A Survey on L_2-Approximation Order From Shift-invariant Spaces

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Abstract. This paper aims at providing a self-contained introduction to notions and results connected with the L_2-approximation power of finitely generated shift-invariant spaces (FSI spaces) $S_\Phi \subset L_2(R^d)$. Here, approximation order refers to a scaling parameter and to the usual scaling of the L_2-projector onto S_Φ, where $\Phi = \{\phi_1, \ldots, \phi_n\} \subset L_2(R^d)$ is a given set of functions, the so-called generators of S_Φ. Special attention is given to the PSI case where the shift-invariant space is generated from the multi-integer translates of just one generator; this case is interesting enough due to its possible applications in wavelet methods. The general FSI case is considered subject to a stability condition being satisfied, and the recent results on so-called superfunctions are developed. For the case of a refinable system of generators the sum rules for the matrix mask and the zero condition for the mask symbol, as well as invariance properties of the associated subdivision and transfer operator are discussed. References to the literature and further notes are extensively given at the end of each section. In addition to this, the list of references is enlarged in order to give a rather comprehensive overview on existing literature in the field.

Key Words and Phrases: shift-invariant spaces, approximation order, L_2-projectors, multiple refinable functions, subdivision.

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§1. Introduction

In this paper we give an overview on recent results concerning the L_2-approximation power of so-called shift-invariant subspaces of $L_2(\mathbb{R}^d)$. We are going to consider only specific shift-invariant spaces, namely principal shift-invariant (PSI) spaces S_ϕ, generated by the multi-integer translates of just one single function $\phi \in L_2(\mathbb{R}^d)$, and more generally finitely generated shift-invariant (FSI) spaces S_Φ with a finite set $\Phi = \{\phi_1, \ldots, \phi_n\} \subset L_2(\mathbb{R}^d)$ of generators. This leads to the following notion of approximation order: Let $P_\Phi : L_2(\mathbb{R}^d) \to S_\Phi$ be the L_2-projector onto the shift invariant space, and let $P_{\Phi,h}$ be its scaled version, *i.e.*, $P_{\Phi,h}(f) := \{P_\Phi(f_h)\}(\frac{\cdot}{h})$ with $f_h(x) := f(h \cdot x)$, for the scale parameter $0 < h \in \mathbb{R}$. Then S_Φ is said to have L_2-approximation order $0 < m \in \mathbb{R}$ for the subspace $W \subset L_2(\mathbb{R}^d)$ if

$$\|f - P_{\Phi,h}(f)\|_2 = \mathcal{O}(h^m) \quad \text{as } h \to 0 ,$$

for any $f \in W$. When W is the Sobolev space $W_2^m(\mathbb{R}^d)$, this definition can be replaced by the following condition on the unscaled operator

$$\|f - P_\Phi(f)\|_2 \leq \text{const.}\ |f|_{m,2} \quad \text{for any } f \in W_2^m(\mathbb{R}^d) ,$$

where $|f|_{m,2}^2 := (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^{2m} \, |f^\wedge(\xi)|^2 \, d\xi$ denotes the usual Sobolev seminorm of order m, referring to the Fourier transform $f^\wedge$ of f and to the Euclidian norm $|\xi|^2 = \xi_1^2 + \cdots + \xi_d^2$ in the frequency domain. It is this notion of approximation order which we will be going to refer to.

One essential concern of this paper is to give a self-contained summary of the subject, where the PSI-case and the FSI-case are developed independently. This concept aims at simplifying the approach to the paper for readers being only interested in the PSI case. At this time, this circle of readers is certainly the bigger part in the community of people interested in the approximation theoretically aspects of wavelets and other related multiresolution methods. Further, we are able to point out the close connections and the basic ideas of the generalization to the FSI case. The main results will be worked out in a form which is perhaps not always most general, but hopefully highly readable, and they can be understood without going back to the original literature. The interested reader, however, will find remarks and extensions at the end of each chapter, including explicit references to the original literature. The bibliography of this survey paper is intended to be even more comprehensive; however, we are aware of the fact of being selective, and we feel unable to compile a list of all papers who have contributed to this field. While we are going to deal with L_2-projectors as approximation methods in this paper only, the reference list gives also information on a bunch of papers treating other linear approximation processes which are quasi-optimal, *i.e.*, having the same approximation power as the L_2-projectors.

The paper is organized as follows. It consists of two subsequent chapters where the first one deals with general shift-invariant spaces, and the second one gives more details for the case where the system of generators is refinable. Both chapters are of course influenced by some basic material from the list of references: Chapter 2 could not have been written this way without recourse to the fundamental work of de Boor, DeVore and Ron, and Chapter 3 uses Jia's important contributions frequently. We think, though, that here and there we could add our own point of looking at the field, and we would like to stress that at least the results for the FSI case in Chapter 3 are new.

Concerning the details of the paper a first orientation is provided by the headings of the subsequent sections. We refrain from repeating this. Notions and notations will be given during the text at adequate places. We should only point to the fact that we always refer to usual multi-index notation: A multi-index is a d-tuple $\mu = (\mu_1, \ldots, \mu_d)$ with its components being nonnegative integers. Further, $|\mu| := \mu_1 + \cdots + \mu_d$, and $\mu! := \mu_1! \cdots \mu_d!$. For two multi-indices $\mu = (\mu_1, \ldots, \mu_d)$ and $\nu = (\nu_1, \ldots, \nu_d)$, we write $\nu \le \mu$ in case $\nu_j \le \mu_j$ for $j = 1, \ldots, d$. In addition, $\binom{\mu}{\nu} := \frac{\mu!}{\nu!(\mu - v)!}$ for $\nu \le \mu$, and D^μ is short for the differential operator $\frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \cdots \partial x_d^{\mu_d}}$.

§2. L_2-projectors onto FSI spaces

2.1. Shift-invariant spaces.

A *shift-invariant space* S is a subspace of $L_2(\mathbb{R}^d)$ which is invariant under multi-integer shifts,

$$s \in S \quad \Longrightarrow \quad s(\cdot - \alpha) \in S \text{ for all } \alpha \in \mathbb{Z}^d .$$

We shall deal with specific shift-invariant spaces spanned by the multi-integer translates of given 'basis functions', or 'generators'. A *principal shift-invariant space* (or *PSI space*) S_ϕ is determined through a *single* generator $\phi \in L_2(\mathbb{R}^d)$ as the closure (with respect to the topology of $L_2(\mathbb{R}^d)$) of

$$S_\phi^0 := \operatorname{span} < \phi(\cdot - \alpha); \alpha \in \mathbb{Z}^d > .$$

Similarly, given a set of *finitely many* generators $\Phi = \{\phi_1, \ldots, \phi_n\} \subset L_2(\mathbb{R}^d)$ its associated *finitely generated shift-invariant space* (or *FSI space*) S_Φ is the closure of

$$S_\Phi^0 := \sum_{i=1}^n S_{\phi_i}^0 .$$

Some preliminary notations follow. Given $f, g \in L_2(\mathbb{R}^d)$, their scalar product can be expressed as

$$(f|g) := \int f(x)\ \overline{g(x)}\ dx = (2\pi)^{-d}\ (f^\wedge | g^\wedge) = (2\pi)^{-d}\ \int_C [f^\wedge | g^\wedge](\xi)\ d\xi$$

with
$$[f^\wedge|g^\wedge] := \sum_{\alpha\in\mathbb{Z}^d} f^\wedge(\cdot+2\pi\alpha)\overline{g^\wedge(\cdot+2\pi\alpha)} \tag{2.1.1}$$
the 2π-periodization of $f^\wedge\overline{g^\wedge}$, now often called the *bracket product* of $f^\wedge$ and $g^\wedge$. Here, we have used Parseval's identity and the Fourier transform with the following normalization,
$$f^\wedge(\xi) = \int f(x)\ e^{-ix\cdot\xi}\ dx\ ,$$
and $x\cdot\xi$ denotes the scalar product of the two vectors in $\mathbb{R}^d$; also, unindexed integrals are taken with respect to the full space $\mathbb{R}^d$, and C stands for the d-dimensional fundamental cube,
$$C := [-\pi, +\pi]^d\ .$$

It is not hard to see that $[f^\wedge|g^\wedge] \in L_1(C)$, hence its Fourier coefficients can be expressed as
$$\begin{aligned}(2\pi)^{-d}\ \int_C [f^\wedge|g^\wedge](\xi)\ e^{+i\alpha\cdot\xi}\ d\xi &= (2\pi)^{-d}\ (f^\wedge|e^{-i\alpha\cdot\xi}g^\wedge)\\ &= (f|g(\cdot-\alpha)) = \int f(x)\ \overline{g(-(\alpha-x))}\ dx\\ &=: (f*g^*)(\alpha)\ ,\quad \alpha\in\mathbb{Z}^d\ ,\end{aligned}$$
with $g^*(x) := \overline{g(-x)}$ denoting the involution of g, and $f*g^*$ the convolution of f and g^* (which is a continuous function). This shows that the bracket product has the Fourier series
$$[f^\wedge|g^\wedge] \quad\sim\quad \sum_{\alpha\in\mathbb{Z}^d} (f*g^*)(\alpha)\ e^{-i\alpha\cdot\xi}\ , \tag{2.1.2}$$
and at the same time verifies the useful fact that *f is orthogonal to the PSI space S_g if and only if $[f^\wedge|g^\wedge]=0$ as an identity in $L_1(C)$*. Specializing to the case where $f=g$, $F := f*f^*$ is called the auto-correlation of f, and
$$[f^\wedge|f^\wedge] = \sum_{\alpha\in\mathbb{Z}^d} |f^\wedge(\cdot+2\pi\alpha)|^2 \quad\sim\quad \sum_{\alpha\in\mathbb{Z}^d} F(\alpha)\ e^{-i\alpha\cdot\xi}\ . \tag{2.1.3}$$

The following useful characterization of FSI spaces in the Fourier transform domain holds true:

Lemma 2.1.4. *For a finite set of generators $\Phi = \{\phi_1,\ldots,\phi_n\} \subset L_2(\mathbb{R}^d)$ and $f\in L_2(\mathbb{R}^d)$ the following are equivalent:*
(i) $f\in S_\Phi$.
(ii) There are 2π-periodic functions $\tau_1,\ldots,\tau_n$ such that $f^\wedge = \sum_{i=1}^n \tau_i\ \phi_i{}^\wedge$.

With this characterization in hand the following is easy to see.

Lemma 2.1.5. *A function $f\in L_2(\mathbb{R}^d)$ is orthogonal to S_Φ, with $\Phi = \{\phi_1,\ldots,\phi_n\}\subset L_2(\mathbb{R}^d)$, if and only if $[f^\wedge|\phi_i{}^\wedge]=0$, $i=1,\ldots,n$, a.e. on C.*

2.2. L_2-projectors onto S_Φ.

The L_2-projector onto S_Φ is the linear (continuous) operator $P_\Phi : L_2(\mathbb{R}^d) \to S_\Phi$ characterized by

$$f - P_\Phi(f) \quad \perp \quad S_\Phi \quad \text{for any } f \in L_2(\mathbb{R}^d). \tag{2.2.1}$$

We give the representation of this projector according to Lemma 2.1.4, with increasing generality.

