Stability Analysis of Nonlinear Systems with Linear Programming

A Lyapunov Functions Based Approach

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Introduction

The Lyapunov theory of dynamical systems is the most useful general theory for studying the stability of nonlinear systems. It includes two methods, Lyapunov's indirect method and Lyapunov's direct method. Lyapunov's indirect method states that the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, has a locally exponentially stable equilibrium point at the origin, if and only if the real parts of the eigenvalues of the Jacobian matrix of $\mathbf{f}$ at zero are all strictly negative. Lyapunov's direct method is a mathematical extension of the fundamental physical observation, that an energy dissipative system must eventually settle down to an equilibrium point. It states that if there is an energy-like function V for (1) that is strictly decreasing along its trajectories, then the equilibrium at the origin is asymptotically stable. The function V is then said to be a Lyapunov function for the system. A Lyapunov function provides via its preimages a lower bound of the region of attraction of the equilibrium. This bound is non-conservative in the sense, that it extends to the boundary of the domain of the Lyapunov function.

Although these methods are very powerful they have major drawbacks. The indirect method delivers a proposition of purely local nature. In general one does not have any idea how large the region of attraction might be. It follows from the direct method, that one can extract important information regarding the stability of the equilibrium at the origin if one has a Lyapunov function for the system, but it does not provide any method to gain it. In this thesis we will tackle these drawbacks via linear programming. The advantage of using linear programming is that algorithms to solve linear programs, like the simplex algorithm used here, are fast in practice. A further advantage is that open source and commercial software to solve linear programs is readily available.

Part I contains mathematical preliminaries.

In **Chapter 1** a brief review of the theory of continuous autonomous dynamical systems and some stability concepts of their equilibrium points is given. We will explain why such systems are frequently encountered in science and engineering and why the concept of stability for their equilibrium points is so important. We will introduce Dini derivatives, a generalization of the classical derivative, and we will prove Lyapunov's direct method with less restrictive assumptions of the Lyapunov function than usually done in textbooks on the topic. Finally, we will introduce the converse theorems in the Lyapunov theory, the theorems that ensure the existence of Lyapunov functions.

Part II includes Linear Program **LP1** and Theorem I, the first main contribution of this thesis.

In **Chapter 2** we will derive a set of linear inequalities for the system (1), dependent on a neighborhood $\mathcal{N}$ of the origin and constants $\alpha > 0$ and $m \geq 1$. An algorithmic description of how to derive these linear inequalities is given in Linear Program **LP1**. Only the images under $\mathbf{f}$ of a discrete set and upper bounds of its partial derivatives up to the third order on a compact set are needed. Theorem I states that if a linear program generated by Linear Program **LP1** does not have a feasible solution, then the origin is not an α, m-exponentially stable equilibrium point of the respective system on $\mathcal{N}$. The linear inequalities are derived from restrictions, that a converse theorem on exponential stability

(Theorem 1.18) imposes on a Lyapunov function of the system, if the origin is an α, m-exponentially stable equilibrium point on $\mathcal{N}$. The neighborhood $\mathcal{N}$, and the constants α and m can be chosen at will.

In **Chapter 3** we will show how this can be used to improve Lyapunov's indirect method, by giving an upper bound of the region of attraction of the equilibrium.

Part III is devoted to the construction of piecewise affine Lyapunov and Lyapunov-like functions for (1) via linear programming. It includes Linear Program **LP2** and Theorem II, the second main contribution of this thesis.

In **Chapter 4** we will show how to partition $\mathbb{R}^n$ into arbitrary small simplices (Corollary 4.12) and then use this partition to define the function spaces CPWA of continuous piecewise affine functions (Definition 4.15). A CPWA space of functions with a compact domain can be parameterized by a finite number of real parameters. They are basically the spaces PWL[**D**] in [19] with more flexible boundary configurations.

In **Chapter 5** we will state Linear Program **LP2**, an algorithmic description of how to derive a linear program for (1). Linear Program **LP2** needs the images under $\mathbf{f}$ of a discrete set and upper bounds of its second order partial derivatives on compact sets. We will use the CPWA spaces and Lyapunov's direct method (Theorem 1.16) to prove, that any feasible solution of a linear program generated by Linear Program **LP2**, parameterizes a CPWA Lyapunov or a Lyapunov-like function for the system. The domain of the wanted Lyapunov or Lyapunov-like function can practically be chosen at will. If the origin is contained in the wanted domain and there is a feasible solution of the linear program, then a true Lyapunov function is the result. If a neighborhood $\mathcal{D}$ of the origin is left out of the wanted domain, then a Lyapunov-like function is parameterized by a feasible solution. This Lyapunov-like function ensures, that all trajectories of the system starting in some (large) subset of the domain are attracted to $\mathcal{D}$ by the dynamics of the system. These results are stated in Theorem II.

In **Chapter 6** we will evaluate the method and compare it to numerous approaches in the literature to construct Lyapunov or Lyapunov-like functions, in particular to the linear program proposed by Julian, Guivant, and Desagesin in [19].

Part IV is the last part of this thesis.

In **Chapter 7** we will shortly discuss the numerical complexity of the simplex algorithm, which was used to solve the linear programs generated by Linear Program **LP1** and Linear Program **LP2** in this thesis, and point to alternative algorithms. We will give examples of CPWA Lyapunov functions generated trough feasible solutions of linear programs generated by Linear Program **LP2** and an example of the use of Linear Program **LP1** to refute the α, m-exponential stability of an equilibrium in several regions.

In **Chapter 8**, the final chapter of this thesis, we give some concluding remarks and ideas for future research.

Symbols

$\mathbb{R}$	the real numbers
$\mathbb{R}_{\geq 0}$	the real numbers larger than or equal to zero
$\mathbb{R}_{>0}$	the real numbers larger than zero
$\mathbb{Z}$	the integers
$\mathbb{Z}_{\geq 0}$	the integers larger than or equal to zero
$\mathbb{Z}_{>0}$	the integers larger than zero
$\mathcal{A}^n$	set of n-tuples of elements belonging to a set $\mathcal{A}$
$\overline{\mathcal{A}}$	the closure of a set $\mathcal{A}$
$\overline{\mathbb{R}}$	$:= \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$
$\partial\mathcal{A}$	the boundary of a set $\mathcal{A}$
$\mathrm{dom}(f)$	the domain of a function f
$f(\mathcal{U})$	the image of a set $\mathcal{U}$ under a mapping f
$f^{-1}(\mathcal{U})$	the preimage of a set $\mathcal{U}$ with respect to a mapping f
$\mathcal{C}(\mathcal{U})$	continuous real valued functions with domain $\mathcal{U}$
$\mathcal{C}^k(\mathcal{U})$	k-times continuously differentiable real valued functions with domain $\mathcal{U}$
$[\mathcal{C}^k(\mathcal{U})]^n$	vector fields $\mathbf{f} = (f_1, f_2, .., f_n)^T$ of which $f_i \in \mathcal{C}^k(\mathcal{U})$ for $i = 1, 2, ., n$
$\mathcal{K}$	strictly increasing functions on $[0, +\infty[$ vanishing at the origin
$\mathfrak{P}(\mathcal{A})$	the power set of a set $\mathcal{A}$
Sym_n	the permutation group of a set $\mathcal{A}$
$\mathrm{con}\,\mathcal{A}$	the convex hull of a set $\mathcal{A}$
$\mathrm{graph}(f)$	the graph of a function f
$\mathbf{e}_i$	the i-th unit vector
$\mathbf{x} \cdot \mathbf{y}$	the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$
$\mathbf{x}^T$	the transpose of a vector $\mathbf{x}$
A^T	the transpose of a matrix A
$\lVert \cdot \rVert_p$	p-norm
$\lVert A \rVert_2$	the spectral norm of a matrix A
$\mathrm{rank}\,A$	the rank of a matrix A
f'	the first derivative of a function f
∇f	the gradient of a scalar field f
$\nabla \mathbf{f}$	the Jacobian matrix of a vector field $\mathbf{f}$
$\chi_{\mathcal{A}}$	the characteristic function of a set $\mathcal{A}$
δ_{ij}	the Kronecker delta, equal to 1 if $i = j$ and equal to 0 if $i \neq j$

Part I

Preliminaries

Chapter 1

Mathematical Background

In this thesis we will consider continuous autonomous dynamical systems. A continuous autonomous dynamical system is a system, of which the dynamics can be modeled by an ordinary differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

This equation is called the *state equation* of the dynamical system. We refer to $\mathbf{x}$ as the *state* of the system and to the domain of the function $\mathbf{f}$ as the *state-space* of the system.

In this chapter we will state a few important theorems regarding continuous autonomous dynamical systems and their solutions. We will introduce some useful notations and the stability concepts for equilibrium points used in this thesis. We will see why one frequently encounters continuous autonomous dynamical systems in control theory and why their stability is of interest. We will introduce Dini derivatives and use them to prove a more general version of the direct method of Lyapunov than usually done in textbooks on the subject. Finally, we will state and prove a converse theorem on exponential stability.

1.1 Continuous Autonomous Dynamical Systems

In order to define the solution of a continuous autonomous dynamical system, we first need to define what we mean with a solution of initial value problems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\xi},$$

and we have to assure, that a unique solution exists for any $\boldsymbol{\xi}$ in the state-space. In order to define a solution of such an initial value problem, it is advantageous to assume that the domain of $\mathbf{f}$ is a *domain* in $\mathbb{R}^n$, i.e. an open and connected subset. The set $\mathcal{U} \subset \mathbb{R}^n$ is said to be *connected* if and only if for every points $\mathbf{a}, \mathbf{b} \in \mathcal{U}$ there is a continuous mapping $\boldsymbol{\gamma} : [0,1] \longrightarrow \mathcal{U}$, such that $\boldsymbol{\gamma}(0) = \mathbf{a}$ and $\boldsymbol{\gamma}(1) = \mathbf{b}$. By a solution of an initial value problem we exactly mean:

Definition 1.1 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain, $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$ be a function, and $\boldsymbol{\xi} \in \mathcal{U}$. We call $\mathbf{y} :]a,b[\longrightarrow \mathbb{R}^n$, $a, b \in \overline{\mathbb{R}}$, $a < t_0 < b$, a solution of the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\xi},$$

if and only if $\mathbf{y}(t_0) = \boldsymbol{\xi}$, $\operatorname{graph}(\mathbf{y}) \subset \mathcal{U}$, $\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t))$ for all $t \in]a,b[$, and neither $\overline{\operatorname{graph}(\mathbf{y}|_{[t_0,b[})}$ nor $\overline{\operatorname{graph}(\mathbf{y}|_{]a,t_0]})}$ is a compact subset of $\mathcal{U}$.

□

One possibility to secure the existence and uniqueness of a solution for any initial state $\boldsymbol{\xi}$ in the state-space of a system, is given by the Lipschitz condition. The function $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$, where $\mathcal{U} \subset \mathbb{R}^m$, is said to be *Lipschitz* on $\mathcal{U}$, with a *Lipschitz constant* $L > 0$, if and only if the *Lipschitz condition*

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\|_2 \leq L\|\mathbf{x} - \mathbf{y}\|_2$$

holds true for all $\mathbf{x}, \mathbf{y} \in \mathcal{U}$. The function $\mathbf{f}$ is said to be *locally Lipschitz* on $\mathcal{U}$, if and only if its restriction $\mathbf{f}|_{\mathcal{C}}$ on any compact subset $\mathcal{C} \subset \mathcal{U}$ is Lipschitz on $\mathcal{C}$. The next theorem states the most important results in the theory of ordinary differential equations. It gives sufficient conditions for the existence and the uniqueness of solutions of initial value problems.

Theorem 1.2 (Peano / Picard-Lindelöf) *Let $\mathcal{U} \subset \mathbb{R}^m$ be a domain, $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$ be a continuous function, and $\boldsymbol{\xi} \in \mathcal{U}$. Then there is a solution of the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\xi}.$$

If $\mathbf{f}$ is locally Lipschitz on $\mathcal{U}$, then there are no further solutions.

PROOF:
See, for example, Theorems VI and IX in §10 in [52].

■

In this thesis we will only consider dynamical systems, of which the dynamics are modeled by an ordinary differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$ is a locally Lipschitz function from a domain $\mathcal{U} \subset \mathbb{R}^n$ into $\mathbb{R}^n$. The last theorem allows us to define the solution of the state equation of such a dynamical system.

Definition 1.3 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain and $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$ be locally Lipschitz on $\mathcal{U}$. For every $\boldsymbol{\xi} \in \mathcal{U}$ let $\mathbf{y}_{\boldsymbol{\xi}}$ be the solution of*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \boldsymbol{\xi}.$$

Let the function

$$\boldsymbol{\phi} : \{(t, \boldsymbol{\xi}) \big| \boldsymbol{\xi} \in \mathcal{U} \text{ and } t \in \mathrm{dom}(\mathbf{y}_{\boldsymbol{\xi}})\} \longrightarrow \mathbb{R}^n$$

be given by $\boldsymbol{\phi}(t, \boldsymbol{\xi}) := \mathbf{y}_{\boldsymbol{\xi}}(t)$ for all $\boldsymbol{\xi} \in \mathcal{U}$ and all $t \in \mathrm{dom}(\mathbf{y}_{\boldsymbol{\xi}})$. The function $\boldsymbol{\phi}$ is called the solution of the state equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

□

It is a remarkable fact, that if $\mathbf{f}$ in (1.1) is a $[\mathcal{C}^m(\mathcal{U})]^n$ function for some $m \in \mathbb{Z}_{\geq 0}$, then its solution $\boldsymbol{\phi}$ and the time derivative $\dot{\boldsymbol{\phi}}$ of the solution are $[\mathcal{C}^m(\mathrm{dom}(\boldsymbol{\phi}))]^n$ functions. This follows, for example, from the corollary at the end of §13 in [52]. We need this fact later in Part II, so we state it as a theorem.

Theorem 1.4 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain, $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$ be locally Lipschitz on $\mathcal{U}$, and $\boldsymbol{\phi}$ be the solution of the state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Let $m \in \mathbb{Z}_{\geq 0}$ and assume that $\mathbf{f} \in [\mathcal{C}^m(\mathcal{U})]^n$, then $\boldsymbol{\phi}, \dot{\boldsymbol{\phi}} \in [\mathcal{C}^m(\mathrm{dom}(\boldsymbol{\phi}))]^n$.*

■

1.2 Equilibrium Points and Stability

The concepts equilibrium point and stability are motivated by the desire to keep a dynamical system in, or at least close to, some desirable state. The term *equilibrium* or *equilibrium point* of a dynamical system, is used for a state of the system that does not change in the course of time, i.e. if the system is in an equilibrium at time t_0, then it will stay there for all times $t \geq t_0$.

Definition 1.5 *A state* $\mathbf{y}$ *in the state-space of (1.1) is called an equilibrium or an equilibrium point of the system if and only if* $\mathbf{f}(\mathbf{y}) = \mathbf{0}$.

□

If $\mathbf{y}$ is an equilibrium point of (1.1), then obviously the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{y}$$

has the solution $\mathbf{x}(t) = \mathbf{y}$ for all t. The solution with $\mathbf{y}$ as an initial value is thus a constant vector and the state does not change in the course of time. By change of variables one can always reach that $\mathbf{y} = \mathbf{0}$ without affecting the dynamics. Hence, there is no loss of generality in assuming that an equilibrium point is at the origin.

A real system is always subject to some fluctuations in the state. There are some external effects that are unpredictable and cannot be modeled, some dynamics that have very little impact on the behavior of the system are neglected in the modeling, etc. Even if the mathematical model of a physical system would be perfect, which is impossible, the system state would still be subject to quantum mechanical fluctuations. The concept of stability in the theory of dynamical systems is motivated by the desire, that the system state stays at least close to an equilibrium point after small fluctuations in the state.

Definition 1.6 *Assume that* $\mathbf{y} = \mathbf{0}$ *is an equilibrium point of (1.1) and let* $\|\cdot\|$ *be an arbitrary norm on* $\mathbb{R}^n$. *The equilibrium point* $\mathbf{y}$ *is said to be stable, if and only if for every* $R > 0$ *there is an* $r > 0$, *such that*

$$\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| \leq R \ \ \textit{for all}\ \|\boldsymbol{\xi}\| \leq r \ \ \textit{and all}\ t \geq 0,$$

where $\boldsymbol{\phi}$ *is the solution of the system.*

□

If the equilibrium $\mathbf{y} = \mathbf{0}$ is not stable in this sense, then there is an $R > 0$ such that any fluctuation in the state from zero, no matter how small, can lead to a state $\mathbf{x}$ with $\|\mathbf{x}\| > R$. Such an equilibrium is called *unstable*. The set of those points in the state-space of a dynamical system, that are attracted to an equilibrium point by the dynamics of the system, is of great importance. It is called the *region of attraction* of the equilibrium.

Definition 1.7 *Assume that* $\mathbf{y} = \mathbf{0}$ *is an equilibrium point of (1.1) and let* $\boldsymbol{\phi}$ *be the solution of the system. The set*

$$\{\boldsymbol{\xi} \in \mathcal{U} \mid \limsup_{t \to +\infty} \boldsymbol{\phi}(t, \boldsymbol{\xi}) = \mathbf{0}\}$$

is called the region of attraction of the equilibrium $\mathbf{y}$.

□

This concept of a stable equilibrium point is frequently too weak for practical problems. One often additionally wants the system state to return, at least asymptotically, to the equilibrium point after a small fluctuation in the state. This leads to the concept of an *asymptotically stable* equilibrium point.

Definition 1.8 *Assume that* $\mathbf{y} = \mathbf{0}$ *is a stable equilibrium point of (1.1). If its region of attraction is a neighborhood of* $\mathbf{y}$*, then the equilibrium point* $\mathbf{y}$ *is said to be asymptotically stable.*

□

Even asymptotic stability is often not strict enough for practical problems. This is mainly because it does not give any bounds of how fast the system must approach the equilibrium point. A much used stricter stability concept is exponential stability. The definition we use is:

Definition 1.9 *Assume that* $\mathbf{y} = \mathbf{0}$ *is an equilibrium point of (1.1), let* $\boldsymbol{\phi}$ *be the solution of the system, and let* $\mathcal{N} \subset \mathcal{U}$ *be a domain in* $\mathbb{R}^n$ *containing* $\mathbf{y}$*. We call the equilibrium* $\mathbf{y}$ $\alpha, m-$*exponentially stable on* $\mathcal{N}$*, where* $m \geq 1$ *and* $\alpha > 0$ *are real constant, if and only if the inequality*

$$\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_2 \leq me^{-\alpha t}\|\boldsymbol{\xi}\|_2$$

is satisfied for all $\boldsymbol{\xi} \in \mathcal{N}$ *and all* $t \geq 0$*. If there is some domain* $\mathcal{N} \subset \mathcal{U}$*, such that the equilibrium at zero is* $\alpha, m-$*exponentially stable on* $\mathcal{N}$*, then we call the equilibrium locally* $\alpha, m-$*exponentially stable.*

□

The interpretation of the constants is as follows. The constant m defies the system of exploding if its initial state $\boldsymbol{\xi}$ is in $\mathcal{N}$. Clearly the solution $\boldsymbol{\phi}$ fulfills the inequality $\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| \leq m\|\boldsymbol{\xi}\|$ for all $\boldsymbol{\xi} \in \mathcal{N}$ and all $t \geq 0$, if the system is $\alpha, m-$exponentially stable on $\mathcal{N}$. The constant α guarantees that the norm of the state reduces in a given time to a fraction of the norm of the initial state, dependent on m, α, and the fraction p. In fact, we have for any $p \in]0, 1]$ that

$$\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_2 \leq p\|\boldsymbol{\xi}\|_2 \quad \text{for all } t \geq \frac{1}{\alpha}\ln(\frac{m}{p}).$$

1.3 Control Systems

One important reason why one frequently encounters systems of the form (1.1) and is interested in the stability of their equilibrium points, is that it is the canonical form of a continuous control system with a closed loop or feedback control. This will be explained in more detail in this section.

Consider a continuous dynamical system, of which the dynamics can by modeled by a differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{1.2}$$

where $\mathbf{f}$ is a locally Lipschitz function from a domain $\mathcal{X} \times \mathcal{Y} \subset \mathbb{R}^n \times \mathbb{R}^l$ into $\mathbb{R}^n$ The function $\mathbf{u}$ in this equation models the possibility to interact with the system. It is called the *control function* of the system and the set $\mathcal{Y}$ is called the *set of control parameters.*

It might be desirable or even necessary to keep the system state close to some point. Why? The system could be a machine or an apparatus that simply gets destroyed if its state is too far from the operating point it was built for, it could be a chemical reactor of which the yields are highest for a certain state, or it might be a rocket of which a non-negligible deviation of the state of the dynamical system, from its preferred value, implies that it misses its target. In fact, there are a lot of systems encountered in engineering and science, which have a preferred value of their respective state. Without loss of generality let us assume that this preferred state is $\mathbf{y} = \mathbf{0}$.

The problem is to determine the function $\mathbf{u}$, such that $\mathbf{y} = \mathbf{0}$ is a stable equilibrium point of (1.2). If the function $\mathbf{u}$ is determined a priori as a function of time t one speaks of *open loop control*. Often, this is not a good idea because errors can accumulate over time and although the actions taken by the control function for a short amount of time are appropriate, this is not necessarily the case for the succeeding time. To overcome these shortcomings, one can take actions dependent on the current state of the system, i.e. $\mathbf{u} := \mathbf{u}(\mathbf{x})$. This is called *closed loop* or *feedback control*. A system of this form is called *feedback system* and the function $\mathbf{u}$ is then called the *feedback*. If one has a constructive method to determine the feedback, such that (1.2) has the stability properties one would like it to have, the problem is solved in a very satisfactory manner.

This is the case in linear control theory, which can be seen as a mature theory. In this theory $\mathbf{f}$ is assumed to be affine in $\mathbf{x}$ and in $\mathbf{u}$ and one searches for a linear feedback that makes the equilibrium at the origin exponentially stable. This means there is an $n \times n$-matrix A and an $n \times l$-matrix B, such that (1.2) has the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}(\mathbf{x}), \tag{1.3}$$

and one wants to determine an $l \times n$-matrix K, such that $\mathbf{u}(\mathbf{x}) = K\mathbf{x}$ makes the equilibrium point at the origin stable. Let $[A|B] = [B, AB, .., A^{n-1}B]$ be the matrix obtained by writing down the columns of $B, AB, .., A^{n-1}B$ consequently. The linear control theory states (see, for example, Theorems 1.2 and 2.9 in Part I of [55]):

> For every $\alpha > 0$ there is an $l \times n$-matrix K and a constant $m \geq 1$, such that the linear feedback $\mathbf{u}(\mathbf{x}) := K\mathbf{x}$ makes the origin a globally α, m-exponentially stable equilibrium of (1.3), if and only if $\operatorname{rank}[A|B] = n$.

If the function $\mathbf{f}$ is not affine, one can still try to linearize the system about equilibrium and use results from linear control theory to make the origin a locally exponentially stable equilibrium (see, for example, Theorem 1.7 in Part II of [55]). The large disadvantage of this method is, that its region of attraction is not necessarily very large and although there are some methods, see, for example, [32], to extend it a little, these methods are far from solving the general problem. The lack of a general analytical method to determine a successful feedback function for nonlinear systems has made less scientific approaches popular. Usually, one uses a non-exact design method to create a feedback $\mathbf{u}$ and then testes if the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))$$

has the wanted stability properties. This test is often done by some non-exact method, like simulation.

The probably most used design method for a feedback when the exact methods fail is fuzzy control design. It has its origin in the theory of fuzzy logic, an attempt to model human logic by allowing more truth values than only true and false. It has the advantage, that the design process is comparatively simple and fast, and that expert knowledge about the system behavior can easily be implemented in the controller. From the mathematical standpoint it has at least two major drawbacks. Firstly, the modeling of human logic is by no means canonical. The fuzzy logic theory gives an infinite family of possible interpretations for **and**, **or**, and **not**. Secondly, the successful real world controllers frequently have no real logical interpretation within the fuzzy logic theory. The action taken by the controller is simply the weighted average of some predefined actions. There are other methods to derive a crisp action of the controller, like (with the nomenclature of [9]) Center-of-Area, Center-of-Sums, Center-of-Largest-Area, First-of-Maxima, Middle-of-Maxima, and Height defuzzification, but none of them can be considered canonical.

For the applications in this thesis, only the mathematical structure of a feedback designed by fuzzy methods is of importance. We take the Sugeno-Takagi fuzzy controller as an important example. The Sugeno-Takagi controller was first introduced in [48]. Examples of its use are, for example, in [47], [45], [49], and [46]. Its mathematical structure is as follows:

> Let $a_1, a_2, .., a_n$ and $b_1, b_2, .., b_n$ be real numbers with $a_i < b_i$ for all $i = 1, 2, .., n$ and let the set $\mathcal{X} \subset \mathbb{R}^n$ be given by
>
> $$\mathcal{X} := [a_1, b_1] \times [a_2, b_2] \times .. \times [a_n, b_n].$$
>
> Let J be a positive integer and let
>
> $$R_{ij} : [a_i, b_i] \longrightarrow [0, 1], \quad i = 1, 2, .., n \text{ and } j = 1, 2, .., J$$
>
> be a family of continuous surjective functions. Let $\wedge_t : [0, 1]^2 \longrightarrow [0, 1]$ be a t-norm, i.e. a binary operator that fulfills the following properties for all $a, b, c \in [0, 1]$:
>
> | | | |
> |---|---|---|
> | *i)* | $a \wedge_t 1 = a$ | (unit element) |
> | *ii)* | $a \leq b \implies a \wedge_t c \leq b \wedge_t c$ | (monotonicity) |
> | *iii)* | $a \wedge_t b = b \wedge_t a$ | (commutativity) |
> | *iv)* | $(a \wedge_t b) \wedge_t c = a \wedge_t (b \wedge_t c)$ | (associativity) |
>
> Define the functions $R_j : \mathcal{X} \longrightarrow [0, 1]$ by
>
> $$R_j(\mathbf{x}) := R_{1j}(x_1) \wedge_t R_{2j}(x_2) \wedge_t .. \wedge_t R_{nj}(x_n) \quad \text{for } j = 1, 2, .., J$$
>
> and assume that
>
> $$\sum_{j=1}^{J} R_j(\mathbf{x}) \neq 0, \text{ for all } \mathbf{x} \in \mathcal{X}.$$
>
> Further, let $\mathbf{g}_j : \mathcal{X} \longrightarrow \mathbb{R}^l$, $j = 1, 2, .., J$, be affine functions. Then the feedback $\mathbf{u} : \mathcal{X} \longrightarrow \mathbb{R}^l$ has the algebraic form
>
> $$\mathbf{u}(\mathbf{x}) := \frac{\sum_{j=1}^{J} R_j(\mathbf{x})\mathbf{g}_j(\mathbf{x})}{\sum_{j=1}^{J} R_j(\mathbf{x})}.$$

The t-norm originally used in the Sugeno-Takagi controller was the minimum norm $a \wedge_t b := \min\{a, b\}$, but others suffice equally. In [45], [49], and [46] for example, Sugeno himself used $a \wedge_t b := ab$ as the t-norm.

Usually, the functions R_{ij} can be assumed to be piecewise infinitely differentiable, e.g. have a trapezoidal or a triangular form. If this is the case and the t-norm used is infinitely differentiable, e.g. the algebraic product, then the feedback $\mathbf{u}$ is infinitely differentiable on the set

$$\mathcal{X} \setminus \bigcup_{i=1}^{n} \bigcup_{j=1}^{J} \bigcup_{k=1}^{K_{ij}} \{\mathbf{x} \in \mathcal{X} \,|\, \mathbf{e}_i \cdot \mathbf{x} = y_{ijk}\},$$

where the $y_{ijk} \in]a_i, b_i[$, $k = 1, 2, .., K_{ij}$, are the points where R_{ij} is not infinitely differentiable, $i = 1, 2, .., n$ and $j = 1, 2, .., J$. One should keep this in mind when he reads the assumptions used by Linear Program **LP2** in this thesis. We are not going into fuzzy methods any further and refer to [9], [27], [4], [54], and [33] for a more detailed treatment of fuzzy control and fuzzy logic.

1.4 Dini Derivatives

The Italian mathematician Ulisse Dini introduced in 1878 in his textbook [8] on analysis the so-called Dini derivatives. They are a generalization of the classical derivative and inherit some important properties from it. Because the Dini derivatives are point-wise defined, they are more suited for our needs than some more modern approaches to generalize the concept of a derivative like Sobolev Spaces (see, for example, [1]) or distributions (see, for example, [53]). The Dini derivatives are defined as follows:

Definition 1.10 *Let $\mathcal{I} \subset \mathbb{R}$, $g : \mathcal{I} \longrightarrow \mathbb{R}$ be a function, and $y \in \mathcal{I}$.*

i) Let y be a limit point of $\mathcal{I} \cap]y, +\infty[$. Then the right-hand upper Dini derivative D^+ of g at the point y is given by

$$D^+g(y) := \limsup_{x \to y+} \frac{g(x) - g(y)}{x - y} := \lim_{\varepsilon \to 0+} \left(\sup_{\substack{x \in \mathcal{I} \cap]y, +\infty[\\ 0 < x - y \leq \varepsilon}} \frac{g(x) - g(y)}{x - y} \right)$$

and the right-hand lower Dini derivative D_+ of g at the point y is given by

$$D_+g(y) := \liminf_{x \to y+} \frac{g(x) - g(y)}{x - y} := \lim_{\varepsilon \to 0+} \left(\inf_{\substack{x \in \mathcal{I} \cap]y, +\infty[\\ 0 < x - y \leq \varepsilon}} \frac{g(x) - g(y)}{x - y} \right).$$

ii) Let y be a limit point of $\mathcal{I} \cap] - \infty, y[$. Then the left-hand upper Dini derivative D^- of g at the point y is given by

$$D^-g(y) := \limsup_{x \to y-} := \lim_{\varepsilon \to 0-} \left(\sup_{\substack{x \in \mathcal{I} \cap]-\infty, y[\\ \varepsilon \leq x - y < 0}} \frac{g(x) - g(y)}{x - y} \right)$$

and the left-hand lower Dini derivative D_- of g at the point y is given by

$$D_-g(y) := \liminf_{x \to y-} \frac{g(x) - g(y)}{x - y} := \lim_{\varepsilon \to 0-} \left(\inf_{\substack{x \in \mathcal{I} \cap]-\infty, y[\\ \varepsilon \leq x - y < 0}} \frac{g(x) - g(y)}{x - y} \right).$$

$\square$

The four Dini derivatives defined in the last definition are sometimes called the *derived numbers* of g at y, or more exactly the *right-hand upper derived number*, the *right-hand lower derived number*, the *left-hand upper derived number*, and the *left-hand lower derived number* respectively.

It is clear from elementary calculus, that if $g : \mathcal{I} \longrightarrow \mathbb{R}$ is a function from a non-empty open subset $\mathcal{I} \subset \mathbb{R}$ into $\mathbb{R}$ and $y \in \mathcal{I}$, then all four Dini derivatives $D^+g(y)$, $D_+g(y)$, $D^-g(y)$, and $D_-g(y)$ of g at the point y exist. This means that if $\mathcal{I}$ is a non-empty open interval, then the functions $D^+g, D_+g, D^-g, D_-g : \mathcal{I} \longrightarrow \overline{\mathbb{R}}$ defined in the canonical way, are all properly defined. It is not difficult to see that if this is the case, then the classical derivative $g' : \mathcal{I} \longrightarrow \mathbb{R}$ exists, if and only if the Dini derivatives are all real valued and $D^+g = D_+g = D^-g = D_-g$.

Using lim sup and lim inf instead of the usual limit in the definition of a derivative has the advantage, that they are always properly defined. The disadvantage is, that because of the elementary

$$\limsup_{x\to y+}(g(x) + h(x)) \leq \limsup_{x\to y+} g(x) + \limsup_{x\to y+} h(x),$$

a derivative defined in this way is not a linear operation anymore. However, when the right-hand limit of the function h exists, then

$$\limsup_{x\to y+}(g(x) + h(x)) = \limsup_{x\to y+} g(x) + \lim_{x\to y+} h(x).$$

This leads to the following lemma, which we will need later.

Lemma 1.11 *Let g and h be real valued functions, the domains of which are subsets of $\mathbb{R}$, and let $D^* \in \{D^+, D_+, D^-, D_-\}$ be a Dini derivative. Let $y \in \mathbb{R}$ be such, that the Dini derivative $D^*g(y)$ is properly defined and h is differentiable at y in the classical sense. Then*

$$D^*[g + h](y) = D^*g(y) + h'(y).$$

■

The reason why Dini derivatives are so useful for the applications in this thesis, is the following generalization of the mean value theorem of differential calculus and its corollary.

Theorem 1.12 *Let $\mathcal{I}$ be a non-empty interval in $\mathbb{R}$, $\mathcal{C}$ be a countable subset of $\mathcal{I}$, and $g : \mathcal{I} \longrightarrow \mathbb{R}$ be a continuous function. Let $D^* \in \{D^*, D_+, D^-, D_-\}$ be a Dini derivative and let $\mathcal{J}$ be an interval, such that $D^*f(x) \in \mathcal{J}$ for all $x \in \mathcal{I} \setminus \mathcal{C}$. Then*

$$\frac{g(x) - g(y)}{x - y} \in \mathcal{J}$$

for all $x, y \in \mathcal{I}$, $x \neq y$.

Proof:
See, for example, Theorem 12.24 in [51].

■

This theorem has an obvious corollary.

Corollary 1.13 *Let $\mathcal{I}$ be a non-empty interval in $\mathbb{R}$, $\mathcal{C}$ be a countable subset of $\mathcal{I}$, $g : \mathcal{I} \longrightarrow \mathbb{R}$ be a continuous function, and $D^* \in \{D^+, D_+, D^-, D_-\}$ be a Dini derivative. Then:*

$$
\begin{array}{lll}
D^*f(x) \geq 0 & \textit{for all } x \in \mathcal{I} \setminus \mathcal{C} & \textit{implies that } f \textit{ is increasing on } \mathcal{I}. \\
D^*f(x) > 0 & \textit{for all } x \in \mathcal{I} \setminus \mathcal{C} & \textit{implies that } f \textit{ is strictly increasing on } \mathcal{I}. \\
D^*f(x) \leq 0 & \textit{for all } x \in \mathcal{I} \setminus \mathcal{C} & \textit{implies that } f \textit{ is decreasing on } \mathcal{I}. \\
D^*f(x) < 0 & \textit{for all } x \in \mathcal{I} \setminus \mathcal{C} & \textit{implies that } f \textit{ is strictly decreasing on } \mathcal{I}.
\end{array}
$$

■

1.5 Direct Method of Lyapunov

The Russian mathematician and engineer Alexandr Mikhailovich Lyapunov published a revolutionary work in 1892 on the stability of motion, where he introduced two methods to study the stability of nonlinear dynamical systems. An English translation of this work can be found in [28].

In the first method, known as *Lyapunov's first method* or *Lyapunov's indirect method*, the local stability of an equilibrium of (1.1) is studied through the Jacobian matrix of $\mathbf{f}$ at the equilibrium. If the real parts of its eigenvalues are all strictly negative, then the equilibrium is locally exponentially stable, and if at least one is strictly positive then it is unstable. A matrix, of which all eigenvalues have a strictly negative real part is said to be *Hurwitz*. A modern presentation of this method can be found in practically all textbooks on nonlinear systems and control theory, e.g. Theorem 3.7 in [21], Theorem 5.14 in [40], and Theorem 3.1 in [44] to name a few.

