the family has r.p and, for all short exact sequences in ${}^*\mathcal{C}(A)$, all connecting homomorphisms in the corresponding long exact sequences are homogeneous. The following crucial fact constitutes the equivalent of what we called the universal property of right derived functors.
Let $(T^i)_{i\in\mathbb{N}}$, be a negative strongly connected sequence of covariant additive functors from $\mathcal{C}(A)$ to $\mathcal{C}(B)$, such that for any graded A-module M the B-module $T^0(M)$ has a grading and T^0 has r.p with respect to these gradings. If $T^i(I) = 0$ for any graded injective A-module I and all $i \in \mathbb{N}$, then there is exactly one grading on $T^i(M)$ such that any of the T^i has r.p. Furthermore, there is a natural equivalence of functors between $T^i|_*$ and the i^{th} right derived functors of $T^0|_*$ for any i.
We recall now that, given graded A-modules M, N, one defines ${}^*\mathrm{Ext}^i_A(\ \ , N)$ as the i^{th} right derived functor of ${}^*\mathrm{Hom}_A(\ \ , N)$, where ${}^*\mathrm{Hom}_A(M, N)$ is the graded A-module whose j^{th} homogeneous component is the group of graded homomorphisms of degree j. Note that a graded module I is injective in ${}^*\mathcal{C}(A)$ if it is injective in this category. We say that I is *injective. If M is finitely generated, ${}^*\mathrm{Ext}^i_A(M, N)$ is, as an A-module, just $\mathrm{Ext}^i_A(M, N)$. One can prove that ${}^*\mathrm{Ext}^i_A(M, I) = 0$, for all $i > 0$ and for all *injective modules I. Clearly, it follows that any graded *injective A-module I is $\Gamma_\mathfrak{a}$-acyclic. In fact, since $R/\mathfrak{a}^n$ is finitely generated, $H^i_\mathfrak{a}(I) = \varinjlim_{n\in\mathbb{N}} {}^*\mathrm{Ext}^i_A(R/\mathfrak{a}^n, I) = 0$, for all $i \in \mathbb{N}$.
It is immediately seen that $\Gamma_\mathfrak{a}$ has the restriction property and that ${}^*\Gamma_\mathfrak{a} = \Gamma|_*$. Thus, applying the previous fact on the family of right derived functors of $\Gamma_\mathfrak{a}$, we deduce that there is exactly one grading on $H^i_\mathfrak{a}(M)$ such that $H^i_\mathfrak{a}$ has r.p. Therefore, with respect to this grading, we have a unique isomorphism between $H^i_\mathfrak{a}|_*$ and ${}^*\mathrm{H}^i_\mathfrak{a}$. One can also verify the family of functors $\varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_A(A/\mathfrak{a}^n,\ \ )$, $i \in \mathbb{N}$, has r.p. Thus, it is straightforward that the graded structure induced on $H^i_\mathfrak{a}(M)$ by that of $\varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i(R/\mathfrak{a}^n, M)$ is exactly that unique grading with respect to which $H^i_\mathfrak{a}$ has the *restriction property.

All the results of the previous section now find a natural counterpart in the graded case. In particular, for any short exact sequence of graded A-modules, all the homomorphisms in the long exact sequence in cohomology are homogeneous. From now on, whenever A, $\mathfrak{a}$ and M are graded and we consider the local cohomology modules of M, then we shall make no distinction between $H^\bullet_\mathfrak{a}(\ \ )$ and ${}^*\mathrm{H}^\bullet_\mathfrak{a}(\ \ )$. We shall simply write $H^\bullet_\mathfrak{a}(\ \ )$ and consider them endowed with their natural graded structure.
We state again the Local Duality Theorem, when $A = R$ is a polynomial ring in n variables over a field K, because it is the case we shall consider henceforth.

Theorem 1.38. *Let R be the polynomial ring $K[X_1, \ldots, X_n]$ with the standard grading and with graded maximal ideal $\mathfrak{m}$. Then, for any finitely generated and* *graded R-module M and for all $i \in \mathbb{N}$, there is a natural homogeneous isomorphism*

$$(H^i_{\mathfrak{m}}(M))^{\vee} \simeq \operatorname{Ext}^{n-i}_R(M, R(-n)).$$

Note that, for any j, $\dim_K H^i_{\mathfrak{m}}(M)_j = \dim_K \operatorname{Ext}^{n-i}_R(M,R)_{-n-j}$, or, equivalently, that

$$\operatorname{Hilb}(H^i_{\mathfrak{m}}(M), t^{-1}) = t^n \operatorname{Hilb}(\operatorname{Ext}^{n-i}_R(M,R), t).$$

Remarks 1.39.

(i) Suppose that the base field K is finite. Let $K' \doteq K[t]_{(0)}$ be the localization of the polynomial ring $K[t]$ at the multiplicative system $K[t] \setminus \{0\}$. Clearly, K' is infinite. Set $R' \doteq R \otimes_K K' \simeq K'[X_1, \dots, X_n]$, $\mathfrak{m}' \doteq \mathfrak{m} \otimes_K K'$ and $M' \doteq M \otimes_K K'$. Since R' is flat over R, we may apply the graded version of the Flat Base Change Theorem and obtain that, for all i, j and graded R-modules M, $H^i_{\mathfrak{m}}(M)_j \otimes_K K' \simeq H^i_{\mathfrak{m}'}(M')_j$. In particular, one has

$$H(H^i_{\mathfrak{m}}(M), j) = H(H^i_{\mathfrak{m}'}(M'), j).$$

Therefore, anytime we investigate the Hilbert function of the local cohomology modules, we may assume without loss of generality, that the base field K is infinite.

(ii) A very important computational tool for local cohomology is the *Čech complex*. Although we have omitted this approach, for our purposes it is sufficient to know that the local cohomology modules of M with respect to $\mathfrak{a}$ are isomorphic to the homology of the Čech complex $C^{\bullet}(M)$ with respect to $x_1, \dots, x_m$, if $\mathfrak{a} = (x_1, \dots, x_m)$.
We recall the following property of the local cohomology modules of a Stanley-Reisner ring $K[\Delta] \doteq R/I_{\Delta}$.
For any multi-homogeneous element x of R, one defines a $\mathbb{Z}^n$-grading on R_x by setting

$$(R_x)_j \doteq \{r/x^h : r \in R_d,\ j = d - h \deg x\}.$$

The Čech-complex of R and, consequently, $C^{\bullet}(K[\Delta])$ and its homology modules, becomes a structure of complex of multi-graded R-modules and multi-homogeneous homomorphisms. Therefore $H^{\bullet}_{\mathfrak{m}}(K[\Delta])$ has a natural $\mathbb{Z}^n$-graded structure (which is compatible with those already mentioned). Since $C^{\bullet}_j \simeq \bigoplus_{\substack{a \in \mathbb{Z}^n \\ |a|=j}} C^i_a$, we have

$$H^i_{\mathfrak{m}}(K[\Delta])_j \simeq \bigoplus_{\substack{a \in \mathbb{Z}^n \\ |a|=j}} H^i_{\mathfrak{m}}(K\Delta)_a.$$

We conclude this section with an example, which anticipates in a special case what we shall prove in Chapter 3.

Example 1.40. Let $R = K[X_1, \ldots, X_n]$, $\mathfrak{m} = (X_1, \ldots, X_n)$ the graded maximal ideal of R and I a monomial ideal of R.
The 0^{th} local cohomology of R/I is $\bigcup_{n\in\mathbb{N}}(0 :_{R/I} \mathfrak{m}^n) \simeq I : \mathfrak{m}^\infty/I$, where $I : \mathfrak{m}^\infty$ denotes the saturation of I as defined in Section 1.3. Therefore the natural grading on $\Gamma_\mathfrak{m}(R/I)$ is the standard grading of $I : \mathfrak{m}^\infty/I$ induced by that of R. In the same way, one has $\Gamma_\mathfrak{m}(R/I^{lex}) \simeq I^{lex} : \mathfrak{m}^\infty/I^{lex}$.
It is interesting to compare the Hilbert functions of $\Gamma_\mathfrak{m}(R/I)$ and $\Gamma_\mathfrak{m}(R/I^{lex})$. For this purpose, observe that, by Lemma 1.25, $\dim_K(I : \mathfrak{m}^\infty)_j \leq \dim_K(I^{lex} : \mathfrak{m}^\infty)_j$ and that, from the very definition, $\dim_K I_j = \dim_K I^{lex}_j$ for any $j \in \mathbb{N}$. Therefore, for any j,

$$\dim_K \Gamma_\mathfrak{m}(R/I)_j \leq \dim_K \Gamma_\mathfrak{m}(R/I^{lex})_j.$$

2 The polarization functor

2.1 Introduction

This chapter is devoted to the definition of a new functor, which we call the *polarization functor* and denote it by $\ ^{\mathbf{p}}$. Polarization for monomial ideals is well-known: It is an algebraic operation associating to any monomial ideal I of a polynomial ring R a squarefree monomial ideal J in a "sufficiently" large polynomial ring S, which is squarefree and still closely related to I. The main properties of J is that it is a "lifting" of I to S and that all of the Betti numbers of J are the same as those of I. The last result is a well-known theorem attributed to Fröberg (see [F]). That J is a lifting of I was proved in [BH1]. There it was proven more generally that any finite multi-graded module M with generators of degrees in $\mathbb{N}^n$ can be lifted to a module N over a new polynomial ring S, such that the minimal resolution of N by multi-graded and free S-modules is squarefree. Here squarefree means that, for any of the shifts appearing in the resolution, all the entries are either 0 or 1.

In the first section we define the functor $\ ^{\mathbf{p}}$ on the category of finite multi-graded R-modules by means of "canonical resolution" and prove that $\ ^{\mathbf{p}}$ is additive. From the very definition, it would be rather hard to compute the polarization of an R-module. Thus, we show that, for any R-module M, one can compute $M^{\mathbf{p}}$ operating on the minimal resolution of M. This fact follows easily from Proposition 2.6, where we compute, for any $a \in \mathbb{Z}^n$, what the polarization of $R(-a)$ is. The next step will be proving that $\ ^{\mathbf{p}}$ is exact. This is performed in Theorem 2.11. As a consequence, it is easy to verify that $\ ^{\mathbf{p}}$ commutes with the functor $\mathrm{Hom}_R(\ \ , R)$, and consequently that $\mathrm{Ext}^i_R(M,R)^{\mathbf{p}} \simeq \mathrm{Ext}^i_{R^{\mathbf{p}}}(M^{\mathbf{p}}, R^{\mathbf{p}})$, for any M and i. Unfortunately, $\ ^{\mathbf{p}}$ does not commute with $\mathrm{Hom}_R(\ \ , N)$ or $\ \otimes_R N$, for an arbitrary R-module N, as one maybe could expect.

In the second section we prove in Proposition 2.14 an extension of the theorem in [BH1] we have cited and recover polarization for monomial ideals by means of what we shall call *complete polarization*. In the last result of this chapter, Corollary 2.18, we explain how we are able to control the Hilbert series of $H^i_{\mathfrak{m}}(M)$, passing from M to one of its complete polarizations. This fact will be of decisive importance in the proof of Theorem 3.15.

2.2 Polarization

In this section we define the polarization functor and study its main properties. We start by defining the polarization of an element of $\mathbb{Z}^n$.

Definition 2.1. Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. We define the *polarization of* a to be the vector $a^{\mathbf{P}} \in \mathbb{Z}^{n+1}$ as follows

$$a^{\mathbf{P}} \doteq \begin{cases} (a, 0), & \text{if } |a_n| \leq 1 \\ (a_1, \ldots, a_{n-1}, a_n - 1, 1), & \text{if } a_n > 1 \\ (a_1, \ldots, a_{n-1}, a_n + 1, -1), & \text{if } a_n < -1. \end{cases}$$

Note that $(0_{\mathbb{Z}^n})^{\mathbf{P}} = 0_{\mathbb{Z}^{n+1}}$.
We proceed by illustrating some easy properties of this operation.

Lemma 2.2. *Let $a, b \in \mathbb{Z}^n$. Then*

(i) $-a^{\mathbf{P}} = (-a)^{\mathbf{P}}$*;*

(ii) if $a - b \in \mathbb{N}^n$ *then* $a^{\mathbf{P}} - b^{\mathbf{P}} \in \mathbb{N}^{n+1}$*;*

(iii) $(a^{\mathbf{P}} - b^{\mathbf{P}})_{n+1} + (a^{\mathbf{P}} - b^{\mathbf{P}})_n = a_n - b_n$.

Proof. (i). Trivial.
To verify (ii) one has to check the six possible cases (three are already excluded from the assumption that $a - b \in \mathbb{N}^n$), but this is an easy task.
(iii) is clear since, for any $c \in \mathbb{Z}^n$, $(c^{\mathbf{P}})_{n+1} + (c^{\mathbf{P}})_n = c_n$. ▲

We consider now $R = K[X_1, \ldots, X_n]$ to be the polynomial ring in n variables with its natural multi-graded structure. Recall that a polynomial f is said to be (multi-)homogeneous of degree $a \in \mathbb{Z}^n$ if $f = \lambda X_1^{a_1} \cdot \ldots \cdot X_n^{a_n}$, where $a = (a_1, \ldots, a_n)$ and $\lambda \in K$. We let $S \doteq R[X_{n+1}]$, also considered with its natural fine grading over $\mathbb{Z}^{n+1}$. For the rest of the section, any module will be considered to be finite and multi-graded. All the maps will be meant to be multi-homogeneous (of degree 0).
Given a homogeneous element $y \in R_c$, $y = \lambda X^c$ with $c \in \mathbb{Z}^n$ and $\lambda \in K$, we let $y^{\mathbf{P}} \doteq \lambda X^{c^{\mathbf{P}}} \in S_{c^{\mathbf{P}}}$.
Now we introduce the canonical resolution of M. For any multi-graded module N, we let $\mathcal{H}(N)$ denote the set of all non-zero multi-homogeneous elements of N. Clearly $\mathcal{H}(N) = \bigcup_{a \in \mathbb{Z}^n} N_a \setminus \{0\}$. We construct the canonical multi-graded free resolution of M as follows. Let

$$K_0(M) \doteq \oplus_{y \in \mathcal{H}(M)} R(-\deg y) \xrightarrow{\alpha_0} M,$$

where α_0 is defined by $e_y^0 \mapsto y$ and e_y^0 denotes the element of the canonical basis such that $(e_y^0)_z = \delta_{yz}$, for any $z \in \mathcal{H}(M)$. The multi-degree of e_y^0 is set to be $\deg y$, for any $y \in \mathcal{H}(M)$. Inductively, for every $i > 0$, let

$$K_i(M) \doteq \oplus_{y \in \mathcal{H}(\ker \alpha_{i-1})} R(-\deg y),$$

and $K_i(M) \xrightarrow{\alpha_i} K_{i-1}(M)$ be defined by $\alpha_i(e^i_y) \doteq y$, for any $y \in \mathcal{H}(\operatorname{Ker}\alpha_{i-1})$. The multi-degree of the element e^i_y is defined to be $\deg y$, for any $y \in \mathcal{H}(\operatorname{Ker}\alpha_{i-1})$. We have thus obtained, in a canonical way, a multi-graded resolution of M, denoted by $K_{\bullet}(M)$.

Now we define $L_{\bullet}(M)$ to be a complex of S-modules associated to M by "lifting" its canonical resolution. To be precise, we let, for any $i \in \mathbb{N}$,

$$L_i(M) \doteq \oplus_{y \in \mathcal{H}(\operatorname{Ker}\alpha_{i-1})} S(-\deg y^{\mathbf{P}}).$$

The maps, denoted by $\tilde{\alpha}_i$, are defined in the natural way from the shifts, in order to obtain multi-homogeneous homomorphisms. We explain the last statement. For any i, the maps α_i of the canonical resolution are determined by the images of the elements of the canonical basis $\alpha_i(e^i_y) = \sum_z a^{i-1}_{yz} X^{\deg y - \deg z} e^{i-1}_z$, for some $a^{i-1}_{yz} \in K$. Let $\tilde{e}^i_y$ denote the element of the canonical basis of $L_i(M)$, for any $y \in \mathcal{H}(\operatorname{Ker}(\alpha_{i-1}))$ and let $\deg \tilde{e}^i_y \doteq \deg y^{\mathbf{P}}$. We define $\tilde{\alpha}_i : L_i(M) \longrightarrow L_{i-1}(M)$ to be

$$\tilde{\alpha}_i(\tilde{e}^i_y) \doteq \sum_z a^{i-1}_{yz} X^{\deg y^{\mathbf{P}} - \deg z^{\mathbf{P}}} \tilde{e}^{i-1}_z.$$

One has to verify that the homomorphism is well-defined, i.e. that $\deg y^{\mathbf{P}} - \deg z^{\mathbf{P}} \in \mathbb{N}^{n+1}$. But since $\deg y - \deg z \in \mathbb{N}^n$, this follows from Lemma 2.2 (ii). It is clear that, for any i, $\tilde{\alpha}_i$ is multi-homogeneous. It remains to be checked that $L_{\bullet}(M)$ is a complex. Since

$$\begin{aligned}
\alpha_i \circ \alpha_{i-1}(e^i_y) &= \sum_z a^{i-1}_{yz} X^{\deg y - \deg z} (\sum_w a^{i-2}_{zw} X^{\deg z - \deg w} e^{i-2}_w) \\
&= \sum_w (\sum_z a^{i-1}_{yz} a^{i-2}_{zw}) X^{\deg y - \deg w} e^{i-2}_w = 0,
\end{aligned}$$

we deduce that $\sum_z a^{i-1}_{yz} a^{i-2}_{zw} = 0$, for every w. On the other hand $\tilde{\alpha}_i \circ \tilde{\alpha}_{i-1}(\tilde{e}^i_y) = \sum_w (\sum_z a^{i-1}_{yz} a^{i-2}_{zw}) X^{\deg y^{\mathbf{P}} - \deg w^{\mathbf{P}}} \tilde{e}^{i-2}_w$, which is also 0 since the sum over z involves the same coefficients.

The next step is, given a homomorphism $\varphi_i : K_i(M) \longrightarrow K_i(N)$ between the canonical resolutions of any two modules M and N, to "lift" it to a new map $\tilde{\varphi}_i : L_i(M) \longrightarrow L_i(N)$. This can be done, in complete analogy with the above procedure, simply by letting

$$\tilde{\varphi}_i(\tilde{e}^i_y) \doteq \sum_z c^i_{yz} X^{\deg y^{\mathbf{P}} - \deg z^{\mathbf{P}}} \tilde{f}^i_z,$$

where $\{e^i_y\}_{y \in \mathcal{H}(\operatorname{Ker}\alpha_{i-1})}$ and $\{f^i_z\}_{z \in \mathcal{H}(\operatorname{Ker}\beta_{i-1})}$ denote the bases of $K_i(M)$ and $K_i(N)$ respectively and $\varphi_i(e^i_y) = \sum_z c^i_{yz} X^{\deg y - \deg z} f^i_z$, with $c^i_{yz} \in K$.