2.2.2. The PSI case. We first assume that for the single generator ϕ the translates are *orthonormal*, *i.e.*, $(\phi|\phi(\cdot - \alpha)) = (\phi * \phi^*)(\alpha) = \delta_{0,\alpha}$ and, equivalently,

$$[\phi^\wedge|\phi^\wedge] = 1 \quad \text{in } L_1(C).$$

In this case it is clear that

$$P_\phi(f) = \sum_{\alpha\in\mathbb{Z}^d} (f|\phi(\cdot - \alpha))\ \phi(\cdot - \alpha),$$

whence $P_\phi(f)^\wedge = \tau_f\ \phi^\wedge$ with $\tau_f = \sum_{\alpha\in\mathbb{Z}^d}(f|\phi(\cdot - \alpha))\ e^{-i\alpha\cdot\xi} \in L_2(C)$ the Fourier series of $[f^\wedge|\phi^\wedge] \in L_1(C)$, and therefore

$$P_\phi(f)^\wedge = \tau_f\ \phi^\wedge \quad \text{with} \quad \tau_f = [f^\wedge|\phi^\wedge]. \tag{2.2.3}$$

Next, we deal with the case that the translates of ϕ form a *Riesz basis* for S_ϕ, *i.e.*, for some constants $0 < A \le B < \infty$ we have

$$A \sum_{\alpha\in\mathbb{Z}^d} |c_\alpha|^2 \quad \le \quad \|\sum_{\alpha\in\mathbb{Z}^d} c_\alpha \phi(\cdot - \alpha)\|^2 \quad \le \quad B \sum_{\alpha\in\mathbb{Z}^d} |c_\alpha|^2$$

for any $\ell_2(\mathbb{Z}^d)$-sequence $c = (c_\alpha)$, where $\|f\| := (f|f)^{1/2}$ denotes the usual $L_2(\mathbb{R}^d)$-norm. Letting $\tau(\xi) = \sum_{\alpha\in\mathbb{Z}^d} c_\alpha\, e^{-i\alpha\cdot\xi}$, it follows that

$$\|\sum_{\alpha\in\mathbb{Z}^d} c_\alpha\, \phi(\cdot - \alpha)\|^2 = (2\pi)^{-d}\, (\tau\, \phi^\wedge \,|\, \tau\, \phi^\wedge) = (2\pi)^{-d} \int_C |\tau(\xi)|^2\, [\phi^\wedge|\phi^\wedge](\xi)\, d\xi$$

and $\sum_{\alpha\in\mathbb{Z}^d} |c_\alpha|^2 = (2\pi)^{-d} \int_C |\tau(\xi)|^2\, d\xi$. Hence the Riesz basis property is equivalent with

$$A \quad \le \quad [\phi^\wedge|\phi^\wedge](\xi) \quad \le \quad B \quad \text{a.e. in } C.$$

By performing the *orthogonalization process*

$$(\phi^\perp)^\wedge := \phi^\wedge / \sqrt{[\phi^\wedge|\phi^\wedge]} \tag{2.2.4}$$

we see that $\phi^\perp \in S_\phi$, hence $S_{\phi^\perp} \subset S_\phi$. From the preceding orthonormal case, $f^\wedge = P_{\phi^\perp}(f)^\wedge$ for $f \in S_\phi$, whence $S_{\phi^\perp} = S_\phi$, and

$$P_\phi(f)^\wedge = \tau_f \ \phi^\wedge \quad \text{with} \quad \tau_f = \frac{[f^\wedge|\phi^\wedge]}{[\phi^\wedge|\phi^\wedge]}. \tag{2.2.5}$$

In the *general PSI case* we use the same formulas with the modification that

$$(\phi^\perp)^\wedge(\xi) := 0 =: \tau_f(\xi) \quad \text{if} \quad [\phi^\wedge|\phi^\wedge](\xi) = 0. \tag{2.2.6}$$

Again, $\phi^\perp \in S_\phi$ and $S_{\phi^\perp} \subset S_\phi$. From

$$[f^\wedge - P_\phi(f)^\wedge|\phi^\wedge] = 0 \quad \text{a.e. in } C \ , \quad \text{for any } f \in L_2(\mathbb{R}^d) \ ,$$

we conclude that P_ϕ is indeed the orthogonal projector onto S_ϕ.

In order to give an error formula, we mention that

$$\|f - P_\phi(f)\|^2 = \|f\|^2 - \|P_\phi(f)\|^2$$

by orthogonality. Therefore, using Parseval's identity we find

$$\|f - P_\phi(f)\|^2 = (2\pi)^{-d} \int \left\{ |f^\wedge(\xi)|^2 - \left| \frac{[f^\wedge|\phi^\wedge](\xi)}{[\phi^\wedge|\phi^\wedge](\xi)} \right|^2 |\phi^\wedge(\xi)|^2 \right\} d\xi \ ,$$

and from this we get

Theorem 2.2.7. *Let P_ϕ denote the orthogonal projector onto the PSI space S_ϕ. Then, for $f \in L_2(\mathbb{R}^d)$ such that supp$f^\wedge \subset C$, we have the error formula*

$$\|f - P_\phi(f)\|^2 = (2\pi)^{-d} \int_C |f^\wedge(\xi)|^2 \left\{ 1 - \frac{|\phi^\wedge(\xi)|^2}{[\phi^\wedge|\phi^\wedge](\xi)} \right\} d\xi.$$

Remark 2.2.8. As noticed before, in this theorem we have to put

$$\Lambda_\phi{}^\wedge(\xi) := \frac{|\phi^\wedge(\xi)|^2}{[\phi^\wedge|\phi^\wedge](\xi)} := 0 \quad \text{if} \quad [\phi^\wedge|\phi^\wedge](\xi) = 0.$$

This shows that $0 \le \Lambda_\phi{}^\wedge(\xi) \le 1$, and

$$\sum_{\alpha \in \mathbb{Z}^d} \Lambda_\phi{}^\wedge(\xi + 2\pi\alpha) = 1 - \chi_{Z_\phi}(\xi),$$

with Z_ϕ the set of all $\xi \in \mathbb{R}^d$ with $[\phi^\wedge|\phi^\wedge](\xi) = 0$, *i.e.*,

$$Z_\phi = \{\xi \in \mathbb{R}^d; \ \phi^\wedge(\xi + 2\pi\alpha) = 0 \text{ for all } \alpha \in \mathbb{Z}^d\}.$$

In case this Z_ϕ has measure 0, we see that Λ_ϕ has the fundamental interpolation property

$$\Lambda_\phi(\alpha) = \delta_{0,\alpha} \quad \text{for} \quad \alpha \in \mathbb{Z}^d.$$

2.2.9. The FSI case. Here, we deal with the stable case only where the translates of the system $\Phi = \{\phi_1, \ldots, \phi_n\}$ form a *Riesz basis* of S_Φ, *i.e.*,

$$A \sum_{i=1}^{n} \sum_{\alpha \in \mathbb{Z}^d} |c_\alpha^{(i)}|^2 \quad \leq \quad \| \sum_{i=1}^{n} \sum_{\alpha \in \mathbb{Z}^d} c_\alpha^{(i)} \phi_i(\cdot - \alpha) \|^2 \quad \leq \quad B \sum_{i=1}^{n} \sum_{\alpha \in \mathbb{Z}^d} |c_\alpha^{(i)}|^2 \tag{2.2.10}$$

for some constants $0 < A \leq B < \infty$. It is opportune to introduce the *Gramian* $\mathbf{G}_\Phi$ *for the system* Φ as

$$\mathbf{G}_\Phi^T = \begin{pmatrix} [\phi_1{}^\wedge, \phi_1{}^\wedge] & [\phi_1{}^\wedge, \phi_2{}^\wedge] & \cdots & [\phi_1{}^\wedge, \phi_n{}^\wedge] \\ [\phi_2{}^\wedge, \phi_1{}^\wedge] & [\phi_2{}^\wedge, \phi_2{}^\wedge] & \cdots & [\phi_2{}^\wedge, \phi_n{}^\wedge] \\ \vdots & \vdots & \ddots & \vdots \\ [\phi_n{}^\wedge, \phi_1{}^\wedge] & [\phi_n{}^\wedge, \phi_2{}^\wedge] & \cdots & [\phi_n{}^\wedge, \phi_n{}^\wedge] \end{pmatrix} \tag{2.2.11}$$

$$= \sum_{\alpha \in \mathbb{Z}^d} \mathbf{\Phi}^\wedge(\cdot + 2\pi\alpha) \big(\mathbf{\Phi}^\wedge(\cdot + 2\pi\alpha) \big)^H ;$$

here, we have used the vector notation

$$\mathbf{\Phi}^\wedge := (\phi_1{}^\wedge \ \phi_2{}^\wedge \ \cdots \ \phi_n{}^\wedge)^T,$$

and the superscripts T and H denote the transpose and the conjugate-transpose, respectively. This Gramian is a 2π-periodic (hermitian) matrix function, and it can be shown that the Riesz basis condition is equivalent to requiring that the spectrum $\{\sigma_1(\xi), \ldots, \sigma_n(\xi)\}$ of $\mathbf{G}_\Phi^T(\xi)$ satisfies

$$A \leq \min_{i=1,\ldots,n} \sigma_i(\xi) \leq \max_{i=1,\ldots,n} \sigma_i(\xi) \leq B \quad \text{a.e. on } C.$$

Without loss of generality we may therefore assume that

$$B \geq \sigma_1(\xi) \geq \sigma_2(\xi) \geq \cdots \geq \sigma_n(\xi) \geq A \,, \tag{2.2.12}$$

and that the Gramian has the spectral decomposition

$$\mathbf{G}_\Phi(\xi) = \sum_{i=1}^{n} \sigma_i(\xi) \ \mathbf{u}_i(\xi) \ \mathbf{u}_i^H(\xi) \tag{2.2.13}$$

with $\mathbf{U} := (\mathbf{u}_1 \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_n)$ a unitary, 2π-periodic matrix function.

Theorem 2.2.14. *The orthogonal projector* $P_\Phi : L_2(\mathbb{R}^d) \to S_\Phi$ *takes the form*

$$P_\Phi(f)^\wedge = (\mathbf{\Phi}^\wedge)^T [f^\wedge | \tilde{\mathbf{\Phi}}^\wedge]$$

with

$$\tilde{\mathbf{\Phi}}^\wedge = \begin{pmatrix} \tilde{\phi}_1{}^\wedge \\ \vdots \\ \tilde{\phi}_n{}^\wedge \end{pmatrix} := (\mathbf{G}_\Phi^T)^{-1} \mathbf{\Phi}^\wedge \quad \text{and} \quad [f^\wedge | \tilde{\mathbf{\Phi}}^\wedge] := \begin{pmatrix} [f^\wedge | \tilde{\phi}_1{}^\wedge] \\ \vdots \\ [f^\wedge | \tilde{\phi}_n{}^\wedge] \end{pmatrix} .$$

Proof: Assuming for the moment that $P_\Phi(f)^\wedge \in L_2(\mathbb{R}^d)$ it is easy to see that $f - P_\Phi(f)$ is orthogonal to S_Φ, for any $f \in L_2(\mathbb{R}^d)$, viz.

$$\begin{aligned} [P_\Phi(f)^\wedge | \mathbf{\Phi}^\wedge] &= \sum_{\alpha \in \mathbb{Z}^d} \overline{\mathbf{\Phi}^\wedge(\cdot + 2\pi\alpha)} \; (\mathbf{\Phi}^\wedge(\cdot + 2\pi\alpha))^T \; [f^\wedge | \tilde{\mathbf{\Phi}}^\wedge] \\ &= \mathbf{G}_\Phi^T \; [f^\wedge | \tilde{\mathbf{\Phi}}^\wedge] = [f^\wedge | \mathbf{G}_\Phi^T \; \tilde{\mathbf{\Phi}}^\wedge] = [f^\wedge | \mathbf{\Phi}^\wedge] \; , \end{aligned}$$

hence $[f^\wedge - P_\Phi(f)^\wedge | \mathbf{\Phi}^\wedge]$ is the zero vector a.e. on C, and the orthogonality follows from Lemma 2.1.5.

In order to prove the made assumption, we use the equivalent representation

$$P_\Phi(f)^\wedge = (\mathbf{\Phi}^{\perp\wedge})^T [f^\wedge | \mathbf{\Phi}^{\perp\wedge}] \quad \text{with} \quad \mathbf{\Phi}^{\perp\wedge} = \begin{pmatrix} \phi_1^{\perp\wedge} \\ \vdots \\ \phi_n^{\perp\wedge} \end{pmatrix} := (\mathbf{G}_\Phi^T)^{-1/2} \mathbf{\Phi}^\wedge \; . \tag{2.2.15}$$

This is the adequate extension of the orthogonalization process (2.2.4). Using the spectral decomposition of the Gramian it is not too hard to see that $\phi_i^{\perp\wedge} \in L_2(\mathbb{R}^d)$, hence $\phi_i^{\perp\wedge} \in S_\Phi$ by Lemma 2.1.4, $i = 1, \ldots, n$. But the Gramian of $\mathbf{\Phi}^{\perp\wedge}$ is the identity matrix, whence the sum $\sum_{i=1}^n S_{\phi_i^\perp}$ is an orthogonal sum of PSI subspaces of S_Φ. Since $P_\Phi = \sum_{i=1}^n P_{\phi_i^\perp}$, the proof is finished. ■

We remark that the proof yields the following corollary that *by the orthogonalization process (2.2.15) we have the orthogonal decomposition*

$$S_\Phi = \sum_{i=1}^n S_{\phi_i^\perp}$$

into PSI spaces. As an easy consequence of Theorem 2.2.14 we get the analogue of Theorem 2.2.7 as follows:

Theorem 2.2.16. *Let* P_Φ *denote the orthogonal projector onto the stable FSI space* S_Φ. *Then, for* $f \in L_2(\mathbb{R}^d)$ *such that supp*$f^\wedge \subset C$, *we have the error formula*

$$\|f - P_\Phi(f)\|^2 = (2\pi)^{-d} \int_C |f^\wedge(\xi)|^2 \left\{ 1 - (\mathbf{\Phi}^\wedge)^T(\xi) \; \mathbf{G}_\Phi^{-1}(\xi) \; \overline{\mathbf{\Phi}^\wedge(\xi)} \right\} d\xi.$$