The second method, known as *Lyapunov's second method* or *Lyapunov's direct method*, enables one to prove the stability of an equilibrium of (1.1) without integrating the differential equation. It states, that if $\mathbf{y} = \mathbf{0}$ is an equilibrium point of the system, $V \in \mathcal{C}^1(\mathcal{U})$ is a *positive definite function*, i.e. $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{U} \setminus \{\mathbf{0}\}$, and $\boldsymbol{\phi}$ is the solution of (1.1). Then the equilibrium is stable, if the inequality

$$\frac{d}{dt}V(\boldsymbol{\phi}(t, \boldsymbol{\xi})) \leq 0$$

is satisfied for all $\boldsymbol{\phi}(t, \boldsymbol{\xi})$ in a neighborhood of the equilibrium $\mathbf{y}$. If $\mathcal{N} \subset \mathcal{U}$ is a domain containing $\mathbf{y}$ and the strict inequality

$$\frac{d}{dt}V(\boldsymbol{\phi}(t, \boldsymbol{\xi})) < 0$$

is satisfied for all $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in \mathcal{N} \setminus \{\mathbf{0}\}$, then the equilibrium is asymptotically stable on the largest compact preimage $V^{-1}([0, c])$ contained in $\mathcal{N}$. In both cases the function V is said to be a *Lyapunov function* for (1.1). Just as Lyapunov's indirect method, the direct method is covered in practically all modern textbooks on nonlinear systems and control theory. Some examples are Theorem 3.1 in [21], Theorem 5.16 in [40], and Theorem 25.1 and Theorem 25.2 in [12].

In this section we are going to prove, that if the time derivative in the inequalities above is replaced with a Dini derivative with respect to t, then the assumption $V \in \mathcal{C}^1(\mathcal{U})$ can be replaced with the less restrictive assumption, that V is locally Lipschitz on $\mathcal{U}$. The same is done in Theorem 42.5 in the standard reference [12] on this subject, but a lot of details are left out. Before starting, we introduce the following notation.

Definition 1.14 *We denote by $\mathcal{K}$ the set of all strictly increasing continuous functions $\psi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, that satisfy $\psi(0) = 0$.*

□

Let $g : \mathcal{V} \longrightarrow \mathbb{R}$ be a function, where $\mathcal{V} \subset \mathbb{R}^n$ is a bounded subset and $g(\mathbf{0}) = 0$. Then g is positive definite, if and only if for any norm $\|\cdot\|$ on $\mathbb{R}^n$, there is a function $\psi \in \mathcal{K}$, such that $g(\mathbf{x}) \geq \psi(\|\mathbf{x}\|)$ for all $\mathbf{x} \in \mathcal{V}$. This alternative characterization of positive definite functions is often more convenient. The first example is the proof of the next lemma, where we prove properties of the preimages $V^{-1}([0, c])$ of a positive definite function V, which will be used in the proof of Lyapunov's direct method.

Lemma 1.15 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a neighborhood of the origin, $V : \mathcal{U} \longrightarrow \mathbb{R}$ be a positive definite function, and $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Then:*

i) For every $c > 0$ the set $V^{-1}([0, c])$ is a neighborhood of $\mathbf{0}$.

ii) For every $R > 0$ there is a $c' > 0$, such that $V^{-1}([0, c])$ is a compact subset of $\mathcal{U} \cap \{\boldsymbol{\xi} \big| \, \|\boldsymbol{\xi}\| \leq R\}$ for all c with $0 < c \leq c'$.

Proof:
The proposition *i)* follows directly from the continuity of V. We prove the proposition *ii)*. From the definition of a positive definite function, we know there is a function $\psi \in \mathcal{K}$, such that $\psi(\|\mathbf{x}\|) \leq V(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{U} \cap \{\boldsymbol{\xi} \big| \, \|\boldsymbol{\xi}\| \leq R\}$. From the definition of a neighborhood it is clear, that there is an $r > 0$, such that $\{\boldsymbol{\xi} \big| \, \|\boldsymbol{\xi}\| \leq r\} \subset \mathcal{U}$. Set $c' := \psi(\min\{r, R\})$. Then $V^{-1}([0, c'])$ is a subset of $\mathcal{U} \cap \{\boldsymbol{\xi} \big| \, \|\boldsymbol{\xi}\| \leq R\}$, because

$$\|\mathbf{x}\| \leq \psi^{-1}(V(\mathbf{x})) \leq \min\{r, R\}$$

for all $\mathbf{x} \in V^{-1}([0, c'])$. The proposition now follows from the facts, that the preimage of a closed set under a continuous function is closed, that bounded and closed is equivalent to compact in $\mathbb{R}^n$, and that a closed subset of a compact set is compact.

■

We come to the direct method of Lyapunov. The next theorem is of central importance for Linear Program **LP2** in Part III of this thesis.

Theorem 1.16 (Direct Method of Lyapunov) *Assume that (1.1) has an equilibrium at the origin and that $V : \mathcal{U} \longrightarrow \mathbb{R}$ is a positive definite continuous function. Let $\boldsymbol{\phi}$ be the solution of (1.1), $D^* \in \{D^+, D_+, D^-, D_-\}$ be a Dini derivative with respect to the time t, and $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then:*

i) If there is a neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin, such that the inequality $D^*[V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))] \leq 0$ is satisfied for all $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in \mathcal{N}$, then the origin is a stable equilibrium of (1.1).

ii) If there is a neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin and a function $\psi \in \mathcal{K}$, such that the inequality $D^*[V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))] \leq -\psi(\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|)$ is satisfied for all $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in \mathcal{N}$, then the origin is a asymptotically stable equilibrium of (1.1) and any compact preimage $V^{-1}([0, c])$ contained in $\mathcal{N}$ is contained in its region of attraction.

PROOF:
Let $R > 0$ be given. It follows form Lemma 1.15 *ii)* that there is a $c > 0$ such that the preimage $V^{-1}([0, c])$ is a compact subset of $\mathcal{N} \cap \{\boldsymbol{\xi} | \|\boldsymbol{\xi}\| \leq R\}$.

Proposition *i)*:
Because $D^*[V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))] \leq 0$ for all $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in \mathcal{N}$, Corollary 1.13 implies that $t \mapsto V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))$ is a decreasing function. This means that if $\boldsymbol{\phi}(t_1, \boldsymbol{\xi}) \in V^{-1}([0, c])$, then $\boldsymbol{\phi}(t_2, \boldsymbol{\xi}) \in V^{-1}([0, c])$ for all $t_2 \geq t_1$. It follows from Lemma 1.15 *i)*, that there is an $r > 0$, such that the set $\{\mathbf{y} | \; \|\mathbf{y}\| \leq r\} \subset V^{-1}([0, c])$. But then $\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| \leq R$ for all $\|\boldsymbol{\xi}\| \leq r$ and all $t \geq 0$, i.e. the equilibrium at the origin is stable.

Proposition *ii)*:
Now let $V^{-1}([0, c])$ be an arbitrary compact preimage contained in $\mathcal{N}$. We have already shown that the equilibrium at the origin is stable, so if we prove that $\limsup_{t \to +\infty} \boldsymbol{\phi}(t, \boldsymbol{\xi}) = \mathbf{0}$ for all $\boldsymbol{\xi} \in V^{-1}([0, c])$, then Lemma 1.15 *i)* implies the proposition *ii)*.

Let $\boldsymbol{\xi}$ be an arbitrary element of $V^{-1}([0, c])$. From $D^*[V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))] \leq -\psi(\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|)$, Lemma 1.11, and the fundamental theorem of integral calculus, it follows that

$$D^* \left[V(\boldsymbol{\phi}(t, \boldsymbol{\xi})) + \int_0^t \psi(\|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|) d\tau \right] \leq 0.$$

By Corollary 1.13 the function

$$t \mapsto V(\boldsymbol{\phi}(t, \boldsymbol{\xi})) + \int_0^t \psi(\|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|) d\tau$$

is decreasing and because $V(\boldsymbol{\phi}(t, \boldsymbol{\xi})) \geq 0$ and $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in V^{-1}([0, c])$ for all $t \geq 0$, we must have

$$\int_0^{+\infty} \psi(\|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|) d\tau < +\infty.$$

The convergence of this integral implies, that if $\tau \mapsto \psi(\|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|)$ is uniformly continuous, then $\lim_{\tau \to +\infty} \psi(\|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|) = 0$. A proof hereof can be found in Lemma 4.4 in [21]. It remains to be shown that $\tau \mapsto \psi(\|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|)$ is uniformly continuous. Because $\boldsymbol{\phi}$ is the solution of (1.1), $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in V^{-1}([0, c])$ for all $t \geq 0$, and $V^{-1}([0, c])$ is compact, we have

$$\|\frac{\partial \boldsymbol{\phi}}{\partial t}(t, \boldsymbol{\xi})\| \leq \max_{\mathbf{y} \in V^{-1}([0,c])} \|\mathbf{f}(\mathbf{y})\| < +\infty,$$

which implies that $\tau \mapsto \boldsymbol{\phi}(\tau, \boldsymbol{\xi})$ is uniformly continuous.

Because $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in V^{-1}([0, c])$ for all $t \geq 0$ and $V^{-1}([0, c])$ is compact, there is a constant $a < +\infty$, such that $a \geq \sup_{t \geq 0} \|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|$. A continuous function with a compact domain is uniformly continuous (see, for example, Theorem 1 in Section 12 in [26]), so the restriction $\psi|_{[0,a]}$ is uniformly continuous. Clearly the composition of uniformly continuous functions is uniformly continuous, so we have proved that

$$\limsup_{t \to +\infty} \boldsymbol{\phi}(t, \boldsymbol{\xi}) = \mathbf{0}$$

for an arbitrary $\boldsymbol{\xi} \in V^{-1}([0, c])$.

■

At the beginning of this section we claimed, that the direct method of Lyapunov can be used without integrating the state equation of a system, but in the corresponding theorem it was assumed that the Dini derivative $D^*[V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))]$ of V, along the trajectories of the system, is negative or strictly negative. Let us consider the case $D^* := D^+$ as an example. Because

$$D^+[V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] := \limsup_{h\to+0} \frac{V(\boldsymbol{\phi}(t+h,\boldsymbol{\xi})) - V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{h}$$

we clearly need another possibility to calculate $D^+[V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))]$. On page 196 in [12] an equality is stated, which translates to

$$\limsup_{h\to 0} \frac{V(\boldsymbol{\phi}(t+h,\boldsymbol{\xi})) - V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{h} = \limsup_{h\to 0} \frac{V(\boldsymbol{\phi}(t,\boldsymbol{\xi}) + h\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))) - V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{h}$$

with the notations used in this thesis. This equality is stated without any restrictions on V, but there is no proof or references given. We close this treatment of Lyapunov's direct method by proving a theorem, that implies that

$$D^+[V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] = \limsup_{h\to+0} \frac{V(\boldsymbol{\phi}(t,\boldsymbol{\xi}) + h\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))) - V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{h}$$

if V is locally Lipschitz.

Theorem 1.17 *Let $\mathcal{V} \subset \mathbb{R}^n$ be a domain, $\mathbf{g} : \mathcal{V} \longrightarrow \mathbb{R}^n$ be a continuous function, and assume that $V : \mathcal{V} \longrightarrow \mathbb{R}$ is locally Lipschitz on $\mathcal{V}$. Let $\mathbf{y}$ be a solution of the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}), \quad \mathbf{x}(0) = \boldsymbol{\xi}.$$

Then

$$\begin{aligned}
D^+[V\circ\mathbf{y}](t) &= \limsup_{h\to 0+} \frac{V(\mathbf{y}(t) + h\mathbf{g}(\mathbf{y}(t))) - V(\mathbf{y}(t))}{h}, \\
D_+[V\circ\mathbf{y}](t) &= \liminf_{h\to 0+} \frac{V(\mathbf{y}(t) + h\mathbf{g}(\mathbf{y}(t))) - V(\mathbf{y}(t))}{h}, \\
D^-[V\circ\mathbf{y}](t) &= \limsup_{h\to 0-} \frac{V(\mathbf{y}(t) + h\mathbf{g}(\mathbf{y}(t))) - V(\mathbf{y}(t))}{h},
\end{aligned}$$

and

$$D_-[V\circ\mathbf{y}](t) = \liminf_{h\to 0-} \frac{V(\mathbf{y}(t) + h\mathbf{g}(\mathbf{y}(t))) - V(\mathbf{y}(t))}{h},$$

for every t in the domain of $\mathbf{y}$.

Proof:
We only prove the theorem for D^+, the other cases can be proved with similar reasoning.

By Taylor's theorem there is a constant $\vartheta_h \in]0,1[$ for any h small enough, such that

$$\begin{aligned} D^+[V\circ\mathbf{y}](t) &= \limsup_{h\to 0+}\frac{V(\mathbf{y}(t+h))-V(\mathbf{y}(t))}{h} \\ &= \limsup_{h\to 0+}\frac{V(\mathbf{y}(t)+h\dot{\mathbf{y}}(t+h\vartheta_h))-V(\mathbf{y}(t))}{h} \\ &= \limsup_{h\to 0+}\frac{V(\mathbf{y}(t)+h\mathbf{g}(\mathbf{y}(t+h\vartheta_h)))-V(\mathbf{y}(t))}{h} \\ &= \limsup_{h\to 0+}\bigg(\frac{V(\mathbf{y}(t)+h\mathbf{g}(\mathbf{y}(t)))-V(\mathbf{y}(t))}{h} \\ &\qquad +\frac{V(\mathbf{y}(t)+h\mathbf{g}(\mathbf{y}(t+h\vartheta_h)))-V(\mathbf{y}(t)+h\mathbf{g}(\mathbf{y}(t)))}{h}\bigg), \end{aligned}$$

so, by Lemma 1.11, it suffices to prove

$$\lim_{h\to 0+}\frac{V(\mathbf{y}(t)+h\mathbf{g}(\mathbf{y}(t+h\vartheta_h)))-V(\mathbf{y}(t)+h\mathbf{g}(\mathbf{y}(t)))}{h}=0.$$

Let $\mathcal{C}$ be a compact neighborhood of $\mathbf{y}(t)$ and $L_{\mathcal{C}}$ be a Lipschitz constant for the restriction of V on $\mathcal{C}$. Then for every h small enough,

$$\begin{aligned} \left|\frac{V(\mathbf{y}(t)+h\mathbf{g}(\mathbf{y}(t+h\vartheta_h)))-V(\mathbf{y}(t)+h\mathbf{g}(\mathbf{y}(t)))}{h}\right| &\leq \frac{L_{\mathcal{C}}}{h}\|h\mathbf{g}(\mathbf{y}(t+h\vartheta_h))-h\mathbf{g}(\mathbf{y}(t))\| \\ &= L_{\mathcal{C}}\|\mathbf{g}(\mathbf{y}(t+h\vartheta_h))-\mathbf{g}(\mathbf{y}(t))\| \end{aligned}$$

and the continuity of $\mathbf{g}$ and $\mathbf{y}$ imply the vanishing of the limit above.

■

1.6 Converse Theorems

In the last section we proved, that the existence of a Lyapunov function V for (1.1) is a sufficient condition for the (asymptotic) stability of an equilibrium of (1.1). There are several similar theorems known, where one either uses more or less restrictive assumptions regarding the Lyapunov function than in Theorem 1.16. Such theorems are often called *Lyapunov-like* theorems. An example for less restrictive assumptions are Theorem 46.5 in [12] and Theorem 4.10 in [21], where the solution of a system is shown to be uniformly bounded, and an example for more restrictive assumptions is Theorem 5.17 in [40], where an equilibrium is proved to be exponentially stable. The Lyapunov-like theorems all have the form:

> If one can find a function V for a dynamical system that satisfies the properties X, then the system has the stability property Y.

A natural question awakened by any Lyapunov-like theorem is whether its converse is true or not, i.e. if there is a corresponding theorem of the form:

> If a dynamical system has the stability property Y, then there is a function V for the dynamical system, that satisfies the properties X.

Such theorems are called the *converse theorems* in the Lyapunov stability theory. For nonlinear systems they are more complicated than the direct methods of Lyapunov and the results came rather late and did not stem from Lyapunov himself. The converse theorems are covered quite thoroughly in Chapter VI in [12]. Some further references are Section 5.7 in [50] and Section 4.3 in [21]. About the techniques to prove such theorems Hahn writes on page 225 in his book [12] :

> In the converse theorems the stability behavior of a family of motions $\mathbf{p}(t, \mathbf{a}, t_0)$ is assumed to be known. For example, it might be assumed that the expression $\|\mathbf{p}(t, \mathbf{a}, t_0)\|$ is estimated by known comparison functions (secs. 35 and 36). Then one attempts to construct by means of a finite or transfinite procedure, a Lyapunov function which satisfies the conditions of the stability theorem under consideration.

In the case of a dynamical system with a stable equilibrium point, an autonomous Lyapunov function may not exist for the system, even for an autonomous system. A proof hereof can be found on page 228 in [12]. For a dynamical system with an asymptotically stable equilibrium point, it is possible to construct a smooth autonomous Lyapunov function, see, for example, Section 51 in [12] or Theorem 24 in Section 5.7 in [50]. The construction of such a Lyapunov function is rather complicated. Both [12] and [50] use Massera's lemma [29] in the construction of a smooth Lyapunov function, which is a pure existence theorem from a practical point of view. For our application in Part II of this thesis, the partial derivatives of the Lyapunov function up to the third order must be bounded and we need formulas for the appropriate bounds. Hence, the converse theorem on asymptotic stability is a bad choice for our application.

Surprisingly, it is very simple to construct a Lyapunov function, for a system with an exponentially stable equilibrium point, from its solution. Before stating and proving a corresponding theorem, let us recall how to differentiate a function of the form

$$\int_{c(x)}^{b(x)} a(x,y)dy.$$

Let $[\alpha, \beta] \subset \mathbb{R}$, $\alpha < \beta$, be a compact interval and assume that the functions $b, c : [\alpha, \beta] \longrightarrow \mathbb{R}$ are continuously differentiable and that $b(x) < c(x)$ for all $x \in [\alpha, \beta]$. Suppose

$$a : \{(x,y) \in \mathbb{R}^2 \big| x \in [\alpha, \beta] \text{ and } b(x) < y < c(x)\} \longrightarrow \mathbb{R}$$

is continuously differentiable with respect to the first argument and that this derivative is bounded. Then

$$\frac{d}{dx}\int_{c(x)}^{b(x)} a(x,y)dy = \int_{c(x)}^{b(x)} \frac{\partial a}{\partial x}(x,y)dy + a(x,b(x))b'(x) - a(x,c(x))c'(x).$$

This formula is the so-called *Leibniz rule.* A proof is, for example, given in Theorem 13.5 in [11].

We come to the last theorem of this chapter, the converse theorem on exponential stability. We state the theorem a little differently than usually done, e.g. Theorem 63 in Chapter 5 in [50], Theorem 4.5 in [21], and Theorem 5.17 in [40], so we will offer a proof.

Theorem 1.18 (Converse Theorem on Exponential Stability) *Assume the origin is an α, m-exponentially stable equilibrium point of (1.1) on an open neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin. Suppose the set $\{\mathbf{y} \in \mathbb{R}^n \big| \|\mathbf{y}\|_2 \leq m \sup_{\mathbf{z}\in\mathcal{N}} \|\mathbf{z}\|_2\}$ is a compact subset of $\mathcal{U}$, and let L be a Lipschitz constant for $\mathbf{f}$ on this compact set. Let $\boldsymbol{\phi}$ be the solution of (1.1) and T be a constant satisfying*

$$T > \frac{1}{\alpha}\ln(m).$$

Then the function $V : \mathcal{N} \longrightarrow \mathbb{R}$,

$$V(\boldsymbol{\xi}) := \int_0^T \|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|_2^2 d\tau$$

for all $\boldsymbol{\xi} \in \mathcal{N}$, satisfies the inequalities

$$\frac{1 - e^{-2LT}}{2L}\|\boldsymbol{\xi}\|_2^2 \leq V(\boldsymbol{\xi}) \leq m^2 \frac{1 - e^{-2\alpha T}}{2\alpha}\|\boldsymbol{\xi}\|_2^2 \tag{1.4}$$

and

$$\nabla V(\boldsymbol{\xi}) \cdot \mathbf{f}(\boldsymbol{\xi}) \leq -(1 - m^2 e^{-2\alpha T})\|\boldsymbol{\xi}\|_2^2 \tag{1.5}$$

for all $\boldsymbol{\xi} \in \mathcal{N}$, and is therefore a Lyapunov function for (1.1).

PROOF:
Proof of (1.4):
By the Cauchy-Schwarz inequality

$$\begin{aligned}
\left|\frac{d}{dt}\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2\right| &= 2\left|\boldsymbol{\phi}(t,\boldsymbol{\xi}) \cdot \dot{\boldsymbol{\phi}}(t,\boldsymbol{\xi})\right| \leq 2\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2 \|\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2 \\
&= 2\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2 \|\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi})) - \mathbf{f}(\mathbf{0})\|_2 \leq 2L\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2,
\end{aligned}$$

which implies

$$\frac{d}{dt}(e^{2Lt}\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2) = e^{2Lt}\left(2L\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2 + \frac{d}{dt}\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2\right) \geq 0,$$

so

$$e^{-2Lt}\|\boldsymbol{\xi}\|_2^2 \leq \|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2$$

for all $t \geq 0$. From this and $\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2 \leq me^{-\alpha t}\|\boldsymbol{\xi}\|_2$ for all $\boldsymbol{\xi} \in \mathcal{N}$ and $t \geq 0$,

$$\begin{aligned}
\frac{1 - e^{-2LT}}{2L}\|\boldsymbol{\xi}\|_2^2 &= \int_0^T e^{-2Lt}\|\boldsymbol{\xi}\|_2^2 dt \leq V(\boldsymbol{\xi}) = \int_0^T \|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2 dt \\
&\leq \int_0^T m^2 e^{-2\alpha t}\|\boldsymbol{\xi}\|_2^2 dt = m^2\frac{1 - e^{-2\alpha T}}{2\alpha}\|\boldsymbol{\xi}\|_2^2,
\end{aligned}$$

follows.

Proof of (1.5):
Let $\boldsymbol{\xi} \in \mathcal{N}$ and t be such that $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{N}$. By the change of variables $a := \tau + t$,

$$V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) = \int_0^T \|\boldsymbol{\phi}(\tau, \boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2^2 d\tau = \int_t^{t+T} \|\boldsymbol{\phi}(a - t, \boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2^2 da,$$

which right-hand side can easily be differentiated using Leibniz rule,

$$\frac{d}{dt}V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) = \|\boldsymbol{\phi}(T,\boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2^2 - \|\boldsymbol{\phi}(0,\boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2^2 + \int_t^{t+T} \frac{d}{dt}\|\boldsymbol{\phi}(a-t,\boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2^2 da.$$

From this, the facts that

$$\boldsymbol{\phi}(0,\boldsymbol{\phi}(t,\boldsymbol{\xi})) = \boldsymbol{\phi}(t,\boldsymbol{\xi}),$$
$$\boldsymbol{\phi}(a-t,\boldsymbol{\phi}(t,\boldsymbol{\xi})) = \boldsymbol{\phi}(a-t+t,\boldsymbol{\xi}) = \boldsymbol{\phi}(a,\boldsymbol{\xi}),$$

and $\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2 \leq me^{-\alpha t}\|\boldsymbol{\xi}\|_2$,

$$\begin{aligned} \frac{d}{dt}V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) &= \|\boldsymbol{\phi}(T,\boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2^2 - \|\boldsymbol{\phi}(0,\boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2^2 + 0 \\ &\leq -(1-m^2e^{-2\alpha T})\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2 \end{aligned}$$

follows. By the chain rule

$$\frac{d}{dt}V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) = [\nabla V](\boldsymbol{\phi}(t,\boldsymbol{\xi})) \cdot \dot{\boldsymbol{\phi}}(t,\boldsymbol{\xi}) = [\nabla V](\boldsymbol{\phi}(t,\boldsymbol{\xi})) \cdot \mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi})),$$

so

$$[\nabla V](\boldsymbol{\xi}) \cdot \mathbf{f}(\boldsymbol{\xi}) = \left.\frac{d}{dt}V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))\right|_{t=0} \leq -(1-m^2e^{-2\alpha T})\|\boldsymbol{\xi}\|_2^2.$$

■

Part II

Refuting α, m-Exponential Stability on an Arbitrary Neighborhood with Linear Programming

The starting point of this part is Lyapunov's indirect method. Suppose that the origin is an equilibrium of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

and that the Jacobian matrix of $\mathbf{f}$ at zero is Hurwitz. Then it follows from Lyapunov's indirect method, that there is a neighborhood $\mathcal{M}$ of zero and constants $\alpha > 0$ and $m \geq 1$, such that the inequality

$$\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_2 \leq m e^{-\alpha t} \|\boldsymbol{\xi}\|_2 \tag{1.6}$$

holds true for the solution $\boldsymbol{\phi}$ of the system for all $t \geq 0$, whenever $\boldsymbol{\xi} \in \mathcal{M}$. Let $\mathcal{M}'$ be the largest set with respect to inclusion, so that (1.6) holds true for all $\boldsymbol{\xi} \in \mathcal{M}'$ and all $t \geq 0$. Lyapunov's indirect method does not deliver any practical estimate of the size of $\mathcal{M}'$.

From Theorem 1.18 we know that the function $V : \mathcal{N} \longrightarrow \mathbb{R}$,

$$V(\boldsymbol{\xi}) := \int_0^T \|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|_2^2 d\tau,$$

is a Lyapunov function for the system above if

$$T > \frac{1}{\alpha} \ln(m)$$

and $\mathcal{N} \subset \mathcal{M}'$. It follows that if the function V is not a Lyapunov function of the system, then $\mathcal{N}$ is not a subset of $\mathcal{M}'$. We will use this to derive a linear program, dependent on $\mathbf{f}$, $\mathcal{N}$, α, and m, with the property, that if there is not a feasible solution of the program, then $\mathcal{N}$ is not a subset of $\mathcal{M}'$.

Chapter 2

Linear Program LP1

In this part of this thesis we will consider a system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.1}$$

where $\mathcal{U} \subset \mathbb{R}^n$ is a domain containing the origin and $\mathbf{f} \in [\mathcal{C}^3(\mathcal{U})]^n$ is a function vanishing at the origin.

We will assume that the origin is an α, m-exponentially stable equilibrium of the system above, on an open neighborhood $\mathcal{N}$ of the origin and we will use this to derive inequalities, that are linear in the values of a Lyapunov function V for the system on a discrete set. These inequalities lead to a linear program and we will show, that if this linear program does not have a feasible solution, then the assumption that the origin is an α, m-exponentially stable equilibrium on $\mathcal{N}$ is contradictory. This linear program uses finite differences as approximation to ∇V. We will derive a formula for bounds of the approximation error in the second section of this chapter. In the third section we will use these bounds to state Linear Program **LP1**. Linear Program **LP1** is an algorithmic description of how to generate a linear program from the function $\mathbf{f}$, the set $\mathcal{N}$, the constants α and m, and the wanted grid steps $\mathbf{h}$, to approximate ∇V. It follows from Theorem I, that if the linear program does not have a feasible solution, then the equilibrium at the origin of the system is not α, m-exponentially stable on $\mathcal{N}$.

2.1 How the Method Works

Consider the dynamical system (2.1). Let $\alpha > 0$, $m \geq 1$, and

$$T > \frac{1}{\alpha} \ln(m)$$

be constants. Let $\mathcal{N} \subset \mathcal{U}$ be an open neighborhood of the origin, such that the set $\{\mathbf{y} \in \mathbb{R}^n \big| \|\mathbf{y}\|_2 \leq m \sup_{\mathbf{z}\in\mathcal{N}} \|\mathbf{z}\|_2\}$ is a compact subset of $\mathcal{U}$, and let L be a Lipschitz constant for $\mathbf{f}$ on this compact set.

Assume the equilibrium at the origin of (2.1) is α, m-exponentially stable on $\mathcal{N}$. Then, by Theorem 1.18, the function $V : \mathcal{N} \longrightarrow \mathbb{R}$,

$$V(\boldsymbol{\xi}) := \int_0^T \|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|_2^2 d\tau \tag{2.2}$$

for all $\boldsymbol{\xi} \in \mathcal{N}$, where $\boldsymbol{\phi}$ is the solution of (2.1), is a Lyapunov function for the system, that satisfies the inequalities

$$\frac{1-e^{-2LT}}{2L}\|\boldsymbol{\xi}\|_2^2 \leq V(\boldsymbol{\xi}) \leq m^2\frac{1-e^{-2\alpha T}}{2\alpha}\|\boldsymbol{\xi}\|_2^2 \tag{2.3}$$

and

$$\nabla V(\boldsymbol{\xi})\cdot\mathbf{f}(\boldsymbol{\xi}) \leq -(1-m^2e^{-2\alpha T})\|\boldsymbol{\xi}\|_2^2 \tag{2.4}$$

for all $\boldsymbol{\xi} \in \mathcal{N}$. Consequently, if the function V does not satisfy (2.3) and (2.4) for all $\boldsymbol{\xi} \in \mathcal{N}$, then the equilibrium at the origin of (2.1) cannot be α, m-exponentially stable on $\mathcal{N}$.

We are going to derive a linear program with the property, that if there is no feasible solution of this program, then the function V defined by (2.2) does not satisfy (2.3) and (2.4) for all $\boldsymbol{\xi} \in \mathcal{N}$. To do this let $\mathbf{h} := (h_1, h_2, .., h_n)$ be a vector with strictly positive real elements. The elements of the vector $\mathbf{h}$ are the grid steps used by the linear program to approximate ∇V. Define the set

$$\mathcal{G}_{\mathbf{h}}^{\mathcal{N}} := \{(i_1h_1, i_2h_2, .., i_nh_n)\big| i_k \in \mathbb{Z},\ k = 1, 2, .., n\} \cap \mathcal{N},$$

and the set

$$\mathcal{H}_{\mathbf{h}}^{\mathcal{N}} := \{\mathbf{y} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}\big|\mathbf{y} + a_i\mathbf{e}_i \in \mathcal{N} \text{ for all } a_i \in [-h_i, h_i] \text{ and all } i = 1, 2, .., n\}.$$

Clearly $\mathbf{y} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ implies that $\mathbf{y} + h_i\mathbf{e}_i \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$ and $\mathbf{y} - h_i\mathbf{e}_i \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$ for all $i = 1, 2, .., n$. For all $\boldsymbol{\xi} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$ we define the constants $V_{\boldsymbol{\xi}}$ by

$$V_{\boldsymbol{\xi}} := V(\boldsymbol{\xi}).$$

Because $\mathbf{f} \in [\mathcal{C}^3(\mathcal{U})]^n$ it follows from Theorem 1.4 that $\boldsymbol{\phi} \in [\mathcal{C}^3(\mathrm{dom}(\boldsymbol{\phi}))]^n$, which in turn implies $V \in \mathcal{C}^3(\mathcal{N})$. A proof of this fact can, for example, be found in Corollary 16.3 in [3]. We use this to define the constants

$$E_{i,\boldsymbol{\xi}} := \frac{\partial V}{\partial \xi_i}(\boldsymbol{\xi}) - \frac{V_{\boldsymbol{\xi}+h_i\mathbf{e}_i} - V_{\boldsymbol{\xi}-h_i\mathbf{e}_i}}{2h_i}$$

for all $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ and all $i = 1, 2, .., n$. The interpretation of the constants $E_{i,\boldsymbol{\xi}}$ is obvious, $E_{i,\boldsymbol{\xi}}$ is the approximation error when

$$\frac{V_{\boldsymbol{\xi}+h_i\mathbf{e}_i} - V_{\boldsymbol{\xi}-h_i\mathbf{e}_i}}{2h_i}$$

is substituted for the derivative of V with respect to the i-th argument at the point $\boldsymbol{\xi}$.

The inequalities (2.3) and (2.4) for all $\boldsymbol{\xi} \in \mathcal{N}$ imply

$$\frac{1-e^{-2LT}}{2L}\|\boldsymbol{\xi}\|_2^2 \leq V_{\boldsymbol{\xi}} \leq m^2\frac{1-e^{-2\alpha T}}{2\alpha}\|\boldsymbol{\xi}\|_2^2$$

for all $\boldsymbol{\xi} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$ and

$$\sum_{i=1}^{n}\left(\frac{V_{\boldsymbol{\xi}+h_i\mathbf{e}_i} - V_{\boldsymbol{\xi}-h_i\mathbf{e}_i}}{2h_i} + E_{i,\boldsymbol{\xi}}\right) f_i(\boldsymbol{\xi}) \leq -(1-m^2e^{-2\alpha T})\|\boldsymbol{\xi}\|_2^2$$

for all $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$.

We cannot calculate the exact values of the constants $E_{i,\boldsymbol{\xi}}$ without integrating (2.1), but in the next section we will give a formula for an upper bound. Let the constants $B_{i,\boldsymbol{\xi}}$ be such that

$$B_{i,\boldsymbol{\xi}} \geq |E_{i,\boldsymbol{\xi}}|$$

for all $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ and all $i = 1, 2, .., n$. Then certainly

$$\sum_{i=1}^{n} \left(\frac{V_{\boldsymbol{\xi}+h_i\mathbf{e}_i} - V_{\boldsymbol{\xi}-h_i\mathbf{e}_i}}{2h_i} f_i(\boldsymbol{\xi}) - B_{i,\boldsymbol{\xi}}|f_i(\boldsymbol{\xi})| \right) \leq -(1 - m^2 e^{-2\alpha T})\|\boldsymbol{\xi}\|_2^2$$

for all $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ and the following proposition is true.

> If there are no real values $W_{\boldsymbol{\xi}}$, $\boldsymbol{\xi} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$, such that inequalities
>
> $$\frac{1 - e^{-2LT}}{2L}\|\boldsymbol{\xi}\|_2^2 \leq W_{\boldsymbol{\xi}} \leq m^2 \frac{1 - e^{-2\alpha T}}{2\alpha}\|\boldsymbol{\xi}\|_2^2$$
>
> for all $\boldsymbol{\xi} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$ and
>
> $$\sum_{i=1}^{n} \left(\frac{W_{\boldsymbol{\xi}+h_i\mathbf{e}_i} - W_{\boldsymbol{\xi}-h_i\mathbf{e}_i}}{2h_i} f_i(\boldsymbol{\xi}) - B_{i,\boldsymbol{\xi}}|f_i(\boldsymbol{\xi})| \right) \leq -(1 - m^2 e^{-2\alpha T})\|\boldsymbol{\xi}\|_2^2$$
>
> for all $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ are satisfied simultaneously, then the origin cannot be an α, m-exponentially stable equilibrium point of (2.1) on $\mathcal{N}$.

These two sets of inequalities for $W_{\boldsymbol{\xi}}$ are the linear constraints of our linear program. In the next section we will derive a formula for the bounds $B_{i,\boldsymbol{\xi}}$.

2.2 Bounds of the Approximation Error

In this section we are going to derive bounds of the absolute values of the constants $E_{i,\boldsymbol{\xi}}$ defined in the last section. All functions, constants, and sets defined in the last section will be used without explanations.