Note that $\widetilde{\mathrm{id}}_{K_i(M)} = \mathrm{id}_{L_i(M)}$.

Lemma 2.3. *Let U, M and N be any R-modules and $K_{\bullet}(\ )$ and $L_{\bullet}(\ )$ be defined as above.*

(i) *Given a commutative diagram of R-homomorphisms*

$$\begin{array}{ccc} K_i(M) & \xrightarrow{\alpha_i} & K_{i-1}(M) \\ \Big\downarrow \varphi_i & & \Big\downarrow \varphi_{i-1} \\ K_i(N) & \xrightarrow{\beta_i} & K_{i-1}(N), \end{array}$$

then the diagram of S-homomorphisms

$$\begin{array}{ccc} L_i(M) & \xrightarrow{\tilde{\alpha}_i} & L_{i-1}(M) \\ \Big\downarrow \tilde{\varphi}_i & & \Big\downarrow \tilde{\varphi}_{i-1} \\ L_i(N) & \xrightarrow{\tilde{\beta}_i} & L_{i-1}(N) \end{array}$$

also commutes.

(ii) *Given any two R-homomorphisms $K_i(U) \xrightarrow{\varphi} K_i(M) \xrightarrow{\psi} K_i(N)$, the diagram*

$$\begin{array}{ccc} L_i(U) & \xrightarrow{\widetilde{\varphi \circ \psi}} & L_i(N) \\ \Big\downarrow \tilde{\varphi} & & \Big\| \\ L_i(M) & \xrightarrow{\tilde{\psi}} & L_i(N) \end{array}$$

is commutative.

Proof. Both statements follow from easy computations. ▲

We are ready to define the polarization functor by means of canonical resolution. Let $M \xrightarrow{\varphi} N$ be an R-homomorphism, and let us consider the canonical resolutions $K_\bullet(M)$ and $K_\bullet(N)$ of M and N, with bases e^i_y and f^i_z, where $y \in \mathcal{H}(\operatorname{Ker} \alpha_{i-1})$ and $z \in \mathcal{H}(\operatorname{Ker} \beta_{i-1})$. One can construct the following diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & K_1(M) & \xrightarrow{\alpha_1} & K_0(M) & \xrightarrow{\alpha_0} & M & \longrightarrow & 0 \\ & & \Big\downarrow \varphi_1 & & \Big\downarrow \varphi_0 & & \Big\downarrow \varphi & & \\ \dots & \longrightarrow & K_1(N) & \xrightarrow{\beta_1} & K_0(N) & \xrightarrow{\beta_0} & N & \longrightarrow & 0, \end{array}$$

where, for any i, φ_i is called the i^{th} *canonical lifting* and is determined recursively so that the squares commute. To be precise, for any $i > 0$,

$$\varphi_0(e^0_x) \doteq f^0_{\varphi(x)}, \;\; \forall x \in \mathcal{H}(M) \qquad \text{and} \qquad \varphi_i(e^i_x) \doteq f^i_{\varphi_{i-1}(x)}, \;\; \forall x \in \mathcal{H}(\operatorname{Ker} \alpha_{i-1}).$$

From the first part of the previous lemma, we obtain a diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & L_1(M) & \xrightarrow{\tilde{\alpha}_1} & L_0(M) & \xrightarrow{\pi} & \operatorname{Coker}\tilde{\alpha}_1 & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \tilde{\varphi}_1} & & \Big\downarrow{\scriptstyle \tilde{\varphi}_0} & & & & \\
\dots & \longrightarrow & L_1(N) & \xrightarrow{\tilde{\beta}_1} & L_0(N) & \xrightarrow{\pi} & \operatorname{Coker}\tilde{\beta}_1 & \longrightarrow & 0,
\end{array}$$

which is commutative as well, observing that $\tilde{\varphi}_i(\tilde{e}^i_x) = \tilde{f}^i_{\varphi_{i-1}(x)}$ for any i. Thus, a homomorphism $\tilde{\varphi} : \operatorname{Coker}\tilde{\alpha}_1 \longrightarrow \operatorname{Coker}\tilde{\beta}_1$ is induced by $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$.

Definition 2.4. With the above notation, we define the *polarization of* M to be the multi-graded S-module

$$M^{\,\mathbf{P}} \doteq \operatorname{Coker}\tilde{\alpha}_1.$$

If $M \xrightarrow{\varphi} N$ is a multi-homogeneous R-module homomorphism, we define the *polarization of* φ to be the multi-homogeneous S-module homomorphism $M^{\,\mathbf{P}} \xrightarrow{\varphi^{\mathbf{P}}} N^{\,\mathbf{P}}$,

$$\varphi^{\,\mathbf{P}} \doteq \tilde{\varphi}.$$

In the following theorem we prove that this construction is functorial.

Theorem 2.5. $^{\mathbf{P}}$ *is an additive covariant functor between the categories of finite multi-graded R-modules and finite multi-graded S-modules.*

Proof. It is obvious that the identity is preserved. Let now $U \xrightarrow{\varphi} M \xrightarrow{\psi} N$ be two multi-homogeneous R-homomorphisms. From Lemma 2.3 (ii) we deduce that the tilde of the 0^{th} canonical lifting $(\varphi\circ\psi)_0$, which is the tilde of $\varphi_0\circ\psi_0$, is $\tilde{\varphi}_0\circ\tilde{\psi}_0$. This implies that $(\varphi\circ\psi)^{\,\mathbf{P}} = \varphi^{\,\mathbf{P}}\circ\psi^{\,\mathbf{P}}$. We have thus proven that $^{\mathbf{P}}$ is a functor. It is clear that $^{\mathbf{P}}$ is covariant. It still must be shown that, if $\varphi, \psi : M \longrightarrow N$ are multi-homogeneous, then $(\varphi+\psi)^{\,\mathbf{P}} = \varphi^{\,\mathbf{P}} + \psi^{\,\mathbf{P}}$. If $x \in \mathcal{H}(M)$, then $(\varphi+\psi)_0(e^0_x) = f^0_{\varphi(x)+\psi(x)}$ and $\varphi_0(e^0_x) + \psi_0(e^0_x) = f^0_{\varphi(x)} + f^0_{\psi(x)}$. The element $\omega \doteq f^0_{\varphi(x)+\psi(x)} - f^0_{\varphi(x)} - f^0_{\psi(x)}$ is multi-homogeneous of degree x. Since $\omega \in \operatorname{Ker}\beta_0$, by definition of $K_{\bullet}(N)$ it follows that $\beta_1(f^1_\omega) = \omega$. This implies that $\tilde{\beta}_1(\tilde{f}^1_\omega) = \tilde{f}^0_{\varphi(x)+\psi(x)} - \tilde{f}^0_{\varphi(x)} - \tilde{f}^0_{\psi(x)}$ and we are done. ▲

What we had in mind when we defined the functor $^{\mathbf{P}}$ is that, in case M is a free R-module, one should be able to compute $M^{\,\mathbf{P}}$ simply by substituting R with S and operating on the shifts in the sense of Definition 2.1. Although it is not immediately evident from the definition that this is possible, since the canonical resolution is rather intricate, one can recover this fact by proving the next proposition, which clarifies what happens to the ring R itself when it is polarized.

Proposition 2.6. *Let $a \in \mathbb{Z}^n$ and $M \simeq R(-a)$. Then $M^{\,\mathbf{P}} \simeq S(-a^{\,\mathbf{P}})$.*

Proof. First, observe that, given an arbitrary $y = \lambda X^c$ in $R(-a)$, the degree of the corresponding basis element e^0_y is $\deg e^0_y = c - a$. If $z \in \mathcal{H}(\operatorname{Ker}\alpha_0)$, then

$z = \alpha_1(e^1_z)$. Therefore z can be expressed through elements of the basis of $K_0(R(-a))$ as $\sum_y a_{zy} X^{\deg z - (\deg y - a)} e^0_y$, for some coefficients $a_{zy} \in K$ and homogeneous elements y of R. For any y appearing in the previous expression we write $y = \lambda_y X^{\deg y}$, with $\lambda_y \in K$. Since z belongs to the kernel of α_0, we have $0 = \alpha_0(z) = \sum_y a_{zy}\lambda_y X^{\deg z}$, which implies $\sum_y a_{zy}\lambda_y = 0$.
Let us now consider, for every $x \in \mathcal{H}(R(-a))$, the element $r_x \doteq e^0_x - x e^0_1$. Clearly $r_x \in \operatorname{Ker}\alpha_0$ for every $x \in \mathcal{H}(R(-a))$. Thus,

$$\begin{aligned} z &= \sum_y a_{zy} X^{\deg z - \deg y + a}(r_y + y e^0_1) \\ &= \sum_y a_{zy} X^{\deg z - \deg y + a} r_y + (\sum_y a_{zy}\lambda_y) X^{\deg z + a} e^0_1. \end{aligned}$$

Therefore $z = \sum_y a_{zy} X^{\deg z - \deg y + a} r_y$, and this shows that $\operatorname{Im}\alpha_1$ is generated by the set of all elements r_y, for $y \in \mathcal{H}(R(-a))$. Since r_y is homogeneous of degree $\deg y - a$ and belongs to $\operatorname{Ker}\alpha_0$, there exists $e^1_{r_y} \in K_1(R(-a))$ such that $\alpha_1(e^1_{r_y}) = r_y = e^0_y - \lambda_y X^{(\deg y - a) + a} e^0_1$. Here we write $(\deg y - a) + a$, because we are going to lift the map to $\tilde{\alpha}_1$. To be even more precise, we should write $(\deg y - a) - (-a)$, but from the first part of Lemma 2.2 $(-a)^{\mathbf{P}} = -a^{\mathbf{P}}$. Thus, $\tilde{\alpha}_1(\tilde{e}^1_{r_y}) = \tilde{e}^0_y - \lambda_y X^{(\deg y - a)^{\mathbf{P}} + a^{\mathbf{P}}} \tilde{e}^0_1$, which is homogeneous of degree $(\deg y - a)^{\mathbf{P}}$, since $\tilde{e}^0_1$ has degree $-a^{\mathbf{P}}$. Let this element be denoted by s_y. It is easy to see that $\operatorname{Im}\tilde{\alpha}_1 = \operatorname{span}_S\{s_y \colon y \in \mathcal{H}(R(-a))\}$. In fact, one has to prove that, for any homogeneous element z of $\operatorname{Ker}\alpha_0$, $\tilde{\alpha}_1(\tilde{e}^1_z)$ can be written as a combination of s_y for some $y \in \mathcal{H}(\operatorname{Ker}\alpha_0)$. But for this purpose one can simply repeat the preceding argument which showed that $\operatorname{Im}(\alpha_1)$ is spanned by $\{r_y \colon y \in \mathcal{H}(R(-a))\}$.
Our final goal is to construct an isomorphism between $\operatorname{Coker}\tilde{\alpha}_1$ and $S(-a^{\mathbf{P}})$. We first define a surjection $\rho : L_0(R(-a)) \longrightarrow S(-a^{\mathbf{P}})$ and then prove that $\operatorname{Ker}\rho \simeq \operatorname{Im}\tilde{\alpha}_1$. In this way we shall have $R(-a)^{\mathbf{P}} \simeq L_0(R(-a))/\operatorname{Ker}\rho \simeq S(-a^{\mathbf{P}})$. We thus define ρ on the basis elements $\tilde{e}^0_y$ to be $\rho(\tilde{e}^0_y) \doteq \lambda_y X^{(\deg y - a)^{\mathbf{P}} + a^{\mathbf{P}}}$. Obviously, $\rho(\tilde{e}^0_1) = 1 \cdot X^{-a^{\mathbf{P}} + a^{\mathbf{P}}} = 1$ and therefore ρ is surjective. Let z be any element in $\operatorname{Ker}\rho$, say $z = \sum_y \mu_y \tilde{e}^0_y$. Then, $0 = \rho(z) = \sum_y \mu_y \lambda_y X^{(\deg y - a)^{\mathbf{P}} + a^{\mathbf{P}}}$. Thus,

$$z = \sum_y \mu_y s_y + (\sum_y \mu_y \lambda_y X^{(\deg y - a)^{\mathbf{P}} + a^{\mathbf{P}}})\tilde{e}^0_1 = \sum_y \mu_y s_y.$$

Since the other inclusion is obvious, we have shown that $\operatorname{Ker}\rho = \operatorname{Im}\tilde{\alpha}_1$.
The induced map $\overline{\rho} : R(-a)^{\mathbf{P}} \longrightarrow S(-a^{\mathbf{P}})$ is the desired isomorphism. ▲

It now follows from the additivity of $^{\mathbf{P}}$ that, if F is a finite free R-module $F = \oplus_i R(-a_i)$, then $F^{\mathbf{P}} = \oplus_i S(-a_i{}^{\mathbf{P}})$.

Proposition 2.7.

(i) If $\varphi : R(-a) \longrightarrow R(-b)$ is the map defined by $\varphi(1) = \lambda X^{a-b}$, with $\lambda \in K$, then $\varphi^{\mathbf{P}}$ can be identified with a map $\Phi : S(-a^{\mathbf{P}}) \longrightarrow S(-b^{\mathbf{P}})$ given by $\Phi(1) = \lambda X^{a^{\mathbf{P}} - b^{\mathbf{P}}}$.

(ii) Let $F \xrightarrow{\varphi} G$ be a multi-homogeneous map between the finite free multi-graded modules $F = \oplus_{i=1}^{k} R(-a_i)$ and $G = \oplus_{j=1}^{h} R(-b_j)$ and let $(\lambda_{ij} X^{a_i - b_j})_{ij}$ be the matrix which represents φ.
Then $\varphi^{\mathbf{P}}$ is represented by the matrix $(\lambda_{ij} X^{a_i{}^{\mathbf{P}} - b_j{}^{\mathbf{P}}})_{ij}$.

Proof. (i) Looking back at the proof of Proposition 2.6, we get a diagram

$$\begin{array}{ccc} S(-a^{\mathbf{P}}) & \xrightarrow{\Phi} & S(-b^{\mathbf{P}}) \\ \uparrow{\scriptstyle \overline{\rho}} & & \uparrow{\scriptstyle \overline{\rho}} \\ R(-a)^{\mathbf{P}} & \xrightarrow{\varphi^{\mathbf{P}}} & R(-b)^{\mathbf{P}}, \end{array}$$

where the vertical arrows are isomorphisms. We prove that the square is commutative. Let [ ] denote the equivalence class of an element in the domain of $\overline{\rho}$. One can immediately see that $\Phi(\overline{\rho}([\tilde{e}_1^0])) = \Phi(1) = \lambda X^{a^{\mathbf{P}} - b^{\mathbf{P}}}$. On the other hand, recalling the definition of $\varphi^{\mathbf{P}}$, one has that $\varphi^{\mathbf{P}}$ acts on the equivalence class of any element of the canonical basis as the tilde of the 0^{th} canonical lifting. Since $\varphi_0(e_x^0) \doteq f_{\varphi(x)}^0$ and, accordingly, $\tilde{\varphi}_0(\tilde{e}_x^0) = \tilde{f}_{\varphi(x)}^0$, we get that $\varphi^{\mathbf{P}}([\tilde{e}_x^0]) = [\tilde{f}_{\varphi(x)}^0]$. Thus, being $\varphi(1) \doteq \lambda X^{a-b}$, we have $\overline{\rho}(\varphi^{\mathbf{P}}([\tilde{e}_1^0])) = \overline{\rho}([\tilde{f}_{\lambda X^{a-b}}^0])$. Now observe that the element $\tilde{f}_{\lambda X^{a-b}}^0 - \lambda X^{(a-b+b)^{\mathbf{P}} - b^{\mathbf{P}}} \tilde{f}_1^0$ is homogeneous of degree $a^{\mathbf{P}}$. But this implies that it belongs to $\operatorname{Im} \tilde{\beta}_1$. Therefore $[\tilde{f}_{\lambda X^{a-b}}^0] = [\lambda X^{(a-b+b)^{\mathbf{P}} - b^{\mathbf{P}}} \tilde{f}_1^0]$ and

$$\overline{\rho}([\tilde{f}_{\lambda X^{a-b}}^0]) = \lambda X^{(a-b+b)^{\mathbf{P}} - b^{\mathbf{P}}} \overline{\rho}([\tilde{f}_1^0]) = \lambda X^{a^{\mathbf{P}} - b^{\mathbf{P}}},$$

i.e. $\overline{\rho}(\varphi^{\mathbf{P}}([\tilde{e}_1^0])) = \lambda X^{a^{\mathbf{P}} - b^{\mathbf{P}}}$, as desired.
(ii) follows directly from (i). ▲

Let A be any graded ring and $x_1, \ldots, x_m$ be a sequence in the Jacobson radical of A. A graded $A/(x_1, \ldots, x_m)A$-module M is said to be *liftable to A* iff there exists a graded A-module N for which $x_1, \ldots, x_m$ is an N-regular sequence and such that $N/(x_1, \ldots, x_m)N$ is isomorphic to M. N is said to be a *lifting of M (to A)*. The following lemma provides a known liftability criterion.

Lemma 2.8. *Let A be a graded ring, x a homogeneous element in the Jacobson radical of A and let $F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$ be an exact sequence of graded $A/(x)A$-modules. Suppose there exists a complex $G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$ of graded A-modules such that $F_i = G_i/(x)G_i$ and φ_i is induced by ψ_i, for $i = 1, 2$. If x is G_0-regular and G_1 is finite, then $G_{\bullet}$ is exact.*

Proof. If we prove that $\operatorname{Ker} \psi_1 = x \operatorname{Ker} \psi_1 + \operatorname{Im} \psi_2$, using Nakayama's Lemma we are done, since G_1 is finite. One needs only to show the inclusion $\operatorname{Ker} \psi_1 \subseteq x \operatorname{Ker} \psi_1 + \operatorname{Im} \psi_2$. Let $y \in \operatorname{Ker} \psi_1$. Its equivalence class in $G_1/(x)G_1$ is in $\operatorname{Ker} \varphi_1 = \operatorname{Im} \varphi_2$. Therefore, there exist elements $z \in \operatorname{Im} \psi_2$ and $w \in G_1$ such that $y = z + xw$. It remains to be proven that $w \in \operatorname{Ker} \psi_1$, but since $0 = \psi_1(y) = x\varphi_1(w)$, and x is G_0-regular, this is clearly the case. ▲

Lemma 2.9. *Let A and x be as in the previous lemma. Assume also that x is A-regular and $(F_\bullet, \varphi_\bullet) : F_\bullet \longrightarrow M \longrightarrow 0$ is a finite, free and graded $A/(x)A$-resolution of M.*

(i) Suppose that there exists a complex $G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$ of finite, free and graded A-modules such that $F_i = G_i/(x)G_i$ and ψ_i induces φ_i, for $i = 1, 2$. Then $N \doteq \operatorname{Coker} \psi_1$ is a lifting of M.