Remark 2.2.17. We may add the useful information that the function

$$A(\xi) := (\mathbf{\Phi}^\wedge)^T(\xi)\ \mathbf{G}_\Phi^{-1}(\xi)\ \overline{\mathbf{\Phi}^\wedge(\xi)}$$

satisfies

$$0 \le \frac{1}{B}(\mathbf{\Phi}^\wedge)^T(\xi)\ \overline{\mathbf{\Phi}^\wedge(\xi)} \le\ A(\xi) = \sum_{i=1}^n |(\phi_i^\perp)^\wedge(\xi)|^2 \le 1\ ,$$

where B is the Riesz constant in (2.1.12). Further

$$\sum_{\alpha\in\mathbb{Z}^d} A(\xi+2\pi\alpha) = \sum_{i=1}^n [(\phi_i^\perp)^\wedge|(\phi_i^\perp)^\wedge] = n\ ,$$

and we arrive at the following *fundamental interpolation property* that

$$\Lambda_\Phi{}^\wedge(\xi) := \frac{1}{n}(\mathbf{\Phi}^\wedge)^T(\xi)\ \mathbf{G}_\Phi^{-1}(\xi)\ \overline{\mathbf{\Phi}^\wedge(\xi)} \quad \Longrightarrow \quad \Lambda_\Phi(\alpha) = \delta_{0,\alpha}\ , \quad \alpha\in\mathbb{Z}^d.$$

2.3. Approximation power.

We first consider the PSI space generated by the famous sinc-function

$$\phi(x) := \prod_{i=1}^d \frac{\sin \pi x_i}{\pi x_i} \quad \text{and} \quad \phi^\wedge(\xi) = \chi_C(\xi) = \begin{cases} 1, & \text{if } \xi\in C, \\ 0, & \text{otherwise.}\end{cases} \tag{2.3.1}$$

Here, by (2.2.3), $P_\phi(f)^\wedge = f^\wedge\chi_C$, whence

$$\|f-P_\phi(f)\|^2 = (2\pi)^{-d}\int_{\mathbb{R}^d\setminus C} |f^\wedge(\xi)|^2\ d\xi \le (2\pi)^{-d}\int |\xi|^{2m}\ |f^\wedge(\xi)|^2\ d\xi$$

for any $m\in\mathbb{R}_+$, with $|\xi|$ the Euclidian norm of $\xi\in\mathbb{R}^d$. This tells

Lemma 2.3.2. *The PSI space generated by the sinc-function (2.3.1) has approximation power m, for $f\in W_2^m(\mathbb{R}^d)$, and any positive real m.*

Next we prove the important fact that the approximation power of an FSI space is already given by a PSI subspace S_ψ as follows:

Theorem 2.3.3. *Let ψ denote the orthogonal projection of the sinc-function onto the (stable) FSI space S_Φ, i.e., $\psi = P_\Phi(\mathrm{sinc})$. Then the approximation power of S_ψ and S_Φ are the same.*

Proof: From Theorem 2.2.14 and (2.2.3) we find that $P_\Phi\circ P_{\mathrm{sinc}} = P_\psi\circ P_{\mathrm{sinc}}$. Writing

$$f-P_\psi(f) = f-P_\Phi(f)+P_\Phi(f)-P_\Phi\circ P_{\mathrm{sinc}}(f)+P_\psi\circ P_{\mathrm{sinc}}(f)-P_\psi(f)$$

we see that (since $P_\psi(f) \in S_\Phi$)

$$\|f - P_\Phi(f)\| \le \|f - P_\psi(f)\| \le \|f - P_\Phi(f)\| + 2\|f - P_{\rm sinc}(f)\|.$$

The theorem now follows from Lemma 2.3.2. ■

A function $\psi \in S_\Phi$ having the property that S_ψ has the same approximation power as the larger space S_Φ, is called a *superfunction* in the FSI space. Finding a superfunction with specific properties is the general background of so-called *superfunction theory*. One such property is ψ being *compactly supported* whenever Φ is. This question will be addressed later.

Remark 2.3.4. By Theorem 2.2.14, the superfunction ψ of Theorem 2.3.3 is given by

$$\psi^\wedge = (\mathbf{\Phi}^\wedge)^T [\chi_C | \tilde{\mathbf{\Phi}}^\wedge],$$

hence satisfies

$$\psi^\wedge = (\mathbf{\Phi}^\wedge)^T \ \overline{\tilde{\mathbf{\Phi}}^\wedge} = A(\xi) \quad \text{for} \quad \xi \in C\ .$$

Also, since $\mathrm{sinc} - \psi$ is orthogonal to S_ψ, we have $[\chi_C - \psi^\wedge | \psi^\wedge] = 0$, whence

$$[\psi^\wedge|\psi^\wedge](\xi) = [\chi_C|\psi^\wedge](\xi) = \overline{\psi^\wedge(\xi)} \quad \text{for} \quad \xi \in C\ .$$

Theorem 2.3.5. *For the (stable) FSI space S_Φ the following are equivalent, for given $0 < m \in \mathbb{R}$:*

(i) S_Φ has approximation power m, for $f \in W_2^m(\mathbb{R}^d)$.

(ii) The function

$$\xi \quad \mapsto \quad |\xi|^{-2m}\Big\{1 - (\mathbf{\Phi}^\wedge)^T(\xi)\ \mathbf{G}_\Phi^{-1}(\xi)\ \overline{\mathbf{\Phi}^\wedge(\xi)}\Big\}$$

lies in $L_\infty(C)$.

(iii) For $\psi := P_\Phi(\mathrm{sinc})$ we have that $|\xi|^{-2m}\Big\{1 - \psi^\wedge(\xi)\Big\} \in L_\infty(C)$.

Proof: With $f = f_1 + f_2$, where $f_1 = f^\wedge \chi_C$ is the orthogonal projection of f onto $S_{\rm sinc}$, we have

$$\|f - P_\Phi(f)\| \le \|f_1 - P_\Phi(f_1)\| + \|f_2 - P_\Phi(f_2)\| \le \|f_1 - P_\Phi(f_1)\| + \|f_2\|.$$

Moreover, for $f \in W_2^m(\mathbb{R}^d)$,

$$\|f_2\|^2 = (2\pi)^{-d} \int_{\mathbb{R}^d \setminus C} |f^\wedge(\xi)|^2 \ d\xi \le |f|_{m,2}^2.$$

The theorem now follows from Theorem 2.2.16 and Remark 2.3.4. ■

Remark 2.3.6. The second statement of this theorem is equivalent to the order relation

$$1 - A(\xi) := 1 - (\mathbf{\Phi}^\wedge)^T(\xi)\ \mathbf{G}_\Phi^{-1}(\xi)\ \overline{\mathbf{\Phi}^\wedge}(\xi) \quad = \quad \mathcal{O}(|\xi|^{2m}) \quad \text{as} \quad \xi \to 0\ ;$$

see Remark 2.2.17. Also, statement (iii) of the theorem is equivalent to the order relation

$$1 - \psi^\wedge(\xi) \quad = \quad \mathcal{O}(|\xi|^{2m}) \quad \text{as} \quad \xi \to 0\ .$$

2.3.7. The (stable) compactly supported case. From now on let us assume that S_Φ is generated by compactly supported functions. In this case, the bracket products $[\phi_i{}^\wedge|\phi_j{}^\wedge]$, $i, j = 1, \ldots, n$, all have finitely many Fourier coefficients (since the functions $\phi_i * \phi_j^*$ are compactly supported as well); *i.e.*, we have identity in (2.1.2), and the entries of the Gramian are all trigonometric polynomials.

In particular, if $\mathbf{G}_\Phi(0)$ is regular - and this is certainly true under the Riesz basis condition assumed here - the function A is holomorphic in a neighborhood of the origin. *Hence the order relations in Remark 2.3.6, at least for $2m \in \mathbb{N}$, can be checked by looking at the power series expansion of $1 - A(\xi)$ at the origin.*

2.3.8. The compactly supported PSI case. This case can be dealt with without recourse to the assumption on stability. Here, the zero set of the trigonometric polynomial $[\phi^\wedge|\phi^\wedge]$ in C has (d-dimensional Lebesgue) measure 0, and Λ_ϕ defined in Remark 2.2.8 has the fundamental interpolation property, indeed. Let us put

$$\Omega(\xi) := 1 - \frac{|\phi^\wedge(\xi)|^2}{[\phi^\wedge|\phi^\wedge](\xi)} \quad = \sum_{0 \neq \alpha \in \mathbb{Z}^d} \frac{|\phi^\wedge(\xi + 2\pi\alpha)|^2}{[\phi^\wedge|\phi^\wedge](\xi)}\ . \tag{2.3.9}$$

From Theorem 2.2.7 we can deduce

Theorem 2.3.10. *Let $\phi \in L_2(\mathbb{R}^d)$ be compactly supported, and $0 < m \in \mathbb{R}$. Then, S_ϕ has approximation power m for $f \in W_2^m(\mathbb{R}^d)$ if and only if*

$$\Omega(\xi) \quad = \quad \mathcal{O}(|\xi|^{2m}) \quad \textit{as} \quad \xi \to 0.$$

This order relation can be expressed in another equivalent way, viz. from (2.3.9) we see that it is equivalent to requiring that

$$|\phi^\wedge(\xi + 2\pi\alpha)|^2 \ = \ \mathcal{O}(|\xi|^{2m}[\phi^\wedge|\phi^\wedge](\xi)) \quad \text{as} \quad \xi \to 0, \quad \text{for all } 0 \neq \alpha \in \mathbb{Z}^d. \tag{2.3.11}$$

A special case of this is

Theorem 2.3.12. *Let $\phi \in L_2(\mathbb{R}^d)$ be compactly supported, and assume that $[\phi^\wedge|\phi^\wedge](0) \neq 0$. Then, for $m \in \mathbb{N}$ the following are equivalent:*

(i) S_ϕ has approximation power m, for $f \in W_2^m(\mathbb{R}^d)$.

(ii) ϕ satisfies the conditions

$$D^\beta \phi^\wedge(2\pi\alpha) = 0 \quad \text{for } 0 \neq \alpha \in \mathbb{Z}^d \text{ and } |\beta| < m \ .$$

2.3.13. Compactly supported superfunction. In case 2.3.7 it is interesting to search after a compactly supported superfunction. With ψ as in Theorem 2.3.3, according to Remark 2.3.4, we see that, on the fundamental cube C, ψ coincides with A as given in Remark 2.2.17. Since the components of $\mathbf{\Phi}^\wedge$ are entire functions, and since the coefficient functions of the Gramian $\mathbf{G_\Phi}$ are trigonometric polynomials, with the determinant non-vanishing due to the Riesz basis property, we see that A (hence ψ) is C^∞ in a neighborhood of the origin. Also, the representation of $\psi^\wedge \in S_\Phi$ according to Lemma 2.1.4 is determined in Remark 2.3.4 as $\psi^\wedge = (\tilde{\mathbf{\Phi}}^\wedge)^H\, \mathbf{\Phi}^\wedge\, \chi_C = (\mathbf{\Phi}^\wedge)^H\, (\mathbf{G}_\Phi^T)^{-1}\, \mathbf{\Phi}^\wedge\, \chi_C$, i.e.,

$$\psi^\wedge = \sum_{i=1}^n \tau_i\ \phi_i{}^\wedge \quad \text{with} \quad \tau_i|_C = \overline{\tilde{\phi}_i{}^\wedge}|_C \ , \ i = 1, \ldots, n.$$

¿From this we see that the 2π-periodic functions τ_i are C^∞ as well in a neighborhood of the origin (in fact everywhere, according to our strong assumptions).

It is now the idea to mimic the behavior of $\psi^\wedge$ at the origin by a function

$$\tilde{\psi}^\wedge = \sum_{i=1}^n \tilde{\tau}_i\ \phi_i{}^\wedge \tag{2.3.14}$$

with *trigonometric polynomials* $\tilde{\tau}_i$ in order to have $\tilde{\psi}$ as a compactly supported function. This can be done, *at least for m an integer*, by forcing the trigonometric polynomials to satisfy the interpolatory conditions

$$D^\beta \tau_i(0) = D^\beta \tilde{\tau}_i(0) \quad \text{for} \quad 0 \leq |\beta| < m \quad \text{and} \quad i = 1, \ldots, n.$$

Using Remarks 2.3.6 and 2.3.4 for ψ, we see that $D^\beta \tilde{\psi}^\wedge(0) = D^\beta \psi^\wedge(0) = \delta_{0\beta}$ and $D^\beta \tilde{\psi}^\wedge(2\pi\alpha) = D^\beta \psi^\wedge(2\pi\alpha) = 0$ for $0 \neq \alpha \in \mathbb{Z}^d$ and $|\beta| < m$. Hence $S_{\tilde{\psi}}$ has approximation power m, for $f \in W_2^m(\mathbb{R}^d)$. We obtain

Theorem 2.3.15. *Given the stable FSI space S_Φ with compactly supported generators $\phi_1, \ldots, \phi_n$, the following are equivalent for $m \in \mathbb{N}$:*

(i) S_Φ has approximation power m, for $f \in W_2^m(\mathbb{R}^d)$.