By Taylor's theorem there are constants $\vartheta_{i,\boldsymbol{\xi}}^+, \vartheta_{i,\boldsymbol{\xi}}^- \in [0, 1]$ for every $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ and every $i = 1, 2, .., n$, such that

$$V_{\boldsymbol{\xi}+h_i\mathbf{e}_i} = V_{\boldsymbol{\xi}} + h_i \frac{\partial V}{\partial \xi_i}(\boldsymbol{\xi}) + \frac{1}{2}h_i^2 \frac{\partial^2 V}{\partial \xi_i^2}(\boldsymbol{\xi}) + \frac{1}{6}h_i^3 \frac{\partial^3 V}{\partial \xi_i^3}(\boldsymbol{\xi} + \vartheta_{i,\boldsymbol{\xi}}^+ h_i \mathbf{e}_i)$$

and

$$V_{\boldsymbol{\xi}-h_i\mathbf{e}_i} = V_{\boldsymbol{\xi}} - h_i \frac{\partial V}{\partial \xi_i}(\boldsymbol{\xi}) + \frac{1}{2}h_i^2 \frac{\partial^2 V}{\partial \xi_i^2}(\boldsymbol{\xi}) - \frac{1}{6}h_i^3 \frac{\partial^3 V}{\partial \xi_i^3}(\boldsymbol{\xi} - \vartheta_{i,\boldsymbol{\xi}}^- h_i \mathbf{e}_i),$$

i.e.

$$\frac{V_{\boldsymbol{\xi}+h_i\mathbf{e}_i} - V_{\boldsymbol{\xi}-h_i\mathbf{e}_i}}{2h_i} = \frac{\partial V}{\partial \xi_i}(\boldsymbol{\xi}) + \frac{h_i^2}{12} \left(\frac{\partial^3 V}{\partial \xi_i^3}(\boldsymbol{\xi} + \vartheta_{i,\boldsymbol{\xi}}^+ h_i \mathbf{e}_i) + \frac{\partial^3 V}{\partial \xi_i^3}(\boldsymbol{\xi} - \vartheta_{i,\boldsymbol{\xi}}^- h_i \mathbf{e}_i) \right).$$

This and the definition of $E_{i,\boldsymbol{\xi}}$ imply

$$|E_{i,\boldsymbol{\xi}}| \leq \frac{h_i^2}{6} \max_{a\in[-h_i,h_i]} \left|\frac{\partial^3 V}{\partial x_i^3}(\boldsymbol{\xi} + a\mathbf{e}_i)\right|. \tag{2.5}$$

Because we do not even know whether the function V is properly defined or not, this bound of $|E_{i,\boldsymbol{\xi}}|$ might seem pointless. But when the origin is an α, m-exponentially stable equilibrium of (2.1) on $\mathcal{N}$, then V is properly defined and

$$\frac{\partial^3 V}{\partial \xi_i^3}(\boldsymbol{\xi}) = \int_0^T \frac{\partial^3}{\partial \xi_i^3}\|\boldsymbol{\phi}(\tau,\boldsymbol{\xi})\|_2^2 d\tau = \int_0^T \left(6\frac{\partial \boldsymbol{\phi}}{\partial \xi_i}(\tau,\boldsymbol{\xi}) \cdot \frac{\partial^2 \boldsymbol{\phi}}{\partial \xi_i^2}(\tau,\boldsymbol{\xi}) + 2\boldsymbol{\phi}(\tau,\boldsymbol{\xi}) \cdot \frac{\partial^3 \boldsymbol{\phi}}{\partial \xi_i^3}(\tau,\boldsymbol{\xi})\right) d\tau.$$

From the assumed α, m-exponential stability we have bounds of $\|\boldsymbol{\phi}(\tau,\boldsymbol{\xi})\|_2$, so if we can find bounds of the partial derivatives

$$\frac{\partial \boldsymbol{\phi}}{\partial \xi_i}(\tau,\boldsymbol{\xi}), \quad \frac{\partial^2 \boldsymbol{\phi}}{\partial \xi_i^2}(\tau,\boldsymbol{\xi}), \quad \text{and} \quad \frac{\partial^3 \boldsymbol{\phi}}{\partial \xi_i^3}(\tau,\boldsymbol{\xi}),$$

without knowing $\boldsymbol{\phi}$, we do have bounds of $|E_{i,\boldsymbol{\xi}}|$ without integrating (2.1).

The next theorem delivers such bounds by use of bounds of the partial derivatives of $\mathbf{f}$. Its proof is long and technical and the following fact from the theory of linear differential equations will be used.

Lemma 2.1 *Let $\mathcal{I} \neq \emptyset$ be an open interval in $\mathbb{R}$ and let $A : \mathcal{I} \longrightarrow \mathbb{R}^{n\times n}$ and $\mathbf{b} : \mathcal{I} \longrightarrow \mathbb{R}^n$ be continuous mappings. Let $t_0 \in \mathcal{I}$ and assume there are real constants M and δ, such that $M \geq \sup_{t\in\mathcal{I}} \|A(t)\|_2$ and $\delta \geq \sup_{t\in\mathcal{I}} \|\mathbf{b}(t)\|_2$. Then the unique solution $\mathbf{y}$ of the initial value problem*

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{b}(t), \quad \mathbf{x}(t_0) = \boldsymbol{\xi},$$

satisfies the inequality

$$\|\mathbf{y}(t)\|_2 \leq \|\boldsymbol{\xi}\|_2 e^{M|t-t_0|} + \delta\frac{e^{M|t-t_0|} - 1}{M}$$

for all $t \in \mathcal{I}$.

Proof:
See, for example, Theorem VI in §14 in [52].

■

In the proof of the next theorem we additionally use the following simple fact about matrices and their spectral norms.

Lemma 2.2 *Let $A = (a_{ij})_{i,j\in\{1,2,..,n\}}$ and $B = (b_{ij})_{i,j\in\{1,2,..,n\}}$ be real $n \times n$-matrices, such that*

$$|a_{ij}| \leq b_{ij}$$

for all $i, j = 1, 2, .., n$. Then $\|A\|_2 \leq \|B\|_2$.

PROOF:
Let $\mathbf{x} \in \mathbb{R}^n$ be a vector with the property that $\|A\mathbf{x}\|_2 = \|A\|_2$ and $\|\mathbf{x}\|_2 = 1$. Define $\mathbf{y} \in \mathbb{R}^n$ by $y_i = |x_i|$ for all $i = 1, 2, .., n$. Then $\|\mathbf{y}\|_2 = 1$ and

$$\begin{aligned}
\|A\|_2^2 &= \|A\mathbf{x}\|_2^2 = \sum_{i,j,k=1}^{n} a_{ij}a_{ik}x_jx_k \leq \sum_{i,j,k=1}^{n} |a_{ij}||a_{ik}||x_j||x_k| \\
&\leq \sum_{i,j,k=1}^{n} b_{ij}b_{ik}|x_j||x_k| = \sum_{i,j,k=1}^{n} b_{ij}b_{ik}y_jy_k = \|B\mathbf{y}\|_2^2 \leq \|B\|_2^2.
\end{aligned}$$

■

We come to the theorem that delivers bounds of the partial derivatives of $\boldsymbol{\phi}$.

Theorem 2.3 *Let $\mathcal{V} \subset \mathbb{R}^n$ be a domain, $\mathcal{M} \subset \mathcal{V}$ be a compact set, $\mathbf{g} \in [\mathcal{C}^3(\mathcal{V})]^n$, and $\boldsymbol{\psi}$ be the solution of the differential equation*

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}).$$

Let the constants a'_{ij}, b'_{ijk}, and c'_{ijkl}, $i, j, k, l = 1, 2, .., n$, be such that

$$\begin{aligned}
a'_{ij} &\geq \sup_{\mathbf{x}\in\mathcal{M}} \left| \frac{\partial g_i}{\partial x_j}(\mathbf{x}) \right|, \\
b'_{ijk} &\geq \sup_{\mathbf{x}\in\mathcal{M}} \left| \frac{\partial^2 g_i}{\partial x_k \partial x_j}(\mathbf{x}) \right|, \\
c'_{ijkl} &\geq \sup_{\mathbf{x}\in\mathcal{M}} \left| \frac{\partial^3 g_i}{\partial x_l \partial x_k \partial x_j}(\mathbf{x}) \right|,
\end{aligned}$$

and let at least one of the a'_{ij} be larger than zero.
Let the $n \times n$-matrices $\widetilde{A} = (a_{ij})$, $\widetilde{B} = (b_{ij})$, and $\widetilde{C} = (c_{ij})$, be given by

$$\begin{aligned}
a_{ij} &:= a'_{ij}, \\
b_{ij} &:= \left(\sum_{k=1}^{n} {b'_{ijk}}^2 \right)^{\frac{1}{2}}, \\
c_{ij} &:= \left(\sum_{k,l=1}^{n} {c'_{ijkl}}^2 \right)^{\frac{1}{2}},
\end{aligned}$$

and set $\mathtt{A} := \|\widetilde{A}\|_2$, $\mathtt{B} := \|\widetilde{B}\|_2$, and $\mathtt{C} := \|\widetilde{C}\|_2$.
Then

i)

$$\|\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha}(t, \boldsymbol{\xi})\|_2 \leq e^{\mathtt{A}t},$$

ii)

$$\|\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\beta \partial \xi_\alpha}(t, \boldsymbol{\xi})\|_2 \leq \mathtt{B}e^{2\mathtt{A}t}\frac{e^{\mathtt{A}t} - 1}{\mathtt{A}},$$

iii)
$$\|\frac{\partial^3 \boldsymbol{\psi}}{\partial \xi_\gamma \partial \xi_\beta \partial \xi_\alpha}(t,\boldsymbol{\xi})\|_2 \leq e^{3\mathtt{A}t}\left(\mathtt{C} + 3\mathtt{B}^2 \frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right)\frac{e^{\mathtt{A}t}-1}{\mathtt{A}},$$

for all $\alpha, \beta, \gamma = 1, 2, .., n$ *and all* $(t, \boldsymbol{\xi}) \in \mathrm{dom}(\boldsymbol{\psi})$*, such that* $t \geq 0$ *and* $\boldsymbol{\psi}(t', \boldsymbol{\xi}) \in \mathcal{M}$ *for all* $t' \in [0, t]$.

PROOF:
Before we start proving the inequalities *i), ii)*, and *iii)* consecutively, recall that $\boldsymbol{\psi}, \dot{\boldsymbol{\psi}} \in [\mathcal{C}^3(\mathrm{dom}(\boldsymbol{\psi}))]^n$ by Theorem 1.4. This means that we can permute the partial derivatives of $\boldsymbol{\psi}$ and $\dot{\boldsymbol{\psi}}$ up to the third order at will. This follows, for example, from Theorem 8.13 in [2].
Proof of *i)*:
Let $\alpha \in \{1, 2, .., n\}$ and $(t, \boldsymbol{\xi}) \in \mathrm{dom}(\boldsymbol{\psi})$ be an arbitrary element, such that $t \geq 0$ and $\boldsymbol{\psi}(t', \boldsymbol{\xi}) \in \mathcal{M}$ for all $t' \in [0, t]$. From $\dot{\boldsymbol{\psi}} = \mathbf{g}(\boldsymbol{\psi})$ and the chain rule for derivatives of vector fields (see, for example, Theorem 8.11 in [2]), we get
$$\frac{d}{dt}\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha} = \frac{\partial}{\partial \xi_\alpha}\dot{\boldsymbol{\psi}} = \frac{\partial}{\partial \xi_\alpha}\mathbf{g}(\boldsymbol{\psi}) = [(\nabla \mathbf{g}) \circ \boldsymbol{\psi}]\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha}, \tag{2.6}$$
where
$$(\nabla \mathbf{g}) \circ \boldsymbol{\psi} := \begin{pmatrix} \frac{\partial g_1}{\partial x_1} \circ \boldsymbol{\psi} & \frac{\partial g_1}{\partial x_2} \circ \boldsymbol{\psi} & \cdots & \frac{\partial g_1}{\partial x_n} \circ \boldsymbol{\psi} \\ \frac{\partial g_2}{\partial x_1} \circ \boldsymbol{\psi} & \frac{\partial g_2}{\partial x_2} \circ \boldsymbol{\psi} & \cdots & \frac{\partial g_2}{\partial x_n} \circ \boldsymbol{\psi} \\ \vdots & \vdots & \ddots & \\ \frac{\partial g_n}{\partial x_1} \circ \boldsymbol{\psi} & \frac{\partial g_n}{\partial x_2} \circ \boldsymbol{\psi} & \cdots & \frac{\partial g_n}{\partial x_n} \circ \boldsymbol{\psi} \end{pmatrix}.$$
Let $t' \in [0, t]$ and $d_{ij}(t')$ be the ij-element of the matrix $[(\nabla \mathbf{g}) \circ \boldsymbol{\phi}](t', \boldsymbol{\xi})$. Because $\boldsymbol{\psi}(t', \boldsymbol{\xi}) \in \mathcal{M}$ we get
$$|d_{ij}(t')| = \left|\frac{\partial g_i}{\partial x_j}(\boldsymbol{\psi}(t', \boldsymbol{\xi}))\right| \leq \sup_{\boldsymbol{\xi}' \in \mathcal{M}}\left|\frac{\partial g_i}{\partial x_j}(\boldsymbol{\xi}')\right| \leq a'_{ij}.$$
It follows from Lemma 2.2 that
$$\mathtt{A} \geq \sup_{t' \in [0,t]} \|[(\nabla \mathbf{g}) \circ \boldsymbol{\psi}](t', \boldsymbol{\xi})\|_2 \tag{2.7}$$
and because
$$\frac{\partial}{\partial \xi_\alpha}\boldsymbol{\psi}(0, \boldsymbol{\xi}) = \frac{\partial}{\partial \xi_\alpha}\boldsymbol{\xi} = \mathbf{e}_\alpha,$$
it follows from (2.6) and Lemma 2.1 that
$$\|\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha}(t, \boldsymbol{\xi})\|_2 \leq e^{\mathtt{A}t}.$$

Proof of *ii)*:
Let $\beta, \alpha \in \{1, 2, .., n\}$ and $(t, \boldsymbol{\xi}) \in \mathrm{dom}(\boldsymbol{\psi})$ be an arbitrary element, such that $t \geq 0$ and $\boldsymbol{\psi}(t', \boldsymbol{\xi}) \in \mathcal{M}$ for all $t' \in [0, t]$. From (2.6) we get
$$\begin{aligned} \frac{d}{dt}\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\beta \partial \xi_\alpha} &= \frac{\partial}{\partial \xi_\beta}\frac{d}{dt}\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha} = \frac{\partial}{\partial \xi_\beta}\left([(\nabla \mathbf{g}) \circ \boldsymbol{\psi}]\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha}\right) \\ &= \left(\frac{\partial}{\partial \xi_\beta}[(\nabla \mathbf{g}) \circ \boldsymbol{\psi}]\right)\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha} + [(\nabla \mathbf{g}) \circ \boldsymbol{\psi}]\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\beta \partial \xi_\alpha} \end{aligned} \tag{2.8}$$

and it follows from

$$\frac{\partial^2}{\partial \xi_\beta \partial \xi_\alpha}\boldsymbol{\psi}(0,\boldsymbol{\xi}) = \frac{\partial^2}{\partial \xi_\beta \partial \xi_\alpha}\boldsymbol{\xi} = \mathbf{0},$$

(2.7), the inequality *i)*, and Lemma 2.1, that

$$\|\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\beta \partial \xi_\alpha}(t,\boldsymbol{\xi})\|_2 \leq \sup_{t'\in[0,t]} \| \left(\frac{\partial}{\partial \xi_\beta}[(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\right)(t',\boldsymbol{\xi})\|_2 e^{\mathtt{A}t}\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}. \tag{2.9}$$

Let $t' \in [0,t]$ and $d'_{ij}(t')$ be the ij-element of the matrix

$$\left(\frac{\partial}{\partial \xi_\beta}[(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\right)(t',\boldsymbol{\xi}).$$

Because $\boldsymbol{\psi}(t',\boldsymbol{\xi}) \in \mathcal{M}$ it follows from the chain rule, the Cauchy-Schwarz inequality, and the inequality *i)*, that

$$\begin{aligned}
|d'_{ij}(t')| &= \left|\left[\frac{\partial}{\partial \xi_\beta}\left(\frac{\partial g_i}{\partial x_j}\circ \boldsymbol{\psi}\right)\right](t',\boldsymbol{\xi})\right| = \left|\left[\left(\nabla \frac{\partial g_i}{\partial x_j}\right)\circ \boldsymbol{\psi}\right](t',\boldsymbol{\xi})\cdot \frac{\partial \boldsymbol{\psi}}{\partial \xi_\beta}(t',\boldsymbol{\xi})\right| \\
&\leq \sum_{k=1}^n \left|\left[\left(\frac{\partial^2 g_i}{\partial x_k \partial x_j}\right)\circ \boldsymbol{\psi}\right](t',\boldsymbol{\xi})\right| \left|\frac{\partial \psi_k}{\partial \xi_\beta}(t',\boldsymbol{\xi})\right| \leq \sum_{k=1}^n b'_{ijk}\left|\frac{\partial \psi_k}{\partial \xi_\beta}(t',\boldsymbol{\xi})\right| \\
&\leq \left(\sum_{k=1}^n {b'_{ijk}}^2\right)^{\frac{1}{2}} \|\frac{\partial \boldsymbol{\psi}}{\partial \xi_\beta}(t',\boldsymbol{\xi})\|_2 \leq b_{ij}e^{\mathtt{A}t}.
\end{aligned}$$

It follows from Lemma 2.2 that

$$\mathtt{B}e^{\mathtt{A}t} \geq \sup_{t'\in[0,t]} \| \left(\frac{\partial}{\partial \xi_\beta}[(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\right)(t',\boldsymbol{\xi})\|_2, \tag{2.10}$$

which together with (2.9) implies the inequality *ii)*.

Proof of *iii)*:
Let $\gamma,\beta,\alpha \in \{1,2,..,n\}$ and $(t,\boldsymbol{\xi}) \in \mathrm{dom}(\boldsymbol{\psi})$ be an arbitrary element, such that $t \geq 0$ and $\boldsymbol{\psi}(t',\boldsymbol{\xi}) \in \mathcal{M}$ for all $t' \in [0,t]$. From (2.8) we get

$$\begin{aligned}
\frac{d}{dt}\frac{\partial^3 \boldsymbol{\psi}}{\partial \xi_\gamma \partial \xi_\beta \partial \xi_\alpha} &= \frac{\partial}{\partial \xi_\gamma}\frac{d}{dt}\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\beta \partial \xi_\alpha} = \frac{\partial}{\partial \xi_\gamma}\left[\left(\frac{\partial}{\partial \xi_\beta}[(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\right)\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha} + [(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\beta \partial \xi_\alpha}\right] \\
&= \left(\frac{\partial^2}{\partial \xi_\gamma \partial \xi_\beta}[(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\right)\frac{\partial \boldsymbol{\psi}}{\partial \xi_\alpha} + \left(\frac{\partial}{\partial \xi_\beta}[(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\right)\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\gamma \partial \xi_\alpha} \\
&\quad + \left(\frac{\partial}{\partial \xi_\gamma}[(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\right)\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\beta \partial \xi_\alpha} + [(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\frac{\partial^3 \boldsymbol{\psi}}{\partial \xi_\gamma \partial \xi_\beta \partial \xi_\alpha},
\end{aligned}$$

and it follows from

$$\frac{\partial^3}{\partial \xi_\gamma \partial \xi_\beta \partial \xi_\alpha}\boldsymbol{\psi}(0,\boldsymbol{\xi}) = \frac{\partial^3}{\partial \xi_\gamma \partial \xi_\beta \partial \xi_\alpha}\boldsymbol{\xi} = \mathbf{0},$$

(2.7), (2.10), the inequalities *i)* and *ii)*, and Lemma 2.1, that

$$\|\frac{\partial^3 \boldsymbol{\psi}}{\partial \xi_\gamma \partial \xi_\beta \partial \xi_\alpha}(t,\boldsymbol{\xi})\|_2 \leq e^{\mathtt{A}t}\left[\sup_{t'\in[0,t]} \| \left(\frac{\partial^2}{\partial \xi_\gamma \partial \xi_\beta}[(\nabla \mathbf{g})\circ \boldsymbol{\psi}]\right)(t',\boldsymbol{\xi})\|_2 + 2\mathtt{B}^2 e^{2\mathtt{A}t}\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right]\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}.$$

Let $t' \in [0,t]$ and $d''_{ij}(t')$ be the ij-element of the matrix

$$\left(\frac{\partial^2}{\partial \xi_\gamma \partial \xi_\beta}[(\nabla \mathbf{g}) \circ \boldsymbol{\psi}]\right)(t', \boldsymbol{\xi}).$$

Because $\boldsymbol{\psi}(t', \boldsymbol{\xi}) \in \mathcal{M}$ it follows from the chain rule, the Cauchy-Schwarz inequality, and the inequalities *i)* and *ii)*, that

$$\begin{aligned}
|d''_{ij}(t')| &= \left|\frac{\partial^2}{\partial \xi_\gamma \partial \xi_\beta}\left(\frac{\partial g_i}{\partial x_j} \circ \boldsymbol{\psi}\right)(t', \boldsymbol{\xi})\right| \\
&= \left|\frac{\partial}{\partial \xi_\gamma}\left(\sum_{k=1}^n \left\{\left[\left(\frac{\partial^2 g_i}{\partial x_k \partial x_j}\right) \circ \boldsymbol{\psi}\right](t', \boldsymbol{\xi})\right\} \frac{\partial \psi_k}{\partial \xi_\beta}(t', \boldsymbol{\xi})\right)\right| \\
&= \left|\sum_{k=1}^n \left[\frac{\partial \psi_k}{\partial \xi_\beta}(t', \boldsymbol{\xi})\left(\left(\nabla \frac{\partial^2 g_i}{\partial x_k \partial x_j}\right) \circ \boldsymbol{\psi}\right)(t', \boldsymbol{\xi}) \cdot \frac{\partial \boldsymbol{\psi}}{\partial \xi_\gamma}(t', \boldsymbol{\xi})\right.\right. \\
&\qquad \left.\left. + \left\{\left[\left(\frac{\partial^2 g_i}{\partial x_k \partial x_j}\right) \circ \boldsymbol{\psi}\right](t', \boldsymbol{\xi})\right\} \frac{\partial^2 \psi_k}{\partial \xi_\gamma \partial \xi_\beta}(t', \boldsymbol{\xi})\right]\right| \\
&\leq \sum_{k=1}^n \left|\frac{\partial \psi_k}{\partial \xi_\beta}(t', \boldsymbol{\xi})\right| \sum_{l=1}^n c'_{ijkl} \left|\frac{\partial \psi_l}{\partial \xi_\gamma}(t', \boldsymbol{\xi})\right| + \sum_{k=1}^n b'_{ijk}\left|\frac{\partial^2 \psi_k}{\partial \xi_\gamma \partial \xi_\beta}(t', \boldsymbol{\xi})\right| \\
&\leq \sum_{k=1}^n \left|\frac{\partial \psi_k}{\partial \xi_\beta}(t', \boldsymbol{\xi})\right| \left(\sum_{l=1}^n {c'_{ijkl}}^2\right)^{\frac{1}{2}} \|\frac{\partial \boldsymbol{\psi}}{\partial \xi_\gamma}(t', \boldsymbol{\xi})\|_2 + \left(\sum_{k=1}^n {b'_{ijk}}^2\right)^{\frac{1}{2}} \|\frac{\partial^2 \boldsymbol{\psi}}{\partial \xi_\gamma \partial \xi_\beta}(t', \boldsymbol{\xi})\|_2 \\
&\leq \|\frac{\partial \boldsymbol{\psi}}{\partial \xi_\beta}(t', \boldsymbol{\xi})\|_2 \left(\sum_{k=1}^n \left(\sum_{l=1}^n {c'_{ijkl}}^2\right)^{\frac{1}{2}\cdot 2}\right)^{\frac{1}{2}} \|\frac{\partial \boldsymbol{\psi}}{\partial \xi_\gamma}(t', \boldsymbol{\xi})\|_2 + b_{ij}\mathtt{B}e^{2\mathtt{A}t}\frac{e^{\mathtt{A}t}-1}{\mathtt{A}} \\
&\leq c_{ij}e^{2\mathtt{A}t} + b_{ij}\mathtt{B}e^{2\mathtt{A}t}\frac{e^{\mathtt{A}t}-1}{\mathtt{A}} \\
&= e^{2\mathtt{A}t}\left(c_{ij} + b_{ij}\mathtt{B}\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right).
\end{aligned}$$

It follows from Lemma 2.2 and the triangle inequality that

$$e^{2\mathtt{A}t}\left(\mathtt{C} + \mathtt{B}^2\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right) \geq \|\left(\frac{\partial^2}{\partial \xi_\gamma \partial \xi_\beta}[(\nabla \mathbf{g}) \circ \boldsymbol{\psi}]\right)(t', \boldsymbol{\xi})\|_2,$$

i.e.

$$\begin{aligned}
\|\frac{\partial^3 \boldsymbol{\psi}}{\partial \xi_\gamma \partial \xi_\beta \partial \xi_\alpha}(t, \boldsymbol{\xi})\|_2 &\leq e^{\mathtt{A}t}\left[e^{2\mathtt{A}t}\left(\mathtt{C} + \mathtt{B}^2\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right) + 2\mathtt{B}^2 e^{2\mathtt{A}t}\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right]\frac{e^{\mathtt{A}t}-1}{\mathtt{A}} \\
&= e^{3\mathtt{A}t}\left(\mathtt{C} + 3\mathtt{B}^2\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right)\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}.
\end{aligned}$$

■

If the equilibrium at the origin of (2.1) is α, m-exponentially stable on $\mathcal{N}$, then

$$\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in \{\mathbf{y} \in \mathbb{R}^n \big| \|\mathbf{y}\|_2 \leq m \sup_{\mathbf{z} \in \mathcal{N}} \|\mathbf{z}\|_2\}$$

for every $\boldsymbol{\xi} \in \mathcal{N}$ and all $t \geq 0$. Because the set $\{\mathbf{y} \in \mathbb{R}^n \big| \|\mathbf{y}\|_2 \leq m \sup_{\mathbf{z}\in\mathcal{N}} \|\mathbf{z}\|_2\}$ is a compact subset of $\mathcal{U}$, this means that if we substitute $\mathcal{U}$, $\mathbf{f}$, $\boldsymbol{\phi}$, and $\{\mathbf{y} \in \mathbb{R}^n \big| \|\mathbf{y}\|_2 \leq m \sup_{\mathbf{z}\in\mathcal{N}} \|\mathbf{z}\|_2\}$ for $\mathcal{V}$, $\mathbf{g}$, $\boldsymbol{\psi}$, and $\mathcal{M}$ in the last theorem respectively, then

$$\|\frac{\partial \boldsymbol{\phi}}{\partial \xi_\alpha}(t,\boldsymbol{\xi})\|_2 \leq e^{\mathtt{A}t},$$

$$\|\frac{\partial^2 \boldsymbol{\phi}}{\partial \xi_\beta \partial \xi_\alpha}(t,\boldsymbol{\xi})\|_2 \leq \mathtt{B}e^{2\mathtt{A}t}\frac{e^{\mathtt{A}t}-1}{\mathtt{A}},$$

and

$$\|\frac{\partial^3 \boldsymbol{\phi}}{\partial \xi_\gamma \partial \xi_\beta \partial \xi_\alpha}(t,\boldsymbol{\xi})\|_2 \leq e^{3\mathtt{A}t}\left(\mathtt{C} + 3\mathtt{B}^2\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right)\frac{e^{\mathtt{A}t}-1}{\mathtt{A}},$$

for all $\alpha, \beta, \gamma = 1, 2, .., n$, all $\boldsymbol{\xi} \in \mathcal{N}$, and all $t \geq 0$, where the constants $\mathtt{A}$, $\mathtt{B}$, and $\mathtt{C}$ are defined as in the theorem. This implies

$$\begin{aligned}
\left|\frac{\partial^3 V}{\partial \xi_i^3}(\boldsymbol{\xi})\right| &= \left|\int_0^T \left(6\frac{\partial \boldsymbol{\phi}}{\partial \xi_i}(t,\boldsymbol{\xi}) \cdot \frac{\partial^2 \boldsymbol{\phi}}{\partial \xi_i^2}(t,\boldsymbol{\xi}) + 2\boldsymbol{\phi}(t,\boldsymbol{\xi}) \cdot \frac{\partial^3 \boldsymbol{\phi}}{\partial \xi_i^3}(t,\boldsymbol{\xi})\right) dt\right| \\
&\leq \int_0^T \left(6\|\frac{\partial \boldsymbol{\phi}}{\partial \xi_i}(t,\boldsymbol{\xi})\|_2\|\frac{\partial^2 \boldsymbol{\phi}}{\partial \xi_i^2}(t,\boldsymbol{\xi})\|_2 + 2\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2\|\frac{\partial^3 \boldsymbol{\phi}}{\partial \xi_i^3}(t,\boldsymbol{\xi})\|_2\right) dt \\
&\leq \int_0^T \left(6e^{\mathtt{A}t}\mathtt{B}e^{2\mathtt{A}t}\frac{e^{\mathtt{A}t}-1}{\mathtt{A}} + 2me^{-\alpha t}\|\boldsymbol{\xi}\|_2 e^{3\mathtt{A}t}\left(\mathtt{C} + 3\mathtt{B}^2\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right)\frac{e^{\mathtt{A}t}-1}{\mathtt{A}}\right) dt \\
&= \frac{6\mathtt{B}}{\mathtt{A}}\left(\frac{e^{4\mathtt{A}T}-1}{4\mathtt{A}} - \frac{e^{3\mathtt{A}T}-1}{3\mathtt{A}}\right) + 2m\|\boldsymbol{\xi}\|_2\frac{\mathtt{C}}{\mathtt{A}}\left(\frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha} - \frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha}\right) \\
&\quad +2m\|\boldsymbol{\xi}\|_2\frac{3\mathtt{B}^2}{\mathtt{A}^2}\left(\frac{e^{(5\mathtt{A}-\alpha)T}-1}{5\mathtt{A}-\alpha} - 2\frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha} + \frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha}\right)
\end{aligned}$$

for all $\boldsymbol{\xi} \in \mathcal{N}$.

From this bound of the third order derivatives of V and (2.5), a formula for an upper bound of $|E_{\boldsymbol{\xi},i}|$ follows,

$$\begin{aligned}
\frac{|E_{\boldsymbol{\xi},i}|}{h_i^2} &\leq \frac{1}{6}\max_{a\in[-h_i,h_i]}\left|\frac{\partial^3 V}{\partial \xi_i^3}(\boldsymbol{\xi}+s\mathbf{e}_i)\right| \\
&\leq \frac{1}{6}\max_{a\in[-h_i,h_i]}\left\{\frac{6\mathtt{B}}{\mathtt{A}}\left(\frac{e^{4\mathtt{A}T}-1}{4\mathtt{A}} - \frac{e^{3\mathtt{A}T}-1}{3\mathtt{A}}\right)\right. \\
&\quad +2m\|\boldsymbol{\xi}+a\mathbf{e}_i\|_2\frac{\mathtt{C}}{\mathtt{A}}\left(\frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha} - \frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha}\right) \\
&\quad \left.+2m\|\boldsymbol{\xi}+a\mathbf{e}_i\|_2\frac{3\mathtt{B}^2}{\mathtt{A}^2}\left(\frac{e^{(5\mathtt{A}-\alpha)T}-1}{5\mathtt{A}-\alpha} - 2\frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha} + \frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha}\right)\right\} \\
&\leq \frac{\mathtt{B}}{\mathtt{A}}\left(\frac{e^{4\mathtt{A}T}-1}{4\mathtt{A}} - \frac{e^{3\mathtt{A}T}-1}{3\mathtt{A}}\right) + \frac{1}{3}m(\|\boldsymbol{\xi}\|_2+h_i)\frac{\mathtt{C}}{\mathtt{A}}\left(\frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha} - \frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha}\right) \\
&\quad +m(\|\boldsymbol{\xi}\|_2+h_i)\frac{\mathtt{B}^2}{\mathtt{A}^2}\left(\frac{e^{(5\mathtt{A}-\alpha)T}-1}{5\mathtt{A}-\alpha} - 2\frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha} + \frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha}\right),
\end{aligned}$$

so we have a formula for the constants $B_{i,\boldsymbol{\xi}}$ in the linear constraints at the end of the last section without integrating (2.1),

$$\begin{aligned} B_{i,\boldsymbol{\xi}} \;:=\; & h_i^2 \left\{ \frac{\mathtt{B}}{\mathtt{A}} \left(\frac{e^{4\mathtt{A}T}-1}{4\mathtt{A}} - \frac{e^{3\mathtt{A}T}-1}{3\mathtt{A}} \right) + \frac{1}{3} m(\|\boldsymbol{\xi}\|_2 + h_i) \frac{\mathtt{C}}{\mathtt{A}} \left(\frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha} - \frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha} \right) \right. \\ & \left. + m(\|\boldsymbol{\xi}\|_2 + h_i) \frac{\mathtt{B}^2}{\mathtt{A}^2} \left(\frac{e^{(5\mathtt{A}-\alpha)T}-1}{5\mathtt{A}-\alpha} - 2\frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha} + \frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha} \right) \right\} \end{aligned}$$

for all $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ and all $i = 1, 2, .., n$.

If the denominator of one of the fractions

$$\frac{e^{(3\mathtt{A}-\alpha)T}-1}{3\mathtt{A}-\alpha}, \quad \frac{e^{(4\mathtt{A}-\alpha)T}-1}{4\mathtt{A}-\alpha}, \quad \text{or} \quad \frac{e^{(5\mathtt{A}-\alpha)T}-1}{5\mathtt{A}-\alpha}$$

is equal to zero, then T should be substituted for the corresponding fraction.

2.3 Linear Program LP1

Let us sum up what we have shown so far in this chapter. Consider the linear program:

Linear Program LP1

Let $\mathbf{f} \in [\mathcal{C}^3(\mathcal{U})]^n$ be a function from a domain $\mathcal{U} \subset \mathbb{R}^n$ containing the origin into $\mathbb{R}^n$, such that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Let $\alpha > 0$ and $m \geq 1$ be constants and $\mathcal{N} \subset \mathcal{U}$ be an open neighborhood of the origin, such that the set $\{\mathbf{y} \in \mathbb{R}^n \big| \|\mathbf{y}\|_2 \leq m \sup_{\mathbf{z}\in\mathcal{N}} \|\mathbf{z}\|_2\}$ is a compact subset of $\mathcal{U}$. Furthermore, let $\mathbf{h} = (h_1, h_2, .., h_n)$ be a vector with strictly positive real elements.

The linear program $\mathbf{LP1}(\mathbf{f}, \alpha, m, \mathcal{N}, \mathbf{h})$ is generated in the following way:

Define the sets $\mathcal{M}$, $\mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$, and $\mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ by

$$\mathcal{M} := \{\mathbf{y} \in \mathbb{R}^n \big| \|\mathbf{y}\|_2 \leq m \sup_{\mathbf{z}\in\mathcal{N}} \|\mathbf{z}\|_2\},$$

$$\mathcal{G}_{\mathbf{h}}^{\mathcal{N}} := \{(i_1 h_1, i_2 h_2, .., i_n h_n) \big| i_k \in \mathbb{Z},\ k = 1, 2, .., n\} \cap \mathcal{N},$$

and

$$\mathcal{H}_{\mathbf{h}}^{\mathcal{N}} := \{\mathbf{y} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}} \big| \mathbf{y} + a_i \mathbf{e}_i \in \mathcal{N} \textit{ for all } i = 1, 2, .., n \textit{ and all } a_i \in [-h_i, h_i]\}.$$

Assign a value to the constant T, such that

$$T > \frac{1}{\alpha} \ln(m),$$

and values to the constants a'_{ij}, b'_{ijk}, and c'_{ijkl}, $i, j, k, l = 1, 2, .., n$, such that

$$a'_{ij} \geq \sup_{\mathbf{x}\in\mathcal{M}} \left| \frac{\partial f_i}{\partial x_j}(\mathbf{x}) \right|,$$

$$b'_{ijk} \geq \sup_{\mathbf{x}\in\mathcal{M}} \left| \frac{\partial^2 f_i}{\partial x_k \partial x_j}(\mathbf{x}) \right|,$$

$$c'_{ijkl} \geq \sup_{\mathbf{x}\in\mathcal{M}} \left| \frac{\partial^3 f_i}{\partial x_l \partial x_k \partial x_j}(\mathbf{x}) \right|,$$

and such that at least one of the a'_{ij} is larger than zero.