(ii) If

$$(G_\bullet, \psi_\bullet) : \ldots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow 0,$$

is a finite, free and graded complex of A-modules such that $F_\bullet = G_\bullet/(x)G_\bullet$ with maps induced by $\psi_\bullet$, then $G_\bullet \longrightarrow N \longrightarrow 0$ is a resolution of N.

Proof. The sequence $G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow N \longrightarrow 0$ is exact by the previous lemma. Applying $\otimes A/(x)$ to the above sequence, we deduce that $M \simeq N \otimes A/(x) \simeq N/(x)N$. In order to prove that N is a lifting of M, one must show that x is N-regular. From the above sequence one can also deduce that $\operatorname{Tor}_1^R(N, A/(x)) = 0$. Since x is A-regular, the sequence $0 \longrightarrow A \xrightarrow{x} A \longrightarrow A/(x) \longrightarrow 0$ is exact. From these last two facts, we obtain a new exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/(x)N \longrightarrow 0$, and this is equivalent to saying that x is N-regular.
(ii) follows from (i) and Lemma 2.8. ▲

Proposition 2.10. *Let M be a multi-graded R-module and let $(F_\bullet, \varphi_\bullet) : F_\bullet \longrightarrow M \longrightarrow 0$ be a finite, free and multi-graded resolution of M. Then, $(F_\bullet^{\mathbf{P}}, \varphi_\bullet^{\mathbf{P}}) : F_\bullet^{\mathbf{P}} \longrightarrow M^{\mathbf{P}} \longrightarrow 0$ is a resolution of $M^{\mathbf{P}}$. Furthermore, if $l \doteq X_{n+1} - X_n$, then l is $M^{\mathbf{P}}$-regular and $M^{\mathbf{P}}/lM^{\mathbf{P}} \simeq M$ (i.e. $M^{\mathbf{P}}$ is a lifting of M to S).*

Proof. It is clear that $F_\bullet^{\mathbf{P}}$ is a complex and that l is S-regular. One may apply the previous lemma and achieve the sought after conclusion if one shows that ${\varphi_i}^{\mathbf{P}}$ induces φ_i for every i. By virtue of Proposition 2.7 the polarized maps are determined by the polarization of the shifts. Thus, it is sufficient to observe that $(a^{\mathbf{P}} - b^{\mathbf{P}})_{n+1} + (a^{\mathbf{P}} - b^{\mathbf{P}})_n = a_n - b_n$, but this is precisely what is shown in Lemma 2.2 (iii). ▲

Note that, if we require $F_\bullet \longrightarrow M \longrightarrow 0$ to be minimal, then $F_\bullet \longrightarrow M \longrightarrow 0$ also has the same property.
The most important result of this section is contained in the next theorem.

Theorem 2.11. *The functor $^{\mathbf{P}}$ is exact.*

We proceed by recalling a standard construction of homological algebra. Let

$$0 \longrightarrow U \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

be a short exact sequence of graded modules over an arbitrary graded ring A. Given free and graded resolutions $(F_\bullet, \alpha_\bullet)$ and $(G_\bullet, \beta_\bullet)$ of U and N respectively, one can construct a graded resolution of M as follows. First observe that the sequence

$$\ldots F_1 \xrightarrow{-\alpha_1} F_0 \xrightarrow{-\varphi \circ \alpha_0} M \xrightarrow{\psi} N \longrightarrow 0$$

is exact. Indeed $\operatorname{Im}\alpha_1 = \operatorname{Ker}\alpha_0 = \operatorname{Ker}\varphi\circ\alpha_0$, since φ is injective, and $\operatorname{Ker}\psi = \operatorname{Im}\varphi = \operatorname{Im}\varphi\circ\alpha_0$, since α_0 is surjective. Secondly, there exist homogeneous homomorphisms ω_i such that in the diagram

$$\begin{tikzcd}
{} \arrow[r] & G_2 \arrow[r, "\beta_2"] \arrow[d, "\omega_2"] & G_1 \arrow[r, "\beta_1"] \arrow[d, "\omega_1"] & G_0 \arrow[r, "\beta_0"] \arrow[d, "\omega_0"] & N \arrow[r] \arrow[d, equal] & 0 \\
{} \arrow[r] & F_1 \arrow[r, "-\alpha_1"'] & F_0 \arrow[r, "-\varphi\circ\alpha_0"'] & M \arrow[r, "\psi"'] & N \arrow[r] & 0
\end{tikzcd}$$

each square is commutative. Now we define $H_i \doteq (F\oplus G)_i = F_i \oplus G_i$, for any $i > 0$. Let $\gamma_0 : H_0 \longrightarrow M$ be $\gamma_0(x,y) \doteq \varphi\circ\alpha_0(x) + \omega_0(y)$ and, for every $i > 0$ $\gamma_i : H_i \longrightarrow H_{i-1}$ be $\gamma_i(x,y) \doteq (\alpha_i(x) + \omega_i(y), \beta_i(y))$. For any i, since ω_i is homogeneous, γ_i is homogeneous as well. One can prove that $(H_{\bullet}, \gamma_{\bullet})$ is a free and graded resolution of M and that the diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & U \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & N \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F_0 \arrow[r] \arrow[u] & F_0 \oplus G_0 \arrow[r] \arrow[u] & G_0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[u] & F_1 \oplus G_1 \arrow[r] \arrow[u] & G_1 \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

is exact and commutative.

Proof of Theorem 2.11. Let $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of R-modules and let us denote by $F_{\bullet}$ and $G_{\bullet}$ two finite multi-graded resolutions of U and N respectively. One constructs a finite multi-graded resolution of M and an exact commutative diagram as illustrated above. Polarizing, we get a new diagram, where, since ${}^{\mathbf{P}}$ is additive, each square still commutes and all rows except possibly the first one are exact. By Proposition 2.10 the polarized columns are also exact and yield resolutions of $U^{\mathbf{P}}$, $M^{\mathbf{P}}$ and $N^{\mathbf{P}}$. It remains to be shown that the first row $0 \longrightarrow U^{\mathbf{P}} \longrightarrow M^{\mathbf{P}} \longrightarrow N^{\mathbf{P}} \longrightarrow 0$ is also exact, but this follows from "diagram chasing". ▲

Note that, given a finite complex of multi-graded free R-modules $(F_{\bullet}, \varphi_{\bullet})$, Lemma 2.2 (i) implies that $\operatorname{Hom}_R(F_{\bullet}, R)^{\mathbf{P}} = \operatorname{Hom}_{R^{\mathbf{P}}}(F_{\bullet}{}^{\mathbf{P}}, R^{\mathbf{P}})$.

Corollary 2.12. *With the same notation as above, for any i,*

$$\operatorname{Ext}^i_{R^{\mathbf{P}}}(M^{\mathbf{P}}, R^{\mathbf{P}}) \simeq \operatorname{Ext}^i_R(M, R)^{\mathbf{P}}.$$

Proof. It follows from the above observation and the exactness of ${}^{\mathbf{P}}$. ▲

Remark and Example 2.13. Observe that in general $M(-a)^{\mathbf{P}} \not\simeq M^{\mathbf{P}}(-a^{\mathbf{P}})$. This is not true even for $M = R(-b)$ and it finds its motivation in the fact that, for arbitrary $a, b \in \mathbb{Z}^n$, $(a-b)^{\mathbf{P}} \neq a^{\mathbf{P}} - b^{\mathbf{P}}$.
Consider for instance $R = K[X_1]$, $S = K[X_1, X_2]$, $M = R(-2)$ and $a = 3$. Then $M^{\mathbf{P}} = S(-1,-1)$ and $M^{\mathbf{P}}(-3^{\mathbf{P}}) = S(-1,-1)(-2,-1) = S(-3,-2)$. On the other hand, $M(-3)^{\mathbf{P}} = R(-5)^{\mathbf{P}} = S(-4,-1)$. Consequently, Corollary 2.12 cannot be extended for $\operatorname{Ext}^{\bullet}_R(\ , N)$, since $\operatorname{Hom}_R(R(-a), R(-b)) \simeq R(a-b)^{\mathbf{P}} = S((a-b)^{\mathbf{P}})$, while $\operatorname{Hom}_S(S(-a^{\mathbf{P}}), S(-b^{\mathbf{P}})) \simeq S(a^{\mathbf{P}} - b^{\mathbf{P}})$.

For the same reason, in general it is not true that $(M \otimes_R N)^{\mathbf{P}} = M^{\mathbf{P}} \otimes_{R^{\mathbf{P}}} N^{\mathbf{P}}$. Let us give an easy example. First of all observe that the polarization of K is $K[X_{n+1}]$. In fact, from the minimal resolution of K,

$$\ldots \longrightarrow \oplus_{i=1}^n R(-e_i) \xrightarrow{\alpha} R \longrightarrow K \longrightarrow 0,$$

one sees immediately that, upon polarizing, the shifts does not change but gets a 0 in the $(n+1)^{st}$ entry. Therefore, the co-kernel of $\alpha^{\mathbf{P}}$, is $S/(X_1, \ldots, X_n) \simeq K[X_{n+1}]$.
Let now $R = K[X_1, X_2]$, $N = K$ and $M = R(-1,-1)$. On the right-hand side one has immediately $K[X_3](-1,-1,0)$, while $(M \otimes_R N)^{\mathbf{P}} \simeq K(-1,-1)^{\mathbf{P}}$. Shifting the minimal resolution of K by $(-1-1)$ and polarizing, one gets that $K(-1,-1)^{\mathbf{P}} = \operatorname{Coker}\big(S(-2,-1,0) \oplus S(-1,-1,-1)) \longrightarrow S(-1,-1,0)\big)$, which is $\big(S/(X_1, X_3)\big)(-1,-1,0) \simeq K[X_2](-1,-1,0)$.

2.3 Complete polarization

In this section we introduce the definition of "complete polarization" of a module M. We show how the classical definition of polarization for monomial ideals can be recovered and prove a useful result which relates the Hilbert series of $H^{\bullet}_{\mathfrak{m}}(M)$ to that of the local cohomology modules of a complete polarization of M.
Let M be a multi-graded R-module. We say that M has a *squarefree resolution* (in R) iff M has a multi-graded resolution by free R-modules $F_{\bullet}$ such that any of the entries of the shifts of $F_{\bullet}$ belongs to $\{-1, 0, 1\}$.

Proposition and Definition 2.14. *Let M be a multi-graded R module. There exists a polynomial ring $S = K[Y_{ij_i}]$ over R, with $i = 1, \ldots, n$ and $j_i = 1, \ldots, c_i$, and an S-module N with a squarefree resolution in S such that*

(i) for $i = 1, \ldots, n$ and $j_i = 2, \ldots, c_i$, the linear forms $Y_{ij_i} - Y_{i1}$ are N-regular;

(ii) if U denotes the ideal of S generated by the regular forms in (i), then $\frac{N}{UN} \simeq M$.

We say that N is a complete polarization of M in S *and we denote it by $M^{\mathbf{P}}$.*

Proof. First we re-write the polynomial ring R as $K[Y_{11}, \ldots, Y_{n1}]$ and note once again that polarization can be computed, according to Definition 2.1, by operating on the shifts of the minimal resolution $F_{\bullet} \longrightarrow M \longrightarrow 0$ of M (cf. Proposition 2.7). The operation introduced in Definition 2.1 acts on the last entry of a vector of $\mathbb{Z}^n$. Clearly one can define similar operations in order to polarize any other entry as well. It is also understandable that the corresponding polarization functors are endowed with the same properties as the functor ${}^{\mathbf{P}}$.
Let us denote the shifts of $F_{\bullet}$ by a_{ij} and denote by $(a_{ij})_k$ their k^{th} entry. We provide an algorithm to prove the proposition.
If all the shifts are squarefree we are already done. Assume that this is not the case. If all of the entries $(a_{ij})_n$ are squarefree, we do nothing and proceed by testing the previous ones. We shall find an h such that not all of the $(a_{ij})_h$ are squarefree. Polarizing $F_{\bullet} \longrightarrow M \longrightarrow 0$ with respect to the h^{th} component, by virtue of Proposition 2.10, we obtain the minimal resolution $F^{\mathbf{P}} \longrightarrow M^{\mathbf{P}} \longrightarrow 0$ of $M^{\mathbf{P}}$. $M^{\mathbf{P}}$ is a lifting to $S = K[Y_{11}, \ldots, Y_{h-11}, Y_{h1}, Y_{h2}, Y_{h+11}, \ldots Y_{n1}]$ of M and $M^{\mathbf{P}}/(Y_{h2} - Y_{h1})M^{\mathbf{P}} \simeq M$. Note that, for any of the $a_{ij}{}^{\mathbf{P}}$, the $(h+1)^{st}$ entry, the one corresponding to the variable Y_{h2}, belongs already to the set $\{-1, 0, 1\}$ and that $|(a_{ij}{}^{\mathbf{P}})_h| < |(a_{ij})_h|$. If $|(a_{ij}{}^{\mathbf{P}})_h| > 1$ we rename the variable Y_{h2}, calling it Y_{h3}, and repeat the polarization of the h^{th} component on $M^{\mathbf{P}}$. Otherwise we apply the whole procedure on $M^{\mathbf{P}}$. Note that, in this case, it would be enough to test the first $h-1$ entries and that after a finite number of steps we achieve the desired squarefree resolution and lifting. ▲

Remarks 2.15.

(i) For $i = 1, \ldots, n$ let

$$c_i \doteq \max\{\max_{h,k}\{|(a_{hk})_i| \colon a_{hk} \text{ is a shift of the minimal resolution of } M\};\ 1\}$$

and $c \doteq \sum_i c_i - n$. Then, in the proof of Proposition 2.14, c applications of ${}^{\mathbf{P}}$ are necessary and sufficient to achieve the desired polarization.

(ii) Proposition 2.14 is an extension of Theorem 2.1 in [BH1], since no condition on the degrees of the homogeneous generators of M is assumed.

(iii) *Polarization for monomial ideals.*
Let I be an arbitrary monomial ideal of R and let $G(I)$ be the set of minimal generators of I. In the literature the *polarization of I* is defined to be the monomial ideal J of S generated by

$$\left\{ \prod_{h=1}^{n} \prod_{k=1}^{\mu_i} Z_{hk} \colon X^{\mu} \in G(I) \right\}.$$

Here S is the polynomial ring $K[Z_{ij}]$, with $i = 1, \dots, n$ and $j = 1, \dots, m$ with $m \gg 0$. It is then understandable that J can be obtained by operating as in the proof of Proposition 2.14, i.e. J is a complete polarization of I in S.

Example 2.16. Let $R = K[X, Y]$ and let

$$\alpha : R(-2, 2) \oplus R(-1, 0) \longrightarrow R(-1, 3) \oplus R(0, 2)$$

be represented by the matrix $\begin{pmatrix} XY & X^2 \\ Y^3 & XY^2 \end{pmatrix}$. Let $M \doteq \operatorname{Coker} \alpha$. We compute a complete polarization of M. It is clear that we have to apply $^{\mathbf{P}}$ at least $(|-2| - 1) + (|3| - 1) = 3$ times. We start operating on the second entry and obtain a map

$$\beta : R'(-2, 1, 1) \oplus R'(-1, 0, 0) \longrightarrow R'(-1, 2, 1) \oplus R'(0, 1, 1),$$

where we set $R' = K[X, Y_1, Y_2]$. Polarizing the second entry again and letting $R'' = K[X, Y_1, Y_2, Y_3]$, we have

$$\gamma : R''(-2, 1, 0, 1) \oplus R''(-1, 0, 0, 0) \longrightarrow R''(-1, 1, 1, 1) \oplus R''(0, 1, 0, 1)$$

Now, we repeat the procedure on the first entry. Letting $S \doteq K[X_1, X_2, Y_1, Y_2, Y_3]$, we obtain

$$\delta : S(-1, -1, 1, 0, 1) \oplus S(-1, 0, 0, 0, 0) \longrightarrow S(-1, 0, 1, 1, 1) \oplus S(0, 0, 1, 0, 1),$$

and $\operatorname{Coker} \delta$ is the desired complete polarization. The matrix which describes δ is $\begin{pmatrix} X_2Y_2 & X_1X_2 \\ Y_1Y_2Y_3 & X_1Y_1Y_2 \end{pmatrix}$.

Example 2.17. Let $I = (X_1^2, X_1X_2, X_2X_3^2) \subset K[X_1, X_2, X_3] = R$. Let $M = R/I$. The minimal resolution of M is

$$\dots \longrightarrow R(-2, 0, 0) \oplus R(-1, -1, 0) \oplus R(0, -1, -2) \xrightarrow{\alpha} R \longrightarrow M \longrightarrow 0.$$

If we set $S \doteq K[X_{11}, X_{12}, X_{21}, X_{31}, X_{32}]$, then a complete polarization of M in S is

$$S/(X_{11}X_{12}, X_{11}X_{21}, X_{21}X_{31}X_{32}),$$

since the first map of the resolution is polarized to

$$S(-1, -1, 0, 0, 0) \oplus S(-1, 0, -1, 0, 0) \oplus S(0, 0, -1, -1, -1) \longrightarrow S.$$

A complete polarization of I in S is the ideal $I^{\mathbf{P}} = (X_{11}X_{12}, X_{11}X_{21}, X_{21}X_{31}X_{32})$.

We conclude this section by proving a result which will be relevant in the next chapter.

Corollary 2.18. *Let N a complete polarization of M in S, where S is a polynomial ring in c variables over R with graded maximal ideal $\mathfrak{n}$. Then,*

$$\operatorname{Hilb}(H^i_{\mathfrak{m}}(M), t) = (t-1)^c \operatorname{Hilb}(H^{i+c}_{\mathfrak{n}}(N), t).$$

Proof. Clearly it is sufficient to consider the case $c = 1$ and, without loss of generality, assume that $S = R[X_{n+1}]$. By Corollary 2.12, $\operatorname{Ext}^i_S(M^{\mathbf{P}}, S) \simeq \operatorname{Ext}_R(M, R)^{\mathbf{P}}$. Thus $l \doteq X_{n+1} - X_n$ is $\operatorname{Ext}^i_S(M^{\mathbf{P}}, S)$-regular, and

$$0 \longrightarrow \operatorname{Ext}^i_S(M^{\mathbf{P}}, S)(-1) \xrightarrow{\cdot l} \operatorname{Ext}^i_S(M^{\mathbf{P}}, S) \longrightarrow \operatorname{Ext}^i_R(M, R) \longrightarrow 0$$

is exact. Accordingly,

$$(1-t) \operatorname{Hilb}(\operatorname{Ext}^i_S(M^{\mathbf{P}}, S), t) = \operatorname{Hilb}(\operatorname{Ext}^i_R(M, R), t).$$

By the Local Duality Theorem,

$$\begin{aligned}
\operatorname{Hilb}(H^i_{\mathfrak{m}}(M), t) &= \operatorname{Hilb}(\operatorname{Ext}^{n-i}_R(M, R(-n)), t^{-1}) = t^{-n} \operatorname{Hilb}(\operatorname{Ext}^{n-i}_R(M, R), t^{-1}) \\
&= (t-1)t^{-n-1} \operatorname{Hilb}(\operatorname{Ext}^{n-i}_S(M^{\mathbf{P}}, S), t^{-1}) \\
&= (t-1) \operatorname{Hilb}(\operatorname{Ext}^{n-i}_S(M^{\mathbf{P}}, S(-n-1)), t^{-1}).
\end{aligned}$$

Applying Local Duality again, we deduce the desired formula. ▲

3 Upper bounds

3.1 Introduction

Macaulay's Theorem can be re-stated in the following way: Let K be a field. Then, for any homogeneous ideal I in a polynomial ring R and for each j,

$$\beta_{1j}(I) \leq \beta_{1j}(I^{lex}).$$

Bigatti and Hulett independently gave a generalization of the aforementioned theorem, proving that, given any homogeneous ideal I in a polynomial ring R over a field K of characteristic 0, all of the graded Betti numbers of I^{lex} are greater than or equal to those of I (see [Bi] or [Hu]). Pardue was able to show how the same result holds true over fields of any characteristic.
Analogously, if I is a squarefree ideal, one can try to compare the graded Betti numbers of I with those of I^{lex} (see Section 1.3.2). This question was studied in [AHHi], where the equivalent of the Bigatti-Hulett Theorem in the squarefree case has been proved. Since, as we have already observed, lexicographic ideals are combinatorial objects, the above theorems also provide a way of computing explicit upper bounds for the graded Betti numbers $\beta_{ij}(I)$ in terms of the Hilbert function of I, in the non-squarefree case, and in terms of the f-vector of I in the squarefree case.