(ii) There exists a unique function $\tilde{\psi} \in S_\Phi$ that has the following properties:

a) $\tilde{\psi}^\wedge$ is of the form (2.3.14), where

$$\tilde{\tau}_i \in \operatorname{span}\{e^{-i\beta\cdot\xi} : \beta \in \mathbb{Z}_+^d \ , \ |\beta| < m\}.$$

b) $\tilde{\psi}^\wedge(0) = 1$ *and* $D^\beta \tilde{\psi}^\wedge(0) = 0$ *for* $\beta \in \mathbb{Z}_+^d \setminus \{0\}$, $|\beta| < m$.

c) $S_{\tilde{\psi}}$ *has approximation power* m, *for* $f \in W_2^m(\mathbb{R}^d)$.

The superfunction $\tilde{\psi}$ of this theorem is called the *canonical superfunction*, and the vector $\mathbf{v} := (\tilde{\tau}_1, \ldots, \tilde{\tau}_n)$ is sometimes referred to as the *canonical* Φ*-vector of order* m.

2.4. Notes and extensions.

2.4.1. Shift-invariant spaces have a long tradition in Signal Processing, and in Approximation Theory. As the most important and widely used examples we mention as specific generators the famous sinc-function (giving rise to expansions of band-limited signals in terms of the Whittaker cardinal series, see [96]), and - as a counterpart to this - Schoenberg's cardinal B-splines generating his cardinal B-spline series, [91]. More recently, shift-invariant spaces have been studied quite thoroughly since they appear in Mallat's setup of multiresolution analysis, see [27, Chapter 5]. Our chapter tries to give an up-to-date discussion of the approximation powers provided by (scaled versions of) such spaces. The main results are worked out in a form which is perhaps not most general, but can be understood without essential recourse to the original literature.

2.4.2. The chapter is very much influenced by the fundamental work of de Boor, DeVore and Ron on L_2-approximation power of shift invariant spaces, [6, 7, 8]. We did not try to include the general L_p-case, where we would like to refer to Jia's survey [43] and the references therein. Much of the original interest in approximation orders stems from understanding the approximation power of box spline spaces, see the book [10, Chapter 3], and the discussion of various notions of approximation orders (like controlled or local approximation orders, see e.g. [11, 34, 52, 53, 71, 72]). Meanwhile, Ron and his coworkers have shown an interesting connection of approximation orders of a PSI space generated by a refinable function to the convergence of the corresponding subdivision scheme [89].

2.4.3. It is probably not useful to attribute the notion of bracket product to any author; periodization techniques have been used for a long time in Fourier analysis, a typical result being Poisson's summation formula which holds true under various assumptions, and with varying interpretation of identities. However, [54] and [6] were the first to use this notion in a way leading, e.g., to the useful characterization of orthogonality of PSI spaces, see [6, Lemma 2.8]. The same paper contains Lemma 2.1.4 for the PSI case, while the general case is given in [7, Theorem 1.7]. A simple proof of the result is due to Jia [47, Theorem 2.1], see also [43, Theorem 1.1].

2.4.4. The construction of the L_2-projector in the stable PSI case is straightforward, and the orthogonalization process 2.2.4 has been used, e.g., in the construction of orthonormal spline wavelets (with infinite support, but exponentially decaying at infinity) by Battle and Lemarié [1, 69]. Much more involved is the general case where formula 2.2.5 and Theorem 2.2.7 (with the agreement 2.2.6) is again due to [6, Theorems 2.9 and 2.20]. In the FSI case the notion of the Gramian was first used by Goodman, Lee and Tang [31, 32]; they also were aware of the orthogonalization process 2.2.15 (see [32, Theorem 3.3]). The representation of the L_2-projector in Theorem 2.2.14 and the error formula of Theorem 2.2.16 are again due to de Boor, DeVore and Ron [7, Theorem 3.9].

2.4.5. Section 2.3 closely follows the ideas and methods in [8], with some slight modifications and also easier arguments due to the fact that we involve the assumption on stability. Concerning the more general (unstable) case, we refer to the Remark stated in that paper after the proof of Theorem 2.2.

2.4.6. The compactly supported PSI case as dealt with in Theorem 2.3.12 is well-understood for polynomial spline functions; first results in this direction go back to the seminal papers [91] of Schoenberg in 1946. The condition given in statement (ii) of the Theorem (together with $\phi^\wedge(0) \neq 0$) is nowadays called the *Strang-Fix conditions*, due to their contribution in [92]. Assertions (i), (ii) of Theorem 2.3.12 are also equivalent to the statement that algebraic polynomials of degree $\leq m-1$ can locally be exactly reproduced in S_ϕ [48, Theorem 2.1].

2.4.7 In the univariate PSI case $d=1$ the following assertion can be shown: Let $\phi \in L^2(\mathbb{R}^d)$ be compactly supported. If S_ϕ has approximation power $m \in \mathbb{N}$, for $f \in W_2^m(\mathbb{R}^d)$, then there exists a compactly supported tempered distribution η such that $\psi = N_m * \eta$, with N_m the cardinal B-spline of order m. Moreover, if $\operatorname{supp}\psi \subseteq [a,b]$, then η can be chosen in such a way that $\operatorname{supp}\eta \subseteq [a, b-m]$; see [84, Proposition 3.6]. A generalization of this idea to the FSI case is treated in [79].

§3. Shift invariant spaces spanned by refinable functions

3.1. Dilation matrices and refinement equations.

Refinable shift-invariant spaces are defined with recourse to a *dilation matrix* M, *i.e.*, a regular <u>integer</u> $(d \times d)$-matrix satisfying

$$\lim_{n\to\infty} M^{-n} = 0 \ .$$

Equivalently, all eigenvalues of M have modulus greater than 1. Given such a matrix, a shift-invariant subspace S of $L_2(\mathbb{R}^d)$ is called M-*refinable*, if

$$s \in S \qquad \Longrightarrow \qquad s(M^{-1}\cdot) \in S \ .$$

In the literature, the dilation matrix is often taken as $M = 2I$ (with I the identity matrix), but we allow here the general case.

A shift-invariant space $S(\Phi) \subset L_2(\mathbb{R}^d)$ generated by $\Phi = \{\phi_1, \ldots, \phi_n\} \subset L_2(\mathbb{R}^d)$, is M-refinable if and only if the function vector $\mathbf{\Phi} := (\phi_1, \ldots, \phi_n)^T$ satisfies a *matrix refinement equation*

$$\mathbf{\Phi} = \sum_{\alpha \in \mathbb{Z}^d} \mathbf{P}_\alpha \, \mathbf{\Phi}(M \cdot - \alpha) \; ; \tag{3.1.1}$$

here the "coefficients" $\mathbf{P}_\alpha$ are real or complex $(n \times n)$-matrices, with n the number of generators of the FSI space. The matrix-valued sequence $\mathbf{P} = (\mathbf{P}_\alpha)_{\alpha \in \mathbb{Z}^d}$ is usually called the *refinement mask*. In case of a single generator, (3.1.1) takes the scalar form

$$\phi = \sum_{\alpha \in \mathbb{Z}^d} p_\alpha \, \phi(M \cdot - \alpha)$$

with the scalar-valued mask $p = (p_\alpha)_{\alpha \in \mathbb{Z}^d}$.

For simplicity, we shall only consider compactly supported function vectors $\mathbf{\Phi}$, and we suppose that the refinement mask $\mathbf{P}$ is finitely supported on $\mathbb{Z}^d$. Then $\Phi \subset L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, and in the Fourier transform domain the refinement equation reads

$$\mathbf{\Phi}^\wedge = \mathbf{H}(M^{-T} \cdot) \, \mathbf{\Phi}^\wedge(M^{-T} \cdot) \tag{3.1.2}$$

with $M^{-T} := (M^T)^{-1}$ and

$$\mathbf{H}(\xi) := \frac{1}{|\det M|} \sum_{\alpha \in \mathbb{Z}^d} \mathbf{P}_\alpha \, e^{-i\alpha \cdot \xi} \; , \quad \xi \in \mathbb{R}^d, \tag{3.1.3}$$

the so-called *refinement mask symbol*. This symbol $\mathbf{H}$ is an $(n \times n)$-matrix of trigonometric polynomials on $\mathbb{R}^d$. Again, in the PSI case we get a scalar-valued symbol which we denote by

$$H(\xi) := \frac{1}{|\det M|} \sum_{\alpha \in \mathbb{Z}^d} p_\alpha \, e^{-i\alpha \cdot \xi} \; , \quad \xi \in \mathbb{R}^d \, . \tag{3.1.4}$$

The goal of this section is a characterization of the approximation power of an M-refinable shift-invariant space S_Φ in terms of the refinement mask $\mathbf{P}$, or of its symbol $\mathbf{H}$, or of the associated subdivision and transfer operators. Here, the structure of the sublattices $M\mathbb{Z}^d$ and $M^T\mathbb{Z}^d$ of $\mathbb{Z}^d$ is important; we have the partitioning

$$\mathbb{Z}^d = \bigcup_{e \in E} (e + M\mathbb{Z}^d) = \bigcup_{e' \in E'} (e' + M^T\mathbb{Z}^d) \; , \tag{3.1.5}$$

where E and E' denote any set of representatives of the equivalence classes $\mathbb{Z}^d / M\mathbb{Z}^d$ and $\mathbb{Z}^d / M^T\mathbb{Z}^d$, respectively. Both E and E' contain $\mu = |\det M|$ representatives, and in a standard form we shall always take

$$E = M([0,1[^d) \cap \mathbb{Z}^d \quad \text{and} \quad E' = M^T([0,1[^d) \cap \mathbb{Z}^d \; . \tag{3.1.6}$$

We also let

$$E_0 := E \setminus \{0\} \quad \text{and} \quad E_0' := E' \setminus \{0\}. \tag{3.1.7}$$

The following fact will be used in the sequel: *Any* $0 \neq \alpha \in \mathbb{Z}^d$ *has a unique representation*

$$\alpha = (M^T)^\ell (e' + M^T\beta) \ , \quad \ell \geq 0, \ e' \in E_0', \ \beta \in \mathbb{Z}^d \ . \tag{3.1.8}$$

This can be seen as follows. Since $\lim_{\ell\to\infty}(M^{-T})^\ell \alpha = 0$, there is a unique minimal integer $\ell \geq 0$ such that $\alpha \in (M^T)^\ell \mathbb{Z}^d \setminus (M^T)^{\ell+1}\mathbb{Z}^d$. Using the partitioning (3.1.5), we find the required unique representation.