Define the $n \times n$-matrices $\widetilde{A} = (a_{ij})$, $\widetilde{B} = (b_{ij})$, and $\widetilde{C} = (c_{ij})$, by

$$a_{ij} := a'_{ij},$$

$$b_{ij} := \left(\sum_{k=1}^{n} {b'_{ijk}}^2 \right)^{\frac{1}{2}},$$

$$c_{ij} := \left(\sum_{k,l=1}^{n} {c'_{ijkl}}^2 \right)^{\frac{1}{2}},$$

and set $\mathtt{A} := \|\widetilde{A}\|_2$, $\mathtt{B} := \|\widetilde{B}\|_2$, and $\mathtt{C} := \|\widetilde{C}\|_2$.

For every $i = 1, 2, .., n$ and every $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$ define the constant $B_{i,\boldsymbol{\xi}}$ by the formula

$$\begin{aligned} B_{i,\boldsymbol{\xi}} \;\; := \;\; & h_i^2 \left\{ \frac{\mathtt{B}}{\mathtt{A}} \left(\frac{e^{4\mathtt{A}T} - 1}{4\mathtt{A}} - \frac{e^{3\mathtt{A}T} - 1}{3\mathtt{A}} \right) + \frac{1}{3} m(\|\boldsymbol{\xi}\|_2 + h_i) \frac{\mathtt{C}}{\mathtt{A}} \left(\frac{e^{(4\mathtt{A}-\alpha)T} - 1}{4\mathtt{A} - \alpha} - \frac{e^{(3\mathtt{A}-\alpha)T} - 1}{3\mathtt{A} - \alpha} \right) \right. \\ & \left. + m(\|\boldsymbol{\xi}\|_2 + h_i) \frac{\mathtt{B}^2}{\mathtt{A}^2} \left(\frac{e^{(5\mathtt{A}-\alpha)T} - 1}{5\mathtt{A} - \alpha} - 2 \frac{e^{(4\mathtt{A}-\alpha)T} - 1}{4\mathtt{A} - \alpha} + \frac{e^{(3\mathtt{A}-\alpha)T} - 1}{3\mathtt{A} - \alpha} \right) \right\}. \end{aligned}$$

If the denominator of one of the fractions

$$\frac{e^{(3\mathtt{A}-\alpha)T} - 1}{3\mathtt{A} - \alpha}, \quad \frac{e^{(4\mathtt{A}-\alpha)T} - 1}{4\mathtt{A} - \alpha}, \quad \text{or} \quad \frac{e^{(5\mathtt{A}-\alpha)T} - 1}{5\mathtt{A} - \alpha}$$

is equal to zero, then substitute T for the corresponding fraction.

For every $\boldsymbol{\xi} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$ let $W[\boldsymbol{\xi}]$ be a variable of the linear program.

The constraints of the linear program are:

LC1) For every $\boldsymbol{\xi} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$:

$$\frac{1 - e^{-2\mathtt{A}T}}{2\mathtt{A}} \|\boldsymbol{\xi}\|_2^2 \leq W[\boldsymbol{\xi}] \leq m^2 \frac{1 - e^{-2\alpha T}}{2\alpha} \|\boldsymbol{\xi}\|_2^2$$

LC2) For every $\boldsymbol{\xi} \in \mathcal{H}_{\mathbf{h}}^{\mathcal{N}}$:

$$\sum_{i=1}^{n} \left(\frac{W[\boldsymbol{\xi} + h_i \mathbf{e}_i] - W[\boldsymbol{\xi} - h_i \mathbf{e}_i]}{2h_i} f_i(\boldsymbol{\xi}) - B_{i,\boldsymbol{\xi}} |f_i(\boldsymbol{\xi})| \right) \leq -(1 - m^2 e^{-2\alpha T}) \|\boldsymbol{\xi}\|_2^2$$

The objective of the linear program is not relevant and can be set equal to zero.

■

The linear constraints of the linear program $\mathbf{LP1}(\mathbf{f}, \alpha, m, \mathcal{N}, \mathbf{h})$ are basically the inequalities at the end of Section 3.1, where we have substituted the constant $\mathtt{A}$ for the arbitrary Lipschitz constant L. From the definition of $\mathtt{A}$ it is clear that it is a Lipschitz constant for $\mathbf{f}$ on $\mathcal{M}$, so this is a valid substitution.

What we have shown for the linear program $\mathbf{LP1}(\mathbf{f}, \alpha, m, \mathcal{N}, \mathbf{h})$ is:

Theorem I

If the linear program $\mathbf{LP1}(\mathbf{f}, \alpha, m, \mathcal{N}, \mathbf{h})$ *does not have a feasible solution, then the dynamical system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

does not have an α, m*-exponentially stable equilibrium at the origin on* $\mathcal{N}$*.*

■

That the linear program $\mathbf{LP1}(\mathbf{f}, \alpha, m, \mathcal{N}, \mathbf{h})$ does not have a feasible solution, means there are no values for the variables $W[\boldsymbol{\xi}]$, $\boldsymbol{\xi} \in \mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$, such that its linear constraints are simultaneously satisfied. That the equilibrium at the origin is not α, m-exponentially stable on $\mathcal{N}$, means that there is an $\boldsymbol{\xi} \in \mathcal{N}$, such that the estimate

$$\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_2 \leq m e^{-\alpha t} \|\boldsymbol{\xi}\|_2$$

on the solution $\boldsymbol{\phi}$ of the system does not apply to all $t \geq 0$.

Chapter 3

Evaluation of the Method

Linear Program **LP1** can be used to improve Lyapunov's indirect method because it gives an estimate of the region of attraction of an equilibrium. To see how this works let us have a look at Lyapunov's indirect method in action.

Assume the origin is an equilibrium point of (2.1) and that the Jacobian matrix A of $\mathbf{f} \in \mathcal{C}^1(\mathcal{U})$ is Hurwitz. Then, for example, by Theorem 3.6 in [21], there exists a unique positive definite symmetric $n \times n$-matrix P, that satisfies the *Lyapunov equation*

$$PA + A^T P = -I,$$

where I is the identity $n \times n$-matrix. Moreover, there are numerically efficient methods to solve such equations (see, for example, [10]). Define the function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$V(\boldsymbol{\xi}) := \boldsymbol{\xi}^T P \boldsymbol{\xi},$$

and let $\lambda_{\min}$ be the smallest and $\lambda_{\max}$ be the largest eigenvalue of P. Because $\mathbf{f} \in \mathcal{C}^1(\mathcal{U})$ there is a function $\mathbf{g} : \mathcal{U} \longrightarrow \mathbb{R}^n$, such that

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x} + \|\mathbf{x}\|_2 \mathbf{g}(\mathbf{x})$$

for all $\mathbf{x} \in \mathcal{U}$ and

$$\lim_{\mathbf{x} \to \mathbf{0}} \mathbf{g}(\mathbf{x}) = \mathbf{0}.$$

Now

$$\lambda_{\min} \|\boldsymbol{\xi}\|_2^2 \leq V(\boldsymbol{\xi}) \leq \lambda_{\max} \|\boldsymbol{\xi}\|_2^2$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$ and

$$\begin{aligned}
\frac{d}{dt} V(\boldsymbol{\phi}) &= \boldsymbol{\phi}^T P \dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\phi}}^T P \boldsymbol{\phi} = \boldsymbol{\phi}^T P \mathbf{f}(\boldsymbol{\phi}) + \mathbf{f}^T(\boldsymbol{\phi}) P \boldsymbol{\phi} \\
&= \boldsymbol{\phi}^T P [A\boldsymbol{\phi} + \|\boldsymbol{\phi}\|_2 \mathbf{g}(\boldsymbol{\phi})] + [A\boldsymbol{\phi} + \|\boldsymbol{\phi}\|_2 \mathbf{g}(\boldsymbol{\phi})]^T P \boldsymbol{\phi} \\
&= \boldsymbol{\phi}^T (PA + A^T P) \boldsymbol{\phi} + \|\boldsymbol{\phi}\|_2 (\boldsymbol{\phi}^T P \mathbf{g}(\boldsymbol{\phi}) + \mathbf{g}^T(\boldsymbol{\phi}) P \boldsymbol{\phi}) \\
&= -\|\boldsymbol{\phi}\|_2^2 + \|\boldsymbol{\phi}\|_2 (\boldsymbol{\phi}^T P \mathbf{g}(\boldsymbol{\phi}) + \mathbf{g}^T(\boldsymbol{\phi}) P \boldsymbol{\phi}),
\end{aligned}$$

where $\boldsymbol{\phi}$ is the solution of (2.1).

Hence,

$$\begin{aligned}\frac{d}{dt}V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) &\leq -\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2 + 2\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2\|P\|_2\|\mathbf{g}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2 \\ &\leq -\left(\frac{1}{\lambda_{\max}} - 2\frac{\lambda_{\max}}{\lambda_{\min}}\|\mathbf{g}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))\|_2\right)V(\boldsymbol{\phi}(t,\boldsymbol{\xi})).\end{aligned}$$

Because $\mathbf{g}$ is continuous at zero, given any $\varepsilon > 0$, there is a $\delta > 0$, such that

$$\|\mathbf{g}(\boldsymbol{\xi})\|_2 < \frac{\lambda_{\min}}{\lambda_{\max}}\varepsilon,$$

whenever $\|\boldsymbol{\xi}\|_2 < \delta$. This implies

$$\frac{d}{dt}V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \leq -\left(\frac{1}{\lambda_{\max}} - 2\varepsilon\right)V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))$$

for all $t \geq 0$, such that $\|\boldsymbol{\phi}(t',\boldsymbol{\xi})\|_2 < \delta$ for all $t' \in [0,t]$, which in turn implies

$$V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \leq V(\boldsymbol{\xi})e^{-\left(\frac{1}{\lambda_{\max}}-2\varepsilon\right)t}$$

for all such t.

By using the upper and lower bounds of V,

$$\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2 \leq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\, e^{-\left(\frac{1}{2\lambda_{\max}}-\varepsilon\right)t}\|\boldsymbol{\xi}\|_2$$

follows. Because this bound of $\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2$ is valid for all $t \geq 0$, such that $\|\boldsymbol{\phi}(t',\boldsymbol{\xi})\|_2 < \delta$ for all $t' \in [0,t]$, it is obviously satisfied for all $t \geq 0$ if

$$\|\boldsymbol{\xi}\|_2 \leq \delta\sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}}.$$

Because $\varepsilon > 0$ was arbitrary, we have shown that for every

$$\alpha < \frac{1}{2\lambda_{\max}}$$

and every

$$m \geq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}},$$

there is a neighborhood $\mathcal{N}_{\alpha,m}$ of zero, such that the equilibrium is an α,m-exponentially stable equilibrium point of (2.1) on $\mathcal{N}_{\alpha,m}$. However, we have no idea how large the sets $\mathcal{N}_{\alpha,m}$ can maximally be.

If $\mathbf{f} \in \mathcal{C}^2(\mathcal{U})$ one can calculate lower bounds of the sets $\mathcal{N}_{\alpha,m}$ by using upper bounds of the second order derivatives of $\mathbf{f}$ on compact neighborhoods of zero, but these bounds are bound to be conservative in the general case. By guessing the maximum size of $\mathcal{N}_{\alpha,m}$, we can use the linear program $\mathbf{LP1}(\mathbf{f},\alpha,m,\mathcal{N}_{\alpha,m},\mathbf{h})$ to refute the hypothesis that the origin is α,m-exponentially stable on $\mathcal{N}_{\alpha,m}$. Further, it seems promising to use the linear program in an iterative algorithm to estimate the size of $\mathcal{N}_{\alpha,m}$. We will discuss this in more detail at the end of this thesis, where we will discuss ideas for future research on the topics covered in this thesis.

Although there are numerous approaches in the literature to construct Lyapunov functions numerically, a search in the excellent *ResearchIndex*[1]*: The NECI Scientific Literature Digital Library* and elsewhere, for a method to exclude classes of positive definite functions from the set of possible Lyapunov functions for a system, remained unsuccessful.

[1]http://citeseer.nj/nec.com/cs

Part III

Lyapunov Function Construction with Linear Programming

In this part we will give an algorithmic description of the derivation of a linear program for a nonlinear system. If this linear program has a feasible solution, then a piecewise affine Lyapunov or Lyapunov-like function for the system can be constructed by using the values of the variables.

The algorithm works roughly as follows:

i) Partition a neighborhood of the equilibrium under consideration in a family $\mathfrak{S}$ of simplices.

ii) Limit the search for a Lyapunov function V for the system, to the class of continuous functions, affine on any $S \in \mathfrak{S}$.

iii) State linear inequalities for the values of V at the vertices of the simplices in $\mathfrak{S}$, so that if they can be fulfilled, then the function V, which is uniquely determined by its values at the vertices, is a Lyapunov function for the system in the whole area.

This algorithm is similar to the procedure presented by Julian *et al.* in [19]. The difference is that the simplicial partition used here is more flexible and that a real Lyapunov function can be constructed from the variables. This will be explained in more detail at the end of this part, where we compare the approaches.

In the first chapter in this part, we will partition $\mathbb{R}^n$ into n-simplices and use this partition to define the function spaces CPWA of continuous piecewise affine functions $\mathbb{R}^n \longrightarrow \mathbb{R}$. A function in CPWA is uniquely determined by its values at the vertices of the simplices in $\mathfrak{S}$. In the second chapter we will state Linear Program **LP2**, an algorithm to generate linear programs for nonlinear systems, and prove that if such a linear program has a feasible solution, then a CPWA Lyapunov or Lyapunov-like function for the corresponding system can be constructed from the variables. Finally, we will evaluate the method and compare it to other approaches in the literature to construct Lyapunov functions for nonlinear systems, in particular to the approach of Julian *et al.*

Chapter 4

Continuous Piecewise Affine Functions

In order to construct a Lyapunov function via linear programming, one needs a class of continuous functions that are easily parameterized. The class of the continuous piecewise affine [1] functions is more or less the canonical candidate. In this chapter we are going to show how to partition $\mathbb{R}^n$ into arbitrary small simplices. The partition will have the property, that the vector space of continuous functions $\mathbb{R}^n \longrightarrow \mathbb{R}$, affine on each of the simplices, is isomorphic to a vector space where the vectors are tuples of real numbers.

4.1 Preliminaries

We start with the definition of a piecewise affine function.

Definition 4.1 *Let $\mathcal{U}$ be a subset of $\mathbb{R}^n$. A function $\mathbf{p} : \mathcal{U} \longrightarrow \mathbb{R}^m$ is called piecewise affine, if and only if there is a set of indices $\mathcal{I}$ and a family of sets $\mathcal{U}_i \subset \mathcal{U}$, $i \in \mathcal{I}$, such that $\bigcup_{i \in \mathcal{I}} \mathcal{U}_i = \mathcal{U}$, and for every $i \in \mathcal{I}$ the function $\mathbf{p}$ is affine on $\mathcal{U}_i$, i.e. for every $i \in \mathcal{I}$ there is an $m \times n$-matrix A_i and a vector $\mathbf{a}_i \in \mathbb{R}^m$, such that $\mathbf{p}(\mathbf{x}) = A_i\mathbf{x} + \mathbf{a}_i$ for all $\mathbf{x} \in \mathcal{U}_i$.*

$\square$

For our applications, piecewise affine functions $\mathbb{R}^n \longrightarrow \mathbb{R}^m$, where the sets $\mathcal{U}_i$ are n-simplices, are of special interest. A simplex is the convex hull of affinely independent vectors in $\mathbb{R}^n$, more exactly:

Definition 4.2 *Let $\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_k \in \mathbb{R}^n$. The set*

$$\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_k\} := \left\{ \mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \,\middle|\, \lambda_i \in [0,1] \ \ \textit{for all } i = 1, 2, .., k, \ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

is called the convex hull of the set $\{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_k\}$.

[1] The popular term for piecewise affine is piecewise linear. In higher mathematics the term linear is reserved for affine mappings that vanish at the origin, so we use the term affine in this thesis to avoid confusion.

If the vectors $\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_k$ *are affinely independent, i.e. if the vectors*

$$\mathbf{x}_1 - \mathbf{x}_j, \mathbf{x}_2 - \mathbf{x}_j, .., \mathbf{x}_{j-1} - \mathbf{x}_j, \mathbf{x}_{j+1} - \mathbf{x}_j, .., \mathbf{x}_k - \mathbf{x}_j$$

are linearly independent for any $j = 1, 2, .., k$, *then the set* $\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_k\}$ *is called a* $(k-1)$*-simplex and the vectors* $\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_k$ *are called the vertices of the simplex.* $\square$

The convex hull of a set $\{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_{n+1}\} \subset \mathbb{R}^n$ has a non-zero volume (n-dimensional Borel measure), if and only if it is an n-simplex. This follows from the well known facts, that the volume of $\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_{n+1}\}$ is the absolute value of

$$\det(\mathbf{x}_1 - \mathbf{x}_j, \mathbf{x}_2 - \mathbf{x}_j, .., \mathbf{x}_{j-1} - \mathbf{x}_j, \mathbf{x}_{j+1} - \mathbf{x}_j, .., \mathbf{x}_{n+1} - \mathbf{x}_j)$$

for any $j = 1, 2, .., n+1$, and that this determinant is non-zero, if and only if the vectors

$$\mathbf{x}_1 - \mathbf{x}_j, \mathbf{x}_2 - \mathbf{x}_j, .., \mathbf{x}_{j-1} - \mathbf{x}_j, \mathbf{x}_{j+1} - \mathbf{x}_j, .., \mathbf{x}_k - \mathbf{x}_j$$

are linearly independent.

A simplex has the nice property, that its elements have unique representations as convex combinations of the vertices of the simplex (see, for example, Exercise 2.28 in [39]). This makes them the optimal sets to define affine functions $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Lemma 4.3 *Let* $\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_{n+1}\}$ *be an* n*-simplex in* $\mathbb{R}^n$ *and* $\mathbf{a}_1, \mathbf{a}_2, .., \mathbf{a}_{n+1} \in \mathbb{R}^m$. *Then a function* $\mathbf{p} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is uniquely defined by:*

i) $\mathbf{p}$ *is affine.*

ii) $\mathbf{p}(\mathbf{x}_i) := \mathbf{a}_i$ *for all* $i = 1, 2, .., n+1$.

For an element $\mathbf{x} \in \mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_{n+1}\}$, $\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i$, $\lambda_i \in [0, 1]$ *for all* $i = 1, 2, .., n+1$, *and* $\sum_{i=1}^{n+1} \lambda_i = 1$, *we have*

$$\mathbf{p}(\mathbf{x}) = \sum_{i=1}^{n+1} \lambda_i \mathbf{a}_i. \tag{4.1}$$

Proof:
Because $\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_{n+1}\}$ is a simplex, the vectors

$$\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, .., \mathbf{x}_{n+1} - \mathbf{x}_1$$

are a basis for $\mathbb{R}^n$. This means that for every $\mathbf{y} \in \mathbb{R}^n$, there are unique real numbers $\mu_2^{\mathbf{y}}, \mu_3^{\mathbf{y}}, .., \mu_{n+1}^{\mathbf{y}}$, such that $\mathbf{y} = \sum_{i=2}^{n+1} \mu_i^{\mathbf{y}} (\mathbf{x}_i - \mathbf{x}_1)$. We define the linear function $\mathbf{l} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\mathbf{l}(\mathbf{y}) := \sum_{i=2}^{n+1} \mu_i^{\mathbf{y}} (\mathbf{a}_i - \mathbf{a}_1)$ for all $\mathbf{y} \in \mathbb{R}^n$. The function $\mathbf{p} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\mathbf{p}(\mathbf{y}) := \mathbf{l}(\mathbf{y} - \mathbf{x}_1) + \mathbf{a}_1$ for all $\mathbf{y} \in \mathbb{R}^n$, is an affine function with the property that $\mathbf{p}(\mathbf{x}_i) = \mathbf{a}_i$ for all $i = 1, 2, .., n+1$.

Assume there are two affine functions $\mathbf{p}_1, \mathbf{p}_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, such that $\mathbf{p}_1(\mathbf{x}_i) = \mathbf{p}_2(\mathbf{x}_i) = \mathbf{a}_i$ for all $i = 1, 2, .., n+1$. Because $\mathbf{p}_1$ and $\mathbf{p}_2$ are affine and $\mathbf{p}_1(\mathbf{x}_1) = \mathbf{p}_2(\mathbf{x}_1) = \mathbf{a}_1$, there exist two $m \times n$-matrices A_1 and A_2 such that $\mathbf{p}_j(\mathbf{x}) = A_j(\mathbf{x} - \mathbf{x}_1) + \mathbf{a}_1$ for all $\mathbf{x} \in \mathbb{R}^n$ and $j = 1, 2$. But this implies that $(A_1 - A_2)(\mathbf{x}_i - \mathbf{x}_1) = \mathbf{0}$ for all $i = 2, ..n+1$, i.e. $A_1 - A_2 = 0$ and then $\mathbf{p}_1 = \mathbf{p}_2$. The affine function $\mathbf{p}$ constructed above is thus unique.

Let A be an $m \times n$-matrix and $\mathbf{b} \in \mathbb{R}^m$ a vector, such that $\mathbf{p}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then (4.1) follows from

$$\mathbf{p}(\mathbf{x}) = A \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i + \mathbf{b} = \sum_{i=1}^{n+1} \lambda_i (A\mathbf{x}_i + \mathbf{b}) = \sum_{i=1}^{n+1} \lambda_i \mathbf{p}(\mathbf{x}_i) = \sum_{i=1}^{n+1} \lambda_i \mathbf{a}_i.$$

$\blacksquare$

In the next theorem we give sufficient restrictions on a family of n-simplices in $\mathbb{R}^n$, to serve as a domain for a continuous piecewise affine function, determined by its values at the vertices of the simplices.

Theorem 4.4 *Let $\mathcal{I}$ be a set of indices and $(S^{(i)})_{i\in\mathcal{I}}$ be a family of n-simplices in $\mathbb{R}^n$, $S^{(i)} := \text{con}\{\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(i)}, .., \mathbf{x}_{n+1}^{(i)}\}$ for every $i \in \mathcal{I}$. Define for every $i, j \in \mathcal{I}$ the set*

$$\mathcal{C}_{(i,j)} := \{\mathbf{c} \in \mathbb{R}^n \big| \text{ there are } r, s \in \{1, 2, .., n+1\}, \text{ such that } \mathbf{c} := \mathbf{x}_r^{(i)} = \mathbf{x}_s^{(j)}\}$$

and the set

$$S^{(i,j)} := \begin{cases} \emptyset, & \text{if} \quad \mathcal{C}_{(i,j)} = \emptyset, \\ \text{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_t\}, & \text{if} \quad \mathcal{C}_{(i,j)} = \{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_t\} \neq \emptyset \end{cases}.$$

Further, let $\{\mathbf{a}_{\mathbf{x}_j^{(i)}} \in \mathbb{R}^m \big| \; i \in \mathcal{I}, \; j = 1, 2, .., n+1\}$ be a set of vectors in $\mathbb{R}^m$ and define the affine functions $\mathbf{p}^{(i)} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $i \in \mathcal{I}$, $\mathbf{p}^{(i)}(\mathbf{x}_j^{(i)}) := \mathbf{a}_{\mathbf{x}_j^{(i)}}$ for all $i \in \mathcal{I}$ and all $j = 1, 2, .., n+1$.

Assume that $S^{(i)} \cap S^{(j)} = S^{(i,j)}$ for every $i, j \in \mathcal{I}$. Then the function $\mathbf{p} : \bigcup_{i\in\mathcal{I}} S^{(i)} \longrightarrow \mathbb{R}^m$, $\mathbf{p}(\mathbf{x}) := \mathbf{p}^{(i)}(\mathbf{x})$ if $\mathbf{x} \in S^{(i)}$, is a properly defined continuous piecewise affine function.

Proof:
The function $\mathbf{p}$ is properly defined, if and only if $\mathbf{p}^{(i)}(\mathbf{x}) = \mathbf{p}^{(j)}(\mathbf{x})$ whenever $\mathbf{x} \in S^{(i)} \cap S^{(j)}$. Lemma 4.3 implies, that for every $\mathbf{x} \in \mathbb{R}^n$, such that $\mathbf{x} \in S^{(i)} \cap S^{(j)}$, we have because of the assumption $S^{(i)} \cap S^{(j)} = S^{(i,j)}$, that

$$\mathbf{p}^{(k)}(\mathbf{x}) = \sum_{l=1}^{t} \lambda_l \mathbf{p}^{(k)}(\mathbf{c}_l) = \sum_{l=1}^{t} \lambda_l \mathbf{a}_{\mathbf{y}_l},$$

where $\mathcal{C}_{(i,j)} = \{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_t\} \neq \emptyset$, $\mathbf{x} = \sum_{l=1}^{t} \lambda_l \mathbf{c}_l$, $\lambda_l \in [0, 1]$ for all $l = 1, 2, .., t$, $\sum_{l=1}^{t} \lambda_l = 1$, and k is either i or j. Because the right hand side of this equation is not dependent of i or j, the function $\mathbf{p}$ is properly defined and obviously continuous.

■

We will use a simplicial partition of $\mathbb{R}^n$, invariable with respect to reflections through the hyperplanes $\mathbf{e}_i \cdot \mathbf{x} = 0$, $i = 1, 2, .., n$, as a domain for continuous piecewise affine functions. We will construct such a partition by first partitioning $\mathbb{R}^n_{\geq 0}$ and then we will extend this partition on $\mathbb{R}^n$ by use of the reflection functions $\mathbf{R}^{\mathcal{J}}$, where $\mathcal{J} \in \mathfrak{P}(\{1, 2, .., n\})$.

Definition 4.5 *For every $\mathcal{J} \in \mathfrak{P}(\{1, 2, .., n\})$, we define the reflection function $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$,*

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i$$

for all $\mathbf{x} \in \mathbb{R}^n$, where $\chi_{\mathcal{J}} : \{1, 2, .., n\} \longrightarrow \{0, 1\}$ is the characteristic function of the set $\mathcal{J}$.

□

Clearly $\mathbf{R}^{\mathcal{J}}$, where $\mathcal{J} = \{j_1, j_2, .., j_k\}$, represents reflections through the hyperplanes $\mathbf{e}_{j_1} \cdot \mathbf{x} = 0$, $\mathbf{e}_{j_2} \cdot \mathbf{x} = 0$, .., and $\mathbf{e}_{j_k} \cdot \mathbf{x} = 0$ in succession.

Let $(S^{(i)})_{i\in\mathcal{I}}$ be a family of n-simplices in $\mathbb{R}^n$ fulfilling the assumptions of Theorem 4.4, such that every simplex $S^{(i)}$, $i \in \mathcal{I}$, is entirely contained in $\mathbb{R}^n_{\geq 0}$. Then for every $\mathcal{J} \in \mathfrak{P}(\{1, 2, .., n\})$, the family $(\mathbf{R}^{\mathcal{J}}(S^{(i)}))_{i\in\mathcal{I}}$ is entirely contained in

$$\{\mathbf{x} \in \mathbb{R}^n \big|\ x_i \in]-\infty, 0] \ \text{ if } i \in \mathcal{J} \text{ and } x_i \in [0, +\infty[\text{ otherwise}\}$$

and trivially, because $\mathbf{R}^{\mathcal{J}}$ is linear and one-to-one, fulfills the assumptions of Theorem 4.4. In the next theorem we will show that even the family $(\mathbf{R}^{\mathcal{J}}(S^{(i)}))_{i\in\mathcal{I},\mathcal{J}\in\mathfrak{P}(\{1,2,..,n\})}$ fulfills the assumptions of Theorem 4.4.

Theorem 4.6 *Let $\mathcal{I}$ be a set of indices and $(S^{(i)})_{i\in\mathcal{I}}$ be a family of n-simplices contained in $\mathbb{R}^n_{\geq 0}$, fulfilling the assumptions of Theorem 4.4. Then the family $(\mathbf{R}^{\mathcal{J}}(S^{(i)}))_{i\in\mathcal{I},\mathcal{J}\in\mathfrak{P}(\{1,2,..,n\})}$ fulfills the assumptions of Theorem 4.4 too.*

PROOF:
Let $\mathcal{J} \in \mathfrak{P}(\{1, 2, .., n\})$ and $i, j \in \mathcal{I}$ be arbitrary. Then clearly $S^{(i)} \cap S^{(j)} = \emptyset$ implies $\mathbf{R}^{\mathcal{J}}(S^{(i)}) \cap S^{(j)} = \emptyset$. Assume that $S^{(i)} \cap S^{(j)} \neq \emptyset$, then the simplices $S^{(i)}$ and $S^{(j)}$ have common vertices, $\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k$, such that $S^{(i)} \cap S^{(j)} = \text{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\}$. Define the set

$$\mathcal{C}' = \{\mathbf{c}'_1, \mathbf{c}'_2, .., \mathbf{c}'_r\} := \{\mathbf{c}' \in \{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\} \ \big|\ \mathbf{R}^{\mathcal{J}}(\mathbf{c}') = \mathbf{c}'\}.$$

One easily sees that if $\mathcal{C}' = \emptyset$, then $\mathbf{R}^{\mathcal{J}}(S^{(i)}) \cap S^{(j)} = \emptyset$ and if $\mathcal{C}' \neq \emptyset$, then $\mathbf{R}^{\mathcal{J}}(S^{(i)}) \cap S^{(j)} = \text{con}\{\mathbf{c}'_1, \mathbf{c}'_2, .., \mathbf{c}'_r\}$. Now let $\mathcal{J}_1, \mathcal{J}_2 \in \mathfrak{P}(\{1, 2, .., n\})$ and $i, j \in \mathcal{I}$ be arbitrary. Because $\mathbf{R}^{\mathcal{J}_2}$ is one-to-one and its own inverse, we have

$$\begin{aligned}
\mathbf{R}^{\mathcal{J}_1}(S^{(i)}) \cap \mathbf{R}^{\mathcal{J}_2}(S^{(j)}) &= \mathbf{R}^{\mathcal{J}_2}(\mathbf{R}^{\mathcal{J}_2}[\mathbf{R}^{\mathcal{J}_1}(S^{(i)}) \cap \mathbf{R}^{\mathcal{J}_2}(S^{(j)})]) \\
&= \mathbf{R}^{\mathcal{J}_2}(\mathbf{R}^{\mathcal{J}_2}[\mathbf{R}^{\mathcal{J}_1}(S^{(i)})] \cap \mathbf{R}^{\mathcal{J}_2}[\mathbf{R}^{\mathcal{J}_2}(S^{(j)})]) \\
&= \mathbf{R}^{\mathcal{J}_2}(\mathbf{R}^{\mathcal{J}_1\Delta\mathcal{J}_2}(S^{(i)})] \cap S^{(j)}),
\end{aligned}$$

where $\mathcal{J}_1\Delta\mathcal{J}_2 := (\mathcal{J}_1 \cup \mathcal{J}_2) \setminus (\mathcal{J}_1 \cap \mathcal{J}_2)$ is the set theoretic symmetric difference. Because of the previous considerations and because $\mathbf{R}^{\mathcal{J}_2}$ is a one-to-one linear mapping, the set $\mathbf{R}^{\mathcal{J}_1}(S^{(i)}) \cap \mathbf{R}^{\mathcal{J}_2}(S^{(j)})$ is either empty or a simplex, determined by the vertices that are common to $\mathbf{R}^{\mathcal{J}_1}(S^{(i)})$ and $\mathbf{R}^{\mathcal{J}_2}(S^{(j)})$. This means, that the family $(\mathbf{R}^{\mathcal{J}}(S^{(i)}))_{i\in\mathcal{I},\mathcal{J}\in\mathfrak{P}(\{1,2,..,n\})}$ fulfills the assumptions of Theorem 4.4.

■

4.2 Simplicial Partition of $\mathbb{R}^n$

From Theorem 4.4 we know what properties of a family of n-simplices in $\mathbb{R}^n$ are sufficient, for them to serve as a domain for a continuous piecewise affine function. In this section we will give a concrete simplicial partition of $\mathbb{R}^n$, where the simplices of the partition fulfill the assumptions of Theorem 4.4.

For every $n \in \mathbb{Z}_{>0}$, we denote by Sym_n the *permutation group* of $\{1, 2, .., n\}$, i.e. Sym_n is the set of the one-to-one mappings from $\{1, 2, .., n\}$ onto itself.

The simplices S_σ, where $\sigma \in \text{Sym}_n$, serve as the atoms of our partition. They are defined in the following way.

Definition 4.7 *For every $\sigma \in \mathrm{Sym}_n$ we define the set*

$$S_\sigma := \{\mathbf{y} \in \mathbb{R}^n \big| \; 0 \leq y_{\sigma(1)} \leq y_{\sigma(2)} \leq .. \leq y_{\sigma(n)} \leq 1\},$$

where $y_{\sigma(i)}$ is the $\sigma(i)$-th component of the vector $\mathbf{y}$.

$\square$

For every $\sigma \in \mathrm{Sym}_n$ the set S_σ is an n-simplex with the volume $1/n!$. That it is an n-simplex follows from the next theorem. That its volume is $1/n!$ follows from straight forward integration

$$\begin{aligned}
\int \chi_{S_\sigma}(\mathbf{x})\, d^n x &= \int_0^1 \left(\int_0^{x_{\sigma(n)}} \left(.. \left(\int_0^{x_{\sigma(2)}} dx_{\sigma(1)} \right) .. \right) dx_{\sigma(n-1)} \right) dx_{\sigma(n)} \\
&= \int_0^1 \left(\int_0^{x_{\sigma(n)}} \left(.. \left(\int_0^{x_{\sigma(3)}} x_{\sigma(2)} dx_{\sigma(2)} \right) .. \right) dx_{\sigma(n-1)} \right) dx_{\sigma(n)} \\
&= \int_0^1 \left(\int_0^{x_{\sigma(n)}} \left(.. \left(\int_0^{x_{\sigma(4)}} \frac{1}{2} x_{\sigma(3)}^2 dx_{\sigma(3)} \right) .. \right) dx_{\sigma(n-1)} \right) dx_{\sigma(n)} \\
&= \frac{1}{n!}.
\end{aligned}$$

Before stating and proving the next theorem we will first state and prove a technical lemma, that will be used in its proof and later on.

Lemma 4.8 *Let $\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_n \in \mathbb{R}^m$ and $\mu_1, \mu_2, .., \mu_n \in \mathbb{R}$. Then*

$$\sum_{i=1}^n \mu_i \sum_{j=i}^n \mathbf{v}_j = \sum_{j=1}^n \mathbf{v}_j \sum_{i=1}^j \mu_i.$$

PROOF:

$$\begin{aligned}
\sum_{i=1}^n \mu_i \sum_{j=i}^n \mathbf{v}_j &= \mu_1 \sum_{j=1}^n \mathbf{v}_j + \mu_2 \sum_{j=2}^n \mathbf{v}_j + .. + \mu_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_n) + \mu_n \mathbf{v}_n \\
&= \mathbf{v}_n \sum_{i=1}^n \mu_i + \mathbf{v}_{n-1} \sum_{i=1}^{n-1} \mu_i + .. + \mathbf{v}_2(\mu_1 + \mu_2) + \mathbf{v}_1 \mu_1 = \sum_{j=1}^n \mathbf{v}_j \sum_{i=1}^j \mu_i.
\end{aligned}$$

$\blacksquare$

The next theorem states that the set S_σ is an n-simplex and provides a formula for its vertices.