Let I be an arbitrary homogeneous ideal in a polynomial ring R with maximal ideal $\mathfrak{m}$. The local cohomology functors $H^i_{\mathfrak{m}}(R/I)$ inherit a natural graded structure as right derived functors of $\Gamma_{\mathfrak{m}}(R/I)$, which is graded as a submodule of R/I (see Section 1.4.2 for more details about the graded structure of $H^{\bullet}_{\mathfrak{m}}$). The homogeneous components of $H^i_{\mathfrak{m}}(R/I)_j$ are K-vector spaces of finite dimension.
The objective of this chapter is to prove the following: Let H be an admissible numerical function (resp. let h be an admissible vector) and let $\mathcal{I}$ denote the family of all homogeneous (resp. squarefree) ideals with Hilbert function H (resp. f-vector h). Let L be the lex-ideal of $\mathcal{I}$. Then, for each $I \in \mathcal{I}$,

$$\dim_K H^i_{\mathfrak{m}}(R/I)_j \leq \dim_K H^i_{\mathfrak{m}}(R/L)_j, \text{ for any } i, j. \tag{3.1}$$

First, as in the proofs of [Bi], [Hu] and [P], we reduce the problem to the monomial case, proving that given any term order $\prec$ on the set of monomials of a polynomial ring R and any ideal $I \subset R$, for all i, j,

$$\dim_K H^i_{\mathfrak{m}}(R/I)_j \leq \dim_K H^i_{\mathfrak{m}}(R/\operatorname{in}_{\prec}(I))_j.$$

The inequality will be proven in Section 3.3 and Section 3.4 separately for the two cases (Theorem 3.10 and Theorem 3.15) with two different strategies.
We remark that, while the result in the non-squarefree case is characteristic free, in the other case we have still to assume the characteristic of the base field to be 0.
In Section 3.3 we shall also discuss briefly a method for computing explicit bounds in the squarefree case by virtue of Theorem 3.10.
Explicit computations of $\dim_K H^i_{\mathfrak{m}}(R/J)_j$ in terms of the Hilbert function H for the non-squarefree case will be provided in the next chapter.

3.2 The reduction to the monomial case

Since I and $\mathrm{in}(I)$ have the same Hilbert function, in order to prove the inequality (3.1), one may assume I to be monomial. Indeed, this assumption causes no loss of generality, since the K-dimension of $H^i_{\mathfrak{m}}(R/I)_j$ is not greater than the K-dimension of $H^i_{\mathfrak{m}}(R/\mathrm{in}(I))_j$. The main goal of this section is to show an argument for this fact. We first make use of a "deformation" argument (see Lemma 3.1) to show in Proposition 3.2 that $\dim_K \mathrm{Ext}^l_R(R/I,R)_i \leq \dim_K \mathrm{Ext}^l_R(R/\mathrm{in}(I),R)_i$ for every l,i. Thus, (3.1) follows immediately from the Local Duality Theorem. For detailed information about deformations of graded ideals and modules we refer to Chapter 15 in [E] and to [P2].

Let $\prec$ be any term order on the monomials of R and $R[t]$ a polynomial ring in one variable over R. By virtue of Proposition 1.20, there exists a non-negative integer vector $0 \neq \omega = (\omega_1,\dots,\omega_n) \in \mathbb{N}^n$ such that $\mathrm{in}_\omega(I) = \mathrm{in}_\prec(I)$.
For every polynomial $f = \sum_i \alpha_i X^{a_i} \in R$, we let $b_f \doteq \max_i\{\omega \cdot a_i\}$. Note that $b_f > 0$ if $f \neq 0$. We set $\tilde{f} = t^{b_f} f(t^{-\omega_1}X_1,\dots,t^{-\omega_n}X_n) \in R[t]$, and, if $I \subset R$ is an ideal of R, we let J be the ideal $J \doteq (\tilde{f} \mid f \in I)$ of $R[t]$.
The following result is well-known.

Lemma 3.1 (Theorem 15.17 in [Ei]). *Let I be any ideal of R and $\prec$ an arbitrary term order on the monomials of R. Let J be defined as above. Then,*

$$(R[t]/J)_t \simeq (R/I)[t,t^{-1}], \tag{3.2}$$

and

$$(R[t]/J)/(tR[t]/J) \simeq R/\mathrm{in}_\prec(I). \tag{3.3}$$

Proof. The automorphism $R[t,t^{-1}] \longrightarrow R[t,t^{-1}]$, defined by $X_i \longrightarrow t^{\omega_i}X_i$, maps $JR[t,t^{-1}]$ into $IR[t,t^{-1}]$. Thus, $R[t,t^{-1}]/JR[t,t^{-1}] \simeq R[t,t^{-1}]/IR[t,t^{-1}]$. Since localization commutes with the formation of quotients, this proves the first isomorphism.
To prove (3.3), it is sufficient to observe that $J \subseteq \mathrm{in}_\omega(I) + (t)$. Indeed, for any $f = \sum_i \alpha_i X^{a_i} \in R$, one has $\tilde{f} = t^{b_f} \sum_i \alpha_i (\prod_{j=1}^n (t^{-\omega_j}X_j)^{(a_i)_j}) = t^{b_f}\sum_i \alpha_i t^{-\omega\cdot a_i} X^{a_i}$. Furthermore, there exists a unique term in the sum, let us say X^{a_h}, such that $t^{-\omega\cdot a_h} = t^{-b_f}$. Since $\mathrm{in}_\omega(I) = \mathrm{in}_\prec(I)$, it is clear that $\alpha_h X^{a_h} \in \mathrm{in}_\omega(I)$. Therefore, $\tilde{f} - \alpha_h X^{a_h} \in (t)$ and we are done. ▲

Note that the isomorphism in (3.3) is homogeneous of degree 0. The isomorphism in (3.2) is homogeneous of degree 0 if we let $\deg t = 0$. We now provide $R[t] = K[X_1, \ldots, X_n, t]$ with a structure of a bi-graded ring by assigning to X_i the bi-degree $(1, \omega_i)$ for $i = 1, \ldots, n$, and to t the bi-degree $(0, 1)$. By construction J is bi-homogeneous.
For the sake of notational simplicity, set $S \doteq R[t]$. Since J is bi-homogeneous, S/J is bi-graded. Moreover, for any M, N bi-graded S-modules, $\operatorname{Ext}^\bullet_S(M, N)$ is bi-graded. From the bi-graduation, one becomes singly graded S-modules by letting

$$\operatorname{Ext}^\bullet_S(M, N)_i \doteq \oplus_j \operatorname{Ext}^\bullet_S(M, N)_{(i,j)}.$$

Since t has bi-degree $(0, 1)$, the multiplication map by $t : M(0, -1) \longrightarrow M$ is bi-homogeneous and induces $\operatorname{Ext}^\bullet_S(M, N)_{(i,j)} \xrightarrow{\cdot t} \operatorname{Ext}^\bullet_S(M, N)_{(i,j+1)}$. This operation provides the module $\operatorname{Ext}^\bullet_S(M, N)_i$ with a natural $K[t]$-module structure. Observe that $\operatorname{Ext}^\bullet_S(M, N)_i$ is a finitely generated $K[t]$-module.
The structure theorem for finitely generated modules over a principal ideal domain now yields that, *a priori*,

$$\operatorname{Ext}^l_S(S/J, S)_i \simeq \left(\oplus_{s=1}^{m_{l,i}} K[t]\right) \oplus \left(\oplus_{s=1}^{h_{l,i}} K[t]/(t^{p_s})\right) \oplus \left(\oplus_{s=1}^{k_{l,i}} K[t]/((t-a_s)^{r_s}))\right),$$

with $a_s \in K \setminus \{0\}$, but, since $\operatorname{Ext}^l_S(S/J, S)_i$ is homogeneous in t, one has

$$\operatorname{Ext}^l_S(S/J, S)_i \simeq \left(\oplus_{s=1}^{m_{l,i}} K[t]\right) \oplus \left(\oplus_{s=1}^{h_{l,i}} K[t]/(t^{p_s})\right), \tag{3.4}$$

for some $m_{l,i}, h_{l,i}$ and p_s in $\mathbb{N}$.

Proposition 3.2. *Let $\prec$ be an arbitrary term order on the monomials of R. Then, for any graded ideal I of R,*

$$\dim_K \operatorname{Ext}^l_R(R/I, R)_i \le \dim_K \operatorname{Ext}^l_R(R/\operatorname{in}(I), R)_i \quad \textit{for all } l, i.$$

Proof. By (3.3), $\operatorname{Ext}^l_R(R/\operatorname{in}(I), R) = \operatorname{Ext}^l_R((S/J)/(tS/J), R)$.
Since t is S-regular and annihilates $(S/J)/(tS/J)$, the hypotheses of Rees' Lemma are verified and therefore $\operatorname{Ext}^l_R((S/J)/(tS/J), R) \simeq \operatorname{Ext}^{l+1}_S((S/J)/(tS/J), S)$. Accordingly,

$$\dim_K \operatorname{Ext}^l_R(R/\operatorname{in}(I), R)_i = \dim_K \operatorname{Ext}^{l+1}_S((S/J)/(tS/J), S)_i. \tag{3.5}$$

Let us now consider the short exact sequence of bi-graded S-modules

$$0 \longrightarrow S/J(0, -1) \longrightarrow S/J \longrightarrow (S/J)/(tS/J) \longrightarrow 0.$$

This leads to the long exact sequence of bi-graded S-modules in cohomology

$$\begin{aligned} \ldots \longrightarrow \operatorname{Ext}^l_S(S/J, S)_{(i,j)} &\xrightarrow{\cdot t} \operatorname{Ext}^l_S(S/J, S)_{(i,j+1)} \longrightarrow \\ &\longrightarrow \operatorname{Ext}^{l+1}_S((S/J)/(tS/J), S)_{(i,j+1)} \longrightarrow \\ &\qquad \longrightarrow \operatorname{Ext}^{l+1}_S(S/J, S)_{(i,j+1)} \xrightarrow{\cdot t} \operatorname{Ext}^{l+1}_S(S/J, S)_{(i,j+2)} \longrightarrow \ldots \end{aligned}$$

for every i, j, from which we become, for every i, the singly graded long exact sequence of S-modules

$$\ldots \longrightarrow \operatorname{Ext}^l_S(S/J,S)_i \xrightarrow{\cdot t} \operatorname{Ext}^l_S(S/J,S)_i \longrightarrow \operatorname{Ext}^{l+1}_S((S/J)/(tS/J),S)_i \longrightarrow \\ \longrightarrow \operatorname{Ext}^{l+1}_S(S/J,S)_i \xrightarrow{\cdot t} \operatorname{Ext}^{l+1}_S(S/J,S)_i \longrightarrow \ldots.$$

Thus,

$$0 \longrightarrow \operatorname{Coker}^{l,i}(\cdot t) \longrightarrow \operatorname{Ext}^{l+1}_S((S/J)/(tS/J),S)_i \longrightarrow \operatorname{Ker}^{l+1,i}(\cdot t) \longrightarrow 0$$

is exact for every l, i. Recalling the structure of $\operatorname{Ext}^l_S(S/J,S)_i$ as expressed in (3.4), one constructs the sequence

$$0 \longrightarrow \operatorname{Ker}^{l,i}(\cdot t) \longrightarrow \left(\oplus_{s=1}^{m_{l,i}} K[t]\right) \oplus \left(\oplus_{s=1}^{h_{l,i}} K[t]/(t^{p_s})\right) \xrightarrow{\cdot t} \\ \xrightarrow{\cdot t} \left(\oplus_{s=1}^{m_{l,i}} K[t]\right) \oplus \left(\oplus_{s=1}^{h_{l,i}} K[t]/(t^{p_s})\right) \longrightarrow \operatorname{Coker}^{l,i}(\cdot t) \longrightarrow 0,$$

which is also exact. It is now easy to see that

$$\operatorname{Ker}^{l,i}(\cdot t) \simeq \oplus_{s=1}^{h_{l,i}} (t^{p_s-1})/(t^{p_s}) \simeq \oplus_{s=1}^{h_{l,i}} K \text{ and } \operatorname{Coker}^{l,i}(\cdot t) \simeq \left(\oplus_{s=1}^{m_{l,i}} K\right) \oplus \left(\oplus_{s=1}^{h_{l,i}} K\right).$$

Thus, $\dim_K \operatorname{Ext}^{l+1}_S((S/J)/t(S/J),S) = \dim_K \operatorname{Coker}^{l,i}(\cdot t) + \dim_K \operatorname{Ker}^{l+1,i}(\cdot t)$, and, by (3.5),

$$\dim_K \operatorname{Ext}^l_R(R/\operatorname{in}(I),R)_i = m_{l,i} + h_{l,i} + h_{l+1,i}. \tag{3.6}$$

On the other hand, in order to study the dimension of $\operatorname{Ext}^l_R(R/I,R)_i$ as a K-vector space, we compute the rank as a $K[t,t^{-1}]$-module of $\operatorname{Ext}^l_R(R/I,R)_i[t,t^{-1}]$.
Since S_t and $R[t,t^{-1}]$ are flat on S and R respectively, from (3.2) one deduces that

$$\begin{aligned} \operatorname{Ext}^l_S(S/J,S)_t &\simeq \operatorname{Ext}^l_{S_t}((S/J)_t,S_t) \\ &\simeq \operatorname{Ext}^l_{R[t,t^{-1}]}(R/I[t,t^{-1}],R[t,t^{-1}]) \\ &\simeq \operatorname{Ext}^l_R(R/I,R)[t,t^{-1}]. \end{aligned}$$

Thus, being $\deg t = 0$,

$$\operatorname{Ext}^l_R(R/I,R)_i[t,t^{-1}] \simeq \operatorname{Ext}^l_R(R/I,R)[t,t^{-1}]_i \simeq (\operatorname{Ext}^l_S(S/J,S)_t)_i \\ \simeq (\operatorname{Ext}^l_S(S/J,S)_i)_t.$$

Again from (3.4), it follows that $(\operatorname{Ext}^l_S(S/J,S)_i)_t \simeq \oplus_{s=1}^{m_{l,i}} K[t,t^{-1}]$, and, consequently,

$$\dim_K \operatorname{Ext}^l_R(R/I,R)_i = m_{l,i}. \tag{3.7}$$

Comparing (3.6) with (3.7) we deduce now the desired conclusion. ▲

The proof also shows that in the formula of Proposition 3.2 equality holds iff $\operatorname{Ext}^l_S(S/J,S)_i$ are free $K[t]$-modules.

Theorem 3.3. *Let R be the polynomial ring $K[X_1,\dots,X_n]$ over a field K and $\prec$ an arbitrary term order on the monomials of R. Then, for any graded ideal I of R,*

$$\dim_K H^i_{\mathfrak{m}}(R/I)_j \le \dim_K H^i_{\mathfrak{m}}(R/\operatorname{in}(I))_j, \text{ for all } i,j.$$

Proof. Recall that, for any M, $H(M,j) = H(M^\vee,-j)$ and combine the Local Duality Theorem with Proposition 3.2. ▲

3.3 The squarefree case

In this section we prove a formula which relates the dimension of the multi-graded components of the local cohomology modules of $K[\Delta]$ with certain (squarefree) Betti numbers of the Stanley-Reisner ring $K[\overline{\Delta}]$. Therefore we shall be able to apply the following result. The notation used here can be found in Section 1.2.

Theorem 3.4 (Aramova-Herzog-Hibi). *Let K be a field of characteristic 0, let I_Δ be a squarefree ideal and $I_{\Delta^{lex}}$ the unique squarefree lex-ideal with the same f-vector as Δ. Then,*

$$\beta_{i,j}(I_\Delta) \le \beta_{i,j}(I_{\Delta^{lex}}), \text{ for all } i,j.$$

Proof. See that of Theorem 2.9 in [AHHi]. ▲

Since we make use of Theorem 3.4 the result will hold true for Stanley Reisner rings over fields of characteristic 0. On the other hand, it is reasonable to think that the inequality of Theorem 3.4 is true independently of the characteristic. If this were the case, the main result of this section, Theorem 3.10, could also be immediately extended to fields of positive characteristic.

Now we state some known results, which will be useful in what follows.