3.1.9. The PSI case. Here, refinability already implies that S_ϕ has some approximation power.

Theorem 3.1.10. *Let $\phi \in L_2(\mathbb{R}^d)$ be compactly supported and M-refinable with finitely supported refinement mask, and assume that $\phi^\wedge(0) \neq 0$. Then ϕ satisfies the Strang-Fix conditions of order one, whence S_ϕ has at least approximation power one, for $f \in W_2^1(\mathbb{R}^d)$.*

Proof: From the scalar refinement equation

$$\phi^\wedge = H(M^{-T}\cdot)\ \phi^\wedge(M^{-T}\cdot) \tag{3.1.11}$$

we conclude that $H(0) = 1$, and due to the periodicity of H we have

$$\phi^\wedge(2\pi M^T\alpha) = \phi^\wedge(2\pi\alpha) \ , \quad \alpha \in \mathbb{Z}^d \ .$$

¿From this, $\phi^\wedge(2\pi\alpha) = \lim_{k\to\infty} \phi^\wedge(2\pi(M^T)^k\alpha) = 0$ for $0 \neq \alpha \in \mathbb{Z}^d$, by an application of the Riemann-Lebesgue Lemma, and the assertion follows from Theorem 2.3.12. ∎

3.1.12. The FSI case. A generalization of this result refers to the refinement equation (3.1.2) at the origin. If $\mathbf{\Phi}^\wedge(0) \neq \mathbf{0}$, then $\mathbf{\Phi}^\wedge(0)$ is a right eigenvector of $\mathbf{H}(0)$, for the eigenvalue 1. Consider a left eigenvector $\mathbf{v}$, say, for the same eigenvalue, and put $\psi^\wedge := \mathbf{v}\mathbf{\Phi}^\wedge$. Then $\psi \in S_\Phi$, and as in the proof before $\psi^\wedge(2\pi\alpha) = 0$ for $0 \neq \alpha \in \mathbb{Z}^d$. In order to apply Theorem 2.3.12 again, we just have to require that $\psi^\wedge(0) \neq 0$. This gives

Theorem 3.1.13. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be compactly supported and M-refinable with finitely supported refinement mask and corresponding mask symbol $\mathbf{H}$. Assume that $\mathbf{\Phi}^\wedge(0) \neq \mathbf{0}$, and that $\mathbf{v}\mathbf{H}(0) = \mathbf{v}$ for a row vector $\mathbf{v}$ satisfying $\mathbf{v}\mathbf{\Phi}^\wedge(0) \neq 0$. Then, $\psi \in S_\Phi$ given by $\psi^\wedge := \mathbf{v}\mathbf{\Phi}^\wedge$ satisfies the Strang-Fix conditions of order one, whence S_ψ and, a fortiori, S_Φ have at least approximation power one, for $f \in W_2^1(\mathbb{R}^d)$.*

3.1.14. The spectral condition on $\mathbf{H}(0)$. The additional assumptions made in this theorem will appear later in a more general form, namely when $\mathbf{v} = (\tilde\tau_1, \ldots, \tilde\tau_n)$ is a row vector of trigonometric polynomials and

$$\psi^\wedge := \mathbf{v}\mathbf{\Phi}^\wedge \ . \tag{3.1.15}$$

Then $\psi \in S_\Phi$ is compactly supported (whenever Φ is), and we point to 2.3.13 where we have constructed compactly supported superfunctions $\tilde{\psi}$ in this way. We say that $\mathbf{v}$ *satisfies the spectral condition (of order* 1*) on* $\mathbf{H}$ *at the origin*, if

(i) $\psi^\wedge(0) = \mathbf{v}(0)\mathbf{\Phi}^\wedge(0) \neq 0$ and

(ii) $\mathbf{v}(0)\,\mathbf{H}(0) = \mathbf{v}(0)$.

For a stronger version of this condition, see Remark 3.2.11 below.

3.2. The zero condition on the mask symbol.

The Strang-Fix conditions can be expressed as a zero condition on the mask symbol. This condition is sufficient for the PSI space S_ϕ having approximation power. The condition is also necessary in the stable case, and even under a weaker assumption than stability.

Theorem 3.2.1. *Let $\phi \in L_2(\mathbb{R}^d)$ be compactly supported and M-refinable with finitely supported refinement mask and the corresponding mask symbol H, and assume that $\phi^\wedge(0) \neq 0$. For $m \in \mathbb{N}$ we have:*

(i) The zero condition of order m on the mask symbol,

$$D^\mu\{H(M^{-T}\cdot)\}(2\pi e') = 0 \quad \text{for all } e' \in E_0' \text{ and } |\mu| < m, \tag{3.2.2}$$

implies that S_ϕ has approximation power m, for $f \in W_2^m(\mathbb{R}^d)$.

(ii) Conversely, if S_ϕ has approximation power m, then (3.2.2) holds true subject that

$$[\phi^\wedge|\phi^\wedge](2\pi M^{-T}e') \neq 0 \quad \text{for any } e' \in E_0' \; . \tag{3.2.3}$$

Proof: First let $m = 1$. In (3.1.11) we substitute $\xi = 2\pi\alpha$ with

$$\alpha = e' + M^T\beta \; , \quad e' \in E', \; \beta \in \mathbb{Z}^d, \; (e', \beta) \neq (0,0) \; ,$$

to give

$$\phi^\wedge(2\pi\alpha) = H(2\pi M^{-T}e')\;\phi^\wedge(2\pi M^{-T}e' + 2\pi\beta) \; .$$

While, for $m = 1$, (i) is always satisfied due to Theorem 3.1.10, we see that assertion (ii) follows since $\{\phi^\wedge(2\pi M^{-T}e' + 2\pi\beta)\}_{\beta\in\mathbb{Z}^d}$ can not be a zero sequence for $e' \in E_0'$ by (3.2.3).

For $m \geq 1$ we use induction and Leibniz' rule in (3.1.11),

$$D^\gamma\phi^\wedge = \sum_{\mu\leq\gamma} \binom{\gamma}{\mu} D^\mu\{H(M^{-T}\cdot)\}\; D^{\gamma-\mu}\{\phi^\wedge(M^{-T}\cdot)\} \; , \tag{3.2.4}$$

with $|\gamma| < m + 1$. In case of statement (i) we see that the zero condition of order $m + 1$ immediately verifies the Strang-Fix conditions $D^\gamma\phi^\wedge(2\pi\alpha) = 0$ for the following situations

$$|\gamma| < m \quad \text{and} \quad 0 \neq \alpha \in \mathbb{Z}^d \quad \text{(via Theorem 2.3.12)} \; ;$$
$$|\gamma| = m \quad \text{and} \quad \alpha = e' + M^T\beta, \quad e' \in E_0' \; , \; \beta \in \mathbb{Z}^d \; .$$

Here, the first situation is the induction assumption. For $|\gamma| = m$ and $0 \neq \alpha \in \mathbb{Z}^d$ as in (3.1.8) with $\ell > 0$, equation (3.2.4) reduces to

$$\begin{aligned}
&D^\gamma \phi\hat{}(2\pi\{(M^T)^\ell(e' + M^T\beta)\}) \\
&= H(M^{-T}2\pi(M^T)^\ell(e' + M^T\beta))\ D^\gamma\{\phi\hat{}(M^{-T}\cdot)\}(2\pi(M^T)^\ell(e' + M^T\beta)) \\
&= H(0) \sum_{|\gamma'|=|\gamma|} c_{\gamma',\gamma}\ \{D^{\gamma'}\phi\hat{}\}(2\pi(M^T)^{\ell-1}(e' + M^T\beta))
\end{aligned}$$

for some constants $c_{\gamma',\gamma}$, and an apparent induction argument with respect to ℓ shows that the full Strang-Fix conditions of order $m+1$ are satisfied, hence S_ϕ has approximation power $m+1$. In case of statement (ii), ϕ satisfies the Strang-Fix conditions of order $m+1$, and due to the induction hypothesis, the symbol satisfies the zero conditions of order m. Hence (3.2.4) yields, for $|\gamma| = m$, $e' \in E_0'$, and any $\beta \in \mathbb{Z}^d$,

$$0 = D^\gamma\phi\hat{}(2\pi(e' + M^T\beta)) = D^\gamma\{H(M^{-T}\cdot)\}(2\pi e')\ \phi\hat{}(2\pi M^{-T}e' + 2\pi\beta)\ .$$

Due to our assumption (3.2.3), this yields an additional order in the zero condition for the mask symbol. ■

Remark 3.2.5. The zero condition on the mask symbol, (3.2.2), is actually equivalent to

$$\{D^\mu H\}(2\pi M^{-T}e') = 0 \quad \text{for all } e' \in E_0' \text{ and } |\mu| < m\ . \tag{3.2.6}$$

In the univariate case, $d = 1$, the refinement equation is of the form

$$\phi = \sum_{\alpha\in\mathbb{Z}} p_\alpha\, \phi(k\cdot - \alpha)$$

for some integer $k \geq 2$. Here, this condition simplifies to

$$D^\mu H(\frac{2\pi j}{k}) = 0\ , \quad j = 1, \dots k-1,\ \mu = 0, \dots, m-1.$$

Since H is a trigonometric polynomial, the zeros can be factored out as

$$H(\xi) = \left(\frac{1}{k} \cdot \frac{1 - e^{-ik\xi}}{1 - e^{-i\xi}}\right)^m\ G(\xi)$$

with another trigonometric polynomial G satisfying $G(0) = 1$.

3.2.7. Condition (Z_m) and the FSI case. The above Theorem 3.2.1 is a special case of the theorem to follow. Let us say that *the matrix symbol* $\mathbf{H}$ *satisfies condition* (Z_m), if there exists a row vector

$$\mathbf{v} = (\tilde{\tau}_1, \dots, \tilde{\tau}_n)$$

of trigonometric polynomials such that

(a) $\mathbf{v}$ satisfies the spectral condition 3.1.14 on $\mathbf{H}$ at the origin and

(b) $D^\mu\{\mathbf{v}\mathbf{H}(M^{-T}\cdot)\}(2\pi e') = 0$ for $|\mu| < m$ and $e' \in E_0'$.

Theorem 3.2.8. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be compactly supported and M-refinable with finitely supported refinement mask and the corresponding mask symbol $\mathbf{H}$. Assume that $\mathbf{\Phi}^\wedge(0) \neq \mathbf{0}$. Then for $m \in \mathbb{N}$ we have:*

(i) If $\mathbf{H}$ satisfies condition (Z_m), then S_Φ has approximation power m, for $f \in W_2^m(\mathbb{R}^d)$.

(ii) Conversely, if S_Φ has approximation power m, and if the Gramian $\mathbf{G}_{\mathbf{\Phi}}$ is regular at $\xi = 2\pi M^{-T} e'$ for any $e' \in E'$, then $\mathbf{H}$ satisfies condition (Z_m).

It should be noted that the condition on the regularity of the Gramian is equivalent to the fact that the sequences

$$\{\phi_j(2\pi M^{-T} e' + 2\pi\beta)\}_{\beta \in \mathbb{Z}^d} , \quad j = 1, \ldots, n ,$$

are linearly independent.

Proof: We can follow the proof of Theorem 3.2.1 with the same notations and the notions according to 3.1.14; in particular, let $\psi^\wedge := \mathbf{v}\,\mathbf{\Phi}^\wedge$. Due to the refinement equation we have

$$\psi^\wedge(2\pi\alpha) = \mathbf{v}(0)\ \mathbf{H}(2\pi M^{-T} e')\ \mathbf{\Phi}^\wedge(2\pi M^{-T} e' + 2\pi\beta) \tag{3.2.9}$$

for any $\alpha = e' + M^T\beta \in \mathbb{Z}^d$, and the induction procedure will use Leibniz' rule in the form

$$D^\gamma \psi^\wedge = \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} D^\mu \{\mathbf{v}\mathbf{H}(M^{-T}\cdot)\}\ D^{\gamma-\mu}\{\mathbf{\Phi}^\wedge(M^{-T}\cdot)\} . \tag{3.2.10}$$

For (i), the induction is based on Theorem 3.1.13 (case $m = 1$), and the induction step of the above proof extends almost literally.

For (ii), let us consider the stronger assumption that the Gramian is regular everywhere. Then according to Theorem 2.3.15 we can find $\mathbf{v} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_n)$ such that $\psi^\wedge = \mathbf{v}\mathbf{\Phi}^\wedge$ satisfies the Strang-Fix condition of order m. In particular, the Strang-Fix conditions of order 1 already show that $\psi^\wedge(0) \neq 0$ and

$$\mathbf{v}(0)\,\mathbf{H}(0)\,\mathbf{\Phi}^\wedge(2\pi\alpha) = \mathbf{v}(0)\,\mathbf{\Phi}^\wedge(2\pi\alpha)$$

for all $\alpha \in \mathbb{Z}^d$ (by putting $e' = 0$ in (3.2.9)), hence $\mathbf{v}$ satisfies the spectral condition on $\mathbf{H}$ at the origin, due to the regularity of $\mathbf{G}_{\mathbf{\Phi}}(0)$. The induction step is now analogous as before.

In order to see (ii) with the weaker assumptions, it should be emphasized that Remark 2.3.4 holds true in a neighborhood of the origin as long as the Gramian is regular there. In this way, also the construction of the superfunction in 2.3.13 can be performed in this neighborhood. ■

Remark 3.2.11. In the statement of Theorem 3.2.8, the row vector $\mathbf{v}$ can be chosen such that its components are linear combinations of the exponentials

$$e^{-i\alpha\cdot\xi} \quad \text{with} \quad \alpha \in \mathbb{Z}_+^d \quad \text{and} \quad |\alpha| < m ,$$

and moreover, such that (besides the zero condition (Z_m)) the *spectral condition of order* m,

$$D^\mu\{\mathbf{v}\,\mathbf{H}(M^{-T}\cdot)\}(0) = D^\mu\{\mathbf{v}(M^{-T}\cdot)\}(0) \tag{3.2.12}$$

is satisfied for $|\mu| < m$.

