NONSTANDARD CHARACTERIZATION OF CONVERGENCE IN LAW FOR D[0,1]-VALUED RANDOM VARIABLES

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Abstract. We prove for random variables with values in the space $D[0,1]$ of cadlag functions — endowed with the supremum metric — that convergence in law is equivalent to nonstandard constructions of internal S-cadlag processes, which represent up to an infinitesimal error the limit process. It is not required as earlier, that the limit process is concentrated on the space $C[0,1]$, and therefore the theory is now applicable to a wider class of limit processes as e.g. to Poisson processes or Gaussian processes. If we consider in $D[0,1]$ the Skorokhod metric — instead of the supremum metric — we obtain a corresponding equivalence to constructions of internal processes with S-separated jumps. We apply these results to functional central limit theorems.

1. Introduction and notations. In [1], [12] and [13] special processes as Brownian motion, Levy's Brownian motion and Brownian bridge were constructed by using nonstandard methods. In [6] it was shown, that all these nonstandard constructions can be derived from a general result by means of suitable invariance principles. An essential assumption in [6] is, that the considered processes a.e. have continuous paths. Therefore a lot of important processes as e.g. the Poisson processes or some Gaussian processes cannot be dealt with these methods. It is the purpose of this paper to close this gap.

In order to apply nonstandard methods we work in this paper with a polysaturated nonstandard model.

Let $D[0,1]$ be the system of all functions $f : [0,1] \to \mathbb{R}$ which are continuous from the right and have left hand limits. Those functions are called cadlag functions. We call a function $g \in {}^*D[0,1]$ an S-cadlag function, if for every standard r and every standard $\varepsilon > 0$ there exists a standard $\delta > 0$, such that

$$\begin{aligned} r < s \le r + \delta \quad &\Rightarrow \quad |g(s) - g(r)| \le \varepsilon, \\ r - \delta \le s, t < r \quad &\Rightarrow \quad |g(s) - g(t)| \le \varepsilon. \end{aligned}$$

If $s \in {}^*\mathbb{R}$ is finite, we denote by ${}^\circ s$ the standard part of s. We call $g \in {}^*D[0,1]$ finite, if $g(s)$ is finite for each $s \in {}^*[0,1]$. If g is finite, we write

1991 *Mathematics Subject Classification.* Primary 28E05, Secondary 60B12.
Key words and phrases. Convergence in law for processes, Nonstandard characterization.

$$°g \text{ for the function } [0,1] \ni t \to °g(t) \in \mathbb{R}.$$

The finite S-cadlag functions $g \in {}^*D[0,1]$ are the nearstandard-points of ${}^*D[0,1]$, where $D[0,1]$ is endowed with the uniform metric δ_∞. The standard part of g — denoted by $st_{\delta_\infty}(g)$ — is given by $°g$ (see Lemma 2.4).

A function $g \in {}^*D[0,1]$ has S-separated jumps, if for every standard $\varepsilon > 0$ there exists a standard $\delta > 0$ and $0 = r_0 < r_1 < \ldots < r_m = 1$ with $r_j - r_{j-1} > \delta$ for $1 \le j \le m$ and $|g(s) - g(r_{j-1})| \le \varepsilon$ for $r_{j-1} \le s < r_j$ and $1 \le j \le m$.

The finite $g \in {}^*D[0,1]$ with S-separated jumps are the nearstandard-points of ${}^*D[0,1]$ with respect to the Skorokhod metric δ_S on $D[0,1]$. The standard part of g — denoted by $st_{\delta_s}(g)$ — is given by ${}^\oplus g$ (see Lemma 2.5), where ${}^\oplus g : [0,1] \to \mathbb{R}$ is defined by ${}^\oplus g(t) = \lim_{r \downarrow t} °g(r)$ for $t \in [0,1]$.

2. The Results. Let (M, δ) be a metric space with Borel-σ-algebra $\mathcal{B}$. If $(\Omega_n, \mathcal{A}_n, P_n)$ are probability spaces and $Y_n : \Omega_n \to M$ are $\mathcal{A}_n, \mathcal{B}$-measurable random variables for $n \in \mathbb{N}$, we denote for $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ by $(\Omega_h, \mathcal{A}_h, P_h, Y_h)$ the nonstandard extension of the sequence $\mathbb{N} \ni n \to (\Omega_n, \mathcal{A}_n, P_h, Y_n)$ at the point h. Then $(\Omega_h, \mathcal{A}_h, P_h)$ is an internal probability space and $Y_h : \Omega_h \to {}^*M$ is an internal function with $Y_h^{-1}(B) \in \mathcal{A}_h$ for $B \in {}^*\mathcal{B}$.

The nonstandard constructions of processes considered in this paper use the concept of a Loeb-measure which was introduced in [7] and gave nonstandard measure theory and probability theory a fresh impetus. We denote the Loeb-measure derived from P_h by P_h^L.

The following theorem is known, if $(\Omega_n, \mathcal{A}_n) \equiv (M, \mathcal{B}), Y_n \equiv id_M$ for all $n \in \mathbb{N}$, where id_M is the identity map on M (see Anderson-Rashid [2] and Loeb [10]). Loeb himself used this special version of Theorem 2.1 and similar results in a series of fundamental papers on axiomatic potential theory and harmonic analysis (see Loeb [8], [9], [11]).