Theorem 3.5 (Hochster). *Let $a \in \mathbb{N}^n$ and let $F = \operatorname{supp} a$. Then*

$$\dim_K \operatorname{Tor}^R_i(K[\Delta],K)_a = \dim_K \tilde{H}_{|F|-i-1}(\Delta_F,K).$$

Proof. See that of Theorem 5.1 in [Ho] ▲

Theorem 3.6 (Hochster). *Let $a \in \mathbb{Z}^n$ and $F = \operatorname{supp} a$. Then*

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta])_a = \begin{cases} \dim_K \tilde{H}_{i-|F|-1}(\operatorname{lk}_\Delta F, K) & \text{if } a \in \mathbb{Z}^n_- \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See for example that of Theorem 5.3.8 in [BH]. ▲

Lemma 3.7 (Eagon-Reiner). *Let Δ be a simplicial complex and G be a non-face of Δ. Then,*

$$\dim_K \tilde{H}_{i-2}(\operatorname{lk}_{\overline{\Delta}} \overline{G}, K) = \dim_K \tilde{H}_{|G|-i-1}(\Delta_G, K).$$

Proof. Recall that $\mathrm{lk}_{\overline{\Delta}}\overline{G}$ is the simplicial complex $\{F\colon F\cup\overline{G}\in\overline{\Delta},\ F\cap\overline{G}=\emptyset\}$. Since G is a non-face of Δ, $\overline{G}$ is a face of $\overline{\Delta}$. Therefore $\mathrm{lk}_{\overline{\Delta}}\overline{G}=\{F\subseteq G\colon F\notin\Delta\}$. Thus, if considered as simplicial complexes on the vertex set G, $\mathrm{lk}_{\overline{\Delta}}\overline{G}$ is the Alexander dual of Δ_G. Applying Alexander Duality, one has $\dim_K\tilde{H}_{i-2}(\mathrm{lk}_{\overline{\Delta}}\overline{G},K)=\dim_K\tilde{H}^{|G|-i-1}(\Delta_G,K)=\dim_K\tilde{H}_{|G|-i-1}(\Delta_G,K)$ and the assertion is proven. ▲

Considering the above, we may now prove the following

Proposition 3.8. *Let $K[\Delta]$ be a Stanley-Reisner ring, $a=(a_1,\dots,a_n)$, $a_i\le 0$ for $i=1,\dots,n$ and $F\in\Delta$, $F=\mathrm{supp}(a)$. If $\overline{a}\doteq(\overline{a_1},\dots,\overline{a_n})\in\mathbb{N}^n$, where*

$$\overline{a_i}\doteq\begin{cases}1 & \text{if } a_i=0\\ 0 & \text{if } a_i<0,\end{cases}$$

then

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta])_a=\beta_{i-|F|+1,\overline{a}}(K[\overline{\Delta}]).$$

Proof. Observe that, if $F\in\Delta$ and $F=\mathrm{supp}(a)$ then $\overline{F}$ is a non-face of $\overline{\Delta}$ and $\mathrm{supp}(\overline{a})=\overline{F}$. By Theorem 3.5, we have

$$\beta_{j,\overline{a}}(K[\overline{\Delta}])\doteq\dim_K\mathrm{Tor}^R_j(K[\overline{\Delta}],K)_{\overline{a}}=\dim_K\tilde{H}_{|\overline{F}|-j-1}(\overline{\Delta}_{\overline{F}},K).$$

Thus, by Lemma 3.7, $\beta_{j,\overline{a}}(K[\overline{\Delta}])=\dim_K\tilde{H}_{j-2}(\mathrm{lk}_\Delta F,K)$. Letting $j=i-|F|+1$, the conclusion follows immediately by virtue of Theorem 3.6. ▲

Recall that $H^i_{\mathfrak{m}}(K[\Delta])_j=\oplus_{|a|=j}H^i_{\mathfrak{m}}(K[\Delta])_a$, where $|a|=\sum_{i=1}^n a_i$ (cf. Remark 1.39 (ii)). Thus, if $j>0$, $H^i_{\mathfrak{m}}(K[\Delta])_j=0$. On the other hand, if $j\le 0$,

$$\begin{aligned}\dim_K H^i_{\mathfrak{m}}(K[\Delta])_j&=\sum_{|a|=j}\dim H^i_{\mathfrak{m}}(K[\Delta])_a=\sum_{|a|=j}\sum_{\substack{F\in\Delta\\ F=\mathrm{supp}(a)}}\beta_{i-|F|+1,\overline{a}}(K[\overline{\Delta}])\\&=\sum_{F\in\Delta}\sum_{\substack{|a|=j\\ \mathrm{supp}(a)=F}}\beta_{i-|F|+1,\overline{a}}(K[\overline{\Delta}]),\end{aligned}$$

Let us fix i and $j\le 0$ and $F\in\Delta$. We define α_F to be the cardinality of the set $\{a\in\mathbb{Z}^n_-\colon \mathrm{supp}(a)=F,\ |a|=j\}$. Clearly, α_F depends on the cardinality of F but not on F itself. Thus we may also write $\alpha_{|F|}$. Moreover, the Betti numbers involved in the sum do not depend on a but on $\mathrm{supp}\,a$. This can be seen for instance in Theorem 3.5. Therefore, since $\mathrm{supp}\,\overline{a}=\overline{F}$, we can re-write the previous formula as follows

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta])_j=\sum_{h=1}^n\alpha_h\beta_{i-h+1,n-h}(K[\overline{\Delta}]).$$

Finally, if $|F|=h$, it is easy to compute that $\alpha_h=\binom{n}{h}\binom{h+|j|-1}{|j|}$. This shows that the dimension of the components of $H^i_{\mathfrak{m}}(K[\Delta])$ is related to the Betti numbers of $K[\overline{\Delta}]$ in the following way.

Lemma 3.9. *Let Δ be a simplicial complex and $\overline{\Delta}$ its Alexander dual. Then, for any i and $j \leq 0$,*

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta])_j = \sum_{h=1}^{n} \binom{n}{h} \binom{h+|j|-1}{|j|} \beta_{i-h+1,n-h}(K[\overline{\Delta}]).$$

Next, we prove (3.1) in the squarefree case.

Theorem 3.10. *Let K be a field of characteristic 0. Let $\mathcal{D}$ be a family of simplicial complexes with a given f-vector and let Δ^{lex} be the lexicographic simplicial complex of $\mathcal{D}$. Then, for all $\Delta \in \mathcal{D}$ and for any i, j,*

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta])_j \leq \dim_K H^i_{\mathfrak{m}}(K[\Delta^{lex}])_j.$$

One more result is needed before proceeding with the proof.

Lemma 3.11. *Let Δ be any simplicial complex and $\overline{\Delta}$ its Alexander dual. Then,*

$$\overline{\Delta^{lex}} = \overline{\Delta}^{lex}.$$

Proof. One has to prove that $I_{\overline{\Delta^{lex}}} = I_{\overline{\Delta}^{lex}}$, or equivalently that $J_{\overline{\Delta^{lex}}} = J_{\overline{\Delta}^{lex}}$ in the exterior algebra E. By Lemma 1.14, one becomes $J_{\overline{\Delta^{lex}}} = 0 :_E J_{\Delta^{lex}}$, which, by definition of Δ^{lex}, is $0 :_E J^{lex}_{\Delta}$. On the other hand, $J_{\overline{\Delta}^{lex}} = J^{lex}_{\overline{\Delta}} = (0 :_E J_{\Delta})^{lex}$. The conclusion follows directly from Lemma 1.31. ▲

Proof of Theorem 3.10. By virtue of Theorem 3.4, for any $\Delta \in \mathcal{D}$, $\beta_{i,j}(K[\Delta]) \leq \beta_{i,j}(K[\Delta^{lex}])$, for every i, j. Applying Lemma 3.9, one has

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta])_j = \sum_{h=1}^{n} \alpha_h \beta_{i-h+1,n-h}(K[\overline{\Delta}]) \leq \sum_{h=1}^{n} \alpha_h \beta_{i-h+1,n-h}(K[\overline{\Delta}^{lex}]),$$

and therefore, from the previous lemma, we obtain

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta])_j \leq \sum_{h=1}^{n} \alpha_h \beta_{i-h+1,n-h}(K[\overline{\Delta^{lex}}]).$$

Applying Lemma 3.9 once again, we deduce the desired inequality. ▲

Remark 3.12. The above theorem provides a way of computing explicit upper bounds for $\dim_K H^i_{\mathfrak{m}}(K[\Delta])_j$, by determining $\dim_K H^i_{\mathfrak{m}}(R/I^{lex}_{\Delta})_j$ in terms of the f-vector of Δ. As we have already seen,

$$\dim_K H^i_{\mathfrak{m}}(R/I^{lex}_{\Delta})_j = \sum_{h=1}^{n} \binom{n}{h} \binom{h+|j|-1}{|j|} \beta_{i-h,n-h}(I_{\overline{\Delta^{lex}}}).$$

The Betti numbers of $I_{\overline{\Delta^{lex}}}$ can be given explicitly, in view of the following formula

$$\beta_{i,i+j}(I_{\overline{\Delta^{lex}}}) = \sum_{u \in G(I_{\overline{\Delta^{lex}}})_j} \binom{m(u)-j}{i},$$

where $G(I_{\overline{\Delta^{lex}}})_j$ denotes the minimal set of generators of $I_{\overline{\Delta^{lex}}}$ in degree j and $m(u) \doteq \max\{i\colon X_i|u\}$ (cf. [AHHi2]). The sum on the right-hand side can be computed combinatorially in terms of the f-vector of $\overline{\Delta^{lex}}$, which is related to that of Δ, as shown in (1.1).

3.4 The non-squarefree case

In this section we prove the inequality (3.1) for non-squarefree ideals. The idea is based on the proof of Pardue that leads to the inequality between the Betti numbers of I and I^{lex} which we discussed in the introduction. For this purpose we show in Lemma 3.13 that we are able to control the Hilbert series of the local cohomology modules when we specialize a complete polarization of R/I by a collection of generic linear forms.

First of all, observe that, if N is an R-module of positive depth and K is infinite, there exists a linear form $l \in R_1$ which is N-regular. More precisely, l can be found by avoiding a finite number of prime ideals, which is an open condition. This implies that, any generic linear form l' is N-regular as well.

Let I be a monomial ideal of R and let $I^{\mathbf{P}}$ be a complete polarization of I in S, where $S \doteq K[Y_{ij}]$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$ for $m \gg 0$ (cf. Remark 2.15 (iii)). Let c denote the number of additional variables needed to achieve such a complete polarization. Let also $\mathcal{L} \doteq \{l_{ij}\}$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$, a collection of linear forms in R, let us say $l_{ij} = \sum_{k=1}^{n} a_k^{ij} X_k$.

Finally let $\sigma_{\mathcal{L}} : S \longrightarrow R$ be the homomorphism defined by $\sigma_{\mathcal{L}}(Y_{ij}) \doteq l_{ij}$, for any i, j, and set $I_{\mathcal{L}} \doteq \sigma_{\mathcal{L}}(I^{\mathbf{P}})$. Clearly, $\sigma_{\mathcal{L}}$ induces a map $\overline{\sigma}_{\mathcal{L}} : S/I^{\mathbf{P}} \longrightarrow R/I_{\mathcal{L}}$.

Note that, if l_{ij} is equal to X_i for any i, j, then $\operatorname{Ker}\sigma_{\mathcal{L}}$ is generated by $Y_{ij} - Y_{i1}$, for $i = 1, \ldots, n$ and $j = 2, \ldots, m$, which form a $S/I^{\mathbf{P}}$-regular sequence of length c. Suppose that $\mathcal{L}$ is generic. Then for any n linear forms of $\mathcal{L}$, let us say l_{i1}, with $i = 1, \ldots, n$, one can find polynomials $f_k(a_h^{i1})$ such that $X_k = \sum_{h=1}^{n} f_k(a_h^{i1}) l_{h1}$, for $k = 1, \ldots, n$. Therefore, there exist polynomials g_h^{ij} in a_k^{ij}, with $i, k = 1, \ldots, n$ and $j = 1, \ldots, m$ such that $l_{ij} = \sum_{h=1}^{n} g_h^{ij} l_{h1}$, for $i = 1, \ldots, n$ and $j = 2, \ldots, m$. In particular, this implies that $\operatorname{Ker}\sigma_{\mathcal{L}}$, which is generated by the linear forms $Y_{ij} - \sum_{h=1}^{n} g_h^{ij} Y_{h1}$, for $i = 1, \ldots, n$ and $j = 2, \ldots, m$, is parameterized in $\mathbb{A}_K^{mn^2}$ by these coefficients. Let us consider this affine space with its Zariski topology. One can prove that, if K is infinite,

$$\operatorname{Tor}_i^S(S/\operatorname{Ker}\sigma_{\mathcal{L}}, S/I^{\mathbf{P}}) = 0 \text{ for all } i > 0$$

is an open property in $\mathbb{A}_K^{mn^2}$ (cf. [P],[P1]). This is equivalent to saying that, if $\mathcal{L}$ is generic, $\operatorname{Ker}\overline{\sigma}_{\mathcal{L}}$ is generated by an $S/I^{\mathbf{P}}$-regular sequence of length c, since we have already provided an example of a collection $\mathcal{L}$ for which this holds true. This is the property we need in order to prove the next lemma.

Lemma 3.13. *Let K be infinite and $\mathcal{L}$ be a collection of generic linear forms as above. Then, if $\mathfrak{n}$ denotes the graded maximal ideal of S,*

$$\operatorname{Hilb}(H^i_{\mathfrak{m}}(R/I_{\mathcal{L}}),t) = (t-1)^c \operatorname{Hilb}(H^{i+c}_{\mathfrak{n}}(S/I^{\mathbf{P}}),t).$$

Proof. As we have seen above, for generic $\mathcal{L}$, $\operatorname{Ker}\overline{\sigma}_{\mathcal{L}}$ is generated by a sequence of linear forms, let us say $l_1,\ldots,l_c$, which are S- and $S/I^{\mathbf{P}}$-regular.
Clearly it is sufficient to consider the case $c=1$. The short exact sequence $0 \longrightarrow S/I^{\mathbf{P}}(-1) \xrightarrow{\cdot l_1} S/I^{\mathbf{P}} \longrightarrow S/(I^{\mathbf{P}},l_1) \longrightarrow 0$, leads to the sequence

$$0 \longrightarrow \operatorname{Ext}^i_S(S/I^{\mathbf{P}},S) \xrightarrow{\cdot l_1} \operatorname{Ext}^i_S(S/I^{\mathbf{P}},S)(1) \longrightarrow \operatorname{Ext}^{i+1}_S(S/(I^{\mathbf{P}},l_1),S) \longrightarrow 0,$$

which is exact. Indeed, by Corollary 2.12, $\operatorname{Ext}^i_S(S/I^{\mathbf{P}},S) \simeq \operatorname{Ext}^i_R(R/I,R)^{\mathbf{P}}$. Therefore, by Proposition 2.10, $\operatorname{depth}\operatorname{Ext}^i_S(S/I^{\mathbf{P}},S) > 0$. Since l_1 is generic, l_1 is $\operatorname{Ext}^i_S(S/I^{\mathbf{P}},S)$-regular and this explains the exactness of the above sequence. Note that, by definition, $S/(I^{\mathbf{P}},l_1) \simeq R/I_{\mathcal{L}}$. Thus, from Rees' Lemma, it follows that $\operatorname{Ext}^{i+1}_S(S/(I^{\mathbf{P}},l_1),S) \simeq \operatorname{Ext}^i_R(R/I_{\mathcal{L}},R)(1)$ and, upon shifting the above sequence by -1, one can argue as in the proof of Corollary 2.18 in order to achieve the desired conclusion. ▲

Given a monomial ideal I and a collection of generic linear forms as above, one defines $\varphi(I) \doteq \operatorname{in}(I_{\mathcal{L}})$. One can verify that φ is well-defined, i.e. $\varphi(I)$ does not depend on the choice of $\mathcal{L}$.

Proposition 3.14 (Pardue). *If I is a monomial ideal and L is the lexicographic ideal with the same Hilbert function as I, then, for $e \gg 0$, $\varphi^e(I) = L$.*

Proof. See Theorem 23 in [P] or Proposition 30 in [P1]. ▲

Theorem 3.15. *Let $R = K[X_1,\ldots,X_n]$ be a polynomial ring in n variables over a field K with its standard grading and $\mathfrak{m} = (X_1,\ldots,X_n)$ be its graded maximal ideal. Let $\mathcal{I}$ be a family of graded ideals of R with a given Hilbert function, and let L be the lexicographic ideal of $\mathcal{I}$. Then, for all $I \in \mathcal{I}$ and for any i,j,*

$$\dim_K H^i_{\mathfrak{m}}(R/I)_j \leq \dim_K H^i_{\mathfrak{m}}(R/L)_j.$$

Proof. By virtue of Remark 1.39 (i), we may assume, without loss of generality, K to be infinite. By Theorem 3.3, eventually passing to initial ideals, we may consider only monomial ideals.
Combining Corollary 2.18 with Lemma 3.13, we obtain for any i,j,

$$\dim_K H^i_{\mathfrak{m}}(R/I)_j = \dim_K H^i_{\mathfrak{m}}(R/I_{\mathcal{L}})_j.$$

Applying Theorem 3.3, we have $\dim_K H^i_{\mathfrak{m}}(R/I)_j \leq \dim_K H^i_{\mathfrak{m}}(R/\operatorname{in}(I_{\mathcal{L}}))_j$, and so $\dim_K H^i_{\mathfrak{m}}(R/I)_j \leq \dim_K H^i_{\mathfrak{m}}(R/\varphi(I))_j$. It is now clear how the conclusion follows directly from Proposition 3.14. ▲

4 On the structure of local cohomology modules of lex-ideals

The aim of this chapter is to describe the local cohomology modules $H^i_{\mathfrak{m}}(R/J)$, in case J is a lex-ideal or a squarefree lex-ideal. By duality, one can study the corresponding Ext groups, which will be examined in the first two sections, with the aid of a reduction argument and of the Eliahou-Kervaire resolution for stable ideals. In the third and last section we shall prove a structure theorem for the local cohomology modules. Moreover, we shall describe their Hilbert function in terms of that of R/J.

4.1 Non-squarefree lex-ideals

Let J be a lex-ideal generated in one degree, let us say $J = (\mathcal{L}(v))$, with $v = X_1^{v_1} \cdot \ldots \cdot X_n^{v_n}$ and with $\deg v = d$. As already observed in the proof of Proposition 1.29, J can be written as $X_1^{v_1}(X_1\mathfrak{m}^{d-v_1-1} + J_1)$, where J_1 is the lex-ideal in the variables $X_2, \ldots, X_n$ generated by $\mathcal{L}(X_2^{v_2} \cdot \ldots \cdot X_n^{v_n})$. Therefore,

$$H^i_{\mathfrak{m}}(R/J) \simeq H^i_{\mathfrak{m}}((R/X_1\mathfrak{m}^{d-v_1-1} + J_1)(-v_1)),$$

for every $i < n-1$. On the other hand, since $((X_1)+J_1)/(X_1\mathfrak{m}^{d-v_1-1}+J_1)$ has finite length, by Lemma 1.33, $H^i_{\mathfrak{m}}(X_1\mathfrak{m}^{d-v_1-1} + J_1) \simeq H^i_{\mathfrak{m}}((X_1) + J_1)$, for every $i > 1$. We thus obtain that

$$H^i_{\mathfrak{m}}(R/J) \simeq H^i_{\mathfrak{m}}((R/X_1\mathfrak{m}^{d-v_1-1} + J_1)(-v_1)) \simeq H^i_{\mathfrak{m}}((R/(X_1) + J_1)(-v_1)),$$

for every $0 < i < n-1$. Applying first the Local Duality Theorem and then Rees' Lemma, we deduce that

$$\begin{aligned}\operatorname{Ext}^i_R(R/J, R) &\simeq \operatorname{Ext}^i_R(R/(X_1) + J_1, R)(v_1) \\ &\simeq \operatorname{Ext}^{i-1}_{R/(X_1)}(R/(X_1) + J_1, R/(X_1))(v_1+1),\end{aligned} \tag{4.1}$$

for every $1 < i < n$.