Theorem 4.9 *For every $\sigma \in \mathrm{Sym}_n$ we have*

$$S_\sigma = \mathrm{con}\{\sum_{j=1}^n \mathbf{e}_{\sigma(j)}, \sum_{j=2}^n \mathbf{e}_{\sigma(j)}, .., \sum_{j=n+1}^n \mathbf{e}_{\sigma(j)}\},$$

where $\mathbf{e}_{\sigma(i)}$ is the $\sigma(i)$-th unit vector in $\mathbb{R}^n$.

PROOF:
We first show, that

$$\mathrm{con}\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)},\sum_{j=2}^{n}\mathbf{e}_{\sigma(j)},..,\sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\}\subseteq S_\sigma.$$

Let

$$\mathbf{y}\in\mathrm{con}\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)},\sum_{j=2}^{n}\mathbf{e}_{\sigma(j)},..,\sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\}.$$

Then there are $\lambda_1,\lambda_2,..,\lambda_{n+1}\in[0,1]$, such that

$$\mathbf{y}=\sum_{i=1}^{n+1}\lambda_i\sum_{j=i}^{n}\mathbf{e}_{\sigma(j)}\quad\text{and}\quad\sum_{i=1}^{n+1}\lambda_i=1.$$

Because

$$y_{\sigma(k)} \;=\; \mathbf{y}\cdot\mathbf{e}_{\sigma(k)}=\left(\sum_{i=1}^{n+1}\lambda_i\sum_{j=i}^{n}\mathbf{e}_{\sigma(j)}\right)\cdot\mathbf{e}_{\sigma(k)}=\sum_{i=1}^{n+1}\sum_{j=i}^{n}\lambda_i\delta_{jk}=\sum_{i=1}^{k}\lambda_i,$$

it follows that $y_{\sigma(k)}\in[0,1]$ and $y_{\sigma(k)}\leq y_{\sigma(l)}$ if $k\leq l$, so $\mathbf{y}\in S_\sigma$.
We now show that

$$S_\sigma\subseteq\mathrm{con}\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)},\sum_{j=2}^{n}\mathbf{e}_{\sigma(j)},..,\sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\}.$$

Let $\mathbf{y}\in S_\sigma$. Then $0\leq y_{\sigma(1)}\leq y_{\sigma(2)}\leq..\leq y_{\sigma(n)}\leq 1$. Set

$$\begin{aligned}\lambda_1&:=y_{\sigma(1)},\\ \lambda_2&:=y_{\sigma(2)}-y_{\sigma(1)},\\ \lambda_3&:=y_{\sigma(3)}-y_{\sigma(2)},\\ &\vdots\\ \lambda_n&:=y_{\sigma(n)}-y_{\sigma(n-1)},\end{aligned}$$

and

$$\lambda_{n+1}=1-y_{\sigma(n)}.$$

Then obviously

$$\sum_{i=1}^{n+1}\lambda_i=1$$

and by Lemma 4.8

$$\mathbf{y}=\sum_{j=1}^{n}\sum_{i=1}^{j}\lambda_i\mathbf{e}_{\sigma(j)}=\sum_{i=1}^{n+1}\lambda_i\sum_{j=i}^{n}\mathbf{e}_{\sigma(j)},$$

i.e.

$$\mathbf{y}\in\mathrm{con}\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)},\sum_{j=2}^{n}\mathbf{e}_{\sigma(j)},..,\sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\}.$$

∎

In the next two theorems we will show, that the family of simplices $(\mathbf{z} + S_\sigma)_{\mathbf{z}\in\mathbb{Z}^n_{\geq 0},\sigma\in\mathrm{Sym}_n}$ partitions $\mathbb{R}^n_{\geq 0}$ and is compatible with the assumptions of Theorem 4.4. We start by showing that the simplices $(S_\sigma)_{\sigma\in\mathrm{Sym}_n}$ partition the set $[0,1]^n$ in the appropriate way.

Theorem 4.10 *Let $\alpha, \beta \in \mathrm{Sym}_n$. Then*

$$S_\alpha \cap S_\beta = \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\},$$

where the $\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k$ are the vertices that are common to S_α and S_β, i.e.

$$\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\} := \{\mathbf{x} \in \mathbb{R}^n \big| \mathbf{x} = \sum_{j=a}^{n} \mathbf{e}_{\alpha(j)} = \sum_{j=a}^{n} \mathbf{e}_{\beta(j)} \text{ for some } a \in \{1, 2, .., n+1\}\}.$$

Proof:
The inclusion

$$S_\alpha \cap S_\beta \supset \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\}$$

is trivial, so we only have to prove

$$S_\alpha \cap S_\beta \subset \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\}. \tag{4.2}$$

To do this define $\sigma \in \mathrm{Sym}_n$, $\sigma := \beta^{-1}\alpha$, and the set

$$\mathcal{A}_\sigma := \{x \in \{1, 2, .., n+1\} \big| \sigma(\{1, 2, .., x-1\}) = \{1, 2, .., x-1\}\}.$$

Clearly

$$\begin{aligned}
\mathcal{A}_\sigma &:= \{x \in \{1, 2, .., n+1\} \big| \sigma(\{1, 2, .., x-1\}) = \{1, 2, .., x-1\}\} \\
&= \{x \in \{1, 2, .., n+1\} \big| \sigma(\{x, x+1, .., n\}) = \{x, x+1, .., n\}\} \\
&= \{x \in \{1, 2, .., n+1\} \big| \beta^{-1}\alpha(\{x, x+1, .., n\}) = \{x, x+1, .., n\}\} \\
&= \{x \in \{1, 2, .., n+1\} \big| \alpha(\{x, x+1, .., n\}) = \beta(\{x, x+1, .., n\})\}
\end{aligned}$$

and

$$\sum_{j=a}^{n} \mathbf{e}_{\alpha(j)} = \sum_{j=b}^{n} \mathbf{e}_{\beta(j)},$$

if and only if $a = b$ and $\alpha(\{a, a+1, .., n\}) = \beta(\{b, b+1, .., n\})$.
Hence

$$\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\} = \{\sum_{j=a_1}^{n} \mathbf{e}_{\alpha(j)}, \sum_{j=a_2}^{n} \mathbf{e}_{\alpha(j)}, .., \sum_{j=a_k}^{n} \mathbf{e}_{\alpha(j)}\},$$

where the $a_1, a_2, .., a_k$ are the elements of $\mathcal{A}_\sigma$. For convenience let $a_1 < a_2 < .. < a_k$.
Let $\mathbf{x}$ be an arbitrary element in $S_\alpha \cap S_\beta$. Then there are $\mu_i, \lambda_i \in [0,1]$, for $i = 1, 2, .., n+1$, such that

$$\sum_{i=1}^{n+1} \mu_i = \sum_{i=1}^{n+1} \lambda_i = 1$$

and

$$\mathbf{x} = \sum_{i=1}^{n+1} \mu_i \sum_{j=i}^{n} \mathbf{e}_{\alpha(j)} = \sum_{i=1}^{n+1} \lambda_i \sum_{j=i}^{n} \mathbf{e}_{\beta(j)}.$$

We prove (4.2) by showing by mathematical induction, that $\mu_i = \lambda_i$ for all $i \in \{1, 2, .., n+1\}$ and that $\mu_i = \lambda_i = 0$ for all $i \in \{1, 2, .., n+1\} \setminus \mathcal{A}_\sigma$.

Lemma 4.8 implies

$$\mathbf{x} = \sum_{j=1}^{n} \mathbf{e}_{\alpha(j)} \sum_{i=1}^{j} \mu_i = \sum_{j=1}^{n} \mathbf{e}_{\beta(j)} \sum_{i=1}^{j} \lambda_i.$$

By comparing the components of the vectors on the left-hand and the right-hand side of this equation, we see that for every pair $(r, s) \in \{1, 2, .., n\}^2$, such that $\alpha(r) = \beta(s)$, we must have

$$\sum_{i=1}^{r} \mu_i = \sum_{i=1}^{s} \lambda_i,$$

i.e.

$$\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{\sigma(m)} \lambda_i \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{\sigma^{-1}(m)} \mu_i \tag{4.3}$$

for all $m = 1, 2, .., n$.

Let $P(r)$ be the proposition: "$\mu_i = \lambda_i$ for all $i \in \{1, 2, .., a_r - 1\}$ and $\mu_i = \lambda_i = 0$ for all $i \in \{1, 2, .., a_r - 1\} \setminus \{a_1, a_2, .., a_{r-1}\}$"

From the definition of $\mathcal{A}_\sigma$ it follows that $a_1 = 1$, so the proposition $P(1)$ is obviously true. There is indeed nothing to prove. We show that for $r < k$, $P(r)$ implies $P(r + 1)$.

Assume that $P(r)$ is true for some $1 \leq r < k$. Then $\mu_i = \lambda_i$ for all $i = 1, 2, .., a_r - 1$ and $\sigma(a_r)$ and $\sigma^{-1}(a_r)$ must be greater than or equal to a_r. If $\sigma(a_r) = \sigma^{-1}(a_r) = a_r$, then trivially $a_r + 1 \in \mathcal{A}_\sigma$, i.e. $a_{r+1} = a_r + 1$, and it follows from (4.3) that $\mu_{a_r} = \lambda_{a_r}$, which implies that $P(r + 1)$ is true.

Suppose $\sigma(a_r)$ and $\sigma^{-1}(a_r)$ are greater that a_r. Then it follows from (4.3), that

$$\sum_{i=1}^{a_r} \mu_i = \sum_{i=1}^{\sigma(a_r)} \lambda_i = \sum_{i=1}^{a_r-1} \mu_i + \lambda_{a_r} + \sum_{i=a_r+1}^{\sigma(a_r)} \lambda_i,$$

i.e.

$$\mu_{a_r} = \lambda_{a_r} + \sum_{i=a_r+1}^{\sigma(a_r)} \lambda_i,$$

and similarly

$$\lambda_{a_r} = \mu_{a_r} + \sum_{i=a_r+1}^{\sigma^{-1}(a_r)} \mu_i.$$

By adding the two last equation, we see that

$$\sum_{i=a_r+1}^{\sigma(a_r)} \lambda_i + \sum_{i=a_r+1}^{\sigma^{-1}(a_r)} \mu_i = 0$$

and because $\mu_i, \lambda_i \in [0, 1]$ for all $i = 1, 2, .., n$, this implies that

$$\mu_{a_r+1} = \mu_{a_r+2} = .. = \mu_{\sigma^{-1}(a_r)} = 0,$$

$$\lambda_{a_r+1} = \lambda_{a_r+2} = .. = \lambda_{\sigma(a_r)} = 0,$$

and

$$\mu_{a_r} = \lambda_{a_r}.$$

Define the integers a and b by

$$a := \max\{s < a_{r+1} \big| \mu_{a_r+1} = \mu_{a_r+2} = .. = \mu_s = 0\}$$

and

$$b := \max\{s < a_{r+1} \big| \lambda_{a_r+1} = \lambda_{a_r+2} = .. = \lambda_s = 0\}.$$

For all $m \in \{a_r + 1, a_r + 2, .., a\}$ we have

$$\sum_{i=1}^{\sigma(m)} \lambda_i = \sum_{i=1}^{m} \mu_i = \sum_{i=1}^{a_r} \mu_i$$

and because $\mu_i = \lambda_i$ for all $i = 1, 2, .., a_r$, this implies that

$$\sum_{i=a_r+1}^{\sigma(m)} \lambda_i = 0,$$

i.e. that

$$\lambda_{a_r+1} = \lambda_{a_r+2} = .. = \lambda_{\sigma(m)} = 0.$$

Therefore

$$b \geq \max\{\sigma(m) \big| m = a_r + 1, a_r + 2, .., a\} = \max\{\sigma(m) \big| m = 1, 2, .., a\},$$

where the equality on the right-hand side is a consequence of $\sigma(\{1, 2, .., a_r - 1\}) = \{1, 2, .., a_r - 1\}$ and $b \geq \sigma(a_r)$.

The set $\{\sigma(m) \big| m = 1, 2, .., a\}$ is a subset of $\{1, 2, .., n\}$ with a distinct elements, so

$$\max\{\sigma(m) \big| m = 1, 2, .., a\} \geq a,$$

i.e. $b \geq a$. With similar reasoning, we can show that $a \geq b$. Hence $a = b$.

We have shown that $\mu_{a_r} = \lambda_{a_r}$, that there is a constant a, with $a_r < a < a_{r+1}$, such that

$$\mu_{a_r+1} = \mu_{a_r+2} = .. = \mu_a = \lambda_{a_r+1} = \lambda_{a_r+2} = .. = \lambda_a = 0,$$

and that $\sigma(\{1, 2, .., a\}) = \{1, 2, .., a\}$. This implies $a+1 = a_{r+1} \in \mathcal{A}_\sigma$ and that the proposition $P(r + 1)$ is true, which completes the mathematical induction.

■

We now apply the last theorem to prove that $(\mathbf{s} + S_\sigma)_{\mathbf{s}\in\mathbb{Z}^n_{\geq 0}, \sigma\in\mathrm{Sym}_n}$ partitions $\mathbb{R}^n_{\geq 0}$ in the appropriate way.

Theorem 4.11 *Let $\alpha, \beta \in \mathrm{Sym}_n$ and $\mathbf{s}_\alpha, \mathbf{s}_\beta \in \mathbb{Z}^n$. Let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\}$ be the set of the vertices common to the simplices $\mathbf{s}_\alpha + S_\alpha$ and $\mathbf{s}_\beta + S_\beta$. Then $(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \emptyset$ if $\mathcal{C} = \emptyset$ and*

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\}$$

if $\mathcal{C} \neq \emptyset$.

PROOF:
Obviously $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\} \neq \emptyset$ implies

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) \supset \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\},$$

and $(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \emptyset$ implies $\mathcal{C} = \emptyset$, so we only have to prove that if $(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) \neq \emptyset$, then $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\} \neq \emptyset$ and

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) \subset \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\}.$$

Assume that $(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) \neq \emptyset$. Clearly

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \mathbf{s}_\alpha + S_\alpha \cap (\mathbf{z} + S_\beta),$$

with $\mathbf{z} := \mathbf{s}_\beta - \mathbf{s}_\alpha$.

Let $\mathbf{x} \in S_\alpha \cap (\mathbf{z} + S_\beta)$. Then there are $\mu_i, \lambda_i \in [0, 1]$ for $i = 1, 2, .., n + 1$, such that

$$\sum_{i=1}^{n+1} \mu_i = \sum_{i=1}^{n+1} \lambda_i = 1$$

and

$$\mathbf{x} = \sum_{i=1}^{n+1} \mu_i \sum_{j=i}^{n} \mathbf{e}_{\alpha(j)} = \sum_{i=1}^{n+1} \lambda_i \sum_{j=i}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z} = \sum_{i=1}^{n+1} \lambda_i \left(\sum_{j=i}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z} \right). \tag{4.4}$$

Because $S_\alpha, S_\beta \subset [0, 1]^n$ the components of $\mathbf{z}$ must all be equal to -1, 0, or 1.

If $z_i = -1$, then the i-th component of the vectors

$$\sum_{j=1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=2}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad .. \quad , \quad \sum_{j=\beta^{-1}(i)}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

is equal to 0 and the i-th component of the vectors

$$\sum_{j=\beta^{-1}(i)+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=\beta^{-1}(i)+2}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad .. \quad , \quad \sum_{j=n+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

is equal to -1. Because $x_i \geq 0$, this implies that

$$\lambda_{\beta^{-1}(i)+1} = \lambda_{\beta^{-1}(i)+2} = \ .. \ = \lambda_{n+1} = 0.$$

By applying this reasoning on all components of $\mathbf{z}$ that are equal to -1, it follows that there is a smallest $s \in \{1, .., n + 1\}$ so that

$$\lambda_{s+1} = \lambda_{s+2} = \ .. \ = \lambda_{n+1} = 0.$$

If $z_i = 1$, then the i-th component of the vectors

$$\sum_{j=1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=2}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad .. \quad , \quad \sum_{j=\beta^{-1}(i)}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

is equal to 2 and the i-th component of the vectors

$$\sum_{j=\beta^{-1}(i)+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=\beta^{-1}(i)+2}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad .. \quad , \quad \sum_{j=n+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

is equal to 1. Because $x_i \leq 1$, this implies that

$$\lambda_1 = \lambda_2 = \ .. \ = \lambda_{\beta^{-1}(i)} = 0.$$

By applying this reasoning on all components of $\mathbf{z}$ that are equal to 1, it follows that there is a largest $r \in \{1, 2, .., n+1\}$ so that

$$\lambda_1 = \lambda_2 = \ .. \ = \lambda_{r-1} = 0.$$

If $r > s$ then $\lambda_1 = \lambda_2 = .. = \lambda_{n+1} = 0$, which is in contradiction to their sum being equal to 1. Therefore $r \leq s$.

From the definitions of r and s it follows, that the components of the vectors

$$\sum_{j=r}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=r+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad .. \quad , \sum_{j=s}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

are all equal to 0 or 1. Let $a_1, a_2, .., a_{t-1}$ be the indices of those components of

$$\sum_{j=r}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

that are equal to 0. By defining the permutation $\gamma \in \mathrm{Sym}_n$ as follows,

$$\begin{aligned}
&\gamma(i) := a_i, \ \text{for all } i = 1, 2, .., t-1, \\
&\gamma(i) := \beta(r + i - t), \ \text{for all } i = t, t+1, .., t+s-r,
\end{aligned}$$

and with $\{b_{t+s-r+1}, b_{t+s-r+2}, .., b_n\} := \{1, 2, .., n\} \setminus \gamma(\{1, 2, .., t+s-r\})$,

$$\gamma(i) := b_i, \ \text{for all } i = t+s-r+1, t+s-r+2, .., n,$$

we obtain,

$$\sum_{i=t+j}^{n} \mathbf{e}_{\gamma(i)} = \sum_{i=r+j}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z}. \tag{4.5}$$

for all $j = 0, 1, .., s-r$. To see this, note that

$$\sum_{i=t}^{n} \mathbf{e}_{\gamma(i)}$$

has all components equal to 1, except for those with an index from the set $\gamma(\{1, 2, .., t-1\})$, which are equal to 0. But from $\gamma(\{1, 2, .., t-1\}) = \{a_1, a_2, .., a_{t-1}\}$ and the definition of the indices a_i, it follows that

$$\sum_{i=t}^{n} \mathbf{e}_{\gamma(i)} = \sum_{i=r}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z}.$$

The equivalence for all $j = 0, 1, .., s-r$ then follows from

$$\begin{aligned}\sum_{i=t+j}^{n} \mathbf{e}_{\gamma(i)} &= \sum_{i=t}^{n} \mathbf{e}_{\gamma(i)} - \sum_{i=t}^{t+j-1} \mathbf{e}_{\gamma(i)} = \sum_{i=r}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z} - \sum_{i=t}^{t+j-1} \mathbf{e}_{\beta(r+i-t)} \\ &= \sum_{i=r}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z} - \sum_{i=r}^{r+j-1} \mathbf{e}_{\beta(i)} = \sum_{i=r+j}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z}.\end{aligned}$$

Because of

$$\mathbf{x} \in S_\alpha \cap (\mathbf{z} + S_\beta)$$

and

$$\mathbf{x} = \sum_{i=r}^{s} \lambda_i \left(\sum_{j=i}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z} \right) = \sum_{i=t}^{t+s-r} \lambda_{i+r-t} \sum_{j=i}^{n} \mathbf{e}_{\gamma(j)}$$

we have

$$\mathbf{x} \in S_\alpha \cap S_\gamma.$$

By Theorem 4.10 and (4.4), it follows that

$$\mathbf{x} = \sum_{i=1}^{l} \mu_{u_i} \sum_{j=u_i}^{n} \mathbf{e}_{\alpha(j)} = \sum_{i=1}^{l} \mu_{u_i} \sum_{j=u_i}^{n} \mathbf{e}_{\gamma(j)},$$

where

$$\{u_1, u_2, .., u_l\} := \{y \in \{1, 2, .., n+1\} \big| \alpha(\{1, 2, .., y-1\}) = \gamma(\{1, 2, .., y-1\})\}.$$

Because the representation of an element of a simplex as a convex sum of its vertices is unique, some of the $u_1, u_2, .., u_l$ must be larger than or equal to t and less than or equal to $t+s-r$, say $u_{m_1}, u_{m_2}, .., u_{m_k}$, and we have

$$\sum_{j=u_i}^{n} \mathbf{e}_{\alpha(j)} = \sum_{j=u_i+r-t}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

for all $i = 1, 2, .., k$, i.e.

$$\mathcal{C} := \{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\} = \{\mathbf{s}_\alpha + \sum_{j=u_1}^{n} \mathbf{e}_{\alpha(j)},\ \mathbf{s}_\alpha + \sum_{j=u_2}^{n} \mathbf{e}_{\alpha(j)},\ \ ..\ \ ,\ \mathbf{s}_\alpha + \sum_{j=u_k}^{n} \mathbf{e}_{\alpha(j)}\}$$

and

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \mathbf{s}_\alpha + S_\alpha \cap (\mathbf{z} + S_\beta) \subset \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, .., \mathbf{c}_k\}.$$

■

From the last theorem and Theorem 4.6 we finally get the simplicial partition of $\mathbb{R}^n$ that we will use in the next section to define the function spaces CPWA.

Corollary 4.12 *The family of simplices* $(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma))_{\mathbf{z} \in \mathbb{Z}^n_{\geq 0},\, \mathcal{J} \in \mathfrak{P}(\{1,2,..,n\}),\, \sigma \in \mathrm{Sym}_n}$ *partitions* $\mathbb{R}^n$ *and fulfills the assumptions of Theorem 4.4.*

■

4.3 The Function Spaces CPWA

In this section we will introduce the function spaces CPWA and derive some results regarding the functions in these spaces, that will be useful when we prove that a feasible solution of a linear program generated by Linear Program **LP2** can be used to define a CPWA Lyapunov function.

A CPWA space is a set of continuous affine functions from a subset of $\mathbb{R}^n$ into $\mathbb{R}$ with a given boundary configuration. If the subset is compact, then the boundary configuration makes it possible to parameterize the functions in the respective CPWA space with a finite number of real parameters. Further, the CPWA spaces are vector spaces over $\mathbb{R}$ in the canonical way. They are thus well suited as a foundation, in the search of a Lyapunov function with a linear program.

We first define the function spaces CPWA for subsets of $\mathbb{R}^n$ that are the unions of n-dimensional cubes.

Definition 4.13 *Let $\mathcal{Z} \subset \mathbb{Z}^n$, $\mathcal{Z} \neq \emptyset$, be such that the set*

$$\mathcal{N} := \bigcup_{\mathbf{z}\in\mathcal{Z}} (\mathbf{z} + [0,1]^n),$$

is connected.

The function space CPWA$[\mathcal{N}]$ *is then defined as follows.*

The function $p : \mathcal{N} \longrightarrow \mathbb{R}$ is in CPWA$[\mathcal{N}]$*, if and only if:*

i) p is continuous.

ii) For every simplex $\mathbf{R}^{\mathcal{J}}(\mathbf{z}+S_\sigma)$, $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$, $\mathcal{J} \in \mathfrak{P}(\{1,2,..,n\})$, and $\sigma \in \mathrm{Sym}_n$, contained in $\mathcal{N}$, the restriction $p|_{\mathbf{R}^{\mathcal{J}}(\mathbf{z}+S_\sigma)}$ is affine.

□

It follows from Corollary 4.12 that the set CPWA$[\mathcal{N}]$ is not empty and that its elements are uniquely determined by their values on the set $\mathcal{N} \cap \mathbb{Z}^n$.

We will need continuous piecewise affine functions, defined by their values on grids with smaller grid steps than one, and we want to use grids with variable grid steps. We achieve this by using images of $\mathbb{Z}^n$ under mappings $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, of which the components are continuous and strictly increasing functions on $\mathbb{R} \longrightarrow \mathbb{R}$, affine on the intervals $[m, m+1]$ where m is an integer, and map zero on itself. We call such $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ mappings *piecewise scaling functions*.

Definition 4.14 *A function $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called a piecewise scaling function, if and only if $\mathbf{PS}(\mathbf{0}) = \mathbf{0}$ and*

$$\mathbf{PS}(\mathbf{x}) = (\mathrm{PS}_1(\mathbf{x}), \mathrm{PS}_2(\mathbf{x}), .., \mathrm{PS}_n(\mathbf{x})) = (\widetilde{\mathrm{PS}}_1(x_1), \widetilde{\mathrm{PS}}_2(x_2), .., \widetilde{\mathrm{PS}}_n(x_n))$$

for all $\mathbf{x} \in \mathbb{R}^n$, where $\widetilde{\mathrm{PS}}_i \in$ CPWA$[\mathbb{R}]$ and is strictly increasing on $\mathbb{R}$ for all $i = 1, 2, .., n$.

□

Note that if $y_{i,j}$, $i = 1, 2, .., n$ and $j \in \mathbb{Z}$, are real numbers such that $y_{i,j} < y_{i,j+1}$ and $y_{i,0} = 0$ for all $i = 1, 2, .., n$ and all $j \in \mathbb{Z}$, then we can define a piecewise scaling function $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, by $\widetilde{\mathrm{PS}_\mathrm{i}}(j) := y_{i,j}$ for all $i = 1, 2, .., n$ and all $j \in \mathbb{Z}$. Moreover, the piecewise scaling functions $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ are exactly the functions, that can be constructed in this way.

In the next definition we use piecewise scaling functions to define general CPWA spaces.

Definition 4.15 *Let* $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a piecewise scaling function and let* $\mathcal{Z} \subset \mathbb{Z}^n$, $\mathcal{Z} \neq \emptyset$, *be such that the set*

$$\mathcal{N} := \bigcup_{\mathbf{z} \in \mathcal{Z}} (\mathbf{z} + [0,1]^n)$$

is connected.

The function space $\mathrm{CPWA}[\mathbf{PS}, \mathcal{N}]$ *is defined as*

$$\mathrm{CPWA}[\mathbf{PS}, \mathcal{N}] := \{p \circ \mathbf{PS}^{-1} \,\big|\, p \in \mathrm{CPWA}[\mathcal{N}]\}$$

and we denote by $\mathfrak{S}[\mathbf{PS}, \mathcal{N}]$ *the set of the simplices in the family*

$$(\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma)))_{\mathbf{z} \in \mathbb{Z}^n_{\geq 0},\ \mathcal{J} \in \mathfrak{P}(\{1,2,..,n\}),\ \sigma \in \mathrm{Sym}_n}$$

that are contained in the image $\mathbf{PS}(\mathcal{N})$ *of* $\mathcal{N}$ *under* $\mathbf{PS}$. □

Clearly

$$\{\mathbf{x} \in \mathbb{R}^n \big| \ \mathbf{x} \text{ is a vertex of a simplex in } \mathfrak{S}[\mathbf{PS}, \mathcal{N})]\} = \mathbf{PS}(\mathcal{N} \cap \mathbb{Z}^n)$$

and every function in $\mathrm{CPWA}[\mathbf{PS}, \mathcal{N}]$ is uniquely determined by its values on the grid $\mathbf{PS}(\mathcal{N} \cap \mathbb{Z}^n)$.

We will use functions from $\mathrm{CPWA}[\mathbf{PS}, \mathcal{N}]$ to approximate functions in $\mathcal{C}(\mathbf{PS}(\mathcal{N}))$, which have bounded second order derivatives on the interiors of the simplices in $\mathfrak{S}[\mathbf{PS}, \mathcal{N}]$. The next lemma gives upper bounds of the approximation error of such an approximation.

Lemma 4.16 *Let* $\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2 .., \mathbf{x}_{n+1}\}$ *be an* n*-simplex in* $\mathbb{R}^n$ *and* $g : \mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2 .., \mathbf{x}_{n+1}\} \longrightarrow \mathbb{R}$ *be a function, such that the partial derivatives*

$$\frac{\partial^2 g}{\partial x_l \partial x_k}$$

exist in the interior of $\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2 .., \mathbf{x}_{n+1}\}$ *for all* $k, l = 1, 2, .., n$. *Further, assume that they are continuous and bounded, i.e. there are constants* $B_{kl} < +\infty$, $k, l = 1, 2, .., n$, *such that*

$$\left| \frac{\partial^2 g}{\partial x_l \partial x_k}(\mathbf{x}) \right| \leq B_{kl}$$

for all $\mathbf{x}$ *in the interior of* $\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2 .., \mathbf{x}_{n+1}\}$ *and all* $k, l = 1, 2, .., n$.

Let $a : \mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2 .., \mathbf{x}_{n+1}\} \longrightarrow \mathbb{R}$ *be an affine function, defined by*

$$a(\mathbf{x}_i) = g(\mathbf{x}_i), \quad \textit{for all } i = 1, 2, .., n+1.$$

Then for any $\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i$, $0 \leq \lambda_i \leq 1$ *for all* $i = 1, 2, .., n+1$, *and* $\sum_{i=1}^{n+1} \lambda_i = 1$, *we have for any* $d \in \{1, 2, .., n+1\}$, *that*

$$|g(\mathbf{x}) - a(\mathbf{x})| \leq \frac{1}{2} \sum_{i=1}^{n+1} \lambda_i \sum_{k,l=1}^{n} B_{kl} |\mathbf{e}_k \cdot (\mathbf{x}_i - \mathbf{x}_d)| (|\mathbf{e}_l \cdot (\mathbf{x} - \mathbf{x}_d)| + |\mathbf{e}_l \cdot (\mathbf{x}_i - \mathbf{x}_d)|).$$

PROOF:
Because simplices are convex sets, there is an $\mathbf{y} \in \mathbb{R}^n$ and for every $\mathbf{x} \in \text{con}\{\mathbf{x}_1, \mathbf{x}_2.., \mathbf{x}_{n+1}\}$ an $\mathbf{y}_\mathbf{x} \in \mathbb{R}^n$, such that $r(\mathbf{x}_d + s\mathbf{y}) + (1-r)(\mathbf{x} + s\mathbf{y}_\mathbf{x})$ is in the interior of $\text{con}\{\mathbf{x}_1, \mathbf{x}_2.., \mathbf{x}_{n+1}\}$ for all $r \in [0,1]$ and all $s \in]0,1]$. It follows from Taylor's theorem that for every $\mathbf{x} \in \text{con}\{\mathbf{x}_1, \mathbf{x}_2.., \mathbf{x}_{n+1}\}$ and every $s \in]0,1]$ there is a $\mathbf{z}_{s,\mathbf{x}} \in \text{con}\{\mathbf{x}_1, \mathbf{x}_2.., \mathbf{x}_{n+1}\}$, such that

$$\begin{aligned} g(\mathbf{x} + s\mathbf{y}_\mathbf{x}) &= g(\mathbf{x}_d + s\mathbf{y}) + \nabla g(\mathbf{x}_d + s\mathbf{y}) \cdot (\mathbf{x} - \mathbf{x}_d + s(\mathbf{y}_\mathbf{x} - \mathbf{y})) \\ &\quad + \frac{1}{2} \sum_{k,l=1}^{n} [\mathbf{e}_k \cdot (\mathbf{x} - \mathbf{x}_d + s(\mathbf{y}_\mathbf{x} - \mathbf{y}))][\mathbf{e}_l \cdot (\mathbf{x} - \mathbf{x}_d + s(\mathbf{y}_\mathbf{x} - \mathbf{y}))] \frac{\partial^2 g}{\partial x_k \partial x_l}(\mathbf{z}_{s,\mathbf{x}}). \end{aligned}$$

Because the second order derivatives of g are bounded on the interior of $\text{con}\{\mathbf{x}_1, \mathbf{x}_2.., \mathbf{x}_{n+1}\}$, we can extend ∇g to a continuous function on $\text{con}\{\mathbf{x}_1, \mathbf{x}_2.., \mathbf{x}_{n+1}\}$, let $s \to 0+$, and get

$$g(\mathbf{x}) = g(\mathbf{x}_d) + \nabla g(\mathbf{x}_d) \cdot (\mathbf{x} - \mathbf{x}_d) + \frac{1}{2} \sum_{k,l=1}^{n} [\mathbf{e}_k \cdot (\mathbf{x} - \mathbf{x}_d)][\mathbf{e}_l \cdot (\mathbf{x} - \mathbf{x}_d)] b_{kl,\mathbf{x}},$$

where $b_{kl,\mathbf{x}} \in [-B_{kl}, B_{kl}]$ for all $k, l = 1, 2, .., n$. Set

$$e_\mathbf{x} := \frac{1}{2} \sum_{k,l=1}^{n} [\mathbf{e}_k \cdot (\mathbf{x} - \mathbf{x}_d)][\mathbf{e}_l \cdot (\mathbf{x} - \mathbf{x}_d)] b_{kl,\mathbf{x}}.$$

Let $\mathbf{x} \in \text{con}\{\mathbf{x}_1, \mathbf{x}_2.., \mathbf{x}_{n+1}\}$ be arbitrary. Then there are real numbers $\lambda_1, \lambda_2, .., \lambda_{n+1} \in [0,1]$ such that

$$\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_i = 1,$$

and we get from Lemma 4.3 and by repeated use of Taylor's theorem

$$\begin{aligned} g(\mathbf{x}) - a(\mathbf{x}) &= g(\mathbf{x}_d) + \nabla g(\mathbf{x}_d) \cdot (\mathbf{x} - \mathbf{x}_d) + e_\mathbf{x} - \sum_{i=1}^{n+1} \lambda_i g(\mathbf{x}_i) \\ &= e_\mathbf{x} + \sum_{i=1}^{n+1} \lambda_i \left[\nabla g(\mathbf{x}_d) \cdot (\mathbf{x}_i - \mathbf{x}_d) - (g(\mathbf{x}_i) - g(\mathbf{x}_d))\right] \\ &= e_\mathbf{x} + \sum_{i=1}^{n+1} \lambda_i [\nabla g(\mathbf{x}_d) \cdot (\mathbf{x}_i - \mathbf{x}_d) - (g(\mathbf{x}_d) + \nabla g(\mathbf{x}_d) \cdot (\mathbf{x}_i - \mathbf{x}_d) + e_{\mathbf{x}_i} - g(\mathbf{x}_d))] \\ &= e_\mathbf{x} - \sum_{i=1}^{n+1} \lambda_i e_{\mathbf{x}_i} \\ &= \frac{1}{2} \sum_{i=1}^{n+1} \lambda_i \sum_{k,l=1}^{n} \left([\mathbf{e}_k \cdot (\mathbf{x}_i - \mathbf{x}_d)][\mathbf{e}_l \cdot (\mathbf{x} - \mathbf{x}_d)] b_{kl,\mathbf{x}} - [\mathbf{e}_k \cdot (\mathbf{x}_i - \mathbf{x}_d)][\mathbf{e}_l \cdot (\mathbf{x}_i - \mathbf{x}_d)] b_{kl,\mathbf{x}_i}\right), \end{aligned}$$

from which

$$|g(\mathbf{x}) - a(\mathbf{x})| \leq \frac{1}{2} \sum_{i=1}^{n+1} \lambda_i \sum_{k,l=1}^{n} B_{kl} |\mathbf{e}_k \cdot (\mathbf{x}_i - \mathbf{x}_d)| (|\mathbf{e}_l \cdot (\mathbf{x} - \mathbf{x}_d)| + |\mathbf{e}_l \cdot (\mathbf{x}_i - \mathbf{x}_d)|)$$

follows.