Remark 3.2.13. Condition (Z_m) can be simplified in the univariate case $d = 1$, and it implies a matrix factorization of the refinement mask symbol $\mathbf{H}$. Here, equation (3.1.1) reads

$$\mathbf{\Phi} = \sum_{\alpha\in\mathbb{Z}} \mathbf{P}_\alpha\,\mathbf{\Phi}(k\cdot - \alpha)$$

for some integer $k \geq 2$. It can be shown that there exists a trigonometric polynomial matrix $\mathbf{A}$ such that

$$\mathbf{H}(\xi) = k^{-m}\,\mathbf{A}(k\xi)\,\mathbf{G}(\xi)\,\mathbf{A}(\xi)^{-1},$$

where $\mathbf{G}$ is another trigonometric polynomial matrix. Moreover, the *factorization matrix* $\mathbf{A}$ necessarily satisfies the following two conditions:

a) $\{D^\mu(\det\mathbf{A})\}(0) = 0$ for $|\mu| < m$.

b) If ψ defined by $\psi^\wedge := \mathbf{v}\,\mathbf{\Phi}^\wedge$ is a superfunction (i.e., S_ψ has approximation power m) then $\{D^\mu(\mathbf{vA})\}(0) = \mathbf{0}$ for $|\mu| < m$.

Further, considering $\mathbf{\Psi}^\wedge(\xi) := (i\xi)^m\,\mathbf{A}(\xi)^{-1}\,\mathbf{\Phi}^\wedge(\xi)$ we obtain

$$\mathbf{\Psi}^\wedge(\xi) = \mathbf{G}(k^{-1}\xi)\,\mathbf{\Psi}^\wedge(k^{-1}\xi)$$

with $\mathbf{G}$ the trigonometric polynomial matrix occurring in the above factorization of $\mathbf{H}$, i.e., $\mathbf{\Psi}$ is a compactly supported refinable distribution vector with refinement mask symbol $\mathbf{G}$.

3.3. The sum rules.

The zero condition on the mask symbol can be given in an equivalent form. Here, the group structure of $E := \mathbb{Z}^d/M\mathbb{Z}^d$ enters the discussion. It is well-known that the dual group is given by $E^\wedge = 2\pi(M^{-T}\mathbb{Z}^d/\mathbb{Z}^d)$, hence *for the Fourier matrix*

$$\mathbf{F}_M \quad := \quad \left(e^{-2\pi i e\cdot(M^{-T}e')}\right)_{e\in E, e'\in E'}$$

we have

$$\frac{1}{|det\ M|}\,\mathbf{F}_M\,\mathbf{F}_M^H = \mathbf{I}\ , \tag{3.3.1}$$

i.e., the matrix is unitary up to the given factor. A simple corollary of this is

Lemma 3.3.2. *For a trigonometric polynomial* $h(\xi) = \sum_{e\in E} b_e\ e^{-ie\cdot\xi}$ *the following are equivalent:*

(i) $h(2\pi M^{-T}e') = 0$ *for all* $e' \in E_0'$.

(ii) $b_e = b_0$ *for all* $e \in E$.

Proof: Condition (i) is equivalent to the fact that the vector $(b_e)_{e\in E}$ is orthogonal to all columns of the Fourier matrix except the first one, and this property is equivalent to the vector being a multiple of the first column. ■

In the PSI case this can be applied to the scalar-valued mask symbol H in the following way: Given any algebraic polynomial q and the corresponding differential operator $q(iD)$, then

$$\{q(iD)\ H\}(\xi) = \frac{1}{|\det \mathrm{M}|} \sum_{\alpha\in\mathbb{Z}^d} p_\alpha\ q(\alpha)\ e^{-i\alpha\cdot\xi}.$$

Rearranging the sum in terms of $\alpha = e + M\gamma$ and inserting the dual lattice gives $\alpha \cdot 2\pi M^{-T}e' = e \cdot 2\pi M^{-T}e'$ (modulo 2π), whence

$$\begin{aligned} &\{q(iD)\ H\}(2\pi M^{-T}e') \\ &= \frac{1}{|\det \mathrm{M}|} \sum_{e\in E} \Big(\sum_{\gamma\in\mathbb{Z}^d} p_{e+M\gamma}\ q(e+M\gamma) \Big)\ e^{-ie\cdot 2\pi M^{-T}e'}. \end{aligned} \tag{3.3.3}$$

Combining this with the above lemma and with (3.2.2) yields the following *sum rules of order* m:

Theorem 3.3.4. *In the PSI case, the zero condition (3.2.2) of order* m *on the mask symbol* H *is equivalent to the fact that the mask* p *satisfies*

$$\sum_{\gamma\in\mathbb{Z}^d} p_{e+M\gamma}\ q(e+M\gamma) = \sum_{\gamma\in\mathbb{Z}^d} p_{M\gamma}\ q(M\gamma)\ , \quad e\in E\ , \tag{3.3.5}$$

for any algebraic polynomial q *of degree less than* m.

In particular, we observe that for ϕ satisfying $\phi^\wedge(0) \neq 0$ the sum rule of order one,

$$\sum_{\gamma\in\mathbb{Z}^d} p_{e+M\gamma} = \sum_{\gamma\in\mathbb{Z}^d} p_{M\gamma}\ , \quad e\in E\ ,$$

is sufficient for S_ϕ having approximation power 1. The converse holds true subject to the additional condition in Theorem 3.2.1 (ii).

Turning to the FSI case, we recognize that Theorem 3.3.4 is a special case of the following statement (when dealing with the scalar case and putting $v_\sigma = \delta_\sigma$):

Theorem 3.3.6. *In the stable FSI case, the zero condition* (Z_m) *on the row vector* $\mathbf{vH}(M^{-T}\cdot)$ *(with* $\mathbf{v}(\xi)=\sum_{\alpha\in\mathbb{Z}^d}\mathbf{v}_\alpha e^{-i\alpha\cdot\xi}$ *a row vector of trigonometric polynomials) is equivalent to the fact that the mask* $\mathbf{P}$ *satisfies the sum rules*

$$\begin{aligned}&\sum_{\sigma\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d}\mathbf{v}_{\gamma-\sigma}\,\mathbf{P}_{e+M\sigma}\,q(e+M\gamma)\\ &=\sum_{\sigma\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d}\mathbf{v}_{\gamma-\sigma}\,\mathbf{P}_{M\sigma}\,q(M\gamma)\ ,\quad e\in E\ ,\end{aligned}\tag{3.3.7}$$

for any algebraic polynomial q *of degree less than* m.

Proof: We have

$$\mathbf{v}(\xi)\ \mathbf{H}(M^{-T}\xi)=\frac{1}{|\det\ M|}\sum_{\sigma\in\mathbb{Z}^d}\sum_{\alpha\in\mathbb{Z}^d}\mathbf{v}_\sigma\ \mathbf{P}_\alpha\ e^{-i(M^{-1}\alpha+\sigma)\cdot\xi}\ ,$$

hence with $\alpha=e+M\gamma$, $e'\in E'$ and q any algebraic polynomial as before,

$$\begin{aligned}&q(iD)\{\mathbf{v}\ \mathbf{H}(M^{-T}\cdot)\}(2\pi e')\\ &=\frac{1}{|\det\ M|}\sum_{e\in E}\Big(\sum_{\sigma\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d}\mathbf{v}_\sigma\ \mathbf{P}_{e+M\gamma}\ q(M^{-1}e+\gamma+\sigma)\Big)e^{-ie\cdot 2\pi M^{-T}e'}.\end{aligned}$$

Applying Lemma 3.3.2, the zero condition on the vector $\mathbf{vH}(M^{-T}\cdot)$, *i.e.*, (Z_m)(b) in 3.2.7, is now equivalent to the fact that the expression within the outer brackets does not depend on e, for any polynomial of degree less than m. Substituting $q(M^{-1}\cdot)$ by q we arrive at the required sum rules. ■

In particular, the sum rules of order 1 are satisfied if there exists a row vector $\mathbf{v}_0\in\mathbb{R}^d$ such that

$$\mathbf{v}_0\sum_{\sigma\in\mathbb{Z}^d}\mathbf{P}_{e+M\sigma}=\mathbf{v}_0\sum_{\sigma\in\mathbb{Z}^d}\mathbf{P}_{M\sigma}\quad\text{for all}\quad e\in E\ .$$

3.4. The subdivision operator and the transfer operator.

In this subsection we consider two linear operators which come with the refinement mask $\mathbf{P}$ in (3.1.1). These operators have been shown to be excellent tools for the characterization of refinable function vectors.

For a given (complex) mask $\mathbf{P}=(\mathbf{P}_\alpha)_{\alpha\in\mathbb{Z}^d}$ of $(n\times n)$-matrices, the *subdivision operator* S_P is the linear operator on the sequence space $X=\big(\ell(\mathbb{Z}^d)\big)^n$ defined by

$$(S_P\mathbf{c})_\alpha:=\sum_{\beta\in\mathbb{Z}^d}\mathbf{P}^H_{\alpha-M\beta}\,\mathbf{c}_\beta\ ,\quad \alpha\in\mathbb{Z}^d\ ;\tag{3.4.1}$$

here, $\mathbf{c} = (\mathbf{c}_\alpha)_{\alpha\in\mathbb{Z}^d} \in \left(\ell(\mathbb{Z}^d)\right)^n$, *i.e.*, $\mathbf{c}$ is a vector of (complex-valued) sequences indexed by $\mathbb{Z}^d$. Restricting the subdivision operator to the case of sequences which are square summable, *i.e.*, $\|\mathbf{c}\|^2 := \sum_{\alpha\in\mathbb{Z}^d} \mathbf{c}_\alpha^H \mathbf{c}_\alpha < \infty$, or $\mathbf{c} \in (\ell_2(\mathbb{Z}^d))^n$ for short, we have the vector-valued Fourier series

$$\mathbf{c}^\wedge(\xi) := \sum_{\alpha\in\mathbb{Z}^d} \mathbf{c}_\alpha \, e^{i\alpha\cdot\xi} \; ,$$

and (3.4.1) leads to

$$(S_P\,\mathbf{c})^\wedge(\xi) = |\det M| \; \mathbf{H}(\xi)^H \, \mathbf{c}^\wedge(M^T\xi) \tag{3.4.2}$$

with the mask symbol $\mathbf{H}$ in (3.1.3).

The *transfer operator* T_P (sometimes also called *transition operator*) associated with the mask $\mathbf{P}$ is the linear operator operating on $\left(\ell_0(\mathbb{Z}^d)\right)^n \subset \left(\ell(\mathbb{Z}^d)\right)^n$, the subspace of compactly supported vector-valued sequences, defined by

$$(T_P\mathbf{d})_\alpha := \sum_{\beta\in\mathbb{Z}^d} \mathbf{P}_{M\alpha-\beta}\, \mathbf{d}_\beta. \tag{3.4.3}$$

The definition naturally extends to $\left(\ell_2(\mathbb{Z}^d)\right)^n$, due to the compact support of the matrix mask, and the Fourier series of the image vector sequence is given by

$$(T_P\mathbf{d})^\wedge(\xi) = \sum_{e'\in E'} \mathbf{H}(M^{-T}(\xi+2\pi e'))\; \mathbf{d}^\wedge(M^{-T}(\xi+2\pi e')). \tag{3.4.4}$$

In case of a scalar-valued mask $p = (p_\alpha)_{\alpha\in\mathbb{Z}^d}$, the linear operators simply read as

$$\begin{aligned}(S_p\,c)_\alpha &= \sum_{\beta\in\mathbb{Z}^d} \overline{p_{\alpha-M\beta}}\, c_\beta \;, \quad c \in \ell(\mathbb{Z}^d),\\ (T_p\,d)_\alpha &= \sum_{\beta\in\mathbb{Z}^d} p_{M\alpha-\beta}\, d_\beta \;, \quad d \in \ell_0(\mathbb{Z}^d),\end{aligned}$$

and for $c,\, d \in \ell_2(\mathbb{Z}^d)$ we observe that

$$\begin{aligned}(S_pc)^\wedge(\xi) &= |\det M|\, \overline{H(\xi)}\, c^\wedge(M^T\xi) \quad \text{and}\\ (T_pd)^\wedge(\xi) &= \sum_{e'\in E'} H(M^{-T}(\xi+2\pi e'))\, d^\wedge(M^{-T}(\xi+2\pi e')).\end{aligned}$$

It is this version of the transfer operator $d^\wedge \mapsto (T_pd)^\wedge$ in the Fourier transform domain and operating on $\left(L_2(C)\right)^n$ which frequently appears in the literature.