2.1 THEOREM. *Let (M, δ) be a metric space with Borel-σ-algebra $\mathcal{B}$. Let Y_n be M-valued random variables. Let furthermore Q be a Radon p-measure on $\mathcal{B}$. Then Y_n converges in law to Q if and only if for each $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ the following conditions are fulfilled:*

(i) $Y_h \in ns_\delta({}^*M)$ *P_h^L-a.e.*

(ii) $P_h^L\{st_\delta(Y_h) \in B\} = Q(B)$ *for all* $B \in \mathcal{B}$.

PROOF. Let $Q_n := (P_n)_{Y_n}$ be the law of Y_n. Then Y_n converges in law to Q if and only if

$$(1) \qquad Q_h = (P_h)_{Y_h} \underset{w}{\approx} Q \qquad \text{for all} \qquad h \in {}^*\mathbb{N} \setminus \mathbb{N},$$

where $\approx_w$ means infinitesimal-closeness with respect to the weak topology. According to a slight modification of Theorem 33.1 of [5], (1) is equivalent to

$$(2) \qquad Q(O) \le Q_h^L({}^*O) \text{ for all open } O \subset M \text{ and all } h \in {}^*\mathbb{N} \setminus \mathbb{N}.$$

As Q is a Radon measure, it follows from Theorem 32.6 of [5], that (2) is equivalent to

$$(3) \qquad \begin{array}{lll} \text{a)} & Q_h^L(ns_\delta({}^*M)) = 1 & \text{for all} \quad h \in {}^*\mathbb{N} \setminus \mathbb{N}, \\ \text{b)} & Q = (Q_h^L)_{st_\delta} & \text{for all} \quad h \in {}^*\mathbb{N} \setminus \mathbb{N}. \end{array}$$

(3) is according to Lemma 2 of [6] equivalent to

$$(4) \quad \begin{array}{lll} \text{a)} & Y_h(\omega) \in ns_\delta({}^*M) \quad \text{for } P_h^L\text{- a.a. } \omega & \text{for all } h \in {}^*\mathbb{N} \setminus \mathbb{N}, \\ \text{b)} & Q = (P_h^L)_{st_\delta \circ Y_h} & \text{for all } h \in {}^*\mathbb{N} \setminus \mathbb{N}. \end{array}$$

This proves the assertion. □

2.2 COROLLARY. *Let $D[0,1]$ be endowed with the uniform metric δ_∞. Let Y_n be $D[0,1]$-valued random variables. Let furthermore Q be a Radon p-measure on the Borel-σ-algebra $\mathcal{B}_\infty$ of $(D[0,1],\delta_\infty)$. Then Y_n converges in law to Q if and only if for each $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ the following conditions are fulfilled:*

(i) Y_h *is P_h^L-a.e. finite and S-cadlag.*

(ii) $P_h^L\{{}^\circ Y_h \in B\} = Q(B)$ *for all $B \in \mathcal{B}_\infty$.*

PROOF. Apply Theorem 2.1 to $(M,\delta) \equiv (D[0,1],\delta_\infty)$ and use Lemma 2.4. □

As $(D[0,1],\delta_\infty)$ is a complete metric space, a probability measure on $\mathcal{B}_\infty$ is a Radon measure if and only if it is τ-smooth. All probability measures on $\mathcal{B}_\infty$ are τ-smooth if and only if each discrete subset of $D[0,1]$ has nonmeasurable cardinal (see Theorem 2 of Billingsley [3]).

If we assume the continuum hypothesis, then $D[0,1]$ and hence all discrete subsets of $D[0,1]$ have nonmeasurable cardinals. Hence in this case each probability measure on $\mathcal{B}_\infty$ is a Radon measure, whence the assumption on Q in Corollary 2.2 is automatically fulfilled.

If we use instead of δ_∞ the Skorokhod metric δ_S on $D[0,1]$, then $(D[0,1],\delta_S)$ is a complete separable metric space (see § 14 of Billingsley [3]) and hence each measure on the Borel-σ-algebra $\mathcal{B}_S$ is a Radon measure. Therefore we obtain

2.3 COROLLARY. *Let $D[0,1]$ be endowed with the Skorokhod metric δ_S. Let Y_n be $D[0,1]$-valued random variables. Let furthermore Q be a p-measure on the Borel-σ-algebra $\mathcal{B}_S$ of $(D[0,1],\delta_S)$. Then Y_n converges in law to Q if and only if for each $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ the following conditions are fulfilled:*

(i) Y_h *is P_h^L-a.e. finite with S-separated jumps.*

(ii) $P_h^L\{{}^\oplus Y_h \in B\} = Q(B)$ *for all $B \in \mathcal{B}_S$.*

PROOF. Apply Theorem 2.1 to $(M,\delta) \equiv (D[0,1],\delta_S)$ and use Lemma 2.5. □