■

Linear Program **LP2** tries to parameterize a CPWA Lyapunov function for a system. Because a CPWA Lyapunov function is not differentiable, one can not use the chain rule to get

$$\frac{d}{dt}V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))\bigg|_{t=0} = \nabla V(\boldsymbol{\xi})\cdot\mathbf{f}(\boldsymbol{\xi}),$$

where $\boldsymbol{\phi}$ is the solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, which makes it possible to use Lyapunov's direct method without integrating the system equation. The next theorem gives a substitute for this equation when $V \in \text{CPWA}$.

Theorem 4.17 *Let* $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a piecewise scaling function,* $\mathcal{N} \subset \mathbb{R}^n$ *be such that the space* $\text{CPWA}[\mathbf{PS},\mathcal{N}]$ *is properly defined, and let* $V \in \text{CPWA}[\mathbf{PS},\mathcal{N}]$*. Define* $\mathcal{H} \subset \mathbb{R}^n$ *as the interior of the set* $\mathbf{PS}(\mathcal{N})$ *and let* $\mathcal{U} \subset \mathbb{R}^n$ *be a domain,* $\mathcal{H} \subset \mathcal{U}$*, and* $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$ *be a locally Lipschitz function. Denote by* $\boldsymbol{\phi}$ *the solution of the system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

Further, let $D^* \in \{D^+, D_+, D^-, D_-\}$ *be a Dini derivative with respect to* t*.*

For every $S^{(i)} \in \mathfrak{S}[\mathbf{PS},\mathcal{N}]$ *let* $\mathbf{w}^{(i)} \in \mathbb{R}^n$ *and* $a_i \in \mathbb{R}$ *be such that* $V(\mathbf{x}) = \mathbf{w}^{(i)}\cdot\mathbf{x} + a_i$ *for all* $\mathbf{x} \in S^{(i)}$*.*

Assume there is a function $\gamma : \mathbb{R}^n \longrightarrow \mathbb{R}$*, such that for every* $S^{(i)} \in \mathfrak{S}[\mathbf{PS},\mathcal{N}]$*, we have for every* $\mathbf{x} \in S^{(i)} \cap \mathcal{H}$*, that*

$$\mathbf{w}^{(i)}\cdot\mathbf{f}(\mathbf{x}) \le \gamma(\mathbf{x}).$$

Then

$$D^*[V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] \le \gamma(\boldsymbol{\phi}(t,\boldsymbol{\xi}))$$

whenever $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{H}$*.*

Proof:
We only prove the theorem for $D^* = D^+$, the cases $D^* = D_+$, $D^* = D^-$, and $D^* = D_-$ follow analogously. Let $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{H}$ be arbitrary. Then there is a simplex $S^{(i)} \in \mathfrak{S}[\mathbf{PS},\mathcal{N}]$ and a $\delta > 0$, such that $\boldsymbol{\phi}(t,\boldsymbol{\xi}) + s\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \in S^{(i)}$ for all $s \in [0,\delta]$ (simplices are convex sets).

It follows from Theorem 1.17, that

$$\begin{aligned}
D^+[V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] &= \limsup_{s\to 0+}\frac{V(\boldsymbol{\phi}(t,\boldsymbol{\xi}) + s\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))) - V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{s} \\
&= \limsup_{s\to 0+}\frac{\mathbf{w}^{(i)}\cdot[\boldsymbol{\phi}(t,\boldsymbol{\xi}) + s\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] + a_i - \mathbf{w}^{(i)}\cdot\boldsymbol{\phi}(t,\boldsymbol{\xi}) - a_i}{s} \\
&= \limsup_{s\to 0+}\frac{s\mathbf{w}^{(i)}\cdot\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}{s} = \mathbf{w}^{(i)}\cdot\mathbf{f}(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \le \gamma(\boldsymbol{\phi}(t,\boldsymbol{\xi})).
\end{aligned}$$

■

The next lemma gives a formula for the gradient of a piecewise affine function, in its values at the vertices of the simplex $\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma))$.

Lemma 4.18 *Let* $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a piecewise scaling function,* $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$*,* $\mathcal{J} \in \mathfrak{P}(\{1, 2, .., n\})$*, and* $\sigma \in \mathrm{Sym}_n$*. Then the simplex* $\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma))$ *has the vertices* $\mathbf{PS}(\mathbf{x}_1)$*,* $\mathbf{PS}(\mathbf{x}_2), .., \mathbf{PS}(\mathbf{x}_{n+1})$*, where*

$$\mathbf{x}_i := \mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)}),$$

for $i = 1, 2, .., n$*.*
Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$*,*

$$V(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x} + a$$

for all $\mathbf{x} \in \mathbb{R}^n$*, be an affine function. Then*

$$\mathbf{w} = \sum_{i=1}^{n} \frac{V(\mathbf{PS}(\mathbf{x}_i)) - V(\mathbf{PS}(\mathbf{x}_{i+1}))}{\mathrm{PS}_{\sigma(i)}(\mathbf{x}_i) - \mathrm{PS}_{\sigma(i)}(\mathbf{x}_{i+1})} \mathbf{e}_{\sigma(i)} = \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(\sigma(i))} \frac{V(\mathbf{PS}(\mathbf{x}_i)) - V(\mathbf{PS}(\mathbf{x}_{i+1}))}{\|\mathbf{PS}(\mathbf{x}_i) - \mathbf{PS}(\mathbf{x}_{i+1})\|_\infty} \mathbf{e}_{\sigma(i)},$$

where $\chi_{\mathcal{J}} : \{1, 2, .., n\} \longrightarrow \{0, 1\}$ *is the characteristic function of the set* $\mathcal{J}$*.*

Proof:
We show that

$$w_{\sigma(i)} = \frac{V(\mathbf{PS}(\mathbf{x}_i)) - V(\mathbf{PS}(\mathbf{x}_{i+1}))}{\mathrm{PS}_{\sigma(i)}(\mathbf{x}_i) - \mathrm{PS}_{\sigma(i)}(\mathbf{x}_{i+1})}$$

and that

$$\mathrm{PS}_{\sigma(i)}(\mathbf{x}_i) - \mathrm{PS}_{\sigma(i)}(\mathbf{x}_{i+1}) = (-1)^{\chi_{\mathcal{J}}(\sigma(i))} \|\mathbf{PS}(\mathbf{x}_i) - \mathbf{PS}(\mathbf{x}_{i+1})\|_\infty$$

for all $i = 1, 2, .., n$.
For any $i \in \{1, 2, .., n\}$ we have

$$V(\mathbf{PS}(\mathbf{x}_i)) - V(\mathbf{PS}(\mathbf{x}_{i+1})) = \mathbf{w} \cdot [\mathbf{PS}(\mathbf{x}_i) - \mathbf{PS}(\mathbf{x}_{i+1})] = \sum_{k=1}^{n} w_{\sigma(k)} [\mathrm{PS}_{\sigma(k)}(\mathbf{x}_i) - \mathrm{PS}_{\sigma(k)}(\mathbf{x}_{i+1})].$$

Because the components of the vectors

$$\mathbf{x}_i = \mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)}) \quad \text{and} \quad \mathbf{x}_{i+1} = \mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{j=i+1}^{n} \mathbf{e}_{\sigma(j)})$$

are all equal, except for the $\sigma(i)$-th one, it follows from the definition of a piecewise scaling function, that

$$\mathrm{PS}_{\sigma(i)}(\mathbf{x}_i) - \mathrm{PS}_{\sigma(i)}(\mathbf{x}_{i+1}) = (-1)^{\chi_{\mathcal{J}}(\sigma(i))} \|\mathbf{PS}(\mathbf{x}_i) - \mathbf{PS}(\mathbf{x}_{i+1})\|_\infty$$

and

$$V(\mathbf{PS}(\mathbf{x}_i)) - V(\mathbf{PS}(\mathbf{x}_{i+1})) = w_{\sigma(i)} [\mathrm{PS}_{\sigma(i)}(\mathbf{x}_i) - \mathrm{PS}_{\sigma(i)}(\mathbf{x}_{i+1})],$$

i.e.

$$w_{\sigma(i)} = \frac{V(\mathbf{PS}(\mathbf{x}_i)) - V(\mathbf{PS}(\mathbf{x}_{i+1}))}{\mathrm{PS}_{\sigma(i)}(\mathbf{x}_i) - \mathrm{PS}_{\sigma(i)}(\mathbf{x}_{i+1})} = (-1)^{\chi_{\mathcal{J}}(\sigma(i))} \frac{V(\mathbf{PS}(\mathbf{x}_i)) - V(\mathbf{PS}(\mathbf{x}_{i+1}))}{\|\mathbf{PS}(\mathbf{x}_i) - \mathbf{PS}(\mathbf{x}_{i+1})\|_\infty}. \qquad \blacksquare$$

In this chapter we have defined the function spaces CPWA and we have proved several properties of the functions in these spaces. In the next chapter we will state Linear Program **LP2** and we will use the results from this chapter to prove, that a CPWA Lyapunov function can be constructed from a feasible solution of a linear program generated by Linear Program **LP2**.

Chapter 5

Linear Program LP2

At the beginning of this chapter we will state Linear Program **LP2**, an algorithmic description of how to generate a linear program for a nonlinear system. In Section 5.1 we will define the functions ψ, γ, and V^{Lya} as continuous affine interpolations of variables of such a linear program. In the sections 5.2-5.5 we will successively derive implications that the constraints of the linear program have on the functions ψ, γ, and V^{Lya}. In Section 5.6 we will state and prove Theorem II, a theorem which states that V^{Lya} is a Lyapunov or a Lyapunov-like function for the system in question.

Linear Program LP2

Let $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^n$ be locally Lipschitz, where $\mathcal{U} \subset \mathbb{R}^n$ is a domain containing zero and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Let $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a piecewise scaling function, $\mathcal{N} \subset \mathcal{U}$ be a bounded domain containing zero, and define the set

$$\mathcal{M} := \bigcup_{\substack{\mathbf{z}\in\mathbb{Z}^n \\ \mathbf{PS}(\mathbf{z}+[0,1]^n)\subset\mathcal{N}}} \mathbf{PS}(\mathbf{z}+[0,1]^n)$$

and assume that it is connected.

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and

$$\mathcal{D} := \mathbf{PS}(]d_1^-, d_1^+[\times]d_2^-, d_2^+[\times .. \times]d_n^-, d_n^+[)$$

be a set, of which the closure is contained in the interior of $\mathcal{M}$, and either $\mathcal{D} = \emptyset$ or $d_i^- \leq -1$ and $1 \leq d_i^+$ for all $i = 1, 2, .., n$.

Finally, assume that for every $\mathbf{z} \in \mathbb{Z}^n$, such that $\mathbf{PS}(\mathbf{z} + [0,1]^n) \subset \mathcal{M} \setminus \mathcal{D}$, the second order derivatives

$$\frac{\partial^2 f_1}{\partial x_k \partial x_l}, \frac{\partial^2 f_2}{\partial x_k \partial x_l}, .. , \frac{\partial^2 f_n}{\partial x_k \partial x_l}$$

are continuous and bounded on $\mathbf{PS}(\mathbf{z}+]0,1[^n)$ for all $k, l = 1, 2, .., n$.

Then the linear program $\mathbf{LP2}(\mathbf{f}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ is constructed in the following way:

i) Define the set

$$\mathcal{X}^{\|\cdot\|} := \{\|\mathbf{x}\| \,\big|\, \mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \mathcal{M}\}.$$

ii) Define for every $\sigma \in \mathrm{Sym}_n$ *and every* $i = 1, 2, .., n+1$, *the vector*

$$\mathbf{x}_i^\sigma := \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)}.$$

iii) Define the set

$$\mathcal{Z} := \{(\mathbf{z}, \mathcal{J}) \in \mathbb{Z}_{\geq 0}^n \times \mathfrak{P}(\{1, 2, .., n\}) \big| \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + [0, 1]^n)) \subset \mathcal{M} \setminus \mathcal{D}\}.$$

iv) For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ *define for every* $\sigma \in \mathrm{Sym}_n$ *and every* $i = 1, 2, .., n+1$, *the vector*

$$\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} := \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \mathbf{x}_i^\sigma)).$$

v) Define the set

$$\mathcal{Y} := \{\{\mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,k+1}^{(\mathbf{z},\mathcal{J})}\} \big| (\mathbf{z}, \mathcal{J}) \in \mathcal{Z} \text{ and } k \in \{1, 2, .., n\}\}.$$

The set $\mathcal{Y}$ *is the set of neighboring grid points in the grid* $\mathbf{PS}(\mathbb{Z}^n) \cap (\mathcal{M} \setminus \mathcal{D})$.

vi) For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ *and every* $k, l = 1, 2, .., n$ *let* $B_{kl}^{(\mathbf{z},\mathcal{J})}$ *be a constant, such that*

$$B_{kl}^{(\mathbf{z},\mathcal{J})} \geq \max_{i=1,2,..,n} \sup_{\mathbf{x} \in \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} +]0,1[^n))} \left| \frac{\partial^2 f_i}{\partial x_k \partial x_l}(\mathbf{x}) \right|$$

and $B_{kl}^{(\mathbf{z},\mathcal{J})} < +\infty$.

vii) For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, *every* $k, i = 1, 2, .., n$, *and every* $\sigma \in \mathrm{Sym}_n$, *define*

$$A_{\sigma,k,i}^{(\mathbf{z},\mathcal{J})} := |\mathbf{e}_k \cdot (\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})|.$$

viii) Define the constant

$$x_{\partial\mathcal{M},\min} := \min\{\|\mathbf{x}\| \big| \mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{M}\},$$

where $\partial\mathcal{M}$ *is the boundary of the set* $\mathcal{M}$.

ix) Let $\varepsilon > 0$ *and* $\delta > 0$ *be arbitrary constants.*

The variables of the linear program are:

$$\begin{aligned}
&\Psi[x], \quad \text{for all } x \in \mathcal{X}^{\|\cdot\|}, \\
&\Gamma[x], \quad \text{for all } x \in \mathcal{X}^{\|\cdot\|}, \\
&V[\mathbf{x}], \quad \text{for all } \mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap (\mathcal{M} \setminus \mathcal{D}), \\
&C[\{\mathbf{x}, \mathbf{y}\}], \quad \text{for all } \{\mathbf{x}, \mathbf{y}\} \in \mathcal{Y}.
\end{aligned}$$

The linear constraints of the linear program are:

LC1) Let $x_1, x_2, .., x_K$ *be the elements of* $\mathcal{X}^{\|\cdot\|}$ *in an increasing order. Then*

$$\Psi[x_1] = \Gamma[x_1] = 0,$$
$$\varepsilon x_2 \leq \Psi[x_2],$$
$$\varepsilon x_2 \leq \Gamma[x_2],$$

and for every $i = 2, 3, .., K-1$*:*

$$\frac{\Psi[x_i] - \Psi[x_{i-1}]}{x_i - x_{i-1}} \leq \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i}$$

and

$$\frac{\Gamma[x_i] - \Gamma[x_{i-1}]}{x_i - x_{i-1}} \leq \frac{\Gamma[x_{i+1}] - \Gamma[x_i]}{x_{i+1} - x_i}.$$

LC2) For every $\mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap (\mathcal{M} \setminus \mathcal{D})$*:*

$$\Psi[\|\mathbf{x}\|] \leq V[\mathbf{x}].$$

If $\mathcal{D} = \emptyset$*, then*

$$V[\mathbf{0}] = 0.$$

If $\mathcal{D} \neq \emptyset$*, then for every* $\mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{D}$*:*

$$V[\mathbf{x}] \leq \Psi[x_{\partial\mathcal{M},\min}] - \delta.$$

LC3) For every $\{\mathbf{x}, \mathbf{y}\} \in \mathcal{Y}$*:*

$$-C[\{\mathbf{x},\mathbf{y}\}] \cdot \|\mathbf{x} - \mathbf{y}\|_\infty \leq V[\mathbf{x}] - V[\mathbf{y}] \leq C[\{\mathbf{x},\mathbf{y}\}] \cdot \|\mathbf{x} - \mathbf{y}\|_\infty.$$

LC4) For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$*, every* $\sigma \in \mathrm{Sym}_n$*, and every* $i = 1, 2, .., n+1$*:*

$$-\Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|] \geq \sum_{j=1}^{n} \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})$$
$$+ \frac{1}{2} \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} (A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})}) \sum_{j=1}^{n} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}].$$

The objective of the linear program is not needed.

■

Note that the values of the constants $\varepsilon > 0$ and $\delta > 0$ do not affect whether there is a feasible solution of the linear program or not. If there is a feasible solution for $\varepsilon := \varepsilon' > 0$ and $\delta := \delta' > 0$, then there is a feasible solution for all $\varepsilon := \varepsilon^* > 0$ and $\delta := \delta^* > 0$. Just multiply all variables with

$$\max\{\frac{\varepsilon^*}{\varepsilon'}, \frac{\delta^*}{\delta'}\}.$$

Assume that the linear program $\mathbf{LP2}(\mathbf{f}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ has a feasible solution, for some particular $\mathbf{f}, \mathcal{N}, \mathbf{PS}$, $\mathcal{D}$, and $\|\cdot\|$, that satisfy the assumptions of the linear program, and let $\boldsymbol{\phi}$ be the solution of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

In the next five sections we will show that if the functions ψ, γ, and V^{Lya} are defined as piecewise affine interpolations of the values of the variables Ψ, Γ, and V respectively, then $\psi, \gamma \in \mathcal{K}$,

$$\psi(\|\mathbf{x}\|) \leq V^{Lya}(\mathbf{x})$$

for all $\mathbf{x} \in \mathcal{M}$, and

$$D^*[V^{Lya}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] \leq -\gamma(\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|),$$

where D^* is an arbitrary Dini derivative with respect to t, for all $\boldsymbol{\phi}(t,\boldsymbol{\xi})$ in the interior of $\mathcal{M}$.

5.1 The Definition of $\psi, \gamma,$ and V^{Lya}

Let $x_1, x_2, .., x_K$ be the elements of $\mathcal{X}^{\|\cdot\|}$ in an increasing order. We define the piecewise affine functions $\psi, \gamma : [x_1, +\infty[\longrightarrow \mathbb{R}$,

$$\psi(y) := \Psi[x_i] + \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i}(y - x_i)$$

and

$$\gamma(y) := \Gamma[x_i] + \frac{\Gamma[x_{i+1}] - \Gamma[x_i]}{x_{i+1} - x_i}(y - x_i),$$

for all $y \in [x_i, x_{i+1}]$ and all $i = 1, 2, .., K-1$. The values of ψ and γ on $]x_K, +\infty[$ do not really matter, but to have everything properly defined, we set

$$\psi(y) := \Psi[x_{K-1}] + \frac{\Psi[x_K] - \Psi[x_{K-1}]}{x_K - x_{K-1}}(y - x_{K-1})$$

and

$$\gamma(y) := \Gamma[x_{K-1}] + \frac{\Gamma[x_K] - \Gamma[x_{K-1}]}{x_K - x_{K-1}}(y - x_{K-1})$$

for all $y > x_K$. Clearly the functions ψ and γ are continuous.

The function $V^{Lya} \in \mathrm{CPWA}[\mathbf{PS}, \mathbf{PS}^{-1}(\mathcal{M})]$ is defined by

$$V^{Lya}(\mathbf{x}) := V[\mathbf{x}]$$

for all $\mathbf{x} \in \mathcal{M} \cap \mathbf{PS}(\mathbb{Z}^n)$.

In the next four sections we will successively show, the implications the linear constraints LC1, LC2, LC3, and LC4 have on ψ, γ, and V^{Lya}.

5.2 The Constraints LC1

The constraints LC1 are

$$\Psi[x_1] = \Gamma[x_1] = 0,$$
$$\varepsilon x_2 \leq \Psi[x_2],$$
$$\varepsilon x_2 \leq \Gamma[x_2],$$

and

$$\frac{\Psi[x_i] - \Psi[x_{i-1}]}{x_i - x_{i-1}} \leq \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i} \tag{5.1}$$

and

$$\frac{\Gamma[x_i] - \Gamma[x_{i-1}]}{x_i - x_{i-1}} \leq \frac{\Gamma[x_{i+1}] - \Gamma[x_i]}{x_{i+1} - x_i}$$

for every $i = 2, 3, .., K-1$, where $\varepsilon > 0$ is a constant and $x_1, x_2, .., x_K$ are the elements of $\mathcal{X}^{\|\cdot\|}$ in an increasing order.

We are going to show that the constraints LC1 imply, that the functions ψ and γ are convex and strictly increasing on $[0, +\infty[$. Because $x_1 = 0$, $\psi(x_1) = \Psi[x_1] = 0$, and $\gamma(x_1) = \Gamma[x_1] = 0$, this means that they are convex $\mathcal{K}$ functions. The constraints are the same for Ψ and Γ, so it suffices to show this for the function ψ.

From the definition of ψ, it is clear that it is continuous and that

$$\frac{\psi(x) - \psi(y)}{x - y} = \frac{\Psi[x_{i+1}] - \Psi[x_i]}{x_{i+1} - x_i} \tag{5.2}$$

for all $x, y \in [x_i, x_{i+1}]$ and all $i = 1, 2, .., K-1$. From $x_1 = 0$, $\Psi[x_1] = 0$, and $\varepsilon x_2 \leq \Psi[x_2]$ we get

$$\varepsilon \leq \frac{\Psi[x_2] - \Psi[x_1]}{x_2 - x_1}.$$

But then we get from (5.1) and (5.2) that $D^+\psi$ is a positive and increasing function on $[x_1, +\infty[$ and it follows from Corollary 1.13, that ψ is a strictly increasing function.

The function ψ is *convex*, if and only if for every $y \in]x_1, +\infty[$ there are constants $a_y, b_y \in \mathbb{R}$, such that

$$a_y y + b_y = \psi(y) \quad \text{and} \quad a_y x + b_y \leq \psi(x)$$

for all $x \in [x_1, +\infty[$ (see, for example, Section 17 in Chapter 11 in [51]). Let $y \in]x_1, +\infty[$. Because the function $D^+\psi$ is increasing, it follows from Theorem 1.12, that for every $x \in [x_1, +\infty[$, there is a $c_{x,y} \in \mathbb{R}$, such that

$$\psi(x) = \psi(y) + c_{x,y}(x - y)$$

and $c_{x,y} \leq D^+\psi(y)$ if $x < y$ and $c_{x,y} \geq D^+\psi(y)$ if $x > y$. This means that

$$\psi(x) = \psi(y) + c_{x,y}(x - y) \geq D^+\psi(y)x + \psi(y) - D^+\psi(y)y$$

for all $x \in [x_1, +\infty[$. Because y was arbitrary, the function ψ is convex.

5.3 The Constraints LC2

The constraints LC2 are

$$\Psi[\|\mathbf{x}\|] \leq V[\mathbf{x}] \tag{5.3}$$

for every $\mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap (\mathcal{M} \setminus \mathcal{D})$,

$$V[\mathbf{0}] = 0$$

if $\mathcal{D} = \emptyset$, and

$$V[\mathbf{x}] \leq \Psi[x_{\partial\mathcal{M},\min}] - \delta$$

for every $\mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{D}$ if $\mathcal{D} \neq \emptyset$.

Define the constant

$$V^{Lya}_{\partial\mathcal{M},\min} := \min_{\mathbf{x}\in\partial\mathcal{M}} V^{Lya}(\mathbf{x})$$

and if $\mathcal{D} \neq \emptyset$ the constant

$$V^{Lya}_{\partial\mathcal{D},\max} := \max_{\mathbf{x}\in\partial\mathcal{D}} V^{Lya}(\mathbf{x}).$$

We are going to show that the constraints LC2 imply, that

$$\psi(\|\mathbf{x}\|) \leq V^{Lya}(\mathbf{x})$$

for all $\mathbf{x} \in \mathcal{M} \setminus \mathcal{D}$, that

$$V^{Lya}(\mathbf{0}) = 0$$

if $\mathcal{D} = \emptyset$, and that

$$V^{Lya}_{\partial\mathcal{D},\max} \leq V^{Lya}_{\partial\mathcal{M},\min} - \delta$$

if $\mathcal{D} \neq \emptyset$.

We first show that (5.3) implies, that

$$\psi(\|\mathbf{x}\|) \leq V^{Lya}(\mathbf{x})$$

for all $\mathbf{x} \in \mathcal{M} \setminus \mathcal{D}$.

Let $\mathbf{x} \in \mathcal{M}\setminus\mathcal{D}$. Then there is a $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, a $\sigma \in \mathrm{Sym}_n$, and constants $\lambda_1, \lambda_2, .., \lambda_{n+1} \in [0, 1]$, such that

$$\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i} \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

By using (5.3), Jensen's inequality for convex functions (see, for example, Section 18 in Chapter 11 in [51]), that V^{Lya} is affine on the simplex $\mathrm{con}\{\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,1}, \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,2}, .., \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,n+1}\}$, that $\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i} \in \mathbf{PS}(\mathbb{Z}^n) \cap (\mathcal{M} \setminus \mathcal{D})$ for all $i = 1, 2, .., n + 1$, and that ψ is increasing, we get

$$\begin{aligned}\psi(\|\mathbf{x}\|) &= \psi(\|\sum_{i=1}^{n+1} \lambda_i \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|) \leq \psi(\sum_{i=1}^{n+1} \lambda_i \|\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|) \leq \sum_{i=1}^{n+1} \lambda_i \psi(\|\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|) = \sum_{i=1}^{n+1} \lambda_i \Psi[\|\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|] \\ &\leq \sum_{i=1}^{n+1} \lambda_i V[\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}] = \sum_{i=1}^{n+1} \lambda_i V^{Lya}(\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}) = V^{Lya}(\sum_{i=1}^{n+1} \lambda_i \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}) = V^{Lya}(\mathbf{x}).\end{aligned}$$

Because $\mathbf{x}$ was arbitrary, this inequality is valid for all $\mathbf{x} \in \mathcal{M} \setminus \mathcal{D}$.

That $V^{Lya}(\mathbf{0}) = 0$ if $\mathcal{D} = \emptyset$ is trivial. Let us consider the case $\mathcal{D} \neq \emptyset$. From the definition of V^{Lya} and the constants $V^{Lya}_{\partial\mathcal{D},\max}$ and $V^{Lya}_{\partial\mathcal{M},\min}$ it is clear, that

$$V^{Lya}_{\partial\mathcal{D},\max} = \max_{\mathbf{x}\in\partial\mathcal{D}\cap\mathbf{PS}(\mathbb{Z}^n)} V[\mathbf{x}]$$

and

$$V^{Lya}_{\partial\mathcal{M},\min} = \min_{\mathbf{x}\in\partial\mathcal{M}\cap\mathbf{PS}(\mathbb{Z}^n)} V[\mathbf{x}].$$

Let $\mathbf{x} \in \partial\mathcal{M} \cap \mathbf{PS}(\mathbb{Z}^n)$ be such that $V[\mathbf{x}] = V^{Lya}_{\partial\mathcal{M},\min}$, then

$$V^{Lya}_{\partial\mathcal{D},\max} \leq \Psi[x_{\partial\mathcal{M},\min}] - \delta = \psi(x_{\partial\mathcal{M},\min}) - \delta \leq \psi(\|\mathbf{x}\|) - \delta \leq V[\mathbf{x}] - \delta = V^{Lya}_{\partial\mathcal{M},\min} - \delta.$$

5.4 The Constraints LC3

The constraints LC3 are

$$-C[\{\mathbf{x},\mathbf{y}\}] \cdot \|\mathbf{x}-\mathbf{y}\|_\infty \leq V[\mathbf{x}] - V[\mathbf{y}] \leq C[\{\mathbf{x},\mathbf{y}\}] \cdot \|\mathbf{x}-\mathbf{y}\|_\infty$$

for every $\{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}$. This means that

$$\left|\frac{V[\mathbf{x}] - V[\mathbf{y}]}{\|\mathbf{x}-\mathbf{y}\|_\infty}\right| \leq C[\{\mathbf{x},\mathbf{y}\}]$$

for every $\{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}$ and these local bounds of the gradient of V^{Lya} will be used in the next section.

5.5 The Constraints LC4

The constraints LC4 are

$$-\Gamma[\|\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|] \geq \sum_{j=1}^{n} \frac{V[\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,j}] - V[\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,j+1}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,j} - \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,j+1})} f_{\sigma(j)}(\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i})$$
$$+ \frac{1}{2}\sum_{r,s=1}^{n} B^{(\mathbf{z},\mathcal{J})}_{rs} A^{(\mathbf{z},\mathcal{J})}_{\sigma,r,i}(A^{(\mathbf{z},\mathcal{J})}_{\sigma,s,i} + A^{(\mathbf{z},\mathcal{J})}_{\sigma,s,1}) \sum_{j=1}^{n} C[\{\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,j}, \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,j+1}\}]$$

for every $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$, every $\sigma \in \mathrm{Sym}_n$, and every $i = 1, 2, .., n+1$.

We are going to show that they imply that

$$-\gamma(\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|) \geq D^*[V^{Lya}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))]$$

for all $\boldsymbol{\phi}(t,\boldsymbol{\xi})$ in the interior of $\mathcal{M}$.

Let $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$ and $\sigma \in \mathrm{Sym}_n$. Then for every $\mathbf{x} \in \mathrm{con}\{\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,1}, \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,2}, .., \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,n+1}\}$, there are constants $\lambda_1, \lambda_2, .., \lambda_{n+1} \in [0,1]$, such that

$$\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i} \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

Because γ is a convex function, we get

$$-\gamma(\|\mathbf{x}\|) \geq -\sum_{i=1}^{n+1} \lambda_i \Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|]. \tag{5.4}$$

Because V^{Lya} is affine on the simplex $\mathrm{con}\{\mathbf{y}_{\sigma,1}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,2}^{(\mathbf{z},\mathcal{J})}, .., \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}\}$, there is a vector $\mathbf{w} \in \mathbb{R}^n$ and a constant $a \in \mathbb{R}$, such that

$$V^{Lya}(\mathbf{y}) = \mathbf{w} \cdot \mathbf{y} + a$$

for all $\mathbf{y} \in \mathrm{con}\{\mathbf{y}_{\sigma,1}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,2}^{(\mathbf{z},\mathcal{J})}, .., \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}\}$. It follows from Hölder's inequality, that

$$\begin{aligned}\mathbf{w} \cdot \mathbf{f}(\mathbf{x}) &= \mathbf{w} \cdot \sum_{i=1}^{n+1} \lambda_i \mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + \mathbf{w} \cdot \left(\mathbf{f}(\mathbf{x}) - \sum_{i=1}^{n+1} \lambda_i \mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right) \\ &\leq \sum_{i=1}^{n+1} \lambda_i \mathbf{w} \cdot \mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + \|\mathbf{w}\|_1 \|\mathbf{f}(\mathbf{x}) - \sum_{i=1}^{n+1} \lambda_i \mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_\infty.\end{aligned} \tag{5.5}$$

By Lemma 4.16,

$$\begin{aligned}&\|\mathbf{f}(\mathbf{x}) - \sum_{i=1}^{n+1} \lambda_i \mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_\infty = \max_{j=1,2,..,n} |f_j(\mathbf{x}) - \sum_{i=1}^{n+1} \lambda_i f_j(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})| \\ &\quad \leq \frac{1}{2} \sum_{i=1}^{n+1} \lambda_i \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} |\mathbf{e}_r \cdot (\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})| \left(|\mathbf{e}_s \cdot (\mathbf{x} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})| + |\mathbf{e}_s \cdot (\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})|\right) \\ &\quad = \frac{1}{2} \sum_{i=1}^{n+1} \lambda_i \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} \left(|\mathbf{e}_s \cdot (\mathbf{x} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})| + A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})}\right) \\ &\quad \leq \frac{1}{2} \sum_{i=1}^{n+1} \lambda_i \sum_{r,s=1}^{n} B_{rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} (A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})}),\end{aligned}$$

where we used

$$|\mathbf{e}_s \cdot (\mathbf{x} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})| \leq \sum_{k=1}^{n+1} \lambda_k |\mathbf{e}_s \cdot (\mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})| \leq \sum_{k=1}^{n+1} \lambda_k |\mathbf{e}_s \cdot (\mathbf{y}_{\sigma,1}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})| = A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})}.$$

We come to the vector $\mathbf{w}$. By Lemma 4.18 and because $V^{Lya}(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}) = V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]$ for all $j = 1, 2, .., n+1$, we get the formula

$$\mathbf{w} = \sum_{j=1}^{n} \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} \mathbf{e}_{\sigma(j)} = \sum_{j=1}^{n} (-1)^{\chi_{\mathcal{J}}(\sigma(j))} \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\|\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\|_\infty} \mathbf{e}_{\sigma(j)}.$$

From this formula and the constraints LC3

$$\|\mathbf{w}\|_1 = \sum_{j=1}^{n} \left| \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\|\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\|_\infty} \right| \leq \sum_{j=1}^{n} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]$$

follows.

We come back to (5.5). By using the results from above, we get

$$\begin{aligned}\mathbf{w}\cdot\mathbf{f}(\mathbf{x}) &\leq \sum_{i=1}^{n+1}\lambda_i\mathbf{w}\cdot\mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + \|\mathbf{w}\|_1\|\mathbf{f}(\mathbf{x}) - \sum_{i=1}^{n+1}\lambda_i\mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_\infty \\ &\leq \sum_{i=1}^{n+1}\lambda_i\Bigg(\sum_{j=1}^{n}\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\ &\quad + \frac{1}{2}\sum_{r,s=1}^{n}B_{rs}^{(\mathbf{z},\mathcal{J})}A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})}(A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})})\sum_{j=1}^{n}C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})},\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\Bigg).\end{aligned}$$

From this, (5.4), and the constraints LC4,

$$\begin{aligned}-\gamma(\|\mathbf{x}\|) &\geq -\sum_{i=1}^{n+1}\lambda_i\Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|] \\ &\geq \sum_{i=1}^{n+1}\lambda_i\Bigg(\sum_{j=1}^{n}\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\ &\quad + \frac{1}{2}\sum_{r,s=1}^{n}B_{rs}^{(\mathbf{z},\mathcal{J})}A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})}(A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})})\sum_{j=1}^{n}C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})},\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\Bigg) \\ &\geq \mathbf{w}\cdot\mathbf{f}(\mathbf{x})\end{aligned}$$

follows.

Because $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$, $\sigma \in \mathrm{Sym}_n$, and $\mathbf{x} \in \mathrm{con}\{\mathbf{y}_{\sigma,1}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,2}^{(\mathbf{z},\mathcal{J})}, .., \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}\}$ were arbitrary, it follows by Theorem 4.17:

For any Dini derivative D^* with respect to t, we have for the solution $\boldsymbol{\phi}$ of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

that

$$D^*[V^{Lya}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] \leq -\gamma(\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|)$$

for all $\boldsymbol{\phi}(t,\boldsymbol{\xi})$ in the interior of $\mathcal{M}\setminus\mathcal{D}$.

5.6 Theorem II

At the beginning of this chapter we stated Linear Program **LP2**. In Section 5.1 we defined the functions ψ, γ, and V^{Lya}, by using the variables of the linear program $\mathbf{LP2}(\mathbf{f},\mathcal{N},\mathbf{PS},\mathcal{D},\|\cdot\|)$. In the sections 5.2-5.5 we derived numerous properties that the functions ψ, γ, and V^{Lya} have, if they are constructed from a feasible solution of the linear program. In Theorem II these properties are summed up and we state and prove several implications they have on the stability behavior of the solution of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. We will use symbols and notations defined in Linear Program **LP2** and Section 5.3, when stating and proving the theorem.