Proposition 4.1. *Let $J = (\mathcal{L}(v))$ be a lex-ideal, with $v = X_1^{v_1} \cdot \ldots \cdot X_n^{v_n}$. Then, for any $i < n$, $\operatorname{Ext}^i_R(R/J, R)$ is cyclic and*

$$\operatorname{Ext}^i_R(R/J, R) \simeq \frac{R}{(X_1, \ldots, X_{i-1}, X_i^{v_i})}\Big(\sum_{h \le i} v_h + i - 1\Big).$$

We recall now a few facts which are useful for the proof. A monomial ideal I of R is called *stable* iff it has the following property: $X_i \frac{u}{X_{m(u)}}$, for all $i \leq m(u) \doteq \max_j\{j \in \operatorname{supp} u\}$ and for any monomial $u \in I$. It is possible to give a complete description of the minimal free resolution $(F_{\bullet}, \partial_{\bullet})$ of R/I, which is usually referred to as the Eliahou-Kervaire resolution of R/I, as done in [AH], which we refer the reader to for more details. Here as in [AH] a basis element of F_i is denoted by $f(\sigma, u)$, where u is a monomial of the minimal set of generators $G(I)$ of I, $\sigma \subset \{1, \ldots, n\}$ with $|\sigma| = i-1$ and $\max(\sigma) < m(u)$. We also let $f(u) \doteq f(\emptyset, u)$. Note that lexicographic ideals are stable. Thus, in order to prove the proposition, we may use the above resolution, dualize and compute the homology. By virtue of the observations which led to (4.1), one has only to compute the first cohomology group. It is known that $\partial_1(f(u)) = u$ and that $\partial_2(f(i,u)) = -X_i f(u) + X_j f(u_i)$, where $u_i \in G(J)$ and $X_i u = u_i X_j$ with $m(u_i) \leq X_j$, since we are assuming J to be generated in one degree. Recall that $\deg f(\sigma, u) \doteq \deg u + |\sigma|$ and denote by $f^*(\sigma, u)$ the elements of the dual basis, whose degree is set to be $-\deg u - |\sigma|$. We describe now the differential maps $\partial_1^* : F_0^* \simeq R \longrightarrow F_1^* \simeq \oplus R(d)$ and $\partial_2^* : F_1^* \simeq \oplus R(d) \longrightarrow F_2^* \simeq \oplus R(d+1)$. Let 1^* be the basis element of F_0^*. Since $\partial_1^*(1^*) \doteq 1^* \circ \partial_1 = \sum_{u \in G(J)} a_u f^*(u)$ and $\partial_1(f(u)) = u$, we obtain

$$\partial_1^*(1^*) = \sum_{u \in G(J)} u f^*(u).$$

Analogously, $\partial_2^*(f^*(u)) = \sum_{\substack{i<m(v)\\ v \in G(J)}} a_{i,v} f^*(i,v)$, where $a_{i,v} = \partial_2^*(f^*(u))(f(i,v)) = f^*(u)(-X_i f(v) + X_{m(v)} f(X_i v / X_{m(v)}))$. Therefore, $a_{i,v} = -X_i$, if $v = u$, or $a_{i,v} = X_{m(v)}$, if $u = \frac{X_i v}{X_{m(v)}}$, which is the case iff $v = \frac{X_j u}{X_i}$, with $i \in \operatorname{supp}(u)$, $j \geq m(u)$ and $\frac{X_j u}{X_i} \in G(J)$. Accordingly,

$$\partial_2^*(f^*(u)) = -\sum_{i<m(u)} X_i f^*(i,u) + \sum_{i \in \operatorname{supp}(u)} \sum_{j \in \mathcal{S}} X_j f^*(i, \frac{u}{X_i} X_j),$$

where $\mathcal{S} = \{j : j \geq m(u),\ j \neq i \text{ with } X_j u / X_i \in \mathcal{L}(v)\}$.

Proof of Proposition 4.1. In view of the observations previous to the proposition, it is sufficient to show that

$$\operatorname{Ext}^1_R(R/J, R) \simeq \frac{R}{(X_1^{v_1})}(v_1).$$

Let us consider a non-zero multi-homogeneous element z of F_1^* of degree $-c$, say $z = \sum_{u \in \mathcal{L}_v} \alpha_u z_u f^*(u)$, for some monomials z_u of R and coefficients $\alpha_u \in K$. Since $f^*(u)$ has degree $-\deg u$, it must hold true that $\deg z_u - \deg u = -c$, and hence $z_u = u/X^c$ for every u with $\alpha_u \neq 0$. But, if z is a cycle, we claim that, for every $u \in \mathcal{L}(v)$, $\alpha_u = a$ for some $a \in K \setminus \{0\}$. In fact only $\partial_2^*(f^*(u))$ and $\partial_2^*(f^*(w))$, with $w \doteq X_i u / X_{m(u)}$, can contribute to the coefficients of $f^*(i,u)$. More precisely, these coefficients are $-\alpha_u X_i u / X^c$ and $\alpha_w X_{m(u)} w / X^c$ and thus $\alpha_u = \alpha_w$. By hypothesis,

the monomials of $G(J)$ form a lex-segment. Thus, the above observations and the fact that $z \neq 0$ imply the claim.
Since X^c divides every $u \in \mathcal{L}(v)$, then $X^c | X_1^d$ and $X^c | v$. Thus $X^c = X_1^k$, for some $k \leq v_1$ and therefore $\operatorname{Ker} \partial_2^*$ is generated by the element $\overline{z} = (1/X_1^{v_1}) \sum_{u \in \mathcal{L}_v} uf^*(u)$. Accordingly, the homomorphism $\varphi : \operatorname{Ext}^1_R(R/J, R) \longrightarrow (R/(X_1^{v_1}))(v_1)$ defined by $\varphi([\overline{z}]) = 1$ provides the desired isomorphism.

▲

4.2 Squarefree lex-ideals

In the squarefree case one proves analogous results on the Ext groups as performed in the previous section. The argumentation is slightly more complicated but follows the same path. In particular, one says that a squarefree ideal I is *squarefree stable* iff $X_i \frac{u}{X_{m(u)}} \in I$ for any $i < m(u)$ such that $i \notin \operatorname{supp} u$ and for every squarefree monomial $u \in I$. Moreover, free resolutions of squarefree stable ideals can be constructed in a similar way to the Eliahou-Kervaire resolutions of non-squarefree stable ideals (see Theorem 2.1 in [AHHi2]). Also in the squarefree case lexicographic ideals are stable.

Lemma 4.2. *Let $A \doteq K[X_1, \ldots, X_r]$ be a polynomial ring and let J be the ideal of A generated by all of the squarefree monomials of A_d. Then $\operatorname{Ext}^i_A(A/J, A) = 0$ for $i \neq r - d + 1$.*

Proof. Since J is squarefree, it defines a simplicial complex Δ. The corresponding Stanley-Reisner ring $K[\Delta]$ is Cohen-Macaulay of dimension $d - 1$. In fact Δ can be viewed as the $d - 1$-skeleton of the simplex generated by $\{1, \ldots, r\}$ and since this is Cohen-Macaulay then Δ is Cohen-Macaulay as well (cf. Exercise 5.1.23 in [BH]). Thus, if $\mathfrak{n}$ denotes the graded maximal ideal of A, $H^{d-1}_{\mathfrak{n}}(K[\Delta]) \neq 0$ and this is the only non-vanishing local cohomology module. The conclusion follows from the Local Duality Theorem. ▲

Lemma 4.3. *Let $A \doteq K[X_1, \ldots, X_r]$ be a polynomial ring and I a squarefree lex-ideal of A generated by $\mathcal{L}(v)$, for some $v \in A_d$. Suppose $X_1 \notin \operatorname{supp} v$. Then, $I = X_1 J + I'A$, where I' is the ideal generated by $\mathcal{L}(v)$ in $A' \doteq K[X_2, \ldots, X_r]$, and, for $0 < i < r - d + 1$,*

$$\operatorname{Ext}^i_A(A/I, A) \simeq \operatorname{Ext}^{i-1}_{A'}(A'/I', A')(1).$$

Proof. The first assertion is clear, since J is the ideal of A generated by all squarefree monomials of A' of degree $d - 1$.
Since J is generated by monomials in the variables $X_2, \ldots, X_r$ and $I'A \subset J$, it is easy to see that $(X_1) \cap I'A = (X_1)J \cap I'A$. Therefore,

$$\frac{(X_1) + I'A}{I} = \frac{(X_1) + I'A}{(X_1)J + I'A} \simeq \frac{(X_1)}{(X_1)J} \simeq \frac{A}{J}(-1). \tag{4.2}$$

On the other hand, from the short exact sequence

$$0 \longrightarrow \frac{(X_1)+I'A}{I} \longrightarrow \frac{A}{I} \longrightarrow \frac{A}{(X_1)+I'A} \longrightarrow 0$$

we derive the long exact sequence in homology

$$\ldots \longrightarrow \operatorname{Ext}_A^{i-1}\left(\frac{(X_1)+I'A}{I}, A\right) \longrightarrow \operatorname{Ext}_A^{i}\left(\frac{A}{(X_1)+I'A}, A\right) \longrightarrow$$
$$\longrightarrow \operatorname{Ext}_A^{i}\left(\frac{A}{I}, A\right) \longrightarrow \operatorname{Ext}_A^{i}\left(\frac{(X_1)+I'A}{I}, A\right) \longrightarrow \ldots.$$

In view of the previous lemma and of (4.2) one can deduce that, for $i \neq r-d+1$, $\operatorname{Ext}_A^{i-1}\left(\frac{(X_1)+I'A}{I}, A\right) = 0$. But this implies that, for $0 < i < r-d+1$,

$$\operatorname{Ext}_A^{i}\left(\frac{A}{I}, A\right) \simeq \operatorname{Ext}_A^{i}\left(\frac{A}{(X_1)+I'A}, A\right).$$

The conclusion follows from Rees' Lemma. ▲

Let v any squarefree monomial of $S \doteq K[X_1, \ldots, X_m]$. Grouping together consecutive variables, we may write

$$v = X_1 X_2 \cdot \ldots \cdot X_{v_1} \cdot X_{v_1+2} \cdot \ldots \cdot X_{v_1+v_2+1} \cdot \ldots = \prod_{j=1}^{h} \prod_{i=j}^{v_j+j-1} X_{v_1+\ldots+v_{j-1}+i},$$

with the standard convention that the empty product is equal to 1. This notation will be useful in what follows. Note that, if I is a squarefree lex-ideal of S generated in degree d, then $\operatorname{proj\,dim} S/I \leq m-d+1$.

Proposition 4.4. *Let $S \doteq K[X_1, \ldots, X_m]$ be a polynomial ring and I be a squarefree lex-ideal generated by $\mathcal{L}(v)$ for some $v \in S_d$. Then, for any $j < m-d+1$, $\operatorname{Ext}_S^j(S/I, S)$ is cyclic and, if $u_j \doteq X_j \cdot \ldots \cdot X_{j+v_j-1}$,*

$$\operatorname{Ext}_S^j(S/I, S) \simeq \frac{S}{(X_1, \ldots, X_{j-1}, u_j)}(\sum_{l \leq j} v_l + j - 1).$$

Proof. We shall prove that, for any $j < m-d+1$,

$$\operatorname{Ext}_S^j(S/I, S) \simeq \frac{S}{(X_{v_1+1}, X_{v_1+v_2+2}, \ldots, X_{v_1+\ldots+v_{j-1}+j-1}, u'_j)}(\sum_{l \leq j} v_l + j - 1),$$

where $u'_j \doteq \prod_{i=j}^{v_j+j-1} X_{v_1+\ldots+v_{j-1}+i}$, which is equivalent to the assertion, upon reindexing the variables.
The case $j > 1$ can be reduced to the case $j = 1$, as suggested by the previous

lemma. In fact, if S_i denotes the polynomial ring $K[X_i,\ldots,X_m]$ for any $i<m$, since $\mathcal{L}(v)=u_1\mathcal{L}(\frac{v}{u_1})$, one has

$$\operatorname{Ext}^j_S(S/I,S)\simeq \operatorname{Ext}^j_{S_{v_1}}\left(\frac{S_{v_1}}{(\mathcal{L}(\frac{v}{u_1}))S_{v_1}},S_{v_1}\right)[X_1,\ldots,X_{v_1}](v_1),$$

and, by Lemma 4.3,

$$\operatorname{Ext}^j_S(S/I,S)\simeq \operatorname{Ext}^{j-1}_{S_{v_1+1}}\left(\frac{S_{v_1+1}}{(\mathcal{L}(\frac{v}{u_1}))S_{v_1+1}},S_{v_1+1}\right)[X_1,\ldots,X_{v_1}](v_1+1).$$

The proof for the case $j=1$ is verbatim that of Proposition 4.1 and therefore we shall omit it. ▲

Remarks 4.5.

(i) Note that Proposition 4.4 provides an alternative and easier method than that suggested in Remark 3.12, to compute an explicit upper bound for the Hilbert series of the local cohomology modules of $K[\Delta]$ when Δ is pure.

(ii) In [K], Kalai introduced an operation, which we denote by σ, that maps a monomial u of R_d into a squarefree monomial u^σ in a polynomial ring with $m(u)+d-1$ variables. Given a monomial ideal I generated by some monomials u_i, with $i=1,\ldots,k$, let I^σ be the squarefree monomial ideal generated by $u_1^\sigma,\ldots,u_k^\sigma$. In [AHHi] it was proven that σ is a bijection between lexicographic ideals and squarefree lexicographic ideals. If $v=X_1^{v_1}\cdot\ldots\cdot X_h^{v_h}$ is a monomial of R_d, it is not difficult to see that $v^\sigma=\prod_{j=1}^h\prod_{i=j}^{v_j+j-1}X_{v_1+\ldots+v_{j-1}+i}$. Now we can consider the ideal I generated by $\mathcal{L}(v)$ in R and the squarefree ideal I^σ generated by the squarefree monomials of $\mathcal{L}(v^\sigma)$ in the polynomial ring S, as described in [AHHi] and compare the formulae of Propositions 4.1 and 4.4. If we let $I_{(i)}$ and $J_{(i)}$ be the ideals such that $\operatorname{Ext}^i_R(R/I,R)\simeq R/I_i$ and $\operatorname{Ext}^i_S(S/I^\sigma,S)\simeq S/J_i$, then

$$I^\sigma_{(i)}=J_i.$$

4.3 Structure theorem and numeric upper bounds

By virtue of the results of the previous two sections, one can describe the R-module structure of local cohomology modules of lex-ideals generated in one degree. In the next proposition J is a non-squarefree lex-ideal, but the squarefree version is analogous.

Proposition 4.6. *Let $i > 0$ and $H^i_{\mathfrak{m}}(R/J) \neq 0$. Then $H^i_{\mathfrak{m}}(R/J)$ is isomorphic as an R-module to $M(-\sum_{h\leq n-i} v_h + i + 1)$, where M is endowed with the R-graded structure of submodule inherited from that of $K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and is defined as $M \doteq \oplus_{c\in\mathbb{Z}_-\cup\{0\}} M_c$ with*

$$M_c = \operatorname{span}_K\big\{X_{n-i}^{a_{n-i}} \cdot \ldots \cdot X_n^{a_n} \big| a_j \leq 0,\ \sum_j a_j = c,\ -a_{n-i} < v_{n-i}\big\}.$$

Proof. From Proposition 4.1 and the Local Duality Theorem, we deduce that, for every $i < n$,

$$H^i_{\mathfrak{m}}(R/J) \simeq \operatorname{Hom}_K\left(\frac{K[X_{n-i}, \ldots, X_n]}{(X_{n-i}^{v_{n-i}})}, K\right)(-\sum_{h\leq n-i} v_h + i + 1). \tag{4.3}$$

Let $L \doteq (X_1, \ldots, X_{i-1}, X_i^a)$, $T \doteq R/L$ and let $M^* \doteq \operatorname{Hom}_K(M, K)$ be the dual over K of a graded R-module M. We also set $(M^*)_i \doteq (M_{-i})^*$. The multiplication map $X_j : M_i \longrightarrow M_{i+1}$ induces the map $X_j^* : (M^*)_{-i-1} \longrightarrow (M^*)_{-i}$ on the homogeneous components of M^* and defines the graded structure of M^* as an R-module. It follows easily that $R^* \simeq K[X_1^{-1}, \ldots, X_n^{-1}]$. Furthermore, the elements of $(T^*)_c$ can be viewed as the K-homomorphisms of R^* which vanish when restricted to $(L^*)_c$. By the definition of L, $(L^*)_c$ is spanned over K by the elements of $\{X_1^{a_1} \cdot \ldots \cdot X_n^{a_n} \colon a_j \leq 0,\ \sum_j a_j = c\}$, where $a_h < 0$ for some $h \leq i-1$ or $-a_i \geq a$. Thus, a K-basis of $(T^*)_c$ is

$$\big\{X_i^{a_i} \cdot \ldots \cdot X_n^{a_n} \big| a_j \leq 0, \sum_j a_j = c,\ -a_i < a\big\}.$$

This fact, together with (4.3), yields the desired isomorphism. ▲

From this point on we shall concentrate on the non-squarefree case.
In the previous proposition we assume J to be a lex-ideal generated in one degree and $i > 0$. On the other hand it is clear that $I/I_{\geq k}$ has finite length for any ideal I and $k \in \mathbb{N}$ (cf. Definition 1.27). Indeed, for any k there exists an $l \in \mathbb{N}$ such that $\mathfrak{m}^l I \subseteq I_{\geq k}$. Therefore, by Lemma 1.33, $H^i_{\mathfrak{m}}(R/I) \simeq H^i_{\mathfrak{m}}(R/I_{\geq k})$ for any k and $i > 0$. It is immediately seen that, if I is a lex-ideal, then $I_{\geq k}$ is a lex-ideal, and if $k \gg 0$, it is generated in one degree. Thus, Proposition 4.6 still works if we let J be an arbitrary lex-ideal and v be the least monomial which belongs to the minimal set of generators of a sufficiently high truncation of J. Note that the preceding is consistent with the statement of the proposition, since the exponent v_n, which is dependent upon the chosen truncation, is not involved there.

Remarks 4.7.

(i) In view of the Local Duality Theorem and Proposition 4.1, for any $A \subset \{1, \ldots, n-1\}$, one can exhibit examples of a graded R-module M such that $H^i_{\mathfrak{m}}(M) = 0$ iff $i \in A$. Indeed it is enough to set $M \doteq R/J$, where J is a lex-ideal generated in one degree by $\mathcal{L}(v)$, and choose a monomial $v \in R$ such that $\operatorname{supp} v = \{j \colon n-j \notin A\}$.