There is a close connection between the two operators. When operating on $\left(\ell_2(\mathbb{Z}^d)\right)^n$ considered as a Hilbert space with scalar product

$$< \mathbf{c} \mid \mathbf{d} > := \sum_{\alpha\in\mathbb{Z}^d} \mathbf{c}_\alpha^H \mathbf{d}_\alpha, \tag{3.4.5}$$

it is easy to see that the adjoint operator S_P^* defined by $< S_P\mathbf{c} \mid \mathbf{d} >=$ $< \mathbf{c} \mid S_P^*\mathbf{d} >$ is given by

$$S_P^* \, \mathbf{d} = \left(T_P(\mathbf{d}\,\breve{}\,)\right)\breve{} \tag{3.4.6}$$

with $\mathbf{d}\,\breve{}_\alpha := \mathbf{d}_{-\alpha}$ denoting the reflection of a sequence. From this it is clear that, *considered as operators on* $X = \left(\ell_2(\mathbb{Z}^d)\right)^n$, *the spectra* $\sigma_X(S_P)$ *and* $\sigma_X(T_P)$ *are connected to each other through complex conjugation, i.e.,*

$$\sigma_X(T_P) = \overline{\sigma_X(S_P)} \; .$$

The situation is a little more involved if we look at eigenvalues of the operators. Here it is opportune to consider (3.4.5) as a sesquilinear form on the dual pairing

$$X \times X' := \left(\ell(\mathbb{Z}^d)\right)^n \times \left(\ell_0(\mathbb{Z}^d)\right)^n \tag{3.4.7}$$

(the canonical bilinear form being (3.4.5) with the superscript 'H' replaced by 'T'). Then formally, (3.4.6) extends to hold true, and the following theorem can be shown along the lines of Theorem 5.1 in [48].

Theorem 3.4.8. *Considered as an operator on* $\left(\ell_0(\mathbb{Z}^d)\right)^n$, T_P *has only finitely many nonzero eigenvalues. In particular,* σ *is an eigenvalue of* T_P *if and only if* $\overline{\sigma}$ *is an eigenvalue of* S_P, *the latter being considered as an operator on* $\left(\ell(\mathbb{Z}^d)\right)^n$.

It is possible to give a characterization of approximation power of $S(\Phi)$ in terms of properties of invariance of the subdivision and the transfer operator. A linear subspace $Y \subset \left(\ell_0(\mathbb{Z}^d)\right)^n$ is called T_P-*invariant*, if

$$\mathbf{d} \in Y \quad \Longrightarrow \quad T_P\mathbf{d} \in Y \; ,$$

and S_P-invariant subspaces $Z \subset \left(\ell(\mathbb{Z}^d)\right)^n$ are defined analogously.

3.4.9. The PSI case. For given $m \in \mathbb{N}$, we put

$$V_m := \{d \in \ell_0(\mathbb{Z}^d) \; ; \; D^\mu d^\wedge(0) = 0 \text{ for all } \mu \in \mathbb{Z}_+^d \text{ with } |\mu| < m\}. \tag{3.4.10}$$

Since $d^\wedge$ is a trigonometric polynomial, the derivatives at the origin are well defined. These zero conditions at the origin are moment conditions for the sequence d, indeed; since $d^\wedge(\xi) = \sum_{\alpha\in\mathbb{Z}^d} d_\alpha e^{i\alpha\cdot\xi}$, we have

$$(-iD)^\mu d^\wedge(0) = 0 \quad \Longleftrightarrow \quad \sum_{\alpha\in\mathbb{Z}^d} d_\alpha \, \alpha^\mu = 0 \; ,$$

whence

Lemma 3.4.11. *The following are equivalent, for* $m \in \mathbb{N}$ *and* $d \in \ell_0(\mathbb{Z}^d)$:

(*i*) $d \in V_m$.

(ii) $\sum_{\alpha \in \mathbb{Z}^d} d_\alpha \, q(\alpha) = 0$ *for any algebraic polynomial* q *of degree less than* m.

We are now ready to present the announced invariance properties which imply an approximation order result, in view of Theorem 3.2.1. Here, (with a slight misuse of notation) we identify the polynomial space P_{m-1} (*i.e.*, polynomials of degree less than m) with the sequence space of polynomial sequences of order m,

$$\{(q(\alpha))_{\alpha \in \mathbb{Z}^d} \; ; \; q \in P_{m-1}\} \; \subset \; \ell(\mathbb{Z}^d).$$

Theorem 3.4.12. *Let* $p \in l_0(\mathbb{Z}^d)$ *be a finitely supported mask with corresponding mask symbol* H. *Then the following assertions are equivalent for* $m \in \mathbb{N}$:

(i) $V_m \subset \ell_0(\mathbb{Z}^d)$ *is invariant under the transfer operator* T_p.

(ii) P_{m-1} *is invariant under the subdivision operator* S_p.

(iii) H *satisfies the zero condition (3.2.2) of order* m.

(iv) p *satisfies the sum rules (3.3.5) of order* m.

Here, the equivalence of (i) and (iii) is obvious from the Fourier transform expression of the transfer operator,

$$(T_p d)^\wedge(\xi) = \sum_{e' \in E'} H(M^{-T}(\xi + 2\pi e')) \; d^\wedge(M^{-T}(\xi + 2\pi e')) \; ,$$

and the equivalence of (iii) and (iv) was established in Theorem 3.3.4. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) can be taken from [48, Theorem 5.2].

3.4.13. The FSI case. Theorem 3.4.12 is a special case of the statements to follow. For a given row vector

$$\mathbf{v}(\xi) = \sum_{\alpha \in \mathbb{Z}^d} \mathbf{v}_\alpha e^{-i\alpha \cdot \xi} \tag{3.4.14}$$

of trigonometric polynomials we extend the notion in (3.4.10) to

$$V_m(\mathbf{v}) := \{\mathbf{d} \in \left(\ell_0(\mathbb{Z}^d)\right)^n ; \; D^\mu \{\mathbf{v}\, \mathbf{d}^\wedge\}(0) = 0 \text{ for all } \mu \in \mathbb{Z}^d_+ \text{ with } |\mu| < m\}. \tag{3.4.15}$$

As before in the PSI case, the zero conditions for $\mathbf{v}\, \mathbf{d}^\wedge$ at the origin are moment conditions for the 'convolved' series $\mathbf{v} * \mathbf{d}^\smile \in \ell_0(\mathbb{Z}^d)$ given by

$$(\mathbf{v} * \mathbf{d}^\smile)_\alpha := \sum_{\beta \in \mathbb{Z}^d} \mathbf{v}_{\alpha-\beta}\, \mathbf{d}^\smile{}_\beta = \sum_{\beta \in \mathbb{Z}^d} \mathbf{v}_{\alpha+\beta}\, \mathbf{d}_\beta \; , \quad \alpha \in \mathbb{Z}^d;$$

note that $\mathbf{d}^\wedge(\xi) = \sum_{\alpha \in \mathbb{Z}^d} \mathbf{d}_\alpha e^{+i\alpha \cdot \xi}$ is a column vector of trigonometric polynomials. Therefore, Lemma 3.4.11 gives

Lemma 3.4.16. *Given the row vector* $\mathbf{v}$ *in (3.4.14), the following are equivalent, for* $m \in \mathbb{N}$ *and* $\mathbf{d} \in \left(\ell_0(\mathbb{Z}^d)\right)^n$*:*

(i) $\mathbf{d} \in V_m(\mathbf{v})$.

(ii) $\mathbf{v} * \mathbf{d}^{\check{}} \in V_m$.

(iii) $\sum_{\gamma\in\mathbb{Z}^d} \Big(\sum_{\beta\in\mathbb{Z}^d} \mathbf{v}_{\gamma+\beta}\, \mathbf{d}_\beta \Big)\, q(\gamma) = 0$ *for any algebraic polynomial* q *of degree less than* m.

Theorem 3.4.17. *Let* $\mathbf{P}$ *be a finitely supported* $(n \times n)$*-matrix mask with mask symbol* $\mathbf{H}$*, and let* $\mathbf{v}$ *be a row vector (3.4.14) of trigonometric polynomials such that the spectral conditions (3.2.12) of order* $m \in \mathbb{N}$ *are satisfied. Then the following assertions are equivalent:*

(i) $V_m(\mathbf{v})$ *is invariant under the transfer operator* T_P*.*

(ii) $\mathbf{H}$ *satisfies the condition* (Z_m) *with* $\mathbf{v}$*, given in 3.2.7.*

Proof: For $\mathbf{d} \in V_m(\mathbf{v})$ we have, using (3.2.12),

$$\begin{aligned}
&\sum_{\nu\le\mu} \binom{\mu}{\nu} D^\nu\{\mathbf{v}\ \mathbf{H}(M^{-T}\cdot)\}(0)\ D^{\mu-\nu}\{\mathbf{d}^\wedge(M^{-T}\cdot)\}(0)\\
&= \sum_{\nu\le\mu} \binom{\mu}{\nu}\ D^\nu\{\mathbf{v}(M^{-T}\cdot)\}(0)\, D^{\mu-\nu}\{\mathbf{d}^\wedge(M^{-T}\cdot)\}(0)\\
&= D^\mu\{(\mathbf{v}\,\mathbf{d}^\wedge)(M^{-T}\cdot)\}(0) = 0\ , \quad |\mu| < m\ .
\end{aligned}$$

On the other hand, by (3.4.4), for any $\mathbf{d} \in \left(\ell_0(\mathbb{Z}^d)\right)^n$,

$$\begin{aligned}
&D^\mu\{\mathbf{v}\,(T_P\mathbf{d})^\wedge\}(0) =\\
&\sum_{e'\in E'} \sum_{\nu\le\mu} \binom{\mu}{\nu} D^\nu\{\mathbf{vH}(M^{-T}(\cdot+2\pi e'))\}(0)\ D^{\mu-\nu}\{\mathbf{d}^\wedge(M^{-T}(\cdot+2\pi e'))\}(0).
\end{aligned}$$

Therefore, since $T_P\mathbf{d}$ is finitely supported whenever $\mathbf{d}$ is, statement (i) is equivalent to the fact that

$$\begin{aligned}
&\sum_{e'\in E'_0} \sum_{\nu\le\mu} \binom{\mu}{\nu} D^\nu\{\mathbf{vH}(M^{-T}(\cdot+2\pi e'))\}(0)\ D^{\mu-\nu}\{\mathbf{d}^\wedge(M^{-T}(\cdot+2\pi e'))\}(0)\\
&= 0 \quad \text{for any } \mathbf{d} \in V_m(\mathbf{v}) \text{ and } |\mu| < m\ .
\end{aligned}$$

It is not too hard to see that this is equivalent to condition (Z_m). ■

As a corollary of this theorem we have

Theorem 3.4.18. *In the stable FSI case, any of the statements in Theorem 3.4.17 can be replaced by the sum rules of Theorem 3.3.6. Here, the adequate form of the spectral conditions (3.2.12) is*

$$|\det M| \sum_{\sigma\in\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d} \mathbf{v}_{\gamma-\sigma}\, \mathbf{P}_{M\sigma}\, q(M\gamma) = \sum_{\alpha\in\mathbb{Z}^d} \mathbf{v}_\alpha\, q(\alpha) \tag{3.4.19}$$

for any polynomial of degree less than m, with $\mathbf{v}(\xi) = \sum_{\alpha\in\mathbb{Z}^d} \mathbf{v}_\alpha e^{-i\alpha\cdot\xi}$.

In order to relate the conditions in Theorem 3.4.17 to an invariance property of the subdivision operator we let

$$\Pi(\mathbb{Z}^d) \;\subset\; \ell(\mathbb{Z}^d)$$

denote the space of all sequences which increase at most polynomially at infinity. In addition, we put

$$W_m(\mathbf{v}) := \{\mathbf{c} \in \left(\Pi(\mathbb{Z}^d)\right)^n;\; <\mathbf{c} \mid \mathbf{d}>= 0 \text{ for all } \mathbf{d} \in V_m(\mathbf{v})\}. \tag{3.4.20}$$

Theorem 3.4.21. *In the stable FSI case, any of the statements in Theorem 3.4.17 can be replaced by any of the following conditions:*

(i) $W_m(\mathbf{v})$ *is invariant under the subdivision operator* S_P.

(ii) $\mathbf{P}$ *satisfies the sum rules (3.3.7) with* $\mathbf{v}$.

Proof: *(ii)* $\Longrightarrow$ *(i)*: Using Theorem 3.3.6 and Theorem 3.4.17, we see that *(ii)* implies the invariance property of T_P. Therefore, $V_m(\mathbf{v})$ is also invariant under the operator S_P^*, by (3.4.6). Also, due to the finite support of the mask $\mathbf{P}$ we have

$$\mathbf{c} \in \left(\Pi(\mathbb{Z}^d)\right)^n \quad\Longrightarrow\quad S_P\mathbf{c} \in \left(\Pi(\mathbb{Z}^d)\right)^n .$$

Therefore, given $\mathbf{c} \in W_m(\mathbf{v})$ we find

$$< S_P\mathbf{c} \mid \mathbf{d} > \;=\; < \mathbf{c} \mid S_P^*\mathbf{d} > \;=\; 0 \,,$$

for any $\mathbf{d} \in V_m(\mathbf{v})$, whence $S_P\mathbf{c} \in W_m(\mathbf{v})$.