Theorem II

Assume that the linear program $\mathbf{LP2}(\mathbf{f}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ *has a feasible solution and let the functions* ψ, γ, *and* V^{Lya} *be defined as in Section 5.1 from this feasible solution. Let* $\boldsymbol{\phi}$ *be the solution of the system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

and let D^* *be an arbitrary Dini derivative with respect to* t.

Then $\psi, \gamma \in \mathcal{K}$,

$$\psi(\|\mathbf{x}\|) \leq V^{Lya}(\mathbf{x}),$$

and

$$D^*[V^{Lya}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))] \leq -\gamma(\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|),$$

for all $\mathbf{x} \in \mathcal{M} \setminus \mathcal{D}$ *and all* $\boldsymbol{\phi}(t, \boldsymbol{\xi})$ *in the interior of* $\mathcal{M} \setminus \mathcal{D}$.

If $\mathcal{D} = \emptyset$, *then* V^{Lya} *is a Lyapunov function for the system, its equilibrium point at the origin is asymptotically stable, and the set*

$$\mathcal{A} := \{\mathbf{x} \in \mathbb{R}^n \big| V^{Lya}(\mathbf{x}) < V^{Lya}_{\partial\mathcal{M},\min}\}$$

is a subset of its region of attraction, i.e.

$$\lim_{t \to +\infty} \boldsymbol{\phi}(t, \boldsymbol{\xi}) = \mathbf{0}.$$

for all $\boldsymbol{\xi} \in \mathcal{A}$.

If $\mathcal{D} \neq \emptyset$, *then*

$$\limsup_{t \to +\infty} \|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| \leq \psi^{-1}(V^{Lya}_{\partial\mathcal{D},\max})$$

for all $\boldsymbol{\xi} \in \mathcal{A}$, *and for every such* $\boldsymbol{\xi}$ *there is a* $t'_{\boldsymbol{\xi}} \geq 0$, *such that* $\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \partial\mathcal{D}$.

Proof:
The stated properties of the functions ψ, γ, and V^{Lya} were proved in the sections 5.1-5.5. The other propositions follow essentially from the direct method of Lyapunov.

The case $\mathcal{D} = \emptyset$:
It follows from Theorem 1.16 that V^{Lya} is a Lyapunov function for the system and that every compact preimage $[V^{Lya}]^{-1}([0, c])$ is contained in the region of attraction of the asymptotically stable equilibrium point at the origin. That the set $\mathcal{A}$ is contained in the region of attraction follows from the definition of the constant $V^{Lya}_{\partial\mathcal{M},\min}$ as the minimum value of the function V^{Lya} on the boundary of its domain.

The case $\mathcal{D} \neq \emptyset$:
We first prove that for every $\boldsymbol{\xi} \in \mathcal{A}$, there is a $t'_{\boldsymbol{\xi}} \geq 0$, such that $\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \partial\mathcal{D}$. Let $\boldsymbol{\xi} \in \mathcal{A}$. If $\boldsymbol{\xi} \in \partial\mathcal{D}$ there is nothing to prove, so assume that $\boldsymbol{\xi}$ is in the interior of $\mathcal{A}$. It follows just as in the proof of Theorem 1.16, that the function

$$g(t) := V^{Lya}(\boldsymbol{\phi}(t, \boldsymbol{\xi})) + \int_0^t \gamma(\|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|) d\tau$$

is decreasing as long as $\boldsymbol{\phi}(t, \boldsymbol{\xi})$ stays in the interior of $\mathcal{A}$. Because $\mathcal{A} \cap \mathcal{D} = \emptyset$ there is a constant $a > 0$, such that $\gamma(\|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|) > a$ for all $\boldsymbol{\phi}(\tau, \boldsymbol{\xi}) \in \mathcal{A}$. Hence $\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \partial\mathcal{A}$ for some $t'_{\boldsymbol{\xi}} > 0$, for else $\lim_{t \to +\infty} g(t) = +\infty$. That $\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \partial\mathcal{A}$ implies, that either

$\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \partial\mathcal{D}$ or $V^{Lya}(\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi})) = V^{Lya}_{\partial\mathcal{M},\min}$, where the second possibility is contradictory to g being decreasing, so we must have $\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \partial\mathcal{D}$.

The second proposition

$$\limsup_{t\to+\infty} \|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| \leq \psi^{-1}(V^{Lya}_{\partial\mathcal{D},\max})$$

for all $\boldsymbol{\xi} \in \mathcal{A}$, follows from $\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \partial\mathcal{D}$. For every $\boldsymbol{\xi} \in \mathcal{A}$ there is a $t'_{\boldsymbol{\xi}} \geq 0$, such that $\boldsymbol{\phi}(t'_{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \partial\mathcal{D}$. Because $V^{Lya}(\mathbf{x}) \leq V^{Lya}_{\partial\mathcal{D},\max}$ for all $\mathbf{x} \in \partial\mathcal{D}$, this implies $V^{Lya}(\boldsymbol{\phi}(t, \boldsymbol{\xi})) \leq V^{Lya}_{\partial\mathcal{D},\max}$ for all $t \geq t'_{\boldsymbol{\xi}}$. But then

$$\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| \leq \psi^{-1}(V^{Lya}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))) \leq \psi^{-1}(V^{Lya}_{\partial\mathcal{D},\max})$$

for all $t \geq t'_{\boldsymbol{\xi}}$, hence

$$\limsup_{t\to+\infty} \|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| \leq \psi^{-1}(V^{Lya}_{\partial\mathcal{D},\max}).$$

■

It is not difficult to see that the set $\mathcal{A}$ in the theorem above can be replaced with the union of all compact $\mathcal{A}_c \subset \mathcal{M}$, where $\mathcal{A}_c$ be the largest connected subset of $\{\mathbf{x} \in \mathbb{R}^n \big| \, V^{Lya}(\mathbf{x}) \leq c\} \cup \mathcal{D}$ containing the origin.

We have proved that a feasible solution of the linear program $\mathbf{LP2}(\mathbf{f}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ leads directly to a Lyapunov function for the system if $\mathcal{D} = \emptyset$ and to a Lyapunov-like function if $\mathcal{D} \neq \emptyset$. In the next chapter we will evaluate this results and compare them to other methods in the literature.

Chapter 6

Evaluation of the Method

In the last chapter we stated an algorithm in Linear Program **LP2**, that generates the linear program $\mathbf{LP2}(\mathbf{f}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$, and in Theorem II we proved, that if this linear program has a feasible solution and $\mathcal{D} = \emptyset$, then a true Lyapunov function $V^{Lya} \in \mathrm{CPWA}[\mathbf{PS}, \mathbf{PS}^{-1}(\mathcal{M})]$ for the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

can be constructed from the solution. From the definition of the set $\mathcal{M}$ in Linear Program **LP2** it is clear, that by a proper choice of the piecewise scaling function **PS**, one can practically reach, that the domain of the Lyapunov function V^{Lya} is $\mathcal{N}$. The set $\mathcal{A}$ from Theorem II is a lower bound, with respect to inclusion, of the region of attraction of the equilibrium at the origin. Hence, the method delivers an estimate of the region of attraction which extends to the boundary of $\mathcal{N}$. Because of the simple algebraic structure of V^{Lya}, this estimate can be calculated very easily with a computer.

If $\mathcal{D} \neq \emptyset$ and the linear program $\mathbf{LP2}(\mathbf{f}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ has a feasible solution, then a Lyapunov-like function $V^{Lya} \in \mathrm{CPWA}[\mathbf{PS}, \mathbf{PS}^{-1}(\mathcal{M})]$ for the system above can be generated. In this case Theorem II states that every $\boldsymbol{\xi} \in \mathcal{A}$ will be attracted to the boundary of $\mathcal{D}$ by the dynamics of the system. At first glance, the method seems more limited when $\mathcal{D} \neq \emptyset$, than the method when $\mathcal{D} = \emptyset$. But if the Jacobian matrix of $\mathbf{f}$ at zero is Hurwitz, then Lyapunov's indirect method delivers a lower bound of the region of attraction of the equilibrium at the origin. Hence, if one combines the methods and assures that $\mathcal{D}$ is contained in the region of attraction, then $\mathcal{A}$ is a less conservative bound of the domain of attraction. The methods are, thus, essentially equivalent in terms of the results, if the Jacobian matrix of $\mathbf{f}$ at zero is Hurwitz, but when $\mathcal{D} \neq \emptyset$ the linear program $\mathbf{LP2}(\mathbf{f}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ is more likely to possess a feasible solution. Note that the Jacobian matrix of $\mathbf{f}$ at zero is Hurwitz, if and only if zero is an exponentially stable equilibrium point, and exponentially stable equilibrium points are robust to perturbation and are thus desirable from the viewpoint of applications.

Because of the lack of a constructive converse theorem for Lyapunov functions, in particular for CPWA Lyapunov functions, Linear Program **LP2** is a trial-and-error search method. A constructive converse theorem for systems with an exponentially stable equilibrium point, where the constructed Lyapunov function is in CPWA, is being worked on. We will discuss this in more detail at the end of this thesis.

6.1 Approaches in Literature

There are a lot of proposals on how to construct Lyapunov functions for nonlinear systems.

In [14], [15], and [16] piecewise quadratic Lyapunov functions are constructed for piecewise affine systems. The construction is based on continuity matrices for the partition of the respective state-space.

In [6], [7], [30], and [31] the Lyapunov function construction for a set of linear systems is reduced to the design of a balanced polytope, fulfilling some properties regarding invariance.

In [5] a convex optimization problem is used to compute a quadratic Lyapunov function for a system linear in a band containing the origin. In [14] there is an illustrating example of its use.

In [34] the stability of an uncertain nonlinear system is analyzed by studying a set of piecewise affine systems.

In [23] piecewise affine Lyapunov function candidates are used to prove the stability of linear systems with piecewise affine control. In [22], [20], [24], and [25] a convex partition is used to prove $G_{H,N}$-stability of time discrete piecewise affine systems. There is no Lyapunov function involved, but the region of attraction can be calculated.

In [42] and [43] a converse theorem is used to construct integral Lyapunov functions for time discrete systems.

In [37] a Lyapunov function of the form $V(\mathbf{x}) = \|W\mathbf{x}\|_\infty$ is parameterized by linear programming.

All these methods have in common, that they cannot be used for general continuous nonlinear systems.

6.2 The Linear Program of Julian *et al.*

In [19] and [17] a linear program for nonlinear systems is proposed by Julian *et al.*. If this linear program has a feasible solution, then there is a constant $b > 0$, such that the solution $\boldsymbol{\phi}$ of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ fulfills

$$\limsup_{t \to +\infty} \|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_\infty \leq b \tag{6.1}$$

for all $\boldsymbol{\xi}$ in a neighborhood of the origin. The linear program is essentially the linear program $\mathbf{LP2}(\mathbf{f}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ with

$$\begin{aligned}
&\mathcal{N} := [-a, a]^n, \quad \text{for an } a \in \mathbb{R}_{>0}, \\
&\mathbf{PS}(\mathbf{x}) := \frac{a}{N}\mathbf{x}, \quad \text{for an } N \in \mathbb{Z}_{>0}, \\
&\mathcal{D} := \emptyset, \\
&\|\cdot\| := \|\cdot\|_\infty,
\end{aligned}$$

and where the linear constraints LC3 are left out and the linear constraints LC4 are simplified to

$$-\Gamma[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|] \geq \sum_{j=1}^{n} \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \tag{6.2}$$

for every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, every $\sigma \in \text{Sym}_n$, and every $i = 1, 2, .., n+1$. The functions γ and V^{Lya} are defined by interpolation as demonstrated in Section 5.1.

Let $\mathbf{f_p} \in \text{CPWA}[\mathbf{PS}, [-N, N]^n]^n$, i.e. every component of $\mathbf{f_p}$ is in $\text{CPWA}[\mathbf{PS}, [-N, N]^n]$, be defined by

$$\mathbf{f_p}(\mathbf{x}) := \mathbf{f}(\mathbf{x}) \quad \text{for every vertex } \mathbf{x} \text{ of a simplex in } \mathfrak{S}[\mathbf{PS}, [-N, N]^n].$$

Note that if $\mathbf{f}|_{[-N,N]^n} = \mathbf{f_p}$, then this linear program is equivalent to ours, but if not then this linear program does not take the approximation error made by replacing $\mathbf{f}$ with $\mathbf{f_p}$ into account. The asymptotic bound (6.1) can be proved in the following way.

Define

$$e := \max_{\mathbf{x} \in \mathcal{N}} \|\mathbf{f}(\mathbf{x}) - \mathbf{f_p}(\mathbf{x})\|_2$$

and

$$W := \max_{\mathbf{x} \in \mathcal{N}} \|\nabla V^{Lya}(\mathbf{x})\|_2.$$

Then (6.2) implies that

$$-\gamma(\|\mathbf{x}\|_\infty) + We \geq \nabla V^{Lya}(\mathbf{x}) \cdot \mathbf{f_p}(\mathbf{x}) + We \geq \nabla V^{Lya}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}).$$

Choose a constant $\eta \in]0, 1[$ and write the last inequality as

$$-(1-\eta)\gamma(\|\mathbf{x}\|_\infty) - \eta\gamma(\|\mathbf{x}\|_\infty) + We \geq \nabla V^{Lya}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}).$$

Hence,

$$-(1-\eta)\gamma(\|\mathbf{x}\|_\infty) \geq \nabla V^{Lya}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$$

if

$$-\eta\gamma(\|\mathbf{x}\|_\infty) + We \leq 0,$$

i.e.

$$\|\mathbf{x}\|_\infty \geq \gamma^{-1}(\frac{We}{\eta}).$$

With similar reasoning as in the proof of Theorem II, one can use this to show that (6.1) is satisfied with

$$b := \alpha_1^{-1}(\alpha_2(\gamma^{-1}(\frac{We}{\eta}))),$$

where $\alpha_1, \alpha_2 \in \mathcal{K}$ and

$$\alpha_1(\|\mathbf{x}\|_\infty) \leq V^{Lya}(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|_\infty).$$

Note that the constant b is not a priori known.

In the linear program of Julian *et al.* a high-level canonical piecewise linear representation of continuous piecewise affine functions is used. A further reference to this representation is [18].

It should be clear from this comparison that Linear Program **LP2** has several advantages over the linear program of Julian *et al.* The boundary configuration is more flexible, a true Lyapunov function can be constructed from a feasible solution, and there is no a posteriori analysis of the quality of a feasible solution.

Part IV

Examples and Concluding Remarks

In this last part we will present examples of CPWA Lyapunov functions generated by using Linear Program **LP2** and we will refute the α, m-exponential stability on several regions, for an equilibrium of a system, with the help of Linear Program **LP1**. Finally, we will give some concluding remarks and suggestions for future research on the topics covered in this thesis.

Finding a Lyapunov function for a nonlinear system with a stable equilibrium point is a trial-and-error process. Linear Program **LP2** is a novel search method, Linear Program **LP1** is a novel method to exclude a large class of Lyapunov functions from the search. The examples are intended to show how the linear programs **LP1** and **LP1** work. It is not the intention at this stage to use the linear programs to systematically solve practical problems.

Chapter 7

Examples

The only non-trivial Lyapunov functions that can be visualized in an easy way without suppressing information, are those with domain in $\mathbb{R}^2$. Because of this we will limit our examples to two-dimensional dynamical systems. The linear programs generated were solved with the linear solver CPLEX® 6.5 from ilog® on a SUN™ Enterprise 450 workstation. CPLEX® uses the simplex algorithm. For a short overview of the simplex algorithm see, for example, Chapter 5 in [41], Section 10.8 in [38], or Chapter 9 in [13], for a more thorough treatment see, for example, [36]. The simplex algorithm needs on average, a number of operations directly proportional to the number of the constraints of the linear program. There are examples known, where the simplex algorithm is much slower and the number of operations needed grows exponentially with the problem size. This theoretical problem, however, does not seem to be of any practical relevance. There are methods to solve linear programs that are better theoretically, the so-called ellipsoid algorithm and the projective algorithm (see, for example, chapters 8 and 9 in [36]), but they are not better at solving practical problems.

In all examples the constant ε from Linear Program **LP2** was set equal to one, the norm used was the infinity norm $\|\cdot\|_\infty$, and $\mathcal{D} = \emptyset$. Further, we always minimized the objective

$$\sum_{\mathbf{z}\in\mathcal{G}_{\mathbf{N}^-,\mathbf{N}^+}} (V[\mathbf{z}] - \Psi[\|\mathrm{PS}(\mathbf{z})\|_\infty])$$

in the linear programs generated by Linear Program **LP2**.

7.1 Example I

Consider the dynamical system $\dot{\mathbf{x}} = \mathbf{f}_{E_1}(\mathbf{x})$, where

$$\mathbf{f}_{E_1}(\mathbf{x}) := \begin{pmatrix} -x_2 \\ x_1 - x_2(1 - x_1^2 + 0.1x_1^4) \end{pmatrix}.$$

This system is taken from Exercise 1.16 (1) in [21].

One easily verifies that $\mathbf{f}_{E_1}(\mathbf{0}) = \mathbf{0}$, that $\mathbf{f}_{E_1}$ is infinitely differentiable on $\mathbb{R}^2$, and that its Jacobian matrix at zero,

$$\nabla\mathbf{f}_{E_1}(\mathbf{0}) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

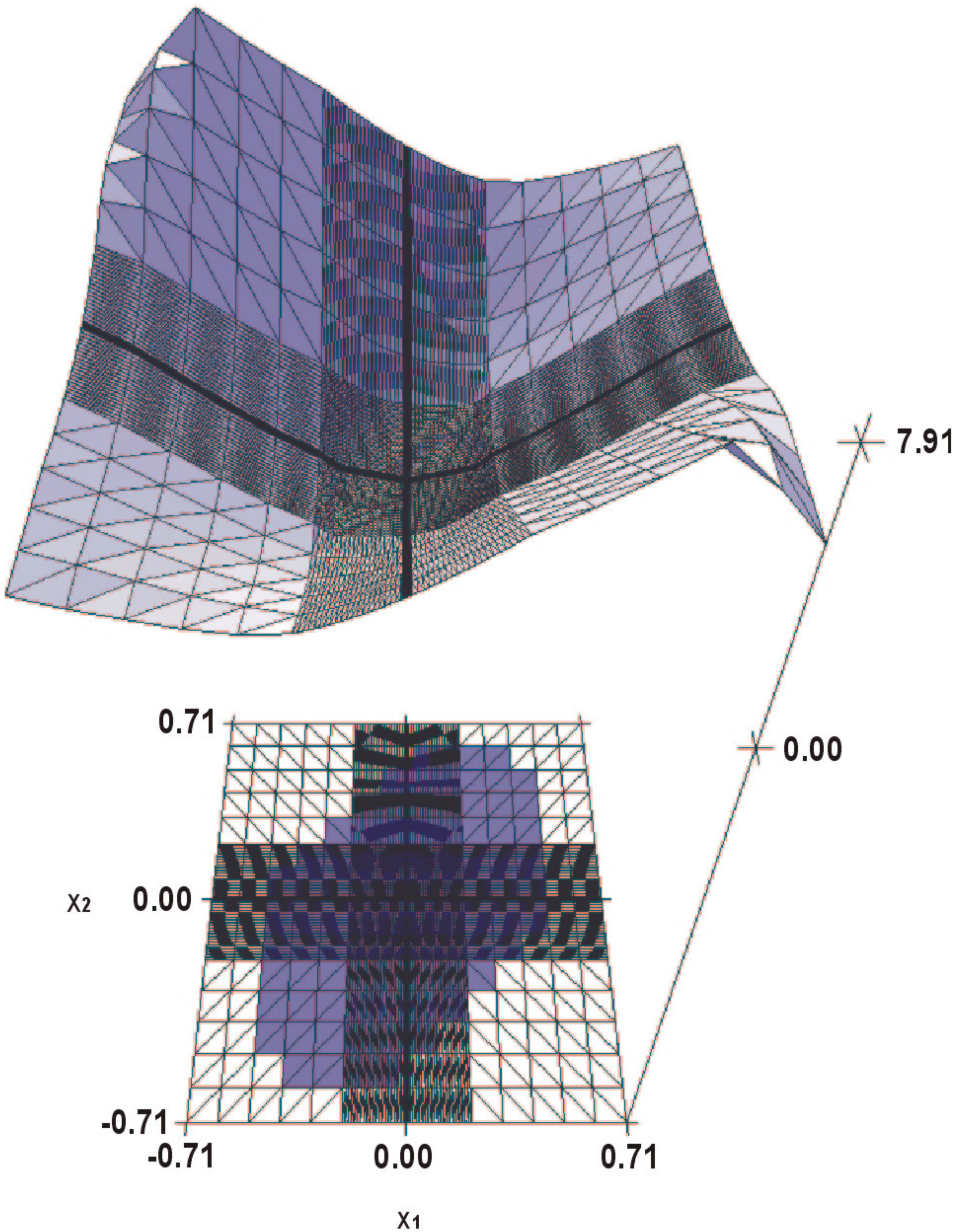

Figure 7.1: A CPWA Lyapunov function for $\dot{\mathbf{x}} = \mathbf{f}_{E_1}(\mathbf{x})$

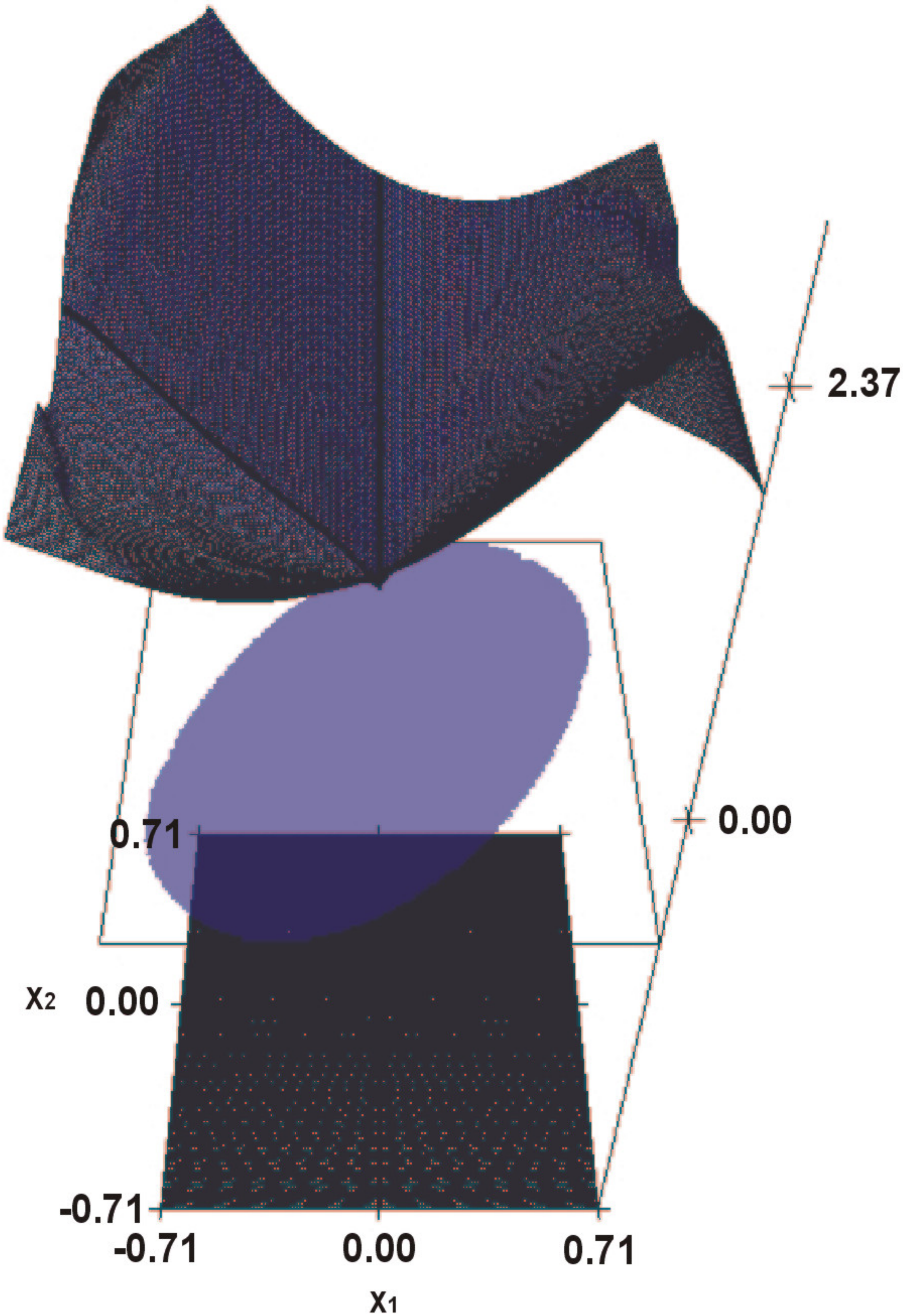

Figure 7.2: A CPWA Lyapunov function for $\dot{\mathbf{x}} = \mathbf{f}_{E_1}(\mathbf{x})$

has the eigenvalues

$$-\frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \text{and} \quad -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The origin is, therefore, a locally exponentially stable equilibrium point of the system (see, for example, Theorem 2.3 in Part II in [55]).

Linear Program **LP2** was used to generate linear constraints for CPWA Lyapunov functions for the system, which CPLEX® was able to satisfy. In Figure 7.1 a CPWA Lyapunov function $[-0.71, 0.71]^2 \longrightarrow \mathbb{R}$, determined by its values on a 71×71 grid ($= 5041$ grid points), is plotted. The partition of the domain of the Lyapunov function is shown at the base of the graph. The grid steps are identical for the x_1- and x_2-axes, 0.001 in the interval $[-0.01, 0.01]$, 0.01 in the intervals $[-0.21, -0.01]$ and $[0.01, 0.21]$, and 0.1 in the intervals $[-0.71, -0.21]$ and $[0.21, 0.71]$. The shaded area on the partition is a lower bound of the region of attraction of the equilibrium.

In Figure 7.2 a CPWA Lyapunov function with the same domain, but determined by its values on a 161×161 grid ($= 25921$ grid points), is drawn. The grid steps are identical for the x_1- and x_2-axes, 0.001 in the interval $[-0.01, 0.01]$ and 0.01 in the intervals $[-0.71, -0.01]$ and $[0.01, 0.71]$. The partition is shown at the base, but because of the small size of the simplices they are hard to identify. A lower bound of the region of attraction is shown as a shaded area above the partition.

7.2 Example II

Consider the dynamical system $\dot{\mathbf{x}} = \mathbf{f}_{E_2}(\mathbf{x})$, where

$$\mathbf{f}_{E_2}(\mathbf{x}) := \begin{pmatrix} x_2 \\ x_1 + x_2 - 3\arctan(x_1 + x_2) \end{pmatrix}.$$

This system is taken from Exercise 1.16 (2) in [21].

One easily verifies that $\mathbf{f}_{E_2}(\mathbf{0}) = \mathbf{0}$, that $\mathbf{f}_{E_2}$ is infinitely differentiable on $\mathbb{R}^2$, and that its Jacobian matrix at zero,

$$\nabla\mathbf{f}_{E_2}(\mathbf{0}) = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix},$$

has the eigenvalues

$$-1 + i \quad \text{and} \quad -1 - i.$$

Hence, zero is a locally exponentially stable equilibrium of the system.

Linear Program **LP2** was used to generate linear constraints for a CPWA Lyapunov function for the system, which CPLEX® was able to satisfy. In Figure 7.3 a CPWA Lyapunov function $[-1.71, 1.71]^2 \longrightarrow \mathbb{R}$ for a 91×91 grid ($= 8281$ grid points) is plotted. As in Example I the partition is drawn at the base and a lower bound of the region of attraction is shown as a shaded area on the partition. The grid steps are identical for the x_1- and x_2-axes, 0.001 in the interval $[-0.01, 0.01]$, 0.01 in the intervals $[-0.21, -0.01]$ and $[0.01, 0.21]$, and 0.1 in the intervals $[-1.71, -0.21]$ and $[0.21, 1.71]$.

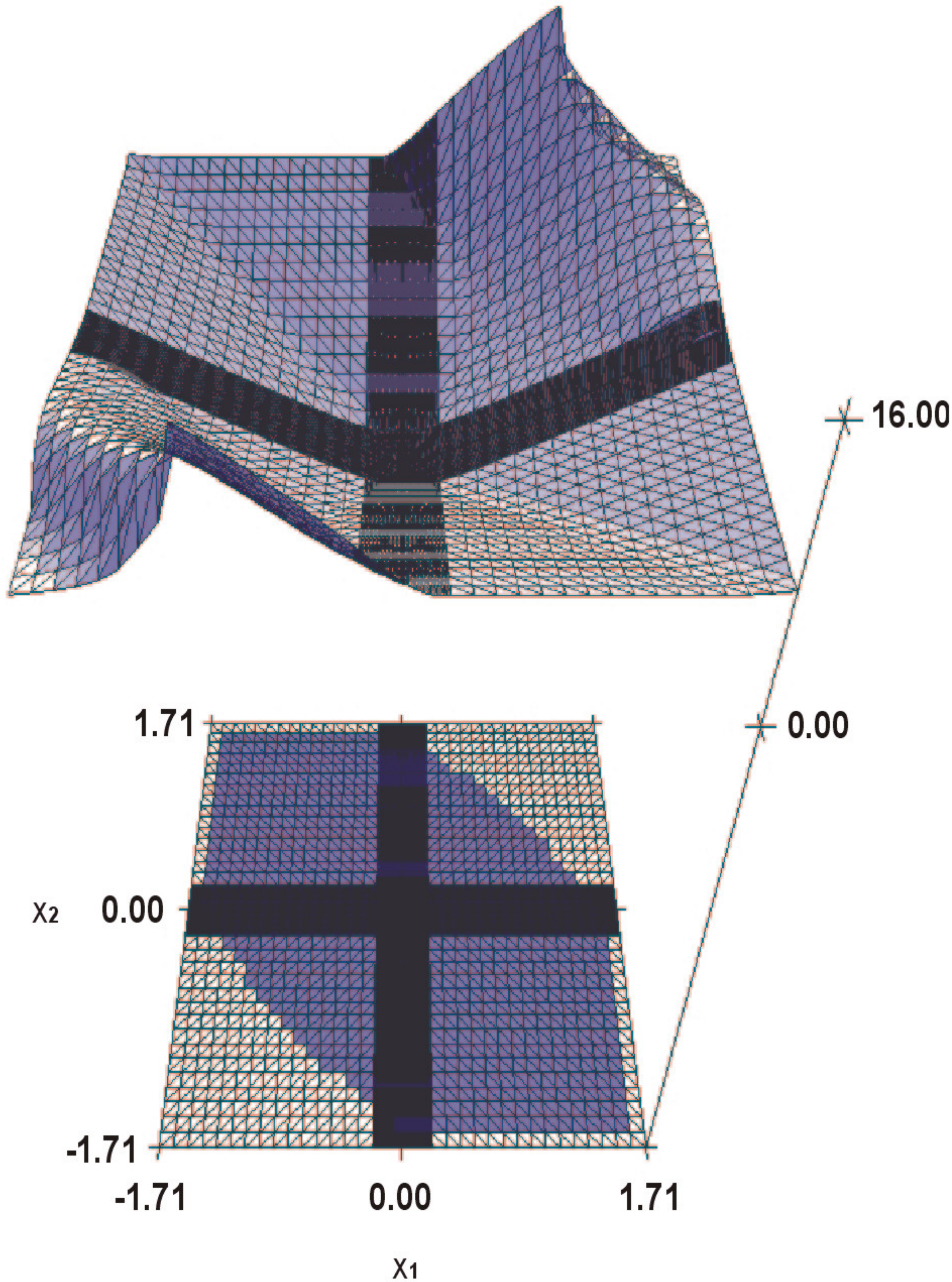

Figure 7.3: A CPWA Lyapunov function for $\dot{\mathbf{x}} = \mathbf{f}_{E_2}(\mathbf{x})$

7.3 Example III

Consider the dynamical system $\dot{\mathbf{x}} = \mathbf{f}_{E_3}(\mathbf{x})$, where

$$\mathbf{f}_{E_3}(\mathbf{x}) := \begin{pmatrix} x_2 \\ -\sin(x_1) - x_2 \end{pmatrix}.$$

It is the state equation of a pendulum with friction, where

$$\begin{aligned} &\frac{g}{l} = \frac{k}{m} = 1, \\ &g := \text{acceleration due to gravity}, \\ &l := \text{the length of the pendulum}, \\ &k := \text{the coefficient of friction}, \\ &m := \text{the mass of the pendulum}. \end{aligned}$$

For a further discussion of this system see, for example, Subsection 1.1.1 and Example 1.3 in [21]. One easily verifies that $\mathbf{f}_{E_3}(\mathbf{0}) = \mathbf{0}$, that $\mathbf{f}_{E_3}$ is infinitely differentiable on $\mathbb{R}^2$, and that its Jacobian matrix at zero,

$$\nabla \mathbf{f}_{E_3}(\mathbf{0}) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

has the eigenvalues

$$-\frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \text{and} \quad -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

This means that zero is a locally exponentially stable equilibrium point of the system.

Linear Program **LP2** was used to generate linear constraints for a CPWA Lyapunov function for the system, which CPLEX® was able to satisfy. In Figure 7.4 a CPWA Lyapunov function $[-1.71, 1.71]^2 \longrightarrow \mathbb{R}$ for a 91×91 grid (= 8281 grid points) is drawn, with the partition used, drawn at the base. The grid steps are identical for the x_1- and x_2-axes, 0.001 in the interval $[-0.01, 0.01]$, 0.01 in the intervals $[-0.21, -0.01]$ and $[0.01, 0.21]$, and 0.1 in the intervals $[-1.71, -0.21]$ and $[0.21, 1.71]$. The shaded area on the partition is a lower bound of the region of attraction.