(ii) Let J be a lex-ideal and $i < n$. Then, $0 \neq H^i_{\mathfrak{m}}(R/J)$ has finite length iff $i = 0$. The "if" part is clear. In order to verify the other part of the assertion one may assume $i > 0$ and J generated in one degree, eventually substituting it with $J_{\geq k}$, for $k \gg 0$. The problem is thus reduced to checking when the Hilbert series of $\mathrm{Ext}^i_R(R/J, R)$ is a polynomial. By virtue of Proposition 4.1, it is easy to see that this is never the case for $i < n$.

We are now interested in determining upper bounds for $\dim_k H^i_{\mathfrak{m}}(R/I)_j$ for every ideal I and for every i, j explicitly in terms of the Hilbert function of R/I. We shall see that, if $i > 0$, one can show that the information brought by the Hilbert polynomial is already sufficient to estimate $\dim_K H^i_{\mathfrak{m}}(R/I)_j$.
Suppose $i = 0$. For any ideal I, $H^0_{\mathfrak{m}}(R/I)$ is Artinian and isomorphic to $I : \mathfrak{m}^\infty/I$. Thus,

$$\begin{aligned} H(H^0_{\mathfrak{m}}(R/I^{lex}), j) &= H(I^{lex} : \mathfrak{m}^\infty, j) - H(I^{lex}, j) \\ &= H(R/I^{lex}, j) - H(R/I^{lex} : \mathfrak{m}^\infty, j), \end{aligned}$$

for every j. Observe that the last expression is entirely determined by the Hilbert function of R/I. More precisely $H(R/I^{lex} : \mathfrak{m}^\infty, j)$ depends only asymptotically on $H(R/I, n)$, i.e. it depends on the Hilbert polynomial of R/I.

Proposition 4.8. *Let I be any graded ideal with $\dim_K R/I > 0$ and with a given Hilbert function. Let*

$$d \doteq \min\{k\colon H(R/I, h+1) = H(R/I, h)^{<h>}, \text{ for every } h \geq k\}.$$

Furthermore, let

$$H(R/I, d) = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \ldots + \binom{k(1)}{1},$$

with $k(d) > k(d-1) > \ldots > k(1) \geq 0$, be the d^{th} binomial expansion of $H(R/I, d)$. Let $v_i \doteq |\{k(h)\colon n - k(h) + h - 1 = i\}|$. Then

$$H(H^0_{\mathfrak{m}}(R/I), j) \leq H(R/I, j) - H(R/K, j),$$

where

$$K = (X_1^{v_1+1}, X_1^{v_1}X_2^{v_2+1}, \ldots, X_1^{v_1} \cdot \ldots \cdot X_{h-2}^{v_{h-2}} X_{h-1}^{v_{h-1}+1}, X_1^{v_1} \cdot \ldots \cdot X_{h-1}^{v_{h-1}} X_h^{v_h}).$$

Proof. By Theorem 3.15, one has $H(H^0_{\mathfrak{m}}(R/I), j) \leq H(R/I, j) - H(R/I^{lex} : \mathfrak{m}^\infty, j)$. Observe that, by definition, d is the greatest degree of a monomial in the minimal set of generators of I^{lex}. Hence $I^{lex}_{\geq d}$ is a lex-ideal generated in one degree, and, by Proposition 1.26, $I^{lex} : \mathfrak{m}^\infty = I^{lex}_{\geq d} : \mathfrak{m}^\infty$. In view of Proposition 1.29, we have that $I^{lex} : \mathfrak{m}^\infty = K$, as desired. ▲

The most interesting case is for $i > 0$.

Theorem 4.9. *Let I be any graded ideal with a given Hilbert polynomial*

$$P_{R/I}(X) = \binom{X+a_1}{a_1} + \binom{X+a_2-1}{a_2} + \ldots + \binom{X+a_l-(l-1)}{a_l},$$

with $a_1 \geq a_2 \geq \ldots \geq a_l \geq 0$ and let

$$v_i \doteq |\{a_j : n - a_j - 1 = i\}|.$$

Then,

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(R/I), t) \preceq \frac{\left(t^{\sum_{h\leq n-i} v_h - i}\right) \sum_{j=1}^{v_{n-i}} t^{-j}}{(1-t^{-1})^i},$$

for every $i > 0$.

Some words of explanation are perhaps required before passing to the proof. We present here the Hilbert polynomial in a non-standard way (see Remark 1.6). That such a presentation is unique is shown in the following lemma.

Lemma 4.10. *Let $P(X) \in \mathbb{Q}[X]$ be a polynomial and let $a_1 \geq \ldots \geq a_l \geq 0$ be integers such that $P(X) = \binom{X+a_1}{a_1} + \binom{X+a_2-1}{a_2} + \ldots + \binom{X+a_l-(l-1)}{a_l}$. Suppose that there exist b_i, for $i = 1, \ldots, l'$, such that $b_1 \geq b_2 \geq \ldots \geq b_{l'} \geq 0$ and $P(X) = \binom{X+b_1}{b_1} + \binom{X+b_2-1}{b_2} + \ldots + \binom{X+b_l-(l-1)}{b_{l'}}$. Then, $l' = l$ and $b_i = a_i$ for $i = 1, \ldots, l$.*

Proof. Since $a_1 \geq \ldots \geq a_l$, $P(X)$ is a polynomial of degree a_1. For the same reason $\deg P(X) = b_1$. Thus, $a_1 = b_1$ and one can apply induction on the polynomial $P(X) - \binom{X+a_1}{a_1}$. ▲

Let us consider the lexicographic ideal $J = \mathcal{L}(v)$, where $J = I^{lex}_{\geq k}$, for $k \gg 0$. We would like express the exponents of v in terms of the Hilbert polynomial of R/J. Let $P_{R/J}(X) = \binom{X+a_1}{a_1} + \binom{X+a_2-1}{a_2} + \ldots + \binom{X+a_l-(l-1)}{a_l}$, with $a_1 \geq a_2 \geq \ldots \geq a_l \geq 0$, be a presentation of the Hilbert polynomial of R/J in terms of $a_1, \ldots, a_l$. We express v_i in terms of $a_1, \ldots, a_l$ for any i. Let $\sum_{h=1}^d \binom{k(h)}{h}$ be the d^{th} binomial expansion of $H(R/J, d)$. Thus, for every $t \geq 0$,

$$H(R/J, d+t) = \sum_{h=1}^{d} \binom{k(h)+t}{h+t} = \sum_{h=1}^{d} \binom{k(h)+t}{k(h)-h} = \sum_{h=r}^{d} \binom{k(h)+t}{k(h)-h},$$

where r is the least integer such that $k(h) - h \geq 0$. Using (1.2), if $b_h \doteq k(h) - h = n - j(d-h+1) - 1$, then $b_d \geq b_{d-1} \geq \ldots \geq b_r \geq 0$. Thus,

$$\begin{aligned}
P_{R/J}(X) &= \sum_{h=r}^{d} \binom{X+k(h)-d}{k(h)-h} \\
&= \sum_{h=r}^{d} \binom{X+k(h)-h+h-d}{k(h)-h} = \sum_{h=r}^{d} \binom{X+b_h+h-d)}{b_h} \\
&= \binom{X+b_d}{b_d} + \binom{X+b_{d-1}-1}{b_{d-1}} + \ldots + \binom{X+b_r-(d-r)}{b_r},
\end{aligned}$$

and hence $a_i = b_{d-i+1} = n - j(i) - 1$. Therefore, $j(i) = n - a_i - 1$ and

$$v_i = |\{a_j \colon n - a_j - 1 = i\}|.$$

Proof of Theorem 4.9. Assume that I and I^{lex} are generated in one degree, eventually substituting I with a sufficiently high truncation. That this causes no loss of generality has been already observed. By the use of Theorem 3.15 and the Local Duality Theorem, one has

$$\begin{aligned}\mathrm{Hilb}(H^i_{\mathfrak{m}}(R/I), t) &\preceq \mathrm{Hilb}(H^i_{\mathfrak{m}}(R/I^{lex}), t)\\ &= \mathrm{Hilb}(\mathrm{Ext}^{n-i}_R(R/I^{lex}, R(-n)), t^{-1}).\end{aligned}$$

Applying Proposition 4.1, we now get

$$\begin{aligned}\mathrm{Hilb}(H^i_{\mathfrak{m}}(R/I), t) &\preceq \mathrm{Hilb}\left(\frac{R}{(X_1, \ldots, X_{n-i-1}, X_{n-i}^{v_{n-i}})}(\sum_{h \leq n-i} v_h - i - 1), t^{-1}\right)\\ &= t^{\sum_{h\leq n-i} v_h - i - 1}\left(\frac{1 + t^{-1} + \ldots + t^{-v_{n-i}+1}}{(1-t^{-1})^i}\right)\\ &= \frac{\left(t^{\sum_{h\leq n-i} v_h - i}\right)\sum_{j=1}^{v_{n-i}} t^{-j}}{(1-t^{-1})^i},\end{aligned}$$

which proves the desired inequality. ▲

It may be convenient to describe explicitly the upper bounds for the Hilbert function.

Corollary 4.11. *Let I be as in Theorem 4.9 and let*

$$b^{n,I}_{i,j} \doteq \begin{cases} \binom{\sum_{h\leq n-i} v_h - j - 1}{i} & \text{if } j \geq \sum_{h<n-i} v_h - i \\ \sum_{k=1}^{v_{n-i}} \binom{i+k+\sum_{h<n-i} v_h - n - j - 1}{2i-n} & \text{otherwise}\end{cases}$$

Then $H(H^i_{\mathfrak{m}}(R/I), j) \leq b^{n,I}_{i,j}$, for all j and for all $i > 0$.

Proof. Apply the first part of the proof of Theorem 4.9 to reduce the problem to the case I^{lex} generated in degree d and note that

$$H(H^i_{\mathfrak{m}}(R/I), j) \leq H(K[X_{n-i}, \ldots, X_n]/(X_{n-i}^{v_{n-i}}), \sum_{h\leq n-i} v_h - j - i - 1).$$

Let us consider $S \doteq K[X_i, \ldots, X_n]/(X_i^{v_i})$. By the same argumentation as in the proof of Proposition 1.22, it is easy to see that

$$H(S, v_i) = |\mathcal{A}(X_i^{v_i})| = \sum_{k=1}^{v_i} \binom{n-2i+k}{k} = \sum_{k=1}^{v_i} \binom{n-2i+k}{n-2i}.$$

Therefore,

$$H(S,j) = \begin{cases} \binom{n+j-i}{n-i} & \text{if } j \leq v_i - 1 \\ \sum_{k=1}^{v_i} \binom{n-2i+k+j-v_i}{n-2i} & \text{otherwise.} \end{cases}$$

It suffices now to substitute $i' = n - i$ and $j' = \sum_{h \leq n-i} v_h - j - i - 1$ in the previous formula to obtain the conclusion. ▲

5 The generalization for modules and for sheaves

5.1 The module case

As we have already seen in Section 3.1, Macaulay's Theorem can be generalized by way of the Bigatti-Hulett Theorem. Hulett proved subsequently the following generalization of Macaulay's Theorem for graded modules: Given an arbitrary graded free module $F = \oplus_{i=1}^{r} Re_i$ and for any graded submodule M of F, there is a lexicographic submodule L such that F/M and F/L have the same Hilbert function.
In the same paper it was also shown that any of the graded Betti numbers of F/L is greater than or equal to that of F/M. The technique of the proof consists of an induction argument on the rank of the module F and a reduction to the Borel-fixed case, and therefore it is still dependent upon the assumption of characteristic 0 (see [Hu1] for more details).
In [P1] this result was proven with a characteristic-free argument, by the use of a generalization of polarization for monomial submodules of F of the form $N = I_{(1)}e_1 \oplus \ldots \oplus I_{(r)}e_r$. This operation, unlike the ideal case, does not coincide with some iteration of the polarization functor defined in Chapter 2, but this will not be required in order to generalize the inequality of Theorem 3.15 as follows:

Theorem 5.1. *Let $F = \oplus_{i=1}^{r} Re_i$ be a free and graded R-module and let $\mathcal{M}$ be a family of graded submodules of F with a given Hilbert function. If L denotes the lexicographic submodule of the family, then, for any $M \in \mathcal{M}$ and for any i, j,*

$$\dim_K H^i_{\mathfrak{m}}(F/M)_j \leq \dim_K H^i_{\mathfrak{m}}(F/L)_j.$$

Let F be a graded free R-module with homogeneous basis $e_1, \ldots, e_r$ and let $\deg e_i = d_i$, for some $d_i \in \mathbb{N}$. We order the basis elements so that $d_1 \leq \ldots \leq d_r$. A *monomial* of F is an element of the form $X^a e_i = X_1^{a_1} \cdot \ldots \cdot X_n^{a_n} e_i$. One defines the *lexicographic order on the monomials* of F, and denotes it by $\prec_{lex}$ (or simply $<$), as follows:

$$X^a e_i \prec_{lex} X^b e_j \quad \text{iff} \quad i > j \text{ or } (i = j \text{ and } X^a \prec_{lex} X^b).$$

A *lexicographic segment (or lex-segment) of degree d* is a subset $\mathcal{L}$ of F_d such that, if $v \in \mathcal{L}$ and $u > v$, then $u \in \mathcal{L}$. A graded submodule L of F is a *lexicographic submodule* iff any of its homogeneous components is spanned as a K-vector space by a lex-segment. Given any graded submodule M of F, let M^{lex} denote the lexicographic submodule of F with the same Hilbert function as M.
Note that, in general, if N is a submodule of F of the form $N = I_{(1)}e_1 \oplus \ldots \oplus I_{(r)}e_r$,

then $N^{lex} \neq I^{lex}_{(1)}e_1 \oplus \ldots \oplus I^{lex}_{(r)}e_r$, although N^{lex} is of the form $J_{(1)}e_1 \oplus \ldots \oplus J_{(r)}e_r$ for some lexicographic ideals $J_{(1)}, \ldots, J_{(r)}$ of R.
Let f be a linear combination over K of monomials of F. The *initial term* $\mathrm{in}(f)$ of f is defined to be the greatest monomial of f in the lexicographic order. Given any graded submodule M of F, the *initial submodule* $\mathrm{in}(M)$ of M is the submodule of F generated by the initial terms of M. Arguing as in the proof of Theorem 1.17, one shows that M and $\mathrm{in}(M)$ have the same Hilbert function. Note that, for any such M, the corresponding initial submodule is of the form $I_{(1)}e_1 \oplus \ldots \oplus I_{(r)}e_r$, for some monomial ideals $I_{(1)}, \ldots, I_{(r)}$ of R. Obviously, it would be easier to work with monomial submodules, and one may do this without loss of generality by virtue of the following proposition.

Proposition 5.2. *Let F be an arbitrary graded R-module and M be any graded submodule of F. Then, given any term order $\prec$ on the monomials of F,*

$$\dim_K H^i_{\mathfrak{m}}(F/M)_j \leq \dim_K H^i_{\mathfrak{m}}(F/\mathrm{in}_{\prec}(M))_j,$$

for any i, j.

The proof is verbatim that of Theorem 3.3 and descends from the following proposition.

Proposition 5.3 (Proposition 8 in [P2]). *Let $M_1 \subseteq \ldots \subseteq M_s \subseteq F$ be a chain of graded submodules of F. Let $\prec$ be a monomial order on F. Then there is a chain of $R[t]$-submodules $\tilde{M}_1 \subseteq \ldots \subseteq \tilde{M}_s \subseteq F \otimes_R R[t] \doteq \tilde{F}$ such that $\tilde{F}/\tilde{M}_i$ is a flat $k[t]$-module, $\tilde{F}/(\tilde{M}_i + (t-1)\tilde{F}) \simeq F/M_i$ and $\tilde{F}/(\tilde{M}_i + t\tilde{F}) \simeq F/\mathrm{in}_{\prec}(M_i)$ for every i.*

Indeed it is immediately seen that $(\tilde{F}/\tilde{M})_t \simeq (F/M) \otimes_R R[t, t^{-1}]$ and $F/\mathrm{in}(M) \simeq (\tilde{F}/\tilde{M})/t(\tilde{F}/\tilde{M})$, which are the analogues of (3.2) and (3.3) respectively.

Proof of Theorem 5.1. Our strategy is similar to that in Section 3.4. Proposition 5.2 and Proposition 30 in [P1] provide the required generalizations of Theorem 3.3 and Proposition 3.14 respectively. Moreover, one can extend the definition of $\sigma_{\mathcal{L}}$, which was introduced previous to Lemma 3.13, via linearity, to a map between free modules of the same rank with basis elements of the same degree. Therefore, we may assume M to be a monomial submodule of F, let us say $M = I_{(1)}e_1 \oplus \ldots \oplus I_{(r)}e_r$, where $\deg e_l = d_l$, for $l = 1, \ldots, r$.
Note that we have not endowed M with a multi-graded structure and, therefore, we may not use the polarization functor as described in Chapter 2.
However, one can define polarization for such a monomial submodule of F to be a monomial submodule of a free P-module F', which has the same rank as F and whose basis element e'_l has degree $\deg e_l$, for $l = 1, \ldots, r$. (Here P is a polynomial ring over R with "sufficiently many" variables; cf. Definition 11 in [P1]). Let us denote by $J_{(l)}$ the ideal of such a polarized submodule on the component e'_l. By Proposition

2.14, it is easy to see that any of the $J_{(l)}$ can be achieved as a complete polarization in P of the corresponding ideal $I_{(l)}$ on the component e_l. Since the local cohomology functors are additive, $\dim_K H^i_{\mathfrak{m}}(F/M)_j = \dim_K H^i_{\mathfrak{m}}(\oplus_{l=1}^r R/I_{(l)}(-d_l))_j = \sum_{l=1}^r \dim_K H^i_{\mathfrak{m}}(R/I_{(l)})_{j+d_l}$. In view of the last observations, we may now argue as in the proof of Theorem 3.15 in order to achieve the desired conclusion.

▲

5.2 The sheaf case

We present here the counterpart for sheaf cohomology of the results of the last two chapters. We recall first a few definitions and facts which are formulated in a wider generality in [BrSh] and [Ha], while we state them here for the polynomial ring $S \doteq K[X_0, \dots, X_n]$ with graded maximal ideal $\mathfrak{n}$. The notation used here can be found in Section 1.4.

To any S-graded module M one can associate a sheaf $\widetilde{M}$ on the n-dimensional projective space $\operatorname{Proj} S = \mathbb{P}^n_K$. Let $\mathcal{O}$ be the structure sheaf of the scheme $\mathbb{P}^n_K$. Given an arbitrary sheaf $\mathcal{F}$ of $\mathcal{O}$-modules, one defines a graded S-module $\Gamma_*(\mathcal{F})$ associated to $\mathcal{F}$ to be $\Gamma_*(\mathcal{F}) \doteq \oplus_j \Gamma(\mathbb{P}^n_K, \mathcal{F}(j))$, where $\mathcal{F}(j)$ denotes the twisted sheaf $\mathcal{F} \otimes_{\mathcal{O}} \widetilde{S(j)}$.