(ii) $\Longleftarrow$ *(i)*: As we have seen in Lemma 3.4.16, $\mathbf{d} \in V_m(\mathbf{v})$ if and only if

$$\sum_{\alpha\in\mathbb{Z}^d} \sum_{\delta\in\mathbb{Z}^d} \mathbf{v}_\delta\, q(\delta-\alpha)\, \mathbf{d}_\alpha = 0$$

for all algebraic polynomials q of degree less than m. Putting here

$$q(\delta-\alpha) = \sum_{|\mu|<m} r_\mu(\delta)(-\alpha)^\mu \quad \text{with} \quad r_\mu := \frac{1}{\mu!} D^\mu q$$

the spectral conditions (3.4.19) yield

$$0 = \sum_{|\mu|<m} \sum_{\alpha\in\mathbb{Z}^d} \left(\sum_{\delta\in\mathbb{Z}^d} \mathbf{v}_\delta \, r_\mu(\delta) \right) (-\alpha)^\mu \, \mathbf{d}_\alpha$$
$$= |\det M| \sum_{|\mu|<m} \left(\sum_{\beta\in\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d} \mathbf{v}_{\gamma-\beta} \, \mathbf{P}_{M\beta} \, r_\mu(M\gamma) \right) \sum_{\alpha\in\mathbb{Z}^d} (-\alpha)^\mu \, \mathbf{d}_\alpha . \tag{3.4.22}$$

Now, let q be any algebraic polynomial of degree less than m. Then the vector sequence $\mathbf{c} = \mathbf{c}(q)$ given by

$$\mathbf{c}_\alpha := \sum_{\beta\in\mathbb{Z}^d} \mathbf{v}^H_{\alpha+\beta} \, \overline{q(M\beta)} \, , \quad \alpha \in \mathbb{Z}^d,$$

is an element of $\left(\Pi(\mathbb{Z}^d)\right)^n$ (since $(\mathbf{v}_\alpha)_{\alpha\in\mathbb{Z}^d}$ is compactly supported), and hence is contained in $W_m(\mathbf{v})$, since for all $\mathbf{d} \in V_m(\mathbf{v})$, using Lemma 3.4.16 again,

$$< \mathbf{c} \,|\, \mathbf{d} > = \sum_{\alpha\in\mathbb{Z}^d} \left(\sum_{\beta\in\mathbb{Z}^d} \mathbf{v}^H_{\alpha+\beta} \, \overline{q(M\beta)} \right)^H \mathbf{d}_\alpha$$
$$= \sum_{\beta\in\mathbb{Z}^d} q(M\beta) \sum_{\alpha\in\mathbb{Z}^d} \mathbf{v}_{\alpha+\beta} \, \mathbf{d}_\alpha = 0 \ .$$

¿From our assumption *(i)* we conclude that $S_P\mathbf{c} \in W_m(\mathbf{v})$ as well, *i.e.*, for any $\mathbf{d} \in V_m(\mathbf{v})$:

$$0 =< S_P\mathbf{c} \mid \mathbf{d} >= \sum_{\alpha\in\mathbb{Z}^d} (S_P\mathbf{c})^H_\alpha \, \mathbf{d}_\alpha$$
$$= \sum_{\alpha\in\mathbb{Z}^d} \left(\sum_{\beta\in\mathbb{Z}^d} \mathbf{P}^H_{\alpha-M\beta} \, \mathbf{c}_\beta \right)^H \mathbf{d}_\alpha$$
$$= \sum_{\alpha\in\mathbb{Z}^d} \left(\sum_{\beta\in\mathbb{Z}^d} \mathbf{P}^H_{\alpha-M\beta} \left(\sum_{\gamma\in\mathbb{Z}^d} \mathbf{v}^H_{\beta+\gamma} \, \overline{q(M\gamma)} \right) \right)^H \mathbf{d}_\alpha$$
$$= \sum_{\alpha\in\mathbb{Z}^d} \left(\sum_{\beta\in\mathbb{Z}^d} \sum_{\gamma\in\mathbb{Z}^d} \mathbf{v}_{\beta+\gamma} \, \mathbf{P}_{\alpha-M\beta} \, q(M\gamma) \right) \mathbf{d}_\alpha .$$

Equivalently, putting

$$q(M\gamma) = q(M\gamma + \alpha - \alpha) = \sum_{|\mu|<m} \frac{1}{\mu!} (D^\mu q)(M\gamma + \alpha) \, (-\alpha)^\mu$$

we have

$$0 = \sum_{|\mu|<m} \sum_{\alpha \in \mathbb{Z}^d} \left(\sum_{\beta \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}^d} \mathbf{v}_{\gamma-\beta}\, \mathbf{P}_{\alpha+M\beta}\, (D^\mu q)(\alpha + M\gamma) \right) (-\alpha)^\mu\, \mathbf{d}_\alpha$$

for all $\mathbf{d} \in V_m(\mathbf{v})$. Comparing this with (3.4.22) shows that the expression within the brackets must be independent of α, for any polynomial q of degree less than m. Whence the sum rules (3.3.7) of order m hold true, as we wanted to show. ■

3.5. Notes and extensions.

3.5.1. A stable, compactly supported M-refinable function or function vector generates a multiresolution analysis for $L_2(\mathbb{R}^d)$, hence allows for a (now) standard construction of a wavelet basis.

The dilation matrix M in the matrix refinement equation (3.1.1) is often chosen as $M = 2I$, with I the $d \times d$ unit matrix. For this special case, refinable functions can be gained from tensor products of refinable univariate functions φ_i, say, as

$$\phi(\mathbf{t}) = \prod_{i=1}^{d} \varphi_i(t_i) , \quad \mathbf{t} = (t_1, \ldots, t_d)^T \in \mathbb{R}^d ,$$

and all results on univariate refinable functions can be simply transferred to the multivariate situation. The corresponding wavelet basis then requires $2^d - 1$ different generating wavelets.

In general, dilation matrices M with smallest possible determinant are of special interest, since here the construction of a corresponding wavelet basis refers to $|\det M| - 1$ wavelets (or multiwavelets). If M is not a diagonal matrix, these wavelets are called *non-separable*, see [23]. An important instant of this is the bivariate construction based on

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

For example, the so-called *Zwart-Powell element* is refinable with respect to the first matrix; see [10, Chapter VII] for more information on approximation power and subdivision of so-called *box splines*.

3.5.2. Considering the refinement equation (3.1.2) in the Fourier transform domain, it follows that

(i) $\mathbf{\Phi}^\wedge(0)$ is either a right eigenvector of $\mathbf{H}(0)$ to the eigenvalue 1 or

(ii) $\mathbf{\Phi}^\wedge(0)$ is the zero vector.

The latter case is not really of interest since then $\mathbf{\Phi}$ can be considered as a derivative of an M-refinable function vector $\tilde{\mathbf{\Phi}}$ with $\tilde{\mathbf{\Phi}}^\wedge \neq \mathbf{0}$. Thus, the assertion that $\mathbf{H}(0)$ has an eigenvalue 1 is the fundamental condition, and the spectral condition on $\mathbf{H}(0)$ in Section 3.1.14 just ensures that $\mathbf{v}(0)$ and $\mathbf{\Phi}(0)$ are non-orthogonal left and right eigenvectors of $\mathbf{H}(0)$ to this eigenvalue. The condition of these eigenvectors being non-orthogonal is clearly vacuous if 1 is a simple eigenvalue.

Moreover, if $\Phi \subset L_2(\mathbb{R}^d)$ generates a stable, M-refinable FSI-space, then the spectral radius of $\mathbf{H}(0)$ necessarily equals 1, with 1 being a simple eigenvalue and the only eigenvalue of absolute value 1, see e.g. [26].

3.5.3. For a given refinement mask the refinement equation (3.1.1) can be interpreted as a functional equation for $\mathbf{\Phi}$. In the Fourier transform domain, the solution vector can be formally written as

$$\mathbf{\Phi}^\wedge(\xi) = \lim_{L\to\infty} \prod_{j=1}^{L} \mathbf{H}((M^{-T})^j\,\xi)\,\mathbf{r},$$

where $\mathbf{r}$ is a right eigenvector of $\mathbf{H}(0)$ to the eigenvalue 1. In particular, it follows that $\mathbf{\Phi}^\wedge(0) = \mathbf{r}$.

The convergence of the infinite product (in the sense of uniform convergence on compact sets) is ensured if the spectral radius of $\mathbf{H}(0)$ is 1, and if there are no further eigenvalues of $\mathbf{H}(0)$ on the unit circle. In this case a non-degenerate eigenvalue 1 defines a solution vector $\mathbf{\Phi}^\wedge$. If the eigenvalue 1 is simple (see Remark 3.5.2.), then this solution vector $\mathbf{\Phi}$ is unique.

3.5.4. This Section 3 often refers to work of Jia on approximation properties of multivariate wavelets. In his paper [48] the PSI-case was completely settled, for general dilation matrices; see also the following remarks. The proof of Theorem 3.1.10 is a trivial extension of the proof of [54, Theorem 2.4].

3.5.5. In the univariate FSI case, $d = 1$, the zero condition (Z_m) and their consequences for the approximation power of S_Φ have been considered by [36] and [76, 77] in the Fourier transformed domain, while [70] has given conditions in time domain.

In particular, in [77] it is shown that approximation power induces a matrix factorization of the symbol $\mathbf{H}$; see Remark 3.2.12. Later on Micchelli and Sauer [74] observed an analogous factorization property for the representing matrix of the subdivision operator. Unfortunately, for $d > 1$ the zero conditions on the mask symbols (3.2.2) and (3.2.6) do not lead to a factorization of the symbols a priori.

The multivariate FSI case with arbitrary dilation matrices is e.g. treated in [15]; the observed conditions relate to the sum rules of Theorem 3.3.6.

3.5.6. The generalized discrete Fourier transform matrix $\mathbf{F}_M$ and its property in (3.3.1) of being unitary are well-known, see e.g. [21]. These properties have been used by Jia [48] again to derive the sum rules as in Theorem 3.3.4.

3.5.7. While the notion of subdivision operator S_P (for general dilation matrices) has been coined by Cavaretta, Dahmen and Micchelli [16], the set-up for the transfer or transition operator is often changed in the literature. We prefer here to say that T_P is (essentially, *i.e.*, modulo reflection) the adjoint of S_P. In this way, Theorem 3.4.12 dealing with the PSI case is identical with [48, Theorem 5.2]. As far as the FSI case is considered, however, our results are new.

It should be noted that for the PSI case and $M = 2I$, the invariance of P_{m-1} under the subdivision operator S_p is equivalent to the property that polynomials of order less than m can be reproduced from multi-integer translates of ϕ. A result along these lines is already contained in [16].

3.5.8. The *symmetrized form of the transfer operator* (in the Fourier transform domain) is given by $\tilde{T}_P{}^\wedge$ operating on $(n \times n)$-matrices $\mathbf{C}$ of trigonometric polynomials as follows,

$$(\tilde{T}_P{}^\wedge \mathbf{C})(\xi) := \\ \sum_{e' \in E'} \mathbf{H}(M^{-T}(\xi + 2\pi e'))\, \mathbf{C}(M^{-T}(\xi + 2\pi e'))\, \mathbf{H}(M^{-T}(\xi + 2\pi e'))^H .$$

For $M = 2I$, Shen [94, Theorem 3.8] has shown that the stability of S_Φ is equivalent to the following condition: *The operator $\tilde{T}^\wedge$ has spectral radius 1, with 1 being a simple eigenvalue and all other eigenvalues lying strictly inside the unit circle; moreover, the eigenmatrix of $\tilde{T}_P{}^\wedge$ corresponding to the eigenvalue 1 is nonsingular on the d-dimensional torus.* We conjecture that this equivalence is also true for arbitrary dilation matrices.

3.5.9. Considering the dilation matrix $M = 2I$, the connection between properties of the subdivision operator S_P and approximation power m provided by the FSI-space S_Φ can be simply given as follows: *The stable FSI space S_Φ has approximation power m, for $f \in W_2^m(\mathbb{R}^d)$, if and only if there exists a nontrivial vector $\mathbf{q}$ of polynomial sequences $q_1, \ldots, q_n \in P_{m-1}$ such that*

$$S_P \mathbf{q} = 2^{-(m-1)}\, \mathbf{q}.$$

In particular, S_P necessarily has the eigenvalues 2^{-k} for $k = 0, \ldots, m-1$.

This result can even be generalized to distribution vectors $\mathbf{\Phi}$ which do not satisfy any conditions of linear independence ([55, Theorem 3.1]).

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