7.4 Example IV

We come to the final example. Consider the dynamical system $\dot{\mathbf{x}} = \mathbf{f}_{E_4}(\mathbf{x})$, where

$$\mathbf{f}_{E_4}(\mathbf{x}) := \begin{pmatrix} -x_2 + x_1(x_1^2 + x_2^2 - 1) \\ x_1 + x_2(x_1^2 + x_2^2 - 1) \end{pmatrix}. \tag{7.2}$$

This system is taken from page 194 in [40]. One easily verifies that $\mathbf{f}_{E_4}(\mathbf{0}) = \mathbf{0}$, that $\mathbf{f}_{E_4}$ is infinitely differentiable on $\mathbb{R}^2$, and that its Jacobian matrix at zero,

$$\nabla \mathbf{f}_{E_4}(\mathbf{0}) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},$$

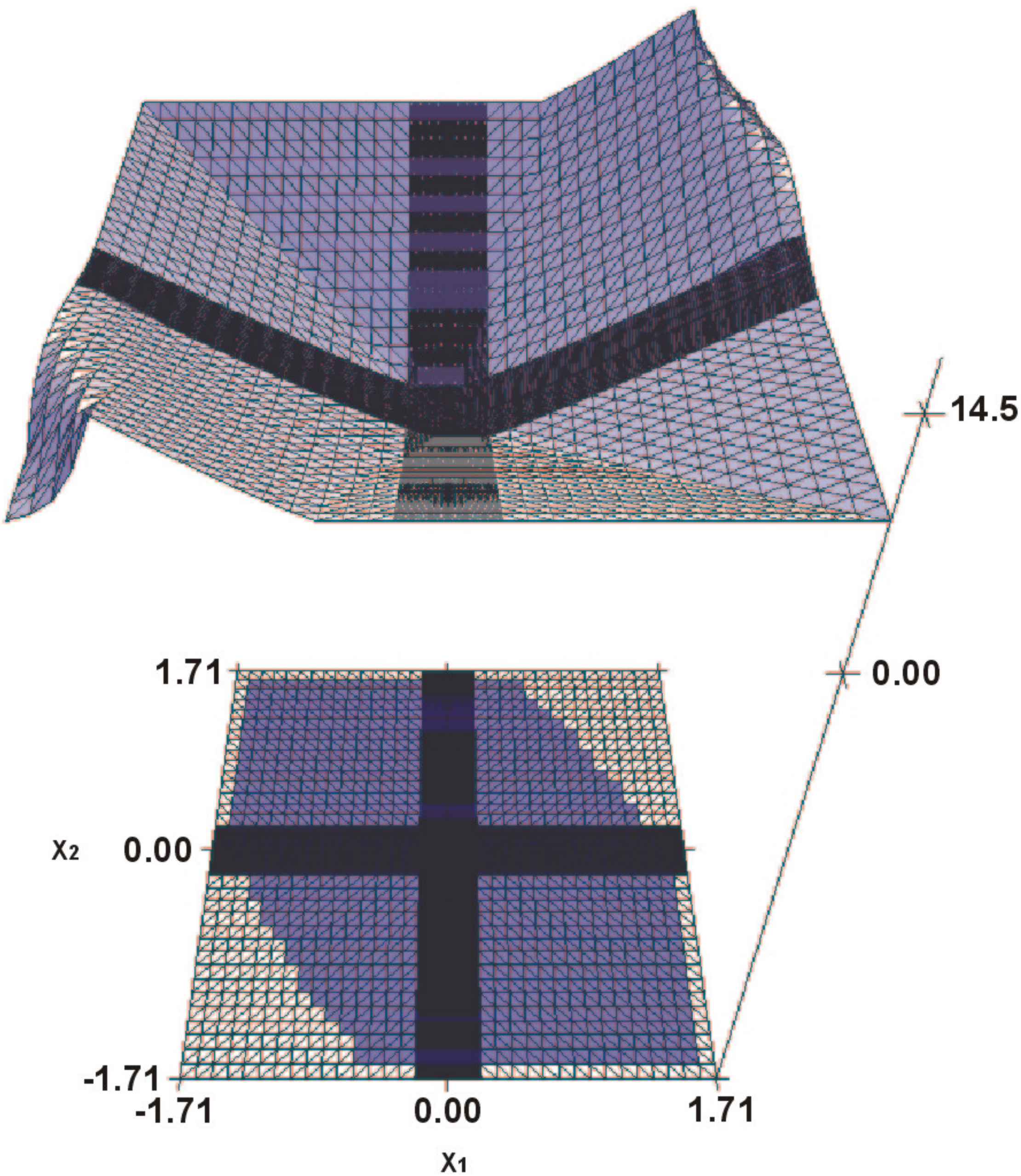

Figure 7.4: A CPWA Lyapunov function for $\dot{\mathbf{x}} = \mathbf{f}_{E_3}(\mathbf{x})$

has the eigenvalues

$$-1+i \quad \text{and} \quad -1-i.$$

Hence, the origin is a locally exponentially stable equilibrium point of the system.

Let us have a closer look at the linearized system. It is well known that the differential equation

$$\dot{\mathbf{x}} = \nabla \mathbf{f}_{E_4}(\mathbf{0})\mathbf{x}$$

has the solution

$$\boldsymbol{\phi}_L(t,\boldsymbol{\xi}) = e^{t\nabla \mathbf{f}_{E_4}(\mathbf{0})}\boldsymbol{\xi}.$$

Routine calculations give

$$\nabla \mathbf{f}_{E_4}(\mathbf{0}) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\boldsymbol{\phi}_L(t,\boldsymbol{\xi}) &= \exp\left[t \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\right]\boldsymbol{\xi} \\
&= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \exp\left[t \begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix}\right] \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\boldsymbol{\xi} \\
&= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1} \begin{pmatrix} e^{(-1+i)t} & 0 \\ 0 & e^{(-1-i)t} \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\boldsymbol{\xi} \\
&= \begin{pmatrix} \frac{e^{(-1+i)t}+e^{(-1-i)t}}{2} & -\frac{e^{(-1+i)t}-e^{(-1-i)t}}{2i} \\ \frac{e^{(-1+i)t}-e^{(-1-i)t}}{2i} & \frac{e^{(-1+i)t}+e^{(-1-i)t}}{2} \end{pmatrix}\boldsymbol{\xi} \\
&= e^{-t}\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\boldsymbol{\xi}.
\end{aligned}$$

The obvious

$$\| \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \|_2 = 1$$

leads to the upper bound

$$\|\boldsymbol{\phi}_L(t,\boldsymbol{\xi})\|_2 \leq e^{-t}\|\boldsymbol{\xi}\|_2$$

of the solution of the linearized system. Hence, the origin is a globally $1,1$-exponentially stable equilibrium of the linearized system.

We come back to the nonlinear system (7.2). The function $V : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\mathbf{x} \mapsto \|\mathbf{x}\|_2^2$, is a Lyapunov function for the system. To see this, note that $V(\mathbf{0}) = 0$ and that

$$\begin{aligned}
\frac{d}{dt}V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))\bigg|_{t=0} &= [\nabla V](\boldsymbol{\xi})\cdot \mathbf{f}_{E_4}(\boldsymbol{\xi}) = 2\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} -\xi_2 + \xi_1[\xi_1^2+\xi_2^2-1] \\ \xi_1 + \xi_2[\xi_1^2+\xi_2^2-1] \end{pmatrix} \\
&= 2\|\boldsymbol{\xi}\|_2^2(\|\boldsymbol{\xi}\|_2^2 - 1),
\end{aligned} \tag{7.3}$$

where $\boldsymbol{\phi}$ is the solution of (7.2). From this it follows that the time-derivative of V along the trajectories of the system is negative definite on the open unit circular disc, so the equilibrium at the origin is asymptotically stable and its region of attraction contains the open unit circular disc (see, for example, Theorem 3.1 in [21]).

Linear Program **LP2** was used to generate linear constraints for CPWA Lyapunov functions for the system, which CPLEX® was able to satisfy. In Figure 7.5 the results for a CPWA Lyapunov function $[-0.71, 0.71]^2 \longrightarrow \mathbb{R}$, determined by its values on a 81×81 grid ($= 6561$ grid points), are drawn. As usual the partition is drawn at the base and the shaded area is a lower bound of the region of attraction. The grid steps are identical for the x_1- and x_2-axes, 0.001 in the interval $[-0.01, 0.01]$, 0.01 in the intervals $[-0.21, -0.01]$ and $[0.01, 0.21]$, and 0.05 in the intervals $[-0.71, -0.21]$ and $[0.21, 0.71]$.

We want to use Linear Program **LP1** to find upper bounds of the region of attraction. To shorten writings we will write $\mathbf{f}$ for $\mathbf{f}_{E_4}$ from now on. To assign values to the constants `A`, `B`, and `C` in Linear Program **LP1** we need upper bounds of all partial derivatives of all components of $\mathbf{f}$ up to the third order.

The first order partial derivatives are:

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1}(\mathbf{x}) &= 3x_1^2 + x_2^2 - 1 \\
\frac{\partial f_1}{\partial x_2}(\mathbf{x}) &= -1 + 2x_1x_2 \\
\frac{\partial f_2}{\partial x_1}(\mathbf{x}) &= 1 + 2x_1x_2 \\
\frac{\partial f_2}{\partial x_2}(\mathbf{x}) &= x_1^2 + 3x_2^2 - 1
\end{aligned}$$

The second order partial derivatives are:

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial x_1 \partial x_1}(\mathbf{x}) &= 6x_1 \\
\frac{\partial^2 f_1}{\partial x_1 \partial x_2}(\mathbf{x}) &= \frac{\partial^2 f_1}{\partial x_2 \partial x_1}(\mathbf{x}) = 2x_2 \\
\frac{\partial^2 f_1}{\partial x_2 \partial x_2}(\mathbf{x}) &= 2x_1 \\
\frac{\partial^2 f_2}{\partial x_1 \partial x_1}(\mathbf{x}) &= 2x_2 \\
\frac{\partial^2 f_2}{\partial x_1 \partial x_2}(\mathbf{x}) &= \frac{\partial^2 f_2}{\partial x_2 \partial x_1}(\mathbf{x}) = 2x_1 \\
\frac{\partial^2 f_2}{\partial x_2 \partial x_2}(\mathbf{x}) &= 6x_2
\end{aligned}$$

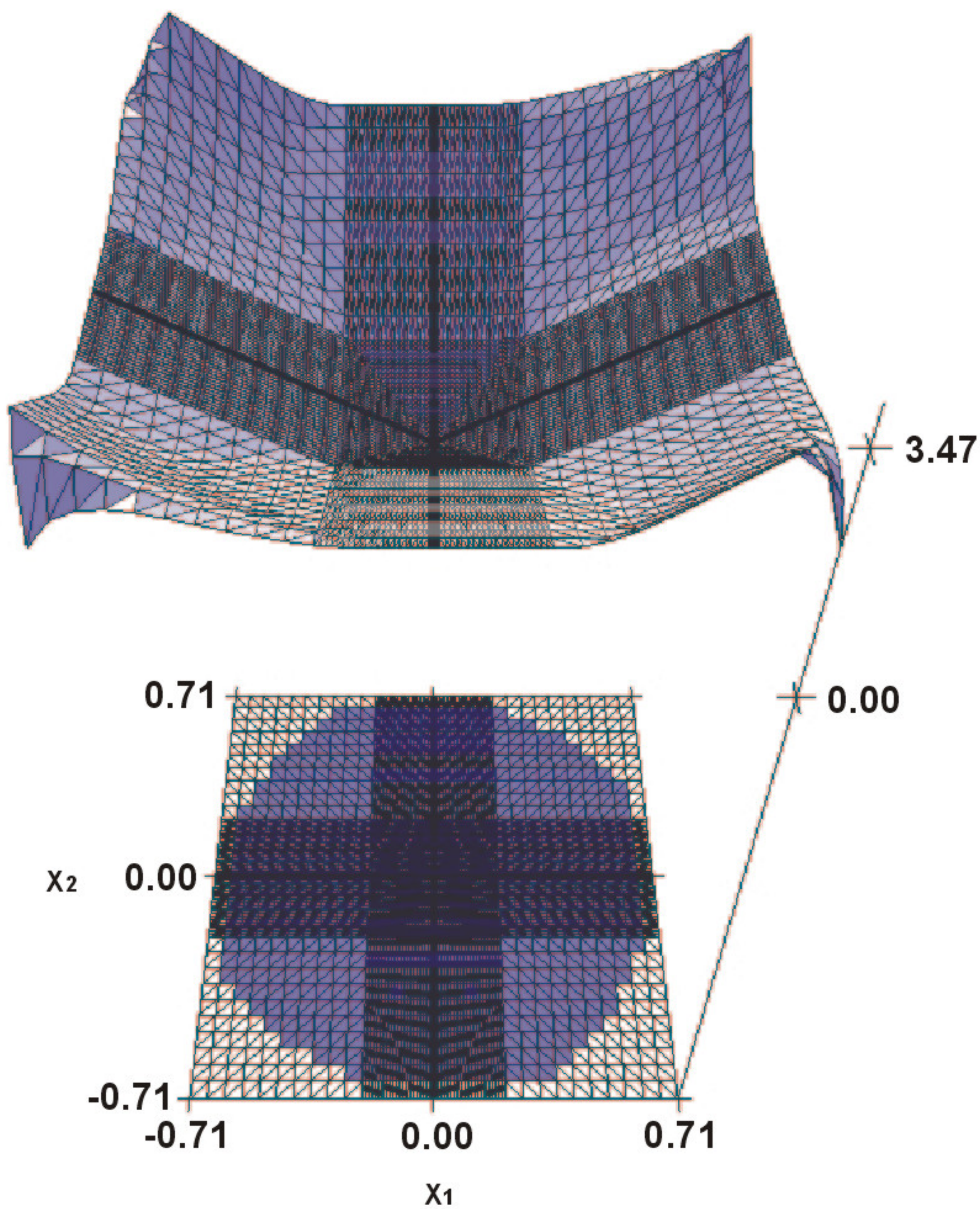

Figure 7.5: A CPWA Lyapunov function for $\dot{\mathbf{x}} = \mathbf{f}_{E_4}(\mathbf{x})$

The third order partial derivatives are:

$$
\begin{aligned}
&\frac{\partial^3 f_1}{\partial x_1 \partial x_1 \partial x_1}(\mathbf{x}) = 6\\
&\frac{\partial^3 f_1}{\partial x_1 \partial x_1 \partial x_2}(\mathbf{x}) = \frac{\partial^3 f_1}{\partial x_1 \partial x_2 \partial x_1}(\mathbf{x}) = \frac{\partial^3 f_1}{\partial x_2 \partial x_1 \partial x_1}(\mathbf{x}) = 0\\
&\frac{\partial^3 f_1}{\partial x_1 \partial x_2 \partial x_2}(\mathbf{x}) = \frac{\partial^3 f_1}{\partial x_2 \partial x_1 \partial x_2}(\mathbf{x}) = \frac{\partial^3 f_1}{\partial x_2 \partial x_2 \partial x_1}(\mathbf{x}) = 2\\
&\frac{\partial^3 f_1}{\partial x_2 \partial x_2 \partial x_2}(\mathbf{x}) = 0\\
&\frac{\partial^3 f_2}{\partial x_1 \partial x_1 \partial x_1}(\mathbf{x}) = 0\\
&\frac{\partial^3 f_2}{\partial x_1 \partial x_1 \partial x_2}(\mathbf{x}) = \frac{\partial^3 f_2}{\partial x_1 \partial x_2 \partial x_1}(\mathbf{x}) = \frac{\partial^3 f_2}{\partial x_2 \partial x_1 \partial x_1}(\mathbf{x}) = 2\\
&\frac{\partial^3 f_2}{\partial x_1 \partial x_2 \partial x_2}(\mathbf{x}) = \frac{\partial^3 f_2}{\partial x_2 \partial x_1 \partial x_2}(\mathbf{x}) = \frac{\partial^3 f_2}{\partial x_2 \partial x_2 \partial x_1}(\mathbf{x}) = 0\\
&\frac{\partial^3 f_2}{\partial x_2 \partial x_2 \partial x_2}(\mathbf{x}) = 6
\end{aligned}
$$

For an $M > 0$ and a domain of the type $]-M, M[^2$ of $\mathbf{f}$, we use these results to assign values to the constants a'_{ij}, b'_{ijk}, and c'_{ijkl}, $i,j,k,l \in \{1,2\}$, from Linear Program **LP2**:

$$
\begin{aligned}
&a'_{11} := \max\{4M^2 - 1, 1\}\\
&a'_{12} := 1 + 2M^2\\
&a'_{21} := 1 + 2M^2\\
&a'_{22} := \max\{4M^2 - 1, 1\}\\
&b'_{111} := 6M\\
&b'_{112} = b'_{121} := 2M\\
&b'_{122} := 2M\\
&b'_{211} := 2M\\
&b'_{212} = b'_{221} := 2M\\
&b'_{222} := 6M\\
&c'_{1111} := 6\\
&c'_{1112} = c'_{1121} = c'_{1211} := 0\\
&c'_{1122} = c'_{1212} = c'_{1221} := 2\\
&c'_{1222} := 0\\
&c'_{2111} := 0\\
&c'_{2112} = c'_{2121} = c'_{2211} := 2\\
&c'_{2122} = c'_{2212} = c'_{2221} := 0\\
&c'_{2222} := 6
\end{aligned}
$$

The constants a'_{ij}, b'_{ijk}, and c'_{ijkl}, $i,j,k,l \in \{1,2\}$, are used to assign values to the entries a_{ij}, b_{ij}, and c_{ij}, $i,j \in \{1,2\}$, of the matrices $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ in Linear Program **LP1**. For

$M \geq 1/\sqrt{2}$ they are:

$$
\begin{aligned}
a_{11} &:= a'_{11} = 4M^2 - 1 \\
a_{12} &:= a'_{12} := 1 + 2M^2 \\
a_{21} &:= a'_{21} := 1 + 2M^2 \\
a_{22} &:= a'_{22} = 4M^2 - 1 \\
b_{11} &:= \sqrt{{b'_{111}}^2 + {b'_{112}}^2} = 2\sqrt{10}M \\
b_{12} &:= \sqrt{{b'_{121}}^2 + {b'_{122}}^2} = 2\sqrt{2}M \\
b_{21} &:= \sqrt{{b'_{211}}^2 + {b'_{212}}^2} = 2\sqrt{2}M \\
b_{22} &:= \sqrt{{b'_{221}}^2 + {b'_{222}}^2} = 2\sqrt{10}M \\
c_{11} &:= \sqrt{{c'_{1111}}^2 + {c'_{1112}}^2 + {c'_{1121}}^2 + {c'_{1122}}^2} = 2\sqrt{10} \\
c_{12} &:= \sqrt{{c'_{1211}}^2 + {c'_{1212}}^2 + {c'_{1221}}^2 + {c'_{1222}}^2} = 2\sqrt{2} \\
c_{21} &:= \sqrt{{c'_{2111}}^2 + {c'_{2112}}^2 + {c'_{2121}}^2 + {c'_{2122}}^2} = 2\sqrt{2} \\
c_{22} &:= \sqrt{{c'_{2211}}^2 + {c'_{2212}}^2 + {c'_{2221}}^2 + {c'_{2222}}^2} = 2\sqrt{10}
\end{aligned}
$$

From this,

$$
\begin{aligned}
\widetilde{A} &= \begin{pmatrix} 4M^2 - 1 & 1 + 2M^2 \\ 1 + 2M^2 & 4M^2 - 1 \end{pmatrix}, \\
\widetilde{B} &= 2\sqrt{2}M \begin{pmatrix} \sqrt{5} & 1 \\ 1 & \sqrt{5} \end{pmatrix},
\end{aligned}
$$

and

$$
\widetilde{C} = 2\sqrt{2} \begin{pmatrix} \sqrt{5} & 1 \\ 1 & \sqrt{5} \end{pmatrix}
$$

follow. The constants `A`, `B`, and `C`, defined as the spectral norms of $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ respectively, can now easily be calculated. The matrix

$$
\widetilde{A}^T \widetilde{A} = \begin{pmatrix} (4M^2 - 1)^2 + (1 + 2M^2)^2 & 2(4M^2 - 1)(1 + 2M^2) \\ 2(4M^2 - 1)(1 + 2M^2) & (4M^2 - 1)^2 + (1 + 2M^2)^2 \end{pmatrix}
$$

has the eigenvalues

$$
4M^4 - 8M^2 + 4 \quad \text{and} \quad 36M^4,
$$

where $36M^4$ is clearly the larger one ($M \geq 1/\sqrt{2}$), and the matrix

$$
\frac{1}{8M^2} \widetilde{B}^T \widetilde{B} = \frac{1}{8} \widetilde{C}^T \widetilde{C} = \begin{pmatrix} 6 & 2\sqrt{5} \\ 2\sqrt{5} & 6 \end{pmatrix}
$$

has the eigenvalues

$$
6 + 2\sqrt{5} \quad \text{and} \quad 6 - 2\sqrt{5},
$$

where $6+2\sqrt{5}$ is the larger one. Hence,

$$\mathtt{A} := \|\widetilde{A}\|_2 = \sqrt{36M^4} = 6M^2,$$
$$\mathtt{B} := \|\widetilde{B}\|_2 = 2\sqrt{2}M\sqrt{6+2\sqrt{5}},$$

and

$$\mathtt{C} := \|\widetilde{C}\|_2 = 2\sqrt{2}\sqrt{6+2\sqrt{5}}.$$

Now that we have assigned values to the constants $\mathtt{A}$, $\mathtt{B}$, and $\mathtt{C}$, we can use Linear Program **LP1** to generate linear constraints, capable of refuting the α, m-exponential stability of the equilibrium at the origin of (7.2) in arbitrary regions.

In all the linear programs generated the set $\mathcal{G}_{\mathbf{h}}^{\mathcal{N}}$ from Linear Program **LP1** was defined by

$$\mathcal{G}_{\mathbf{h}}^{\mathcal{N}} := \{(ih, jh) \in \mathbb{R}^2 \big| i, j \in \mathbb{Z},\ |i|, |j| \leq S\},$$

where $h = 0.01$ and S is a positive integer. The results are shown in the following table:

	$\alpha = 0.9$ $m = 1.1$ $S = 110$	$\alpha = 0.9$ $m = 1.2$ $S = 110$	$\alpha = 0.8$ $m = 1.1$ $S = 110$	$\alpha = 0.8$ $m = 1.1$ $S = 120$	$\alpha = 0.8$ $m = 1.1$ $S = 130$
Feasible solution	No	Yes	Yes	Yes	No

Table 7.1: Results from Linear Program **LP1**

The simple algebraic structure of (7.2) enables an exact analysis of the system, in order to compare the results with those of the linear programs **LP1** and **LP2**. For this let $\boldsymbol{\phi}$ be the solution of (7.2). It follows from (7.3) that

$$\frac{d}{dt}\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2 = 2\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2(\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2 - 1).$$

This differential equation is equivalent to the integral equation

$$\int_{\|\boldsymbol{\xi}\|_2^2}^{\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2} \frac{dy}{y(1-y)} = 2\int_0^t \tau d\tau,$$

which can easily be solved

$$\ln\left(\frac{\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2 - 1}{\|\boldsymbol{\xi}\|_2^2 - 1} \cdot \frac{\|\boldsymbol{\xi}\|_2^2}{\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2^2}\right) = 2t,$$

i.e.

$$\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2 = \frac{e^{-t}\|\boldsymbol{\xi}\|_2}{\sqrt{1-\|\boldsymbol{\xi}\|_2^2(1-e^{-2t})}}.$$

From this formula for $\boldsymbol{\phi}$ one immediately sees that:

i) For every $0 < r \leq 1$, the origin is a $1, \frac{1}{\sqrt{1-r^2}}$ - exponentially stable equilibrium point on the set $\{\mathbf{y} \in \mathbb{R}^2 \big| \|\mathbf{y}\|_2 < r\}$.

ii) If $\|\boldsymbol{\xi}\|_2 = 1$, then $\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_2 = 1$ for all $t \geq 0$.

iii) For $\|\boldsymbol{\xi}\|_2 > 1$ the solution $\boldsymbol{\phi}(t, \boldsymbol{\xi})$ has a finite escape time

$$t_{esc} = \frac{1}{2} \ln \left(\frac{\|\boldsymbol{\xi}\|_2^2}{\|\boldsymbol{\xi}\|_2^2 - 1} \right).$$

This is, of course, fully compatible with our numerical results.

Chapter 8

Concluding Remarks

In this thesis, stability properties of equilibrium points of general continuous autonomous nonlinear systems are analyzed via linear programming. The novel methods presented in this thesis do not need the nonlinear system to be of any specific type, like piecewise affine, and show that it is possible to extract a lot of non-trivial information regarding stability by using linear constraints.

8.1 Summary of Contributions

The main contributions of this thesis are the algorithms Linear Program **LP1** and Linear Program **LP2**, and the corresponding theorems Theorem I and Theorem II.

Linear Program **LP1** gives explicit linear programs for dynamical systems and all constants $\alpha > 0$ and $m \geq 1$, such that if there is no feasible solution of the program, then the equilibrium of the system under consideration cannot be α, m-exponentially stable on the region in question. The theory in Part II regarding Linear Program **LP1** seems to be substantially closed and complete. The bounds of the derivatives of the Lyapunov function

$$V(\boldsymbol{\xi}) := \int_0^T \|\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\|_2^2 d\tau,$$

from the standard proof of the converse theorem on exponential stability, are fairly good and it hardly seems possible to greatly improve them, without making major restrictions concerning the system in question.

Linear Program **LP2** gives explicit linear programs for dynamical systems too. Such a linear program, generated for a particular system, has the property that a feasible solution of it defines a CPWA Lyapunov function for the system or when $\mathcal{D} \neq \emptyset$, a CPWA Lyapunov-like function which guarantees that the trajectories of the system are attracted to $\mathcal{D}$. In both cases a lower bound of the region of attraction is provided by the preimages of the Lyapunov or Lyapunov-like function. Because these are in CPWA, the calculation of their preimages is a trivial task.

8.2 Some Ideas for Future Research

It might be interesting to use Linear Program **LP1** as the evaluating step in an iterative algorithm to estimate the region of attraction of an equilibrium of a nonlinear system. One could start with a small grid around the equilibrium, such that there is a feasible solution of the linear program generated by Linear Program **LP1**, and then successively add grid points to the grid until Linear Program **LP1** generates a linear program, which no longer has a feasible solution.

The theory regarding Linear Program **LP2** wakes some very interesting questions. They can, in essence, be categorized in two types, one regarding the existence of a CPWA Lyapunov(-like) function for a particular system, and another regarding the fine tuning of the CPWA Lyapunov(-like) function by using the objective in the linear program.

First, we will discuss the fine tuning of the Lyapunov(-like) function. One recalls that Linear Program **LP2** states a group of linear constraints for a system, such that a feasible solution thereof defines a CPWA Lyapunov(-like) function for the system. The objective of the linear program is not needed. Therefore, it is logical to try to use the objective to select a Lyapunov(-like) function that is well suited to the problem at hand. A well suited Lyapunov(-like) function might, for example, have one or more of the following properties:

i) It secures a large region of attraction.

ii) It secures some robustness properties of the stability.

iii) It gives bounds of the rate of convergence to equilibrium of the trajectories of the system.

The problem is to formulate these properties as objectives that are linear in the variables from the constraints of Linear Program **LP2**. In [19] there is a solution concept to the third point. They define the performance of the Lyapunov function, in accordance with [35], as the largest positive constant λ, such that

$$D^+[V(\boldsymbol{\phi}(t,\boldsymbol{\xi}))] \leq -\lambda V(\boldsymbol{\phi}(t,\boldsymbol{\xi})).$$

Then they formulate a linear objective, which gives a lower bound of λ.

We come to the second question. For the time being Linear Program **LP2** is a trial-and-error search method for Lyapunov functions, but can it be used to prove a constructive converse theorem? The problem "find values for the variables in Linear Program **LP2** that satisfy the constraints" seems much simpler to solve than "find a positive definite function that is strictly decreasing along all trajectories of the system". Consider the following quotations from the modern literature about the value of such a theorem:

Khalil writes on page 180 in [21]:

> Most of these converse theorems are provided by actually constructing auxiliary functions that satisfy the conditions of the respective theorems. Unfortunately, almost always this construction assumes the knowledge of the solutions of the differential equation. Therefore, these theorems do not help in the practical search for an auxiliary function. The mere knowledge that a function exists is, however, better than nothing. At least we know that our search is not hopeless.

Slotline and Li write on page 120 in [44]:

> A number of interesting results concerning the existence of Lyapunov functions, called *converse Lyapunov theorems*, have been obtained in this regard. For many years, these theorems were thought to be of no practical value because, like the previously described theorems, they do not tell us how to generate Lyapunov functions for a system to be analyzed.

Vidyasagar writes on pages 235-236 in [50]:

> Since the function V is constructed in terms of the solution trajectories of the system, the converse theorems cannot really be used to construct an explicit formula for the Lyapunov function, except in special cases (e.g., linear systems; see Section 5.4)

Currently, such a theorem is being worked on. It looks promising to combine the results from Part II and Part III of this thesis to prove a constructive converse theorem on exponential stability.

Bibliography

[1] R. Adams. *Sobolev Spaces.* Acad. Press, New York, 1975.

[2] T. Apostol. *Calculus: Volume 2.* John Wiley & Sons, New York, 2. edition, 1969.

[3] H. Bauer. *Maß- und Integrationstheorie.* de Gruyter, Berlin, 2. edition, 1990.

[4] H. Berenji. Fuzzy logic controllers. In R. Yager and L. Zadeh, editors, *An Introduction to Fuzzy Logic Applications in Intelligent Systems*, pages 69–96. Kluwer, Boston, 1992.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, 1994.

[6] R. Brayton and C. Tong. Stability of dynamical systems: a constructive approach. *IEEE Transactions on Circuits and Systems*, **CAS-26**:224–234, 1979.

[7] R. Brayton and C. Tong. Constructive stability and asymptotic stability of dynamical systems. *IEEE Transactions on Circuits and Systems*, **CAS-27**:1121–1130, 1980.

[8] U. Dini. *Fondamenti per la teoria delle funzioni di variabili reali.* Pisa, 1878.

[9] D. Driankov, H. Hellendorn, and R. Reinfrank. *An Introduction to Fuzzy Control.* Springer, Berlin, 2. edition, 1996.

[10] G. Golub and C. van Loan. *Matrix Computations.* The John Hopkins University Press, Baltimore, 1983.

[11] H. Grauert and I. Lieb. *Differential- und Integralrechnung III.* Springer, Berlin, 2. edition, 1977.

[12] W. Hahn. *Stability of Motion.* Springer, New York, 1967.

[13] G. Hämmerlin and K.-H. Hoffmann. *Numerische Mathematik.* Springer, Berlin, 3. edition, 1992.

[14] M. Johansson. *Piecewise Linear Control Systems.* Ph.D. Thesis: Lund Institute of Technology, Lund, 1999.

[15] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. In *Proceedings of the European Control Conference ECC'97*, 1997.

[16] M. Johansson and A. Rantzer. On the computation of piecewise quadratic Lyapunov functions. In *Proceedings of the 36th IEEE Conference on Decision and Control*, 1997.

[17] P. Julian. *A High Level Canonical Piecewise Linear Representation: Theory and Applications.* Ph.D. Thesis: Universidad Nacional del Sur, Bahia Blanca, 1999.

[18] P. Julian, A. Desages, and O. Agamennoni. High-level canonical piecewise linear representation using a simplicial partition. *IEEE Transactions on Circuits and Systems*, **46**:463–480, 1999.

[19] P. Julian, J. Guivant, and A. Desages. A parametrization of piecewise linear Lyapunov function via linear programming. *Int. J. Control*, **72**:702–715, 1999.

[20] D. Karweina. *Rechnergetützte Stabilitätsanalyse für nichtlineare zeitdiskrete Regelungssysteme, basierend auf der Methode der Konvexen Zerlegung.* Ph.D. Thesis: VDI Verlag, Düsseldorf, 1989.

[21] H. Khalil. *Nonlinear Systems.* Macmillan, New York, 1992.

[22] H. Kiendl. Stability analysis for advanced fuzzy systems. In *Proceedings of EUFIT'99 (CD-ROM)*, Aachen, 1999. Verlag Mainz.

[23] H. Kiendl and J. Ruger. Stability analysis of fuzzy control systems using facet functions. *Fuzzy Sets and Systems*, **70**:275–285, 1995.

[24] R. Knicker. Stability analysis of fuzzy and other nonlinear systems based on the method of convex decomposition. In B. Reusch, editor, *Computational Intelligence, Theory and Applications*, pages 391–400. Springer, Berlin, 1999.

[25] R. Knicker and P. Krause. The method of convex decomposition for stability analysis with stepwise mapping. In *Proceedings of EUFIT'99 (CD-ROM)*, Aachen, 1999. Verlag Mainz.

[26] A. Kolmogorov and S. Fomin. *Introductory Real Analysis.* Dover, New York, 1970.

[27] R. Kruse, J. Gebhardt, and F. Klawonn. *Foundations of Fuzzy Systems.* John Wiley & Sons, Chichester, 1994.

[28] A. M. Lyapunov. The general problem of the stability of motion. *Int. J. Control*, **55**:531–773, 1992.

[29] J. L. Massera. On Liapunoff's conditions of stability. *Ann. Math.*, **50**:705–721, 1949.

[30] A. Michel, B. Nam, and V. Vittal. Computer generated Lyapunov functions for interconnected systems: Improved Results with Applications to Power Systems. *IEEE Transactions on Circuits and Systems*, **CAS-31**:189–198, 1984.

[31] A. Michel, N. Sarabudla, and R. Miller. Stability analysis of complex dynamical systems: some computational methods. *Circuits and Systems Signal Processing*, **1**:171–202, 1982.

[32] T. Möllers. *Stabilität von Fuzzy-Reglern: Analysekriterien und explizite Syntheseverfahren.* Ph.D. Thesis: Shaker-Verlag, Aachen, 1998.

[33] H. Nguyen and E. Walker. *A First Course in Fuzzy Logic.* Chapman & Hall/CRC, Boca Raton, 2. edition, 2000.

[34] Y. Ohta, H. Imanishi, L. Gong, and H. Haneda. Computer generated Lyapunov functions for a class of non-linear systems. *IEEE Transactions on Circuits and Systems*, **CAS-40**:343–354, 1993.

[35] A. Olas. Construction of optimal Lyapunov functions for systems with structured uncertainties. *IEEE Transactions on Automatic Control*, **AC-39**:167–171, 1994.

[36] M. Padberg. *Linear Optimization and Extensions*. Springer, Berlin, 2. edition, 1999.

[37] A. Polanski. Lyapunov function construction by linear programming. *IEEE Trans. Autom. Control*, **42**:1013–1016, 1997.

[38] W. Press, S. Teukolsky, W. Vetterling, and B. Flannery. *Numerical Recipes in C*. Cambridge University Press, Cambridge, 2. edition, 1992.

[39] R. Rockafellar and R. Wets. *Variational Analysis*. Springer, Berlin, 1998.

[40] S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control*. Springer, New York, 1999.

[41] R. Schaback and H. Werner. *Numerische Mathematik*. Springer, Berlin, 4. edition, 1992.

[42] T. Scheel. *Integrale Ljapunov-Funktionen zur rechnergestützten Stabilitätsanalyse nichtlinearer zeitdiskreter Regelungssysteme*. Ph.D. Thesis: VDI Verlag, Düsseldorf, 1997.

[43] T. Scheel and H. Kiendl. Stability analysis of fuzzy and other nonlinear systems using integral Lyapunov functions. In *Proceedings of EUFIT'95*, pages 765–770, Aachen, 1995. Augustinus Buchhandlung.

[44] J.-J. Slotline and W. Li. *Applied Nonlinear Control*. Prentice-Hall, New Jersey, 1991.

[45] M. Sugeno and G. Kang. Fuzzy modeling and control of multilayer incinerator. *Fuzzy Sets and Systems*, **18**:329–346, 1986.

[46] M. Sugeno and G. Kang. Structure identification of fuzzy model. *Fuzzy Sets and Systems*, **28**:15–33, 1988.

[47] M. Sugeno and M. Nishida. Fuzzy control of model car. *Fuzzy Sets and Systems*, **16**:103–113, 1985.

[48] M. Sugeno and T. Takagi. Fuzzy identification of systems and its applications to modeling and control. *IEEE Trans. Syst. Man and Cybern.*, **15**:116–132, 1985.

[49] M. Sugeno and K. Takana. Successive identification of a fuzzy model and its applications to prediction of a complex system. *Fuzzy Sets and Systems*, **42**:315–334, 1991.

[50] M. Vidyasagar. *Nonlinear System Analysis*. Prentice Hall, Englewood Cliffs, 2. edition, 1993.

[51] W. Walter. *Analysis I*. Springer, Berlin, 2. edition, 1990.

[52] W. Walter. *Gewöhnliche Differentialgleichungen*. Springer, Berlin, 5. edition, 1993.

[53] W. Walter. *Einführung in die Theorie der Distributionen*. BI-Wiss.-Verl., Mannheim, 3. edition, 1994.

[54] R. Yager and D. Filev. *Essentials of Fuzzy Modeling and Control*. John Wiley & Sons, New York, 1994.

[55] J. Zabczyk. *Mathematical Control Theory: An Introduction*. Birkhäuser, Boston, 1992.