Abstract

Let R be a polynomial ring in n variables over a field K with graded maximal ideal $\mathfrak{m} = (X_1, \ldots, X_n)$ and consider the lexicographic term order on the monomials of R induced by the assignment $X_1 > X_2 > \ldots > X_n$.

We prove the following Upper Bound Theorem for local cohomology:
Let $\mathcal{I}$ be a family of graded (resp. squarefree) ideals with a given Hilbert function and let L denote the (unique) lexicographic (resp. squarefree lexicographic) ideal of $\mathcal{I}$. Then, for any $I \in \mathcal{I}$,

$$\dim_K H^i_{\mathfrak{m}}(R/I)_j \leq \dim_K H^i_{\mathfrak{m}}(R/L)_j,$$

for any i, j.

We provide a Structure Theorem for the local cohomology modules of lexicographic ideals, by means of which we prove that:
Let $\mathcal{I}$ be the family of ideals of $S \doteq K[X_0, \ldots, X_n]$ with Hilbert polynomial P and let L be the (unique) saturated lexicographic ideal of the family. Then, for any i and for any $I \in \mathcal{I}$ the Hilbert function of

$$H_*(\mathbb{P}^n_K, \tilde{I}) \doteq \oplus_{j \in \mathbb{Z}} H^i(\mathbb{P}^n_K, \tilde{I}(j))$$

admits a sharp upper bound depending only on P, which is reached for $I = L$.

The main results have been generalized for graded submodules of a free module.