2.4 LEMMA. *Let $D := D[0,1]$ be endowed with the uniform metric δ_∞. Then $g \in ns_{\delta_\infty}({}^*D)$ if and only if g is finite and S-cadlag. If $g \in ns_{\delta_\infty}({}^*D)$, then $st_{\delta_\infty}(g) = {}^\circ g$.*

PROOF. Let $g \in ns_{\delta_\infty}({}^*D)$, then there exists $f \in D$ with $g \underset{\delta_\infty}{\approx} f$. As $g(s) \approx {}^*f(s)$ for all $s \in {}^*[0,1]$, $g(s)$ is finite for all $s \in {}^*[0,1]$ and $st_{\delta_\infty}(g) = {}^\circ g$. As ${}^*f$ is S-cadlag and $g(s) \approx {}^*f(s)$ for all $s \in {}^*[0,1]$, g is S-cadlag, too.

Let now conversely $g \in {}^*D$ be S-cadlag such that g is finite. Put $f(t) := {}^\circ g(t)$ for $t \in [0,1]$. Then $f \in D$ and it suffices to show

$$(1) \qquad s \in {}^*[0,1] \wedge {}^\circ s < s \Rightarrow g(s) \approx {}^*f(s).$$

$$(2) \qquad s \in {}^*[0,1] \wedge s < {}^\circ s \Rightarrow g(s) \approx {}^*f(s).$$

As g and ${}^*f$ are S-cadlag, (1) follows from $g(s) \approx g({}^\circ s) \approx f({}^\circ s) \approx {}^*f(s)$.

To prove (2), let $s \in {}^*[0,1]$ and $\varepsilon \in \mathbb{R}_+$ be given. As g and ${}^*f$ are S-cadlag, there exists $\delta \in \mathbb{R}_+$ with $|{}^*f(s_1) - {}^*f(s_2)| \leq \varepsilon$ and $|g(s_1) - g(s_2)| \leq \varepsilon$ for all $s_1, s_2 \in {}^*[0,1]$ with ${}^\circ s - \delta \leq s_1, s_2 < {}^\circ s$. As $f({}^\circ s - \delta) \approx g({}^\circ s - \delta)$ we obtain $|g(s_2) - {}^*f(s_2)| \leq 3\varepsilon$ for all $s_2 \in {}^*[0,1]$ with ${}^\circ s - \delta \leq s_2 < {}^\circ s$. This implies (2). □

2.5 LEMMA. *Let $D := D[0,1]$ be endowed with the Skorokhod metric δ_S. Then $g \in ns_{\delta_s}({}^*D)$ if and only if g is finite and has S-separated jumps. If $g \in ns_{\delta_s}({}^*D)$, then $st_{\delta_s}(g) = {}^\oplus g$.*

PROOF. See 5.37 and 5.36 of Stroyan-Bayod [14]. □

Now we give applications of Corollary 2.2 and 2.3 to special stochastic processes.

2.6 APPLICATION OF 2.2. *For each $n \in \mathbb{N}$ let $\xi_{n,1}, \ldots, \xi_{n,n}$ be independent random variables with $P_n\{\xi_{n,k} = 1\} = p_n$ and $P_n\{\xi_{n,k} = 0\} = 1 - p_n$ for $k = 1, \ldots, n$. Define Y_n by $(Y_n(\omega))(t) := \sum_{k \leq nt} \xi_{n,k}(\omega)$. Assume that $np_n \to \lambda$. Then for each $h \in {}^*\mathbb{N} \setminus \mathbb{N}$*

(i) *Y_h is P_h^L-a.e. finite and S-cadlag.*

(ii) *${}^\circ Y_h$ is a Poisson-process with parameter λ, where*
$${}^\circ(Y_h(\omega))(t) = {}^\circ(\textstyle\sum_{k \leq ht} \xi_{h,k}(\omega)).$$

PROOF. According to Billingsley [3, p. 143, Problem 3] Y_n converges in law to a measure Q which is the law of a Poisson process with parameter λ. Hence the assertion follows from Corollary 2.2. □

2.7 APPLICATION OF 2.3. *Let $\xi_n : \Omega \to [0,1], n \in \mathbb{N}$, be independent random variables with common distribution function F. Let*
$$(Y_n(\omega))(t) := \sqrt{n}(F_n(t,\omega) - F(t)),$$
where F_n is the empirical distribution function of $\xi_1, \ldots, \xi_n$. Then for each $h \in {}^*\mathbb{N} \setminus \mathbb{N}$

(i) *Y_h is P^L-a.e. finite with S-separated jumps.*

(ii) *${}^\oplus Y_h$ is a Gaussian process in D with zero means and $\operatorname{cov}({}^\oplus Y_h(s), {}^\oplus Y_h(t)) = F(s)(1 - F(t))$ for $0 \leq s \leq t \leq 1$.*

PROOF. According to Billingsley [3, p. 141, Theorem 16.4] Y_n converges in law to a measure Q on the Borel-σ-algebra $\mathcal{B}_S$, which is the law of a Gaussian process in $D[0,1]$ with zero means and covariances $F(s)(1 - F(t)), s \leq t$. Hence the assertion follows from Corollary 2.3. □

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