Recall the definition of the functor $D_{\mathfrak{n}}$ in Section 1.4. It is known that there is a natural equivalence of functors from $\mathcal{C}(S)$ to itself between $D_{\mathfrak{n}}(\ )$ and $\Gamma_*(\tilde{\ })$, and that this equivalence has the *restriction property, i.e. it can be seen as an equivalence of functors from $^*\mathcal{C}(S)$ to itself. In particular $\Gamma_*(\widetilde{M})_j = \Gamma(\mathbb{P}^n_K, \widetilde{M}(j)) \simeq D_{\mathfrak{n}}(M)_j$, for any j and for any graded S-module M. Furthermore, one can check that $H^i_*(\mathbb{P}^n_K, \widetilde{M}) \doteq \oplus_j H^i(\mathbb{P}^n_K, \widetilde{M}(j))$ has a natural graded structure as an S-module. This fact, in view of the functorial isomorphism known as the Serre-Grothendieck Correspondence Theorem

$$H^i(\mathbb{P}^n_K, \widetilde{M}(j)) \cong R^i D_{\mathfrak{n}}(M)_j, \text{ for any } i, j,$$

implies that, for any i,

$$H^i_*(\mathbb{P}^n_K, \widetilde{M}) \simeq R^i D_{\mathfrak{n}}(M) \tag{5.1}$$

as graded S-modules.

Observe that from what was said before the proof of Theorem 4.9, it is clear that, given a family of graded ideals with a fixed Hilbert polynomial, there exists a unique saturated lexicographic ideal with the same Hilbert polynomial, this ideal being described explicitly in Proposition 1.29.

Theorem 5.4. *Let $\mathcal{J}$ be the family of ideals of S with a given Hilbert polynomial P and let L be the saturated lexicographic ideal of the family. Then, for any i and any $I \in \mathcal{J}$, the Hilbert function of $H^i_*(\mathbb{P}^n_K, \widetilde{I})$ admits a sharp upper bound depending only on P, which is reached for $I = L$.*

Proof. Let $I \in \mathcal{J}$ and let us consider the short exact sequence $0 \longrightarrow I \longrightarrow S \longrightarrow S/I \longrightarrow 0$. From the corresponding long exact sequence in cohomology it is easy to see that (a) $H^0_{\mathfrak{n}}(I) = 0$, (b) $H^{i-1}_{\mathfrak{n}}(S/I) \simeq H^i_{\mathfrak{n}}(I)$ for any $i < n+1$ and (c) there exists a short exact sequence $0 \longrightarrow H^n_{\mathfrak{n}}(S/I) \longrightarrow H^{n+1}_{\mathfrak{n}}(I) \longrightarrow H^{n+1}_{\mathfrak{n}}(S) \longrightarrow 0$. Clearly, $\mathfrak{n}$ denotes here the graded maximal ideal of S. Recall that, in view of Theorem 1.34, the sequence

$$0 \longrightarrow H^0_{\mathfrak{n}}(I) \longrightarrow I \longrightarrow D_{\mathfrak{n}}(I) \longrightarrow H^1_{\mathfrak{n}}(I) \longrightarrow 0$$

is exact and the i^{th} right derived functor $R^i D_{\mathfrak{n}}(\ )$ of $D_{\mathfrak{n}}$ is naturally equivalent to $H^{i+1}_{\mathfrak{n}}(\ )$, for any $i > 0$. Since the 0^{th} local cohomology module of I is 0 and $H^1_{\mathfrak{n}}(I) \simeq H^0_{\mathfrak{n}}(S/I)$, one obtains a short exact sequence

$$0 \longrightarrow I \longrightarrow D_{\mathfrak{n}}(I) \longrightarrow H^0_{\mathfrak{n}}(S/I) \simeq \frac{I : \mathfrak{n}^\infty}{I} \longrightarrow 0.$$

Thus, $D_{\mathfrak{n}}(I) \simeq I : \mathfrak{n}^\infty$, and $R^i D_{\mathfrak{n}}(I) \simeq H^{i+1}_{\mathfrak{n}}(I)$ for any $i > 0$. By virtue of Theorem 3.15, $H(I : \mathfrak{n}^\infty, j) \le H(L, j)$ for any $I \in \mathcal{J}$ and for any j. In fact,

$$\begin{aligned} H(I : \mathfrak{n}^\infty, j) &= (H(I : \mathfrak{n}^\infty, j) - H(I, j)) + H(I, j) = H(H^0_{\mathfrak{n}}(S/I), j) + H(I, j) \\ &\le H(H^0_{\mathfrak{n}}(S/I^{lex}), j) + H(I^{lex}, j) = H(I^{lex} : \mathfrak{n}^\infty, j) = H(L, j). \end{aligned}$$

Accordingly,

$$H(D_{\mathfrak{n}}(I), j) \le H(L, j) \ \text{ for any } j. \tag{5.2}$$

Let now $0 < i < n$. Since $H^{i+1}_{\mathfrak{n}}(I) \simeq H^i_{\mathfrak{n}}(S/I)$, one has, again by Theorem 3.15, $H(R^i D_{\mathfrak{n}}(I), j) = H(H^i_{\mathfrak{n}}(S/I), j) \le H(H^i_{\mathfrak{n}}(S/L), j)$. Therefore, applying Corollary 4.11,

$$H(R^i D_{\mathfrak{n}}(I), j) \le b^{n+1,L}_{i,j} \ \text{ for any } j. \tag{5.3}$$

On the other hand, since $H(H^{n+1}_{\mathfrak{n}}(S), j) = H(S(-n-1), -j) = H(S, -n-j-1) = \binom{n+1-n-j-1-1}{-n-j-1} = \binom{-j-1}{-n-j-1}$, if $i = n$ then

$$\begin{aligned} H(R^n D_{\mathfrak{n}}(I), j) &= H(H^n_{\mathfrak{n}}(S/I), j) + H(H^{n+1}_{\mathfrak{n}}(S), j) \\ &\le H(H^n_{\mathfrak{n}}(S/L), j) + \tbinom{-j-1}{-n-j-1} \\ &= b^{n+1,L}_{n,j} + \tbinom{-j-1}{-n-j-1}. \end{aligned} \tag{5.4}$$

Now the conclusion of the theorem follows from (5.2), (5.3) and (5.4) by virtue of (5.1). Note that the bounds depend only on the Hilbert polynomial P (cf. Corollary 4.11). Clearly, the bounds are reached iff $I = L$. ▲

We conclude by observing that an analogous, but weaker result can be shown for families of quasi-coherent sheaves on $\mathbb{P}^n_K$ with a given Euler characteristic, in view of Theorem 5.1, which extends the validity of Theorem 3.15 to the module case.

6 Ext groups of principal strongly stable ideals

In Section 4.1 we have shown that given a lex-ideal J generated by $\mathcal{L}(v)$ one can compute the Ext groups of R/J in terms of the exponents of v. In particular, we have seen that $\operatorname{Ext}^i_R(R/J, R)$ is cyclic for any i and does not vanish if and only if $i \in \operatorname{supp} v$. In the interest of completeness, we investigate here the behaviour of Ext groups of principal strongly stable ideals. It will be proven that the i^{th} Ext group vanishes iff the i^{th} variable does not belong to the support of the monomial which determines the ideal, in analogy to the lex-ideal case.

Let us recall the following definitions. A monomial ideal I of R is *strongly stable* iff, for all $u \in G(I)$, one has $X_j u/X_i \in I$ for all $j < i$, $i \in \operatorname{supp} u$. Let v be a monomial of R_d and I the smallest strongly stable ideal which contains v. We say that v is *the principal generator of* I. It is clear that I is generated in one degree and that the minimal set of generators of I, denoted here by $\mathcal{S}(v)$, is given by those monomials of R which can be obtained by exchanging variables in the support of v with greater ones. Furthermore, $\mathcal{S}(v) \subseteq \mathcal{L}(v)$ in general, while, clearly, $\mathcal{S}(X_1^a) = \mathcal{L}(X_1^a)$ and $\mathcal{S}(X_n^b) = \mathcal{L}(X_n^b)$.
It is interesting to find necessary and sufficient conditions for $\mathcal{S}(v)$ to be a lex-segment.

Lemma 6.1. *Let v be a monomial of R. Then $\mathcal{S}(v) = \mathcal{L}(v)$ iff $v = X_1^a X_n^b$ or $v = X_1^a X_j X_n^b$.*

Proof. We start proving the "if" part. Let $[X_i, \ldots, X_{i+h}]_c$ denote the set of monomials of R_c in the variables $X_i, \ldots, X_{i+h}$ and let $v = X_1^a X_n^b$. The assertion is trivial if $b = 0$. Thus, we may assume $b \geq 1$ and write $\mathcal{L}(v)$ as the (disjoint) union of the sets $X_1^{a+1}[X_1, \ldots, X_n]_{b-1}$ and $X^a[X_2, \ldots, X_n]_b$. Both of these sets are clearly contained in $\mathcal{S}(v)$. Analogously, let $v = X_1^a X_j X_n^b$ and observe that the case $b = 0$ is again trivial. Thus, one may write $\mathcal{L}(v) = X_1^{a+1}[X_2, \ldots, X_n]_b \cup X_1^a \mathcal{L}(X_j X_n^b) = X_1^{a+1}[X_2, \ldots, X_n]_b \cup X_1^a X_j^2 [X_{j+1}, \ldots, X_n]_{b-1} \cup X_1^a X_j [X_{j+1}, \ldots, X_n]_b$, and deduce the conclusion from the previous case.
Conversely, we show that in all other cases there exists a monomial greater than v which does not belong to $\mathcal{S}(v)$. Since we already studied the case $v = X_1^a$, we may present v in the following way: $v = X_1^a X_{i_1}^{v_{i_1}} \cdot \ldots \cdot X_{i_h}^{v_{i_h}}$, where $a \geq 0$, $h \geq 1$, $i_{j+1} > i_j > 1$ for $j = 1, \ldots h-1$ and $v_{i_j} > 0$ for $j = 1, \ldots, h$. Observe that, if $h = 1$ and $v_{i_1} > 1$ or if $h = 2$, $v_{i_1} = 1$ and $i_2 < n$ then $\mathcal{S}(v)$ is strictly contained in $\mathcal{L}(v)$. Thus we may assume that $h > 1$ and, therefore, that $d \doteq \deg v > v_{i_h}$. We set $w \doteq X_1^{d - v_{i_h} - 1} X_{i_h}^{v_{i_h}+1}$. Clearly w does not belong to $\mathcal{S}(v)$, since the exponent of the

last variable is greater than that of v. In order to test when $w > v$, it is sufficient to check the inequality $X_1^{v_{i_1}+\ldots+v_{i_{h-1}}-1} X_{i_h} > X_{i_1}^{v_i} \cdot \ldots \cdot X_{i_{h-1}}^{v_{i_{h-1}}}$, which is always verified if $h > 2$, or if $h = 2$ and $v_{i_1} > 1$. This exhausts also the last possible case and the proof is completed. ▲

The following proposition is the analogue, for principal strongly stable ideals, of the result on the structure of Borel-fixed ideals proven in [P], Chapter VI, Proposition 1.

Proposition 6.2. *Let I be a principal strongly stable ideal generated by $\mathcal{S}(X^\nu)$, for some monomial $X^\nu \in R$. Then*

$$I = \prod_k (X_1, \ldots, X_k)^{\nu_k}.$$

Proof. The assertion is clear if the support of X^ν consists of only one element. Let l be the largest integer in the support of X^ν. It is sufficient to prove that $I = J(X_1, \ldots, X_l)^{\nu_l}$, where J is the principal strongly stable ideal with principal generator $X^\mu \doteq \frac{X^\nu}{X_l^{\nu_l}}$. Observe that $\mathcal{S}(X^\nu) = \mathcal{S}(X^\mu)X_l^{\nu_l} \cup B$, where B is the set of monomials of $\mathcal{S}(X^\nu)$ which are not divided by $X_l^{\nu_l}$. Moreover, if w is any monomial of $\mathcal{S}(X^\mu)$, since $\mathcal{S}(X^\nu)$ is strongly stable, then $wX_l^{\nu_l} \in \mathcal{S}(X^\mu)X_l^{\nu_l} \subset \mathcal{S}(X^\nu)$. Accordingly, with the same notation as in the proof of the previous lemma, $w[X_1, \ldots, X_l]_{\nu_l} \in \mathcal{S}(X^\nu)$ and, therefore, $\mathcal{S}(X^\nu) \supseteq \mathcal{S}(X^\mu)[X_1, \ldots, X_l]_{\nu_l}$. On the other hand, $B \subseteq \mathcal{S}(X^\mu)([X_1, \ldots, X_l]_{\nu_l} \setminus [X_l]_{\nu_l})$. Thus, $\mathcal{S}(X^\nu) = \mathcal{S}(X^\mu)X_l^{\nu_l} \cup B \subseteq \mathcal{S}(X^\mu)[X_1, \ldots, X_l]_{\nu_l}$ and we are done. ▲

Let $A = K[X_1, \ldots, X_n]$ and $B = K[X_1, \ldots, X_m]$ be two polynomial rings over the same field K and let $n \geq m$. Let I be an ideal of B and denote by IA its extension to A. Recall that, since A is flat over B, one has, for any i, $\mathrm{Ext}_A^i(A/IA, A) \simeq \mathrm{Ext}_B^i(B/I, B) \otimes_B A$, which is isomorphic to $\mathrm{Ext}_B^i(B/I, B)[X_{m+1}, \ldots, X_n]$. This is to say that, in this situation, one may study the Ext groups of R/I in the smaller ring without loss of information. In particular, one has $\mathrm{Ext}_A^i(A/IA, A) = 0$ for $i > m$. It is now clear also how we are going to apply the previous proposition. Let v be any monomial of R such that the principal strongly stable ideal generated by $\mathcal{S}(v)$ is not a lex-ideal (this case was already treated before, and is therefore uninteresting). Write v as $X_1^a X_{i_1}^{v_i} \cdot \ldots \cdot X_{i_h}^{v_{i_h}}$, with the notation as in the proof of Lemma 6.1, let $v' \doteq v/X_{i_h}^{v_{i_h}}$ and consider polynomial rings $S \doteq K[X_1, \ldots, X_{i_h}]$ and $T \doteq K[X_1, \ldots, X_{i_{h-1}}]$. Furthermore, let J be the ideal of T with principal generator v' and I the ideal of S with principal generator v. We know from the proof of the proposition that $I = JS \cdot (X_1, \ldots, X_{i_h})^{v_{i_h}}$. Since JS/I has finite length, for any $i > 0$, there is a short exact sequence

$$0 \longrightarrow JS/I \longrightarrow H^0_{(X_1, \ldots, X_{i_h})}(S/I) \longrightarrow H^0_{(X_1, \ldots, X_{i_h})}(S/JS) \longrightarrow 0. \tag{6.1}$$

and, for any $i > 0$, an isomorphism $H^i_{(X_1,\dots,X_{i_h})}(S/I) \simeq H^i_{(X_1,\dots,X_{i_h})}(S/JS)$. Thus, for any $i < i_h$,

$$\begin{aligned}\operatorname{Ext}^i_R(R/IR, R) &\simeq \operatorname{Ext}^i_S(S/I, S) \otimes_S R \\ &\simeq \operatorname{Ext}^i_S(S/JS, S) \otimes_S R \simeq \operatorname{Ext}^i_T(T/J, T) \otimes_T S \otimes_S R \\ &\simeq \operatorname{Ext}^i_T(T/J, T) \otimes_T R.\end{aligned} \tag{6.2}$$

Moreover, the above short exact sequence shows that $H^0_{(X_1,\dots,X_{i_h})}(S/I)$, and therefore $\operatorname{Ext}^{i_h}_S(S/I, S)$ does not vanish. In view of these observations $\operatorname{Ext}^{i_h}_R(R/IR, R) \neq 0$ and $\operatorname{Ext}^i_R(R/IR, R) = 0$ if $i_{h-1} < i < i_h$ or $i > i_h$, while for $i < i_{h-1}$ the problem is reduced to that of a principal strongly stable ideal in a ring with less variables. Using induction the next proposition follows immediately.

Proposition 6.3. *Let I be a principal strongly stable ideal of R with principal generator v. Then,*

$$\operatorname{Ext}^j_R(R/I, R) = 0 \quad \textit{iff} \quad j \notin \operatorname{supp} v.$$

The next proposition completes the results of this section.

Proposition 6.4. *Let I be a principal strongly stable ideal of R with minimal system of generators $\mathcal{S}(v) \neq \mathcal{L}(v)$.*
Let $v = X_1^a X_{i_1}^{v_i} \cdot \ldots \cdot X_{i_h}^{v_{i_h}}$ with $a \geq 0$, $h \geq 1$, $i_{j+1} > i_j > 1$ for $j = 1, \dots, h-1$ and $v_{i_j} > 0$ for $j = 1, \dots, h$.
Set $c \doteq \begin{cases} 1 \text{ if } v_{i_1} > 1 \\ 2 \text{ if } v_{i_1} = 1 \end{cases}$ *and, for any $c \leq j \leq h$, let $R_{i_j} \doteq K[X_1, \dots, X_{i_j}]$.*
Furthermore, let I_{i_c} be the ideal of R_{i_c} generated by $\mathcal{S}(X_1^a X_{i_1}^{v_{i_1}})$ if $v_{i_1} > 1$ or by $\mathcal{S}(X_1^a X_{i_1} X_{i_2}^{v_{i2}})$ if $v_{i_1} = 1$. Finally, for any $j = c+1, \dots, h$, let

$$I_{i_j} \doteq I_{i_c} \prod_{l=c+1}^{j} (X_1, \dots, X_{i_l})^{v_{i_l}} \cdot R_{i_j}.$$

Then,

(i) for any $j \leq i_c$,

$$\operatorname{Ext}^j_R(R/I, R) \simeq \operatorname{Ext}^j_{R_{i_c}}(R_{i_c}/I_{i_c}, R_{i_c}) \otimes_{R_{i_c}} R;$$

(ii) for $j = c+1, \dots, h$,

$$\operatorname{Ext}^{i_j}_R(R/IR, R) \simeq \left(\frac{I_{i_{j-1}} R_{i_j}}{I_{i_j}}\right)^{\vee} \otimes_{R_{i_j}} R.$$

Proof. (i) It follows immediately from what was said before Proposition 6.3. But note that, by virtue of Lemma 6.1, I_{i_c} is a lex-ideal and, thus, its Ext groups can be computed as in Proposition 4.1.
(ii) By Lemma 1.33 (i), $H^0_{(X_1,\ldots,X_{i_j})}(R_{i_j}/I_{i_{j-1}}R_{i_j}) = 0$, since X_{i_j} is a non-zerodivisor of $R_{i_j}/I_{i_{j-1}}R_{i_j}$. Thus, from (6.1), $\mathrm{Ext}^{i_j}_{R_{i_j}}(R_{i_j}/I_{i_j}, R_{i_j}) \simeq (I_{i_{j-1}}R_{i_j}/I_{i_j})^\vee$, and the conclusion follows from (6.2). ▲

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